FINITE-ELEMENT ANALYSIS OF NONLINEAR MAGNETIC FIELDS

Madabushi V. Chari, B.E.(India), M.Sc. (Lond.), D.I.C.

FINITE-ELEMENT ANALYSIS OF NONLINEAR MAGNETIC FIELDS IN ELECTRIC MACHINES

Electrical Madabushi V. Chari, B.E.(India), M.Sc.(Lond.),D.I.C. Ph.D.

ABSTRACT

A new method for solving two-dimensional saturable magnetic field problems, such as are encountered in the analysis of electric machines, is presented. Full account is taken of saturation of the iron parts, complicated geometry of the region, distribution of current sources and the presence of slots. The new technique is based on a general variational formulation of the field problem in terms of an energy functional, which is discretised by first order triangular finite elements. By minimising the functional by a set of trial functions defined in the discretised region, a unique solution to the magnetic field problem is obtained. This process results in a set of nonlinear algebraic equations which is solved iteratively by a multi-dimensional Newton-Raphson scheme. The field analysis is applied to a transformer, a turbo-generator and a D.C. generator, and their performance characteristics are predicted, neglecting eddy-current and hysteresis effects. Comparison of the computed values and test results shows satisfactory agreement, thus demonstrating the efficacy of the method and its general applicability to electric machines. In developing the computer algorithm, emphasis was laid on fast execution and utmost economy with the degree of accuracy desired.

FINITE-ELEMENT ANALYSIS OF NONLINEAR MAGNETIC FIELDS IN ELECTRIC MACHINES

by

Madabushi V. Chari, B.E.(India), M.Sc.(Lond.), D.I.C.

Department of Electrical Engineering

McGill University,

Montreal, Quebec.

FINITE-ELEMENT ANALYSIS OF NONLINEAR MAGNETIC FIELDS IN ELECTRIC MACHINES

by

Madabushi V. Chari, B.E. (India), M.Sc., (London), D.I.C. (Imp. Col.)

A thesis submitted to the Faculty of Graduate Studies and Research
in partial fulfillment of the requirements for the degree of
Doctor of Philosophy.

Department of Electrical Engineering,
McGill University,
Montreal, Quebec,

March, 1970.

ABSTRACT

A new method for solving two-dimensional saturable magnetic field problems, such as are encountered in the analysis of electric machines, is presented. Full account is taken of saturation of the iron parts, complicated geometry of the region, distribution of current sources and the presence of slots. The new technique is based on a general variational formulation of the field problem in terms of an energy functional, which is discretised by first order triangular finite elements. By minimising the functional by a set of trial functions defined in the discretised region, a unique solution to the magnetic field problem is obtained. This process results in a set of nonlinear algebraic equations which is solved iteratively by a multi-dimensional Newton-Raphson scheme. This field analysis is applied to a transformer, a turbogenerator and a D.C. generator, and their performance characteristics are predicted, neglecting eddy-current and hysteresis effects. Comparison of the computed values and test results shows satisfactory agreement, thus demonstrating the efficacy of the method and its general applicability to electric machines. In developing the computer algorithm, emphasis was laid on fast execution and utmost economy consistent with the degree of accuracy desired.

ACKNOWLEDGEMENTS

The author is deeply indebted to Dr. P. Silvester for his able guidance and active encouragement throughout the course of this research and the preparation of the thesis. His constant enthusiasm, thirst for knowledge and friendly disposition has been a great source of inspiration.

The author wishes to thank Mr. Z. Csendes and Mr. P. Benedek for their comments on the work and helpful suggestions from time to time. Thanks are also due to all his colleagues in the Electrical Engineering department of McGill University for their assistance and Mrs. P. Hyland for her excellent typing.

The author is forever grateful to his dear wife, Padma, who showed patience and provided encouragement and motivation far beyond the call of duty.

Grateful acknowledgement is made to McGill University and the National Research Council of Canada for financial support in making this research possible. The author also wishes to thank the management of Heavy Electricals Ltd., Bhopal, India, for permission to undertake the research.

TABLE OF CONTENTS

			Page
ABSTRACT			i
ACKNOWLEDGEMENTS			ii
TABLE OF CONTENTS			iii
CHAPTER	I	INTRODUCTION	1
	1.1	General Background	1
	1.2	Present Work	5
		References	8
CHAPTER	II	THE NONLINEAR ELLIPTIC PROBLEM	10
	2.1	Introduction	10
	2.2	Statement of the Problem	10
	2.3	Past Methods for Solving the Electro-Magnetic Field Problem	14
	2.3.1	Transformation Methods	14
	2.3.2	Magnetic Circuit Analysis	18
	2.3.3	Finite Difference Methods	19
	2.3.4	Restricted Variational Formulation	21
	2.4	General Variational Formulation	22
	2.4.1	Discussion of the Variational Method	24
	2.4.2	Conditions for Extremum of the Functional, the Euler Equation, Covariance and Self-Sufficiency Property	26
	2.4.3	The Second Variation of the Functional and Sufficient Conditions for its Minimisation	29
	2.4.4	Legendre Condition for Functional Minimum	31
	2.5	The Energy Functional for the Nonlinear Poisson's Equation	34
		References	43
CHAPTER	III	FINITE ELEMENT FORMULATION	46
	3.1	Description of Direct Methods	46
	3.2.1	Finite Element Method	47
	3.2.2	Discrete Representation of the Nonlinear Energy Functional by Finite Elements and Derivation of the Coefficient Matrix	48
		Polynomial Approximation to Functional Minimisation	51
		Evaluation of the Polynomial Coefficients	57
		Calculation of Element Matrices for First Order Elements	58

			Page
	3.3	Boundary Conditions and their Effect on the Variational Formulation	62
	3.4	Continuity of the Magnetic Field across the Interface Between Finite Elements	64
	3.5	Minimisation of the Functional in the Discretised Region and Uniqueness of the Approximate Solution	66
		References	70
CHAPTER	IV	ITERATIVE SOLUTION OF NONLINEAR ALGEBRAIC EQUATIONS	71
	4.1	Introduction	71
	4.1.1	Functional Iteration Method	72
	4.2	Explicit Iteration Scheme or the Chord Method	76
	4.3	Second Order Method	78
	4.3.1	The Generalised Newton-Raphson Method	79
		Newton-Raphson Method for a System of Equations	81
		Existence and Convergence	83
		Rate of Convergence of the Newton Method	84
	4.3.2	Some Modifications of the Newton Method	86
		i. Whittaker's Method	86
		ii. Regula Falsi	86
		iii. Muller's Method	87
		iv. Accelerated Iteration Method	88
	4.3.3	Application of the Generalised Newton-Raphson Method to the Multi-Dimensional Field Problem using First Order Finite Elements	89
	4.4	Solution of Linear Algebraic Equations	93
	4.4.1	Conditioning of the Matrix and Accuracy of the Solution Obtained	94
		References	96
CHAPTER	V	PRACTICAL COMPUTATIONAL ASPECTS OF THE FINITE ELEMENT METHOD	98
	5.1	Introduction	98
	5.2.1	Calculation of the Geometrical Coefficients and Current Vectors	100
	5.2.2	Special Indexing Routine	101
	5.3.1	Assembly of the Total Coefficient Matrix and the Current Vector	106
	5.3.2	Chord and Newton-Raphson Iteration Schemes	107
	5.3.3	Matrix Reduction	108
	5.3.4	Solution of Linear Equations for a Coefficient Matrix with a Band Structure	110

