Serial Manipulator Kinematics with Dual Quaternions and Grassmannians

by Pasquale Gervasi

B.Eng. (McGill University), 1995

Department of Mechanical Engineering
Centre for Intelligent Machines
McGill University
Montréal, Québec, Canada

September 1999

A thesis submitted to the Faculty of Graduate Studies and Research
in partial fulfilment of the requirements of the degree of
Master of Engineering

0-612-55022-2

Abstract

Inverse kinematics (IK) of serial manipulators is related to reflection-free Euclidean spaces where distances and orientations remain invariant. These direct isometries can be derived from simpler forms known as quaternions and dual quaternions. Combining them with Hermann Grassmann's Extension Principle, the IK of serial manipulators is formulated entirely in projective geometry free of any metric. The only rules governing this geometry is the preservation of ratios and incidence. Actually, the holonomicity of these IK problems can be described using incidence relations alone.

The algebraic constraints, derived from incident relations, define a *manifestly* holonomic system as opposed to general holonomic systems that need only satisfy Frobenius' Theorem for integrability using *Pfaffian* forms. The solution to these algebraic equations will require an introduction to algebraic geometry and commutative algebra. Definitions of basic geometric and algebraic objects along with a study of their respective properties are included. Pertinent theorems are proved and illustrated with simple examples to establish a dictionary between algebra and geometry. By this means, kinematic analysis is conveniently subjected to the theories of algebraic geometry and commutative algebra.

More precisely, the inverse kinematics (IK) of a 6R serial manipulator (6Rsm) is formulated in the homogeneous projective space of dual quaternions (DQ). This leads to a robust algorithm because the 8-space of DQ together with the Grassmannian extension ensures that only valid solutions, which satisfy all of the constraint equations, are admitted. Numerical examples, based on two real 6Rsm architectures, are presented to illustrate the efficacy of the new algorithm. Its *fiber* (solutions) and *critical loci* (singularities) are described.

Résumé

Le modèle géométrique inverse (MGI) des manipulateurs sériels est lié à l'espace euclidien des déplacements dans lequel les distances et les orientations demeurent invariantes. Ces isométries directes peuvent être obtenues à partir de formes plus simples comme les quaternions et les quaternions duaux. En les associant avec le principe d'extension d'Hermann Grassmann, le (MGI) des manipulateurs sériels est entièrement exprimé en terme de géométrie projective, donc sans nécessité d'une métrique. Les seules lois régissant cette géométrie sont la préservation des rapports et des incidences. En fait, l'holonomicité du (MGI) peut être représentée en utilisant seulement les relations d'incidence.

Les contraintes algébriques issues des relations d'incidence définissent un système manifestement holonome par opposition à un système holonome général qui ne respecte que le Théorème de Frobenius d'intégration en utilisant les formes différentielles de Pfaff. La résolution de ces équations algébriques nécessitera l'introduction de notions de géométrie algébrique et d'algèbre commutative. Les définitions de base de la géométrie et des objets algébriques ainsi qu'une étude de leurs propriétés respectives sont données. Des théorèmes bien adaptés aux problèmes sont démontrés et illustrés à l'aide d'exemples simples pour établir un lexique de référence entre l'algèbre et la géométrie. De cette manière la modélisation géométrique est formulée en terme de géométrie algébrique et d'algèbre commutative.

Plus précisément, le (MGI) d'un manipulateur sériel à six articulations rotoïdes est exprimé dans l'espace de projection homogène des quaternions duaux (QD). Il en résulte un algorithme robuste car l'espace de dimension huit des (QD) ajouté à l'extension de Grassmann assure que seules les solutions valides, qui tiennent compte de toutes les équations de contraintes, sont obtenues. Des exemples numériques, reposant sur l'analyse des architectures de deux manipulateurs réelles sériels à six articulations rotoïdes sont présentés pour illustrer l'efficacité du nouvel algorithme. Leurs solutions et leurs singularités sont aussi décrites.

Acknowledgements

I am eternally grateful to Professor Paul J. Zsombor-Murray for supplying an infinite amount of enthusiasm, support, and patience; without it I could never have completed my studies. But his most humanitarian attribute must be the art of arriving at deviously simple ideas inspired by the need to transfer that knowledge to others.

I'd like to extend my gratitude to my parents Celestino and Lauretta Gervasi. Their love and support was always there and contributed greatly to the completion of my work. A special thanks goes out to David Delnick, a personal friend and full-time tennis partner, who showed me that as long as you have *footwork* any *shot* is conceivable.

Claim of Originality

The inverse kinematic problem(IKP) of the 6R serial manipulator is solved in a geometry where one can identify an invariant property between all conics thus rendering their respective *shape* superfluous. The statement may seem irrelevant but it does portray the events which are to follow quite accurately, not to mention my claim of originality.

At present, formulations of the IKP are dependent on the ability to measure distance in the space where the manipulator is embedded (*i.e.* a "measuring stick" which enables one to tell the difference between an ellipse and a circle). The following arguments will show that the solution to the IKP does not require a measure. In other words one need not differentiate among different conics but need only acknowledge a conic "section". The derivation of the algebraic constraints in this *newer* geometry, namely projective geometry, requires a novel interpretation of the quaternion and incidence relations.

The handling of any set of algebraic constraints adheres to the laws of algebraic geometry and commutative algebra which will be presented.

An algorithm based on these mathematical tools is proposed. It quickly and safely retrieves *all* solutions to the IKP.

Table of Contents

List of Figures

List of Tables

List of Symbols

Symbol	Description
$\mathcal{D}^6$	Six space of Euclidean displacements
D	An element of $\mathcal{D}^6$
SO^3	Three space of reflection-free rotations
$\mathcal{A}^3$	Three space of point displacements
q	Quaternion
$\hat{q}$	Dual quaternion
$\mathcal{P}^n$	n dimensional projective space
$\mathcal{M}_n$	n dimensional manifold
Π	A projection map
$\mathbf{V}$	Variety - see Appendix A
I	Ideal - see Appendix A
$\mathrm{I}(\mathbf{V})$	Gröbner basis of $\mathbf{V}$ - see Appendix A

List of Abbreviations

Symbol	Description
6Rsm	Six revolute serial manipulator
DH	Denavit-Hartenberg
DKP	Direct kinematics problem
EE	End effector
FF	Base frame
FK	Forward kinematics
IK	Inverse kinematics
IKP	Inverse kinematics problem
P	Prismatic joint
R	Revolute joint
sm	Serial manipulator

Chapter 1

Introduction

There are two types of kinematic problems associated with robotic manipulators, namely the *direct kinematic problem*(DKP) and the *inverse kinematic problem* (IKP). The DKP of serial manipulators (sm) is known to be quite simple, but the IKP is more complex because it leads to a system of non-linear algebraic equations. The IKP solution is usually, if not always, associated with transformations, called direct isometries, whose invariant properties are found in metric geometry. In the literature, algorithms for solving the IK of a 6Rsm use these invariants to derive the algebraic constraints. Then the elimination process begins but not without more metric constraints. By outlining the critical steps of two such algorithms one can argue that although a metric geometry, finds a solution its diversity in invariant properties leads to a myriad of definitions and different types of constraints. The objective, therefore, is to solve the IKP using another kind of geometry with fewer invariant properties. Affine geometry and projective geometry are two such geometries, with the latter having fewer invariants than the former. The separation of these *other geometries* from metric geometry will constitute a starting point in preparing the reader to understand the rules which govern their structure.

1.1. Background

The kinematic analysis of serial manipulators begins by uniquely determining its architecture using *Denavit-Hartenberg*(DH) notation. The DH parameters also determine the manipulator's *configuration* or *posture* by defining a frame for each link. The frame associated with the last link is commonly referred to as the pose of the end-effector(EE). *Forward kinematics*(FK) or *direct kinematics*(DK) involves finding the pose of the end-effector given the DH parameters and actuator variables values. The IKP consists of finding the configuration of the manipulator given the pose of the end-effector.

1.1.1. The Denavit-Hartenberg Notation. The DH parameters can unambiguously represent a kinematic model of a robotic manipulator. The model is a *kinematic chain* consisting of rigid bodies or links coupled by joints called *kinematic pairs.* A kinematic pair constrains the relative motion between two rigid bodies. There are two basic types, namely, *upper* and *lower* kinematic pairs. Upper kinematic pairs are those articulations where contacts between rigid bodies occur along a line or at a point common to both bodies as in roller bearings element. Lower kinematic pairs constrain the relative motion to act along a surface common to both bodies. One can derive five of the six[1] different types of lower kinematic pairs by combining two basic types:

- **Revolute** joint, **R**, allows two rigid bodies to rotate relative to each other about the axes of the revolute; defining a common cylinder between the links.
- **Prismatic** joint, **P**, allows the two bodies to move along a common surface of contact known as a prism of arbitrary cross section. The two bodies can only move in pure translation parallel to edges of the prism.

[1] The helical joint, i.e., a screw-in-nut motion, is the special case which cannot be so derived.

The kinematic chain of a serial manipulator is *simple*, every link is coupled to, at most, two other links. The chain is considered *open* when there is a link, located at either chain end, which is connected to only one other link as opposed to a *closed* chain where every link is joined to two other links to create a *linkage*. The 6Rsm consists of a simple kinematic open chain with six **R** joints. The following is a procedure to determine the relative location of the axes which in turn attach frames to the links.

- Links are numbered $0, 1, \ldots, 7$, where link 0 is the base and link 7 is the EE. The i^{th} **R** joint couples the $(i-1)^{th}$ link with the i^{th} link.
- A frame $\mathcal{F}_i$ is attached to the $(i-1)^{th}$ link.
- Z_i axis, of frame $\mathcal{F}_i$, is coincident with the ith joint axes.
- X_i is the common perpendicular between Z_{i-1} and Z_i, directed from the former to the latter. If Z_{i-1} and Z_i intersect then we adopt the *right hand* rule. If Z_{i-1} and Z_i are parallel then X_i can be chosen to go through the origin of $\mathcal{F}_{i-1}$.
- Define a_i to be the distance between Z_i and Z_{i+1}. From the definition of X_{i+1} it is easy to see that a_i is *nonnegative*
- Define b_i to be the signed distance between X_i and X_{i+1} measured along the positive Z_i direction.
- Define α_i to be the *twist* angle separating Z_i and Z_{i+1} and measured about the positive direction X_{i+1}.
- Define the actuator angle θ_i separating X_i and X_{i+1} and measured about the positive Z_i.

Fig. 1.1 illustrates the coordinate frames of the Puma robot. Note that the orientation of the last frame (*i.e.*, $\mathcal{F}_7$) is chosen arbitrarily and its origin is usually placed at the *operation point* P of the EE. The IK algorithms illustrated here require a complete mathematical description of a link's frame $\mathcal{F}_{i+1}$. The following rotation matrix $\mathbf{Q}_i$ and the position vector $\mathbf{a}_i$ carry the frame $\mathcal{F}_i$ into $\mathcal{F}_{i+1}$.

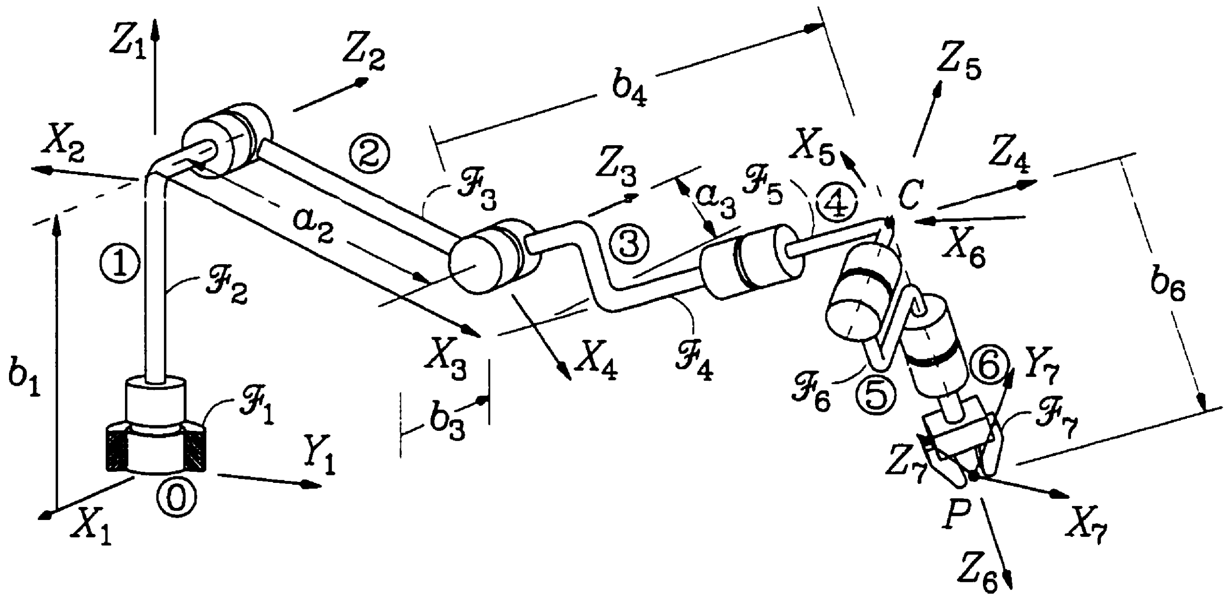


FIGURE 1.1. Coordinate frames of a Puma robot

$$\mathbf{Q}_i = \begin{bmatrix} \cos\theta_i & -\cos\alpha_i\sin\theta_i & \sin\alpha_i\sin\theta_i \\ \sin\theta_i & \cos\alpha_i\cos\theta_i & \sin\alpha_i\cos\theta_i \\ 0 & \sin\alpha_i & \cos\alpha_i \end{bmatrix}, \quad \mathbf{a}_i = \begin{bmatrix} a_i\cos\theta_i \\ a_i\sin\theta_i \\ b_i \end{bmatrix} \tag{1.1.1}$$

The successive application of this transformation, or direct isometry, from the base to the EE can be represented by

$$\mathbf{Q}_1\mathbf{Q}_2\mathbf{Q}_3\mathbf{Q}_4\mathbf{Q}_5\mathbf{Q}_6 = \mathbf{Q} \tag{1.1.2}$$

$$\sum_1^6 \mathbf{a}_i = \mathbf{p} \tag{1.1.3}$$

where $\mathbf{Q}$ and $\mathbf{p}$ are the orientation and origin of the EE frame with respect to the base frame. Constraints eqn. (1.1.2) and eqn. (1.1.3), are used to solve the IKP of a 6Rsm. One should also note that each element of the matrix $\mathbf{Q}_i$ and the vector $\mathbf{a}_i$ is a linear expression in the terms $[\cos(\theta_i), \sin(\theta_i)]$. By introducing six vectors

$$\mathbf{x}_i = \begin{bmatrix} \cos\theta_i \\ \sin\theta_i \end{bmatrix}, \; i = 1 \ldots 6$$

we can now say that the product $\Pi_{i=1}^{6}\mathbf{Q}_i$ is *hexalinear* or, simply, *multi-linear* in vectors $\{\mathbf{x}_i\}_1^6$.

