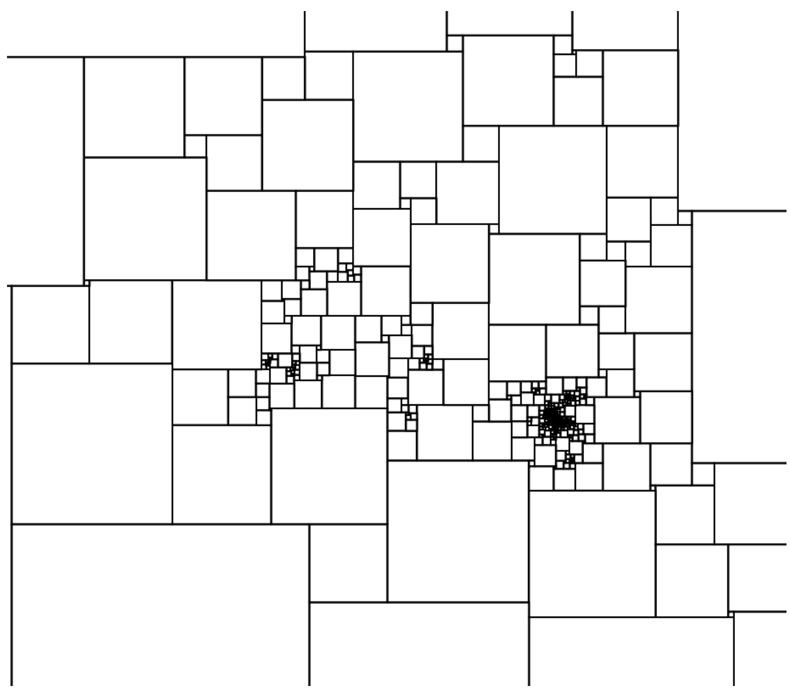
A Master's Thesis

ASPECTS OF RANDOM SQUARINGS

September 21, 2015



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I would also like to thank the Mathematics Department at McGill University for, first of all, accepting me into their graduate program with a great supervisor and secondly, generously funding my studies throughout.

If I had to do it all over again, I would want it the exact same way.

Motivation

What can one expect the surface of a randomly chosen 3D shape to look like? This seemingly innocuous question is driving an enormous amount of modern research in an ever-expanding list of subjects, most notably in probability theory and statistical physics. An analogous question has already been asked in one-dimension with respect to paths, to which the answer is the celebrated Brownian motion. However, the two dimensional case does not have such a straightforward answer. In fact, even measures on such objects aren't very well understood.

The most popular theory of areas on these random surfaces is called the Liouville Quantum Gravity (LQG). In a recent paper, Bertrand Duplantier and Scott Sheffield have attempted to tackle LQG by using a modified version of the Gaussian Free Field (GFF). The result is a very elegant and powerful formula which relates fractal dimensions in LQG with those in Euclidean space. Their method is closely related to the mathematical physics derivation of LQG and relies on constructing a binary random square tiling of the plane based on the areas produced by the GFF.

Another, more approximative approach to LQG has been taken by Gregory Miermont and Jean-Francois Le Gall who used random maps to approximate such random surfaces. By sampling uniformly from special subsets of planar maps with a given number of edges and letting the number of edges go to infinity, upon rescaling we find convergence to what is called the Brownian map. It has been conjectured that the areas of the Brownian map are simply a rescaled version of the modified GFF presented by Duplantier and Sheffield. If this were true, then one could simply study the limits of such graphs to understand the random surfaces.

This work describes an object sitting at the intersection of both of these approaches: a random infinite square tilings of rectangles. The construction of this object is based on a procedure originally presented by Brooks et al. in 1940. By considering rectangles tiled by infinitely many squares (hereto referred to as squarings), one finds an object which not only has the simplicity and concreteness of the random graphs but which also has a well behaved canonical embedding, an important fact used in the GFF approach.

The random infinite squarings are constructed by studying the behaviour of random walks on random infinite planar maps. By doing so, one gets an explicit embedding of a planar graph as a squaring in the plane whose properties are conjecturally closely related to the GFF and to the underlying structure of the graph. This thesis will primarily be concerned with proving the convergence of random squarings based on 3-connected graphs, discussing open problems as well as showing some properties of the infinite squaring.

1 Introduction

As described in the previous section, the majority of this thesis consists of setting up, and consequently proving, the below Theorem.

Theorem 1 ([2], Theorem 1.1). *There exists an explicitly defined sequence* $(\mathbf{S}_n, n \geq 1)$ *of random squarings of rectangles with the following properties.*

1. $\mathbf{S}_n = S(\mathbf{G}_n)$*, where* $\mathbf{G}_n$ *is an* $(n+4)$*-edge random planar map whose law is given in Section 3.1,*
2. $\mathbf{S}_n$ *converges almost surely for the Hausdorff distance to a compact limit* $\mathbf{S}_\infty$*, which a.s. has exactly one point of accumulation*[1].
3. $\mathbf{S}_\infty$ *has the law of* $\mathcal{S}(\mathbf{G}_\infty)$*, where* $\mathbf{G}_\infty$ *itself has the law of the uniform infinite* 3*-connected planar map.*

We begin with some basic graph theoretical and probabilistic tools in Section 1. In Section 2 we explore two mappings from graphs to squarings and briefly discuss other geometric representations of graphs. Section 3 contains the bulk of the thesis and tackles the proof of Theorem 39, which itself proves Theorem 1, this all being proved in Section 3.2. Afterwards, we discuss some relevant properties of the squarings and in Section 4 present a list of open problems, with some discussion.

1.1 Graph Theory

We now introduce some basic graph theoretical definitions. Throughout the thesis, all graphs are assumed to be simple, connected and locally-finite (all vertex degrees are finite) unless we specify otherwise. We let $v(G)$ and $e(G)$ denote the vertex set and edge set of $G = (v(G), e(G))$ respectively, and let $\deg_G(v)$ (or simply $\deg(v)$) be the degree of a vertex $v \in v(G)$. By $\vec{e}(G)$ we mean the set of all directed edges of G. If $e = st$ is a directed edge from s to t then we say that s is the *source* and t is the *target*. We will write bold faced letters $\mathbf{G}$ for rooted graphs (G, ρ) specifying the way in which it is rooted when there is ambiguity. In particular, a graph $\mathbf{G}$ is edge rooted if $\mathbf{G} = (G, st)$ and $st \in \vec{e}(G)$ (the root) is a directed edge from s to t in G, it is vertex rooted if $\mathbf{G} = (G, \rho)$ where $\rho \in v(G)$ is called the root of G.

A path in G from $v \in v(G)$ to $w \in v(G)$ is a sequence of vertices $p = (p_1, p_2, \dots, p_n)$ such that $p_1 = v$, $p_n = w$ and with $\{p_i, p_{i+1}\} \in e(G)$. The *length* of a path p, written $|p|$ is the number of vertices in the path. To any connected graph G there exists a natural metric $d_G : V \times V \to \mathbb{R}$ given by,

$$d_G(v, w) = \inf\{|p| - 1 \ : \ p \text{ is a path from } v \text{ to } w\}.$$

[1] A accumulation point in $\mathbb{R}^2$ is a point such that any open neighborhood of it contains infinitely many squares.

In other words $d_G(v, w)$ is the minimum number of edges in a path from v to w. We will simply write d, foregoing the subscript, for the metric when there is no ambiguity. If $F \subset e(G)$, the graph $G - F$ is the graph defined by $G - F = (v(G), e(G) - F)$. If $U \subset v(G)$, the subgraph of G induced by U is the subgraph $G[U] = (U, e(G[U]))$ such that for any $u, v \in U$ the edge $e = \{u, v\} \in e(G[U])$ if and only if $e \in e(G)$. The ball $B_G(v, r)$ centered at v with radius r is the subgraph of G induced by

$$\{w \in v(G) \ : \ d_G(v, w) \leq r\}$$

where once again we shall drop the subscript when there is no risk of confusion. We will write $N_G(v) = v(B_G(v, 1)) - \{v\}$ for the set of neighbours of v and $w \sim v$ if $w \in N_G(v)$. We call a graph *k-vertex-connected* (or simply *k-connected*) if no removal of less than k vertices disconnects the graph. We say that G is *one-ended* if $|v(G)| = \infty$ and the graph obtained by removing any finite number of vertices has at most one infinite component.

The metric on graphs also induces a metric called the *local-weak metric* (first introduced by Aldous in [3]) on the set $\mathcal{G}$ of rooted isomorphism classes of (not necessarily finite) vertex-rooted graphs as follows: Given $\mathbf{G} = (G, \rho)$ and $\mathbf{G}' = (G', \rho')$, two vertex-rooted[2] graphs, define $D : \mathcal{G} \times \mathcal{G} \to \mathbb{R}$ by

$$D(\mathbf{G}, \mathbf{G}') = \inf \left\{ \frac{1}{r+1} \ : \ B_G(\rho, r) \cong B_{G'}(\rho', r) \right\}$$

where here $\cong$ denotes isomorphism between rooted graphs. In particular, $D(\mathbf{G}, \mathbf{G}')$ "measures" the difference between balls centered at ρ and ρ' in $\mathbf{G}$ and $\mathbf{G}'$. Furthermore, such a metric allows us to discuss sequences of rooted graphs in the following way: Let $(\mathbf{G}_n, n \geq 1)$ be a sequence of vertex-rooted graphs, we say that the sequence converges *locally-weakly* if there exists a graph $\mathbf{G}_\infty \in \mathcal{G}$ such that $D(\mathbf{G}_n, \mathbf{G}_\infty) \to 0$ as $n \to \infty$ or equivalently if for any $r > 0$ there exists an N sufficiently large that $B_{G_\infty}(\rho_\infty, r) \cong B_{G_n}(\rho_n, r)$ for all $n \geq N$. Note that it does not matter which representative of an equivalence class we take, as the metric is clearly invariant under isomorphisms of graphs.

Theorem 2. *The metric space $(\mathcal{G}, D)$ is separable and complete.*

Proof. To show that the space is separable simply note that the set of all *finite* vertex-rooted graphs in $\mathcal{G}$ is dense. For completeness construct the graph $\mathbf{G}_\infty$ iteratively by defining the subgraph $B_{\mathbf{G}_\infty}(\rho, r)$ whenever the same ball is fixed in $\mathbf{G}_n$ for n large. □

Recall that a graph G is a *planar graph* if it can be embedded on the sphere $\mathbb{S}^2$ (or equivalently in the plane $\mathbb{R}^2$) in such a way that no two edges overlap, except at their endpoints. Let $\Phi = \{\phi_e\}_{e \in e(G)}$ and $\Phi'_G = \{\phi'_e\}_{e \in e(G)}$ be two embeddings of G in $\mathbb{S}^2$, we define the equivalence relation $\sim$ on embeddings of G by setting $\Phi_G \sim \Phi'_G$ if there exists an orientation preserving homeomorphism $\psi : \mathbb{S}^2 \to \mathbb{S}^2$ such

[2] If the graphs are edge-rooted, then we can take the source of the root-edge as the root vertex.

that $\psi(\Phi_G) = \Phi'_G$ (see Figure 1). Under this identification we call the equivalence class of embeddings of a planar graph G a *planar map*.

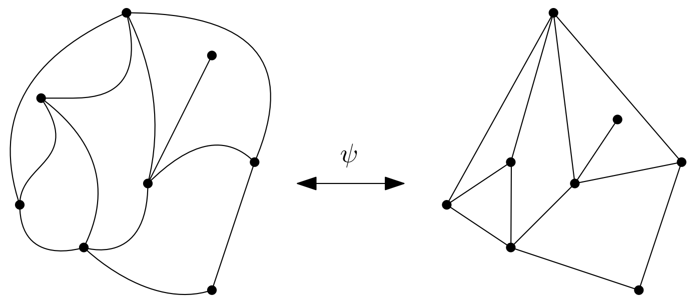

Figure 1: An embedding of a graph and an equivalent (under $\sim$) embedding. A planar map refers to either of these.

A *face* of a planar map is a connected component of $\mathbb{S}^2 - \Phi(G)$, for any representative embedding Φ of G. We write $f(G)$ for the set of faces of G and say that vertices and edges are *incident* to (or are in) a face if they lie on its boundary. Furthermore, two faces are incident if there is an edge is adjacent to both of them.

1.2 Probability on Graphs

We will now briefly describe some important topics relating both probability theory and graph theory. Notably we discuss results in random walks, probability measures on graphs and electrical networks. Most of these results can be found in the "standard" references on these subjects, for example [8, 12, 18, 20, 26].

Random Walks

As usual, let G be a locally-finite simple connected graph, a *simple random walk* on G starting at v is a Markov chain $X = (X_n, n \geq 0)$ on the vertices of G where $X_0 = v$ and which has transition probabilities,

$$\mathbb{P}(X_{n+1} = w \mid X_n = u) = \begin{cases} \frac{1}{\deg(u)} & \text{if } \{u, w\} \in e(G) \\ 0 & \text{otherwise} \end{cases}.$$

We will write $\mathbb{P}_v(\cdot)$ for the probability measure associated to the simple random walk on G starting at v. Let $c : e(G) \to \mathbb{R}^+$ be a function assigning *weights* to the edges of G, the *random walk with edge weight* c starting at v is once again the Markov chain $(X_n, n \geq 0)$ on the vertices of G with $X_0 = v$ and such that

$$\mathbb{P}(X_{n+1} = w \mid X_n = u) = \begin{cases} \frac{c(\{u,w\})}{C_u} & \text{if } \{u,w\} \in e(G) \\ 0 & \text{otherwise} \end{cases}$$

where

$$C_u = \sum_{v \in N(u)} c(\{u, v\}).$$

A simple random walk on G is then an edge weighted random walk with $c(e) \equiv 1$ for all $e \in e(G)$. For a random walk $X = (X_n,\ n \geq 0)$ on G, we define the hitting time of $u \in v(G)$ as

$$\tau_u(X) = \inf\{n \in \mathbb{N} \ : \ X_n = u\}$$

as the first time that the random walk $(X_n, n \geq 0)$ hits u. By convention we set $\tau_u = \infty$ if the random walk never lands on v. We call the simple random walk on G *recurrent* if for all $v \in v(G)$ the random walk started at v almost surely returns to v, more technically,

$$\mathbb{P}_v(\tau_v < \infty) = 1,$$

and in this case G is called a *recurrent graph*. If

$$\mathbb{P}_u(\tau_u < \infty) = 1$$

for some $u \in v(G)$, then it must also be the case for all other $v \in v(G)$ by connectedness, as there is a non-zero probability that a random walk travels from u to any other vertex. The simple random walk on G is *transient* if it is not recurrent (and G is then called a *transient graph*). For any set $A \subset v(G)$ we define in a similar manner the hitting time $\tau_A(X) = \inf\{n \geq 1 \ : \ X_n \in A\}$ as the first time the random walk lands on a vertex of A.

Harmonic Functions

Let $\varphi : v(G) \to \mathbb{R}$ be a function on the vertices of G. If for all $v \in v(G)$

$$\varphi(v) = \frac{1}{\deg(v)} \sum_{u \in N(v)} \varphi(u)$$

we call the function φ *harmonic* on G. If φ satisfies the above equality save for vertices in some set $B \subset v(G)$, we say that φ is harmonic *with boundary* B. A graph G has the *Liouville property* if any bounded harmonic function over all of G is constant.

Proposition 3. *If G is a recurrent graph, then G has the Liouville property.*

Proof. Let $(X_n, n \geq 0)$ be a random walk on G starting at v, let $\varphi : v(G) \to \mathbb{R}$ be a bounded harmonic function and let $\mathcal{F}_n$ be the σ-algebra generated by X_n. It is first claimed that the random variable defined by

$$Y_n = \varphi(X_n)$$

is a martingale adapted to $\mathcal{F}_n$. Indeed, we know that given Y_n the expected value of Y_{n+1} is precisely Y_n, since suppose we are at position X_n then

$$\mathbb{E}(Y_{n+1} \mid X_n) = \frac{1}{\deg(X_n)} \sum_{v \sim X_n} \varphi(v) = Y_n$$

so that in particular

$$\mathbb{E}(Y_{n+1} \mid \mathcal{F}_n) = Y_n.$$

By the martingale convergence theorem (see for example [26]) since φ is bounded, there exists a random variable Φ such that $\Phi = \lim_{n\to\infty} Y_n$ almost surely. However, recurrence implies that the random walk visits every vertex of G infinitely often so that Φ must be constant, and hence so is f. $\square$

The good news is that when we consider harmonic functions *with boundary* on recurrent graphs we have, in general, a unique non-constant function determined entirely by its value on the boundary. The finite case follows from the following maximum principle.

Proposition 4 (Maximum Principle). *Let G be a finite graph, and let $\varphi : v(G) \to \mathbb{R}$ be a harmonic function with boundary $B \subset v(G)$. Then for all $v \in v(G)$,*

$$\inf_{u \in B} \varphi(u) \leq \varphi(v) \leq \sup_{u \in B} \varphi(u)$$

so that φ attains its maximum and minimum on the boundary.

Proof. First note that the value of φ at any vertex can be at most (or at least) the largest (or smallest) value of φ on its neighbours. Suppose $m \in v(G) - B$ were an interior maximum of φ, say $\varphi(m) = M$, then by the previous observation all of the values of φ on the neighbors of m must be equal to M. By iterating this argument over a path from m to B, we find that some point in B must have value M, and the claim holds. An identical argument holds for the minimum. $\square$

This proposition does not apply to general infinite graphs, even recurrent ones. Indeed, consider the following counter-example: On the graph $\mathbb{Z}$, take $B = \{0\}$ and define the harmonic function φ with boundary B by $\varphi(n) = n$. Then,

$$\varphi(n) = \frac{1}{2}(n - 1 + n + 1) = \frac{1}{2}(\varphi(n-1) + \varphi(n+1))$$

so that φ is indeed harmonic, but it does not attain a maximum nor minimum on the boundary. In fact, this counter-example shows the importance of boundedness in the Liouville property for recurrent graphs. It furthermore shows that uniqueness is also not guaranteed for harmonic functions on infinite graphs by simply rescaling. However, if we assume *a priori* that φ is bounded, then we do have a unique harmonic function with prescribed boundary. Before we prove this, we require some more machinery.

Let $B \subset v(G)$ and let $\psi : B \to \mathbb{R}$ be a real-valued function. Suppose that G is recurrent and as before let $\mathbb{P}_v$ denote the probability measure associated to the simple random walk on G starting at v. Define a new function $\hat{\psi} : v(G) \to \mathbb{R}$ by

$$\hat{\psi}(v) = \mathbb{E}_v(\psi(X_{\tau_B}))$$

ie. $\hat{\psi}(v)$ is the expected value of ψ for a random walk hitting B. We now claim that this defines a bounded harmonic function which coincides with ψ on B. That this function is always well defined follows from the recurrence assumption, that it is also bounded and agrees with ψ is also immediate. We must now verify that it is harmonic outside of B. By the Markov property and law of total expectation,

$$\hat{\psi}(v) = \mathbb{E}_v(\psi(X_{\tau_B})) = \sum_{u \in N(v)} \mathbb{E}_u(\psi(X_{\tau_B})) \cdot \mathbb{P}_v(X_1 = u) = \frac{1}{\deg(v)} \sum_{u \in N(v)} \hat{\psi}(u)$$

as desired. We can now prove uniqueness.

Proposition 5 ([20], Theorem 4.2.3). *Let G be a recurrent graph and let $\varphi : v(G) \to \mathbb{R}$ be a bounded harmonic function with boundary B, then φ is uniquely determined by its values on B.*

Proof. In view of the above construction, it suffices to show that

$$\varphi(v) = \mathbb{E}_v(\psi(X_{\tau_B})).$$

By definition we have,

$$\varphi(v) = \frac{1}{\deg(v)} \sum_{u \sim v} \varphi(u) = \sum_{u \in B} \varphi(u) \cdot \mathbb{P}_v(X_1 = u) + \sum_{u \in v(G) - B} \varphi(u) \cdot \mathbb{P}_v(X_1 = u).$$

Write $p_{v,u} = \mathbb{P}_v(X_1 = u)$ and iterate the right hand side n-times to get

$$\begin{aligned}\varphi(v) = &\sum_{u_1 \in B} \varphi(u_1) \cdot p_{v,u_1} + \sum_{u_1 \in B^c} \sum_{u_2 \in B} \varphi(u_2) \cdot p_{v,u_1} \cdot p_{u_1,u_2} + \cdots \\ &+ \sum_{u_1 \in B^c} \sum_{u_2 \in B^c} \cdots \sum_{u_n \in B} \varphi(u_n) \cdot p_{v,u_1} \cdots p_{u_{n-1},u_n} \\ &+ \sum_{u_1 \in B^c} \sum_{u_2 \in B^c} \cdots \sum_{u_n \in B^c} \varphi(u_n) p_{v,u_1} \cdots p_{u_{n-1},u_n} \\ &= \mathbb{E}_v(\varphi(X_1)\mathbf{1}_{\tau_B=1}) + \mathbb{E}_v(\varphi(X_2)\mathbf{1}_{\tau_B=2}) + \cdots + \mathbb{E}_v(\varphi(X_n)\mathbf{1}_{\tau_B=n}) + \mathbb{E}_v(\varphi(X_n)\mathbf{1}_{\tau_B \geq n}).\end{aligned}$$

Let $M = \sup_{v \in v(G)} |\varphi(v)|$ and note that by the Cauchy-Schwarz inequality,

$$\mathbb{E}_v(\varphi(X_n)\mathbf{1}_{\tau_B \geq n})^2 \leq \mathbb{E}_v(\varphi(X_n)^2) \cdot \mathbb{E}_v(\mathbf{1}_{\tau_B \geq n}) \leq M^2 \cdot \mathbb{P}_v(\tau_B \geq n) \to_n 0$$

since G is recurrent by assumption (ie. $\tau_B < \infty$ almost surely). It now follows that the last line in the equality above tends to $\mathbb{E}_v(\varphi(X_{\tau_B}))$ as $n \to \infty$, which was to be shown. $\square$

Note further that we also have extended the maximum principle of finite graphs to the recurrent case: if φ is bounded on a recurrent graph, then it assumes its maximum and minimum on the boundary.

