The botanical beauty of random binary trees:

a method for the synthetic imagery of botanical trees

by

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Abstract

We describe a general method for producing computer images of botanical trees. We use techniques from probabilistic analysis to generate random combinatorial trees and then model them as three dimensional geometric trees. We choose modelling functions that are based on results from theoretical biology. By changing the underlying distribution used to generate the random combinatorial trees, we are able to produce images of a wide variety of botanical trees. The bifurcation ratio of Horton's first law has been calculated for several tree species by previous researchers. A new result on the HORTON-STRAHLER number for random binary tries (also presented here) allows the generation of random combinatorial trees with the appropriate bifurcation ratios. We experiment with generating images of several different tree species. We implement our algorithm in both L-systems and PostScript.

Keywords: botanical tree modelling, random binary trees, branching patterns, bifurcation ratio, Horton's law, random tries, HORTON-STRAHLER number, L-systems, PostScript.

Résumé

On présente une méthode générale pour produire à l'ordinateur des images des arbres botaniques. On utilise des techniques probabilistiques pour construire des arbres aléatoires randomisés qui sont modelés comme des arbres geometriques à trois dimensions. On choisit des fonctions de modelage qui sont basés sur la biologie théoretique. En changeant la distribution des arbres aléatoires, on est capable de produire des images de plusieurs types d'arbres botaniques. La proportion de Horton a été déjà calculée par des autres scientifiques pour plusieur éspèces d'arbres. On présente un nouveau résultat pour le nombre de HORTON-STRAHLER qui permet la génération des arbres aléatoires randomisés avec la proportion de Horton desirée. On fait des experiences pour produire des images de plusieurs éspèces d'arbres différentes. Notre algorithme est réalisé en "L-systems" et PostScript.

Acknowledgements

First and foremost, I thank Sue Whitesides for accepting me as her student—*without her supportive supervision, this thesis would not exist.* I thank Luc Devroye for teaching me about the beautiful world of random combinatorial trees. I thank the remaining members of my committee: David Avis, Greg Dudek, and R. Rigelhof; and the external examiner Jules Bloomenthal for taking their time to review my work. I thank Przemek Prusinkiewicz for his inspiring book, stimulating conversations and helpful `vlab` software (thanks also to his team at the University of Calgary). I thank Marty Lechowicz for his valuable discussions on tree biology. I thank M. Guevara from the Department of Physiology for pointing out some key papers concerning bifurcation ratios for lungs.

In terms of authorship, science is rarely an individual endeavor and thus it is only fitting to acknowledge the work of those who have directly influenced this thesis. As described in Chapter 4.2.4, this work arose out of my Master's thesis under the supervision of Luc Devroye. Chapter 2 is joint work with Przemek Prusinkiewicz and Sue Whitesides; Chapters 3 and 4 are joint work with Sue Whitesides; Appendices B and C are joint work with Luc Devroye. Naturally, any errors in this manuscript are my own fault.

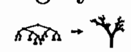
On a financial level, I thank the following groups for their support: NSERC for a 1967 Post-Graduate Scholarship; FCAR pour une bourse de doctorat; the School of Computer Science for my teaching assistantship and my technical reports librarian position; the Computational Geometry Group for student funding from their team grant; Sue Whitesides, Luc Devroye and Przemek Prusinkiewicz for financial travel assistance.

Finally, I thank Zita and Tykeroo for their unfailing love, support and perspective.

Contents

Overview

> ...the entire [botanical] tree is a pure mathematical function...[but] I have never had the opportunity to prove it. (LE CORBUSIER)[1]

This thesis is about trees, many different kinds of trees. It can be seen as an attempt to explore LE CORBUSIER's observation. More specifically, this thesis concerns itself with generating realistic-looking images of botanical trees by computer. Our method is first to generate random combinatorial trees; then to model them as geometric trees that are unions of cylinders; and finally to render these geometric trees by computer imagery techniques to produce realistic-looking images of botanical trees (e.g., ).

By the nature of the approach taken, this thesis is rather interdisciplinary and attempts to synthesize results from the seemingly disjoint fields of probabilistic analysis, computer graphics, and theoretical biology. It is organized in the following way. In Chapter 1 we introduce the problem by way of its motivations and general approaches. In Chapter 2 we explain our basic model and our choice of modelling functions. In Chapter 3 we show how to combinatorially and visually model several species of botanical trees. In Chapter 4 we review previous work and try to show how many of these techniques can be seen as special cases of our general technique. In Chapter 5 we discuss our contributions and propose further work. In Appendix A we discuss the implementation details of our algorithm. In Appendix B we prove a tight asymptotical bound on the Horton-Strahler number for random binary tries, which we used in Chapter 3. A glossary and a table of symbols are also provided.

This thesis contributes to the state-of-the-art in the following ways. In terms of computer graphics, we present a simple yet powerful algorithm that is capable of producing a wide variety of tree-like images. Furthermore, we have consciously postponed the review of previous work until after the presentation of our general model so that the previous work in Chapter 4—the first comprehensive and unified survey—could be presented in a consistent manner. In terms of biology, we uncover the only known technique that allows for the simulation of the combinatorial structure of many natural branching

[1] As reported on p. 178 in LIONNAIS (1971).

patterns such as botanical trees. Finally, we derive the first known results on the HORTON-STRAHLER number for random binary tries.

Chapter 1

Introduction

Long before CAYLEY coined the term *tree* in 1889 to denote a connected, acyclic graph, people have been intrigued by the mathematical structure of botanical trees.

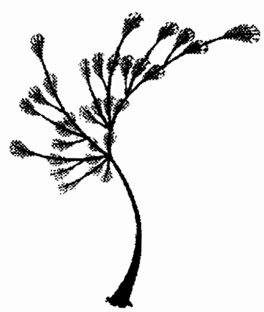


Figure 1.1: *Taken from (*KNUTH 1973A*).*

Our goal is to provide a simple and general method for generating realistic-looking images of botanical trees. Our method is first to generate *random combinatorial trees* (e.g., ), and next to model them as *geometric trees* that are unions of cylinders (e.g., ); finally these geometric trees are *rendered* by computer imagery techniques to produce natural-looking images of *botanical trees* (e.g., ).

We use the term *tree* in the following four distinct ways. A *botanical tree* is defined by the *Concise Oxford Dictionary* as

> a perennial plant with a single woody self-supporting stem or trunk usually unbranched for some distance above ground.

A *combinatorial tree* is a connected, acyclic graph. A *geometric tree* consists of simple three dimensional objects such as cylinders connected together in an acyclic fashion. A *rendered tree* is a two dimensional image intended to look like a botanical tree.

We are not the first to take this basic approach. In 1966, ULAM was the first to attempt to describe tree growth via random branching patterns. In 1971, LEOPOLD simulated random three-dimensional tree branching with "Tinker-Toys" and various production rules (randomized by a deck of playing cards). In 1971, HONDA produced the first computer images of botanical trees. STEVENS (1974) was the first to interpret random binary trees as drawings of botanical trees. Motivated by the similarity of trees and river networks, VIENNOT ET AL. (1989) used the HORTON-STRAHLER number to generate random drawings of trees. Our present approach is an extension and refinement of the probabilistic techniques of KRUSZEWSKI (1994) and DEVROYE AND KRUSZEWSKI (1995).

The synthetic imagery of botanical trees has applications in many diverse fields such as:

- *flight and scene simulation* (e.g., BROOKS ET AL. 1974; GARDNER 1984; WEBER AND PENN 1995);
- *computerized landscaping* (e.g., AONO AND KUNII 1984; DANAHY 1987; DANAHY AND WRIGHT 1988; AGUI ET AL. 1991; LITTLEHALES 1992; PRUSINKIEWICZ ET AL. 1994);
- *film production* (e.g., LUCASFILMLTD 1984; REEVES AND BLAU 1985);
- *graphics experimentation* (e.g., BLOOMENTHAL 1985; OPPENHEIMER 1986;
- *biologic modelling* (e.g., HONDA 1971; HONDA ET AL. 1981; BORCHET AND HONDA 1984, NIKLAS 1986; DE REFFYE ET AL. 1988; PRUSINKIEWICZ AND LINDENMAYER 1990; PRUSINKIEWICZ ET AL. 1996);
- *combinatorial visualization* (e.g., VIENNOT ET AL. 1989; KRUSZEWSKI 1993; KRUSZEWSKI 1994; DEVROYE AND KRUSZEWSKI 1995).

Chapter 2

Our Basic Methodology

We first describe a general model of a geometric tree. We then describe our choices of modelling functions. In Chapter 4, we will show how many previous techniques can be viewed as special cases of this general model.

2.1 A general geometric model

As previously mentioned, we model botanical trees as combinatorial trees which in turn are visualized as geometric trees whose renderings resemble botanical trees. We now give the main details of our algorithm but first we need to introduce some more terminology.

A *combinatorial tree* is a *rooted binary tree* consisting of nodes connected by edges such that each node has at most one left and at most one right *child* node. A *geometric tree* is a three dimensional object consisting of cylinders (of varying radii and lengths) attached together in an acyclic fashion. Two cylinders are attached together by identifying center points of their end disks. To fully describe a geometric tree, the dimensions of the cylinders, the attachment information which center points to identify and the angles determined by the long axes of the cylinders must be given.

2.1.1 From botanical trees to combinatorial trees

For our purposes, a *botanical tree* consists of *branch segments* and sometimes *leaves*. A branch segment is a section of the tree between two consecutive bifurcations. In addition, the *base* of the tree, which starts at the ground and ends at the first bifurcation, is also a branch segment. Thus, underground structures such as roots will not be considered. For example, Figure 2.1 shows a photo of a tree taken in front of the McConnell Engineering Building at McGill University, Montreal in the late spring of 1996.

The combinatorial representation of a botanical tree is typically obtained as follows (e.g., see MacDonald 1983 for an overview). Following Honda

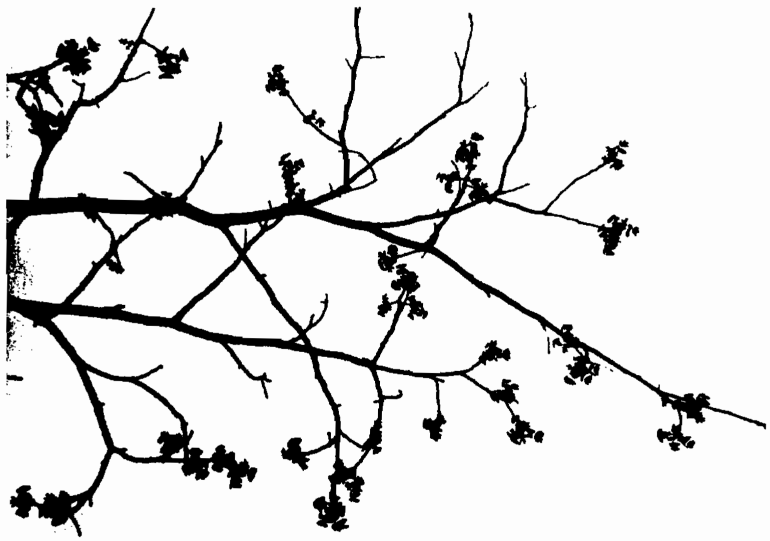


Figure 2.1: *Sections of a botanical tree.*

(1971), all pieces of the tree such as twigs, boughs and trunk are abstracted to straight branches of zero width. The base of the tree is mapped to the root of a combinatorial tree. The branch segments originating from the base are mapped to children of the root of the combinatorial tree, and so forth. Thus far in our research, we consider only the case of a branch segment splitting into exactly two subsequent branch segments and consequently we work with rooted binary trees. This common technique in biology arose from hydrology where river systems are mapped to combinatorial trees (e.g., see HORTON 1945). For example, Figure 2.2 shows a combinatorial representation of part of the botanical tree from Figure 2.1.

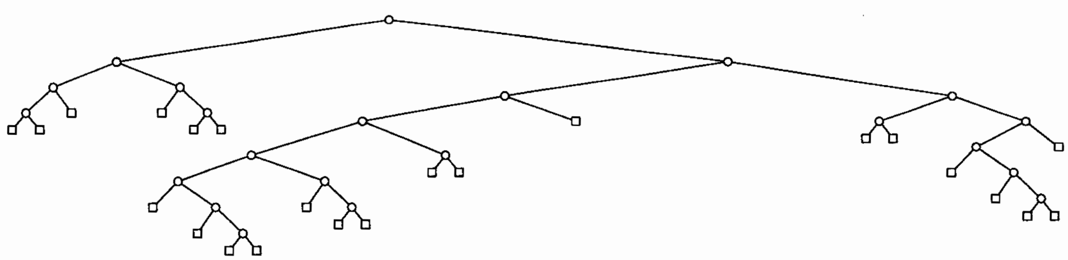


Figure 2.2: *A combinatorial representation of the botanical tree in Figure 2.1.*

In our work, we use the following restricted set of combinatorial trees. For us, a *combinatorial tree* is a *rooted tree* such that each node in the tree has either zero or two child nodes. Nodes with two children are called *internal nodes* (denoted by ○) and nodes with zero children are called *external nodes* (denoted by □) For example, Figure 2.3 shows the five possible rooted trees with exactly three internal and four external nodes. These external nodes can

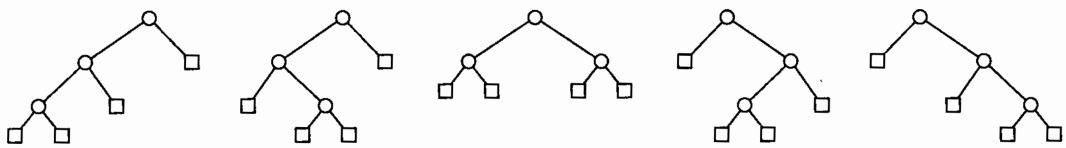


Figure 2.3: *All possible rooted binary trees with three internal nodes and four external nodes.*

be seen to correspond to the ends of the "working pipes" in the *pipe model* proposed by SHINOZAKI ET AL. (1964A) (and extended by SHINOZAKI ET AL. 1964B—see also Chapter 16 of MACDONALD 1983). Each unit pipe supports a "unit amount of photosynthetic organs," e.g., a constant number of leaves. For us, each pipe connects a leaf to the tree's trunk, i.e., the pipe starts at a corresponding leaf and goes down the entire length of the tree to its trunk. Thus, internal cylinders can be seen as bundles of corresponding pipes, much like individual wires in a thick communications cable. We reserve the word *root* for combinatorial meaning, i.e., the only node with no parent.

Let T be a combinatorial tree of this type. For a node $u \in T$, we denote by $|u|$ the number of nodes in the subtree rooted at u. For example, if u is an external node then $|u| = 1$; if u is an internal node then $|u| \geq 3$. This is similar to the ordering scheme by SHREVE (1966) in which only the number of external nodes in the subtrees are counted. Let both $||T||$ and n denote the number of internal nodes in T. It is easy to see the following well-known fact.

FACT 2.1 *Any binary tree T with n internal nodes has exactly $n+1$ external nodes.*

PROOF. We argue by induction on n.
BASIS. For $n = 0$, T is by definition, a single external node $\square$.
INDUCTIVE STEP. We assume that the hypothesis is true for all trees with fewer than n internal nodes and proceed to prove it for each tree T with exactly n internal nodes. Let T_L and T_R be the left and right subtrees of the root of T. By the inductive hypothesis, T_L and T_R have $||T_L|| + 1$ and $||T_R|| + 1$ external nodes, respectively. Thus, the number of external nodes in T is

$$\begin{gathered} ||T_L|| + 1 + ||T_R|| + 1 \\ (||T_L|| + ||T_R||) + 2 \\ n - 1 + 2 \\ n + 1 \,. \end{gathered}$$

Therefore, as the hypothesis holds for a tree consisting of a single external node and for all trees with less than n internal nodes, we conclude by induction that it holds for all trees with n internal nodes. $\square$

We denote by $d(u)$ the *depth* of u, that is, the number of nodes on the path in T from the root to u (including the root and u). For example, Figure 2.4 shows the tree from Figure 2.2 with the node depths added. We note that this

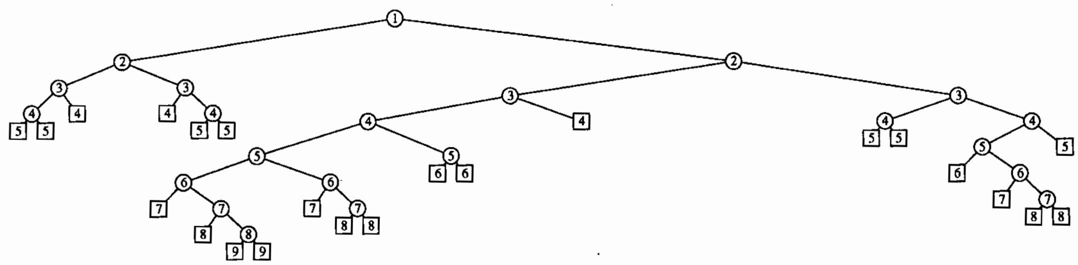
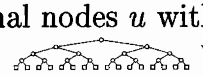


Figure 2.4: *Nodes are labelled by depth.*

is identical to the *botanical tree ordering scheme* proposed by WEIBEL (1963). The *height* $H(T)$ of the tree T is the maximum node depth in T. For example, the tree in Figure 2.4 has height nine. For any node u, $H(T) - d(u) + 1$ gives the *Horsfield ordering scheme* (HORSFIELD ET AL. 1971). A *complete tree* with k levels (and thus height k) is a combinatorial tree such that all internal nodes u, with depth $d(u) < k - 1$, have two internal nodes as children and all internal nodes u with depth $d(u) = k - 1$ have two external nodes as children (e.g., ).

2.1.2 Geometric trees

In 1971, HONDA introduced the following geometric model which all subsequent research essentially either explicitly or implicitly adhere to (Honda's model lacks only non-zero cylinder radii). We will use this geometric model together with our combinatorial model as a general model of botanical trees.

Let a *geometric tree* $\mathcal{T}$ consist of n *internal cylinders* each having two child cylinders and $n + 1$ *external cylinders* each having no children. Two cylinders are attached together by identifying the center points of their end disks. A cylinder μ has a *radius* $R(\mu)$ and a *length* $L(\mu)$ (e.g., see Figure 2.5).[1]

[1] N.B. Cylinders can also be tapered; however, for simplicity we choose to view this as a rendering issue.

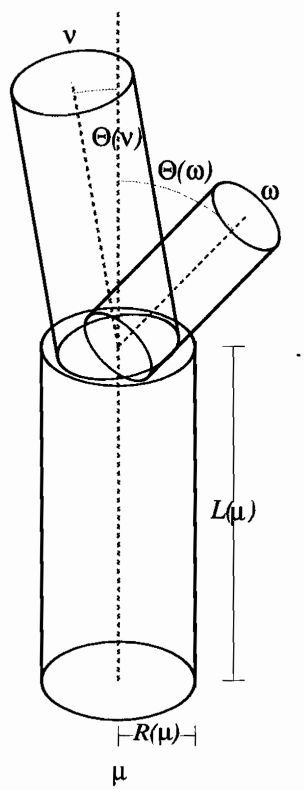


Figure 2.5: *Layout of parent cylinder μ with child cylinders ν and ω.*

Each internal cylinder μ with child cylinders ν and ω is associated with a plane Π_μ, namely the plane that contains the axis of the cylinder and the axes of its child cylinders. According to Honda (and simple observation), these three axes should generally be coplanar in geometric models. The plane to be associated with the root cylinder is arbitrarily chosen. Once the plane Π_μ has been chosen, the axes of the child cylinders of μ can be specified by giving the angles they form in Π_μ with the axis of the parent. The axis of ν deviates from the axis of μ by a *deviation angle* of $\Theta(\nu)$ degrees. The plane Π_ν (to be associated with ν) is obtained from the plane Π_μ by rotating Π_ν about the axis of ν by a *divergence angle* of $\alpha(\nu)$ degrees (e.g., Figure 2.6).

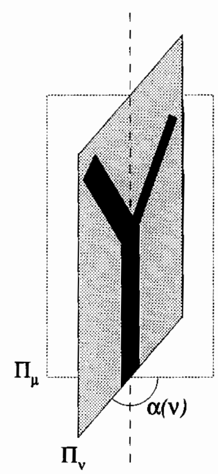


Figure 2.6: *Plane Π_ν diverges from parent plane Π_μ by angle $\alpha(\nu)$.*

2.1.3 From combinatorial trees to geometric trees

Each node u in combinatorial tree T is represented as a cylinder μ in geometric tree $\mathcal{T}$ (e.g., see Figure 2.7). To fully specify a geometric tree arising from a combinatorial tree, for each node one must give *modelling functions* that compute the corresponding cylinder's dimensions and location in space.

Thus, most methods can be seen to differ only in *how the underlying combinatorial trees are generated and what modelling functions are used.* We will compare our method to others in Chapter 4. Accordingly, there are two problems to solve: how to generate the combinatorial tree and how to develop modelling functions to produce the geometric trees. As artists have been studying these same problems for centuries, we feel it useful to first examine some of their ideas.

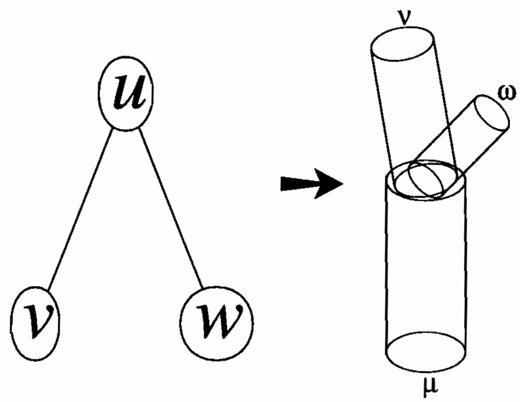


Figure 2.7: *Nodes u, v, and w are modelled as cylinders μ, ν, and ω.*

2.2 Modelling heuristics

Painters and biologists have been modelling botanical trees for centuries. In this section, we examine several references which have indirectly influenced our choice of modelling functions.

2.2.1 From painting

In his book *Botany for Painters* (circa 1513), LEONARDO DA VINCI established rules to guide artists in representing trees. Although DA VINCI attempted to give scientific explanations as to why things look as they do, his observations are first and foremost concerned with how things *should* look. This is often the approach of computer graphics. That is, often one is concerned with developing a model which produces convincing synthetic images rather than actually articulating how nature works.

The following is a collection of DA VINCI's maxims (numbered NOTES) on drawing trees taken from his book *Botany for Painters.* As we will be referring to them from time to time, we list them here. We have placed our own titles (in SMALL CAPS) on how the notes relate to botanical tree drawing.

394 RADIUS AS RECURSIVE FUNCTION

All the branches of a tree at every stage of its height when put together are equal in thickness to the trunk.

396 RELATIONSHIP OF CYLINDER SIZE AND DEVIATION ANGLE

... The branches of trees or plants have a twist wherever a minor branch is given off; and this giving off the branch forms a fork; this said fork occurs

between two angles, of which the largest will be that which is on the side of the larger branch, and in proportion, unless accident has spoilt it.

397 RELATIONSHIP BETWEEN CYLINDER SIZE AND NODE DEPTH

... The lower shoots on the branches of trees grow more than the upper ones and this occurs only because the sap that nourishes them, being heavy, tends downwards more than upwards; ...

400 CYLINDER LAYOUT

The beginning of the ramification [the shoot] always has the central line [axis] of its thickness directed to the central line [axis] of the plant itself.

403 DEVIATION ANGLE AS FUNCTION OF NODE DEPTH

The plants which spread very much have the angles of the spaces which divide their branches more obtuse in proportion as their point of origin is lower down; that is, nearer to the thickest and oldest portion of the tree. Therefore in the youngest portions of the tree the angles of ramification are more acute.

404 HELIOTROPISM

The tips of the boughs of plants [and trees], unless they are borne down by the weight of their fruits, turn towards the sky as much as possible....

404 DIVERGENCE ANGLE AND LEAF ORIENTATION

... The rule of the leaves produced on the last shoot of the year will be that they will grow in a contrary direction on the twin branches; that is, that the insertion of the leaves turns round each branch in such a way as the sixth leaf above is produced over the sixth leaf below, and the way they turn is that if one turns towards its companion to the right, the other turns to the left ...

405 TROPISM AS A FUNCTION OF NODE DEPTH

The lowest branches of those trees which have large leaves and heavy fruits, such as nut-trees, fig-trees, and the like, always droop towards the ground....

407 DEVIATION ANGLE

The lowest branches, after they have formed the angle of their separation from the parent stem, always bend downwards so as not to crowd against the other branches which follow them on the same stem ...

408 DEVIATION ANGLE

... The main branches of the lower part bend down more than those above, so as to be more oblique than those upper ones, and also because they are larger and older, and to seek light and flee shadow.

409 HELIOTROPISM

In general almost all the upright portions of trees curve somewhat turning the convexity towards the south; and their branches are longer and thicker and more abundant towards the south than towards the north. ...

410 OVERALL TREE SHAPE

The cherry-tree is of the character of the fir-tree as regards its ramification which is placed in stages round its main stem; and its branches spring, four

> *or five or six (together), opposite each other; and the tips of the topmost shoots form a pyramid from the middle upwards; and the walnut and oak form a hemisphere from the middle upwards.*
>
> 417 LEAF SIZE
>
> *... its largest leaves are on the thickness part of the stem and the smallest on the slenderest part, that is, towards the top...*

These notes are very insightful. For example, NOTE 400 already reaffirms the choice of coplanar parent and sibling axes and NOTE 404 anticipates the deviation angle. For a more modern reference we have found COLE's book *The artistic anatomy of trees* to be an informative guide.

2.2.2 From biology

SUGDEN (1984) provides a good introduction to the biology of trees. Biologists, HORN (1971), HALLÉ ET AL. (1978), and WILSON (1984) explained how the overall shape of a tree is a function of its biology in terms of overall tree architecture. MATTHECK (1991) and (NIKLAS 1992) both explained tree growth from a mechanical perspective. Finally, FARRAR (1995) is a good guide for identifying trees in Canada.

2.3 Our parameterization

We now present our choice of modelling functions. Whenever possible, we rely on results from theoretical biology.

2.3.1 Radius

In NOTE 394, LEONARDO DA VINCI suggested that the cross-sectional area of the parent cylinder is equal to the sum of the cross-sectional areas of its child cylinders. That is, the radius of cylinder μ with children ν and ω is:

$$\pi(R(\mu))^2 = \pi(R(\nu))^2 + \pi(R(\omega))^2.$$

DA VINCI's intuition is essentially correct; however, physical measurements by MURRAY (1927) showed that in practice the correct exponent in the formula is about 2.49 for large trees (rather than 2) and about 3 for small trees.[2] As the majority of our trees are "small," we currently use value 3 for our exponent.

