ON THE LIMIT DISTRIBUTIONS OF HIGH LEVEL CROSSINGS BY A STATIONARY PROCESS

by

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ABSTRACT

This paper is concerned with real valued stationary processes having continuous sample paths with probability one. When such a process is Gaussian, a well known result says that if the sample paths are not too badly behaved and if an appropriate mixing condition holds, then under suitable normalization the number of upcrossings of a high level is asymptotically Poisson distributed.

Recently this result has been extended to general stationary processes, not necessarily Gaussian. In the first part of this paper we obtain this general theorem for the upcrossings of a stationary process. We first consider the discrete parameter case. We show that the point process of exceedances of a stationary sequence converges weakly to a Poisson point process, provided some appropriate dependence conditions are satisfied. We then consider the continuous parameter case and we obtain a general Poisson point process limit theorem for the upcrossings of a stationary process, not necessarily Gaussian. We also treat the case of ε-upcrossings.

In the second part of this paper, the classical limit theorem (Gaussian case) is obtained via the general theorem, thus bringing the Gaussian case within the general framework.

Throughout the paper, the discussion will be carried on in terms of the maximum of the process and we will obtain several important results on the limit distributions of the maximum.

RESUME

Ce mémoire traite des processus stochastiques stationnaires à valeurs réelles ayant, avec probabilité un, des trajectoires aléatoires continues. Lorsqu'un tel processus est Gaussien, un résultat bien connu nous dit que si les trajectoires aléatoires ne sont pas trop irrégulières et si le processus est assez bien mélangé, alors le nombre de passages d'un haut niveau, convenablement normalisé, est asymptotiquement distribué selon une loi de Poisson.

Récemment ce résultat a été obtenu pour une plus grande classes de processus stationnaires, pas nécessairement Gaussien. Dans le première partie de ce mémoire nous obtenons ce théorem général sur la distribution limite du nombre de passages de niveau. Nous considerons d'abord le cas d'une suite stationnaire et nous montrons que le processus ponctuel engendré par l'excédent d'un haut niveau converge faiblement vers un processus ponctuel de Poisson. Nous considérons ensuite le cas d'un processus stationnaire à espace paramètre continu et nous montrons que le processus ponctuel engendré par les passages d'un haut niveau converge faiblement vers un processus ponctuel de Poisson. Nous considérons également le cas des ε-passages de niveau.

Dans la deuxième partie de ce mémoire, le théorem classique (cas Gaussien) est obtenu à l'aide du théorem général, amenant ainsi le cas Gaussien dans le cadre général.

Tout au long de ce mémoire, la discussion se fera principalement en termes du maximum du processus et nous obtiendrons plusieurs résultats importants sur la distribution limite du maximum.

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CHAPTER 1

INTRODUCTION

1.1 The upcrossing problem

In this paper we shall be concerned primarily with asymptotic distributional properties of the number of upcrossings of a continuous parameter stationary process. Consider $(\xi(t);\ t \in \mathbb{R})$, a real valued standard stationary Gaussian process defined on a probability space (Ω, F, P). By Gaussian we mean that for each positive integer n and for each $(t_1, t_2, \ldots, t_n)$ in $\mathbb{R}^n$ the random vector $(\xi(t_1), \ldots, \xi(t_n))$ has a Gaussian joint distribution. By stationary we mean that for each positive integer n and for each $(t_1, t_2, \ldots, t_n)$ in $\mathbb{R}^n$, the random vectors $(\xi(t_1), \ldots, \xi(t_n))$ and $(\xi(t_1+\tau), \ldots, \xi(t_n+\tau))$ have the same joint distribution, for every τ. By standard we mean that the distribution of $\xi(t)$ (which, by stationarity, is the same for every t) has mean 0 and variance 1. $\mathbb{R}$, of course, denotes the set of real numbers.

If $\omega \in \Omega$, its image by $\xi(t)$ will be denoted by $\xi(t,\omega)$. For a fixed $\omega \in \Omega$, the function $t \mapsto \xi(t,\omega)$ is called a sample path. We shall assume that the process has continuous sample paths with probability 1, i.e.

$$P[\omega \in \Omega:\ \xi(t,\omega) \text{ is everywhere continuous}] = 1 .$$

We say that $\xi(t)$ has an upcrossing of the level u at time t_0 if there exists a $\delta > 0$ such that $\xi(t) \le u$ for all t in $(t_0-\delta, t_0)$, $\xi(t) \ge u$ for all t in $(t_0, t_0+\delta)$, and for every $\eta > 0$, $\xi(t) > u$ for some t in $(t_0, t_0+\eta)$. Let $U_u(T)$ be the number of upcrossings of the level u in the time interval $(0,T]$. That $U_u(T)$ is a random variable can be shown as follows. Let

$$H_n(\omega) = \sum_{k=1}^{2^n} H_{n,k}(\omega)$$

where $H_{n,k}(\omega)$ equals 1 if

$$\xi((k-1)T/2^n,\omega) < u < \xi(kT/2^n,\omega)$$

and 0 otherwise. It is easily seen that H_n increases almost surely to $U_u(T)$ as $n \to \infty$. The H_n are measurable so that $U_u(T)$ is measurable. We only need $U_u(T) < \infty$ almost surely in order to say that $U_u(T)$ is a random variable. For the moment assume this is so.

The general problem concerning the number of upcrossings $U_u(T)$ is to find its probability distribution

$$P[U_u(T) = k] , \quad k = 0,1,2,\dots .$$

This remains an unsolved problem. Most results in this direction have been to either compute the moments of the distribution or compute the limit distribution as $u \to \infty$ and $T \to \infty$. This paper is concerned with the limit distribution problem.

It is intuitively clear, and actually easily checked, that if T is fixed and $u \to \infty$, then $U_u(T) \to 0$ for almost every $\omega \in \Omega$. However, if for each $T > 0$ we can choose a u_T in such a way that $E[U_{u_T}(T)]$ is independent of T, then it is reasonable to hope that the distribution of $U_{u_T}(T)$ will converge to some non-degenerate distribution, as $T \to \infty$. We now suppose that we are given such a family $(u_T;\ T > 0)$. It will be convenient to slightly modify our notation: we will write $Z_T((a,b])$ for the number of upcrossings of the level u_T in the time interval $(Ta,Tb]$ (i.e. $Z_T((a,b]) = U_{u_T}(Tb) - U_{u_T}(Ta)$). Hence our hypothesis is that for some $\tau > 0$ we have

$$(1.1)\qquad E[Z_T((0,1])] = \tau \quad \text{for all} \quad T > 0,$$

where $E[Z_T((0,1])]$ denotes the expectation of the random variable $Z_T((0,1])$. The following result holds.

THEOREM 1.1.1. *Let* $(\xi(t);\ t\in\mathbb{R})$ *be a standard stationary Gaussian process defined on a probability space* (Ω,F,P) *and having, with probability one, continuous sample paths. Let* τ *be a fixed positive constant. Let* $Z_T((a,b])$ *denote the number of upcrossings of the level* u_T *in the time interval* $(Ta,Tb]$, *where the* u_T*'s are chosen in such a way that* (1.1) *holds. Assume the following conditions hold:*

(M) *A certain mixing condition.*

(L) *A certain local condition.*

Then, if $(a_1,b_1],(a_2,b_2],\ldots,(a_j,b_j]$ *are disjoint subintervals of* $(0,1]$ *and if* $k_1,k_2,\ldots,k_j$ *are non-negative integers, we have*

$$(1.2)\qquad \lim_{T\to\infty} P[Z_T((a_i,b_i]) = k_i;\ i=1,\ldots,j] = \prod_{i=1}^{j} \frac{e^{-\tau(b_i-a_i)}[\tau(b_i-a_i)]^{k_i}}{k_i!}.$$

////

The theorem says that the $Z_T((a_i,b_i])$'s are asymptotically independent Poisson random variables. In order to get this Poisson behaviour we have to assume a mixing condition (M) and a local condition (L). By mixing we mean a certain dependency decay: $\xi(t)$ and $\xi(t+\tau)$ will be in some sense approximately independent as τ becomes large. Under an appropriate mixing condition we will have almost independency between upcrossings occurring far apart. A local condition will ensure us that the sample paths are not too badly behaved. This will bound the probability of having more than one upcrossing in a short time interval.

These conditions will be written in terms of the so-called covariance function defined as

(1.3) $$r(t) = E[\xi(s)\xi(s+t)] .$$

Note that since the process is Gaussian, $E[\xi(s)\xi(s+t)]$ is well defined (i.e. $E[|\xi(s)\xi(s+t)|] < \infty$) and since the process is stationary $E[\xi(s)\xi(s+t)]$ does not depend on s. Thus $r(t)$ is well defined.

This theorem was first obtained by Volkonski and Rozanov [1961] as an application of their general results [1959]. They assumed the following conditions.

(M) $\alpha(t) = O(t^{-\varepsilon})$ as $t \to \infty$, for some $\varepsilon > 0$

where $\alpha(t)$ is defined by

$$\alpha(t) = \sup_{s \in \mathbb{R}} \sup_{\substack{A \in F_{-\infty}^{s} \\ B \in F_{s+t}^{+\infty}}} |P[A \cap B] - P[A]P[B]| ,$$

F_a^b being the σ-algebra generated by $(\xi(t);\ a < t < b)$, i.e. the smallest σ-algebra of subsets of Ω for which $\xi(t)$ is measurable for all $t \in (a,b)$. (This condition is called a strong mixing condition and $\alpha(t)$ is referred to as the mixing function).

(L) $r^{(iv)}(t)$ exists.

These conditions were subsequently weakened by several authors:

Cramer [1966] (M) $r(t) = O(t^{-\varepsilon})$ as $t \to \infty$, for some $\varepsilon > 0$.

(L) $r^{(iv)}(t)$ exists.

Belayev [1967] (M) $r(t)\log t \to 0$ and $r'(t)(\log t)^{1/2} \to 0$, as $t \to \infty$.

(L) $r''(t)$ exists and, for some $b > 1$,

$r''(t) - r''(0) = O(|\log|t||^{-b})$, as $t \to 0$.

Qualls [1967] (M) $r(t) = O(t^{-\varepsilon})$ as $t \to \infty$, for some $\varepsilon > 0$.

(L) $r''(t)$ exists and, for some $\delta > 0$,

$$\int_0^{\delta} \frac{r''(t)-r''(0)}{t}\,dt < \infty.$$

Berman [1971a] (M) $r(t)\log t \to 0$ as $t \to \infty$.

(L) $r''(t)$ exists.

Berman conditions can hardly be improved. On one side, if $r(t)\log t \to \gamma > 0$ then (1.2) does not hold. This emerges from the work of Mittal and Ylvisaker [1975] (and can be shown using Theorem 5.2 of Leadbetter, Lindgren and Rootzen [1979]). On the other side, if $r''(t)$ does not exist, then $E[N_T((a,b])] = +\infty$ for all $T > 0$ and all $0 \leq a < b < \infty$.

More recently Leadbetter [1980] considered stationary processes which are not necessarily Gaussian. He obtained a general Poisson limit distribution theorem for the upcrossings. The objective of this paper is twofold. Firstly we give a comprehensive exposition of Leadbetter's recent results. Then we show that for a standard stationary Gaussian process whose covariance function satisfies Berman conditions, Leadbetter's general theorem applies. This brings the Gaussian case within the general framework.

1.2 The point process of upcrossings

Our discussion will be carried on in terms of point processes. Let (Ω, F, P) be a probability space. Let $\mathcal{B}$ be the family of Borel subsets of the interval $(0,1]$. A point process on the interval $(0,1]$ is a function

$$\begin{aligned} Z:\ \Omega \times \mathcal{B} &\longrightarrow \bar{N} \\ (\omega, B) &\longmapsto Z(\omega, B) \end{aligned}$$

(where N is the set of non negative integers and where $\bar{N}$ is the set of non-negative integers to which we added $+\infty$) such that for each fixed B in $\mathcal{B}$ the function

$$\begin{aligned} Z(B):\ \Omega &\longrightarrow \bar{N} \\ \omega &\longmapsto Z(\omega, B) \end{aligned} ,$$

is a random variable (i.e. it is F-measurable and finite a.e.), and for almost every $\omega \in \Omega$ the function

$$\begin{aligned} Z(\omega):\ \mathcal{B} &\longrightarrow \bar{N} \\ B &\longmapsto Z(\omega, B) \end{aligned}$$

is a finite positive measure. (This is actually a special type of point process. For a general definition see Kallenberg [1976] or Grandell [1977]).

We shall be concerned with weak convergence of point processes. Let $Z_1, Z_2, \ldots,$ and Z be point processes on $(0,1]$. We say that Z_n converges weakly to Z, and we write

$$Z_n \xrightarrow{w} Z$$

if for every positive integer k and for every choice of k disjoint

Borel subsets of $[0,1]$, say $B_1, B_2, \ldots, B_k$, with $\Lambda(\partial B_i) = 0$, the random vector $(Z_n(B_1), \ldots, Z_n(B_k))$ converges in distribution to the random vector $(Z(B_1), \ldots, Z(B_k))$. (∂B_i is the topological boundary of B_i and Λ is the Lebesgue measure).

A point process Z on $(0,1]$ is called a Poisson point process with intensity $\tau > 0$ if for every $B \in \mathcal{B}$, $Z(B)$ is Poisson distributed with mean $\tau\Lambda(B)$ and for every positive integer k, $Z(B_1), Z(B_2), \ldots, Z(B_k)$ are independent whenever $B_1, B_2, \ldots, B_k$ are disjoint Borel subsets of $(0,1]$.

The following theorem will play a very important role. It is a special case of Theorem 4.7 in Kallenberg [1976].

THEOREM 1.2.1. *Let* $Z_1, Z_2, \ldots$ *and* Z *be point processes on* $(0,1]$, Z *being Poisson with intensity* τ. *Suppose that*

(a) $E[Z_n((a,b])] \to \tau(b-a)$ *for all* $0 \le a < b \le 1$.

(b) $P[Z_n(B) = 0] \to e^{-\tau\Lambda(B)}$ *for all* B *of the form* $\bigcup_{i=1}^{k} (a_i, b_i]$ *where* $0 \le a_1 < b_1 < a_2 < b_2 < \cdots < b_k \le 1$.

Then $Z_n \xrightarrow{w} Z$. ////

Now let us go back to our upcrossings problem. Consider a stationary stochastic process $(\xi(t);\ t \in \mathbb{R})$ defined on a probability space (Ω, F, P), and having, with probability one, continuous sample paths. Let $(u_T;\ T > 0)$ be a given family of constants (typically $u_T \to +\infty$ as $T \to +\infty$). For $\omega \in \Omega$ and $B \in \mathcal{B}$, let $Z_T(\omega, B)$ be the number of upcrossings of the level u_T by the sample path $t \mapsto \xi(t,\omega)$ within the time set $TB = \{Tb;\ b \in B\}$. Under general conditions, Z_T will be a point process, as defined above. It will be shown that for a suitable choice of constants u_T and under appropriate conditions, the point process of upcrossings Z_T converges weakly to a Poisson point process. This will be done using Theorem 1.2.1.

1.3 Summary

In Chapter 2 we will consider the case of a stationary sequence $(\xi_n;\ n=1,2,\ldots)$. We shall say that an exceedance of the level u occurs at i_0 if $\xi_{i_0} > u$. It will be shown that the point process of exceedances converges weakly to a Poisson point process provided the stationary sequence satisfies certain dependence conditions.

In Chapter 3 we will use the results and ideas of Chapter 2 to treat the continuous parameter case. We will obtain Leadbetter's general Poisson point process limit theorem for the upcrossings of a stationary process, not necessarily Gaussian. We shall also consider the point process of ε-upcrossings.

Chapter 4 is devoted to stationary Gaussian processes. In the first part of this chapter we investigate the relationship between the covariance function and the spectral distribution function and we review the available literature, especially more recent work, concerning the sample path analytical properties, and the moments of the number of upcrossings of a level. Then we show that the general Poisson point process limit theorem of Chapter 3 can be applied to stationary Gaussian processes whose covariance function satisfies Berman local and mixing conditions. This gives us Theorem 1.1.1 via the general framework.

CHAPTER 2

CONVERGENCE OF THE POINT PROCESS OF EXCEEDANCES OF A STATIONARY SEQUENCE

2.1 Introduction

A random sequence $(\xi_i;\ i=1,2,\ldots)$ is called a stationary sequence if for every positive integer m and for every choice of positive integers $i_1,i_2,\ldots,i_m$ the random vectors $(\xi_{i_1},\xi_{i_2},\ldots,\xi_{i_m})$ and $(\xi_{i_1+k},\xi_{i_2+k},\ldots,\xi_{i_m+k})$ have the same distribution, for every positive integer k. The joint distribution function of $\xi_{i_1},\xi_{i_2},\ldots,\xi_{i_m}$ will be denoted by $F_{i_1,i_2,\ldots,i_m}$. Hence for $(x_1,\ldots,x_m)$ in $\mathbb{R}^m$ we write

$$F_{i_1,i_2,\ldots,i_m}(x_1,x_2,\ldots,x_m) = P[\xi_{i_1}\le x_1,\ \xi_{i_2}\le x_2,\ldots,\xi_{i_m}\le x_m]$$

and the stationarity condition is that

$$F_{i_1,i_2,\ldots,i_m} = F_{i_1+k,i_2+k,\ldots,i_m+k}\ .$$

In particular F_i is independent of i and it will be denoted simply by F; it is called the marginal distribution function of the sequence.

Let $(\xi_i;\ i=1,2,\ldots)$ be a stationary sequence defined on a probability space (Ω,F,P). Let $(u_n;\ n=1,2,\ldots)$ be a given sequence of real constants. For each positive integer n we define the point process of exceedances of the level u_n as follows:

$$\begin{aligned} Z_n:\ \Omega\times\mathcal{B} &\to N \\ (\omega,B) &\mapsto Z_n(\omega,B) \end{aligned}$$

where $Z_n(\omega,B)$ is the number of instant j in the time set nB for which we have $\xi_j(\omega) > u_n$. For each n, Z_n is a point process as defined in Chapter 1: for any fixed ω in Ω, the function

$B \mapsto Z_n(\omega,B)$ is a finite positive measure on the family $\mathcal{B}$ of Borel subsets of $(0,1]$ and for any fixed B in $\mathcal{B}$, the function $\omega \mapsto Z_n(\omega,B)$ is a random variable on Ω (denoted by $Z_n(B)$).

Our main goal in this chapter is to show that for a suitable choice of constants u_n, the sequence of point processes Z_n converges weakly to a Poisson point process. We will have to consider stationary sequences satisfying certain dependence restrictions. We now examine different types of dependence restrictions.

Mixing conditions

As mentioned in Chapter 1, a mixing condition is one which gives asymptotic independency for ξ_i's living far apart. Next to independence itself, the strongest type of mixing condition one can think of is the so-called m-dependence: the sequence $(\xi_i;\ i=1,2,\ldots)$ is m-dependent if ξ_i and ξ_j are independent whenever $|i-j| > m$. Independence is simply m-dependence with $m = 0$.

Rosenblatt [1956] introduced the notion of strong mixing. The sequence $(\xi_i;\ i=1,2,\ldots)$ is said to satisfy a strong mixing condition if there is a sequence $\alpha_\ell \geq 0$, with $\alpha_\ell \to 0$ as $\ell \to \infty$, such that

$$|P[A \cap B] - P[A]P[B]| \leq \alpha_\ell \tag{2.1}$$

whenever $A \in F(\xi_1,\xi_2,\ldots,\xi_p)$ and $B \in F(\xi_{p+\ell+1},\xi_{p+\ell+2},\ldots)$ for some positive integer p. Here $F(\ldots)$ denotes the σ-algebra generated by the indicated random variables. $(\alpha_\ell;\ \ell=1,2,\ldots)$ is usually called the mixing sequence. Note that m-dependence is simply strong mixing with $\alpha_\ell = 0$ for $\ell \geq m$.

Roughly speaking, strong mixing says that events living far apart are uniformly asymptotically independent. In proving the Poisson point process limit theorem for the exceedances of a stationary sequence we will be dealing with events of the form $\bigcap_{j\in J}\{\xi_j \le u_n\}$, where J is a finite subset of N. It will be enough to have asymptotic independence for events of that form. In the sequel it will be convenient to write $F_{i_1 i_2 \cdots i_m}(u)$ instead of $F_{i_1 i_2 \cdots i_m}(u,u,\ldots,u)$ in order to simplify the notation. We shall say that the condition D holds if for any integers $1 \le i_1 < i_2 < \cdots < i_p < j_1 < j_2 < \cdots < j_q$, with $j_1 - i_p \ge \ell$, and for any real number u, we have

$$(2.2) \qquad |F_{i_1 \cdots i_p j_1 \cdots j_q}(u) - F_{i_1 \cdots i_p}(u) F_{j_1 \cdots j_q}(u)| \le \alpha_\ell$$

where $\alpha_\ell \to 0$ as $\ell \to \infty$. Clearly strong mixing implies condition D: if (2.1) holds, then (2.2) certainly holds since it can be written as $|P[A \cap B] - P[A]P[B]| \le \alpha_\ell$ with $A = \{\xi_{i_1} \le u, \ldots, \xi_{i_p} \le u\} \in F(\xi_1, \xi_2, \ldots, \xi_{i_p})$ and $B = \{\xi_{j_1} \le u, \ldots, \xi_{j_q} \le u\} \in F(\xi_{i_p+\ell+1}, \xi_{i_p+\ell+2}, \ldots)$.

Finally, if $(u_n;\ n = 1,2,\ldots)$ is a sequence of real constants, we say that condition $D(u_n)$ holds if for any integers $1 \le i_1 < i_2 < \cdots < i_p < j_1 < j_2 < \cdots < j_q \le n$ for which $j_1 - i_p \ge \ell$, we have

$$(2.3) \qquad |F_{i_1 \cdots i_p j_1 \cdots j_q}(u_n) - F_{i_1 \cdots i_p}(u_n) F_{j_1 \cdots j_q}(u_n)| \le \alpha_{n,\ell}$$

where $\alpha_{n,\ell_n} \to 0$ as $n \to \infty$, for some sequence $\ell_n = o(n)$. Without loss of generality we may (and will) assume that $\alpha_{n,\ell}$ form a non-increasing sequence in ℓ, for each fixed n. If this is so, it is easily seen that the condition $\alpha_{n,\ell_n} \to 0$ as $n \to \infty$, for some $\ell_n = o(n)$, is equivalent to the condition $\alpha_{n,n\beta} \to 0$ as $n \to \infty$, for each $\beta > 0$. Clearly if condition D holds, then condition $D(u_n)$ holds for every sequence (u_n).

Local condition

To obtain the main result of this chapter we will need the following local condition which rules out cases where nearby ξ_j's are too highly dependent. Given a sequence of real constants (u_n), we say that the condition $D'(u_n)$ holds if

$$\lim_{k\to\infty} \limsup_{n\to\infty} n \sum_{j=2}^{[n/k]} P[\xi_1 > u_n, \xi_j > u_n] = 0 , \tag{2.4}$$

where $[t]$ denotes the integer part of t.

We now state the main result of this chapter.

THEOREM 2.1.1. *Let* $(\xi_i;\ i=1,2,\dots)$ *be a stationary sequence on* (Ω,F,P). *Let* $(u_n;\ n=1,2,\dots)$ *be a sequence of real constants. Let* Z_n *be the point process of exceedances of the level* u_n. *If for some* $\tau > 0$

$$n(1-F(u_n)) \longrightarrow \tau \quad \text{as} \quad n \longrightarrow \infty , \tag{2.5}$$

where F *is the marginal distribution function of the sequence, and if the conditions* $D(u_n)$ *and* $D'(u_n)$ *both hold, then*

$$Z_n \xrightarrow{w} Z \tag{2.6}$$

where Z *is a Poisson point process with parameter* τ. ////

Thanks to Theorem 1.2.1 it will be enough to show that

(a) $E[Z_n((a,b])] \longrightarrow \tau(b-a)$ for $0 \le a < b \le 1$

(b) $P[Z_n(A)=0] \longrightarrow e^{-\tau\Lambda(A)}$ for any A of the form $A = \bigcup_{i=1}^{k} (a_i,b_i]$ where $0 \le a_1 < b_1 < a_2 < b_2 < \cdots < a_k < b_k \le 1$.

It is easy to see that (a) follows from (2.5). Writing

$$x_i^{(n)} = \begin{cases} 1 & \text{if } \xi_i' > u_n \\ 0 & \text{if } \xi_i \leq u_n \end{cases}$$

we have

$$Z_n((a,b]) = \sum_{i=[na]+1}^{[nb]} x_i^{(n)}$$

so that

$$E[Z_n((a,b])] = ([nb]-[na])(1-F(u_n))$$

from which (a) follows, using (2.5). To obtain (b) we will have to work harder. Consider for the moment the simple case where $(\xi_i;\ i=1,2,\ldots)$ is a sequence of independent and identically distributed random variables. We have

$$\begin{aligned} P[Z_n(A)=0] &= P[Z_n(\bigcup_{i=1}^{k}(a_i,b_1])=0] \\ &= P[Z_n((a_1,b_1])=0,\ldots,Z_n((a_k,b_k])=0] \\ &= \prod_{i=1}^{k} P[Z_n((a_i,b_i])=0] \end{aligned}$$

and we have

$$e^{-\tau\Lambda(A)} = \prod_{i=1}^{k} e^{-\tau(b_i-a_i)} .$$

Hence to get (b) it suffices to show that

$$P[Z_n((a,b])=0] \to e^{-\tau(b-a)}$$

for all $0 \leq a < b \leq 1$. But this follows easily from (2.5) since

$$
\begin{aligned}
P[Z_n((a,b])=0] &= P[\xi_i \le u_n;\ i=[na]+1,\dots,[nb]] \\
&= \prod_{i=1}^{[nb]-[na]} P[\xi_{[na]+i} \le u_n] \\
&= (F(u_n))^{[nb]-[na]} \\
&= \{(1-(1-F(u_n)))^n\}^{([nb]-[na])/n} \\
&= \{(1-(\tfrac{\tau}{n}+o(\tfrac{1}{n})))^n\}^{([nb]-[na])/n} \\
&\to e^{-\tau(b-a)} \quad \text{as} \quad n\to\infty .
\end{aligned}
$$

Thus Theorem 2.1.1 is proven for the simple case where $(\xi_i;\ i=1,2,\dots)$ is a sequence of independent and identically distributed random variables. The independence assumption was used at two stages. Firstly we wrote

$$(2.7) \qquad P[Z_n((a_i,b_i])=0;\ i=1,\dots,k] = \prod_{i=1}^{k} P[Z_n((a_i,b_i])=0] .$$

Secondly we showed that (2.5) implies

$$(2.8) \qquad P[Z_n((a,b])=0] \to e^{-\tau(b-a)} .$$

In the following sections we shall obtain Theorem 2.1.1 by showing that if, instead of independence, the conditions $D(u_n)$ and $D'(u_n)$ hold, then (2.8) still holds and the difference of the two terms in (2.7) goes to 0 as $n\to\infty$.

Note that we have

$$Z_n((a,b])=0 \Leftrightarrow \max_{[na]<i\le[nb]} \xi_i \le u_n .$$

For brevity we will write

$$M_n = \max\{\xi_1,\xi_2,\dots,\xi_n\}$$

and our discussion will be carried on in terms of the random variable M_n.

The results presented in the next three sections are due to Leadbetter, except for one half of Theorem 2.3.2 (namely the fact that $P[M_n \le u_n] \to e^{-\tau}$ implies that $n(1-F(u_n)) \to \tau$) which is due to R. Davis [1979]. The results are presented essentially as in Leadbetter [1974, 1976, 1978] and Leadbetter, Lindgren and Rootzen [1979].

2.2 Asymptotic independence of the maximum over juxtaposed intervals

In the next three sections, $(\xi_i;\ i = 1,2,\ldots)$ will be a stationary sequence defined on some given probability space (Ω,F,P). For any set K of positive integers we will write

$$M(K) = \max_{i \in K} \xi_i .$$

If $K = \{1,2,\ldots,n\}$ we write simply M_n. It will be convenient to talk of an interval to mean any finite set K of consecutive integers $\{j_1, j_1+1, \ldots, j_2\}$ say. We then say that K has length j_2-j_1+1. If $F = \{k_1, k_1+1, \ldots, k_2\}$ with $j_2 < k_1$, we say that K and F are separated by k_1-j_2.

LEMMA 2.2.1. *Suppose that the condition* $D(u_n)$ *holds for some given sequence* (u_n). *Let* n, r *and* k *be fixed positive integers. Suppose* $K_1, K_2, \ldots, K_r$ *are subintervals of* $\{1,2,\ldots,n\}$ *such that* K_i *and* K_j *are separated by at least* k *whenever* $i \neq j$. *Then*

$$\left| P\left[\bigcap_{j=1}^{r} \{M(K_j) \le u_n\}\right] - \prod_{j=1}^{r} P[M(K_j) \le u_n] \right| \le (r-1)\alpha_{n,k} \tag{2.9}$$

where $\alpha_{n,r}$ *is as in* (2.3).