			Page
	5.3.5	Formula for the Total Number of Arithmetic Operations Required	111
	5.3.6	Comparison of the Arithmetic Operations Required by the Triangular Decomposition Method and by an Inverse Routine for a Band Matrix and also for a Full Matrix	114
	5.4.1	Convergence Criteria and Rate of Convergence	114
	5.5	Flux Plotting Routine	115
		References	118
CHAPTER	VI	APPLICATION OF THE METHOD TO ELECTRICAL MACHINES	119
	6.1.1	The Transformer Problem	119
	6.1.2	Determination of the Magnetising Current	125
	6.2.1	Evaluation of Turbogenerator Performance	129
	6.2.2	No-Load Saturation Curves and Iron Losses	136
	6.2.3	Short-Circuit and Zero Power Factor Characteristics	139
	6.2.4	Waveform Analysis of the No-Load Voltage	139
	6.2.5	Evaluation of Sequence Reactances	143
	6.3.1	Determination of the Performance Characteristics of a D.C. Generator	146
	6.3.2	Open-Circuit Characteristics of the D.C. Generator	148
	6.3.3	Voltage Regulation On-Load	151
	6.3.4	Flux Distribution	157
CHAPTER	VII	CONCLUSIONS	158
APPENDIX	I	SURFACE INTEGRALS IN AREA CO-ORDINATES	163
APPENDIX	II	COTANGENT IDENTITY FOR TRIANGULAR FINITE ELEMENTS	168
APPENDIX	III	EVALUATION OF THE CHANGES TO THE DIFFERENTIAL OF THE FUNCTIONAL DUE TO A LOADED BOUNDARY	171
APPENDIX	IV	THE NEWTON-RAPHSON FORMULATION FOR FIRST ORDER TRIANGULAR FINITE ELEMENTS	173
APPENDIX	V	EXPRESSIONS FOR THE NEGATIVE AND ZERO SEQUENCE REACTANCES OF A TURBOGENERATOR	178
APPENDIX	VI	PERIODICITY CONDITION FOR ROTATING ELECTRICAL MACHINERY	182

CHAPTER I

INTRODUCTION

1.1 General Background

The present decade has witnessed phenomenal growth in electrical power systems and sizes of electrical plant such as transformers, turbo-generators, salient pole alternators, direct current machines and other devices. An accurate prediction of their performance has, therefore, become increasingly important in order to meet stringent specifications, to effect economy in design and to ensure reliability of operation. Some of the performance indicators that machine designers and power systems engineers are vitally concerned with are the excitation requirements under open-circuit, short-circuit and full load conditions, sequence reactances, transient characteristics, short-circuit ratio, iron and stray load losses, end-field and eddy current effects in the case of A.C. machines ; load regulation and commutation characteristics in D.C. machines and others.

Saturation of the iron parts considerably affects all of these quantities, by introducing nonlinearities in the magnetic field. Until recently, the magnetic field distribution in electrical machines was explored by linear analytical techniques [1.1], application of conformal and other transformation methods [1.2] and magnetic circuit analysis [1.3] based on linear theory ; the results thus obtained were modified by introducing empirical design constants to account for nonlinear effects. Simplicity of such methods coupled with the superposition principle [1.4] so widely used in machine analysis, had paid off well on small and medium size machines. However, with growing complexity

of electro-mechanical devices, a rigorous nonlinear analysis of the field problem has become increasingly important.

Nonlinear solutions of the magnetic field problem by analytical methods [1.5] or transformation techniques [2.1] for linearising the nonlinearities have been used, but are limited to cases with idealised magnetisation characteristics and simplified geometry of the region. Measurement techniques by analogue models [1.6] have been employed for two dimensional problems and in some cases nomographic methods [2.3] were also applied. In view of the limitations and inadequacies of all the earlier methods, the need for numerical solutions was recognised even in the early stages of the design art. Nevertheless, only with the advent of large scale digital computers could such methods be developed and extensively used for solving the field distribution in electrical machines in the presence of magnetic saturation.

All of the numerical methods in present day use fall under three principal headings :

(a) divided difference schemes,

(b) integral equation techniques and

(c) variational formulations.

In one case, besides the aforesaid, a nonlinear magnetic circuit analysis [2.4] was carried out based on a circuit representation of the magnetic characteristics of the media, fluxes and mmfs, and an unaccelerated relaxation technique was adopted for obtaining a solution to the field problem. Its chief limitation is that the circuit re-

presentation is not of general application and its usefulness for solving the field problem under different conditions of machine operation has not been established.

A vast majority of the numerical methods in use belong to the divided difference class [1.7 - 1.9] wherein the partial differential equations are replaced by a set of difference equations and a solution is obtained satisfying the specified boundary conditions. The discretisation process by finite differences yields a large number of equations. Invariably, many redundant nodes are required which enormously increase the computational work. In this method, the boundary conditions have to be explicitly specified at material interfaces and outside boundaries by a set of equations, thereby enhancing the complexity of the problem. Further since the permeabilities of the different regions often differ considerably, the convergence of the iteration scheme is necessarily slow despite the use of acceleration techniques [2.11].

Very recently an integral equation approach to two and three dimensional field problems has been proposed [1.10]. In this method the material inhomogeneity in the region of interest is replaced by an equivalent distribution of sources in free space, resulting in a field distribution which corresponds to the magnetic field in the original problem. The contribution of each of these sources is then considered as a solution of Maxwell's equations in free space and the summation of such contributions from all the sources yields the required solution. The method is attractive, since it purports to solve end-field problems of finite length, and the boundary conditions are implicit in the integral formulation. However, it has so far been applied to very simple cases and the convergence scheme used does not seem to have been optimised. It is therefore difficult to assess the computational advantage of this method and the accuracies that would result in

solving practical field problems in electrical machines with a high degree of saturation.

A third possibility which is coming into the fore is the variational method. It consists of formulating the partial differential equations of the field problem in terms of an integral expression called the energy functional. In most engineering applications, this expression can be identified with stored energy in the system. In general, the Euler equation of this functional will yield the original differential equation. The solution to the field problem is then obtained by choosing a function amongst a set of triai functions which minimises the energy functional satisfying the specified boundary conditions.

The method was first used for analysing saturation effects in accelerator magnets [2.12], for which a restricted functional formulation based on the assumption of fixed reciprocal permeability was used. Further the set of trial functions was defined in a discretised region consisting of finite rectangular meshes or triangular elements of variable geometry, but fixed topology. This restriction coupled with the slow convergence of the iteration method used did not result in any computational gain over the divided difference approach.