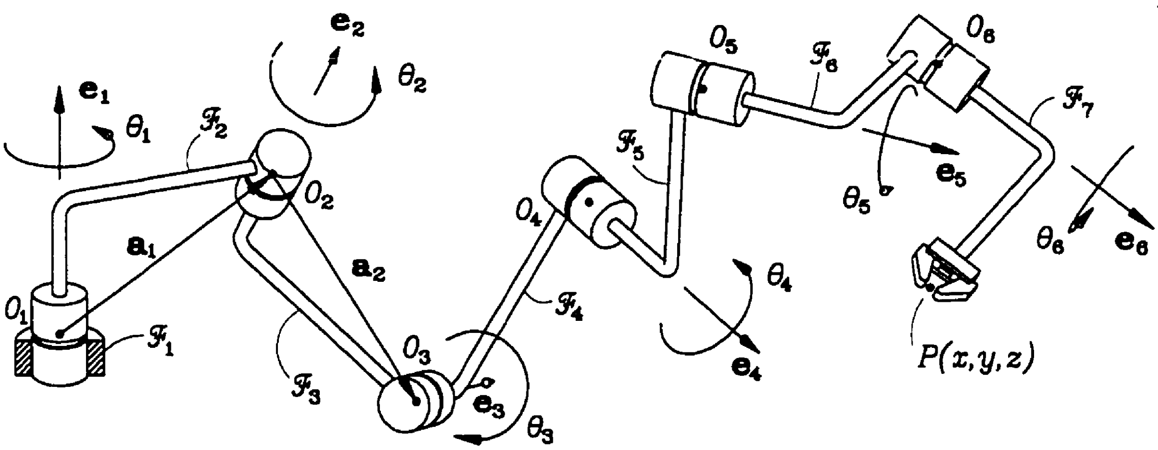


FIGURE 1.2. Axis and displacement vectors of a Puma robot

They represent an algebraic system of twelve nonlinear equations, nine of them for orientation and three for position. However, since $\mathbf{Q}_i$ is orthogonal, we only have three independent equations from eqn. (1.1.2) and three from eqn. (1.1.3) to solve for the six unknowns, namely, θ_i, $for\ i = 1 \ldots 6$. We need a few more definitions before continuing. First re-express eqn. (1.1.1) in two forms

$$\mathbf{Q}_i \equiv \begin{bmatrix} \mathbf{m}_i^T \\ \mathbf{n}_i^T \\ \mathbf{o}_i^T \end{bmatrix} \equiv \begin{bmatrix} \mathbf{p}_i & \mathbf{q}_i & \mathbf{u}_i \end{bmatrix} \tag{1.1.4}$$

From eqn. (1.1.1) one should note that the third row of $\mathbf{Q}_i$, $\mathbf{o}_i$, is independent of θ_i; later it will be shown how this observation helps eliminate one of the unknown angles. In Fig. 1.2 the unit vector $\mathbf{e}_i$ defines the direction of the i^{th} joint axis. Its components in $\mathcal{F}_i$ are

$$\mathbf{e} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

by definition from the DH notation. Thus we have

$$\mathbf{Q}_i\mathbf{o}_i \equiv \mathbf{Q}_i^T\mathbf{u}_i = \mathbf{e} \tag{1.1.5}$$

After introducing the following definition

$$\mathbf{a}_i = \mathbf{Q}_i\mathbf{b}_i$$

we can rewrite eqns. (1.1.2) & (1.1.3)

$$\begin{aligned}\mathbf{Q}_3\mathbf{Q}_4\mathbf{Q}_5 &= \mathbf{Q}_2^T\mathbf{Q}_1^T\mathbf{Q}\mathbf{Q}_6^T \\ \mathbf{Q}_3\left(\mathbf{b}_3 + \mathbf{Q}_4\mathbf{b}_4 + \mathbf{Q}_4\mathbf{Q}_5\mathbf{b}_5\right) &= \mathbf{Q}_2^T\mathbf{Q}_1^T\left(\mathbf{p} - \mathbf{Q}\mathbf{b}_6\right) - \left(\mathbf{b}_2 + \mathbf{Q}_2^T\mathbf{b}_1\right)\end{aligned}$$

Next, we derive six equations free of θ_6 by first introducing the following equations

$$\begin{aligned}\boldsymbol{\sigma} &\equiv \mathbf{Q}\mathbf{o}_6 \\ \boldsymbol{\rho} &\equiv \mathbf{p} - \mathbf{Q}\mathbf{b}_6 = \mathbf{p} - \mathbf{Q}\mathbf{Q}_6^T\mathbf{a}_6\end{aligned}$$

where $\mathbf{o}_6$ is the last row of $\mathbf{Q}_6$ matrix. So we obtain

$$\mathbf{Q}_3\mathbf{Q}_4\mathbf{u}_5 = \mathbf{Q}_2^T\mathbf{Q}_1^T\boldsymbol{\sigma} \tag{1.1.6}$$

$$\mathbf{Q}_3\left(\mathbf{b}_3 + \mathbf{Q}_4\mathbf{b}_4 + \mathbf{Q}_4\mathbf{Q}_5\mathbf{b}_5\right) = \mathbf{Q}_2^T\mathbf{Q}_1^T\boldsymbol{\rho} - \mathbf{b}_2 - \mathbf{Q}_2^T\mathbf{b}_1 \tag{1.1.7}$$

where the RHS of eqn. (1.1.7) is bilinear in $\mathbf{x}_1$ and $\mathbf{x}_2$ and the LHS is trilinear in $\mathbf{x}_3, \mathbf{x}_4$ and $\mathbf{x}_5$. If we separate the LHS and the RHS of both equations, we obtain the following

$$\begin{aligned}\mathbf{f} &\equiv \mathbf{Q}_3\left(\mathbf{b}_3 + \mathbf{Q}_4\mathbf{b}_4 + \mathbf{Q}_4\mathbf{Q}_5\mathbf{b}_5\right) \\ \mathbf{g} &\equiv \mathbf{Q}_2^T\mathbf{Q}_1^T\boldsymbol{\rho} - \left(\mathbf{b}_2 + \mathbf{Q}_2^T\mathbf{b}_1\right) \\ \mathbf{h} &\equiv \mathbf{Q}_3\mathbf{Q}_4\mathbf{u}_5 \\ \mathbf{i} &\equiv \mathbf{Q}_2^T\mathbf{Q}_1^T\boldsymbol{\sigma}\end{aligned}$$

However, not all of the six scalar equations, free of θ_6, are independent because LHS and the RHS of the second pair have to fulfill the following

$$\mathbf{h}\cdot\mathbf{h} = 1, \quad \mathbf{i}\cdot\mathbf{i} = 1$$

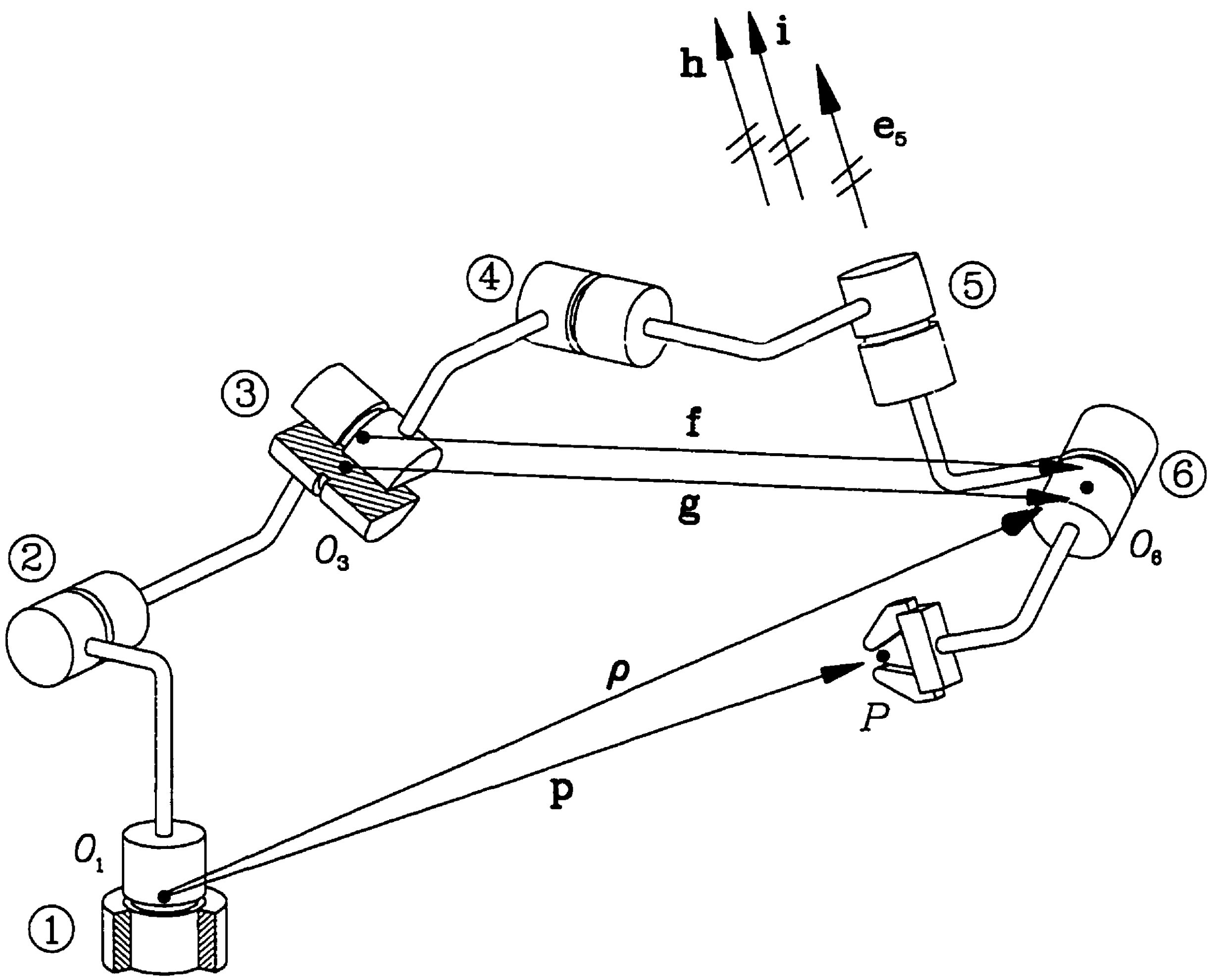


FIGURE 1.3. Key vectors associated with solving the IKP of a Puma robot

and so we have five independent equations in the two vector eqns. (1.1.6) and (1.1.7). There exists three methods used to eliminate four of the five unknowns from this system. Li, Woernle and Hiller (1991) as well as Raghavan and Roth (1990, 1993) approached the problem numerically so as to obtain a 16^{th} degree mono-variate polynomial in terms of the half-angle of the 5^{th} unknown joint angle. The 3^{rd} method by Angeles, and Etemadi Zangane (1992), used a semi-graphical approach to obtain two bivariate equations in two unknowns, from which they generated a set of contours.

1.1.2. A Semi-Graphical Method. From eqns. (1.1.8) it is possible to obtain the three vector equations

$$\mathbf{f} = \mathbf{g} \tag{1.8a}$$

$$\mathbf{h} = \mathbf{i} \tag{1.8b}$$

$$\mathbf{f} \times \mathbf{h} = \mathbf{g} \times \mathbf{i} \tag{1.8c}$$

To obtain a 4^{th} vector equation, the condition that the reflection of the vector **h** onto a plane perpendicular to vector **f** is equal to the reflection of **i** onto a plane perpendicular to **g** was used so

$$\mathbf{h} - 2\frac{\mathbf{f} \cdot \mathbf{h}^2}{\|\mathbf{f}\|}\mathbf{f} = \mathbf{i} - 2\frac{\mathbf{g} \cdot \mathbf{i}^2}{\|\mathbf{g}\|}\mathbf{g}$$

and since $\|\mathbf{f}\|^2 = \|\mathbf{g}\|^2$ we have

$$\|\mathbf{f}\|^2\mathbf{h} - 2(\mathbf{f} \cdot \mathbf{h})\mathbf{f} = \|\mathbf{g}\|^2\mathbf{i} - 2(\mathbf{g} \cdot \mathbf{i})\mathbf{g} \tag{1.8d}$$

The two additional scalar equations are derived as

$$\mathbf{f} \cdot \mathbf{f} = \mathbf{g} \cdot \mathbf{g} \tag{1.8e}$$

$$\mathbf{f} \cdot \mathbf{h} = \mathbf{g} \cdot \mathbf{i} \tag{1.8f}$$

LHS of all four vector equations are trilinear in $\mathbf{x}_3$, $\mathbf{x}_4$ and $\mathbf{x}_5$ and RHS are bilinear in $\mathbf{x}_1$ and $\mathbf{x}_2$. LHS of the both scalar equations are bilinear in $\mathbf{x}_4$, $\mathbf{x}_5$ and the RHS are linear in $\mathbf{x}_1$. This system is then separated into two groups of scalar equations:

- the first group contains the first two scalar equations of each of the four vector equations
- the second group is composed of the third scalar equations of each vector equation together with the two remaining scalar equations.