Proof. For brevity we write $A_j = \{M(K_j) \le u_n\}$. Let $K_j = \{k_j, \ldots, \ell_j\}$ where $k_1 \le \ell_1 < k_2 \le \ell_2 < \cdots < k_r \le \ell_r$ (after renumbering if necessary). Then, since $k_2 - \ell_1 \ge k$, condition $D(u_n)$ tells us that

$$|P[A_1 \cap A_2] - P[A_1]P[A_2]| \le \alpha_{n,k} .$$

Similarly we have

$$\begin{aligned}
&|P[A_1 \cap A_2 \cap A_3] - P[A_1]P[A_2]P[A_3]| \\
&\quad \le |P[A_1 \cap A_2 \cap A_3] - P[A_1 \cap A_2]P[A_3]| + P[A_3]|P[A_1 \cap A_2] - P[A_1]P[A_2]| \\
&\quad \le 2\alpha_{n,k}
\end{aligned}$$

since $K_1 \cup K_2 \subset \{k_1, \ldots, \ell_2\}$ and $k_3 - \ell_2 \ge k$. Proceeding in this way we obtain (2.9). ///

Now let k be a fixed positive integer and for any positive integer n let $[n/k]$ denote the integer part of n/k. Clearly we have $[n/k]k \le n < ([n/k]+1)k$. In the following we shall approximate $P[M_n \le u_n]$ by $P^k[M_{[n/k]} \le u_n]$.

For a given n, choose m so that $k < m < [n/k]$ and let us divide the interval $\{1,2,\ldots,[n/k]k\}$ into $2k$ consecutive intervals in the following way:

$$\begin{aligned}
I_1 &= \{1,2,\ldots,[n/k]-m\} \\
I_1^* &= \{[n/k]-m+1,\ldots,[n/k]\} \\
I_2 &= \{[n/k]+1,\ldots,2[n/k]-m\} \\
I_2^* &= \{2[n/k]-m+1,\ldots,2[n/k]\} \\
&\vdots \\
I_k &= \{(k-1)[n/k]+1,\ldots,k[n/k]-m\} \\
I_k^* &= \{k[n/k]-m+1,\ldots,k[n/k]\} .
\end{aligned}$$

Finally write

$$I_{k+1} = \{(k-1)[n/k]+m+1,\ldots,k[n/k]\}$$
$$I^*_{k+1} = \{k[n/k]+1,\ldots,k[n/k]+m\} .$$

To approximate $P[M_n \le u_n]$ by $P^k[M_{[n/k]} \le u_n]$ we proceed as follows. First we approximate $P[M_n \le u_n]$ by $P[\bigcap_{j=1}^{k} \{M(I_j) \le u_n\}]$, i.e. we disregard the small intervals I^*_j. Then, using Lemma 2.2.1 we approximate $P[\bigcap_{j=1}^{k} \{M(I_j) \le u_n\}]$ by $\prod_{j=1}^{k} P[M(I_j) \le u_n]$ which, by stationarity, is equal to $P^k[M_{[n/k]-m} \le u_n]$. Finally we approximate $P^k[M_{[n/k]-m} \le u_n]$ by $P^k[M_{[n/k]} \le u_n]$, i.e. we throw back in the small interval I^*_1. Specifically we have:

LEMMA 2.2.2. *With the above notation, and assuming that the condition* $D(u_n)$ *holds, we have*

(i) $0 \le P[\bigcap_{j=1}^{k} \{M(I_j) \le u_n\}] - P[M_n \le u_n] \le (k+1)P[M(I_1) \le u_n < M(I^*_1)]$

(ii) $|P[\bigcap_{j=1}^{k} \{M(I_j) \le u_n\}] - P^k[M(I_1) \le u_n]| \le k\alpha_{n,m}$

(iii) $|P^k[M(I_1) \le u_n] - P^k[M_{[n/k]} \le u_n]| \le kP[M(I_1) \le u_n < M(I^*_1)]$.

Hence by combining (i), (ii) *and* (iii) *we have*

$$(2.10) \qquad |P[M_n \le u_n] - P^k[M_{[n/k]} \le u_n]| \le (2k+1)P[M(I_1) \le u_n < M(I^*_1)] + k\alpha_{n,m} .$$

Proof. (i) follows at once since $\{M_n \le u_n\} \subset \bigcap_{j=1}^{k} \{M(I_j) \le u_n\}$ and their difference implies $M(I_j) \le u_n < M(I^*_j)$ for at least one j, $1 \le j \le k+1$. We have

$$\begin{aligned} 0 &\le P[\bigcap_{j=1}^{k} \{M(I_j) \le u_n\}] - P[M_n \le u_n] \\ &= P[(\bigcap_{j=1}^{k} \{M(I_j) \le u_n\}) - \{M_n \le u_n\}] \\ &\le P[\bigcup_{j=1}^{k+1} \{M(I_j) \le u_n < M(I^*_j)\}] \end{aligned}$$

$$\leq \sum_{j=1}^{k+1} P[M(I_j) \leq u_n < M(I_j^*)]$$
$$= (k+1)P[M(I_1) \leq u_n < M(I_1^*)]$$

where we used stationarity in the last statement. (ii) follows from Lemma 2.2.1 and the fact that $\prod_{j=1}^{k} P[M(I_j) \leq u_n]$ is equal to $P^k[M(I_1) \leq u_n]$ by stationarity. To obtain (iii) we note that

$$0 \leq P[M(I_1) \leq u_n] - P[M_{[n/k]} \leq u_n] = P[M(I_1) \leq u_n < M(I_1^*)] .$$

Put $y = P[M(I_1) \leq u_n]$ and $x = P[M_{[n/k]} \leq u_n]$. (iii) follows from the obvious inequalities

$$0 \leq y^k - x^k \leq k(y-x)$$

which holds whenever $0 \leq x \leq y \leq 1$. (2.10) follows easily. ////

LEMMA 2.2.3. *With the above notation, and assuming that the condition* $D(u_n)$ *holds, let* r *be a fixed positive integer. Then if* $n > (2r+1)mk$ *we have*

$$P[M(I_1) \leq u_n < M(I_1^*)] \leq \frac{1}{r} + 2r\alpha_{n,m} . \tag{2.11}$$

Proof. Since $[n/k] \geq (2r+1)m$, we may choose intervals $K_1, K_2, \ldots, K_r$, each of length m, from $I_1 = \{1,2,\ldots,[n/k]-m\}$ so that they are separated from each other and from I_1^* by at least m. Then

$$P[M(I_1) \leq u_n < M(I_1^*)] \leq P[\bigcap_{s=1}^{r} \{M(K_s) \leq u_n\} \cap \{M(I_1^*) > u_n\}]$$
$$= P[\bigcap_{s=1}^{r} \{M(K_s) \leq u_n\}] - P[\bigcap_{s=1}^{r} \{M(K_s) \leq u_n\} \cap \{M(I_1^*) \leq u_n\}]$$

By stationarity, $P[M(K_s) \leq u_n] = P[M(I_1^*) \leq u_n] = p$, say, and by Lemma 2.2.1 the two terms on the right hand side differ from p^r and

p^{r+1} (in absolute magnitude) by no more than $(r-1)\alpha_{n,m}$ and $r\alpha_{n,m}$ respectively. Hence

$$P[M(I_1) \le u_n < M(I_1^*)] \le p^r - p^{r+1} + 2r\alpha_{n,m}$$

from which (2.11) follows since $p^r - p^{r+1} \le \frac{1}{r+1} < \frac{1}{r}$. ////

From these lemmas we easily obtain the following theorem which is an essential step leading to the Poisson point process limit theorem for the exceedances.

THEOREM 2.2.4. *If* $(\xi_i;\ i=1,2,\ldots)$ *is a stationary sequence and if* (u_n) *is a sequence of real constants for which the condition* $D(u_n)$ *holds, then for every positive integer* k *we have*

$$P[M_n \le u_n] - P^k[M_{[n/k]} \le u_n] \to 0 \quad \textit{as} \quad n \to \infty .$$

Proof. Fix k. If m, r, n are positive integers satisfying

(2.12) $$k < m < [n/k] \quad \text{and} \quad n > (2r+1)mk$$

then by Lemma 2.2.2 and Lemma 2.2.3 we have

(2.13) $$|P[M_n \le u_n] - P^k[M_{[n/k]} \le u_n]| \le \frac{2k+1}{r} + (4kr+2r+k)\alpha_{n,m} .$$

Now fix r. Take $m = \ell_n$, as in our statement of condition $D(u_n)$. Since $\ell_n = o(n)$, (2.12) is satisfied for large enough n. Thus, since $\alpha_{n;\ell_n} \to 0$ as $n \to \infty$, (2.13) gives us

$$\limsup_{n\to\infty} |P[M_n \le u_n] - P^k[M_{[n/k]} \le u_n]| \le \frac{2k+1}{r} .$$

Since r was arbitrary, the proof is complete. ////

2.3 A weak convergence result for the maximum

If $(\xi_i;\ i=1,2,\dots)$ is a sequence of independent and identically distributed random variables with marginal distribution function F and if (u_n) is a sequence of real constants such that $n(1-F(u_n)) \to \tau$ as $n \to \infty$, then we have

$$\begin{aligned}
P[M_n \le u_n] &= P[\max_{1\le i\le n} \xi_i \le u_n] \\
&= P[\xi_1 \le u_n,\ \xi_2 \le u_n, \dots, \xi_n \le u_n] \\
&= \prod_{i=1}^{n} P[\xi_i \le u_n] \\
&= (F(u_n))^n \\
&= (1-(1-F(u_n)))^n \\
&= (1-(\tfrac{\tau}{n}+o(\tfrac{1}{n})))^n .
\end{aligned}$$

Hence $P[M_n \le u_n] \to e^{-\tau}$ as $n \to \infty$. Conversely, $P[M_n \le u_n] \to e^{-\tau}$ can be written as $(F(u_n))^n \to e^{-\tau}$. This implies $F(u_n) \to 1$ and $n \log[1-(1-F(u_n))] \to -\tau$ from which we get $n(1-F(u_n)) \to \tau$. Thus we have:

THEOREM 2.3.1. *Let* $(\xi_i;\ i=1,2,\dots)$ *be a sequence of independent and identically distributed random variables with marginal distribution function* F, *let* (u_n) *be a sequence of real constants, and let* τ *be a positive constant. Then*

$$n(1-F(u_n)) \to \tau \quad \textit{as} \quad n \to \infty$$

if and only if

$$P[M_n \le u_n] \to e^{-\tau} \quad \textit{as} \quad n \to \infty . \qquad ////$$

Using the result of Section 2.2 we now show that this theorem also holds for stationary sequences satisfying the conditions $D(u_n)$ and $D'(u_n)$.

THEOREM 2.3.2. *Let* $(\xi_i;\ i = 1,2,\ldots)$ *be a stationary sequence with marginal distribution function* F, *let* (u_n) *be a sequence of real constants for which the conditions* $D(u_n)$ *and* $D'(u_n)$ *hold, and let* τ *be a positive constant. Then*

$$n(1-F(u_n)) \to \tau \quad \text{as} \quad n \to \infty \tag{2.14}$$

if and only if

$$P[M_n \le u_n] \to e^{-\tau} \quad \text{as} \quad n \to \infty . \tag{2.15}$$

Proof. Let k be a fixed positive integer. From

$$\{M_{[n/k]} > u_n\} = \bigcup_{j=1}^{[n/k]} \{\xi_j > u_n\}$$

we get

$$\sum_{j=1}^{[n/k]} P[\xi_j > u_n] - \sum_{1 \le i < j \le [n/k]} P[\xi_i > u_n,\ \xi_j > u_n] \le P[M_{[n/k]} > u_n] \le \sum_{j=1}^{[n/k]} P[\xi_j > u_n] ,$$

and using stationarity we get

$$1 - [n/k](1-F(u_n)) \le P[M_{[n/k]} \le u_n] \le 1 - [n/k](1-F(u_n)) + S_{n,k} \tag{2.16}$$

where $S_{n,k} = [n/k] \sum_{j=2}^{[n/k]} P[\xi_1 > u_n,\ \xi_j > u_n]$. Since condition $D'(u_n)$ holds, we have

$$\limsup_{n\to\infty} S_{n,k} = o(\tfrac{1}{k}) \quad \text{as} \quad k \to \infty .$$

Suppose (2.14) holds. Taking $n \to \infty$ in (2.16) we get

$$1-\frac{\tau}{k} \leq \liminf_{n\to\infty} P[M_{[n/k]} \leq u_n] \leq \limsup_{n\to\infty} P[M_{[n/k]} \leq u_n] \leq 1-\frac{\tau}{k}+o(\frac{1}{k}) .$$

Now taking the k-th power and then using Theorem 2.2.4 we get

$$(1-\frac{\tau}{k})^k \leq \liminf_{n\to\infty} P[M_n \leq u_n] \leq \limsup_{n\to\infty} P[M_n \leq u_n] \leq (1-\frac{\tau}{k}+o(\frac{1}{k}))^k .$$

Taking $k\to\infty$ we obtain (2.15). Now suppose (2.15) holds. Then by Theorem 2.2.4 we have

$$P[M_{[n/k]} \leq u_n] \to e^{-\tau/k} \text{ as } n\to\infty .$$

Writing (2.16) as

$$1-P[M_{[n/k]} \leq u_n] \leq [n/k](1-F(u_n)) \leq 1-P[M_{[n/k]} \leq u_n]+S_{n,k}$$

and letting $n\to\infty$, we get

$$1-e^{-\tau/k} \leq \frac{1}{k}\liminf_{n\to\infty} n(1-F(u_n)) \leq \frac{1}{k}\limsup_{n\to\infty} n(1-F(u_n)) \leq 1-e^{-\tau/k}+o(\frac{1}{k}) .$$

Multiplying by k and letting $k\to\infty$, we get (2.14). ////

Theorem 2.3.2 is interesting on its own. As we shall see later, it can be used to obtain interesting results concerning the limit distribution of $a_n(M_n-b_n)$ for suitable choice of a_n, b_n. For the moment our goal is to use Theorem 2.3.2 to obtain Theorem 2.1.1. First we need a technical result. So far we have been dealing with just one sequence $(u_n;\ n=1,2,\ldots)$. Given such a sequence we now define a family of sequences $((u_n(\theta);\ n=1,2,\ldots);\ \theta>0)$ by writing

$$u_n(\theta) = u_{[n/\theta]} .$$

Note that $(u_n(1))$ is simply the original sequence (u_n).

LEMMA 2.3.3.

(a) *If* $n(1-F(u_n)) \to \tau$, *then* $n(1-F(u_n(\theta))) \to \theta\tau$ *for all* $\theta > 0$.

(b) *If condition* $D(u_n)$ *holds, then condition* $D(u_n(\theta))$ *holds for* $0 < \theta \le 1$.

(c) *If condition* $D'(u_n)$ *holds, then condition* $D'(u_n(\theta))$ *holds for* $0 < \theta \le 1$.

Proof. If $n(1-F(u_n)) \to \tau$, then $[n/\theta](1-F(u_{[n/\theta]})) \to \tau$, which gives $n(1-F(u_n(\theta))) \to \theta\tau$. Thus (a) is proved. Assume condition $D(u_n)$ holds. Take $1 \le i_1 < \cdots < i_p < j_1 < \cdots < j_q \le n$ with $j_1 - i_p \ge \ell$. Writing $\underset{\sim}{i} = (i_1,\ldots,i_p)$ and $\underset{\sim}{j} = (j_1,\ldots,j_q)$, we have

$$(2.17)\quad |F_{\underset{\sim}{i}\underset{\sim}{j}}(u_n(\theta)) - F_{\underset{\sim}{i}}(u_n(\theta))F_{\underset{\sim}{j}}(u_n(\theta))| = |F_{\underset{\sim}{i}\underset{\sim}{j}}(u_{[n/\theta]}) - F_{\underset{\sim}{i}}(u_{[n/\theta]})F_{\underset{\sim}{j}}(u_{[n/\theta]})| .$$

If $0 < \theta \le 1$, then $n \le [n/\theta]$. Hence, since condition $D(u_n)$ holds, the right hand side of (2.17) does not exceed

$$\alpha'_{n,\ell} = \alpha_{[n/\theta],\ell}$$

where $\alpha_{n,\ell}$ is as in (2.3). Since $\alpha_{n,\ell_n} \to 0$ for some $\ell_n = o(n)$, we have $\alpha'_{n,\ell'_n} \to 0$ with $\ell'_n = \ell_{[n/\theta]}$. Clearly $\ell_{[n/\theta]} = o(n)$. Hence condition $D(u_n(\theta))$ holds. (b) is proved. For $0 < \theta \le 1$ we have

$$n\sum_{j=2}^{[n/k]} P[\xi_1 > u_n(\theta),\ \xi_j > u_n(\theta)] = n\sum_{j=2}^{[n/k]} P[\xi_1 > u_{[n/\theta]},\ \xi_j > u_{[n/\theta]}]$$

$$\le [n/\theta]\sum_{j=2}^{[[n/\theta]/k]} P[\xi_1 > u_{[n/\theta]},\ \xi_j > u_{[n/\theta]}] .$$

By condition $D'(u_n)$, the upper limit of this expression as $n \to \infty$ (or equivalently as $[n/\theta] \to \infty$) tends to 0 as $k \to \infty$. Hence condition $D'(u_n(\theta))$ holds. (c) is proved. ////

THEOREM 2.3.4. *Let* $(\xi_i; i=1,2,\ldots)$ *be a stationary sequence with marginal distribution function* F, *let* (u_n) *be a sequence of real constants for which the conditions* $D(u_n)$ *and* $D'(u_n)$ *hold and let* τ *be a positive constant such that* (2.14) *holds. If* $(\nu(n); n=1,2,\ldots)$ *is a sequence of positive integers such that for some* $0<\theta\leq 1$

$$\frac{\nu(n)}{n} \longrightarrow \theta \quad as \quad n \longrightarrow \infty$$

then

$$P[M_{\nu(n)} \leq u_n] \rightarrow e^{-\theta\tau} \quad as \quad n \rightarrow \infty . \tag{2.18}$$

Proof. By Lemma 2.3.3 we have

(a) $n(1-F(u_n(\theta))) \rightarrow \theta\tau$ as $n \rightarrow \infty$

(b) condition $D(u_n(\theta))$ holds

(c) condition $D'(u_n(\theta))$ holds.

Hence from Theorem 2.3.2 (with $\theta\tau$ instead of τ) we have

$$P[M_n \leq u_n(\theta)] \rightarrow e^{-\theta\tau} \quad \text{as} \quad n \rightarrow \infty$$

Thus

$$P[M_{\nu(n)} \leq u_{\nu(n)}(\theta)] \rightarrow e^{-\theta\tau} \quad \text{as} \quad n \rightarrow \infty .$$

Hence (2.18) will follow from

$$P[M_{\nu(n)} \leq u_n] - P[M_{\nu(n)} \leq u_{\nu(n)}(\theta)] \rightarrow 0 \quad \text{as} \quad n \rightarrow \infty \tag{2.19}$$

which is to be expected since $u_{\nu(n)}(\theta) = u_{[\nu(n)/\theta]}$ and by hypothesis $\nu(n)/\theta \sim n$. (2.19) is obtained as follows: If $u_n \geq u_{\nu(n)}(\theta)$, then

$$\begin{aligned} 0 &\leq P[M_{\nu(n)} \leq u_n] - P[M_{\nu(n)} \leq u_{\nu(n)}(\theta)] \\ &= P[u_{\nu(n)}(\theta) < M_{\nu(n)} \leq u_n] \end{aligned}$$

$$\leq P\Big[\bigcup_{j=1}^{\nu(n)} \{u_{\nu(n)}(\theta) < \xi_j \leq u_n\}\Big]$$

$$\leq \nu(n)[F(u_n) - F(u_{\nu(n)}(\theta))] .$$

Similarly, if $u_n < u_{\nu(n)}(\theta)$ then

$$0 \leq P[M_{\nu(n)} \leq u_{\nu(n)}(\theta)] - P[M_{\nu(n)} \leq u_n]$$

$$\leq \nu(n)[F(u_{\nu(n)}(\theta)) - F(u_n)] .$$

Thus we have

$$\begin{aligned} |P[M_{\nu(n)} \leq u_n] - P[M_{\nu(n)} \leq u_{\nu(n)}(\theta)]| &\leq \nu(n)|F(u_n) - F(u_{\nu(n)}(\theta))| \\ &= \nu(n)|(1-F(u_n)) - (1-F(u_{\nu(n)}(\theta)))| \\ &= \nu(n)\left|\frac{\tau}{n}(1+o(1)) - \frac{\theta\tau}{\nu(n)}(1+o(1))\right| \\ &= o(1) \quad \text{as} \quad n \to \infty . \end{aligned}$$

Thus (2.19) holds and the proof is complete. ////

The Poisson point process limit theorem for the exceedances of a stationary sequence will follow easily from the following result.

THEOREM 2.3.5. *Let* $(\xi_i;\ i = 1,2,\ldots)$ *be a stationary sequence with marginal distribution function* F, *let* (u_n) *be a sequence of real constants for which the conditions* $D(u_n)$ *and* $D'(u_n)$ *hold and let* τ *be a positive constant such that* (2.14) *holds. If* $0 \leq a_1 < b_1 < a_2 < b_2 < \cdots < a_r < b_r \leq 1$ *and if* $K_{n,i} = \{[na_i]+1, [na_i]+2, \ldots, [nb_i]\}$, *then*

$$(2.20) \qquad P\Big[\bigcap_{j=1}^{r} \{M(K_{n,i}) \leq u_n\}\Big] \to e^{-\tau\sum_{1}^{r}(b_i - a_i)} \quad \text{as} \quad n \to \infty .$$

Proof. By Lemma 2.2.1 we have

$$(2.21) \qquad \Big|P\Big[\bigcap_{j=1}^{r}\{M(K_{n,i}) \leq u_n\}\Big] - \prod_{i=1}^{r} P[M(K_{n,i}) \leq u_n]\Big| \leq (r-1)\alpha_{n,n\beta}$$

where $\beta = \min\limits_{1<i<r-1} (a_{i+1}-b_i)$ and where $\alpha_{n,k}$ is as in (2.3). Since condition $D(\overline{u}_n)$ holds, the right hand side of (2.21) goes to 0 as $n \to \infty$. By stationarity we have

$$P[M(K_{n,i}) \le u_n] = P[M([nb_i]-[na_i]) \le u_n] ,$$

and clearly $\dfrac{([nb_i]-[na_i])}{n} \to (b_i-a_i)$ as $n \to \infty$. Thus from Theorem 2.3.4 we have

$$P[M(K_{n,i}) \le u_n] \to e^{-\tau(b_i-a_i)} \quad \text{as} \quad n \to \infty$$

and hence
$$\prod_{i=1}^{r} P[M(K_{n,i}) \le u_n] \to e^{-\tau\sum_1^r (b_i-a_i)} \quad \text{as} \quad n \to \infty .$$

This, combined with (2.21), gives us (2.20). ////

2.4 The Poisson point process limit theorem for the exceedances

The convergence of the point process of exceedances of a stationary sequence to a Poisson point process, Theorem 2.1.1, is now easily obtained.

Let $(\xi_i;\ i = 1,2,\ldots)$ be a stationary sequence with marginal distribution function F, let (u_n) be a sequence of real constants for which the conditions $D(u_n)$ and $D'(u_n)$ hold and let Z_n be the point process of exceedances of the level u_n. If $n(1-F(u_n)) \to \tau$, for some $\tau > 0$, then $Z_n \xrightarrow{w} Z$ where Z is a Poisson point process with intensity τ.

Proof. By Kallenberg's result, it is enough to show that

(a) $E[Z_n((a,b])] \to \tau(b-a)$ as $n \to \infty$, for all $0 \le a < b \le 1$.

(b) $P[Z_n(A) = 0] \to e^{-\tau\Lambda(A)}$ for all A of the form $\bigcup_{i=1}^{r}(a_i, b_i]$ with $0 \le a_1 < b_1 < a_2 < b_2 < \cdots < a_r < b_r \le 1$, where $\Lambda(A)$ is the Lebesgue measure of A.

As we saw in Section 2.1, (a) is immediate:

$$E[Z_n((a,b])] = ([nb]-[na])(1-F(u_n)) \to (b-a)\tau \; .$$

Now let A be as described above. Then

$$\begin{aligned} P[Z_n(A) = 0] &= P[Z_n(\bigcup_{i=1}^{r}(a_i, b_i]) = 0] \\ &= P[\sum_{i=1}^{r} Z_n((a_i, b_i]) = 0] \\ &= P[\bigcap_{i=1}^{r}\{Z_n((a_i, b_i]) = 0\}] \\ &= P[\bigcap_{i=1}^{r}\{M(K_{n,i}) \le u_n\}] \end{aligned}$$

where $K_{n,i} = \{[na_i]+1, [na_i]+2, \ldots, [nb_i]\}$. The last equality holds since the sets $\{Z_n((a_i, b_i]) = 0\}$ and $\{M(K_{n,i}) \le u_n\}$ are the same. By Theorem 2.3.5 we have

$$P[\bigcap_{i=1}^{r}\{M(K_{n,i}) \le u_n\}] \to \exp\left(-\tau \sum_{i=1}^{r}(b_i - a_i)\right) \quad \text{as} \quad n \to \infty \; .$$

Hence (b) holds. Theorem 2.1.1 is proved. ///

2.5 Connected results

As before, $(\xi_i;\ i = 1,2,\ldots)$ will denote a stationary sequence with marginal distribution function F, M_n will be the maximum of $\{\xi_1, \xi_2, \ldots, \xi_n\}$, (u_n) will be a sequence of real constants, Z_n will be the point process of exceedances of the level u_n and Z will be a Poisson point process with intensity τ. The last three sections can be summarized as follows:

A. If condition $D(u_n)$ holds and if k is a (fixed) positive integer then

$$P[M_n \le u_n] - P^k[M_{[n/k]} \le u_n] \to 0 \quad \text{as} \quad n \to \infty .$$

B. If conditions $D(u_n)$ and $D'(u_n)$ hold, then

$$n(1-F(u_n)) \to \tau \;\Rightarrow\; P[M_n \le u_n] \to e^{-\tau} .$$

C. If conditions $D(u_n)$ and $D'(u_n)$ hold, then

$$n(1-F(u_n)) \to \tau \;\Rightarrow\; Z_n \xrightarrow{w} Z .$$

We used A to obtain B and then B to obtain C. The converse of B was not used to obtain C but was presented for completeness:

B'. If conditions $D(u_n)$ and $D'(u_n)$ hold, then

$$P[M_n \le u_n] \to e^{-\tau} \;\Rightarrow\; n(1-F(u_n)) \to \tau .$$

These results are interesting in their own. But they also lead to some very important theorems as we shall now see.

Gnedenko's theorem

To begin with, we shall discuss a famous result concerning the asymptotic behaviour of the distribution of M_n.

THEOREM 2.5.1. *If* $(\xi_i;\ i=1,2,\ldots)$ *is a sequence of independent and identically distributed random variables and if for some constants* $a_n > 0$ *and* b_n *we have*

$$P[a_n(M_n-b_n) \le x] \xrightarrow{\bullet} G(x)$$

for some non-degenerate distribution function G *(where* $\xrightarrow{\bullet}$ *means convergence at the continuity points of the limiting function), then* G *is of one of the three types listed below:*

Type 1 $\quad G(x) = \exp(-e^{-x}) \qquad -\infty < x < \infty$

Type 2 $$G(x) = \begin{cases} 0 & -\infty < x \le 0 \\ \exp(-x^{-\alpha}) & 0 < x < \infty \end{cases}$$ for some $\alpha > 0$

Type 3 $$G(x) = \begin{cases} \exp(-(-x)^{\alpha}) & -\infty < x \le 0 \\ 1 & 0 < x < \infty \end{cases}$$ for some $\alpha > 0$ ////

(If G_1 and G_2 are distribution functions and if for some $a > 0$ and some b we have $G_2(x) = G_1(ax+b)$ for every x, then G_1 and G_2 are said to be of the same type. Observe that type 2 and type 3 are in fact families of types indexed by the parameter α.)

This result is referred to as Gnedenko's theorem even though it was discovered long before Gnedenko. Frechet [1927] found that the possible limit laws for $a_n M_n$, with a suitable choice of $a_n > 0$, were only laws of types 2 and 3. Fisher and Tippet [1928] established that the limit laws for $a_n(M_n-b_n)$, with suitable choice of $a_n > 0$ and b_n, were only laws of types 1, 2 and 3. Later DeMises [1939] found conditions on the distribution function of the ξ_i's for $a_n(M_n-b_n)$ to converge to a law of types 1, 2, or 3. However Gnedenko [1943] was the first to prove the theorem in complete generality. Its proof may be displayed as the two following results:

LEMMA 2.5.2. *Let* $M_1, M_2, \ldots$ *be random variables (which in this Lemma may be maxima or not) and suppose* $a_n > 0$ *and* b_n *are constants such that*

$$P[a_n(M_n - b_n) \leq x] \xrightarrow{w} G(x) \tag{2.22}$$

for some non-degenerate distribution function G. *Furthermore suppose that for* $k = 2,3,\ldots$ *we have*

$$P[a_{nk}(M_n - b_{nk}) \leq x] \xrightarrow{w} G^{1/k}(x) \ . \tag{2.23}$$

Then, corresponding to each k, *there are constants* $\alpha_k > 0$ *and* β_k *such that*

$$G^k(\alpha_k x + \beta_k) = G(x) \ , \quad \text{for all } x \in \mathbb{R}. \qquad //// \tag{2.24}$$

LEMMA 2.5.3. *If* G *is a non-degenerate distribution function such that* (2.24) *holds for* $k = 1,2,3,\ldots$ *(for some constants* $\alpha_k > 0$ *and* b_k*) then* G *is one of the three extreme value types listed in Theorem 2.5.1.* ////

One can easily check that if M_n is the maximum of the first n terms of a sequence of independent and identically distributed random variables, then (2.23) holds (for $k=2,3,\ldots$) whenever (2.22) holds. Hence Theorem 2.5.1 follows at once from Lemma 2.5.2 and Lemma 2.5.3. The first of these lemmas is essentially a result of Khintchine and its proof may be found in Gnedenko and Kolmogorov [1954, section 10, theorem 1].