Fact 6. Let φ_1 and φ_2 be harmonic with boundary B_1 and B_2 respectively, then $a_1\varphi_1 + a_2\varphi_2$ is harmonic with boundary $B_1 \cup B_2$.

Electrical Networks

Suppose for now that G is a finite graph. Let us view G as a network of resistors where each edge behaves as a resistor of unit resistance. If we apply a potential voltage difference between two vertices, say s and t, then we can define a flow (defined by the current) on G given by Kirchoff's laws. More technically, let $\phi : v(G) \to \mathbb{R}$ be a function on the vertices of G, and let $j : \vec{e}(G) \to \mathbb{R}$ be an anti-symmetric ($j_{uv} = -j_{vu}$) function on the directed edges of G, then the pair (ϕ, j) *satisfies Kirchoff's laws with source-set* B if it satisfies:

1. **Kirchoff's Current Law**: For all $v \notin B$,

$$\sum_{u \in N(v)} j_{uv} = 0.$$

2. **Ohm's Law**: For any directed edge $uv \in \vec{e}(G)$,

$$j_{uv} = \phi(u) - \phi(v).$$

If (ϕ, j) satisfies Kirchoff's laws, then ϕ is called a *potential function*, j is called a *current flow* and (ϕ, j) is called a *Kirchoff pair*. Any pair which satisfy these laws is in fact uniquely determined by the value of ϕ on B.

Theorem 7. *There exists a unique Kirchoff pair (ϕ, j) for any choice of source-set B and value of ϕ thereon. Furthermore, ϕ is harmonic with boundary B, and for any harmonic function ϕ with boundary B, there exists a unique j such that (ϕ, j) is a Kirchoff pair.*

Proof. It is first claimed that for a Kirchoff pair (ϕ, j), the potential ϕ is harmonic with boundary B. Indeed, for $v \notin B$ we have

$$0 = \sum_{u \in N(v)} j_{uv} = \sum_{u \in N(v)} \phi(u) - \phi(v) = -\deg(v)\phi(v) + \sum_{u \in N(v)} \phi(u)$$

solving for $\phi(v)$ shows the desired equality. By Proposition 5 if two Kirchoff pairs (ϕ_1, j_1) and (ϕ_2, j_2) have the same source-set and agree on B, then $\phi_1 = \phi_2$. As the current j is completely determined by Ohm's law it is therefore completely determined by ϕ. In particular, for ϕ harmonic, define the function j by $j_{uv} = \phi(u) - \phi(v)$. It is now trivial to check that if j is defined by Ohm's law and ϕ is harmonic, then (ϕ, j) is a Kirchoff pair, and we are done. $\square$

From Proposition 5 and Theorem 7, it follows that we in fact have

$$\phi(v) = \mathbb{E}_v(\phi(X_{\tau_B}))$$

which gives us a nice interpretation of electrical potential in terms of random walks on G.

Let $s \neq t$ be vertices in G and recall that an st-flow on G is a function $j : \vec{e}(G) \to \mathbb{R}$ on the set of directed edges of G which satisfies the following:

1. $j_{vu} = -j_{uv}$,
2. $j_{uv} = 0$ if $\{uv\} \notin e(G)$,
3. for all $u \neq s, t$ we have
$$\sum_{v \in N(u)} j_{uv} = 0.$$

It is easy to check that current (as in Kirchoff's laws) satisfies these properties if $B = \{s, t\}$. We say that j is a *unit* st-flow if the total flow out of s (or t) is equal to 1.

Fact 8 ([8]). If j is a unit st-flow which satisfies Kirchoff's laws for some pair (ϕ, j) with source-set $\{s, t\}$, then

$$j_e = \mathbb{E}(\text{net number of crossings in a random walk from } s \text{ to } t).$$

Proof. Let $\eta(v)$ be the expected number of visits to vertex v in a walk from s to t, then for $v \neq s, t$

$$\eta(v) = \sum_{w \sim v} \eta(w) \cdot \frac{1}{\deg(w)} = \deg(v) \frac{1}{\deg(v)} \sum_{w \sim v} \frac{\eta(w)}{\deg(w)}$$

so that $\varphi(v) = \eta(v) / \deg(v)$ is harmonic. Let i be the unique current flow corresponding to φ then

$$i_{xy} = \varphi(x) - \varphi(y) = \frac{\eta(x)}{\deg(x)} - \frac{\eta(y)}{\deg(y)} = \eta(x)\mathbb{P}_x(X_1 = y) - \eta(y)\mathbb{P}_y(X_1 = x)$$

but $\eta(x)\mathbb{P}_x(X_1 = y)$ is the expected number of crossings from x to y (and vice versa). It remains to show that $i = j$, ie. that i is a unit st-flow. However, since a random walk starts at, but must always leave, s, the total net number of exits from s must be 1, and we are done. $\square$

The *effective resistance* between two vertices $u, v \in v(G)$ is the value

$$R(u, v) := \frac{1}{\sum_{w \in v(G)} j_{uw}}$$

where we take the pair (ϕ, j) satisfying Kirchoff's laws with $\phi(u) = 1$ and $\phi(v) = 0$. Writing $\lambda(j)$ for the total current exiting u allows us to succinctly write,

$$R(u, v) = \frac{1}{\lambda(j)}.$$

We may similarly define the effective resistance between a vertex and a set: the *effective resistance between v and $A \subset v(G)$* is the value

$$R(v, A) := \frac{1}{\lambda(j)}$$

where the pair (ϕ, j) this time satisfies Kirchoffs laws with $\phi(v) = 1$, $\phi(a) = 0$ for all $a \in A$ and λ is defined again by the total current exiting v. If v is not connected to A then we take $R(v, A) = \infty$.

Theorem 9 ([12]). *A graph G is recurrent if and only if for some $w \in v(G)$, we have $R(w, v(G) - v(B_G(w, n))) \to \infty$ as $n \to \infty$.*

Proof. First note that if ϕ is bounded and harmonic with boundary $\phi(w) = 1$ and $\phi(a) = 0$ for all $a \in A$, then by Proposition 5 (where here $B = A \cup \{w\}$),

$$\phi(u) = \mathbb{E}_u(\phi(X_{\tau_B})) = \mathbb{P}_u(\tau_B = w)$$

for all $u \in v(G)$. If we take j to be the Kirchoff current corresponding to ϕ we have

$$\lambda(j) = \sum_{v \in N(s)} 1 - \mathbb{P}_v(\tau_B = s) = \sum_{v \in N(s)} \mathbb{P}_v(X_{\tau_B} \in A)$$

which is the probability that a random walk starting at s hits A before returning to s. Now let (ϕ_n, j_n) be the Kirchoff pair satisfying $\phi_n(w) = 1$ and $\phi_n(b) = 0$ for $b \in v(G) - v(B(w,n))$. From the definition, the resistance grows to infinity if and only if $\lambda(j_n) \to 0$ as $n \to \infty$. In particular, from the above observation, this translates to: the resistance grows to infinity if and only if the random walk returns to w almost-surely, and we are done. □

Given any flow j on G, define the energy of j as

$$\mathcal{E}(j) = \frac{1}{2} \sum_{e \in \vec{e}(G)} j_e^2$$

and we define $\mathcal{E}(\phi)$ for a harmonic function ϕ as

$$\mathcal{E}(\phi) = \frac{1}{2} \sum_{uv \in \vec{e}(G)} (\phi(u) - \phi(v))^2.$$

Lemma 10 ([12]). *Let (ϕ, j) be the pair satisfying Kirchoff's laws with source set $\phi(s) = 1$ and $\phi(t) = 0$, then*

$$R(s,t) = \mathcal{E}(j)^{-1}.$$

Furthermore, if instead we rescale ϕ so that $\lambda(j) = 1$ and $\phi(t) = 0$ then

$$R(s,t) = \mathcal{E}(j).$$

Proof. Using Ohm's law and then the current law, we can write the energy as

$$\mathcal{E}(j) = \frac{1}{2} \sum_{v \in v(G)} \sum_{u \in N(v)} j_{v,u}^2 = \sum_{v \in v(G)} \sum_{u \in N(v)} (\phi(v) - \phi(u)) j_{v,u} = \phi(s)\lambda(j) - \phi(t)\lambda(j) = \phi(s)\lambda(j)$$

from which both cases of the lemma follow. □

In fact, an even stronger relation exists between unit st-flows and energy.

Proposition 11. *Let (ϕ, j) be a Kirchoff pair where j is a unit st-current, then j is the unique st-flow minimizing $\mathcal{E}(j)$ over all unit st-flows.*

Proof. Let i be a unit st-flow and let j be the unit flow from Kirchoff's laws. Write $h = i - j$ so that $\lambda(h) = 0$ and

$$\begin{aligned}\mathcal{E}(i) = \frac{1}{2}\sum_{e\in\vec{e}(G)}(h_e - j_e)^2 &= \mathcal{E}(j) + \mathcal{E}(h) - \sum_{e\in\vec{e}(G)} j_e h_e \\ &= \mathcal{E}(j) + \mathcal{E}(h) - \sum_{e\in\vec{e}(G)}(\phi(e_+) - \phi(e_-))h_e \\ &= \mathcal{E}(j) + \mathcal{E}(h) - \phi_j(s)\lambda(h) \geq \mathcal{E}(j)\end{aligned}$$

with equality precisely when $i_e = j_e$ for all $e \in \vec{e}(G)$. $\square$

This proposition allows us to prove the famous Rayleigh Monotonicity theorem.

Theorem 12 (Rayleigh Monotonicity). *Let $e \in e(G)$ and let $G' = G - e$, then $R_G(s,t) \leq R_{G'}(s,t)$ for all $s, t \in v(G)$.*

Proof. Let j be the unique minimal energy unit st-flow on G', then we can promote j to a flow of equal energy on G by setting $j_e = 0$. $\square$

We have an important result due to Kirchoff relating effective resistance to spanning trees of a graph. Recall that a *spanning tree* of G is a tree $T \subset G$ such that $v(T) = v(G)$.

Theorem 13 ([18], P. 116). *Let $\mathcal{T}$ be the set of spanning trees of an edge rooted graph $\mathbf{G} = (G, st)$ and let T be a uniformly chosen tree from $\mathcal{T}$. Then,*

$$\mathbb{P}(st \in T) = R(s,t).$$

Proof. Note by Ohm's law that it suffices to show that $\mathbb{P}(st \in T) = i_{st}$ where i is the unique energy minimizing unit st flow. From Fact 8, i_{st} is the expected net number of edge crossings from s to t in a random walk from s to t. Since st is an edge, if $X = (X_n, n \geq 0)$ is a random walk then the expected net number of crossings is in fact equal to $\mathbb{P}_s(X \text{ crosses } st)$ as the edge st is crossed once or not at all (and never in the direction ts as the random walk stops at t). We finally claim that $\mathbb{P}(st \in T) = \mathbb{P}_s(X \text{ crosses } st)$. By applying Wilson's Method (see [18] Chapter 4) we generate a uniformly random spanning tree T' by rooting at t and starting the process at s. In this process, we find that $st \in T'$ if and only if the random walk crosses st, precisely as desired. $\square$

Remark 14. We can apply this theorem to the case where st is not an edge of G as follows: Connect s and t by an edge and call this new graph G', then by the energy formulation of resistance, $R_{G'}(s,t)^{-1} = R_G(s,t)^{-1} + 1$. In particular,

$$R_G(s,t) = \frac{R_{G'}(s,t)}{1 - R_{G'}(s,t)} = \frac{\mathbb{P}(st \in T)}{1 - \mathbb{P}(st \in T)}$$

where T is a uniformly chosen spanning tree in G'.

For a planar map G, its *dual* G^* is defined as the graph $G^* = (f(G), e(G^*))$ where an edge connects two faces if and only if they are adjacent. We write e^* for the edge of G^* crossing $e \in e(G)$. Let $\mathbf{G} = (G, st)$ be an edge-rooted graph, and define the dual $\mathbf{G}^* = (G^*, s^*t^*)$ where s^*t^* is oriented so that s^*t* travels from the let to the right of st, when it is oriented according to the current flow, see Figure 2.

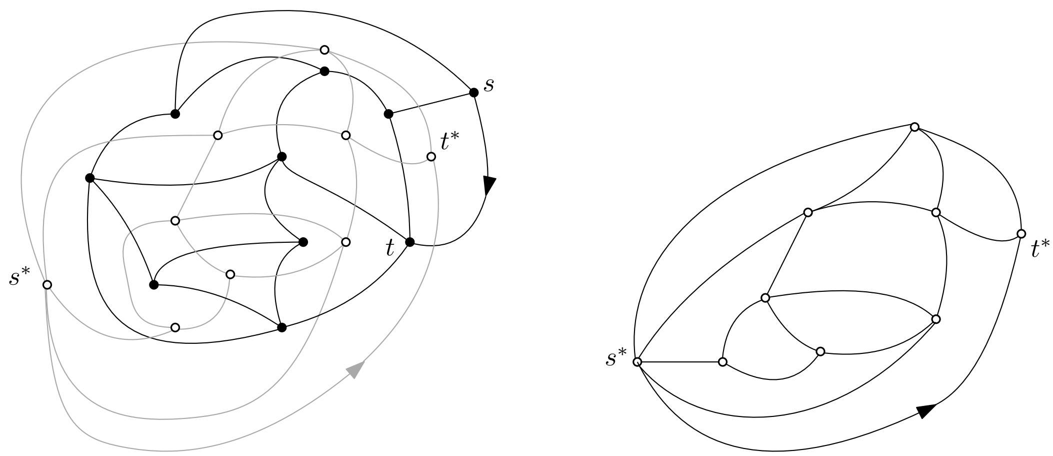

Figure 2: Constructing the dual (in grey) on the left, and a cleaned up version on the right.

Lemma 15. *Let $\mathbf{G} = (G, st)$ be an edge rooted planar map, and let $\mathbf{G}^*$ be its dual. Then,*

$$R_G(s,t) = 1 - R_{G^*}(s^*, t^*)$$

where s^* is the source of $(st)^*$ and t^* is the target.

Proof. Let T be a spanning tree in G, and let $T_* = \{e^* \in e(G^*) \; : \; e \notin T\}$ be the dual edge in G^* to the set consisting of all edges not belonging to T. It is claimed that T_* is a spanning tree of G^*. Since T contains no cycles, T_* must be connected (if it were not connected then T must contain a cycle). Furthermore, $v(T_*) = v(G^*)$ since every face of G has an edge not belonging to T, again as T contains no cycles. It remains to show that T_* is a tree. If T_* contained a cycle, then some face of G^* belongs inside the cycle. However, in this case a vertex of G could not be reached by T, contradicting the fact that it was spanning. We now have a bijection between the set of spanning trees of G and those of G^*, given by $T \mapsto T_*$. In particular,

$$\mathbb{P}(st \in T) = \mathbb{P}((st)^* \notin T_*) = 1 - \mathbb{P}((st)^* \in T^*)$$

where T^* is a uniformly chosen spanning tree in G^*. Applying Theorem 13 yields the result. $\square$

To conclude this subsection we will prove two useful facts.

Fact 16. For any two vertices $s, t \in v(G)$, if (ϕ, j) is a pair satisfying Kirchoff's laws with source-set $\phi(s) = 1$ and $\phi(t) = 0$, then

$$\lambda(j) = \deg(s) - \sum_{v \in N(s)} \phi(v) = \sum_{v \in N(t)} \phi(v).$$

This fact follows immediately from Kirchoff's current law. The second fact requires a little more work. For a vertex $v \in v(G)$ and flow j, we say that the edge $e = uv$ is *incoming* at v if $j_e > 0$ and outgoing if $j_e \leq 0$.

Fact 17 ([21]). Let (ϕ, j) be a Kirchoff pair on a planar map G where j is an st-flow. Then for any $v \in v(G)$ and planar embeddings of G there exists an angle in the plane at v which contains all incoming edges, whose complement contains all outgoing edges of v.

Proof. It is clear that any vertex of G can be reached via a path of increasing (resp. decreasing) potential to s (resp. to t). Suppose that two incoming edges separate two outgoing edges at v. Then there exists a path of increasing potential for each incoming edge back to s and paths for each outgoing edge of decreasing potential to t. Now at least one increasing path must intersect a decreasing path in the embedding, and they cannot intersect at an edge by planarity. It follows that they they must meet at the same vertex, say w. But then the potential at w is simultaneously strictly above and strictly below the potential at v, an impossibility. $\square$

Probability Measures on Graphs

We now consider probability measures on the set of graphs. Recall the local-weak metric on rooted graphs defined by

$$D(\mathbf{G}, \mathbf{G}') = \inf\left\{\frac{1}{r+1} \;:\; B_G(\rho, r) \cong B_{G'}(\rho', r)\right\}.$$

If $\mathbf{G}$ is a random variable taking values in $\mathcal{G}$, the set of all vertex rooted graphs, then we can write $\mathbf{G} = (G, \rho)$ where G is a random graph and ρ is a random vertex in G. We shall call the random variable $\mathbf{G}$ a *finite uniformly rooted random graph* if it is finite almost surely and conditional on G, ρ is chosen according to the law,

$$\mathbb{P}(\rho = v \mid G) = \frac{\deg_G(v)}{2|e(G)|}.$$

Now, let $(\mathbf{G}_n, n \geq 1)$ be a sequence of random rooted graphs. The metric D on $\mathcal{G}$ induces a notion of convergence in distribution. In particular, we say that the sequence $(\mathbf{G}_n, n \geq 1)$ converges in distribution if there exists a random graph (U, ρ) such that for all $r > 0$ and finite graphs B,

$$\mathbb{P}(B_{G_n}(\rho_n, r) = B) \to \mathbb{P}(B_U(\rho, r) = B)$$

as $n \to \infty$ and such that for all $r > 0$,

$$\sum_B \mathbb{P}(B_U(\rho, r) = B) = 1$$

where the sum is taken over all finite graphs B. This essentially means that any fixed ball around the root of the random graphs $\mathbf{G}_n$ converges in distribution to the ball around the root in U.

Recall that a real valued random variable X has exponential tail if for all $k > 0$,

$$\mathbb{P}(X \geq k) \leq e^{-ck}$$

for some constant $c > 0$. This brings us to a beautiful theorem by Gurel-Gurevich and Nachmias.

Theorem 18 ([13], Theorem 1.1). *Let (U, ρ) be a random graph which is the distributional limit of finite uniformly rooted random planar maps such that $\deg_U(\rho)$ has exponential tail, then U is almost-surely recurrent.*

This marvellous theorem relating the vertex degrees of a random graph to its recurrence plays a crucial role in the main result of the thesis.

2 Squaring Constructions

We finally bring our attention to the squaring of rectangles in the plane. Formally, a *squaring of a rectangle* is a non-empty collection $S = (s_i, i \geq 0)$ of sets $s_i \subset \mathbb{R}^2$ such that each s_i is a closed square, any two squares have pairwise disjoint interiors and the closure of the union of the squares is a compact rectangle. We will freely interchange the word square to mean either the set $s_i \in S$ or ∂s_i (a hollow square), specifying which when there is ambiguity.

A squaring S is called *composite* if there is a strict subset S' of S which is also a squaring of a rectangle, otherwise it is *non-composite*. If four squares of S have non-empty intersection $s_i \cap s_j \cap s_k \cap s_\ell \neq \emptyset$ then we shall call the squaring *degenerate* and the point at which these intersect is called a *point of degeneracy*, see Figure 3. We call a squaring S *good* if it is non-composite and non-degenerate.

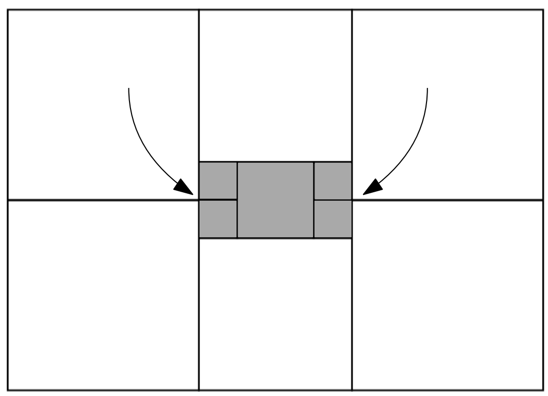

Figure 3: The grey squares correspond to a sub-squaring, making the squaring composite. Arrows point to degeneracy points in the squaring.

2.1 Triangulations

In this section we detail a procedure to generate squarings of a rectangle based on work by Oded Schramm [23]. The "power" in this construction is that these squarings can be given almost any desired contact graph structure directly. It shall be useful throughout this section to call points $x \in \mathbb{R}^2$ *trivial squares*.

Let G be a finite planar map, we say that G is a *triangulation* if all of its faces have degree 3. If all of the faces of G have degree 3, except for one, then we call G a *triangulation with boundary* and we say that the vertices belonging to that face are the boundary ∂G. Suppose now that we have a finite squaring S of a rectangle, say with squares $s_1, s_2, ..., s_n$, we construct the *contacts graph* $R(S)$ of S as follows: Let $v(R(S)) = \{s_1, ..., s_n\}$ and connect each pair of vertices by an edge precisely if $|s_i \cap s_j| > 1$. Finally, if four squares intersect at a point $x \in \mathbb{R}^2$ add the point to $v(R(S))$ and add the edges $\{x, s_i\}, \{x, s_j\}, \{x, s_k\}, \{x, s_\ell\}$ to $R(S)$, see Figure 4.