First group is represented in the following form

$$\mathbf{A}\mathbf{x}_{12} = \mathbf{b}$$

where $\mathbf{x}_{12}$ is bilinear in $\mathbf{x}_1$ and $\mathbf{x}_2$ and $\mathbf{b}$ is trilinear in $\mathbf{x}_3$, $\mathbf{x}_4$ and $\mathbf{x}_5$. The six equations of the second group are represented as follows

$$\mathbf{C}\mathbf{x}_1 = \mathbf{d}$$

where $\mathbf{d}$ is bilinear in $\mathbf{x}_4$ and $\mathbf{x}_5$. From these two groups we obtained six linear homogeneous equations of the form

$$\mathbf{D}\mathbf{y}_3 = \mathbf{0}_6$$

where

$$\mathbf{y}_3 \equiv \begin{bmatrix} \cos\theta_3 \\ \sin\theta_3 \\ 1 \end{bmatrix}$$

and $\mathbf{D}$ is a 6×3 matrix with entries bilinear in $\mathbf{x}_4$ and $\mathbf{x}_5$. After partitioning $\mathbf{D}$ into two 3×3 matrices $\mathbf{D}_u$ and $\mathbf{D}_l$ we obtained two conditions for the singularity of the matrix $\mathbf{D}$, namely

$$\begin{aligned} det\,(\mathbf{D}_l) \equiv f_1\,(\theta_4, \theta_5) &= 0 \\ det\,(\mathbf{D}_u) \equiv f_2\,(\theta_4, \theta_5) &= 0 \end{aligned}$$

and, since the non-trivial solution $\mathbf{y}_3$ has to satisfy both of the above equations, the third equation is imposed, namely

$$det\,(\mathbf{D}_u - \mathbf{D}_l) \equiv f_3\,(\theta_4, \theta_5) = 0$$

The set of contours c_1, c_2, and c_3 is defined by f_1, f_2, and f_3. The intersections of all three contours are valid solutions of the problem, while the points of intersection of only two contours are spurious solutions.

1.1.3. The Monovariate Polynomial Approach.

Procedure of Raghavan and Roth. This procedure begins with the equation

$$\mathbf{\Sigma x}_{45} = \mathbf{0}_9 \tag{1.9}$$

which is similar to the equation

$$\mathbf{D y}_3 = \mathbf{0}_6$$

used by Angeles and Etemadi Zanganeh. Here $\mathbf{\Sigma}$ is a 6 × 9 matrix whose entries are linear in $\mathbf{x}_3$ and with $\mathbf{x}_{45}$ defined as

$$\mathbf{x}_{45} \equiv \begin{bmatrix} s_4 s_5 & s_4 c_5 & c_4 s_5 & c_4 c_5 & s_4 & c_4 & s_5 & c_5 & 1 \end{bmatrix}^T$$

After using the relations for s_i and c_i in the form $\tau_i \equiv \tan\left(\frac{\theta_i}{2}\right)$, for $i = 4, 5$, and after multiplying LHS and RHS of eqn. (1.9) with $(1+\tau_4^2)(1+\tau_5^2)$,

$$\mathbf{\Sigma}'\mathbf{x}'_{45} = 0 \tag{1.10}$$

is obtained. After introducing the same trigonometric identities for $i = 3$ and multiplying the first four scalar equations by $(1+\tau_3^2)$ eqn. (1.10 becomes

$$\mathbf{\Sigma}''\mathbf{x}'_{45} = 0$$

where $\mathbf{\Sigma}''$ is a 6 × 9 matrix. The determinant of *any* 6 × 6 sub-matrix of matrix $\mathbf{\Sigma}''$ is an 8^{th} degree polynomial in τ_3. With the application of dialytic elimination [**1**] both τ_4 and τ_5 are removed. A new system of equations is then obtained in the form

$$\mathbf{S}\tilde{\mathbf{x}}_{45} = \mathbf{0}_{12}$$

where $\mathbf{S}$ is a 12 × 12 matrix. To obtain nontrivial solutions,

$$det\,(\mathbf{S}) = 0$$

must be satisfied and from there one obtains a 16^{th} degree mono-variate polynomial in τ_3.

Procedure of Li et al. Lee's procedure differs somewhat from the previous one. A matrix

$$\mathbf{T}_i = \begin{bmatrix} -\tau_i & 1 & 0 \\ 1 & \tau_i & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

multiplies a rotational matrix

$$\mathbf{C}_i = \begin{bmatrix} \cos(\theta_i) & -\sin(\theta_i) & 0 \\ \sin(\theta_i) & \cos(\theta_i) & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

$$\mathbf{T}_i\mathbf{C}_i \equiv \mathbf{U}_i = \begin{bmatrix} \tau_i & 1 & 0 \\ 1 & -\tau_i & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The matrix $\mathbf{U}_i$ is obtained using the identities

$$s_i - \tau_i c_i \equiv \tau_i, \quad \tau_i s_i + c_i \equiv 1$$

where $\tau_i = \tan(\frac{\theta_i}{2})$, $s_i = \sin(\theta)$, and $c_i = \cos(\theta_i)$. Then two new vectors, namely $\bar{\mathbf{f}}$ and $\bar{\mathbf{h}}$, are introduced

$$\begin{aligned} \bar{\mathbf{f}} &\equiv \mathbf{\Lambda}_3\left(\mathbf{b}_3 + \mathbf{Q}_4\mathbf{b}_4 + \mathbf{Q}_4\mathbf{Q}_5\mathbf{b}_5\right) \\ \bar{\mathbf{h}} &\equiv \mathbf{\Lambda}_3\left(\mathbf{Q}_4\mathbf{u}_5\right) \end{aligned}$$

where

$$\mathbf{\Lambda}_i = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\alpha_i) & -\sin(\alpha_i) \\ 0 & \sin(\alpha_i) & \cos(\alpha_i) \end{bmatrix}$$

such that $\mathbf{Q}_i = \mathbf{C}_i\mathbf{\Lambda}_i$. Then the four vector eqns. (1.8a - 1.8d) are multiplied by $\mathbf{T}_3$ to obtain the following set

$$\mathbf{U}_3\bar{\mathbf{f}} = \mathbf{T}_3\mathbf{g} \tag{1.11a}$$

$$\mathbf{U}_3\bar{\mathbf{h}} = \mathbf{T}_3\mathbf{i} \tag{1.11b}$$

$$\mathbf{U}_3\left(\bar{\mathbf{f}} \times \bar{\mathbf{h}}\right) = \mathbf{T}_3\left(\mathbf{g} \times \mathbf{i}\right) \tag{1.11c}$$

$$\mathbf{U}_3\left[\left(\mathbf{f} \cdot \mathbf{f}\right)\bar{\mathbf{h}} - 2\left(\mathbf{f} \cdot \mathbf{h}\right)\bar{\mathbf{f}}\right] = \mathbf{T}_3\left[\left(\mathbf{g} \cdot \mathbf{g}\right)\mathbf{i} - 2\left(\mathbf{g} \cdot \mathbf{i}\right)\mathbf{g}\right] \tag{1.11d}$$

The twelve equations above together with two additional scalar equations eqns. (1.8e & 1.8f) form the set of fourteen scalar equations. By using the fact that the third scalar equations in all four vector equations remain the same, because of the form of matrices $\mathbf{T}_3$ and $\mathbf{U}_3$, we can write the following

$$\bar{\mathbf{f}}_3 = \bar{\mathbf{g}}_3 \quad (1.12a)$$

$$\bar{\mathbf{f}}_3 = \bar{\mathbf{g}}_3 \quad (1.12b)$$

$$\bar{\mathbf{h}}_3 = \bar{\mathbf{i}}_3 \quad (1.12c)$$

$$(\bar{\mathbf{f}}_3 \times \bar{\mathbf{h}}_3) = (\bar{\mathbf{g}}_3 \times \bar{\mathbf{i}}_3) \quad (1.12d)$$

$$(\mathbf{f} \cdot \mathbf{f})\,\bar{\mathbf{h}}_3 - 2\,(\mathbf{f} \cdot \mathbf{h})\,\bar{\mathbf{f}}_3 = (\mathbf{g} \cdot \mathbf{g})\,\bar{\mathbf{i}}_3 - 2\,(\mathbf{g} \cdot \mathbf{i})\,\bar{\mathbf{g}}_3 \quad (1.12e)$$

all of which are free of θ_3. Then, by multiplying both sides of eqns. (1.12a - 1.12e) and eqns. (1.8e & 1.8f) with τ_3 we obtain six new scalar equations, which will give us a set of twenty scalar equations. This set can be cast in form

$$\mathbf{A}\mathbf{x} = \boldsymbol{\beta}$$

where the vector $\boldsymbol{\beta}$ is trilinear in τ_3, $\mathbf{x}_1$ and $\mathbf{x}_2$ and matrix $\mathbf{A}$ is a 20×16 matrix and it is a function of design parameters only. $\mathbf{A}$ and $\boldsymbol{\beta}$ are then partitioned as

$$\mathbf{A} \equiv \left[\begin{array}{cc} \mathbf{A}_u & \mathbf{A}_l \end{array}\right], \quad \boldsymbol{\beta} \equiv \left[\begin{array}{cc} \boldsymbol{\beta}_u & \boldsymbol{\beta}_l \end{array}\right]$$

After solving for $\mathbf{x}$ from $\mathbf{A}_u$ we obtain

$$\mathbf{A}_l\mathbf{A}_u^{-1}\boldsymbol{\beta}_u = \boldsymbol{\beta}_l$$

These equations can be presented in the form

$$(A_i c_2 + B_i s_2 + C_I)\,\tau_3 + D_i c_2 + E_i s_2 + F_I = 0, \; i = 1, 2, 3, 4$$

which after expressing s_i and c_i in the form of τ_i, we obtain

$$(C_i - A_i)\,\tau_2^2\tau_3 + 2B_i\tau_2\tau_3 + (A_i + C_i)\,\tau_3 + (F_i - D_i)\,\tau_2^2 + 2E_i\tau_2 + D_i = 0$$

Then with the multiplication of the above equations with τ_2 we will obtain 4 more equations resulting in the system of 8 equations which can be represented in the form

$$\mathbf{Mz} = 0$$

and whose nontrivial solution is obtained by setting

$$det\,(\mathbf{M}) = 0$$

This will give us a polynomial octic in $\mathbf{x}_1$, *i.e.*, of 16th order in $\tan\frac{\theta_1}{2}$, which is the mono-variate polynomial in mind. Other elimination methods have been introduced by others but all use eqns. (1.1.2 & 1.1.3) as the basis for the constraints. Hence, elimination proceeds, by definition, in a metric geometry; the geometry of direct isometries. It is argued that the elimination procedure can be *simplified* if one adopted a more *general* geometry.

1.2. Systematic Discussion of Geometry and Its Foundations

It will be illustrated, using DQs and Grassmannians, that the *special* structure of a metric geometry is not necessary to solve the IKP. It is *special* because metric geometry is a specific case of other *kinds of geometry*. Paraphrasing Klein [**2**], to extract these other geometries we must proceed as "chemists" who apply stronger reagents to isolate more of the valuable ingredients from raw material. Our reagents will be affine, followed by projective transformations. It is by means of this process that an understanding of projective geometry, where the formulation of the IK of the 6Rsm will take place, will be gained.

We begin with the following general point transformation:

$$\begin{aligned} x' &= a_1x + b_1y + c_1z + d_1 \\ y' &= a_2x + b_2y + c_2z + d_2 \\ z' &= a_1x + b_3y + c_3z + d_3 \end{aligned} \tag{1.1}$$

If it is a direct isometry, the following two conditions are required

$$\mathbf{Q}\mathbf{Q}^T = \mathbf{I}_{3\times 3} \tag{1.2}$$

$$det\,(\mathbf{Q}) = +1 \tag{1.3}$$

where

$$\mathbf{Q} = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} \tag{1.4}$$

If we relax the above conditions such that

$$\mathbf{Q}\mathbf{Q}^T = \lambda^2 \mathbf{I}_{3\times 3}$$

$$det\,(\mathbf{Q}) = \pm\lambda$$

then we include two more transformations, namely reflections and similarity transformations with the origin as center. The result is a geometry in which invariant properties include the fixed distance between two points given (x_1, y_1, z_1), and (x_2, y_2, z_2), $\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2}\lambda$ remains unchanged (*i.e.* independent of the location of the points). Similarly, one can say the same about angle measures subtended by three such points or that five points in a plane define a conic with particular principal axis and inter-focal length. The totality of these geometric properties belongs to metric geometry.

If we begin again with eqn. (1.1) but this time unconstrain the relations among the coefficients, we obtain the group of affine transformations. Notice that metric geometry is therefore a subgroup of affine transformations. The aggregate of concepts and theorems which remain invariant under affine transformations form *affine geometry*. The notion of distance and angle disappear as do the distinctions between circle and ellipse. But one can distinguish between finite and infinite space and everything which depends on these two spaces such as parallelism of two lines or distinction of conics among ellipses, hyperbolas, parabolas, etc.