In this thesis, a general nonlinear variational formulation is presented with a view to overcoming some of the shortcomings of the earlier methods and achieving economy, efficiency and fast programming of the field problem. Triangular finite elements of unrestricted geometry, topology and containing material inhomogeneities are used for discretising the field region. The finite element method is well known in the field of elasticity and structural mechanics. The type of nonlinearities generally met with in such problems are in the main caused by iarge displacements and are therefore termed geometrical non-

linearities [1.11]. Material nonlinearities encountered in elasto-plastic analysis have been dealt with much in the same way as the restricted variational method for magneto-static problems described earlier. Only in one case of a homogeneous medium, a nonlinear variational formulation was derived by Kachanov [1.12] which is comparable to the general variational formulation presented in this thesis, for inhomogeneous and nonlinear media.

1.2 Present Work

To the best of the author's knowledge, this is the first time that a method of solving the two-dimensional nonlinear field problem in electrical machines by a variational method using finite elements, is presented.

The main objectives of the present study are

(a) Formulation of the nonlinear electro-magnetic field problem in electrical machines in general variational terms and obtaining a solution by minimising the resulting energy functional, by finite element analysis.

(b) Derivation of a practical, efficient and economical algorithm for determining the magnetic field distribution in a transformer, turbo-generator and a direct current generator.

(c) Prediction of the performance characteristics of these machines under open-circuit, short-circuit and full load conditions and the determination of other steady state machine parameters.

(d) Verification of the efficacy of the method by experiment and by comparison with factory test results.

The aforesaid objectives were accomplished as follows.

In Chapter II, the variational method is presented in detail and the general energy functional is derived from Maxwell's equations for the field problem. The conditions for functional minimisation are described and applied to the nonlinear energy functional to ascertain minimality. The criteria for a unique solution are discussed and the basis of constructing a minimising sequence of trial solutions is presented.

A theoretical analysis of the general order finite element method is presented in Chapter III and it is specialised to obtain a first order solution to the magnetic field problem. The test for functional minimisation is carried out in the discretised region and proof of the validity of the finite element method as a minimising sequence is furnished.

The methods used in this study for solving nonlinear algebraic equations are described in Chapter IV with particular reference to the first order chord method and the quadratically convergent Newton-Raphson scheme. Existence and uniqueness of the solution obtained are ensured by reference to well known theorems of numerical analysis and the rate of convergence of the iteration schemes is established. The Newton-Raphson scheme is formulated for a case of N independent variables and the necessary equations are derived for the recursion algorithm. Methods for solving linear equations are briefly discussed and their merits and limitations are stated.

In Chapter V, the practical computational aspects of the field solution are discussed in detail with reference to a flow chart for the computer program. The solution algorithm suitable for sparse band-structured matrices and the necessary indexing

scheme are fully described. The rate of convergence of the iteration process is illustrated by the reduction in the error norm for a transformer problem. An automatic flux-plotting routine by means of a digital x - y plotter is fully described.

The application of the finite element method for solving field problems in a transformer, a 30 MW turbogenerator and a 5 kW direct current machine is described in Chapter VI. Using the material characteristics obtained from core samples or manufacturers' catalogues, extensive field analysis is carried out on these machines. An improved algorithm for compact storage of sparse matrices and an efficient and economical routine for solving algebraic equations are used. For the load analysis of the D.C. machine, a new connection matrix is derived to include the so-called periodicity condition, which permits field representation of the region under one pole pitch only instead of the entire problem. In all these cases, the computed performance characteristics and parameters are compared with test results.

To summarise the aforesaid, a new technique for solving two-dimensional nonlinear field problems in electrical machines in an efficient and economical manner compared to other existing methods is presented in this thesis. The algorithms developed, it is hoped, will assist machine designers in predicting the steady state performance characteristics of various types of electrical machines at the design stage accurately. Some of the areas that may yet be explored using the variational method are

(a) field solution of machines under transient conditions,

(b) eddy current effects in solid iron parts and

(c) end-field problems.

REFERENCES

[1.1] G.W. Carter, The Electro-magnetic Field in its Engineering Aspects, Longmans Green & Co., London, 1954.

[1.2] W.J. Gibbs, Conformal Transformations in Electrical Engineering, Chapman and Hall Ltd., London, 1958.

[1.3] A. Gray, Electrical Machine Design, McGraw-Hill, New York, pp. 45 - 56, 1926.

[1.4] B. Adkins, The Generalised Theory of Electrical Machines, Chapman and Hall Ltd., London, p. 22, 1964.

[1.5] H.M. McConnell, "Polyphase Induction Machine with Solid Rotor", AIEE Trans., Vol. 72, Part III, pp. 343 - 349, 1953.

[1.6] O.V. Tozoni, Mathematical Models for the Evaluation of Electric and Magnetic Fields , Iliffe Books Ltd., London, 1968.

[1.7] F.C. Trutt, E.A. Erdélyi and R.F. Jackson, "The Nonlinear Potential Equation and Numerical Solution for Highly Saturated Machines", IEEE Trans. Aerospace, Vol. 1, No. 2, pp. 430-440, August 1963.

[1.8] M. Ivanes and J.C. Sabonnadière, "Calcul du champ d'induction magnétique dans un angle de transformateur compte tenu de la saturation", REVUE GÉNÉRALE DE L'ÉLECTRICITÉ, Vol. 1, No. 2, pp. 206 - 212, February 1967.

[1.9] K. Reichert, "Ein numerisches Verfahren zur Berechnung magnetischer Felder insbesondere in Anordnungen mit Permanentmagneten", ARCHIV FÜR ELECTROTECHNIK, Vol. 52, No. 3, pp. 176 - 195, February 1968.

[1.10] S.G. Zaky, "An Integral Equation Approach to the Solution of the Three-Dimensional Magnetic Field Problems", Ph.D. Thesis, University of Toronto, 1969.

[1.11] C. Brebbia and J. Connor, "Geometrically Nonlinear Finite-Element Analysis", Journal of the Engineering Mech. Div., Proc. ASME, EM2, pp. 463 - 481, April 1969.

[1.12] L.M. Kachanov, "Variational Methods of Solution of Plasticity Problems", Applied Mech. and Math., Vol. 23, pp. 880 - 883, 1959.

CHAPTER II

THE NONLINEAR ELLIPTIC PROBLEM

2.1 Introduction

In this chapter, the nonlinear Poisson's equation is derived for a quasi-stationary field in a two dimensional continuum satisfying specified boundary conditions. Various past methods for solving the field equations based on analytical and numerical techniques such as transformation methods, magnetic circuit analysis, finite difference methods and restricted variational formulation , are discussed and their merits and shortcomings are stated. A theoretical analysis of the general variational method and its application to the solution of nonlinear field equations are presented covering the following aspects :

1. The necessary and sufficient conditions for a functional minimum.
2. Extremum, Euler equation, self-sufficiency and covariance.
3. The derivation of the nonlinear energy functional for the two dimensional electro-magnetic field problem and its minimisation.

