The Spectrum of the Laplacian on Fractals

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Contents

Abstract

In this work, we review some of the major results concerning dimension and spectral theory on self-affine and self-similar fractal sets. After a focused exposition on the underlying iterated function systems, measure theoretic results are used to determine the Hausdorff dimension of arbitrary self-similar sets. The dimension is then generalized to classes of self-affine sets, using the singular value function as well as probabilistic results. A detailed analysis of the Laplacian is given on the Sierpinski Gasket, to develop extensions of the classical Szego Limit Theorem. This analysis is predicated upon existence of localized eigenfunctions, something which cannot be guaranteed on the Koch Snowflake. This work hence finishes with a review of numerical studies on the Dirichlet and Neumann boundary value problems for the snowflake domain.

Abrégé

Dans ce travail, il sera question de certains résultats majeurs sur la dimension et la théorie spectrale des ensembles fractals auto-affines et auto-similaires. Après une exposition sur la définition des fonctions itérées sous-jacentes, des résultats de la théorie des mesures sont utilisés afin de déterminer la dimension Hausdorff d'ensembles auto-similaires arbitraires. La dimension est ensuite généralisée aux classes d'ensembles auto-affines par le biais de la fonction valeur singulière ainsi que par des résultats probabilistiques. Une analyse détaillée du Laplacien est ensuite donnée sur le "Gasket" de Sierpinski afin d'étendre le théorème classique de Szego sur les limites. Cette analyse repose sur l'existence de fonctions propres localisées, ce qui ne peut être garanti sur le "flocon de neige" de Koch. Ce travail conclut donc sur une revue des études numériques des problèmes aux limites de Dirichlet et Neumann pour le domaine "flocon de neige".

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Chapter 1

Introduction

This thesis explores the topic of self-similar and self-affine sets, with a focus on dimension theory as well as the spectrum of the Laplacian on such sets. These fractal sets are a very rich and beautiful object to study, due to their presence in many fields of research. Fractals provide a description of many real world objects such as coastlines, mountains, and rivers. In addition, fractals model many biological objects including blood vessels, the nervous system, and the human brain. Analysis on fractals has applications in computer graphics, communications, medical imaging, and economics. Despite such connections to other fields, the mathematical theory surrounding fractal sets is interesting by itself. In this thesis, connections will be made between microlocal analysis, graph theory, spectral theory, and probability.

The second chapter provides the basic infrastructure needed to understand and analyze fractals. In particular, the various notions of dimension - Hausdorff, Box-Counting, and Packing, will be introduced. After describing the iterated function systems which define fractals, the properties of such systems are used to calculate the dimension of self-similar sets.

In chapter 3, we consider the more general self-affine sets. Via the singular value function, the dimension of typical self-affine sets is determined, based on the seminal works of Solomyak and Falconer ([18],[17]). However, there are many exceptional cases, and thus the second half of this chapter is devoted to the study of such a family of planar sets - Sierpinski Carpets. McMullen's [16] analysis uses some probability theory notions and results.

Chapter 4 details the relatively recent papers by Okoudjou et al. on a specific fractal - the Sierpinski Gasket. After reviewing the well-established theory of the Laplacian on the gasket, the existence of localized eigenfunctions is used to develop the analogue of the classical Szego Limit Theorem. This work demonstrates how the spectrum of the Laplacian can be a critical tool in developing strong theoretical results. Finally, this chapter culminates with the application of the theory of pseudodifferential operators to generalize the Szego Limit Theorem.

The fifth chapter of this thesis focuses on the spectrum of the Laplacian on another specific fractal - the Koch Snowflake. The current most accurate numerical results are briefly outlined, to give an idea of the behaviour of the spectrum. Unlike the previous chapter, the theory is much more difficult and underdeveloped, highlighting the criticality of the localized eigenfunctions. More comments on this issue, and resulting questions will be explored in the conclusion.

Chapter 2

Fractals Basics

In this chapter, we present a review of the basic tools needed to understand fractals. This framework follows that of Chapters 2,3, and 9 of [10].

2.1 Hausdorff Measure and Hausdorff Dimension

We first must introduce the notion of dimension of any set, which is a key property in being able to understand fractals.

Definition 2.1.1. For $U \subset \mathbb{R}^n$ nonempty, we define the *diameter* of U as

$$|U| := \sup\{|x-y| : x, y \in U\}$$

Definition 2.1.2. Let $F \subset \mathbb{R}^n$. If $F \subset \cup_{i=1}^{\infty} U_i$ where $0 \leq |U_i| \leq \delta$ for each i, then we call $\{U_i\}$ a δ-*cover* of F.

Suppose $F \subset \mathbb{R}^n$ and $s \geq 0$. For any $\delta > 0$, we define

$$\mathcal{H}^s_\delta(F) := \inf\left\{\sum_{i=1}^{\infty} |U_i|^s : \{U_i\} \text{ is a } \delta - \text{cover of } F\right\} \tag{2.1}$$

Then, we have the s-dimensional Hausdorff measure:

$$\mathcal{H}^s(F) = \lim_{\delta\to 0} \mathcal{H}^s_\delta(F) \tag{2.2}$$

First, we note that the limit in (2.2) exists for any $F \subset \mathbb{R}^n$. Now, from (2.1), it is clear that for any $F \subset \mathbb{R}^n$ and $\delta < 1$, $\mathcal{H}^s_\delta(F)$ is non-increasing in s. Hence by (2.2), $\mathcal{H}^s(F)$ is non-increasing in s. Moreover, letting $t > s$ and $\{U_i\}$ be a δ-cover of F, then

$$\sum_{i=1}^{\infty} |U_i|^t = \sum_{i=1}^{\infty} |U_i|^{t-s}|U_i|^s \leq \delta^{t-s} \sum_{i=1}^{\infty} |U_i|^s$$

So we can take infima and get that $\mathcal{H}^t_\delta(F) \leq \delta^{t-s}\mathcal{H}^s_\delta(F)$. Letting $\delta \to 0$ we see an important property: if $\mathcal{H}^s(F) < \infty$ then $\mathcal{H}^t(F) = 0$ for $t > s$. Hence there is some critical value of s where $\mathcal{H}^s(F)$ jumps from ∞ to 0, and we call this critical value the *Hausdorff dimension* of F, denoted by $\dim_H F$ (see fig. 2.1 below).

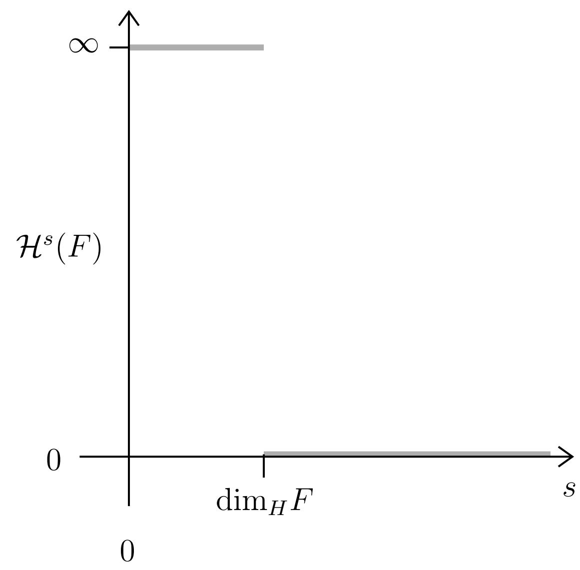

Figure 2.1: Plot of $\mathcal{H}^s(F)$ versus s for a set $F \subset \mathbb{R}^n$.

Formally, we can write

$$\dim_H F = \inf\{s \geq 0 : \mathcal{H}^s(F) = 0\} = \sup\{s \geq 0 : \mathcal{H}^s(F) = \infty\} \tag{2.3}$$

so that

$$\mathcal{H}^s(F) = \begin{cases} \infty \ \text{ if } \ 0 \leq s < \dim_H(F) \\ 0 \ \text{ if } \ s > \dim_H(F) \end{cases}$$

and if $s = \dim_H(F)$, then $\mathcal{H}^s(F)$ is 0, ∞, or $0 < \mathcal{H}^s(F) < \infty$.

2.2 Box - Counting Dimensions

The Hausdorff dimension we introduced in section 2.1 is the principal definition of dimension for understanding fractals. However, a significant disadvantage of Hausdorff dimension is that it can often be difficult to compute (for example if the associated IFS, to be introduced later, does not consist of contracting similarities; see section 2.4 and section 2.5). Hence, it is useful to introduce another notion of dimension which is widely used and is relatively easy to calculate - the box-counting dimension.

Let $F \subset \mathbb{R}^n$ be nonempty and bounded and let $N_\delta(F)$ be the smallest number of sets of diameter at most δ that cover F. Then, the *lower box-counting dimension* of F is given by $\underline{dim}_B F = \underline{\lim}_{\delta\to 0} \frac{\log N_\delta F}{-\log \delta}$ (a lower limit). Analogously, the *upper box-counting dimension* of F is given by $\overline{dim}_B F = \overline{\lim}_{\delta\to 0} \frac{\log N_\delta F}{-\log \delta}$ (an upper limit). If these two quantities are equal, then the common value is the *box-counting dimension* of F:

$$dim_B F = \lim_{\delta\to 0} \frac{\log N_\delta F}{-\log \delta} \tag{2.4}$$

Remark 2.2.1. We assume that $\delta > 0$ is sufficiently small (< 1) so that $-\log \delta$ is strictly positive.

There are many equivalent ways of defining the box-counting dimension; namely we can take $N_\delta F$ to be different quantities. For example, consider the collection of cubes in the δ-coordinate mesh of $\mathbb{R}^n$; i.e. cubes of form $[m_1\delta, (m_1+1)\delta] \times ... \times [m_n\delta, (m_n+1)\delta]$ where $m_1, ..., m_n \in \mathbb{Z}$. Now, we let $N'_\delta(F)$ denote the number of δ-mesh cubes that intersect F. The cubes provide a collection of $N'_\delta(F)$ sets of diameter $\sqrt{n}\cdot\delta$ that cover F. Hence we have that $N_{\delta\sqrt{n}}(F) \leq N'_\delta(F)$. We can take logarithms to get that $\frac{\log N_{\delta\sqrt{n}}(F)}{-\log(\delta\sqrt{n})} \leq \frac{\log N'_\delta(F)}{-\log\sqrt{n}-\log\delta}$, and then limits as $\delta \to 0$ to conclude:

$$\underline{dim}_B F \leq \underline{\lim}_{\delta\to 0} \frac{\log N'_\delta(F)}{-\log \delta} \tag{2.5}$$

and

$$\overline{dim}_B F \leq \overline{\lim}_{\delta\to 0} \frac{\log N'_\delta(F)}{-\log \delta} \tag{2.6}$$

Now, we note that any set of diameter at most δ is contained in 3^n mesh cubes of side δ (by choosing a cube containing some point of the set as well as its neighboring cubes).

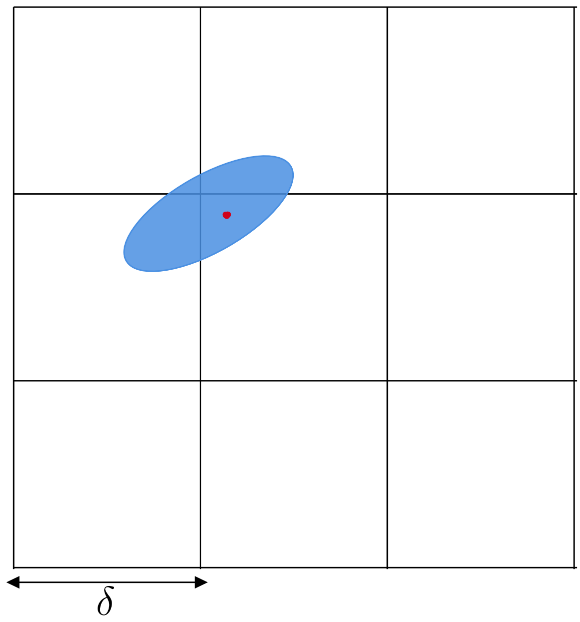

Figure 2.2: An example for $n = 2$. The set (shown in blue) of diameter $\leq \delta$ is contained in $3^2 = 9$ mesh cubes of side δ.

Hence, we have that $N'_\delta(F) \leq 3^n N_\delta(F)$ and taking logarithms and the limit as $\delta \to 0$ we arrive at the opposite inequalities of (2.5) and (2.6). So we conclude that

$$dim_B F = \lim_{\delta \to 0} \frac{\log N'_\delta F}{-\log \delta}$$

and hence we can indeed take $N_\delta F$ to be the number of δ-mesh cubes that intersect F. The purpose of introducing this particular equivalent definition is that it gives meaning to the name "box-counting" - to find the box-counting dimension of a plane set F, we just draw a mesh of squares of side δ and count the amount that overlap the set for various small δ. The number of cubes of side δ that intersect a set F gives an indication of how "spread out" or irregular the set is when examined at scale δ. The box-counting dimension reflects how rapidly the irregularities develop as $\delta \to 0$.

Another useful equivalent definition is to take $N_\delta F$ to be the smallest number of cubes of side δ that cover F. This definition will be useful in calculating the box-counting dimension of the Koch Snowflake (see chapter 5). The fact that this definition is equivalent follows similarly to the previous mesh definition. We simply note that any cube of side δ

has diameter $\delta\sqrt{n}$ and that any set of diameter at most δ is contained in a cube of side δ.

2.3 Packing Measures and Packing Dimension

In the previous section (section 2.2), one may notice that the box-counting dimension was not defined in terms of measures, as the Hausdorff dimension was. This obviously can present an issue, since we lose access to powerful measure theory results. To this end, we shall introduce in this section the Packing Dimension, which acts as a "dual" to the Hausdorff dimension, due to the inherent dual role that coverings and packings play.

Suppose $F \subset \mathbb{R}^n$ and $s \geq 0$. For any $\delta > 0$, we define

$$\mathcal{P}^s_\delta(F) := \sup\left\{\sum_i |B_i|^s\right\}$$

where $\{B_i\}$ is a collection of disjoint balls with centers in F and radii at most δ. We note that since $\mathcal{P}^s_\delta(F)$ decreases in δ, we have that the limit $\lim_{\delta\to 0}\mathcal{P}^s_\delta(F) := \mathcal{P}^s_0(F)$ exists. Then we define:

$$\mathcal{P}^s(F) := \inf\left\{\sum_i \mathcal{P}^s_0(F_i) : F \subset \cup_{i=1}^{\infty} F_i\right\} \tag{2.7}$$

, which we call the s-dimensional Packing Measure.