Proceeding to projective transformations, we begin by introducing linear fractional transformations

$$
\begin{aligned}
x' &= (a_1x + b_1y + c_1z + d_1) : (a_4x + b_4y + c_4z + d_4), \\
y' &= (a_2x + b_2y + c_2z + d_2) : (a_4x + b_4y + c_4z + d_4), \qquad (1.5) \\
z' &= (a_3x + b_3y + c_3z + d_3) : (a_4x + b_4y + c_4z + d_4),
\end{aligned}
$$

which includes affine transformations as a subgroup. Here, the role of infinity and the concepts connected with it in affine geometry disappear. There is only one *proper* conic because all others can be derived from it.

By introducing homogeneous coordinates $(\xi : \eta : \zeta : \tau)$ such that

$$x = \frac{\xi}{\tau} \quad y = \frac{\eta}{\tau} \quad z = \frac{\zeta}{\tau}$$

eqn. (1.5) can be written as

$$
\begin{aligned}
\rho'\xi' &= (a_1\xi + b_1\eta + c_1\zeta + d_1\tau), \\
\rho'\eta' &= (a_2\xi + b_2\eta + c_2\zeta + d_2\tau), \qquad (1.6) \\
\rho'\zeta' &= (a_3\xi + b_3\eta + c_3\zeta + d_3\tau), \\
\rho'\tau' &= (a_4\xi + b_4\eta + c_4\zeta + d_4\tau),
\end{aligned}
$$

The origin in the homogeneous space of $\{\xi, \eta, \zeta, \tau\}$ is removed or nonexistent; at least one homogeneous coordinate must be non-zero. The group of projective transformations is termed collineations because three collinear points remain collinear after transformation by eqn. (1.6). A collineation is an affine transformation if it sends the plane at infinity into itself, which means that, to every point with $\tau = 0$, there corresponds another with $\tau' = 0$. From eqn. (1.6) this happens when $a_4 = b_4 = c_4 = 0$. By dividing eqn. (1.6) by $\rho'\tau'$ and replacing $a_1 : d_4$ by a_1, $b_1 : d_4$ by b_1, and so on we obtain eqn. (1.1). In order to obtain the group of similarity transformations from the projectivities, we leave invariant not only the plane at infinity (i.e. $\tau = 0$) but also the imaginary spherical circle (i.e. $\xi^2 + \eta^2 + \zeta^2 = 0$). To prove this, let $\tau = 0$

and replace (ξ, η, ζ) by (ξ', η', ζ') in the quadratic condition

$$\begin{aligned}
(\xi')^2 + (\eta')^2 + (\zeta')^2 &= (a_1\xi + b_1\eta + c_1\zeta)^2 + (a_2\xi + b_2\eta + c_2\zeta)^2 + (a_3\xi + b_3\eta + c_3\zeta)^2 \\
&= (a_1^2 + a_2^2 + a_3^2)\xi^2 + (b_1^2 + b_2^2 + b_3^2)\eta^2 + (c_1^2 + c_2^2 + c_3^2)\zeta^2 \\
&\quad +2(a_1b_1 + a_2b_2 + a_3b_3)\xi\eta + 2(a_1c_1 + a_2c_2 + a_3c_3)\xi\zeta \\
&\quad +2(c_1b_1 + c_2b_2 + c_3b_3)\eta\zeta \\
&= \lambda^2\xi^2 + \lambda^2\eta^2 + \lambda^2\zeta^2 \\
&= \xi^2 + \eta^2 + \zeta^2 \\
&= 0
\end{aligned}$$

Hence we can quickly point out that the six conditions which leave the imaginary spherical circle invariant define the group of similarity transformations. Finally, the direct isometries used to solve the IKP are just particular cases of projectivities. It turns out that as long as one preserves the isotropy of Euclidean space, the IKP can be formulated entirely in a projective space.

Chapter 2

Dual Quaternions, A Map, Fiber, Critical Locus

2.1. Repositioning the Quaternion

Conventional notions of the quaternion are attributed to A. Cayley's [**3**] view that the rotation of a vector, by means of a quaternion, is in general a bilinear operation.

But Sir W. R. Hamilton, who invented quaternions, saw them to be much more than just an artificial method in reconstructing known results. Hamilton used the word quaternion[1] to denote a new mathematical method "as an extension, or improvement, of Cartesian geometry, which the artifices of coordinate axes, are got rid of, all directions in space being treated on precisely the same terms. It is therefore, except in some degraded forms, possessed of the perfect isotropy of Euclidean space [4]." Such was his view that he questioned the necessity to have coordinate axes, *i.e.*, the columns in the orthogonal matrix **Q**, when expressing a rotation of Euclidean space. According to his biographer, Peter Guthrie Tait, Hamilton refrained from trying to convince his fellow mathematicians of this intrinsic nature of the quaternion - represented by the novel use of the three imaginaries **i**, **j**, and **k** - because he feared his

[1] meaning a set of four

method wouldn't be recognized by those steeped in Cartesianism [5]. It is believed that these arguments justify a short review of the quaternion as seen by its originator.

Geometrically, the quaternion q can be viewed as the quotient of two vectors in space

$$q = \mathbf{v}_2\mathbf{v}_1^{-1}$$

where $\mathbf{v}\mathbf{v}^{-1} = \|\mathbf{v}\|^2$. How Hamilton derived the representation of a vector using the imaginary quantity $\sqrt{-1}$ is an interesting story worth reading [4]. For our purposes we will only mention the result. Given the vectors

$$\begin{aligned} \mathbf{v}_2 &= x'\mathbf{i} + y'\mathbf{j} + z'\mathbf{k} \\ \mathbf{v}_1 &= x\mathbf{i} + y\mathbf{j} + z\mathbf{k} \\ q &= \mathbf{v}_2\mathbf{v}_1^{-1} \\ &= (xx' + yy' + zz') + (z'y - y'z)\mathbf{i} + (x'z - z'x)\mathbf{j} + (y'x - x'y)\mathbf{k} \end{aligned}$$

The three distinct space units $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ follow the multiplication rules $\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = -1$, $\mathbf{ij} = -\mathbf{ji} = \mathbf{k}$, $\mathbf{jk} = -\mathbf{kj} = \mathbf{i}$, $\mathbf{ki} = -\mathbf{ik} = \mathbf{j}$. It is the use of these units which allows one to represent a rotation without artifices coordinate axes. Using the notation of the previous section, the columns of $\mathbf{Q}$ represent the new coordinate axes resulting from the rotation of the inertial frame; but one can assume to know everything about the rotation using only a quaternion q. If the rotation were to happen normal to the vector one can write the transformation as

$$q\mathbf{v}_1 \rightarrow \mathbf{v}_2$$

In this case q is a linear operator which rotates $\mathbf{v}_1$ into $\mathbf{v}_2$ and therefore its axis is perpendicular to both vectors. Rotating a vector about an arbitrary axis is expressed as a bilinear operation

$$\mathbf{v}_2 = q\mathbf{v}_1q^{-1}$$

which was popularized by Cayley [3], even though Hamilton knew of the results independently [3] [4]. In essence, Cayley views rotation, by means of a quaternion, as a bilinear operation, whereas Hamilton views it as a linear operation. By adopting insights from both camps we obtain a quaternion which represents general rotations while its multiplication is limited to a linear operation.

2.2. Dual Quaternions as General Displacements

A quaternion, whether Cayley's or Hamilton's, represents a rotation. A general displacement may also include the displacement of the origin represented by a position vector **d**. By multiplying **d** by a quaternion q

$$y = -\frac{\mathbf{d}}{2}q$$

we have another quaternion y. A spatial displacement can now be represented by a pair of quaternions or, more concisely, by a dual quaternion

$$\hat{q} = q + \epsilon y$$

One can find a detailed account of this representation in [6], pp. 521-524. The eight terms which make up $\hat{q}$ are the eight homogeneous coordinates known as the Study soma. The soma maps the set of Euclidean displacements into a six-dimensional manifold in a seven dimensional projective space [7]. Consider the toroidal 6-space, $\mathcal{T}^6$, of six R-joint displacements, θ, where θ is an element of $\mathcal{T}^6$. Consider also the 6-space of Euclidean displacements $D \in \mathcal{D}^6 = SO^3 \times \mathcal{A}^3$ where D is an element of $\mathcal{D}^6$, the product of the 3-space of rotations SO^3 and the 3-space of point displacements $\mathcal{A}^3$. This represents an end effector (EE) displacement. $\psi : \mathcal{T}^6 \rightarrow \mathcal{D}^6$ maps the six joint angles θ into an EE pose D via the product of six dual quaternions $\theta \mapsto \hat{q}_1\hat{q}_2\hat{q}_3\hat{q}_4\hat{q}_5\hat{q}_6$. $\hat{q}$ is a point which represents a screw in a homogeneous projective 8-space. $\hat{q}_i(\theta)$ transforms a rigid-body reference frame $\mathcal{F}_i \rightarrow \mathcal{F}_{i+1}$ via joint angle θ_i, while D is expressed as $\hat{q}_7$, which transforms the base frame (FF) $\mathcal{F}_1 \rightarrow \mathcal{F}_7$ directly into EE frame. Recall that

$$\hat{q}_i = (r_0 + \epsilon y_0) + (r_1 + \epsilon y_1)\mathbf{i} + (r_2 + \epsilon y_2)\mathbf{j} + (r_3 + \epsilon y_3)\mathbf{k}$$

where $\epsilon^2 = 0$, $\mathbf{ii} = \mathbf{jj} = \mathbf{kk} = -1$, $\mathbf{ij} = -\mathbf{ji} = \mathbf{k}$, $\mathbf{ki} = -\mathbf{ik} = \mathbf{j}$, and $\mathbf{jk} = -\mathbf{kj} = \mathbf{i}$. The Euler-Rodrigues parameters are $\{r_i :\}_0^3$ while $\{y_i :\}_0^3 = -\mathbf{a}(r_0 + r_1\mathbf{i} + r_2\mathbf{j} + r_3\mathbf{k})/2$ accounts for position vector $\mathbf{a}$ directed from origin O_i of frame $\mathcal{F}_i$ to O_j of frame $\mathcal{F}_j$. Sets of eight coordinates belong to Study's soma S_6^2 and are hence screws if they belong to the varieties $\mathbf{V}(\Sigma_{i=0}^3 y_i r_i = 0)$ and $\mathbf{V}'(\Sigma_{i=0}^3 r_i^2 \neq 0)$. A *generic fiber* of map ψ consists of a number of points, counted with multiplicity, in $\mathcal{T}^6$, which map into the same point in S_6^2. A fiber of ψ admits 16 solutions, fewer if it belongs to a critical locus, *i.e.*, where the Jacobian matrix $\mathbf{J}$ is singular. For a 6R serial manipulator $\mathbf{J}$ is a 6×6 matrix, with columns arranged in frame sequence from fixed (FF) to end effector (EE) frame, of Plücker line coordinates of the six R-joint axes with respect to the operating point P on EE. If the rank of $\mathbf{J}$, $\rho < 6$, then the lines belong to a complex ($\rho = 5$), congruence ($\rho = 4$) or series ($\rho = 3$). Throughout the critical loci, the axes assume such configurations. But a fiber (solution set) containing critical loci (singularities) may also contain noncritical elements (nonsingular solutions) and a good algorithm should find the 16^{th}-order univariate polynomial or detect if ψ is singular. If it is singular, then the algorithm must separate the solution set into its critical and noncritical elements. This requires the facility to decompose a higher-dimensional projective space into incident linear subspaces. For this reason it is useful to examine Klein's [2] Grassmannian treatment of incidence relations of simplectic segments of planes, lines and points in the plane and these three as well as the tetrahedral simplex in 3-space.

Chapter 3

Grassmannians

The proposed algorithm makes extensive use of Grassmannians. Their meaning, as far as the algorithm is concerned, can be best explained by constructing Grassmannians in the 2D plane and in 3D space.

3.1. Grassmannians in the Plane

Consider 3×3 matrices with rows of homogeneous point coordinates.

$$\mathbf{E}_2 = \begin{bmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{bmatrix} \tag{3.1}$$

If the rank of $\mathbf{E}_2$ is $\rho = 3$, then the determinant is $2!$ times the area of the triangle on the three points. If $\rho = 2$, three 2×2 subdeterminants, taking any two rows, are the coordinates of the line on the three collinear points. If $\rho = 1$, all three coordinates are the same point and multiple combinations of each other. Note that if $\rho = 2$ (or $\rho = 1$) one may manipulate the matrix to obtain one (or two) row(s) of $(0 : 0 : 0)$.

Example: Line Equation as an Incidence Relation

In the plane, one can view three collinear points as spanning a triangle with zero area

$$\begin{aligned} det\,(\mathbf{E}_2) &= 0 \\ (y_1 - y_2)x - (x_1 - x_2)y + (x_1y_2 - x_2y_1) &= 0 \end{aligned}$$

Incidence relations are not restricted to linear configurations.