The derivation of the second lemma constitutes the major part of Gnedenko's paper [1943]. A simple proof is presented in deHaan [1976].

Using the result of Section 2.2 it is easy to obtain Gnedenko's theorem for stationary sequences.

THEOREM 2.5.4. *Let* $(\xi_i;\ i=1,2,\ldots)$ *be a stationary sequence and suppose that for some constants* $a_n > 0$, b_n, *we have*

$$P[a_n(M_n - b_n) \le x] \xrightarrow{\cdot} G(x)$$

for some non-degenerate distribution function G. *Suppose that for each real number* x *the condition* $D(u_n)$ *holds for the sequence* $u_n = x/a_n + b_n$. *Then* G *has one of the three extreme value forms listed in Theorem 2.5.1.*

Proof. Let x be a continuity point of G. Writing $u_n = x/a_n + b_n$ we have

$$P[M_n \le u_n] \longrightarrow G(x) .$$

Since condition $D(u_n)$ holds, Theorem 2.2.4 tells us that

$$P[a_{nk}(M_n - b_{nk}) \le x] \longrightarrow G^{1/k}(x) \qquad k = 1,2,3,\ldots .$$

Hence we get

$$P[M_n \le u_{nk}] \xrightarrow{\cdot} G^{1/k} \qquad k = 1,2,3,\ldots .$$

Thus the result follows from Lemmas 2.5.2 and 2.5.3. /////

This theorem can be found in Leadbetter [1974]. It was previously obtained by Loynes [1965] for stationary sequences satisfying a strong mixing condition. Loynes investigations had its origin in Watson's [1954] paper on the maximum of m-dependent stationary sequences. The asymptotic distribution of M_n in the case of a stationary Gaussian sequence was studied by Berman [1964]. His dependence conditions were written in terms of the covariance function of the sequence.

The associated sequence of independent random variables

As a second application of the results of this chapter, consider a stationary sequence $(\xi_i;\ i = 1,2,\ldots)$ with marginal distribution function F and let $(\eta_i;\ i = 1,2,\ldots)$ be a sequence of independent and identically distributed random variables having the same marginal distribution function F ($(\eta_i;\ i = 1,2,\ldots)$ is sometimes called the associated sequence of independent random variables). Let M_n be, as usual, the maximum of $\{\xi_1,\xi_2,\ldots,\xi_n\}$, and let $\hat{M}_n$ be the maximum of $\{\eta_1,\eta_2,\ldots,\eta_n\}$. Then:

THEOREM 2.5.5. *With the above notation, if* (u_n) *is such that the conditions* $D(u_n)$ *and* $D'(u_n)$ *hold (for the sequence* $(\xi_i;\ i = 1,2,\ldots)$*) then for* $0 < \theta \leq 1$ *we have*

$$P[M_n \leq u_n] \rightarrow \theta \quad \textit{if and only if} \quad P[\hat{M}_n \leq u_n] \rightarrow \theta .$$

Proof. This follows trivially Theorem 2.3.1, which says that

$$P[\hat{M}_n \leq u_n] \rightarrow \theta \quad \text{if and only if} \quad n(1-F(u_n)) \rightarrow -\log \theta ,$$

combined with Theorem 2.3.2, which says that

$$P[M_n \leq u_n] \rightarrow \theta \quad \text{if and only if} \quad n(1-F(u_n)) \rightarrow -\log \theta. \quad ////$$

THEOREM 2.5.6. *With the above notation, if* $a_n > 0$ *and* b_n *are given constants such that for every* x *the conditions* $D(u_n)$ *and* $D'(u_n)$ *hold (for the sequence* $(\xi_i;\ i = 1,2,\ldots)$*) with* $u_n = x/a_n + b_n$, *and if* G

is a non-degenerate distribution function, then

(2.25) $$P[a_n(M_n-b_n) \le x] \xrightarrow{\cdot} G(x)$$

if and only if

(2.26) $$P[a_n(\hat{M}_n-b_n) \le x] \xrightarrow{\cdot} G(x) .$$

Proof. We will show that (2.25) implies (2.26). The converse follows in exactly the same way. If (2.25) holds, then by Gnedenko's theorem G is one of the three extreme value types. Hence G is continuous everywhere. Thus in (2.25) and (2.26) the notation $\xrightarrow{\cdot}$ means convergence at every point. If x is such that $G(x) > 0$, then $P[a_n(M_n-b_n) \le x] \to G(x)$ implies $P[a_n(\hat{M}_n-b_n) \le x] \to G(x)$ by Theorem 2.5.5. If $G(x) = 0$ then for every y with $G(y) > 0$ we have

$$P[a_n(\hat{M}_n-b_n) \le x] \le P[a_n(\hat{M}_n-b_n) \le y] .$$

Hence we have

$$\limsup_{n\to\infty} P[a_n(\hat{M}_n-b_n) \le x] \le G(y)$$

for all y with $G(y) > 0$. Letting y decrease to $\inf\{y: G(y) > 0\}$ and using continuity of G we get

$$\lim_{n\to\infty} P[a_n(\hat{M}_n-b_n) \le x] = 0 .$$

This holds for all x. Hence (2.26) holds. ////

The k-th largest value

We conclude with a third application. Let $(\xi_i;\ i = 1,2,\ldots)$ be a stationary sequence with marginal distribution function F and for any positive integer k, let $M_n^{(k)}$ denote the k-th largest value of

$\{\xi_1, \xi_2, \ldots, \xi_n\}$. $M_n^{(1)}$ is our usual M_n.

THEOREM 2.5.7. *If* (u_n) *is a sequence of constants such that the conditions* $D(u_n)$ *and* $D'(u_n)$ *hold and if for some* $\tau > 0$ *we have* $n(1-F(u_n)) \to \tau$ *as* $n \to \infty$, *then*

$$P[M_n^{(k)} \leq u_n] \to e^{-\tau} \sum_{s=0}^{k-1} \frac{\tau^s}{s!} \quad \text{as} \quad n \to \infty .$$

Proof. Observe that

$$\{M_n^{(k)} \leq u_n\} = \{Z_n((0,1]) \leq k-1\}$$

where Z_n is as usual the point process of exceedances of the level u_n. Thus

$$\begin{aligned} P[M_n^{(k)} \leq u_n] &= P[Z_n((0,1]) \leq k-1] \\ &= \sum_{s=0}^{k-1} P[Z_n((0,1]) = s] . \end{aligned}$$

But from Theorem 2.1.1 we have

$$P[Z_n((0,1]) = s] \to e^{-\tau} \frac{\tau^s}{s!} \qquad s = 0,1,\ldots,k-1$$

which gives us the desired result. ////

COROLLARY 2.5.8. *The theorem still holds if the assumption* $n(1-F(u_n)) \to \tau$ *is replaced by* $P[M_n \leq u_n] \to e^{-\tau}$ *(this follows from Theorem 2.3.2) or by* $P[\hat{M}_n \leq u_n] \to e^{-\tau}$ *(this follows from Theorems 2.3.2 and 2.5.5).* ////

THEOREM 2.5.9. *Suppose that we are given constants* $a_n > 0$ *and* b_n *such that for every* x *the conditions* $D(u_n)$ *and* $D'(u_n)$ *hold with* $u_n = x/a_n + b_n$. *Let* G *be a non-degenerate distribution function. If*

$$P[a_n(M_n-b_n) \leq x] \xrightarrow{\cdot} G(x)$$

then for $k = 1,2,3,\ldots$ *we have*

$$P[a_n(M_n^{(k)}-b_n) \leq x] \xrightarrow{\cdot} G_k(x)$$

where $$G_k(x) = \begin{cases} G(x) \sum_{s=0}^{k-1} (-\log G(x))^s/s! & \text{if } G(x) > 0 \\ 0 & \text{if } G(x) = 0 . \end{cases}$$

Proof. This follows from Theorem 2.5.7 (and its corollary) just like Theorem 2.5.6 was obtained from Theorem 2.5.5. ////

These examples illustrate how the results of this chapter can be used. Although we shall not do so here, it is also possible to consider exceedances of several levels, considered jointly. We would then obtain a Poisson result which in turn could be used to obtain the joint distribution of various quantities of interest, such as two or more $M_n^{(k)}$'s. These problems and more are treated in Leadbetter, Lindgren and Rootzén [1979].

CHAPTER 3

CONVERGENCE OF THE POINT PROCESS OF UPCROSSINGS OF A STATIONARY PROCESS

3.1 Introduction

In this chapter, $(\xi(t),\ t \in \mathbb{R})$ will denote a stationary stochastic process defined on a probability space (Ω, F, P) and having, with probability one, continuous sample paths. As in the discrete case, $F_{t_1 t_2 \cdots t_n}$ will denote the joint distribution function of the random variables $\xi(t_1), \xi(t_2), \ldots, \xi(t_n)$ and, for brevity, we will write $F_{t_1 t_2 \cdots t_n}(u)$ instead of $F_{t_1 t_2 \cdots t_n}(u,u,\ldots,u)$. F will denote the marginal distribution function of the process. In order to avoid complicated notations and technical lemmas, we will assume that F is continuous and that $F(u) < 1$ for all u (this includes the Gaussian case). The results of this chapter are still true if we don't make these assumptions.

Our main goal in this chapter is to obtain the Poisson point process limit theorem for the upcrossings. We shall use essentially the same approach we used for the discrete case. For $T > 0$ we write

$$M(T) = \max_{0 \leq t \leq T} \xi(t) .$$

The value of $M(T)$ for a particular $\omega \in \Omega$ is denoted by $M(t,\omega)$,

$$M(T,\omega) = \max_{0 \leq t \leq T} \xi(t,\omega) .$$

It is easily seen that $M(T)$ is a random variable on (Ω, F, P). It is well defined and finite a.e. since for almost every ω in Ω the sample path $t \mapsto \xi(t,\omega)$ is continuous. It is measurable since, again by continuity of the sample paths, we have

$$M(T,\omega) = \lim_{k\to\infty} \max\{\xi(\frac{iT}{2^k});\ i = 0,1,2,\ldots,2^k\}$$

for almost every ω in Ω. We shall assume, when needed, that for some $h_0 > 0$

$$P[M(h) > u] \sim h\mu_u \quad \text{as} \quad u \to \infty \quad \text{for} \quad 0 < h < h_0$$

where μ_u is the mean number of upcrossings of the level u in the time interval $(0,1]$. The function μ_u will play the role played by $1-F(u_n)$ in the discrete case. Given a family of real constants $(u_T;\ T > 0)$, let Z_T be the point process of upcrossings of the level u_T. We shall follow the lines of Chapter 2 and prove the following results:

A_c. $$P[M(T) \le u_T] - P^k[M(T/k) \le u_T] \to 0 \quad \text{as} \quad T \to \infty$$

B_c. $$T\mu_{u_T} \to \tau \;\Rightarrow\; P[M(T) \le u_T] \to e^{-\tau}$$

C_c. $$T\mu_{u_T} \to \tau \;\Rightarrow\; Z_T \xrightarrow{w} \underset{\sim}{Z}$$

where Z is a Poisson point process with intensity τ. For completeness we will also obtain the converse of B_c,

B'_c. $$P[M(T) \le u_T] \underset{T}{\to} e^{-\tau} \;\Rightarrow\; T\mu_{u_T} \to \tau$$

The results of this chapter are presented as in Leadbetter [1980].

3.2 Asymptotic independence of the maximum over juxtaposed intervals

In this section we shall obtain a continuous time parameter version of Theorem 2.2.4. In addition to a $D(u_n)$-type condition we shall need some regularity conditions to make sure the sample paths are relatively smooth.

Let a positive function $\psi(u)$ and a family of positive constants $q_a(u)$, $a > 0$, $u > 0$, be given. We say that the condition R1 holds if for some $h_0 > 0$

$$(3.1) \qquad P[M(h) > u] \sim h\psi(u) \quad \text{as} \quad u \to \infty, \quad \text{for all} \quad 0 < h < h_0 .$$

We say that the condition R2 holds if

$$(3.2) \qquad q_a(u) \to 0 \quad \text{as} \quad u \to \infty, \quad \text{for each} \quad a > 0, \quad \text{and}$$

$$(3.3) \qquad \limsup_{u\to\infty} \frac{P[\xi(0) < u,\ \xi(q_a(u)) < u,\ M(q_a(u)) > u]}{q_a(u)\psi(u)} \to 0 \quad \text{as} \quad a \to 0 .$$

LEMMA 3.2.1.

(i) *If the condition* R1 *holds for* $\psi(u)$ *and if* $d(u) \to 0$ *as* $u \to \infty$, *then*

$$P[M(d(u)) > u] = o(\psi(u)) \quad \textit{as} \quad u \to \infty .$$

(ii) *If the condition* R1 *holds for* $\psi(u)$ *then*

$$P[\xi(0) > u] = o(\psi(u)) \quad \textit{as} \quad u \to \infty .$$

(iii) *If the conditions* R1 *and* R2 *hold for* $\psi(u)$ *and* $q_a(u)$ *and if* I *is an interval of length* h, $0 < h < h_0$ *(*h_0 *as in* (3.1)*), then there are constants* γ_a *such that*

$$(3.4) \qquad 0 \le \limsup_{u\to\infty} \frac{P[\xi(jq_a(u)) \le u;\ jq_a(u) \in I] - P[M(I) \le u]}{\psi(u)} \le \gamma_a ,$$

where $\gamma_a \to 0$ *as* $a \to 0$. *The convergence is uniform in all intervals of this fixed length* h.

Proof. Fix $h \in (0,h_0)$, where h_0 is as in (3.1). For u large enough we have $0 < d(u) < h$. Hence $P[M(d(u)) > u] \leq P[M(h) > u]$ from which we get, since condition R1 holds,

$$\limsup_{u\to\infty} \frac{P[M(d(u)) > u]}{\psi(u)} \leq h .$$

We get (i) by letting h decrease to 0. (ii) follows at once from (i) since $P[\xi(0) > u] \leq P[M(1/u) > u]$. Let I be an interval of length h $(0 < h < h_0)$. It consists of no more than $[h/q_a(u)]$ subintervals of the form $((j-1)q_a(u), jq_a(u)]$ together with (possibly) a shorter interval at each end. The difference in probability in (3.4) is clearly non-negative and (using stationarity) is dominated by

$$\gamma_{a,u} = \frac{h}{q_a(u)} P[\xi(0) < u,\ \xi(q_a(u)) < u,\ M(q_a(u)) > u] + 2P[M(q_a(u)) > u] .$$

Hence (3.4) holds with

$$\gamma_a = \limsup_{u\to\infty} \frac{\gamma_{a,u}}{\psi(u)} .$$

When the conditions R1 and R2 hold, we get $\gamma_a \to 0$ as $a \to 0$ by using (3.3) and (i). This proves (iii). ////

We now introduce the continuous parameter version of condition $D(u_n)$. Let $(u_T;\ T > 0)$ be a family of constants such that $u_T \to \infty$ as $T \to \infty$. Let $(q_T;\ T > 0)$ be a family of positive constants such that $q_T \to 0$ as $T \to \infty$. We say that the condition $D_c(u_T)$ holds with respect to q_T if for any

(3.5) $\quad s_1 < s_2 < \cdots < s_p < t_1 < t_2 < \cdots < t_{p'} \in \{kq_T;\ 0 \leq kq_T \leq T\}$,

with $t_1 - s_p \geq \gamma$, we have

(3.6) $$\left|F_{s_1\ldots s_p t_1\ldots t_{p'}}(u_T) - F_{s_1\ldots s_p}(u_T)F_{t_1\ldots t_{p'}}(u_T)\right| \le \alpha_{T,\gamma}$$

where $\alpha_{T,\gamma_T} \to 0$ for some $\gamma_T = o(T)$ as $T \to \infty$. Without loss of generality we may (and will) assume that $\alpha_{T,\gamma}$ is non-increasing in γ. When this is so, the condition $\alpha_{T,\gamma_T} \to 0$ for some $\gamma_T = o(T)$ as $T \to \infty$ is equivalent to the condition $\alpha_{T,\theta T} \to 0$ for each $\theta > 0$.

The following theorem is obtained from Theorem 2.2.4 by writing $M(T)$ as the M_n of a suitable stationary sequence.

THEOREM 3.2.2. *Assume that the conditions* R1 *and* R2 *hold for some given* $\psi(u)$ *and* $q_a(u)$. *Let* $(u_T; T > 0)$ *be a family of constants, with* $u_T \to \infty$ *as* $T \to \infty$, *such that*

(a) *for each* $a > 0$, *the condition* $D_c(u_T)$ *holds with respect to* $q_T = q_a(u_T)$.

(b) $T\psi(M_T)$ *is bounded.*

Then for $0 < h \le h_0$ (h_0 *as in* (3.1)) *and for every positive integer* k,

(3.7) $$P[M(nh) \le u_{nh}] - P^k[M([n/k]h) \le u_{nh}] \to 0 \quad \text{as} \quad n \to \infty .$$

Proof. Fix $h \in (0, h_0)$. Consider the random sequence $\zeta_1, \zeta_2, \ldots$ defined by

$$\zeta_n = \max\{\xi(t);\ (n-1)h \le t \le nh\}$$

and write $M_n = \max\{\zeta_1, \zeta_2, \ldots, \zeta_n\}$. Note that $M(nh) = M_n$. It is easily seen that the ζ_n-sequence is stationary. Let $v_n = u_{nh}$. If we can show that the condition $D(v_n)$ holds for the sequence $(\zeta_i;\ i = 1, 2, \ldots)$, then by Theorem 2.2.4 we will have for every positive integer k,

$$P[M_n \le v_n] - P^k[M_{[n/k]} \le v_n] \to 0 \quad \text{as} \quad n \to \infty ,$$

which is exactly the same as (3.7). Hence it suffices to show that the condition $D(\nu_n)$ holds for $(\zeta_i;\ i=1,2,\dots)$. Consider integers

$$1 \le i_1 < i_2 < \cdots < i_p < j_1 < j_2 < \cdots < j_{p'} \le n \tag{3.8}$$

with $j_1 - i_p \ge \ell$. For all such choice of positive integers we would like to have

$$\begin{aligned}\Big|P[\zeta_{i_1} \le \nu_n, \dots, \zeta_{i_p} \le \nu_n, \zeta_{j_1} \le \nu_n, \dots, \zeta_{j_{p'}} \le \nu_n] \\ - P[\zeta_{i_1} \le \nu_n, \dots, \zeta_{i_p} \le \nu_n] P[\zeta_{j_1} \le \nu_n, \dots, \zeta_{j_{p'}} \le \nu_n]\Big| \le \alpha^*_{n,\ell}\end{aligned} \tag{3.9}$$

with $\alpha^*_{n,\ell_n} \to 0$ for some $\ell_n = o(n)$ as $n \to \infty$. Put $I_r = [(i_r-1)h, i_r h]$ and $J_s = [(j_s-1)h, j_s h]$. Write q for $q_a(u_{nh})$. Consider the following subsets of Ω:

$$A_q = A_q(a,n,p,i_1,i_2,\dots,i_p) = \bigcap_{r=1}^{p} \{\xi(jq) \le u_{nh};\ jq \in I_r\},$$

$$A = A(n,p,i_1,i_2,\dots,i_p) = \bigcap_{r=1}^{p} \{\zeta_{i_r} \le u_{nh}\},$$

$$B_q = B_q(a,n,p',j_1,j_2,\dots,j_{p'}) = \bigcap_{s=1}^{p'} \{\xi(jq) \le u_{nh};\ jq \in J_s\},$$

$$B = B(n,p',j_1,j_2,\dots,j_{p'}) = \bigcap_{s=1}^{p'} \{\zeta_{j_s} \le u_{nh}\}.$$

The left hand side of (3.9) is just $|P[A \cap B] - P[A]P[B]|$ and for each $a > 0$ we have

$$\begin{aligned}|P[A\cap B] - P[A]P[B]| \le\ & |P[A\cap B] - P[A_q \cap B_q]| \\ & + |P[A_q \cap B_q] - P[A_q]P[B_q]| \\ & + P[A_q]\,|P[B_q]-P[B]| + P[B]\,|P[A_q]-P[A]|.\end{aligned} \tag{3.10}$$

As in the proof of Lemma 3.2.1(iii), we have

$$(3.11) \qquad 0 \le P[A_q \cap B_q] - P[A \cap B] \le \gamma^{(1)}_{a,u_{nh}}$$

where $\gamma^{(1)}_{a,u_{nh}} = \frac{nh}{q_a(u_{nh})} P[\zeta(0) < u_{nh},\ \xi(q_a(u_{nh})) < u_{nh},\ M(q_a(u_{nh})) > u_{nh}]$
$+ 2nP[M(q_a(u_{nh})) > u_{nh}]$

and
$$\limsup_{n\to\infty} \frac{\gamma^{(1)}_{a,u_{nh}}}{n\psi(u_{nh})} \to 0 \quad \text{as} \quad a \to 0 .$$

(Note that (3.11) holds for all i_r's and j_s's satisfying (3.8), while $\gamma^{(1)}_{a,u_{nh}}$ does not depend on the i_r's and j_s's). Similarly we have

$$0 \le P[A_q] - P[A] \le \gamma^{(2)}_{a,u_{nh}}$$
$$0 \le P[B_q] - P[B] \le \gamma^{(3)}_{a,u_{nh}}$$

where
$$\limsup_{n\to\infty} \frac{\gamma^{(2)}_{a,u_{nh}}}{n\psi(u_{nh})} \to 0 \quad \text{as} \quad a \to 0$$

and
$$\limsup_{n\to\infty} \frac{\gamma^{(3)}_{a,u_{nh}}}{n\psi(u_{nh})} \to 0 \quad \text{as} \quad a \to 0 .$$

Put $\gamma_{a,u_{nh}} = \gamma^{(1)}_{a,u_{nh}} + \gamma^{(2)}_{a,u_{nh}} + \gamma^{(3)}_{a,u_{nh}}$. From (3.10) we get

$$(3.12) \qquad |P[A \cap B] - P[A]P[B]| \le \gamma_{a,u_{nh}} + |P[A_q \cap B_q] - P[A_q]P[B_q]|$$

where

$$(3.13) \qquad \limsup_{n\to\infty} \frac{\gamma_{a,u_{nh}}}{n\psi(u_{nh})} \to 0 \quad \text{as} \quad a \to 0 .$$

Since the largest jq in any I_r is at most $i_p h$, and the smallest jq in any J_s is at least $(j_1-1)h$, their difference is at least $(\ell-1)h$. Also the largest jq does not exceed nh. Thus by (3.6) and (3.12)

$$(3.14) \qquad |P[A \cap B] - P[A]P[B]| \leq \gamma_{a,u_{nh}} + \alpha^{(a)}_{nh,(\ell-1)h}$$

in which the dependence of $\alpha_{T,\ell}$ on a is explicitly indicated. Note that the left hand side of (3.14) does not depend on a. Hence

$$|P[A \cap B] - P[A]P[B]| \leq \alpha^{*}_{n,\ell}$$

where
$$\alpha^{*}_{n,\ell} = \inf_{a>0}(\gamma_{a,u_{nh}} + \alpha^{(a)}_{nh,(\ell-1)h}) \quad .$$

which is precisely what we need, provided we can show that $\lim\limits_{n\to\infty} \alpha^{*}_{n,\theta n} = 0$, for any $\theta > 0$. But for any $a > 0$ we have

$$\alpha^{*}_{n,\theta n} \leq \gamma_{a,u_{nh}} + \alpha^{(a)}_{nh,(\theta n-1)h} \leq \gamma_{a,u_{nh}} + \alpha^{(a)}_{nh,\frac{1}{2}\theta nh}$$

for large enough n (we are using the fact that $\alpha^{(a)}_{T,\ell}$ decrease in ℓ). Hence for all $a > 0$ we have

$$\limsup_{n\to\infty} \alpha^{*}_{n,\theta n} \leq \limsup_{n\to\infty} \gamma_{a,u_{nh}} + \lim_{n\to\infty} \alpha^{(a)}_{nh,\frac{1}{2}\theta nh}$$
$$= \limsup_{n\to\infty} \gamma_{a,u_{nh}} \quad .$$

But, since $T\psi(u_T)$ is assumed to be bounded, (3.13) implies

$$\limsup_{n\to\infty} \gamma_{a,u_{nh}} \longrightarrow 0 \quad \text{as} \quad a \longrightarrow 0 \ .$$

Thus we have $\lim\limits_{n\to\infty} \alpha^{*}_{n,\theta n} = 0$. This completes the proof. ////

It is now a simple matter of technical calculation to go from the above result to the main result of this section.

THEOREM 3.2.3. *Assume that the conditions* R1 *and* R2 *hold for some given* $\psi(u)$ *and* $q_a(u)$. *Let* $(u_T;\ T>0)$ *be a family of constants, with* $u_T \to \infty$ *as* $T \to \infty$, *such that*

(a) *for each* $a > 0$, *the condition* $D_c(u_T)$ *holds with respect to* $q_T = q_a(u_T)$.

(b) $T\psi(u_T)$ *is bounded*

(c) $\psi(u_T) \sim \psi(u_{[T/h]h})$ *as* $T \to \infty$, *for some* $0 < h < h_0$, h_0 *as in condition* R1

Then for every positive integer k *we have*

$$P[M(T) \le u_T] - P^k[M(T/k) \le u_T] \to 0 \quad \text{as} \quad T \to \infty . \tag{3.15}$$

Proof. Let $h \in (0,h_0)$ be such that (c) holds. Clearly we have

$$\begin{aligned}
&|P[M(T) \le u_T] - P^k[M(T/k) \le u_T]| \\
&\quad \le |P[M(T) \le u_T] - P[M(n_T h) \le u_T]| \\
&\qquad + |P[M(n_T h) \le u_T] - P[M(n_T h) \le u_{n_T h}]| \\
&\qquad + |P[M(n_T h) \le u_{n_T h}] - P^k[M([n_T/k]h) \le u_{n_T h}]| \\
&\qquad + |P^k[M([n_T/k]h) \le u_{n_T h}] - P^k[M([n_T/k]h) \le u_T]| \\
&\qquad + |P^k[M([n_T/k]h) \le u_T] - P^k[M(T/k) \le u_T]| .
\end{aligned}$$

where we wrote $n_T = [T/h]$. (3.15) will follow if we can show that each one of those five terms goes to 0 as $T \to \infty$.

1. $|P[M(T) \le u_T] - P[M(n_T h) \le u_T]| \to 0$ as $T \to \infty$.

Since $n_T h \le T < (n_T+1)h$, we have

$$\{M(n_T h) \le u_T\} = \{M(n_T h) \le u_T \text{ and } \xi(t) > u_T \text{ for some } t \in [n_T h, T]\} \cup \{M(T) \le u_T\}$$

where the two sets on the right are disjoint. It follows that

$$\begin{aligned}0 &\le P[M(n_T h) \le u_T] - P[M(T) \le u_T] \\ &\le P[\xi(t) > u_T \text{ for some } t \in [n_T h, T]] \\ &\le P[M(h) > u_T] \,.\end{aligned}$$

Since $u_T \to \infty$ as $T \to \infty$ and since $M(h)$ is finite almost everywhere, we have $P[M(h) > u_T] \to 0$ as $T \to \infty$. This proves 1.

2. $|P[M(n_T h) \le u_T] - P[M(n_T h) \le u_{n_T h}]| \to 0$ as $T \to \infty$.

If $u_{n_T h} > u_T$, we have

$$\begin{aligned}0 &\le P[M(n_T h) \le u_{n_T h}] - P[M(n_T h) \le u_T] \\ &= P[u_T < M(n_T h) \le u_{n_T h}] \\ &\le n_T P[u_T < M(h) \le u_{n_T h}] \\ &= n_T (P[M(h) > u_T] - P[M(h) > u_{n_T h}]) \,.\end{aligned}$$

Similarly, if $u_{n_T h} < u_T$ then

$$\begin{aligned}0 &\le P[M(n_T h) \le u_T] - P[M(n_T h) \le u_{n_T h}] \\ &= P[u_{n_T h} < M(n_T h) \le u_T] \\ &\le n_T P[u_{n_T h} < M(h) \le u_T] \\ &= n_T (P[M(h) > u_{n_T h}] - P[M(h) > u_T]) \,.\end{aligned}$$

Hence, using condition R1, we get

$$\begin{aligned}&|P[M(n_T h) \le u_T] - P[M(n_T h) \le u_{n_T h}]| \\ &\le n_T |P[M(h) > u_{n_T h}] - P[M(h) > u_T]| \\ &= n_T |h\psi(u_{n_T h})(1+o(1)) - h\psi(u_T)(1+o(1))| \\ &= |n_T h\psi(u_{n_T h})(1+o(1)) - (\frac{n_T h}{u_T}) u_T \psi(u_T)(1+o(1))| \,.\end{aligned}$$

Using (b) and (c) it is easily seen that this last expression goes to 0 as $T \to \infty$. This proves 2.