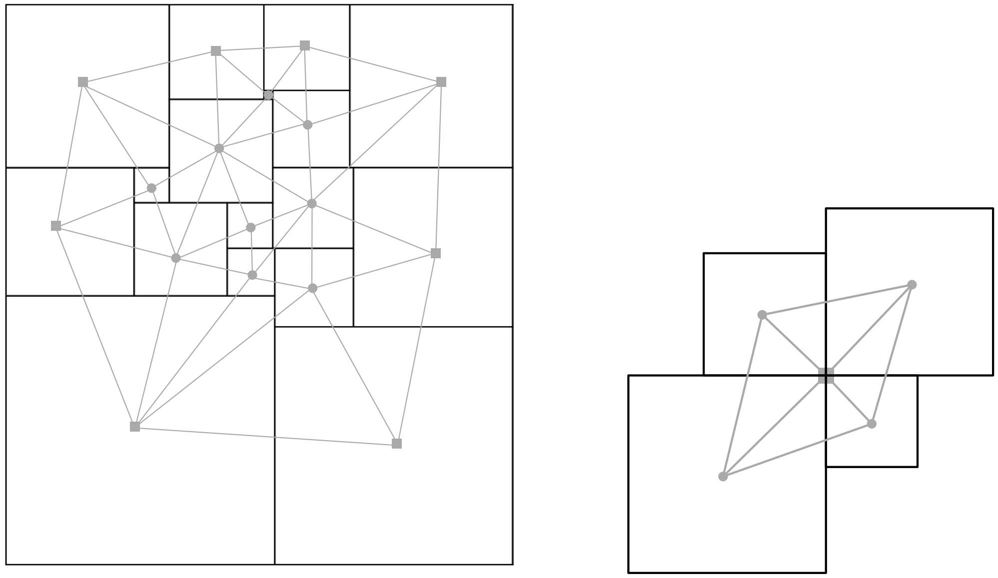

Figure 4: On the left denotes the contacts graph (in grey) structure for the squaring. The right shows the construction of the contacts graph at a point of degeneracy, with the square vertex denoting a trivial square.

It is easy to see that $R(S)$ is a proper triangulation with boundary for any squaring S. Let us now distinguish four regions of the boundary of $R(S)$: Let D_1, D_2, D_3 and D_4 be the sets of squares along the top, right, bottom and left of S respectively.

A 5-tuple (G, D_1, D_2, D_3, D_4) is called a *triangulation of a quadrilateral* if G is a triangulation with boundary, $D_1 \cup D_2 \cup D_3 \cup D_4 = \partial G$, each D_i is connected and $|D_i \cap D_{i+1}| = 1 \bmod 4$, so that in particular $(R(S), D_1, D_2, D_3, D_4)$ as constructed above is a triangulation of a quadrilateral for any squaring S.

Theorem 19 ([23], Theorem 1.3 and 5.1). *There exists an explicit mapping $\Phi : \mathcal{T}_Q \to \mathcal{S}$ from the set $\mathcal{T}_Q$ of triangulations of quadrilaterals to squarings of rectangles $\mathcal{S}$ with the following properties:*

(i) $\Phi(R(S)) = S$ *for all* $S \in \mathcal{S}$*,*

(ii) for all squarings $S \in \mathcal{S}$ *and graphs* $Q \in \Phi^{-1}(S) \subset \mathcal{T}_Q$ *we have* $\Phi(Q) = \Phi(R(S))$*,*

(iii) vertices $v \in v(Q)$ *are mapped bijectively to squares* S_v*, and* $\{u, v\} \in e(Q)$ *if and only if* $S_u \cap S_v \neq \emptyset$*.*

This essentially says that provided a given triangulation of a quadragulation is well-behaved (which will be discussed later), then the squaring of Q under Φ has the same contacts graph as Q.

Before delving into the details of the construction, we shall require the notion of extremal length. Let G be a graph and let $m : v(G) \to \mathbb{R}^+$ be a non-negative function on the vertices of G (also called a metric on G). The *mass* of m, $M(m)$ is defined as

$$M(m) = \sum_{v \in v(G)} m(v)^2$$

and the *weight* of a path γ is

$$m(\gamma) = \sum_{v \in \gamma} m(v).$$

Write $\|m\| = \sqrt{M(m)}$, for the L^2 norm on $\{m : v(G) \to \mathbb{R}^+$. For $U \subset v(G)$ and $V \subset v(G)$ disjoint sets of vertices, we define the mass of (U, V) to be

$$m(U, V) = \inf_{\gamma} m(\gamma)$$

where the infimum is taken over all paths γ which start in U and end in V. A metric $e : v(G) \to \mathbb{R}^+$ is called a (U, V)*-extremal metric* if,

$$\frac{e(U,V)}{\|e\|} = \sup \left\{ \frac{m(U,V)}{\|m\|} \ : \ m \text{ is a metric} \right\}.$$

Note that the set of positive functions m which satisfy $m(U, V) \geq 1$ is non-empty, closed and convex in a Hilbert space and that $\|\cdot\|$ is a norm on this set. Since it is closed and convex there exists a unique element of least norm in this set and it is simple to check that this element is in fact (U, V)-extremal. In fact, such an extremal metric is unique up to rescaling. These extremal metrics are key in going back and forth between triangulations of quadrilaterals and squarings of rectangles. First of all, squarings correspond to extremal metrics on their contacts graph.

Theorem 20 ([23], Theorem 1.3). *Let S be a squaring of a rectangle with squares $s_1, s_2, ..., s_n$ and let $(R(S), D_1, D_2, D_3, D_4)$ be the derived triangulation. If $m : v(R(S)) \to \mathbb{R}^+$ is defined by $m(v) = |s_v|$, then m is a (D_1, D_3)-extremal metric on $R(S)$.*

By forcing the rectangle to have area 1, say to be a squaring of the rectangle $[0, h^{-1}] \times [0, h]$, we get a *unique* extremal metric on $R(S)$ defined by the square sizes. Furthermore, such an extremal metric has unit mass, and satisfies $m(D_1, D_3) = h$. By rotating the squarings clockwise 90 degrees, we preserve the contacts graph $R(S)$ but send the boundary sets D_i to D_{i+1} mod 4. It is then easy to see that not only does m define a (D_1, D_3)-extremal metric, but also a (D_2, D_4)-extremal metric on the same graph where now $m(D_2, D_4) = h^{-1}$. We now state the converse to the above theorem.

Theorem 21 ([23], Theorem 5.1). *Let $T = (G, D_1, D_2, D_3, D_4)$ be a triangulation of a quadrilateral, let $m : v(G) \to \mathbb{R}^+$ be a unit mass (D_1, D_3)-extremal metric and let*

$$S(T) = \{[x(v) - m(v), x(v)] \times [y(v) - m(v), y(v)] \ : \ v \in V(G)\}$$

where $x(v) = m(D_2, v)$ and $y(v) = m(D_1, v)$. The set $S(T)$ is a squaring of the rectangle $[0, h^{-1}] \times [0, h]$ where $h = m(D_1, D_3)$.

Note that the number of squares (including the trivial ones) is equal to the number of vertices of G, and squares have positions defined by their mass-distance to D_1 and D_2. See Figure 5 for an explicit correspondence.

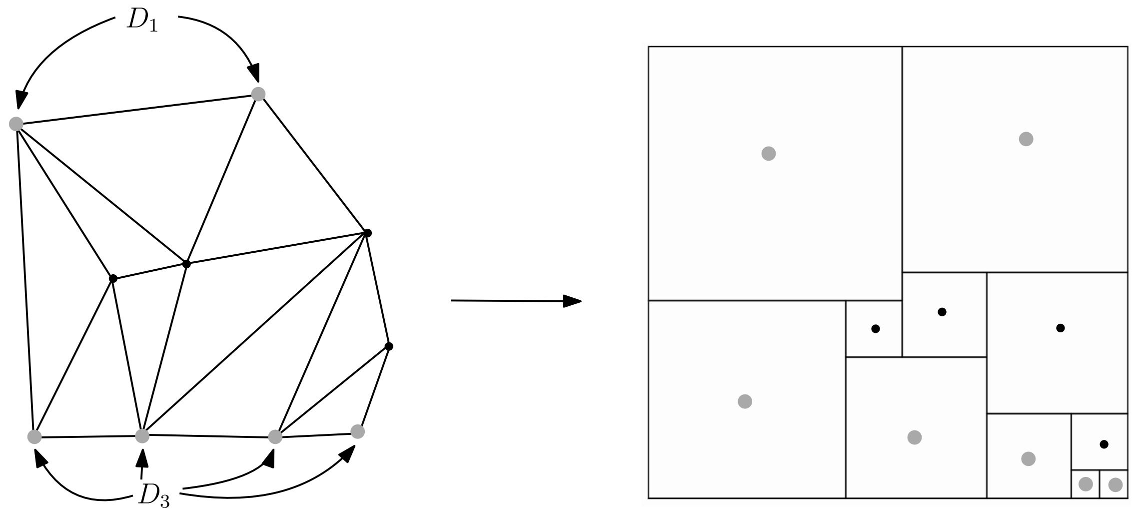

Figure 5: A triangulation of a quadrilateral with its corresponding squaring. The grey vertices denote the boundary D_1 and D_3 (top and bottom resp.).

It is particularly non-trivial to find the composite or degeneracy structure of S directly from the structure of G. In fact, such a result would allow us to immediately answer an open question regarding the distributional limit of good squarings and gain asymptotics for the number of such squarings. We now pose this as an open question.

Question 22. Is there an "easy" way to determine whether or not $S(T)$ is *good* directly from T?

To check that $S(T)$ is non-degenerate boils down to verifying that T does not have vertices on which $m = 0$. This condition should be easily verifiable, however the composite structure of $S(T)$ has no obvious phrasing in terms of m, nor in terms of the structure of T.

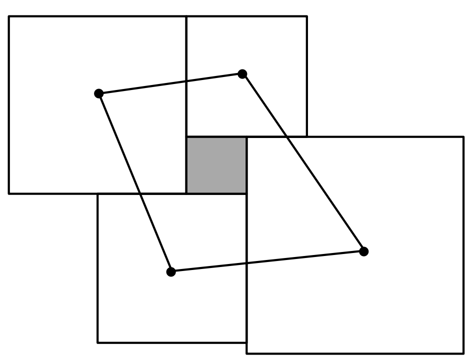

Figure 6: A 4-cycle necessarily has this structure, if the grey box contains vertices then it is a sub-squaring.

In fact, being non-composite would imply part of the non-degeneracy condition: if $S(T)$ were non-composite then any 4-cycle in G containing more than a single point must introduce a degenerate point (see Figure 6), in particular it would then suffice only to check those 4-cycles only containing one point.

2.2 Random Walks and Networks

In this section we present another way to construct squarings from graphs, using a more probabilistic approach. This construction is a slightly modified version of the one introduced by Brooks, Stone, Tutte and Smith (hereto referred to as the *BSST squaring*) in [21], and is the cornerstone of the results in Chapter 3 of this thesis.

Throughout section 2.2, we let $\mathbf{G} = (G, st)$ be an edge-rooted planar graph such that $G - st$ is connected. We view $\mathbf{G}$ as an electrical circuit in the following manner: remove the edge st and apply a unit potential difference of one between s and t, ie. set $\phi(s) = 1$ and $\phi(t) = 0$. Whenever we *view* $\mathbf{G}$ *as a circuit*, we shall mean the previous. Given the pair (ϕ, i) satisfying Kirchoffs laws on $\mathbf{G}$ when viewed as a circuit, for $e = \{u, v\} \in e(G)$ we write $e = e_+e_-$ labeled so that $\phi(e_+) = \max(\phi(u), \phi(v))$ and $\phi(e_-) = \min(\phi(u), \phi(v))$. Note that $\phi(e_+) = |i_e| + \phi(e_-)$.

Suppose now that $\mathbf{G}$ is a planar map, and define the dual $\mathbf{G}^*$ of $\mathbf{G}$ as before: $\mathbf{G}^* = (G^*, s^*t^*)$ where G^* is the graph constructed from G by connecting an edge between two vertices in G^* if the corresponding faces in G are adjacent. We root G^* by setting the root edge in G^* to be the corresponding root edge in G and oriented such that the root edge in G^* crosses st from left to right when st is oriented according to i. We may also view $\mathbf{G}^*$ as a circuit in the same way as above.

Theorem 23 ([21], Theorem 4.31). *Let $\mathbf{G}$ be an edge-rooted planar map such that $G - st$ is connected. Define the set*

$$\mathcal{S}(\mathbf{G}) = \{[\phi^*(e_+^*), \phi^*(e_-^*)] \times [\phi(e_+), \phi(e_-)] \ : \ e \in G\}$$

where (ϕ^*, i^*) is the unique pair satisfying Kirchoff's laws on G^* with $\phi^*(s^*) = \lambda(i)$ and $\phi^*(t^*) = 0$. The set $\mathcal{S}(\mathbf{G})$ is a squaring of the rectangle $[0, \lambda(i)] \times [0, 1]$.

In such a construction, the edges (instead of vertices as in the previous section) correspond to squares. For the Kirchoff pair (ϕ, i) on $\mathbf{G}$ when viewed as a circuit, we define a dual current i_* by setting $i_*(e^*) = i(e)$ and oriented in such a way that e^* travels from left to right of e when e is oriented according to the flow i.

Lemma 24. *The dual current i_* is a unit s^*t^*-current flow on G^* and furthermore $i_* = i^*$ where i^* is defined as in Theorem 23.*

Proof. It is first claimed that i_* is in fact a flow. Note first that the algebraic sum of currents clockwise around a face in G is 0 by Ohm's law. By construction of the dual flow, an edge e^* is outgoing if it travels counter-clockwise around the face and incoming if it travels clockwise. But then for all vertices except for s^* and t^* the flow in is equal to the flow out. Furthermore, the outgoing edges adjacent to s^* in G^* define a path in G from s to t so that i_* is a unit s^*t^*-flow (see Figure 7). Recall that the unit Kirchoff current i^* minimizes the energy over all unit flows so that from the energy formulation of resistance

$$R(\mathbf{G}^* - (st)^*) = \mathcal{E}(i^*) \leq \mathcal{E}(i_*) = \mathcal{E}(i) = R(\mathbf{G} - st)^{-1}.$$

But from Remark 14 and Lemma 15 we also have

$$R(\mathbf{G}^* - (st)^*) = \frac{R(\mathbf{G}^*)}{1 - R(\mathbf{G}^*)} = \frac{1 - R(\mathbf{G})}{R(\mathbf{G})} = \frac{1}{R(\mathbf{G} - st)}$$

so that in particular $\mathcal{E}(i^*) = \mathcal{E}(i_*)$ and we are done by uniqueness. □

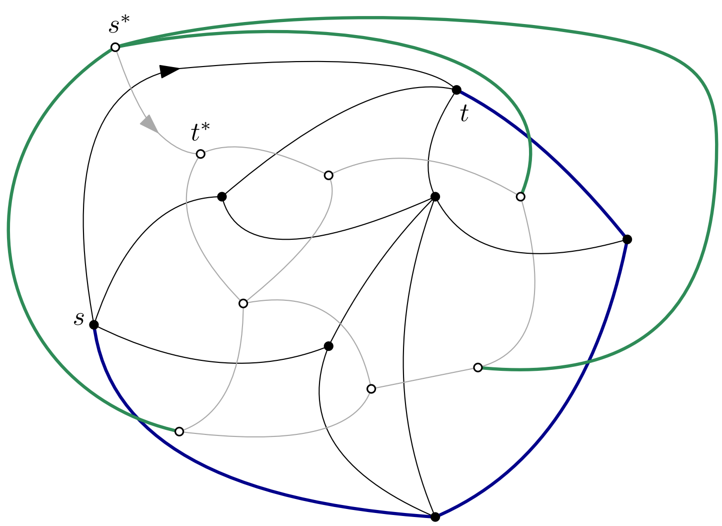

Figure 7: The dual of G is given in grey. Green lines denote outgoing edges of s^* in G^* which induce a dark blue path from s to t in G.

In view of Fact 17, by duality we find that for any point in a face of G there exists an angle about the point in which the flow travels only clockwise and the complement of this angle has flows travelling only counter-clockwise.

Proof of Theorem 23. By the previous lemma, each square in Theorem 23 is a proper square (ie. its height is equal to its width). It thus remains to show that the set $\mathcal{S}(\mathbf{G})$ defines a tiling of a rectangle. We first show that it covers the entire rectangle $R := [0, \lambda(i)] \times [0, 1]$. Fix a point $(x, y) \in R$ and let $\mu_y = \{e \in e(G) \;:\; y \in [\phi(e_+), \phi(e_-)]\}$. It is claimed that μ_y is a cut-set separating s from t, indeed any path $p = \{p_1, ..., p_n\}$ defines a collection of potential $\{\phi(p_1), ...\phi(p_n)\}$ with $\phi(p_1) = 1$ and $\phi(p_n) = 0$ so that we must have $y \in [\phi(p_k), \phi(p_{k+1})]$ for some k and the claim is proved. It furthermore follows that the dual $\mu_y^* = \{e^* \;:\; e \in \mu_y\}$ defines a path from s^* to t^* (see Figure 8).

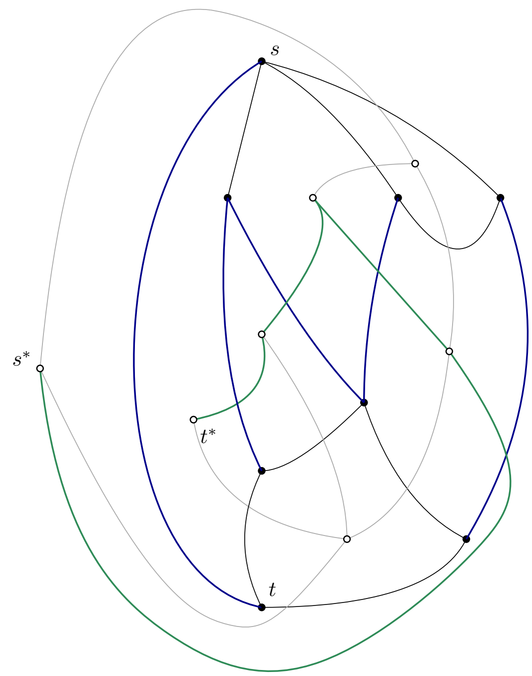

Figure 8: The dual of G is given in grey. Green lines denote the path in G^* induced by the cutset in blue.

Let $\lambda_x = \{e^* \in e(G^*) \,:\, x \in [\phi^*(e^*_+), \phi^*(e^*_-)]\}$ so that once again λ_x is a cut-set in G^* separating s^* from t^*. But then $\mu^*_y \cap \lambda_x \neq \emptyset$ and there exists a square containing the point (x, y). To show that each square has disjoint interior simply note that

$$\text{area}(R) = \lambda(i) = \mathcal{E}(i) = \sum_{e \in e(G)} |s_e|^2$$

so that the area of the rectangle is exactly the sum of the areas of the squares. □

We shall now describe a converse to the previous theorem. Let S be a squaring of a rectangle, say of $[0, \lambda] \times [0, 1]$. Let $L(S) = \{\ell_i\}_{i=1}^N$ be the set of all maximal (in terms of length) horizontal unbroken lines in the edges of squares in S. Define the *p-net* of S as the edge-rooted planar map $\mathcal{G}(S)$ constructed as follows: place a vertex at the center of each line in $L(S)$. For each square s, add an edge connecting the vertices corresponding to the upper and lower borders of s. Finally, let s be the vertex corresponding to the top line and t the bottom line, and connect them via an edge. See Figure 9 for an example.

Theorem 25 ([21], Theorem 4.31). *Let S be a squaring and $\mathcal{G}(S)$ be its p-net, then $\mathcal{S}(\mathcal{G}(S)) = S$.*

Proof. By rescaling we may suppose that S is a squaring of $[0, \lambda] \times [0, 1]$ and we let $\mathbf{G} = \mathcal{G}(S)$. Define a function $\phi : v(G) \to [0, 1]$ by setting $\phi(v)$ to be the height of the line corresponding to v in the p-net construction. We now claim that ϕ is a harmonic function with boundary $\phi(s) = 1$ and $\phi(t) = 0$.

Indeed, first note that the horizontal length of the line corresponding to a vertex v is equal to the sum of the sizes of all squares lying directly above or directly below the line (see Figure 9 for an example). Furthermore, for an edge $e \in \vec{e}(G)$ the value $\phi(e_+) - \phi(e_-)$ is equal to the size of the square in S corresponding to e. Combining these two observations tells us that for any $v \in v(G)$ such that $v \neq s, t$,

$$0 = \sum_{w \sim v} \phi(w) - \phi(v) = \sum_{w \sim v} \phi(w) - \deg(v)\phi(v)$$

which shows that ϕ is harmonic, as claimed. Now we may define the unique Kirchoff pair (ϕ, i) and it follows immediately by Proposition 24 that $\mathcal{S}(\mathbf{G}) = S$. $\square$

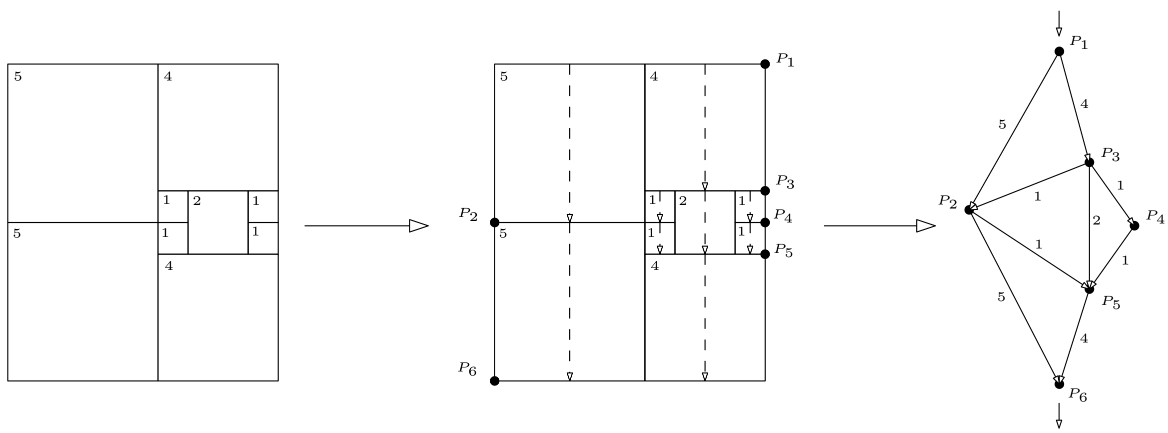

Figure 9: A squaring and its corresponding graph $\mathcal{G}(S)$, re-scaled so that current is an integer.