Remark 2.3.1. One can show that $\mathcal{P}^s(F)$ indeed is a measure on $\mathbb{R}^n$.

Now similarly to when we defined the Hausdorff dimension, we shall write

$$\dim_P F = \inf\{s \geq 0 : \mathcal{P}^s(F) = 0\} = \sup\{s \geq 0 : \mathcal{P}^s(F) = \infty\} \tag{2.8}$$

where $\dim_P F$ is the *Packing dimension* of F.

2.4 Iterated Function Systems

Iterated Function systems (or IFSs for short) are important as they allow us to define fractals and calculate their dimensions in a simple way, as we will see in the next section.

Definition 2.4.1. Let $D \subset \mathbb{R}^n$ be closed (often it will be that $D = \mathbb{R}^n$). We call a mapping $S : D \to D$ a *contraction mapping* (or simply a contraction) if there exists $c \in (0,1)$ such that for all $x, y \in D$,

$$|S(x) - S(y)| \leq c|x - y|$$

In the case that $|S(x) - S(y)| = c|x - y|$, we call S a *contracting similarity* (with ratio c) and note that in this case S transforms sets into geometrically similar sets.

Definition 2.4.2. An IFS is a finite family of contractions $\{S_1, ..., S_m\}$ (where $m \geq 2$).

Definition 2.4.3. A nonempty compact subset $F \subset D$ is an *attractor* for the IFS if

$$F = \cup_{i=1}^m S_i(F)$$

We would like to examine the fundamental property of IFSs. To do so, first we shall let $\mathcal{S}$ denote the class of all non-empty compact subsets of D. Let $A_\delta = \{x \in D : |a - x| \leq \delta$ for some $a \in A\}$ be the δ-neighborhood of A. Now, we have a complete metric on $\mathcal{S}$ given by

$$d(A, B) := \inf\{\delta : A \subset B_\delta \ \& \ B \subset A_\delta\} \tag{2.9}$$

This metric is called the Hausdorff metric or *Hausdorff distance* on $\mathcal{S}$.

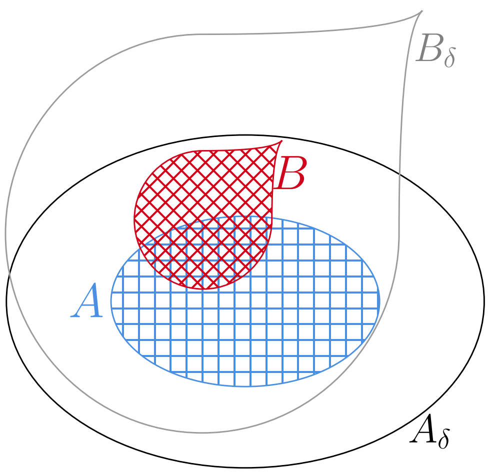

Figure 2.3: The Hausdorff distance/metric between A and B is the least δ such that $A \subset B_\delta$ and $B \subset A_\delta$

Now, we shall state and prove the fundamental property of IFSs

Theorem 2.4.1 (Fundamental Property of IFSs). *Let $\{S_1, ..., S_m\}$ be an IFS. There exists a unique attractor F for the IFS. Moreover, if we define S on $\mathcal{S}$ by $S(E) = \cup_{i=1}^m S_i(E)$ for $E \in \mathcal{S}$ and write S^k for the k-th iterate of S then*

$$F = \cap_{k=0}^{\infty} S^k(E) \tag{2.10}$$

for every set $E \in \mathcal{S}$ such that $S_i(E) \subset E$ $\forall i$.

Proof. First, we note that the mapping S takes elements of $\mathcal{S}$ into other elements of $\mathcal{S}$. Now, for $A, B \in \mathcal{S}$ we have that :

$$\begin{aligned} d(S(A), S(B)) &= d(\cup_{i=1}^m S_i(A), \cup_{i=1}^m S_i(B)) \\ &\leq \max_{1 \leq i \leq m} d(S_i(A), S_i(B)) \\ &\leq \max_{1 \leq i \leq m} c_i d(A, B) \\ &:= \alpha d(A, B) \end{aligned}$$

where $\alpha := \max_{1 \leq i \leq m} c_i$ is such that $0 < \alpha < 1$. Note that in the penultimate line above, we have used that the $S_1, ..., S_m$ are contractions. Hence we have shown that S is a contraction mapping on the complete metric space $(\mathcal{S}, d)$. By the Banach Fixed Point Theorem (also known as the Contraction Mapping Theorem), S has a unique fixed point i.e. $\exists! F \in \mathcal{S}$ such that $S(F) = F$. Hence, F is a unique attractor for the IFS. Moreover, the Contraction Mapping Theorem tells us that $S^k(E) \to F$ as $k \to \infty$. So if $S_i(E) \subset E$ for all i then $S(E) \subset E$ and $S^k(E)$ is a decreasing sequence of non-empty compact sets containing F and have intersection that must equal F; i.e. $F = \cap_{k=0}^{\infty} S^k(E)$.

$\square$

So, each IFS determines a unique attractor, which is usually a fractal. But, how do we actually compute the attractor of an IFS? It turns out that the mapping S in theorem 2.4.1 is the key. Recall in the above proof to theorem 2.4.1 that the sequence

$S^k(E)$ converges to F (the attractor) for any $E \in \mathcal{S}$ with $S_i(E) \subset E$. These increasingly good approximations $S^k(E)$ to F are called *pre-fractals* for F if F is a fractal. Now, we introduce the following notation:

$$\mathcal{I}_k := \{(i_1, ..., i_k) : 1 \le i_j \le m\} \tag{2.11}$$

Then for each k, we have that $S^k(E) = \cup_{\mathcal{I}_k} S_{i_1} \circ \cdots \circ S_{i_k}(E)$. Moreover, if $S_i(E) \subset E$ for all i and $x \in F$ (a point), from (2.10) we have that there is a sequence $(i_1, i_2, ...)$ such that $x \in S_{i_1} \circ \cdots \circ S_{i_k}(E)$ for all k. So we have a coding for x:

$$x = x_{i_1, i_2, ...} = \cap_{k=1}^{\infty} S_{i_1} \circ \cdots \circ S_{i_k}(E) \tag{2.12}$$

and $F = \cup\{x_{i_1, i_2, ...}\}$.

2.5 Self-Similar and Self-Affine Sets

The advantage of IFSs is that it allows us to calculate the dimension of the associated attractor in terms of the defining contractions of the IFS. Here the contractions $S_1, ..., S_m : \mathbb{R}^n \to \mathbb{R}^n$ will be similarities (hence recalling the previous section, we have $|S_i(x) - S_i(y)| = c_i|x - y|$ for $x, y \in \mathbb{R}^n$ where $0 < c_i < 1$). We introduce the following terminology:

Definition 2.5.1. The attractor for an IFS given by similarities $\{S_1, ..., S_m\}$ is called a (strictly) *self-similar set*.

A standard example of a self-similar set is the Sierpinski Gasket (see chapter 4). We will also need the following definition:

Definition 2.5.2. The contractions S_i satisfy the *open set condition* if there exists a non-empty bounded open set V such that $\cup_{i=1}^{m} S_i(V) \subset V$, with the union being disjoint.

To prove the main result of this section, we require the following geometrical result, whose short proof is not of interest:

Lemma 2.5.1. *Let $\{V_i\}$ be a collection of disjoint open subsets of $\mathbb{R}^n$ such that each V_i contains a ball of radius $a_1 r$ and is contained in a ball of radius $a_2 r$. Then any ball B of radius r intersects at most $(1+2a_2)^n a_1^{-n}$ of the closures $\bar{V}_i$.*

Theorem 2.5.1 (Dimension of Self-Similar Sets)**.** *Suppose the open set condition holds for the similarities $S_1, ..., S_m$. If F is the attractor of the IFS $\{S_1, ..., S_m\}$ then $\dim_H F = s$ where s is given by*

$$\sum_{i=1}^{m} c_i^s = 1$$

Moreover, for this value of s we have that $0 < \mathcal{H}^s(F) < \infty$.

Proof. Let s satisfy $\sum_{i=1}^m c_i^s = 1$. We need both an upper and lower bound on the s-dimensional Hausdorff measure $\mathcal{H}^s(F)$.

For the upper bound: Let $\mathcal{I}_k$ denote the set of sequences $(i_1, ..., i_k)$ where $1 \leq i_j \leq m$ (as in eq. (2.11)). Also, for any set A and $(i_1, ..., i_k) \in \mathcal{I}_k$, we write $A_{i_1,...,i_k} := S_{i_1} \circ \cdots \circ S_{i_k}(A)$. Then, since F is the attractor of the IFS $\{S_1, ..., S_m\}$, we know that $F = \cup_{i=1}^m S_i(F)$ and hence:

$$F = \cup_{\mathcal{I}_k} F_{i_1,...,i_k}$$

But, we note that $S_{i_1} \circ \cdots \circ S_{i_k}$ is a similarity with ratio $c_{i_1} \cdots c_{i_k}$ hence:

$$\begin{aligned}
\sum_{\mathcal{I}_k} |F_{i_1,...,i_k}|^s &= \sum_{\mathcal{I}_k} (c_{i_1} \cdots c_{i_k})^s |F|^s \\
&= \left(\sum_{i_1} c_{i_1}^s\right) \cdots \left(\sum_{i_k} c_{i_k}^s\right) |F|^s \\
&= |F|^s
\end{aligned}$$

where in the last step we have used that $\sum_{i=1}^m c_i^s = 1$. So, noting that for any $\delta > 0$ we can choose k such that $|F_{i_1,...,i_k}| \leq (\max_i c_i)^k |F| \leq \delta$, we conclude that $\mathcal{H}_\delta^s(F) \leq |F|^s$ and thus:

$$\mathcal{H}^s(F) \leq |F|^s$$

For the lower bound: let $\mathcal{I}$ denote the set of infinite sequences $(i_1, i_2, ...)$ where

$1 \le i_j \le m$ and let

$$I_{i_1,...,i_k} := \{(i_1, ..., i_k, q_{k+1}, ...) : 1 \le q_j \le m\}$$

be the sequences in $\mathcal{I}$ with initial terms $(i_1, ..., i_k)$. We then define a mass distribution μ on $\mathcal{I}$ given by $\mu(I_{i_1,...,i_k}) = (c_{i_1} \cdots c_{i_k})^s$. Note that since $\sum_{i=1}^m c_i^s = 1$, we have $(c_{i_1} \cdots c_{i_k})^s = \sum_{i=1}^m (c_{i_1} \cdots c_{i_k} c_i)^s$ and hence $\mu(I_{i_1,...,i_k}) = \sum_{i=1}^m \mu(I_{i_1,...,i_k,i})$. Consequently, μ is indeed a mass distribution on $\mathcal{I}$ (with $\mu(\mathcal{I}) = 1$).

Next, we extend μ to a mass distribution $\tilde{\mu}$ on F by

$$\tilde{\mu}(A) := \mu\{(i_1, i_2, ...) : x_{i_1,i_2,...} \in A\}$$

for $A \subset F$ (refer to section 2.4 for the definition of $x_{i_1,i_2,...}$). Our goal will be to apply the Mass Distribution Principle (see chapter 6).

Since the open set condition holds for the similarities $S_1, ..., S_m$, we have a non-empty bounded open set V such that $\cup_{i=1}^m S_i(V) \subset V$ (disjoint union). Hence $\cup_{i=1}^m S_i(\bar{V}) = S(\bar{V}) \subset \bar{V}$ and so the decreasing sequence $S^k(\bar{V})$ converges to F (recall theorem 2.4.1). In particular, we have that $\bar{V} \supset F$ and $\bar{V}_{i_1,...,i_k} \supset F_{i_1,...,i_k}$ for each finite sequence $(i_1, ..., i_k)$.

Now let B be any ball of radius $r < 1$. Truncate each (infinite) sequence $(i_1, i_2, ...) \in \mathcal{I}$ after the first term i_k such that

$$\left(\min_{1 \le i \le m} c_i\right) r \le c_{i_1} \cdots c_{i_k} \le r \tag{2.13}$$

We introduce the notation $\mathcal{Q}$ for the set of the (finite) sequences obtained in this fashion. Note that for every (infinite) sequence $(i_1, i_2, ...) \in \mathcal{I}$, there is exactly one value of k with $(i_1, ..., i_k) \in \mathcal{Q}$. Now, since the $V_1, ..., V_m$ are disjoint, so are $V_{i_1,...,i_k,1}, ..., V_{i_1,...,i_k,m}$ for each sequence $(i_1, ..., i_k)$. Consequently, the collection $\{V_{i_1,...,i_k} : (i_1, ..., i_k) \in \mathcal{Q}\}$ is disjoint and we have that

$$F \subset \cup_{\mathcal{Q}} F_{i_1,...,i_k} \subset \cup_{\mathcal{Q}} \bar{V}_{i_1,...,i_k}$$

Now, choose a_1, a_2 such that V contains a ball of radius a_1 and is contained in a ball of radius a_2. Then for all sequences $(i_1, ..., i_k) \in \mathcal{Q}$, $V_{i_1,...,i_k}$ contains a ball of radius $c_{i_1} \cdots c_{i_k} a_1$ and is contained in a ball of radius $c_{i_1} \cdots c_{i_k} a_2$. So recalling eq. (2.13), $V_{i_1,...,i_k}$ contains a ball of radius $(\min_i c_i) a_1 r$ and is contained in a ball of radius $a_2 r$. Let $\tilde{\mathcal{Q}}$ denote the set of sequences $(i_1, ..., i_k) \in \mathcal{Q}$ such that B intersects $\bar{V}_{i_1,...,i_k}$. Since we had that $\{V_{i_1,...,i_k} : (i_1, ..., i_k) \in \mathcal{Q}\}$ is disjoint, we can apply lemma 2.5.1 to conclude that there are at most $q := (1 + 2a_2)^n a_1^{-n} (\min_i c_i)^{-n}$ sequences in $\tilde{\mathcal{Q}}$.

