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GENERAL FORM OF THE STRING EFFECTIVE ACTION IN FOUR DIMENSIONS

by

David Aubin

Physics Department, McGill University

3600 University Street, Montréal

Québec, Canada, H3A 2T8.

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Abstract

General Form of the String Effective Action in Four Dimensions: We study the effective action that governs heterotic string theory compactified on a four-dimensional, $N = 1$ supersymmetric background. A scaling symmetry of the four-dimensional field T that represents the breathing mode on of internal manifold is introduced. This symmetry, being valid at each order in both perturbative series of the theory (string-loop and sigma model expansions), is used to find restrictions on the terms that can correct the truncated supersymmetric functions entering the effective action. We derive the general form of these corrections and show that, contrary to earlier conjectures in the literature, loops are not *a priori* counted by the ratio T/S, in which T and S represent the standard four-dimensional breathing mode and dilaton fields. Some other symmetry considerations allow us to restrict even more the corrections to the truncated results. Our conclusions are extended to the cases with matter fields, and more than one moduli for the internal manifold.

Some regularization techniques for problems related to those encountered in the quantization of string theories are also presented in an appendix.

Résumé

Forme générale de l'action effective des cordes en quatre dimensions

Nous étudions l'action effective, $N = 1$-supersymétrique, qui gouverne les cordes hétérotiques compactifiées dans un espace-temps à quatre dimensions. Nous introduisons une symétrie de changement d'échelle pour le champs T, qui est défini sur l'espace à quatre dimensions et qui représente le rayon typique de la variété interne. Cette symétrie, étant valide pour tous les ordres perturbatifs des deux expansions de la théorie (expansion des boucles de corde et du modèle sigma), nous permet de poser des restrictions sur les termes qui sont susceptibles de corriger les functions supersymétriques tronquées se retrouvant dans l'action effective. Nous identifions la forme générale que ces corrections peuvent prendre et montrons que, contrairement à ce qui a été conjecturé auparavant, les boucles ne sont pas *a priori* comptées par le ratio T/S, où T et S représentent les champs quadri-dimensionnels décrivant respectivement le rayon typique de la variété interne et le dilaton. D'autres symétries nous permettent de restreindre encore plus la forme des termes correctifs. Nos conclusions sont appliquées au cas avec des champs de matière et avec plus d'un module.

Des techniques de régularisation s'appliquant à des problemes rencontrés dans la quantification des cordes sont aussi développées en appendice.

En guise de préambule[1]

L'apparance des bateaux qui arrivent à bon port après un long périple, ne raconte que quelques bribes des multiples embûches, tempêtes et jours sans vent, mutineries et maladies, qu'il aura fallu surmonter pour triompher aujourd'hui. La foule qui s'amasse le long des quais sous les fanions et les drapeaux, assourdie par les coups de canons, acclame les braves matelots. La foule n'est pas si bête, et ses cris de joie sont le reflet de son admiration pour ces gens qui ont souffert elle le sait.

J'achève aussi mon périple. Et, comme pour les navires, mon gréement ne semble pas trop magané. Pourtant, ce mémoire représente deux annés de labeur qui a laissé ses marques. Regardez bien! Vous verrez une écorchure zébrant le grand-mat. Approchez-vous et voyez combien il manque de poulies et de cordages, admirez ces planches cassées et ces voiles rapiècées! Ne frissonnez-vous point en découvrant ce grand trou noir à même la coque?

Aussi ardu que puisse être ces voyages, les matelots se retrouvent toujours la semaine ou le mois suivant prêts à reprendre la mer. Pourquoi donc? Pour l' amour de Thalassa la Belle, la Grande, tout simplement.

Jours sombres et nuits de découvertes éblouissante, frustrations et joies en-

[1] If you don't read French, I'm very sad for you! However, what's contained in this little non-formal introduction isn't crucial for the understanding of the thesis. In any case, I'd like to thank here my supervisor who's susceptible to be unable to understand the following. I am very grateful for all the knowledge he made me acquire and for the fantastic experience of working with him. *Thanx* Cliff!

nivrantes, ont rempli une partie de ces deux années. Je saisis la présente occasion pour avouer que, comme les marins, j'ai souffert. Mais, pareil à eux, je replongerai très bientôt par amour pour cette quête infinie et combien passionnante.

Je veux dédier ce travail à ma nouvelle épouse Corinne Le Quéré qui, par son amour et par la merveilleuse preuve qu'elle m'en donne aujourd'hui, m'a soutenue tout le long de ces deux ans.

À mon père Gilles, qui m'enseigna, alors je n'avais pas encore cinq ans, la différence entre masse et poids, et qui est donc le premier responsable de la passion de tout connaître qui me dévore.

À ma mère Louise Lavoie, dont je suis redevable pour (presque) tout le reste qui m'offre un merveilleux exemple de force morale et de paix intérieure.

À Clifford Burgess, mon directeur de thèse,

À tous mes amis et collègues... Merci! [2]

"Il y a un spectacle plus grand que la mer: c'est le ciel;
Il y a un spectacle plus grand que le ciel: c'est l'intérieur de l'âme."

Victor Hugo, *Les misérables*.

[2] Cette recherche a été rendue possible grâce à une Bourse en Science et Génie 1967 du Conseil de Recherche en Sciences Naturelles et Génie (C.R.S.N.G.) du Canada. Je lui suis grandement reconnaissant de l'aide inestimable ainsi accordée.
Je remercie de même l'auteur de la référence [1], ouvrage de base qui m'a été essentiel tout au long de mon écriture.

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Chapter 1

Introduction

Superstring theory [2, 3, 4] has emerged as the most promising candidate to unify into one theoretical framework the two fundamental theories of modern physics: quantum field theory and general relativity. This would solve an outstanding problem that has been puzzling the physics community for more than fifty years. From the phenomenological point of view, the theory of heterotic strings [5], which include explicit Yang-Mills fields, is the theory that seems the closer to a complete description of all the known interactions.

The most pressing problem facing heterotic string theory is the question of breaking a ten-dimensional theory down to a realistic four-dimensional theory Until such a dimensional reduction can be made, the theory lacks any real contact with physically measurable quantities. Unfortunately, a completely satisfactory mechanism for spontaneous dimensional breaking to a four dimensional theory is yet to be found.

The best that can be done in the present knowledge, is to study perturbatively various classical vacua of the theory and see whether reasonable phenomenology can be extracted from them. With relatively mild assumptions on compactification (*e.g.* preservation of an $N = 1$ supersymmetry [6]), it is possible to obtain reasonable phenomenological predictions. But the number of such theories is huge: many thousands of Calabi-Yau spaces are possible vacua!

The effective Lagrangian contains massless and very massive fields. The low-energy four-dimensional Lagrangian should not be sensitive directly to the massive fields: they must be integrated out. This integration is unfortunately very difficult to perform rigorously. However, if we set the massive fields to zero in the ten-dimensional effective action [7, 8, 9], then we can obtain an effective four-dimensional theory that approximates compactification on a Calabi-Yau space in a quite satisfactory manner. In this work, it is made very explicit how this approximation is in fact quite precise.

As one would expect, this effective theory is expressible by a $D = 4$, $N = 1$ supergravity action. The approach of this paper is to find the general form that corrections to this action can take, using symmetry considerations. The work done in reference [9] indicates some symmetries that restrict the form of the corrections at lowest-order in the string-loop expansion and in the α'-expansion.

We identify a symmetry of the full ten-dimensional supergravity action that is valid at any finite order in both perturbative series. We then exploit it to specify the general form that is allowed for terms entering the effective action governing heterotic strings in a four-dimensional, $N = 1$ supersymmetric background. The study of this symmetry has some important consequences for the effective theory.

In particular, it has been conjectured [10] that the two expansions of string theory were counted only by one parameter S/T, where S and T represent, respectively, the dilaton and the scalar field associated with the first breathing mode of the internal manifold. This conjecture is shown to be false by of our more general analysis.

To the best of our knowledge, this is the first time that this symmetry Has been presented in this context, and that its implication for the four-dimensional string action are discussed.

The rest of this paper is divided as follows: chapter **2** presents a review of bosonic and heterotic string theory, including massless fields defined on the D-dimensional background spacetime ($D = 26$ or 10). We present a derivation of the ten-dimensional effective action for heterotic strings, following reference [11]. Our original work is presented in chapter **3**. In the first section, a non-renormalization theorem for the effective theory in ten dimensions is presented. Focussing our attention on the four-dimensional theory, we first give the truncated Lagrangian [7, 8, 9], and then present our symmetry. The rest of the chapter is devoted to a discussion of its implications on the form that can take the corrections to the truncated action. We also include other symmetry considerations, in order to be as complete as possible. The conclusions are summarized in chapter **4**.

You will also find an appendix at the end of the paper. The work presented in it is not directly related to the main body of the thesis. Nevertheless, it is included because it contains solutions to some regularization problems that arise in string theory. Some of these results are not restricted to two-dimensional theories and are valid in general.

Chapter 2

String Theory with Background Fields

In order to achieve the goal of unification of all the phenomena of nature into the structure of string theory, we must know how to include in the theory a description of the behavior of a string in a curved spacetime. For example, spacetime in string theories must always have a large number of dimensions (10 and 26, for the theories of, respectively, superstrings and bosonic strings). In order to recover the only four macroscopically observable dimensions of spacetime, we need to compactify the others. Typically such a compactification involves backgrounds with non-trivial curvatures. This example illustrates the importance of the understanding of string theory in the presence of background fields.

This chapter introduces string theory and the massless string fields whose background values are important at low energies. We first describe bosonic string theory in a flat background and its quantization. We then treat bosonic strings propagating in a curved spacetime, and describe the derivation of its effective action in the critical spacetime dimension D. The last section of this chapter repeats the discussion for the superstring. We choose to work in the framework of heterotic string theory, since this is the theory which seems to have the most promising phenomenology at low-energy.

2.1 Introduction to String Theory

2.1.1 Basic Ingredients

Our approach to string theory is a first-quantized path-integral formalism, *i.e.* a quantum theory for one-dimensional extended objects (for an introduction, see references [2, 3, 4]). This has the advantage of making manifest many of the symmetries of the theory, as was first introduced for strings by Polyakov in the early 1980s [12].

The ingredients that are used in this functional framework are the world-sheet, the target space, a mapping $X^M(\sigma)$ between these two spaces and a string action. We adopt the convention to always work with manifolds whose metric has Euclidean signature. As is usually the case for quantum field theory (first-quantized string theory can be considered as a two-dimensional QFT, with the worldsheet as a base space), the physical conclusions may be obtained by continuing to Minkowski signature at the end of a calculation. In this paper, the explicit calculations of appendix A are done in Euclidean space, but the rest of the discussion is not sensitive to the signature of the metric.

The above ingredients are defined as follows:

- The physical meaning of the *worldsheet* is best expressed with Minkowskian convention; the worldsheet is then the two-dimensional manifold, denoted by M, that is swept out by the string as it evolves in time.

  For our purpose, the study of the *partition function*, *i.e.* vacuum-to-vacuum amplitudes, suffices. According to our convention to work with metric with Euclidean signature, the manifold is, in this case, compact and orientable, since we are always concerned about closed and oriented strings. We introduce the metric $h_{\alpha\beta}(\sigma)$ to describe it with respect to the coordinates (σ^1, σ^2).

- The ***target space*** (also called the ***background space***) is the ordinary real spacetime through which the string propagates. In bosonic string theory, the target spacetime seems to necessarily have $D = 26$ dimensions. This is imposed by the necessity to preserve ***Weyl invariance*** on the worldsheet even after quantization. Weyl invariance is a scaling property of the action about which more will be said later (see equation (2.4)). For heterotic strings (see section **2.3**), the analogous ***critical dimension*** of the target space is $D = 10$.

- We represent the embedding of the worldsheet into the D-dimensional target space by a set of D functions called $x^M(\sigma)$. We use capital letters taken from the middle of the Latin alphabet (*e.g.* M, N, P) to indicate the indices of the D-dimensional tangent space. These functions defined on the worldsheet are the coordinates of the points of the worldsheet in spacetime. In the case of heterotic strings (see section **2.3**), there exist fermionic two-dimensional fields also: left-handed superpartners $\lambda^M(\sigma)$ of target space coordinates $x^M(\sigma)$, and right-handed spinors $\psi^s(\sigma)$ that couple to the Yang-Mills background field.

- The ***string action*** $I[x, h]$ is a functional of the two-dimensional fields, like $x^M(\sigma)$ and $h_{\alpha\beta}(\sigma)$, defined by an integration over the worldsheet. It enters in the theory for the computations of amplitudes, or the S-matrix (see below equation (2.1)).

2.1.2 Bosonic Strings in Flat Target Space

Polyakov's idea for string quantization is to consider the metric of the worldsheet $h_{\alpha\beta}(\sigma)$, as well as the other string field $x^M(\sigma)$, as dynamical variables of a two-dimensional field theory [12]. Within the framework of path-integral quantization, every amplitude is expressed as a functional integral. The S-matrix elements for n scattering (string) particles, with momenta p_i^M in D-space and quantum numbers j_i, are given by (this more modern point of view than Polyakov's is presented in [13, 14, 15]):

$$S_{\mathrm{M}}(p_1, j_1; \ldots; p_n, j_n) = \sum_{\gamma=0}^{\infty} C_\gamma \int \frac{[dh][dx]}{\Omega} \exp(-I[x,h]) \times V_1[x,h;p_1,j_1] \ldots V_n[x,h:p_n,j_n]. \quad (2.1)$$

Here the functional integration over the worldsheet metric $\int [dh]/\Omega$ can be gauged away and replaced by a Jacobian and an integration over a finite number of ***moduli*** [16]. The definition of the functional measure $[dx]$ is discussed in the appendix. $I[x,h]$ is the bosonic string action as will be described below (see equation (2.5)); and $V_k[x,h;p,j]$ is the ***vertex function***, a functional of $x^M(\sigma)$ and $h_{\alpha\beta}(\sigma)$ that also depends on p^M and j. There is a vertex function associated with every scattering string state (see references [2, 14]).

The infinite sum in equation (2.1) is a sum over the genii, γ, of the different worldsheets. The constants C_γ are proportional to $\lambda^{\chi(\mathrm{M})}$, where λ is the string coupling constant, and $\chi(\mathrm{M})$ is the Euler characteristic of the surface M. This latter quantity is related to the genus γ of the surface by the relation $\chi(\mathrm{M}) = 2 - 2\gamma$.

The string coupling constant λ that appears above and the Regge slope α', that is presented below, are the two parameters of string theory. We consider that the string coupling constant, which is dimensionless, is small enough so that the perturbation series in powers of λ will give asymptotical value for the S-matrix. This expansion is the ***string loop expansion*** for which each order in labelled by the genus γ of the corresponding worldsheet.

In the same way, we expand the S-matrix in powers of α'. This expansion is not really a perturbation series in α' alone, since this parameter is dimensionful. Rather it is based on the assumption that the parameter $\alpha'^{-1/2}$ is large compared to the other dimensionful parameters entering the theory, such as the momenta p_i^M. For heterotic string theory in curved space, we form small dimensionless parameters using α' together with r_4^2 and r_6^2, the typical radii of, respectively, our four-dimensional spacetime and the six-dimensional internal manifold of compactified dimensions.

For bosonic string theory in a flat target space (*i.e.* with metric δ_{MN}), the string action $I_{\rm flat}[x,h]$ is completely determined by three symmetries that are assumed for the theory [13, 14]: D-dimensional Poincaré invariance, worldsheet general covariance and Weyl invariance.

- Poincaré transformation of the coordinates x^M is defined as:

$$x^M \to \Lambda^M_N x^N + a^M, \tag{2.2}$$

  with Λ^M_N an arbitrary O(D) matrix (Euclidean space).

- Worldsheet general covariance is just invariance under a general two-dimensional reparameterization f^α : M $\to$ M, with:

$$\begin{aligned} x^M &\to f^* x^M(\sigma) = x^M(f(\sigma)), \\ h^{\alpha\beta} &\to f^* h^{\alpha\beta}(\sigma) = \frac{\partial f^\alpha}{\partial \sigma^\gamma}\frac{\partial f^\delta}{\partial \sigma^\delta} h^{\gamma\delta}(f(\sigma)). \end{aligned} \tag{2.3}$$

- Finally, Weyl invariance is a symmetry of the action which ensures that the theory is not dependent on the metric of the worldsheet. Given a fixed scalar function $\phi(\sigma)$ on the worldsheet, a Weyl transformation is defined to be the following:

$$h_{\alpha\beta}(\sigma) \to \exp[\phi(\sigma)] h_{\alpha\beta}(\sigma), \tag{2.4}$$

  and leaving the embedding $x^M(\sigma)$ unchanged.

These three symmetries completely fix the flat bosonic string action to be given by the following functional:

$$I_{\rm flat}[x,h] = \frac{T}{2}\int d^2\sigma \sqrt{h} h^{\alpha\beta}\frac{\partial x^M}{\partial \sigma^\alpha}\frac{\partial x^N}{\partial \sigma^\beta}\delta_{MN}, \tag{2.5}$$

where $h(\sigma)$ is the determinant $h = \det(h_{\alpha\beta})$, and T is the *string tension*, which can be identified with the classical tension of the string. The Regge slope, mentioned earlier is related to T by $\alpha' = 1/2\pi T$. In the present discussion, we assume that the worldsheet coordinates $\{\sigma^\alpha\}$ are dimensionless and that the D-dimensional spacetime coordinates $x^M(\sigma)$ carry dimensions of *Length*. In order for the string action to be dimensionless, the string tension must have dimensions of $(Mass)^2$. The Regge slope α' carries therefore dimensions of $(Length)^2$. In order to correctly reproduce Newton's constant for the gravitational action, we choose $\alpha'^{-1/2}$ to be of the order of the Planck mass, $M_\text{P} \sim 10^{19}$ GeV.

In the framework of string theory, the quantization of this action can be understood as two-dimensional quantum field theory. The embedding $x^M(\sigma)$ is, in this point of view, a set of D operators, or quantum fields, on the worldsheet. They are scalars in the two-dimensional space, though $x^M(\sigma)$ is still a vector with respect to the real D-dimensional background space.

Let us now turn to a description of bosonic string theory which incorporates the effects of massless background fields, such as gravity, for example, but also other fields.

2.2 Bosonic Strings with Background Fields

2.2.1 Non-Linear Sigma Model

Background fields are D-dimensional quantum fields that are produced by string field theory. In our first-quantized language, they appear as external fields in our action. However, it is possible to compute the mass spectrum corresponding to these fields (see reference [2]). Perturbing around flat spacetime, we get full towers of different states having very large masses (of the order of the Planck mass M_P) and a few massless states, as well as a *tachyon*, *i.e.* a particle of imaginary mass. These masses are background dependent but, for slowly-varying fields, the masses of the 'massless' fields remain very small compared to

M_P, and the 'tachyon' remains a tachyon. Throughout this paper, we therefore call the light background fields, massless fields.

When we allow our theory to have background fields, the string action that we must consider is a bit more complicated than that of equation (2.5). The terms involving heavy fields turn out to be nonrenormalizable as two-dimensional quantum field theory, while the term involving massless background fields and the tachyon are renormalizable. The conditions that are imposed on the string action become renormalizability and the worldsheet symmetries that we had before: invariance under a general coordinate transformation and Weyl invariance. We can no longer require the D-dimensional Poincaré invariance since one of the background fields is the curved metric of the D-dimensional target spacetime. General covariance for the D-dimensional target space is not required, but it comes as a consequence of the other conditions.

The most general classical bosonic string action satisfying these criteria (renormalizability, general covariance on the worldsheet and Weyl invariance) is:

$$I[G,B] = \frac{1}{4\pi\alpha'} \int d^2\sigma \left[\sqrt{h}h^{\alpha\beta}G_{MN}(x) + \epsilon^{\alpha\beta}B_{MN}(x)\right] \partial_\alpha x^M \partial_\beta x^N. \quad (2.6)$$

Here, the Levi-Cività tensor $\epsilon^{\alpha\beta}(\sigma)$ is a tensor density (not a covariant tensor), so we must be aware that while ϵ^{12} takes the value 1, the tensor density $\epsilon_{\alpha\beta}$ is proportional to $h = \det(h_{\alpha\beta})$. Actions of the form (2.6) are called non-linear sigma models and were originally used to describe the coupling of mesons. Many of the techniques that have been developed for those theories may be applied here [17]. The limit $\alpha' \to 0$ is the semi-classical limit for the σ-model; for that reason, the α'-expansion, in the context of string theory, is often called the sigma-model expansion.

In the string action (2.6), the fields $G_{MN}(x)$ and $B_{MN}(x)$ are massless fields of the background spacetime. They are, respectively, the spacetime metric and the antisymmetric tensor field. As we shall see using the effective action, they correspond to the usual D-dimensional quantum fields, up to some scaling.

Surprisingly enough, this is not the whole story. The action (2.6) has an anomaly associated with Weyl symmetry, and we add a local term to the action in such a way that the anomaly is cancelled [18] (this is still not the whole story, we will study Weyl invariance further in the next section and see how it relates to the equations of motion of the background fields). Two more background fields are introduced through a string action that is *not* classically Weyl invariant, but still renormalizable and generally covariant on the worldsheet. They are the ***tachyon*** $\Psi(x)$ and the ***dilaton*** $\Phi(x)$. They appear through the following string action:

$$I[\Psi,\Phi] = \frac{1}{4\pi}\int d^2\sigma\sqrt{h}\,[\Psi(x) + R_h\Phi(x)]\,, \tag{2.7}$$

where $R_h(\sigma)$ is the 2-dimensional Ricci scalar of the worldsheet manifold, which is related to its Gaussian curvature k by the relation $R_h = 2k$. Both of the above fields are D-dimensional scalars. We will not discuss any longer the tachyon term, since it does not appear in heterotic string theory. It will simply be omitted, even in our bosonic string discussion.

An important feature of the action (2.7) is that it does not have the factor α'^{-1} as the other term (2.6) does. The reason for this is that it is a renormalization term that must cancel 'quantum' effects (*i.e.* of order α') of the action $I[G,B]$.

Furthermore, if the dilaton were shifted by a constant, a, then the action (2.7) acquires an additional term:

$$\frac{a}{4\pi}\int d^2\sigma\sqrt{h}\,R_h = a\chi(M), \tag{2.8}$$

where we used the Gauß-Bonnet theorem to relate the worldsheet curvature to its Euler characteristic. Putting this action into the expression (2.1) for the S-matrix, we see that the string coupling constant may be absorbed into the vacuum expectation value (v.e.v.) of the dilaton field. This can be expressed as:

$$\lambda = \exp < \Phi(x) > . \tag{2.9}$$

2.2.2 *D*-Dimensional Effective Action

The effective action is the integration over D-dimensional space of a Lagrangian density involving only the background fields ($G_{MN}(x)$, $B_{MN}(x)$ and $\Phi(x)$) and *not* the string coordinates. The effective action is meant to produce the same equations of motion and scattering amplitudes as the direct string calculations. We will study later a four-dimensional effective action, *i.e.* in the same way, involving only background fields defined on the physical four-dimensional spacetime.