[2] According to MURRAY (1926A), blood arteries in cats can also be formulized with exponent 3. Cf. SUWA ET AL. (1963) and MACDONALD (1983).

That is, we assign constant radius $c_R > 0$ to each external cylinder and derive the following recursive formula for the radius of cylinder μ:

$$R(\mu) = \begin{cases} c_R & \text{if } |u| = 1; \\ (R(\nu)^3 + R(\omega)^3)^{1/3} & \text{otherwise.} \end{cases}$$

Figure 2.8 plots the range of values of $R(\mu)$ if we set $R(\omega) = cR(\nu)$, for $0 \le c \le 1$ and $R(\nu) = 1$ for various exponents.

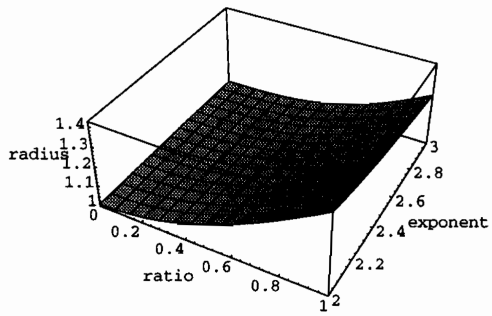


Figure 2.8: *The effect of exponent on radius.*

2.3.2 Length

Based on findings by SUWA ET AL. (1963) for arterial systems, we propose length as a function of radius, i.e.,

$$L(\mu) = c_{L_1} R(\mu)^{c_{L_2}},$$

where c_{L_1} and $c_{L_2} > 0$ are constants. For simplicity, we currently let $c_{L_2} = 1$.

2.3.3 Deviation angle

In a botanical tree it is a common observation (e.g., NOTE 396 of DA VINCI) that for two sibling branch segments the larger sibling deviates less from the parent branch than its smaller sibling; i.e., $\Theta(\nu) \le \Theta(\omega)$ if $R(\nu) \ge R(\omega)$. In 1926, MURRAY formalized this observation for blood vessels by applying the physiological principle of minimum work to determine a mathematical formula for the deviation angles of arteries.[3] In 1927, he extended this theory to botanical trees. We use his formula for deviation angle:

$$\Theta(\nu) = \arccos\left(\frac{R(\mu)^4 + R(\nu)^4 - R(\omega)^4}{2R(\mu)^2 R(\nu)^2}\right).$$

[3]This is also reported on pp. 948–957 by THOMPSON (1942) and on pp. 116–123 by STEVENS (1974). See MACDONALD (1983) for a more up to date survey of similar results.

It is interesting to note that in this formula the deviation angle does not depend on cylinder length.

Figure 2.9 shows the range of $\Theta(\nu)$. The horizontal axis shows the ratio $0 \leq c \leq 1$. The bottom plot shows for $R(\nu) = cR(\mu)$ and the top show shows for $R(\mu) = cR(\nu)$.

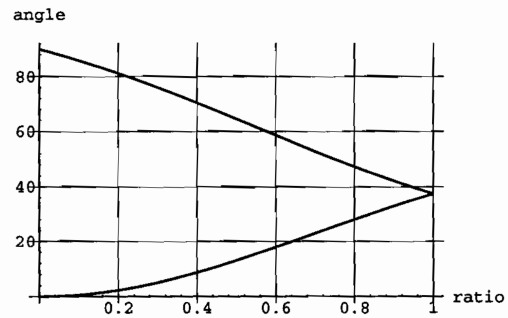


Figure 2.9: *Radius ratio versus deviation angle.*

2.3.4 Divergence angle

The late nineteenth century *Schimper-Braun law* states that the divergence angle α for any botanical plant must be one of the following angles:

$$360° \times \left\{\frac{1}{2}, \frac{1}{3}, \frac{2}{5}, \frac{3}{8}, \frac{5}{13}, \frac{8}{21}, \frac{13}{34}, \ldots\right\} \qquad (\text{JEAN } 1978).$$

The numerators and denominators of the fractions form two *Fibonacci sequences*, as each new numerator and denominator is created by adding the two previous numerators and denominators respectively. The limit of this sequence of fractions is $\left(\frac{2}{\sqrt{5}+1}\right)^2$. For simplicity, we also let

$$\alpha(\mu) = 137.5° \simeq \left(\frac{2}{\sqrt{5}+1}\right)^2 \times 360°$$

for all $\mu \in \mathcal{T}$. In Chapter 4, we will see that almost all other techniques also use this limit.

2.4 Rendering geometric trees

Once the geometric tree has been constructed, it must be rendered to better resemble a botanical tree. First, environmental influences called *tropisms* deform the layout of the original geometric tree. Next, the three dimensional object must be projected onto a particular two dimensional plane whose silhouette must somehow be created. That is, the cylinders must be drawn as to resemble branch segments. For added realism bark is simulated by texture maps and finally, leaves are added.

In 1985, BLOOMENTHAL addressed this specific problem. That is, he assumed that someone else would invent an algorithm which produces acceptable geometric trees and thus concerned himself with how to render these geometric trees as convincing images.[4] The computer program `vlab` (see Appendix A for a complete description) utilizes many of his techniques. Whereas most previous approaches represent branches as polygons or simple cylinders, he formed sophisticated model of the physical branches as generalized cylinders, (i.e., a spline with disks, e.g., the human backbone) with very smooth joins at the ramiforms (where the branches meet, e.g., see Figure 2.10). BLOOMENTHAL applied tensions to these splines to simulate twisted and gnarled tree branches. He drew very realistic-looking maple trees in part by texture mapping the bark from a "bump map" which comes from an actual tree. Finally, the leaves were drawn according to a simple hinged polygon that is texture mapped with a photograph of an actual leaf. For other sophisticated models of leaves, see OPPENHEIMER (1986); DEMKO ET AL. (1985); VIENNOT ET AL. (1989); PRUSINKIEWICZ AND LINDENMAYER (1990); NORTON ET AL. (1990); WEJCHERT AND HAUMANN (1991); and HAUMANN ET AL. (1991).

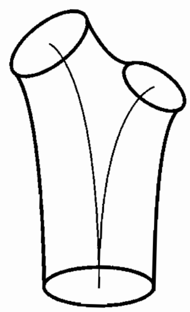


Figure 2.10: *A smooth ramiform (*BLOOMENTHAL 1985*).*

Figure 2.11 shows a two dimensional rendering of the geometric tree which results from the combinatorial tree in Figure 2.2. Here, we render the internal cylinders as branch segments and the external cylinders as leaves (the actual rendering is done in PostScript). From now on in this chapter external cylinders will not be rendered. Hence, images for the most part will be leafless.

[4]N.B. See also FOLEY ET AL. (1996) for a general survey of computer graphic techniques.

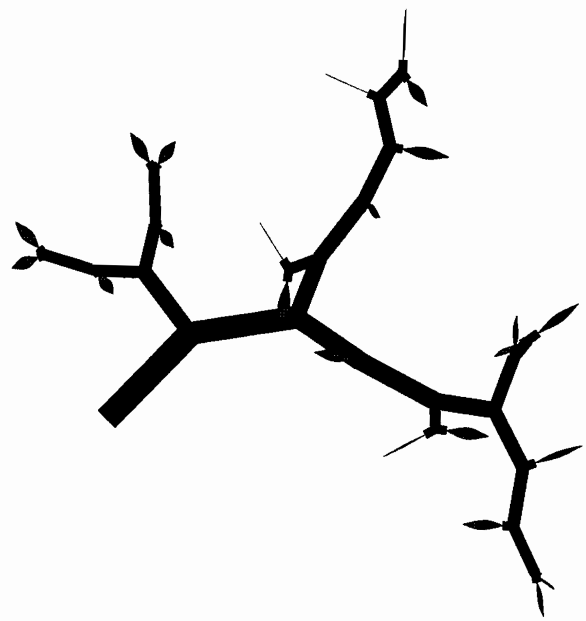


Figure 2.11: *A two dimensional rendered tree.*

2.5 Combinatorial split trees

The previous section described how we model combinatorial trees as geometric trees. We presented an example of a combinatorial tree that was generated by examining an actual botanical tree. In this section, we discuss our method of generating combinatorial trees.

2.5.1 Generating split trees

DEVROYE (1986B) introduced random split trees as a probabilistic model of random binary search trees. A *split tree* $T(n)$ with exactly n internal nodes and $n+1$ external nodes, for $0 < X < 1$, is recursively defined as follows:

$$T(n) = \begin{cases} \square, & \text{if } n = 0; \\ \underset{T(\lfloor nX \rfloor) \qquad T(n-1-\lfloor nX \rfloor)}{\overset{\bigcirc}{/ \qquad \backslash}}, & \text{otherwise.} \end{cases}$$

At each recursive call, X may or may not maintain its previous value (e.g., in Chapter 2.5.2, we discuss X as a random variable).

A split tree with $n = 2^k - 1$ internal nodes and X fixed at 0.5 produces a perfectly balanced tree (i.e., complete binary tree) with exactly $k+1$ levels as in Figure 2.12. We call this complete tree a *deterministic symmetric tree.*

Choosing constant $X \neq 0.5$ results in a *deterministic asymmetric tree* (e.g., see Figure 2.13). Again, this model can be thought of as a variation of the

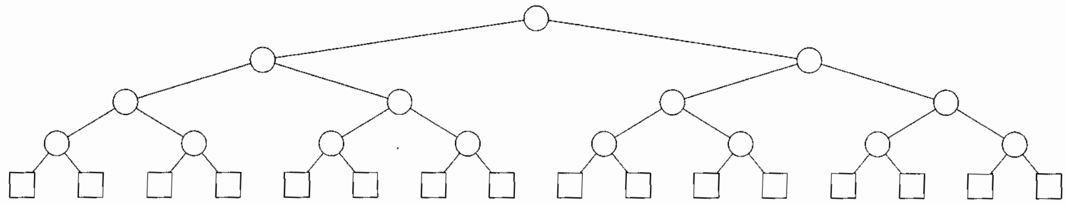


Figure 2.12: *A deterministic symmetric combinatorial tree.*

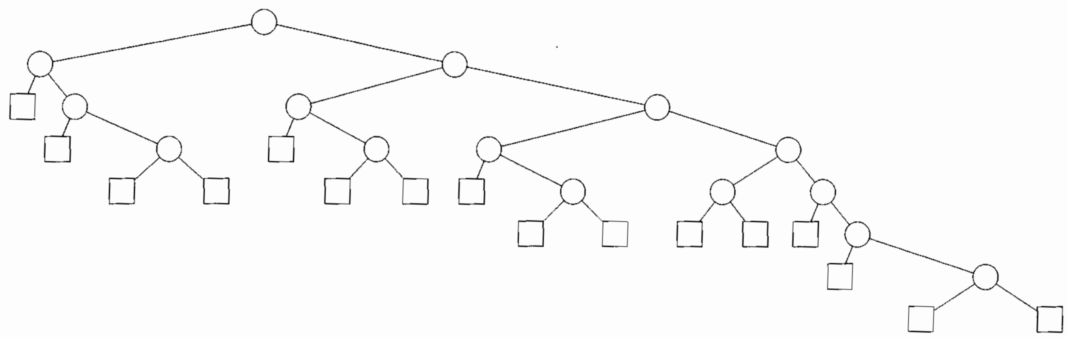
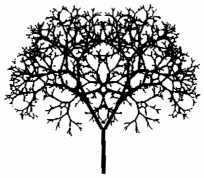


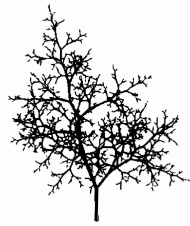
Figure 2.13: *A deterministic asymmetric combinatorial tree with* $X = 0.25$ *and* $n = 15$.

pipe model where the resources are deterministically partitioned.

Table 2.1 shows the result of decreasing X (cf. the renderings shown on pages 146–147 of MacDonald 1983). Each tree consists of 2000 internal cylinders. Note however that all rendered trees have been scaled to fit into the same-sized bounding box.



$X = \frac{1}{2}$



$X = \frac{1}{3}$

$X = \frac{1}{4}$

$X = \frac{1}{5}$

$X = \frac{1}{6}$

$X = \frac{1}{7}$

$X = \frac{1}{8}$

$X = \frac{1}{9}$

Table 2.1: *Renderings of deterministic asymmetric trees for decreasing but constant X.*

We note that for these rendered trees neither the modelling functions nor the rendering method from the previous section have been changed—only the underlying combinatorial trees have been changed. Thus it is interesting how the model goes from *dichotomous branching* (the parent segment splits into two more or less equal sibling segments) to *monopodial branching* (one sibling is seen to be an extension of the parent branch whereas the other sibling is seen to be an offshoot of the parent branch).

As this is a three dimensional model, we note that various viewpoints can be chosen for the rendering. For example, Table 2.2 shows the tree with $X = \frac{1}{2}$ from Table 2.1 from various viewpoints.

Furthermore, we note that the *different flow rate* model by HONDA ET AL. (1981) can be simulated in the following way. Let N be the time variable; let $n = 2^N$ be the number of internal nodes; let $X = \frac{1}{2f}$ where $0 \leq f \leq 1$ is the "flow rate"; and let T be the resulting split tree. Let T' be the tree that remains after all nodes $u \in T$ whose depth $d(u) > N$ from T. The resulting combinatorial tree T' is now a "flow rate tree".

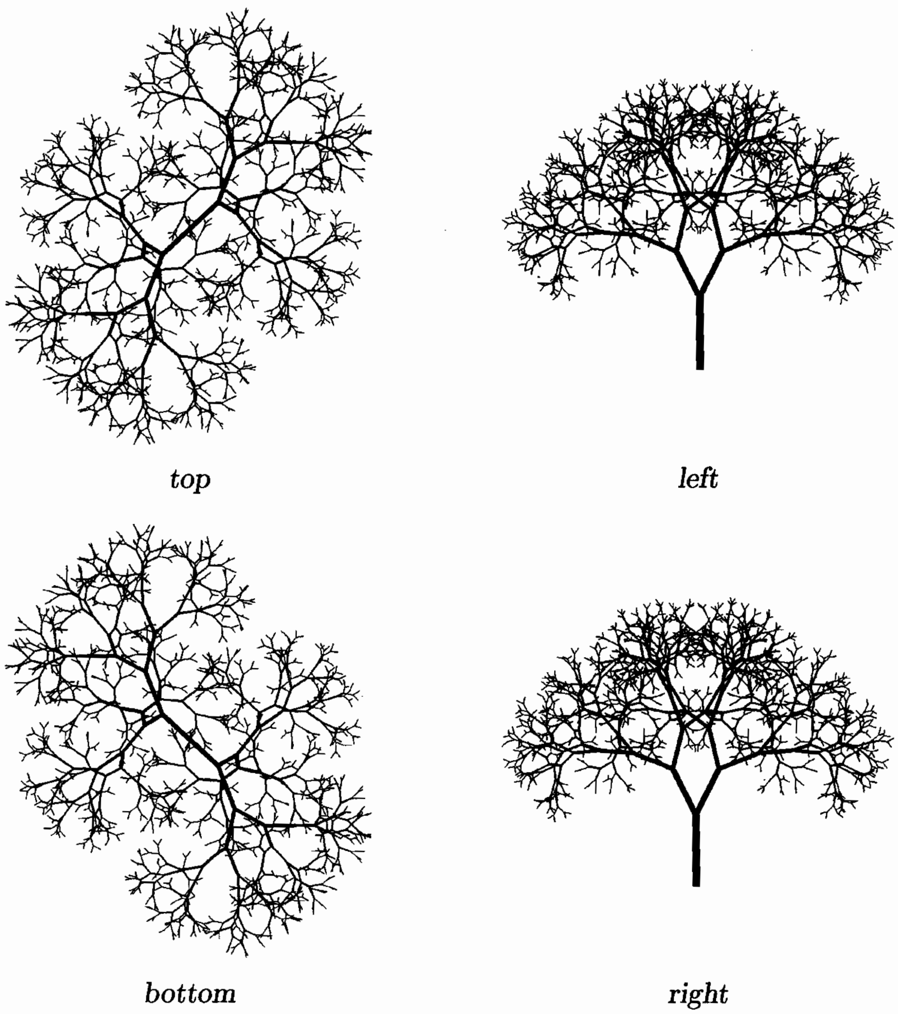


Table 2.2: *The same tree viewed from various spatial orientations.*

2.5.2 Randomization techniques

By considering X to be a $[0,1]$-valued random variable (and by generating a new value at each stage in the construction of T), we can produce a richer set of combinatorial trees. The beta distribution beta(a,b) is an ideal choice for the distribution of X (as explained below). Formally, the beta(a,b) distribution has density

$$f(x) = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)}\, x^{a-1}(1-x)^{b-1}, \qquad 0 < x < 1,$$

where $a, b > 0$ are parameters and Γ is the gamma function (see DEVROYE 1986A on how to generate this random variable by computer). The expected value (MENDENHALL ET AL. 1986) of random variable X is then

$$\mathbf{E}X = \frac{a}{a+b}$$

and its variance is

$$\mathbf{Var}X = \frac{ab}{(a+b)^2(a+b+1)}.$$

For example, Figure 2.14 shows the first few levels in a random split tree with 15 internal nodes produced by choosing X to be a random variable having a

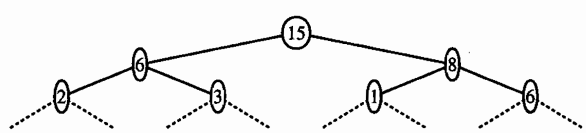


Figure 2.14: *The first few levels in the split tree.*

beta(1,1) distribution. Figure 2.15 shows the resulting combinatorial tree.

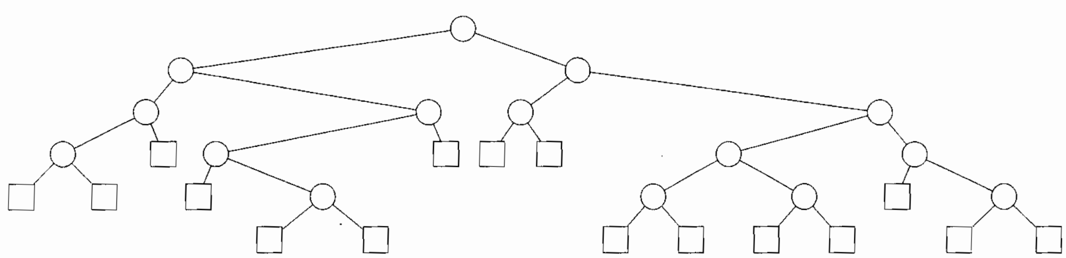


Figure 2.15: *A random symmetric combinatorial tree.*

If we continue to follow the pipe model of SHINOZAKI ET AL. (1964A) we can now think of the resources being randomly partitioned. That is, for each internal cylinder, a certain random portion of the pipes goes through the left and right child cylinder.

For our purposes of synthetic imagery, the beta distribution has several advantages: its range is from 0 to 1; its expected value and variance can be chosen independently; and the expected height and depth of the resulting split trees are highly predictable. That is, the bushiness and elongation of the trees can be controlled by varying the parameters. More formally DEVROYE (1996) showed the following theorems for random split trees in general.

THEOREM 2.1 *Let D_n be the depth of the last node in a random split tree with n nodes. Then*

$$\frac{D_n}{\log n} \to \frac{1}{\mu} \qquad \text{in probability as } n \to \infty,$$

and $\mathbf{E}\{D_n\}/\log n$ tends to the same limit, where $\mu = 2\mathbf{E}\ \{Y \log(1/Y)\}$, $Y \in [0,1]$ is X and $1-X$ with equal probability, and X is the branch-splitting random variable introduced earlier.

THEOREM 2.2 *Let H_n be the height of a random split tree with n nodes. Then*

$$\frac{H_n}{\log n} \to \gamma \qquad \text{in probability as } n \to \infty,$$

where $\gamma = \inf\left\{c : e^{t^*}(2m(t^*))^c < 1\right\}$, $m(t) = \mathbf{E}\{Y^t\}$, $t \geq 0$, t^* is the unique solution of $m'(t)/m(t) = -1/c$, and Y is as in Theorem 2.1.

With random beta trees, one can choose the desired expected depth and height and solve the above formulas to determine explicit values for a and b. Note for example that $1/\mu$ can take any value between $1/\log 2$ and ∞.

2.5.3 Random "families"

Roughly speaking the varying of a and b in the choice of the beta(a,b) distribution (to be used in the random generation of the underlying combinatorial trees) controls the overall shape of the corresponding geometric trees. Recall that as Table 2.1 shows, varying the shape of the combinatorial trees varies the appearance of the rendered trees. Table 2.3 shows how randomization allows the creation of random "families" of trees. These families can be thought to intuitively correspond to species of botanical trees. In each column, $\mathbf{E}X$ is constant while $\mathbf{Var}X$ increases from left to right. For each row, $\mathbf{E}X$ increases from the top to the bottom columns.

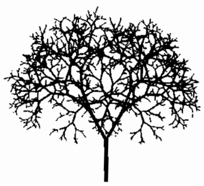
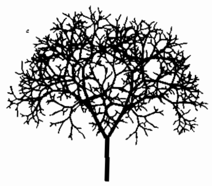
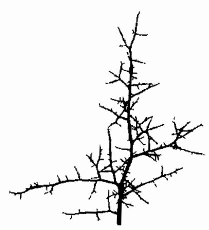
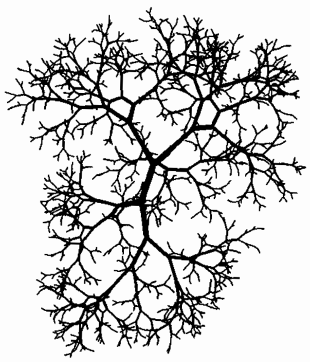
			
beta(80,80) **E**$X = 0.5$ **Var**$X = 0.001$	beta(40,40) **E**$X = 0.5$ **Var**$X = 0.003$	beta(10,10) **E**$X = 0.5$ **Var**$X = 0.012$	beta(1,1) **E**$X = 0.5$ **Var**$X = 0.083$
			
beta(80,40) **E**$X = 0.6$ **Var**$X = 0.002$	beta(20,10) **E**$X = 0.6$ **Var**$X = 0.011$	beta(8,4) **E**$X = 0.6$ **Var**$X = 0.017$	beta(4,2) **E**$X = 0.6$ **Var**$X = 0.032$
			
beta(80,10) **E**$X = 0.9$ **Var**$X = 0.001$	beta(40,5) **E**$X = 0.9$ **Var**$X = 0.002$	beta(8,1) **E**$X = 0.9$ **Var**$X = 0.010$	beta(4,0.5) **E**$X = 0.9$ **Var**$X = 0.018$

Table 2.3: *Random symmetrical and asymmetrical trees.*

As before with the deterministic case (e.g., Table 2.2), this is a three dimensional model. Thus, we note that various viewpoints can be chosen for the rendering. For example, Table 2.4 shows the tree with beta(10,10) from Table 2.1 from various viewpoints.



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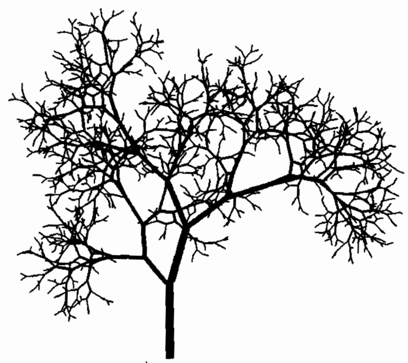

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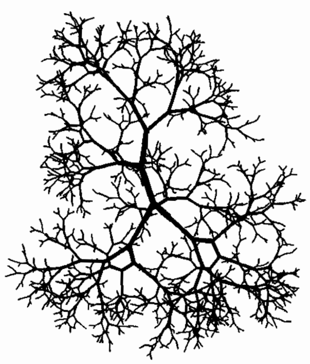

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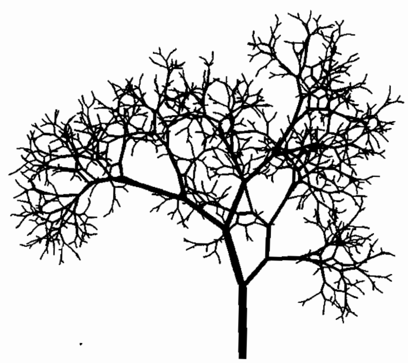


right

Table 2.4: *The same tree viewed from various spatial orientations.*

2.5.4 Random "individuals"

If a and b are fixed and n is large, the resulting renderings are similar without being identical in appearance. That is, one can generate random "individuals" from within a certain "family." Table 2.5 shows the rendered trees resulting from eight random combinatorial trees whose X is a beta(10,80) random variable.

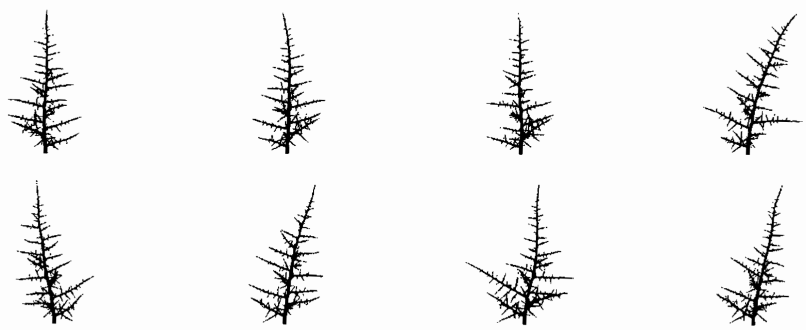
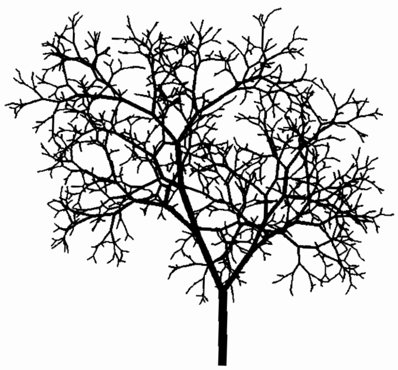


Table 2.5: *Eight "individuals" from the beta(10,80) "family."*

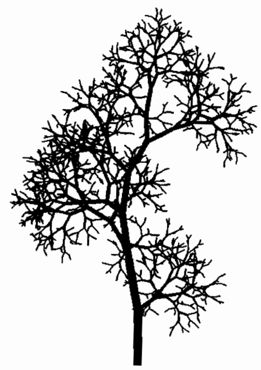
2.5.5 Variable distributions

In Chapter 2.5.3, the distribution of the splitting variable is fixed throughout the entire tree. However we are not limited to this. Table 2.6 shows the effect of making the splitting variable a function of the node u.