So, noting that if $x_{i_1,i_2,...} \in F \cap B \subset \cup_{\tilde{\mathcal{Q}}} \bar{V}_{i_1,...,i_k}$ then there is a k with $(i_1, ..., i_k) \in \tilde{\mathcal{Q}}$, we have that:

$$\begin{aligned}
\tilde{\mu}(B) &= \tilde{\mu}(B \cap F) \\
&= \mu\{(i_1, i_2, ...) : x_{i_1,i_2,...} \in F \cap B\} \\
&\leq \mu\{\cup_{\tilde{\mathcal{Q}}} I_{i_1,...,i_k}\} \\
&\leq \sum_{\tilde{\mathcal{Q}}} \mu(I_{i_1,...,i_k}) \\
&= \sum_{\tilde{\mathcal{Q}}} (c_{i_1} \cdots c_{i_k})^s \\
&\leq \sum_{\tilde{\mathcal{Q}}} r^s \\
&\leq q r^s
\end{aligned}$$

where in the penultimate step we have used (2.13). Hence we can finally apply the Mass Distribution Principle (see theorem 6.0.3) to arrive at

$$\mathcal{H}^s(F) \geq \frac{\tilde{\mu}(F)}{q} = \frac{1}{q} > 0$$

and $dim_H F = s$ (note that it is easily seen that $\tilde{\mu}(F) = 1$).

□

Self-similar sets are actually a particular case of an important class of sets called self-affine sets.

Definition 2.5.3. An *affine transformation* $S : \mathbb{R}^n \to \mathbb{R}^n$ is a transformation having form

$$S(x) = T(x) + b$$

, where T is a linear transformation on $\mathbb{R}^n$ represented by an $n \times n$ matrix, and b is a vector in $\mathbb{R}^n$.

Definition 2.5.4. The attractor for an IFS given by affine contractions $\{S_1, ..., S_m\}$ is called a *self-affine set.*

Unlike similarities, these affine transformations contract with differing ratios in different directions. The dimension of the resulting self-affine sets will be explored in the next chapter.

Chapter 3

On the Dimension of Self-Affine Sets

In the previous chapter, we established the dimension for the attractor of an IFS consisting of similarities, and briefly commented on the more general class of attractors - namely self-affine sets. A natural question then is whether we can establish a similar formula for the dimension of arbitrary self-affine sets.

3.1 Dimension via the Singular Value Function

Ideally, one would like to generalize theorem 2.5.1 to self-affine sets. However, this is not so straightforward. To illustrate the difficulties which may arise, we begin with a simple example which shows discontinuous behavior of the dimension. Let S_1, S_2 be affine contractions on $\mathbb{R}^2$ that map the unit square onto the rectangles R_1 and R_2, where each rectangle has sides of length $\frac{1}{2}$ and ϵ for $0 < \epsilon < \frac{1}{2}$. Rectangle R_1 is against the y-axis, and R_2 is a distance $0 \leq \lambda \leq \frac{1}{2}$ from the y-axis (refer to fig. 3.1).

If we let F be the attractor for the IFS $\{S_1, S_2\}$, one can easily show that for $\lambda > 0$, we have that $\dim_H F \geq 1$. However, for $\lambda = 0$, we have that $\dim_H F = \frac{\log 2}{-\log \epsilon} < 1$. Hence, we have affine transformations that can be varied in a continuous way, but the dimension of the resulting self-affine set does not change continuously. This just scratches the surface of the difficulties with this problem, as one can imagine worse behaviour for more involved sets.

Nevertheless, in this section we shall determine the dimension of self-affine sets which

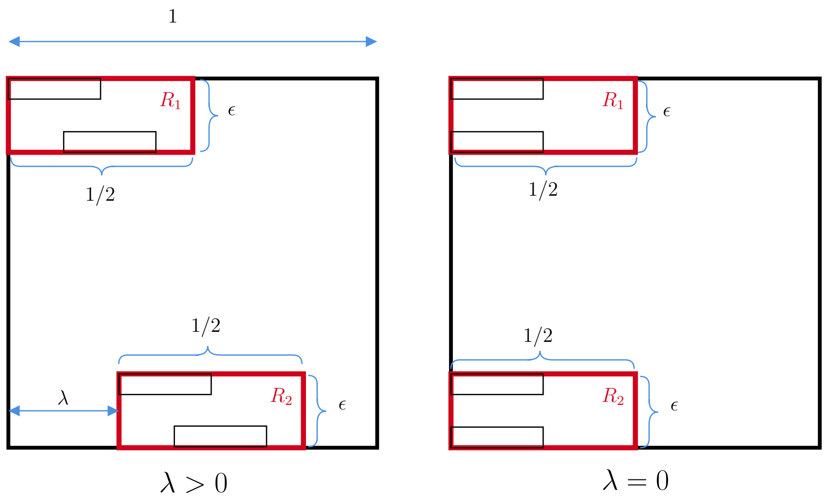

Figure 3.1: The contractions S_1 and S_2 map the unit square to the rectangles R_1 and R_2. The case for both $\lambda > 0$ and $\lambda = 0$ is shown.

are attractors of the affine contractions $S_i(x) = T_i(x) + b_i$ $(i = 1, ..., m)$, for almost every $(b_1, ..., b_m)$. We shall follow the works of Falconer [17] and Solomyak [18]. To this end, we begin with two definitions:

Definition 3.1.1. Let T be a linear, non-singular contraction on $\mathbb{R}^n$. Then, the *singular values* $1 > \alpha_1 \geq \alpha_2 \geq ... \geq \alpha_n > 0$ of T are defined in 2 equivalent ways:

1. They are the positive square roots of the eigenvalues of T^*T. Here T^* denotes the adjoint of T.

2. Letting B denote the unit ball in $\mathbb{R}^n$, the singular values are the lengths of the principal semi-axes of the ellipsoid $T(B)$.

Remark 3.1.1. Considering the second part of the above definition, the singular values are clearly related to the contraction in different directions by T.

Definition 3.1.2. For $0 \leq s \leq n$, the *singular value function* is given by:

$$\varphi^s(T) := \alpha_1\alpha_2 \cdots \alpha_{r-1}\alpha_r^{s-r+1}$$

where $r \in \mathbb{Z}$ is such that $r - 1 < s \leq r$.

We note that by definition, $\varphi^s(T)$ is continuous and decreasing in s. In addition, we have submultiplicity: for fixed s and linear mappings T & U, we have that $\varphi^s(TU) \leq$

$\varphi^s(T)\varphi^s(U)$. Now, letting $\mathcal{I}_k$ denote the set of all sequences $(i_1, ..., i_k)$ where $1 \leq i_j \leq m$, $j = 1, ..., k$, we define the sums $\sum_k^s := \sum_{\mathcal{I}_k} \varphi^s(T_{i_1} \circ \cdots \circ T_{i_k})$. These sums are submultiplicative in k, since for s fixed:

$$\begin{aligned}
\sum_{k+l}^{s} &= \sum_{\mathcal{I}_{k+l}} \varphi^s(T_{i_1} \circ \cdots \circ T_{i_{k+l}}) \\
&\leq \sum_{\mathcal{I}_{k+l}} \varphi^s(T_{i_1} \circ \cdots \circ T_{i_k})\varphi^s(T_{i_{k+1}} \circ \cdots \circ T_{i_{k+l}}) \\
&= \left(\sum_{\mathcal{I}_k} \varphi^s(T_{i_1} \circ \cdots \circ T_{i_k})\right)\left(\sum_{\mathcal{I}_l} \varphi^s(T_{i_1} \circ \cdots \circ T_{i_l})\right) \\
&= \sum_{k}^{s} \sum_{l}^{s}
\end{aligned}$$

where in the second line we have used the submultiplicity of the singular value function. Hence, we know that $(\sum_k^s)^{\frac{1}{k}}$ converges to some $\sum_\infty^s$ as $k \to \infty$. But since φ^s is decreasing in s, we have that $\sum_\infty^s$ is decreasing in s. So, provided that $\sum_\infty^n \leq 1$, there exists unique s such that $1 = \sum_\infty^s = \lim_{k\to\infty}(\sum_k^s)^{\frac{1}{k}}$. We denote such s as $d(T_1, ..., T_m)$, and note that we can equivalently write:

$$d(T_1, ..., T_m) = \inf\left\{ s : \sum_{k=1}^{\infty} \sum_{\mathcal{I}_k} \varphi^s(T_{i_1} \circ \cdots \circ T_{i_k}) < \infty \right\}$$

We are now ready to introduce the main result of this section:

Theorem 3.1.1. *Let F be the attractor for the IFS $\{S_1, ..., S_m\}$ where $S_i = T_i + b_i$ $(i = 1, ..., m)$ are affine contractions on $\mathbb{R}^n$. Then, $dim_H F \leq d(T_1, ..., T_m)$. Moreover, if the contraction ratios are less than $\frac{1}{2}$ (i.e. if $|T_i(x) - T_i(y)| \leq c|x - y|$ for all $i = 1, ..., m$ with $0 < c < \frac{1}{2}$), then $dim_H F = d(T_1, ..., T_m)$ for Lebesgue almost every $(b_1, ..., b_m) \in \mathbb{R}^{nm}$.*

Remark 3.1.2. Initially, Falconer [17] proved the result for contraction ratios strictly less than $\frac{1}{3}$. Solomyak [18] then later extended the bound to $\frac{1}{2}$. We also note that due to Przytycki and Urbanski [19], the bound $\frac{1}{2}$ is sharp.

Here we shall only prove the bound $\dim_H F \leq d(T_1, ..., T_m)$. To that end, let $B \in \mathbb{R}^n$ be a ball that is large enough such that $S_i(B) \subset B$ for all $i = 1, ..., m$. Let $\delta > 0$, and take k sufficiently large such that $|S_{i_1} \circ \cdots \circ S_{i_k}(B)| < \delta$ for each $(i_1, ..., i_k) \in \mathcal{I}_k$. By the work done at the end of section 2.4, we know that

$$F \subset \cup_{\mathcal{I}_k} S_{i_1} \circ \cdots \circ S_{i_k}(B) \tag{3.1}$$

Now, let $\alpha_1, ..., \alpha_n$ denote the singular values of $T_{i_1} \circ \cdots \circ T_{i_k}$. We know that $S_{i_1} \circ \cdots \circ S_{i_k}(B)$ is just a translation of the ellipsoid $T_{i_1} \circ \cdots \circ T_{i_k}(B)$, which has principal axes of lengths $\alpha_1|B|, ..., \alpha_n|B|$. Hence, $S_{i_1} \circ \cdots \circ S_{i_k}(B)$ is contained in a rectangular parallelepiped $\mathcal{P}$, which has sides of lengths $\alpha_1|B|, ..., \alpha_n|B|$.

Now, for $0 \leq s \leq n$ and $r \in \mathbb{Z}$ such that $r-1 < s \leq r$, we can divide the parallelepiped $\mathcal{P}$ into at most

$$\left(\frac{2\alpha_1}{\alpha_r}\right)\left(\frac{2\alpha_2}{\alpha_r}\right)\cdots\left(\frac{2\alpha_{r-1}}{\alpha_r}\right) \leq 2^n \alpha_1 \cdots \alpha_{r-1}\alpha_r^{1-r}$$

cubes which have side length $\alpha_r|B| < \delta$. Therefore, we can cover $S_{i_1} \circ \cdots \circ S_{i_k}(B)$ by cubes U_i having $|U_i| < \delta\sqrt{n}$. Also,

$$\begin{aligned}\sum_i |U_i|^s &\leq 2^n \alpha_1 \cdots \alpha_{r-1}\alpha_r^{1-r}\alpha_r^s|B|^s \\ &= 2^n|B|^s\varphi^s(T_{i_1} \circ \cdots \circ T_{i_k})\end{aligned}$$

where in the last line we have used the definition of the singular value function. So, taking these covers of $S_{i_1} \circ \cdots \circ S_{i_k}(B)$ for each $(i_1, ..., i_k) \in \mathcal{I}_k$, and recalling (3.1), we have that:

$$\mathcal{H}^s_{\delta\sqrt{n}}(F) \leq 2^n|B|^s \sum_{\mathcal{I}_k} \varphi^s(T_{i_1} \circ \cdots \circ T_{i_k})$$

However, as $\delta \to 0$, we must have that $k \to \infty$ and hence by the above and the definition of $d(T_1, ..., T_m)$, we have that $\mathcal{H}^s(F) = 0$ for $s > d(T_1, ..., T_m)$. So, $\dim_H F \leq d(T_1, ..., T_m)$ as desired.

3.2 Dimension of Sierpinski Carpets

In the previous section, we have determined the dimension of typical self-affine sets. However, there are many exceptional cases. Hence in this section, we determine the Hausdorff dimension of an important family of planar sets called *Sierpinski Carpets*, which are generalizations of the Cantor set in two dimensions. We shall follow the framework of the seminal work by McMullen [16].

To define Sierpinski Carpets, we begin by letting $n > m$, and letting R denote the set consisting of pairs of integers (i, j) where $0 \leq i < n$, $0 \leq j < m$. Then the (general) *Sierpinski Carpet* is the plane fractal:

$$SC := \left\{ \left(\sum_{k=1}^{\infty} \frac{x_k}{n^k}, \sum_{k=1}^{\infty} \frac{y_k}{m^k} \right) : (x_k, y_k) \in R \ \forall k \right\}$$

Letting $r := |R|$, we note that $SC = \cup_{i=1}^{r} F_i(SC)$, where the F_i are affine transformations, contracting SC by a factor of m vertically and n horizontally. Hence, the SC is a true self-affine set in the sense introduced in section 2.5.

Remark 3.2.1. If we had that $n = m$, then the resulting SC would be a self-similar set.

We would like to determine $\dim_H(SC)$. To do so, we begin with carefully selected rectangles:

$$R_k(p, q) := \left[\frac{p}{n^l}, \frac{p+1}{n^l} \right] \times \left[\frac{q}{m^k}, \frac{q+1}{m^k} \right]$$

where $l := \lfloor k \log_n m \rfloor \in \mathbb{Z}$ is such that $n > \frac{m^k}{n^l} \geq 1$. So we have coverings $C = \{R_k(p, q)\}$ of SC, and we introduce the integers $N_k :=$ # of $R_{k'}(p, q) \in C$ with $k' = k$. Our first lemma then expresses the dimension of SC in terms of coverings by the specific rectangles:

Lemma 3.2.1. *The s-dimensional Hausdorff measure $\mathcal{H}^s(SC) = 0$ iff for any $\epsilon > 0$, there exists a covering $C = \{R_k(p, q)\}$ of SC such that*

$$\sum_k N_k \frac{1}{m^{sk}} < \epsilon$$

Although we do not prove this lemma, we note that it is fairly straightforward if we note that $m^{-k} \approx |R_k(p,q)|$ (where $|\cdot|$ denotes diameter) and we recall the definitions introduced in section 2.1.