However, it is not possible to compute the full effective action: the best that can be done is a calculation order by order in α' [19, 20]. This provides an expression for the D-dimensional effective action at each order. However, this procedure involves quite lengthy calculations, and another equivalent method based on beta functions and Weyl invariance was developed [11]. The two methods are shown to be equivalent in references [21].

Let us present the beta function method in the following. In ordinary quantum field theory, when we have a theory with n coupling constants, we have n beta functions, one for each coupling constant, giving the dependence of the coupling as some arbitrary scale, introduced by regularization, is varied. For instance, the $\lambda\phi^4$ theory, with dimensional regularization gives (as presented by Ramond in reference [22]):

$$\beta(\lambda) = \lim_{\epsilon\to 0} \mu \frac{\partial\lambda}{\partial\mu}, \tag{2.10}$$

where λ is the coupling constant, μ the arbitrary scale and ϵ the dimensional regularization parameter, *i.e.* this is computed in $4-2\epsilon$ dimensions.

Analogously, our action contains, not a finite number of coupling constants, but a *coupling function* $G_{MN}(x)$. Thus, we get a beta functional $\beta[G_{MN}(x)] \equiv \beta^{G}_{MN}(x)$. Other beta functions are also present for the other coupling functions of our non-linear σ-model, namely $\beta^{B}_{MN}(x)$ and $\beta^{\Phi}(x)$. In the above analogy, the arbitrary scale μ corresponds to the parameter for Weyl transformations.

To have a theory invariant under Weyl scalings, it is sufficient to ask that the beta functionals vanish [23].

The β-functionals are computed at leading-order in α' in [11]; they are given by:

$$\begin{aligned}
\beta^{G}_{MN} &= R_{MN} - \frac{1}{4}H_{M}{}^{PQ}H_{NPQ} + 2\nabla_M\nabla_N\Phi + o(\alpha'), \\
\beta^{B}_{MN} &= \nabla_P H^{P}{}_{MN} - 2\nabla_P\Phi H^{P}{}_{MN} + o(\alpha'), \\
\beta^{\Phi} &= \frac{D-26}{48\pi^2} + \frac{\alpha'}{16\pi^2}\left\{4(\nabla_M\Phi\nabla^M\Phi) - 4(\Box\Phi) - R + \frac{1}{12}H^2\right\} + o(\alpha'^2),
\end{aligned} \tag{2.11}$$

where

$$H_{MNP}(x) = \partial_M B_{NP} + \partial_P B_{MN} + \partial_N B_{PM}, \tag{2.12}$$

and $H^2 = H_{MNP}H^{MNP}$. Also, the symbol ∇_M represents the covariant derivative (using Christoffel symbols $\Gamma^{P}_{MN}(x)$ obtained from the metric $G_{MN}(x)$). We also have $\Box\Phi = \nabla^M\nabla_M\Phi$. R_{MN} and R are, respectively, the Ricci curvature tensor and the Riemann scalar curvature of D-dimensional spacetime.

It is important to note that the first non-trivial contribution of $\beta^{\Phi}(x)$ when the spacetime dimension is restricted to its critical value $D = 26$ is not of the same order as the corresponding terms in other beta functions. As discussed in [11], this comes from the fact that the coefficient of the action (2.7), involving the dilaton, is one order higher in α' than $I[G, B]$ (equation (2.6)).

The conditions for Weyl invariance of the theory then turns out to be: $\beta^{G}_{MN} = \beta^{B}_{MN} = \beta^{\Phi} = 0$, providing the equations of motion for the massless fields of the theory. These equations may be interpreted as Euler-Lagrange equations of the following 26-dimensional effective action:

$$S_{\text{eff}} = \frac{1}{2\kappa^2}\int d^Dx\sqrt{G}e^{-2\Phi}\left(R + 4(\nabla_M\Phi)^2 - \frac{1}{12}H^2\right), \tag{2.13}$$

where $G(x)$ is the determinant $\det(G_{MN})$ and κ is Newton's constant for gravity in D dimensions. Since string theory contains only one single dimensionful parameter, we can relate κ and α'. In reference [14], it is shown that the gravitational coupling must be proportional to the string coupling constant λ.

Dimensional analysis implies then that $\kappa = c\lambda\alpha'^{(D-2)/4}$, for some constant of proportionality c.

The non-linear Φ-dependence of the action (2.13) makes explicit the fact that $e^{2\Phi}$ is the string-loop expansion parameter and that this effective action is an approximation coming uniquely from the sphere-term of the genus expansion. The n-loop contribution is computed on a worldsheet with genus $\gamma = n$ and is proportional to $e^{-\Phi(x)\chi(M)} = e^{2(n-1)\Phi}$, as can be seen from equations (2.7) and (2.8).

It is important to emphasize that in the sigma-model variables, *i.e.* the very ones we have been using up to now, the expansion parameters are then uniquely given by the dilaton and the Regge slope α' (more correctly dimensionless quantities involving α' and, for example, r_6, the typical radius of the compactified manifold). The other fields give no indications of the order in the perturbative expansions.

We can however find an Hilbert-like action from (2.13). Let us perform the following conformal scaling of $G_{MN}(x)$ to its canonical form:

$$G^{\rm can}_{MN}(x) = \varphi^{-1}(x)\, G_{MN}(x), \tag{2.14}$$

where, we changed variables for the dilaton field:

$$\varphi = \exp[4\Phi/(D-2)], \tag{2.15}$$

each string loop term being down by a factor of φ^{-4} with respect to the preceding one. This new canonical metric also counts, in part, string loops. We then get the following equivalent effective action in term of the canonical metric [11]:

$$S_{\rm eff} = \frac{1}{2\kappa^2}\int d^Dx\sqrt{G}\left(R - \frac{4}{D-2}(\nabla_M\Phi)^2 - \frac{1}{12}e^{-8\Phi/(D-2)}H^2\right), \tag{2.16}$$

where R is the scalar curvature constructed from the new canonical metric; all contraction are also performed using it. This action is essentially the bosonic part of an action involving a graviton, a dilaton field and an antisymmetric tensor field as was derived with the techniques of supergravity by Chamseddine, Chapline and Manton (see references [25] and equation (3.1)). So, using string theory at the lowest order in the α'-expansion and in the string-loop expansion, we recover the principal features of the bosonic part of supergravity

Notice that, in these variables, it is still possible to see that this action is a contribution of the sphere: the quantities R_{MN}, $\nabla_M\Phi\nabla_N\Phi$ and $H_{MPQ}H_N{}^{PQ}$ are not scaled by the transformation (2.14), while in canonical variables, the factor $\sqrt{G}G^{MN} \sim \varphi^{-4}$ asymptotically as $\varphi \to 0$. Using the relations (2.9) and (2.15) between the string coupling constant and the dilaton, this shows that the Einstein term is a sphere contribution to the effective action.

It is possible to find string-theoretic corrections to the action (2.16), with higher powers of α'. This is done in references [19]. As an illustration of this, let us state the beta functional $\beta^G_{MN}(x)$ at the next order in the α'-expansion, when we set the other background fields to zero:

$$\beta_{MN}(x) = R_{MN} - \frac{\alpha'}{2} R_{MPQR}R_N{}^{PQR}. \tag{2.17}$$

This implies a 'stringy' correction to general relativity. But if the typical dimension of $R_{MPQR}(x)$ becomes very small compared with the string tension T, then this correction is negligible. In other words, this correction is not important when the typical radius of spacetime is a lot larger than the Regge slope $\alpha'^{1/2}$. This provides us with an upper limit on the value that the parameter α' can take with respect to the other dimensionful quantities of the theory. The effective Lagrangian, for which (2.17) is an Euler-Lagrange equation, thus acquires an additional stringy term proportional to $\alpha' R_{MNPQ}R^{MNPQ}$, compared with the previous effective action (2.16).

Bosonic string theory could possibly explain very well a world of bosons, but in order to include fermions, the most common matter particles of the universe, we must enlarge the theory. The usual way to do this uses the very properties of supersymmetry, providing in this way, as an additional advantage, theories without tachyons [2]. This gives rise to two new groups of theories: superstring and heterotic string theories. We shall treat only the latter.

2.3 Heterotic Strings with Background Fields

2.3.1 Flat Target Space

Supersymmetry is a symmetry in the action (here, the 2-dimensional one) which relates bosons to fermions. Essentially, this is done by the addition to the string coordinates $x^M(\sigma)$ of superpartners $\lambda^M(\sigma)$ which are Grassmann variables.

The properties of heterotic strings are more easily expressible for a space with Minkowskian signature, we will therefore express them in this framework. Heterotic string theory (see references [5]) has an interesting $N = \frac{1}{2}$ Majorana-Weyl worldsheet supersymmetry: meaning that only the left-moving bosons have fermionic superpartners and the supersymmetry of opposite chirality is absent. Right-moving fermions do exist, but they are used to form a chiral $E_8 \times E_8$ or $\mathrm{Spin}(32)/\mathbf{Z}_2$ current algebra. By left- or right-moving, we mean that the fields depend only on one of the combinations $\tau \pm \sigma$, where (τ, σ) parameterize the Minkowskian worldsheet. For fermions, it turns out that the Dirac equation implies that they are left- or right-handed Majorana-Weyl spinors, respectively.

The worldsheet action for heterotic string in a flat target space is written in the covariant language as [5] (we do not write the very complicated coupling of the gravitino with the right-handed fermions ψ^s; the gravitino will be gauged away very soon and we introduce it here only for the completeness of the argument):

$$\begin{aligned} I_{\mathrm{het}} &= \frac{1}{4\pi\alpha'} \int d^2\sigma \sqrt{h} \left[h^{\alpha\beta} \partial_\alpha x^M \partial_\beta x^M + i\bar{\lambda}^M \gamma^\alpha \partial_\alpha \lambda^M + i\bar{\psi}^s \gamma^\alpha \partial_\alpha \psi^s \right. \\ & \qquad \left. + i(\bar{\chi}_\alpha \gamma^\alpha \gamma^\beta \lambda^M) \partial_\beta x^M + \text{gravitino terms} \right]. \end{aligned} \tag{2.18}$$

The field content of this equation, $x^M(\sigma)$, $\lambda^M(\sigma)$, $\psi^s(\sigma)$, $h_{\alpha\beta}(\sigma)$ and $\chi_\alpha(\sigma)$, is explained below.

We define the two-dimensional gamma matrices in flat Minkowskian 2-space, as in four dimensions, by the requirement that they obey the Clifford algebra[1]:

$$\{\gamma^a, \gamma^b\} = 2\eta^{ab}, \tag{2.19}$$

where $\eta^{ab} = \mathrm{diag}(1, -1)$.

For a general curved worldsheet, the procedure is the same as in four-dimensional curved spacetime (see [26], for example). We define a *zweibein* $e^a_\alpha(\sigma)$, a non-coordinate basis of the tangent space (the latin index refers to the Euclidean tangent space), which has the following properties of completeness and orthonormality:

$$\begin{aligned} (e_a)^\alpha (e_b)^\beta \eta^{ab} &= h^{\alpha\beta}, \\ (e_a)^\alpha (e_b)^\beta h_{\alpha\beta} &= \eta_{ab}. \end{aligned} \tag{2.20}$$

The γ-matrices in curved space are then defined by:

$$\gamma^\alpha(\sigma) = (e_a)^\alpha \gamma^a. \tag{2.21}$$

They are, as indicated, dependent on their position on the worldsheet

The x^M's of equation (2.18) are, as before, the embedding of the worldsheet in the D-dimensional target spacetime, although here, anomaly cancellation for Weyl transformation obliges us to work in critical dimension $D = 10$, *i.e.* $M = 0, 1, \ldots, 9$.

The λ^M's of equation (2.18) are two-component left-handed Majorana-Weyl spinors. With a Minkowskian worldsheet, we may choose to work in the Majorana representation for the two dimensional γ-matrices:

$$\gamma^0 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \qquad \gamma^1 = \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}, \tag{2.22}$$

and define

$$\gamma_P = \gamma^0\gamma^1 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}. \tag{2.23}$$

[1] **The corresponding gamma matrices for a Euclidean two-dimensional space, denoted γ^α, will be defined and used thoroughly in appendix A.**

Of course, those Dirac γ-matrices satisfy the Minkowski anticommutation rule $\{\gamma^a, \gamma^b\} = 2\eta^{ab}$. In this representation, the Weyl and the Majorana conditions for the left-handed spinor λ^M read, respectively, as follows:

$$(1 - \gamma_P)\lambda^M = 0, \tag{2.24}$$

$$(\lambda^M)^* = \lambda^M. \tag{2.25}$$

The ψ^s's are right-handed Majorana-Weyl fermions where $s = 1, \ldots, 32$. The linearly-realized sigma-model gauge group is either SO(32) for heterotic strings having the gauge group Spin(32)/$\mathbf{Z}_2$, or a SO(16)×SO(16) subgroup for heterotic strings with gauge group $\mathrm{E}_8 \times \mathrm{E}_8$, depending on the boundary conditions imposed on the spinor field. $\chi_\alpha(\sigma)$ in equation (2.18) represents a two-dimensional spin-$\frac{3}{2}$ field; this is what we call in four-dimensional field theory the *gravitino*, since it is the superpartner of the zweibein $e^a_\alpha(\sigma)$.

The theory of heterotic strings has the supersymmetry ('$N = \frac{1}{2}$'):

$$\begin{aligned}
\delta x^M &= \bar{\varepsilon}\lambda^M, \\
\delta\lambda^M &= i\gamma^\alpha[\partial_\alpha x^M + \chi_\alpha\lambda^M]\varepsilon, \\
\delta\chi_\alpha &= -2\nabla_\alpha\varepsilon, \\
\delta e^a_\alpha &= i\bar{\varepsilon}\gamma^a\chi_\alpha.
\end{aligned} \tag{2.26}$$

ε is the parameter of the symmetry, which has the properties of a Majorana-Weyl spinor.

The complete heterotic string action (not the equation (2.18)) also has a *super-Weyl invariance*:

$$\delta\chi_\alpha = i\gamma_\alpha\eta. \tag{2.27}$$

Here also η is a Majorana-Weyl fermionic parameter. This symmetry leaves the other string fields invariant. We can fix the worldsheet metric to some gauge choice using the two bosonic symmetries: reparameterization and Weyl invariance. Now, these two fermionic symmetries (supersymmetry (2.26) and super-Weyl (2.27)) can be used locally to set the four components of $\chi_\alpha = 0$.

There are some complications with non-trivial solutions that arise in the process (super-Teichmüller parameters and superconformal Killing spinors), but they do not affect our conclusions. From now on, we will omit the worldsheet gravitino contribution to the theory.

2.3.2 Gauge Field and Heterotic Effective Action

Heterotic string theory incorporates as a massless bosonic background field (besides the graviton, the dilaton and the antisymmetric tensor field, as we encountered them in section **2.2**), a gauge vector boson $A^{\mathrm{a}}_M(x)$ for the current algebra SO(32) or SO(16)×SO(16). The most general renormalizable action for the heterotic string in the presence of an arbitrary background fields and satisfying general covariance, Weyl invariance and $N = \frac{1}{2}$ supersymmetry, is derived in reference [27] (in the gauge where $\chi_\alpha = 0$):

$$
\begin{aligned}
I[G,B,\Phi,A] = \frac{T}{2}\int d^2\sigma\sqrt{h}\Big\{&G_{MN}(x)\left[\partial_\alpha x^M\partial^\alpha x^N + i\bar{\lambda}^M\gamma^\alpha(D_\alpha\lambda)^N\right] \\
&+\frac{\epsilon^{\alpha\beta}}{\sqrt{h}}B_{MN}(x)\partial_\alpha x^M\partial_\beta x^N - \frac{i\epsilon_{\alpha\beta}}{h^{3/2}}H_{MNP}(x)\bar{\lambda}^M\gamma^\alpha\lambda^N\partial^\beta x^P \\
&+\frac{i}{2}\partial_\alpha\left[\bar{\lambda}^M\gamma^\alpha\lambda^N B_{MN}(x)\right] + i\bar{\psi}^s\gamma^\alpha\partial_\alpha\psi^s \\
&+\bar{\psi}^s T^{\mathrm{a}}_{st}\gamma^\alpha\psi^t\left[A^{\mathrm{a}}_M(x)\partial_\alpha x^M - \frac{i}{4}F^{\mathrm{a}}_{MN}(x)\bar{\lambda}^N\gamma_\alpha\lambda^M\right]\Big\} \\
&+\frac{1}{2}\int d^2\sigma\sqrt{h}\,R_h\Phi(x).
\end{aligned}
\tag{2.28}
$$

In this equation, as before, indices from the beginning of the Greek alphabet (*e.g.* α or β) refer to the worldsheet parameterized by $\{\sigma^\alpha\}$. Of course, all the contractions in (2.28) involve the worldsheet metric $h_{\alpha\beta}(\sigma)$, or the zweibein $e^\alpha_a(\sigma)$ for the gamma matrices. Indices taken from the middle of Latin alphabet (M, N, P, etc.) refer to the D-dimensional target space. $G_{MN}(x)$ is the metric of the background space (10 dimensions). $B_{MN}(x)$ is the antisymmetric tensor field.

The covariant derivative of the spinors λ^M is defined by:

$$(D_\alpha \lambda)^M = \partial_\alpha \lambda^M + \Gamma^M_{NP}(x)\partial_\alpha x^N \lambda^P, \tag{2.29}$$

where, as usual, $\Gamma^M_{NP}(x)$ represents the Christoffel connection on the ten-dimensional space. T^{a} denotes an arbitrary generator of one of these groups. Finally $A^{\mathrm{a}}_M(x)$ is the background gauge field with, as usual:

$$F^{\mathrm{a}}_{MN} = \partial_N A^{\mathrm{a}}_M - \partial_M A^{\mathrm{a}}_N + f^{\mathrm{abc}} A^{\mathrm{b}}_M A^{\mathrm{c}}_N. \tag{2.30}$$

Here f^{abc} are the structure constants of the group defined through the relation

$$[T^{\mathrm{a}}, T^{\mathrm{b}}] = -i f^{\mathrm{abc}} T^{\mathrm{c}}. \tag{2.31}$$

As for the bosonic string, the S-matrix may be represented by a ten-dimensional effective action. The ten-dimensional action that reproduces the scattering amplitudes obtained from (2.28) at lowest order in α' has the form of the Chamseddine-Chapline-Manton supergravity action with matter fields (refcrences [25] and equation (3.1)). This time, there are also fermionic matter fields in the effective action. This is derived in references [11, 19].

Two things are to be noticed about the effective action for heterotic strings, in contrast with the bosonic case. Firstly, we must include Yang-Mills interactions through a term in the effective heterotic string Lagrangian proportional to $\alpha' \mathrm{tr}(F_{MN}F^{MN})$, where the trace is over gauge indices. This term is of the same order as the R^2 term, coming from stringy corrections to the effective action for bosonic strings (see section **2.2.2**).

Secondly, interactions of the gauge fields with the antisymmetric tensor field are introduced by the mean of a redefinition of the field strength H_{MNP} by a Chern-Simons completion. In the language of differential forms:

$$H = dB + \frac{\alpha'}{8}\mathrm{tr}(F \wedge A) - \frac{\alpha'}{8}\mathrm{tr}(R \wedge \omega), \tag{2.32}$$

with ω the connection one-form; the Yang-Mills trace is taken in the fundamental representation (for the adjoint representation, we have to multiply the Yang-Mills term by a factor $\frac{1}{30}$ [3]).

The bosonic part of the effective action keeping only the terms having two or less derivatives has thus the form of:

$$
\begin{aligned}
S_{\text{eff}} &= \frac{1}{2\alpha'^4}\int d^Dx\sqrt{G}\left\{R - \frac{1}{D-2}(\Phi_{;M})^2 - \frac{1}{12}e^{-8\Phi/(D-2)}H^2\right. \\
&\qquad \left. -\frac{\alpha'}{2}e^{-4\Phi/(D-2)}\mathrm{tr}(F^2),\right\} \qquad (2.33)
\end{aligned}
$$

where $F^2 = F^{\mathrm{a}}_{MN}F^{\mathrm{a}MN}$, the trace is over the gauge index a, and H_{MNP} is given by equation (2.32). We also used explicitly the fact that the only dimensionful constant was α'.

This effective action governs the light background bosons of the heterotic string theory at low energy. In order to have a more complete description of the low-energy theory, we must include fermions and cut down the number of space-time dimensions to four. The fermion content at low energy is determined by supersymmetry. The compactification causes more problems; the next chapter addresses these issues.

Chapter 3

Four-Dimensional String Effective Action

A realistic low-energy effective theory should be written in a language that we can interpret easily: an effective action of a four-dimensional quantum field theory. Of course, an unusual property of string theories is the rather large number of spacetime dimensions. The problem of expressing the low-energy theory addresses the question of ***compactification***, *i.e.* the fact that six dimensions of spacetime are so tightly curled up that we cannot even see them!

Compactification is a difficult problem. In our case, we are only interested in the light background particles (that we called 'massless'; light is meant to be understood by comparison with both the Planck mass M_{P} and the compactification scale $M_{\mathrm{C}} = 1/r_6$). The formal way of obtaining an effective action for them would be to integrate over all the very massive states of the theory that can contributes to the scattering amplitudes. It is a lot easier to first approximate the result by setting every of these massive states to zero. This is what is called ***truncation*** and we shall see that its predictions are very close to the full theory at the lowest order in the perturbative expansions. This method was first that the scaling of the harmonic forms is the following:

$$\omega^K_{mn} \to \lambda^{-1}\omega^K_{mn}, \quad \pi^L_{mn} \to \lambda^{-1}\pi^L_{mn}. \tag{3.16}$$

In general, we can do the same thing also for the forms or tensors defined on K that are eigenstates of the corresponding kinetic operator with non-zero eigenvalue. By this, we mean that for a given form $\omega_{n_1 n_2 \cdots n_q}(y)$, the transformation rule will be:

$$\omega_{n_1 n_2 \cdots n_q} \to \lambda^{-q/2}\omega_{n_1 n_2 \cdots n_q}. \tag{3.17}$$

The antisymmetric covariant tensor $\epsilon_{mnpqrs}/\sqrt{g}$ also scale with the proper factor λ^{-3}, as a consequence of (3.14). This implies that every scalar combination of such forms and tensors will be invariant under the BM symmetry. An example of this is given by the Yukawa coupling constants λ_{JKL} and ξ_{JKL} defined in equation (3.11) that are indeed invariant.

Being a ten-dimensional field, the dilaton $\varphi(x)$ will not scale. Equations (3.3) and (3.16) then show that the moduli k^K or h^L do not scale either. The other fields that transform under the BM symmetry (3.14) are (following from equation (3.2) and the normalization of the gauge group generators T^{Jp}):

$$A^K \to \lambda A^K, \quad B^L_\xi \to \lambda^{1/2} B^L_\xi, \quad C^{K\xi} \to \lambda^{1/2} C^{K\xi}. \tag{3.18}$$

Also, the coefficients $d_{\xi\zeta\chi}$ are invariant under the BM transformation (3.14), as may be derived from the equation (3.12).