3. $|P[M(n_T h) \le u_{n_T h}] - P^k[M([n_T/k]h) \le u_{n_T h}]| \to 0$ as $T \to \infty$.

This is the content of Theorem 3.2.2.

4. $|P^k[M([n_T/k]h) \le u_{n_T h}] - P^k[M([n_T/k]h) \le u_T]| \to 0$ as $T \to \infty$.

If $u_{n_T h} > u_T$, we have, as in the proof of 2,

$$\begin{aligned} 0 &\le P[M([n_T/k]h) \le u_{n_T h}] - P[M([n_T/k]h) \le u_T] \\ &\le [n_T/k](P[M(h) > u_T] - P[M(h) > u_{n_T h}]) . \end{aligned}$$

From the fact that

$$0 \le x \le y \le 1 \quad \text{implies} \quad 0 \le y^k - x^k \le k(y-x) \tag{3.16}$$

we get

$$\begin{aligned} 0 &\le P^k[M([n_T/k]h) \le u_{n_T h}] - P^k[M([n_T/k]h) \le u_T] \\ &\le k[n_T/k](P[M(h) > u_T] - P[M(h) > u_{n_T h}]) . \end{aligned}$$

Similarly, if $u_{n_T h} < u_T$ then

$$\begin{aligned} 0 &\le P^k[M([n_T/k]h) \le u_T] - P^k[M([n_T/k]h) \le u_{n_T h}] \\ &\le k[n_T/k](P[M(h) > u_{n_T h}] - P[M(h) > u_T]) . \end{aligned}$$

Hence we have

$$\begin{aligned} &|P^k[M([n_T/k]h) \le u_{n_T h}] - P^k[M([n_T/k]h) \le u_T]| \\ &\quad \le k[n_T/k]\,|P[M(h) > M_T] - P[M(h) > u_{n_T h}]| \end{aligned}$$

which goes to 0 as $T \to \infty$, just as in the proof of 2. This proves 4.

5. $|P^k[M([n_T/k]h) \le u_T] - P^k[M(T/k) \le u_T]| \to 0$ as $T \to \infty$.

Since $0 \le T/k - [n_T/k]h \le h$, we have, as in 1,

$$0 \le P[M([n_T/k]h) \le u_T] - P[M(T/k) \le u_T] \le P[M(h) > u_T] .$$

Using (3.16) we get

$$0 \le P^k[M([n_T/k]h) \le u_T] - P^k[M(T/k) \le u_T] \le kP[M(h) > u_T]$$

which goes to 0 as $T \to \infty$, as in the proof of 1. This proves 5. Theorem 3.2.3 is proved. ////

3.3 A weak convergence result for the maximum

In order to obtain a continuous parameter version of Theorem 2.3.2, we will need a continuous parameter version of the condition $D'(u_n)$. This condition, we shall call it condition $D_c'(u_T)$, will be phrased in terms of the so-called ε-upcrossings of a level by the process. This concept was originally introduced by Pickands [1969a,1969b] to deal with processes whose sample paths were so irregular that the ordinary upcrossings could be infinite in number in a finite interval. Here we use this concept whether the number of ordinary upcrossings is infinite or not.

Let ε be positive. The stationary process $(\xi(t); t \in \mathbb{R})$ is said to have an ε-upcrossing of the level u at the point t_0 if it has an ordinary upcrossing of the level u at t_0 and if $\xi(t) \le u$ for all t in $(t_0-\varepsilon, t_0)$. (This is not exactly Pickands original definition but it leads to the same asymptotic results and it is easier to handle). Note that if there is an ε-upcrossing at t_0, then the interval $(t_0-\varepsilon, t_0)$

contains no ε-upcrossings. Thus the number of ε-upcrossings in a unit interval does not exceed $1/\varepsilon$. We will write $\mu_{\varepsilon,u}$ for the mean number of ε-upcrossings of the level u in the time interval $(0,1]$. Using stationarity one can check that $t\mu_{\varepsilon,u}$ is the mean number of ε-upcrossings of the level u in the time interval $(s,s+t]$, for any s.

Let $\psi(u)$ be a given function for which the condition R1 holds. We say that the condition $D_c'(u_T)$ holds for the family of constants $(u_T;\ T>0)$, with $u_T \to \infty$ as $T \to \infty$, if

$$\limsup_{T\to\infty} T|\mu_{\varepsilon T,u_T} - \psi(u_T)| \to 0 \quad \text{as} \quad \varepsilon \to 0 .$$

THEOREM 3.3.1. *Assume that the conditions* R1 *and* R2 *hold for some given* $\psi(u)$ *and* $q_a(u)$. *Let* $(u_T;\ T>0)$ *be a family of constants, with* $u_T \to \infty$ *as* $T \to \infty$, *such that for each* $a>0$ *the condition* $D_c(u_T)$ *holds with respect to* $q_T = q_a(u_T)$, *and such that the condition* $D_c'(u_T)$ *holds with respect to* $\psi(u)$. *Let* τ *be a positive constant. Then*

$$(3.17) \qquad T\psi(u_T) \to \tau \quad as \quad T \to \infty$$

implies

$$(3.18) \qquad P[M(T) \le u_T] \to e^{-\tau} \quad as \quad T \to \infty .$$

Conversely, with the additional assumption that for some $0<h<h_0$, h_0 *as in condition* R1, *we have* $\psi(u_T) \sim \psi(u_{[T/h]h})$ *as* $T\to\infty$, *then* (3.18) *implies* (3.17).

Proof. Let k be a positive integer. Since there is either 0 or 1 (T/k)-upcrossing of the level u_T in the time interval $(0,T/k]$, we have

$$(3.19)\quad (T/k)\mu_{(T/k),u_T} = \text{mean number of } (T/k)\text{-upcrossings in } (0,T/k]$$
$$= P[\text{exactly one } (T/k)\text{-upcrossing in } (0,T/k]]$$
$$\leq P[M(T/k) > u_T]\ .$$

To get the last inequality we used the fact that the probability of having a (T/k)-upcrossing of the level u_T at the point T/k is 0 (This follows from the fact that the marginal distribution function of the process is assumed to be continuous). On the other hand, using stationarity, we have

$$(3.20)\qquad P[M(T/k) > u_T] \leq ([T/kh]+1)P[M(h) > u_T]\ ,\qquad 0 < h < h_0\ .$$

Combining (3.19) and (3.20) and using

$$[T/kh]+1 \sim T/kh \quad \text{and} \quad P[M(h) > u_T] \sim h\psi(u_T)$$

we get

$$(T/k)\mu_{(T/k),u_T} \leq P[M(T/k) > u_T] \leq (T/k)\psi(u_T)(1+\theta_T)$$

or, equivalently,

$$(3.21)\quad 1-(T/k)\psi(u_T)(1+\theta_T) \leq P[M(T/k) \leq u_T]$$
$$\leq 1-(T/k)\psi(u_T) + (T/k)(\psi(u_T)-\mu_{(T/k),u_T})$$

where $\theta_T \to 0$ as $T \to \infty$.

Assume (3.17) holds. Taking $T \to \infty$ in (3.21) and using condition $D_c'(u_T)$ we get

$$(1-\tau/k) \leq \liminf_{T\to\infty} P[M(T/k) \leq u_T]$$
$$\leq \limsup_{T\to\infty} P[M(T/k) \leq u_T] \leq (1-\tau/k+o(1/k)) \quad \text{as} \quad k \to \infty\ .$$

Taking the k-th power and using Theorem 3.2.3, this becomes

$$(1-\tau/k)^k \le \liminf_{T\to\infty} P[M(T)\le u_T]$$
$$\le \limsup_{T\to\infty} P[M(T)\le u_T] \le (1-\tau/k+o(1/k))^k \quad \text{as} \quad k\to\infty .$$

Letting $k\to\infty$ we get (3.18).

Conversely assume that (3.18) holds and that for some $0<h<h_0$, h_0 as in condition R1, we have $\psi(u_T) \sim \psi(u_{[T/h]h})$ as $T\to\infty$. The inequalities in (3.21) can be written as

(3.22) $$1-P[M(T/k)\le u_T] \le (T/k)\psi(u_T)(1+\theta_T)$$

and

(3.23) $$(T/k)\psi(u_T) \le 1-P[M(T/k)\le u_T] + (T/k)(\mu_{(T/k),u_T}-\psi(u_T))$$

Note that when the condition $D_c'(u_T)$ holds, $T\psi(u_T)$ is bounded so that the conditions of Theorem 3.2.3 are satisfied. Let $T\to\infty$ in (3.22) and (3.23). Using Theorem 3.2.3 and condition $D_c'(u_T)$ we get

$$1-e^{-\tau/k} \le (1/k)\liminf_{T\to\infty} T\psi(u_T)$$
$$\le (1/k)\limsup_{T\to\infty} T\psi(u_T) \le 1-e^{-\tau/k}+o(1/k) .$$

Multiplying by k and letting $k\to\infty$ we obtain (3.17). ////

Following the lines of Section 2.3 we will now show that, under the conditions of Theorem 3.3.1, (3.17) implies

$$P[M(\theta T)\le u_T] \to e^{-\theta\tau} \quad \text{as} \quad T\to\infty, \quad 0<\theta\le 1 .$$

Suppose we are given a family of constants $(u_T;\ T>0)$. For $\theta>0$ we define $u_T(\theta) = u_{(T/\theta)}$.

LEMMA 3.3.2.

(a) *If* $T\psi(u_T) \to \tau$, *then* $T\psi(u_T(\theta)) \to \theta\tau$ *for all* $\theta > 0$.

(b) *If condition* $D_c(u_T)$ *holds (with respect to some* q_T) *then condition* $D_c(u_T(\theta))$ *holds (with respect to the same* q_T) *for* $0 < \theta \leq 1$.

(c) *If condition* $D_c'(u_T)$ *holds (with respect to some* $\psi(u)$) *then condition* $D_c'(u_T(\theta))$ *holds (with respect to the same* $\psi(u)$) *for all* $\theta > 0$.

Proof. Clearly if $T\psi(u_T) \to \tau$, then $(T/\theta)\psi(u_{(T/\theta)}) \to \tau$ and hence $T\psi(u_T(\theta)) \to \theta\tau$. Thus (a) is proved. Now suppose that the condition $D_c(u_T)$ holds with respect to some q_T. Take

$$s_1 < s_2 < \cdots < s_p < t_1 < t_2 < \cdots < t_{p'} \in \{kq_T;\ 0 \leq kq_T \leq T\}$$

with $t_1 - s_p \geq \gamma$. If $0 < \theta \leq 1$, then $T \leq T/\theta$. Hence, writing $\alpha'_{T,\gamma} = \alpha_{T/\theta,\gamma}$,

$$\left|F_{s_1\cdots s_p t_1\cdots t_{p'}}(u_{T/\theta}) - F_{s_1\cdots s_p}(u_{T/\theta})F_{t_1\cdots t_{p'}}(u_{T/\theta})\right| \leq \alpha_{T/\theta,\gamma} = \alpha'_{T,\gamma}$$

We have $\alpha_{T,\gamma_T} \to 0$ as $T \to \infty$, for some $\gamma_T = o(T)$. Take $\gamma'_T = \gamma_{T/\theta}$. Then $\gamma'_T = o(T)$ and

$$\alpha'_{T,\gamma'_T} = \alpha_{T/\theta,\gamma_{T/\theta}} = o(T) \quad \text{as} \quad T \to \infty .$$

Hence the condition $D_c(u_T(\theta))$ holds with respect to q_T. This proves (b). Finally assume condition $D_c'(u_T)$ holds with some given $\psi(u)$, i.e. we have

$$\text{(3.24)} \qquad \limsup_{T\to\infty} T\left|\mu_{\varepsilon T,u_T} - \psi(u_T)\right| \to 0 \quad \text{as} \quad \varepsilon \to 0 .$$

For any fixed $\theta > 0$, $\varepsilon \to 0$ is equivalent to $\varepsilon\theta \to 0$. Hence (3.24) can be written as

$$\text{(3.25)} \qquad \limsup_{T\to\infty} T\left|\mu_{\varepsilon\theta T,u_T} - \psi(u_T)\right| \to 0 \quad \text{as} \quad \varepsilon \to 0 .$$

Similarly, $T \to \infty$ is equivalent to $T/\theta \to \infty$. Hence (3.25) can be written as

$$\limsup_{T\to\infty} (T/\theta)\left|\mu_{\varepsilon T,u_{T/\theta}} - \psi(u_{T/\theta})\right| \to 0 \quad \text{as} \quad \varepsilon \to 0 .$$

Hence we have

$$\limsup_{T\to\infty} T\left|\mu_{\varepsilon T,u_T(\theta)} - \psi(u_T(\theta))\right| \to 0 \quad \text{as} \quad \varepsilon \to 0 ,$$

i.e. condition $D_c'(u_T(\theta))$ holds for $\psi(u)$. This completes the proof. ////

THEOREM 3.3.3. *Assume that the conditions* R1 *and* R2 *hold for some given* $\psi(u)$ *and* $q_a(u)$. *Let* $(u_T; T > 0)$ *be a family of constants, with* $u_T \to \infty$ *as* $T \to \infty$, *such that for each* $a > 0$ *the condition* $D_c(u_T)$ *holds with respect to* $q_T = q_a(u_T)$, *and such that the condition* $D_c'(u_T)$ *holds with respect to* $\psi(u)$. *Let* τ *be a positive constant. Then, for* $0 < \theta \leq 1$,

$$T\psi(u_T) \to \tau \quad \textit{as} \quad T \to \infty$$

implies

$$\text{(3.26)} \qquad P[M(\theta T) \leq u_T] \to e^{-\theta\tau} \quad \textit{as} \quad T \to \infty .$$

Proof. Fix $\theta \in (0,1]$. By the lemma we have

$$T\psi(u_T(\theta)) \to \theta\tau$$

and the conditions $D_c(u_T(\theta))$ and $D_c'(u_T(\theta))$ hold. Hence by Theorem 3.3.1

we get

$$P[M(T) \le u_T(\theta)] \to e^{-\theta\tau} \quad \text{as} \quad T \to \infty$$

from which (3.26) follows easily. ////

The Poisson point process limit theorem for the upcrossings of a stationary process will follow easily from the following result, as in Chapter 2.

THEOREM 3.3.4. *Assume that the conditions* R1 *and* R2 *hold for some given* $\psi(u)$ *and* $q_a(u)$. *Let* $(u_T; T > 0)$ *be a family of constants, with* $u_T \to \infty$ *as* $T \to \infty$, *such that for each* $a > 0$ *the condition* $D_c(u_T)$ *holds with respect to* $q_T = q_a(u_T)$, *and such that the condition* $D_c'(u_T)$ *holds with respect to* $\psi(u)$. *Let* τ *be a positive constant and let* $I_1, \ldots, I_k$ *be subintervals of* $(0,1]$, $I_j = (a_j, b_j]$ *with*

$$0 \le a_1 < b_1 < a_2 < b_2 < \cdots < a_k < b_k \le 1 .$$

Then

$$T\psi(u_T) \to \tau \quad \text{as} \quad T \to \infty$$

implies

$$P\left[\bigcap_{j=1}^{k} \{M(TI_j) \le u_T\}\right] \to \exp\left(-\tau \sum_{i=1}^{k} (b_i - a_i)\right) \quad \text{as} \quad T \to \infty ,$$

where $M(TI_j)$ *is the maximum of the process over the time set* $TI_j = (Ta_j, Tb_j]$.

Proof. For simplicity we write $q = q_a(u_T)$. For each $a > 0$ we have

$$\begin{aligned}
&\left|P\left[\bigcap_{j=1}^{k} \{M(TI_j) \le u_T\}\right] - \prod_{j=1}^{k} P[M(TI_j) \le u_T]\right| \\
&\le \left|P\left[\bigcap_{j=1}^{k} \{M(TI_j) \le u_T\}\right] - P\left[\bigcap_{j=1}^{k} \{\xi(iq) \le u_T;\ iq \in TI_j\}\right]\right| \\
&\quad + \left|P\left[\bigcap_{j=1}^{k} \{\xi(iq) \le u_T;\ iq \in TI_j\}\right] - \prod_{j=1}^{k} P[\xi(iq) \le u_T;\ iq \in TI_j]\right| \\
&\quad + \left|\prod_{j=1}^{k} P[\xi(iq) \le u_q;\ iq \in TI_j] - \prod_{j=1}^{k} P[M(TI_j) \le u_T]\right| .
\end{aligned}$$

Let $x(a,T)$, $y(a,T)$, $z(a,T)$ be the three terms on the right hand side of the inequality, in the order in which they appear. Then

$$(3.27)\qquad \limsup_{T\to\infty}\left|P\left[\bigcap_{j=1}^{k}\{M(TI_j)\le u_T\}\right]-\prod_{j=1}^{k}P[M(TI_j)\le u_T]\right|$$
$$\le \limsup_{T\to\infty} x(a,T)+\limsup_{T\to\infty} y(a,T)+\limsup_{T\to\infty} z(a,T)\ .$$

The left hand side of (3.27) does not depend on a. Hence if we can show that each term on the right hand side of (3.27) goes to 0 as $a\to 0$, it will follow that

$$(3.28)\qquad \left|P\left[\bigcap_{j=1}^{k}\{M(TI_j)\le u_T\}\right]-\prod_{j=1}^{k}P[M(TI_j)\le u_T]\right|\to 0 \quad \text{as}\quad T\to\infty\ .$$

By stationarity we have $M(TI_j)=M((b_j-a_j)T)$. Hence (3.28) combined with Theorem 3.3.3 will give us

$$P\left[\bigcap_{j=1}^{k}\{M(TI_j)\le u_T\}\right]\to \exp\left(-\tau\sum_{i=1}^{k}(b_i-a_i)\right)\quad \text{as}\quad T\to\infty\ .$$

Thus it remains only to show that each term on the right hand side of (3.27) goes to 0 as $a\to 0$.

1. $$\limsup_{T\to\infty} x(a,T)\to 0 \quad\text{as}\quad a\to 0\ .$$

Let us write
$$A=A(T)=\bigcap_{j=1}^{k}\{M(TI_j)\le u_T\}$$
$$B=B(a,T)=\bigcap_{j=1}^{k}\{\xi(iq)\le u_T;\ iq\in TI_j\}\ .$$

For each a we have $A\subset B$. The set $\bigcup_{j=1}^{k}TI_j$ is covered by no more than $T\theta/q$ subintervals of the form $[(i-1)q,iq]$ together with possibly $2k$ shorter intervals, where we wrote θ for $\sum_{j=1}^{k}(b_j-a_j)$. Using stationarity we get .

$$x(a,T) = |P[A]-P[B]| = P[B-A]$$
$$\leq \frac{T\epsilon}{q} P[\xi(0) < u_T,\ \xi(q) < u_T,\ M(q) > u_T] + 2kP[M(q) > u_T] \ .$$

From Lemma 3.2.1 we have

$$P[M(q) > u_T] \to 0 \quad \text{as} \quad T \to \infty \ .$$

Hence from the above inequality and from the fact that $T\psi(u_T) \to \tau$ as $T \to \infty$, we obtain

$$\limsup_{T\to\infty} x(a,T) \leq \theta\tau \limsup_{T\to\infty} \frac{P[\xi(0) < u_T,\ \xi(q) > u_T,\ M(q) > u_T]}{q\psi(u_T)} \ .$$

Since condition R2 holds we get

(3.29) $$\limsup_{T\to\infty} x(a,T) \to 0 \quad \text{as} \quad a \to 0 \ .$$

2. $$\limsup_{T\to\infty} y(a,T) \to 0 \quad \text{as} \quad a \to 0 \ .$$

Let us write $D_j = D_j(a,T) = \{\xi(iq) \leq u_T;\ iq \in TI_j\}$

Using ideas of Lemmas 2.2.1, 2.2.2 and 2.2.3 we can show that

$$\limsup_{T\to\infty} y(a,T) = \limsup_{T\to\infty} \left| P[\bigcap_{j=1}^{k} D_j] - \prod_{j=1}^{k} P[D_j] \right| = 0 \quad \text{for each } a \ ,$$

and hence $\limsup_{T\to\infty} y(a,T) \to 0$ as $a \to 0$, trivially.

3. $$\limsup_{T\to\infty} z(a,T) \to 0 \quad \text{as} \quad a \to 0 \ .$$

Let us write
$$K_j = K_j(a,T) = \{\xi(iq) \leq u_T;\ iq \in TI_j\}$$
$$F_j = F_j(T) = \{M(TI_j) \leq u_T\} \ .$$

As in the proof of (3.29) we have

(3.30) $$\limsup_{T\to\infty} |P[K_j]-P[F_j]| \to 0 \quad \text{as} \quad a\to 0, \quad \text{for } j=1,2,\ldots,k .$$

Clearly

$$\begin{aligned} &|P[K_1]P[K_2]-P[F_1]P[F_2]| \\ &\quad \le P[K_2]|P[K_1]-P[F_1]| + P[F_1]|P[K_2]-P[F_2]| . \end{aligned}$$

Hence from (3.30) we get

$$\limsup_{T\to\infty} |P[K_1]P[K_2]-P[F_1]P[F_2]| \to 0 \quad \text{as} \quad a \to 0 .$$

Proceeding in this manner we obtain

$$\limsup_{T\to\infty} \left|\prod_{j=1}^{k} P[K_j] - \prod_{j=1}^{k} P[F_j]\right| \to 0 \quad \text{as} \quad a \to 0 ,$$

i.e. $$\limsup_{T\to\infty} z(a,T) \to 0 \quad \text{as} \quad a \to 0 .$$

This completes the proof of Theorem 3.3.4. ////

3.4 The Poisson point process limit theorem for the upcrossings

As in the previous sections, $(\xi(t);\ t\in\mathbb{R})$ will be a stationary stochastic process defined on some probability space $(\Omega,\mathcal{F},P)$ and having, with probability one, continuous sample paths. $(u_T;\ T>0)$ will be some given family of constants, with $u_T \to \infty$ as $T\to\infty$. As in Chapters 1 and 2, $\mathcal{B}$ will be the σ-field of Borel subsets of $(0,1]$ and $\bar{N}$ will be the set of non-negative integers to which we add the point $+\infty$. For each $T>0$, consider

$$\begin{aligned} Z_T\colon\ &\Omega\times\mathcal{B} \to \bar{N} \\ &(\omega,B) \mapsto Z_T(\omega,B) \end{aligned}$$

where $Z_T(\omega,B)$ is the number of upcrossings of the level u_T by the sample path $\xi(t,\omega)$ within the time set TB. Our first concern is to show that Z_T is a point process, as defined in Section 1.2.

LEMMA 3.4.1. *For each* $B \in \mathcal{B}$, *the function*

$$\begin{aligned} Z_T(B): \Omega &\longrightarrow \bar{N} \\ \omega &\longmapsto Z_T(\omega,B) \end{aligned}$$

is measurable.

Proof. Let $\mathcal{E}$ be the family of all sets B in $\mathcal{B}$ for which $Z_T(B)$ is measurable. That $\mathcal{E}$ contains the sets of the form $(a,b]$ (with $0 \leq a < b \leq 1$) was proved (using a different notation) in Section 1.1. It follows easily that $\mathcal{E}$ contains the field of all finite disjoint unions of such half open intervals. The σ-field generated by this field is $\mathcal{B}$. Hence, using the fact that the σ-field generated by a field coincides with the monotone class generated by that field, we get $\mathcal{E} = \mathcal{B}$ by showing that $\mathcal{E}$ is a monotone class. This is easily checked. ////

Let μ_u be the mean number of upcrossings of the level u in the time interval $(0,1]$. When needed, we will assume that

$$\mu_u < \infty \quad \text{for all } u . \tag{3.31}$$

Under this additional condition, Lemma 3.4.1 becomes

LEMMA 3.4.2. *If* (3.31) *holds, then for each* $B \in \mathcal{B}$ *the function*

$$\begin{aligned} Z_T(B): \Omega &\longrightarrow \bar{N} \\ \omega &\longmapsto Z_T(\omega,B) \end{aligned}$$

is a random variable.

Proof. We know already that $Z_T(B)$ is measurable. Using our additional hypothesis we get

$$E[Z_T(B)] \le E[Z_T((0,1])] = T\mu_{u_T} < \infty .$$

Hence $Z_T(B)$ is finite almost everywhere. Thus it is a random variable. ////

We now fix ω and look at Z_T as a function of B.

LEMMA 3.4.3. *If* (3.31) *holds, then for almost all* ω *in* Ω, *the function*

$$Z_T(\omega):\ \mathcal{B} \to \bar{N}$$
$$B \mapsto Z_T(\omega,B)$$

is a finite positive measure.

Proof. Clearly, for all $\omega \in \Omega$, $Z_T(\omega)$ is a positive measure on $\mathcal{B}$ since for disjoint B_i's we have $Z_T(\omega,\bigcup_{i=1}^{\infty} B_i) = \sum_{i=1}^{\infty} Z_T(\omega,B_i)$. As in Lemma 3.4.2, we have $Z_T(\omega,(0,1]) < \infty$ for almost all ω. For those ω, $Z_T(\omega)$ is a finite positive measure on $\mathcal{B}$. ////

The last two lemmas combined give

THEOREM 3.4.4. *If* (3.31) *holds, then* Z_T *is a point process.* ////

Using Kallenberg's result, the main theorem of this chapter follows easily from the results of the previous sections.

THEOREM 3.4.5. *Assume that the conditions* R1 *and* R2 *hold for the function* $\psi(u) = \mu_u$ *and some given* $q_a(u)$. *Let* $(u_T;\ T>0)$ *be a family of constants, with* $u_T \to \infty$ *as* $T \to \infty$, *such that for each* $a > 0$ *the condition* $D_c(u_T)$ *holds with respect to* $q_T = q_a(u_T)$, *and such that the* *condition* $D_c'(u_T)$ *holds with respect to* $\psi(u) = \mu_u$. *If for some positive constant* τ *we have*

(3.32)
$$T\mu_{u_T} \to \tau \quad as \quad T \to \infty$$

then
$$Z_T \xrightarrow{w} Z \quad as \quad T \to \infty$$

where Z *is a Poisson point process with intensity* τ.

Proof. By Kallenberg's result (Theorem 1.2.1) it suffices to show that

(a) $E[Z_T((a,b])] \to \tau(b-a)$ for all $0 \le a < b \le 1$

(b) $P[Z_T(B) = 0] \to e^{-\tau\Lambda(B)}$ for all B of the form $\bigcup_{i=1}^{k}(a_i,b_i]$ with $0 \le a_1 < b_1 < a_2 < b_2 < \cdots < a_k < b_k \le 1$, where $\Lambda(B)$ is the Lebesgue measure of B.

Now (a) follows at once since using (3.32) we get

$$E[Z_T((a,b])] = T(b-a)\mu_{u_T} \to \tau(b-a) .$$

To obtain (b) note that

$$\begin{aligned} 0 &\le P[Z_T(B) = 0] - P[M(TB) \le u_T] \\ &= P[Z_T(B) = 0,\ M(TB) > u_T] \\ &\le \sum_{i=1}^{k} P[\xi(Ta_i) > u_T] \\ &= kP[\xi(0) > u_T] \end{aligned}$$

since if the maximum in $TB = \bigcup_{i=1}^{k}(Ta_i,Tb_i]$ exceeds u_T, but there are no upcrossings of u_T in these intervals, then $\xi(t)$ must exceed u_T at the initial point of at least one such interval. Thus we have

$$0 \le P[Z_T(B) = 0] - P[M(TB) \le u_T] \to 0 \quad \text{as} \quad T \to \infty .$$

By Theorem 3.3.4 we have

$$P[M(TB) \le u_T] = P[\bigcap_{j=1}^{k} \{M((Ta_j, Tb_j]) \le u_T\}] \to \exp\Big(-\tau \sum_{i=1}^{k} (b_i - a_i)\Big)$$
as $T \to \infty$.

Thus $$P[Z_T(B) = 0] \to e^{-\tau\Lambda(B)} \quad \text{as } T \to \infty . \qquad ////$$

Certain stationary processes have sample paths so badly behaved that $\mu_u = +\infty$ for all u. For these processes it is clear that Theorem 3.4.5 cannot apply. It may however be possible to show that the point process of ε-upcrossings converges to some Poisson point process.

For $\omega \in \Omega$ and $B \in \mathcal{B}$, let $Z_T^{(\varepsilon)}(\omega,B)$ be the number of ε-upcrossings of the level u_T within the time set TB. As in Section 3.3, $\mu_{\varepsilon,u}$ will denote the mean number of ε-upcrossings of the level u within the time interval $(0,1]$. As we did for ordinary upcrossings, we can easily obtain the following:

THEOREM 3.4.6. $Z_T^{(\varepsilon)}$ *is a point process.* ////

Note that in the present situation we always have $\mu_{\varepsilon,u} < \infty$.

In Theorem 3.4.5, the ordinary upcrossings case, we had to assume that the condition R1 hold with $\psi(u) = \mu_u$. In the ε-upcrossings case, it is enough to assume that it holds for some $\psi(u)$.