2.2 Statement of the Problem

Consider a two dimensional finite region R of the (x, y) plane bounded by S (see Figure 2.1) containing some prescribed distribution of current sources, combination of material inhomogeneities and nonlinearities. In the air and iron regions, since

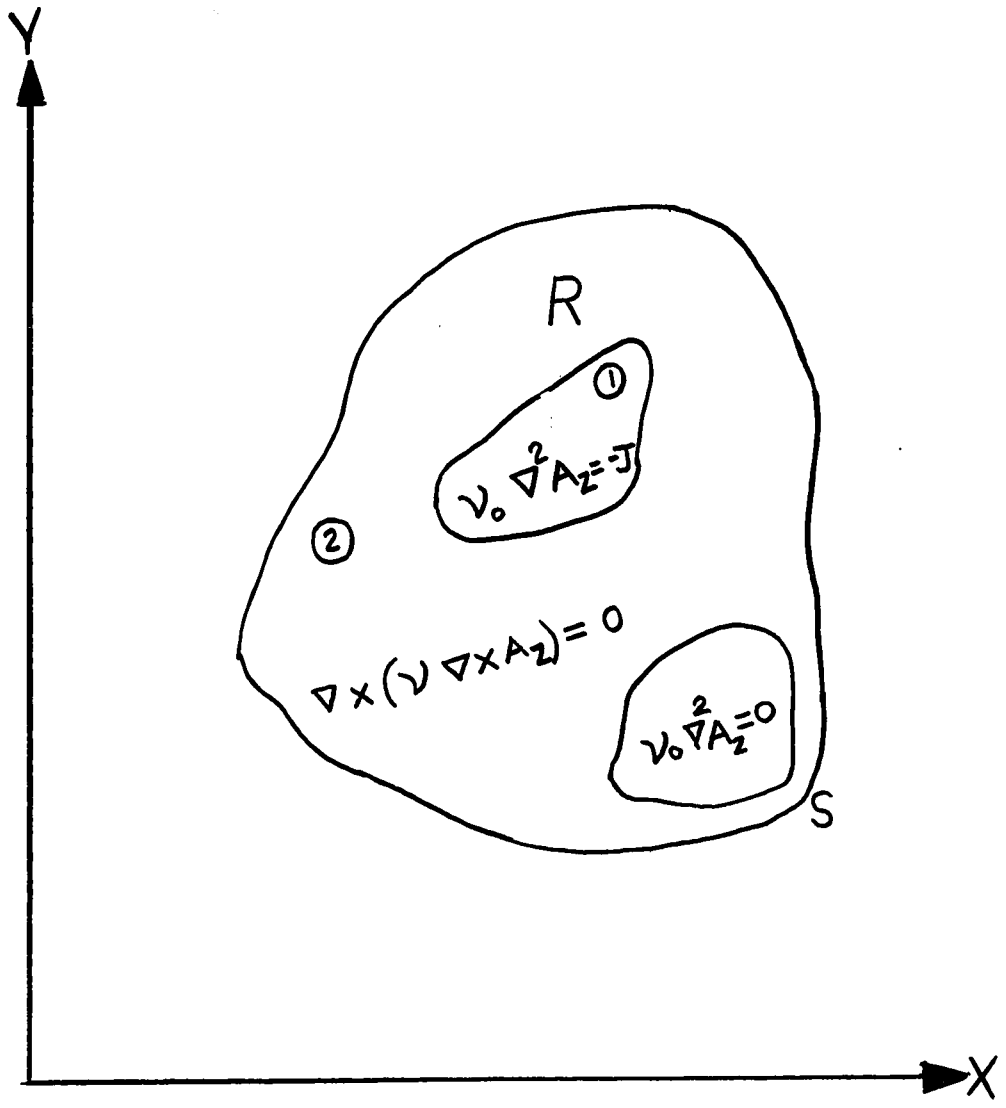


FIGURE 2.1. TWO DIMENSIONAL REGION R .

(1) Current Region.

(2) Iron Region.

(3) Air and Other Non-Iron Current Free Region.

there are no current sources, Laplace's equation holds. If we assume that

(a) The electro-magnetic field is quasi-stationary (i.e., displacement currents can be neglected at power-frequencies).

(b) Time-harmonic effects are absent.

(c) That the dimension along the Z direction is very large so that the magnetic vector potential and the current density vector have components only along the Z direction and are invariant in that direction, then the following field equations apply

$$\text{curl } \bar{H} = \bar{J} \tag{2.1}$$

$$\text{div } \bar{B} = 0 \tag{2.2}$$

The constitutive relations are

$$\bar{B} = \text{curl } \bar{A} \tag{2.3}$$

$$\bar{H} = \nu \cdot \bar{B} \tag{2.4}$$

where ν, the reciprocal of the permeability termed the reluctivity of the medium, is assumed single valued and is both position and field dependent. It is this latter property of field dependence that gives rise to the nonlinearity of the field problem.

By Coulomb's convention, if we assume

$$\text{div } \bar{A} = 0 \tag{2.5}$$

and substituting for $\bar{H}$ in Equation (2.1) from Equations (2.3) and (2.4), one obtains

$$\text{curl } (\nu \text{ curl } \bar{A}) = -\bar{J} \tag{2.6}$$

Since A has only a Z directed component A_z (x, y) , the following further relation is obtained.

$$\text{curl } \bar{A} = \left(\frac{\partial A}{\partial y} \hat{i} - \frac{\partial A}{\partial x} \hat{j} \right) \tag{2.7}$$

Using Equation (2.7) in Equation (2.6) and expanding the result, there is

$$\frac{\partial}{\partial x}\left(\nu \frac{\partial A}{\partial x}\right) \hat{k} + \frac{\partial}{\partial y}\left(\nu \frac{\partial A}{\partial y}\right) \hat{k} = -J \hat{k} \tag{2.8}$$

which reduces to

$$\frac{\partial}{\partial x}\left(\nu \frac{\partial A}{\partial x}\right) + \frac{\partial}{\partial y}\left(\nu \frac{\partial A}{\partial y}\right) = -J \tag{2.9}$$

For the different regions shown in Figure 2.1, the field problem can be expressed by the following partial differential equations, which have been combined in Equation (2.9).

For Region 1 (current carrying zone)

$$\nu_o \left(\frac{\partial^2 A}{\partial x^2} + \frac{\partial^2 A}{\partial y^2} \right) = -J \tag{2.10}$$

In Region 2 (iron parts, where saturation is present but there are no current sources), the pseudo-Laplacian

$$\text{Curl } (\nu \text{ Curl } A) = 0 \tag{2.11}$$

In Region 3 (air and other non-iron current-free zones)

$$\nu_o \left(\frac{\partial^2 A}{\partial x^2} + \frac{\partial^2 A}{\partial y^2} \right) = 0 \tag{2.12}$$

Equations (2.10) to (2.12) are the required field equations for the nonlinear problem of the two dimensional region R satisfying the boundary conditions defined by

$$C = C_1 (A) + C_3 \left(\frac{\partial A}{\partial x} \hat{i} + \frac{\partial A}{\partial y} \hat{j} \right) = 0 \qquad (x, y) \in R \tag{2.13}$$

2.3 Past Methods for Solving the Electro-Magnetic Field Problem

In order to solve the respective field equations in the various sections of the two-dimensional region, certain analytical and numerical methods were developed in the past, whose merits and limitations will now be discussed.