The complex scalar fields S and T must also get scaled by the BM symmetry (3.14). The scaling is directly read from their definition (3.7). The dilaton φ is kept invariant, so:

$$S \to \lambda^3 S, \quad T \to \lambda T. \tag{3.19}$$

It is straightforward to show from the definitions of $\theta(x)$ and $\eta(x)$ (see equation (3.8)) that this scaling symmetry is consistent for each of the constituent fields of the complex scalars S and T.

applied to string theory on a toy background by Witten [7]. The full truncation of the Chamseddine-Chapline-Manton action in the context of heterotic string was later developed by the authors of reference [9].

The procedure that we adopt in order to describe the low-energy effective theory in four dimensions is to work within the framework of a $D = 4$, $N = 1$ supersymmetric theory. The residual supersymmetry is *assumed* to have survived the process of compactification. One way that assures that we have such a theory is to choose the background internal space to be a Calabi-Yau manifold *i.e.* Ricci-flat with SU(3) holonomy [6]. The truncated Lagrangian is naturally put into the form of such a four-dimensional theory [7]. Our approach for finding the full low-energy effective action is then to use symmetry considerations that allow us to dictate the form that the full low energy action must take.

This chapter is divided as follows: we first derive a nonrenormalization theorem for the effective string action in ten dimensions. We proceed to the identification of the massless modes of the four-dimensional effective theory, and then use the standard $D = 4$, $N = 1$ supergravity action to match the Lagrangian obtained using these massless fields. We then introduce a symmetry, that we call breathing mode (BM) symmetry, that is valid at each order in both perturbative expansions (string loops and sigma model expansion). This BM symmetry is crucial for our discussion, it allows us to state the general form for the effective Lagrangian at each order in the perturbative series. We derive it at first in a special case (no matter field and only one modulus for the internal manifold) that is generalized at the end. We are then in a position to extract some interesting implications of this property: nonrenormalization theorems and specific prescriptions.

3.1 Ten-Dimensional Supergravity

The Chamseddine-Chapline-Manton ten-dimensional supergravity action is unique: it is the most general action with two or less derivatives and with the given field content for supergravity in ten dimensions coupled to Yang-Mills inter. ctions. We have seen that the bosonic part of the effective action governing heterotic string at low-energy was consistent with it (see section **2.3.2**). The effective ten-dimensional fermionic fields of heterotic string theory match the fermions of this supergravity theory. The supergravity action is given by [25, 9]:

$$\begin{aligned} S_{\mathrm{SUGRA}} = \int d^{10}x\sqrt{G}\Bigg\{ & \frac{R}{2\kappa^2} - \frac{3}{4\kappa^2}\frac{H^2}{\varphi^2} - \frac{1}{\kappa^2}\frac{(\partial_M\varphi)^2}{\varphi^2} \\ & -\frac{1}{4\kappa^{3/2}}\frac{\mathrm{tr}(F^2)}{\varphi} - \frac{1}{2}\bar{\psi}_M\Gamma^{MNP}D_N\psi_P - \frac{1}{2}\bar{\psi}\Gamma^M D_M\psi \\ & -\frac{1}{2}\mathrm{tr}(\bar{\lambda}\Gamma^M D_M\lambda) - \frac{\kappa^{1/4}}{4}\frac{1}{\varphi^{1/2}}\mathrm{tr}(\bar{\lambda}\Gamma^M\Gamma^{NP}F_{NP})\left(\psi_M + \frac{1}{6\sqrt{2}}\Gamma_M\psi\right) \\ & +\frac{1}{8\sqrt{2}}\frac{H_{NMP}}{\varphi}\Big[\bar{\psi}_Q\Gamma^{QMNPR}\psi_R + 6\bar{\psi}^M\Gamma^N\psi^P - \sqrt{2}\bar{\psi}_Q\Gamma^{MNP}\Gamma^Q\psi \\ & \qquad\qquad +\mathrm{tr}(\bar{\lambda}\Gamma^{MNP}\lambda)\Big] \\ & -\frac{1}{\sqrt{2}}\bar{\psi}_M\Gamma^M\Gamma^N\psi\left(\frac{\partial_N\varphi}{\varphi}\right) + \text{four-Fermi terms}\Bigg\}. \end{aligned} \tag{3.1}$$

The gauge group is here chosen to be $E_8 \times E_8$ for phenomenological reasons [3].

This action is the one that we take as the lowest-order effective action for heterotic strings in a ten-dimensional background space. We want to compute from this action a four-dimensional effective theory. In order to do this, we must introduce the four-dimensional massless modes that come from the ten-dimensional fields entering the above action (3.1). It will be done in section **3.1.2**, but before, we may illustrate the kind of arguments that we will use in four dimensions by stating a nonrenormalization theorem.

3.1.1 A Nonrenormalization Theorem in Ten Dimensions

The full effective action governing the heterotic string in an $N = 1$ supersymmetric ten-dimensional background is an expansion in term of two parameters that were already introduced in chapter **2**. They are the string coupling constant λ, which can be interpreted as the constant part of the dilaton expectation value (we have $\lambda \sim \varphi^2$ according to the equation (2.15)) and the Regge slope α', that carries dimensions of $(Length)^2$.

We know, from the analysis presented in section **2.3.2**, that the action (3.1) is the ten-dimensional effective action for heterotic string theory *at lowest order in both the string-loop and the sigma-model expansions.* This action might therefore be corrected by higher order effects.

A nonrenormalization theorem is a statement that the action gets no correction to certain orders of the perturbative expansions. We are in position to state such a nonrenormalization theorem concerning the ten-dimensional string effective theory:

NR Theorem 1 *At lowest order in the α'-expansion, the effective action (3.1) does not get corrected by effects of higher order in the string-loop expansion.*

The proof goes as follows: take the effective action (2.33) or the bosonic part of (3.1). Things are arranged in such a way that the only dimensionful components are derivatives ∂_M, gauge field A^a_M and the Regge slope α'. Thus, the α'-expansion is essentially the expansion in the number of derivatives of the fields. The bosonic part of the action (3.1) is the most general $D = 10$, $N = 1$ supersymmetric action with the given field content and with at most two derivatives and its coefficients are completely fixed by supersymmetry, including its dependence on the dilaton.

Let us consider now a correction term not of the lowest order in the string-loop expansion (*i.e.* not a sphere contribution). If it is at the lowest order in the σ-model expansion, then it must involve at most two derivatives. But, since it is of higher order in the genus expansion, the dilaton $\varphi(x)$ must couple differently to the other field than (3.1). This is forbidden by supersymmetry and this term cannot exist. It follows that string-loop corrections cannot renormalize the action (3.1). Let us now truncate the theory to four dimensions.

3.1.2 Four-Dimensional Massless Modes

If we demand that in four dimensions, an $N = 1$ supersymmetry survives, we must compactify the six internal dimensions of heterotic string theory on a Ricci-flat manifold, K, of SU(3) holonomy (see [6, 28]).

We expand the ten-dimensional fields in term of the eigenfunctions of the operator $\Box$ (or the corresponding kinetic operator) defined on the compact manifold K. The coefficients multiplying these functions depend on the position in the four-dimensional spacetime and are the four-dimensional fields of the theory. We will express the zero-modes (*i.e.* eigenstates with eigenvalue zero) in the language of differential forms; they turn out to correspond to harmonic forms.

The manifold K possesses a definite number of harmonic forms defined on it. This space may be parameterized by complex coordinates, and the harmonic forms are chosen to have a definite number of holomorphic and antiholomorphic indices $(i, \bar{j})$. The number of independent harmonic forms of each type is given by the Hodge numbers $b_{i,\bar{j}}$. In general, these numbers are all obtained from $b_{1,\bar{1}} \geq 1$ and $b_{1,\bar{2}}$, which can be larger or equal to 0.

We express the harmonic $(1,\bar{1})$-forms $\omega^K_{i,\bar{j}}(y)$ with $K = 1, \ldots, b_{1,\bar{1}}$, and the $(1,\bar{2})$ forms $\omega^L_{i,\bar{j}\bar{k}}(y)$ with $L = 1, \ldots, b_{1,\bar{2}}$. We write $\pi^L_{ij} = \epsilon_j{}^{\bar{k}\bar{l}} \omega^K_{i,\bar{k}\bar{l}}$, where ϵ_{ijk} is the SU(3)-invariant antisymmetric tensor.

Then the four-dimensional massless fields can be derived from the corresponding massless modes of the full ten-dimensional action (3.1). We write everything in the canonical variables, *i.e.* using the metric $G^{\mathrm{can}}_{MN}(x, y)$ defined in equation (2.14) for spacetime which produces a canonical Einstein term in the ten-dimensional Lagrangian. The massless modes are then [9]:

$$\begin{aligned}
\varphi(x,y) &= \varphi(x) + \text{heavy modes}, \\
B_{\mu\nu}(x,y) &= B_{\mu\nu}(x) + \text{heavy modes}, \\
A^a_\mu(x,y) &= A^a_\mu(x) + \text{heavy modes}, \\
B_{i,\bar{j}}(x,y) &= A^K(x)\, \omega^K_{i,\bar{j}}(y) + \text{heavy modes},
\end{aligned} \tag{3.2}$$

$$
\begin{aligned}
A_i^{\rm a}(x,y) &= C^{K\xi}(x)\,\omega_i^{K\jmath}(y)\,T^{\rm a}_{\jmath\xi} + B^L_\xi(x)\,\pi^L_{ij}(y)\,T^{{\rm a}j\xi} + \ldots, \\
G^{\rm can}_{MN}(x,y) &= \begin{bmatrix} e^{-3\sigma(x)}g_{\mu\nu}(x) & 0 \\ 0 & e^{\sigma(x)}\tilde{g}_{mn}(x,y) \end{bmatrix} + \ldots,
\end{aligned}
$$

where $\tilde{g}_{mn}(x,y) = \bar{g}_{mn}(y) + h_{mn}(x,y)$ with $\bar{g}^{mn}(y)\ h_{mn}(x,y) = 0$ and

$$
\begin{aligned}
h_{i,\bar{\jmath}}(x,y) &= \varphi^{-1}(x)\ k^K(x)\ \omega^K_{i,\bar{\jmath}}(y), \quad K = 2,\ldots,b_{1,\bar{1}}, \\
h_{ij}(x,y) &= \varphi^{-1}(x)\ h^L(x)\ \pi^L_{ij}(y), \quad L = 1,\ldots,b_{1,\bar{2}}.
\end{aligned} \tag{3.3}
$$

Here the massless fields $A_\mu(x)$ take values in the adjoint representation of the unbroken gauge group, while the matter fields $C^{K\xi}$ (resp. B^K_ξ) transforms as $\overline{\mathbf{27}}$ (resp. $\mathbf{27}$) of the E_6 subgroup of E_8 left unbroken by Calabi-Yau compactification. $T^{j\xi}$ are the corresponding generators of E_8; they are normalized by tr $(T_{i\xi}T^*_{\bar{\jmath}\zeta}) = \bar{g}_{i,\bar{\jmath}}\delta_{\xi\zeta}$. The fields $k^K(x)$ and $h^L(x)$ are the so-called *moduli* fields of the compact manifold. Their expectation values correspond to the moduli that describe the internal manifold [2].

From now on, we will use the following notations for the coordinates of spacetime: x^μ are the coordinates of the four-dimensional observable spacetime M_4 ($\mu = 0,1,2,3.$) and y^m describe the six dimensional compact manifold K ($m = 1,\ldots,6.$). In the complex coordinates for K, we use, as before, the indices $i = 1,2,3$ and $\bar{\jmath} = \bar{1},\bar{2},\bar{3}$ for, respectively, holomorphic and antiholomorphic coordinates. We denote as $\tilde{g}_{mn}(x,y)$ the six-dimensional metric of the internal space K and the four-dimensional metric is $g_{\mu\nu}(x)$, which has a canonical kinetic term of the Hilbert (or Einstein) form, *i.e.* $\mathcal{L}_4 \sim \sqrt{g}R_4$.

The *truncation* of string theory, as presented in reference [9], is the expression of the string action in terms of these massless fields (3.2) and (3.3). Defining $\mathcal{L}_4 = \int_{\rm K} \mathcal{L}_{10}$, it gives an approximation for what should be the low-energy Lagrangian governing heterotic strings in a four-dimensional supersymmetric background.

3.2 Truncation of the Ten-Dimensional Action

3.2.1 $D = 4$, $N = 1$ Supergravity

We know that Calabi-Yau compactification is supposed to preserve an $N = 1$ supersymmetry for the four-dimensional compactified Lagrangian [6]. A realistic assumption to be made is thus that the correct Lagrangian in four dimensions is the $D = 4$, $N = 1$ supergravity Lagrangian coupled to Yang-Mills and matter that has been worked out by Cremmer *et al.* [29]. It is totally described by only two functions depending on the complex scalars z^i of the chiral superfields representing matter [30]:

- A real gauge invariant function $\mathcal{G}(z, z^*)$, that we call the *Kähler function*, written as:

$$\mathcal{G}(z, z^*) = K(z, z^*) + \log |W(z)|^2, \tag{3.4}$$

  where the real function $K(z, z^*)$ and the analytic function $W(z)$ are respectively called the *Kähler potential* and the *superpotential.* They are defined completely once it is specified that there is no analytic term in the Kälher potential.

- And an analytic function $f_{\mathrm{ab}}(z)$ with a and b being gauge indices; this is the *gauge coupling function* (see the action (3.5) below).

The bosonic part of this $D = 4$, $N = 1$ supergravity effective action is thus given by [29]:

$$\begin{aligned} S &= \frac{1}{2\alpha'} \int d^4x \sqrt{g} \, \{R - 2\mathcal{G}_{zz^*} D_\mu z D^\mu z^* + 2V[\mathcal{G}(z, z^*)] \\ &\quad - \frac{\alpha'}{2} \mathrm{Re}\,(f_{\mathrm{ab}}) F^{\mathrm{a}}_{\mu\nu} F^{\mathrm{b}\mu\nu} + \frac{i\alpha'}{2} \mathrm{Im}\,(f_{\mathrm{ab}}) F^{\mathrm{a}}_{\mu\nu} \tilde{F}^{\mathrm{b}\mu\nu} \Big\}, \end{aligned} \tag{3.5}$$

where the potential V is:

$$V[\mathcal{G}] = \alpha'^{-1} e^{\mathcal{G}} \left(3 - \frac{\mathcal{G}_{z_i} \mathcal{G}_{z_j^*}}{\mathcal{G}_{z_i^* z_j}} \right) - \frac{1}{2} \mathrm{Re}\,(f^{-1}_{\mathrm{ab}}) \left(\mathcal{G}_{z_i^*} T^{\mathrm{a}}_{ij^*} z_j\right) \left(\mathcal{G}_{z_k^*} T^{\mathrm{b}}_{kl^*} z_l\right) \tag{3.6}$$

where $\tilde{F}^a_{\mu\nu}$ is the dual field strength $\frac{1}{2}\epsilon_{\mu\nu\rho\lambda}F^{a\rho\lambda}$, $T^a_{ij^*}$ is a gauge group generator matrix and $\mathcal{G}_{z_i}$ and $\mathcal{G}_{z_i^*}$ denote, respectively, the derivatives $\partial\mathcal{G}/\partial z_i$ and $\partial\mathcal{G}/\partial z_i^*$.

As a supersymmetric theory, the effective action (3.5) also includes fermionic fields. However, they do not play an important role for our discussion, since supersymmetry indicates that they do not require the introduction of any new function to characterize further the theory. Moreover, the symmetry considerations that we look at in the next sections may be extended to fermionic fields without any complications. We will therefore omit the fermions for the rest of the discussion.

3.2.2 The Truncated Lagrangian

In the case of the heterotic effective action, there are many scalar fields in the Kähler function (3.4) and the gauge coupling function (both entering the action corresponding to (3.5) for heterotic strings). In fact, their number is related to the number of harmonic forms defined on the internal manifold (*i.e.* $b_{1,\bar{1}}$ and $b_{1,\bar{2}}$). As we can see from equations (3.2) and (3.3), we have: the dilaton field $\varphi(x)$, the *breathing mode* $e^{\sigma(x)}$ (corresponding to the $K=1$ modulus) the other moduli $k^K(x)$ and $h^L(x)$ (for respectively $K=2,\ldots,b_{1,\bar{1}}$ and $L=1,\ldots,b_{1,\bar{2}}$), the matter fields $C^{K\xi}(x)$ and $B^L_\xi(x)$ (for all K and L) and the scalars defined by the antisymmetric tensor field on the internal space $A^K(x)$. They all combine to form complex scalar fields of chiral supermultiplet.

We now concentrate on the dilaton and the breathing mode field. In most of the rest of the discussion, we focus our attention on the case where $b_{1,\bar{1}}=1$ and $b_{1,\bar{2}}=0$. In this case, we omit the index K that should appear, since only the value $K=1$ is present. We will however state results for the other moduli whenever it is possible.

For this special case, apartfrom the matter fields B and C, the theory involves only two complex scalar fields, S and T, that are defined as follows in order to have the standard $D=4$, $N=1$ supersymmetry transformation rules

[7, 9]:

$$\begin{aligned} S &= \varphi^{-1}e^{3\sigma} + i\theta, \\ T &= \varphi e^{\sigma} + \alpha' \left(C^{*K\xi} C^{\xi}_{K} + B^{*}_{L\xi} B^{\xi}_{L} \right) + i\eta, \end{aligned} \tag{3.7}$$

where $\theta(x)$ and $\eta(x)$ are two axion fields arising from $B_{MN}(x,y)$ as defined below. η is simply given by the relation $\eta(x) = \sqrt{2}A^{K}(x)$, with $K = 1$. In our case (where there is just one $(1,\bar{1})$- and no $(1,\bar{2})$-moduli), we have $B_{mn}(x,y) = \eta(x)\ \omega_{mn}(y)$, where ω_{mn} is the unique harmonic form expressed in the real coordinate basis $\{y^m\}$. We define $\theta(x)$, in the language of differential forms, by $d\theta = *H$ in four dimensions, with a non-conventional $*$-dualisation:

$$H_{\mu\nu\rho} = -\frac{1}{3\sqrt{2g}} (\varphi e^{-3\sigma})^2 \epsilon_{\mu\nu\rho\lambda} \partial^{\lambda}\theta. \tag{3.8}$$

At the truncation level, we know the exact form of both the Kähler and the gauge coupling functions, when we omit to take in consideration the other moduli fields $k^K(x)$ with $k \geq 2$ and $h^L(x)$. They are found in [7, 9]:

$$\begin{aligned} \mathcal{G}_{\text{trunc}} &= -\log(S + S^*) - 3\log(T + T^*) + \log|W(z)|^2, \\ f_{(\text{trunc})\text{ab}} &= S\delta_{\text{ab}}. \end{aligned} \tag{3.9}$$

where δ_{ab} is the Kronecker delta and the superpotential is given by:

$$W(z) = 8\sqrt{2}\alpha'^{3/2} \left(\lambda_{JKL} d_{\xi\zeta\chi} C^{\xi}_{J} C^{\zeta}_{K} C^{\chi}_{L} + \xi_{JKL} d^{\xi\zeta\chi} B_{J\xi} B_{K\zeta} B_{L\chi} \right). \tag{3.10}$$

Here, the couplings are defined by (in the real coordinate basis of K):

$$\lambda_{JKL} = \frac{1}{3!} \left(\frac{1}{\int d^6y \sqrt{g}} \right) \int d^6y\ \epsilon^{mnpqrs} \omega^{J}_{mn} \omega^{K}_{pq} \omega^{L}_{rs}, \tag{3.11}$$

and ξ_{JKL} is the corresponding expression in term of the harmonic $(1,\bar{2})$-forms $\pi^{L}_{mn}(y)$ instead of the $(1,\bar{1})$-forms $\omega^{K}_{mn}(y)$. The coefficients $d_{\xi\zeta\chi}$ are defined through the following equation:

$$\text{tr}\ (T_{i\xi} T_{j\zeta} T_{k\chi}) = \epsilon_{ijk} d_{\xi\zeta\chi}. \tag{3.12}$$

Notice that λ_{JKL} and ξ_{JKL} are interesting quantities that look just like ordinary constants as for the four-dimensional theory, but, in fact, depend on the internal space in a very crucial way. The effective theory in four-dimension is full of these constants that carry informations about the internal space.

The truncated Kähler and gauge coupling functions, as expressed in (3.9) above, are those that we want to correct at all orders in the two expansion parameters (φ and α'), with the objective of finding a more complete low-energy effective Lagrangian. In order to do this, we introduce in the next section a symmetry valid at each order of the perturbative expansions that will help us to give restrictions on the form of the general correction terms.

3.3 Breathing Mode Symmetry

The breathing mode (BM) symmetry relies on the crucial observation that there is some ambiguity in the definition of the four-dimensional metric as expressed by (3.2). Indeed, the truncation of the ten-dimensional metric defines the metrics of the four- and the six-dimensional spaces, as well as the breathing mode by the following expression (still in the case where there are no $(1,\bar{2})$ and only one $(1,\bar{1})$ modulus, namely the breathing mode $\sigma(x)$):

$$G^{\text{can}}_{MN}(x,y) = \begin{pmatrix} e^{-3\sigma(x)} g_{\mu\nu}(x) & 0 \\ 0 & e^{\sigma(x)} \bar{g}_{mn}(y) \end{pmatrix}. \tag{3.13}$$

We therefore see that the four- and six-dimensional metrics are defined up to an arbitrary constant that can be absorbed into the breathing mode e^{σ}. In effect, we may write the following symmetry:

$$e^{\sigma} \to \lambda e^{\sigma}, \quad g_{\mu\nu} \to \lambda^3 g_{\mu\nu}, \quad \bar{g}_{mn} \to \lambda^{-1} \bar{g}_{mn}, \tag{3.14}$$

that gives $G_{MN} \to G_{MN}$.

The important thing for the transformation is to keep the ten-dimensional fields invariant, it will then be a symmetry of the ten-dimensional effective action. Furthermore, since the ten-dimensional fields come directly from the string action in which they appear as 'massless' background fields, the full theory will be invariant under the BM symmetry. We can see that no anomaly can ruin the symmetry because the string functional measures are also constructed with these invariant background fields (see the appendix). If all the ten-dimensional fields are kept unaffected by the BM symmetry, then the symmetry should be respected by the four-dimensional effective action to all order in the two perturbative series (sigma model and string loop expansions). However, our knowledge of the genus expansion is very small. And, while at the sphere level every four-dimensional fields will have a simple scaling under the BM symmetry, loop effects will in general ruin this property and the transformation laws may become more complicated. Nevertheless the symmetry should remain exact, since it does not involve any anomaly.

We may then exploit the BM symmetry to give restrictions on the possible form that the correction terms to the truncated effective Lagrangian can take, in a very convincing way at the tree-level, and more hypothetically for string-loop terms. To see that it is possible to arrange the transformation laws of the four-dimensional fields in such a way that the string massless background fields stay invariant, we have to notice that we have enough scalar fields in equation (3.2) to counterbalance every scaling imposed by the transformation rules (3.14). In what follows, we derive explicitly the scaling rules for all the fields introduced in equations (3.2) and (3.3).