We can now see that the sizes of the squares of a squaring are determined by the amount of current flowing through them when a potential difference of 1 is applied to the root vertices, and that the height of the squares are determined by the potential at the source of each directed edge. This interpretation allows us to view the squaring as corresponding to a random walk on $\mathcal{G}(S)$. In particular, the y-coordinate of each square is $\mathbb{P}_{e_+}(\tau_s > \tau_t)$, the probability of hitting s before t in a random walk starting at the source of $\vec{e}$. This interpretation in terms of random walks plays an absolutely crucial role in the later convergence proofs for random squarings.

We now briefly discuss the issues of degeneracy and composition. From the p-net construction, we see that we may associate to each vertex $v \in v(G)$ the horizontal line in the edges of $\mathcal{S}(\mathbf{G})$. In particular, for $v \in v(G)$ let $\ell(v)$ be the maximal horizontal line in the bottom edges of squares which correspond to incoming edges of v (see Figure 9). The dual to $\ell(v)$ is an association between faces of G and vertical lines of $\mathcal{S}(\mathbf{G})$. For $f \in f(G)$ we let $\ell(f) = \ell(f^*)$ where f^* is the vertex corresponding to f in $\mathbf{G}^*$.

Lemma 26 ([2], Lemma 2.3). *If $\mathbf{G} = (G, st)$ is finite and 3-connected, then for all $v \in v(G)$ and $f \in f(G)$, the lines $\ell(v)$ and $\ell(f)$ have non-zero length.*

Proof. Suppose that for some $w \in v(G)$ the line $\ell(w)$ were a single point $z \in \mathbb{R}^2$. Any square s_e connected to w must also have 0 width and hence must be the trivial square z. Let $U = \{u \in v(G) : z \in \ell(u) \text{ and } \ell(u) \text{ has non-zero length}\}$, clearly we must have $|U| \leq 2$ since at most two non-zero horizontal lines can share an endpoint. Now U must be a cut-set separating w from s and t again since any square whose corresponding edge is incident to a zero length vertex must itself be of length zero. Since there exists a cut-set of size 2, G is not 3-connected, a contradiction. The facial argument follows by duality. □

In this squaring, the degeneracy condition is essentially reversed when compared to the Schramm construction, the composition is "easy" while the degeneracy condition is difficult to check. The non-composition condition can be seen to be equivalent to the following fact, which follows from the previous lemma.

Fact 27. Let S be a squaring, if S is composite then there exists a graph $\mathbf{G}$ which is not 3-connected satisfying $\mathcal{S}(\mathbf{G}) = S$ and if G has a 2-connected subgraph then $\mathcal{S}(\mathbf{G})$ is composite.

If S were non-degenerate and $\mathcal{G}(S)$ was 3-connected then this would immediately imply that S were non-composite. It is also easy to see that a squaring has a unique inverse given by its p-net if it is non-degenerate. Indeed, all maximal horizontal lines could be bijectively mapped to vertices of G and vertical lines would correspond precisely to faces of G. We furthemore have, again by Lemma 26, the following fact.

Fact 28. For a squaring S, the number of 3-connected graphs in $\mathcal{S}^{-1}(S)$ is less than 3^d where d is the number of degenerate points of S.

Proof. Let $G \in \mathcal{S}^{-1}(S)$ be 3-connected, and suppose that $z \in \mathbb{R}^2$ were a point of degeneracy of S. Since G is 3-connected, $\ell(v)$ is non-degenerate for any v so that either two lines $\ell(u)$ and $\ell(w)$ intersect z, or only one does. In the first case, only two cases are possible by 3-connectedness, the first is that no edge connects u to w, or the second case is that a single edge connects the two (note that such an edge must have 0 current). If only one line $\ell(u)$ intersect z then the structure of G around u is completely determined by S and we are done. □

This raises the following question.

Question 29. Is there an easy way to determine the degeneracy of $\mathcal{S}(\mathbf{G})$ directly from $\mathbf{G}$?

Provided one could find a straightforward way to compare $R(S)$ to $\mathcal{G}(S)$, we would have a positive answer to both this question and the similar one in the previous section. A quick attempt at the problem shows that this is a decidedly non-trivial task.

2.3 Squarings of Cylinders

The BSST squaring is slightly limiting in the fact that it requires us to choose an edge instead of any two arbitrary vertices. We now present a slight generalization of the BSST squaring in such a way that allows us to construct squarings of *cylinders*, and where G can be arbitrarily bi-rooted. This process was first described by Benjamini and Schramm in [6] to prove properties of harmonic functions on transient graphs. We sketch the construction, without proving its correctness.

Let $S = \{s_1, s_2, ...\}$ be a collection of closed 2-dimensional smooth manifolds in $\mathbb{R}^3$ which are isometric to squares in $\mathbb{R}^2$. In other words, for any s_i, there exists a smooth function $\phi_i : s_i \to \mathbb{R}^2$ such that $\phi_i(s_i) = [0, s] \times [0, s]$ for some s and $d(\phi_i(x), \phi_i(y)) = d_{s_i}(x, y)$ for all $x, y \in s_i$ where d_{s_i} is the restriction of the Euclidean metric onto s_i viewed as a sub-manifold of $\mathbb{R}^3$ (ie. restricting paths to s_i). We say that S is a squaring of a cylinder if

$$\overline{\bigcup_{i=1}^{\infty} s_i}$$

is isometric to $r \cdot \mathbb{S}_1 \times [0, h]$ for some constants $r, h > 0$ and for all i, j either $s_i \cap s_j = \emptyset$ or $s_i \cap s_j = \partial s_i \cap \partial s_j$ ie. squares only intersect at their boundary.

Let $\mathbf{G} = (G, s, t)$ be a finite bi-vertex-rooted planar map, and consider the unique energy minimizing unit st-flow on G. Let (G^*, f, g) be the doubly *face-rooted* dual graph where the roots are taken to be the corresponding root vertices in G. Let $c^* : \vec{e}(G^*) \to \mathbb{R}$ be a flow on G^* defined as follows: $|c^*_{e^*}|$ is equal to the flow $|c_e|$ in the edge e directed in such a way that e^* travels from the left of e to the right when e is oriented with the flow c. Let $\phi^* : v(G^*) \to \mathbb{R}$ be defined by setting $\phi^*(w) = 0$ for some vertex $w \sim f$ (the choice here is irrelevant up to rotations of the cylinder). Finally, for $v \in v(G)$, let

$$\phi^*(v) = \sum_{e_i \in p} c(\vec{e}_i) \pmod 1$$

where p is a path from w to v. We now construct a squaring of a cylinder based on $\mathbf{G}$ which we shall denote by $\mathcal{C}(\mathbf{G})$. Let $\mathcal{C}(\mathbf{G}) = \{s_e\}_{e \in e(G)}$ where each s_e is isometric to a square of size $|c_e|$, has its top left corner at position $(\phi^*(e_+), \phi(e_+))$ and has appropriate curvature to fit on the cylinder of radius r where r is given by the flow out of s (or equivalently into t). Note that in this case the radius corresponds analogously to the horizontal length of the BSST squaring. This defines a proper squaring of a cylinder (the details of which can be found in [6]) which is essentially the same as the BSST construction, but with root vertices instead of a root edge. Furthermore, this squared cylinder can be viewed as identifying together the vertical borders of the BSST squaring to make a cylinder, in the case that s and t lie on the same face.

We can alternatively construct $\mathcal{C}(\mathbf{G})$ by setting the vertical location and sizes of each square according to the values given by ϕ and requiring that a clockwise walk around a given radial line must preserve the corresponding orientation of the edges around the vertex in G. This is slightly easier to explain, but the fact that this is always possible itself requires the above construction.

Next say an infinite planar map G is *end-convergent* if it is one ended and transient. We know by Section 1.3 that the potential function with $\phi(s) = 1$ and $\phi(t) = 0$ can be written as $\phi(v) = \mathbb{E}_v(\phi(X_{\tau_{st}}))$ and this becomes $\phi(v) = \mathbb{P}_v(\tau_s < \tau_t)$. If $\mathbf{G} = (G, \rho)$ is end-convergent, then it should be true that setting $\phi(v) = \mathbb{P}_v(\tau_\rho = \infty)$ would yield an infinite squaring of a finite cylinder since $\phi(v)$ would be non-constant and the single "end" would correspond to the analogous sink $\phi(t) = 0$. Indeed Benajimini and Schramm proved the following.

Theorem 30 ([6], Theorem 4.1). *Let* $\mathbf{G} = (G, \rho)$ *be a end-convergent rooted planar map, and let* $\phi(v) = \mathbb{P}_v(\tau_\rho = \infty)$*. The mapping* $\mathcal{C}(\mathbf{G})$ *is an infinite squaring of a cylinder* $\eta\mathbb{S}_1 \times [0, 1)$.

In fact, the boundary $\eta\mathbb{S}_1 \times \{1\}$ behaves very much like a discrete version of the Poisson boundary for continuous functions in the case where G has bounded degree vertices. In particular, for $(x, t) \in \mathcal{C}$ let $v(x, t)$ be the lower potential vertex corresponding to the square in the cylinder containing (x, t).

Theorem 31 ([6], Theorem 5.1). *Let* $\psi : \mathbb{S}_1 \to \mathbb{R}$ *be a continuous function on the circle. If* $\mathbf{G}$ *is end-convergent and has bounded degrees, then there exists a bounded harmonic function* $g : v(G) \to \mathbb{R}$ *such that*

$$\lim_{t \to 1} g(v(x, t)) = \psi(x)$$

for almost all (with respect to Lebesgue measure) $x \in \mathbb{S}_1$.

Note in particular that such a theorem *cannot* hold in the recurrent case since any harmonic function would necessarily be bounded (and hence constant by Proposition 3). Furthermore, note that the transience property of G plays a crucial role in guaranteeing that $\mathcal{C}(\mathbf{G})$ is a finite cylinder as the resistance to infinity is finite.

Finally, for squarings of cylinders for recurrent graphs, one might attempt to mimick Theorem 30 for the case of a one-ended recurrent graph. Let $\mathbf{B}_n = (B_G(\rho, n)/\sim, \rho, \delta)$ where $\sim$ is the identification of all boundary points to a single vertex δ in such a way that $B_{B_n}(\rho, n-1) \cong B_G(\rho, n-1)$ as rooted planar maps, be the doubly rooted planar map which is the ball of radius n around ρ in G rooted at ρ and at the new point δ. We shall call the unit circumference cylinder $\mathcal{C}_1(\mathbf{G})$ of G the set defined as the limit (in the local Hausdorff sense in $\mathbb{R}^3$)

$$\mathcal{C}_1(G) = \lim_{n \to \infty} \mathcal{C}(\mathbf{B}_n)$$

whenever this limit exists. Discussion of this cylinder is postponed until section 4.1.

2.4 Other Geometric Representations of Graphs

Circle Packings

While squarings of rectangles have been of interest to mathematicians for quite some time, the related study of *circle packings* is significantly more mature and has been the target of much more research, in general due to its relation to the Riemann mapping theorem [22].

Let G be a finite planar map, a *circle packing* P for G is a collection $P = \{c_v\}_{v \in v(G)}$ of circles such that any two circles which intersect only do so at their boundary, and two circles intersect if and only if their corresponding vertices are adjacent. For a packing P we define as before its contacts graph $R(P)$. We now sketch a proof (due to [7] but presented by [14]) that for a large class of graphs, there exists an essentially unique circle packing $\mathcal{P}(T)$ such that $R(\mathcal{P}(T)) = T$ for any T in this class.

A triangulation with boundary T is called a *complex* if two adjacent faces share only a single edge. A *label* $L : v(T) \to \mathbb{R}_+ \cup \{\infty\}$ for T is a function mapping each vertex to a positive number, which we shall view as a corresponding circle's radius. We allow the labels to take the value infinity if we are working in hyperbolic space. We also do not require that L corresponds to a real packing, and call L a *packing label* if it does.

It is easy to see that for any choice of radii r_1, r_2, r_3 there exists a triangle $T(r_1, r_2, r_3)$ such that for any two vertices v_i, v_j we have $d(v_i, v_j) = r_i + r_j$ and that this triangle is unique up to congruency. Given a label L for T, the *angle sum* function for L is the function $\theta_L : v(T) \to \mathbb{R}_+$ defined by

$$\theta_L(v) = \sum_{i=1}^{n} \theta_i$$

where θ_i is the angle at vertex v in the triangle $T(L(v), L(w), L(u))$ and where $u \sim v$, $w \sim v$ and w, u lie on the same face, see Figure 10.

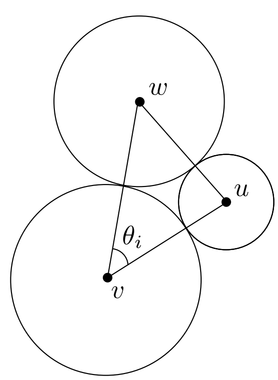

Figure 10: The definition of θ_i, construct the triangle using the labels for neighbours of v.

We furthermore have a stronger result:

Lemma 32 ([14], Lemma 2.1). *Let C_1, C_2, C_3 be a circle packing of a triangle with radii r_1, r_2, r_3 respectively. Let $\alpha = \angle 123$, $\beta = \angle 312$ and $\gamma = \angle 132$. If r_2 and r_3 are fixed and r_1 is increased continuously, then α and γ will increase continuously. If r_1 is fixed and r_2 or r_3 are increased, α decreases continuously.*

The proof is a simple straightforward calculation in both Euclidean and hyperbolic settings. We now arrive to the main characterization of packing labels, for which the proof appears in any good source on circle packings (for example [7]). We focus solely on the case of circle packings in the *hyperbolic plane* throughout the rest of this section.

Theorem 33 ([14], Lemma 2.3). *Let T be a complex, and let L be a label for T. The label L is a packing label for T if and only if $\theta_L(v) = 2\pi$ for every internal vertex $v \in v(T)$.*

We call a packing label L *maximal* if for each boundary vertex v we have $R(v) = \infty$.

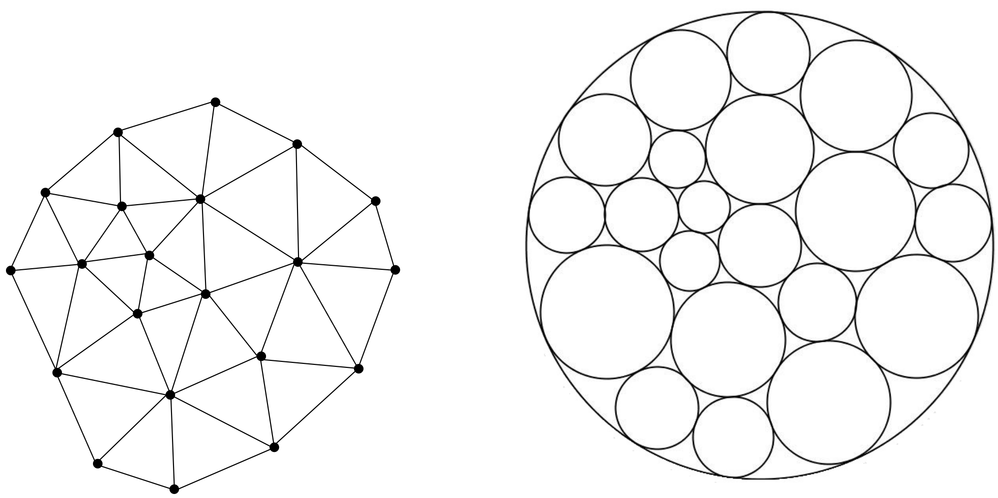

Figure 11: A complex on the left, and its corresponding maximal packing in hyperbolic space on the right.

Theorem 34. *For any complex T there exists a maximal hyperbolic packing label L which is unique up to Möbius transformation.*

The proof of this theorem is based on a discrete version of the Perron Method for constructing harmonic functions with prescribed boundary. Let T be a complex, we define $\mathcal{L}(T)$ to be the set of all labels L such that $\theta_L(v) \geq 2\pi$ for all $v \in v(T)$, this is the analogous Perron Family.

Lemma 35 ([14], Lemma 2.10). *Let T be a complex, and let L_1 and L_2 be labels in $\mathcal{L}(T)$, then*

1. $\max\{L_1, L_2\} \in \mathcal{L}(T)$,

2. $L_1(v) < \infty$ *for all interior vertices,*

3. $\mathcal{M}(v) := \sup_{L\in\mathcal{L}(T)} L(v)$ *is a maximal packing label in the hyperbolic plane.*

Proof. (1) By Lemma 32, fixing one radius in $T(r_1, r_2, r_3)$ and increasing the others increases the angle at v_1. In particular, $2\pi \le \theta_{L_i}(v) \le \theta_{\max\{L_1,L_2\}}(v)$ so that $\max\{L_1, L_2\} \in \mathcal{L}(T)$.

(2) Fix a vertex v and let $\{r_1, r_2, ..., r_k\}$ be the radii of the circles adjacent to the circle c_v, fix $i \in \{1, 2, ..k\}$ so that $\theta_{L_1}(v)$ is increasing in r_i and so is maximized when $r_i = \infty$. As all adjacent circles have equal radii, each angle is equal to $2\pi/k$ and hence $L_1(v) \le r_v = -\log(\sin(\pi/k)) < \infty$.

(3) Suppose that for some exterior vertex we have $\theta_{\mathcal{M}}(v) < \infty$ then by monotonicity we can increase it slightly and this will only increase the value of the angle sums of interior vertices so that we have in particular that $\mathcal{M}(v) = \infty$ for exterior vertices. Now suppose that for some interior v we have $\theta_{\mathcal{M}}(v) > 2\pi$, then by the continuity in Lemma 32 we can increase $\mathcal{M}(v)$ slightly, without decreasing $\theta_{\mathcal{M}}(v)$ to below 2π, contradicting maximality. $\square$

It now remains to show that the set $\mathcal{L}(T)$ is non-empty, for then we would have a maximal packing by the above lemma. The proof is a straightforward two case induction argument: in the first case there are no interior edges of K where both vertices are boundary vertices and in the second case there is at least one such edge. The details can be found in [14] and are omitted.

Now we have constructed a circle packing on the hyperbolic plane, what happens if we want a circle packing in the Euclidean plane? This is easy since we can "flatten" the hyperbolic plane to gain proper Euclidean circles. Indeed, simply embed the hyperbolic plane into the Poincare hyperbolic disk. We in fact have the Koebe-Andreev-Thurston theorem.

Theorem 36. *Let T be a complex, then there exists a circle packing in $\mathbb{R}^2$, $\mathcal{P}(T)$ such that $R(\mathcal{P}(T)) = T$ and this packing is unique up to Möbius transformations.*

Now let G be an arbitrary planar map and consider the graph $\mathcal{T}(G)$ constructed by adding a new vertex to each face of G and connecting this new vertex to each vertex incident to the corresponding face. It is easy to check now that $\mathcal{T}(G)$ is in fact a complex and thus the circle packing $\mathcal{P}(\mathcal{T}(G))$ is well-defined. Now from $\mathcal{P}(\mathcal{T}(G))$, remove those circles corresponding to the new facial vertices in $\mathcal{T}(G)$ to yield a new circle packing, call it $\mathcal{P}(G)$. It is immediate that $R(\mathcal{P}(G)) = G$, however, if G is not a triangulation then there is no guarantee of uniqueness for the packing $\mathcal{P}(G)$, so at best we can conclude that for any planar map G there exists a circle packing whose tangency graph is G.

Tutte Embedding

While not a packing type embedding, the Tutte embedding is quite similar in construction to that of the BSST squaring.

Consider the following problem: We are given a planar map G and wish to embed G in the plane in such a way that each edge of G is a straight line and all faces are convex. We call such an embedding a *convex straight-line embedding*. Of particular interest are those graphs G which are 3-vertex-connected in which case we have the following theorem due to Tutte.

THEOREM 37. *[Tutte Embedding] Let G be a 3-connected planar map, then for any specification of the external face as a convex polygon, there exists a straight-line embedding of G whose external face is as prescribed and every interior face is convex.*

This embedding is precisely specified by harmonic functions on G. In particular, let $\Phi : v(G) \to \mathbb{R} \times \mathbb{R}$ be a function on the vertices of G, we shall call Φ *harmonic* if it is harmonic in both coordinates. We know that for any specified choice of boundary, there exists a unique harmonic function Φ with the desired boundary. Let Φ be a harmonic function with boundary given by the positions of the external face, and place each vertex $v \in v(G)$ in the plane at coordinates given by $\Phi(v)$. We must now show that if the external face is convex then this embedding is the desired embedding in the theorem, and we follow the presentation by [24].

Proof of Theorem 37. We first prove that all interior vertices are mapped to the interior of the prescribed polygon. Fix a boundary edge and write n for the inward pointing normal. Consider the function $g(x) = \langle x, n \rangle + c$ where c is chosen such that $g(x) = 0$ for any point on the chosen boundary edge. As the boundary face is convex, for each other boundary vertex we have $g(\Phi(v)) > 0$. We now claim that $g(\Phi(\cdot))$ is harmonic on G. By a simple calculation,

$$\begin{aligned} \frac{1}{\deg(v)} \sum_{w \sim v} g(\Phi(w)) &= \frac{1}{\deg(v)} \sum_{w \sim v} (\langle \Phi(w), n \rangle + c) \\ &= \left\langle \frac{1}{\deg(v)} \sum_{w \sim v} \Phi(w), n \right\rangle + c \\ &= \langle \Phi(v), n \rangle + c = g(\Phi(v)). \end{aligned}$$

for internal vertices v. By the Maximum principle we must have that $g(\Phi(\cdot)) \geq 0$ for all internal vertices v and applying this to every boundary edge yields the claimed property.

We now prove that each face is convex. Suppose that some face f were not convex, then there exists a line ℓ passing through 4 edges of f. Let $g(x)$ be as before, but this time with the line ℓ, and oriented arbitrarily. Now each of the four edges travel from one side of ℓ to the other. This implies that g orients the edges so that traveling around the face clockwise, at least two clockwise oriented edges separate two counter-clockwise oriented edges. However, this contradicts the dual to Fact 17 and f must be convex.