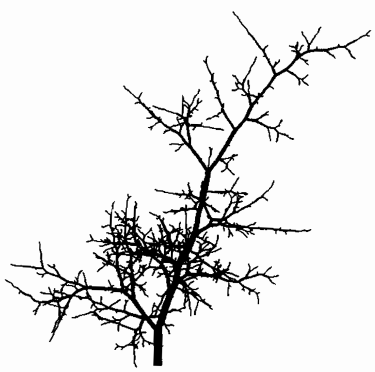
For example in the first figure, random variable X is distributed as beta($d(u)$,$d(u)$) where $d(u)$ is the depth of u in the tree. As discussed in Chapter 2.5.2, $\mathbf{E}X = \frac{d(u)}{2d(u)} = \frac{1}{2}$ and $\mathbf{Var}X = \frac{d(u)^2}{2d(u)^2(2d(u)+1)} \simeq \frac{1}{4d(u)} \to 0$ as $d \to \infty$. Thus the splits become more and more balanced higher up in the tree. In the second figure, $\frac{n}{|u|}$ increases as $|u|$ decreases and thus the splits become more and more symmetric and stable deeper in the tree. In the third figure, the splits become more and more lopsided however the divergence angle helps the tree to spiral upwards. In the final figure, at each split for node u, X is distributed as beta(1,1) if $|u| \geq \frac{n}{5}$ and as beta(10,10) otherwise.



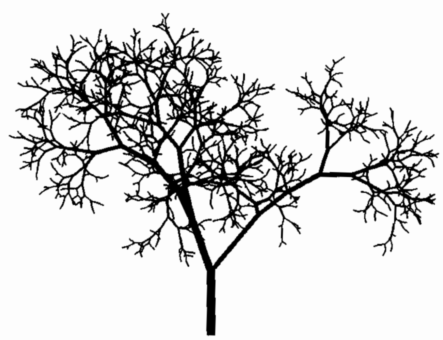
beta(d(u),d(u))



beta$(\frac{n}{|u|},\frac{n}{|u|})$



beta(d,1)



beta(1,1) or *beta(10,10)*

Table 2.6: *Variable distributions.*

2.5.6 A large example

Finally, we end this chapter with a large example.

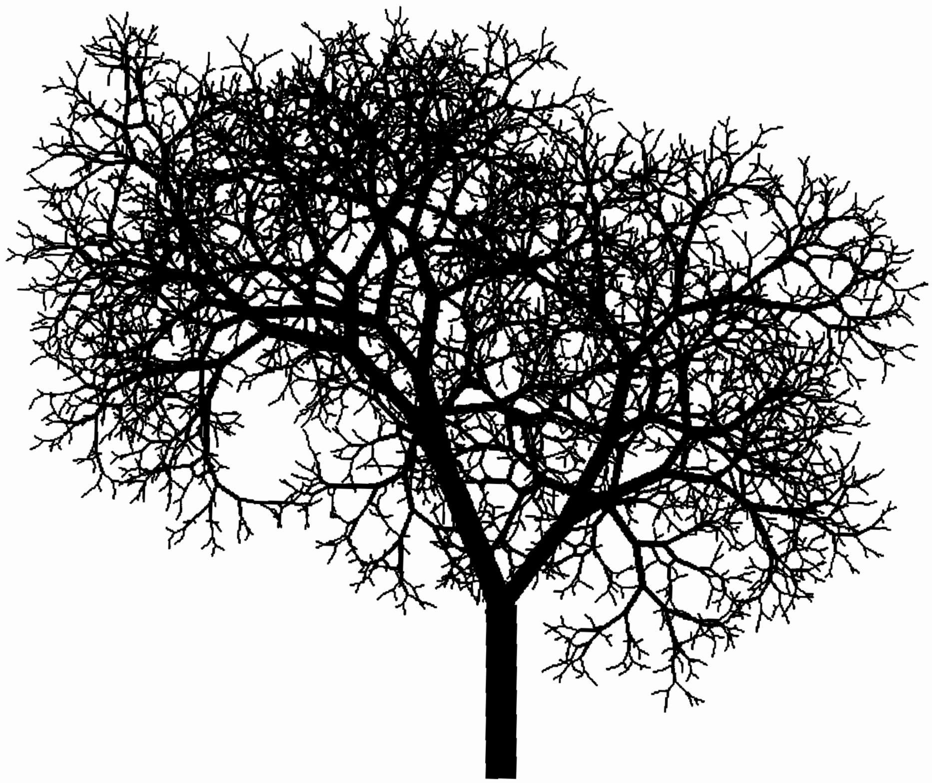


Figure 2.16: *A tree with 10000 nodes from the beta(10,10) family.*

Chapter 3

Modelling according to Horton's first law

In Chapter 2 we introduced a general combinatorial-geometric model of botanical trees. We generate combinatorial trees according to a splitting algorithm. Each node in a combinatorial tree is modelled as a cylinder in the corresponding geometric tree. Cylinders are rendered to resemble branches in botanical trees. We now examine a combinatorial method to model actual species of botanical trees according to Horton's first law.

The rest of this chapter is organized as follows. In Chapter 3.1 we explain the origins and applications of the HORTON-STRAHLER ordering and bifurcation ratios. In Chapter 3.2 we explain how to generate random combinatorial trees for specific expected bifurcation ratios. In Chapter 3.3 we attempt to model specific species based on known bifurcation ratios.

3.1 HORTON-STRAHLER orderings/bifurcation ratios

We present a method for modelling particular botanical species according to their respective HORTON-STRAHLER numbers. Accordingly, we first introduce the notation of the HORTON-STRAHLER number.

3.1.1 Definition

For a node u in a binary (combinatorial) tree T, let the HORTON-STRAHLER *order* $S(u)$ be defined as

$$S(u) = \begin{cases} 0 & \text{if } |u| = 1\ , \\ \max\{S(v), S(w)\} + \mathrm{I}_{[S(v)=S(w)]} & \text{if } |u| \geq 3 \text{ and} \\ & \quad u \text{ has children } v \text{ and } w, \end{cases}$$

where I_A is the indicator of the event A (i.e, $I_A = 1$ if A is true and 0 otherwise). We define $S(T)$ as the HORTON-STRAHLER *number* of the root of tree T.

That is, we use the word *order* for nodes and the word *number* for trees. For example, Figure 3.1 shows the HORTON-STRAHLER ordering for the tree from Figure 2.2. Note that all external nodes have order zero.

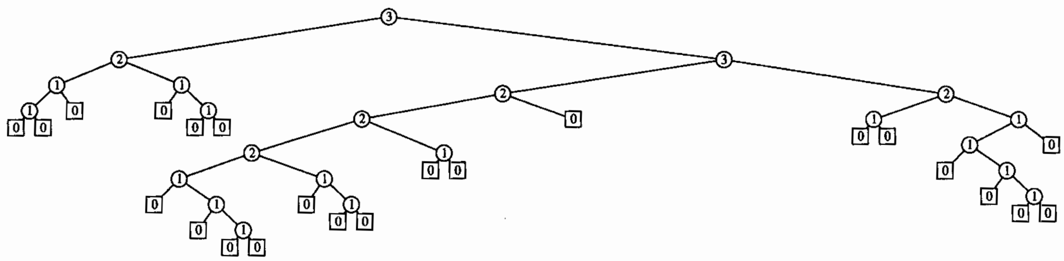


Figure 3.1: *The* HORTON-STRAHLER *ordering of a binary tree.*

The two extreme values for the HORTON-STRAHLER number are immediately apparent. At the one extreme is a single chain of n nodes. Let T be a chain of nodes such that each internal node (except the deepest one) has exactly one internal node and one external node as children (e.g., see Figure 3.2). A chain is also called a *"gourmand de la vigne"* by VIENNOT (1990) and a "herringbone tree" by FITTER (1985). At the other extreme is the complete

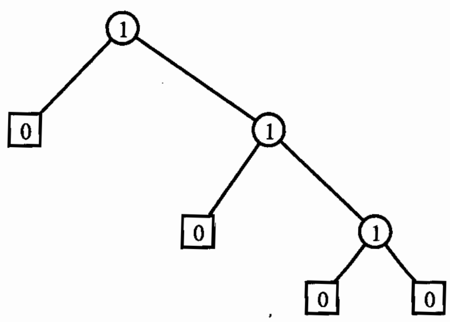


Figure 3.2: *A "gourmand de la vigne" with* HORTON-STRAHLER *number one.*

tree with k levels, $2^k - 1$ nodes and HORTON-STRAHLER number $k - 1$ (e.g., see Figure 3.3). Generalizing this, it is clear that for each binary tree T with n internal nodes, $S(T) \leq \log_2 n + 2$ (e.g., KRUSZEWSKI 1993). For a more detailed combinatorial discussion, see Appendix B.

3.1.2 Bifurcation ratios—Horton's first law

The HORTON-STRAHLER number arises from geography where it has been successfully used as a metric of flow in river networks (e.g., see HORTON 1945

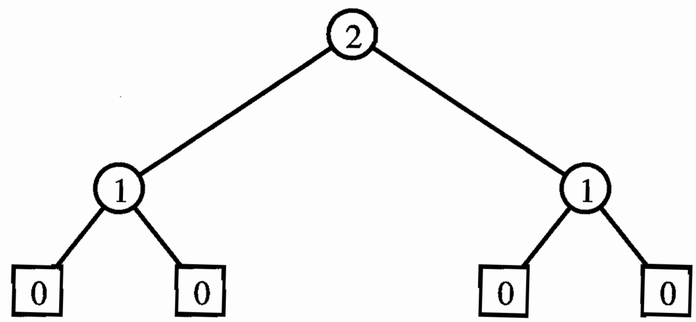


Figure 3.3: *A complete tree with* HORTON-STRAHLER *number two.*

and STRAHLER 1952). The classification goes as follows. For a combinatorial tree T with HORTON-STRAHLER number $S(T)$, let N_m be the number of order m nodes. Thus, $N_0 = n+1$ for all T. For example, in Figure 3.1, $N_0 = 23$, $N_1 = 15$, $N_2 = 5$ and $N_3 = 2$; in a chain, $N_0 = n+1$ and $N_1 = n$; and in a complete tree $N_i = 2^{k-i}$ for $0 \geq i \geq k$. Let the *order ratio* be defined as

$$R_{i+1} = \frac{N_i}{N_{i+1}}.$$

For example, in a chain $R_2 = \frac{n+1}{n}$ and $R_i = 0$ for $i > 2$ and in a complete tree $R_i = 2$ for $1 < i \leq k$. Let the *bifurcation ratio* R be the average of the R_i, i.e.,

$$R = \frac{1}{S(T)} \sum_{i=1}^{S(T)} R_i.$$

Horton's first law of hydrology estimates N_i as

$$N_i = R^{S(T)-i}.$$

In 1971, LEOPOLD and OOHATA AND SHIDEI were concurrently the first to use the HORTON-STRAHLER ordering on botanical trees. They, along with BARKER ET AL. (1973), MCMAHON AND KRONAUER (1976), WHITNEY (1976), TOMLINSON (1978), HONDA ET AL. (1981) and others, have measured the bifurcation ratio for various species of botanical trees (see Chapter 13 of MACDONALD 1983 for a comprehensive listing of these results.) Table 3.1 lists the results that we will use.

common name	latin name	researchers	max order	R	c_L (mm)	c_R (mm)
apple	*malus*	BARKER ET AL. (1973)	5	4.35	22	2.9
white oak	*Quercus alba*	MCMAHON AND KRONAUER (1976)	6	4.11	25	1
white fir	*Abies concolor*	LEOPOLD (1971)	5	4.8	16	3.5
sugar maple (forest grown)	*Acer saccharum*	STEINGRAEBER ET AL. (1979)	4-7	3.19	17	2.1
sugar maple (in open ground)	*Acer saccharum*	STEINGRAEBER ET AL. (1979)	3-5	7.05	17	2.1

Table 3.1: *Parameterization data for various species.*

3.1.3 Relationship between the HORTON-STRAHLER number and bifurcation ratios

Horton showed that for large enough n,

$$\begin{aligned}\sum_{i=1}^{S(T)} N_i &= \sum_{i=1}^{S(T)} R^{S(T)-i} \\ &= \sum_{i=0}^{S(T)-1} R^i \\ &= \frac{R^{S(T)}-1}{R-1}.\end{aligned}$$

Thus, taking logarithms, we have the following fact.

FACT 3.1 *If T obeys Horton's first law,*

$$\log_R n \sim S(T).$$

3.1.4 The HORTON-STRAHLER number for deterministic trees

Let $||T||$ be the number of internal nodes in tree T. We mimic a proof by KRUSZEWSKI (1993) to show the following fact.

FACT 3.2 *For any deterministic split binary tree T with constant $X \leq \frac{1}{2}$,*

$$S(T) \leq \log_{\frac{1}{X}} ||T|| + 1 \; .$$

PROOF. We argue by induction on n, the number of internal nodes in T.

BASIS. For $n = ||T|| = 1$, thus $S(T) = 1 \leq \log_{\frac{1}{X}} 1 + 1 = 1$.

INDUCTIVE STEP. We assume that the hypothesis is true for all trees with fewer than n internal nodes and proceed to prove it for each tree T with n internal nodes. Let T_L and T_R be the subtrees of T. Without loss of generality, we assume $S(T_L) \geq S(T_R)$.

If $S(T_L) > S(T_R)$ then $S(T) = S(T_L)$ and by the inductive hypothesis $S(T) \leq \log_{\frac{1}{X}} ||T_L|| + 1 \leq \log_{\frac{1}{X}} n + 1$.

If $S(T_L) = S(T_R)$ then, again without loss of generality, we assume $||T_L|| \leq ||T_R||$ so that $||T_L|| \leq Xn$. Thus,

$$
\begin{aligned}
S(T) &= 1 + S(T_L) \\
&\leq 1 + \log_{\frac{1}{X}} ||T_L|| + 1 \qquad \text{(by the inductive hypothesis)} \\
&\leq 2 + \left(\log_{\frac{1}{X}}(n) - 1\right) \\
&\leq \log_{\frac{1}{X}} n + 1\ .
\end{aligned}
$$

Therefore, as the hypothesis holds for both the single-internal node tree and for all trees with less than n internal nodes, we conclude by induction that it holds for all trees with n internal nodes. □

Thus, $\log_R n \sim S(T) \leq \log_{\frac{1}{X}} n$ implies that following fact.

FACT 3.3 *For any each deterministic split binary tree T with constant $X \leq \frac{1}{2}$ and bifurcation ratio R,*

$$R \sim \frac{1}{X}.$$

It is interesting to compare this to the experimental result by OOHATA AND SHIDEI (1971) that

$$\frac{N_1}{n} \sim 1 - \frac{1}{R}.$$

3.1.5 A deterministic example

Thus, the deterministic split tree algorithm with $X = 1/R$ produces combinatorial trees with bifurcation ratios of approximately R. This allows us to *model combinatorially* specific species of botanical trees. For example, BARKER ET AL. (1973) measured an apple tree with 579 branches and calculated its bifurcation ratio as 4.35. Figure 3.4 shows the visualization of a deterministic split tree with $X = 1/4.35$ and 579 internal nodes.

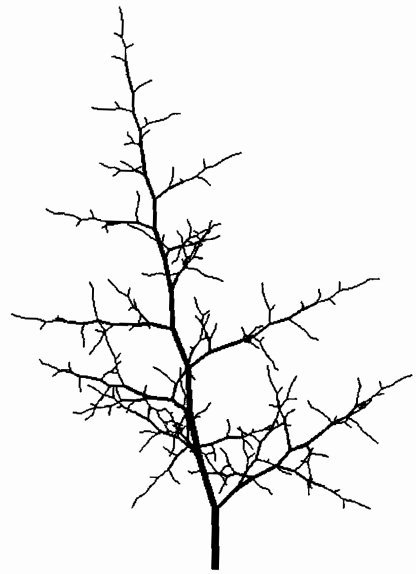


Figure 3.4: *A deterministic apple tree based on constant* $X = 0.23$.

We base the length and radius functions on the data in BARKER ET AL. (1973) in the following way. They reported that an order one branch segment has an average radius of $r = 1.45$ mm. We solve for c_R, the value of an order zero branch, as

$$\begin{aligned} r &= \left({c_R}^3 + {c_R}^3\right)^{\frac{1}{3}} \\ r &= 2^{\frac{1}{3}} c_R \\ c_R &= \frac{r}{2^{\frac{1}{3}}} \\ c_R &\sim 1.15 \text{ mm} \end{aligned}$$

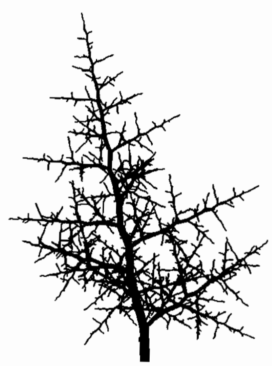
Similarly, an order one branch segment has average length $l = 22.0$ mm. We currently let $c_{L_2} = 1$ and thus for an order zero branch we have

$$\begin{aligned} l &= c_{L_1} r^{c_{L_2}} \\ c_L &= \frac{l}{r} \\ c_L &\sim 19.1 \text{ mm} \end{aligned}$$

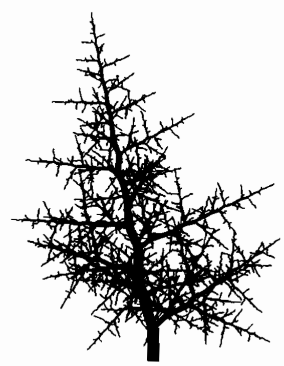
We note that this method of generating trees however is not limited to that of the measured data. If we assume that Horton's law holds for a particular species, then we can generate trees with as many branches as we like. For example, Table 3.2 shows the resulting trees for an increasing series of nodes.

$n = 1000$

$n = 2000$



$n = 4000$



$n = 8000$

Table 3.2: *A series of deterministic "apple" trees with an increasing number of nodes.*

3.2 Expected bifurcation ratios

The chief limitation of this technique is that the trees produced are deterministic. That is, only one individual of a particular family for a fixed n can be generated. For image synthesis and other purposes this is very undesirable. We now rely on a result from probabilistic analysis to provide a method of generating combinatorial trees with specific *expected bifurcation ratios.*

3.2.1 Expected HORTON-STRAHLER number

DEVROYE AND KRUSZEWSKI (1996)[1] showed for the random tries constructed from n independent identically distributed (i.i.d.) sequences of independent Bernoulli (p) random variables $0 < p \leq \frac{1}{2}$ that for the HORTON-STRAHLER number S_n,

$$\frac{S_n}{\log n} \to \frac{1}{\log \frac{1}{p}}$$

in probability as $n \to \infty$. From this, we have the following fact.

FACT 3.4 *For a random* PATRICIA *tree with* $n-1$ *internal nodes and* n *external nodes and* HORTON-STRAHLER *number* S_n,

$$\mathbf{E}S_n \sim \log_{\frac{1}{p}} n.$$

3.2.2 Binomial distribution

In Chapter 2.5.2 we used the beta distribution as the basis of our random split variables. We now use the *binomial distribution* which in the following way is a restricted set of the beta(a,b), since if a and b tend to infinity such that $\frac{a}{a+b} = p$, the $\lfloor n\text{beta}(a,b)\rfloor$ behaves like binomial(n,p). Formally, the binomial(n,p) distribution has probability density

$$p(y) = \binom{n}{y} p^y (1-p)^{n-y}, \qquad 0 \leq y \leq n,$$

where $n, p \geq 0$ are parameters (again, see DEVROYE 1986A on how to generate this random binomial variable by computer). The expected value of random binomial(n,p) variable X is then

$$\mathbf{E}X = np$$

(e.g., see MENDENHALL ET AL. 1986) and its variance is

$$\mathbf{Var}X = np(1-p).$$

[1]For completeness, this paper reappears as Appendix B in this thesis.

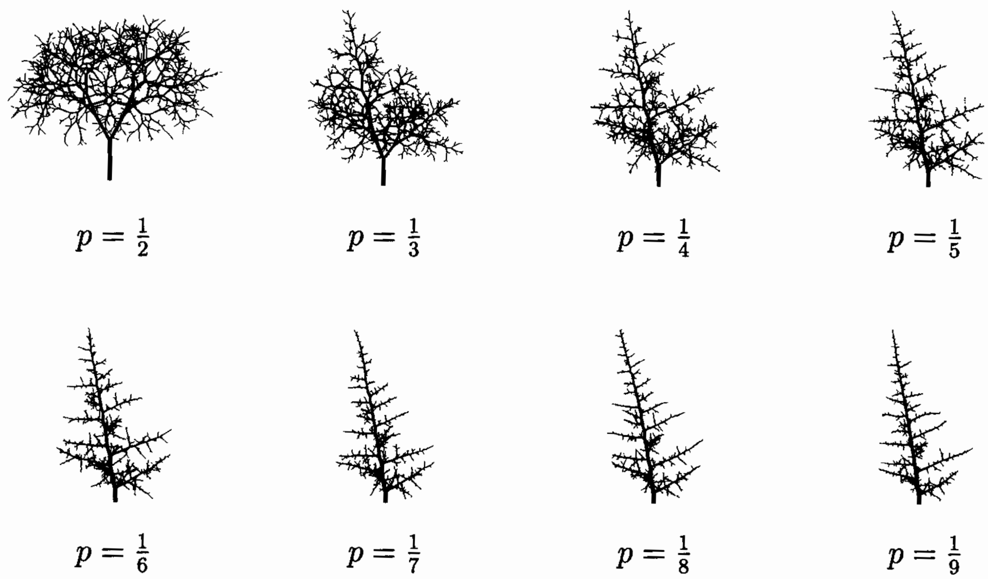
As X is now a discrete random variable depending on n, we modify our splitting algorithm $T(n)$ for generating a tree with n internal nodes in the appropriate way:

$$T(n) = \begin{cases} \square, & \text{if } n = 0;\\ \underset{T(X)\qquad T(n-1-X)}{\overset{\bigcirc}{/\qquad\backslash}}, & \text{otherwise.}\end{cases}$$

If we continue to follow the pipe model of SHINOZAKI ET AL. (1964A), we can now think of each resource as being somehow randomly and independently allocated. That is, for each internal cylinder, each pipe going through that cylinder flips a biased coin to decide whether it goes through the left or right child cylinder.

We have chosen the binomial distribution because for this style of split tree the resulting trees are random PATRICIA trees on $n + 1$ strings. Thus, we can make use of Fact 3.4 to combinatorially model botanical trees.

Table 3.3 shows how each value of p determines a new "family" of tree. As before, each tree has 2000 internal nodes. It is useful to compare this table with Tables 2.1 and 2.3. Table 3.3 can be seen as a randomization of Table 2.1 and as a restriction on the variance of Table 2.3.



$p = \frac{1}{2}$ $p = \frac{1}{3}$ $p = \frac{1}{4}$ $p = \frac{1}{5}$

$p = \frac{1}{6}$ $p = \frac{1}{7}$ $p = \frac{1}{8}$ $p = \frac{1}{9}$

Table 3.3: *Trees from the binomial distribution.*

However, it is important not to make the tempting generalization that

$$\mathbf{E}S(T) = \log_{\frac{1}{\mathbf{E}X}} n,$$

for *any* X. For example, beta(1,1) provides a counter-example, since $\mathbf{E}X = \frac{1}{1+1} = \frac{1}{2}$ but $\mathbf{E}S(T) \leq \log_3 n$ (e.g., Theorem C.3 in Appendix C).

3.3 Combinatorial and visual modelling of specific trees

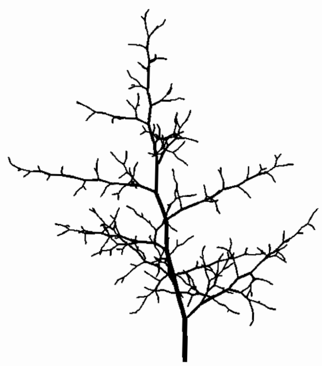
In Section 3.1.2 we explained how researchers have calculated the bifurcation ratios of various species of botanical trees. We used one bifurcation ratio to generate deterministic "apple" trees. In Section 3.2.2 we showed how to generate trees according to specific expected bifurcation ratios. We now attempt to combinatorially and visually model particular species of botanical trees. We can do this by using the binomial distribution along with Fact 3.4 and the bifurcation ratios from Table 3.1. We have chosen these particular trees due to the availability of their data and their suitability for landscaping purposes.

3.3.1 Apple

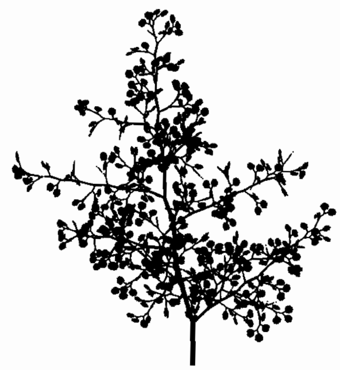
We now present a randomized version of the apple tree model (e.g., Figure 3.4) from Chapter 3.1.5. The only parameter which we change in the first figure from Table 3.4 is X which is now a binomial random variable with $p = 0.23$. The second figure shows the tree from the first figure now with leaves. That is, each external node is modelled as a leaf. In the third figure each internal cylinder with exactly two external child cylinders is modelled as a blossom (the corresponding external child cylinders are dropped). Finally, in the fourth figure each blossom gives rise to a fruit.

As before with Tables 2.2 and 2.4, Table 3.5 shows the first figure from Table 3.4 from various viewpoints.

Since we are now using random split trees, we can now generate different individuals of a particular species for a particular size. For example, Table 3.6 shows four different apple trees, each with 1000 internal nodes.