In the general case of more than one single modulus, this BM symmetry also scales the harmonic forms ω^K_{mn} and π^L_{mn}. Being eigenstates of the operator $\Box_6$, their normalization is not fixed and we may choose:

$$\int d^6y\sqrt{\bar{g}}\,\bar{g}^{mp}\bar{g}^{nq}\omega^K_{mn}\omega^L_{pq} = \Omega_K\delta^{KL}, \tag{3.15}$$

where $\Omega_K = \int_K d^6y\sqrt{\bar{g}}$ is the volume of the manifold K. And similarly for π^L_{mn}. The volume factor transforms under the BM symmetry (3.14) by: $\Omega_K \to \lambda^{-3}\Omega_K$. The requirement that the above normalization should be invariant implies then

As one expects, the truncated functions (3.9) have the right behavior under our symmetry. Assuming that the gauge coupling function f_{ab} has the form (3.9), we compute the scaling of the following typical term (see equation (3.5)):

$$\mathcal{L}_{YM} = -\frac{1}{4}\sqrt{g}\,\mathrm{Re}\,(f_{ab})\,g^{\mu\nu}g^{\rho\lambda}F^{a}_{\mu\nu}F^{b}_{\rho\lambda}\left(\alpha'^{-3}\int d^6y\sqrt{\bar{g}}\right). \tag{3.20}$$

The two inverse metrics cancel the factor λ^6 coming from the four-dimensional determinant $\sqrt{g}$, and the truncated gauge function acquires a scale factor of λ^3 from the S-field that gets cancelled by the six-dimensional metric determinant $\sqrt{\bar{g}}$. This term of the Lagrangian is therefore invariant as it should be and we see that the gauge coupling function f_{ab} scales in the same way as does the field S does under the BM transformation.

To check that the truncated form (3.9) of the Kähler function $\mathcal{G}(z, z^*)$ has the right behavior under the symmetry, we take the part of potential term, which has the following form (neglecting the part proportional to the derivatives of $\mathcal{G}$):

$$\mathcal{L}_{\text{pot}} = -3\sqrt{g}\alpha'^{-2}e^{K}|W|^2\int d^6y\sqrt{\bar{g}}. \tag{3.21}$$

Again, we see that all the λ-dependence of this term get cancelled at the truncated level after the BM transformation is performed. The determinants of the four- and the six-dimensional metrics carry together a factor λ^3 which is cancelled by the factor λ^{-6} coming from the Kähler potential e^K and the factor λ^3 of the superpotential W. The above term is therefore invariant under the BM symmetry. From this, we see that under the BM transformation, we must have $e^{\mathcal{G}} \rightarrow \lambda^{-3}e^{\mathcal{G}}$.

In the next section, we start studying the implications of this symmetry to find correction terms to the truncated Lagrangian.

3.4 Corrections Allowed by BM Symmetry

The upshot of our analysis is the fact that the use of the symmetry (3.14) and (3.19) allows us to read the possible form that can be taken by the higher-loop corrections to the two truncated functions (3.9) defining the supersymmetric four-dimensional effective action. We define the full supersymmetric functions to be given by:

$$\begin{aligned} \mathcal{G}(z,z^*) &= \mathcal{G}_{\text{trunc}}(z,z^*) + \log \bar{\mathcal{G}}(z,z^*), \\ f_{\text{ab}}(z) &= S\left[\delta_{\text{ab}} + \bar{f}_{\text{ab}}(z)\right]. \end{aligned} \tag{3.22}$$

Then, since the truncated term has already the right behavior under the transformation (3.19), then the correction $\bar{\mathcal{G}}(z,z^*)$ must be invariant under BM symmetry. The same reasoning shows that the gauge coupling correction $f_{\text{ab}}(z)$ must not scale under the action of the symmetry transformation. In other words, we get:

$$\bar{\mathcal{G}}(z,z^*) \to \bar{\mathcal{G}}(z,z^*), \qquad \bar{f}_{\text{ab}}(z) \to \bar{f}_{\text{ab}}(z), \tag{3.23}$$

with respect to the transformation of the fields (3.19) We emphasize that the transformation rule for the fields z^i are different for each scalar, *e.g.* S and T do not scale in the same way. Also note that dimensional analysis shows that $\bar{\mathcal{G}}$ and $\bar{f}_{\text{ab}}$ must be dimensionless.

If the theory were to contain only the four-dimensional fields, then (3.23) would restrict very much the corrections $\bar{\mathcal{G}}$ and $\bar{f}_{\text{ab}}$. For instance, it is possible to evaluate the effective action at the stationary point $B = C = 0$ [9] (still omitting the moduli other than the global radius of K), in which case the only possible corrections must be functions of the unique invariant combination of the scalar fields, T^3/S. This correction is however incompatible with what is known of the large S and T limits (see below equation (3.31) and what follows).

3.4.1 Internal Degrees of Freedom

The effective theory in four dimensions depends on the the fields defined on the six-dimensional internal space K. For example, we saw that the terms (3.20) and (3.21) were proportional to $\sqrt{\bar{g}(y)}$ through the volume of K, Ω_{K}. We also encountered the constants λ_{JKL} and ξ_{JKL} that were constructed out of the harmonic forms of the compact space. These kinds of constants that are coefficients in the four-dimensional theory have a crucial impact on our understanding of the theory in four dimensions. Another possibility is that these coefficients involve six-dimensional curvatures $R^p{}_{mqn}(y)$ or $F^{a}_{mn}(y)$.

Let us introduce the notation $\mathcal{V}_i^{(d)}$ for these constants in the four-dimensional theory. The superscript d is the engineering dimension of the constant (in other words, $\mathcal{V}_i^{(d)}$ has dimensions of $(Mass)^d$) and the subscript i is just to distinguish between various kind of constants that have the same dimension.

We claim that no matter what six-dimensional fields or forms are hidden in the expression of $\mathcal{V}_i^{(d)}$, then its scaling under the BM symmetry (3.14) is the following:

$$\mathcal{V}_i^{(d)} \rightarrow \lambda^{d/2} \mathcal{V}_i^{(d)}, \tag{3.24}$$

To see this, we must just notice that our fields are defined such that only derivatives ∂_M or gauge vector fields $A^{a}_{M}(x,y)$ have dimensions of $Mass$. The same will true for the six-dimensional space K. Apart from the coordinates, the only quantities defined on K that carry dimensions are ∂_m and $A^{a}_{m}(x,y)$. Both of these quantities have one Lorentz index that will need to be contracted with the inverse metric $\bar{g}^{mn}$, or with the antisymmetric tensor, or with any of the forms we need to consider, *etc.* The engineering dimensions of $\mathcal{V}_i^{(d)}$ correspond to the number of indices that have to be contracted using a tensor that scale by a factor $\lambda^{d/2}$, according to equations (3.14), (3.16) and (3.17). In the same way that we followed for λ_{JKL} and ξ_{JKL}, we introduce a factor Ω_{K}^{-1} for each integration over K that we must perform in the expression of $\mathcal{V}_i^{(d)}$, then these constants have the claimed scaling under the BM symmetry.

In order to fix ideas, we study explicitly the case of the sphere. Here, we will always have one integration. We only consider the constants $\mathcal{V}_i^{(d=2c)}$ that involve six-dimensional curvatures and no other derivatives. Considered as quantum fields, terms in the action are proportional to integrals over K of vacuum expectation values (v.e.v.) of different combinations of these curvatures. As before, the subscript $i = (i_0, i_1, \ldots, i_c)$ indicates that there are usually more than one possible combination. Then $\mathcal{V}_i^{(2c)}$ is of the following normalized form (remember that it is only a special case):

$$\mathcal{V}_i^{(2c)} = \frac{1}{\int d^6y\sqrt{\bar{g}}} \int d^6y\sqrt{\bar{g}}\, t_{i_0}^{m_1 \cdots m_{2c}} < T^{i_1}_{m_1 m_2} T^{i_2}_{m_3 m_4} \cdots T^{i_c}_{m_{2c-1} m_{2c}} > . \quad (3.25)$$

In the above expression, $T^{i_k}_{mn}$ represents either $F^a_{mn}(y)$ or $R^p{}_{mqn}(y)$. It is understood that the gauge indices (a for the Yang-Mills gauge group and, p and q for the Lorentz group) are properly contracted or summed over in such a way that the v.e.v. is a gauge-invariant Lorentz scalar. Both $F^a_{mn}(y)$ and $R^p{}_{mqn}(y)$ are invariant under the BM symmetry (3.14). More genrally the tensors $T^{i_k}_{mn}$ can also represent a pair of derivatives applied either to the curvatures or to some part of the contraction tensor $t_{i_0}^{m_1 \cdots m_{2c}}(y)$.

A contraction of c such curvatures necessitates a tensor $t_{i_0}^{m_1 \cdots m_{2c}}(y)$ with $2c$ indices, constructed with the covariant tensors $\bar{g}_{mn}(y)$ or $(1/\sqrt{\bar{g}})\epsilon^{mnpqrs}$, which are the only invariant ones available. But we have:

$$\bar{g}_{mn} \rightarrow \lambda^{-1}\bar{g}_{mn}, \quad \frac{\epsilon^{mnpqrs}}{\sqrt{\bar{g}}} \rightarrow \lambda^3 \frac{\epsilon^{mnpqrs}}{\sqrt{\bar{g}}}. \quad (3.26)$$

Of course, tensors as ω^K_{mn} or π^L_{mn} can be used to form the contraction tensor $t_{i_0}^{m_1 \cdots m_{2c}}(y)$, but their scaling laws are indirectly determined by the metric $\bar{g}_{mn}$

The transformation laws (3.26) show that the power of λ in the scaling is simply given by half the number of upper indices. Hence the transformation law for the contraction tensor $t_{i_0}^{m_1 \cdots m_{2c}}(y)$ is a scaling by a factor λ^c. The behavior of the v.e.v. $\mathcal{V}_i^{(2c)}$ under the BM transformation is therefore (consistently with (3.24)):

$$\mathcal{V}_i^{(2c)} \rightarrow \lambda^c \mathcal{V}_i^{(2c)}, \quad (3.27)$$

since neither F^a_{mn}, nor $R^p{}_{mqn}$ scale under the BM symmetry.

It is essential to include these v.e.v.'s since they appear as coupling constants of the terms in the action that have two or less four-dimensional derivatives, even when the v.e.v.'s included have a large number of derivatives. They then contribute to the low-energy effective Lagrangian in four dimensions and are responsible for some correction terms that must be include in the functions $\bar{\mathcal{G}}$ and $\bar{f}_{\mathrm{ab}}$.

The next part of the discussion is devoted to the derivation of the most general correction term that is allowed by our BM symmetry (3.19).

3.4.2 General Correction Term

For now, we study only the special case where there is one single modulus and where the matter fields vanish: $B_\xi = C^\xi = 0$. The general case where we allow the matter fields to take non-zero values is quite similar to the following; it will not be treated in detail here. We will state the solution in the last section of this chapter, the discussion can be found in [31]. Some results for the inclusion of other moduli are also given in the last section and the above reference.

Let us investigate the possible correction terms that may be obtained from these v.e.v.'s, together with the scalar fields S and T. We define $\mathcal{C}_i^{(d)}$ to be the correction term of $\bar{\mathcal{G}}(S,S^*;T,T^*)$ (or $f_{\mathrm{ab}}(S,T)$) that arise from the coefficients $\mathcal{V}_i^{(d)}$. Since the fields S and T are indistinguishable from their respective conjugates S^* and T^* from the point of view of their properties under the BM transformation (3.19), we may at first concentrate only on corrections $\mathcal{C}_i^{(d)}(S,T)$ to the Kähler function (see however the implication of the axion symmetry (3.37), presented in section **3.5**). It will be understood that we can replace every field by its complex conjugate in the case of the Kähler function corrections.

$\mathcal{V}_i^{(d)}$ has dimension of $(Mass)^d$. So, in order to arrive at a dimensionless factor, we must introduce the Regge slope α'. The correction terms take the following form:

$$\mathcal{C}_i^{(d)}(S,T;\alpha') = \alpha'^{d/2}\mathcal{V}_i^{(d)}T^{-d/2}\mathcal{F}_i^{(d)}(S,T). \tag{3.28}$$

By construction, we must require that, under the BM scaling $S \to \lambda^3 S$ and $T \to \lambda T$, the correction be invariant $\mathcal{C}_i^{(d)} \to \mathcal{C}_i^{(d)}$. The factor $T^{-d/2}$ is introduced in (3.28) to counterbalance the scaling the v.e.v. under the BM symmetry. Hence, the correction is invariant under our symmetry if and only if the function $\mathcal{F}_i^{(d)}(S,T)$ is also. In other words, by virtue of our symmetry, $\mathcal{F}_i^{(d)}(S,T)$ must be function of the scalar fields only in the invariant ratio T^3/S. *A priori,* this last function of T^3/S is unrestricted. Expanding in powers of $1/S$, which is the string coupling constant expansion as we will see (equation (3.31)), we get:

$$\mathcal{F}_i^{(d)}(S,T) = \sum_{n=0}^{\infty} \mathcal{A}_{i,n}^{(d)} \left(\frac{T^3}{S}\right)^n , \tag{3.29}$$

for some set of arbitrary constants $\mathcal{A}_{i,n}^{(d)}$. We will also write:

$$\mathcal{C}_{i,n}^{(d)} = \alpha'^{d/2} \mathcal{V}_{i,n}^{(d)} T^{3n-d/2} S^{-n}, \tag{3.30}$$

for the general correction term, putting the constant $\mathcal{A}_{I,n}^{(d)}$ in the definition of the coefficient $\mathcal{V}_{i,n}^{(d)}$.

In order to understand the reasons that gives us the range of the parameter n in equation (3.29), we have to use some of our knowledge of the two perturbative expansions arising in the theory: *(i)* the string loop expansion, which is written in terms of a series in λ, the string coupling constant that was assimilated to the exponential of the v.e.v. of the string dilaton field in chapter 1 (so $\lambda \sim \varphi^2$ according to equation (2.15)) and *(ii)* the sigma model expansion, which is expanded in term of α'/r_6^2, where r_6, the typical radius of the internal manifold, is the only dimensionful constant that can be combined with α' in order to give a parameter without dimension. It will be possible to identify the order of the correction term (3.30) in these expansion for each d and n.

The scalar fields S and T depend on these parameter and can be used to replace them. To see this, let us write $s = \mathrm{Re}\,(S) = \varphi^{-1}e^{3\sigma}$ and $t = \mathrm{Re}\,(T) = \varphi e^{\sigma}$ (without the matter fields), then in the sigma model variables:

$$G_{MN}^{\sigma-\mathrm{mod}} = \begin{pmatrix} \frac{1}{s} g_{\mu\nu} & 0 \\ 0 & t\bar{g}_{mn} \end{pmatrix} . \tag{3.31}$$

We know that in these variables, the string loop expansion parameter is φ^4 only, because the other fields are string background fields which are explicitly written without any dependence on the string coupling constant (see chapter **1**).

Let us take the limit $\varphi \to 0$. According to the above, G_{MN} and $\bar{g}_{mn}$ are kept constant, so equation (3.31) implies that t also is. Therefore, in this limit, $e^{-\sigma} \sim \varphi$. This means that $s^{-1} \sim \varphi^4$ is the string-loop expansion parameter.

On the other hand, the σ-model variables are very convenient for the α'-expansion since no fields involve any dependence on α' in this case. Also, the order in α' is then only dimensional analysis, because the Regge slope carries dimensions. It is possible in ten-dimensional field theory to transform to a set of variables where the parameter α' is included in the definition of the fields (*e.g.* $G_{MN} \to \alpha'^{-1} G_{MN}$). In this case, we can choose t to be equivalent to the sigma-model parameter. But, some other fields like s and $g_{\mu\nu}$ will then acquire some α'-dependence. We should be careful when we say that the breathing mode and the Regge slope are equivalent; it is only true in some cases. For this reason, we prefer not to count α' with t, but directly by dimensional analysis.

All the corrections $\bar{\mathcal{G}}$ and $\bar{f}_{ab}$ are normalized in such a way that the term $(1/S)^0$ corresponds to the sphere; a surface of genus γ will contribute to corrections of order $(1/S)^\gamma$. Since $1/S$ is the only string-loop counting parameter in equation (3.28), this explains why we begin the series in equation (3.29) at $n = 0$ only.

Secondly, $\mathcal{V}_i^{(d)}$ is of the order $(1/r_6)^d$, for it depends only on the six-dimensional manifold. We require the sigma model perturbation series to start with order $(\alpha'/r_6^2)^0$ and then continue with terms involving higher power of the ratio α'/r_6^2. The power series (3.29) of the function $\mathcal{F}_i^{(d)}(T^3/S)$ is therefore unrestricted for large n, since the power in α' comes only from the coefficient $\mathcal{V}_i^{(d)}$. For convenience, we will denote the order in the string-loop expansion by the genus γ, or the letter n, and the order in the sigma model expansion by the letter $c = d/2$. This indicates that, in general, d should be an even integer.

This analysis shows that if we restrict ourselves to the case $d = 0$ and $c = 0$

then the correction $\mathcal{C}^{(0)}$ can only be a constant and cannot depend on the scalar fields S and T. This is what we already knew: the truncated Lagrangian is the lowest-order approximation for the heterotic effective action.

Still in the case where $d = 0$, *i.e.* zeroth order in the α'-expansion, the higher-loop contributions to the Lagrangian are thus seen to be abitrary analytical function of the invariant T^3/S with dimensionless coefficients. At the genus n level, the correction is of the form: $\mathcal{V}^{(0)}_{i,n} T^{3n}/S^n$.

We will now explicitly treat the very restrictive case that we already talked about in the preceding section, where all the coefficients $\mathcal{V}^{(d)}_i$ are interpreted as v.e.v.'s of curvatures (Yang-Mills and gravitational). Although this is sufficient to understand the case of string tree level, it need not be general enough to describe all string-loop effects. It will nonetheless give good insights on the form of the corrections even in this case.

3.4.3 V.E.V. Corrections

We will study first the cases for small d's, considering only v.e.v.'s $\mathcal{V}^{(d)}_i$ of the form (3.25), for which $d = 2c$. We need not consider the case $c = 1$ because the compact internal manifold K is supposed to be Ricci-flat, so the v.e.v. $< R >$ vanishes. Moreover the other possible v.e.v. $\sum_{\rm a} < \omega^{mn} F^{\rm a}_{mn} >$ also vanishes because $F^{\rm a}_{mn}$ can be chosen to be proportional to traceless generators of the gauge group. We thus get no correction as function of only one internal curvature.

The first important correction terms arise for $c = 2$. In this case, there exists only one possible v.e.v. with an antisymmetric contraction tensor, since the curvatures satisfy the following condition on a Calabi-Yau space [28]: $\int_{\rm K} {\rm tr} R \wedge R = \int_{\rm K} {\rm tr} F \wedge F$. However, there is also the term coming from the v.e.v.'s $R^{mn} R_{mn}$ and $F^{mn} F_{mn}$. As we said before all these different possible v.e.v.'s are denoted by $\mathcal{V}_i^{(4)}$. We have:

$$\begin{aligned}
\mathcal{V}_1^{(4)} &= \Omega_K^{-1} \int_K d^6 y \epsilon^{mnpqrs} \omega_{mn} < R^{m'}{}_{pn'q} R^{n'}{}_{rm's} >, \\
&= \Omega_K^{-1} \int_K d^6 y \epsilon^{mnpqrs} \omega_{mn} < F^a_{pq} F^a_{rs} >, \qquad (3.32) \\
\mathcal{V}_2^{(4)} &= \Omega_K^{-1} \int_K d^6 y R^{mn} R_{mn}, \qquad etc...
\end{aligned}$$

We derive the form of the correction terms of the form (3.30) involving these v.e.v.'s, *i.e.* still in the case where $c = 2$. The first term ($n = 0$) of the power series of equation (3.29) is responsible for a correction $\mathcal{C}_{i,0}^{(4)} \sim \alpha'^2 \mathcal{V}_i^{(4)}/T^2$. This is coming from the sphere, or order $(1/S)^0$, still at the order $c = 2$ in the sigma-model expansion. The second term ($c = 2$, $n = 1$) arises at torus-level in the string loop expansion, $(1/S)$, and at order $(\alpha'/r_6^2)^2$ in the σ-model expansion. It corresponds to the axion term that is found by truncating the Green-Schwarz anomaly cancellation term. The next section (**3.4.4**) is devoted to an explicit derivation of this term. It is easy to derive the two-loop term: $\mathcal{V}_i^{(4)} T^4/S^2$ and the following for higher loop contributions.

In the case where $c = 3$, the function $\mathcal{F}_i^{(2c)}(T^3/S)$ gives rise to three different terms for each possible v.e.v. Omitting the α'-dependence that is easily recovered by dimensional analysis, we get:

$$\begin{array}{lll}
n = 0: & \mathcal{C}_{i,0}^{(6)} \sim \mathcal{V}_{i,0}^{(6)}/T^3 & \text{(sphere)} \\
n = 1: & \mathcal{C}_{i,1}^{(6)} \sim \mathcal{V}_{i,1}^{(6)}/S & \text{(torus)} \\
n = 2: & \mathcal{C}_{i,2}^{(6)} \sim \mathcal{V}_{i,2}^{(6)} T^3/S & \text{(2-torus)} \\
etc... & &
\end{array}$$

We summarize first v.e.v.correction terms allowed by our symmetry for the two expansions (string loops and sigma model) in table 1.

In general, we see that *there is some v.e.v. correction to the truncated Lagrangian at the each order* (γ, c) *except for* $c = 1$ *and it is proportional to:*

$$\mathcal{C}_{i,n}^{(2c)}(S,T) = \alpha'^c \mathcal{V}_{i,n}^{(2c)} T^{3n-c} S^{-n}, \qquad (3.33)$$

First v.e.v. correction terms $C^{(2c)}_{i,n}$ in string loop and sigma model expansions

$C^{(2c)}_{i,n}$	sphere $\gamma = 0$	torus $\gamma = 1$	2-torus $\gamma = 2$	3-torus $\gamma = 3$	$\cdots$
$(\alpha'/r_6^2)^0$	constant	$\frac{\mathcal{V}_i^{(0)}T^3}{S}$	$\frac{\mathcal{V}_i^{(0)}T^6}{S^2}$	$\frac{\mathcal{V}_i^{(0)}T^9}{S^3}$	$\cdots$
$(\alpha'/r_6^2)^1$	$(\mathcal{V}^{(2)} = 0)$	$\cdots$	$\cdots$	$\cdots$	$\cdots$
$(\alpha'/r_6^2)^2$	$\alpha'^2 \frac{\mathcal{V}_i^{(4)}}{T^2}$	$\alpha'^2 \frac{\mathcal{V}_i^{(4)}T}{S}$	$\alpha'^2 \frac{\mathcal{V}_i^{(4)}T^4}{S^2}$	$\alpha'^2 \frac{\mathcal{V}_i^{(4)}T^7}{S^3}$	$\cdots$
$(\alpha'/r_6^2)^3$	$\alpha'^3 \frac{\mathcal{V}_i^{(6)}}{T^3}$	$\alpha'^3 \frac{\mathcal{V}_i^{(6)}}{S}$	$\alpha'^3 \frac{\mathcal{V}_i^{(6)}T^3}{S^2}$	$\alpha'^3 \frac{\mathcal{V}_i^{(6)}T^6}{S^3}$	$\cdots$
$(\alpha'/r_6^2)^4$	$\alpha'^4 \frac{\mathcal{V}_i^{(8)}}{T^4}$	$\alpha'^4 \frac{\mathcal{V}_i^{(8)}}{TS}$	$\alpha'^4 \frac{\mathcal{V}_i^{(8)}T^2}{S^2}$	$\alpha'^4 \frac{\mathcal{V}_i^{(8)}T^5}{S^3}$	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

Table 1.

where $n = \gamma$.