Now, recalling that $r = |R|$, we can write the enumeration $(x_i, y_i)_{i=0}^{r-1}$ for R. Let $S_r := \prod_1^\infty \{0,1,2,...,r-1\}$. We then have a surjective map $\psi : S_r \to SC$ given by $(i_1, i_2, i_3, ...) \mapsto (\sum_{k=1}^\infty \frac{x_{i_k}}{n^k}, \sum_{k=1}^\infty \frac{y_{i_k}}{m^k})$. We shall write:

$$p := \sum_{j=1}^{l} \tilde{x}_j n^{l-j} \quad (0 \leq \tilde{x}_j < n, l = \lfloor k \log_n m \rfloor)$$
$$q = \sum_{j=1}^{k} \tilde{y}_j m^{k-j} \quad (0 \leq \tilde{y}_j < m)$$

Then we can define:

$$A_k(p,q) := \{(i_1, i_2, ..., i_k) : x_{i_j} = \tilde{x}_j \text{ for } j = 1,..,l \text{ and } y_{i_j} = \tilde{y}_j \text{ for } j = 1,...,k\}$$
$$B_k := \prod_{k+1}^{\infty} \{0,1,...,r-1\}$$

Hence, we have sets of the form $A_k(p,q) \times B_k \in S_r$ which under the mapping ψ correspond roughly to the $R_k(p,q) \in SC$. Additionally, we have coverings $C = \{A_k(p,q) \times B_k\}$ of S_r, and we introduce the integers $N_k := \#$ of $A_{k'}(p,q) \times B_{k'} \in C$ with $k' = k$. Our second lemma then expresses the dimension of SC in terms of coverings of S_r:

Lemma 3.2.2. *The s-dimensional Hausdorff measure $\mathcal{H}^s(SC) = 0$ iff for any $\epsilon > 0$, there exists a covering $C = \{A_k(p,q) \times B_k\}$ of S_r such that*

$$\sum_k N_k \frac{1}{m^{sk}} < \epsilon$$

Proof. Since

$$A_k(p,q) \times B_k \subset \psi^{-1}(R_k(p,q)) \subset \cup_{i=-1,0,1\ j=-1,0,1} A_k(p+i, q+j) \times B_k$$

we can pass between covers of S_r and SC by modifying N_k up to a bounded constant. Then by lemma 3.2.1, we are finished.

$\square$

The main reason we have done this is that coverings of S_r by cylinders $(A_k(p,q) \times B_k)$ are easier to study than coverings of SC by rectangles $(R_k(p,q))$. To that end, we now will briefly analyze the size of the cylinders. As before, we write the enumeration $(x_i, y_i)_{i=0}^{r-1}$ for R. Then, for $i = 0, 1, ..., r-1$ we define $a_i := \#$ of j such that $y_i = y_j$. So the cardinality of $A_k(p,q)$ is $a_{i_{l+1}} a_{i_{l+2}} \cdots a_{i_k}$ for any $(i_1, ..., i_k) \in A_k(p,q)$. We note that $n > m$ and hence $l = \lfloor k \log_n m \rfloor < k$.

We shall now let

$$\delta := \log_m \left(\sum_{j=0}^{m-1} t_j^{\log_n m} \right) \tag{3.2}$$

where t_j is the number of i such that $(i,j) \in R$ (our final goal will be to show that δ is actually equal to $\dim_H(SC)$). We can then write $m^\delta = \sum_{i=0}^{r-1} a_i^{\log_n m - 1}$. So if we let

$$b_i := \frac{a_i^{\log_n m - 1}}{m^\delta} \tag{3.3}$$

we can use the b_i to define a measure; let μ be the unique probability measure on the Borel subsets of S_r which satisfies: for any $(i_1, ..., i_k)$, we have that $\mu((i_1, i_2, ..., i_k) \times B_k) = b_{i_1} b_{i_2} \cdots b_{i_k}$. This unique probability measure exists as a consequence of Kolmogorov Extension Theorem.

Now, we define the sequence of functions f_k on S_r, given by:

$$f_k(i_1, i_2, i_3, ...) := \left[\frac{(a_{i_1} a_{i_2} \cdots a_{i_k})^{\log_n m}}{a_{i_1} a_{i_2} \cdots a_{i_l}} \right]^{\frac{1}{k}}$$

Lemma 3.2.3. *If $z \in A_k(p,q) \times B_k$, then $\mu(A_k(p,q) \times B_k) = (f_k(z) m^{-\delta})^k$.*

Proof. For each $(i_1, i_2, ..., i_k) \in A_k(p,q)$, by definition we have that

$$\begin{aligned} \mu((i_1, i_2, ..., i_k) \times B_k) &= b_{i_1} b_{i_2} \cdots b_{i_k} \\ &= \frac{(a_{i_1} a_{i_2} \cdots a_{i_k})^{\log_n m}}{(a_{i_1} a_{i_2} \cdots a_{i_k})(m^{\delta k})} \end{aligned}$$

However, recall that the cardinality of $A_k(p,q)$ is $a_{i_{l+1}} a_{i_{l+2}} \cdots a_{i_k}$ for any $(i_1, ..., i_k) \in A_k(p,q)$.

Hence:

$$\begin{aligned}
\mu(A_k(p,q)\times B_k) &= (a_{i_{l+1}}a_{i_{l+2}}\cdots a_{i_k})\frac{(a_{i_1}a_{i_2}\cdots a_{i_k})^{\log_n m}}{(a_{i_1}a_{i_2}\cdots a_{i_k})(m^{\delta k})} \\
&= \frac{(a_{i_1}a_{i_2}\cdots a_{i_k})^{\log_n m}}{(a_{i_1}\cdots a_{i_l})}m^{-\delta k} \\
&= (f_k(z)m^{-\delta})^k
\end{aligned}$$

where in the last line we have used the definition of the f_k.

$\square$

Lemma 3.2.4. *We have that for all $z \in S_r$, $\overline{\lim}_{k\to\infty} f_k(z) \geq 1$. Also, $f_k \to 1$ almost everywhere (with respect to μ).*

Proof. First, we shall simplify the problem. Note that we may write $f_k(z) = h_k(z)g_k(z)^{\log_n m}$ where

$$\begin{aligned}
g_k(i_1,i_2,...) &:= \frac{(a_{i_1}a_{i_2}\cdots a_{i_k})^{\frac{1}{k}}}{(a_{i_1}a_{i_2}\cdots a_{i_l})^{\frac{1}{l}}} \\
h_k(i_1,i_2,...) &:= (a_{i_1}a_{i_2}\cdots a_{i_l})^{(\frac{1}{l})(\log_n m-\frac{l}{k})}
\end{aligned}$$

Moreover, since $1 \leq a_i \leq n$ we have that $1 \leq h_k \leq n^{\log_n m-\frac{l}{k}}$. Then since

$$\log_n m - \frac{l}{k} = \log_n m - \frac{\lfloor k\log_n m\rfloor}{k} \to 0$$

as $k \to \infty$, we deduce that $h_k(z) \to 1$. So , we only need to show that for all $z \in S_r$, $\overline{\lim}_{k\to\infty} g_k(z) \geq 1$ and $g_k \to 1$ almost everywhere (with respect to μ).

For the first part, fix $z = (i_1, i_2, ...)$. Since $l = \lfloor k\log_n m\rfloor < k$ and $1 \leq a_i \leq n$, it is clear that:

$$\overline{\lim}_{k\to\infty} g_k(z) = \overline{\lim}_{k\to\infty}\frac{(a_{i_1}a_{i_2}\cdots a_{i_k})^{\frac{1}{k}}}{(a_{i_1}a_{i_2}\cdots a_{i_l})^{\frac{1}{l}}} \geq 1$$

For the second part, note that the functions $(i_1, i_2, ...) \to b_{i_k}$ $(k = 1, 2, 3, ...)$ are iid random variables with respect to μ. Hence by the Strong Law of Large Numbers, the sequence $(b_{i_1}b_{i_2}\cdots b_{i_k})^{\frac{1}{k}}$ $(k = 1, 2, 3, ...)$ converges for almost every $(i_1, i_2, ...) \in S_r$. Writing

g_k as

$$g_k(i_1, i_2, ...) = \left(\frac{(b_{i_1} b_{i_2} \cdots b_{i_k})^{\frac{1}{k}}}{(b_{i_1} b_{i_2} \cdots b_{i_l})^{\frac{1}{l}}} \right)^{(\log_n m - 1)^{-1}}$$

(recall the definition of the b_i in (3.3)) , we thus have that $g_k \to 1$ almost everywhere.

□

The idea now is to use the first part of the previous lemma 3.2.4 to bound the dimension of SC from above by δ (recall the definition (3.2)):

Lemma 3.2.5. *We have that* $dim_H(SC) \leq \delta$.

Proof. Let $\epsilon > 0$ and C_k be the nonempty sets of form $A_k(p,q) \times B_k$ with $f_k(z) > m^{-\epsilon}$ (for $z \in A_k(p,q) \times B_k$). These sets are disjoint, and by lemma 3.2.3:

$$\mu(A_k(p,q) \times B_k) = (f_k(z) m^{-\delta})^k > m^{-(\delta+\epsilon)k}$$

Recalling that $\mu(S_r) = 1$, we then must have that the cardinality of C_k, denoted by M_k, is $< m^{(\delta+\epsilon)k}$.

Now for any $z \in S_r$, by lemma 3.2.4 we have that $\overline{\lim}_{k\to\infty} f_k(z) \geq 1 > m^{-\epsilon}$. Hence, any $z \in S_r$ is covered by C_k for infinitely many k, and consequently $C := \cup_{k \geq K} C_k$ forms a covering of S_r for any choice of K. In particular, we shall take K large enough so that $\sum_{k \geq K} m^{-\epsilon k} < \epsilon$. Then, the N_k associated to C (as seen in lemma 3.2.2) satisfy:

$$\begin{aligned} \sum_k N_k m^{-(\delta+2\epsilon)k} &= \sum_{k \geq K} M_k m^{-(\delta+2\epsilon)k} \\ &< \sum_{k \geq K} m^{(\delta+\epsilon)k} m^{-(\delta+2\epsilon)k} \\ &= \sum_{k \geq K} m^{-\epsilon k} \\ &< \epsilon \end{aligned}$$

So by lemma 3.2.2 we are finished.

□

The second part of lemma 3.2.4 then allows us to bound the dimension of SC from below by δ:

Lemma 3.2.6. *We have that $dim_H(SC) \geq \delta$.*

Proof. Let $\beta < \delta$. We shall first show that there is $\epsilon > 0$ such that $\sum_k N_k m^{-\beta k} > \epsilon$ for any covering C of S_r. To that end, let $E_K := \{z \in S_r : f_k(z) < m^{\delta-\beta} \ \forall k \geq K\}$. We have that $m^{\delta-\beta} > 1$ and also by lemma 3.2.4, we know that $f_k \to 1$ almost everywhere. Hence, we may pick K such that $\mu(E_K) > 0$.

Now, take $\epsilon := \min\{\mu(E_K), m^{-\beta K}\} > 0$ and let C be any covering of S_r. We first note that if $N_k \neq 0$ for some $k < K$ then we have $\sum_k N_k m^{-\beta k} > m^{-\beta K} \geq \epsilon$. Hence, from now on we assume that $N_k = 0$ for all $k < K$.

Now, for elements of C with $A_k(p,q) \times B_k \cap E_K \neq \emptyset$, we have that (where $z \in A_k(p,q) \times B_k \cap E_K$):

$$\begin{aligned} \mu(A_k(p,q) \times B_k) &= (f_k(z) m^{-\delta})^k \\ &< (m^{\delta-\beta} m^{-\delta})^k \\ &= m^{-\beta k} \end{aligned}$$

where in the first line we have used lemma 3.2.3. Then since C covers E_K, we conclude that $\sum_k N_k m^{-\beta k} > \mu(E_K) \geq \epsilon$. The purpose of this is that by lemma 3.2.2, we have that $\mathcal{H}^\beta(SC) \neq 0$, where our $\beta < \delta$ is arbitrary. So our result follows.

□

Combining the results of lemma 3.2.5 and lemma 3.2.6, we arrive at the final conclusion of this section:

Theorem 3.2.1 (Dimension of Sierpinski Carpets)**.** *The Hausdorff dimension of SC is given by*

$$dim_H(SC) = \log_m \left(\sum_{j=0}^{m-1} t_j^{\log_n m} \right)$$

where t_j is the number of i such that $(i,j) \in R$.

Chapter 4

The Sierpinski Gasket

In this chapter we focus on a single fractal - the Sierpinski Gasket. After reviewing the spectral properties of the Laplacian on the Sierpinski Gasket, we use the existence of localized eigenfunctions to develop analogues of the Szegö Limit Theorem (following [2],[9]).

4.1 Construction

The Sierpinski Gasket (SG) is a fractal having an overall shape of an equilateral triangle, which is subdivided recursively into smaller equilateral triangles (see figure below). It is a typical post-critically finite fractal, but is special in that it has a well developed theory of the Laplacian. Here we shall describe the SG via an approximation from within by a sequence of graphs.

To that end, let

$$\begin{cases} F_1(x) := \frac{1}{2}x \\ F_2(x) := \frac{1}{2}x + (\frac{1}{2}, 0) \\ F_3(x) := \frac{1}{2}x + (\frac{1}{4}, \frac{\sqrt{3}}{4}) \end{cases}$$

be contractions on $\mathbb{R}^2$. We will need the following definition: for $\omega = (\omega_1, ..., \omega_m)$, the set $F_\omega(SG) := F_{\omega_1} \circ ... \circ F_{\omega_m}(SG)$ where $\omega_i \in \{1, 2, 3\}$ is called an $m-$cell.