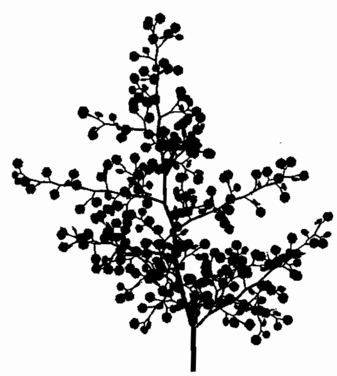


no leaves



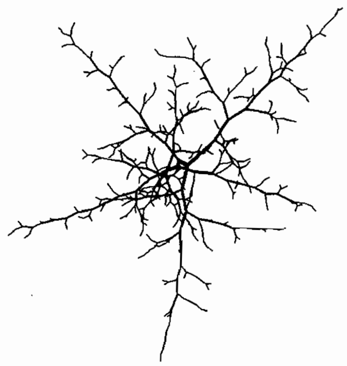
leaves

leaves and blossoms

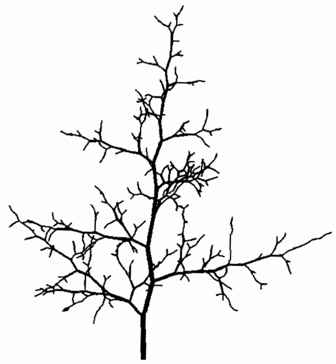


leaves and fruit

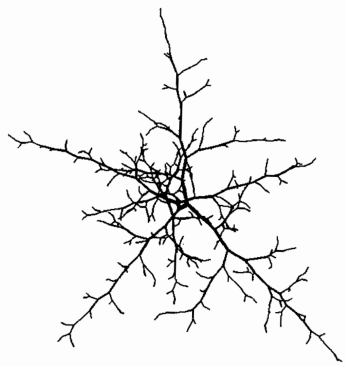
Table 3.4: *The life of an apple tree.*



top



left



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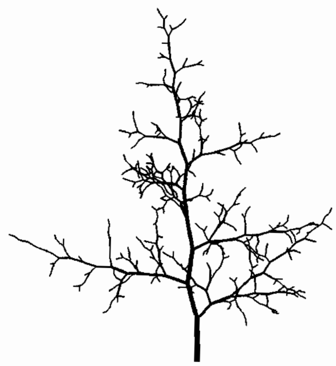


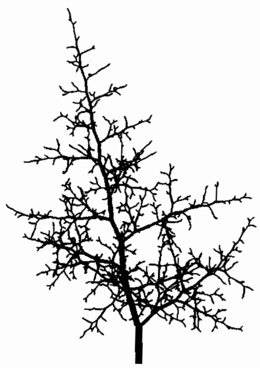
right

Table 3.5: *Various view points of an apple tree.*

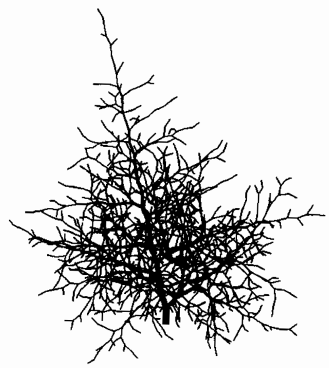
Table 3.6: *A series of "random" apple trees.*

3.3.2 White oak

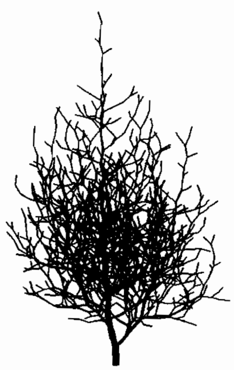
White oak trees are commonly found in Montreal as popular landscaping trees. The first figure in Table 3.7 shows the result of our model being parameterized according to Table 3.1. The radius and length constants were derived from measurements taken from a white oak tree outside the McConnell Engineering building at McGill University in the spring of 1996. We have observed that in many small white oak trees the main branch is often shorter than its sibling branch. We model this by setting $c_{L_2} < 1$ in our length function. For example, the second figure in Table 3.7 shows the result of $c_{L_2} = 0.25$. In this figure, we note that the secondary branches spread out too far and in real white oak trees that the branches tend to bend towards the sky. We simulate this *heliotropism* by deflecting the axis of each geometric cylinder by a vector $c[0, 1, 0]$ where c is a constant. The third figure shows the result of our simulated tropism. Finally, the fourth figure shows the result of modelling each external cylinder as a simple oak leaf.



basic tree



changed power law



with heliotropism

with leaves

Table 3.7: *An increasingly more realistic looking white oak tree.*

3.3.3 White fir

We are unsure how researchers such as LEOPOLD (1971) calculated the HORTON-STRAHLER orders for white firs since these trees are rarely binary trees. Rather, they are ternary trees, that is, a node u has three children: one main node v_1 and two secondary nodes v_2, v_3 (e.g., see Figure 3.5).

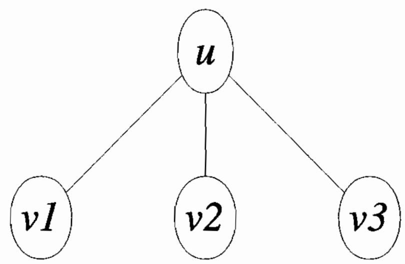


Figure 3.5: *A ternary tree.*

We propose the following extension of our model to include these trees. The first problem is to generate a ternary combinatorial tree which maintains the correct bifurcation ratio. Let X be distributed as binomial(n,p) where $p \leq \frac{1}{2}$ (here $p = 1/4.8$). For a node u such that $|u| = n \geq 1$, we propose the following. Let $|v_3| = X$; $|v_2| = \max(|v_3| - 1, 0)$; and $|v_1| = n - 1 - |v_2| - |v_3|$.

We now alter our geometric modelling functions in the appropriate way. We use results from MURRAY (1926B) for *cylinder radii*[2] as:

$$R(\mu) = \begin{cases} c_R & \text{if } |u| = 1; \\ \left(R(\nu_1)^3 + R(\nu_2)^3 + R(\nu_3)^3\right)^{1/3} & \text{otherwise.} \end{cases}$$

The length function remains unchanged.

For fir trees, we suppose that the three sibling cylinders are coplanar with the main cylinder ν_3 in between the other two ν_1 and ν_2. That is, $\Theta(\nu_3) = 0$. For ν_1 and ν_2,

$$\Theta(\nu_i) = c_\Theta \left(\frac{R(\mu) - R(\nu_i)}{R(\mu)}\right),$$

where $c_\Theta = 90$. The first figure in Table 3.8 shows a rendering of a white fir tree with 1000 nodes. The second figure shows the result of rendering each cylinder with its needles.

[2]Cf. For another species of conifer, the Scots pine, PERRTUNEN ET AL. (1996) also used this formula but with exponent 2 rather than 3.

a simple fir tree *with needles*

Table 3.8: *An increasingly more realistic looking white fir tree.*

3.3.4 Sugar maple

Sugar maples are deciduous ternary trees. We model them in a similar way to white firs. However, although all three child buds begin their development coplanar, for larger branches the three child axes are not coplanar. Instead, for our purposes each child plane differs from the parent plane Π_μ by divergence angle $\frac{360°}{3} = 120°$. We modify the deviation angle in the following way:

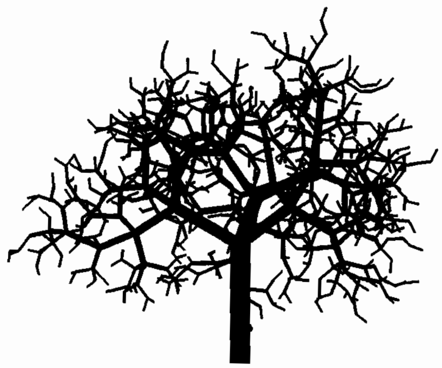
$$\Theta(\nu_i) = c_\Theta \left(\frac{R(\mu) - R(\nu_i)}{R(\mu)} \right),$$

where $c_\Theta = 90$.

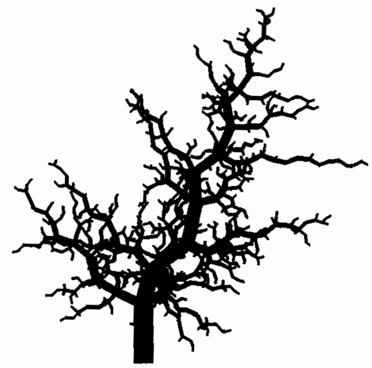
STEINGRAEBER ET AL. (1979) reported that the ratio changes considerably for sugar maples depending on whether they grow in the open ground ($R = 7.05$) or in a shady forest ($R = 3.19$) (cf. WHITNEY 1976). Table 3.9 shows two trees which differ only in bifurcation ratio.

Furthermore, STEINGRAEBER (1982) reported that the R_m for sugar maples varied significantly in the lower orders and thus are partial exceptions to Horton's first law. That is, order ratios R_1, R_2 and R_3 differ for the leader shoots and the lower branches. For $i > 3$, R_i do not differ greatly.

We can interpret these results in the following combinatorial way. The distribution of the splitting variable is a function of the node's depth and subtree size. Since $p = 1/R$, and R seems to increase with depth for nodes of a certain size, we propose the following possible "parameterization" of STEINGRAEBER's



grown in forest



grown on open ground

Table 3.9: *Different bifurcation ratios.*

observation. Figure 3.6 shows a tree based on varying bifurcation ratio such that the split variable X for node u is distributed as binomial($|u|$,$p(u)$) where

$$p(u) = \begin{cases} \frac{1}{3.19} & \text{if } |u| > 100; \\ \frac{1}{3.19+\log(d(u))} & \text{otherwise.} \end{cases}$$

According to KOIKE (1990), "[l]ittle is known about the relationship between growth characteristics of deciduous broad-leaved trees and their pattern of autumn coloring." KOIKE (1987) reported that for another maple *Acer mono* that the autumn colouring of the leaves began at the outer part of the tree crown. In Figure 3.7, we make the leaf colour a function of combinatorial depth such that the leaves become lighter in colour the deeper their corresponding external nodes are located in the combinatorial tree.

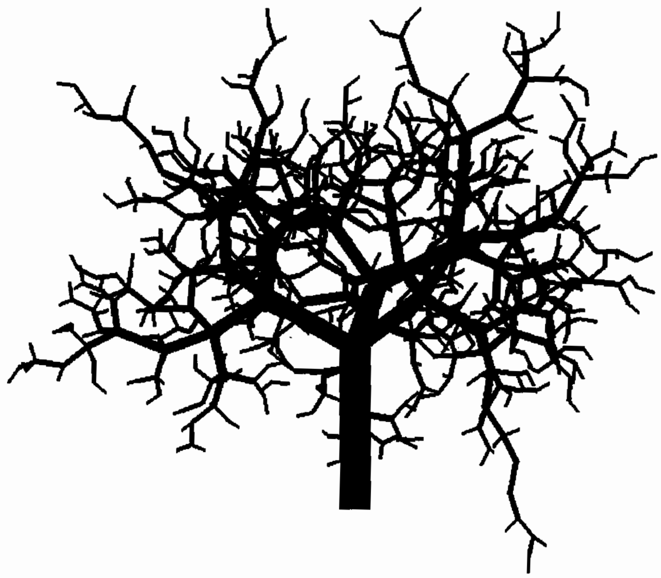


Figure 3.6: *Bifurcation ratio as a function of depth and subtree size.*

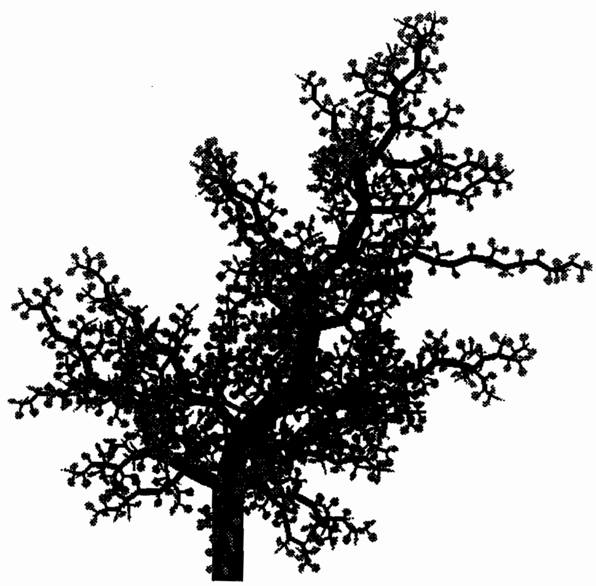


Figure 3.7: *Leaf colour as function of depth.*

Chapter 4

Related work

In Chapter 2 we laid out our basic model together with our general parameterization. In Chapter 3 we showed how to use the probabilistic trie model as a combinatorial model of several species of botanical trees. In this chapter, we review previous techniques for visually simulating botanical trees. We divide the previous work into two basic categories: simple and complex geometric models. In a *simple geometric model* the general silhouette of the botanical tree is modelled as a single geometric object. In a *complex geometric model* individual branch segments are modelled as cylindrical geometric objects. For example, our basic model is a complex geometric model. In fact, we hope to show that many of the previous techniques can be viewed as special cases of our general model.

4.1 Simple Geometric Models

Simple geometric models have the chief advantage of being computationally simple (and therefore fast) and are accordingly used in many real-time applications such as flight simulation where impressionistic painting-like images suffice. That is, when viewed from a distance a simple representation is often sufficient.

4.1.1 Billboarding

For our purposes, a *billboard* is a two dimensional geometric tree usually in the shape of a simple polygonal (e.g., a triangle for a conifer) that is rendered using texture mapping to appear as a painting of a botanical tree. This billboard is made three dimensional (more properly two and one half dimensional) by always rotating it around a fixed vertical axis so that it always faces the viewer (e.g., PITT 1995; RATZER 1995). This technique is so popular that computer graphics vendors such as Silicon Graphics now include built-in billboarding functions (e.g., `pfBillboard` described on p. 386 of Rohlf and Helman 1994).

4.1.2 Textured quadratic surfaces

Gardner (1984) makes this technique fully three dimensional by taking quadratic surfaces such as ellipsoids (e.g., see Note 394 of Chapter 2.2) and "painting" them to resemble botanical trees. By adding a mixture of natural light and transparency textures, he produces trees that appear realistic when viewed from a great distance such as from an aircraft.

4.2 Complex Geometric Models

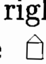
As already mentioned, a complex geometric tree is a two or three dimensional object consisting of cylinders which represent individual branch segments. We now describe previous techniques to produce combinatorial trees and their modelling functions.

4.2.1 Handcrafted trees

Computer aided design (CAD) systems provide a convenient method for constructing geometric trees (e.g., Hoinkes 1995). The trunk of the geometric tree consists of a cylinder created by the designer using the CAD system who then creates the corresponding child cylinders that are connected to the parent cylinder in a manner that is pleasing to the designer's eye. Obviously, this technique severely limits the number of cylinders in a particular tree as well as the overall number of possible geometric trees actually produced.

4.2.2 Fractal techniques

Simple fractals

In 1957, Bosman described how to draw tree-like shapes called *Pythagoras trees* by using a compass, a straight-edge and a drafting board.[1] The construction is as follows. First, draw a square □ and attach a right-sided triangle (whence the name Pythagoras tree) to the top of the square  so that the hypotenuse borders the square. Now, take the remaining two sides of the triangle and attach two separate squares . Repeat the above process on the two new squares. Continue the entire construction in parallel until either the figures are too small to be seen or the pencil wears out (e.g., see Figure 4.1). Now of

[1] This is also described on pp. 67–74 of (Lauwerier 1991) and on pp. 126–131 of (Peitgen et al. 1992).

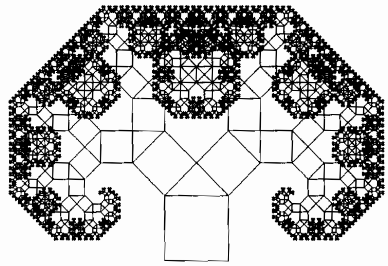


Figure 4.1: *A Pythagoras tree with twelve levels (*BOSMAN 1957*).*

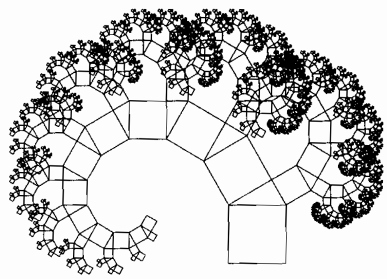


Figure 4.2: *A non-isosceles Pythagoras tree (*BOSMAN 1957*).*

course as Figure 4.2 shows, the triangle does not necessarily have to be isosceles. Flipping the angles of the triangle at every even level (e.g., see Figure 4.3) produces the surprising result that trees which have seemingly different appearances (such as the bush-like Figure 4.2 and the pine-like Figure 4.3) have very similar underlying structures. Figure 4.4 shows how non-right-sided isosceles triangles produce broccoli-like trees. LAUWERIER (1991) randomizes the angles by perturbing them slightly to produce more realistic looking trees.

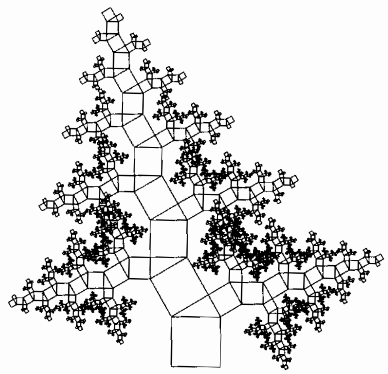


Figure 4.3: *The previous tree now perturbed by flipping angles (*Bosman 1957*).*

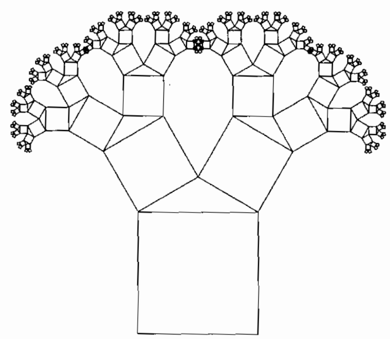


Figure 4.4: *A broccoli-like Pythagoras tree (*Peitgen et al. 1992*).*

A pythagoras tree is an example of a *fractal*. In 1977, MANDELBROT coined the term "fractal" to refer to a geometric object which consists of identical patterns that repeat themselves on an ever-reducing scale, i.e., "a shape made of parts similar to the whole in some way" (MANDELBROT 1977B; MANDELBROT 1982). For example, Figure 4.5 is constructed by starting with an initial |

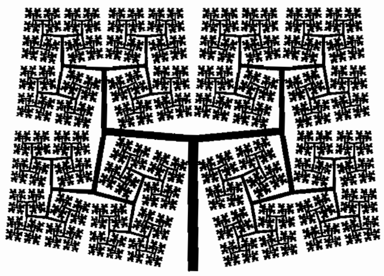


Figure 4.5: *A simple fractal tree (Mandelbrot 1977b).*

branch. Two scaled copies are added to the end of the parent to form children branches. Under infinite recursion, all fractals share the characteristic that at whatever level it is examined, the viewed portion has a similar appearance, i.e. every part (no matter how small) is representative of the whole.

Fractals have been enormously popular in modelling natural phenomena such as trees. Figure 4.6 shows a space filling tree based on |; the fractal

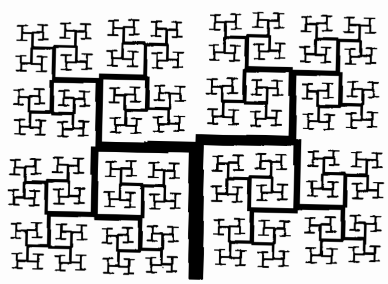


Figure 4.6: *A space filling tree intended to model the lung (*MANDELBROT 1982*).*

grows to fill in the entire rectangle.[2] Such a fractal has been thought to model the space filling capacities of the lung (e.g., see pp. 140–150 of MACDONALD 1983). Furthermore, the two-dimensional version has been used to describe the layout of certain leaves.

[2]N.B. For example, according to our definition the H-tree drawings in VLSI layout of MEAD AND REM (1979) and LEISERSON (1980) are classified as fractals.

McGuire (1991) provides a graphic and photographic essay about the relationship between fractals and nature which presents breath-taking photographs of fractal-like structures such as trees.

Complete combinatorial trees

Simple inspection of the renderings from Figures 4.1 to 4.6 reveals that in each case that the underlying combinatorial tree is a complete tree (e.g., ) on k levels with $2^k - 1$ internal nodes and 2^k external nodes. For example, the two dimensional rendering in Figure 4.6 can essentially be described as a complete tree T with $k = 10$ that is interpretated under the following modelling functions:

$$L(\mu) = r_L^{d(u)},$$

$$R(\mu) = r_R^{d(u)},$$

$$\Theta(\mu) = c_\Theta,$$

and

$$\alpha(\mu) = c_\alpha,$$

where $r_L = r_R = 0.686$ (the scaling ratios), $c_\Theta = 90°$ and $c_\alpha = 0°$ are constants.

Honda

In 1971, Honda was the first to use computers to generate images of botanical trees. Under the rubric of geometric trees, Honda's simplest model can be described in the following way.[3] The underlying combinatorial tree T is a complete tree with k levels. For each pair of sibling nodes $v, w \in T$, one (say v) is deemed the *main child node* (i.e., the continuation of the parent) and the other (say w) is deemed the *secondary child node* (i.e., an offshoot of the parent). Let $m(u)$ be the number of main nodes on the path from the root of T to node u (the root is a main node). Let $s(u)$ be the number of secondary nodes on the path from the root to u. Thus, $d(u) = m(u) + s(u)$. Each node $u \in T$ is modelled as a cylinder μ in the geometric tree $\mathcal{T}$ according to the following functions:

$$L(\mu) = c_L r_{L_m}^{m(u)} r_{L_s}^{s(u)},$$

$$R(\mu) = c_R,$$

$$\Theta(\mu) = \begin{cases} c_{\Theta_m} & \text{if } u \text{ is a main node,} \\ c_{\Theta_s} & \text{if } u \text{ is a secondary node,} \end{cases}$$

[3] As already noted in Chapter 2.1.2, Honda's model has had a fundamental influence on almost all complex geometric models including our own.

and

$$\alpha(\mu) = 137.5°,$$

where c_L, $1 \geq r_{l_m} \geq r_{L_s} \geq 0$, $c_R = 0$, $c_{\Theta_m} \leq c_{\Theta_s}$ are constants. For example, Figure 4.7 shows a rendering of a tree parameterized as $k = 9$, $r_{L_m} = r_{L_s} = 0.85$, $c_{\Theta_m} = 16.7°$, and $c_{\Theta_s} = 33.3°$. Figure 4.8 shows a projection of this tree

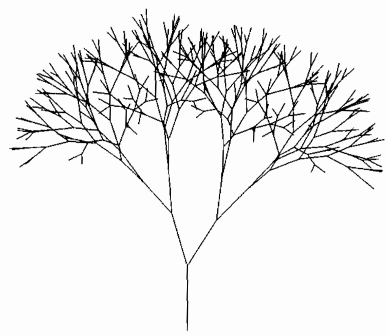


Figure 4.7: *A simple botanical tree (*HONDA 1971*).*

onto the horizontal plane.

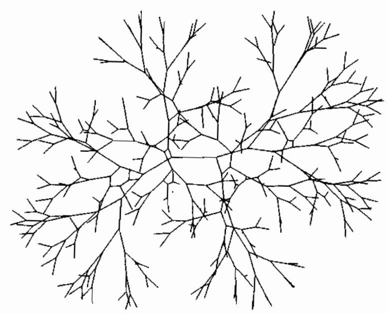


Figure 4.8: *The previous figure viewed from above (*HONDA 1971*).*

Strict monopodial branching can be modelled by letting $c_{\Theta_m} = 0°$. For example, Figure 4.9 shows an rendering of a tree parameterizes as $k = 9$, $r_{L_m} = 0.9$, $r_{L_s} = 0.8$ and $c_{\Theta_s} = 45°$.

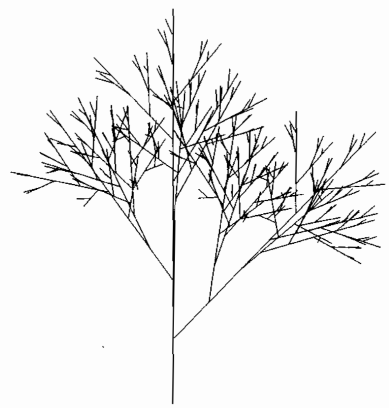


Figure 4.9: *A monopodial tree (*HONDA 1971*).*

Aono and Kunii

AONO AND KUNII (1984) developed HONDA's work by introducting of cylinder radii and *tropisms* (environmental influences). They presented four different geometric models (GMT). GMT1 is a basic model, similar to HONDA's (e.g., $\alpha(\mu) = 140°$) however A-systems[4] are used to generate the combinatorial trees and the cylinders have non-zero radii, i.e.,

$$R(\mu) = c_R r_{R_m}^{m(u)} r_{R_s}^{s(u)},$$

where $r_{R_m} \geq r_{R_s}$ are constants. For example, Figure 4.10 shows a rendering based on GMT1 parameterized as $k = 9$.

GMT2 is an extension of GMT1 with the addition of various tropisms: wind, sunlight and gravity. That is, branch positions are influenced by various attractors which simulate various tropisms. In addition, branch positions are now slightly randomized.

GMT3 is the ternary version of GMT1. Each internal cylinder μ now has *three* child cylinders, one main child and two secondary children (placed on opposite sides of μ's axis in plane Π_μ). Accordingly, the underlying combinatorial tree is now a complete ternary tree on k levels with $3^{k-1} - 1$ internal nodes and 3^{k-1} external nodes. For example, Figure 4.11 shows a rendering of GMT3 parameterized as $k = 9$, $r_{L_m} = 0.9$, $r_{L_s} = 0.6$, $c_{\Theta_m} = 0°$ and $c_{\Theta_s} = 60°$. Figure 4.12 shows the previous figure from above.

DA VINCI, among others (e.g., see NOTE 403 in Chapter 2.2), remarked that the deviation angle in botanical trees typically decreases higher up in the tree. Thus, under our general geometric model,

$$\Theta(\mu) = f(d(u))$$

[4] A-systems are essentially parametric L-systems.

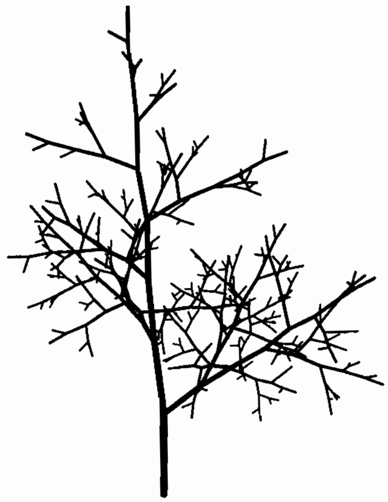


Figure 4.10: *An example of* GMT1 (AONO AND KUNII 1984).