This exhibits the failure of the naive α'-counting by $1/T$. In fact, this property is only true at string tree-level. For higher loops, we must also take in account the field S which is equivalent to a factor α'^3.

The following section is intended to present in details a special case ($c = d/2 = 2$, $n = 1$), that has been already studied from a different approach. Our results are in complete agreement with the previous one.

3.4.4 Green-Schwarz Axion Term: an Example

Green and Schwarz found that a certain term has to arise as a one-string-loop counter-term in the 10-dimensional action, in order to cancel all the anomalies (Yang-Mills and gravitational) [32]. This term is then responsible for a correction to the gauge coupling function [33]:

$$\bar{f}_{\rm ab}(S,T) \sim \frac{T}{S}\delta_{\rm ab}. \tag{3.34}$$

This would seem to be in contradiction with our BM symmetry because the only invariant combination of the scalar fields alone as been found to be T^3/S. Nevertheless, as we saw in the above discussion, it is not, because of the v.e.v.'s of the six-dimensional curvature fields. It is an explicit counter-example which exhibits the failure of the naive argument that says that only T counts the σ-model perturbative theory.

In more details, let us take the example of one of the anomaly-cancellation terms of Green and Schwarz (Yang-Mills). It looks like:

$$\mathcal{L}_{\rm ct} \sim \alpha'^{-1}\epsilon^{M_1M_2\ldots M_{10}}B_{M_1M_2}F^{\rm a}_{M_3M_4}F^{\rm a}_{M_5M_6}F^{\rm b}_{M_7M_8}F^{\rm b}_{M_9M_{10}}. \tag{3.35}$$

Upon truncation, we get $\mathcal{L}_4 \sim \mathcal{V}\alpha'^2\eta\ \epsilon^{\mu\nu\lambda\rho}F^{\rm a}_{\mu\nu}F^{\rm a}_{\lambda\rho}$ in $D=4$ where the six-dimensional fields $F^{\rm b}_{mn}(y)$ are replaced by their v.e.v. and:

$$\mathcal{V} = \left(\frac{1}{\int d^6y\sqrt{\bar{g}}}\right)\int d^6y\ \epsilon^{mnpqrs}\omega_{mn} < F^{\rm b}_{pq}(y)F^{\rm b}_{rs}(y) > \tag{3.36}$$

($\mathcal{V}$ was one of the previously called $\mathcal{V}^{(4)}$ in equation (3.32)). Under our symmetry, $\mathcal{V} \to \lambda^2\mathcal{V}$ and then the combination $\mathcal{V}T/S$ is invariant. Therefore, the upshot is that we actually have $\bar{f}_{\rm ab}(S,T) \sim \alpha'^2(\mathcal{V}T/S)\delta_{\rm ab}$.

This behavior as $1/S$, according to (3.31), is a clear indication that this correction is coming from the one-string-loop sector. The fact that it is proportional to T should not surprise us. It does not mean that the term is an α'^{-1}-contribution in the σ-model expansion, because in our convention we chose $\mathcal{V} \sim 1/r_6^4$, so $\alpha'^2\mathcal{V}T/S$ is a two-loop term in the sigma model expansion.

3.5 Other Symmetry Considerations

In order to restrict even more the form they can take, it convenient to consider some other known symmetries: the axion symmetries that is valid to all orders in α' and the string-loop expansion, and the $D = 4$, $N = 2$ supersymmetry, which applies only at string tree-level and with matter fields B and C set to zero.

3.5.1 Axion Symmetries

The axion (or Peccei-Quinn) symmetries were exhibited by Witten [28, 34] and hold for all finite order in both perturbative expansions. It is valid for both axions of the theory. These symmetry state that the action is invariant under a shift of the axions by constants:

$$\theta \to \theta + \alpha_1, \qquad \eta \to \eta + \alpha_2, \tag{3.37}$$

where α_1 and α_2 are independent constants. In terms of the scalar fields T and S (as defined in equation (3.7)), this means that the Kähler potential should depend on the scalar fields only through their real parts:

$$s = (S + S^*), \quad t = (T + T^*). \tag{3.38}$$

This symmetry has the important consequence of restricting even more the correction terms of table 1. While every substitution of T^* (resp. S^*) to T (resp. S) was ***a priori*** possible in the general correction term (3.30), now we know that we must replace every entry of table 1, and the general term (3.30), by the corresponding function of s and t, instead of S and T.

The same symmetry can be used to state a non-renormalization theorem for the gauge coupling function f_{ab}. The axion symmetry indeed requires that the real part of the gauge coupling function Re (f_{ab}) be function of t and s, uniquely. However, this holomorphic function can satisfy this condition only if it linear in T and S. The upshot is then:

NR Theorem 2 *The only correction term to the gauge coupling $f_{\mathrm{ab}}(S,T)$ at finite order in the perturbative expansions is proportional to T and comes from the $(\gamma = 1, k = 1)$-level. This is the Green-Schwarz term, as we have seen it,* *which is a one-string-loop term, $o(1/S)$, and of order α'/r_6^2 in the sigma model expansion.*

The conclusion of this theorem was already taken for granted in the past, but the 'proof' given by Cecotti *et al.* was based on an incorrect conjecture formulated by Nilles [10]. The assumption is that, under the transformation $S \to r^{-1/2}S$ and $T \to r^{1/2}T$, the n-string-loop effective Lagrangian had the following scaling property: $\mathcal{L}_n \to r^{n-1/2}\mathcal{L}_n$, being also valid at tree-level ($n = 0$). This behavior would oblige each correction term to be of the form $(T/S)^n$. Our general term (3.30) shows that this is only a special case and the next section will exhibit a term in the effective Lagrangian that has to exist and does not possess the above property. Our demonstration closes the loophole in the precedent argument.

Applied to the superpotential, this symmetry also has the virtue of giving us the possibility of deriving another nonrenormalization theorem. The axion symmetry acting on the superpotential $W(z)$ restricts its form. As before, we demand that it should be a function of the real parts of the scalar fields only, *i.e.* s and t. The solution to this problem is found in reference [9]. The superpotential may be of the following form (we include matter fields C and B, and other moduli G and H; see section **3.6.1**):

$$W(S,T,B,C,G,H) = \exp(aS + bT + c_K G_K)w(C,B,H), \tag{3.39}$$

where a, b and c_K are some constant to be determined.

Let us neglect the other moduli fields G_K and H_L and concentrate on the S- and T-dependence of the superpotential. The exponent looks like a non-perturbative effect. It produces a term proportional to $as + bt$ in the correction $\bar{\mathcal{G}}$ to the Kähler function.

But, we see from the general form (3.30) of the terms in the expansions that the exponent of s is always negative: it is not possible to get a power series for the Kähler function that would be expressed in terms of positive powers of s. If the superpotential has some S-dependence, it is therefore not a perturbative one. On the other hand, once we know that we have no perturbative S-dependence in the superpotential W, the T-dependence can only arise from the sigma-model expansion at the sphere level. The series will not involve positive powers of T as can be seen from (3.30) and cannot contribute to a factor $\exp(bT)$. The nonrenormalization theorem is thus:

NR Theorem 3 *The superpotential W, that is independent of S and T at the truncated level, will not acquire any dependence on these fields at any order in the perturbative expansions.*

The above theorem was proven at string tree-level by Witten [28] and extended to string loops by Dine and Seiberg [34]. Their proofs are essentially repeated here, but we emphasized the fact that since the superpotential appear through its modulus only, an exponential dependence over the fields S and T would have been *a priori* permitted by the axion symmetries. It is only our knowledge of the expansion parameters that allow us to conclude.

The previous considerations now permit us to say that, in the case of the gauge coupling function $f_{ab}(S,T)$ the only perturbative correction is proportional to $V^{(2)}T/S$. For the Kähler function, however, we know that the general correction term $C_n^{(d)}(s,t)$ is given by:

$$C_n^{(d)}(s,t) = \sum_i \alpha'^{d/2} V_{i,n}^{(d)} t^{3n-d/2} s^{-n}, \tag{3.40}$$

where $n = \gamma$. And the superpotential does not get renormalized.

3.5.2 $D = 4$, $N = 2$ Supersymmetry

In the string-loop perturbation series, the terms in the Lagrangian that are contributions from the sphere have a particular property. Using this, we are able to show that the terms coming from surfaces of genus $\gamma = 0$ does not get corrected at any order in the sigma model expansion. This property is the preservation of an $N = 2$ supersymmetry for the sphere-term of the four-dimensional effective action governing heterotic strings [35], instead of the $N = 1$ supersymmetry which has been seen to be an effect of Calabi-Yau compactification and, thus, to be a general symmetry for all order in both perturbative expansions.

This symmetry implies that the Kähler potential is written as a sum of two terms $K(s,t) = K_1(t) + K_2(s)$. On the sphere, the dilaton part $K_2(s)$ cannot be modified at higher order in the sigma model expansion because the correction terms of table 1 corresponding to the sphere do not involve the scalar field s.

The (1,1)-moduli part of the Kähler potential is of the form (see [35, 36]):

$$K_1(t) = -\log Y, \tag{3.41}$$

where $Y(T, T^*)$ is completely determined by a complex holomorphic function $F(T)$ called ***prepotential***:

$$Y(T,T^*) = F + F^* - \frac{1}{2}\left[\frac{dF}{dT} + \left(\frac{dF}{dT}\right)^*\right](T + T^*). \tag{3.42}$$

We can show that, if we require $Y(T, T^*)$ to be function of $t = T + T^*$ uniquely to respect the axion symmetry, then the general solution for the prepotential $F(T)$ is the following:

$$F(T) = -2AT^3 + BT + \alpha, \qquad (A, B \in \mathbf{R}; \alpha \in \mathbf{C}), \tag{3.43}$$

giving: $Y = At^3 + 2\mathrm{Re}\,\alpha$. Notice that the linear term is irrelevant for the Kähler potential $K_1(t)$. This implies that the general form of the Kähler potential at the sphere level is:

$$K(s,t) = -\log(s) - \log(At^3 + a) = -\log(s) - 3\log(t) - \log\left(A + \frac{a}{t^3}\right) \tag{3.44}$$

where $a = 2\mathrm{Re}\,\alpha$ is a constant.

The number of corrections to the truncated Kähler potential is only two. The constant term A is clearly at lowest order in both expansions, since it does not depend on s and t. Truncation implies that it is one.

The other correction that we get is proportional to t^{-3}. Comparing this with the general form (3.30), we observe that it corresponds to $n = 0$ (of course, we are on the sphere!) and $d = 6$. We know that the order in the sigma-model expansion is given by $c = d/2 = 3$. It is an $o(\alpha'^3)$ effect (four loops) and it is the only correction term coming from higher loops.

NR Theorem 4 *The sphere-term of the truncated prepotential does not get corrected at the higher levels in the perturbative expansion in α'/r_6^2, the four-loop orders, at order α'^3. The Kähler potential acquires then corrections at orders α'^3, α'^6,* etc...

This correction to the Kähler potential is exactly the same as the one found by Candelas *et al.* [37] in a special case. It corresponds to the fourth-loop counter-term to the metric that is found in direct sigma-model calculations [38]. This counterterm $\delta\bar{g}_{ij} \sim \partial_i\partial_j\mathcal{V}_\sigma$, where $\mathcal{V}_\sigma$ is a particular case of v.c.v. similar to equation (3.25) with $c = 3$. The counter-term, being at four loops, the constant $\mathcal{V}_\sigma$ is an $o(\alpha'^3)$ term in the sigma-model expansion, by dimensional analysis. It comes directly from a counter-term in the prepotential, so this latter term must be a third-order one. The correction that we have found is thus consistent with the results found in [38].

3.6 Other Moduli and Matter Fields

We can now extend our analysis to the more general cases where there is more than one modulus, *i.e.* some number of $(1,\bar{1})$- and $(2,\bar{1})$-moduli. In addition, we can treat the case where the matter fields are not set to zero. We will not write the detailed analysis for those cases; they are presented in [31], but the results go as follows.

3.6.1 Other Moduli

The inclusion of other moduli is also possible. The corresponding scalar fields that appear in the $D = 4$, $N = 1$ effective Lagrangian are $G^K(x)$ and $H^L(x)$, which are defined as follows [9]:

$$
\begin{aligned}
H^L &= h^L + \text{non-linear terms}, \\
G^K &= k^K + i\sqrt{2}\,\Omega_K^{1/3} A^K + \text{non-linear terms},
\end{aligned}
\tag{3.45}
$$

where $K = 2, \dots, b_{1,\bar{1}}$; we write $G^{K=1} = T$.

Then we have $H^L \to H^I$ and $G^K \to G^K$ (except for $K = 1$ of course!) under the BM symmetry (3.14). It is therefore to be expected that this symmetry give no new information to restrict the appearance of the moduli fields in the effective action: as far as the BM symmetry is concerned they could appear in arbitrary manner.

However, since an axion symmetry is also valid for the other $(1,\bar{1})$-moduli, the fields $G^K(x)$ [9], the correct combination to incorporate is $(G^K + G^{*K})$. On the other, since we do not know the general form of the terms involving G^K, for $K \geq 2$, then we cannot conclude that the perturbative expansion does not produce the exponential term in the superpotential of equation (3.39). This term however would be a tree-level term in the sigma-model expansion, because G^K does not count α'. The upshot of the superpotential NR theorem remains: *W is not renormalized in the σ-model expansion.*

If we require the Kähler potential to depend only on the real part of the $(1,\bar{1})$-moduli, T and G^K, the same $N = 2$ supersymmetric arguments as before, indicates that the prepotential $F(G^K)$ (corresponding to (3.43)) has to be of the form :

$$
\begin{aligned}
F(T, G^K) &= AT^3 + A_K G^K T^2 + A_{JK} G^J G^K T + A_{JKL} G^J G^K G^L \\
&\quad + BT + B_K G^k + \alpha,
\end{aligned}
\tag{3.46}
$$

for $J, K, L = 2, \dots, b_{1,\bar{1}}$. In order for each term of this function to scale consistently under the BM symmetry, we must require that the constants A and B

transform under the symmetry by a factor λ for each their indices. For instance:

$$A \to A, \qquad A_{JKL} \to \lambda^3 A_{JKL}, \qquad B_K \to \lambda B_K. \tag{3.47}$$

This indicates that the constants in equation (3.46) carry a counting of the Regge slope: *e.g.* $A = o(\alpha'^0)$, $A_{JKL} = o(\alpha'^3)$, *etc.* The corrections to the Kähler potential would then be of each order: first, second and third. But, the direct calculations of the counter-term gives only a four-loop contribution to the metric as do general arguments in [38], so we must have uniquely a third order correction term in the Kähler potential and $A_K, A_{JK} = 0$. The Kähler potential is thus in general:

$$K_1(t, g^K) = -\log(A't^3 + A'_{JKL}g^J g^K g^L + a), \tag{3.48}$$

with A' a constant involving no power of α', while A'_{JKL} and a are constants of the type $\alpha'^3\mathcal{V}_i^{(6)}$. We also defined $g^K = G^K + G^{*K}$, and we must keep in mind that $J, K, L = 2, 3, \ldots, b_{1,\bar{1}}$. This shows that, in the argument of the logarithm of the Kähler potential, completely decouples the breathing mode and the other $(1,\bar{1})$-moduli completely decouple.

3.6.2 Matter Fields

For the case of the single moduli internal space, the fields $C^\xi \to \lambda^{1/2}C^\xi$ under our BM symmetry (equation (3.18)), and they carry the dimension of *Mass*, so they must be coupled to $\alpha'^{1/2}$. The dimensionless invariant combination under the BM symmetry is thus $\alpha'^{1/2}C^\xi/\sqrt{t}$ or its complex conjugate. These terms are of order $c = 1/2$ in the sigma-model expansion. This means that the general correction term (3.40) can be multiplied by some power of these combinations arbitrary large.

$$C_n^{(d)}(s,t) = \sum_i \alpha'^d \mathcal{V}_{i,n}^{(d)} t^{3n-d/2} s^{-n} \mathcal{H}\left(\frac{C^\xi}{\sqrt{t}}, \frac{C_\xi^*}{\sqrt{t}}\right), \tag{3.49}$$

where $\mathcal{H}$ is an arbitrary function. Moreover, $C_n^{(d)}$ is of order n in the string-loop expansion, and, depending on the powers of the arguments the function $\mathcal{H}$, the order in the α' expansion changes. If we have a power m for the ratio $C^{\xi}/\sqrt{t}$ (or with C^*_{ξ}), then the order in the sigma model expansion is: $\frac{1}{2}(d+m)$.

This result generalizes with no further complications for the case with many moduli, with the matter fields $C^{K\xi}$ and $B\xi^L$ taking the place of C^{ξ}.

The truncated Kähler potential exhibit an interesting SL(2,**R**) symmetry (see [39]) that would indicate that the lowest-order corrections need to be functions of $C^*_{\xi}C^{\xi}/t$ uniquely. This symmetry, however, is not *a priori* a general property of all perturbative orders. Moreover, this symmetry seems to be broken by the inclusion of the other $(1,\bar{1})$-moduli, if they exist. This particular combination of the fields should not be considered as the most general term appearing in the effective action.

Chapter 4

Conclusion

This study was motivated by the question: What are the manifestations of string theory at low-energy? We answer at least some part of this question. We are allowed to say that, if we take the framework of an $N = 1$, $D = 4$ supergravity coupled with Yang-Mills, the corrections to the well-known lowest-order effective Lagrangian are given by corrections to the Kähler potential expressed in equation (3.40). The exact coefficient of each term is, of course, unknown at this stage, and moreover, is expected to be function of the topology of the internal manifold. However, the identification of the terms that are allowed is a step in the forward direction for the understanding of the low-energy manifestations of heterotic string theory.

Let us summarize the procedure we use to derive this result. Our key-observation was that the definition of the metrics of four-dimensional spacetime and six-dimensional internal manifold, hides an ambiguity. We extend this symmetry to hold for the full effective theory in ten-dimensions, and therefore for the four-dimensional action at each order in both perturbative expansions of the theory.

Our general conclusion is that we need to take in account constants that are formed by internal degrees of freedom. They were called $\mathcal{V}_i^{(2c)}$ and we saw that c was at the same time half of the engineering dimension of the coefficients and the order in the sigma-model perturbative expansion. The level in the string-loop expansion is given by S^{-1}, as it is known for already some time. The new information is that once both of these orders are chosen the field $T(x)$ must appear to a specific power T^{3n-c}. The general correction term is given by equation (3.30) that we repeat here:

$$C_n^{(2c)}(s,t) = \sum_i \alpha'^c V_{i,n}^{(2c)} t^{3n-c} s^{-n}, \tag{4.1}$$

With this and other symmetry considerations, we may state four nonrenormalization theorems, plus another one that is derived without the use of the above formula in section **3.1.1**:

1. At lowest order in the α'-expansion, the heterotic string effective action is the Chamseddine-Chapline-Manton action and it does not get corrected by effects of higher order in the string-loop expansion.

2. The only correction term to the gauge coupling function $f_{ab}(S,T)$ at finite order in the perturbative expansions is proportional to T and comes from the ($\gamma = 1, c = 2$)-level. This is an effect of the anomaly cancellation term of Green and Schwarz.

3. The superpotential W, that is independent of S and T at the truncated level, will not acquire any dependence on these fields at any order in the perturbative expansions.

4. The prepotential of the sphere-term of the truncated Kähler potential, for $B = C = 0$, does not get corrected at the higher levels in the perturbative expansion in α', except at $o(\alpha'^3)$ (four loops in the sigma model).

The extension of the BM symmetry to the other moduli and to the matter fields does not restrict their appearance in the correction terms to which they contribute. However, some of the above theorems still apply or are slightly modified.

The very lowest order term ($\gamma = 0$, $c = 0$) can be renormalized by a function of the other moduli. The symmetry does not restrict the appearance of these moduli fields in the gauge coupling function either. The superpotential could depend on H, but it does not get renormalized at any finite order. Finally, the NR theorem 4 is not modified by the inclusion of other moduli, but the counter-term is.

An interesting extension of these conclusions would be to verify that they hold for orbifolds, a case where the corrections are sometimes explicitly computable. We could then determine the above unknown coefficients for some of these special cases. Also, it would be interesting to how these conclusions can be applied to the case of four-dimensional strings with non-linear realization of the worldsheet supersymmetry.

The results that we will find in the following appendix are summarized in their own conclusion at the end of the appendix (section **A.6**).

Appendix A

Regularization and String Theory

This appendix is ***not closely*** related with the rest of the thesis! However, the little history of this research made us focus our attention on that subject in first place and for some period of time, we thus want to share our experience, here.

As it is expressed by the first section of this appendix, the functional measures, $[dx]$, $[d\lambda]$ and $[d\psi]$, that enter the Polyakov's framework for quantization of strings contains many ambiguous expressions: functional determinants or infinite products over a constant. One may be, with reason, bothered by them. This appendix is intended to make explicit these ambiguities. The key tool that we use here is regularization. This concept is relatively new in physics (fifty years) and it appears with the infinities that are involved in perturbative language and in functional analysis framework. It is related to the well-known mathematical study of analytical continuation of holomorphic functions in the framework of classical complex analysis.

We have included the following discussion within this paper, for at least two more reasons: first, some of these regularization schemes are very general and can be applied to various cases, not restricted to string theory and two-dimensional physics. Next, we investigate several methods for regularizing the same quantities and we feel that the comparison between them, that perfectly agree, is worthwhile.

The structure of the discussion is the following: we begin by motivate the study of partial measures for our two-dimensional fields, that do not involve the string tension T [1]. This leads us to the study of ratios of determinants and anomalies for the partial measures. We compute these, both in general and in particular cases, using different methods (Weyl anomaly, Gilkey procedure and explicit computations), showing some relation between zeta function of operators that involve zero-modes and those for which they are removed. Finally, we conclude by a small section where the regularized values of those expressions are stated.