2.3.1 Transformation Methods

Unlike in the case of linear partial differential equations, the principle of superposition does not apply to nonlinear equations and, therefore, they must be linearized or

solved in some other way. Methods used for this purpose are known as transformations, which are powerful analytical tools for solving nonlinear equations in general. Typically these techniques linearize the system of equations (for example the Kirchhoff and Hodograph transformations), reduce the partial differential equations to nonlinear ordinary differential equations (e.g. the similarity transformation), transform the system to one already solved or perform some other reduction of complexity.

In general these transformations can be classified into three groups:

(a) change of the dependent variables,

(b) change of the independent variables,

(c) change of both the dependent and independent variables.

In the Kirchhoff transformation, a new dependent variable is introduced so as to linearize the nonlinear equations.

Let us consider the pseudo-Laplacian which occurs in diffusion, heat-conduction and magnetic field problems.

$$\nabla [f (\varphi) \nabla \varphi] = 0 \tag{2.14}$$

Introducing a new dependent variable such that

$$\begin{aligned} \psi &= \psi (\varphi) \\ \frac{d \psi}{d \varphi} &= f (\varphi) \end{aligned} \tag{2.15}$$

Equation (2.14) reduces to the linear Laplace's equation

$$\nabla^2 \psi = 0 \tag{2.16}$$

The boundary conditions are also changed likewise and it can be shown that for the Dirichlet problem, the boundary conditions transform to yet another Dirichlet form. For the Neumann type boundary conditions, however, the Kirchhoff transformation introduces nonlinearities [2.1, p. 22], resulting in complicated boundary conditions. This transformation on the dependent variable has the feature that the physical range of the independent variable is unchanged, but the method is limited in its application to very simple geometries and boundaries.

An illustration of the transformation of the independent variable is the "Similarity transformation" due to Boltzmann [2.2], which transforms the independent variable such that the partial differential equation is changed into an ordinary differential equation. The technique was applied to a one dimensional diffusion equation of the form

$$\frac{\partial C}{\partial t} = \frac{\partial}{\partial x}\left[D(C) \frac{\partial C}{\partial x} \right] \tag{2.17}$$

Choosing a function of the independent variables x and t given by

$$\eta = x^{\alpha} t^{\beta}$$

where α and β are to be determined, Equation (2.17) is modified as an ordinary differential equation in η free of x and t so that

$$\frac{d}{d\eta}\left[D(C) \frac{dC}{d\eta} \right] + \frac{\eta}{2} \frac{dC}{d\eta} = 0 \tag{2,18}$$

This transformation can be used effectively only if the boundary and initial conditions are consolidated, the medium is homogeneous and the geometry of the region and boundary is a simple one.

The Hodograph transformation so named by Hamilton in 1869 [2.1, p. 171] is a typical example of the mixed method and it permits a certain amount of flexibility in the geometry of the field region. A set of quasi-linear equations of the form

$$F_1 \frac{\partial u}{\partial x} + F_2 \frac{\partial u}{\partial y} + F_3 \frac{\partial v}{\partial x} + F_4 \frac{\partial v}{\partial y} = F(u, v, x, y)$$

$$G_1 \frac{\partial u}{\partial x} + G_2 \frac{\partial u}{\partial y} + G_3 \frac{\partial v}{\partial x} + G_4 \frac{\partial v}{\partial y} = G(u, v, x, y) \qquad (2.19)$$

where F_i , G_i are functions of u, v, x and y, together representing second order equations, are transformed by changing the independent variables x and y as functions of u and v, so that

$$F_1 y_v - F_2 x_v - F_3 y_u + F_4 x_u = 0$$

$$G_1 y_v - G_2 x_v - G_3 y_u + G_4 x_u = 0 \qquad (2.20)$$

Hence the solution of the modified set of Equations (2.20) leads to the solution of (2.19) provided the Jacobian $j = x_u y_v - x_v y_u \neq 0$. This transformation has been successfully applied in fluid mechanics problems where the geometry of the region of interest may not be simple or regular. The advantage of linearity gained by this hodograph is, however, paid for by complicated boundary conditions.

The above examples of transformations are but a few of a large number of such techniques in use and illustrate their usefulness for solving quasi-linear partial differential

equations. There is no general way of obtaining the required transformation, and imagination, ingenuity and good fortune play a major role in their choice. The chief limitations of the methods are that, in general, they lead to complexity of boundary conditions and are really suitable only for cases wherein the material medium is homogeneous and the boundaries and geometry are simple.

2.3.2 Magnetic Circuit Analysis

This is a forerunner of numerical methods and uses a relaxation technique for the solution of the field problem. In this method, a magnetic circuit is developed with lumped reluctances representing various parts of the field region and the flux densities and mmf drops are determined for an initial estimate of core flux. With the values of flux density obtained, the appropriate permeabilities are determined from the B-H curve and the new reluctances are estimated. The iterative cycle is continued until the total mmf drops over a pole pitch (in the case of an electrical machine) attain an acceptable minimum.

The merits of the method are that it is an advance over nomographic techniques of field plotting [2.3] and with the aid of digital computers, the field region can be faithfully represented by an equivalent magnetic circuit. The technique was employed by Binns [2.4] for the estimation of the open-circuit saturation curve of a turbo-alternator and it is claimed that only 6 to 8 iterations are required to obtain a solution of an acceptable degree of accuracy.

The chief limitation of the method, however, is that the circuit representation of the region is a unique one for each problem and cannot, therefore, be generalised. Further,

since the flux paths are restricted to the branches containing lumped reluctances, the network representation should be sufficiently fine in order to obtain a useful flux-plot. As such, the computational advantage gained by the small number of iterations is offset by the large number of branches of the magnetic circuit and the corresponding number of equations to be solved. Lastly, load analysis of an electrical machine cannot be effectively carried out by this method except, perhaps, by the use of innumerable number of branches covering the entire machine region, and not just the region over a pole-pitch.

2.3.3 Finite Difference Methods

Historically these methods can be traced back to Gauss [2.5], and one of the oldest iterative schemes "the Gauss-Seidel method" dates back to 1873 [2.6]. The word "relaxation" was introduced by Southwell [2.7], who described a method of solving stresses in jointed frames by the systematic relaxation of the strains. As the name implies, the finite difference method is based on replacing the partial differential equations of the field problem by a number of difference approximations and then solving the resulting large number of algebraic equations. The solutions to the approximate system represent solution values at discrete points in the region of interest. The continuous differential operator is replaced, or approximated by, a matrix operator. The necessary and sufficient conditions for this approximation may be found in text books on linear spaces [2.8]. The elliptic operators under consideration here conform to these conditions and, therefore, may be approximated by matrices.