Example: The Circle Equation as an Incident Relation

Find the circle on three given points.

$$\begin{vmatrix} x^2+y^2 & x & y & 1 \\ x_1^2+y_1^2 & x_1 & y_1 & 1 \\ x_2^2+y_2^2 & x_2 & y_2 & 1 \\ x_3^2+y_3^2 & x_3 & y_3 & 1 \end{vmatrix} = 0$$

$$\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} (x^2+y^2) - \begin{vmatrix} x_1^2+y_1^2 & y_1 & 1 \\ x_2^2+y_2^2 & y_2 & 1 \\ x_3^2+y_3^2 & y_3 & 1 \end{vmatrix} x +$$
$$\begin{vmatrix} x_1^2+y_1^2 & x_1 & 1 \\ x_2^2+y_2^2 & x_2 & 1 \\ x_3^2+y_3^2 & x_3 & 1 \end{vmatrix} y - \begin{vmatrix} x_1^2+y_1^2 & x_1 & y_1 \\ x_2^2+y_2^2 & x_2 & y_2 \\ x_3^2+y_3^2 & x_3 & y_3 \end{vmatrix} 1 = 0$$

It is interesting to note that the determinant multiplying the term (x^2+y^2) is 2! times the area of the triangle defined by the three points. If the three points are collinear then this term disappears and we are left with a line equation!

3.2. Grassmannians in Space

In space, points have four homogeneous coordinates; so, consider 4×4 matrices. If $\rho = 4$, the determinant is 3! times the volume of the tetrahedron on the four points. If $\rho = 3$, four 3×3 determinants, taken from any three rows, give the coordinates of the plane on three points. If $\rho = 2$, from any two rows, six 2×2 matrices give the Plücker ray coordinates of the line on two points, while if $\rho = 1$, there are four versions of the same point. It will now be shown how the deficient rows provide the key to incident relations. For example, three points don't define a plane if they are collinear or coincident. If they don't define a plane, they can only produce the nonexistent coordinates $(0:0:0:0)$.

Example: Incident Relations of a Line on a Plane

Incidence relations of a line $\mathcal{L} = (p_{01} : p_{02} : p_{03} : p_{23} : p_{31} : p_{12})$ and a plane $\pi = (\eta_0 : \eta_1 : \eta_2 : \eta_3)$ are summarized below.

Lemma 1 *The intersection* $\mathcal{L} \cap \pi = (x_0 : x_1 : x_2 : x_3)$ *either defines a unique point in* $\mathcal{P}^3$ *or* $\mathcal{L} \subset \pi$.

Proof By Bezout's Theorem the point is unique because varieties $\mathcal{L}$ and π are linear. The only nonexistent invalid point is $\{0 : 0 : 0 : 0\}$ because even if π is parallel to $\mathcal{L}$ they nevertheless intersect at a unique valid point at infinity (*i.e.* $x_0 = 0$ and $x_1^2 + x_2^2 + x_3^2 \neq 0$). Therefore, to determine $\mathcal{L} \subset \pi$ one "finds a point" with the invalid coordinates by evaluating a singular 4×4 "point-equation" determinant. The first row contains the linearly-dependent, variable plane coordinates of a fourth plane which is on the point usually defined at the intersection of the three given planes whose coordinates populate the three lower rows. Recall that the coefficients of the linear equation that results from expanding the first row minors are the coordinates of the point of intersection. In what follows ζ_i are the fourth variable plane coordinates, η_i are the coordinates of π, and η_i' and η_i'' are the coordinates of any two planes on $\mathcal{L}$ so as to generate its axial Plücker coordinates.

$$\begin{vmatrix} \zeta_0 & \zeta_1 & \zeta_2 & \zeta_3 \\ \eta_0 & \eta_1 & \eta_2 & \eta_3 \\ \eta_0' & \eta_1' & \eta_2' & \eta_3' \\ \eta_0'' & \eta_1'' & \eta_2'' & \eta_3'' \end{vmatrix} = 0$$

We expand on first row minors, collect terms, note that the coefficients generated by the three numerical rows, $\eta_i, \eta_i', \eta_i''$, are x_i, and axially convert the ray Plücker coordinates of $\mathcal{L}$ for the lower two rows. With the *Kronecker-*δ_{ij}, this produces four sufficient incident relationships.

$$x_i = \sum_{j=0}^{3} (1 - \delta_{ij}) \eta_j p_{ij} = 0 \;:\; i = 0 \ldots 3,\; p_{ij} = -p_{ji}$$

As explained by Klein, [**2**], the *Grassmannian Extension Principle*, [**8**], illustrated above for 2- on 3-dimensional subspaces incidence, both on a vector 4-space, is valid in higher dimension too.

Chapter 4

Synopsis of the IKP of a 6R Serial Manipulator

Applying the tools introduced earlier and including theories in algebraic geometry and commutative algebra, as outlined in Appendix A, formulation and solution of two examples will be treated as follows.

- Converting a DH table to dual quaternions
- The algorithm
  - (i) The constraint equations
  - (ii) First elimination step
  - (iii) Second elimination step
  - (iv) The characteristic polynomial
  - (v) The extension step

There exists a simple geometric problem which reveals the algorithm's intent. The intersection of two circles in the plane is a problem in high school mathematics but it is not obvious that the problem is equivalent to the intersection of a paraboloid and two planes in 3D space. Let us begin with the general homogeneous equation of two circles

$$A(x^2+y^2)+Bxz+Cyz+Dz^2 = 0 \tag{4.1}$$

$$A'(x^2+y^2)+B'xz+C'yz+D'z^2 = 0 \tag{4.2}$$

Now we introduce a polynomial map

$$\begin{aligned}\eta_1 &= xz\\ \eta_2 &= yz\\ \eta_3 &= x^2+y^2\\ \eta_4 &= z^2\end{aligned} \tag{4.3}$$

The above map obviously constrains the new coordinates η_i, for $i = 1 \ldots 4$:

$$\eta_1^2+\eta_2^2-\eta_3\eta_4 = 0 \tag{4.4}$$

Substituting the monomials of eqns. (4.1) and (4.2) according to eqns. (4.3) produces

$$A\eta_1+B\eta_2+C\eta_3+D\eta_4 = 0 \tag{4.5}$$

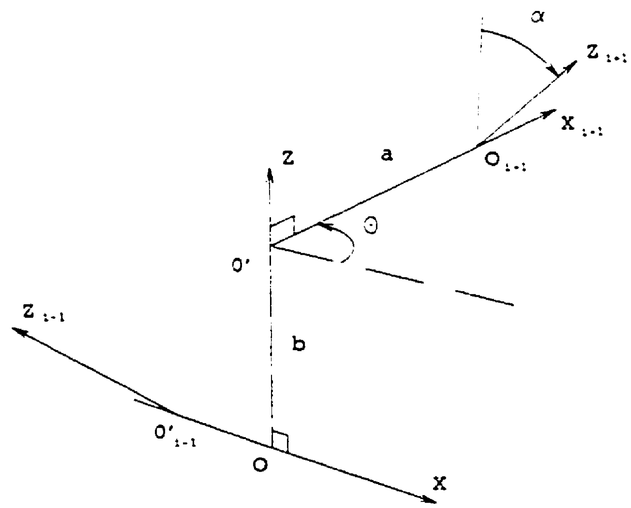
$$A'\eta_1+B'\eta_2+C'\eta_3+D'\eta_4 = 0 \tag{4.6}$$

In summary, eqn. (4.3) transforms the 2D plane to a paraboloid, eqn. (4.4), in 3D space and turns circles, eqns. (4.1) & (4.2), into 3D planes, eqns. (4.5) & (4.6). By extending the space of solutions to a manifold, having the same dimensions, embedded in a higher space, nonlinear constraints can be made linear. The extension can also lead to a clearer understanding of the original problem. In the circle example above, one can easily show that two concentric circles map into a line at ∞ in the new 3D space.

Chapter 5

Converting a DH Table to Dual Quaternions

Constraint equations arise as a product of dual quaternions, six of which are obtained from joint coordinates and Denavit-Hartenberg (DH) parameters. The seventh represents the given pose of EE. The figure below illustrates how a row in the DH table effects a frame change.



Frame $\mathcal{F}_i$ is transformed to $\mathcal{F}_{i+1}$ by a dual quaternion composed of four primitive ones. The first is a pure translation b_i along Z_i:

$$r_0 + r_1\mathbf{i} + r_2\mathbf{j} + r_3\mathbf{k} = 1,\ y_0 + y_1\mathbf{i} + y_2\mathbf{j} + y_3\mathbf{k} = -\frac{b_i\mathbf{k}}{2}1,\ \hat{q}_{b_i} = 1 - \frac{b_i}{2}\epsilon\mathbf{k}$$

The second is a pure rotation about Z_i by θ_i:

$$r_0 + r_1\mathbf{i} + r_2\mathbf{j} + r_c hapter5.auxchapter5.tex3\mathbf{k} = \cos(\frac{\theta_i}{2}) + \sin(\frac{\theta_i}{2})\mathbf{k},\ y_0 + y_1\mathbf{i} + y_2\mathbf{j} + y_3\mathbf{k} = 0$$

so $\hat{q}_{\theta_i} = \cos(\theta_i/2) + \sin(\theta_i/2)\mathbf{k}$. Similarly, $\hat{q}_{a_i} = 1 - (a_i/2)\epsilon\mathbf{i}$, $\hat{q}_{\alpha_i} = \cos(\alpha_i/2) + \sin(\alpha_i/2)\mathbf{i}$

Finally,

$$\hat{q}_i = \hat{q}_{b_i}\hat{q}_{\theta_i}\hat{q}_{a_i}\hat{q}_{\alpha_i} \tag{5.7}$$

Now we convert a rotation matrix $\mathbf{Q}$ and vector $\mathbf{a} = \mathbf{d}$ into a dual quaternion. The Euler-Rodriguez parameters are computed as

$$\{r_1\ r_2\ r_3\}^T = \text{vect}(\sqrt{\mathbf{Q}}) \tag{5.8a}$$

$$r_0 = [\text{tr}(\sqrt{\mathbf{Q}}) - 1]/2 \tag{5.8b}$$

and the rest of the Study soma is

$$-\frac{\mathbf{d}}{2}(r_0 + r_1\mathbf{i} + r_2\mathbf{j} + r_3\mathbf{k}) \tag{5.8c}$$

thereby expressing a dual quaternion $\hat{q}_{EE}$ which transforms FF into EE. Now we illustrate the IK algorithm with two examples, *i.e.*, the Fanuc Arc Mate and Diestro. The second is given a pose which will require treatment of a fiber with singular and non-singular values. Design parameters, here and elsewhere, are from Angeles [9]. Furthermore one should note that the group structure of a general displacement also manifests itself in these special dual quaternions. Furthermore we have the conjugate $\bar{\hat{q}}$ of a dual quaternion $\hat{q}$ such that

$$\hat{q} = \hat{S} + \hat{\mathbf{V}}$$

$$\bar{\hat{q}} = \hat{S} - \hat{\mathbf{V}}$$

$$\hat{q}\bar{\hat{q}} = \hat{S}^2 + ||\hat{\mathbf{V}}||^2$$

where $\hat{S}$ is a dual number and $\hat{\mathbf{V}}$ is a dual vector. If $\hat{q}$ lies on the Study soma, S_6^2, then

$$\hat{q}\bar{\hat{q}} = r_0^2 + r_1^2 + r_2^2 + r_3^2 \neq 0$$

Chapter 6

The Algorithm

The algorithm is divided into five steps:

- Step 1. The constraint equations.
- Step 2. First elimination step.
- Step 3. Second elimination step.
- Step 4. The characteristic polynomial.
- Step 5. The extension step.

Step 1. The constraint equations

Constraint equations arise as a product of dual quarternions, $\hat{q}_i$, six of which are obtained from joint coordinates and Denavit-Hartenberg (DH) parameters and the seventh, $\hat{q}_{EE}$, is given by the pose of the end-effector:

$$\begin{aligned} \hat{q}_1\hat{q}_2\hat{q}_3\hat{q}_4\hat{q}_5\hat{q}_6\tilde{\hat{q}}_{EE} &= \hat{q}_{EE}\tilde{\hat{q}}_{EE} \\ &= (f_0 + \epsilon f_1) + (f_2 + \epsilon f_3)\mathbf{i} \\ &\quad +(f_4 + \epsilon f_5)\mathbf{j} + (f_6 + \epsilon f_7)\mathbf{k} \end{aligned}$$

to produce seven homogeneous equations:

$$I = \langle f_1 = 0, f_2 = 0, f_3 = 0, f_4 = 0, f_5 = 0, f_6 = 0, f_7 = 0 \rangle.$$

Step 2. First Elimination Step

Introduce a polynomial map $\mathcal{M}_3$:

$$\mathbf{x} = \begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \end{bmatrix} \equiv \begin{bmatrix} c_4c_5c_6 \\ c_4c_5s_6 \\ c_4s_5c_6 \\ c_4s_5s_6 \\ s_4c_5c_6 \\ s_4c_5s_6 \\ s_4s_5c_6 \\ s_4s_5s_6 \end{bmatrix}$$

Substitute the expressions above into I and compute the Grassmannian:

$$\begin{aligned} \begin{vmatrix} \eta \\ \frac{\partial I}{\partial \mathbf{x}} \end{vmatrix} &= 0 \\ x_0\eta_0 + x_1\eta_1 + x_2\eta_2 + x_3\eta_3 + x_4\eta_4 + x_5\eta_5 + x_6\eta_6 + x_7\eta_7 &= 0 \end{aligned}$$

where $\eta = [\eta_0, \ldots, \eta_7]^T$. The implicit representation of $\mathcal{M}_3$ is:

$$\begin{aligned} \mathbf{I}(\mathcal{M}_3) = & \langle x_7x_0 - x_3x_4, x_7x_4 - x_6x_5, x_7x_2 - x_6x_3, \\ & x_5x_0 - x_1x_4, x_6x_0 - x_2x_4, x_7x_1 - x_5x_3, \\ & x_7x_0 - x_6x_1, x_7x_0 - x_5x_2, x_3x_0 - x_1x_2\rangle \end{aligned}$$