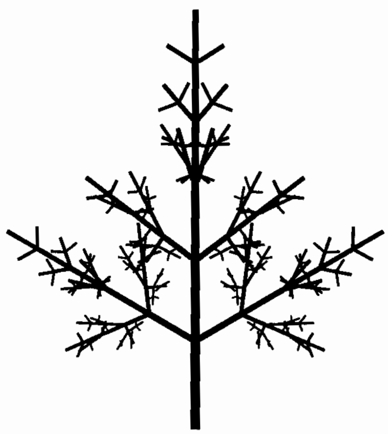


Figure 4.11: *An example of* GMT3 (AONO AND KUNII 1984).

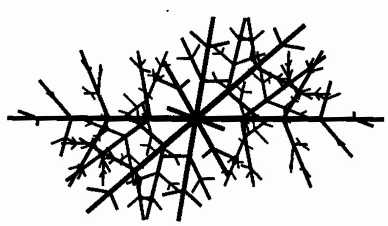


Figure 4.12: *The previous figure viewed from above* (AONO AND KUNII 1984).

where $f(x)$ is a decreasing function. GMT4 (e.g., see Figure 4.13) implements this for secondary nodes u as

$$\Theta(\mu) = \max(110 - 20 * d(u), 0)^{\circ}.$$

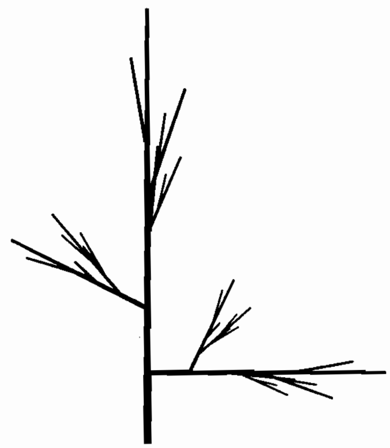


Figure 4.13: *An example of* GMT4 *(*AONO AND KUNII 1984*).*

Oppenheimer

The basic method of OPPENHEIMER (1986) can be seen as an extension of GMT3 from AONO AND KUNII (1984) where $r_{L_m} = 0.8$, $r_{L_s} = 0.4$, $c_{\Theta_m} = 0^{\circ}$ and $c_{\Theta_s} = 60^{\circ}$. OPPENHEIMER creates realistic-looking twisted and gnarled branches by carefully *pruning branch segments.* That is, for a particular section of the tree, the side branches are removed to leave only the stem.

For consistency we view pruning as an action on the underlying combinatorial tree. Let T be a chain of nodes as defined in Chapter 3.1.1. By rendering only the corresponding internal cylinders in $\mathcal{T}$, OPPENHEIMER generates four different effects (e.g., see Table 4.1). A *tapered cylinder* results from $c_{\Theta_m} = 0^{\circ}$ and $\alpha(\mu) = 0^{\circ}$; a *spiral* results from $c_{\Theta_m} \neq 0^{\circ}$ and $\alpha(\mu) = 0^{\circ}$; a *helix* results from $c_{\Theta_m} \neq 0^{\circ}$ and $\alpha(\mu) \neq 0^{\circ}$. Finally a *"squiggle"* results from randomly choosing between a tapered cylinder, a spiral, and a helix for each node in the chain.[5]

A random instance of a particular "species" is generated by slightly randomizing the parameters of the DNA (cf. REEVES AND BLAU 1985). Therefore, since the underlying topology is never changed, these are not stochastic

[5] Although seemingly unaware of the fractal nature of his work, KAWAGUCHI (1982) essentially uses fractal techniques (e.g., cf. BOSMAN 1957) to simulate the growth of shells, horns, tusks, and spiral plants (cf. THOMPSON 1942).

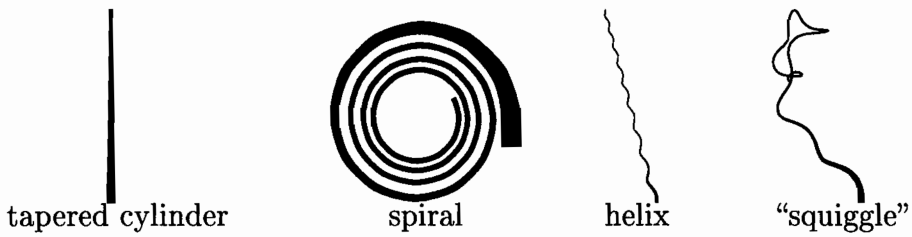


Table 4.1: *Progressively more and more twisted branches (*OPPENHEIMER 1986*).*

fractals. Each instantiation of a particular species has the same topology; different trees are the result of different geometric interpretations. So rather than being strictly self-similar, the trees are statistically self-similar.

Realistic-looking bark is textured mapped from sawtooth waves modulated by Brownian fractal noise to form squiggly waves (cf. BLOOMENTHAL 1985) so that when closely examined, the bark resembles a series of mountain ranges viewed from an airplane.

L-systems

In 1968, LINDENMAYER developed *string rewriting grammars* called *L-systems* to model plant growth mathematically. For our purposes, a grammar G (e.g., see LINZ 1990 for a more formal and general explanation) is defined as a quadruple

$$G = (S, P, I, N)$$

where S is a set of objects called *symbols*, P is a finite ordered set of productions, I is the initial string (a string is an ordered set of symbols), and N is the limit on the number of iterations (during which a string is rewritten). An L-system is a string rewriting grammar in which all symbols of the string are replaced at the same time (during the same *iteration*).[6]

For example, suppose that $S = \{\texttt{A}\}$, $P = \{p_1 : \texttt{A->AA}\}$, and $I = \texttt{A}$. During the first iteration the production p_1 rewrites instances of string `A` in `I` as `AA`. Thus after the first iteration, `I` is rewritten by output string $O_1 = \texttt{AA}$. After the second iteration O_1 is rewritten as output string $O_2 = \texttt{AAAA}$. After iteration i, O_i consists of a string of 2^i `A`'s. The rewriting terminates after the $\texttt{N}^{\text{th}}$ iteration.

Suppose that $S = \{\texttt{A,B,C}\}$, $P = \{p_1 : \texttt{A->B};\;\; p_2 : \texttt{B->AC}\}$, and $I = \texttt{A}$. During the first iteration the symbol `A` is rewritten by symbol `B` according to production p_1 to result in output string $O_1 = \texttt{B}$. During the second iteration

[6] N.B. This parallelism is felt to more closely reflect how plants grow.

each symbol in string O_1 is replaced according to a production rule and thus O_1 is replaced by string $O_2 =$ AC. A third iteration results in $O_3 =$ BC. We note that symbol C is rewritten since there is no production rule which matches it. As in the first example, string rewriting is terminated after iteration N.

PRUSINKIEWICZ AND LINDENMAYER (1990) provided a detailed survey and explanation of the use of L-systems for modelling plants (see PRUSINKIEWICZ ET AL. 1996 for an updated report). The elegance of L-systems becomes apparent when the strings are *interpreted* as drawings. PRUSINKIEWICZ AND LINDENMAYER (1990) (see also SMITH 1984 who did this implicitly) used the following system *turtle interpretation of strings* (e.g., see ABELSON AND DISESSA 1982):

- F move forward and draw a line of length d,
- + turn the turtle to the left by angle θ,
- - turn the turtle to the right by angle θ,

where d and θ are constants. The additional symbols

- [save current graphics state,
-] restore previous graphics state,

form *bracketed OL-systems* (cf. AONO AND KUNII 1984 who do this implicitly and cf. LEOPOLD 1971 whose simulation technique is essentially grammar-based).

Under this scheme, $S = \{\texttt{F}, \texttt{+}, \texttt{-}, \texttt{[}, \texttt{]}\}$. Suppose, for example,

$$P = \{p_1 : \texttt{F} \rightarrow \text{FF-[-F+F+F]+[+F-F-F]}\}$$

and $I =$ F. After the first iteration, we have the string FF-[-F+F+F]+[+F-F-F] which under turtle graphic interpretation looks like . Figure 4.14 shows the result of five iterations. This style of production is called an *edge-rewriting system*.

Typically each iteration grows the combinatorial tree one additional level. As Figure 4.15 shows, drawings of successive iterations can be used to form an animation of plant growth (see also SMITH 1984; NIKLAS 1986; PRUSINKIEWICZ ET AL. 1988; DE REFFYE ET AL. 1988; PRUSINKIEWICZ ET AL. 1993). As a biological necessity the filming of actual plant's growth must take place over a long period of time during which many environmental factors may interfere. The use of artificial animation according to accurate development models avoids these problems.

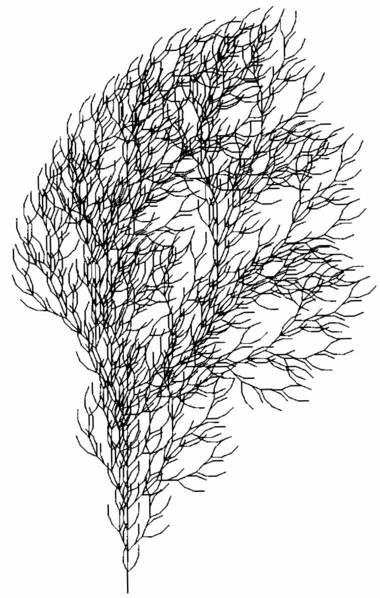


Figure 4.14: *An example of an L-system under turtle interpretation ($\theta = 22.5°$).*

By using an intermediary symbol, say X, a *node-rewriting system* is created (e.g. $S = \{\mathtt{F}, \mathtt{+}, \mathtt{-}, \mathtt{[}, \mathtt{]}, \mathtt{X}\}$. For example, Figure 4.16 shows the grammar starting with string $I = \mathtt{X}$ and production rules

$$P = \{p_1 : \mathtt{X} \to \mathtt{F[+X][-X]FX}; \quad p_2 : \mathtt{F} \to \mathtt{FF}\}.$$

An advantage of node-rewriting systems is that the branches can have variable lengths in a manner similar to HONDA.

SMITH (1984) calls these images created by L-systems *graftals* as they are fractal-like but strictly speaking they are not fractals (e.g., see pp. 152–3 of MANDELBROT 1982). Smith also introduces the important idea of *database amplification*, i.e., the ability to generate complex images from small databases (collections of rules). This characteristic is considered very important by many researchers (e.g., BLOOMENTHAL 1985; REEVES AND BLAU 1985; OPPENHEIMER 1986; PRUSINKIEWICZ AND LINDENMAYER 1990).

Finally, the assignment of probabilities to the production rules introduces a specific kind of randomness into L-systems. For example, Figure 4.17 shows a series of branching structures produced by the same *stochastic L-system* where

$$P = \left\{p_1 : \mathtt{F} \xrightarrow{0.33} \mathtt{F[+F]F[-F]F}; \quad p_2 : \mathtt{F} \xrightarrow{0.33} \mathtt{F[+F]F}; \quad p_3 : \mathtt{F} \xrightarrow{0.34} \mathtt{F[-F]F}\right\}.$$

That is, production p_1 is applied to string F with probability 0.33.

The extension of L-systems to *parametric L-systems* by PRUSINKIEWICZ AND HANAN (1990) renders L-systems as a complete programming language.

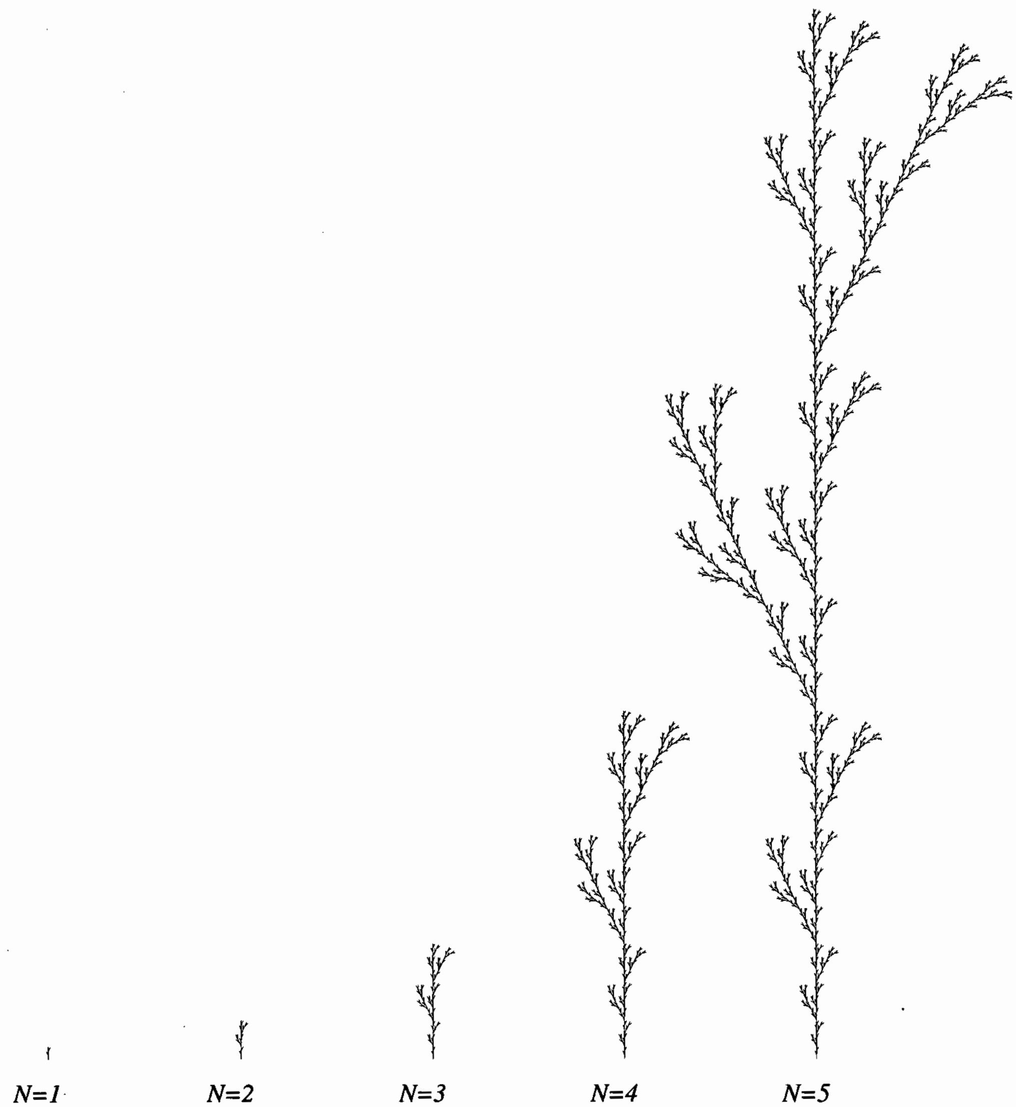


Figure 4.15: *Using L-systems to animate plant growth (*PRUSINKIEWICZ AND LINDENMAYER 1990*).*

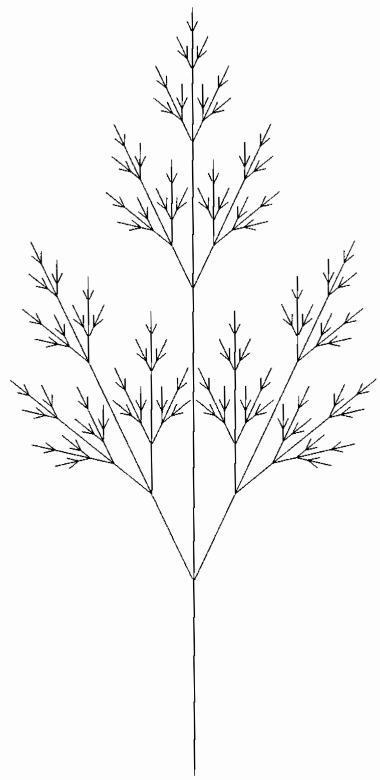


Figure 4.16: *An example of node-rewriting ($N = 6$, $\theta = 25.7°$) (*PRUSINKIEWICZ AND LINDENMAYER 1990*).*

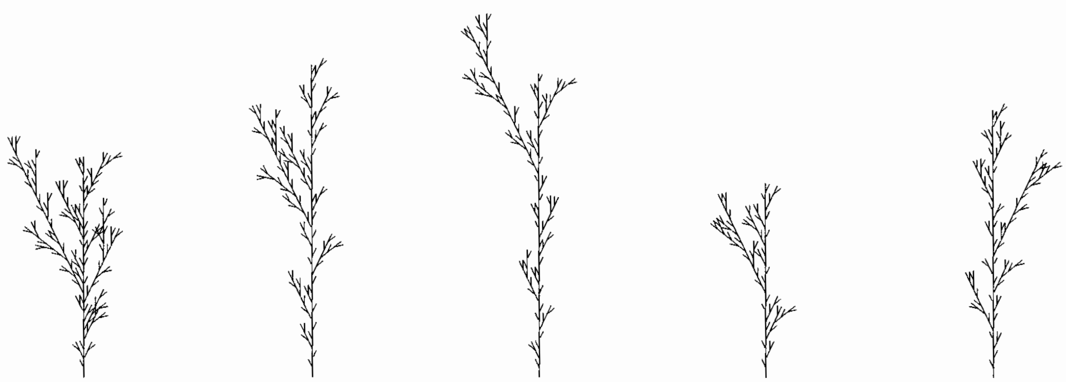


Figure 4.17: *An example of stochastic L-systems (*PRUSINKIEWICZ AND LINDENMAYER 1990*).*

That is, if a standard L-system can be thought of as a simple recursive program, then a parametric L-system can be thought of as a recursive program which allows parameter passing. For example, the symbol `F(a)` now means step forward a distance of length a. This is a very powerful extension. For example, the models of AONO AND KUNII (1984) and REEVES AND BLAU (1985) can now be modelled by L-systems (e.g., ORTH 1993).

4.2.3 Particle systems

REEVES AND BLAU (1985) utilized *particle systems* to create large, complex landscapes populated by various species of trees. Under their model, each tree starts as a single particle which carves out the trunk. Each particle moves in a fixed direction, velocity and direction, etc., according to a set of stochastic laws based upon the modelled species. After a particle dies, it gives birth to new

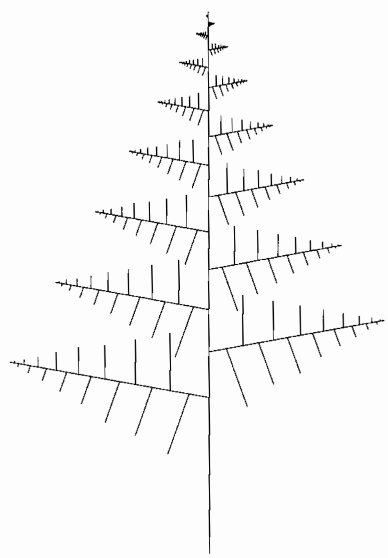


Figure 4.18: *A simple 2D particle system (*REEVES AND BLAU 1985*).*

particles which may inherit all or only some of its characteristics. Overall tree shape, either triangular or elliptic, is controlled by carefully choosing particle movement. Each particle is seen to be the base of its own tree. REEVES AND BLAU (1985) produced one of the most complete and comprehensive uses of randomization: including a fully randomized particle system, planting follows a random distribution according to terrain (e.g., evergreen are more likely than populars to be on hills) and the branching patterns (e.g., length, width, angles) are taken from distributions of real trees.

GREEN AND SUN (1988) created a special language for procedural modelling and motion. For example, with this new language they implemented a particle system for tree drawing; they draw the trees and then add tropisms such as wind to form an animation of a tree swaying back and forth in the wind.

ORTH (1993) compared particle systems and L-systems with regards to tree and grass drawing and concluded that ultimately the differences between them are more syntatic than semantic.

WEBER AND PENN (1995) provided a particle system-like method for rendering realistic images at any distance in real-time. They followed REEVES AND BLAU but specified for a much wider variety of plants. Furthermore, the recursive structure need not remain constant, i.e., it can change with levels. This is akin to the changing distributions in Chapter 2.5.5.

4.2.4 Combinatorial methods

As already mentioned in the introduction from Chapter 1, we are not the first to use combinatorial trees to generate images of botanical trees.

Ramification matrices

VIENNOT ET AL. (1989) were the first to use the HORTON-STRAHLER number to draw botanical trees.[7] VIENNOT ET AL. used the following analogy of flow in a river and sap flow in a tree. As there is a relationship between the size of a river segment and its corresponding HORTON-STRAHLER value in the rivers underlying binary tree, VIENNOT ET AL. took binary trees and used HORTON-STRAHLER orders as a gauge of sap flow to determine branch layout. What separates this technique from previous ones is the ability to separate shape from development, i.e., direct control over the combinatorial tree. This control is provided by a *ramification matrix* which is used to generate the underlying combinatorial tree. The underlying combinatorial tree is generated in the following way. As each node with HORTON-STRAHLER order k has bi-order either (k, j) when $k > j$ or $(k-1, k-1)$. Let $p_{i,j}$ be the probability of a node with order k having bi-order (i, j). For example, consider the matrix

$$\begin{bmatrix} 0.4 & 0.6 & & & & \\ 0.2 & 0.3 & 0.5 & & & \\ 0.1 & 0.2 & 0.3 & 0.4 & & \\ 0.05 & 0.1 & 0.2 & 0.3 & 0.35 & \\ 0.025 & 0.05 & 0.1 & 0.2 & 0.3 & 0.325 \end{bmatrix}.$$

Starting at the root node with order k, one uses the above matrix to randomly select its bi-order (i, j) (and thus generate the root's corresponding children

[7] This is also reported in EYROLLES (1986), VIENNOT (1990), LE MÉHAUTÉ (1991) and ALONSO AND SCHOTT (1995).

nodes with orders i and j). One recurses on these children until the nodes with order one (i.e., the external nodes) are reached. By shuffling matrices together (e.g., by swapping rows), hybrid combinatorial trees can be generated (cf. the varying distribution technique from Chapter 2.5.5 and WEBER AND PENN 1995).

These combinatorial trees are modelled as geometric trees according to the following functions. For cylinder μ corresponding to node u with order $S(u)$, the length function is

$$L(\mu) = c_L S(u) \quad \text{or} \quad c_L S(u)^2,$$

and the radius function is

$$R(k) = c_R S(u)^{c_{R_2}} \quad \text{or} \quad c_2 {c_{R_2}}^{S(u)},$$

where c_L, c_{R_1}, and c_{R_2} are constants. The deviation angle for cylinder ν with sibling cylinder ω and parent cylinder μ is:

$$\Theta(\nu) = \begin{cases} c_{\Theta_m} \frac{S(w)}{S(v)-1}, & \text{if } S(v) > S(w); \\ c_{\Theta_s} \frac{S(w)-S(v)}{S(w)-1}, & \text{if } S(v) < S(w); \\ c_{\Theta_f}, & \text{if } S(v) = S(w); \end{cases}$$

where c_{Θ_m}, c_{Θ_s} and c_{Θ_f} (the forking angle) are constants typically 10°, 30°, and 30° respectively. As these geometric trees are two dimensional, $\alpha(\mu) = 0°$. For example, Figure 4.19 shows a drawing of a tree (generated by the above ramification matrix) whose root has HORTON-STRAHLER order six. By drawing the child cylinders as rectangles that touch along the sides rather than along the axis, they are able to fill in the resulting notches with small triangles  .

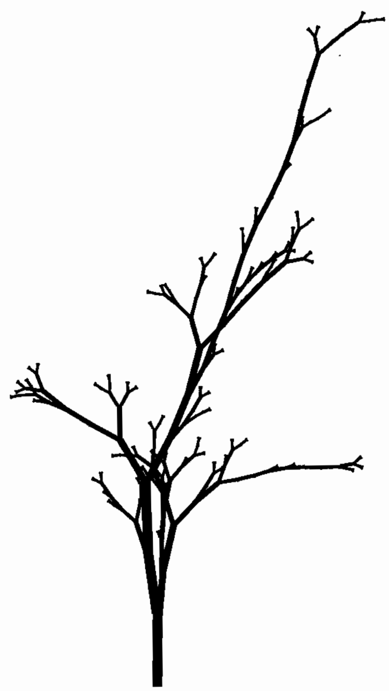


Figure 4.19: *A binary tree drawn according to the* HORTON-STRAHLER *number (*VIENNOT ET AL. 1989*).*

Random binary trees

As already shown in Chapter 2.5.2, a ramification matrix is not the only way to generate random combinatorial trees. Probabilistic analysis techniques (e.g., see DEVROYE 1994 or MAHMOUD 1992) provide algorithms for randomly generating many distributions of computer science trees (e.g., binary search trees, tries and PATRICIA trees). KRUSZEWSKI (1994) used these algorithms to generate random binary trees and then applied the modelling rules from VIENNOT ET AL. to produce geometric trees (see also KRUSZEWSKI 1993).[8] For example, one way of generating a random *binary search tree* is to use random split trees such that X is a a $[0, 1]$-value *uniform* random. Similarly, as already discussed in Chapter 3, random PATRICIAs follow the binomial distribution.

Once T has been generated, the HORTON-STRAHLER orders are recursively calculated. Finally, T is traversed in preorder during which the corresponding geometric tree is created from the modelling rule of VIENNOT ET AL. For example, Figure 4.20 shows a rendering of a random binary search with 500 internal nodes. Figures 4.20 and 4.19 are similar in appearance because the ramification matrix used to generate Figure 4.19 was based on experimental HORTON-STRAHLER orders for random binary search trees and the same modelling rules were applied.

DEVROYE AND KRUSZEWSKI (1995) generalize the technique by KRUSZEWSKI (1994) both combinatorially and geometrically. They introduced the beta trees which we used in 2.5.2. In terms of modelling functions, DEVROYE AND KRUSZEWSKI (1995) can be seen as the most general technique, since any function is now permissible, particularly, functions based on subtree size. This strategy arises from the results in probabilistic analysis (e.g., DEVROYE AND KRUSZEWSKI 1994 and DEVROYE AND KRUSZEWSKI 1996) that for many families of random binary trees, the *expected* HORTON-STRAHLER number is logarithmic with respect to the number of nodes in the tree. Thus, DEVROYE AND KRUSZEWSKI (1995) proposed to use $c \log |u|$ as a replacement for $S(u)$. Experimentation showed that the æsthetically most pleasing results for length and radius functions are

$$L(\mu) = c_L \sqrt{|u|}$$

and

$$R(\mu) = c_R \sqrt{|u|}.$$

However as Table 4.2 shows, many length and radius functions are possible (note: in the bottom row, the radius function is fixed).

[8] N.B. ARQUÈS ET AL. (1991) also visualized random tries with these modelling rules, however, the combinatorial trees were generated according to a technique by FLAJOLET ET AL. (1985).