LEMMA 3.4.7. *If the condition* R1 *holds for some function* $\psi(u)$, *then*

(3.33) $$\frac{\mu_{\varepsilon,u}}{\psi(u)} \to 1 \quad \text{as} \quad u \to \infty, \quad \text{for all small enough } \varepsilon.$$

Proof. By hypothesis we have, for some $h_0 > 0$,

(3.34) $$\frac{P[M(h) > u]}{h\psi(u)} \to 1 \quad \text{as} \quad u \to \infty, \quad 0 < h < h_0 .$$

Let us write $Z^*(\varepsilon,u,T)$ for the number of ε-upcrossings of the level u in the time interval $(0,T]$. Clearly $Z^*(\varepsilon,u,\varepsilon)$ is either zero or one. Hence

$$\varepsilon\mu_{\varepsilon,u} = E[Z^*(\varepsilon,u,\varepsilon)] = P[Z^*(\varepsilon,u,\varepsilon)=1] \le P[M(\varepsilon)>u]$$

so that

$$\mu_{\varepsilon,u} \le \frac{P[M(\varepsilon)>u]}{\varepsilon} .$$

Hence, for $0<\varepsilon<h_0$ we get, using (3.34),

$$\limsup_{u\to\infty} \frac{\mu_{\varepsilon,u}}{\psi(u)} \le 1 . \tag{3.35}$$

Clearly we have

$$\begin{aligned} P[M(2\varepsilon)>u] &\le P[M(\varepsilon)>u] + P[Z^*(\varepsilon,u,2\varepsilon)-Z^*(\varepsilon,u,\varepsilon)=1] \\ &= P[M(\varepsilon)>u] + \varepsilon\mu_{\varepsilon,u} . \end{aligned}$$

Hence we get

$$2\,\frac{P[M(2\varepsilon)>u]}{2\varepsilon\psi(u)} - \frac{P[M(\varepsilon)>u]}{\varepsilon\psi(u)} \le \frac{\mu_{\varepsilon,u}}{\psi(u)} .$$

Letting $u\to\infty$ and using (3.34), we get, for $0<\varepsilon<h_0/2$,

$$1 \le \liminf_{u\to\infty} \frac{\mu_{\varepsilon,u}}{\psi(u)} .$$

Together with (3.35) this gives us (3.33). ////

THEOREM 3.4.8. *Assume that the conditions* R1 *and* R2 *hold for some given* $\psi(u)$ *and* $q_a(u)$. *Let* $(u_T;\ T>0)$ *be a given family of constants, with* $u_T\to\infty$ *as* $T\to\infty$, *such that for each* $a>0$ *the condition* $D_c(u_T)$ *holds with respect to* $q_T=q_a(u_T)$; *and such that the condition* $D_c'(u_T)$ *holds with respect to* $\psi(u)$. *If for some positive constant* τ *we have*

$$T\psi(u_T) \to \tau \quad \text{as} \quad T\to\infty \tag{3.36}$$

then for all sufficiently small ε *we have*

(3.37) $$Z_T^{(\varepsilon)} \xrightarrow{w} Z \quad as \quad T \to \infty,$$

where Z *is a Poisson point process with intensity* τ.

Proof. By Lemma 3.4.7 we have

$$\frac{\mu_{\varepsilon,u}}{\psi(u)} \to 1 \quad \text{as} \quad u \to \infty, \quad \text{for all small enough} \quad \varepsilon.$$

It follows easily that the conditions R1, R2 and $D_c'(u_T)$ hold for $\mu_{\varepsilon,u}$ as well as for $\psi(u)$ and that (3.36) holds with μ_{ε,u_T} instead of $\psi(u_T)$, for all small enough ε. (3.37) is then easily obtained by repeating verbatim the proof of Theorem 3.4.5. ////

3.5 Connected results

As in the discrete case, we can obtain from the theorems of the last three sections some interesting connected results.

Gnedenko's theorem

In Chapter 2, Gnedenko's theorem on the possible limit distributions of the normalized maximum of a sequence of independent and identically distributed random variables was extended to the case of a stationary sequence. It is reasonable to believe that such a result should hold for the maximum of a stationary process. It does indeed.

THEOREM 3.5.1. *Assume that the conditions* R1 *and* R2 *hold for some given* $\psi(u)$ *and* $q_a(u)$. *Suppose that for some families of constants* $a_T > 0$ *and* b_T *we have*

$$(3.38) \qquad P[a_T(M(T)-b_T) \leq x] \xrightarrow{w} G(x) \quad \text{as} \quad T \to \infty$$

for some nondegenerate distribution function G. *Suppose that for each* x, *the family* $u_T = x/a_T + b_T$ *is such that* $T\mu(u_T)$ *is bounded and for each* $a > 0$ *the condition* $D_c(u_T)$ *holds for* $q_T = q_a(u_T)$. *Then* G *is one of the three extreme value types listed in Theorem 2.5.1.*

Proof. Take $0 < h < h_0$, h_0 as in condition R1. From (3.38) we have

$$P[a_{nh}(M(nh)-b_{nh}) \leq x] \xrightarrow{w} G(x) \quad \text{as} \quad n \to \infty .$$

This can be written as

$$P[\alpha_n(M_n-\beta_n) \leq x] \xrightarrow{w} G(x) \text{ , as} \quad n \to \infty$$

where we write

$$\alpha_n = a_{nh} , \qquad \beta_n = b_{nh}$$
$$M_n = \max\{\zeta_1, \zeta_2, \ldots, \zeta_n\}$$
$$\zeta_i = \max\{\xi(t);\ (i-1)h \leq t \leq ih\} .$$

The sequence $(\zeta_i;\ i = 1,2,\ldots)$ is stationary and, as in the proof of Theorem 3.2.2, for each x the condition $D(\nu_n)$ holds for the sequence $\nu_n = x/\alpha_n + \beta_n$. Thus by Theorem 2.5.4, G is one of the three extreme value types listed in Theorem 2.5.1. ////

Associated sequence of independent random variables

Theorem 3.3.1, on the convergence of $P[M(T) \leq u_T]$, may be related to the corresponding result for sequences of independent and identically distributed random variables in the following way.

THEOREM 3.5.2. *Assume that the conditions* R1 *and* R2 *hold for some given* $\psi(u)$ *and* $q_a(u)$. *Let* $(u_T;\ T>0)$ *be a family of constants, with* $u_T \to \infty$ *as* $T \to \infty$, *such that for each* $a > 0$ *the condition* $D_c(u_T)$ *holds with respect to* $q_T = q_a(u_T)$, *and such that the condition* $D_c'(u_T)$ *holds with respect to* $\psi(u)$. *Assume moreover that for some fixed* h, $0 < h < h_0$, h_0 *as in condition* R1, *we have* $\psi(u_T) \sim \psi(u_{[T/h]h})$ *as* $T \to \infty$. *Then, for* $0 < \rho \leq 1$,

$$P[M(T) \leq u_T] \to \rho \quad as \quad T \to \infty$$

if and only if

$$P[\hat{M}_n \leq u_{nh}] \to \rho \quad as \quad T \to \infty,$$

where $\hat{M}_n = \max\{\zeta_1, \zeta_2, \ldots, \zeta_n\}$ *for some sequence* $(\zeta_i;\ i=1,2,\ldots)$ *of independent and identically distributed random variables whose marginal distribution function* $\hat{F}$ *satisfies*

$$1 - \hat{F}(u) \sim h\psi(u) \quad as \quad u \to \infty .$$

Proof. This is easily obtained using Theorem 2.3.1 and Theorem 3.3.1. ////

The k-th largest local maxima

Suppose that with probability one the sample paths of our stationary process $(\xi(t);\ t \in \mathbb{R})$ are continuously differentiable. Let $(u_T;\ T>0)$ be a given family of constants. Let $\tilde{Z}_T$ be the number of local maxima within the interval $(0,T)$, for which the process value exceeds the level u_T. For the rest of this section let us write Z_T for $Z_T((0,1])$, the number of upcrossings of the level u_T within the interval $(0,T]$. Clearly there is at least one local maximum between any two upcrossings. Hence

(3.39) $$\tilde{Z}_T \geq Z_T - 1 \ .$$

If the sample paths are not too irregular and if T is large, so that u_T is large, it is reasonable to hope that $\tilde{Z}_T$ and Z_T will be approximately equal. Let us write $\tilde{\mu}_u$ for the mean number of local maximum above the level u_T within the interval $(0,1)$. By stationarity we have $E[\tilde{Z}_T] = T\tilde{\mu}_{u_T}$.

LEMMA 3.5.3. *Suppose* $T\mu_{u_T} \to \tau > 0$, *as* $T \to \infty$, *and suppose that* $\tilde{\mu}_{u_T} \sim \mu_{u_T}$ *as* $T \to \infty$. *Then* $E[|\tilde{Z}_T - Z_T|] \to 0$. *Thus for every non-negative integer* k *we have*

$$P[\tilde{Z}_T = k] - P[Z_T = k] \to 0 \quad \textit{as} \quad T \to \infty \ .$$

Proof. From (3.39) we get

$$E[|\tilde{Z}_T - Z_T|] = E[\tilde{Z}_T - Z_T] + 2P[\tilde{Z}_T = Z_T - 1] \ .$$

If $\tilde{Z}_T = Z_T - 1$, then $\xi(T) > u_T$. Hence

$$E[|\tilde{Z}_T - Z_T|] \leq T\tilde{\mu}_{u_T} - T\mu_{u_T} + 2P[\xi(T) > u_T]$$

from which we get $E[|\tilde{Z}_T - Z_T|] \to 0$ as $T \to \infty$. This in turn implies that $(\tilde{Z}_T - Z_T) \to 0$ in probability, giving $P[\tilde{Z}_T \neq Z_T] \to 0$, and hence $(P[\tilde{Z}_T = k] - P[Z_T = k]) \to 0$. ////

Now let us write $M^{(k)}(T)$ for the k-th largest local maximum in $(0,T)$. Clearly

(3.40) $$\{M^{(k)}(T) \leq u_T\} = \{\tilde{Z}_T < k\} \ .$$

THEOREM 3.5.4. *Assume that the conditions* R1 *and* R2 *hold for the function* $\psi(u) = \mu_u$ *and some given* $q_a(u)$. *Let* $(u_T; T > 0)$ *be a family of constants, with* $u_T \to \infty$ *as* $T \to \infty$, *such that for each* $a > 0$ *the condition* $D_c(u_T)$ *holds with respect to* $q_T = q_a(u_T)$, *and such that the condition* $D'_c(u_T)$ *holds with respect to* $\psi(u) = \mu_u$. *Suppose moreover that* $\tilde{\mu}_{u_T} \sim \mu_{u_T}$ *as* $T \to \infty$. *If for some* $\tau > 0$

$$T\mu_{u_T} \to \tau \quad as \quad T \to \infty$$

then

$$P[M^{(k)}(T) \le u_T] \to e^{-\tau} \sum_{s=0}^{k-1} \tau^s/s! \ . \tag{3.41}$$

Proof. By (3.40) we have

$$P[M^{(k)}(T) \le u_T] = P[\tilde{Z}_T < k] = \sum_{s=0}^{k-1} P[\tilde{Z}_T = s] \ . \tag{3.42}$$

By Lemma 3.5.3 we have $(P[\tilde{Z}_T = s] - P[Z_T = s]) \to 0$ and by Theorem 3.4.5 we have $P[Z_T = s] \to e^{-\tau}\tau^s/s!$ Combining these two results with (3.42) we get (3.41). ////

We conclude this short series of applications with a continuous parameter version of Theorem 2.5.9.

THEOREM 3.5.5. *Assume that the conditions* R1 *and* R2 *hold for the function* $\psi(u) = \mu_u$ *and some given* $q_a(u)$. *Suppose that for some families of constants* $a_T > 0$ *and* b_T *we have*

$$P[a_T(M(T) - b_T) \le x] \to G(x) \quad as \quad T \to \infty$$

for some non-degenerate distribution function G. *For each* $x > 0$, *suppose that for* $u_T = x/a_T + b_T$ *the condition* $D_c(u_T)$ *holds with respect* *to* $q_T = q_a(u_T)$, *for each* $a > 0$, *and the condition* $D_c'(u_T)$ *holds with respect to* $\psi(u) = \mu_u$. *Suppose moreover that for some* h, *with* $0 < h < h_0$, h_0 *as in condition* R1, *we have* $\psi(u_T) \sim \psi(u_{[T/h]h})$ *as* $T \to \infty$, *and that* $\tilde{\mu}_{u_T} \sim \mu_{u_T}$ *as* $T \to \infty$. *Then any positive integer* k *we have*

(3.43) $$P[a_T(M^{(k)}(T) - b_T) \le x] \xrightarrow{\cdot} G_k(x) \quad \text{as} \quad T \to \infty$$

where
$$G_k(x) = \begin{cases} G(x) \sum_{s=0}^{k-1} (-\log G(x))^s/s! & \text{if} \quad G(x) > 0 \\ 0 & \text{if} \quad G(x) = 0 . \end{cases}$$

Proof. First observe that the conditions of Theorem 3.5.2 are satisfied (if condition $D_c'(u_T)$ holds with respect to $\psi(u_T)$, then $T\psi(u_T)$ is bounded). Hence G is one of the three extreme value types listed in Theorem 2.5.1. Thus G and G_k are everywhere continuous so that $\xrightarrow{\cdot}$ means convergence at every point x. Let x be such that $G(x) > 0$. Put $G(x) = e^{-\tau}$. By Theorem 3.3.1 we have $T\psi(u_T) \to \tau$, where $u_T = x/a_T + b_T$. Hence by Theorem 3.5.4 we have

$$P[M^{(k)}(T) \le u_T] \to e^{-\tau} \sum_{s=0}^{k-1} \tau^s/s!$$

or, equivalently

$$P[a_T(M^{(k)}(T) - b_T) \le x] \to G(x) \sum_{s=0}^{k-1} (-\log G(x))^s/s! .$$

If x is such that $G(x) = 0$, then for every y with $G(y) > 0$ we have, since for such y we have $x < y$,

$$P[a_T(M^{(k)}(T) - b_T) \le x] \le P[a_T(M^{(k)}(T) - b_T) \le y] .$$

Letting $T \to \infty$ we get

$$(3.44) \qquad \limsup_{T\to\infty} P[a_T(M^{(k)}(T)-b_T) \le x] \le G(y) \sum_{s=0}^{k-1} (-\log G(y))^s/s!\ .$$

As y decreases to $y_0 = \inf\{y: G(y) > 0\}$, $G(y)$ decreases to 0. Hence the right hand side of (3.44) goes to 0. Thus

$$P[a_T(M^{(k)}(T)-b_T) \le x] \to 0 \quad \text{as} \quad T \to \infty$$

whenever $G(x) = 0$. Therefore (3.43) holds. ////

CHAPTER 4

STATIONARY GAUSSIAN PROCESSES

4.1 Introduction

The results of Chapter 3 were obtained for stationary processes satisfying certain conditions. These conditions were written, more or less directly, in terms of the distribution of the process, i.e. in terms of the family of finite dimensional distribution functions of the process. We now turn our attention to the case where the process is Gaussian. It turns out that the distribution of a standard stationary Gaussian process is completely characterized by the covariance function of the process. Hence a condition on the distribution of such a process can, at least in principle, be written as a condition on the covariance function, and vice versa.

Berman [1971a] obtained Theorem 1.1.1 for standard stationary Gaussian processes, having continuous sample paths with probability one, and whose covariance function $r(t)$ satisfies

$$(4.1) \qquad r''(t) \text{ exists,}$$

$$(4.2) \qquad r(t)\log t \longrightarrow 0 \text{ as } t \longrightarrow \infty .$$

The main goal of this chapter is to show that under Berman's hypothesis we have

(a) the condition R1 holds with $\psi(u) = \mu_u$

(b) the condition R2 holds with $\psi(u) = \mu_u$ and $q_a(u) = \frac{a}{u}$

(c) for each $a > 0$, the condition $D_c(u_T)$ holds with respect to the family $q_T = q_a(u_T) = a/u_T$, whenever u_T is such that $T\mu_{u_T} \longrightarrow \tau$ for some $\tau > 0$.

(d) the condition $D_c'(u_T)$ holds with respect to $\psi(u) = \mu_u$ whenever u_T is such that $T\mu_{u_T} \to \tau$ for some $\tau > 0$.

Thus the results of Chapter 3 hold for a standard stationary Gaussian process, having continuous sample paths with probability one, and whose covariance function satisfies Berman conditions (4.1) and (4.2). In particular, Theorem 1.1.1 holds under Berman conditions.

This approach does not reduce the amount of work required, but does bring the Gaussian case within the general framework. We will see that Berman conditions are almost necessary for Theorem 1.1.1 to hold. This shows that the conditions of Chapter 3 (conditions R1, R2, $D_c(u_T)$, $D_c'(u_T)$) are not too restrictive.

4.2 The covariance function and the spectral distribution function

Before we attack the core of this chapter, we have to explain what it means to say that the distribution of a standard stationary Gaussian process is completely characterized by the covariance function of the process.

Let $(\xi(t);\ t \in \mathbb{R})$ be a standard stationary Gaussian process (SSGP) defined on some probability space (Ω, F, P). As far as probability theory is concerned, the value taken by $\xi(t)$ at a given point $\omega \in \Omega$ is not really important. What really matters is the family of finite dimensional distribution functions (fddf's) of the process, i.e. the family

$$\{F_{t_1,t_2,\ldots,t_k};\ (t_1,t_2,\ldots,t_k) \in \mathbb{R}^k;\ k = 1,2,\ldots\}$$

where $F_{t_1,t_2,\ldots,t_k}$ is the joint distribution function of $(\xi(t_1),\ldots,\xi(t_k))$, i.e.

$$F_{t_1,\ldots,t_k}(x_1,\ldots,x_k) = P[\xi(t_1) \le x_1,\ldots,\xi(t_k) \le x_k] \,.$$

It is important to characterize those families of fddf's which are the family of fddf's of some SSGP.

A well known result of elementary probability theory says that a function F on $\mathbb{R}$ satisfies the conditions

- F is non-decreasing
- $\lim_{x\to-\infty} F(x) = 0, \quad \lim_{x\to+\infty} F(x) = 1$
- F is right continuous

if and only if there exists a random variable X, defined on some probability space (Ω,F,P), such that

$$F(x) = P[X \le x] \quad \text{for all} \quad x \in \mathbb{R} \,.$$

Such an F is called a (1-dimensional) distribution function. Similarly a function F on $\mathbb{R}^k$ satisfies the conditions

- F is non-decreasing in each of its variables and $\Delta_{b-a}F(a) \ge 0$ for all $a \le b$; $a, b \in \mathbb{R}^k$ (where, for $a = (a_1,\ldots,a_k)$ and $b = (b_1,\ldots,b_k)$, $a \le b$ means $a_i \le b_i$ for $i = 1,\ldots,k$, and where $\Delta_{b_i-a_i}F(a) = F(a_1,\ldots,a_{i-1},b_i,a_{i+1},\ldots,a_k) - F(a)$ and $\Delta_{b-a}F(a) = \Delta_{b_1-a_1}\Delta_{b_2-a_2}\cdots\Delta_{b_k-a_k}F(a)$)
- $\lim_{x_j\to-\infty} F(x_1,\ldots,x_k) = 0$ for each j and $\lim_{\substack{x_1\to\infty \\ \vdots \\ x_k\to\infty}} F(x_1,\ldots,x_k) = 1$
- F is right continuous in each of its variables

if and only if there exists a random vector $(X_1,\ldots,X_k)$, defined on some probability space (Ω,F,P), such that

$$F(x_1,\ldots,x_k) = P[X_1 \le x_1,\ldots,X_k \le x_k] \quad \text{for all} \quad (x_1,\ldots,x_k) \in \mathbb{R}^k .$$

Such an F is called a k-dimensional distribution function. Now consider a family

$$\Gamma = (F_{t_1 \cdots t_k};\ (t_1,\ldots,t_k) \in \mathbb{R}^k,\ k = 1,2,\ldots) \tag{4.3}$$

where each $F_{t_1 \cdots t_k}$ is a k-dimensional distribution function, as above. A question arises: when is Γ the family of fddf's of some stochastic process $(\xi(t);\ t \in \mathbb{R})$? The answer is given by Kolmogorov's theorem.

THEOREM 4.2.1. *The family* Γ *given in* (4.3), *where each* $F_{t_1 \cdots t_k}$ *is a* k-*dimensional distribution function, is the family of fddf's of some stochastic process* $(\xi(t);\ t \in \mathbb{R})$ *if and only if the following two conditions are satisfied:*

I. The symmetry condition. *If* $(t_1,\ldots,t_k)$ *and* $(x_1,\ldots,x_k) \in \mathbb{R}^k$ *and if* σ *is a permutation of* $\{1,2,\ldots,k\}$, *then*

$$F_{t_1 \cdots t_k}(x_1,\ldots,x_k) = F_{t_{\sigma(1)} \cdots t_{\sigma(k)}}(x_{\sigma(1)},\ldots,x_{\sigma(k)}) .$$

II. The consistency condition. *For* $(t_1,\ldots,t_k)$ *and* $(x_1,\ldots,x_k) \in \mathbb{R}^k$ *and for* $t_{k+1} \in \mathbb{R}$, *we have*

$$F_{t_1 \cdots t_k}(x_1,\ldots,x_k) = \lim_{x_{k+1} \to \infty} F_{t_1 \cdots t_k t_{k+1}}(x_1,\ldots,x_k,x_{k+1}) . \qquad ////$$

The proof can be found in Cramer and Leadbetter [1967]. The following result follows at once.

THEOREM 4.2.2. *The family* Γ *given in* (4.3), *where each* $F_{t_1 \cdots t_k}$ *is a* k-*dimensional distribution function, is the family of fddf's of some SSGP if and only if*

(i) *the symmetry and consistency conditions are satisfied,*

(ii) *for* $(t_1,\ldots,t_k) \in \mathcal{R}^k$ *and* $\tau \in \mathcal{R}$, $F_{t_1+\tau,\ldots,t_k+\tau} = F_{t_1,\ldots,t_k}$,

(iii) *for* $(t_1,\ldots,t_k) \in \mathcal{R}^k$, $F_{t_1 \cdots t_k}$ *is a* k-*dimensional Gaussian distribution function with mean vector* $(0,0,\ldots,0)$ *and covariance matrix* $(\sigma_{t_i t_j};\ 1 \le i,j \le k)$ *satisfying* $\sigma_{t_i t_i} = 1$ *for* $i = 1,2,\ldots,k$. ////

Theorem 4.2.2 characterizes those families of fddf's which are the family of fddf's of some SSGP. Let us write $\mathcal{E}^*$ for the class of all such families.

We now turn our attention to the covariance function. If $\{\xi(t);\ t\in\mathcal{R}\}$ is a SSGP, then its covariance function is

$$r(t) = E[\xi(s)\xi(s+t)]\ , \tag{4.4}$$

as defined in Section 1.1. As we did for the families of fddf's, we would like to characterize those functions $r(t)$ which are the covariance function of some SSGP. This is easily done.

THEOREM 4.2.3. *A real valued function* $r(t)$, *defined on* $\mathcal{R}$, *is the covariance function of some SSGP if and only if*

(i) $r(0) = 1$

(ii) $r(t) = r(-t)$

(iii) *for all positive integer* k, *and for all* $(t_1,\ldots,t_k)$ *and* $(x_1,\ldots,x_k)$ *in* $\mathcal{R}^k$, *we have*

$$\sum_{j=1}^{k} \sum_{i=1}^{k} r(t_j-t_i)x_j x_i \ge 0\ .$$

Proof. If $r(t)$ is the covariance function of some SSGP then (i) and (ii) follow easily from the definitions whereas (iii) follows from

$$\sum_{j=1}^{k} \sum_{i=1}^{k} r(t_j - t_i) x_j x_i = E[|\sum_{\ell=1}^{k} x_\ell \xi(t_\ell)|^2] \geq 0 .$$

Conversely, suppose $r(t)$ is a real valued function, defined on $\mathcal{R}$, satisfying (i), (ii) and (iii). We associate to $r(t)$ a family Γ of fddf's in the following way: given $(t_1, t_2, \ldots, t_k) \in \mathcal{R}^k$ let $F_{t_1 \cdots t_k}$ be the k-dimensional Gaussian distribution function with mean vector $(0, \ldots, 0)$ and covariance matrix $(\sigma_{ij}) = (r(t_i - t_j))$. Note that from (ii) and (iii) the matrix $(\sigma_{ij};\ 1 \leq i, j \leq k)$ is symmetric and nonnegative definite so that $F_{t_1 \cdots t_k}$ is a well defined k-dimensional Gaussian distribution function (see Cramer and Leadbetter [1967, page 26]). By (i) the diagonal elements of this covariance matrix are all equal to one. It is easily checked that the conditions (i), (ii) and (iii) of Theorem 4.2.2 are then satisfied. Thus our family Γ is the family of fddf's of some SSGP. This SSGP has covariance function $r(t)$. ////

It is clear from (4.4) that the covariance function of the process is uniquely determined by the family of fddf's. The proof of Theorem 4.2.3 shows that the converse is also true: given the covariance function, the family of fddf of the process is uniquely determined. Hence we have a natural one-to-one and onto correspondence between the class, say $\mathcal{L}^*$, of all covariance functions of SSGP's and the class $\mathcal{E}^*$ of all families of fddf's of SSGP's.

Let us now restrict ourselves to a smaller class of processes. We say that a SSGP is continuous in quadratic mean (QM) at the point t if

$$\lim_{s \to t} E[|\xi(s)-\xi(t)|^2] = 0 \ .$$

This limitation is by no means too restrictive. In the sequel we will only consider SSGP's having, with probability one, continuous sample paths, and QM continuity is a necessary condition for sample path continuity.

It is easily seen (see Cramer and Leadbetter [1967]) that if $(\xi(t);\ t \in \mathbb{R})$ is a SSGP with covariance function $r(t)$, then the following conditions are equivalent:

- $\xi(t)$ is QM continuous on $\mathbb{R}$
- $\xi(t)$ is QM continuous at 0
- $r(t)$ is continuous on $\mathbb{R}$
- $r(t)$ is continuous at 0

Thus when we restrict ourselves to QM continuous SSGP's, Theorem 4.2.3 becomes:

THEOREM 4.2.4. *A real valued function* $r(t)$, *defined on* $\mathbb{R}$, *is the covariance function of some QM continuous SSGP if and only if*

(i) $r(0) = 1$

(ii) $r(t) = r(-t)$

(iii) *for all positive integer* k, *and for all* $(t_1,\ldots,t_k)$ *and* $(x_1,\ldots,x_k)$ *in* $\mathbb{R}^k$, *we have*

$$\sum_{j=1}^{k} \sum_{i=1}^{k} r(t_j-t_i)x_j x_i \geq 0$$

(iv) $r(t)$ *is continuous at* 0. ////

The one-to-one and onto correspondence between $\mathcal{L}^*$ and $\mathcal{E}^*$ reduces to a one-to-one and onto correspondence between $\mathcal{L}$, the class of all covariance functions of QM continuous SSGP's, and $\mathcal{E}$, the class of all families of fddf's of QM continuous SSGP's.

The reader with basic knowledge of Fourier transforms, as applied to probability theory, has observed that Theorem 4.2.4 says merely that the class $\mathcal{L}$ is precisely the class of all real valued so-called characteristic functions. It follows (see Lukacs [1964]) that the mapping

$$G \longrightarrow \int_{-\infty}^{\infty} \cos \lambda t \; dG(\lambda)$$

is a one-to-one and onto correspondence between the family N of all symmetric distribution functions G (i.e. G is a 1-dimensional distribution function and $G(-x) = 1 - \lim_{t \uparrow x} G(t)$, for each x) and the family $\mathcal{L}$ of all covariance functions of QM continuous SSGP's. Thus if $r(t)$ is the covariance function of some QM continuous SSGP, then it can be written as

$$r(t) = \int_{-\infty}^{\infty} \cos \lambda t \; dG(\lambda) \tag{4.5}$$

for some G in N. This representation is unique. G is then called the spectral distribution function of the process.

Thus any two of $\mathcal{E}$, $\mathcal{L}$ and N are, in a very natural way, in a one-to-one and onto correspondence. Hence the distribution of a QM continuous SSGP, i.e. its family of fddf's, is completely determined by its covariance function, as well as by its spectral distribution function.

4.3 Analytical properties of the sample paths and finiteness of the moments of the number of upcrossings.

In this section we will give necessary and/or sufficient conditions for a QM continuous SSGP to have certain properties. In the literature, although they are usually given in terms of the covariance function, these conditions are often given in terms of the spectral distribution function. Since (4.5) defines a one-to-one and onto correspondence between N and $\mathcal{L}$ it is clear that to a condition on the covariance function corresponds a condition on the spectral distribution function, and vice-versa.