In this method, regular rectangular meshes are super-imposed on the continuous plane so that the function values ϕ at the nodes of each mesh represent the continuous

function A at that point in the continuous plane. Also the current densities, flux densities and reluctivities are assumed to have a constant value in the respective meshes. A five point regular star is defined in each mesh as shown in Figure 2.2 and approximations to the partial derivatives are obtained. As an illustration, the difference formulation of the Laplacian is derived as

$$\frac{\partial^2 A}{\partial x^2} + \frac{\partial^2 A}{\partial y^2}\Big|_o = \frac{1}{h^2}(\varphi_N + \varphi_E + \varphi_S + \varphi_W - 4\varphi_o) \qquad (2.21)$$

If this process is continued over all nodes of the various meshes, a large number of equations are obtained which must be solved simultaneously. Written in matrix form, the resulting system is of a very high order, but sparse, i.e., the coefficient matrix contains a large number of zero elements. Direct methods, such as inversion of the large matrix are, as a rule, inconvenient and make extensive demands on computer storage. Iterative solution of such a system of equations is more common since the coefficient matrix need not be stored, but generated as and when required. One such scheme known as "alternating relaxation" was used by Erdelyi and Ahmed [2.9] for solving the electro-magnetic field problem in a D.C. machine on no load. In this scheme, as a first step, the reluctivities are assumed constant and the vector potentials are relaxed. Subsequently the reluctivities are recalculated from the φ's and the process is continued until the vector potentials converge sufficiently.

The advantage of this technique lies in its general applicability to different geometries, inhomogeneous media and nonlinear problems. The more recent versions of this method [2.10] accommodate different co-ordinate systems, varying mesh sizes and use of improved iterative techniques [2.11].

The chief disadvantages of the finite difference schemes, however, are

(a) the mesh sub-divisions, despite care and diligence of the user, result in a large number of number of nodes and enhance the number of equations to be solved and computational time,

(b) the matching of different co-ordinate systems and specification of boundary conditions are complex and cumbersome,

(c) the convergence process is highly sensitive to the correct choice of under-relaxation factors and to the path of line integration used, and

(d) slow convergence of the iterations despite the use of an accelerated block relaxation technique.

2.3.4 Restricted Variational Formulation

This method was first used by Winslow [2.12] and Concus [2.13] for solving saturable magnetic-field problems in accelerator magnets, and it consists of formulating the problem in variational terms by an integral expression known as a functional. A solution A (x, y) differentiable in R is sought such that it minimises the functional satisfying the required boundary conditions. The integral formulation of the functional is then discretised by finite difference methods either by the use of regular rectangular meshes or a set of six triangular meshes of variable geometry but fixed topology meeting at each node. The difference formulation of the variational problem is then minimised by setting its first

derivative to zero for each of the nodal values of A. The resulting set of equations is then solved either by linearised over-relaxation [2.12, p. 172] or non-linear over-relaxation schemes [2.14].

Although this method provides a concise alternative formulation to the divided difference schemes, the restricted variational formulation [2.15], wherein the reciprocal permeability is held fixed, does not lead to a true energy functional for the nonlinear Poisson's equation. Further the finite difference discretisation of the functional either by the use of rectangular meshes or a set of triangular meshes of fixed topology results in a large number of nodes and equations. Consequently, the method suffers from the same draw-backs of excessive computational work and convergence difficulties as the earlier finite difference schemes.

2.4 The General Variational Formulation

In view of the limitations of the foregoing schemes, a general variational approach is developed herein for obtaining numerical solutions to the field problem, which is free of topological and geometrical restrictions. Also inhomogeneities and nonlinearities of the field region do not pose difficulties in any way and the iterative methods employed are not plagued by convergence problems common to all finite difference schemes. The present work differs from the earlier methods reported, in the following principal aspects.

(i) A general unrestricted nonlinear functional formulation is derived which eliminates the need for special equations to represent boundary conditions.

(ii) Triangular finite elements of variable geometry, topology and material parameters are employed for approximating the field solution.

(iii) A multi-dimensional Newton-Raphson formulation is developed for obtaining nearly quadratic convergence of the iteration process.

In addition to the above, the present work includes :

- Rigorous solution of the nonlinear field problem without altering the geometry or approximating the current density in the conductors by current sheets is carried out for a transformer, a turbo-generator and a direct-current machine.
- Formulation of the problem such that standard iteration methods may be applied for its solution and the proof of convergence and uniqueness of solution obtained is presented.
- Development of a practical algorithm, using the band and symmetry properties of the coefficient matrix including acceleration techniques for the iteration process.
- Complete analysis of the no-load operation of a transformer, evaluation of the open-circuit, short-circuit, zero-power factor and wave-form characteristics of a large turbo-alternator are carried out. Field requirements, iron-losses, sequence-reactances and the harmonic components of the voltage wave are

predicted accurately and in an economical way from outline drawings at the design stage.

- The so called 'periodicity condition' is described and the necessary connection matrix is derived for analysing the on-load operation of rotating electrical-machines. The method is applied for evaluating the load characteristics of a D.C. machine.

- In all the above cases, the computed and test results are compared.

2.4.1 Discussion of the Variational Method

By using variational methods, it has been shown in text books of mathematical physics [2.16] that a unique solution for Poisson's, Laplace's or Helmholz's equation can be obtained by minimising the appropriate energy functional. Since many of the problems are usually linear, the operators are positive definite and the functionals of the variational problem are quadratic. In such cases, it is shown in Reference [2.17, p.16-6] that any trial function which sets the first variation of the functional to zero, or in other words, makes it attain a stationary value, also minimises the functional. However, in the case of nonlinear operators, the functional is not always quadratic and, therefore, other criteria must be found for ensuring the minimisation of the functional.

In a two dimensional continuum R bounded by S, if we define a set of all functions $\varphi(x, y)$, subject to the boundary conditions $\varphi = f(S)$ on S, then any quantity

which takes a specific numerical value corresponding to each function in the set is said to be a functional on the set of all φ . In illustration (Reference 18)

$$\mathfrak{F}(\varphi) = \int_R \int f(x, y, \varphi, \varphi_x, \varphi_y)\, dx\, dy \tag{2.22}$$

where $\varphi_x = \partial\varphi / dx$ and $\varphi_y = \partial\varphi / \partial y$.