Substitute expressions for $\mathbf{x}$ into $\mathbf{I}(\mathcal{M}_3)$ to obtain the projection:

$$\Pi_1(\mathcal{M}_3) = S = \mathbf{V}(s_1, \ldots, s_9) \subset \mathcal{P}^1 \times \mathcal{P}^1 \times \mathcal{P}^1$$

Step 3. Second Elimination Step

Introduce a polynomial map $\mathcal{M}_2$:

$$\mathbf{y} = \begin{bmatrix} y_0 \\ y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \\ y_6 \\ y_7 \\ y_8 \end{bmatrix} \equiv \begin{bmatrix} c_2^2 c_1 s_1 \\ c_2^2 s_1^2 \\ c_2 s_2 c_1^2 \\ c_2 s_2 c_1 s_1 \\ c_2 s_2 s_1^2 \\ s_2^2 c_1^2 \\ s_2^2 s_1 c_1 \\ s_2^2 s_1^2 \\ c_2^2 c_1^2 \end{bmatrix}$$

Substitute the expressions above into S and compute the Grassmannian:

$$\begin{aligned} \begin{vmatrix} \xi \\ \frac{\partial S}{\partial \mathbf{y}} \end{vmatrix} &= 0 \\ y_0\xi_0 + y_1\xi_1 + y_2\xi_2 + y_3\xi_3 + y_4\xi_4 + y_5\xi_5 + y_6\xi_6 + y_7\xi_7 + y_8\xi_8 &= 0 \end{aligned}$$

where $\xi = [\xi_0, \ldots, \xi_8]^T$. The implicit representation of $\mathcal{M}_2$ consists of 20 homogeneous equations in $\mathbf{y}$.

$$\mathbf{I}(\mathcal{M}_3) = \langle t_1 \ldots t_{20} \rangle$$

Step 4. The Characteristic Polynomial

Substituting expressions for $\mathbf{y}$ into $\mathbf{I}(\mathcal{M}_2)$ results in a projection

$$\Pi_2(\mathcal{M}_2) = T = \mathbf{V}(t_1, \ldots, t_{20}) \subset \mathcal{P}^1$$

Each generator of $I_T = \langle t_1, \ldots, t_{20} \rangle \subset k[c_3, s_3]$ factors into two 16^{th} order univariates $t_i = hg_i$; h is the characteristic univariate and the other is different for each generator t_i. Geometrically, we can express hg_i as the union of two hypersurfaces:

$$\mathbf{V}(hg_i) = \mathbf{V}(h) \cup \mathbf{V}(g_i)$$

therefore:

$$
\begin{aligned}
T &= \mathbf{V}(h) \cup \mathbf{V}(g_1) \cap \ldots \cap \mathbf{V}(h) \cup \mathbf{V}(g_{20}) \\
&= \mathbf{V}(h) \cup (\mathbf{V}(g_1) \cap \ldots \cap \mathbf{V}(g_{20})) \\
&= \mathbf{V}(h) \cup \{\emptyset\} \\
&= \mathbf{V}(h)
\end{aligned}
$$

Step 5. Extension Step

Looking at both polynomial maps introduced earlier we see that:

$$
\begin{aligned}
x_2 : x_6 &= c_4 : s_4, \\
x_5 : x_6 &= c_5 : s_5, \\
x_2 : x_3 &= c_6 : s_6, \\
y_0 : y_1 &= c_1 : s_1, \\
y_2 : y_5 &= c_2 : s_2
\end{aligned}
$$

Chapter 7

Examples

7.1. Example 1: Fanuc Arc Mate

We will solve the IK of the Fanuc Arc Mate for the pose

$$\mathbf{R} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \quad \mathbf{d} = \begin{bmatrix} 130 \\ 850 \\ 1540 \end{bmatrix} \text{(mm)}$$

The DH parameters are given in TABLE 7.1. Now the pose generated by the actuators must be the one given as $\lambda \hat{q}_1 \hat{q}_2 \hat{q}_3 \hat{q}_4 \hat{q}_5 \hat{q}_6 = \hat{q}_{EE}$, where λ indicates that only the ratios of eight homogeneous coordinates are preserved. Using eqns. (5.8a - 5.8c) the right-hand side is

$$= [(1 - \epsilon 630) - (1 + \epsilon 205)\mathbf{i} - (1 - \epsilon 140)\mathbf{j} - (1 + \epsilon 565)\mathbf{k}]/2 \qquad (7.1)$$

Then it is easy to show that the product of the six dual quaternions will form eight hexalinear polynomial expressions in

$$\mathbf{X} = \{[\cos(\theta_i/2), \sin(\theta_i/2)] \; : \; i = 1 \ldots 6\}$$

TABLE 7.1. DH Parameters of the Fanuc Arc Mate Manipulator

i	b_i (mm)	θ_i	a_i (mm)	α_i
1	810	θ_1	200	90°
2	0	θ_2	600	0°
3	30	θ_3	130	90°
4	550	θ_4	0	90°
5	100	θ_1	0	90°
6	100	θ_1	0	0°

We now multiply both sides of eqn. (7.1) by $\tilde{\hat{q}}_{EE}$

$$\begin{aligned} \hat{q}_1\hat{q}_2\hat{q}_3\hat{q}_4\hat{q}_5\hat{q}_6\tilde{\hat{q}}_{EE} &= \hat{q}_{EE}\tilde{\hat{q}}_{EE} \\ &= (f_0+\epsilon f_1)+(f_2+\epsilon f_3)\mathbf{i}+(f_4+\epsilon f_5)\mathbf{j}+(f_6+\epsilon f_7)\mathbf{k} \end{aligned} \quad (7.2)$$

to produce seven homogeneous equations that form an *ideal* as defined in Appendix A.

$$I = \langle f_1=0, f_2=0, f_3=0, f_4=0, f_5=0, f_6=0, f_7=0\rangle.$$

The algorithm will introduce projective maps which will bring the variety $\mathbf{V}(I)$ onto $T \subset \mathcal{P}^1$. The characteristic univariate is T.

7.1.1. First Elimination Step. First one must decide which three angles should be eliminated first and then introduce the appropriate polynomial map. The following map represents the decision to eliminate θ_4, θ_5, and θ_6:

$$\mathbf{x} = \begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \end{bmatrix} \equiv \begin{bmatrix} c_4c_5c_6 \\ c_4c_5s_6 \\ c_4s_5c_6 \\ c_4s_5s_6 \\ s_4c_5c_6 \\ s_4c_5s_6 \\ s_4s_5c_6 \\ s_4s_5s_6 \end{bmatrix} \quad (7.3)$$

where $c_i = \cos\frac{\theta_i}{2}$ and $s_i = \sin\frac{\theta_i}{2}$. The polynomial map describes a parametrization of a 3-dimensional manifold $\mathcal{M}_3$ in a 7-dimensional projective space $\mathcal{P}^7$. The complete elimination of θ_4, θ_5, and θ_6 requires a description of $\mathcal{M}_3$ in terms of homogeneous equations in the polynomial ring $k[x_0,\ldots,x_7]$ (see eqn. 4.4). It is very difficult to decide this by just looking at the map even though some are readily apparent such as

$$x_0x_7 - x_1x_6 = 0,\ x_1x_3 - x_0x_4 = 0,\ \ldots$$

We need to compute a basis, $\mathbf{I}(\mathcal{M}_3) \subset k[x_0,\ldots,x_7]$, which is a set of generators that can span all polynomials that vanish on $\mathcal{M}_3$. A *Gröbner* basis, was computed using the **grobner** package in MapleV

$$
\begin{aligned}
\mathrm{I}(\mathcal{M}_3) &= \langle x_7x_0 - x_3x_4, x_7x_4 - x_6x_5, x_7x_2 - x_6x_3, \\
&\quad x_5x_0 - x_1x_4, x_6x_0 - x_2x_4, x_7x_1 - x_5x_3, \\
&\quad x_7x_0 - x_6x_1, x_7x_0 - x_5x_2, x_3x_0 - x_1x_2\rangle
\end{aligned}
$$

The polynomial map, eqns. (7.3), also transforms I into a homogeneous system of seven linear equations in $[x_0, \ldots, x_7]$. Solving for $[x_0, \ldots, x_7]$ by forming a Grassmannian using I

$$
\begin{aligned}
\begin{vmatrix} \eta \\ \frac{\partial \mathbf{f}}{\partial \mathbf{x}} \end{vmatrix} &= 0 \\
x_0\eta_0 + x_1\eta_1 + x_2\eta_2 + x_3\eta_3 + x_4\eta_4 + x_5\eta_5 + x_6\eta_6 + x_7\eta_7 &= 0
\end{aligned}
$$

where $\eta = [eta_0, \ldots, \eta_7]^T$ and $\mathbf{f} = [f_1, \ldots, f_7]^T$. Each expression x_i has the non-zero factor $(s_1^2 + c_1^2)^3(s_2^2 + c_2^2)^3(s_3^2 + c_3^2)^3$ in common. Once factored out we then substitute expressions for $\mathbf{x}$ into $\mathcal{M}_3$ to obtain the projection

$$
\Pi_1(\mathcal{M}_3) = S = \mathbf{V}(s_1, \ldots, s_9) \subset \mathcal{P}^1 \times \mathcal{P}^1 \times \mathcal{P}^1
$$

The projective variety S implies an ideal whose generators are $I_S = \langle s_1, \ldots, s_9\rangle \subset k[c_1, s_1, c_2, s_2, c_3, s_3]$. Note that these generators do not necessarily form a basis. We show here the first generator of I_S.

$$
\begin{aligned}
x_7x_0 - x_3x_4 &= 0 \\
21900c_3^2s_2c_1c_2s_1 - 7872c_3^2s_2^2c_1s_1 + 3972s_2^2c_1s_3^2s_1 + 6192s_2c_1^2s_3^2c_2 & \\
-9168s_2^2c_1^2s_3c_3 - 14028c_3^2c_2^2s_1c_1 - 21900c_3s_2^2c_1s_1s_3 + 7008c_3^2s_2c_1^2c_2 & \\
+11504c_3s_2c_1^2c_2s_3 + 3212s_2s_1^2c_3^2c_2 - 5372s_2^2s_1^2c_3s_3 + 12312s_2s_1c_3c_2c_1s_3 & \\
+11224c_2s_1^2s_3s_2c_3 - 12148c_2^2s_1^2s_3c_3 + 9988c_2s_1^2s_3^2s_2 + 21900c_2^2s_1s_3c_1c_3 & \\
+10128c_2^2s_1s_3^2c_1 - 21900c_2c_1s_3^2s_2s_1 - 8352c_2^2c_1^2s_3c_3 - 5008c_3^2s_2^2c_1^2 - 3180s_2^2s_1^2c_3^2 & \\
-4288c_2^2c_1^2s_3^2 - 32c_3^2c_2^2s_1^2 - 2000c_2^2c_1^2c_3^2 - 5552s_2^2s_1^2s_3^2 - 4400s_2^2c_1^2s_3^2 - 5580c_2^2s_1^2s_3^2 &= 0
\end{aligned}
$$

7.1.2. Second Elimination Step. Similar to the first elimination step we introduce the following polynomial map:

$$\mathbf{y} = \begin{bmatrix} y_0 \\ y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \\ y_6 \\ y_7 \\ y_8 \end{bmatrix} \equiv \begin{bmatrix} c_2^2 c_1 s_1 \\ c_2^2 s_1^2 \\ c_2 s_2 c_1^2 \\ c_2 s_2 c_1 s_1 \\ c_2 s_2 s_1^2 \\ s_2^2 c_1^2 \\ s_2^2 s_1 c_1 \\ s_2^2 s_1^2 \\ c_2^2 c_1^2 \end{bmatrix}$$

which describes a 2-dimensional manifold $\mathcal{M}_2 \subset \mathcal{P}^8$. The implicit representation of $\mathcal{M}_2$ is found by computing a basis $\mathbf{I}(\mathcal{M}_2) \subset k[y_0, \ldots, y_8]$ which consists of 20 homogeneous equations. The redundancy is necessary in order to span any polynomial which vanishes on $\mathcal{M}_2$ under the rules of the polynomial ring $k[y_0, \ldots, y_8]$. The new polynomial map transforms S into nine hyperplanes $U_i \subset \mathcal{P}^8$ that intersect in a point $\mathbf{y} \in \mathcal{P}^8$ for a given θ_3. Again we construct the first order Grassmannian to find the intersection point

$$\begin{aligned} \begin{vmatrix} \eta \\ \frac{\partial \mathbf{U}}{\partial \mathbf{y}} \end{vmatrix} &= 0 \\ y_0\xi_0 + y_1\xi_1 + y_2\xi_2 + y_3\xi_3 + y_4\xi_4 + y_5\xi_5 + y_6\xi_6 + y_7\xi_7 + y_8\xi_8 &= 0 \end{aligned}$$

where $\mathbf{U} = [U_1, U_2, U_3, U_4, U_5, U_6, U_7, U_9]$ and $\eta = [\eta_0, \ldots, \eta_7]$. Note that we require only eight independent hyperplanes, U_i, to define a point $\mathbf{y} \in \mathcal{P}^8$. Below we have the expression for y_1:

$$\begin{aligned} y_1 = \; & 200586202227189c_3^{12} + 330659577836546s_3^{11}c_3 - 257794773972732s_3^{12} \\ & -384376879381591 3s_3^8c_3^4 + 1777678601538204s_3^{10}c_3^2 - 97070619962883s_3^9c_3^3 \\ & -480752219251780c_3^7s_3^5 + 151377553612602s_3^7c_3^5 + 5129825197784826c_3^6s_3^6 \\ & +177692488608410c_3^{10}s_3^2 - 1994744166551904c_3^8s_3^4 + 731975465044212c_3^9s_3^3 \\ & -147277751031177c_3^{11}s_3 \end{aligned}$$