Remark 4.1.1. We have that $\dim_H(SG) = \frac{\log 3}{\log 2}$ since the solution to $\sum_{i=1}^{3}(\frac{1}{2})^s = 1$ is

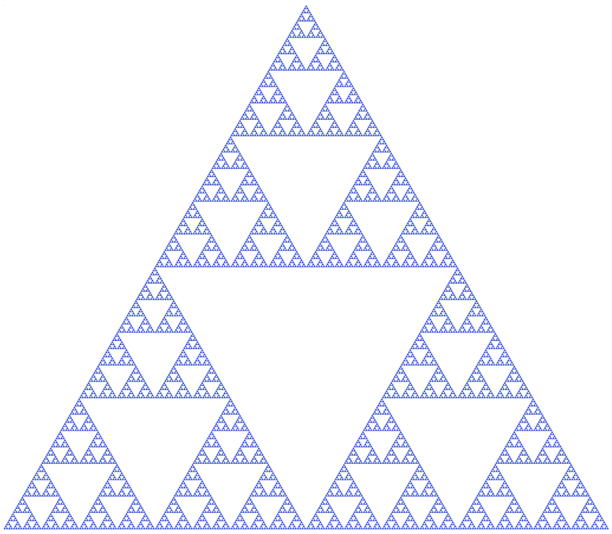

Figure 4.1: The Sierpinski Gasket.

$s = \frac{\log 3}{\log 2}$. See theorem 2.5.1

Remark 4.1.2. We equip the SG with a probability measure μ that assigns measure 3^{-m} to each m-cell. This will become important later.

Now, let $V_0 := \{(0,0),(1,0),(\frac{1}{2},\frac{\sqrt{3}}{2})\}$, which forms the boundary of the SG. Let

$$V_n := \cup_{i=1}^{3} F_i V_{n-1} \ , \ n \geq 1$$

Our sequence of graphs $\{\Gamma_m\}$ with vertices in V_m and edge relation $x \sim_m y$ is obtained inductively : we let Γ_0 be the complete graph on V_0 and $x \sim_m y$ iff x and y belong to the same $m-$cell.

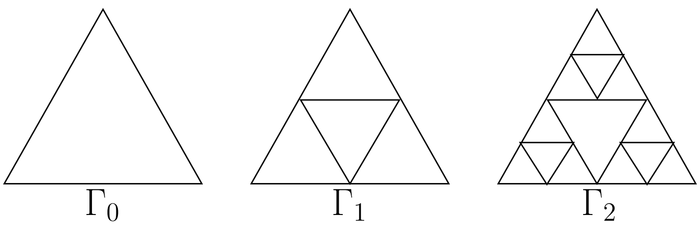

Figure 4.2: The first three graphs which approximate the Sierpinski Gasket

4.2 Theory of the Laplacian on SG

In this section, we define the Laplacian on the Sierpinski Gasket as a renormalized limit of graph Laplacians, and examine its spectrum. We start with the graph Laplacian Δ_m on the graph Γ_m, which is given by

$$\Delta_m f(x) = \sum_{y \sim_m x} f(y) - 4f(x)$$

where $x \in V_m \setminus V_0$. Then, our Laplacian on SG is simply the renormalized limit

$$\Delta := \frac{3}{2} \lim_{m\to\infty} 5^m \Delta_m$$

In [5], a full description of the spectrum of the Laplacian Δ on the SG is presented, and here we recall the main results. For each Dirichlet eigenvalue λ of Δ, there is an integer $j \geq 1$ such that if u is an eigenfunction corresponding to eigenvalue λ and $k \geq j$, then $u|_{V_k}$ is an eigenfunction of the graph Laplacian Δ_k with eigenvalue γ_k. The integer j we shall refer to as the *generation of birth*. Moreover, the only possible values of γ_j are $2, 5, 6$, and further values for $k > j$ are obtained by the following equation:

$$\gamma_k = \frac{1}{2}\left[5 + \epsilon_k \sqrt{25 - 4\gamma_{k-1}}\right] \tag{4.1}$$

where $\epsilon_k \in \{-1, 1\}$. The relation between λ and γ_k is given by

$$\lambda = \frac{3}{2} \lim_{k\to\infty} 5^k \gamma_k \tag{4.2}$$

The upshot is that convergence in (4.2) tells us that $\epsilon_k = 1$ for at most a finite number of k values. This allows us to define the *generation of fixation* l:

$$l := \min\{k : \epsilon_k = -1\}$$

So, the spectrum of Δ is completely determined by the spectrum of the graph Laplacian. The generation of birth j and generation of fixation l are critical, as they determine the size and multiplicity of an eigenvalue. Namely, there exists a constant κ such that the $\frac{1}{2}(3^{m+1}-3)$ smallest eigenvalues of Δ have size at most $\kappa 5^m$ and generation of fixation $l \leq m$. Moreover:

Theorem 4.2.1 (Spectrum of Laplacian on the Sierpinski Gasket)**.**

1. *The so called "2-series eigenvalues" are the eigenvalues obtained from* (4.1) *&* (4.2) *with* $j = 1$ *and* $\gamma_j = 2$*. Each 2-series eigenvalue has multiplicity 1.*

2. *The "5-series eigenvalues" are the eigenvalues obtained from* (4.1) *&* (4.2) *with any* $j \geq 1$ *and* $\gamma_j = 5$*. There are* 2^{m-j} *5-series eigenvalues for each* $1 \leq j \leq m$*, each with multiplicity* $\frac{1}{2}(3^{j-1}+3)$*. For each 5-series eigenvalue, there is a basis for the corresponding eigenspace in which all but two of the basis functions have support in a collection of* $(j-1)$*-cells arranged in a loop around a hole of scale at least* $(j-1)$ *in the SG. There is one eigenfunction per hole and a total of* $\frac{1}{2}(3^{j-1}-1)$ *holes.*

3. *The "6-series eigenvalues" are the eigenvalues obtained from* (4.1) *&* (4.2) *with any* $j \geq 2$*,* $\gamma_j = 6$*,* $\epsilon_{j+1} = 1$*. There are* 2^{m-j-1} *6-series eigenvalues for each* $2 \leq j < m$ *and* 1 *for* $j = m$*. Each 6-series eigenvalue has multiplicity* $\frac{1}{2}(3^j - 3)$*. For each 6-series eigenvalue, there is a basis for the corresponding eigenspace that is indexed by points in* $V_{j-1} \setminus V_0$*, and in which each basis element is supported on the union of the two* j*-cells that intersect at the corresponding point in* $V_{j-1} \setminus V_0$*.*

4.3 Szegö Limit Theorem on the SG

Using our full description of the spectrum of the Laplacian on the SG seen in theorem 4.2.1, we proceed to develop the Szegö Limit Theorem.

4.3.1 Special Case

First we shall develop the Szegö Limit Theorem on the Sierpinski Gasket for a specific case (for a quick introduction to the classical Szegö Limit Theorem, see Appendix A) . Namely, we will only be looking at a single 5-series or 6-series eigenspace of the Laplacian.

We will first need a preliminary definition: we say that a function is *localized* at scale N if its support is contained in a single N-cell. Now, let λ_j denote a 6-series eigenvalue with generation of birth $j \geq 2$, and denote its corresponding eigenspace by E_j. Let E_j^N be the span of the eigenfunctions (corresponding to λ_j) that are localized at scale $N < j$. We have that:

$$\begin{aligned} dim(E_j^N) &:= d_j^N \\ &= \frac{1}{2}\left(3^j - 3^{N+1}\right) \end{aligned}$$

where we have used theorem 4.2.1. The dimension of the complementary space (in E_j) is then $\alpha_j^N = \frac{1}{2}(3^{N+1} - 3)$ and the number of eigenfunctions supported on a single cell is $m_j^N = \frac{1}{2}(3^{j-N} - 3)$ (since there are 3^N cells having scale N).

Remark 4.3.1. We can deduce the same information for a 5-series eigenvalue; in this case we will have $d_j^N = \frac{1}{2}(3^{j-1} - 3^N)$, $\alpha_j^N = \frac{1}{2}(3^N - 3)$, and $m_j^N = \frac{1}{2}(3^{j-N-1} - 1)$.

Now, for each N-cell, apply Gram-Schmidt to orthonormalize the eigenfunctions supported on the cell. Taking the union over the N-cells, we obtain an orthonormal basis $\{\tilde{u}_k\}_{k=1}^{d_j^N}$ for E_j^N (note that eigenfunctions on seperate cells are orthogonal). If we add the remaining basis elements v_k of E_j (so the v_k are not localized at scale N) and apply Gram-Schmidt again, we obtain an orthonormal basis $\{u_k\}_{k=1}^{d_j} = \{\tilde{u}_k\}_{k=1}^{d_j^N} \cup \{v_k\}_{k=1}^{\alpha_j^N}$ for E_j.

Remark 4.3.2. We let $d_j := dim(E_j)$.

For a real valued function f on the SG, we let $[f]$ be the operator corresponding to pointwise multiplication by f. Also, for $g \in L_2(SG)$, we have

$$\begin{aligned} P_j g(x) &:= \sum_{k=1}^{d_j} \langle g, u_k \rangle u_k(x) \\ &= \sum_{k=1}^{d_j} g_k u_k(x) \end{aligned}$$

In other words, P_j is the projection of $L_2(SG)$ onto E_j. We can now state the following lemma which we will later need:

Lemma 4.3.1. *Consider the simple function* $f = \sum_{k=1}^{3^N} a_k \mathbb{1}_{C_k}$ *on the SG, where* $a_k > 0$ *and* C_k *is an* N*-cell. Then,*

$$\lim_{j\to\infty} \frac{1}{d_j} \log det P_j[f]P_j = \int_{SG} \log f(x) d\mu(x)$$

and for j *large,*

$$\frac{1}{d_j} \log det P_j[f]P_j - \int_{SG} \log f(x) d\mu(x) = O\left(\frac{1}{d_j}\right)$$

Proof. Note that

$$P_j[f]P_j = \begin{bmatrix} R_j & 0 \\ 0 & N_j \end{bmatrix}$$

is a $d_j \times d_j$ matrix where R_j is a $d_j^N \times d_j^N$ matrix that corresponds to the localized part, and N_j is a $\alpha_j^N \times \alpha_j^N$ matrix corresponding to the nonlocalized part. Since $\langle N_j g, g\rangle \leq ||f||_\infty$ for all $g \in E_j$ with $||g||_2 = 1$, we have the following (to be used later in this proof as well as in the next theorem):

$$\log det N_j \leq \alpha_j^N ||f||_\infty \tag{4.3}$$

Now, note that R_j is a block diagonal matrix whose blocks are $m_j^N \times m_j^N$ matrices that correspond to a single N-cell C_k. Hence the blocks are just $a_k \mathbb{I}_{m_j^N}$ (where $\mathbb{I}_n$ is the

$n \times n$ identity matrix). So we have:

$$\begin{aligned}
\log det P_j[f]P_j &= \log det R_j + \log det N_j \\
&= \log \left(\prod_{k=1}^{3^N} a_k^{m_j^N} \right) + \log det N_j \\
&= m_j^N \left(\sum_{k=1}^{3^N} \log a_k \right) + \log det N_j \\
&= m_j^N 3^N \left(\sum_{k=1}^{3^N} 3^{-N} \log a_k \right) + \log det N_j \\
&= d_j^N \left(\sum_{k=1}^{3^N} 3^{-N} \log a_k \right) + \log det N_j
\end{aligned}$$

where in the last line we have used that $d_j^N = 3^N m_j^N$. Dividing by d_j, we arrive at:

$$\begin{aligned}
\frac{1}{d_j} \log det P_j[f]P_j &= \frac{d_j^N}{d_j} \sum_{k=1}^{3^N} 3^{-N} \log a_k + \frac{1}{d_j} \log det N_j \\
&= \frac{d_j^N}{d_j} \int_{SG} \log f d\mu + \frac{1}{d_j} \log det N_j
\end{aligned}$$

where we recall that we have equipped the SG with a probability measure μ that assigns measure 3^{-m} to each m-cell.

But, $d_j - d_j^N = \alpha_j^N$, and hence we obtain:

$$\frac{1}{d_j} \log det P_j[f]P_j - \int_{SG} \log f d\mu = -\frac{\alpha_j^N}{d_j} \int_{SG} \log f d\mu + \frac{1}{d_j} \log det N_j \tag{4.4}$$

Finally, recalling (4.3) and using (4.4), we conclude that:

$$\begin{aligned}
\left| \frac{1}{d_j} \log det P_j[f]P_j - \int_{SG} \log f d\mu \right| &\leq \frac{\alpha_j^N}{d_j} \left(|| \log f ||_1 + ||f||_\infty \right) \\
&= O\left(\frac{3^N}{3^j} \right)
\end{aligned}$$

$\square$

Now with the lemma in hand, we prove the main result of this section:

Theorem 4.3.1 (Szegö Limit Theorem on SG - Single Eigenspace)**.** *Let $f > 0$ be continuous on the SG. Then,*

$$\lim_{j\to\infty} \frac{1}{d_j} \log det P_j[f]P_j = \int_{SG} \log f(x) d\mu(x)$$

Proof. Since the SG is compact, we have $m := \min_{x\in SG} f(x) > 0$. Hence letting $\epsilon > 0$, by continuity there exists N and a simple function $f_N = \sum_{k=1}^{3^N} a_k \mathbb{1}_{C_k}$ (see previous lemma) such that $||f - f_N||_\infty < \min\{\frac{m}{2}, \frac{\epsilon m}{2}, \frac{1}{2}\}$. Thus, we have that

$$1 - \epsilon \leq \frac{f(x)}{f_N(x)} \leq 1 + \epsilon \tag{4.5}$$

and

$$\log(1-\epsilon) \leq \log\left(\frac{f(x)}{f_N(x)}\right) \leq \log(1+\epsilon) \tag{4.6}$$

Now,

$$P_j[f]P_j = \begin{bmatrix} R_j & \eta \\ 0 & N_j \end{bmatrix}$$

is a $d_j \times d_j$ matrix where R_j is a $d_j^N \times d_j^N$ matrix that corresponds to the localized part, and N_j is a $\alpha_j^N \times \alpha_j^N$ matrix corresponding to the nonlocalized part. By the previous lemma and (4.5), we have that

$$\left| \frac{1}{d_j} \log det R_j - \frac{d_j^N}{d_j} \int_{SG} \log f_N d\mu \right| \leq 2\epsilon \tag{4.7}$$

Also, (4.6) implies that

$$\left| \int_{SG} \log f d\mu - \int_{SG} \log f_N d\mu \right| \leq 2\epsilon \tag{4.8}$$

Hence if we use the bound $\log detN_j \leq \alpha_j^N ||f||_\infty$ (see (4.3)), we have that:

$$\begin{aligned}
&\left|\frac{1}{d_j}\log detP_j[f]P_j - \int_{SG}\log f(x)d\mu(x)\right| \leq \left|\frac{1}{d_j}\log detR_j - \frac{d_j^N}{d_j}\int_{SG}\log f_N d\mu\right| \\
&+\frac{1}{d_j}\log detN_j + \left|\int_{SG}\log f_N - \log f d\mu\right| + \frac{\alpha_j^N}{d_j}\left|\int_{SG}\log f_N d\mu\right| \\
&\leq 4\epsilon + \frac{1}{d_j}\log detN_j + \frac{\alpha_j^N}{d_j}\left|\int_{SG}\log f_N d\mu\right| \\
&\leq 4\epsilon + \frac{\alpha_j^N}{d_j}||f||_\infty + \frac{\alpha_j^N}{d_j}||\log f||_1 \\
&\leq 4\epsilon + 3^{N-j}(||f||_\infty + ||\log f||_1)
\end{aligned}$$

where in the second inequality we have used (4.7) and (4.8) .