If we now change the function $\varphi(x, y)$ into a new function $\varphi(x, y) + \epsilon\eta(x, y)$, the change $\epsilon\eta(x, y)$ in $\varphi(x, y)$ is called the variation of φ and is denoted by

$$\delta\varphi = \epsilon\eta(x, y) \tag{2.23}$$

Here $\eta(x, y)$ is also a function of the same set. Corresponding to this change in φ and sufficiently small ϵ, the functional will change by the amount

$$\Delta\mathfrak{F} = \int_R \int [f(x, y, \varphi + \epsilon\eta, \varphi_x + \epsilon\eta_x, \varphi_y + \epsilon\eta_y) - f(x, y, \varphi, \varphi_x, \varphi_y)]\, dx\, dy \tag{2.24}$$

Expanding the right hand member by Taylor expansion in powers of ϵ , there follows

$$\delta\mathfrak{F} = \int_R \int [\frac{\partial f}{\partial\varphi}\epsilon\eta + \frac{\partial f}{\partial\varphi_x}\epsilon\eta_x + \frac{\partial f}{\partial\varphi_y}\epsilon\eta_y + (\text{terms including higher powers of } \epsilon)]\, dx\, dy \tag{2.25}$$

In analogy with the definition of the differential, the first three terms in the right-hand side member are defined to be the first variation of the functional for all sufficiently small ϵ, so that after substituting for $\epsilon\eta$ from (2.23) and some algebra

$$\delta \mathfrak{F} = \int_R \int \left[\frac{\partial f}{\partial \varphi} \delta \varphi + \frac{\partial f}{\partial \varphi_x} \delta \varphi_x + \frac{\partial f}{\partial \varphi_y} \delta \varphi_y \right] dx\, dy \qquad (2.26)$$

2.4.2 Conditions for Extremum of the Functional, the Euler Equation, Covariance and Self-Sufficiency Property

If φ in Equation (2.22) is a continuously differentiable function of (x, y) and the integral is carried over the two dimensional region, then the necessary condition for an extremum is given by setting the first variation to zero, so that Equation (2.26) becomes

$$\delta \mathfrak{F} = 0 = \int_R \int \left[\frac{\partial f}{\partial \varphi} \delta \varphi + \frac{\partial f}{\partial \varphi_x} \delta \varphi_x + \frac{\partial f}{\partial \varphi_y} \delta \varphi_y \right] dx\, dy \qquad (2.27)$$

Here the variation $\delta\varphi$ is to be continuously differentiable over R and is to vanish on the boundary S, when φ is prescribed on S, but is otherwise completely arbitrary. Under these conditions $\delta\varphi$ is termed an admissible variation of φ.

The second and third terms of Equation (2.27) can be expanded using Green's theorem [2.19] and by substituting the resulting values for these terms in Equation (2.27), it is shown in [2.18, p. 136, line 5], that

$$\delta \mathfrak{F} = 0 = \oint_S \left(\frac{\partial f}{\partial \varphi_x} \cos\theta + \frac{\partial f}{\partial \varphi_y} \sin\theta\right) \delta\varphi \, ds + \int_R \int \left[\frac{\partial f}{\partial \varphi} - \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial \varphi_x}\right) - \frac{\partial}{\partial y}\left(\frac{\partial \varphi}{\partial \varphi_y}\right)\right] \delta\varphi \, dx \, dy \tag{2.28}$$

where θ represents the angle between the positive x - axis and the outward normal at a point on the boundary S of R, and s is the arc length along S as shown in Figure 2.3. The closed integral term of (2.28) vanishes when φ is prescribed on S. (Dirichlet boundary condition). Otherwise the natural boundary condition (Neumann type) must be assumed to be satisfied so that the term becomes zero. Also the integrand of the double integral must vanish in R giving the well known Euler Equation.

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial \varphi_x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial \varphi_y}\right) - \frac{\partial f}{\partial \varphi} = 0 \tag{2.29}$$

In order that variational methods can be applied to the solution of field problems without the introduction of additional functions, it is necessary to obtain a functional that yields the differential equation of the field problem as its Euler equation. If such a functional exists, then the equation is termed self-sufficient [2.20], analogous to the self-adjointness property in the case of linear problems.

One further property of interest associated with the Euler equation is covariance, by virtue of which its form remains unchanged when the co-ordinates are changed. For example if we introduce new independent variables, ξ and η then the Euler equation (2.29) can be written as [Reference 2.17, p. 16-4, Equation (16.23)],

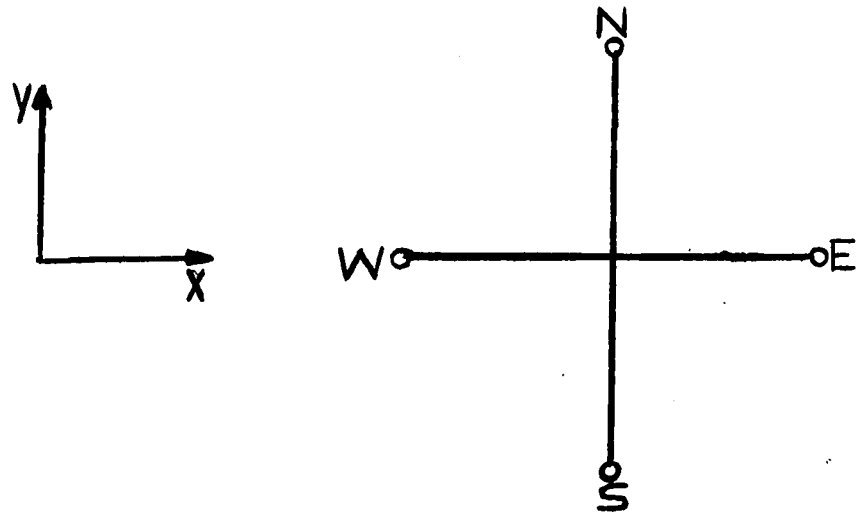


FIGURE 2.2. REGULAR FIVE-POINT STAR.

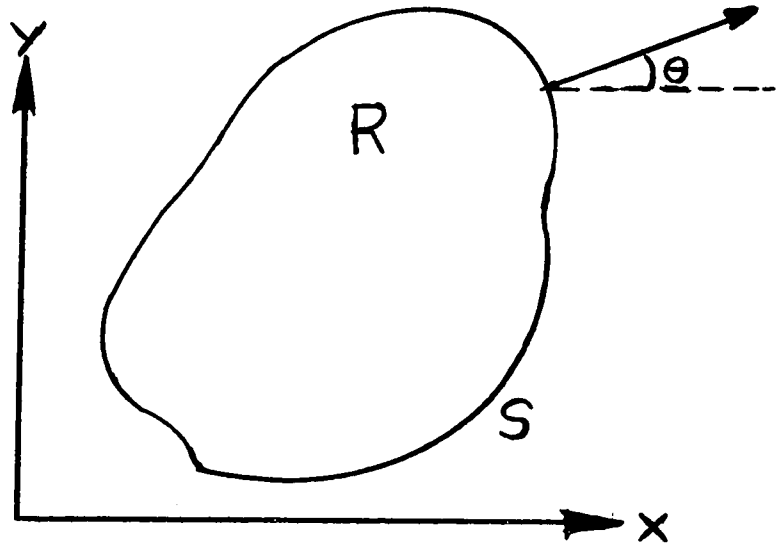


FIGURE 2.3. NORMAL DERIVATIVE ACROSS THE BOUNDARY OF REGION R.