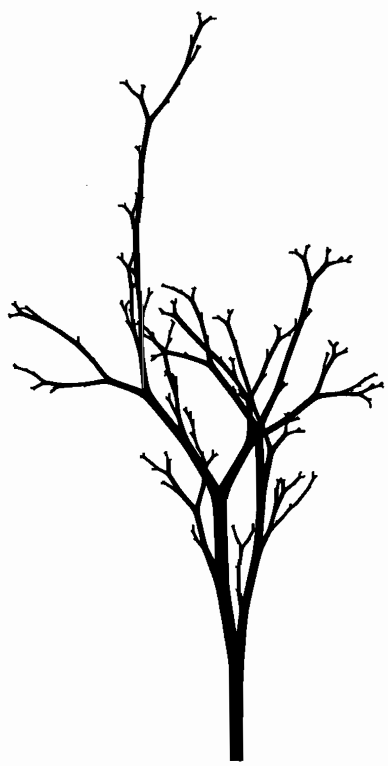


Figure 4.20: *A random binary search tree with 500 internal nodes (*KRUSZEWSKI *1994).*

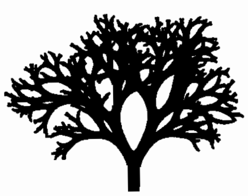
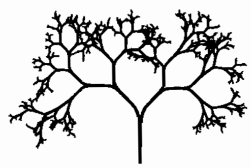
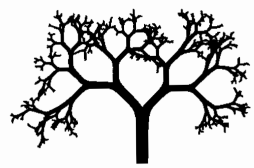
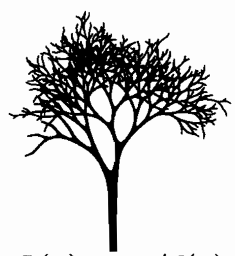
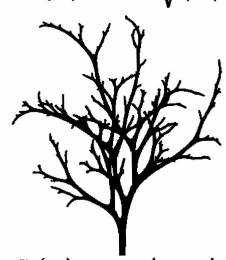
			
$L(\mu) = c_L \log\lvert u\rvert$ $R(\mu) = c_R \log\lvert u\rvert$	$L(\mu) = c_L \log\lvert u\rvert$ $R(\mu) = c_R\sqrt{\lvert u\rvert}$	$L(\mu) = c_L\sqrt{\lvert u\rvert}$ $R(\mu) = c_R \log\lvert u\rvert$	$L(\mu) = c_L\sqrt{\lvert u\rvert}$ $R(\mu) = c_R\sqrt{\lvert u\rvert}$
			
$L(\mu) = c_L/\sqrt{\lvert u\rvert}$	$L(\mu) = c_L/\log\lvert u\rvert$	$L(\mu) = c_L/d(u)$	$L(\mu) = c_L\lvert\sin\lvert u\rvert\rvert$

Table 4.2: *Various length and radius functions for the same tree (*DEVROYE AND KRUSZEWSKI 1995*).*

As with the length and radius functions, any deviation function is possible. For example, the first figure in Table 4.3 shows the effect of the following function for cylinder ν

$$\Theta(\nu) = \begin{cases} c_{\Theta_m}, & \text{if } |v| > |w|; \\ c_{\Theta_s}, & \text{if } |v| < |w|; \\ c_{\Theta_f}, & \text{if } |v| = |w|; \end{cases}$$

where c_{Θ_m}, c_{Θ_s} and c_{Θ_f} (the forking angle) are constants typically 10°, 25°, and 30° respectively. This function is clearly inspired by the one by VIENNOT ET AL. The second figure shows the following choice of deviation angle

$$\Theta(\mu) = \frac{23^\circ}{d(u)}$$

which is similar to GMT4 from AONO AND KUNII (1984). However, DEVROYE AND KRUSZEWSKI (1995) find the following ratio (based on Da Vinci's NOTE 396) produces the most pleasing results

$$\Theta(\nu) = \frac{|w|}{|v| + |w|}.$$

This can be seen as an intuitive version of Murray's deviation angle function described in Chapter 2.3.3.

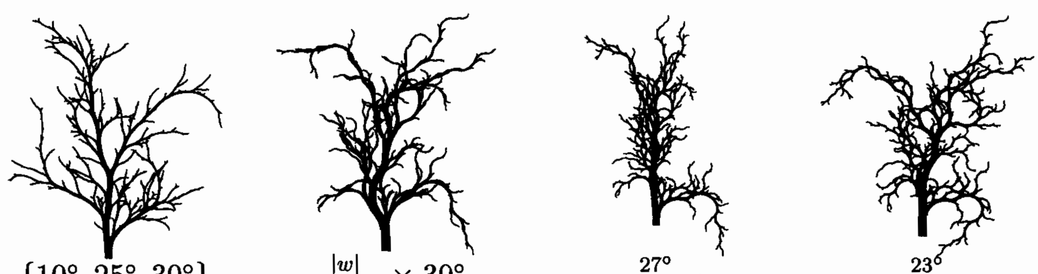


Table 4.3: *Various deviation angle functions for the same tree (*DEVROYE AND KRUSZEWSKI 1995*).*

This technique has the advantage over the original one by VIENNOT ET AL. (1989) in that the HORTON-STRAHLER number distribution need not be established before generating the tree. KRUSZEWSKI (1993) avoided notches completely by drawing the branches as quadrilaterals whose bases are attached directly to the tops of the corresponding parents  . KRUSZEWSKI (1994) padded the joint between the parent-child quadrilateral pair with Bézier curves for smoother branches  . However, whereas Viennot et al. have exact control over the tree order, Kruszewski has exact control over the number of branches. Viennot et al. can generate the tree in one pass whereas Kruszewski must first construct the tree in order to calculate the Horton-Strahler numbering and then draw the tree, thus requiring two passes.

As this is a two dimensional model, $\alpha(\mu) = 0°$. Three dimensionality is simulated by multiplying $\Theta(\mu)$ by $\cos(2\pi U)$ where U is a uniform random variable (e.g., see Figure 4.21).

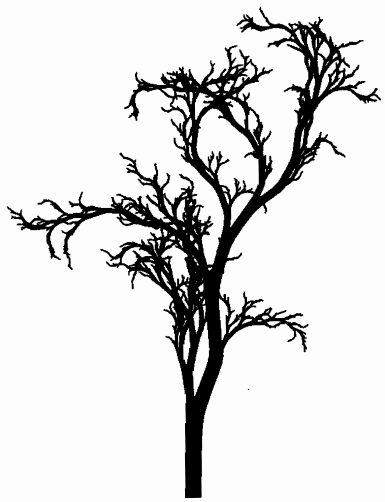


Figure 4.21: *A simulated 3D tree (*Devroye and Kruszewski 1995*).*

Tropism is the property by which an organism turns in a certain direction in response to external stimulus. In plants, this stimulus is primarily the sun and hence heliotropism has been incorporated into many models (e.g., Chiba et al. 1994). We simulate heliotropism according to sun position and intensity. With respect to intensity, we use the admittedly naïve idea that the larger the branch the more light it receives over its lifetime and thus the more it reacts by changing its angle. That is, for node u after θ_u is determined, θ_u is multiplied by an intensity factor (based on $|u|$) which pulls branch u closer to the sun. In Table 4.4, we take the beta(1,5) tree with 500 nodes and subject it to increasing sun intensity with the sun directly overhead. However, as Table 4.4 shows, we neglect to consider that leaves tend to spread out to maximize coverage.

Wind is also an important environmental factor. Both Viennot et al. 1989 and Kruszewski (1994) simulate wind by changing the underlying structure; the former always flips the larger branch to one side while the latter uses asymmetric tries. As Table 4.5 shows, by placing the sun directly overhead (i.e., perpendicular to the ground) and inverting the intensity function (i.e.,

Table 4.4: *A tree under increasingly intense sun (*DEVROYE AND KRUSZEWSKI 1995*).*

larger branches should bend less than smaller ones), wind can be reasonably simulated.

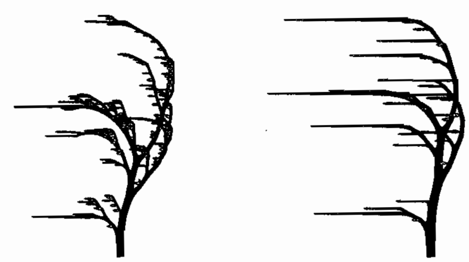


Table 4.5: *A tree under increasingly intense wind (*DEVROYE AND KRUSZEWSKI 1995*).*

Finally, if we set the wind to blow from above, we can simulate the effect of droughts or flexible branches such as those found in weeping willows.

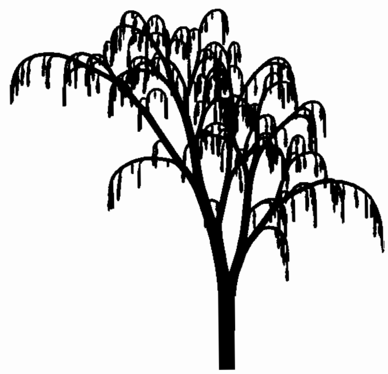


Figure 4.22: *A weeping willow* (DEVROYE AND KRUSZEWSKI 1995).

4.2.5 Branching processes

Split trees are not the only way to generate combinatorial trees: *branching processes* are another (e.g., see DEVROYE 1994). For our purposes, we limit our definition of a branching process to the following. A node gives birth to a number of children according to a fixed probability distribution where p_i is the probability of giving birth to exactly i children (all nodes follow the same distribution). Thus, $p_0 + p_1 + p_2 = 1$ and $p_i = 0$ for all $i > 2$. We start with a single node–the root, the first *generation* and let it reproduce. We then let the nodes of the second generation (the nodes at depth two) to reproduce. If $p_2 = 1$, then an infinite tree is produced such that if we stop after k generations, a complete tree with exactly $2^k - 1$ nodes results. If the expected number of children per node $m = p_1 + 2p_2 < 1$ then the branching process dies out with probability one and if $m > 1$ it will live forever. Hence, the necessity to limit the branching process to k generations.

Computer-simulated plant evolution

NIKLAS (1986) used genetic-like algorithms to model the evolution of plants based on fossils (see also NIKLAS AND KERCHNER 1984; cf. MACKENZIE 1993 who used L-systems to model plant evolution). As an evolutionary biologist, he desired to test various theories of plant development. Although computer simulations do not "prove" theories in the sense of a rigorous mathematical proof, they do facilitate the *hypothetic-deductive method*. That is, once a hypothesis is formed, its consequences are deduced and compared with the observed phenomena. If they are similar, the hypothesis is said to be "confirmed." As an unexpected bonus, his incremental growth model allows the creation of animations of the growth of plants which have been extinct for millions of years.

NIKLAS started with a simple plant model based on HONDA (1971) and parameterized it to correspond to a fossil approximately 410 million years old. There are three parameters: deviation angle, divergence angle and p_2 the probability of giving birth to exactly two children. The first two parameters correspond to those in HONDA. Growth is terminated after $k = 10$ generations. Finally, $L(\mu) = c_L$ and $R(\mu) = c_R$.

For each simulation he began with one plant and by generating random variations on the values of the three parameters, he produced a series of "mutations." He calculated the amount of sunlight each plant received and allowed the one plant which received the most sunlight to reproduce again (leaving the others to die out). By starting with a plant based on a fossil approximately 410 million years old, the sequence of plants generated by these simulations resemble actual fossils over a 60 million year period.

Stochastic Modelling

As the title *Plant Models Faithful to Botanical Structure and Development* suggests, DE REFFYE ET AL. (1988) tried to model plant growth as accurately as possible. With the goal of creating a simulation tool for agronomy and botany researchers, they tried to incorporate as much current biological information from these fields as possible (e.g., see HALLÉ ET AL. 1978). For example, they modelled seasonal tree-aging and the effects of the physical world such as wind, pestilence, fertilizer. Plant simulation is based on stochastic bud metamorphisms (thereby allowing such events as producing fruit and aging). The simulation starts with a single bud. At each time unit, each existing bud can undergo one of the following transformations:

- become a flower and die (and thus disappear);
- become dormant;
- become an internode (a branch with one or more new buds);
- simply die (and disappear).

These events occur with probabilities based on known stochastic laws for each species. Furthermore, the geometric parameters for the branch layout, such as length, width, deviation and divergence angles, are calculated according to specific and known stochastic laws (cf. REEVES AND BLAU 1985). The bending of branches by gravity is simulated according to known material strength laws (e.g., MCMAHON AND KRONAUER 1976; MATTHECK 1991). It should be noted that this approach to tropisms is much more sophisticated than the approaches such as AONO AND KUNII (1984) where simple attractors are used.

4.2.6 Environmental Modelling

PRUSINKIEWICZ (1993) divided visual plant models into two types: *structure-oriented* and *space-oriented.* In a structure-oriented model, a plant grows according to specific rules which do not directly take the environment into account. For example, most of the models presented in the chapter, along with our own model, are structure-oriented. In a space-oriented model, the environment surrounding the plant takes on importance in its development. Thus, for structure-oriented models tropisms are seen to influence the final arrangement of the geometric cylinders while for space-oriented models, tropisms directly affect and determine the underlying combinatorial trees. Indeed, it is difficult to separate the combinatorial from the geometric trees. Finally, we note that since this method implies relatively sophisticated simulations, we restrict our attention to only visual plant models. Consequently, we ignore biologically intricate work such as the LIGNUM model by PERRTUNEN ET AL. (1996).

Voxel space automata

GREENE (1989) used volume elements called *voxels* to generate realistic-looking vegetation that is capable of reacting to its environment. Voxels allow efficient ray-casting which in turn allows a simple method of determining how much light hits an object. In terms of plants, this allows sophisticated simulation of heliotropism. Via various growth rules, heliotropism allow vines to creep over three dimensional surfaces such as walls and houses.

L-systems

PRUSINKIEWICZ ET AL. (1994) introduced an "environmentally-sensitive" extension to L-systems in which a plant can grow within the limits of a geometric shape. This can simulate the pruning of a human gardener to create a "synthetic topiary."

PRUSINKIEWICZ ET AL. (1996) proposed a framework to study the competition of plants for land and light and the interaction of roots with water in the soil.

Imaginary plant hormones as growth regulators

Rather than using a centralized control, CHIBA ET AL. (1994) regulated tree growth by simulating imaginary plant hormones. In botany various plant hormones are believed to control plant growth (e.g., see WAREING AND PHILLIPS 1978 for an introduction to the pertinent biology). However, according to CHIBA ET AL. these mechanisms are not exactly understood so they invented their own virtual plant hormone. This "hormone" is produced by active buds and is transmitted throughout the entire tree (this is perhaps analogous to idea of the flow of Chi throughout the human body.) At each growth step, the hormone is redistributed until a new equilibrium is obtained (via an iterative technique). Under this model heliotropism affects not only the branch position but also branch life. For example, if a leaf does not receive enough sunlight, it dies and its corresponding branch withers. The amount of hormone that a branch receives affects its ability to reproduce. Thus, the "hormone" controls the bushiness of the tree.

Chapter 5

Conclusion

In this thesis, we have proposed a general combinatorial-geometric model for creating synthetic images of botanical trees. In Chapter 2 we explained our basic model and showed how to parameterize it for specific modelling functions and split distributions (both deterministic and random beta). In Chapter 3 we reviewed the biological research on the bifurcation ratios for specific species of botanical trees. Here (and in Appendix B), we showed how random combinatorial trees with specific expected bifurcation ratios can be generated by choosing the corresponding p from the binomial distribution for the split trees. This allows us to model *combinatorially* several species of botanical trees. Finally, we explored geometric functions and modelling techniques to model these trees better. In Chapter 4 we reviewed previous techniques in the field from the perspective of our general model. Under our model almost all of the structure-oriented models can be seen as specific parameterizations of our general model. Before evaluating the significance of this thesis, we will examine future work and applications, as these influence the evaluation.

5.1 Future work

In this section, we outline possible directions for further research that arise from work presented in this thesis.

5.1.1 A tool for architects and artists

As mentioned in Chapter 1, trees form a significant part of most scene animations. Our model provides an effective way of generating a wide variety of tree-like shapes. As it is highly parameterizable, we feel that it will provide a solid basis for a highly interactive editor that could allow users such as artists and architects to design interesting looking trees quickly. Not only can three dimensional objects be generated but their two dimensional renderings can also be used as texture for billboarding techniques.

5.1.2 A modelling tool

Our approach to modelling specific species of botanical trees is admittedly simple. It would be interesting to take our basic model and to attempt a detailed parameterization of it for a particular species of botanical tree. Furthermore, not only would the visual aspects of the tree be modelled but also it would be important to generate data on the geometric aspects of it, e.g., average geometric height, volume, etc.

If one assumes that Horton's first law holds for certain species of botanical trees[1] then we have presented a method which models these botanical trees *combinatorially*. A more rigorous combinatorial test would be to find a branching pattern that would simultaneously produce, for example, both the correct HORTON-STRAHLER and Horsfield orderings.

As we have not gathered physical statistics from our simulations, we can only judge the validity of our physical modelling functions on a visual basis. It would be interesting to explore Horton's second law, that cylinder radius and length also obey geometric laws (e.g. see PARK 1985) with this model. In terms of geometric trees, it would be interesting to calculate the average values such as cylinder length and radius and compare them with known data. For example, with regard to cylinder radius we currently use the exponent 3, but when should we use 2.49? Our current length function produces good visual results but we would like to know the reasons why.

Finally, as many botanical trees are not strictly k-ary, it is important to extend this (or any model) to model trees whose branches have an arbitrary number of child branches.

5.1.3 Animation of tree growth

Our current method of generating combinatorial trees does not facilitate the animation of tree development. However, we could animate the development by building up our combinatorial trees node by node. It is well-known (e.g., see DEVROYE 1994) that PATRICIA trees can also be built in the following way. We start with an initial random string that corresponds to an external node. We build up the tree one node at a time by inserting random strings consecutively. For example, ARQUÈS ET AL. (1991) followed a similar approach to animate the development of the trees in VIENNOT ET AL. (1989).

[1] This is still an unresolved matter (e.g., see BERNTSON 1995, p. 280), since it remains unclear which ordering system (i.e, Horton, Horsfield or Wiebel) is more accurate biologically.

5.1.4 Combinatorial visualization

LE MÉHAUTÉ (1991) reported that the botanical tree drawing technique of VIENNOT ET AL. (1989) has been used as a tool for speech analysis. The possible pronunciations of a particular word are modelled by a Markov chain (e.g., see FOURNOT ET AL. 1989 or SABAH 1989) which can in turn be represented as a stochastic matrix of transition probabilities. By using this matrix as input to the algorithm, a possible pronunciation can be visualized. Physical simulation techniques such as simulated annealing (e.g., CRUZ AND TWAROG 1995) and spring algorithms (e.g., EADES 1984) have already been successfully applied to graph drawing. Furthermore, we note the similarity between MANDELBROT's fractal model of the lung and the space-filling H-trees of VLSI layout. Due to the biological necessity of seeking out sunlight, the growth of botanical trees can be regarded as nature's way of laying out combinatorial trees. It would be interesting to explore the idea of whether or not it is sometimes helpful to visualize a combinatorial tree as a botanical one.

We propose a novel method of visualizing massive combinatorial trees. We envision two different types of applications: animation of algorithms and navigation of large hierarchies such as file systems.

For the first type of application, the idea is to provide some kind of shape analysis. Researchers working with large combinatorial trees are often concerned about tree shape and how the shape changes over time (e.g., over a sequence of insertions and deletions). Typically, one is concerned whether or not a tree is bushy or sparse, e.g., bushy implies more balanced and a shorter height. Another interest is in the changes in the tree's bushiness. For example, in some search problems a bushy search tree is less desirable than a sparse one. Perhaps this technique can be used as a tool to remark visually how changing the search algorithm results in a better tree.

A second type of application would be to use the layout as a navigation tool that aids the user to better understand the tree by providing an informative layout (e.g., see REISS 1994 or the cone trees of ROBERTSON ET AL. 1991). Unlike simple balanced search trees, many rooted combinatorial trees such as file system hierarchies have nodes with *arbitrary fan-out*, i.e., k child nodes where $k \geq 0$. We propose the following generalization. An internal node u now has k child nodes v_1 to v_k. Correspondingly, an internal cylinder μ now has k child cylinders ν_1 to ν_k. We generalize Murray's results for *cylinder radii* as:

$$R(\mu) = \begin{cases} c_{R_1} & \text{if } |u| = 1; \\ \left(\sum_{i=1}^{k} R(\nu_i)^{c_{R_2}}\right)^{1/c_{R_2}} & \text{otherwise.} \end{cases}$$

We can think of the axis layout of the two children in terms of a spring algorithm. Place the children on a circle with equal repulsive force between children. The two children end up at opposite sides of the circle, which is equivalent to being coplanar. We now generalize this circle for k children so that each child is $\frac{360°}{k-1}$ away from its two consecutive siblings. We now apply the following deviation angle to it:

$$\Theta(\nu_j) = c_\Theta \times \frac{\sum_{i=1}^{k} R(\nu_i) - R(\nu_j)}{\sum_{i=1}^{k} R(\nu_i)}.$$

The length and divergence functions remain unchanged.

5.1.5 Lightning

As noted by REED AND WYVILL (1994), the synthetic[2] reproduction of lightning for visual purposes has been overlooked by the computer graphics community. This lack of work is particularly curious given the relatively large amount of work in the field of image synthesis of botanical trees. REED AND WYVILL (1994) were the first to generate synthetic images of lightning. A particle system was used to generate the geometric structure of the lightning bolts. We propose a synthesis and extension of this work with our own.

We first show how the model by REED AND WYVILL can be parameterized according to our model. For lightning segment μ, the radius function is

$$R(\mu) = c_R r_{R_m}^{m(u)} r_{L_s}^{s(u)}$$

where $c_R \geq 0$ (initially chosen at random), $r_{R_m} = 0.95$, $r_{R_s} = 0.5$ are constants. The length function is

$$L(\mu) = c_{L_1} * U(c_{L_2}, c_{L_3}),$$

where c_1, $c_{L_2} \leq 1 \leq c_{L_3}$ are constants and $U(x, y)$ is a uniform random variable ranging from x to y. The deviation angle is

$$\Theta(\mu) = N(16, 1)°,$$

where $N(16, 1)$ is a normal random variable with mean 16 and standard deviation 1. The divergence angle is mentioned but unspecified. They used a branching process to generate the underlying combinatorial tree which they found to be very sensitive to the seed selected for the number generator and therefore they wanted to formalize the branching algorithm used to generate lightning channels. These two points can be addressed by using random split trees. Furthermore, REED AND WYVILL (1994) also mentioned that in actual lightning branching is frequently near the ground. This could be modelled by making a and b of the beta distribution a function of node depth such as Table 2.6. Finally, for our own simple rendering in Figure 5.1 we replace straight branch segments with random walks.

[2] By synthetic, we mean that the method used did not necessarily follow known physical laws of lightning but rather were aimed at producing convincing images.

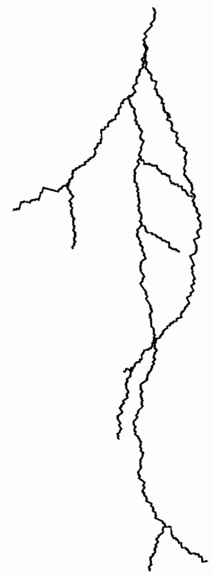


Figure 5.1: *A simple lightning bolt.*

5.1.6 Modelling the lung

In 1970, Cumming et al. applied the Horton-Strahler ordering to the pulmonary artery and the bronchial tree in the human (e.g., see Table 5.1).

	Max Order	Branching Ratio	Diameter Ratio	D_{17} (cms)	Length Ratio	L_{17} (cms)
Pulmonary Artery	17	3.39	1.71	1.86	1.80	12.2
Bronchial Tree	17	2.74	1.40	1.61	1.49	5.0

Table 5.1: Horton-Strahler *data on the human lung (*Cumming et al. 1970*).*

Many researchers have experimented with random branching patterns in the hope of finding a method of generating random combinatorial trees with a specific expected bifurcation ratio (e.g., Berry et al. 1975, Hollingworth and Berry 1975, Berry and Bradley 1976, Horsfield 1980, Pelt and Verwer 1983, Pelt and Verwer 1984, Horsfield and Woldenberg 1986a, Horsfield and Woldenberg 1986b)

As described in Horsfield and Woldenberg (1986a), attempts have been made to simulate the corresponding combinatorial trees of the bronchial tree by *random segmental* and *random terminal growth*. In both cases, the experimental work showed that neither of these two methods produced the correct expected bifurcation ratios. From a probabilistic analysis perspective, it is clear why.

HORSFIELD AND WOLDENBERG (1986A) (and others before them) determined that segmental branching produces an expected bifurcation ratio $\mathbf{E}R = 4.0$. As segmental branching produces equiprobable binary trees, this agrees with the theoretical results reviewed in Appendix B.1.2.

For terminal branching, they and others show that $\mathbf{E}R = 3$. As terminal branching can be shown to be equivalent to generating random binary search trees, this can be partially explained by Theorem C.3 in Appendix C which states that for a random binary search tree with n nodes and HORTON-STRAHLER number S_n,

$$\lim_{n\to\infty} \mathbf{P}\left\{ S_n \geq \frac{(1+\epsilon)}{\log 3} \log n \right\} = 0,$$

for all $\epsilon > 0$.

HORSFIELD AND WOLDENBERG (1986A) concluded that "some other growth process must therefore be operative in the bronchial tree." From Fact 3.4, we now propose to combinatorially model the bronchial tree as a random patricia tree with $p = \frac{1}{2.74}$.

The alveoli are arbitrarily defined as of HORTON-STRAHLER order one and it is known that there are approximately 3×10^8 alveoli. Therefore, for our purposes, it is impractical to model each alveolus. We propose the following approximation. We generate a combinatorial tree with 1000 nodes that represents the bronchial tree from the largest segment downward.

CUMMING ET AL. (1970) found that not only did Horton's first law apply to the human lung but also Horton's second law. *Horton's second law* states that a segment (originally river segments but here artery and lung segments) with order k has expected diameter $(c_D)^k$ where c_D is the diameter of an order one segment and expect length $(c_L)^k$ where c_L is the length of an order one segment. Thus, for lung segment μ,

$$R(\mu) = \left(\frac{1.61}{2}\right) \times 1.40^{S(u)}$$

and

$$L(\mu) = 5 \times 1.49^{S(u)}.$$

Beginning with the work of MURRAY (1926A,B;1927), many researchers have suggested different models for deviation angle in natural branching structures such as arteries (e.g., ROY AND WOLDENBERG 1982, WOLDENBERG AND HORSFIELD 1983, WOLDENBERG AND HORSFIELD 1986). However, for the time being, we continue to use our deviation angle function. As we could not find information on the divergence angle for either lungs or arteries, we use our constant of 137.5°. Barring this, we can simulate the lung both *combinatorially* and *geometrically*. Figure 5.2 shows an image of one half of a bronchial tree. Figure 5.3 shows a corresponding image of the artery system configured in a similar manner.