A.1 String Field Measures

First, let us recall a result of standard differential geometry on finite-dimensional manifolds. It is possible to define the measure on a manifold K by the measure on the tangent bundle TK. This is because a set of coordinates $\{q^i\}$ of K defines a natural set of coordinates $\{p^i\}$ on the tangent space $\mathrm{T}_q\mathrm{K}$. They are explicitly given by the parameterization: $v = p^i(\partial/\partial q^i)_q$ for any $v \in \mathrm{T}_q\mathrm{K}$. If we have another set $\hat{q}^i$ on K, then we get $\hat{p}^i$ as new coordinates of the tangent space, with Jacobian

$$J(q) = \det\left|\frac{\partial \hat{p}^i}{\partial p^j}\right|_q = \det\left|\frac{\partial \hat{q}^i}{\partial q^j}\right| \tag{A.1}$$

This shows that $J(q)$ depends indeed only on the position q on the manifold and not on where we are on $\mathrm{T}_q\mathrm{K}$. A volume form $\omega = f(q)dq^1 \cdots dq^n$ on K

[1] **Important notice:** through all this appendix, we use the string tension T, instead of the Regge slope. This is not to be confused with the breathing mode scalar field T in the four dimensional effective action (see chapter **2**)

thus induces an $\Omega = f(q)dp^1 \cdots dp^n$ on $\mathrm{T}_q\mathrm{K}$ which transforms with the same Jacobian $J(q)$. If for some reasons, it is more convenient to work on TK, we can turn this argument around and define on the tangent bundle the measure of K and its transformation law under a change of variables.

There is nothing that forbids us to extend this argument to our case of an infinite-dimensional manifold. The measures of the worldsheet metric $[dh]$ and of the gravitino $[d\chi]$ are formally defined this way [16]. But we can use worldsheet symmetries (general covariance, supersymmetry, Weyl and super-Weyl scalings) to gauge away all the freedom contain in the functional measures. There remain, then, a Jacobian and a integration over a finite number of super-moduli (or super-Teichmüller parameters). The measures of the metric and the gravitino are completely independent of the ten-dimensional fields and of the string tension. We will use the above remark to define the string field measures $[dx]$, $[d\lambda]$ and $[d\psi]$ that do depend on the background field and/or the Regge slope.

A.1.1 Functional Measure for $x^M(\sigma)$

The measure for $x^M(\sigma)$ is much easier to define than that for the worldsheet metric (see references [16]). We define the norm of an element $\delta x^M(\sigma)$ of the tangent space, $\mathrm{T}\mathcal{E}$ of the space of embeddings, $\mathcal{E}$. We ask that it must be invariant under worldsheet reparameterization, and ultra-local, meaning that we do not couple different locations on the worldsheet in the definition of the norm (the metric over $\mathrm{T}\mathcal{E}$ must be proportional to $\delta(\sigma - \sigma')$ [40]). If moreover, we want this expression to be generally covariant with respect to background spacetime, then we are forced to define this norm as follows:

$$\| \delta x \|_T^2 = AT \int d^2\sigma \sqrt{h}\, G_{MN}(x) \delta x^M \delta x^N, \tag{A.2}$$

where the constant A is unphysical and arbitrarily fixed; in fact, we will neglect it! The subscript T is included to emphasize the dependence of the norm on string tension.

The dependence on T in equation (A.2) is crucial for our purpose. It comes in as a consequence of the requirement that the norm be dimensionless, since we implicitly define the x-measure in the exponential of a Gaussian integral. Let us note the following:

$$\int \left(\prod_{\sigma,M} d(\delta x^M(\sigma)) \right) e^{-\|\delta x\|_T^2/2} = \mathrm{Det}^{-1/2}[TG_{MN}(x(\sigma))\delta(\sigma - \sigma')]. \quad \text{(A.3)}$$

We require that the measure $[dx]_T$ satisfies $1 = \int [dx] \exp(- \| \delta x \|_T^2 /2)$, just as previously. This condition and equation (A.3) imply that the x-measure is:

$$[dx]_T = \mathrm{Det}^{1/2}[TG_{MN}(x(\sigma))\delta(\sigma - \sigma')] \prod_{\sigma,M} dx^M(\sigma). \quad \text{(A.4)}$$

The expression $\mathrm{Det}[TG_{MN}(x(\sigma))\delta(\sigma - \sigma')]$ is a functional determinant. It results in the introduction within the measure $[dx]_T$ of the following factor: $\prod_\sigma \det^{1/2}[G_{MN}(x(\sigma))]$, as well as some T-dependence requiring some kind of regularization. The former factor is essential in order to assure the general covariance of the partition function (see [41]), and it remains to be interpreted.

A finite dimension example could perhaps be more clear for the implications of the T-dependence of the measure $[dx]_T$. We choose for simplicity a flat background metric and we take the variable of integration to be y^i, where $i = 1, 2, \ldots, n$. Then, according to (A.4), the measure is $[dy] = T^{n/2} d^n y$, implying that:

$$Z = \int [dy] e^{-\frac{T}{2} y^T \mathbf{M} y} = \det(\mathrm{M}), \quad \text{(A.5)}$$

and ***not*** $\det(T\mathbf{M})$, as we could have thought. This is consistent with the requirement that the partition function be dimensionless, as can be deduced from the S-matrix expression (2.1).

A.1.2 Polyakov Measure for Fermions

The usual way to treat fermions in a path-integral formalism is to use Grassmann variables. The integration over these anti-commuting variables is quite peculiar.

As an illustration of this strange behavior, let us derive the Jacobian of the transformation $\eta \rightarrow \theta = a\eta$, which will be of some importance to us (here, a is some fixed real or complex number). We know that Grassmann integration implies

$$\int d\theta\, \theta = 1 \qquad \int d\theta 1 = 0. \tag{A.6}$$

Defining $J(a)$ to be the Jacobian, we trivially see that $J(a) = 1/a$; it is precisely the inverse of the ordinary result for commuting real variables.

This rule has one important consequence: when we define the norm of some spinor field to be $\| \delta\psi \|_T^2 = \mathcal{G}_{ij}\delta\psi^i\delta\psi^j$, where i and j represent all the indices of the spinor (gauge or target spacetime index, position on the worldsheet, etc.), the measure is then given by $\mathrm{Det}^{-1/2}(\mathcal{G}_{ij})\prod_i d\psi^i$. This is to be compared with the Riemann rule in which the square root of the determinant appears in the numerator.

Once this is noted, we may set the spinor norms as before, demanding that they be ultra-local, dimensionless, invariant under worldsheet reparameterization and generally covariant with respect to background ten-dimensional spacetime. We know the dimensionality of λ^M and ψ^s from the action (2.28), so the measures are given by:

$$\begin{aligned} \| \delta\lambda \|_T^2 &= T\int d^2\sigma\sqrt{h}\, G_{MN}(x)\delta\lambda^M\delta\lambda^N, \\ \| \delta\psi \|_T^2 &= T\int d^2\sigma\sqrt{h}\, \delta_{st}\delta\psi^s\delta\psi^t. \end{aligned} \tag{A.7}$$

Note that T-dependence is required to make the norms dimensionless. The fermionic measures are then implicitly defined by the usual Gaussian integrals. The result is therefore very similar to the previous one for $x^M(\sigma)$, except that one gets the inverse power, since the integrations are over Grassmann variables.

$$\begin{aligned} [d\lambda]_T &= \mathrm{Det}^{-1/2}[TG_{MN}(x(\sigma))\delta(\sigma-\sigma')]\prod_{\sigma,M} d(\delta\lambda^M(\sigma)), \\ [d\psi]_T &= \prod_{\sigma,s}\{T^{-1/2}d(\delta\psi^s(\sigma))\} \end{aligned} \tag{A.8}$$

The latter infinite product over a constant may cause some problems in the integration, but as we will see, it is not the case. Regularization techniques applicable in this case will also be developed in the following sections.

A.2 Jacobians for the DS Transformation

In the previous section, equations (A.4) and (A.8), we introduced the functional determinant: $\mathrm{Det}[TG_{MN}(x(\sigma))\delta(\sigma-\sigma')]$ and the infinite product $\prod_{s,\sigma} T^{1/2}$. These two expressions are basically the same thing, for the determinant is also an infinite product. In order to interpret these quantities, we need to know how to regularize them.

For the moment, we will be more interested in their T-dependence. This leads us naturally to introduce a norm for the fields independent of the string tension in order to single out this dependence. For example, in the case of $\psi^s(\sigma)$, which is simpler because the part of the action (2.28) involving ψ is quadratic in this field, and the definition of its norm does not involve G_{MN}. Similarly to (A.7), the ψ-norm is defined as:

$$\| \delta\psi \|_1^2 = \int d^2\sigma\sqrt{h}\delta\psi^s\delta\psi^s. \tag{A.9}$$

Consistently with the previous notation, the subscript 1 in (A.9) indicates that the norm is independent of T. We will write as $[d\psi]_1$ the corresponding measure verifying:

$$1 = \int [d\psi]_1 e^{-\|\delta\psi\|_1^2/2}. \tag{A.10}$$

This measure cannot be taken as the full measure for $\psi^s(\sigma)$ for partition function evaluation, because it is equivalent to (see equation (A.8)):

$$[d\psi]_T = \prod_{s,\sigma} T^{1/2}[d\psi]_1. \tag{A.11}$$

Clearly, $[d\psi]_1$ has dimensions and is not equivalent to $[d\psi]_T$.

Let us introduce a dimensionful scaling (DS) transformation: we define $\tau = T^{-1}\Lambda^2$. Then, the transformation is:

$$\left.\begin{array}{l} \tilde{G}_{MN} = \tau^{-1}G_{MN} \\ \tilde{B}_{MN} = \tau^{-1}B_{MN} \\ \tilde{\psi} = \tau^{-1/2}\psi \end{array}\right\} \Longrightarrow I(T) = I(\Lambda^2). \tag{A.12}$$

This transformation cannot have an anomaly since it is equivalent to dimensional analysis: each tensor of type (r,s) then carries a dimension of $(Mass)^{r-s}$.

However, the norm defined in equation (A.9) has not the right behavior under a DS transformation (A.12). The net result of such a transformation is to scale the norm by a factor of τ. It is then to be expected that the measure $[d\psi]_1$ acquire some Jacobian $J_\psi(\tau)$ under the DS transformation (A.12). If the transformation has no anomaly, then this Jacobian will cancel exactly the factor coming from the infinite product, namely $\prod_{s,\sigma}\tau^{-1/2}$.

We can compute this Jacobian $J_\psi(\tau)$ noticing that

$$\begin{aligned} N_1 \mathrm{Det}^{1/2}(\Lambda^2 \not{D}) &= \int [d\tilde{\psi}]_1 \; e^{-I_\psi[\tilde{\psi},\Lambda^2]} \\ &= J_\psi(\tau)\int [d\psi]_1 \; e^{-I_\psi[\psi,T]} \\ &= N_1 J_\psi(\tau)\, \mathrm{Det}^{1/2}(T\not{D}). \end{aligned} \tag{A.13}$$

where N_1 is some numerical constant. In (A.13), we denoted I_ψ the part of the heterotic string action (2.28) involving $\psi^s(\sigma)$ and we let:

$$I_\psi[\psi, T\not{D}] = \frac{T}{2}\int d^2\sigma\sqrt{h}\;\bar{\psi}\not{D}\psi, \tag{A.14}$$

Hence, the Jacobian $J_\psi(\tau)$ is given by the formula:

$$J_\psi(\tau) = \frac{\mathrm{Det}^{1/2}(\Lambda^2\not{D})}{\mathrm{Det}^{1/2}(T\not{D})}. \tag{A.15}$$

Obviously, this can be also applied to the other two-dimensional fields x^M and λ^M. In the same way, we define the norms $\| \delta x \|_1^2$ and $\| \delta\lambda \|_1^2$ that are independent of string tension T, and then we set the measures $[dx]_1$ and $[d\lambda]_1$ using the same condition as before. We then get:

$$[dx]_T = \frac{\mathrm{Det}^{1/2}[G_{MN}(x(\sigma))\delta(\sigma - \sigma')]}{\mathrm{Det}^{1/2}[TG_{MN}(x(\sigma))\delta(\sigma - \sigma')]}[dx]_1. \tag{A.16}$$

And similarly for $[d\lambda]$, apart that the relation is inverse due to the spinorial nature of this field. Such a ratio of determinant is something we would like to interpret.

Again, the knowledge of the fact that there is no anomaly for the full measures $[dx]_T$ and $[d\lambda]_T$ allows us to relate this with some Jacobians in the same way as before. For example, working with bosonic strings in flat Euclidean background spacetime the Jacobian $J_x(\tau)$ for the DS transformation (A.12) on the measure $[dx]_1$ is given by:

$$J_x(\tau) = \frac{\mathrm{Det}^{-1/2}(-\Lambda^2\Box)}{\mathrm{Det}^{-1/2}(-T\Box)}, \tag{A.17}$$

$\Box = \delta_{MN}\Box_0$, and $\Box_0$ is defined by:

$$\Box_0 = \frac{1}{\sqrt{h}}\partial_\alpha(\sqrt{h}h^{\alpha\beta}\partial_\beta). \tag{A.18}$$

The Jacobian of the measure $[d\lambda]_1$ is given by some expression more similar to $J_\psi(\tau)$ (equation (A.15)). Those Jacobians should cancel exactly the scaling coming from the ratio of functional determinants.

This kind of ratios of functional determinants is what we will be interested to compute in the next sections using several methods. This problem can be generally stated as the regularization of an expression like: $\mathrm{Det}(\tau\Delta)/\mathrm{Det}(\Delta)$, where the determinant is an functional one, τ is some parameter (usually the one defined in equation (A.12)) and Δ some (at most quadratic) operator. In the finite case, Δ is a $n \times n$ matrix and the above expression is equal to τ^n, but as $n \longrightarrow \infty$, some regularization is needed. This is what we will do in the next sections using different methods.

A.3 Weyl Anomaly

The first of our methods uses the fact the effect of the DS symmetry on the norm defined without the string tension in them is very similar to the action of a Weyl transformation. We can then exploit this in order to compute the Jacobians for each of the partial measures $[dx]_1$, $[d\psi]_1$ and $[d\lambda]_1$.

As we already mentioned several times, it happens that, when an action I is invariant under some symmetry transformation of the fields, the partition function is not. This is due to the fact that the functional measures of the fields are not always invariant under the symmetry themselves. Weyl transformation, for example, induces an anomaly, that has been a central issue in string theory ever its debut. It was first computed using functional methods by Polyakov [12].

A Weyl scaling of the worldsheet metric, $h_{\alpha\beta} \to \xi h_{\alpha\beta}$ also makes the norm (A.9) scale by some power of ξ. So, it will not notice the difference between those two transformations (DS and Weyl) provided that we choose ξ to be the required power of τ.

For convenience, we will also denote $\xi = e^{\epsilon}$ so that the Weyl scaling of the worldsheet metric will be equivalently written as $\delta h_{\alpha\beta}(\sigma) = \epsilon h_{\alpha\beta}(\sigma)$, for infinitesimal variation. In the same way, we will write $\tau = e^{\omega}$ and the DS transformation law (A.12) becomes for the metric of target spacetime: $\delta G_{MN}(x) = \omega G_{MN}(x)$, where ω is also small.

We know that the zweibein is such that: $(e_a)^{\alpha}(e^a)^{\beta} = h^{\alpha\beta}$. So under a Weyl transformation, $\delta(e_a)^{\alpha} = -\frac{\epsilon}{2}(e_a)^{\alpha}$. Now if the action is to be Weyl invariant, then the fields must transform as follow:

$$\begin{aligned}
\delta x^M &= 0, \\
\delta\lambda^M &= -\tfrac{\epsilon}{4}\lambda^M, \\
\delta\psi^s &= -\tfrac{\epsilon}{4}\psi^s,
\end{aligned} \tag{A.19}$$

On the other hand, the effect of the DS transformation (A.12) is the same for all norms $\| \delta x \|_1^2$, $\| \delta\psi \|_1^2$ and $\| \delta\lambda \|_1^2$. It consists simply in the scaling of those norms by a factor of τ.

We will symbolically write $\mathcal{W}(\xi)$ for the Weyl transformation (A.19) with parameter $\xi = e^{\epsilon}$ and $\mathcal{S}(\tau)$ for the transformation (A.12) with parameter τ. Comparing the effect of those two transformations of the fields on the measures defined with the T-independent norms, we find that, even for finite change:

$$\begin{aligned} \text{for} \quad [dx]_1 :& \quad \mathcal{S}(\tau) \Longleftrightarrow \mathcal{W}(\tau), \\ \text{for} \quad [d\lambda]_1 :& \quad \mathcal{S}(\tau) \Longleftrightarrow \mathcal{W}(\tau^2), \\ \text{for} \quad [d\psi]_1 :& \quad \mathcal{S}(\tau) \Longleftrightarrow \mathcal{W}(\tau^2). \end{aligned} \tag{A.20}$$

The symbol '$\Longleftrightarrow$' in the above expression do not mean that the effects of each of the above field transformations on the full measures are equivalent, but only the changes of the T-independent norms of the fields are.

The author of [42] computes the effect of a Weyl transformation on the measures for superstrings (using implicitly the heat-kernel procedure that is the subject of the next section). Under a Weyl scaling $\mathcal{W}(e^{\epsilon})$ they change as follow:

$$\delta[dx]_1 = -\epsilon\left(\frac{D}{48\pi}\int d^2\sigma\sqrt{h}R_h\right)[dx]_1 = -\epsilon\frac{D}{12}\chi(\mathrm{M})[dx]_1, \tag{A.21}$$

and

$$\delta[d\lambda]_1 = -\epsilon\left(\frac{D}{192\pi}\int d^2\sigma\sqrt{h}R_h\right)[d\lambda]_1 = -\epsilon\frac{D}{48}\chi(\mathrm{M})[d\lambda]_1, \tag{A.22}$$

where, using Gauß-Bonnet theorem, we made explicit the dependence over $\chi(\mathrm{M})$, which is the topological Euler characteristic of the worldsheet M.

The effect on the λ-measure of the Weyl transformation in equation (A.22) differs from the result in [42] by a factor one half. This is due to the fact that we are here dealing with Majorana-Weyl fermions, while, in the paper, the fermions had only to verify Majorana condition. Hence, the trace over spin indices contributes to a factor half less.

For ψ-spinors, we have a similar action, thus a similar result. The only change is in the number of fermions, here $N_F = 32$ instead of D:

$$\delta[d\psi]_1 = -\epsilon\left(\frac{N_F}{192\pi}\int d^2\sigma\sqrt{h}R_h\right)[d\psi]_1 = -\epsilon\frac{N_F}{48}\chi(\mathrm{M})[d\psi]_1. \tag{A.23}$$

We thus get the Jacobians that we were looking for:

$$J_x(\tau) = \tau^{-D\chi(\mathrm{M})/12}, \qquad J_\lambda(\tau) = \tau^{-D\chi(\mathrm{M})/24}, \qquad J_\psi(\tau) = \tau^{-N_F\chi(\mathrm{M})/24}. \tag{A.24}$$

In the next section, we will examine a way to compute exactly the $[d\psi]_1$-anomaly. This way is more direct than the previous one and makes use of the explicit change in ψ under the transformation (A.12). Also, this procedure will allow us to do the regularizations that we need. This is what we call the Gilkey heat-kernel procedure.

A.4 Gilkey Heat-Kernel Procedure

A.4.1 Zeta Function

We will use the ζ-function trick to regularize the ratio of functional determinants that gives the above Jacobians. If we define μ_n to be the eigenvalues of the operator Δ, then we say that the functional determinant of Δ is the product over all μ_n's. However, there could be difficulties in evaluating this infinite product, so the regularization process goes as follows. First let the function $\zeta_\Delta(s)$ be defined by:

$$\zeta_\Delta(s) = \sum_n \frac{1}{\mu_n^s}, \tag{A.25}$$

for all s contained in the region of the complex plane where the series converge. The ζ-function is then analytically continued to $s = 0$ and we obtain that:

$$\mathrm{Det}(\Delta) = e^{-d\zeta/ds}\Big|_{s=0}. \tag{A.26}$$

But the essential property to remark for our purpose is that we may now express $\zeta_{\lambda\Delta}(s)$ in term of $\zeta_\Delta(s)$, for any constant λ. This is done as follows:

$$\zeta_{\lambda\Delta}(s) = \sum_n (\lambda\mu_n)^{-s} = \lambda^{-s}\zeta_\Delta(s). \tag{A.27}$$

Differentiating the two sides of the last equality, we get:

$$\zeta'_{\lambda\Delta}(s) = -\lambda^{-s}\log(\lambda)\zeta_\Delta(s) + \lambda^{-s}\zeta'_\Delta(s). \tag{A.28}$$

Now, evaluating the last equation at $s = 0$, using the definition (A.26) for determinants and, say, the expression (A.15) that we have got for $J_\psi(\tau)$ (for which Δ is $\not{D}$, we are in position to write, in term of the function $\zeta_\Delta(s)$, the Jacobian that we are looking for. This gives:

$$\begin{aligned} J_\psi(\tau) &= \left(e^{\frac{1}{2}\zeta_\Delta(0)\log\Lambda^2 - \frac{1}{2}\zeta'_\Delta(0)}\right)\left(e^{-\frac{1}{2}\zeta_\Delta(0)\log T + \frac{1}{2}\zeta'_\Delta(0)}\right), \\ &= \tau^{\frac{1}{2}\zeta_\Delta(0)}, \end{aligned} \tag{A.29}$$

remembering that we defined $\tau = T^{-1}\Lambda^2$. Therefore, the only important quantity to regularize this kind of ratio of functional determinants is $\zeta_\Delta(0)$. Gilkey heat-kernel procedure allows us to do exactly this.

A.4.2 Gilkey coefficients

In this section, we will adopt the convention to simply write $\zeta(s)$ for $\zeta_\Delta(s)$. Now let us turn to Gilkey's method, that will allow us to compute $\zeta(0)$. It is in fact a general procedure to obtain the ζ-function of a second order differential operator. Let us explain how it is done.

For μ_n and $\psi_n(\sigma)$, respectively eigenvalues and eigenfunctions of the operator Δ, let us define the ***heat-kernel function*** $K(\sigma, \sigma'; t)$ to be the following:

$$K(\sigma, \sigma'; t) = \sum_n e^{-\mu_n t}\psi_n(\sigma)\psi_n^\dagger(\sigma'). \tag{A.30}$$

Then we can show that:

$$\zeta(s) = \frac{1}{\Gamma(s)}\int_0^\infty dt\, t^{s-1}\int d^2\sigma\ \mathrm{tr}K(\sigma, \sigma; t). \tag{A.31}$$

The trace is to be taken over all indices (spin, gauge, etc.) of the eigenstates $\psi_n(\sigma)$. Then, Gilkey has shown that it is possible to evaluate the ζ-function of an operator written in the form:

$$\Delta = m^2 - h^{\alpha\beta}\partial_\alpha\partial_\beta - P^\alpha(\sigma)\partial_\alpha - Q(\sigma), \tag{A.32}$$

where, as we can see, $Q(\sigma)$ does not involve the regularization factor m^2. For a operator of the above form, we can explicitly compute the *Gilkey coefficients* $a_k(\sigma)$ which are defined by the coincidence limit of the heat-kernel function[2]:

$$K(\sigma,\sigma;t) = \frac{1}{4\pi}\frac{\sqrt{h}}{t}e^{-m^2 t}\sum_{k=0}^{\infty} a_k(\sigma)t^k, \tag{A.33}$$

where, as before, $h = \det(h_{\alpha\beta})$ is the determinant of the worldsheet metric.