□

4.3.2 Full Result

Now, instead of focusing on a single eigenspace as in the previous section, here we will look at all the eigenvalues up to a certain value Λ. Hence we shall fix the scale N, and let E_Λ denote the span of the eigenfunctions corresponding to eigenvalues λ for which $\lambda \leq \Lambda$. Similarly to before, we let $dim(E_\Lambda) = d_\Lambda$ and P_Λ be the projection of $L_2(SG)$ onto E_Λ. We have the following:

Theorem 4.3.2 (Szegö Limit Theorem on SG). *Let $f > 0$ be continuous on the SG. Then,*

$$\lim_{\Lambda\to\infty}\frac{1}{d_\Lambda}\log detP_\Lambda[f]P_\Lambda = \int_{SG}\log f(x)d\mu(x)$$

Proof. Let $\epsilon > 0$. First, note that $P_\Lambda[f]P_\Lambda$ forms a block diagonal matrix M_Λ with one block M_λ per eigenvalue λ. Moreover, $\log detM_\Lambda = \sum_{\lambda\leq\Lambda}\log detM_\lambda$. From theorem 4.3.1, if λ is a 6-series or $5-$series eigenvalue with generation of birth $j > N$, we have that

$$\left|\log detM_\lambda - d_\lambda\int_{SG}\log f d\mu\right| \leq 4\epsilon d_\lambda + 3^N\left(||f||_\infty + ||\log f||_1\right) \tag{4.9}$$

Let $\mathcal{S}_N$ denote the set of eigenvalues $\lambda \leq \Lambda$ with generation of birth $j > N$. Summing over the eigenvalues and noting that $\frac{1}{d_\Lambda}\sum_{\{\lambda\in\mathcal{S}_N\}} d_\lambda \leq 1$, from (4.9) we conclude that

$$\begin{aligned}&\left|\frac{1}{d_\Lambda}\log det M_\Lambda - \int_{SG}\log f d\mu\right| \\ &\leq 4\epsilon + (||f||_\infty + ||\log f||_1)\left(\left(\frac{3^N}{d_\Lambda}\sum_{\{\lambda\in\mathcal{S}_N\}} 1\right) + \left(\sum_{\{\lambda\notin\mathcal{S}_N\}}\frac{d_\lambda}{d_\Lambda}\right)\right)\end{aligned}$$

Looking at the above, it is clear that we need to find both the number of $\lambda \in \mathcal{S}_N$, as well as the sum of the d_λ for $\lambda \notin \mathcal{S}_N$. The key tool we will use is the theory of the Laplacian on the SG developed in section 4.2 (namely theorem 4.2.1). First, let $m \in \mathbb{N}$ be such that $\kappa 5^{m-1} \leq \Lambda < \kappa 5^m$ (note that if $N \geq m$, then $\mathcal{S}_N$ is empty, hence we suppose that $N < m$).

For $\lambda \in \mathcal{S}_N$ such that $\lambda \leq \kappa 5^m$, by theorem 4.2.1, we have that:

$$\begin{aligned}\sum_{\{\lambda\in\mathcal{S}_N\}} &= \sum_{j=N+1}^{m} 2^{m-j} + \sum_{j=N+1}^{m} 2^{m-j-1} \\ &= O(2^{m-N})\end{aligned}$$

Hence, we have that:

$$\frac{3^N}{d_\Lambda}\sum_{\{\lambda\in\mathcal{S}_N\}} = O\left(\frac{3^N}{d_\Lambda}2^{m-N}\right) \tag{4.10}$$

Now, for $\lambda \notin \mathcal{S}_N$ (having generation of birth $j \leq N$), theorem 4.2.1 gives:

$$\begin{aligned}\sum_{\{\lambda\notin\mathcal{S}_N:\lambda\leq\Lambda\}} d_\lambda &\leq \sum_{\{\lambda\notin\mathcal{S}_N:\lambda\leq\kappa 5^m\}} d_\lambda \\ &= 2^{m-1} + \sum_{j=1}^{N} 2^{m-j-1}(3^{j-1}+3) + \sum_{j=2}^{N} 2^{m-j-2}(3^j-3) \\ &= O(2^{m-N}3^N)\end{aligned}$$

i.e.

$$\sum_{\{\lambda\notin\mathcal{S}_N\}} d_\lambda = O(2^{m-N}3^N) \tag{4.11}$$

Hence, we can put everything together (also noting that $d_\Lambda \geq \frac{1}{2}(3^m - 3)$ since

$\Lambda \geq \kappa 5^{m-1}$) to get:

$$\begin{aligned}\left|\frac{1}{d_\Lambda}\log detM_\Lambda - \int_{SG}\log f d\mu\right| &\leq 4\epsilon + (||f||_\infty + ||\log f||_1)\left(C\left(\frac{3}{2}\right)^{N-m}\right)\\ &= 4\epsilon + \tilde{C}\left(\frac{3}{2}\right)^{N-m}\end{aligned}$$

where in the first line we have used (4.10) and (4.11). But since $\epsilon > 0$ is arbitrary and $\Lambda \to \infty$ iff $m \to \infty$, we conclude that

$$\lim_{\Lambda\to\infty}\frac{1}{d_\Lambda}\log detP_\Lambda[f]P_\Lambda = \int_{SG}\log f(x)d\mu(x)$$

□

4.4 Extension to Pseudo-Differential Operators

In this section, we would like to replace $[f]$ (pointwise multiplication by f) with a pseudo-differential operator, following [9].

4.4.1 Defining pseudo-differential operators on the SG

First, we shall note that the IFS defining the SG can be generalized via the contractions $\{F_j(x) = \frac{1}{2}(x - a_j) + a_j\}$ where $j = 1, 2, 3$ and the $\{a_j\}$ are not co-linear in $\mathbb{R}^2$. Now, as in the previous section, we equip the SG with a probability measure μ that assigns measure 3^{-m} to each m-cell. In addition, we shall introduce the symmetric self-similar resistance form $\mathcal{E}$ (see [13]), which is a Dirichlet form on $L^2(\mu)$. By standard theory on such forms (see [14]), there exists a negative definite self-adjoint Laplacian Δ defined by

$$\mathcal{E}(u, v) = \int(-\Delta u)v d\mu$$

for all v in the domain of $\mathcal{E}$ with $v(a_j) = 0$ ($j = 1, 2, 3$). We shall refer to this as the *Dirichlet Laplacian.* This Laplacian indeed coincides with that of the previous section, for which we have described its spectrum. Now, the spectrum of $-\Delta$ (we shall write

$sp(-\Delta))$ can be written as $sp(-\Delta) = \{\lambda_1 \leq \lambda_2 \leq ... \leq \lambda_n \leq ...\}$ where $\lim_{n\to\infty} \lambda_n = \infty$. We let $\{\varphi_n\}_{n\in\mathbb{N}}$ be an orthonormal basis of $L^2(\mu)$, where φ_n is an eigenfunction with corresponding eigenvalue λ_n.

Remark 4.4.1. We shall let D denote the set of finite linear combinations of the φ_n. Note that D is dense in $L^2(\mu)$.

To define pseudo-differential operators on the SG, we shall briefly recall the theory developed in [15], splitting into constant and variable coefficient cases:

Constant Coefficient Pseudo-Differential Operators

For $p : (0, \infty) \to \mathbb{C}$, let

$$p(-\Delta)u := \sum_n p(\lambda_n)\langle u, \varphi_n\rangle\varphi_n \tag{4.12}$$

where $u \in D$. This defines an operator on $L^2(\mu)$ which we refer to as a *constant coefficient pseudo-differential operator*. By the Spectral Theorem , if p is bounded then $p(-\Delta)$ extends to a bounded linear operator on $L^2(\mu)$. If p is a 0-symbol (the precise definition is given in [15]), then $p(-\Delta)$ is an order 0 pseudo-differential operator. In addition, by Theorem 3.6 of [15], we know that $p(-\Delta)$ is a singular integral operator on $L^2(\mu)$ and hence extends to a bounded operator on $L^q(\mu)$ for $q \in (1, \infty)$.

Variable Coefficient Pseudo-Differential Operators

For $p : SG \times (0, \infty) \to \mathbb{C}$ measurable, we let

$$p(x, -\Delta)u(x) := \sum_n \int_{SG} p(x, \lambda_n)P_{\lambda_n}(x, y)u(y)d\mu(y) \tag{4.13}$$

where $u \in D$ and $\{P_{\lambda_n}\}_{n\in\mathbb{N}}$ denotes the spectral resolution/projection of $-\Delta$. This defines what we refer to as a *variable coefficient pseudo-differential operator*. If p is a 0-symbol, then by Theorem 9.3 of [15], $p(x, -\Delta)$ extends to a bounded operator on $L^2(\mu)$. Moreover, by Theoreom 9.6 ([15]), $p(x, -\Delta)$ extends to a bounded operator on $L^q(\mu)$ for $q \in (1, \infty)$.

4.4.2 Extended Szegö Limit Theorem on the SG

In this subsection, we aim to generalize theorem 4.3.1 and theorem 4.3.2. As in section 4.3, for an eigenvalue λ of $-\Delta$, we shall denote its corresponding eigenspace by E_λ. We let P_λ denote the orthogonal projection onto E_λ and $d_\lambda := dim(E_\lambda)$. Similarly, we let E_Λ denote the span of the eigenfunctions corresponding to eigenvalues λ for which $\lambda \leq \Lambda$. Lastly, we let $dim(E_\Lambda) = d_\Lambda$ and P_Λ be the orthogonal projection onto E_Λ.

Now, we fix $p : SG \times (0, \infty) \to \mathbb{R}$ measurable and let $p(x, -\Delta)$ be defined by eq. (4.13). We also impose that $p(\cdot, \lambda)$ is continuous for all eigenvalues λ (of $-\Delta$) and that there exists a continuous map $q : SG \to \mathbb{R}$ such that the limit $\lim_{\lambda \in sp(-\Delta), \lambda \to \infty} p(x, \lambda) = q(x)$ exists and is uniform in x. In all of the following, we may take $||p||_\infty > 0$, otherwise the results are trivial.

We begin with a Lemma whose proof can be found in [9].

Lemma 4.4.1. *For all $\Lambda > 0$, the eigenvalues of $P_\Lambda p(x, -\Delta) P_\Lambda$ are contained in a bounded interval $[A, B]$.*

Now,

Lemma 4.4.2. *Let $\Lambda > 0$. Let $p : SG \times (0, \infty) \to \mathbb{R}$ be a bounded measurable function such that $p(\cdot, \lambda)$ is continuous for all eigenvalues λ. Then, the map on $C[A, B]$ given by*

$$F \mapsto \frac{1}{d_\Lambda} Tr F(P_\Lambda p(x, -\Delta) P_\Lambda)$$

is a nonnegative continuous functional. A and B are given by lemma 4.4.1.

Proof. The map is trivially non-negative and linear. Moreover, since $P_\Lambda L^2(SG)$ is finite dimensional, we immediately get continuity. □

Special Case

Here we are in the setting of section 4.3.1, hence we let λ_j denote a 6-series eigenvalue with generation of birth $j \geq 2$, and denote its corresponding eigenspace by E_j. Recall that letting $dim(E_j) = d_j$, we have an orthonormal basis $\{u_k\}_{k=1}^{d_j} = \{\tilde{u}_k\}_{k=1}^{d_j^N} \cup \{v_k\}_{k=1}^{\alpha_j^N}$

for E_j. We note that as remarked in section 4.3.1, the same can be deduced for a 5-series eigenvalue, hence the following two results hold for 5-series eigenvalues with similar proofs.

Let P_j denote the projection onto E_j. Then, the matrix $\Gamma_j := P_j p(x, -\Delta) P_j$ is $d_j \times d_j$ with entries

$$\gamma_j(m, l) := \int p(x, \lambda_j) u_m(x) u_l(x) d\mu(x)$$

We can now state the following lemma which we will later need:

Lemma 4.4.3. *Consider the simple function $f = \sum_{i=1}^{3^N} a_i \mathbb{1}_{C_i}$ on the SG where C_i is an N-cell. Then, for all $k \geq 0$,*

$$\lim_{j \to \infty} \frac{1}{d_j} Tr(P_j[f]P_j)^k = \int_{SG} f(x)^k d\mu(x)$$

Proof. First, note that if $k = 0$, since $Tr(\mathbb{I}_{E_j}) = d_j$ the result is trivial. Hence, we shall let $k > 0$ and fix $j > N$. As in lemma 4.3.1, we have that

$$P_j[f]P_j = \begin{bmatrix} R_j & 0 \\ 0 & N_j \end{bmatrix}$$

is a $d_j \times d_j$ matrix where R_j is a $d_j^N \times d_j^N$ matrix that corresponds to the localized part, and N_j is a $\alpha_j^N \times \alpha_j^N$ matrix corresponding to the nonlocalized part. So,

$$Tr(P_j[f]P_j)^k = Tr(R_j)^k + Tr(N_j)^k \tag{4.14}$$

Now, note that R_j is a block diagonal matrix whose blocks are $m_j^N \times m_j^N$ matrices that correspond to a single N-cell C_i. Hence the blocks are just $a_i \mathbb{I}_{m_j^N}$ $(i = 1, ..., 3^N)$ and therefore:

$$\begin{aligned}
Tr(R_j)^k &= \sum_{i=1}^{3^N} m_j^N a_i^k \\
&= d_j^N \sum_{i=1}^{3^N} \frac{a_i^k}{3^N} \\
&= d_j^N \int_{SG} f(x)^k d\mu(x)
\end{aligned}$$

where we recall that $d_j^N = 3^N m_j^N$. On the other hand, since each element in N_j is smaller in absolute value than $||f||_\infty$, we conclude that

$$|Tr(N_j)^k| \leq (\alpha_j^N)^k ||f||_\infty^k$$

So,

$$\begin{aligned}
\frac{1}{d_j} Tr(P_j[f]P_j)^k - \int f(x)^k d\mu(x) &= \frac{1}{d_j}[Tr(R_j)^k + Tr(N_j)^k] - \int f(x)^k d\mu(x) \\
&= \frac{d_j^N}{d_j} \int_{SG} f(x)^k d\mu(x) + \frac{1}{d_j} Tr(N_j)^k - \int_{SG} f(x)^k d\mu(x) \\
&= -\frac{\alpha_j^N}{d_j} \int f(x)^k d\mu(x) + \frac{1}{d_j} Tr(N_j)^k
\end{aligned}$$

where in the first line we have used (4.14), and in the last line we have used that $d_j - d_j^N = \alpha_j^N$. Finally:

$$\left| \frac{1}{d_j} Tr(P_j[f]P_j)^k - \int f(x)^k d\mu(x) \right| \leq \frac{\alpha_j^N}{d_j} \int |f(x)|^k d\mu(x) + \frac{(\alpha_j^N)^k}{d_j} ||f||_\infty^k$$

So we have proved the result since f is bounded on the compact SG and $\lim_{j\to\infty} \frac{(\alpha_j^N)^k}{d_j} = 0$ for all $k > 0$.