It finally remains to prove that the graph is in fact planar. The proof of this fact is not technical, but tedious so a simple sketch will be presented (details can be found in [24]). First, it can be shown that every vertex of G is a planar wheel in the embedding, again by an appeal to Fact 17. This shows that any edge belongs to two disjoint faces. It thus remains to show that *every* face is disjoint, which follows immediately from the observation that the boundary faces are disjoint and the disjointness of adjacent faces. $\square$

The existence of an embedding of a planar graph G where edges are straight lines is not difficult to show (this claim is also known as Faye's Theorem). Indeed, suppose that every face of G is a triangle (so that it is maximally planar) then we can show that it has the desired embedding by induction. For any face x, y, z, it is claimed that we can embed G so that the external face is x, y, z. It is easy to check by Euler's formula that some vertex v must have at most degree 5 which is not one of x, y, z. Let G' be $G - v$ which is again made maximally planar by adding edges. The graph G' has an embedding with x, y, z as the outer face by induction, remove the new edges that were added to $G - v$ and add v again. There clearly exists a point in the face where v can connect to all other vertices by straight lines (since there are at most 5 vertices), so we are done. The problem with this method is that there is very little control on the size of edges of the embedding so that certain areas of the embedding can be extremely "clumped" while others can be quite sparse. The Tutte embedding is in general a suitable compromise between computability and visual appeal.

Tutte showed in [25] that his embedding can be found in another way. Embed G arbitrarily (except for the outer face which we require to be a convex polygon) in the plane and imagine that each edge is a spring with stiffness 1, the total energy of the system is then given by

$$\frac{1}{2} \sum_{e \in e(G)} \ell(e)^2$$

where $\ell(e)$ is the length of the edge e in the embedding.

Theorem 38 (Tutte's Spring Theorem). *Tutte's embedding minimizes the energy above over all embeddings of G in the plane with prescribed outer face.*

For large graphs, Tutte's spring theorem tells us that we can approximate Tutte's embedding by simulating springs as edges. This is often faster than calculating the harmonic functions explicitly, especially for graphs with extremely large vertex numbers when an approximation will suffice.

3 Random Squarings

This section comprises the bulk of this thesis and is based on original work by Louigi Addario-Berry and the author [2]. The observations in the electrical network squaring section suggest that we focus our attention on 3-connected planar maps. Write $\mathcal{H}_n$ for the set of 3-connected, edge-rooted planar maps with $n+4$ edges. We consider a random edge-rooted graph with $n+4$ edges, say $\mathbf{G}_n$ chosen uniformly over a set $\mathcal{G}_n \supset \mathcal{H}_n$ which contains non-3-connected graphs. However, 3-connected graphs will be dense enough in $\mathcal{G}_n$ that $\mathbf{G}_n$ has uniformly positive probability of belonging to $\mathcal{H}_n$ and conditional on being 3-connected, is uniformly distributed over all such graphs. We prove the following theorem.

Theorem 39 ([2], Theorem 1.1). *There exists an explicitly defined sequence* $(\mathbf{S}_n, n \geq 1)$ *of random squarings of rectangles with the following properties.*

1. $\mathbf{S}_n = S(\mathbf{G}_n)$*, where* $\mathbf{G}_n$ *is an* $(n+4)$*-edge random planar map whose law is given in Section 3.1,*

2. $\mathbf{S}_n$ *converges almost surely for the Hausdorff distance to a compact limit* $\mathbf{S}_\infty$*, which a.s. has exactly one point of accumulation*[3].

3. $\mathbf{S}_\infty$ *has the law of* $\mathcal{S}(\mathbf{G}_\infty)$*, where* $\mathbf{G}_\infty$ *itself has the law of the uniform infinite* 3*-connected planar map.*

In particular, (3) implies that the graph $\mathbf{G}_\infty$ is the distributional local-weak limit of uniformly random 3-connected planar maps. This theorem is proved in 3 steps. We first prove the almost sure convergence of a sampling procedure used to generate the infinite uniform 3-connected planar map. From such convergence will yield convergence of a sequence of harmonic functions which will then imply the Hausdorff convergence of the squarings. Finally, the almost sure existence of a single point of accumulation is proved using a result by He and Schramm.

3.1 Sampling

In this section we will describe the process which will generate the squarings with the desired law. Most importantly, we will need to sample 3-connected maps *uniformly* over a fixed number of edges. The main benefit of the sampling process is its concreteness, the construction can be carried out by hand if required. This allows us to perform an analysis showing that we can couple $\mathbf{G}_n$ and $\mathbf{G}_{n+1}$ so that $\mathbf{G}_{n+1}$ is obtained from $\mathbf{G}_n$ in a "local" way.

Binary Tree Closure

A *corner* of a planar map G is an ordered pair of edges (e, f) which share a face and vertex, and such that f follows from e clockwise around the shared vertex. We write $C(G)$ for the set of corners of G.

[3] A accumulation point in $\mathbb{R}^2$ is a point such that any open neighborhood of it contains infinitely many squares.

A planar map is a *quadrangulation* if all of its faces have degree 4 and is a *quadrangulation of a hexagon* if it is a quandrangulation with a single face of degree 6. It is *irreducible* if any cycle of length 4 delimits a face.

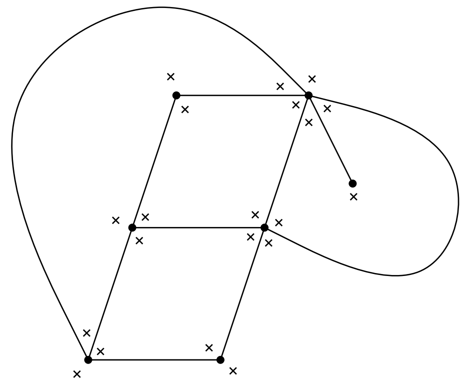

Figure 12: A planar map with corners denoted by $\times$.

A *counter-clockwise walk* around a tree T is a sequence of corners $\{c_i\}_{i=1}^N$ such that $c_N = c_1$, and if $c_i = (e_i, f_i)$ then $(e_i, f_i) = (e_{i+1}, e_i)$.

If a tree $\mathbf{T} = (T, st)$ is edge-rooted, then we define an orientation of the edges of T by setting for $e = \{u, v\}$, $e = uv$ if $d(u, s) < d(v, s)$ (ie. edges are oriented pointing *away* from s). For a vertex $v \in v(T)$, the set of vertices belonging to outgoing edges of v are called its *children*. An edge-rooted planar tree $\mathbf{T}$ is called a *binary tree* if the root edge has no children and each vertex has degree 3 or 1. If a corner is around a degree 1 vertex it is called a *bud corner* or else it is *internal*.

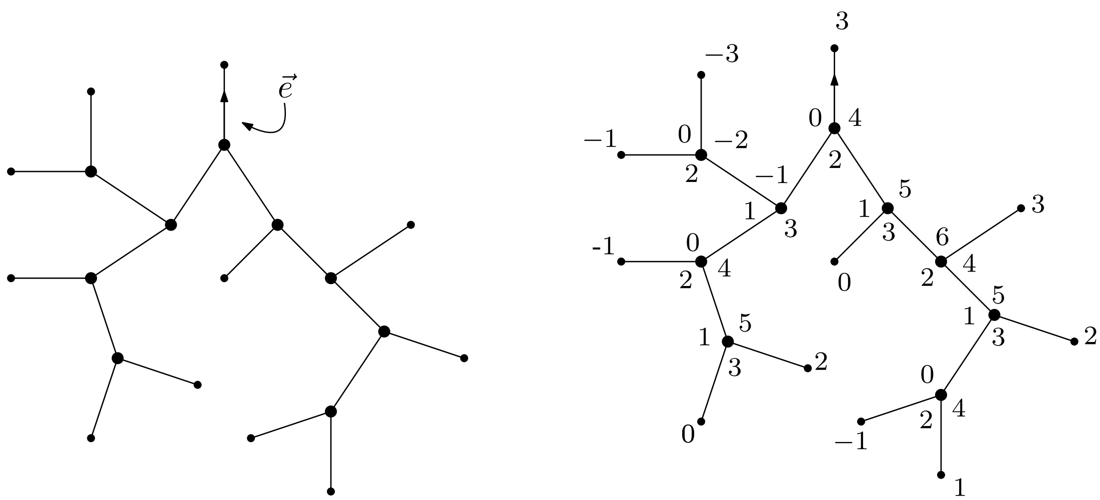

Figure 13: On the left is an example of a binary tree, small vertices denote leaves. On the right shows the corresponding labeling σ.

We now describe a "closure" operation which sends such binary trees to quadrangulations of hexagons. This operation is based on work by Fusy et al. [11], following the presentation from

Addario-Berry [1].

Let $\mathbf{T} = (T, \vec{e})$ be an edge-rooted binary tree, let $\{c_i\}_{i=1}^{N}$ be a counter-clockwise walk around the outside of $\mathbf{T}$ starting with $(f, \vec{e})$. Define a labeling $\sigma : C(T) \to \mathbb{Z}$ of the corners of $\mathbf{T}$ as follows:

(i) $\sigma(c_1) = 0$,

(ii) if c_i is a bud corner, $\sigma(c_{i+1}) = \sigma(c_i) + 3$,

(iii) if c_i is an internal corner, $\sigma(c_{i+1}) = \sigma(c_i) - 1$.

Let κ be a bud corner, if there exists an internal corner κ' such that $\sigma(\kappa') \leq \sigma(\kappa)$, we let $a(\kappa)$ be the first such corner following κ in the walk $\{c_i\}_{i=1}^{N}$. We now surround $\mathbf{T}$ by a hexagon H (ie. a cyclic graph with 6 vertices). The *closure* of $\mathbf{T}$ is the edge-rooted graph $\mathbf{G}$ constructed by identifying all bud corners κ with $a(\kappa)$ if it exists and identifying all other bud corners to the hexagon in such a way that all faces, save the unbounded one, have degree 4 and keeping the same root edge.

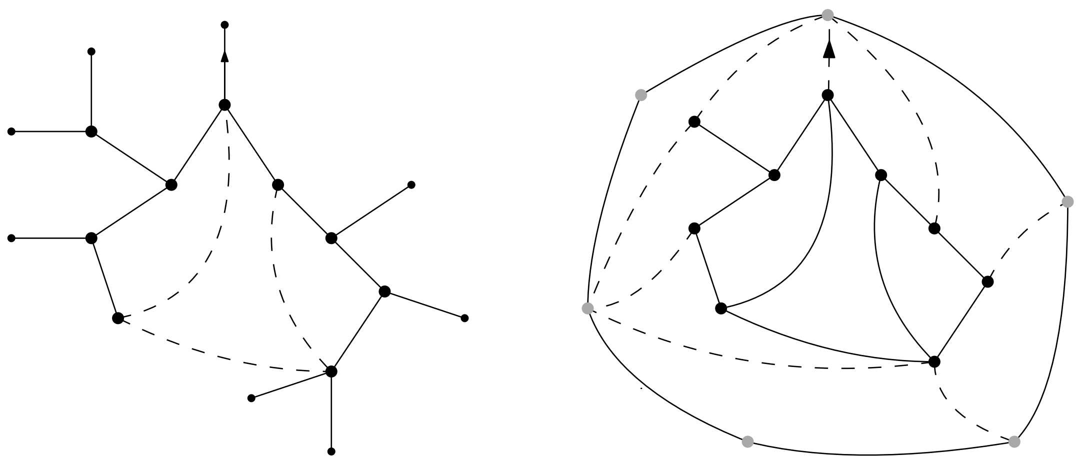

Figure 14: The closure of the previous example binary tree on the left and its closure in the hexagon on the right (grey vertices form the hexagon).

Theorem 40 ([11], Theorem 1.1). *The closure operation is a bijection from the set of edge-rooted binary trees to the set of edge rooted irreducible quadrangulations of hexagons.*

The inverse is explicitly defined in [11] but it will not be required for the proof of the Theorem 39.

Uniformly Random Trees

We now turn our attention to the *random* aspect of the theorem. We would like to explicitly sample binary plane trees in such a way that conditional on the number of edges, our binary tree is uniformly distributed over the set of all such binary trees. Our first result is a counting one, recall that a vertex-rooted binary tree is *full* if all vertices have degree 3 or 1 and the root has degree 2.

Lemma 41. *The number of vertex-rooted full binary trees with n internal nodes is the nth Catalan number c_n. More specifically,*

$$c_n = \frac{1}{n+1}\binom{2n}{n}.$$

Proof. Let t_i be the number of full rooted binary trees with i internal nodes, and let

$$T(x) = \sum_{i=0}^{\infty} t_i x^i$$

be the generating function for t_i. For any tree with i internal nodes, we can remove the root node to create two new trees, which we root at the node which was adjacent to the original root node. In particular, we now have two trees of size k and $i-k-1$ and since this can also be reversed (with two trees of size k and $i-k-1$ to get a tree of size i), we have the following relation:

$$T(x) = x(1 + T(x)^2).$$

Solving for $T(x)$ gives

$$T(x) = \frac{1+\sqrt{1-4x^2}}{2x}$$

and hence we find that

$$t_i = \frac{1}{i+1}\binom{2i}{i}$$

upon Taylor expansion of $T(x)$. $\square$

Let $(\mathbf{T}_i, i \geq 1)$ be a sequence of vertex-rooted full binary trees, we call $\mathbf{T}_i$ a *growth sequence* if the number of internal nodes of $\mathbf{T}_{i+1}$ is the number of internal nodes of $\mathbf{T}_i$ plus 1, and $\mathbf{T}_i \subset \mathbf{T}_{i+1}$. In other words, the sequence is a growth sequence if we can go from $\mathbf{T}_i$ to $\mathbf{T}_{i+1}$ by adding some leaves.

Theorem 42 ([17]). *There exists a growth sequence $(\mathbf{T}_i, i \geq 1)$ of full binary trees such that $\mathbf{T}_k$ is uniformly distributed over the set of vertex-rooted full binary trees with k internal nodes.*

Proof. For numbers $j_1, j_2 \in \mathbb{N}$, we let $B(j_1, j_2)$ be the set of binary trees with j_1+j_2+1 internal nodes, j_1 internal nodes lying in the left subtree of the root and j_2 internal nodes in the right subtree (not including the root). Let $\mathbb{T}_n = \{B(j_1, j_2) : j_1 + j_2 + 1 = n\}$ and consider the graph $\mathbb{G}_n$ constructed as follows: $v(\mathbb{G}_n) = \mathbb{T}_n \cup \mathbb{T}_{n+1}$ and an edge connects two vertices $B(j_1, j_2)$ and $B(j_1', j_2')$ if $j_1 \leq j_1'$ and $j_2 \leq j_2'$.

Suppose that there existed a flow $\phi_n : \vec{e}(\mathbb{G}_n) \to \mathbb{R}_{\geq 0}$ such that the out-flow of each vertex $B(j_1, j_2) \in \mathbb{T}_n$ is $\mathbb{P}(\mathbf{T}_n \in B(j_1, j_2))$ and the in-flow of each vertex $B(j_1', j_2') \in \mathbb{T}_{n+1}$ is $\mathbb{P}(\mathbf{T}_{n+1} \in B(j_1', j_2'))$. Then conditional on $\mathbf{T}_n$ we can construct the tree $\mathbf{T}_{n+1}$ by recursively choosing which (left

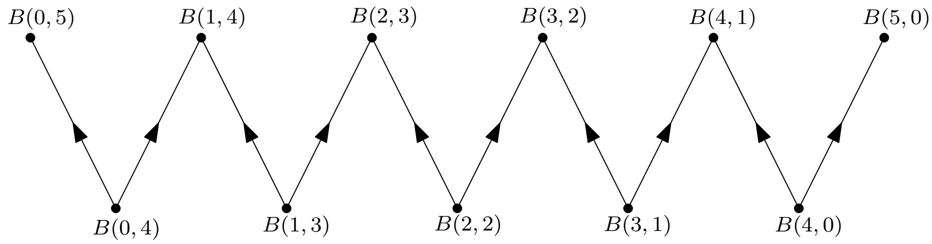

Figure 15: The graph $\mathbb{G}_4$ with desired flow direction, notice the zig-zag pattern will persist in $\mathbb{G}_n$ for any n.

or right) sub-tree to extend based on the $B(\cdot,\cdot)$ structure of $\mathbf{T}_n$. In particular, extend the left subtree with probability

$$\frac{\phi_n((B(j_1,j_2),B(j_1+1,j_2)))}{\lambda(B(j_1,j_2))}$$

where λ is the out-flow and similarly for the right tree. Repeat this process with each subsequent sub-tree to generate $\mathbf{T}_{n+1}$. For the base case, we simply start with the root vertex with two leaves attached. It is easy to check that $\mathbf{T}_{n+1}$ constructed in this way has the desired distribution. It thus remains to show that for all n such a flow ϕ_n exists. By attempting to construct the flow, it is enough to have that

$$\sum_{i=0}^{k}\mathbb{P}(\mathbf{T}_{n+1}\in B(i,n-i))\leq\sum_{i=0}^{k}\mathbb{P}(\mathbf{T}_n\in B(i,n-i-1))\leq\sum_{i=0}^{k+1}\mathbb{P}(\mathbf{T}_{n+1}\in B(i,n-i)).$$

Suppose that for all $i+j=n-1$,

$$\mathbb{P}(\mathbf{T}_{n+1}\in B(i,j))<\mathbb{P}(\mathbf{T}_n\in B(i-1,j))$$

and

$$\mathbb{P}(\mathbf{T}_{n+1}\in B(i,j))<\mathbb{P}(\mathbf{T}_n\in B(i,j-1))$$

then

$$\sum_{i=0}^{k}\mathbb{P}(\mathbf{T}_{n+1}\in B(i,n-i))<\sum_{i=0}^{k}\mathbb{P}(\mathbf{T}_n\in B(i,n-i-1))$$

and

$$\begin{aligned}\sum_{i=0}^{k}\mathbb{P}(\mathbf{T}_n\in B(i,n-i-1)) &= 1-\sum_{i=k+1}^{n-1}\mathbb{P}(\mathbf{T}_n\in B(i,n-i-1))\\ &< 1-\sum_{i=k+2}^{n+1}\mathbb{P}(\mathbf{T}_{n+1}\in B(i,n-i)=\sum_{i=0}^{k+1}\mathbb{P}(\mathbf{T}_{n+1}\in B(i,n-i))\end{aligned}$$

so that the flow ϕ_n would in fact exist. Now, computing directly,

$$\mathbb{P}(\mathbf{T}_{n+1} \in B(i+1,j)) = \frac{|B(i+1,j)|}{c_{n+1}} = \frac{c_{i+1}c_j}{c_{n+1}}$$

and similarly,

$$\mathbb{P}(\mathbf{T}_n \in B(i,j)) = \frac{c_i c_j}{c_n}$$

so we are reduced to showing that

$$\frac{c_{i+1}}{c_{n+1}} < \frac{c_i}{c_n}$$

which is trivial to check by Lemma 41. The reverse inequality follows similarly, and we can thus construct the flow ϕ_n so the theorem is proved. □

Now let $(\mathbf{T}_n, n \geq 1)$ be the growth sequence as in Theorem 42

$$\mathbf{T}_\infty = \lim_{n \to \infty} \mathbf{T}_n$$

converges almost surely. Furthermore, $\mathbf{T}_\infty$ is almost-surely one-ended: it has at least one end as it is locally finite, but can be modeled as a critical Galton-Watson tree [17] where any branch is finite with positive probability, implying that there can be at most one end.

Convergence of Closure

Throughout this section we let $\mathbf{T} = (\mathbf{T}_n, n \geq 1)$ be the growth sequence of uniformly random binary plane trees as described above, but which are made to be edge-rooted by adding an edge st to the left of the left child of the root, rooting so that it points away from the original root vertex. We now apply the closure mapping to the sequence $\mathbf{T}_n$ to yield a sequence $(\mathbf{M}_n, n \geq 0)$ of edge-rooted quandrangulations of hexagons and show that this sequence converges almost surely, once again with respect to the local-weak metric. Only a rough idea is presented here, the details can be found in [1].

We first focus on the effect of the growth of the binary tree in the closure mapping. As the growth procedure is applied to leaf nodes, it suffices to focus on these. The process is much more easily understood visually.

Let η be a leaf node which will change into an internal node in the growth process, Figure 16 shows a typical connection in the closure mapping, with an arrow indicating the closure target.

After adding two leave nodes, say μ_1 and μ_2 to η the only changes which can happen involve only η, the closure target and any other leaves which have the same target as η.

In particular, the leaves which closed on the old target now close onto η (ℓ in Figure 17 now closes at η instead) since there are now 3 consecutive internal corners in the counter-clockwise walk from η

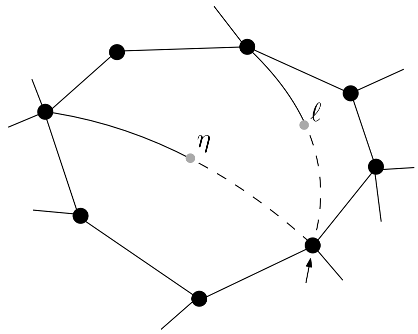

Figure 16: η denotes a leaf node to become an internal node in the growth process, the arrow points to closure target of η and ℓ is a leaf with the same target as η.