Gilkey's results for the coefficients $a_k(\sigma)$ can be found in [43]. For our purpose, as we are about to see, the knowledge of the second coefficient $a_1(\sigma)$ is sufficient. Following Gilkey, we first define:

$$K^\alpha = \frac{1}{2}\left(P^\alpha + g^{\beta\gamma}\Gamma^\alpha_{\beta\gamma}\right) \tag{A.34}$$

and

$$\varepsilon = Q - \partial^\alpha K_\alpha - K^\alpha K_\alpha + g^{\beta\gamma}\Gamma^\alpha_{\beta\gamma}K_\alpha. \tag{A.35}$$

with P^α and Q as in equation (A.32). Then Gilkey has shown that:

$$a_1(\sigma) = \frac{1}{6}(R_h + 6\varepsilon), \tag{A.36}$$

where R_h is the Riemann scalar curvature or the worldsheet M.

Let us see how we get that the ζ-function, as expressed in (A.31), reduces to the only $k = 1$ term of the heat-kernel expansion (A.33), when we evaluate it at $s = 0$. For convenience, we introduce the following functions:

$$\begin{aligned} c_k(s) &= \int_0^\infty dt\; t^{s+k-2}e^{-m^2 t} \\ &= m^{2(1-s-k)}\Gamma(s+k-1). \end{aligned} \tag{A.37}$$

Using this definition, it is easy to rewrite our zeta function as:

$$\zeta(s) = \frac{1}{\Gamma(s)}\sum_{k=0}^{\infty} c_k(s)\int d^2\sigma \frac{\sqrt{h}}{4\pi}\,\mathrm{tr}[a_k(\sigma)] \tag{A.38}$$

[2]In the following, all the calculations are in general explicitly done for a two-dimensional manifold (the worldsheet), but Gilkey's procedure applies equally well for operators defined in a space of arbitrary dimension.

In this expression, all the dependence over s is in the ratio $\rho_k(s) \equiv c_k(s)/\Gamma(s)$. What is important to us is the behavior of this ratio $\rho_k(s)$ in the limit $s \to 0$.

As $z \to 0$, the Euler gamma function $\Gamma(z)$ has a simple pole. But it also has a simple pole for every negative integer $z = -n$. From this, we clearly see that, $\Gamma(s+k-1)$ also gets a pole when $s \to 0$, but only for the values $k = 0$ or 1. In fact, the Laurent series of $\Gamma(z)$ near any of these poles is well known.

$$\begin{aligned} k=0: \quad & \Gamma(s-1) = -\frac{1}{s} + (\gamma - 1) + o(s), \\ k=1: \quad & \Gamma(s) = \frac{1}{s} - \gamma + o(s), \end{aligned} \tag{A.39}$$

$\gamma = 0.5772\ldots$ being the Euler-Mascheroni constant. The limit of the ratios $\rho_k(s)$ as $s \to 0$ thus vanishes for all k, except for the two values $k = 0$ and 1. Their value are given by the following limits:

$$\begin{aligned} \rho_0(0) = \quad & \lim_{s\to 0}(sc_0) = \quad -m^2, \\ \rho_1(0) = \quad & \lim_{s\to 0}(sc_1) = \quad 1, \end{aligned} \tag{A.40}$$

showing that the $k = 0$ contribution to the ζ-function evaluated at zero vanishes in the limit $m \to 0$. Finally, the expression for $\zeta(0)$ only involves the Gilkey coefficient $a_1(\sigma)$:

$$\zeta(0) = \frac{1}{4\pi}\int d^2\sigma\sqrt{h}\ \mathrm{tr}[a_1(\sigma)]. \tag{A.41}$$

With this, we have restricted our problem to the finding of the Gilkey coefficient $a_1(\sigma)$ for the two operators that we have considered, namely $\not{D}$ and $\Box$.

A.4.3 Jacobian for $[d\psi]_1$

Let us first define very explicitly what is meant by $\not{D}$. To be the most general as possible, we will work in a reparameterization invariant language. The covariant derivative $D_\alpha(x,\lambda)$ acting on the ψ-field thus involves a general covariant derivative ∇_α, plus a gauge connection $a_\alpha(x,\lambda)$ coming from the gauge group,*i.e.* we have set $D_\alpha = \nabla_\alpha + a_\alpha$.

All of those operators will be expressed explicitly in a moment, but, before, we need to define $\mathcal{J}_{ab}$ to be spin-$\frac{1}{2}$ representation of the generator J_{ab} of the

Lorentz group (or rather of SO(2), since we are working in Euclidean space), then:

$$\mathcal{J}_{ab} = \frac{1}{4}[\bar{\gamma}_a, \bar{\gamma}_b]. \tag{A.42}$$

Here, $\bar{\gamma}^a$ is the notation for the γ-matrices defined in flat Euclidean space. They differ from the gamma matrices of equation (2.22) only by: $\bar{\gamma}^1 = -i\gamma^1$ and $\bar{\gamma}^2 = \gamma^0$. We also define $\bar{\gamma}_P \equiv -i\bar{\gamma}^1\bar{\gamma}^2 = \gamma_P$, as we can see in equation (2.23). Notice that they are then simply given by the corresponding Pauli matrices, $\bar{\gamma}_P$ being the third one. Since in two dimensions, only four matrices are independent, namely, I, $\bar{\gamma}^1$, $\bar{\gamma}^2$ and $\bar{\gamma}_P$, the spin-$\frac{1}{2}$ representation of the group generator J_{ab} can be written in term of those, as:

$$\mathcal{J}_{ab} = \frac{1}{2}\epsilon_{ab}\bar{\gamma}_P. \tag{A.43}$$

Once this is settled, we may express precisely what we meant by the covariant derivative D_α in equation (A.14). First, let us precise that $\not{D}$ is a short-hand notation for $\bar{\gamma}^\alpha D_\alpha$. Now, the general covariant derivative on the worldsheet ∇_α is given by:

$$\nabla_\alpha = \partial_\alpha + \Omega_\alpha \left(+\Gamma^\gamma_{\alpha\beta} \quad \text{as necessary}\right), \tag{A.44}$$

Ω_α being the spin connection defined by:

$$\Omega_\alpha = \frac{i}{2}\omega_\alpha^{ab}\mathcal{J}_{ab} = \frac{i}{4}\omega_\alpha^{ab}\epsilon_{ab}\bar{\gamma}_P. \tag{A.45}$$

The quantities $\omega_\alpha^{ab}(\sigma)$ are a connection that can be computed from the requirement that the covariant derivative of the zweibein vanishes, *i.e.* $\nabla_\alpha e^a_\beta = 0$. Solving this equation, and writing everything in term of the zweibein, we get [2]:

$$\omega_\alpha^{ab} = \frac{1}{2}e^{\beta a}(\partial_\alpha e^b_\beta - \partial_\beta e^b_\alpha) - \frac{1}{2}e^{\beta b}(\partial_\alpha e^a_\beta - \partial_\beta e^a_\alpha) - \frac{1}{2}e^{\beta a}e^{\gamma b}(\partial_\beta e_{\gamma c} - \partial_\gamma e_{\beta c})e^c_\alpha \tag{A.46}$$

Finally, since equation (A.14) is the ψ-part of the action (2.28) for heterotic string theory, it is clear that the previously introduced gauge connection a_α is the following:

$$(a_\alpha)_{st}(x,\lambda) = T^j_{st}\left[A^j_M(x)\partial_\alpha x^M - \frac{i}{4}F^j_{MN}(x)\lambda^N\bar{\gamma}_\alpha\lambda^M\right]. \tag{A.47}$$

Before going any further, we must be careful with the *zero-modes* of $\not{D}$. A zero-mode of an operator is an eigenstate of this operator that has eigenvalue zero. This, in a Grassmannian integration, causes obviously some troubles, since, when integrating the zero-mode part of the exponential, we are, in fact, integrating a constant over a Grassmann measure. And it vanishes, as it is well known. In order to get rid of them, we introduce a small 'mass' term m that we will take to zero at the end. Do not worry about the justification of this step since we will need to come back to this in sections **A.5** and **A.6**. Explicitly, we define $\Delta_m = \not{D} + m$, and we suppose then that Δ_m has no zero-modes at all.

However, in order to be able to use Gilkey's results [43] to compute the value of the ζ-function, we need to find a way to relate Δ_m to a second order operator (see equation (A.32)). In order to achieve this, it is useful to remark that:

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} 1 & BD^{-1} \\ 0 & 1 \end{bmatrix}\begin{bmatrix} A - BD^{-1}C & 0 \\ 0 & D \end{bmatrix}\begin{bmatrix} 1 & 0 \\ D^{-1}C & 1 \end{bmatrix}, \tag{A.48}$$

so

$$\mathrm{Det}\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \det(A - BD^{-1}C)\,\det D. \tag{A.49}$$

Similarly, we can get:

$$\mathrm{Det}\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \det A\,\det(D - CA^{-1}B). \tag{A.50}$$

Therefore we see that:

$$\begin{aligned} \mathrm{Det}(\Delta_m) &= \det\left[m^2 - D_+D_-\right], \\ &= \det\left[m^2 - D_-D_+\right], \end{aligned} \tag{A.51}$$

where we defined $D_\pm = D_1 \pm iD_2$ with D being the covariant derivative defined in equation (A.14). On the other hand, we can easily see that:

$$\hat{\Delta}^2 \equiv m^2 - \not{D}^2 = \begin{bmatrix} m^2 - D_- D_+ & 0 \\ 0 & m^2 - D_+ D_- \end{bmatrix}. \quad \text{(A.52)}$$

This has the obvious consequence that we can compute the determinant of Δ in term of the one of $\hat{\Delta}^2$, explicitly we get: $\mathrm{Det}(\Delta) = \mathrm{Det}^{1/2}(\hat{\Delta}^2)$. Using the definition (A.26) for determinants, this shows that $\zeta'_{\hat{\Delta}^2}(0) = 2\zeta'_{\Delta}(0)$. And it is not hard to see that it comes from the fact that:

$$\zeta_{\hat{\Delta}^2}(s) = \zeta_{\Delta}(2s), \quad \text{(A.53)}$$

since the eigenvalues of $\hat{\Delta}^2$ are the square of those of Δ_m. This means that we are in position to compute the Jacobian $J_\psi(\tau)$ with the following expression:

$$J_\psi(\tau) = \lim_{m \to 0} \tau^{\frac{1}{2}\zeta_{\hat{\Delta}^2}(0)}. \quad \text{(A.54)}$$

At this point, the last step is to put the operator $\hat{\Delta}^2$ in the desired form (A.32) and to compute the coefficient $a_1(\sigma)$. Remember that we defined $\hat{\Delta}^2$, in equation (A.52) to be given by: $\hat{\Delta}^2 = m^2 - \not{D}^2$. Let us express $\not{D}^2 = (\bar{\gamma} \cdot D)^2$ in a different form:

$$(\bar{\gamma} \cdot D)^2 = \frac{1}{2}\{\bar{\gamma}^\alpha, \bar{\gamma}^\beta\} D_\alpha D_\beta + \frac{1}{2}\bar{\gamma}^{\alpha\beta}[D_\alpha, D_\beta], \quad \text{(A.55)}$$

with $\bar{\gamma}^{\alpha\beta} = \frac{1}{2}[\bar{\gamma}^\alpha, \bar{\gamma}^\beta]$. We then write these terms in a very explicit way. The first of these is given by:

$$h^{\alpha\beta} D_\alpha D_\beta = \Box + 2a^\alpha \partial_\alpha + 2a^\alpha \Omega_\alpha - h^{\alpha\beta}\Gamma^\gamma_{\alpha\beta} a_\gamma + \partial^\alpha a_\alpha + a^2, \quad \text{(A.56)}$$

where $\Box \equiv h^{\alpha\beta}\nabla_\alpha\nabla_\beta$ (equivalent to the previous definition in two dimensions) which can be rewritten as:

$$\Box = h^{\alpha\beta}\partial_\alpha\partial_\beta + \left(2\Omega^\alpha - h^{\beta\gamma}\Gamma^\alpha_{\beta\gamma}\right)\partial_\alpha - h^{\alpha\beta}\Gamma^\gamma_{\alpha\beta}\Omega_\gamma + \partial^\alpha\Omega_\alpha + \Omega^2. \quad \text{(A.57)}$$

The second term of equation (A.55), involving the commutator, requires the knowledge of the commutator $[\nabla_\alpha, \nabla_\beta]$ of the general covariant derivatives.

Remember that it is a very common statement in differential geometry and in general relativity that this last commutator, when applied to a vector field v_α takes the value:

$$[\nabla_\alpha, \nabla_\beta] v_\gamma = R^\delta{}_{\alpha\beta\gamma} v_\delta, \tag{A.58}$$

where, $R^\delta{}_{\alpha\beta\gamma}$ is the Riemann curvature tensor on the worldsheet. In order to express the commutator $[\nabla_\alpha, \nabla_\beta]$, we find the equivalent of equation (A.58) for an arbitrary group representation. When we notice that the representation of the SO(2) generator $(J_{ab})_{cd}$ is simply $\delta_{ac}\delta_{bd} - \delta_{ad}\delta_{bc}$ for a vector field, we find that an obvious generalization of the formula (A.58) for an arbitrary spin field is:

$$[\nabla_\alpha, \nabla_\beta] = \frac{1}{2} R_{\alpha\beta}{}^{ab} \mathcal{J}_{ab}, \tag{A.59}$$

giving $[\nabla_\alpha, \nabla_\beta] = \frac{1}{4} R_{\alpha\beta}{}^{ab} \epsilon_{ab} \bar{\gamma}_P$, when applied on a spin-$\frac{1}{2}$ spinor field, as we can find it from equation (A.43). This enables us to derive the whole commutator $[D_\alpha, D_\beta]$. Before, let us define the field strength tensor $f_{\alpha\beta}(x, \lambda)$ of the gauge connection $a_\alpha(x, \lambda)$ to be the following:

$$f_{\alpha\beta} = \nabla_\alpha a_\beta - \nabla_\beta a_\alpha + [a_\alpha, a_\beta]. \tag{A.60}$$

And then we get:

$$[D_\alpha, D_\beta] = \frac{1}{4} R_{\alpha\beta}{}^{ab} \epsilon_{ab} \bar{\gamma}_P + f_{\alpha\beta}. \tag{A.61}$$

Here, we introduce specific properties of the two-dimensional worldsheet that make the above expression (A.61) simpler. First, let us us remark that in a two-dimensional space, like our worldsheet, Riemann tensor has only one independent component, namely the scalar curvature R_h.

$$R_{\alpha\beta\gamma\delta} = \frac{R_h}{2} \left(h_{\alpha\gamma} h_{\beta\delta} - h_{\alpha\delta} h_{\beta\gamma} \right). \tag{A.62}$$

With this and the fact that $\epsilon_{ab}\epsilon^{ab} = 2$ in two-dimension, we get:

$$\frac{1}{2} \bar{\gamma}^{\alpha\beta} [D_\alpha, D_\beta] = \frac{R_h}{4} + \frac{1}{2} \bar{\gamma}_P \epsilon^{\alpha\beta} f_{\alpha\beta}. \tag{A.63}$$

Therefore, putting all together the results of equations (A.56), (A.57) and (A.63), we finally get the desired form (A.32) for the operator $\hat{\Delta}^2$, where $P^\alpha(\sigma)$ and $Q(\sigma)$ read explicitly as follows:

$$
\begin{aligned}
P^\alpha(\sigma) &= +2a^\alpha + 2\Omega^\alpha - h^{\beta\gamma}\Gamma^\alpha_{\beta\gamma} \\
Q(\sigma) &= +2a^\alpha\Omega_\alpha - h^{\alpha\beta}\Gamma^\gamma_{\alpha\beta}a_\gamma + \partial^\alpha a_\alpha + a^2 - h^{\alpha\beta}\Gamma^\gamma_{\alpha\beta}\Omega_\gamma + \partial^\alpha\Omega_\alpha + \Omega^2 \\
&\quad -\frac{R_h}{4} - \gamma_P\epsilon^{\alpha\beta}f_{\alpha\beta}.
\end{aligned}
\tag{A.64}
$$

Remark that this and equation (A.57) show that:

$$\not{\nabla}^2 = \Box - \frac{R_h}{4}. \tag{A.65}$$

Using equations (A.34) and (A.35), all this, fortunately, reduces to very simple expressions: $K^\alpha = a^\alpha + \Omega^\alpha$, and

$$\varepsilon = -\frac{R_h}{4} - \frac{1}{2}\bar{\gamma}_P\epsilon^{\alpha\beta}f_{\alpha\beta}. \tag{A.66}$$

Therefore, we finally get the value of the Gilkey coefficient $a_1(\sigma)$.

$$\mathrm{tr}[a_1(\sigma)] = -\mathrm{tr}\left(\frac{R_h}{12} + \frac{1}{2}\bar{\gamma}_P\epsilon^{\alpha\beta}f_{\alpha\beta}\right). \tag{A.67}$$

Here the trace is over spin and gauge indices. We know that the field strength $f_{\alpha\beta}$ is proportional to the generator T^M. We know that we can choose a representation in which all the matrices T^M_{st} are traceless, so this term drops out of the expression. The spin trace will not have any effect on our answer because, dealing with Weyl spinors, we are summing over only one index. Thus, the final expression for the ζ-function will be:

$$\zeta(0) = -\frac{N_F}{48\pi}\int d^2\sigma\sqrt{h}R_h = -\frac{N_F}{12}\chi(\mathrm{M}). \tag{A.68}$$

Therefore, according to equation (A.54), the change in the product part $[d\psi]_1$ of the measure of ψ due to the action of the DS transformation (A.12) will be given by the Jacobian $J_\psi(\tau)$.

$$J_\psi(\tau) = \tau^{-N_F\chi(\mathrm{M})/24}. \tag{A.69}$$

Explicitly, that means that under the DS transformation (A.12), the change in the product part of the functional measure of the spinor field $\psi(\sigma)$ is the following:

$$\delta[d\psi] = -\omega\frac{N_F}{24}\chi(\mathrm{M})\,[d\psi], \qquad \text{(A.70)}$$

where, as before, $\omega = \log\tau$. We see that this result agrees with the one obtained previously (see equation (A.23) and (A.24)),

A.4.4 Jacobian for $[dx]_1$

When we want to apply the above heat-kernel procedure to the other fields of the heterotic string action (2.28), the things are not as clear as before, since the fields $x^M(\sigma)$ and $\lambda^M(\sigma)$ involve operator which are much more complicated than a Dirac operator like $\not{D}$ (equation (A.14)). In fact, the terms involving the embedding $x^M(\sigma)$ are not quadratic, and not even *a priori* defined.

However, the Gilkey procedure apply for the bosonic case in some approximation. It is possible to expand the coordinates $x^M(\sigma)$ around some fixed point $\bar{x}^M$ that does not depend on the position on the worldsheet σ^α. Writing $\xi^M(\sigma) = x^M(\sigma) - \bar{x}^M$, we can expand the target space metric as follows, x^M being reparameterized in Gaussian coordinates:

$$G_{MN}(x) = \bar{G}_{MN} - \frac{1}{3}R_{MPNQ}(\bar{x})\xi^P\xi^Q + o(\xi^3), \qquad \text{(A.71)}$$

where we have introduced the notation $\bar{G}_{MN} = G_{MN}(\bar{x})$.

We will consider the case when we can neglect all corrections and assume that the metric of the target space is simply $\bar{G}_{MN}$. Then, we may integrate by part one of the derivatives $\partial_\alpha x^M$ and get the action of bosonic string theory in a curved background spacetime (see equation (2.6) or the first term of the heterotic string action (2.28)) in a quadratic form $I_x = -\bar{G}_{MN}(x^M, \Box x^N)$, where (,) represents the scalar product between worldsheet scalars (an integration over the worldsheet M).

This action, since it is the first term of the heterotic string action (2.28), is clearly seen to classically obey the transformation rule (A.12) under a DS transformation of the background metric. We want to find how the product part of the x-measure scales for any worldsheet genus. If we neglect the conformal Killing vectors, the result is quite easy to get. Using the same argument as in equations (A.13), we find that the Jacobian $J_x(\tau)$ of the measure $[dx]_1$ is given by the ratio of determinants:

$$J_x(\tau) = \frac{\mathrm{Det}^{-1/2}(-\Lambda^2 \bar{G}_{MN}\Box)}{\mathrm{Det}^{-1/2}(-T\bar{G}_{MN}\Box)}. \tag{A.72}$$

The target space metric contributes to the some regularized factor $f(\bar{G})$ in the evaluation of the functional determinant $\mathrm{Det}(-\bar{G}_{MN}\Box) = f(\bar{G})\mathrm{Det}^D(-\Box_0)$, where we defined $\Box_0$ to be the Laplacian $\Box$ acting on a worldsheet scalar:

$$\Box_0 = \frac{1}{\sqrt{h}}\partial_\alpha(\sqrt{h}h^{\alpha\beta}\partial_\beta). \tag{A.73}$$

Using the equation (A.29) of section **A.4.1**, one find that the Jacobian $J_x(\tau)$ is expressed in the following way in term of the ζ-function of the operator $\Box_0$:

$$J_x(\tau) = \tau^{-\frac{D}{2}\zeta_{\Box_0}(0)}. \tag{A.74}$$

When we apply heat-kernel procedure of section **A.4.2** to the operator $\Box_0$, we can easily compute the Gilkey coefficients using equations (A.56) and Gilkey's method presented in (A.34) and (A.35). The result is that $\varepsilon = 0$ and therefore we get $a_1(\sigma) = R_h/6$. This gives us:

$$\zeta_{\Box_0}(0) = \chi(\mathrm{M})/6. \tag{A.75}$$

and

$$J_x(\tau) = \tau^{-\frac{D}{12}\chi(\mathrm{M})}. \tag{A.76}$$

Again, this is in accord with the result of section **A.3**, equation (A.21). In conclusion to this section, we may say that the Gilkey procedure is quite efficient to find the ζ-function evaluated at zero, associated with different second order operators. But previously, the problem of zero-mode was neglected and we said that we would have to study it more carefully. It is what we will do in the next section by computing explicitly the zeta function of $\Box_0$ and $\nabla\!\!\!\!/$ on the sphere and the torus as representatives of Riemann surfaces of genus $\gamma = 0$ or 1.

A.5 Explicit Calculation of Zeta Functions

We will only study scalar operator ζ-functions (with respect to the target space: $\Box_0$ instead of $\delta_{MN}\Box_0$). The motivation for studying this case is simple: the last sections considerations allow us to infer that the effect of the number of dimensions of the target space is simply to multiply the zeta function by D.