7.1.3. The Characteristic Polynomial. Substituting expressions for $\mathbf{y}$ into $\mathbf{I}(\mathcal{M}_2)$ results in a projection

$$\Pi_2(\mathcal{M}_2) = T = \mathbf{V}(t_1, \dots, t_{20}) \subset \mathcal{P}^1$$

Each generator t_i of $I_T = \langle t_1, \dots, t_{20} \rangle \subset k[c_3, s_3]$ factors into two 16^{th} order univariates $t_i = hg_i$; h is the characteristic univariate. We can prove this geometrically by expressing $t_i = hg_i$ as the union of two projective varieties

$$\mathbf{V}(hg_i) = \mathbf{V}(h) \cup \mathbf{V}(g_i)$$

Therefore,

$$\begin{aligned} T &= (\mathbf{V}(h) \cup \mathbf{V}(g_1)) \cap \dots \cap (\mathbf{V}(h) \cup \mathbf{V}(g_{20})) \\ &= \mathbf{V}(h) \cup (\mathbf{V}(g_1) \cap \dots \cap \mathbf{V}(g_{20})) \\ &= \mathbf{V}(h) \cup \{\emptyset\} \\ &= \mathbf{V}(h) \end{aligned}$$

The result is the homogeneous univariate

$$\begin{aligned} h = {} & s_3^2(-13s_3 + 55c_3)(-3333274256606544s_3^{13} + 37320642465920s_3^{12}c_3 \\ & + 3576771782752974s_3^{11}c_3^2 + 5831345240633715s_3^{10}c_3^3 \\ & - 2301395695910640s_3^9c_3^4 - 9687157192302605s_3^8c_3^5 \\ & + 1517088882732216s_3^7c_3^6 + 11831667725709290s_3^6c_3^7 \\ & - 1304583569462508s_3^5c_3^8 - 5535502172563110s_3^4c_3^9 \\ & + 1314494925664602s_3^3c_3^{10} + 1296973546412035s_3^2c_3^{11} \\ & - 248914020827316s_3c_3^{12} + 261036838514835c_3^{13}) \end{aligned} \tag{7.4}$$

7.1.4. The Extension Step. Using the above ratios we can *extend*, or solve for, the complete set of unknowns by expressing the five pairs of ratios $\{(c_i : s_i)\}$ in terms of (c_3, s_3). Looking at both polynomial maps introduced earlier we see that

$$x_2 : x_6 = c_4 : s_4,\ x_5 : x_6 = c_5 : s_5,\ x_2 : x_3 = c_6 : s_6$$
$$y_0 : y_1 = c_1 : s_1,\ y_2 : y_5 = c_2 : s_2 \tag{7.5}$$

One solution of eqn. (7.4), is $(c_3 : s_3) = (1 : 0)$. Extending to the other angles using eqn. (7.5)

$$\begin{aligned}
(y_0 : y_1) &= (45431253558700617498624E07 : 45431253558700617498624E07) \\
&= (1 : 1) = (c_1 : s_1) \text{ or } \theta_1 = 90° \\
(y_2 : y_5) &= (1 : 1) = (c_2 : s_2) \text{ or } \theta_2 = 90° \\
(x_2 : x_6) &= (0 : -20) = (c_4 : s_4) \text{ or } \theta_4 = 180° \\
(x_5 : x_6) &= (0 : -20) = (c_5 : s_5) \text{ or } \theta_5 = 180° \\
(x_2 : x_3) &= (0 : 0) = (c_6 : s_6) \text{ undefined}
\end{aligned}$$

To understand what happened with the last ratio let us look at $\mathbf{x}$

$$(x_0 : x_1 : x_2 : x_3 : x_4 : x_5 : x_6 : x_7) = (0 : 0 : 0 : 0 : 0 : 0 : -20 : 0)$$

which says that to solve for θ_4, θ_5, and θ_6, one must use the coordinate x_6. A closer look at the first polynomial map reveals that

$$(c_6 : s_6) = (x_2 : x_3) = (x_6 : x_7) = (-20 : 0) \text{ or } \theta_6 = 0°$$

The other real solutions to the IK of the Fanuc Arc Mate are computed similarly to finally arrive at the results of Table 6.2.

TABLE 7.2. IK Solutions of the Fanuc Arc Mate

Solution	θ_1	θ_2	θ_3	θ_4	θ_5	θ_6
1 & 2	90°	90°	0°	180°	180°	0°
3	90°	16.0095°	153.403°	180°	100.588°	0°
4	75.1566°	15.3252°	150.851°	15.2657°	−103.353°	176.393°

TABLE 7.3. DH Parameters of Diestro

i	b_i (mm)	θ_i	a_i (mm)	α_i
1	50	θ_1	50	90°
2	50	θ_2	50	−90°
3	50	θ_3	50	90°
4	50	θ_4	50	−90°
5	50	θ_5	50	90°
6	50	θ_6	50	−90°

7.2. Example 2: Diestro

We will solve the IKP of the Diestro for the pose

$$\mathbf{R} = \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ 1 & 0 & 0 \end{bmatrix} \quad \mathbf{d} = \begin{bmatrix} 0 \\ -50 \\ 50 \end{bmatrix} \text{(mm)}$$

The DH parameters of the manipulator are given in TABLE 7.3. In this second example the solution contains both critical and non-critical values. The singularity was detected as we proceeded with the algorithm to obtain the characteristic polynomial and found that projecting $\mathcal{M}_2$ onto $\mathcal{P}^1$ gave the zero function; more precisely,

$$\Pi_1(\mathcal{M}_2) = \mathcal{P}^1$$

which describes a one-parameter set of solutions. Extending to the complete solution,

$$(c_1 : s_1) = (2c_3 : -(c_3 - s_3)) \tag{7.1a}$$

$$(c_2 : s_2) = (c_3 - s_3 : -(s_3 + 3c_3)) \tag{7.1b}$$

$$(c_4 : s_4) = (s_3 + c_3 : 3c_3 - s_3) \tag{7.1c}$$

$$(c_5 : s_5) = ((c_3 - s_3)^2 : (s_3 + c_3)(s_3 + 3c_3)) \tag{7.1d}$$

$$(c_6 : s_6) = (s_3 + c_3 : c_3 - s_3) \tag{7.1e}$$

This curve describes the critical values of the fiber, the non-critical values are found by satisfying incidence; recall that the excluded origin $(0 : \ldots : 0)$ in projective space defines incidence. Non-critical values exist if there are values for θ_3 such that

$$(y_0 : y_1 : y_2 : y_3 : y_4 : y_5 : y_6 : y_7 : y_8) = (0 : 0 : 0 : 0 : 0 : 0 : 0 : 0 : 0) \tag{7.2}$$

It turns out that if

$$(c_3 - s_3)(s_3 + c_3)(s_3^6 - 3s_3^4c_3^2 + 51s_3^2c_3^4 + 15c_3^6) = 0$$

then eqn. (7.2) is satisfied. From the above equation there are only two real solutions, namely $\theta_3 = 90°$ and $\theta_3 = -90°$. Extending these solutions will require eqns. (7.1a - 7.1e) and the projection $\Pi_1(\mathcal{M}_3)$. Recall that the first elimination step resulted in a projective variety $\Pi_1(\mathcal{M}_3) = S \subset k[c_1, s_1, c_2, s_2, c_3, s_3]$. Let us set $\theta_3 = 90°$ and compute the Gröbner basis

$$\mathbf{I}(S_{\theta_3=90°} - \mathbf{V}(c_1^2 + s_1^2) \cup \mathbf{V}(c_2^2 + s_2^2)) = \langle c_1 s_1 = c_1 c_2 = s_1(c_2 + s_2) = c_2(c_2 + s_2) = 0 \rangle$$

The above equations have both critical and non-critical solutions

$$c_2 = 0, \qquad s_1 = 0 \tag{7.3}$$

$$c_2 + s_2 = 0, \qquad c_1 = 0 \tag{7.4}$$

Eqn. (7.3) represent the singular solutions because they satisfy eqns. (7.1a -7.1e) and therefore eqn. (7.4) represent the non-singular solutions.

Chapter 8

Conclusion

Projective geometry fully described the IK, finding fibers with finite or infinite solutions, without formulation singularity. The IK was formulated with dual quaternions, the eight homogeneous coordinates of the 6-space of rigid body displacement. Thus reformulated, other problems in kinematics may be better solved as well. Though we ask questions about objects manipulated in metric space, answers may come from projective geometry where, Jacob Steiner thought, **[2]**, that order is brought to chaotic theorems and everything arranges itself there naturally.

Appendix A

Algebraic Geometry & Commutative Algebra

A.1. Varieties and Ideals

This section is abstracted from [**10**] pp. 5-36.

We begin by defining geometric objects called **affine varieties**.

DEFINITION A.1.1. *Let k be a field, and let $f_1, \ldots, f_s$ be polynomials in $k[x_1, \ldots, x_n]$. Then we set*

$$\mathbf{V}(f_1, \ldots, f_s) = \{(a_1, \ldots, a_n) \in k^n : f_i(a_1, \ldots, a_n) = 0 \; \forall \; 1 \leq i \leq s\}$$

We call $\mathbf{V}(f_1, \ldots, f_s)$ the **affine variety** *defined by $f_1, \ldots, f_s$.*

For example, $\mathbf{V}(x^2 + y^2 - 1)$ is a unit circle centered at the origin. Polynomials $f_1, \ldots, f_s$ also define an algebraic object.

DEFINITION A.1.2. *$I \subset k[x_1, \ldots, x_n]$ is an* **ideal** *if $0 \in I$, if $f, g \in I$ then $f + g \in I$, and if $f \in I$ and $h \in k[x_1, \ldots, x_n]$, then $hf \in I$.*

DEFINITION A.1.3. *Let $f_1, \ldots, f_s$ be polynomials in $k[x_1, \ldots, x_n]$. Then we set*

$$\langle f_1, \ldots, f_s \rangle = \left\{ \sum_{i=1}^{s} h_i f_i : h_1, \ldots, h_s \in k[x_1, \ldots, x_n] \right\}$$

The crucial fact is that $\langle f_1, \dots, f_s \rangle$ is an ideal.

LEMMA A.1.1. *If $f_1, \dots, f_s \in k[x_1, \dots, x_n]$, then $\langle f_1, \dots, f_s \rangle$ is an ideal of $k[x_1, \dots, x_n]$. We will call $\langle f_1, \dots, f_s \rangle$ the* **ideal generated by** *$f_1, \dots, f_s$.*

An ideal is similar to a subspace. Both are closed under addition and multiplication. For a subspace, we multiply by scalars, for an ideal, by polynomials. Further, notice that the ideal generated by polynomials $f_1, \dots, f_s$ is similar to basis vectors $v_1, \dots, v_n$. An affine variety $V = \mathbf{V}(f_1, \dots, f_s) \subset k^n$ says that $f_1, \dots, f_s$ vanishes on V. But are there other polynomials that vanish on V? Consider $V = \mathbf{V}(y - x^2, z - x^3)$ and $W = \mathbf{V}(z - xy, y^2 - xz)$. Does $V = W$?

$$\begin{aligned} 1(z - x^3) - x(y - x^2) &= z - xy \\ &= 1(0) - x(0) = 0 \\ (y + x^2)(y - x^2) - x(z - x^3) &= y^2 - xz \\ &= (y + x^2)(0) - x(0) = 0 \end{aligned}$$

Yes, because the two polynomials of V span the two polynomials of W. But how do we find or express all other such polynomials that vanish on the same geometric object called V or W? Consider the set of *all* polynomials that vanish on a given variety.

DEFINITION A.1.4. *Let $V \subset k^n$ be an affine variety and*

$$\begin{aligned} \mathbf{I}(V) = \{f \in k[x_1, \dots, x_n] : f(a_1, \dots, a_n) = 0 \\ \forall \ (a_1, \dots, a_n) \in V\} \end{aligned}$$

LEMMA A.1.2. *If $f_1, \dots, f_s \in k[x_1, \dots, x_n]$, then $\langle f_1, \dots, f_s \rangle \subset \mathbf{I}(\mathbf{V}(f_1, \dots, f_s))$, although equality need not occur.*

Which states that:

$$\mathbf{I}(\mathbf{V}(f_1, \dots, f_s)) = \langle f_1, \dots, f_s \rangle$$

need not be true. How does one find the polynomials $g_1, \ldots, g_t$ that generate $\mathbf{I}(\mathbf{V}(f_1, \ldots, f_s))$? The Hilbert Basis Theorem states that the generating set is finite

$$I = \langle g_1, \ldots, g_t \rangle$$

so $g_1, \ldots, g_t$ is a *basis* of I. Finally, one may find such a basis, called a Gröbner basis, given a variety $V = \mathbf{V}(f_1, \ldots, f_s)$ by implementing Buchberger's algorithm [**10**]. We can think of $\langle g_1, \ldots, g_t \rangle$ as a basis for spanning any polynomial or any set of polynomials vanishing on V.

A.2. Sums, Products, and Intersections of Ideals

This section is abstracted from [**10**] pp.180-188.
Sum, Product, and Intersection are operations on a pair of ideals to produce another ideal.

A.2.1. Sums of Ideals.

DEFINITION A.2.1. *If I and J are ideals of the ring $k[x_1, \ldots, x_n]$, then the* **sum** *of I and J, denoted $I + J$, is the set*

$$I + J = f + g : f \in I \ \ and \ \ g \in J$$

Geometrically, this is what happens. Let $I = \langle x^2 + y \rangle$ and $J = \langle z \rangle$ be ideals in $\mathcal{R}^3$. Then $I + J = \langle x^2 + y, z \rangle$ contains both $x^2 + y$ and z. That is, it must be the intersection of $\mathbf{V}(I)$ and $\mathbf{V}(J)$.