Conditions on the covariance function and the spectral distribution function

THEOREM 4.3.1. *Let* $G(\lambda)$ *be in* N *and let* $r(t)$ *be the corresponding element in* $\mathcal{L}$, *i.e.* $r(t)$ *is given by* (4.5). *Then*

(4.6) $$\int_0^\infty \lambda^4 dG(\lambda) < \infty$$

$$\Leftrightarrow$$

(4.7) $r(t)$ has a fourth derivative

$$\Rightarrow$$

(4.8) for some $a > 1$ we have $\int_0^\infty \lambda^2[\log(1+\lambda)]^a dG(\lambda) < \infty$

$$\Leftrightarrow$$

(4.9) $r(t)$ has a second derivative, and for some $b > 1$

$$r''(t)-r''(0) = O(|\log|t||^{-b}) \quad \text{as} \quad t \to 0$$

$\Rightarrow$

(4.10) $$\int_0^\infty \lambda^2 \log(1+\lambda)\,dG(\lambda) < \infty$$

$\Leftrightarrow$

(4.11) $r(t)$ has a second derivative, and for some $\delta > 0$
$$\int_0^\delta \frac{r''(t)-r''(0)}{t}\,dt < \infty$$

$\Rightarrow$

(4.12) $$\int_0^\infty \lambda^2\,dG(\lambda) < \infty$$

$\Leftrightarrow$

(4.13) $r(t)$ has a second derivative

$\Rightarrow$

(4.14) for some $a > 1$ we have
$$\int_0^\infty [\log(1+\lambda)]^a\,dG(\lambda) < \infty$$

$\Leftrightarrow$

(4.15) for some $b > 1$ we have
$$r(t) - r(0) = O(|\log|t||^{-b}) \quad \text{as} \quad t \to 0 \, .$$ ////

Note that (4.7), (4.9), (4.11) and (4.13) are respectively Cramer, Belayev, Qualls and Berman local conditions for Theorem 1.1.1 to hold. Clearly we have (4.6) $\Rightarrow$ (4.8) $\Rightarrow$ (4.10) $\Rightarrow$ (4.12) $\Rightarrow$ (4.14). The equivalences (4.6) $\Leftrightarrow$ (4.7) and (4.12) $\Leftrightarrow$ (4.13) are well known results (see Lukacs [1964]). Qualls [1967] obtained (4.10) $\Leftrightarrow$ (4.11) and Belayev

[1961] obtained (4.8) $\Leftrightarrow$ (4.9) and (4.14) $\Leftrightarrow$ (4.15).

In the literature, the conditions (4.7), (4.9), (4.11) and (4.13) often appear under an equivalent form. We state those equivalent forms that are commonly used.

(A) Condition (4.7) is equivalent to each one of

- the fourth derivative of $r(t)$ exists at $t = 0$
- the fourth derivative of $r(t)$ exists and is continuous everywhere

(4.16) • $r(t) = 1 - \frac{\lambda_2 t^2}{2} + \frac{\lambda_4 t^4}{4!} + o(t^4)$, as $t \to 0$, for some $0 \leq \lambda_2, \lambda_4 < \infty$

(B) Condition (4.9) is equivalent to

(4.17) • $r(t) = 1 - \frac{\lambda_2 t^2}{2} + O(t^2|\log|t||^{-b})$, as $t \to 0$, for some $1 < b < \infty$ and some $0 \leq \lambda_2 < \infty$

(C) Condition (4.11) is equivalent to each one of

- $r(t)$ has a second derivative, and for some $\delta > 0$

$$\int_0^\delta \frac{r'(t) - tr''(0)}{t^2}\, dt < \infty$$

- $r(t)$ has a second derivative, and for some $\delta > 0$

$$\int_0^\delta \frac{r(t) - r(0) - r''(0)t^2/2}{t^3}\, dt < \infty$$

(D) Condition (4.13) is equivalent to each one of

- the second derivative of $r(t)$ exists at $t = 0$
- the second derivative of $r(t)$ exists and is continuous everywhere

(4.18) $$r(t) = 1 - \frac{\lambda_2 t^2}{2} + o(t^2) \quad \text{for some} \quad 0 \le \lambda_2 < \infty$$

(E) Moreover if (4.17) or (4.18) holds, then

$$\lambda_2 = -r''(0) = \int_{-\infty}^{\infty} \lambda^2 dG(\lambda) ,$$

and if (4.16) holds, then in addition we have

$$\lambda_4 = r^{(iv)}(0) = \int_{-\infty}^{\infty} \lambda^4 dG(\lambda) .$$

The results (A), (D) and (E) can be found in Lukacs [1964], (B) can be found in Cramer and Leadbetter [1967] and (C) can be found in Qualls [1967].

Sample path analytical properties

We say that two stochastic processes $(\xi(t);\ t \in \mathcal{R})$ and $(\eta(t);\ t \in \mathcal{R})$, defined on the same probability space, are equivalent if for each t in $\mathcal{R}$ we have $\xi(t) = \eta(t)$ almost everywhere. The following theorem summarizes some results concerning sample path continuity and differentiability.

THEOREM 4.3.2. *Let* $(\xi(t);\ t \in \mathcal{R})$ *be a SSGP with covariance function* $r(t)$. *Then the following results hold.*

(a) *If* $r(t)$ *is continuous, then* $(\xi(t);\ t \in \mathcal{R})$ *has an equivalent version such that either*

  (i) *with probability one, the sample paths are continuous, or*

  (ii) *with probability one, the sample paths have discontinuities of the second kind at every point*

(b) *If* $r(t)$ *satisfies* (4.15), *then* $(\xi(t);\ t \in \mathcal{R})$ *has an equivalent version having, with probability one, continuous sample paths.*

(c) *If* $r(t)$ *satisfies* (4.13), *then* $(\xi(t); t \in \mathbb{R})$ *has an equivalent version having, with probability one, almost everywhere differentiable sample paths.*

(d) *If* $r(t)$ *satisfies* (4.9), *then* $(\xi(t); t \in \mathbb{R})$ *has an equivalent version having, with probability one, continuously differentiable sample paths.*

Hunt [1951] obtained sample path differentiability and sample path continuity under conditions (4.8) and (4.14) respectively. These conditions on the spectral distribution function are often called Hunt's differentiability condition and Hunt's continuity condition. They are known to be close to necessary. Belayev [1960,1961] obtained (b) and (d) above by translating Hunt's conditions on the spectral distribution function into conditions on the covariance function (this is a part of Theorem 4.3.1). He also obtained (c). Davies and Dowson [1975] showed that almost everywhere differentiability cannot be replaced by everywhere differentiability in (c). (a) is due to Dobrushin [1960].

The moments of the number of upcrossings

We now discuss some results on the moments of the number of upcrossings $Z_u(T)$ of a level u by a SSGP, say $(\xi(t), t \in \mathbb{R})$, during the time interval $(0,T]$.

If $(\xi(t+h)-\xi(h))/h$ converges in quadratic mean (QM) as $h \to 0$, then the limiting random variable, which is unique up to equivalence, is denoted by $\xi'(t)$ and is called the QM derivative of the process at the point t. It is easily seen that $\xi'(t)$ exists if and only if $r''(t)$ exists. (Hence if $\xi'(t_0)$ exists at the point $t = 0$ then it exists at every point t.) Let

$$M_k(T) = E[Z_u(T)(Z_u(T)-1)\cdots(Z_u(T)-(k-1))] .$$

The following theorem was obtained by Cramer and Leadbetter [1965] under the assumption that the process has sample path derivative. Ylvisaker [1966] showed that the existence of a QM derivative is sufficient.

THEOREM 4.3.3. *Let* $(\xi(t);\ t \in \mathbb{R})$ *be a SSGP whose spectral distribution function* $G(\lambda)$ *possesses a continuous component and a finite second moment* λ_2. *(From the above results it follows that* $(\xi(t);\ t \in \mathbb{R})$ *has an equivalent version with continuous sample paths. We assume that we are dealing with such a version). Let* $Z_u(T)$ *and* $M_k(T)$ *be as defined above. Then for* $k = 1,2,\ldots$ *the following equality holds whether both sides are finite or not:*

$$(4.19) \quad M_k(T) = \int_0^T \cdots \int_0^T \int_0^\infty \cdots \int_0^\infty y_1 \cdots y_k p_{t_1 \cdots t_k}(u,\ldots,u,y_1,\ldots,y_k) \cdot dy_1 \cdots dy_k dt_1 \cdots dt_k$$

where $p_{t_1 \cdots t_k}(x_1,\ldots,x_k,y_1,\ldots,y_k)$ *is the joint density of* $\xi(t_1),\ldots,\xi(t_k),\xi'(t_1),\ldots,\xi'(t_k)$. ////

The spectral distribution function $G(\lambda)$ is assumed to have a continuous component only to guarantee the existence of the mentioned joint density. For $k = 1$, $(\xi(t),\xi'(t))$ always have a joint density so that we don't need this extra assumption on $G(\lambda)$.

Formula (4.19) can in principle be evaluated in terms of the covariance function $r(t)$ and its derivatives. Of course, in general the expression so obtained is not very usable. For $k = 1$ or 2 we may however obtain a manageable form.

If $k = 1$, then formula (4.19) reduces to

(4.20) $$E[Z_u(T)] = \int_0^T \int_0^\infty y p_t(u,y)\,dy\,dt \qquad \text{(finite or not)}$$

where $p_t(x,y)$ is a bivariate Gaussian density with mean vector $(0,0)$ and covariance matrix

$$\Sigma_2 = \begin{bmatrix} 1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$$

so that

$$p_t(u,y) = \frac{1}{\sqrt{2\pi}} e^{-u^2/2} \frac{1}{\sqrt{2\pi\lambda_2}} e^{-y^2/2\lambda_2} .$$

Thus (4.20) becomes, after integration,

(4.21) $$E[Z_u(T)] = T \frac{\sqrt{\lambda_2}}{2\pi} e^{-u^2/2}$$

which is always finite. Note that in the statement of Theorem 4.3.3 it is assumed that $\lambda_2 < \infty$. Ylvisaker [1965] showed that if $\lambda_2 = +\infty$ then $E[Z_u(T)] = +\infty$. Thus (4.21) always holds.

If $k = 2$, then formula (4.19) reduces to

(4.22) $$E[Z_u(T)(Z_u(T)-1)] = \int_0^T \int_0^T \int_0^\infty \int_0^\infty y_1 y_2 p_{t_1 t_2}(u,u,y_1,y_2)\,dy_1\,dy_2\,dt_1\,dt_2 .$$

This time $p_{t_1 t_2}(x_1,x_2,y_1,y_2)$ is a 4-dimensional Gaussian density with mean vector $(0,0,0,0)$ and covariance matrix

$$\Sigma_4 = \begin{bmatrix} 1 & r(\tau) & 0 & r'(\tau) \\ r(\tau) & 1 & -r'(\tau) & 0 \\ 0 & -r'(\tau) & \lambda_2 & -r''(\tau) \\ r'(\tau) & 0 & -r''(\tau) & \lambda_2 \end{bmatrix} \qquad \tau = t_2 - t_1 .$$

Qualls [1967] has shown that if (4.10) holds (or equivalently, if (4.11) holds) then $E[Z_u(T)(Z_u(T)-1)] < \infty$ and (4.22) can be written as

$$(4.23)\qquad E[Z_u(T)(Z_u(T)-1)] = 2\int_0^T (T-t)\exp\{-u^2/(1+r(t))\}B(t)I(b,h)dt$$

where
$$B(t) = \frac{\lambda_2(1-r^2(t)) - (r'(t))^2}{2\pi(1-r^2(t))^{3/2}} > 0$$

$$b = \frac{r''(t)(1-r^2(t)) + r(t)(r'(t))^2}{\lambda_2(1-r^2(t)) - (r'(t))^2}, \qquad |b| < 1,$$

$$h = \frac{ur'(t)}{1+r(t)}\left\{\frac{1-r^2(t)}{\lambda_2(1-r^2(t)) - (r'(t))^2}\right\}^{1/2}$$

$$I(b,h) = \frac{1}{2\pi\sqrt{1-b^2}}\int_h^\infty\int_{-h}^\infty (x-h)(y+h)\exp\left\{-\frac{x^2+y^2+2bxy}{2(1-b^2)}\right\}dxdy .$$

In his paper Qualls raised the question of whether (4.10) is a necessary condition for finiteness of the second moment of $Z_u(T)$. Geman [1972] answered this question positively.

The following theorem summarizes the results on finiteness of the first and second moment of $Z_u(T)$.

THEOREM 4.3.4. *Let* $(\xi(t);\ t\in\mathbb{R})$ *be a SSGP having, with probability one, continuous sample paths. Let* μ_u *and* σ_u^2 *be respectively the mean and the variance of the number of upcrossings of the level* u *in the time interval* $(0,1]$ *and let* G *be the spectral distribution function of the process. Then,*

$$\mu_u < \infty \Leftrightarrow \int_0^\infty \lambda^2 dG(\lambda) < \infty .$$

If in addition G *has a continuous component, then*

$$\sigma_u^2 < \infty \Leftrightarrow \int_0^\infty \lambda^2\log(1+\lambda)dG(\lambda) < \infty . \qquad ////$$

One can show that if $r(t) \to 0$ as $t \to \infty$, then $G(\lambda)$ is everywhere continuous. Thus Theorem 4.3.4 applies to the SSGP's considered in Theorem 1.1.1 and we conclude that, in their respective versions of that theorem, Volkonskii and Rozanov, Cramer, Belayev, and Qualls were all assuming $\sigma_u^2 < \infty$ whereas Berman only assumes $\mu_u < \infty$.

We conclude this section with some comments on the expressions (4.21) and (4.23) for the first and second moments of the number of upcrossings. Formula (4.21) is called Rice's formula. It was first obtained by Rice [1945] for the case where the spectral distribution function is discrete with only a finite number of jumps. In 1957 Grenander and Rosenblat gave a simpler proof for the same case. In 1960 Ivanov proved the result for the case where the spectral distribution function has a fourth moment. In 1961 Bulinskaya proved the result under Hunt's condition guaranteeing $\xi(t)$ to have continuous sample derivatives. Finally in 1964 and 1965 both Itô and Ylvisaker proved the result that Rice's formula holds, whether λ_2 is finite or not.

A formula for the variance of the number of zeros during the time interval $(0,T]$ was given by Steinberg, Schultheiss, Wogrin and Zweig [1955]. Volkonskii and Rozanov [1961], in a footnote, obtained under certain conditions, mainly that the covariance function has a sixth derivative, a formula for the variance of the number of upcrossings of a given level. In the case of the zero level, their formula reduces to the one of Steinberg et al. Finally Leadbetter and Cryer [1965] obtained Steinberg's formula for the zero level under the assumption that the covariance function $r(t)$ satisfies (4.11). This formula, written in terms of $r(t)$, $r'(t)$ and $r''(t)$, is derived in Cramer and Leadbetter [1967]. One can check that Quall's formula (4.23) also reduces to the formula of Steinberg et al. when $u = 0$.

4.4 Stationary Gaussian processes satisfying Berman local and mixing conditions

We shall now prove the results announced in Section 4.1, namely that the conditions R1, R2, $D_c(u_T)$ and $D_c'(u_T)$ hold for SSGP's satisfying Berman conditions. ϕ and Φ will denote the standard Gaussian density and the standard Gaussian distribution, respectively, i.e.

$$\phi(u) = \frac{1}{\sqrt{2\pi}} e^{-u^2/2}$$

and

$$\Phi(u) = \int_{-\infty}^{u} \phi(t)dt .$$

The condition R1

Let $(\xi(t);\ t \in \mathbb{R})$ be a SSGP, defined on some probability space (Ω, F, P), whose covariance function satisfies Berman local condition (4.1), or equivalently

$$(4.24) \qquad r(t) = 1 - \frac{\lambda_2 t^2}{2} + o(t^2) \quad \text{as} \quad t \to 0$$

where λ_2 is the second spectral moment. From Section 4.3 we know that $(\xi(t);\ t \in \mathbb{R})$ is equivalent to a process having, with probability one, continuous sample paths. We assume that $(\xi(t);\ t \in \mathbb{R})$ is itself such a process. We want to show that under these assumptions the condition R1 holds with $\psi(u) = \mu_u$, i.e. for some $h_0 > 0$ we have

$$(4.25) \qquad \frac{P[M(h) > u]}{h\mu_u} \to 1 \quad \text{as} \quad u \to \infty, \quad 0 < h < h_0$$

where $M(h)$ and μ_u have the usual meaning. We shall first obtain (4.25) for a very particular process. We will then extend the result to the general case by showing that the maximum of the general process over $[0,h]$ and the maximum of the particular process over $[0,h]$ behave similarly.

Let A and ϕ be independent random variables on (Ω,F,P), A being Rayleigh distributed with density

$$f_A(x) = xe^{-x^2/2}, \quad x \geq 0,$$

and ϕ being uniformly distributed over $[0,2\pi]$, and let $(\xi^*(t);\ t\in\mathcal{R})$ be the process defined by

$$\xi^*(t) = A\cos(\sqrt{\lambda_2}\,t+\phi)$$

where λ_2 is as in (4.24). It is easily seen that $(\xi^*(t);\ t\in\mathcal{R})$ is a SSGP with continuous sample paths and with covariance function $r^*(t)$ satisfying

$$(4.24)^* \qquad r^*(t) = 1-\frac{\lambda_2 t^2}{2}+o(t^2) \text{ as } t\to 0.$$

$(\xi^*(t);\ t\in\mathcal{R})$ is sometimes called the trigonometric process associated with $(\xi(t);\ t\in\mathcal{R})$. It has the same second spectral moment. For this process, the distribution of the maximum over the interval $[0,h]$, say $M^*(h)$, is easily computed. We get, for $0<h<\pi/\sqrt{\lambda_2}$,

$$P[M^*(h)\leq u] = \Phi(u)-\frac{h\sqrt{\lambda_2}}{2\pi}e^{-u^2/2}.$$

Hence, using Rice's formula, we have, for $0<h<\pi/\sqrt{\lambda_2}$,

$$\frac{P[M^*(h) > u]}{h\mu_u^*} = \frac{\frac{h\sqrt{\lambda_2}}{2} e^{-u^2/2} + (1-\Phi(u))}{\frac{h\sqrt{\lambda_2}}{2\pi} e^{-u^2/2}}$$

$$= 1 + \frac{\sqrt{2\pi}}{h\sqrt{\lambda_2}} \frac{1-\Phi(u)}{\phi(u)}$$

where of course μ_u^* is the mean number of upcrossings of the level u by the process $(\xi^*(t);\ t\in\mathbb{R})$. From the well known relation

$$\text{(4.26)} \qquad \frac{u(1-\Phi(u))}{\phi(u)} \to 1 \quad \text{as} \quad u\to\infty$$

we get, writing $h_0 = \pi/\sqrt{\lambda_2}$,

$$\text{(4.27)} \qquad \frac{P[M^*(h) > u]}{h\mu_u^*} \to 1 \quad \text{as} \quad u\to\infty, \quad 0 < h < h_0 .$$

Since $(\xi(t);\ t\in\mathbb{R})$ and $(\xi^*(t);\ t\in\mathbb{R})$ have the same spectral moment, Rice's formula gives us $\mu_u = \mu_u^*$. Hence (4.25) will follow from (4.27) if we can show that

$$\frac{P[M(h) > u] - P[M^*(h) > u]}{h\mu_u} \to 0 \quad \text{as} \quad u\to\infty$$

or, equivalently, if we can show that

$$\text{(4.28)} \qquad \frac{|P[M(h) \le u] - P[M^*(h) \le u]|}{h\mu_u} \to 0 \quad \text{as} \quad u\to\infty .$$

We will now obtain (4.28). For each $a > 0$ we have

$$\begin{aligned} |P[M(h) \le u] - P[M^*(h) \le u]| \le{} & \Big|P[M(h) \le u] - P[\bigcap_{0\le \frac{ja}{u}\le h} \{\xi(\tfrac{ja}{u}) \le u\}]\Big| \\ & + \Big|P[\bigcap_{0\le \frac{ja}{u}\le h} \{\xi(\tfrac{ja}{u}) \le u\}] \\ & - P[\bigcap_{0\le \frac{ja}{u}\le h} \{\xi^*(\tfrac{ja}{u}) \le u\}]\Big| \end{aligned}$$

$$+ \left| P\left[\bigcap_{0 \le \frac{ja}{u} \le h} \{\xi^*(\tfrac{ja}{u}) \le u\}\right] - P[M^*(h) \le u] \right|$$

so that (4.28) will follow if we can show that each one of the following three expressions goes to 0 as a decreases to 0, for small enough h:

$$\limsup_{u\to\infty} \frac{\left| P[M(h) \le u] - P\left[\bigcap_{0 \le \frac{ja}{u} \le h} \{\xi(\tfrac{ja}{u}) \le u\}\right] \right|}{\mu_u},$$

$$\limsup_{u\to\infty} \frac{\left| P\left[\bigcap_{0 \le \frac{ja}{u} \le h} \{\xi(\tfrac{ja}{u}) \le u\}\right] - P\left[\bigcap_{0 \le \frac{ja}{u} \le h} \{\xi^*(\tfrac{ja}{u}) \le u\}\right] \right|}{\mu_u},$$

$$\limsup_{u\to\infty} \frac{\left| P\left[\bigcap_{0 \le \frac{ja}{u} \le h} \{\xi^*(\tfrac{ja}{u}) \le u\}\right] - P[M^*(h) \le u] \right|}{\mu_u}.$$

THEOREM 4.4.1. *With the above notation and assumptions we have*

$$(4.29) \qquad \lim_{a\to 0} \limsup_{u\to\infty} \frac{\left| P[M(h) \le u] - P\left[\bigcap_{0 \le \frac{ja}{u} \le h} \{\xi(\tfrac{ja}{u}) \le u\}\right] \right|}{\mu_u} = 0,$$

for all small enough h.

Proof. The first step of the proof is to estimate $P[\xi(0) < u < \xi(\frac{a}{u})]$ for large u. For convenience we assume $\lambda_2 = 1$ (the general case reduces to this case by a change of time scale). Let us write $q = a/u$. If

$$\zeta = \frac{\xi(q)-\xi(0)}{q},$$

then we have

$$P[\xi(0) < u < \xi(q)] = P[u-q\zeta < \xi(0) < u]$$
$$= \int_0^\infty \int_{u-qz}^{u} p(x,z)\,dx\,dz$$
$$= q\int_0^\infty \int_0^z p(u-qy,z)\,dy\,dz$$

where p is the joint density of $(\xi(0),\zeta)$. It is easily checked that $(\xi(0),\zeta)$ has a bivariate normal distribution with mean $(0,0)$ and with covariance matrix

$$\Sigma = \begin{bmatrix} 1 & (r(q)-1)/q \\ (r(q)-1)/q & 2(1-r(q))/q^2 \end{bmatrix}.$$

Writing out this density we find that

$$e^{u^2/2}p(u-qy,z) = (2\pi|\Sigma|)^{-1/2}\exp\left(\tfrac{1}{2}u^2\left(1-\frac{2(1-r(q))}{q^2|\Sigma|}\right)\right)$$
$$\times \exp\left\{-\frac{1}{2|\Sigma|}\left(\frac{2}{q^2}(1-r(q))(-2uqy+q^2y^2)\right.\right.$$
$$\left.\left. - \frac{2}{q}(r(q)-1)(u-qy)z + z^2\right)\right\}$$

where $|\Sigma|$ denotes the absolute value of the determinant of Σ. It is easily seen that for fixed a we have

$$|\Sigma| \to 1 \quad \text{as} \quad u \to \infty .$$

Using (4.24), with $\lambda_2 = 1$, and recalling that $q = a/u$, one can check that for each $a > 0$ we have

$$e^{u^2/2}p(u-qy,z) \to \frac{1}{2\pi}\exp\{-\frac{a^2}{8} - \frac{1}{2}(-2ay+az+z^2)\} \quad \text{as} \quad u \to \infty .$$

It may also be checked that over the integration range we have

$$e^{u^2/2}p(u-qy,z) \leq K_1 e^{-K_2 z^2}$$

where K_1 and K_2 are positive constants, not depending on u. Hence by dominated convergence we get

$$2\pi \lim_{u\to\infty} e^{-u^2/2}\int_0^\infty\int_0^z p(u-qy,z)\,dy\,dz$$

$$= e^{-a^2/8}\int_0^\infty\int_0^z \exp\left(-\tfrac{1}{2}(-2ay+az+z^2)\right)dy\,dz$$

$$= \frac{\Phi(a/2) - \Phi(-a/2)}{a\sqrt{2\pi}}$$

i.e.

$$\text{(4.30)} \qquad \lim_{u\to\infty} \frac{P[\xi(0) < u < \xi(a/u)]}{(a/u)\mu_u} = \frac{\Phi(a/2)-\Phi(-a/2)}{a\sqrt{2\pi}}$$

for each $a > 0$. Now let $Z_u(h)$ be, as usual, the number of upcrossings of the level u by $\xi(t)$ in the time interval $(0,h]$. Similarly, for any $q > 0$, let $N^*(u,q,h)$ be the number of points iq in $(0,h]$ for which $\xi((i-1)q) < u < \xi(iq)$. By stationarity we get

$$E[N^*(u,q,h)] = [h/q]P[\xi(0) < u < \xi(q)] .$$

Thus, if $q = a/u$, then using (4.30) we get

$$\text{(4.31)} \qquad \lim_{u\to\infty} \frac{E[N^*(u,q,h)]}{h\mu_u} = \frac{\Phi(a/2)-\Phi(-a/2)}{a\sqrt{2\pi}} \quad \text{for each } a > 0 .$$

Clearly $\{M(h) \leq u\} \subset \bigcap_{0\leq ja/u\leq h} \{\xi(\frac{ja}{u}) \leq u\}$ and it is easy to check that their difference implies that either $\xi(0) > u$, or $Z_u(h) \geq 1$ and $N^*(u,\frac{a}{u},h) = 0$. Thus

$$\left|P[M(h) \le u] - P[\bigcap_{0 \le ja/u \le h} \{\xi(\tfrac{ja}{u}) \le u\}]\right|$$

$$\le P[\xi(0) > u] + P[Z_u(h) \ge 1 \text{ and } N^*(u,\tfrac{a}{u},h) = 0]$$

$$\le 1 - \Phi(u) + P[Z_u(h) - N^*(u,\tfrac{a}{u},h) \ge 1]$$

$$\le 1 - \Phi(u) + E[Z_u(h) - N^*(u,\tfrac{a}{u},h)]$$

$$= 1 - \Phi(u) + E[Z_u(h)] - E[N^*(u,\tfrac{a}{u},h)]$$

$$= 1 - \Phi(u) + h\mu_u - E[N^*(u,\tfrac{a}{u},h)] ,$$

where we used the fact that $Z_u(h) - N^*(u,\frac{a}{u},h)$ is nonnegative to get the third inequality. Dividing by μ_u and letting $u \to \infty$, we get, using (4.26) and (4.31),

$$(4.32) \quad \limsup_{u\to\infty} \frac{\left|P[M(h) \le u] - P[\bigcap_{0 \le \frac{ja}{u} \le h} \{\xi(\frac{ja}{u}) \le u\}]\right|}{\mu_u} \le h\left(1 - \frac{\Phi(\frac{a}{2}) - \Phi(\frac{-a}{2})}{a\sqrt{2\pi}}\right) .$$

But clearly

$$\lim_{a\downarrow 0} \frac{\Phi(a/2) - \Phi(-a/2)}{a\sqrt{2\pi}} = 1 .$$

Hence (4.29) follows from (4.32) by taking the limit as a decreases to 0. ////

In order to show that as a decreases to 0, the second of the three expressions listed before the statement of Theorem 4.4.1 goes to 0, we need the following lemma. It is stated and proved as in Leadbetter, Lindgren and Rootzen [1979]. The method used in the proof was introduced by Slepian [1962] and was later developed by Berman [1964,1971a,1971b] and Cramer and Leadbetter [1967].

LEMMA 4.4.2. *Suppose* $\xi_1, \xi_2, \ldots, \xi_n$ *are standard Gaussian random variables with covariance matrix* $\Lambda^1 = (\Lambda^1_{ij})$, *and* $\eta_1, \eta_2, \ldots, \eta_n$ *similarly with covariance matrix* $\Lambda^0 = (\Lambda^0_{ij})$, *and let* $\rho_{ij} = \max(|\Lambda^1_{ij}|, |\Lambda^0_{ij}|)$. *Further, let* $\underline{u} = (u_1, \ldots, u_n)$ *be a vector of real numbers and write* $u = \min(|u_1|, \ldots, |u_n|)$. *Then*

$$(4.33) \quad \begin{aligned} &P[\xi_j \le u_j;\ j=1,\ldots,n] - P[\eta_j \le u_j;\ j=1,\ldots,n] \\ &\quad \le \frac{1}{2\pi} \sum_{1 \le i < j \le n} (\Lambda^1_{ij} - \Lambda^0_{ij})^+ (1-\rho_{ij})^{-1/2} \exp(-u^2/(1+\rho_{ij})) \end{aligned}$$

where $(x)^+ = \max(0,x)$.