We introduced a bosonic real scalar $\varphi(\sigma)$ and a fermionic Majorana spinor $\psi(\sigma)$, both defined on a two-dimensional space: the worldsheet M. These fields will help us to compute explicitly the ζ-functions, evaluated at zero, for the operator $\Box_0$ (as defined in (A.73)) and $\nabla\!\!\!\!/ \equiv \bar{\gamma}^\alpha \nabla_\alpha$ (with ∇_α given in (A.44)). We will find the value of $\zeta_{-\Box_0}(0)$ and $\zeta_{\nabla\!\!\!\!/}(0)$ on a unit sphere and a flat torus, doing the explicit sum over the eigenvalues of the operators, just as it is stated in the definition (A.25).

A.5.1 Sphere

Zeta-function of $\Box_0$*:* We will at first concentrate on the scalar field $\varphi(\sigma)$. That is we focus our attention on the calculation of

$$\zeta_\Box(s) = \sum_n \frac{1}{\mu_n^s}, \tag{A.77}$$

where

$$-\Box_0 \varphi_n = \mu_n \varphi_n. \tag{A.78}$$

The first step is to determine what are the eigenvalues μ_n's. That means we have to solve the above eigenvalue equation (A.78). Since the operator $\Box_0$ explicitly contains a metric dependence, the equation, that we want to solve, differs for different worldsheet metrics. We have a good clue from section **A.3** that the value of the zeta function at $s = 0$ is a topological invariant. Therefore, it will be sufficient to specialize to the unit sphere and the flat torus, when studying the cases where $\gamma = 0$ or 1.

Let us begin with the unit sphere S^2, we choose to parameterize it with the usual spherical coordinates: θ for the latitude and ϕ as the azimuthal angle. The metric $h_{\alpha\beta}(\theta, \phi)$ on S^2 is then given by:

$$h^{\alpha\beta} = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{\sin^2\theta} \end{bmatrix}. \tag{A.79}$$

Using that metric, we find that the only non-zero components of the Christoffel symbol are (except for the obvious symmetry in the lower indices):

$$\begin{aligned} \Gamma^{\theta}_{\phi\phi} &= -\sin\theta\cos\theta, \\ \Gamma^{\phi}_{\theta\phi} &= \cot\theta. \end{aligned} \tag{A.80}$$

So, the left-hand side of the eigenvalue equation (A.78) becomes explicitly:

$$\Box_0\varphi_n = \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\varphi}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2\varphi}{\partial\phi^2}. \tag{A.81}$$

In this last expression, we immediately recognize the operator $-\mathbf{L}^2$, the angular momentum very common and useful in non-relativistic quantum mechanics. The solution for the eigenvalue problem is well known:

$$\mathbf{L}^2 Y^l_m(\theta, \phi) = l(l+1) Y^l_m(\theta, \phi), \tag{A.82}$$

with degeneracy $2l + 1$. The functions $Y^l_m(\theta, \phi)$ are called spherical harmonics.

Omitting the zero mode (when $\mu_n = 0$) which makes it divergent, one thus gets the following expression for the ζ-function:

$$\tilde{\zeta}^0_\Box(s) = \sum_{l=1}^{\infty} \frac{2l+1}{[l(l+1)]^s}. \tag{A.83}$$

In the last equation, the superscript 0 represents the genus of the surface that we are working on; *i.e.* on the sphere, $\gamma = 0$. The tilde just says that we are neglecting zero-modes. It will be possible to evaluate explicitly $\tilde{\zeta}^0_{\Box}(0)$ using Newton binomial expansion of the denominator [44].

$$(l+1)^{-s} = l^{-s} \sum_{k=0}^{\infty} l^{-k} \frac{\Gamma(1-s)}{k!\Gamma(1-s-k)}. \qquad \text{(A.84)}$$

The above expansion is mathematically justified for all l, except $l = 1$. Nevertheless, a careful treatment of this term shows that, in the limit $s \to 0$, our expression is correct. The next step consists in interchanging the sum order:

$$\begin{aligned} \tilde{\zeta}^0_{\Box}(0) &= \sum_{l,k} \left[2l^{1-2s-k} + l^{-2s-k}\right] \frac{\Gamma(1-s)}{k!\Gamma(1-s-k)}, \\ &= \sum_{k=0}^{\infty} \left[2\zeta_R(2s+k-1) + \zeta_R(2s+k)\right] \frac{\Gamma(1-s)}{k!\Gamma(1-s-k)}, \end{aligned} \qquad \text{(A.85)}$$

where $\zeta_R(z)$ is the Riemann ζ-function, which is defined by the following series:

$$\zeta_R(z) = \sum_{n=1}^{\infty} \frac{1}{n^z}, \qquad \text{(A.86)}$$

and its analytic continuation. This function is found to be analytic over the entire complex plane with the exception of a simple pole in $z = 1$.

We now take the limit $s \to 0$ in the expression of $\tilde{\zeta}^0_{\Box}(s)$ of equation (A.85). In this limit, $\Gamma(1-s-k)$ diverges for all $k \geq 1$ while $\Gamma(1-s)$ stays finite. This implies that in the summation, we keep only the terms where $k = 0$, and those where the Riemann ζ-function diverge also, thus cancelling the gamma function divergence. In other words, we may express only the important part of $\tilde{\zeta}^0_{\Box}(s)$ when $s \to 0$ by the following:

$$\begin{aligned} \tilde{\zeta}^0_{\Box}(s) &= \Gamma(1-s)\left\{ 2\left(\frac{\zeta_R(2s-1)}{\Gamma(1-s)} + \frac{1}{2}\frac{\zeta_R(2s+1)}{\Gamma(-1-s)}\right) \right. \\ &\qquad \left. + \frac{\zeta_R(2s)}{\Gamma(1-s)} + \frac{\zeta_R(2s+1)}{\Gamma(-s)} + o(s)\right\}, \end{aligned} \qquad \text{(A.87)}$$

In the above expression, the first two terms come from the first Riemann ζ-function of (A.85) with $k = 0$ and 2; while the $\zeta_R(2s+k)$ of (A.85) with, respectively, $k = 0$ and 1, is responsible for the third and fourth term of (A.87).

Now, using the well known property of the Euler function: $\Gamma(n+1) = n\Gamma(n)$, we finally get:

$$\begin{aligned} \zeta_{\Box}(0) &= \lim_{s\to 0}\left\{2\zeta_R(2s-1) + \zeta_R(2s) + s^2\zeta_R(2s+1)\right\}, \\ &= 2\zeta_R(-1) + \zeta_R(0). \end{aligned} \tag{A.88}$$

No matter how awkward this last expression may look, the values of $\zeta_R(z)$ for $z < 1$ are well-defined by analytic continuation of the definition (A.86). And for $z = 0$ and -1, they are found to be $\zeta_R(0) = -\frac{1}{2}$ and $\zeta_R(-1) = -\frac{1}{12}$. Therefore, we obtain in the case of a scalar on a sphere, neglecting the zero-mode:

$$\tilde{\zeta}^0_{\Box_0}(0) = -\frac{2}{3}. \tag{A.89}$$

We will interpret this result and the subsequent ones in section **A.4.3** after that we will have done the calculations for the other cases that we consider here. We now turn to the evaluation of the ζ-function of the operator $\nabla\!\!\!/$, again on the sphere.

Zeta function of $\nabla\!\!\!/$*:* We want to compute explicitly the value of $\zeta^0_{\nabla\!\!\!/}(0)$, taking the unit sphere as the worldsheet. For that case, we could solve the eigenvalue equation, as we did for the scalar, but instead, we take the eigenvalues that are directly found in the appendix **B** of reference [45].

However, it is not possible to solve the eigenvalue problem for the $\nabla\!\!\!/$, since this operator acting on a field changes its chirality. What is usually done is to find the eigenvalues of the squared Dirac operator, $\nabla\!\!\!/^2$, that is called the *Dirac Laplacian.* This is exactly what we had to do in the last section.

The eigenvalues of $-\nabla\!\!\!/^2$ are given in [45] for a sphere of arbitrary dimensionality S^N. For $l = 0, 1, 2, \ldots$, they are:

$$\mu_l^2 = (l + \frac{1}{2}N)^2, \tag{A.90}$$

with degeneracies (recalling that tr $\mathbf{1} = 2^{N/2}$ for N even and that we are dealing with Weyl fermions, so the following degeneracy differs from [45]):

$$d_l = 2\frac{(l+N-1)!}{l!(N-1)!}. \tag{A.91}$$

Therefore, in our case, a two-dimensional sphere S^2, we may consider that the 'eigenvalues' of $\nabla\!\!\!/$ are simply the square root of those of the Dirac Laplacian, namely $l+1$ with degeneracy $2(l+1)$. Notice that there are no zero-modes. This may be understood by noticing that, as we implicitly found in the last section (equation (A.64)), we have $\nabla\!\!\!/^2 = \Box - R_h/4$. Since the eigenvalues of $\Box$ are negative or zero, as in the scalar case, and $R_h = 2$ for the unit sphere, the eigenvalues of the Dirac Laplacian could by no mean be less or equal to zero.

The ζ-function of the operator $\nabla\!\!\!/$ is thus very easy to evaluate. According to the definition (A.77), the zeta function is expressed as follows:

$$\zeta_{\nabla\!\!\!/}(s) = \sum_{l=0}^{\infty} 2\frac{l+1}{(l+1)^s}. \tag{A.92}$$

With a shift in the dummy index $l \to l+1$, it is obvious that this reduces to $\zeta_{\nabla\!\!\!/}(s) = 2\zeta_R(s-1)$, where $\zeta_R(z)$ still represents the Riemann zeta function. Remembering that its value at $z = 1$ is $-\frac{1}{12}$, we finally get in the case of a spinor on a $\gamma = 0$ Riemann surface:

$$\zeta_{\nabla\!\!\!/}(0) = -\frac{1}{6}. \tag{A.93}$$

We will relate those results with the ones we can obtain using Gilkey heat-kernel procedure, but, in order to make this connection more explicit, we will study another case: the flat torus as a representative of a surface of Euler characteristic $\chi(\mathrm{M}) = 0$ $(\gamma = 1)$.

A.5.2 Torus

Zeta function of $\Box_0$*:* The techniques used in the previous section apply as well in the case of the torus. The eigenfunctions of the operator $\Box$ acting on a scalar are restricted by periodic boundary conditions. The corresponding eigenvalues are given in [15]:

$$\omega_{mn} = \left(\frac{2\pi}{y}\right)^2 \left[m^2y^2 + (n-mx)^2\right], \tag{A.94}$$

where the real numbers x and y are the Teichmüller parameters of the torus. They appear in the following expression, quite common and useful in string theory, for the metric $h_{\alpha\beta}(\sigma)$ of the flat torus with fundamental domain given by the unit square:

$$h_{\alpha\beta} = \begin{bmatrix} 1 & x \\ x & x^2 + y^2 \end{bmatrix}. \tag{A.95}$$

We have the following expression for the ζ-function (the sum will be over $\mathbf{Z}_*^2 = \{(m,n) : m, n \in \mathbf{Z}; (m,n) \neq (0,0)\}$ since we once again forget the diverging zero-mode part):

$$\tilde{\zeta}^1_\square(s) = \sum_{(m,n)\in\mathbf{Z}_*^2} \omega_{mn}^{-s}. \tag{A.96}$$

Now, we note that under the transformation $m \to -m$ and $n \to -n$ the eigenvalues ω_{mn} are invariant. We thus can express the zeta-function in the following way:

$$\begin{aligned} \tilde{\zeta}^1_\square(s) &\equiv 2\zeta_+(s) + \zeta_0(s), \\ &= 2\sum_{m=-\infty}^{+\infty} \sum_{n=1}^{+\infty} \omega_{mn}^{-s} + \sum_{m\in\mathbf{Z}_*} \omega_m^{-s}, \end{aligned} \tag{A.97}$$

$\mathbf{Z}_*$ represents the set of non-zero integers. In the last equation, we have defined ω_n to be:

$$\omega_m \equiv \omega_{m0} = \left(\frac{2\pi}{y}\right)^2 m^2(x^2 + y^2) \tag{A.98}$$

Again, we remark that w_m does not change if $m \to -m$, so it is easy to compute $\zeta_0(s)$ in the above expression. Since $\omega_0 = 0$, we get:

$$\begin{aligned} \zeta_0(s) &= 2\sum_{m=1}^{\infty} \omega_m^{-s}, \\ &= 2\left(\frac{2\pi}{y}\right)^{-2s} (x^2 + y^2)^{-s}\zeta_R(2s), \end{aligned} \tag{A.99}$$

where $\zeta_R(s)$ is again the Riemann ζ-function. Remembering that $\zeta_R(0) = -\frac{1}{2}$ and taking the limit, we obtain the intermediate result that $\zeta_0(0) = -1$.

It remains to compute the $n \neq 0$ part of the zeta function. In order to do that, we must know how to expand the following expression:

$$(n^2 + na_m + b_m)^{-s} = n^{-2s} \sum_{k=0}^{\infty} \left(\frac{na_m + b_m}{n^2} \right)^k \frac{\Gamma(1-s)}{k!\Gamma(1-s-k)}. \qquad (A.100)$$

Expanding the binomial $(na_m + b_m)^k$ in term of the Newton coefficient, this gives:

$$(n^2 + na_m + b_m)^{-s} = \sum_{k=0}^{\infty} \sum_{j=0}^{k} n^{-2s-k-j} b_m^j a_m^{k-j} \frac{\Gamma(1-s)}{j!(k-j)!\Gamma(1-s-k)}. \quad (A.101)$$

The coefficients a_m and b_m are the only numbers depending on m, in our case:

$$\begin{aligned} a_m &= m\alpha &&= -2mx, \\ b_m &= m^2\beta &&= m^2(x^2 + y^2). \end{aligned} \qquad (A.102)$$

We therefore have:

$$\zeta_+(s) = 2 \left(\frac{2\pi}{y} \right)^{-2s} \sum_{k=0}^{+\infty} \sum_{j=0}^{k} \beta^j \alpha^{k-j} \frac{\zeta_R(2s+k+j)\Gamma(1-s)}{j!(k-j)!\Gamma(1-s-k)} \left(\sum_{m=-\infty}^{+\infty} m^{k+j} \right). \qquad (A.103)$$

We compute the last summation in the above expression using Riemann ζ-function (again!).

$$\begin{aligned} \sum_{m=-\infty}^{+\infty} m^p &= \sum_{m=1}^{+\infty} m^p + \sum_{m=-1}^{-\infty} m^p + \delta_p, && (A.104) \\ &= \zeta_R(-p)\,[1 + (-1)^p] + \delta_p. && (A.105) \end{aligned}$$

where δ_p is the Kronecker delta for p and 0 (*i.e.* taking values 0 for $p \neq 0$ and 1 for $p = 0$). This leads us to a final expression for the ζ-function of $\Box_0$ in terms of an infinite summation over k. But this can be reduced to only one term noticing that $\Gamma(1-s-k)$ is divergent when $s \to 0$ for all k's except $k = 0$, and that the term proportional to $\zeta_R(1)$ vanishes. Therefore

$$\zeta_+(s) = 2 \left(\frac{2\pi}{y} \right)^{-2s} \zeta_R(2s)\,[2\zeta_R(0) + 1] + o(s). \qquad (A.106)$$

So, using our result for $\zeta_0(0)$, we finally get that, when we neglect the zero-modes, the ζ-function for a scalar on a torus is:

$$\tilde{\zeta}^1_\Box(0) = 2\zeta_R(0)[2\zeta_R(0) + 1] - 1 = -1. \qquad (A.107)$$

Showing that the zeta function corresponding to the operator $\Box_0$ defined on a flat torus (or any genus $\gamma = 1$ surface) is -1.

Zeta function of $\not{\nabla}$*:* It is quite easy to extend our analysis of the flat torus to the operator $\not{\nabla}$. We already emphasized the link between $\Box$ and $\not{\nabla}$, deriving the relation: $\not{\nabla}^2 = \Box - R_h/4$. Thus, for a flat torus, $\not{\nabla}^2$ is simply the operator $\Box$, since the torus has no curvature. Moreover the operator $\Box$ is the same acting on either field, scalar or spinor, because there is no non-zero connections.

This remark allows us to use the same results as for the bosonic case, but this time, the boundary conditions on the eigenstates of $\Box$ can be either periodic or anti-periodic, giving four different possible combinations, labelled by $(\pm, \pm)$. The eigenvalues of the operator $\Box$ corresponding to these combinations are given by [15]:

$$\omega_{mn}^{(ab)} = \left(\frac{2\pi}{y}\right)^2 \left[\mu_a^2 y^2 + (\nu_a - \mu_a x)^2\right], \tag{A.108}$$

with

$$\begin{aligned} \mu_+ &= m, \qquad & \mu_- &= m + \tfrac{1}{2}, \\ \nu_+ &= n, \qquad & \nu_- &= n + \tfrac{1}{2}. \end{aligned} \tag{A.109}$$

This shows that there is one zero-mode for the periodic boundary conditions $(+,+)$, and none in the other cases.

Now, there is four different ζ-functions $\zeta^{(ab)}(s)$ corresponding to each choice of boundary condition. Using the same method as before, we easily get that the zeta-functions computed with the anti-periodic boundary conditions in either direction or in both vanishes giving:

$$\tilde{\zeta}_{\not{\nabla}}^{(ab)}(0) = \begin{cases} -1 & \text{if } (a,b) = (+,+), \\ 0 & \text{otherwise.} \end{cases} \tag{A.110}$$

A.5.3 Contact with Gilkey Procedure

This section **A.4** is interesting only if we can make the contact with the Gilkey procedure we used in section **A.3**. We will see that the difference between the values of the various ζ-function found in the present section and the ones that we derived in the previous section comes only from the inclusion or not of the zero-modes of the operator under consideration.

Let us first express the zeta function of section **A.3**, in a form that will permit us to compare. In this section, we had $\zeta_\Delta(0)$ for Δ being $\not{D}^2$ (or, in the same time $\not{\nabla}^2$) and $-\Box_0$, depending on the topology of the worldsheet. Writing, this times, the ζ-function on a surface with specific genus, we get that Gilkey's method had led us to:

$$\zeta_{-\Box_0}(0) = \begin{cases} \frac{1}{3} & \text{for a sphere} \\ 0 & \text{for a torus} \end{cases} \qquad \text{(Gilkey)} \tag{A.111}$$

And we have gotten in the last section, by explicit evaluation:

$$\tilde{\zeta}_{-\Box_0}(0) = \begin{cases} -\frac{2}{3} & \text{for a sphere} \\ -1 & \text{for a torus} \end{cases} \tag{A.112}$$

We therefore see that this does not correspond with what we obtained just before. But, in fact, it is due to the introduction of the cut-off parameter m in the Gilkey procedure of section **A.3**.

En effet, this mass term has the consequence that the zero-modes of the operator $\Box_0$ are included in the sum over eigenvalues in the definition (A.77) of $\zeta(s)$ as an extra factor of one when $m \to 0$ before s does. Since the operator $\Box$ is known to have only one zero-mode per dimension when acting on scalars, the two ways to evaluate zeta functions agree when we compare equivalent objects, in that case ζ-functions without considering the zero-modes.

In the case of the Dirac operator $\not{\nabla}$, when there are no zero-modes, so the result of section **A.3** agrees with the one found just above. Namely, Gilkey's result are (see equation (A.41)):

$$\zeta_{\not{\nabla}}(0) = \begin{cases} -\frac{1}{6} & \text{for a sphere} \\ 0 & \text{for a torus} \end{cases} \qquad \text{(Gilkey)} \tag{A.113}$$

This is also what we have found in the last explicit calculations. However, for the case of periodic boundary conditions on the eigenspinors defined on the flat torus, the result was:

$$\tilde{\zeta}^{(++)}_{\not\nabla}(0) = -1. \tag{A.114}$$

Let us mention that this is of small importance for the theory since the partition function associated with these boundary conditions for spinors in known to vanish on a torus [15].

Finally, we can state the following prescription:

$$\tilde{\zeta}^{\gamma,s}(0) = \zeta^{\gamma,s}_{\text{Gilkey}}(0) - N_0^{\gamma,s}, \tag{A.115}$$

where $\tilde{\zeta}^{\gamma,s}(0)$ is the zeta function, defined on the Riemann surface with genus γ and spin structure s, that does not take in account the zero-modes of the operator considered, and $N_0^{\gamma,s}$ is the number of those modes.

The reason for this prescription is rather clear: $\zeta_\Delta(0)$ can be thought of as the number of modes of the operator Δ. In the expression $\tilde{\zeta}(0)$, we do not count the zero-modes, while in Gilkey procedure, we count them as as a small mass term. The relation (A.115) follows trivially.

In the case of the operator $\Box_0$, there is a unique zero-mode for every Riemann surfaces, whatever is the genus. It is the constant mode independent of σ^α. So, the zeta-function without the zero-mode is given by:

$$\tilde{\zeta}^{\gamma}_{\Box_0} = \chi(\mathrm{M})/6 - 1, \tag{A.116}$$

for all Euler characteristic. The general result for $\not\nabla$ require the knowledge of the number of zero-modes of this operator, defined on every Riemann surfaces with every spin structure. We will not express them explicitly here.

A.6 Conclusion: Regularization

The purpose of all those calculation was to know how to regularize the T-dependence of the two-dimensional field measures $[dx]_T$, $[d\lambda]_T$ and $[d\psi]_T$, as defined in section **A.1**. Since it was given by products over all values of σ, we must take the results that we have got including the zero-modes.

We can finally express the regularized power of T arising in the two-dimensional field measures. They are given by the power of τ in each of the Jacobians (A.24) obtained for the partial measures $[dx]_1$, $[d\lambda]_1$ and $[d\psi]_1$. They depend on the worldsheet genus and they are explicitly given by:

$$\begin{aligned} [dx]_T &= T^{-D\chi(\mathrm{M})/12}[dx]_1, \\ [d\lambda]_T &= T^{-D\chi(\mathrm{M})/24}[dx]_1, \\ [dx]_T &= T^{-N_F\chi(\mathrm{M})/24}[dx]_1. \end{aligned} \tag{A.117}$$

As secondary results, we also obtain the following useful formulas:

$$\mathrm{Det}(\tau\Delta) = \tau^{-\zeta_\Delta(0)}\mathrm{Det}(\Delta). \tag{A.118}$$

And, for $\Delta = -(\Box + \varepsilon)$;

$$\zeta_\Delta(0) = \frac{1}{4\pi}\int d^2\sigma\sqrt{h}\,\mathrm{tr}[a_1(\sigma)], \tag{A.119}$$

where $a_1(\sigma) = R_h/6 + \varepsilon$ is one of Gilkey coefficients.

Finally, we derive that for $\tilde{\zeta}(s)$ the zeta-function of the operator without considering the zero-modes, we have:

$$\tilde{\zeta}^{\gamma,s}(0) = \zeta^{\gamma,s}_{\mathrm{Gilkey}}(0) - N_0^{\gamma,s}, \tag{A.120}$$

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