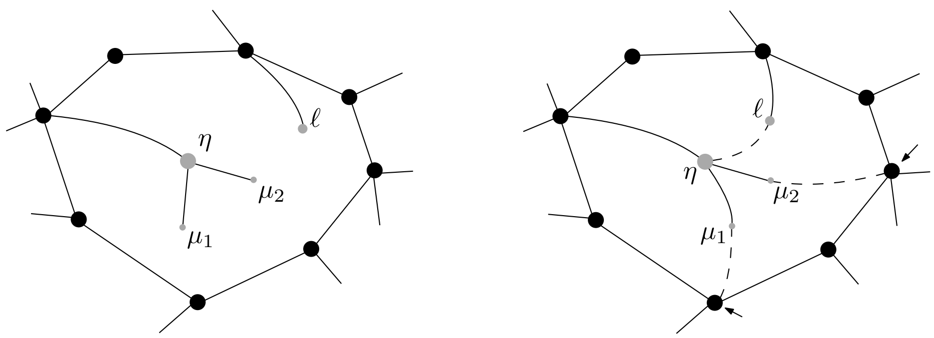

Figure 17: The left figure shows the local modifications before the closure operation, while the right figure shows the binary tree post-closure, with arrows indicating new closure targets.

to these leaves. The right leaf of η connects to the clockwise neighbor of the original target and the left child of η connects to the counter-clockwise neighbor of η. One can verify that any modification behaves in exactly this matter. The technical details to prove this are not too difficult, and can be found in [1] once again.

Because the growth operation induces a *local* modification of $\mathbf{M}_n$, we gain the first part of the following theorem.

Theorem 43. *The sequence of random quadrangulations* $(\mathbf{M}_n, n \geq 1)$ *converges almost surely to a random map* $\mathbf{M}_\infty$. *Furthermore,* $\mathbf{M}_\infty$ *is the closure operation applied to* $\mathbf{T}_\infty$ *and* $\mathbf{M}_\infty$ *is almost-surely locally finite.*

The second part of this Theorem boils down to showing that for any corner κ there almost-surely exists a corner $a(\kappa)$ with label less than κ. For details see [1].

Angular Mapping

We now describe a classical operation on planar graphs which is used extensively in planar map convergence theorems.

Let $\mathbf{G}$ be a vertex-rooted quadrangulation, bi-color the vertices of G white and black, taking the root to be black. The *angular closure* of $\mathbf{G}$ is the graph $A(\mathbf{G})$ with vertex set $v(A(\mathbf{G})) = \{v \in v(G) :$

v is black$\}$ and connecting two vertices together if they lie on the same face. The inverse is just as simple, add a vertex to each face of $A(\mathbf{G})$ and connect this vertex to every vertex in the same face, and finally removing the original edges. Once again we follow the exposition of [1] by Addario-Berry, applying the angular mapping to the sequence of quadrangulations $\mathbf{M}_n$.

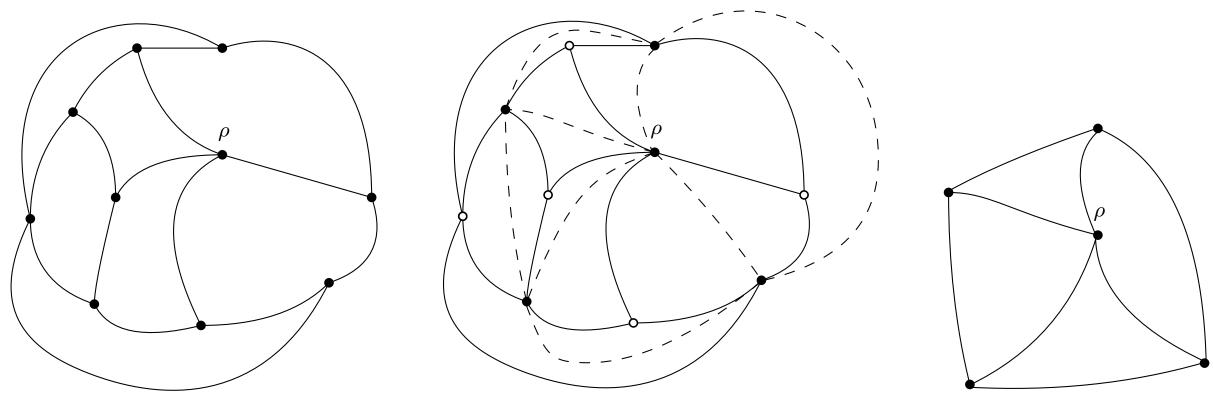

Figure 18: On the left a quadrangulation. The center figure denotes intermediate steps in the angular mapping (and its inverse). The right figure is the resulting planar map.

It is a result by Tutte that a quadrangulation $\mathbf{Q}$ is irreducible if and only if $A(\mathbf{Q})$ is 3-connected. To see this, suppose that $A(\mathbf{Q})$ were 2-connected, then we can view two faces of $A(\mathbf{Q})$ as separating some subgraph $H \subset A(\mathbf{Q})$ from the rest of $A(\mathbf{Q})$ (see Figure 19). Applying the inverse angular mapping connects the two cut vertices of H by a cycle of length 4, and hence $\mathbf{Q}$ is not irreducible.

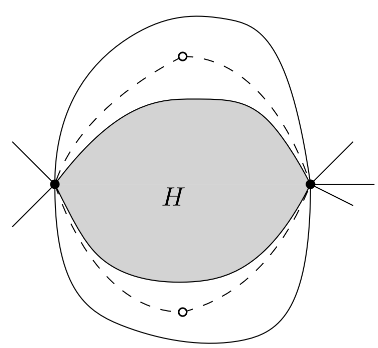

Figure 19: The subgraph H is connected at only two vertices which creates a 4-cycle in the angular map (dotted line).

To prove the converse, let $U \subset \mathbf{Q}$ be a subgraph contained in a cycle of length 4. As the white vertices in the angular mapping of $\mathbf{Q}$ will become faces, and two white vertices share two black vertices in the cycle around U it follows that the two black vertices belong to two faces, which separate the image of U from $A(\mathbf{Q})$ which immediately implies that $A(\mathbf{Q})$ is not 3-connected.

Let $\mathbf{Q}_n$ be the quadrangulation of a hexagon $\mathbf{M}_n$ from the previous section, augmented with an edge connecting opposing vertices of the outer hexagon, chosen uniformly at random from the 3 possible such edges. Note that $\mathbf{Q}_n$ need not be irreducible (but $\mathbf{M}_n$ is [11]), as the added edge could connect a path of length 3 from one vertex of the hexagon to the opposing vertex. However, it is reasonably likely that $\mathbf{Q}_n$ is irreducible, as the following theorem shows.

Theorem 44 ([11]). *Let $\mathbf{Q}_n$ be as before, then $\mathbb{P}(\mathbf{Q}_n$ is irreducible$) \to 2^8/3^6$ as $n \to \infty$ and conditional on irreducibility, $A(\mathbf{Q}_n)$ is uniformly distributed over the set of rooted 3-connected graphs with $n+4$ edges.*

Let $\mathbf{G}_n = A(\mathbf{Q}_n)$ so that from the almost-sure convergence of the $\mathbf{M}_n$ we have the following.

Proposition 45. *The sequence $\mathbf{G}_n$ converges almost surely to a random graph $\mathbf{G}_\infty$.*

Proof. Let $\mathbf{Q}$ be a quadrangulation and let $\{u, v\} = e \in e(A(\mathbf{Q}))$ be an edge in $A(\mathbf{Q})$. If we view u and v as belonging to $\mathbf{Q}$ then these vertices are joined by a path of length 2 in $\mathbf{Q}$ since u and v must both be black vertices belonging to the same face, and $\mathbf{Q}$ is a quadrangulation. In particular, any path in $A(\mathbf{Q})$ induces a path twice as long in $\mathbf{Q}$ so that for $x, y \in v(A(\mathbf{Q}))$ we have $2d_{A(\mathbf{Q})}(x, y) \geq d_{\mathbf{Q}}(x, y)$. However, this implies that balls in $A(\mathbf{Q})$ are entirely determined by balls of twice the radius in $\mathbf{Q}$ and since the root of $\mathbf{G}_n$ is the same as the root in $\mathbf{Q}_n$, local convergence of $\mathbf{G}_n$ follows immediately from convergence of $\mathbf{Q}_n$. $\square$

From the previous theorem, $\mathbb{P}(\mathbf{G}_n$ is 3-connected$) \to 2^8/3^6$ as $n \to \infty$ and conditional on 3-connectedness $\mathbf{G}_n$ is uniform over the set of 3-connected vertex-rooted graphs with $n+4$ edges. It turns out (see [1]) that the event that $\mathbf{G}_n$ is 3-connected is asymptotically independent[4] to the local structure around the root st of $\mathbf{M}_n$ (hence the subtitle *Comment s'enfuire de l'hexagone* in [1]). Let $\hat{\mathbf{G}}_n$ be uniformly distributed in the set of all 3-connected vertex-rooted planar maps with $n+4$ edges, then

Theorem 46 ([2], Theorem 7). *The sequence of random 3-connected graphs $(\hat{\mathbf{G}}_n, n \geq 1)$ converges in distribution to $\mathbf{G}_\infty$, and furthermore $\mathbf{G}_\infty$ is almost surely 3-connected.*

Furthermore, from [5] we have,

Lemma 47 ([5], Theorem 2.1 (a)). *For all $\epsilon > 0$ there exists a $B > 0$ such that for all $n \geq 1$,*

$$\mathbb{P}(\deg_{\hat{\mathbf{G}}_n}(\rho_n) = d) < B \cdot \left(\frac{1}{2} + \epsilon\right)^d.$$

Since this bound applies uniformly over n, it immediately implies that the degree of the root in $\mathbf{G}_\infty$ has exponential tail. Furthermore, note that since a planar graph G is 3-connected if and only if its dual is 3-connected, the random variable $\hat{\mathbf{G}}_n^*$ has the same distribution as $\hat{\mathbf{G}}_n$ (implying $\mathbf{G}_\infty = \mathbf{G}_\infty^*$ in distribution).

[4] Events A and B are asymptotically independent if $\mathbb{P}_n(A \cap B) \to \mathbb{P}_n(A) \cdot \mathbb{P}_n(B)$.

3.2 Convergence

Suppose now that we had an infinite recurrent graph G and a bounded harmonic function ϕ with finite boundary B. As the harmonic function is completely determined by a random walk on G, it should follow from the recurrence that any modification to G sufficiently "far away" from a vertex v should not affect the value of $\varphi(v)$ too much. It is a precise formulation of this observation which yields the following theorem.

Theorem 48 ([2], Theorem 3.2). *Let $(\mathbf{H}_n, n \geq 1)$ be a sequence of rooted graphs such that $\mathbf{H}_n = (H_n, \rho_n)$ converges to $\mathbf{H}_\infty = (H_\infty, \rho_\infty)$ with respect to the local weak metric and suppose that H_∞ is recurrent. Fix a finite subset $B \subset v(H_\infty)$ and let N be large enough that $B \subset v(H_n)$ for all $n \geq N$. Let $\varphi_n : v(H_n) \to \mathbb{R}$ be bounded harmonic functions which agree with each other on B for all $n \geq N$. Then for all $v \in v(H_\infty)$, $\varphi_n(v) \to \varphi_\infty(v)$ as $n \to \infty$.*

Proof. Throughout we assume that $n \geq N$ as described in the statement of the theorem. Fix a vertex $v \in v(H_\infty)$ and take N_0 so large that $v \in v(H_n)$ for $n \geq N_0$. Let $X^{(n)}$ be a random walk on $\mathbf{G}_n$ started at v and recall that the unique bounded harmonic function with boundary B is given by $\varphi_n(v) = \mathbb{E}_v(\varphi(X^{(n)}_{\tau_B}))$. Since B is finite,

$$\varphi_n(v) = \sum_{b \in B} \mathbb{P}_v\left(X^{(n)}_{\tau_B(X^{(n)})} = b\right) \cdot \varphi_\infty(b)$$

so that it suffices to show that $\mathbb{P}_v\left(X^{(n)}_{\tau_B(X^{(n)})} = b\right) \to \mathbb{P}_v\left(X^{(\infty)}_{\tau_B(X^{(\infty)})} = b\right)$ as $n \to \infty$.

Let $E_n(r)$ be the event that the random walk $X^{(n)}$ leaves the ball $B(v, r)$ before hitting B, ie.

$$E_n(r) = \{\tau_B > \tau_{B(v,r)^c}\}.$$

Let $\epsilon > 0$ be arbitrary and note that since H_∞ is recurrent we have $\mathbb{P}(E_\infty(r)) \to 0$ as $r \to \infty$, so now choose R sufficiently large that $\mathbb{P}(E_\infty(r)) < \epsilon$ for all $r \geq R$. Let N_1 be large enough that $B_{H_n}(v, R+1) \cong B_{H_\infty}(v, R+1)$ for all $n \geq N_1$ and note furthermore that for such n,

$$\mathbb{P}(E_n(R)) = \mathbb{P}(E_\infty(R)) < \epsilon.$$

Now,

$$\begin{aligned}\mathbb{P}(X^{(n)}_{\tau_B} = b) &= \mathbb{P}(E_n(R)^c, X^{(n)}_{\tau_B} = b) + \mathbb{P}(E_n(R), X^{(n)}_{\tau_B} = b)\\ &= \mathbb{P}(E_n(R)^c, X^{(n)}_{\tau_B} = b) + \mathbb{P}(E_\infty(R), X^{(n)}_{\tau_B} = b)\\ &< \mathbb{P}(E_\infty(R)^c, X^{(n)}_{\tau_B} = b) + \epsilon\\ &= \mathbb{P}(E_n(R)^c, X^{(\infty)}_{\tau_B} = b) + \epsilon\\ &\le \mathbb{P}(X^{(\infty)}_{\tau_B} = b) + \epsilon\end{aligned}$$

and a symmetric argument shows the converse inequality. Furthermore, since $\epsilon > 0$ was arbitrary the convergence follows. $\square$

Let A and B be two closed subsets of $\mathbb{R}^2$, we define the *Hausdorff distance* between A and B by,

$$d_H(A, B) = \max\{\sup_{a\in A}\inf_{b\in B} d(a, b), \sup_{b\in B}\inf_{a\in A} d(a, b)\}$$

in other words, the Hausdorff distance between the two sets is the maximum smallest distance between two points in A and B. Because A and B are closed, $d_H(A, B) = 0$ if and only if $A = B$ so that d_H defines a metric on closed subsets of $\mathbb{R}^2$.

By noting that the heights (and hence sizes) of squares corresponding to the BSST squaring are given by harmonic functions with boundary st, we can then apply the previous theorem and show that the squarings converge in the Hausdorff sense.

Proposition 49 ([2], Proposition 3.3). *Let* $(\mathbf{H}_n, n \ge 1)$ *be a sequence of locally finite, edge-rooted recurrent planar maps* $\mathbf{H}_n = (H_n, st)$ *such that* $\mathbf{H}_n - \{s, t\}$ *is connected for all* $1 \le n \le \infty$ *and such that* $\mathbf{H}_n \to \mathbf{H}_\infty$ *in the local weak sense. Then* $\mathcal{S}(\mathbf{H}_\infty)$ *is a squaring of a rectangle, and* $\mathcal{S}(\mathbf{H}_n) \to \mathcal{S}(\mathbf{H}_\infty)$ *as* $n \to \infty$*, for the Hausdorff distance.*

Proof. Suppose that $\mathbf{H}_n$ are finite and let $\lambda_n = \lambda(i_n)$ where i_n is the flow on $\mathbf{G}_n$ when viewed as a circuit. Let $x_n = \phi^*_{H^*_n}$ and $y_n = \phi_{H_n}$ be the respective dual and original harmonic functions defined as in Theorem 19. For $e \in e(H_\infty)$ let n be sufficiently large that $e \in e(H_n)$, then in this case the square s_e is completely determined by the values of $x_n(e^*_+), x_n(e^*_-)$ and $y_n(e_+), y_n(e_-)$ where we know that these last two sequences converge as $n \to \infty$ by Theorem 48. Recall furthermore from Fact 16 that

$$\lambda_n = \deg_{H_n}(s) - \sum_{w\sim s} y_n(s) - 1$$

so we furthermore have that $\lambda_n \to \lambda$ as $n \to \infty$. Let $\hat{x}_n$ be the unique harmonic function on H^*_n such that $\hat{x}_n(s^*) = 1$ and $\hat{x}_n(t^*) = 0$. By uniqueness and linearity of harmonic functions $\lambda_n\hat{x}_n = x_n$ which converges point-wise since both λ_n and $\hat{x}_n$ do. It follows then that $\mathcal{S}(\mathbf{H}_n) \to \mathcal{S}(\mathbf{H}_\infty)$ in the

Hausdorff metric. Furthermore, as all squares are disjoint in $\mathcal{S}(\mathbf{H}_n)$ the same must be true in $\mathcal{S}(\mathbf{H}_\infty)$. It thus remains to show that $\mathcal{S}(\mathbf{H}_\infty)$ is in fact a square tiling of a rectangle. By the energy formulation of resistance, the area of the "squaring" $\mathcal{S}(\mathbf{H}_\infty)$ is equal to the sum of all currents squared in the edges of $\mathbf{H}_\infty$, which is precisely λ_∞. Since square sizes are disjoint, the total area of the squarings converge to λ_∞ and the rectangle $[0, \lambda_n] \times [0, 1]$ converge, it must be that $\mathcal{S}(\mathbf{H}_\infty) = \lim_{n\to\infty} \mathcal{S}(\mathbf{H}_n)$ and that $\mathcal{S}(\mathbf{H}_\infty)$ is a squaring of the rectangle $[0, \lambda_\infty] \times [0, 1]$.

In the case that $\mathbf{H}_n$ are infinite, we first view $\mathbf{H}_n$ as the limit of finite graphs so that $\mathcal{S}(\mathbf{H}_n)$ are in fact squarings of rectangles and an almost verbatim argument as above shows that $\mathcal{S}(\mathbf{H}_n) \to \mathcal{S}(\mathbf{H}_\infty)$ in the Hausdorff distance. $\square$

We now write $\mathbf{S}_n = \mathcal{S}(\mathbf{G}_n)$ where $\mathbf{G}_n$ is the sequence of random planar maps described in the previous section.

Corollary 50. *The squarings $\mathbf{S}_n$ converge almost-surely to a squaring $\mathbf{S}_\infty$ as $n \to \infty$. Furthermore, $\mathbf{S}_\infty$ has infinitely many non-trivial squares and for each vertex v, $\ell(v)$ has non-zero length.*

Proof. We know that the vertex degree of $\mathbf{G}_\infty$ has exponential tail by Lemma 47 so that Theorem 18 implies that $\mathbf{G}_\infty$ is almost-surely recurrent. In particular, Theorem 48 then implies the convergence $\mathbf{S}_n \to \mathbf{S}_\infty$. Furthermore, $\mathbf{G}_\infty$ is almost-surely 3-connected so that the lines $\ell(v)$ or $\ell(f)$ have positive length by Lemma 26 and since any positive length line must border an non-trivial square, there must be infinitely many of these. $\square$

3.3 Local Properties

In this section we shall prove that the random squaring $\mathbf{S}_\infty$ presented in the previous section almost surely has a single point of accumulation. Before doing so, we must introduce some more terms.

Recall from before that a *packing* P is a collection of measurable subsets of $\mathbb{R}^2$ such that any two P_i have disjoint interiors. The *contacts graph* of P is the graph $R(P)$ with vertex set $\{P_i : i \in I\}$ and edges connecting vertices if the corresponding sets intersect. The packing P is *locally finite* if $R(P)$ is. Note that this contacts graph differs from the one in 2.1 in that degenerate points are not given a vertex, and we simply cross the edges. A measurable set $A \subset \mathbb{R}^2$ is called *δ-fat* if for any $x \in A$ and $r > 0$,

$$\mathrm{Vol}(A \cap B(x, r)) \geq \delta \cdot \mathrm{Vol}(B(x, r))$$

in otherwords, no part of any set is too "skinny". A packing P is the called *fat* if there exists a $\delta > 0$ such that all sets in P are δ-fat. It is called *well-separated* if for each $P_i \in P$, the set of all neighbours of P_i contains a Jordan curve containing P_i in its interior. A *point of accumulation* of a packing P is a point x such that any open set containing x contains infinitely many sets of P. Note furthermore that $\mathbf{S}_\infty$ almost-surely has at least one limit point since it is an infinite squaring of a *compact* rectangle.

We now turn our attention to graphs. Again, a graph is *one-ended* if for any finite subset $U \subset v(G)$, the graph $G - U$ has a single infinite component. Intuitively it would seem that in order for a packing to have a single point of accumulation, it should suffice to have that its contacts graph be locally finite and one-ended, however it is easy to see that one can conclude that the set of limit points is at best connected. Indeed, if it weren't then a finite number of squares could separate two limit points, implying that $R(S)$ were not one-ended, but the converse does not hold.

In order to use a result by He and Schramm, we must introduce the notions of edge and vertex parabolicity. We say that a graph G is *vertex-parabolic* if there exists a function $m : v(G) \to \mathbb{R}_{\geq 0}$ such that for any infinite path γ in G, we have

$$\sum_{v \in \gamma} m(v) = \infty.$$

and

$$\sum_{v \in v(G)} m(v)^2 < \infty.$$

A graph is *edge-parabolic* if there exists a function $m : e(G) \to \mathbb{R}_{\geq 0}$ such that the above hold for edges instead of vertices. A theorem by He and Schramm gives us a sufficient condition for a packing to have a single limit point.

Theorem 51 ([15], Theorem 1.2). *Let $P = \{P_i \ : \ i \in I\}$ be a well-separated fat packing, and suppose that $R(P)$ is locally finite and one-ended. If $R(P)$ is vertex-parabolic then P contains a single point of accumulation.*

As the graph $\mathbf{G}_\infty$ is almost surely locally finite, is easy to see that the random squaring $\mathbf{S}_\infty$ is almost surely locally finite. Each set in the packing is a square so it is fat and well-separated. To apply the above theorem to $\mathbf{S}_\infty$ we must show that $R(\mathbf{S}_\infty)$ is vertex-parabolic and one-ended.