□

Now with the lemma, in hand, we prove the extended version of theorem 4.3.1:

Theorem 4.4.1. *Let $p : SG \times (0, \infty) \to \mathbb{R}$ be a bounded measurable function such that $p(\cdot, \lambda)$ is continuous for all eigenvalues λ. Assume that the limit* $\lim_{\lambda \in sp(-\Delta), \lambda \to \infty} p(x, \lambda) =$

$q(x)$ exists and is uniform in x. Then, for all $k \geq 0$,

$$\lim_{j\to\infty} \frac{1}{d_j} Tr(P_j p(x,-\Delta)P_j)^k = \int_{SG} q(x)^k d\mu(x) \tag{4.15}$$

Hence, for any continuous F supported on $[A,B]$ (where A, B are as in lemma 4.4.1) we have that

$$\lim_{j\to\infty} \frac{1}{d_j} Tr F(P_j p(x,-\Delta)P_j) = \int_{SG} F(q(x)) d\mu(x) \tag{4.16}$$

Proof. First, we shall note that if $k = 0$, since $Tr(\mathbb{I}_{E_j}) = d_j$, the result is trivial. Hence, we shall let $k > 0$ and suppose that $\exists C > 0$ such that $p(x,\lambda) \geq C$ for all $(x,\lambda) \in SG \times (0,\infty)$. In addition, let $\epsilon > 0$ be such that $\epsilon < \frac{C}{2}$.

Since $\lambda \mapsto \lambda^k$ is uniformly continuous on $[A,B]$, there exists $0 < \delta < \epsilon$ such that if $|\lambda - \tilde{\lambda}| < \delta$ then $|\lambda^k - (\tilde{\lambda})^k| < \epsilon$. We shall come back to this later.

Now, since we have that q is continuous, there exists a simple function $f_N = \sum_{i=1}^{3^N} a_i \mathbb{1}_{C_i}$ where the C_i are N-cells, such that $||q - f_N||_\infty < \frac{\delta}{2}$. Thus, since $\mu(SG) = 1$, we have:

$$\left| \int_{SG} q(x)^k d\mu(x) - \int_{SG} f_N(x)^k d\mu(x) \right| < \epsilon \tag{4.17}$$

Now by uniform convergence, there exists $J \geq 1$ such that $||p(\cdot,\lambda_j) - q(\cdot)||_\infty < \frac{\delta}{2}$ for all $j \geq J$. Hence, for all $x \in SG$ and $j \geq J$ we have that:

$$\frac{C}{2} \leq f_N(x) - \delta \leq p(x,\lambda_j) \leq f_N(x) + \delta \tag{4.18}$$

By lemma 4.4.3 (increasing J if necessary), we have that for all $j \geq J$,

$$\left| \frac{Tr(P_j[f_N]P_j)^k}{d_j} - \int_{SG} f_N(x)^k d\mu(x) \right| < \epsilon \tag{4.19}$$

So now, let $j \geq J$. (4.18) implies that $0 \leq P_j[f_N - \delta]P_j \leq P_j p(x,-\Delta)P_j \leq P_j[f_N + \delta]P_j$. Hence, letting σ_m^j denote the eigenvalues of $P_j p(x,-\Delta)P_j$ and $\sigma_{m,N}^j$ denote the eigenvalues of $P_j[f_N]P_j$, we have that $|\sigma_m^j - \sigma_{m,N}^j| < \delta$ for all $m = 1,...,d_j$. However, then recalling the remarks in the second paragraph of this proof, we have that:

$$\left|(\sigma_m^j)^k - (\sigma_{m,N}^j)^k\right| < \epsilon \;\; \forall m = 1, ..., d_j \tag{4.20}$$

So finally, since $Tr(P_j p(x,-\Delta)P_j)^k = \sum_m (\sigma_m^j)^k$ and $Tr(P_j[f_N]P_j)^k = \sum_m (\sigma_{m,N}^j)^k$ we have that:

$$\begin{aligned}
\left|\frac{Tr(P_j p(x,-\Delta)P_j)^k}{d_j} - \int_{SG} q(x)^k d\mu(x)\right| &\leq \left|\frac{Tr(P_j p(x,-\Delta)P_j)^k}{d_j} - \frac{Tr(P_j[f_N]P_j)^k}{d_j}\right| \\
&+ \left|\frac{Tr(P_j[f_N]P_j)^k}{d_j} - \int_{SG} (f_N(x))^k d\mu(x)\right| \\
&+ \left|\int_{SG} (f_N(x))^k d\mu(x) - \int_{SG} q(x)^k d\mu(x)\right| \\
&\leq \left|\frac{\sum_{m=1}^{d_j}\left[(\sigma_m^j)^k - (\sigma_{m,N}^j)^k\right]}{d_j}\right| + \epsilon + \epsilon \\
&\leq 3\epsilon
\end{aligned}$$

where in the last two lines we have used (4.19),(4.17), and (4.20). In conclusion, we have shown that (4.15) holds. For (4.16), we simply use lemma 4.4.2 and apply Stone-Weierstrass Approximation Theorem.

□

Full Result

Now, we are ready to develop the full result, namely the extended version of theorem 4.3.2.

Theorem 4.4.2. *Let $p : SG \times (0,\infty) \to \mathbb{R}$ be a bounded measurable function such that $p(\cdot,\lambda)$ is continuous for all eigenvalues λ. Assume that the limit $\lim_{\lambda\in sp(-\Delta),\lambda\to\infty} p(x,\lambda) = q(x)$ exists and is uniform in x. Then, for any continuous function F supported on $[A,B]$, we have:*

$$\lim_{\Lambda\to\infty} \frac{1}{d_\Lambda} TrF(P_\Lambda p(x,-\Delta)P_\Lambda) = \int_{SG} F(q(x))d\mu(x) \tag{4.21}$$

where A, B are from lemma 4.4.1.

Proof. In this proof, we shall let $gob(\lambda)$ denote the generation of birth of the eigenvalue λ.

Now, by the finishing remarks in the proof of theorem 4.4.1, we know by Stone-Weierstrass that is suffices to show the following:

$$\lim_{\Lambda\to\infty}\frac{1}{d_\Lambda}Tr(P_\Lambda p(x,-\Delta)P_\Lambda)^k = \int_{SG} q(x)^k d\mu(x)$$

Well, for $k = 0$, it holds trivially, hence we shall let $k > 0$ and $\epsilon > 0$. First, note that $Tr(P_\Lambda p(x,-\Delta)P_\Lambda)^k = \sum_{\lambda\leq\Lambda} Tr(P_\lambda p(x,-\Delta)P_\lambda)^k$ and $d_\Lambda = \sum_{\lambda\leq\Lambda} d_\lambda$. Now, by theorem 4.4.1 (recalling that it holds for both 5 and 6-series eigenvalues) there exists $J > 1$ such that if λ is a 6-series (or 5-series) eigenvalue with $gob(\lambda) \geq J$ then:

$$\left|\frac{Tr(P_\lambda p(x,-\Delta)P_\lambda)^k}{d_\lambda} - \int_{SG} q(x)^k d\mu(x)\right| < \epsilon \tag{4.22}$$

For $\Lambda > 0$, let $\Gamma_J(\Lambda)$ denote the set of eigenvalues $\lambda \leq \Lambda$ such that $gob(\lambda) > J$. Similarly, let $\tilde{\Gamma}_J(\Lambda)$ denote the set of eigenvalues $\lambda \leq \Lambda$ such that $gob(\lambda) \leq J$. We remark that since 2-series eigenvalues have generation of birth equal to 1, $\Gamma_J(\Lambda)$ consists only of 5 and 6 -series eigenvalues. Now, fixing $\Lambda_1 > 0$ such that $\Gamma_J(\Lambda_1) \neq \emptyset$ and $\tilde{\Gamma}_J(\Lambda_1) \neq \emptyset$ we have for all $\Lambda > \Lambda_1$:

$$\begin{aligned}
&\left|\frac{1}{d_\Lambda}Tr(P_\Lambda p(x,-\Delta)P_\Lambda)^k - \int_{SG} q(x)^k d\mu(x)\right| \\
&\leq \frac{1}{d_\Lambda}\sum_{\lambda\in\Gamma_J(\Lambda)}\left|Tr(P_\lambda p(x,-\Delta)P_\lambda)^k - d_\lambda\int_{SG} q(x)^k d\mu(x)\right| \\
&+ \frac{1}{d_\Lambda}\sum_{\lambda\in\tilde{\Gamma}_J(\Lambda)}\left|Tr(P_\lambda p(x,-\Delta)P_\lambda)^k - d_\lambda\int_{SG} q(x)^k d\mu(x)\right|
\end{aligned}$$

For convenience, we shall let the first term above be denoted by $\mathcal{A}$ and the second by $\mathcal{B}$.

First we deal with term $\mathcal{A}$. Well, by (4.22), we have that

$$\mathcal{A} \leq \frac{\epsilon}{d_\Lambda}\sum_{\lambda\in\Gamma_J(\Lambda)} d_\lambda \leq \epsilon \tag{4.23}$$

Now for term $\mathcal{B}$ we recall the proof of theorem 4.3.2, which implies that

$$\lim_{\Lambda \to \infty} \frac{\sum_{\lambda \in \tilde{\Gamma}_J(\Lambda)} d_\lambda}{d_\Lambda} = 0$$

Hence there exists $\Lambda_2 > \Lambda_1$ such that if $\Lambda > \Lambda_2$ then:

$$\frac{\sum_{\lambda \in \tilde{\Gamma}_J(\Lambda)} d_\lambda}{d_\Lambda} < \frac{\epsilon}{||p||_\infty^k + ||q||_\infty^k} \tag{4.24}$$

So, since $Tr(P_\lambda p(x, -\Delta) P_\lambda)^k \leq d_\lambda ||p||_\infty^k$ and $\mu(SG) = 1$ we get for $\Lambda > \Lambda_2$:

$$\mathcal{B} \leq (||p||_\infty^k + ||q||_\infty^k) \frac{\sum_{\lambda \in \tilde{\Gamma}_J(\Lambda)} d_\lambda}{d_\Lambda} < \epsilon$$

where we have used (4.24). Bringing it all together, we arrive at:

$$\begin{aligned} \left| \frac{1}{d_\Lambda} Tr(P_\Lambda p(x, -\Delta) P_\Lambda)^k - \int_{SG} q(x)^k d\mu(x) \right| &\leq \mathcal{A} + \mathcal{B} \\ &\leq \epsilon + \epsilon \end{aligned}$$

□

Chapter 5

The von Koch Snowflake

In this chapter, we examine another specific fractal - the Koch Snowflake. After reviewing the basic properties of the snowflake, we present numerical results which demonstrate the behaviour of the eigenfunctions of the Laplacian on the snowflake domain.

5.1 Construction and Basic Properties

The Koch Snowflake is a fractal composed of three *Koch Curves* glued together in a triangle. Each Koch Curve is obtained by starting with a line segment (we shall take the segment to have length 1), and dividing it into thirds. From the middle third, we draw an equilateral triangle of side length 1/3 pointing outwards. We then delete the middle third of the original line segment (thus deleting the base of the new equilateral triangle). Recursively doing this results in the Koch Curve:

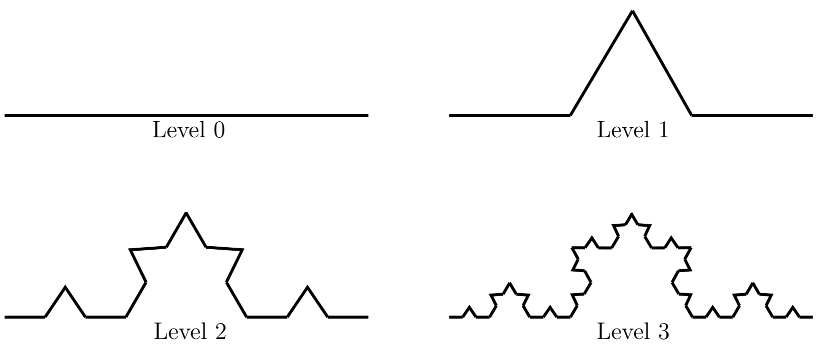

Figure 5.1: The first few iterations of obtaining a Koch Curve.