In particular, if $\eta_1, \ldots, \eta_n$ *are independent and if* $\delta = \max_{i \ne j} |\Lambda^1_{ij}| < 1$, *then for any real* u *and integers* $1 \le \ell_1 < \cdots < \ell_s$

$$(4.34) \quad \left| P[\xi_{\ell_j} \le u;\ j=1,\ldots,s] - \Phi^s(u) \right| \le K \sum_{1 \le i < j \le s} |r_{ij}| \exp(-u^2/(1+|r_{ij}|))$$

where $r_{ij} = \Lambda^1_{\ell_i \ell_j}$ *is the correlation between* ξ_{ℓ_i} *and* ξ_{ℓ_j}, *and* K *is a constant (depending on* δ*). If furthermore* $(\xi_n;\ n=1,2,\ldots)$ *is a stationary standard Gaussian sequence with covariance function* $r(n)$ *and if* $1 \le \ell_1 < \cdots < \ell_s \le n$, *then*

$$(4.35) \quad \left| P[\xi_{\ell_j} \le u;\ j=1,\ldots,s] - \Phi^s(u) \right| \le Kn \sum_{i=1}^{n} |r(i)| \exp(-u^2/(1+|r(i)|)) .$$

Proof. We shall suppose that Λ^1 and Λ^0 are positive definite (as opposed to nonnegative definite) and hence that $(\xi_1, \ldots, \xi_n)$ and $(\eta_1, \ldots, \eta_n)$ have joint densities f_1 and f_0, respectively. (The nonnegative definite case is easily dealt with by considering $\xi_i + \varepsilon_i$ and $\eta_i + \varepsilon_i$, where the ε_i are independent Gaussian variables with mean 0, and then letting $\mathrm{Var}(\varepsilon_i) \to 0$, using continuity.) Clearly

$$P[\xi_j \le u_j;\ j=1,\ldots,n] = \int_{-\infty}^{u}\!\!\cdots\!\int f_1(\underline{y})d\underline{y}$$

$$P[\eta_j \le u_j;\ j=1,\ldots,n] = \int_{-\infty}^{u}\!\!\cdots\!\int f_0(\underline{y})d\underline{y}$$

where f_1 and f_0 are the standard Gaussian densities based on the covariance matrices Λ^1 and Λ^0. If we write $\Lambda^h = h\Lambda^1 + (1-h)\Lambda^0$, for $0 \le h \le 1$, the matrix Λ^h is positive definite with units down the main diagonal and elements $\Lambda^h_{ij} = h\Lambda^1_{ij} + (1-h)\Lambda^0_{ij}$ for $i \ne j$. Let f_h be the standard Gaussian density based on Λ^h, and write

$$F(h) = \int_{-\infty}^{u}\!\!\cdots\!\int f_h(\underline{y})d\underline{y} \ .$$

The left hand side of (4.33) is then easily recognized as $F(1)-F(0)$. Now

$$F(1) - F(0) = \int_0^1 F'(h)dh$$

where
$$F'(h) = \int_{-\infty}^{u}\!\!\cdots\!\int \frac{\partial f_h(\underline{y})}{\partial h}\, d\underline{y} \ .$$

The density f_h depends on h only through the elements Λ^h_{ij} of Λ^h (regarding f_h as a function of Λ^h_{ij} for $i \le j$, say). We have $\Lambda^h_{ii} = 1$ independent of h, while for $i < j$, $\Lambda^h_{ij} = h\Lambda^1_{ij} + (1-h)\Lambda^0_{ij}$ so that

$$\frac{\partial \Lambda^h_{ij}}{\partial h} = \Lambda^1_{ij} - \Lambda^0_{ij} \ .$$

Thus

$$F'(h) = \int_{-\infty}^{u}\!\!\cdots\!\int \sum_{i\le j} \frac{\partial f_h}{\partial \Lambda^h_{ij}} \frac{\partial \Lambda^h_{ij}}{\partial h}\, d\underline{y}$$

$$= \sum_{i<j} (\Lambda^1_{ij} - \Lambda^0_{ij}) \int_{-\infty}^{u}\!\!\cdots\!\int \frac{\partial f_h}{\partial \Lambda^h_{ij}}\, d\underline{y} \ .$$

Now a useful property of the multidimensional normal density is that its derivative with respect to a covariance Λ_{ij} is the same as the second mixed derivative with respect to the corresponding variables y_i, y_j (cf. Cramer and Leadbetter [1967]). Thus

$$\frac{\partial f_h}{\partial \Lambda_{ij}} = \frac{\partial^2 f_h}{\partial y_i \partial y_j} .$$

Thus
$$F'(h) = \sum_{i<j} (\Lambda^1_{ij} - \Lambda^0_{ij}) \int_{-\infty}^{u} \cdots \int \frac{\partial^2 f_h}{\partial y_i \partial y_j} \, dy .$$

The y_i and y_j integrations may be done at once to give

$$F'(h) = \sum_{i<j} (\Lambda^1_{ij} - \Lambda^0_{ij}) \int_{-\infty}^{u'} \cdots \int f_h(y_i = u_i,\ y_j = u_j) \, dy' \tag{4.36}$$

where $f_h(y_i = u_i,\ y_j = u_j)$ denotes the function of $n-2$ variables formed by putting $y_i = u_i$, $y_j = u_j$, the integration being over the remaining variables. We can dominate the last integral by letting the variables run from $-\infty$ to $+\infty$. But

$$\int_{-\infty}^{\infty} \cdots \int f_h(y_i = u_i,\ y_j = u_j) \, dy'$$

is just the bivariate density, evaluated at (u_i, u_j), of two standard Gaussian random variables with correlation Λ^h_{ij}, and may therefore be written

$$\frac{1}{2\pi(1-(\Lambda^h_{ij})^2)^{1/2}} \exp\left(-\frac{u_i^2 - 2\Lambda^h_{ij} u_i u_j + u_j^2}{2(1-(\Lambda^h_{ij})^2)}\right) .$$

Since $|\Lambda^h_{ij}| = |h\Lambda^1_{ij} + (1-h)\Lambda^0_{ij}| \le \max(|\Lambda^1_{ij}|, |\Lambda^0_{ij}|) = \rho_{ij}$

and $$u = \min(|u_1|,\ldots,|u_n|) \le \min(|u_i|,|u_j|) ,$$

it may be easily shown that the above expression is not greater than

$$\frac{1}{2\pi(1-\rho_{ij}^2)^{1/2}} \exp(-u^2/(1+\rho_{ij})) .$$

Eliminating the possible negative terms in (4.36), we have

$$F'(h) \le \frac{1}{2\pi} \sum_{i<j} (\Lambda_{ij}^1-\Lambda_{ij}^0)^+(1-\rho_{ij}^2)^{-1/2}\exp(-u^2/(1+\rho_{ij}))$$

and since $F(1) - F(0) = \int_0^1 F'(h)dh$, we get (4.33). (4.34) and (4.35) follow easily. ////

THEOREM 4.4.3. *With the above notation and assumption we have*

$$(4.37) \qquad \lim_{a\to 0} \lim_{u\to\infty} \sup \frac{\left|P[\bigcap_{0\le \frac{ja}{u}\le h} \{\xi(\frac{ja}{u}) \le u\}] - P[\bigcap_{0\le \frac{ja}{u}\le h} \{\xi^*(\frac{ja}{u}) \le u\}]\right|}{\mu_u} = 0$$

for all small enough h.

Proof. Writing

$$\xi_j = \xi(ja/u)$$
$$\xi_j^* = \xi^*(ja/u)$$
$$s = s(a,u) = [hu/a]$$

and using the above lemma, we get

$$(4.38) \quad \left|P[\bigcap_{0\le \frac{ja}{u}\le h} \{\xi(\frac{ja}{u}) \le u\}] - P[\bigcap_{0\le \frac{ja}{u}\le h} \{\xi^*(\frac{ja}{u}) \le u\}]\right|$$
$$= \left|P[\bigcap_{j=1}^{s} \{\xi_j \le u\}] - P[\bigcap_{j=1}^{s} \{\xi_j^* \le u\}]\right|$$
$$\le \frac{1}{2\pi} \sum_{1\le i<j\le s} |\Lambda_{ij}-\Lambda_{ij}^*|(1-\rho_{ij}^2)^{-1/2}\exp(-u^2/(1+\rho_{ij}))$$

where $\Lambda_{ij} = r(t_i - t_j)$, $\Lambda^*_{ij} = r^*(t_i - t_j)$, $t_i = ia/u$,

$\rho_{ij} = \max(|\Lambda_{ij}|, |\Lambda^*_{ij}|) = \max(|r(t_i - t_j)|, |r^*(t_i - t_j)|) = \rho(t_i - t_j)$,

$\rho(t) = \max(|r(t)|, |r^*(t)|)$.

From (4.24) and (4.24)* we get

$$1 - \rho(t) = \frac{\lambda_2 t^2}{2} + o(t^2) .$$

Thus for some $h_0 > 0$ and some $\alpha > 0$ we have

(4.39) $$1 - \rho(t) \geq 4\alpha t^2 , \quad \text{for } |t| < h_0 .$$

Since $0 \leq \rho(t) \leq 1$, we get

$$1 - \rho^2(t) \geq 1 - \rho(t) \geq 4\alpha t^2 , \quad \text{for } |t| < h_0 ,$$

and hence

(4.40) $$\frac{1}{\sqrt{1-\rho^2(t)}} \leq \frac{1}{2\sqrt{\alpha}|t|} , \quad \text{for } 0 < |t| < h_0 .$$

In the following, K will denote a positive constant which may change from line to line. Writing $\psi(t) = r(t) - r^*(t)$ and using (4.40), we see that the right hand side of (4.38) is bounded by

$$K \sum_{1 \leq i < j \leq s} \frac{|\psi(t_i - t_j)|}{|t_i - t_j|} \exp\left(-u^2/(1+\rho(t_i - t_j))\right) , \quad 0 < h < h_0,$$

which in turn is bounded by

$$Ks \sum_{j=1}^{s} \frac{|\psi(t_j)|}{|t_j|} \exp\left(-u^2/(1+\rho(t_j))\right) = Ks \sum_{j=1}^{s} \frac{|\psi(ja/u)|}{(ja/u)} \exp\left(-u^2/(1+\rho(ja/u))\right)$$

$$= Ks \sum_{j=1}^{s} (ja/u)\theta(ja/u)\exp\left(-u^2/(1+\rho(ja/u))\right)$$

where $\theta(t) = |\psi(t)|/t^2$. It is easily seen that this last expression is bounded by

$$(4.41) \qquad Kse^{-u^2/2}\sum_{j=1}^{s}(ja/u)\theta(ja/u)\exp\{-(u^2/4)(1-\rho(ja/u))\} \ .$$

Now for $0 < h < h_0$ and $1 \leq j \leq s = [hu/a]$, we have $(ja/u) \leq h < h_0$.
Hence from (4.39) we get

$$(u^2/4)(1-\rho(ja/u)) \geq \alpha j^2 a^2 \ .$$

It follows that (4.41) is bounded by

$$Ke^{-u^2/2}\sum_{j=1}^{s} j\theta(ja/u)e^{-\alpha j^2 a^2}$$

which in turn is bounded by

$$Ke^{-u^2/2}\sum_{j=1}^{s} j\theta(ja/u)\eta^j$$

where $\eta = e^{-\alpha a^2} \in (0,1)$.

We have shown that if a is a fixed positive number, then for all small enough h we have, using Rice's formula,

$$\frac{\left|P[\bigcap_{0\leq\frac{ja}{u}\leq h}\{\xi(\frac{ja}{u})\leq u\}]-P[\bigcap_{0\leq\frac{ja}{u}\leq h}\{\xi^*(\frac{ja}{u})\leq u\}]\right|}{\mu_u} \leq K\sum_{j=1}^{[hu/a]} j\theta(ja/u)\eta^j$$

where $K > 0$ and $0 < \eta < 1$ do not depend on u and where

$$\theta(t) = \frac{|r(t)-r^*(t)|}{t^2} \ .$$

By (4.24) and $(4.24)^*$ we have $\theta(t) \to 0$ as $t \to 0$. Since $\theta(ja/u)$ is uniformly bounded ($1 \leq j \leq [hu/a]$, $u > 1$, say), by dominated convergence we get

$$\lim_{u\to\infty} \sum_{j=1}^{[hu/a]} j\theta(ja/u)\eta^j = \sum_{j=1}^{[hu/a]} \lim_{u\to\infty} j\theta(ja/u)\eta^j = 0$$

Hence for $a > 0$ and for $0 < h < h_0$ (h_0 independent of a), we have

$$\lim_{u\to\infty} \frac{\left| P[\bigcap_{0\le\frac{ja}{u}<h} \{\xi(\frac{ja}{u}) \le u\}] - P[\bigcap_{0\le\frac{ja}{u}<h} \{\xi^\star(\frac{ja}{u}) \le u\}] \right|}{\mu_u} = 0$$

from which (4.37) follows trivially. ////

THEOREM 4.4.4. *With the above notation and assumptions we have*

$$\lim_{a\to 0} \limsup_{u\to\infty} \frac{\left| P[\bigcap_{0\le\frac{ja}{u}<h} \{\xi^\star(\frac{ja}{u}) \le u\}] - P[M^\star(h) \le u] \right|}{\mu_u} = 0$$

for all small enough h.

Proof. The proof is the same as for Theorem 4.4.1. ////

Combining the results of the last three theorems we get (4.28), and hence (4.25). Thus the condition R1 holds for SSGP's satisfying Berman local condition (provided we are dealing with a version having, with probability one, continuous sample paths).

The condition R2

Again we consider a SSGP $(\xi(t);\ t \in \mathbb{R})$ whose covariance function satisfies Berman local condition (4.1), or equivalently (4.24). For such a process the condition R2 is satisfied with $\psi(u) = \mu_u$ and $q_a(u) = a/u$, i.e.

$$(4.42) \qquad \lim_{a\to 0} \limsup_{u\to\infty} \frac{P[\xi(0) < u,\ \xi(a/u) < u,\ M(a/u) > u]}{(a/u)\mu_u} = 0 \ .$$

This can be shown as follows. From

$$\{\xi(0) > u\} = \{\xi(0) > u,\ \xi(a/u) \geq u\} \cup \{M(a/u) > u\}$$
$$\cup \{\xi(0) < u,\ \xi(a/u) < u,\ M(a/u) > u\}$$

and using the fact that $P[\xi(0) = u] = P[\xi(0) < u,\ \xi(a/u) = u] = P[M(a/u) = u] = 0$, we get

$$\frac{P[\xi(0) < u,\ \xi(a/u) < u,\ M(a/u) > u]}{(a/u)\mu_u}$$

$$= \frac{P[\xi(0) \leq u]}{(a/u)\mu_u} - \frac{P[\xi(0) < u < \xi(a/u)]}{(a/u)\mu_u} - \frac{P[M(a/u) \leq u]}{(a/u)\mu_u}$$

$$(4.43) \qquad = -\frac{1 - \Phi(u)}{(a/u)\mu_u} - \frac{P[\xi(0) < u < \xi(a/u)]}{(a/u)\mu_u} + \frac{P[M(a/u) > u]}{(a/u)\mu_u} .$$

Rice's formula combined with (4.26) gives us

$$\lim_{u\to\infty} \frac{1 - \Phi(u)}{(a/u)\mu_u} = \frac{1}{a}\sqrt{2\pi/\lambda_2}$$

and from the proof of Theorem 4.4.1 we have (equation (4.30))

$$\lim_{u\to\infty} \frac{P[\xi(0) < u < \xi(a/u)]}{(a/u)\mu_u} = \frac{\Phi(a/2)-\Phi(-a/2)}{a\sqrt{2\pi}} .$$

Thus if we can show that

$$(4.44) \qquad \lim_{u\to\infty} \frac{P[M(a/u) > u]}{(a/u)\mu_u} = 1 + \frac{1}{a}\sqrt{2\pi/\lambda_2}$$

then from (4.43) we will have

$$\lim_{u\to\infty} \frac{P[\xi(0) < u,\ \xi(a/u) < u,\ M(a/u) > u]}{(a/u)\mu_u} = 1 - \frac{\Phi(a/2)-\Phi(-a/2)}{a\sqrt{2\pi}}$$

from which we get (4.42).

We proceed as we did in obtaining condition R1. Let $(\xi^*(t);\ t\in\mathbb{R})$ be the trigonometric process associated with $(\xi(t);\ t\in\mathbb{R})$. From

$$P[M^*(a/u) \le u] = \Phi(u) - \frac{a\sqrt{\lambda_2}}{2\pi u} e^{-u^2/2}, \quad 0 < \frac{a}{u} < \frac{\pi}{\sqrt{\lambda_2}},$$

we get, for each $a > 0$,

$$\lim_{u\to\infty} \frac{P[M^*(a/u) > u]}{(a/u)\mu_u} = 1 + \frac{1}{a}\sqrt{2\pi/\lambda_2}\ .$$

Hence (4.44) will follow from

$$\lim_{u\to\infty} \frac{P[M(a/u) > u] - P[M^*(a/u) > u]}{(a/u)\mu_u} = 0\ , \quad \text{for each } a > 0$$

or, equivalently, from

$$\lim_{u\to\infty} \frac{|P[M(a/u) \le u] - P[M^*(a/u) \le u]|}{(a/u)\mu_u} = 0\ , \quad \text{for each } a > 0\ .$$

This can be obtained by arguments similar to those leading to (4.28).

The condition $D_c(u_T)$

Let us now assume that our SSGP $(\xi(t);\ t\in\mathbb{R})$ has a covariance function satisfying both Berman local condition and Berman mixing condition, i.e.

$$r(t) = 1 - \frac{\lambda_2 t^2}{2} + o(t^2)\ , \quad \text{as } t\to 0\ ,$$

(4.45) $$r(t)\log t \to 0 \quad \text{as } t\to\infty\ .$$

Moreover, let us assume that u_T is chosen in such a way that for some $\tau > 0$,

(4.46) $$T\mu_{u_T} \to \tau \quad \text{as } T\to\infty\ .$$

(As before we assume our process has continuous sample paths. Hence μ_u is well defined. Furthermore, from Berman local condition it follows that $\mu_u < \infty$ for every u. Thus (4.46) makes sense.) Under these assumptions, we shall show that for each $a > 0$ the condition $D_c(u_T)$ holds with respect to the family $q_T = q_a(u_T) = a/u_T$.

Fix $a > 0$. We must show that there exists some $\alpha_{T,\gamma}$ satisfying, for each $\theta > 0$,

$$(4.47) \qquad \alpha_{T,\theta T} \to 0 \text{ as } T \to \infty ,$$

such that for any choice of $0 \leq s_1 < s_2 < \cdots < s_p < t_1 < t_2 < \cdots < t_{p'}$ in $\{ja/u_T;\ 0 \leq ja/u_T \leq T\}$ with $t_1 - s_p \geq \gamma > 0$ we have

$$(4.48) \qquad |F_{s_1 \cdots s_p t_1 \cdots t_{p'}}(u_T) - F_{s_1 \cdots s_p}(u_T) F_{t_1 \cdots t_{p'}}(u_T)| \leq \alpha_{T,\gamma} .$$

The left hand side of (4.48) stands for

$$\begin{aligned} &\big|P[\xi(s_1) \leq u_T, \ldots, \xi(s_p) \leq u_T, \xi(t_1) \leq u_T, \ldots, \xi(t_{p'}) \leq u_T] \\ &\quad - P[\xi(s_1) \leq u_T, \ldots, \xi(s_p) \leq u_T] P[\xi(t_1) \leq u_T, \ldots, \xi(t_{p'}) \leq u_T]\big| \end{aligned}$$

and hence can be written as

$$(4.49) \qquad |P[\xi_1 \leq u_T, \ldots, \xi_{p+p'} \leq u_T] - P[\eta_1 \leq u_T, \ldots, \eta_{p+p'} \leq u_T]|$$

where $(\xi_1, \ldots, \xi_{p+p'})$ is a zero mean Gaussian random vector based on the covariance matrix

$$\Lambda = \left[\begin{array}{cccc|cccc}
r(0) & r(s_1-s_2) & \cdots & r(s_1-s_p) & r(s_1-t_1) & r(s_1-t_2) & \cdots & r(s_1-t_{p'}) \\
r(s_2-s_1) & r(0) & \cdots & r(s_2-s_p) & r(s_2-t_1) & r(s_2-t_2) & \cdots & r(s_2-t_{p'}) \\
\vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\
r(s_p-s_1) & r(s_p-s_2) & \cdots & r(0) & r(s_p-t_1) & r(s_p-t_2) & \cdots & r(s_p-t_{p'}) \\
\hline
r(t_1-s_1) & r(t_1-s_2) & \cdots & r(t_1-s_p) & r(0) & r(t_1-t_2) & \cdots & r(t_1-t_{p'}) \\
r(t_2-s_1) & r(t_2-s_2) & \cdots & r(t_2-s_p) & r(t_2-t_1) & r(0) & \cdots & r(t_2-t_{p'}) \\
\vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\
r(t_{p'}-s_1) & r(t_{p'}-s_2) & \cdots & r(t_{p'}-s_p) & r(t_{p'}-t_1) & r(t_{p'}-t_2) & \cdots & r(0)
\end{array}\right],$$

and $(\eta_1,\ldots,\eta_{p+p'})$ is a zero mean Gaussian random vector based on the covariance matrix

$$\Lambda^* = \left[\begin{array}{cccc|cccc}
r(0) & r(s_1-s_2) & \cdots & r(s_1-s_p) & 0 & 0 & \cdots & 0 \\
r(s_2-s_1) & r(0) & \cdots & r(s_2-s_p) & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\
r(s_p-s_1) & r(s_p-s_2) & \cdots & r(0) & 0 & 0 & \cdots & 0 \\
\hline
0 & 0 & \cdots & 0 & r(0) & r(t_1-t_2) & \cdots & r(t_1-t_{p'}) \\
0 & 0 & \cdots & 0 & r(t_2-t_1) & r(0) & \cdots & r(t_2-t_{p'}) \\
\vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\
0 & 0 & \cdots & 0 & r(t_{p'}-t_1) & r(t_{p'}-t_2) & \cdots & r(0)
\end{array}\right]$$

Hence, by Lemma 4.4.8, the quantity in (4.49) is bounded by

$$\frac{1}{2\pi}\sum_{1\le i<j\le p+p'} |\Lambda_{ij}-\Lambda^*_{ij}|(1-\rho^2_{ij})^{-1/2}\exp(-u_T^2/(1+\rho_{ij}))$$

where $\rho_{ij} = \max(|\Lambda_{ij}|,|\Lambda^*_{ij}|)$. Looking up at the covariance matrices we are dealing with, we see that the last expression is simply

$$\frac{1}{2\pi}\sum_{\substack{1\le i\le p\\ 1\le j\le p'}} |r(s_i-t_j)|(1-r^2(s_i-t_j))^{-1/2}\exp(-u_T^2/(1+|r(s_i-t_j)|)) .$$

For each (i,j) in the above summation, s_i-t_j is of the form ka/u_T

for some positive integer k. For each such k there are certainly no more than $[Tu_T/a]$ couples (i,j) for which $s_i-t_j = ka/u_T$, since $p+p' \leq [Tu_T/a]$. Moreover for each (i,j) in the summation we have $|s_i-t_j| \geq \gamma$. Hence the last expression is bounded above by

$$(4.50) \quad \frac{1}{2\pi}\frac{Tu_T}{a}\sum_{\gamma\leq\frac{ka}{u_T}\leq T} |r(ka/u_T)|(1-r^2(ka/u_T))^{-1/2}\exp(-u_T^2/(1+|r(ka/u_T)|)) .$$

From Berman mixing condition (4.45) we have $r(t)\to 0$ as $t\to\infty$. Combined with the fact that $r(t)$ is continuous, this implies that the supremum of $|r(t)|$, as t ranges over $[\gamma,\infty)$, is obtained at some point t_γ of $[\gamma,\infty)$. If $|r(t_\gamma)| = 1$, then $\xi(0)$ and $\xi(t_\gamma)$ are linearly related, since they are Gaussian random variables, as are, by stationarity, $\xi(t_\gamma)$ and $\xi(2t_\gamma)$, and hence so are $\xi(0)$ and $\xi(2t_\gamma)$. Thus $|r(2t_\gamma)| = 1$. In this way it follows that $|r(kt_\gamma)| = 1$ for all k, contradicting the requirement that $r(t)\to 0$ as $t\to\infty$. Thus we have $|r(t_\gamma)| < 1$. It follows that

$$\sup_{t\geq\gamma}(1-r^2(t))^{-1/2} = (1-r^2(t_\gamma))^{-1/2} < \infty .$$

Hence (4.50) is bounded above by

$$(4.51) \qquad \frac{1}{2\pi}K(\gamma)\frac{Tu_T}{a}\sum_{\gamma\leq\frac{ka}{u_T}\leq T} |r(ka/u_T)|\exp(-u_T^2/(1+|r(ka/u_T)|))$$

where we write $K(\gamma) = (1-r^2(t_\gamma))^{-1/2}$. Thus we have shown that (4.48) holds with $\alpha_{T,\gamma}$ given by the expression (4.51). It remains to show that for each $\theta > 0$ the condition (4.47) is satisfied. Since $r(t)\to 0$ as $t\to\infty$, we have $K(\gamma)\to 1$ as $\gamma\to\infty$. Hence $K(\gamma) < 2$ for large enough γ. Now fix $\theta > 0$. For large enough T we have

$$(4.52)\qquad \alpha_{T,\theta T} \le \frac{Tu_T}{\pi a} \sum_{\theta T \le \frac{ka}{u_T} \le T} |r(ka/u_T)| \exp\left(-u_T^2/(1+|r(ka/u_T)|)\right) .$$

If we write $\delta_t = \sup_{s \ge t} |r(s)|$, then the right hand side of (4.52) is not greater than

$$\frac{1}{\pi}(Tu_T/a)\delta_{\theta T} \sum_{\theta T \le \frac{ka}{u_T} \le T} \exp\left(-u_T^2/(1+|r(ka/u_T)|)\right) .$$

This last expression is equal to

$$\frac{1}{\pi}(Tu_T/a)\delta_{\theta T} e^{-u_T^2} \sum_{\theta T \le \frac{ka}{u_T} \le T} \exp\left(u_T^2|r(ka/u_T)|/(1+|r(ka/u_T)|)\right)$$

since

$$-u_T^2 + \frac{u_T^2|r(ka/u_T)|}{(1+|r(ka/u_T)|)} = \frac{-u_T^2}{(1+|r(ka/u_T)|)} .$$

Now for each $\theta T \le ka/u_T \le T$ we have

$$|r(ka/u_T)|/(1+|r(ka/u_T)|) \le \delta_{\theta T}$$

so that $\quad \exp\left(u_T^2|r(ka/u_T)|/(1+|r(ka/u_T)|\right) \le \exp(u_T^2\delta_{\theta T})$

for each term in the last summation. Since there are no more then $[Tu_T/a]$ terms in the summation, the right hand side of (4.52) is bounded by

$$\frac{1}{\pi}(Tu_T/a)^2\delta_{\theta T} e^{-u_T^2} e^{u_T^2\delta_{\theta T}} .$$

Hence for large enough T we have

$$(4.53)\qquad \alpha_{T,\theta T} \le (1/\pi)(1/a)^2(u_T^2\delta_{\theta T})(Te^{-u_T^2/2})^2 e^{-u_T^2\delta_{\theta T}} ,$$

Using (4.46) and Rice's formula we get

$$u_T^2/2\log T \to 1 \quad \text{as} \quad T \to \infty .$$

Hence
$$u_T^2 \delta_{\theta T}/2\delta_{\theta T}\log T \to 1 \quad \text{as} \quad T \to \infty .$$

But
$$\begin{aligned} \delta_{\theta T}\log T &= \delta_{\theta T}\log \theta T - \delta_{\theta T}\log \theta \\ &= \sup_{s \geq \theta T}|r(s)|\log \theta T - \log \theta \sup_{s \geq \theta T}|r(s)| \\ &\leq \sup_{s \geq \theta T}|r(s)\log s| - \log \theta \sup_{s \geq \theta T}|r(s)| . \end{aligned}$$

Thus from our mixing condition we get

$$\delta_{\theta T}\log T \to 0 \quad \text{as} \quad T \to \infty$$

and hence

(4.54)
$$\lim_{T\to\infty} u_T^2 \delta_{\theta T} = 0 .$$

From (4.46) combined with Rice's formula we also get

(4.55)
$$\lim_{T\to\infty} T e^{-u_T^2/2} = 2\pi\tau/\sqrt{\lambda_2} .$$

Combining (4.53), (4.54) and (4.55) we get

$$\lim_{T\to\infty} \alpha_{T,\theta T} = 0 .$$

We have shown that for each $a > 0$ the condition $D_c(\dot{u}_T)$ holds with respect to the family $q_T = a/u_T$.

The condition $D'_c(u_T)$

As above, we let $(\xi(t);\ t\in\mathbb{R})$ be a SSGP whose covariance function satisfies Berman local condition and Berman mixing condition

$$(4.56)\qquad r(t) = 1-\frac{\lambda_2 t^2}{2}+o(t^2) \quad \text{as} \quad t\to 0\ ,$$

$$(4.57)\qquad r(t)\log t \to 0 \quad \text{as} \quad t\to\infty\ ,$$

and we assume that we are dealing with a version having, with probability one, continuous sample paths. Moreover we assume that u_T is chosen in such a way that for some $\tau > 0$ we have

$$(4.58)\qquad T\mu_{u_T} \to \tau \quad \text{as} \quad t\to\infty\ .$$

Under these assumptions, the condition $D'_c(u_T)$ is satisfied with the function $\psi(u) = \mu_u$, i.e.

$$(4.59)\qquad \limsup_{T\to\infty} T|\mu_{\varepsilon T,u_T}-\mu_{u_T}| \to 0 \quad \text{as} \quad \varepsilon\to 0\ .$$

This may be obtained from arguments of Pickands [1969a,1969b]. Such a proof should appear in a forthcoming paper by Lindgren and Rootzen (in preparation for Institute of Statistics Mimeo Series, University of North Carolina at Chapel Hill). Unfortunately we failed in trying to produce such a proof. We will thus content ourselves with a somewhat undirect proof.