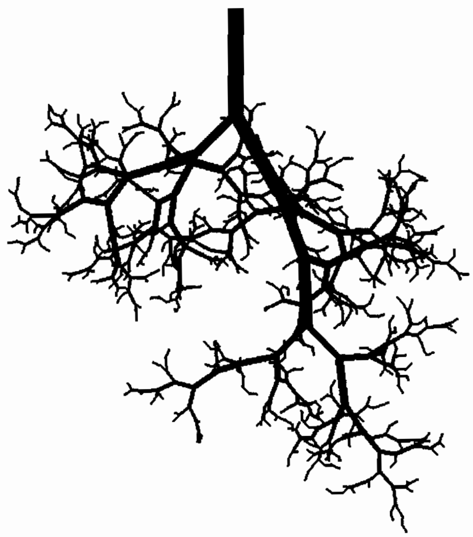


Figure 5.2: *A portion of the bronchial tree.*

Figure 5.3: *A portion of the pulmonary artery.*

5.2 Contributions of this thesis

Here we summarize by application area what we view are the main contributions of this thesis.

Graphics: We have described a method to generate three dimensional models of botanical trees that is simple to understand and implement, fast running, and with high database amplification. Apart from brief treatment in PRUSINKIEWICZ AND LINDENMAYER (1990) and Chapter 20 of FOLEY ET AL. (1996) on *Advanced Modelling Techniques*, there has been little unifying work in this field. By using the rubic of our basic combinatorial-geometri model from Chapter 2, Chapter 4 can be seen as one of the most comprehensive listings so far. Furthermore, our basic model from Chapter 2 facilitates the comparison and understanding of many previous techniques. Thus, this thesis has introduced an important technique; furthermore, as Chapter 5.1.5 suggests, this technique can be applied to branching phenomena such as lightning.

Biology: The generation of random binary trees with specific expected bifurcation ratios has been an open problem in many fields of theoretical biology ranging from botany to physiology. Not only does our method of binomial split trees provide such a technique but it also corresponds nicely with the biologic-theoretic pipe model of (SHINOZAKI ET AL. 1964A).

Finally, we have expressed the ordering systems of biologists in terms of combinatorics.

Probability: Although the Horton-Strahler number has been well studied for the equiprobable binary tree model, there has been little theoretical work on other models. Our result for random tries from Appendix B is the first for other random data structures.

L-systems: We have demonstrated in Appendix A that L-systems can model split trees.

Appendix A

Implementation Details

A.1 L-systems

As mentioned in Sections 4.2.2 and 4.2.6, L-systems have been extended into a powerful modelling language. The majority of the original images in this thesis were produced using *The Virtual Laboratory Environment*. This software package, called `vlab`, is an interactive environment for creating and conducting simulated experiments using L-systems. `vlab` is an on-going research project by PRUSINKIEWICZ's team (e.g., see MERCER ET AL. 1990) at the University of Calgary. It runs on Silicon Graphics workstations and is available by anonymous `ftp` (see `http://www.cpsc.ucalgary.ca/projects/bmv/- index.htlm`). For a more detailed description, see Appendix A of PRUSINKIEWICZ AND LINDENMAYER (1990).

An L-system typically consists of two parts: a series of *definitions* (constants and external procedure calls) and a series of *production rules*. We now explain the basic L-system used in our thesis. Table A.1 shows the necessary definitions which, although technically not part of the L-system language allow for efficient external function calls. This is a variation of the *development controlled by resource allocation* in Section 7.3.3 of PRUSINKIEWICZ ET AL. (1996).

Definition d_1 declares the number of internal cylinders. Definition d_2 is the limit on the number of derivations. Definition d_3 is the random variable function. For example, for deterministic trees, we define it as `#define RAND 0.5`. Definitions d_5 to d_{11} define our geometric functions.

Table A.2 shows the productions. A geometric tree is built in the following way. We start with the axiom `CYLINDER(N,0,0,0,0,0)`. Production p_1 expands this axiom. That is, p_1 generates the combinatorial tree from string `CYLINDER(n,m,r,t,a,b)` where `n` is the number of internal cylinders at this subtree; `m` is a message marker; `r` is the radius of this cylinder; `t` denotes whether the cylinder is the left or right child of its parent; `a` and `b` are the deviation angles for the cylinder's children. Production p_2 stops the growth

d_1	:	`#define N 100`
d_2	:	`#define STEPS 1300`
d_3	:	`#define RAND bran(10,10)`
d_4	:	`#define SPLIT(n) (floor(n*RAND))`
d_5	:	`#define unit_radius 1`
d_6	:	`#define RADIUS(r1,r2) ((r1^3+r2^3)^(1/3))`
d_7	:	`#define LENGTH(r) (2*(r^1.2))`
d_8	:	`#define DEVIATION(t,a1,a2) (t>0 ? t*a1 : t*a2)`
d_9	:	`#define left_theta 1`
d_{10}	:	`#define right_theta -1`
d_{11}	:	`#define THETA(r,r1,r2) (acos((r^4+r1^4-r2^4)/(2*r^2*r1^2)))`

Table A.1: *Definitions for our generic tree L-system.*

of the combinatorial subtree when an external node is created. Production p_3 collects the child cylinders' radii and uses them to calculate the parent cylinder's own radius and the child cylinders' deviation angles. Production p_4 passes the deviation angles from the parent cylinders to their child cylinders. Production p_5 signals to the node that all information has been processed for its subtree and hence it is ready to be rendered. Production p_6 renders the internal cylinder, here as a simple column of radius `r` and length `LENGTH(r)`. Production p_7 renders the external cylinder, which in this case means simply eliminating the corresponding string.

```
p1 : CYLINDER(n,m,r,t,a,b) : m==0 && n>0
         {n1=SPLIT(n); n2=n-1-n1;}-->
         CYLINDER(n,1,r,t,a,b)
         [CYLINDER(n1,0,0,left_theta,0,0)]
         [CYLINDER(n2,0,0,right_theta,0,0)]
p2 : CYLINDER(n,m,r,t,a,b) : m==0 && n==0 -->
         CYLINDER(n,2,unit_radius,t,a,b)
p3 : CYLINDER(n,m,r,t,a,b) >
         [CYLINDER(n1,m1,r1,t1,a1,b1)][CYLINDER(n2,m2,r2,t2,a2,b2)] :
         m==1 && m1==2 && m2==2
         {r=RADIUS(r1,r2); a=THETA(r,r1,r2); b=THETA(r,r2,r1);} -->
         CYLINDER(n,2,r,t,a,b)
p4 : CYLINDER(n,m,r,t,a,b) < CYLINDER(n1,m1,r1,t1,a1,b1) :
         m==3 && m1==2  --> CYLINDER(n1,3,r1,DEVIATION(t1,a,b),a1,b1)
p5 : CYLINDER(n,m,r,t,a,b) : m==2 && n==N -->
         CYLINDER(n,3,r,t,a,b)
p6 : CYLINDER(n,m,r,t,a,b) : m==3 && n>0 -->
         !(2*r)+(t)/(137.5);F(LENGTH(r))
p7 : CYLINDER(n,m,r,t,a,b) : m==3 && n==0 -->
```

Table A.2: *Productions of our generic tree L-system.*

A.2 PostScript

Many of the figures from Chapter 4 were generated with the PostScript language. The basic program `botanical.tree.ps` is available by anonymous `ftp` at `ftp.cs.mcgill.ca` in the directory `pub/tech-reports/libarary/botanical.trees/`. IGA AND MONSARRAT (1991) provide a full set of PostScript macros for drawing in three dimensions. We used them early in our research; however, we stopped using them for several reasons, primarily because the `linmap` macros for mapping 2D planes onto 3D which we used for leaves in DEVROYE AND KRUSZEWSKI (1995) did not work when the PostScript file was incorporated into a TeX document using any known macro such as `\psfig`, etc. Moreover, the IGA AND MONSARRAT (1991) macro set did not allow us to either make our own 3D curves or to have control over the Bézier curves.

Originally we implemented a set of macros like IGA AND MONSARRAT (1991) following the scheme described on pp. 245–258 of FOLEY ET AL. (1996) and MONSARRAT (1994). This code, `3D.matrices.ps`, is also available by `ftp` at the same location as above. A.3 lists the implemented commands.

However, as PRUSINKIEWICZ AND LINDENMAYER (1990) showed, most styles of tree drawing, including our own, lend themselves nicely to the turtle

inputs	command	output	description
—	matrix3	*matrix*	create a 4 by 4 identity matrix
—	initmatrix3	—	set CTM3 to device default
matrix	identmatrix3	*matrix*	fill *matrix* with identity transform
matrix	defaultmatrix3	*matrix*	fill *matrix* with device default matrix
matrix	currentmatrix3	*matrix*	fill *matrix* with CTM3
matrix	setmatrix3	—	replace CTM3 by *matrix*
tx ty tz	translate3	—	translate user space by *(tx, ty, tz)*
sx sy sz	scale3	—	scale user space by *sx*, *sy* and *z*
angle	rotatex3	—	rotate user space about the x-axis by *angle* degrees
angle	rotatey3	—	rotate user space about the y-axis by *angle* degrees
angle	rotatez3	—	rotate user space about the x-axis by *angle* degrees
matrix	concat3	—	replace CTM3 by *matrix*×CTM
matrix1 matrix2 matrix3	concatmatrix3	*matrix3*	fill *matrix3* with *matrxi1*×*matrix2*
—	newpath3	—	initialize current path to be empty
—	currentpoint3	*x y z*	return current point coordinate
x y z	moveto3	—	set current point to *(x,y,z)*
—	closepath3	—	connects projected sub path back to its starting point
—	strokepath3	—	strokes the projection of the current path
x1 y1 x2 y2 x3 y3	curveto3	—	appends a Bézier curve in the xy-plane

Table A.3: *Matrix based 3D macros for PostScript.*

graphics approach as described by ABELSON AND DISESSA (1982). Tables A.4 and A.5 list the respective two and three dimensional turtle commands. These macros are contained in `turtle.ps` and are available by `ftp` at the same location as above.

inputs	command	output	description
distance	FORWARD2	—	move the turtle forward by *distance* units and draw a line
distance	GOTO2	—	move the turtle forward by *distance* units without drawing a line
angle	LEFT2	—	rotate the turtle to the left by *angle* degrees
angle	RIGHT2	—	rotate the turtle to the right by *angle* degrees
—	CURRENT_POINT2	*x y*	return the current point coordinate of the turtle
—	GSAVE2	—	save current graphics state of turtle
—	GRESTORE2	—	restore previous graphics state of turtle
—	MOVETO2	—	set current point to turtle's location
distance	LINETO2	—	draw a line of length *distance* without moving the turtle

Table A.4: *Two-dimensional turtle commands for PostScript.*

inputs	command	output	description
distance	FORWARD3	—	move the turtle forward by *distance* units and draw a line
distance	GOTO3	—	move the turtle forward by *distance* units without drawing a line
angle	ROTATE_X	—	rotate the turtle about the x-axis by *angle* degrees
angle	ROTATE_Y	—	rotate the turtle about the y-axis by *angle* degrees
angle	ROTATE_Z	—	rotate the turtle about the z-axis by *angle* degrees
—	CURRENT_POINT3_2	*x y*	return the current point coordinate of the turtle
—	GSAVE3	—	save current graphics state of turtle
—	GRESTORE3	—	restore previous graphics state of turtle
—	MOVETO3	—	set current point to turtle's location
distance	LINETO3	—	draw a line of length *distance* without moving the turtle

Table A.5: *Three-dimensional turtle commands for PostScript.*

Appendix B

The Horton-Strahler number for random binary tries

In this appendix we prove the necessary theorems that imply Fact 3.4 from Chapter 3.2 that allowed us to model combinatorially specific species of botanical trees. We note that this appendix is joint work with Luc Devroye and appears in DEVROYE AND KRUSZEWSKI (1996).

B.1 Introduction

B.1.1 The trie model

In 1960, FREDKIN coined the term *trie* for an efficient data structure to store and re*trie*ve strings. This concept was further developed and modified by KNUTH (1973B), LARSON (1978), FAGIN ET AL. (1979), LITWIN (1981), AHO ET AL. (1983) and others. The tries considered here are constructed from n independent infinite binary strings $X_1, \ldots, X_n$. Each string defines an infinite path in a binary tree: a 0 forces a move to the left, and a 1 forces a move to the right. An *infinite p-trie* is a random binary tree obtained by highlighting n infinite paths (from the root down). These paths are independent and are described by independent, identically distributed (i.i.d.) sequences of Bernoulli (p) random variables, $0 < p < 1$. For example, Figure B.1 shows an infinite p-trie built from the infinite strings 01001…, 01011…, 10011…, 10100… and 11100…. The tree is now pruned so that it has just n leaves[1] at the n representative nodes (e.g., see Figure B.2). That is, the *finite p-trie* is the infinite p-trie maximally trimmed so that each of the n infinite paths is finite and visits at least one node not visited by any other path (that node is necessarily a leaf of the future p-trie). Observe that no representative node is allowed to be an ancestor of any other representative node. This implies

[1] N.B. Here in this appendix only, *leaf* refers to a node with no children.

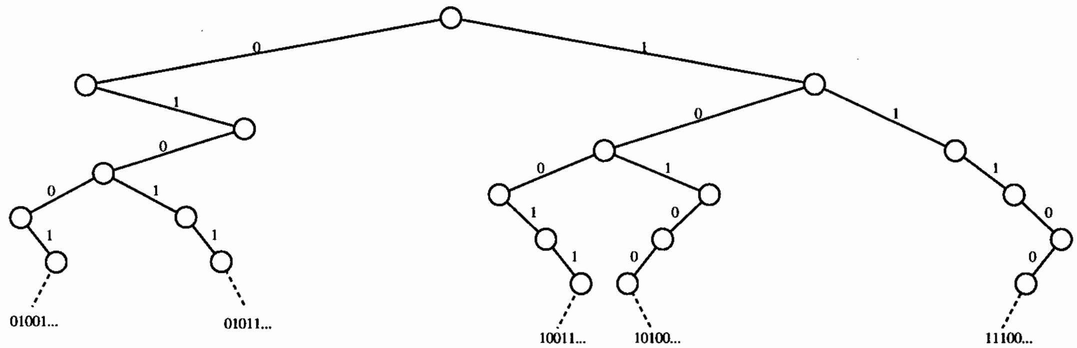


Figure B.1: *An infinite p-trie.*

that every internal (non-leaf) node has at least two leaves in its collection of descendants.

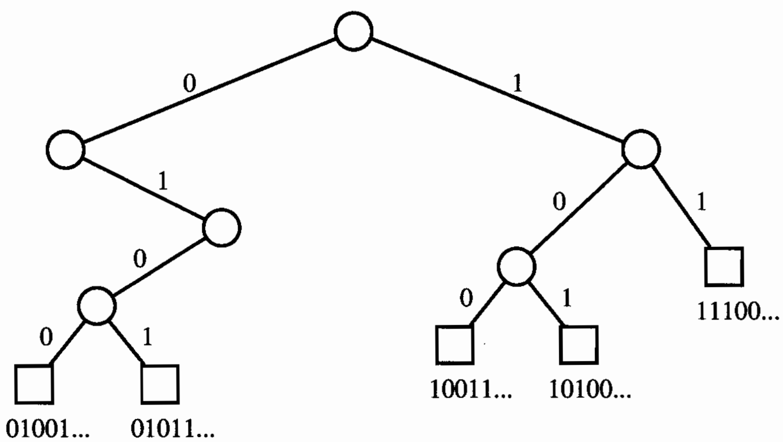


Figure B.2: *The p-trie is a trimmed-down version of the infinite p-trie in which the strings are associated with the leaves.*

B.1.2 The HORTON-STRAHLER number

We refer to Chapter 3.1.1 for the definition of the HORTON-STRAHLER number. Figure B.3 shows the HORTON-STRAHLER labelling of the trie from Figure B.2. The HORTON-STRAHLER number arises in computer science because of its relationship to expression evaluation. In a computer, an arithmetic expression is evaluated by micro-operations using registers. To facilitate this process the expression is stored as an expression tree with the operators in the internal nodes and the operands in the external nodes. The arithmetic expression is

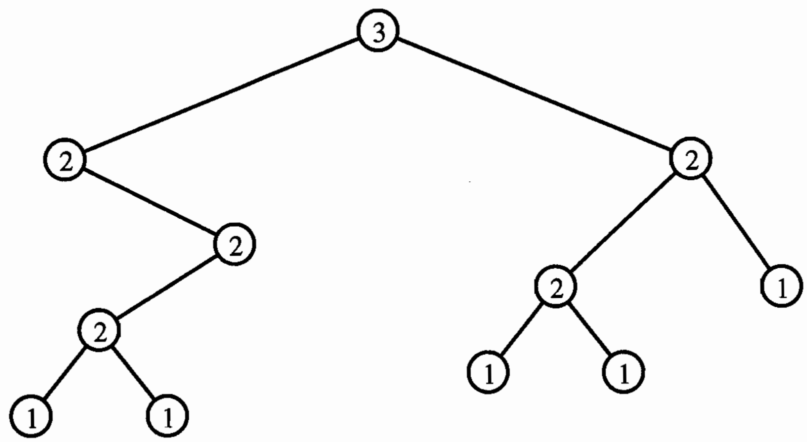


Figure B.3: *The binary trie with the* HORTON-STRAHLER *labelling.*

evaluated by traversing the corresponding tree. In 1958, ERSHOV showed that by always traversing the child node with the lower HORTON-STRAHLER number first, the corresponding register use is minimal (note however that this does not minimize time). Furthermore, the minimum number of registers required to evaluate an expression tree T is exactly $S(T)+1$. As expression evaluation is a special type of postorder traversal, the same paradigm shows that the minimum stack size required for a postorder traversal of a binary tree T is $S(T)+1$ (e.g., see FRANÇON 1984). In fact, the HORTON-STRAHLER number occurs in almost every field involving some kind of natural branching pattern. For example, VAUCHAUSSADE DE CHAUMONT AND VIENNOT (1985) studied it for RNA structures and VANNIMENUS AND VIENNOT (1989) experimentally studied the ramification matrix of *injection patterns.* VIENNOT (1990) provides a thorough overview.

The properties of the HORTON-STRAHLER number have only been studied for one model of random binary trees, equiprobable binary trees. These are random binary trees with n nodes drawn uniformly and at random from all possible rooted binary trees with n nodes. Let S_n be the HORTON-STRAHLER number of a random equiprobable binary tree with n nodes so that $\mathbf{E}S_n$ and $\mathbf{Var}S_n$ are the corresponding expected value and variance. It is well-known (e.g., FLAJOLET ET AL. 1979, KEMP 1979, MEIR AND MOON 1980, MEIR ET AL. 1980, MOON 1980, DEVROYE AND KRUSZEWSKI 1994, and PRODINGER 1995) that

$$\mathbf{E}S_n \sim \log_4 n \quad \text{and} \quad \mathbf{Var}S_n = O(1) .$$

PENAUD (1991) proved the conjecture by VIENNOT ET AL. (1989) on the structure of the ramification matrix for EBTS.

B.1.3 Our approach

We first define two tree metrics, the BALANCE number and the FILL LEVEL, which serve as deterministic upper and lower bounds for the HORTON-STRAHLER number. We then derive the upper and lower bounds respectively of these two metrics and show that they converge to the same value, thereby squeezing the HORTON-STRAHLER number between them.

B.2 The BALANCE number

We first define an infinite trie T^* as the infinite complete binary tree. A position of a node in T^* is addressed by two integers, (i, l), where l is the level number ($l \geq 0$), and $0 \leq i \leq 2^l - 1$ is an integer indicating the node at level l. For example, the root is at level 0, so $i = l = 0$ for the root. The integer i, when expanded into l bits, describes the path from the root to the node (0 forces a left turn, 1 forces a right turn). Let $|i|_l$ denote the number of one bits in the last l bits of i.

If we take an i.i.d. sequence of Bernoulli (p) random variables, say Z_1, Z_2, Z_3, ..., and write the bits backwards to form integers, then we obtain the integers

$$Z_1 + 2^1 Z_2 + 2^2 Z_2 + \cdots \ .$$

These are precisely the integers visited on the path from the root by our sequence. At level 0, we visit 0. At level 1, Z_1, at level 2, $Z_1 + 2^1 Z_2$, and so forth. When we refer to node (i, l), and $i \geq 2^l$, we are in fact referring to $(i \bmod 2^l, l)$. Therefore, we allow such references modulo 2^l.

The probability that a random i.i.d. sequence of Bernoulli (p) random variables carves out a path that reaches (i, l) is $q_{i,l} = p^{|i|_l}(1-p)^{l-|i|_l}$. We call this the probability of node (i, l). For every node (i, l) we record its cardinality $C_{i,l}$, the number of the n strings $X_1, \ldots, X_n$ that go through it, i.e., those strings that have in their first l bits the integer i written backwards. If $|i|_l = k$, then $C_{i,l}$ is binomial $(n, p^k(1-p)^{l-k})$. The sibling of a node (i, l) is (i', l) where i' and i differ in the last bit only. We define the BALANCE number of (i, l) as

$$B_{i,l} = \sum_{j=1}^{l} \mathrm{I}_{[1 \leq C_{i,j} \leq C_{i',j}]}$$

where (i, j) denotes $(i \bmod 2^j, j)$. The BALANCE number B_n of the p-trie is

$$B_n = \sup_{(i,l)} B_{i,l} \ ,$$

where the supremum is only over those nodes (i, l) that are in the p-trie. For example, Figure B.4 shows our trie with the edges labelled by the indicator function $\mathrm{I}_{[1 \leq C_{i,j} \leq C_{i',j}]}$ and the nodes labelled by BALANCE number.

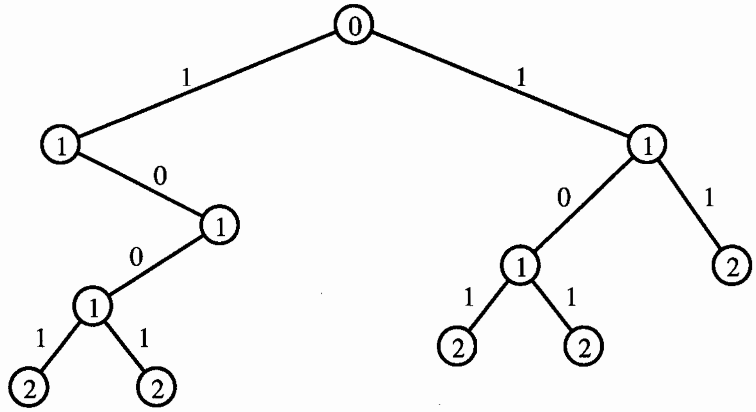


Figure B.4: *The trie with* BALANCE *number labelling.*

We note that since nodes with no siblings have the same BALANCE numbers as their parents, the finite and infinite p-tries (and the corresponding PATRICIA tree—a PATRICIA tree is a trie in which all internal nodes with one child are removed and recursively replaced by that sole child, e.g., see Figure B.5) all have the same BALANCE number.

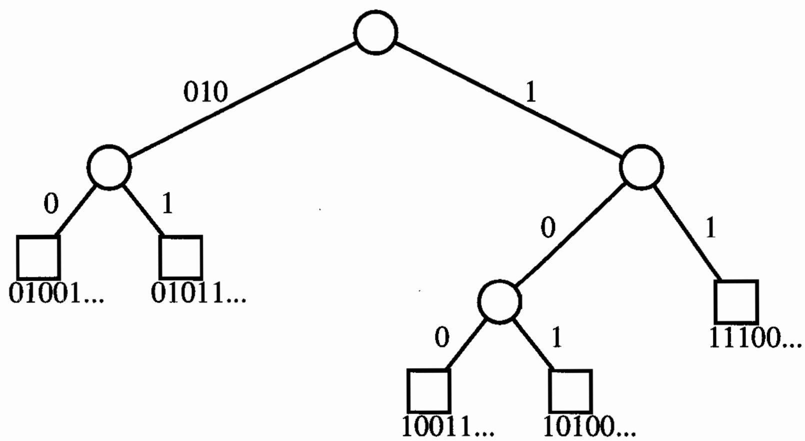


Figure B.5: *The p-trie is compressed into a* PATRICIA *tree.*

We now show the following upper bound on B_n.

THEOREM B.1 *For $0 < p < \frac{1}{2}$ and $\epsilon > 0$,*

$$\lim_{n\to\infty} \mathbf{P}\left\{B_n > (1+\epsilon)\log_{\frac{1}{p}} n\right\} = 0 \ .$$

PROOF. The nodes are separated into three categories (e.g. see Figure B.6):

$$A = \{(i,l) : nq_{i,l} \geq n^{\epsilon}\},$$

$$B = \{(i,l) : n^{-\epsilon} < nq_{i,l} < n^{\epsilon}\},$$

$$C = \{(i,l) : nq_{i,l} \leq n^{-\epsilon}\}.$$

Let A_0 be the event that for all $(i,l) \in A$, $l \geq 1$, $C_{i,l} < C_{i',l}$ if and only if $q_{i,l} < q_{i',l}$. For $p < \frac{1}{2}$, we will see in Lemma B.1 that, if A_0^c is the complement of A_0,

$$\mathbf{P}\{A_0^c\} \to 0 \, .$$

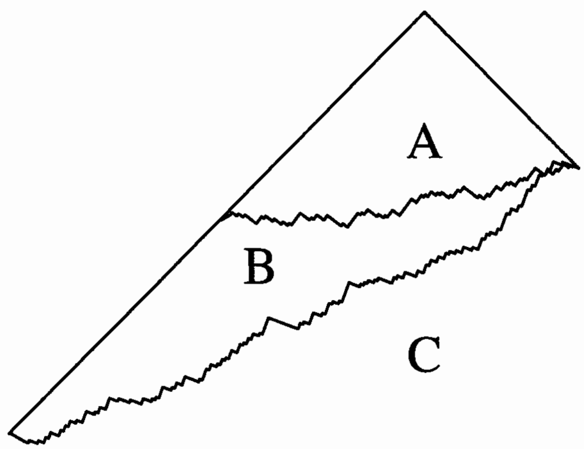


Figure B.6: *An abstract trie.*

On any path, the number of nodes that belong to B is not more than $2 + 2\epsilon \log_{\frac{1}{1-p}} n$ (assuming still $p < 1/2$, then paths of the form $(\ldots 000 \ldots)$ maximize the path length). Finally, let B^* be the subset of nodes in B with at least one child in C. We show in Lemma B.2 that

$$\mathbf{P}\{\exists (i,l) \in B^* : C_{i,l} > M\} \to 0 \text{ if } M \geq 1 + \frac{1}{\epsilon} \, . \tag{B.1}$$

Collecting all this, we note that for any (i,l), with probability tending to one,

$$B_{i,l} \leq M + \left(2 + 2\epsilon \log_{\frac{1}{1-p}} n\right) + \left| \left\{ \begin{array}{l} (m,j) \text{ on path from } (i,l) \text{ to root,} \\ (m,j) \in A, \; q_{m,j} < q_{m',j} \end{array} \right\} \right| \tag{B.2}$$

As any path visits B^*, and every node of B^* has cardinality $\leq M$ with probability tending to one, the contribution to $B_{i,l}$ from all nodes below that node of B^* is $\leq M$. Observe that the last quantity of Equation B.2 is maximized by choosing i with binary expansion $(111\ldots)$.