Proposition 52. *If G is edge-parabolic then G is vertex-parabolic.*

Proof. Let $m : e(G) \to \mathbb{R}_{\geq 0}$ be a function satisfying the definition for edge-parabolicity. Define $m' : v(G) \to \mathbb{R}_{\geq 0}$ as,

$$m'(v) = \sup\{m(e) \ : \ e \sim v\}$$

so that $m'(v)$ is the maximum over all m-edge values of edges incident to v. Now,

$$\sum_{v \in v(G)} m'(v)^2 = \sum_{v \in v(G)} (\sup\{m(e) \ : \ e \sim v\})^2 \leq \sum_{v \in v(G)} \sum_{e \sim v} m(e)^2 \leq 2 \sum_{e \in e(G)} m(e)^2 < \infty$$

so that it has finite mass. Now let γ be an infinite path in G,

$$\sum_{v \in \gamma} m'(v) = \sum_{v \in \gamma} \sup\{m(e) \ : \ e \sim v\} \geq \sum_{e \in \gamma} m(e) = \infty$$

so that m' satisfies the required property for vertex-parabolicity. $\square$

We now claim that a graph G is edge-parabolic if and only if it is recurrent. This will follow from the below theorem by Duffin [9] after a couple of technical details.

Theorem 53 ([9], Theorem 2). *The edge-extremal length between two nodes of a network is equal to the effective resistance between those nodes.*

Here we use edge-extremal length exactly as in Section 2.1 with edges instead of vertices. Define the (v, ∞)-extremal length of a graph G as

$$E(v,\infty) = \left\{\sup_m \inf_\gamma \frac{m(\gamma)}{\|m\|} \ : \ m : e(G) \to \mathbb{R}_{\geq 0},\ 0 < \|m\| < \infty\right\}$$

where γ is an infinite path starting at v.

Proposition 54. *For any $v \in v(G)$, there exists a metric $w : e(G) \to \mathbb{R}_{\geq 0}$ such that*

$$\frac{\inf_\gamma w(\gamma)}{\|w\|} = E(v,\infty)$$

satisfying $0 < \|w\| < \infty$.

Proof. Let

$$\mathcal{M} = \{m : e(G) \to \mathbb{R}_{\geq 0} \ : \ \inf_\gamma m(\gamma) \geq 1,\ \|m\| < \infty\}$$

and let $(m_n, n \geq 0)$ be a sequence of metrics such that

$$\frac{\inf_\gamma m_n(\gamma)}{\|m_n\|} \to E(v,\infty)$$

as $n \to \infty$ and $0 < \|m_n\| < \infty$. By rescaling the m_n (for n large $\inf_\gamma m_n(\gamma) > 0$) so that $\inf_\gamma m_n(\gamma) = 1$ (if $\inf_\gamma m_n(\gamma) = \infty$ for all n then we are done), we may suppose that they belong to $\mathcal{M}$. If $\|m_n\| > c > 0$ for all n, then the sequence belongs to a closed convex subset of $\mathcal{M}$ (take those elements such that $\|m\| \geq c$) and such a set has a unique element w of minimal norm. In this case,

$$\frac{\inf_\gamma m_n(\gamma)}{\|m_n\|} = \frac{1}{\|m_n\|} \leq \frac{1}{\|w\|} \leq \frac{\inf_\gamma w(\gamma)}{\|w\|}$$

and we are done. Now suppose that $\|m_n\| \to 0$ (so that $E(v,\infty) = \infty$) and let $(m_{n_k}, k \geq 0)$ be a sub-sequence such that $\|m_{n_k}\| \leq 2^{-k}$. Define,

$$w(e) = \sum_{k=0}^{\infty} m_{n_k}(e)$$

so that $\|w\| < \infty$ but $\inf_\gamma w(\gamma) = \infty$, and hence w satisfies the above equality. □

Combining this proposition with the previous theorem we get a nice equivalence.

Theorem 55. *An infinite graph G is recurrent if and only if it is edge-parabolic.*

Given a planar map G, we construct its *augmented graph* $U(G)$ as follows (see Figure 20): First, subdivide each edge of G into two by adding a new vertex in the middle of it (henceforth called a subdivision vertex). Next, add a new vertex to each face of G and connect it to all subdivision vertices incident to that face. Finally, in the resulting graph connect each pair of subdivision vertices lying on a common face within each face.

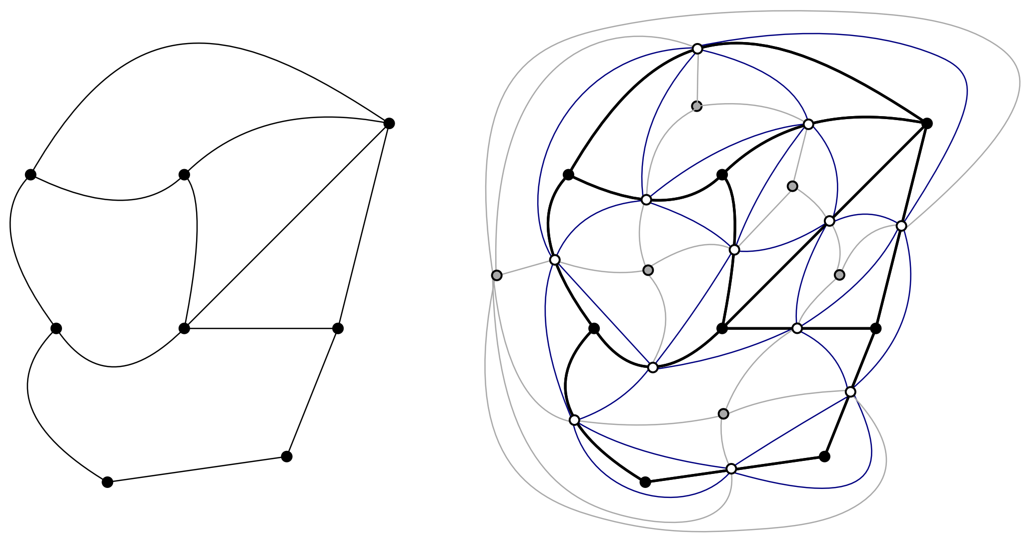

Figure 20: On the left a planar map. On the right is its augmented map. Dark blue edges denote edges between subdivision vertices, and grey edges are between facial and subdivision vertices. Subdivision vertices are coloured white, black vertices belong to G and grey vertices denote the facial vertices.

We define the square of a graph G to be the graph $G^2 = (v(G), e(G^2))$ where $e(G^2) = \{\{v, w\} : d_G(v, w) \leq 2\}$.

Proposition 56. *Let $\mathbf{G}$ be a 3-connected planar map, then $R(\mathcal{S}(\mathbf{G}))$ is isomorphic to a subgraph of $U(G)^2$.*

Proof. The vertices of the contacts graph correspond to squares, which in turn correspond to edges in G so we associate to each vertex of the contacts graph the corresponding subdivision vertex in $U(G)^2$. If two squares do not only meet at a point of degeneracy then they either share a horizontal or vertical line. Because of the identification of horizontal lines with vertices in G and vertical lines with faces of G, the two squares either share the same vertex or the same face, in which case they are connected by a path of length two in $U(G)$.

Suppose now that two squares share a point of degeneracy, then there are two cases: (i) an unbroken horizontal or vertical line passes through the point, in which case both squares share a vertex in G or a

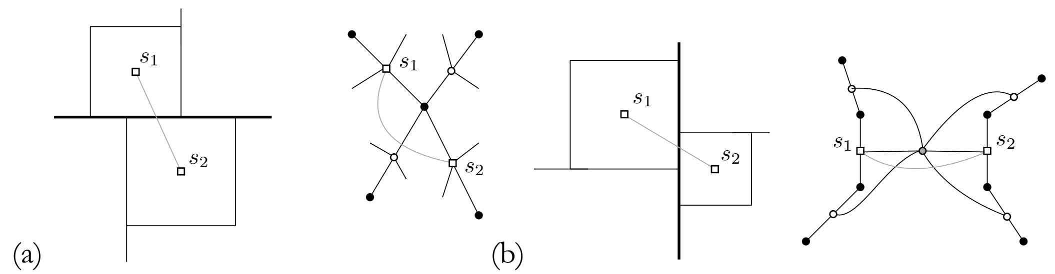

Figure 21: Two cases for the contacts graph structure in $U(G)$: (a) the squares share a proper horizontal edge or (b) the squares share a vertical edge.

face in G, otherwise (ii) no unbroken line passes through the point of degeneracy. Because the graph is 3-connected, it follows from Lemma 26 that we must only have an edge separating a face in G and connecting to two equipotential vertices (see Figure 22). In this case, the subdivision vertices are connected by a path of length two in $U(G)$ and we are done. □

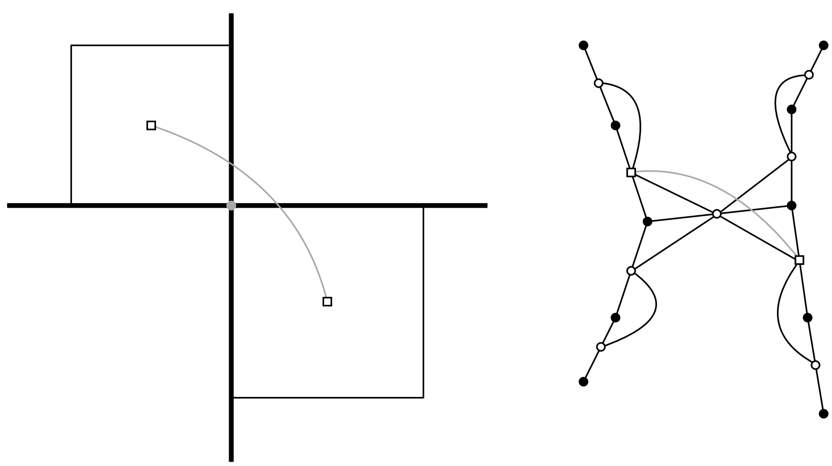

Figure 22: The case where two squares share a degeneracy point. Note that since G is 3-connected there must be only a single edge with 0 current.

The last piece required to show that $R(\mathcal{S}(\mathbf{G}))$ is vertex-parabolic is that $U(G_\infty)$ is in fact recurrent.

Theorem 57. *$U(G_\infty)$ is almost surely recurrent.*

Proof. Let $\hat{\mathbf{G}}_n = (\hat{G}_n, s_n t_n)$ be uniformly sampled from all 3-connected planar maps with $n+4$ edges and construct a new random graph $\hat{\mathbf{U}}_n = (U(\hat{G}_n), \hat{\nu}_n)$ where $\nu_n = s_n$ with probability 1/3 and $\nu_n = v_{s_n t_n}$ where $v_{s_n t_n}$ is the subdivision vertex of the edge $s_n t_n$ with probability 2/3. Now since $\hat{\mathbf{G}}_n$ converges in distribution to $\mathbf{G}_\infty$ it follows that $\hat{\mathbf{U}}_n$ converges in distribution to $\mathbf{U}_\infty = (U(G_\infty), \rho_\infty)$ (since the map $U : \mathcal{G} \to \mathcal{G}$ is continuous). It now remains to show that ν_n is chosen according to the stationary distribution on $\hat{U}_n$, since the recurrence will then follow immediately from Theorem 18. First note that the number of edges of $U(G)$ is $6|e(G)|$, since every edge of G corresponds to 6 other edges (two subdivision edges, two facial edges and two edges connecting subdivision vertice). As the dual of $\hat{G}_n$ has the same distribution as $\hat{G}_n$, the probability that we root at a facial or primal vertex is the probability we chose the edge upon which it lies, oriented away from the vertex (facial or primal) and rooting at the source. In particular, for $v \in v(G)$ the probability that $\hat{\nu_n} = v$ is $\deg(v)/12|e(G)|$ and for $f \in f(G)$ the same holds (which is precisely as required). It thus remains to check that subdivision vertices are chosen as roots with probability $8/12|e(G)|$. A subdivision vertex lies on two edges, one in $\hat{G}$ and one in $\hat{G}_n^*$ so the probability that we root $\hat{G}_n$ at one of those is $1/|e(G)|$ and regardless of the orientation, we choose $\hat{\rho}_n$ with probability $2/3$, in particular ρ_n is a subdivision vertex with probability $2/3|e(G)|$ and we are done.

□

Combining Proposition 56 and Theorem 57 gives us the first piece in our puzzle.

Proposition 58. *The graph $R(\mathcal{S}(\mathbf{G}_\infty))$ is almost-surely vertex parabolic.*

Proof. We know that $U(G_\infty)$ is almost-surely recurrent, hence edge-parabolic so let $m : e(D(G_\infty)) \to \mathbb{R}$ be a function satisfying $m(\gamma) = \infty$ for any infinite path γ and $\|m\| < \infty$. Define a new function $w : v(R(\mathcal{S}(\mathbf{G}_\infty)) \to \mathbb{R}$ by setting

$$w(v) = \sum_{w \sim v} m(\{w, v\})$$

to be the sum of the masses of neighbours of v in the embedding of $R(\mathbf{S}_\infty)$ in $U(G_\infty)$. We now claim w satisfies the properties of vertex parabolicity. Let γ be an infinite path in $R(\mathcal{S}(\mathbf{G}_\infty))$, then again by Proposition 56 we know that $\gamma \subset U(G_\infty)^2$ so that the neighbours of γ in $U(G_\infty)$ induce an infinite path in $U(G_\infty)$, hence

$$w(\gamma) = \sum_{w \sim v} e(\{w, v\}) \geq \sum_{e \in \gamma \subset e(U(G_\infty))} m(e) = \infty.$$

It thus remains to show that w has finite mass. For this, simply note that by Cauchy-Schwarz

$$\|w\|^2 = \sum_{v \in v(R(\mathcal{S}(\mathbf{G}_\infty)))} w(v)^2 = \sum_{v \in v(R(\mathcal{S}(\mathbf{G}_\infty)))} \left(\sum_{w \sim v} m(\{w, v\}) \right)^2 \leq 8\|m\|^2,$$

as any subdivision vertex has degree 8 and any edge is incident to at most one subdivision vertex. □

In order to finally use Theorem 51 we must prove that $R(\mathcal{S}(\mathbf{G}_\infty))$ is almost-surely one ended. It is well known that $\mathbf{T}_\infty$ from Section 3.1 is almost-surely one-ended and since $\mathbf{Q}_\infty$ is constructed via vertex identifications of $\mathbf{T}_\infty$ and is furthermore locally finite, it follows immediately that $\mathbf{Q}_\infty$ is also almost-surely one-ended.

Furthermore, the angular mapping sends white vertices of $\mathbf{Q}$ to faces of $A(\mathbf{Q})$ and black vertices to vertices of $A(\mathbf{Q})$ so that we have an identification of faces in $\mathbf{G}$ with white vertices in $\mathbf{Q}$ and of vertices of $\mathbf{G}$ with black vertices in $\mathbf{Q}$.

Theorem 59 ([2], Theorem 4.3). *The graph $R(\mathcal{S}(\mathbf{G}_\infty))$ is almost-surely one-ended.*

Proof. Suppose for a contradiction that $R(\mathcal{S}(\mathbf{G}_\infty))$ had more than one end, let $\mathcal{C} = \{s_1, s_2, ..., s_n\}$ be a finite cycle of squares in the contacts graph whose removal has two infinite graph theoretically connected components. Let I_1 and I_2 denote the set of squares lying in one infinite component and in the other. Now, there exists a simple closed path γ in $\bigcup \partial s_i$ such that I_1 lies in the interior of γ and I_2 lies in the exterior. Because of the identification of horizontal and vertical lines in the squaring with vertices in Q_∞, such a path γ induces a finite cycle K in Q_∞ (choose those vertices whose corresponding lines intersect γ) see Figure 23.

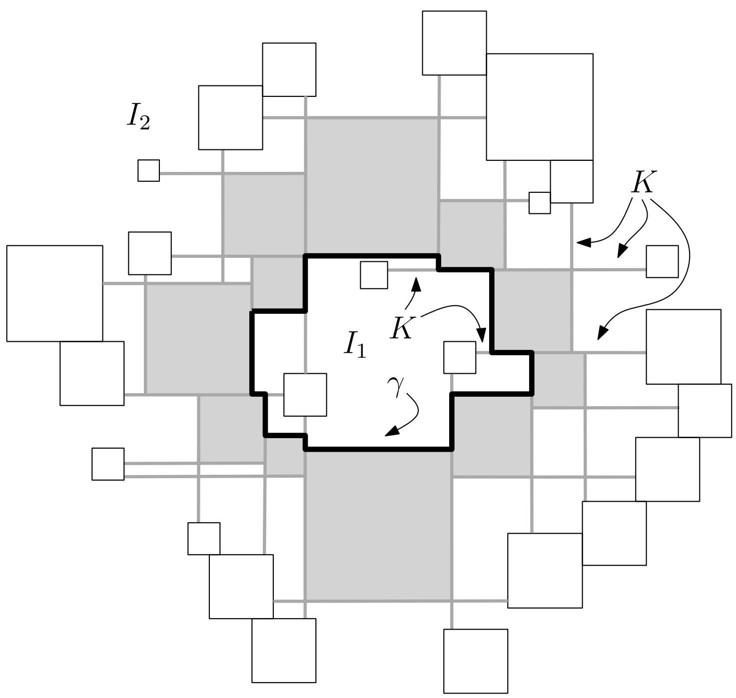

Figure 23: The set $\mathcal{C}$ is given in grey squares. The thick black line is the path γ, and the thin grey lines denote corresponding vertices in the set K.

It is now claimed that K separates Q_∞ into two infinite connected components. As squares in the squaring correspond to faces in Q_∞ it follows that all but finitely many squares in I_1 must lie in a different component of Q_∞ to those squares in I_2, which proves the claim. But this shows that Q_∞ has two infinite components, a contradiction, and we are done. □

In particular we have shown, by Theorem 51, that $R(\mathcal{S}(G_\infty))$ almost-surely has one single point of accumulation. Now, by Corollary 50 the squarings $\mathbf{S}_n = \mathcal{S}(\mathbf{G}_n)$ converge almost-surely to a compact limit $\mathbf{S}_\infty$ which almost-surely has a single point of accumulation. Furthermore, the limit squaring $\mathbf{S}_\infty$ has the law of $\mathcal{S}(\mathbf{G}_\infty)$ which together finally prove the claims of Theorem 39.

In view of Theorem 48, we have also shown that if ϕ is a harmonic function on G, a 3-connected recurrent planar map, then there exists some $L \in \mathbb{R}$ such that $\phi(v_n) \to L$ for any choice of path $(v_n, n \geq 0)$ and in particular that Tutte's embedding for G has only one point of accumulation.

3.4 Resistance Results

The fact that resistance to infinity is infinite in recurrent graphs has been used several times throughout the thesis but in a fairly imprecise way. Let $\mathbf{B}_n$ be as in section 2.3 so that the height of cylinder $\mathcal{C}(\mathbf{B}_n)$ is equal to the resistance from s to $\partial B_n(s)$. If the resistance grows sublinearly, then provided that the cylinder's behavior is close to the behavior of the squaring $\mathbf{S}_\infty$ near the limit point, it should be true that a linear scaling around the limit point "converges" to $\mathbb{R}^2$ (see section 4.1 for more details). The following was an attempt to make precise this idea, however a link between the cylinder and limit point proved difficult (see again section 4.1 for more details). While these results did not end up yielding the desired property, we have decided to include them as we believe the approach and techniques may yet be useful for the study of the squarings.

We now prove some growth bounds on the resistance. We shall first consider *k-level trees*, which are defined to be rooted plane trees such that the graph distance to each leaf is exactly k. A *split* in a tree is a vertex of degree at least 3 and we *collapse* a split by identifying all outwards edges of a split together. It is easy to see that collapsing a split in a k-level tree preserves the k-level tree structure. For a k-level tree T we write $R(T)$ for the resistance between the root and the leaves when these leaves are identified to a single vertex. A fundamental result which we shall require is the Collapsing Lemma.

Lemma 60 (Collapsing Lemma). *Let T be a k-level tree, and let T' be the tree constructed from T by collapsing a split. Then,*

$$R(T) \leq R(T').$$

Proof. Let $j_{T'} : \vec{e}(T') \to \mathbb{R}$ be the unique unit flow from the root to the leaves of T' minimizing energy. Construct a flow j_T on T be setting $j_T(e) = j_{T'}(e)$ if e was not collapsed and set the flow on the collapsed edges to be that which makes j_T a proper flow (this is clearly always possible). Now,

$$\begin{aligned} R(T) - R(T') \leq \mathcal{E}(j_T) - \mathcal{E}(j_{T'}) &= \frac{1}{2} \sum_{e \in \vec{e}(T)} j_T(e)^2 - \frac{1}{2} \sum_{e \in \vec{e}(T')} j_{T'}(e)^2 \\ &= \left[\sum_{e \in C(T,T')} j_T(e)^2 \right] - j_{T'}(e')^2 \end{aligned}$$

where $C(T,T')$ is the set of collapsed edges and e' is the collapsed edge. Now we know that

$$j_{T'}(e')^2 = \left(\sum_{e \in C(T,T')} j_T(e) \right)^2 \geq \sum_{e \in C(T,T')} j_T(e)^2$$

so that from the first equation $R(T) - R(T') \leq 0$ as desired. $\square$

Another proof of the Collapsing lemma can be carried out using simple properties of resistances and the inequality

$$\left(\sum_{i=1}^{n} \frac{1}{x_i + 1} \right)^{-1} \leq 1 + \left(\sum_{i=1}^{n} \frac{1}{x_i} \right)^{-1}$$

for positive numbers $x_i > 0$. By the Rayleigh monotonicity, in order to find an upper bound on the resistance for k-level trees, it suffices to consider only those trees which have at most one split at each level. In this view we have a nice theorem.

Theorem 61. *Let T be a tree and Δ_i the difference in height between the ith and $i+1$th split. Then,*

$$R_{n_k}(T) \leq \sum_{i=0}^{\ell} \Delta_i + \frac{1}{\ell} \sum_{i=\ell+1}^{k} \Delta_i$$

for all $0 \leq \ell \leq k$ where n_k is the level of the kth split.