THEOREM A.2.1. *If I and J are ideals in $k[x_1, \ldots, x_n]$, then $\mathbf{V}(I + J) = \mathbf{V}(I) \cap \mathbf{V}(J)$*

A.2.2. Products of Ideals.

A product of two ideals corresponds to the union of the varieties:

$$\mathbf{V}(f_1, \ldots, f_r) \cup \mathbf{V}(g_1, \ldots, g_s) =$$
$$\mathbf{V}(f_i g_j, 1 \le i \le r, 1 \le j \le s)$$

For instance, the variety $\mathbf{V}(xz, yz)$ corresponds to the product of the generators of the ideals, $\langle x, y\rangle$ and $\langle z\rangle$ in $k[x, y, z]$, which in turn implies $\mathbf{V}(xz, yz) = \mathbf{V}(x, y) \cup \mathbf{V}(z)$. Algebraically,

DEFINITION A.2.2. *If I and J are two ideals in $k[x_1, \ldots, x_n]$, then their* **product**, *denoted by IJ, is the ideal generated by all polynomials fg where $f \in I$ and $g \in J$.*

Computing a set of generators for the product of ideals IJ, given sets of generators for I and J, is now shown.

PROPOSITION A.2.1. *Let $I = \langle f_1, \ldots, f_r\rangle$ and $J = \langle g_1, \ldots, g_s\rangle$. The IJ is generated by the set of all products of both generators:*

$$IJ = \langle f_i g_j : 1 \le i \le r, 1 \le j \le s\rangle$$

A.2.3. Intersections of Ideals.

DEFINITION A.2.3. *The* **intersection** *$I \cap J$ of two ideals is the set of polynomials belonging to both I and J.*

N.B., $IJ \subset (I \cap J)$ because IJ implies polynomials of the form fg where $f \in I$ and $g \in J$ and therefore $fg \in I \cap J$. However, IJ can be strictly contained in $I \cap J$. Let $I = J = \langle x, y\rangle$, then $IJ = \langle x^2, xy, y^2\rangle$ is strictly contained in $I \cap J = I = J = \langle x, y\rangle$ ($x \in I \cap J$, but $x \notin IJ$). An algorithm to compute $I \cap J$ requires a theorem and some new notation. If we have $f(t) \in k[t]$ and $I \in k[x_1, \ldots, x_n]$ then fI is the ideal in $k[x_1, \ldots, x_n, t]$ generated by the set of polynomials $fh : h \in I$. Notice that each ideal f and I lie in different rings; in fact $I \in k[x_1, \ldots, x_n]$ is not an ideal in $k[x_1, \ldots, x_n, t]$ since multiplying a polynomial $f \in I$ by t will make it a polynomial in $k[x_1, \ldots, x_n, t] \notin k[x_1, \ldots, x_n]$ and, therefore, I is not an ideal in $k[x_1, \ldots, x_n, t]$ by definition. From now on we will write a polynomial $h \in k[x_1, \ldots, x_n]$ as $h(x)$ but $g(x,t) \in k[x_1, \ldots, x_n, t]$. So $fI = f(t)I = \langle f(t)h(x) : h(x) \in I \rangle$.

LEMMA A.2.1.

- *If I is generated as an ideal in $k[x_1, \ldots, x_n]$ by $p_1(x), \ldots, p_r(x)$, then $f(t)I = \langle fp_1, \ldots, fp_r \rangle$.*
- *If $g(x,t) \in fI$ and $a \in k$ then $g(x,a) \in I$*

Finally, we can state the following theorem:

THEOREM A.2.2. *Let I, J be ideals in $k[x_1, \ldots, x_n]$; then,*

$$I \cap J = (tI + (1-t)J) \cap k[x_1, \ldots, x_n]$$

The above theorem leads to an algorithm for computing the Gröbner basis of the intersection of two ideals: if $I = \langle f_1, \ldots, f_r \rangle$ and $J = \langle g_1, \ldots, g_s \rangle$ are ideals in $k[x_1, \ldots, x_n]$, then consider:

$$\langle tf_1, \ldots, tf_r, (1-t)g_1, \ldots, (1-t)g_s \rangle \subset k[x_1, \ldots, x_n]$$

First compute the Gröbner basis of the above ideal in $k[x_1, \ldots, x_n, t]$, with respect to a lexicographical monomial ordering where t is greater than x_i. Elements of this basis not in terms of t form a basis for $I \cap J$. Using Elimination Theory it can be proven that a basis for $I \cap J$ can always be computed in this way. Geometrically,

THEOREM A.2.3. *If I and J are ideals in $k[x_1, \ldots, x_n]$, then $\mathbf{V}(I \cap J) = \mathbf{V}(I) \cup \mathbf{V}(J)$*

PROPOSITION A.2.2. *If V and W are varieties with $V \subset W$, then $W = V \cup \left(\overline{W - V}\right)$. Where $\overline{W - V}$ is the smallest algebraic variety containing $W - V$; $\overline{W - V}$ is called the* **Zariski closure** *of $W - V$.*

A.3. A Discussion on Kinematic Analysis as Application of Algebraic Geometry and Commutative Algebra

This section is abstracted from [**10**] pp. 190-206.

Although the ideas presented above may seem abstract, their relevance to the study of kinematics will be illustrated with a simple example. Obviously, one can only study *manifestly* holonomic systems when applying the theorems introduced here. In other words, one models the system as a set of polynomial constraints $f_1 = 0, \ldots, f_n = 0$ which defines an ideal I; we write

$$I = \langle f_1, \ldots, f_n \rangle$$

But this ideal, using the Algebra *-vs.-* Geometry dictionary, defines a variety (*i.e.*, a geometric object):

$$W = \mathbf{V}(I)$$

At this point one can compute $\mathbf{I}(W)$ (*i.e.*, a Gröbner basis) using lexicographical ordering to eliminate variables and arrive at the best algebraic representation of the system. But an interesting alternative is to remove those constraints which define the metric geometry in the system and interpret them differently. In kinematic analysis one usually must deal with equations like:

$$\begin{aligned} \sin(\theta)^2 + \cos(\theta)^2 &= 1 \\ q_0^2 + q_1^2 + q_2^2 + q_3^2 &= 1 \end{aligned}$$

which are independent of the design parameters of the system. By removing them from the original definition of the ideal I and having them define an ideal J we can still arrive at the desired result but through a different algorithm. First of all the ideal J would look like:

$$J = \left\langle \sin(\theta)^2 + \cos(\theta)^2 = 0, q_0^2 + q_1^2 + q_2^2 + q_3^2 = 0 \right\rangle$$

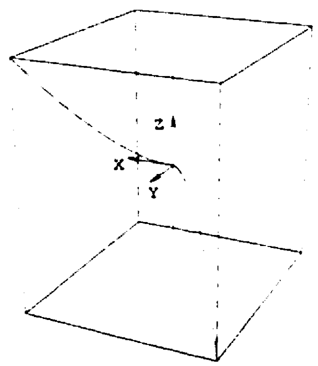
Then the **ideal quotient** of I by J is computed as

$$L = I : J \tag{1.1}$$

where $W = \mathbf{V}(L)$ is the variety describing the solution set of the original system. The ideal quotient, in this example, interprets the polynomials defining J as **non-zero conditions**. In general, there exists a computational way of *removing* unwanted solutions if one can bundle them into a variety. The following is a simple but non-trivial example. Consider the variety:

$$V = \mathbf{V}(xz - y^2, x^3 - yz)$$

The variety is sketched below. V is union of two curves. It is apparent that the z-axis,



$\mathbf{V}(x, y)$, vanishes on V; so, how do we express $V - \mathbf{V}(x, y)$ algebraically? According to eqn. (1.1) we need to compute

$$\left\langle xz - y^2, x^3 - yz \right\rangle : \langle x, y \rangle$$

The above can be written as

$$(I : x) \cap (I : y)$$

where $I = \langle xz - y^2, x^3 - yz\rangle$. To compute $I : x$ we first calculate $I \cap \langle x\rangle$ using Theorem A.2.2 with lexicographical order $z > y > x$

$$\begin{aligned} I : x &= \left\langle \frac{x^2z - xy^2}{x}, \frac{x^4 - xyz}{x}, \frac{x^3y - xz^2}{x} \right\rangle \\ &= \left\langle xz - y^2, x^3 - yz, x^2y - z^2 \right\rangle \\ &= I + \left\langle x^2y - z^2 \right\rangle \end{aligned}$$

Similarly, if we compute $I : \langle y\rangle$ we find that $I : \langle y\rangle = I : \langle x\rangle$; therefore,

$$(I : x) \cap (I : y) = \left\langle xz - y^2, x^3 - yz, x^2y - z^2 \right\rangle$$

$W = \mathbf{V}(xz - y^2, x^3 - yz, x^2y - z^2)$ is a curve parametrized by (t^3, t^4, t^5). So the equation $x^2y - z^2$ may be redundant but it *weeds* out all of the z-axis except the origin. So W is the Zariski closure of $V - \mathbf{V}(x, y)$ (*i.e.*, $W = \overline{V - \mathbf{V}(x, y)}$).

Establishing a language between algebra and geometry allows one to map geometric operations to their algebraic equivalent.

We finish this section with the ALGEBRA *-vs.-* GEOMETRY dictionary as published in [**10**].

ALGEBRA		**GEOMETRY**
radical ideals		varieties
I	$\longrightarrow$	$\mathbf{V}(I)$
$\mathbf{I}(V)$	$\longleftarrow$	V
addition of ideals		intersection of varieties
$I + J$	$\longrightarrow$	$\mathbf{V}(I) \cap \mathbf{V}(J)$
$\sqrt{\mathbf{I}(V) + \mathbf{J}(W)}$	$\longleftarrow$	$V \cap W$
product of ideals		union of varieties
IJ	$\longrightarrow$	$\mathbf{V}(I) \cup \mathbf{V}(J)$
$\sqrt{\mathbf{I}(V)\mathbf{V}(W)}$	$\longleftarrow$	$V \cup W$
intersection of ideals		union of varieties
$I \cap J$	$\longrightarrow$	$\mathbf{V}(I) \cup \mathbf{V}(J)$
$\mathbf{I}(V) \cap \mathbf{I}(W)$	$\longleftarrow$	$V \cup W$
quotient of ideals		difference of varieties
$I : J$	$\longrightarrow$	$\overline{\mathbf{V}(I) - \mathbf{V}(J)}$
$\mathbf{I}(V) : \mathbf{I}(W)$	$\longleftarrow$	$\overline{V - W}$
elimination of variables		projection of varieties
$\sqrt{\mathbf{I} \cap k[x_{l+1}, \ldots, x_n]}$	$\longleftrightarrow$	$\overline{\pi_l(\mathbf{V}(I))}$

A.4. Projective Algebraic Geometry

This section is abstracted from [10] pp. 349-390.

Projective spaces extend systems of polynomial equations to completion. Consider the trivial example of an affine variety, its partial solutions and their projection map.

$$(xy - 1 = 0, xz - 1 = 0) \rightarrow y - z = 0 \rightarrow \{(1/a, a, a) : a \neq 0\}$$

A qualification is required to exclude the spurious member $a = 0$ from the one parameter solution family (x, y, z). Using w as a fourth homogeneous coordinate, we have

$$(xy - w^2 = 0, xz - w^2 = 0) \rightarrow y - z = 0 \rightarrow \{(1 : a^2 : a^2 : a)\}$$

The *Projection Extension Theorem*, proven in Cox, Little, and O'Shea [10], says that systems of homogenous polynomial equations always extend to completion..

REFERENCES

[1] Salmon, G., 1885, "Higher Algebra", Chelsea Publishing Co., New York.

[2] Klein, F., 1948, "Elementary Mathematics from an Advanced Standpoint (Geometry)", Translated by Hedrick & Noble, Dover.

[3] Shuster, M. D., 1993, "A Survey of Attitude Representations", *Journal of the Astronautical Sciences*, Vol. 4, No. 4, pp. 439-517.

[4] Tait, P. G., 1900, "Quaternions", *Scientific Papers*, Cambridge University Press, Vol. 2, pp.445-456.

[5] Tait, P. G., 1900, "On The Intrinsic Nature of the Quaternion Method", *Scientific Papers*, Cambridge University Press, Vol. 2, pp.392-398.

[6] Bottema, O. and Roth, B., 1990, "Theoretical Kinematics", Dover.

[7] Husty, M., 1994, "An Algorithm for Solving the Direct Kinematics of Stewrt-Gough-type Platforms," Dept. of Mechanical Engineering and Centre for Intelligent Machines Technical Report, McGill University, Montreal.

[8] Grassmann, H., 1995, "A New Branch of Mathematics", Translated by Kannenberg, Open Court.

[9] Angeles, J., 1996, "Fundamentals of Robotic Mechanical Systems", Springer-Verlag.

[10] Cox, D., Little, J. and O'Shea, D., 1996, "Ideals, Varieties, and Algorithms", Springer-Verlag.

[11] Clifford, W. K., 1873, "Preliminary Sketch of Bi-Quaternions," *Proc. London Mathematical Society*, Vol. 4, pp. 381-395.

[12] Daniali, H. R. M., Zsombor-Murray, P. J., and Angeles, J., 1993, "Dual Numbers", Dept. of Mechanical Engineering and McGill Research Centre for Intelligent Machines, McGill University, Montreal.

[13] Ravani, B. and Roth, B., 1984, "Mappings of Spatial Kinematics", *Journal of Mechanisms, Transmissions, and Automation in Design*, Vol. 106, pp. 341-346.

[14] Yaglom, I., 1996, "Felix Kein and Sophus Lie", Birkhäuser.