Then we must have, for any $(m,j) \in A$ on the path to (i,l), $np^m \geq n^\epsilon$, or $m \leq (1-\epsilon)\log_{\frac{1}{p}} n$. Therefore, as we may take $M = 1 + \frac{1}{\epsilon}$,

$$\begin{aligned} B_{i,l} &\leq 1 + \frac{1}{\epsilon} + \left(2 + 2\epsilon \frac{\log \frac{1}{p}}{\log \frac{1}{1-p}} \log_{\frac{1}{p}} n\right) + (1-\epsilon)\log_{\frac{1}{p}} n \\ &\leq 3 + \frac{1}{\epsilon} + \left(1 + 2\epsilon \frac{\log \frac{1}{p}}{\log \frac{1}{1-p}}\right) \log_{\frac{1}{p}} n \ . \end{aligned}$$

Thus we will have shown that

$$\mathbf{P}\left\{\sup_{(i,l)} B_{i,l} > \left(1 + 2\epsilon \frac{\log \frac{1}{p}}{\log \frac{1}{1-p}}\right) \log_{\frac{1}{p}} n\right\} \to 0 \ ,$$

for all $\epsilon > 0$. □

We are left with two technical lemmas.

LEMMA B.1

$$\mathbf{P}\{A_0^c\} \to 0 \ .$$

PROOF. Take $(i,l) \in A$ and let (i^*, l^*) denote its parent (note: $l^* = l-1$, $i^* = i \bmod 2^{l^*}$). Given C_{i^*,l^*}, we know that $C_{i,l}$ is binomial $(C_{i^*,l^*}, 1-p)$ or binomial (C_{i^*,l^*}, p) depending upon whether its is left or right child. Now, if $q_{i,l} < q_{i',l}$

$$\begin{aligned} [C_{i,l} \geq C_{i',l}] &= [C_{i,l} \geq C_{i^*,l^*} - C_{i,l}] \\ &= \left[C_{i,l} \geq \frac{1}{2} C_{i^*,l^*}\right] \\ &= \left[C_{i,l} - pC_{i^*,l^*} \geq \left(\frac{1}{2} - p\right) C_{i^*,l^*}\right] \ . \end{aligned}$$

Thus, by Hoeffding's inequality (HOEFFDING 1963),

$$\mathbf{P}\left\{C_{i,l} \geq C_{i',l} \,\middle|\, C_{i^*,l^*}\right\} \leq \exp\left\{-2\left(\frac{1}{2} - p\right)^2 C_{i^*,l^*}\right\} \ .$$

We argue similarly for $q_{i,l} > q_{i',l}$, $[C_{i,l} \leq C_{i',l}]$, and note that

$$\begin{aligned} \mathbf{P}\left\{\left[\begin{array}{c} C_{i,l} \geq C_{i',l}, q_{i,l} < q_{i',l} \\ \text{or} \\ C_{i,l} \leq C_{i',l}, q_{i,l} > q_{i',l} \end{array}\right]\right\} &\leq 2\mathbf{E}\left\{e^{-2(\frac{1}{2}-p)^2 C_{i^*,l^*}}\right\} \\ &\stackrel{\text{def}}{=} 2\mathbf{E}\left\{\delta^{C_{i^*,l^*}}\right\} \qquad (\text{where } 0 < \delta < 1) \\ &= 2\left(1 - q_{i^*,l^*} + q_{i^*,l^*}\delta\right)^n \\ &\leq 2e^{-(1-\delta)q_{i^*,l^*}n} \\ &\leq 2e^{-(1-\delta)n^\epsilon} \end{aligned}$$

as $nq_{i^*,l^*} \geq n^\epsilon$ because $(i,l) \in A$. Thus, by Boole's inequality,

$$\mathbf{P}\left\{ \bigcup_{(i,l)\in A} \left[\begin{array}{c} C_{i,l} \geq C_{i',l}, q_{i,l} < q_{i',l} \\ \text{or} \\ C_{i,l} \leq C_{i',l}, q_{i,l} > q_{i',l} \end{array} \right] \right\} \leq |A| 2e^{-(1-\delta)n^\epsilon} . \tag{B.3}$$

Clearly, $|A|$ is not more than the number of leaves in the tree pruned to A times the height of A. But as the leaves are disjoint, their probabilities cannot sum to more than one, and each individual probability is at least $n^{-(1-\epsilon)}$, the number is not more than $n^{1-\epsilon}$. The height of A is not more than $1 + \log_{\frac{1}{1-p}} n$, by a trivial argument. Thus, Equation B.3 is not larger than

$$2\left(1 + \log_{\frac{1}{1-p}} n\right) n^{1-\epsilon} e^{-(1-\delta)n^\epsilon} \to 0 \,. \,\square$$

LEMMA B.2

$$\mathbf{P}\left\{ \sup_{(i,l)\in B^*} C_{i,l} > M \right\} \to 0 \quad \textit{for} \quad M \geq 1 + \frac{1}{\epsilon} \,.$$

PROOF. First we count the number of nodes in B^*. Clearly, for any node in B^*, $nq_{i,l} > n^{-\epsilon}$ and $nq_{i,l}p \leq n^{-\epsilon}$ because one of its children must be in C. Let C^* be the collection of all the rightmost ("p") children of nodes in B^* (i.e., all nodes in C^* have probability p times that of their parent in B^*). Note that the nodes in C^* are disjoint, hence their probabilities sum to at most one. But for $(i,l) \in C^*$,

$$nq_{i,l} = nq_{i^*,l^*}p > n^{-\epsilon}p \,,$$

or $q_{i,l} > p/n^{1+\epsilon}$. Therefore, $|C^*|n^{-(1+\epsilon)}p < 1$. Thus, $|B^*| < n^{1+\epsilon}/p$. Fix $(i,l) \in B^*$. Recall that $q_{i,l} \leq 1/pn^{1+\epsilon}$. Then

$$\begin{aligned}
\mathbf{P}\{C_{i,l} > M\} &\leq \sum_{j>M}^{n} \binom{n}{j} (q_{i,l})^j (1 - q_{i,l})^{n-j} \\
&\leq \binom{n}{m} (q_{i,l})^m \left(1 + nq_{i,l} + (nq_{i,l})^2 + \cdots\right) \\
&\qquad\qquad\qquad \text{(where } m = \lfloor M + 1 \rfloor\text{)} \\
&\leq (nq_{i,l})^m \frac{1}{1 - nq_{i,l}} \\
&\leq \frac{1}{(pn^\epsilon)^m} \frac{1}{1 - \frac{1}{pn^\epsilon}} \\
&\leq \frac{2}{(pn^\epsilon)^m}
\end{aligned}$$

for n large enough. Thus

$$\mathbf{P}\left\{\sup_{(i,l)\in B^*} C_{i,l} > M\right\} \le \frac{2|B^*|}{(pn^\epsilon)^m} \le \frac{2n^{1+\epsilon}}{p(pn^\epsilon)^m}$$

for all n large enough. This tends to zero if $\epsilon m > 1+\epsilon$. That is, if $m > 1+1/\epsilon$. This holds if $M = 1 + 1/\epsilon$. □

We can now derive the result in Theorem B.1 for all $p \in (0, 1)$.

COROLLARY B.1 *For all* $\epsilon > 0$, $0 < p < 1$,

$$\lim_{n\to\infty} \mathbf{P}\left\{B_n > (1+\epsilon)\log_{\frac{1}{\min(p,1-p)}} n\right\} = 0 \ .$$

PROOF. We note that for $p = 1/2$, the same proof works throughout, except for the following. From Equation B.2, regardless of whether Lemma B.1 holds or not,

$$\begin{aligned} B_{i,l} &\le M + 2 + 2\epsilon \log_2 n + (1-\epsilon)\log_2 n \\ &\le 1 + \frac{1}{\epsilon} + (1+\epsilon)\log_2 n \ . \end{aligned}$$

So, we need not bother with Equation B.2 nor an extension of Lemma B.1. In the proof of Lemma B.2, the fact that $p < 1/2$ was not used. We thus see that for all $\epsilon > 0$, $0 < p < 1$,

$$\lim_{n\to\infty} \mathbf{P}\left\{B_n > (1+\epsilon)\log_{\frac{1}{\min(p,1-p)}} n\right\} = 0 \ . \ \square$$

B.3 The FILL LEVEL

The FILL LEVEL or saturation level of a binary tree is the deepest level l in the tree such that all possible 2^l nodes at that level exist. For example, the trie of Figure B.2 has FILL LEVEL 2. In 1992, DEVROYE showed that for random PATRICIA trees constructed from n i.i.d. sequences of independent equiprobable bits and FILL LEVEL F_n that

$$\frac{F_n - \log_2 n}{\log_2 \log n} \to -1$$

almost surely. We let F_n be the FILL LEVEL of a p-trie with n strings and show the following lower bound—the short proof is included here for completeness. For a much larger class of random tries, F_n was studied by PITTEL (1985), whose results imply the bound given below (KRUSZEWSKI 1993).

THEOREM B.2 *For $\epsilon > 0$ and $0 < p < 1$,*

$$\lim_{n\to\infty} \mathbf{P}\left\{F_n < (1-\epsilon)\log_{\frac{1}{\min(p,1-p)}} n\right\} = 0 \ .$$

PROOF. Without loss of generality, we assume that $p \le 1/2$. We note that

$$[F_n < l] \equiv \left[\min_{0\le i\le 2^l-1} C_{i,l} = 0\right] \ .$$

Equivalently, by Bonferroni's inequality, we have

$$\begin{aligned}
\mathbf{P}\{F_n < l\} &\le \mathbf{P}\left\{\min_{0\le i\le 2^l-1} C_{i,l} = 0\right\} \\
&\le \sum_{i=0}^{2^l-1} \mathbf{P}\{C_{i,l} = 0\} \\
&= 2^l \max_{0\le i\le 2^l-1} \mathbf{P}\{C_{i,l} = 0\} \\
&= 2^l\left(1 - \min_{1\le i\le 2^l-1} q_{i,l}\right)^n \\
&\le 2^l\left(1-p^l\right)^n \\
&\le 2^l e^{-np^l} \ .
\end{aligned}$$

This tends towards 0 with n if we take $l \sim (1-\epsilon)\log n/\log(1/p)$ for any $\epsilon > 0$.
□

It is equally easy to show that in fact $F_n/\log_{\frac{1}{\min(p,1-p)}} n \to 1$ in probability (see KRUSZEWSKI 1993 and Corollary B.2 below).

B.4 The HORTON-STRAHLER number

We introduce another metric related to the BALANCE number. For a node u in a binary tree, we set

$$B_u^* = \begin{cases} 0 & \text{if } |u| = 0 \ , \\ \max\left(B_v^* + \mathrm{I}_{[|v|\le|w|]},\ B_w^* + \mathrm{I}_{[|w|\le|v|]}\right) & \text{if } |u| \ge 1 \text{ and} \\ & u \text{ has children } v \text{ and } w \ , \end{cases}$$

(see Figure B.7). We call B_u^* the alternate BALANCE number of u. It is easy to see that $B_u^* = 1$ for all leaves u. If B_n is the BALANCE number of any binary tree with root u, then $B_n = B_u^*$ because B_u^* is the maximum number of 1's (from the $\mathrm{I}_{[]}$'s) along any path in the tree. Note however that the BALANCE number of individual nodes—the $B_{i,l}$'s in the second section—are *not* equal to the quantities B_u^*.

We note that the BALANCE number provides an upper bound on the HORTON-STRAHLER number.

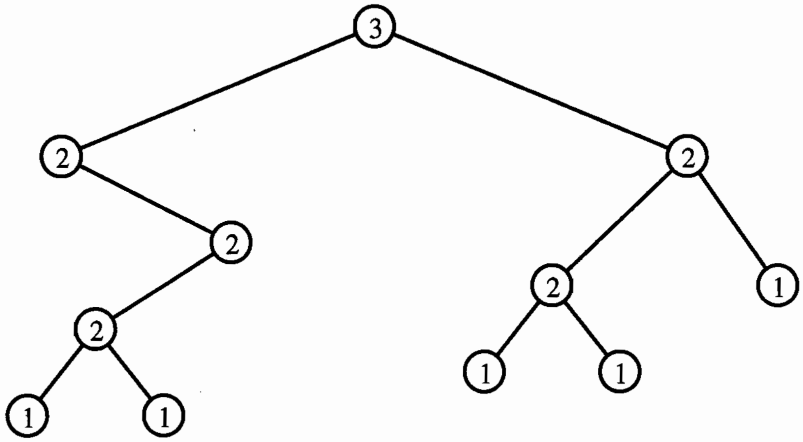


Figure B.7: *Alternate* BALANCE *number labelling.*

LEMMA B.3 *For each binary trie with root u, $S(u) \leq B^*_u$.*

PROOF. For a particular tree, this follows by induction on h, the height of a node (distance from its furthest descendant leaf). At leaves u, $S(u) = B^*_u = 1$. Assume that the assertion holds for all nodes of height less than h. At height h we take a node u with children v and w. We have $S(v) \leq B^*_v$, $S(w) \leq B^*_w$ by assumption. If $S(v) = S(w)$, then, assuming $|v| \leq |w|$, we have $B^*_u \geq B^*_v + 1 \geq S(v) + 1 = S(u)$. If $S(v) \neq S(w)$, then $S(u) = \max(S(v), S(w)) \leq \max(B^*_v, B^*_w) \leq B^*_u$, and we are done. □

We observe that the FILL LEVEL provides a lower bound for the HORTON-STRAHLER number.

LEMMA B.4 *For each binary tree with root u, $S(u) \geq F_u$.*

PROOF. Straightforward. □

We conclude the following tight bound on the HORTON-STRAHLER number S_n for p-tries.

THEOREM B.3 *For a p-trie with n strings,*

$$\frac{S_n}{\log n} \to \frac{1}{\log \frac{1}{\min(p,1-p)}} \qquad \textit{in probability.}$$

PROOF. The upper bound follows from Lemma B.3 and Corollary B.1. The lower bound follows from Lemma B.4 and Theorem B.2. □

This theorem together with Lemmas B.3 and B.4 allow us to conclude the following.

COROLLARY B.2 *For a p-trie with n strings,*

$$\frac{B_n}{\log n} \to \frac{1}{\log \frac{1}{\min(p,1-p)}} \qquad \text{in probability}$$

and

$$\frac{F_n}{\log n} \to \frac{1}{\log \frac{1}{\min(p,1-p)}} \qquad \text{in probability.}$$

Finally, we note that as p-tries and their corresponding PATRICIA trees have the same HORTON-STRAHLER numbers, our bound also holds for PATRICIA trees, hence Fact 3.4.

Appendix C

The Horton-Strahler number for random binary search trees

In this appendix, we derive an upper bound (believed to be tight) on the HORTON-STRAHLER number for *random binary search trees* (BSTs). This result implies that the random binary search tree model is incorrect as a combinatorial model of human lung as described in Chapter 5.1.6. We note that this appendix is joint work with Luc Devroye.

C.1 Introduction

C.1.1 The binary search tree model

According to KNUTH (1973B) (pp. 446–7), binary search trees were independently discovered by several researchers during the 1950's. As with tries, BSTs provide a method for storing and retrieving (i.e. searching for) unique elements. Whereas tries are more suitable for static data sets, BSTs lend themselves to dynamic deletion and insertion of elements. An element e is inserted into a BST by searching the tree for that element. First, e is compared to r, the value stored at the root. If $e \leq r$ then the search continues down the left subtree, otherwise it continues down the right subtree et cetera, until the current subtree is empty where e is inserted. Thus, given a list of elements the corresponding BST is built by repeated insertion on an initially empty tree. For example, the list (`b e f h a g c d`) results in the tree in Figure C.1.

From a theoretical viewpoint, random binary search trees can be simulated by random split trees with $[0, 1]$ uniform (or equivalently beta(1,1)) split distribution (e.g., see DEVROYE 1986B).

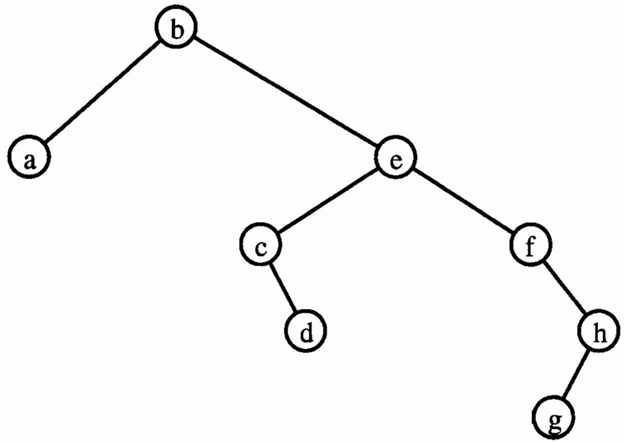


Figure C.1: *A* BST *build by insertions.*

C.1.2 The HORTON-STRAHLER number

Unlike equiprobable binary trees, the HORTON-STRAHLER number for random binary search trees has never been studied from a theoretical viewpoint. However as mentioned in Chapter 5.1.6, the HORTON-STRAHLER number has been examined from an experimental viewpoint. For example, HORSFIELD AND WOLDENBERG (1986A) and others concluded that the expected bifurcation ratio R was

$$\mathbf{E}R \sim 3.$$

That is, given Fact 3.1 we have

$$\mathbf{E}S_n \sim \log_3 n \sim \frac{1}{\log 3} \log n,$$

where S_n is the HORTON-STRAHLER number for a random BST with n nodes. (This experimental result was also calculated by KRUSZEWSKI 1993).

C.1.3 Our approach

We would like to take an approach similar to the one taken in Appendix B in which we squeezed the HORTON-STRAHLER number between the BALANCE number and the FILL LEVEL. However, the following theorem by DEVROYE (1986B) shows why this does not work for random BSTs.

THEOREM C.1 *For a random binary search tree with* n *nodes and* FILL LEVEL F_n,

$$\mathbf{P}\left\{\lim_{n\to\infty} \frac{F_n}{\log n} = 0.3733\ldots\right\} = 1.$$

Nonetheless, the upper bound provided by the BALANCE number is sufficient to show that random BSTs are unsuitable for simulating the combinatorial structure of human lungs.

C.2 The BALANCE number

In this section, we show the following upper bound on B_n, the BALANCE number for a random BST with n nodes.

THEOREM C.2 *For a random binary search tree with n nodes and* BALANCE *number B_n,*

$$\lim_{n\to\infty} \mathbf{P}\left\{B_n \geq \frac{1+\epsilon}{\log 3}\log n\right\} = 0,$$

for all $\epsilon > 0$.

PROOF. Let T be a random BST. Let node (i,k,l) be defined as the i-th node at level l in T that has exactly k ones (according to its BALANCE number labelling) on the path from the root of T to that node. Let $|(i,k,l)|$ be the size of the subtree rooted at (i,k,l). We observe that if $B_n \geq k+m$ then there exists at least one node (i,k,l) with $|(i,k,l)| \geq 2^m$. Summing over all possible levels gives us

$$\begin{aligned}
\mathbf{P}\{B_n \geq k+m\} &\leq \mathbf{P}\left\{\bigcup_{l=k}^{n}\bigcup_{i=1}^{\binom{l}{k}}[|(i,k,l)| \geq 2^m]\right\} \\
&\leq \sum_{l=k}^{n}\binom{l}{k}\mathbf{P}\left\{n\prod_{j=1}^{k}\frac{1-U_j}{2}\prod_{j=k+1}^{l}\frac{1+U_j}{2} \geq 2^m\right\}
\end{aligned}$$

where the U_j's are i.i.d. uniform $[0,1]$ random variables. We need to find the smallest k such that this probability approaches zero as $n \to \infty$. Markov's inequality gives us

$$\begin{aligned}
\mathbf{P}\left\{n\prod_{j=1}^{k}\frac{1-U_j}{2}\prod_{j=k+1}^{l}\frac{1+U_j}{2} \geq 2^m\right\} &\leq \frac{1}{2^m}\mathbf{E}\left\{n\prod_{j=1}^{k}\frac{1-U_j}{2}\prod_{j=k+1}^{l}\frac{1+U_j}{2}\right\} \\
&= \frac{n}{2^m}\prod_{j=1}^{k}\mathbf{E}\frac{1-U_j}{2}\prod_{j=k+1}^{l}\mathbf{E}\frac{1+U_j}{2} \\
&\qquad \text{(by independence)} \\
&= \frac{n}{2^m}\left(\mathbf{E}\frac{1-U}{2}\right)^k\left(\mathbf{E}\frac{1+U}{2}\right)^{l-k} \\
&\qquad \text{(identical distribution)}
\end{aligned}$$

Appendix D

Glossary

BALANCE **number** A deterministic upper bound on the HORTON-STRAHLER **number** for any **tree** T.

base The **branch segment** of a **botanical tree** that starts at the ground and ends at the first **bifurcation**.

Bernoulli (p) **variable** A random variable that takes value 1 with probability p and value 0 with probability $1 - p$.

bifurcation The point at which a **branch segment** splits into two child branch segments.

bifurcation ratio The average order ratio for a **binary tree**.

billboard A two dimensional **simple geometric tree** that is made to appear three dimensional by rotating it around a fixed vertical axis so that it always faces the viewer.

botanical tree A perennial plant with a single woody self-supporting stem or trunk usually unbranched for some distance above ground. For our purposes, a botanical tree consists of **branch segments** and sometimes **leaves**.

branching process A method of generating **combinatorial trees** by which **nodes** give birth to child nodes according to a fixed probability distribution.

branch segment A section of the **botanical tree** between two consecutive **bifurcations**. In addition, the **base** of a tree is also a branch segment.

combinatorial tree A connected, acyclic graph. A **rooted binary tree** consists of **nodes** connected by **edges** such that each node has at most one left and at most one right child node.

pipe A theoretical connection in a **botanical tree** from the **base** to a constant number of **leaves**.

production A rule in a **grammar** for replacing one **string** with another.

p**-trie** A **trie** whose random **strings** are sequences of **Bernoulli** (p) **variables**.

ramiform A smooth join where the parent and child **cylinders** meet.

root A **node** in a **tree** that has no parent.

Schimper-Braun law A botanical law which states that the **divergence angle** for plants and **trees** follows a specific **Fibonacci**-like sequence.

Shreve order The number of **external nodes** in a **node**'s **subtree**.

size The number of **nodes** in a **combinatorial tree**.

split distribution The probabilistic distribution that determines the partioning of **nodes** into two **subtrees**.

split tree A **combinatorial tree** generated by recursive allocation of **nodes** according to a **split distribution**.

string A sequence of symbols.

structure-oriented model A model that grows a **tree** according to specific rules which do not directly take in account environmental factors.

space-oriented model A model that grows a **tree** according to environmental influences.

subtree of a node The **tree** rooted at particular **node**.

tree — see **botanical tree**, **combinatorial tree**, **complete tree**, **geometric tree** or **split tree**.

trie An efficient data structure to store and re*trie*ve **strings**. More specifically, it is **combinatorial tree** constructed from a sequence of strings of bits.

tropism An environment influence on the growth of a botanical tree.

Weibel order(ing) The **depth** of a **node**.

Appendix E

Table of Symbols

$|u|$ The number of nodes in the subtree rooted at u, i.e., the subtree size.

(i, l) The position of a node in T^*.

$|i|_l$ The number of one bits in the last l bits of integer i.

$[X]$ The event X.

$\alpha(\mu)$ The divergence angle function of cylinder μ.

μ A cylinder.

Π_μ The plane that contains the axis of internal cylinder μ and the axes of μ's child cylinders.

$\Theta(\mu)$ The deviation angle function of cylinder μ.

B_n The BALANCE number of a random p-trie.

$B_{i,l}$ The BALANCE number of node (i, l).

c_α The constant for the divergence angle function, typically 137.5°.

c_Θ The constant for deviation angle function.

c_{Θ_f} The constant for the deviation angle function for forking cylinders.

c_{Θ_m} The constant for the deviation angle function for main cylinders.

c_{Θ_s} The constant for the deviation angle function for secondary cylinders.

c_L The constant for the length function.

c_R The constant for the radius function.

$C_{i,l}$ The number of strings that pass through node (i, l).

$d(u)$ The depth of node u.

$\mathbf{E}X$ The expected value of random variable X.

$H(T)$ The height of tree T.

$\mathrm{I}_{[]}$ The indicator function.

$L(\mu)$ The length function for cylinder μ.

$m(u)$ The number of main nodes on the path from the root to the node u.

n The number of internal nodes in a combinatorial tree.

N_i The number of nodes of order i.

$\mathbf{P}\{X\}$ The probability of event X occuring.

$q_{i,l}$ The probability that node (i, l) exists in a random trie.

r_L The scaling ratio for the length function.

r_R The scaling ratio for the radius function.

$R(\mu)$ The radius function for cylinder μ.

R The bifurcation ratio.

R_i The order ratio.

$s(u)$ The number of secondary nodes on the path from the root to the node u.

$S(u)$ The HORTON-STRAHLER order of node u.

S_n The HORTON-STRAHLER number for a random binary tree with n nodes.

T A combinatorial tree.

$\mathcal{T}$ A geometric tree.

T^* The infinite complete binary tree.

$T(n)$ The split tree algorithm.

u A node.

$\mathbf{Var}X$ The variance of random variable X.

X The splitting variable.

X_i An infinite binary string.

Z_i A Bernoulli (p) random variable.

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Colophon

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