We have already shown that when (4.56), (4.57) and (4.58) hold, the conditions R1, R2 and $D_c(u_T)$ are satisfied. If we could also obtain the condition $D'_c(u_T)$ then it would follow from the theorems of Chapter 3 that

$$(4.60)\qquad P[M(T)\le u_T] \to e^{-\tau} \quad \text{as} \quad T\to\infty\ ,$$

and

(4.61) $$Z_T \xrightarrow{w} Z \quad \text{as} \quad T \to \infty ,$$

where $M(T)$, Z_T and Z are, as usual, the maximum over $[0,T]$, the point process of upcrossings of the level u_T and a Poisson point process with intensity τ, respectively. Our proof that under (4.56), (4.57) and (4.58) the condition $D_c'(u_T)$ holds will be indirect in the sense that we will actually obtain (4.60) as an intermediate step. Of course a more direct proof would be desirable for it would then make the results of Chapter 3 useful; they would provide a nice way of obtaining (4.60) and (4.61) for the Gaussian case, whereas they are rather useless once (4.60) has been obtained. But our main goal is not to exhibit nicer or simpler proofs of Berman results. Our goal in this chapter is simply to show that the general conditions of Chapter 3 are satisfied by SSGP's, with covariance function satisfying Berman conditions.

LEMMA 4.4.5. *If both Berman local condition and Berman mixing condition hold and if* u_T *is chosen so that* (4.58) *holds for some* $\tau > 0$, *then* (4.60) *holds.*

Proof. Most of the ideas and techniques involved in this proof have already been used in this paper. So we shall simply give an outline of the proof.

We know that under the present assumptions the condition R1 is satisfied with $\psi(u) = \mu_u$. Thus for some $h_0 > 0$,

(4.62) $$\frac{P[M(h) > u]}{h\mu_u} \to 1 \quad \text{as} \quad u \to \infty, \quad 0 < h < h_0 .$$

Consider a fixed $h > 0$ for which (4.62) holds. Choose $\varepsilon > 0$, small enough so that $0 < \varepsilon < h$. For each $T > 0$, write

(4.63) $$n_T = [T/h]$$

and divide the time interval $[0,T]$ as follows:

$$\begin{aligned} I_1 &= [0,h-\varepsilon] & I_1^* &= [h-\varepsilon,h] \\ I_2 &= [h,2h-\varepsilon] & I_2^* &= [2h-\varepsilon,2h] \\ &\vdots & &\vdots \\ I_{n_T} &= [(n_T-1)h,n_Th-\varepsilon] & I_{n_T}^* &= [n_Th-\varepsilon,n_Th] \\ \tilde{I} &= [n_Th,T] \end{aligned}$$

Finally choose $q = q(n_T)$ such that $qu_T \to 0$ as $T \to \infty$. For convenience choose q so that h is always a multiple of q, so that the same number of points jq will lie in each of $I_1, I_2, \ldots,$ and in each of $I_1^*, I_2^*, \ldots$. We then proceed as follows:

(i) approximate $P[M(T) \le u_T]$ by $P[M(n_Th) \le u_T]$

(ii) approximate $P[M(n_Th) \le u_T]$ by $P[M(\bigcup_{k=1}^{n_T} I_k) \le u_T]$

(iii) approximate $P[M(\bigcup_{k=1}^{n_T} I_k) \le u_T]$ by $P[\xi(jq) \le u_T;\ jq \in \bigcup_{k=1}^{n_T} I_k]$

(iv) approximate $P[\xi(jq) \le u_T;\ jq \in \bigcup_{k=1}^{n_T} I_k]$ by $P^{n_T}[\xi(jq) \le u_T;\ jq \in I_1]$

(v) approximate $P^{n_T}[\xi(jq) \le u_T;\ jq \in I_1]$ by $P^{n_T}[M(I_1) \le u_T]$

(vi) approximate $P^{n_T}[M(I_1) \le u_T]$ by $P^{n_T}[M(h) \le u_T]$.

More specifically we have the following approximations:

(i) $$0 \le P[M(n_Th) \le u_T] - P[M(T) \le u_T] \to 0 \quad \text{as} \quad T \to \infty .$$

Clearly $\{M(T) \le u_T\} \subset \{M(n_Th) \le u_T\}$ and the difference of these two sets is included in $\{\xi(t) > u_T$ for some $t \in [n_Th,(n_T+1)h]\}$. Hence using stationarity we get

$$0 \le P[M(n_T h) \le u_T] - P[M(T) \le u_T] \le P[M(h) > u_T] .$$

By (4.62) the right hand side of this expression goes to 0 as T goes to ∞. This gives us (i).

(ii) $$0 \le P[M(\bigcup_{k=1}^{n_T} I_k) \le u_T] - P[M(n_T h) \le u_T] \le K_1 \varepsilon$$

Again the difference of the probabilities is clearly nonnegative. It is dominated by $P[\bigcup_{k=1}^{n_T} \{M(I_k^*) > u_T\}]$. By stationarity we get

$$\begin{aligned} P[\bigcup_{k=1}^{n_T} \{M(I_k^*) > u_T\}] &\le \sum_{k=1}^{n_T} P[M(I_k^*) > u_T] \\ &= n_T P[M(\varepsilon) > u_T] . \end{aligned}$$

Hence we have

$$0 \le \frac{P[M(\bigcup_{k=1}^{n_T} I_k) \le u_T] - P[M(n_T h) \le u_T]}{n_T \varepsilon \mu_{u_T}} \le \frac{P[M(\varepsilon) > u_T]}{\varepsilon \mu_{u_T}} .$$

From (4.62), the right hand side goes to 1 as T goes to ∞. Hence for large enough T we have

$$0 \le P[M(\bigcup_{k=1}^{n_T} I_k) \le u_T] - P[M(n_T h) \le u_T] \le 2 n_T \mu_{u_T} \varepsilon .$$

Combining (4.58) and (4.63) we get $n_T \mu_{u_T} \to \tau/h$. Therefore (ii) holds, say with $K_1 = 4\tau/h$, for all large enough T. The important point is that K_1 does not depend on ε.

(iii) $$0 \le P[\xi(jq) \le u_T;\ jq \in \bigcup_{k=1}^{n_T} I_k] - P[M(\bigcup_{k=1}^{n_T} I_k) \le u_T] \to 0 \quad \text{as} \quad T \to \infty .$$

This is obtained using arguments similar to those used in obtaining Theorem 4.4.1.

(iv) $P[\xi(jq) \leq u_T;\ jq \in \bigcup_{k=1}^{n_T} I_k] - P^{n_T}[\xi(jq) \leq u_T;\ jq \in I_1] \to 0$ as $T \to \infty$.

Repeating the argument used in obtaining condition $D_c(u_T)$, one can check that the absolute value of the above difference is bounded above by

$$K \frac{T}{q} \sum_{\varepsilon \leq jq \leq T} |r(jq)| \exp(-u_T^2/(1+|r(jq)|)) \quad . \tag{4.64}$$

This expression is very similar to

$$K \frac{T}{q} \sum_{\theta T \leq jq \leq T} |r(jq)| \exp(-u_T^2/(1+|r(jq)|)) \quad . \tag{4.65}$$

In obtaining condition $D_c(u_T)$ we have shown that if $q = q_a(u_T) = \frac{a}{u_T}$ where $a > 0$ is fixed, then the expression (4.65) goes to 0 as T goes to ∞. If (4.64) the summation ranges over a larger interval and instead of having $qu_T = a$ fixed, we have $qu_T \to 0$ as $T \to \infty$. Nevertheless one can show that the expression (4.64) does go to 0 as T goes to ∞, as long as $qu_T \to 0$ at an appropriately slow rate.

(v) $0 \leq P^{n_T}[\xi(jq) \leq u_T;\ jq \in I_1] - P^{n_T}[M(I_1) \leq u_T] \to 0$ as $T \to \infty$.

As in (iii), this is obtained using arguments similar to those used in obtaining Theorem 4.4.1.

(vi) $$0 \leq P^{n_T}[M(I_1) \leq u_T] - P^{n_T}[M(h) \leq u_T] \leq K_2 \varepsilon \quad .$$

$M(I_1)$ is simply $M(h-\varepsilon)$. Clearly $\{M(h) \leq u_T\} \subset \{M(h-\varepsilon) \leq u_T\}$ and the difference of these two sets is contained in $\{M(I_1^*) > u_T\}$. Thus, using stationarity,

$$0 \leq P[M(I_1) \leq u_T] - P[M(h) \leq u_T] \leq P[M(\varepsilon) > u_T] \quad .$$

We recall that $0 \le x \le y \le 1 \Rightarrow 0 \le y^n - x^n \le n(y-x)$. Thus

$$0 \le P^{n_T}[M(I_1) \le u_T] - P^{n_T}M[(h) \le u_T] \le n_T P[M(\varepsilon) > u_T] .$$

From (4.58), (4.62) and (4.63) we have $n_T P[M(\varepsilon) > u_T] \to (\tau/h)\varepsilon$ as $T \to \infty$. Hence we can find K_2 so that (vi) holds for all large enough T.

Combining these approximations we get

$$\limsup_{T\to\infty} \left| P[M(T) \le u_T] - P^{n_T}[M(h) \le u_T] \right| \le K\varepsilon$$

where $K = K_1 + K_2$ does not depend on ε. Hence

(4.66) $$P[M(T) \le u_T] - P^{n_T}[M(h) \le u_T] \to 0 \quad \text{as} \quad T \to \infty .$$

Again from (4.58), (4.62) and (4.63) we have

$$P[M(h) > u_T] = \tau/n_T + o(1/n_T) \quad \text{as} \quad T \to \infty .$$

Thus $$P^{n_T}[M(h) \le u_T] = \left(1 - \frac{\tau}{n_T} + o\left(\frac{1}{n_T}\right)\right)^{n_T} \to e^{-\tau} \quad \text{as} \quad T \to \infty .$$

Combined with (4.66), this gives us (4.60). This completes the proof. ////

COROLLARY 4.4.6. *Under the hypothesis of the lemma we have*

(4.67) $$P[M(\varepsilon T) \le u_T] \to e^{-\varepsilon\tau} \quad \text{as} \quad T \to \infty ;$$

for each positive ε.

Proof. Put $v_T = u_{T/\varepsilon}$. Then, using Rice's formula,

$$T\mu_{v_T} = T\frac{\sqrt{\lambda_2}}{2\pi}e^{-v_T^2/2} = T\frac{\sqrt{\lambda_2}}{2\pi}e^{-u^2_{(T/\varepsilon)}/2} .$$

Thus $(T/\varepsilon)\mu_{v_T} \to \tau$ as $T \to \infty$, or $T\mu_{v_T} \to \varepsilon\tau$ as $T \to \infty$. Hence by the lemma we get

$$P[M(T) \le v_T] \to e^{-\varepsilon\tau}$$

or

$$P[M(T) \le u_{T/\varepsilon}] \to e^{-\varepsilon\tau} .$$

Hence

$$P[M(\varepsilon T) \le u_T] \to e^{-\varepsilon\tau} . \qquad ////$$

THEOREM 4.4.7. *Let* $(\xi(t);\ t \in \mathbb{R})$ *be a SSGP having, with probability one, continuous sample paths, and let the constants* u_T *be chosen so that* (4.58) *holds. If the covariance function satisfies Berman local condition and Berman mixing condition, then the condition* $D_c'(u_T)$ *holds with* $\psi(u) = \mu_u$.

Proof. As in the proof of Lemma 3.4.7 we have, for all small enough positive ε,

$$\text{(4.68)} \qquad \varepsilon T\mu_{\varepsilon T, u_T} \le P[M(\varepsilon T) \ge u_T] ,$$

$$\text{(4.69)} \qquad \varepsilon T\mu_{\varepsilon T, u_T} \ge P[M(2\varepsilon T) > u_T] - P[M(\varepsilon T) > u_T] .$$

From (4.68) we get

$$T\mu_{\varepsilon T, u_T} - \tau \le \frac{1}{\varepsilon}(e^{-\varepsilon\tau} - P[M(\varepsilon T) \le u_T]) + \frac{1}{\varepsilon}(1 - e^{-\varepsilon\tau} - \varepsilon\tau) .$$

Hence from Corollary 4.4.6 we have

$$\text{(4.70)} \qquad \limsup_{T\to\infty}\left(T\mu_{\varepsilon T, u_T} - \tau\right) \le \frac{1}{\varepsilon}\left(1 - e^{-\varepsilon\tau} - \varepsilon\tau\right) .$$

Similarly from (4.69) we get

$$T\mu_{\varepsilon T, u_T} - \tau \ge \frac{1}{\varepsilon}\left(P[M(\varepsilon T) \le u_T] - e^{-\varepsilon\tau}\right) + \frac{1}{\varepsilon}\left(e^{-2\varepsilon\tau} - P[M(2\varepsilon T) \le u_T]\right)$$
$$+ \frac{1}{\varepsilon}\left(e^{-\varepsilon\tau} - e^{-2\varepsilon\tau} - \varepsilon\tau\right) .$$

Hence from Corollary 4.4.6 we have

(4.71) $$\liminf_{T\to\infty}(T\mu_{\varepsilon T,u_T}-\tau) \geq \frac{1}{\varepsilon}(e^{-\varepsilon\tau}-e^{-2\varepsilon\tau}-\varepsilon\tau) .$$

Since the right hand side of (4.70) and the right hand side of (4.71) both go to 0 as ε decreases to 0, we get

$$\limsup_{T\to\infty}|T\mu_{\varepsilon T,u_T}-\tau| \to 0 \quad \text{as} \quad \varepsilon\to 0 . \qquad ////$$

CHAPTER 5

CONCLUSION

In Chapter 3 we have presented some general theorems on the weak convergence of the maximum and on the weak convergence of the point process of upcrossings, for stationary processes satisfying certain conditions. In Chapter 4 we showed that these conditions are fulfilled by standard stationary Gaussian processes, having with probability one continuous sample paths, with covariance function satisfying Berman local and mixing conditions. It follows that for such processes we have, with the usual notation,

$$(5.1) \qquad P[M(T) \leq u_T] \longrightarrow e^{-\tau}$$

and

$$(5.2) \qquad Z_T \xrightarrow{w} Z$$

whenever u_T is chosen so that $T\mu_{u_T} \longrightarrow \tau > 0$. One also obtains the following important result: if $u_T = x/a_T + b_T$, with

$$(5.3) \qquad \begin{aligned} a_T &= (2 \log T)^{1/2} \\ b_T &= (2 \log T)^{1/2} + (2 \log T)^{-1/2} \log\left(\frac{\sqrt{\lambda_2}}{2\pi}\right) , \end{aligned}$$

then $T\mu_{u_T} \longrightarrow e^{-x}$ so that (5.1) becomes

$$(5.4) \qquad P[a_T(M(T)-b_T) \leq x] \longrightarrow \exp(-e^{-x}) .$$

(In virtue of Gnedenko's theorem, this double exponential limit is not too surprising.)

Can Berman conditions be weakened? Let us first consider the local condition. Pickands [1969a, 1969b] has considered SSGP whose covariance function satisfies

(5.5) $$r(t) = 1 - C|t|^{\alpha} + o(|t|^{\alpha}) \quad \text{as} \quad t \to 0$$

for some $0 < C < \infty$ and some $0 < \alpha \leq 2$. Let $(\xi(t); t \in \mathbb{R})$ be such a process. It is easily seen that under (5.5) Belayev continuity condition is satisfied. Hence $(\xi(t); t \in \mathbb{R})$ is equivalent to a process having, with probability one, continuous sample paths. We may assume that $(\xi(t); t \in \mathbb{R})$ is itself such a process. If $\alpha = 2$ then (5.5) is simply Berman local condition. If $0 < \alpha < 2$ then $r''(t)$ does not exist and hence μ_u, the mean number of upcrossings of the level u per unit interval, is infinite. Hence there is no hope for a Poisson limit theorem for the distribution of the number of upcrossings. Even though it is continuous, a typical sample path is so badly behaved that when it crosses the level u, it does so too often. Pickands showed that a Poisson limit distribution is still appropriate if one counts only the significant upcrossings, the ε-upcrossings defined in Chapter 3. His main results are a limiting expression for $\mu_{\varepsilon,u}$, a Poisson limit theorem for the distribution of the number of ε-upcrossings and a double exponential limit distribution for the (suitably normalized) maximum of the process. More specifically, Pickands [1969a,1969b] proved the following results:

(P1) If (5.5) holds for some $0 < C < \infty$ and some $0 < \alpha \leq 2$, then for all small enough $\varepsilon > 0$

(5.6) $$\mu_{\varepsilon,u} \sim \frac{C^{1/\alpha} H_{\alpha}}{\sqrt{2\pi}} u^{2/\alpha - 1} e^{-u^2/2} \quad \text{as} \quad u \to \infty$$

where $\mu_{\varepsilon,u}$ is, as before, the mean number of ε-upcrossings of the level u per unit time interval and where H_{α} is a constant depending only on α. (This constant is given by

(5.7) $$H_\alpha = \lim_{T\to\infty} \frac{1}{T} \int_0^\infty e^s P[\sup_{0\le t\le T} Y(t) > s] ds$$

where $Y(t)$ is a nonstationary Gaussian process with means and covariances given by

$$E[Y(t)] = -|t|^\alpha , \quad \mathrm{cov}[Y(t_1), Y(t_2)] = |t_1|^\alpha + |t_2|^\alpha - |t_2 - t_1|^\alpha ,$$

and it satisfies $0 < H_\alpha < \infty$).

(P2) If, in addition, Berman mixing condition holds, and if u_T is chosen so that $T\mu_{\varepsilon, u_T} \to \tau > 0$, then

$$\lim_{T\to\infty} P[Z_T^{(\varepsilon)}((0,1]) = k] = \frac{e^{-\tau}\tau^k}{k!} , \quad k = 0,1,2,\ldots$$

where $Z_T^{(\varepsilon)}$ is the point process of ε-upcrossings corresponding to u_T.

(P3) If

$$a_T = (2 \log T)^{1/2}$$

and if

(5.8) $$b_T = (2 \log T)^{1/2} + (2 \log T)^{-1/2}((\tfrac{1}{\alpha} - \tfrac{1}{2}) \log \log T + \log[(2\pi)^{-1/2} 2^{(1-\alpha)/\alpha} C^{1/\alpha} H_\alpha])$$

then $P[a_T(M(T) - b_T) \le x] \to \exp(-e^{-x})$ as $T \to \infty$.

Observe that the right hand side of (5.6) does not depend on ε. Hence for all (small enough) $\varepsilon_1, \varepsilon_2 > 0$, we have

$$\mu_{\varepsilon_1, u} \sim \mu_{\varepsilon_2, u} \quad \text{as} \quad u \to \infty .$$

Moreover in the case $\alpha = 2$ we have, in addition,

$$\mu_{\varepsilon,u} \sim \mu_u \quad \text{as} \quad u \to \infty \, .$$

This follows from the fact that condition R1 holds with both $\mu_{\varepsilon,u}$ and μ_u. If we combine this with Rice's formula we get

$$H_2 = 1/\sqrt{\pi} \, .$$

(This can also be obtained from (5.7)). In particular this tells us that (5.8) reduces to (5.3) when $\alpha = 2$, as one would expect.

Pickands' proofs are analog to the proofs of Cramer [1966], Qualls [1967] and Berman [1971a]. The ideas are similar but the computations are much more complicated. (Let us mention that Qualls and Watanabe [1972] obtained the same results under slightly more general assumptions, namely that the covariance function satisfies

$$r(t) = 1 - H(t)|t|^{\alpha} + o(H(t)|t|^{\alpha}) \quad \text{as} \quad t \to 0$$

where $H(t)$ satisfies $\lim_{t\to 0} \frac{H(ct)}{H(t)} = 1$ for all $c > 0$).

Again all these results can be obtained via the theorems of Chapter 3. This is done by showing that if $(\xi(t);\, t \in R)$ is a SSGP (having, with probability one, continuous sample paths) whose covariance function satisfies Pickands' local condition (5.5) and Berman mixing condition, then:

(a) The condition R1 holds with $\psi(u)$ equal to the right hand side of (5.6).

(b) The condition R2 holds with $\psi(u)$ as in (a) and with $q_a(u) = au^{-2/\alpha}$.

(c) For each $a > 0$, the condition $D_c(u_T)$ holds with respect to the family $q_T = q_a(u_T) = au_T^{-2/\alpha}$, whenever u_T is chosen so that $T\iota(u_T) \to \tau > 0$, ψ as in (a).

(d) The condition $D_c'(u_T)$ holds with ψ as in (a), whenever u_T is chosen so that $T\psi(u_T) \to \tau > 0$.

These results are stated in Leadbetter [1980] and the proofs should appear in Lindgren and Rootzen (in preparation for Institute of Statistics Mimeo Series, University of North Carolina at Chapel Hill).

Let us now turn our attention to the mixing condition. So far we have always assumed Berman mixing condition

$$r(t)\log t \to 0 \quad \text{as} \quad t \to \infty . \tag{5.9}$$

As mentioned in Section 1.1, if

$$r(t)\log t \to \gamma > 0 \quad \text{as} \quad t \to \infty , \tag{5.10}$$

then (5.2) does not hold. Thus (5.9) is close to being a necessary condition for (5.2) and also (5.1) and (5.4) to hold. Berman actually showed that (5.9) can be replaced by

$$\int_0^\infty r^2(t)dt < \infty . \tag{5.11}$$

Conditions (5.9) and (5.11) are not comparable; some covariance functions satisfy (5.9) but not (5.11) and some satisfy (5.11) but not (5.9). Mittal [1979] showed that the condition

$$\Lambda\{t:\ 0 \le t \le T;\ |r(t)\log t| > f(t)\} = o(T^\beta) \quad \text{as} \quad T \to \infty , \tag{5.12}$$

for some $0 \le \beta < 1$ and some $f(t) = o(1)$ as $T \to \infty$, is strictly weaker than (5.9) or (5.11), and that combined with

$$(5.13) \qquad r(t) \to 0 \quad \text{as} \quad t \to \infty$$

it is a sufficient condition for (5.4) to hold. (Λ is, as usual, the Lebesgue measure). It is our conjecture that (a), (b), (c), (d) above are still true if Berman mixing condition is replaced by Mittal mixing condition ((5.12) combined with (5.13)).

REFERENCES

Belayev, Yu. K. [1960]. Local properties of the sample functions of stationary Gaussian processes. *Th. Prob. Appl.* 5, 117-119 (English translation).

Belayev, Yu. K. [1961]. Continuity and Hölder's conditions for sample functions of stationary Gaussian processes. *Proc. Fourth Berk. Symp. on Math. Stat. and Probability* 2, 23-33.

Belayev, Yu. K. [1966]. On the number of intersections of a level by a Gaussian stochastic process I. *Th. Prob. Appl.* 11, 106-113 (English translation).

Belayev, Yu. K. [1967]. On the number of intersections of a level by a Gaussian stochastic process II. *Th. Prob. Appl.* 12, 392-404 (English translation).

Berman, S. M. [1964]. Limit theorems for the maximum term in stationary sequences. *Ann. Math. Statist* 35, 502-516.

Berman, S. M. [1971a]. Asymptotic independence of the number of high and low level crossings of stationary Gaussian processes. *Ann. Math. Statist.* 42, 927-945.

Berman, S. M. [1971b]. Maxima and high level excursions of stationary Gaussian processes. *Trans. Amer. Math. Soc.* 160, 67-85.

Bulinskaya, E. V. [1961]. On the mean number of crossings of a level by a stationary Gaussian process. *Th. Prob. Appl.* 6, 435-438 (English translation).

Cramer, H. [1966]. On the intersections between the trajectories of a normal stationary process and a high level. *Arkiv For Mat.* 6, 337-349.

Cramer, H. and Leadbetter, M. R. [1965]. The moments of the number of crossings of a level by a stationary normal process. *Ann. Math. Statist.* 36, 1656-1663.

Cramer, H. and Leadbetter, M.R. [1967]. *Stationary and related stochastic processes.* Wiley, New York.

Davies, P. L. and Dowson, D. C. [1975]. On the differentiability of a class of stationary Gaussian processes. *Stoch. Proc. Appl.* 3, 283-286.

Davis, R. A. [1979]. Maxima and minima of stationary sequences. *Ann. Prob.* 3, 453-460.

Dobrushin, R. L. [1960]. Properties of sample functions of a stationary Gaussian process. *Th. Prob. Appl.* 5, 120-122 (English translation).

Fisher, R. A. and Tippett, L. H. C. [1928]. Limiting forms of the frequency distribution of the largest or smallest member of a sample. *Proc. Camb. Phil. Soc.* 24, 180-190.

Fréchet, M. [1927]. Sur la loi de probabilité de l'écart maximum. *Ann. Soc. Polonaise de Math.* 6, 93.

Gnedenko, B. V. [1943]. Sur la distribution limite du terme maximum d'une série aléatoire. *Ann. Math.* 44, 423-453.

Gnedenko, B. V. and Kolmogorov, A. N. [1954]. *Limit distributions for sums of independent random variables.* Addison-Wesley, New York.

Geman, D. [1972]. On the variance of the number of zeros of a stationary Gaussian process. *Ann. Math. Statist.* 3, 977-982.

Grenander, U. and Rosenblatt, M. [1957]. *Statistical analysis of stationary time series.* Wiley, New York.

Grandell, J. [1977]. Point processes and random measures. *Adv. Appl. Prob.* 9, 502-526.

deHaan, L. [1976] Sample extremes: an elementary introduction. *Statist. Neerlandica* 30, 161-172.

Hunt, G. A. [1951]. Random fourier transforms. *Trans. Amer. Math. Soc.* 71, 38-69.

Itô, K. [1964]. The expected number of zeros of continuous stationary Gaussian processes. *J. Math. Kyoto Univ.* 3, 2, 207-216.

Ivanov, V. A. [1960]. On the average number of crossings of a level by sample functions of a stochastic process. *Th. Prob. Appl.* 5, 319-323 (English translation).

Kallenberg, O. [1976]. *Random measures.* Academic Press, London and New York.

Leadbetter, M. R. [1972]. Point processes generated by level crossings. From *Stochastic point processes*, Peter Lewis, ed., Wiley Interscience, New York.

Leadbetter, M. R. [1974]. On extreme values in stationary sequences. *Z. Warh. verw. Geb.* 28, 289-303.

Leadbetter, M. R. [1976]. Weak convergence of high level exceedances by a stationary sequence. *Z. Warh. verw. Geb.* 34, 11-15.

Leadbetter, M. R. [1978]. Extreme value theory under weak mixing conditions, M.A.A. Studies in Mathematics 18, *Studies in Probability Theory*, Murray Rosenblatt, ed.

Leadbetter, M. R. [1980]. Extreme value theory for continuous parameter stationary processes. Institute of Statistics Mimeo Series # 1276, University of North Carolina at Chapel Hill.

Leadbetter, M. R. and Cryer, J. D. [1965]. The variance of the number of zeros of a stationary normal process. *Bull. Amer. Math. Soc.* 71, 561-563.

Leadbetter, M. R., Lindgren, G. and Rootzén, H. [1978]. Conditions for the convergence in distribution of maxima of stationary normal processes. *Stoch. Proc. Appl.* 8, 131-139.

Leadbetter, M. R., Lindgren, G. and Rootzen, H. [1979]. Extremal and related properties of stationary processes. Part I: extremes of stationary sequences, University of Umea Technical Report, Umea, Sweden.

Loynes, R. M. [1965]. Extreme values in uniformly mixing stationary stochastic processes. *Ann. Math. Statist.* 36, 993-999.

Lukacs, E. [1964]. *Characteristic functions*, London, C. Griffin, Griffin's statistical monographs and courses, No. 5.

DeMises [1939]. La distribution de la plus grande de n valeurs. *Revue Math. Union Interbaltique* 1, 1, 141-160.

Mittal, Y. [1979]. A new mixing condition for stationary Gaussian processes. *Ann. Prob.* 7, 727-730.

Mittal, Y. and Ylvisaker, D. [1975]. Limit distributions for maxima of stationary Gaussian processes. *Stoch. Proc. Appl.* 3, 1-18.

Pickands, J. [1969a]. Upcrossing probabilities for stationary Gaussian processes. *Trans. Amer. Math. Soc.* 145, 51-73.

Pickands, J. [1969b]. Asymptotic properties of the maximum in a stationary Gaussian process. *Trans. Amer. Math. Soc.* 145, 75-86.

Qualls, C. [1967]. On the limit distribution of high level crossings of a stationary Gaussian process. Ph.D. dissertation, University of California at Riverside.

Qualls, C. and Watanabe, H. [1972]. Asymptotic properties of Gaussian processes. *Ann. Math. Statist.* 43, 580-596.

Rice, S. O. [1945]. Mathematical analysis of random noise. *Bell System Tech. J.* 24, 46-156.

Rosenblatt, M. [1956]. A central limit theorem and a strong mixing condition. *Proc. Nat. Acad. Sci. USA* 42, 43-47.

Steinberg, H., Schultheiss, P. M., Wogrin, C. A. and Zweig, F. [1955]. Short time frequency measurements of narrow-band random signals by means of a zero counting process. *J. Appl. Phys.* 26, 195-201.

Slepian, D. [1962]. The one-sided barrier problem for Gaussian noise, *Bell System Tech. J.* 41, 463-501.

Volkonski, V. A. and Rozanov, Yu. A. [1959]. Some limit theorems for random functions I. *Th. Prob. Appl.* 4, 178-196 (English translation).

Volkonski, V. A. and Rozanov, Yu. A. [1961]. Some limit theorems for random functions II. *Th. Prob. Appl.* 6, 186-197 (English translation).

Watson, G. S. [1954]. Extreme values in samples from m-dependent stationary stochastic processes. *Ann. Math. Statist.* 25, 798-800.

Ylvisaker, D. [1965]. The expected number of zeros of a stationary Gaussian process. *Ann. Math. Statist.* 36, 1043-1046.

Ylvisaker, D. [1966]. On a theorem of Cramer and Leadbetter. *Ann. Math. Statist.* 37, 682-685.