Formally, the Koch Curve is the attractor of the IFS

$$\begin{cases} F_1(x) = \begin{pmatrix} \frac{1}{3} & 0 \\ 0 & \frac{1}{3} \end{pmatrix} x \\ F_2(x) = \begin{pmatrix} \frac{1}{6} & \frac{-\sqrt{3}}{6} \\ \frac{\sqrt{3}}{6} & \frac{1}{6} \end{pmatrix} x + \begin{pmatrix} \frac{1}{3} \\ 0 \end{pmatrix} \\ F_3(x) = \begin{pmatrix} \frac{1}{6} & \frac{\sqrt{3}}{6} \\ \frac{-\sqrt{3}}{6} & \frac{1}{6} \end{pmatrix} x + \begin{pmatrix} \frac{1}{2} \\ \frac{\sqrt{3}}{6} \end{pmatrix} \\ F_4(x) = \begin{pmatrix} \frac{1}{3} & 0 \\ 0 & \frac{1}{3} \end{pmatrix} x + \begin{pmatrix} \frac{2}{3} \\ 0 \end{pmatrix} \end{cases}$$

The Koch Snowflake thus is an equilateral triangle, whose three sides are developed as Koch Curves.

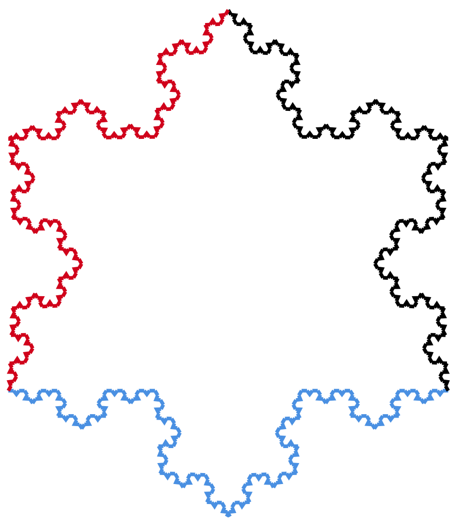

Figure 5.2: The Koch Snowflake. The three Koch curves which form the snowflake have been given different colors.

A first property of the Koch Snowflake is that it has a finite area:

Theorem 5.1.1 (Area of Snowflake)**.** *The area of the classical Koch Snowflake of side length one is* $A = \frac{2\sqrt{3}}{5}$.

Proof. We start with the level zero/iteration zero snowflake; i.e. just an equilateral triangle of side length one - the area is $A_0 = \frac{\sqrt{3}}{4}$. Now, at level m, we have added $3 \cdot 4^{m-1}$ triangles.

Moreover, the area of each triangle added at level m is $\frac{1}{9}$ the area of the triangle added at level $m-1$. Hence, the area of each triangle added at level m is $\frac{1}{9^m}A_0$. So, the total new area added at level m is $(\frac{1}{9}A_0)(3 \cdot 4^{m-1}) = (\frac{4}{9})^m A_0.\frac{3}{4}$. Hence, we calculate the area of the level m snowflake:

$$\begin{aligned}
A_m &= A_0\left(1+\frac{3}{4}\sum_{k=1}^{m}\left(\frac{4}{9}\right)^k\right)\\
&= A_0\left(1+\frac{1}{3}\sum_{k=0}^{m-1}\left(\frac{4}{9}\right)^k\right)\\
&= A_0\left(1+\frac{1}{3}\cdot\frac{9}{5}\left(1-\left(\frac{4}{9}\right)^m\right)\right)\\
&= A_0\left(1+\frac{3}{5}\left(1-\left(\frac{4}{9}\right)^m\right)\right)\\
&= A_0\left(\frac{8}{5}-\frac{3}{5}\left(\frac{4}{9}\right)^m\right)
\end{aligned}$$

Finally, the area of the Koch Snowflake is

$$A = \lim_{m\to\infty} A_m = \frac{8}{5}A_0 = \frac{8}{5}\frac{\sqrt{3}}{4} = \frac{2\sqrt{3}}{5}$$

□

Despite having finite area, the Koch Snowflake has infinite perimeter:

Theorem 5.1.2 (Perimeter of Snowflake)**.** *The perimeter of the classical Koch Snowflake of side length one is infinite.*

Proof. At each level , the number of sides of the snowflake is multiplied by 4, hence the number of sides of the level m snowflake is $3 \cdot 4^m$. However, the length of each side at level m is $\frac{1}{3^m}$. Hence, the perimeter of the level m snowflake is $(3 \cdot 4^m)(\frac{1}{3^m}) = 3 \cdot (\frac{4}{3})^m$. But as $m \to \infty$, this quantity tends to infinity. □

Theorem 5.1.3 (Dimension of Snowflake)**.** *The box-counting dimension of the classical Koch Snowflake of side length one is* $\frac{\log 4}{\log 3}$

Proof. We consider a Koch Curve, so a third of the boundary of the snowflake. If we take boxes having side length $\frac{1}{3}$, we need 3 total boxes to cover the curve. Now, if we take boxes of side length $\frac{1}{9}$, we will need $4 \cdot 3 = 12$ boxes to cover the curve. Continuing this process, we have that the number of boxes of side length $\frac{1}{3^n}$ necessary to cover the curve is $N_n = 3 \cdot 4^{n-1}$. Hence using the last equivalent definition of box-counting dimension introduced in section 2.2, we have that

$$\begin{aligned}
\dim_B &= \lim_{n\to\infty} \frac{\log N_n}{-\log(\frac{1}{3^n})} \\
&= \lim_{n\to\infty} \frac{\log(3 \cdot 4^{n-1})}{n \log 3} \\
&= \lim_{n\to\infty} \frac{\log 3 + (n-1)\log 4}{n \log 3} \\
&= \lim_{n\to\infty} \left(\frac{\log 3 - \log 4}{n \log 3} + \frac{n \log 4}{n \log 3} \right) \\
&= \frac{\log 4}{\log 3}
\end{aligned}$$

$\square$

5.2 Theory of the Laplacian on Snowflake

Let $\Omega \subset \mathbb{R}^2$ be such that $\partial\Omega$ is the Koch snowflake. In this section, we shall briefly summarize the numerical results in [11] , [12] concerning the eigenvalue problems

$$\Delta u + \lambda u = 0 \ \text{ in } \Omega \ , \ u = 0 \ \text{ on } \partial\Omega$$

(i.e. with Dirichlet boundary conditions) and

$$\Delta u + \lambda u = 0 \ \text{ in } \Omega \ , \ \frac{\partial u}{\partial \nu} = 0 \ \text{ on } \partial\Omega$$

(i.e. with Neumann boundary conditions).

Recall that the eigenvalues of the Laplacian may be written as $0 \leq \lambda_1 \leq \lambda_2 \leq ... \to \infty$ with corresponding eigenfunctions $\{\varphi_k\}_{k=1}^{\infty}$. It is important to first remark that the object of study for this section is fundamentally different than that of the previous chapter (the Sierpinski Gasket). Here we are considering a domain whose boundary is fractal, as opposed to working strictly on a fractal itself. Hence, the key observation of the previous chapter (the localization of eigenfunctions on cells) cannot be used. Instead, to study the eigenvalues/eigenfunctions on the snowflake domain, Neuberger et al. approximate the Laplacian by a symmetric matrix (a discretized Laplacian) and numerically compute the eigenvalues and corresponding eigenfunctions of the Laplacian matrix via ARPACK. The results can then be extrapolated for grid spacing tending to 0.

For the Dirichlet boundary conditions, the eigenfunctions from a one dimensional eigenspace exhibit D_6 symmetry:

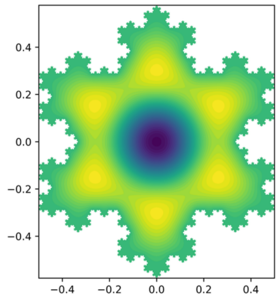

Figure 5.3: The 6th eigenfunction φ_6 on the Koch Snowflake with Dirichlet boundary conditions. Figure is due to Strichartz and Wiese [11].

Eigenfunctions from a two dimensional eigenspace however are symmetric under reflections:

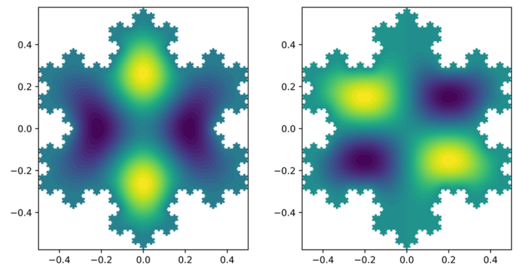

Figure 5.4: The eigenfunctions φ_4, φ_5 on the Koch Snowflake with Dirichlet boundary conditions. Figure is due to Strichartz and Wiese [11].

For the Neumann boundary conditions, the eigenfunctions from a one dimensional eigenspace also exhibit D_6 symmetry as in the Dirichlet case. Eigenfunctions from a two dimensional eigenspace however exhibit symmetric properties:

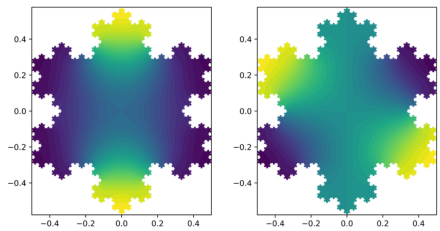

Figure 5.5: The eigenfunctions φ_4, φ_5 on the Koch Snowflake with Neumann boundary conditions. Figure is due to Strichartz and Wiese [11].

Chapter 6

Conclusion

Starting with the basic infrastructure, this thesis provides an introduction to analysis on fractal sets that is relatively self-contained. The dimension theory of such sets is built up and motivated to introduce the works of Solomyak, Falconer, and McMullen, which determines the Hausdorff dimension of general classes of self-affine sets. Two ubiquitous fractal sets are then examined in depth from a spectral theory viewpoint. The technical results by Okoudjou et al. on the Sierpinski Gasket were reviewed, which develop analogues and extensions of the Szego Limit Theorem. This highlights the importance of localized eigenfunctions in developing strong theoretical results on fractal sets.

To this end, a natural research question concerns whether or not such limit theorems may exist for more general classes of fractals. The existence of localized eigenfunctions of the Laplacian seems to be sufficient, and conditions for such existence may be found in [6], giving merit to this question.

In addition to this question, the author of this thesis is particularly interested in the spectral gap of the Laplacian on fractals. The size of the first eigenvalue of the Laplacian has already been examined on surfaces including the sphere, torus, projective plane, surfaces of genus 2, and the Klein bottle. The analogous problem on fractals seems to be a promising research direction, and may be approached perhaps via approximations by discrete graphs.

The author hopes that this thesis has motivated interest in analysis on fractals, by detailing strong theoretical results that have recently been developed. The thesis also should be helpful for its expository content on fractals, as well as its application of various tools from a range of mathematical disciplines.

Appendix A: Classical Szegö Limit Theorem

Let $\phi : \mathbb{T} \to \mathbb{C}$ be a complex-valued function defined on the complex unit circle $\mathbb{T}$. We define the Fourier coefficients of ϕ by

$$\hat{\phi}(n) := \frac{1}{2\pi} \int_0^{2\pi} \phi\left(e^{i\theta}\right) e^{-in\theta} d\theta$$

The Szegö Limit Theorems (first proved by Gábor Szegö) concern the limiting behaviour of the determinants of the Toeplitz matrices $T_n(\phi)$ defined by $T_n(\phi) = (\hat{\phi}(j-k))_{j,k=0}^{n-1}$. Namely, if we define the geometric mean of ϕ as

$$GM(\phi) := \exp\left[\frac{1}{2\pi} \int_0^{2\pi} \log \phi(e^{i\theta}) d\theta\right]$$

then the first Szegö Limit Theorem is as follows:

Theorem 6.0.1 (Szegö Limit Theorem 1). *Let $\phi > 0$ with $\phi \in L_1(\mathbb{T})$. Then,*

$$\lim_{n\to\infty} \frac{detT_n(\phi)}{detT_{n-1}(\phi)} = GM(\phi)$$

Under the additional assumption that the derivative of ϕ is Hölder continuous of order $\alpha > 0$, we have the second (also called the "strong") Szegö Limit Theorem:

Theorem 6.0.2 (Strong Szegö Limit Theorem). *Let $\phi > 0$ with $\phi \in L_1(\mathbb{T})$ and suppose that the derivative of ϕ is Hölder continuous of order $\alpha > 0$. Then,*

$$\lim_{n\to\infty} \frac{detT_n(\phi)}{(GM(\phi))^n} = \exp\left[\sum_{k=1}^{\infty} k|(\widehat{\log \phi})(k)|^2\right]$$

The majority of the proofs of the classical Szegö Limit Theorems are quite long and rather indirect. The first clear proof was due to H. Widom (see [1]), who used ideas from operator theory to generalize the theorems to matrix-valued functions. Namely, if we let $H(\phi) = (\hat{\phi}(j+k+1))_{j,k\geq 0}$ be the Hankel matrices and write $\tilde{\phi}(z) = \phi(z^{-1})$, Widom made great use of the following identity:

$$T_n(\phi\psi) - T_n(\phi)T_n(\psi) = P_n H(\phi)H(\tilde{\psi})P_n + Q_n H(\tilde{\phi})H(\psi)Q_n$$

where $P_n(f_0, f_1, ...) = (f_0, ..., f_n, 0, ...)$ and $Q_n(f_0, f_1, ...) = (f_n, ..., f_0, 0, ...)$.

Appendix B : The Mass Distribution Principle

Definition 6.0.1. A mass distribution μ is a measure on a bounded subset of $\mathbb{R}^n$ for which $0 < \mu(\mathbb{R}^n) < \infty$.

Theorem 6.0.3 (Mass Distribution Principle)**.** *Let μ be a mass distribution on F and suppose that for some s , there exists $c > 0$ and $\delta > 0$ such that $\mu(U) \leq c|U|^s$ for all sets U with $|U| \leq \delta$. Then we have that*

$$\mathcal{H}^s(F) \geq \frac{\mu(F)}{c}$$

and $dim_H F \geq s$.

Proof. Let $\{U_i\}$ be any δ-cover of F. Then, using the monotonicity and subadditivity of measures we have:

$$\begin{aligned}\mu(F) &\leq \mu(\cup_{i=1}^{\infty} U_i)\\ &\leq \sum_{i=1}^{\infty} \mu(U_i)\\ &\leq c\sum_{i=1}^{\infty} |U_i|^s\end{aligned}$$

where in the last line we have used that $\mu(U_i) \leq c|U_i|^s$ (recalling that $|U_i| \leq \delta$). Hence, taking infima we obtain $\mathcal{H}^s_\delta(F) \geq \frac{\mu(F)}{c}$, and therefore conclude that $\mathcal{H}^s(F) \geq \frac{\mu(F)}{c}$. Lastly, since $\mu(F) > 0$ we have that $dim_H(F) \geq s$. □

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