Proof. Fix ℓ and let T' be the tree constructed from T by collapsing the first ℓ splits. It suffices now to bound the resistance on T'. For the collapsed part it is clear from the additive property of resistance that this has resistance exactly

$$\sum_{i=1}^{\ell} \Delta_i$$

so we now need to bound the non-collapsed part and again apply the series property of resistance. As we collapsed ℓ splits, the ℓ-th level vertex has out-degree at least ℓ, which in particular implies that there are at least ℓ disjoint paths from ℓ to any level $k > \ell$. Each of these paths has length

$$\sum_{i=\ell+1}^{k} \Delta_i$$

to the n_kth level. Thus applying the properties of resistances in parallel, we find that the non-collapsed part of T' has resistance bounded by

$$\frac{1}{\ell} \sum_{i=\ell+1}^{k} \Delta_i$$

and thus

$$R_{n_k}(T) \leq R_{n_k}(T') \leq \sum_{i=1}^{\ell} \Delta_i + \frac{1}{\ell} \sum_{i=\ell+1}^{k} \Delta_i$$

which was to be shown. □

If we look at k-level trees with a split at each level, the we have a simple corollary.

Corollary 62. *Let T be a k-level tree with a split at each level, then*

$$R(T) \leq 2\sqrt{k}.$$

Proof. Take $\ell = \sqrt{k}$ in the above theorem, then using $\Delta_i = 1$ yields the desired result. □

We now digress slightly to focus on the above bound. Define a sequence of k-level binary tree B_k inductively as follows: B_0 is a single vertex and B_{k+1} is constructed by starting with a root vertex, on the left attaching a copy of B_k and on the right attaching a length k path. See Figure 24 for an example.

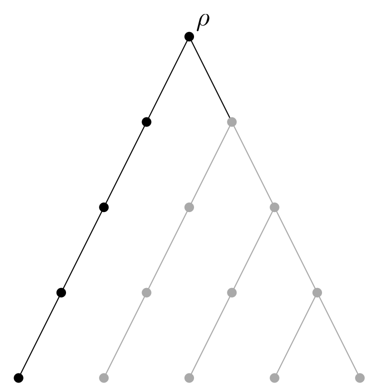

Figure 24: An example of B_4, note the B_3 on the right subtree.

Intuitively, this should in fact be the tree with the largest resistance over all those trees with a split at each level, however it does not seem trivial to prove. We rephrase this as a conjecture:

Conjecture 63. Let T be a k-level tree with a split at each level, then $R(T) \leq \sqrt{k}$, and this bound is asymptotically tight for B_k.

It is easy to check, however, that the resistance of B_k is in fact asymptotically $\sqrt{k}$ since we have,

$$R(B_k) = \left(\frac{1}{R(B_{k-1}) + 1} + \frac{1}{k+1} \right)^{-1} = \frac{kR(B_{k-1}) + R(B_{k-1}) + k + 1}{R(B_{k-1}) + k + 2}$$

and one easily verifies by induction that $\sqrt{k + 1 - 2\sqrt{k+1}} \leq R(B_k) \leq \sqrt{k}$. Thus our bound in the Corollary should only be off asymptotically by a constant factor of 2.

Let us now turn out attention back to rooted graphs and apply the above results to these. An immediate consequence of Corollary 62 and Rayleigh monotonicity, we have the following.

Corollary 64. *If $B_G(r, \rho)$ contains a rooted sub-graph isomorphic to an r-level tree rooted at ρ with a split at each level, then $R_r(G) \leq 2\sqrt{r}$.*

It may be tempting to relate the resistance bound to the size of $\partial B_G(r, \rho)$, the boundary of a radius r ball, however it is easy to construct a tree which has balls of arbitrarily large perimeter, but whose resistance grows linearly. In fact, even degree bounds may not guarantee a non-trivial growth bound.

For a graph G, let $(X_e, e \in e(G))$ be i.i.d. Bernoulli-p random variables on the edges of G. Let G_p be the random subgraph of G constructed by setting $v(G_p) = v(G)$ and $e \in e(G_p)$ if and only if $X_e = 1$. The *bond percolation probability* is the quantity,

$$p_c = \inf\{p \in [0, 1] \ : \ G_p \text{ has an infinite connected component}\}$$

ie. the smallest probability such that G_p has an infinite connected component. Note that G_{p_c} may not necessarily have such a component. We shall need a theorem of Menger.

Theorem 65 (Menger's Theorem). *Let G be a finite graph and let U and V be two disjoint sets of vertices. The minimum number of edges required to disconnect U from V in G is equal to the number of edge-disjoint paths from U to V.*

An application of this theorem in G_p is the basis for the proceeding lemma. By the kth layer of a graph we mean the set of vertices at distance exactly k from ρ.

Lemma 66. *Let G be a rooted graph such that the bond percolation probability p_c satisfies $p_c < 1$, then the number of edge-disjoint paths from layers 2^r to 2^{r+1} is $\omega(1)$ and in particular there exists a subsequence r_k such that the number of disjoint paths between 2^{r_k} and 2^{r_k+1} is greater than $\log(r_k)/\log(1 - p_c)$.*

Proof. Consider a bond percolation process on G with probability $p \in (p_c, 1)$, and let E_r be the event that the layers 2^r and 2^{r+1} are disconnected. By Menger's theorem, $\mathbb{P}(E_r) \geq (1 - p)^{F_r}$ where F_r is the minimum number of edge-disjoint paths from layer 2^r to 2^{r+1}. Suppose that $F_r \leq C \log(r)$ for some $C > 0$, then

$$\sum_{r=1}^{\infty} \mathbb{P}(E_r) \geq \sum_{r=1}^{\infty} (1 - p)^{F_r} \geq \sum_{r=1}^{\infty} r^{\log(1-p)C}$$

which is infinite if $C \leq 1/\log(1-p)$. However, by choice of layer size, the events E_r are independent so we may apply the Borel-Cantelli lemma to find that if $C \leq 1/\log(1 - p)$ then

$$\mathbb{P}(E_r \text{ infinitely often}) = 1.$$

But if the layers 2^r and 2^{r+1} are disconnected for infinitely many r, then it is impossible to have an infinite component, a contradiction. $\square$

The a simple consequence of this lemma relates to cycles, however we require that the faces of G have bounded degree. In fact, such a condition can be slightly loosened provided we control the face sizes in between "layers" but we present the simplest case instead.

Corollary 67. *Let G be as in the previous lemma and let G be planar and have bounded degree faces. Then the size of the separating cycles between layers 2^r and 2^{r+1} is $\omega(1)$.*

Proof. Let d be the maximal face degree of any face in G and let F_r denote a set of edges whose removal disconnects layers 2^r from 2^{r+1} so by the preceeding lemma, $|F_r| = \Omega(\log(r))$. As F_r disconnects two components of G, it induces a cycle in G^* so we define $\{v_1, ..., v_{|F_r|}\}$ to be such a cycle. Vertices in G^* correspond to faces in G so let $V_i = \{f \in f(G) \ : \ f \sim v_i\}$ be the set of edges incident to the faces v_i in G. Since $\{v_i\}_{i=1}^{|E_r|}$ was a cycle, $\bigcup F_i$ also contains a cycle of length at most d times as long and we are done. $\square$

We would like to apply this lemma in order to prove the following conjecture:

Conjecture 68. If $\mathbf{G}$ is a rooted graph such that the bond percolation probability p_c satisfies $p_c < 1$, then G has sub-linear resistance.

An important issue arises: Menger's theorem does not estimate the *length* of the independent paths. In particular, this causes a problem as we cannot directly apply Theorem 61 (as we have in the previous theorems) to some subtree of $\mathbf{G}$ without knowledge about the length of the independent paths. Furthermore, some path connecting layer 2^r to 2^{r+1} could be extremely long (relative to 2^r) which induces an essentially negligible resistance in the tree. This is yet another issue which we faced while attempting to prove the convergence property, as we explain more thoroughly in the next chapter.

4 Discussion and Open Problems

We shall now focus on a few open problems which relate to the squaring convergence. We first focus on the squaring itself, trying to prove some results based on processes which depend on the random squaring and subsequently present a list of open problems relating to the squaring.

4.1 Convergence

Squarings of Cylinders

We finally return to the squaring of cylinders for recurrent graphs, alluded to in Section 3.3. We introduce some more definitions and prove a nice theorem relating squarings of cylinders to the transience and recurrence structure of a graph. Let $j : \vec{e}(G) \to \mathbb{R}$ be a flow on G, we call j a *path-independent* flow if the algebraic sum of current around a closed loop is 0. If $\mathbf{G} = (G, \rho)$ is a rooted graph, a flow j on $\mathbf{G}$ is called *perfect* if it is a path-independent unit flow with *sink* ρ. The following fact is easy to prove.

Fact 69. A flow j is path-independent if and only if there exists a positive harmonic function ϕ with $\phi(\rho) = 0$ and such that (ϕ, j) is a Kirchoff pair.

A consequence of the above fact is that a finite graph admits no perfect flow due to the bounded uniqueness of Proposition 4. We thus assume that $\mathbf{G}$ is an infinite graph throughout the rest of this section.

A rooted graph $\mathbf{G} = (G, \rho)$ can be *embedded on a cylinder* if there exists a non-trivial squaring of a cylinder $\mathcal{C} = \{s_i\}_{e \in e(G)}$ (we allow squares to be points) indexed by the edges of G and a harmonic function $\phi : v(G) \to \mathbb{R}^+$ with boundary $\phi(\rho) = 0$ such that for any edge $e \in e(G)$, the square s_e has size $\phi(e_+) - \phi(e_-)$ and is at height $\phi(e_+)$, and if e and e' are two edges incident to the same vertex, then the squares s_e and $s_{e'}$ share a radial line.[5]

A more intuitive way to view this is as follows: a graph can be embedded on a cylinder if we can associate radial lines to vertices and squares to edges which preserve incidences. Our first Theorem is simply an accumulation of previous results.

Theorem 70. *Let $\mathbf{G}$ be a one-ended vertex-rooted planar map, the following are equivalent:*

(1) $\mathbf{G}$ *is transient,*

(2) $\mathbf{G}$ *can be embedded on a finite cylinder of unit circumference,*

(3) $\mathbf{G}$ *admits a perfect flow of finite energy.*

[5] This definition of embedding on a cylinder is in some sense cheating, as it is more of a condition on the vertices of $\mathbf{G}$ than on the cylinder, but it is essentially equivalent to any other similar definition.

Proof. That (1) implies (2) is precisely Theorem 30, furthermore that (3) implies (1) follows from energy minimization combined with Theorem 9. We must now prove that (2) implies (3). Let $\phi : v(G) \to \mathbb{R}^+$ be the function satisfying the necessary embedding properties and define a flow $j : \vec{e} \to \mathbb{R}$ by setting $j_{uv} = \phi(u) - \phi(v)$. By harmonicity j is a unit flow which is path-independent. Furthermore, $\mathcal{E}(j) = \mathcal{E}(\phi) = \text{area}(\mathcal{C}) < \infty$ and we are done. $\square$

In fact, there is a partial converse to the above theorem which will require a little more work to prove but it is not particularly difficult.

Theorem 71. *Let* $\mathbf{G}$ *be a one-ended vertex-rooted planar map, then* $\mathbf{G}$ *is recurrent if and only if it can be embedded on an infinite cylinder.*

Proof. We first prove that recurrence implies $\mathbf{G}$ is embeddable on an infinite cylinder. For each n let $j_n : \vec{e}(B_G(\rho, n)) \to \mathbb{R}$ be the unique energy minimizing unit flow from ρ to $\partial B_G(\rho, n)$ when this last set is identified to a single vertex. Recall that the Cartesian product of countably many compact intervals is itself compact so that in particular the set

$$I := \prod_{e \in \vec{e}(G)} [-1, 1]$$

is compact. Now, $j_n(e) \in I$ whenever $e \in B_G(\rho, n)$ so that there exists a subsequence n_k such that $j_{n_k}(e) \to j_e$ for all $e \in \vec{e}(G)$ (for k large enough that $j_{n_k}(e)$ is well defined). From Section 3.3 the j_{n_k} can each be used to define a squaring of a cylinder, say $\mathcal{C}_{n_k}$. As each square of $\mathcal{C}_{n_k}$ converges (since the squares are entirely determined by j_{n_k}), the limit (in the local Hausdorff sense) $\mathcal{C}_\infty = \lim_{k \to \infty} \mathcal{C}_{n_k}$ exists. It remains to show that $\mathcal{C}_\infty$ is a squaring of an infinite cylinder. The flow $j_\infty = \lim_{k \to \infty} j_{n_k}$ is a unit flow from ρ to ∞ since for any fixed $v \in \mathbf{G}_\infty$ the vertex v is eventually inside all balls $B_r(\rho)$ for all $r > d(v, \rho)$ and hence the in-flow at v is equal to the out-flow at v for all r sufficiently large. Furthermore, the flow j necessarily has infinite energy and hence $\mathcal{C}_\infty$ has infinite area. It furthermore is connected since each $\mathcal{C}_{n_k}$ is, and it follows that $\mathcal{C}_\infty$ is an infinite cylinder.

The reverse direction follows from Doob's optional stopping lemma. Let $\phi : v(G) \to \mathbb{R}^+$ be the vertical position of the vertices on the infinite cylinder, let $X = (X_n, n \geq 0)$ be a random walk on G, let $\tau_R^+ = \inf_k \{\phi(X_k) \geq R\}$ and similarly let $\tau_R^- = \inf_k \{\phi(X_k) \leq R\}$. Fix vertex X_0 with $r := \phi(x_0)$ so large so that for all $n \leq \min(\tau_{r/2}^-, \tau_{2^j r}^+)$ and $j > 0$ the random variable $\phi(X_n)$ is a martingale, as ϕ is harmonic everywhere except for ρ. Let $\tau = \min(\tau_{r/2}^-, \tau_{2^j r}^+)$ so that τ is a stopping time which is finite almost-surely by recurrence. By Doob's optional stopping lemma, $\mathbb{E}(\phi(X_\tau)) = \mathbb{E}(X_0) = r$ so that by Markov's inequality

$$\mathbb{P}(\phi(X_\tau) \geq 2^j r) \leq \frac{\mathbb{E}(X_\tau)}{2^j r} = \frac{1}{2^j}$$

so that the probability that a random walk on G hits layer $2^j r$ before $r/2$ for any j is less than $1/2^j$.

Now, since the cylinder is infinite, one-ended and locally-finite (as $\mathbf{G}_\infty$ is) we note that along any infinite path $\{v_i\}_{i=0}^{\infty}$ with $d(v_i, \rho) \to \infty$ and with $v_0 = \rho$ we have $\phi(v_i) \to \infty$ so that $\{v : \phi(v) \geq M\}$ is a finite set for all M and hence $\mathbb{P}(\tau = \infty) = 0$. Moreover, if a random walk $(X_n, n \geq 0)$ is transient then $\phi(X_n)$ must get arbitrarily large before returning to ρ. Let $\lambda = \sup_{\phi(v) \,:\, v \sim \rho}$ be the largest square incident to the vertex ρ and note that

$$\sum_{j=0}^{\infty} \mathbb{P}(\phi(X_\tau) \geq 2^j \lambda) \leq \sum_{j=0}^{\infty} \frac{1}{2^j} < \infty$$

where $X_0 \sim \rho$ and thus by Borel-Cantelli,

$$\mathbb{P}(\phi(X_\tau) \geq 2^j \lambda \text{ for infinitely many } j) = 0.$$

However, by the previous observation the transience of the random walk depends on the walk reaching infinite potential before returning to ρ, which happens with probability 0, as desired. □

The proof of Theorem 71 shows that *some* sub-sequence in the definition of $\mathcal{C}_1$ converges defined as in Section 3.3, but makes no promise on the uniqueness of the limit. The results from the previous section suggest that under the recurrence and one-ended assumption the limit $\mathcal{C}_1$ should very well exist, but one does not have much (if any) control over the behaviour of the graph "far" from the root. Furthermore, in order to have a full converse to Theorem 70 we would need to show that a graph is recurrent if and only if it has a perfect flow of *infinite* energy. The forward direction is easy, the converse direction could be proved by using the flow to embed $\mathbf{G}$ on a cylinder, however, there is no immediate guarantee that the flow is "uniform" far from the root, which is an issue similar to that of before.

Local Structure of Limit Point

We know that our squaring (based on the 3-connected graphs) $\mathbf{S}_\infty$ almost surely has one point of accumulation, call this point $z(\mathbf{S}_\infty)$. For a set $S \subset \mathbb{R}^2$ let $S - z = \{s - z : s \in S\}$ and for $a \in \mathbb{R}$ let $aS = \{as : s \in S\}$. For each t define the random set $S_t = t(\mathbf{S}_\infty - z(\mathbf{S}_\infty))$ to be the set $\mathbf{S}_\infty$ translated so that the limit point is at the origin and scaled by t. We are concerned with the limit $S_\infty = \lim_{t\to\infty} S_t$ if it exists under the local Hausdorff metric.

We conjecture that S_∞ converges almost surely to $\mathbb{R}^2$ (or at least in probability) in the local-Hausdorff sense. Evidence for this fact follows from the following observation. Fix a ball $B(0, \epsilon)$ for any $\epsilon > 0$ and let E_t be the set of edges in G whose corresponding square in S_t intersects $B(0, \epsilon)$. The set E_t is almost surely finite and separates $\mathbf{G}_\infty$ into at least two components. We know from Section 4.1 that $\mathbf{G}_\infty$ is one-ended and it is likely that it has bond percolation probability less than 1 as several popular uniformly infinite graphs have such a property, for example see [16, 4]. Lemma 66 would then suggest that $|E_t| \to \infty$ as $t \to \infty$ almost surely. So the boundary of any ball will intersect infinitely many squares, in particular, if the limit does exist it should look something like $\mathbb{R}^2$.

This problem boils down to understanding two rather difficult relationships; the first being the relation between balls $B_G(\rho, n)$ in G and those balls $B_{\mathbb{R}^2}(0, \epsilon)$ in the plane, the second is the rate of convergence for harmonic functions $\varphi : v(G) \to \mathbb{R}^2$ to their limiting values along an infinite ray in G. In particular, provided that balls in G are not poorly behaved in relation to balls in $\mathbb{R}^2$ and that harmonic functions converge sufficiently fast, it should be true that $S_\infty = \mathbb{R}^2$ in the local Hausdorff sense. Even though these properties are easy to state, they are certainly not easy to solve.

A related problem, and intuitively easier to prove is that of re-rooting. Let $\mathbf{G}_\infty$ be as before and let $\mathbf{G}'_\infty$ be the graph (G_∞, st') where t' is chosen uniformly randomly from the set of neighbors of s. We are interested in the Hausdorff distance $d((\mathbf{S}_\infty - z(\mathbf{S}_\infty)) \cap B(0, \epsilon)\ ,\ (\mathbf{S}'_\infty - z(\mathbf{S}'_\infty)) \cap B(0, \epsilon))/\epsilon$ where $\mathbf{S}'_\infty = \mathcal{S}(\mathbf{G}'_\infty)$. In particular, does this limit tend to 0 as $\epsilon \to 0$ almost-surely? In other words, if we change the root of $\mathbf{G}_\infty$ slightly, does the structure around the limit point change in a significant way? A positive answer to the convergence of S_t to $\mathbb{R}^2$ would immediately imply a positive answer to this question, but the converse does not immediately hold.

4.2 Open Problems

We now present a list of open problems pertaining primarily to the third chapter of this thesis. For more open questions, see [1] or [2].

1. Is there a simple way to check the composition or degeneracy structure directly from the contacts graph of a squaring? What about from its p-net?

2. The box-counting dimension of the random squaring $\mathbf{S}_\infty$ is conjectured to be almost-surely well-defined and, in fact, constant. Recall the box counting dimension is defined as the limit $\log(n_\epsilon)/\log(1/\epsilon) \to c$ as $\epsilon \to 0$ where n_ϵ is the minimum number of balls of radius ϵ required to cover $\mathbf{S}_\infty$. Is c is well defined and constant a.s.?

3. As before let $z(\mathbf{S}_\infty)$ be the a.s. unique limit point of $\mathbf{S}_\infty$. Can the law of $z(\mathbf{S}_\infty)$ be explicitly defined? If yes, what is it?

4. Does the limit $\lim_{t\to\infty} S_t$ (defined as above) exist almost surely in the local Hausdorff sense? If yes, is it equal to $\mathbb{R}^2$?

5. Does $d((\mathbf{S}_\infty - z(\mathbf{S}_\infty)) \cap B(0,\epsilon), (\mathbf{S}'_\infty - z(\mathbf{S}'_\infty)) \cap B(0,\epsilon))/\epsilon$ converge to 0 almost surely as $\epsilon \to 0$?

6. By fixing the potential at s to be 1 and at t to be 0, we fix the height of the rectangle $\mathbf{S}_\infty$ to be 1. Let W be the random variable denoting the width of $\mathbf{S}_\infty$, as the squaring of the dual of a graph is the same squaring but rotated, we know that $W = 1/W$ in distribution. What is the law of W? Even easier, is $\mathbb{P}(W = 1) > 0$?

7. Let $\hat{\mathbf{S}}_n$ be uniformly selected from non-composite and non-degenerate squarings of rectangles of height 1. Does $\hat{\mathbf{S}}_n$ converge in distribution to $\mathbf{S}_\infty$?

8. Let $\mathbf{G}$ be a rooted planar map with percolation probability $p_c < 1$. We know that the number of disjoint paths grows at least as $\omega(1)$. Under what conditions does the resistance grow sub-linearly?

9. Translate and rescale $\mathbf{S}_n$ so that it is centered at the origin. Stereographically project $\mathbf{S}_n$ onto the unit sphere in such a way that the image of the unbounded area $\mathbb{R}^2 - \mathbf{S}_n$ has area $1/\sqrt{n}$. Let μ_n be a measure on $\mathbb{S}^2$ defined by setting the measure of each square to be $1/n$. Does the measure μ_n converge weakly to a measure μ on the sphere which is some version of the Liouville Quantum Gravity?

10. Squarings of other types of planar maps may also be of interest. In particular, let $\mathbf{Q}_n = (Q_n, st)$ be a uniformly chosen edge-rooted quadrangulation over those graphs with n edges. Does the limit $\mathcal{S}(\mathbf{Q}_n)$ have some interesting properties? The distributional limit $\mathbf{Q}_\infty = \lim_{n\to\infty} \mathbf{Q}_n$ has been of considerable interest to graph theorists as of late, perhaps the squaring will yield some new information.

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