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial \varphi_x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial \varphi_y}\right) - \frac{\partial f}{\partial \varphi} = \frac{1}{\partial(x, y) / \partial(\xi, \eta)} \cdot \left[\frac{\partial}{\partial \xi}\left(\frac{\partial \psi}{\partial \varphi_\xi}\right) + \right.$$

$$\left. \frac{\partial}{\partial \eta}\left(\frac{\partial \psi}{\partial \varphi_\eta}\right) - \frac{\partial \psi}{\partial \varphi}\right] \tag{2.30}$$

where

$$f(x, y, \varphi, \varphi') = \psi(\xi, \eta, \varphi, \varphi')\left[\frac{dx}{d\xi} + \frac{dy}{d\xi}\right] \tag{2.31}$$

This property is of value, in case we wish to use other co-ordinate systems such as polar co-ordinates, or change the frame of reference in any other way.

2.4.3 The Second Variation of the Functional and Sufficient Conditions for its Minimisation

If $\mathfrak{F}$ is the functional, φ an admissible function and $\delta\varphi$ an admissible variation, then $\mathfrak{F}(\varphi + \epsilon\,\delta\varphi)$ is a function of ϵ. If we expand this functional by Taylor's expansion as before, the coefficient of ϵ will be the first variation $\delta\mathfrak{F}(\varphi, \delta\varphi)$. The coefficient of $\epsilon^2/2$ is called the second variation of $\mathfrak{F}$ and is denoted by $\delta^2\mathfrak{F}[\varphi, \delta\varphi]$, so that

$$\mathfrak{F}[\varphi + \epsilon\,\delta\varphi] = \mathfrak{F}[\varphi] + \epsilon\,\delta\mathfrak{F}[\varphi, \delta\varphi] + \frac{\epsilon^2}{2!}\delta^2\mathfrak{F}[\varphi, \delta\varphi] \tag{2.32}$$

or

$$\delta^2\mathfrak{F}[\varphi, \delta\varphi] = \frac{d^2}{d\epsilon^2}\mathfrak{F}[\varphi + \epsilon\,\delta\varphi]\Big|_{\epsilon = 0} \tag{2.33}$$

The second variation is analogous to the second derivative. Supposing $\mathfrak{F}$ has a minimum value at φ, then

$$\delta \mathfrak{F} [\varphi, \delta \varphi] = 0 \tag{2.34}$$

so that from Equation (2.32) we have

$$\frac{1}{2} \delta^2 \mathfrak{F} [\varphi, \delta \varphi] + ... = \frac{\mathfrak{F} [\varphi + \epsilon \delta \varphi] - \mathfrak{F} [\varphi]}{\epsilon^2} \geq 0 \tag{2.35}$$

where the omitted terms on the left vanish, when $\epsilon = 0$. Letting $\epsilon = 0$, we conclude

$$\delta^2 \mathfrak{F} [\varphi, \delta \varphi] \geq 0 \tag{2.36}$$

for all admissible variations $\delta \varphi$. If we now substitute for

$$\mathfrak{F} = \int \int f (x, y, \varphi, \varphi') dx dy \quad \text{in Equation (2.36), then}$$

by setting $\varphi = \varphi + \epsilon \delta \varphi$, and expanding the whole by Taylor's theorem, there is obtained [Reference 2.17, p. 16-5. (16.33)]

$$\delta^2 \mathfrak{F} [\varphi, \delta \varphi] = \int \int [\frac{\partial^2 f}{\partial \varphi^2} (\delta \varphi)^2 + 2 \frac{\partial^2 f}{\partial \varphi \partial \varphi'} \delta \varphi \delta \varphi' + \frac{\partial^2 f}{\partial \varphi'^2} (\delta \varphi')^2] dx dy \tag{2.37}$$

The sufficient condition for minimising the functional, therefore, is

$$\delta^2 \mathfrak{F} \geq 0 \tag{2.38}$$

i.e., the second variation is non-negative. A functional which satisfies inequality (2.38), for all admissible functions φ and admissible variations $\delta\varphi$ is called convex. If strict inequality of the second variation holds, then the functional is termed strictly convex.

2.4.4 Legendre Condition for Functional Minimum

If we add to the second variation the integral

$$\epsilon^2 \int\int (2\eta\eta' w + \eta^2 w')\, dx\, dy \tag{2.39}$$

where w is an arbitrary function of x, y of class C^1 in (x_o, x_1, y_o, y_1), it is shown in [2.21, p. 46] that $\delta^2 \mathfrak{F}$ reduces to the form

$$\delta^2 \mathfrak{F} = \epsilon^2 \int\int [f_{\varphi\varphi} + w')\, \eta^2 + 2\,(f_{\varphi\varphi'} + w)\, \eta\eta' + f_{\varphi'\varphi'} \eta'^2]\, dx\, dy \tag{2.40}$$

since the integrand of (2.39) is equal to zero for all admissible variations $\eta = \delta\varphi$ that vanish on the boundary. This can be readily seen by re-writing Equation (2.39) and setting $\epsilon\eta = 0$, so that

$$\epsilon^2 \int\int \frac{d}{dx}(\eta^2 w)\, dx + \frac{d}{dy}(\eta^2 w)\, dy = \epsilon^2 [\eta^2 w] = 0 \tag{2.41}$$

The arbitrary function w is determined by the condition that the discriminant of the quadratic form in η, η' under the integral shall vanish, so that

$$(f_{\varphi\varphi'} + w)^2 - f_{\varphi'\varphi'} \cdot (f_{\varphi\varphi} + w') = 0 \tag{2.42}$$

which reduces $\delta^2 \mathfrak{F}$ to the form

$$\delta^2 \mathfrak{F} = \epsilon^2 \int\int f_{\varphi'\varphi'} \left[\eta' + \frac{f_{\varphi\varphi'} + w}{f_{\varphi'\varphi'}} \eta \right]^2 dx\, dy \tag{2.43}$$

The above transformation is due to Legendre [2.22], who inferred that $f_{\varphi'\varphi'}$ must not change sign in $(x_o\, x_1\, ,\, y_o\, y_1)$ and that $\delta^2 \mathfrak{F}$ has always the same sign as $f_{\varphi'\varphi'}$. Thus the necessary condition for the functional minimum becomes

$$f_{\varphi'\varphi'} = \frac{\partial^2 f}{\partial \varphi'^2} \geq 0 \tag{2.44}$$

According to Lagrange [2.23], this is true if and only if

(a) the differential equation $(f_{\varphi\varphi'} + w)^2 - f_{\varphi'\varphi'} (f_{\varphi\varphi} + w') = 0$ has an integral which is finite and continuous in the interval $(x_o\, x_1\, ,\, y_o\, y_1)$,

(b) $f_{\varphi'\varphi'}$ does not vanish in $(x_o\, x_1\, ,\, y_o\, y_1)$.

The Legendre condition provides a weak minimum to the functional which will mean that the minimum is obtained for local variations of $\delta\varphi$ which differ from

zero in a small neighbourhood of an arbitrary point $(x_o , y_o) \in S$. In Reference [2.24, p. 60] it is shown that if the Legendre condition holds, the functional attains a value

$$\int\int_S \frac{a}{4} \sum (\delta \varphi')^2 \, dS \tag{2.45}$$

where $a > 0$.

For a strong minimum, however, Equation (2.38) must be satisfied, which according to Berg [2.17, p. 16-5, (16.35)] yields

$$\frac{\partial^2 f}{\partial \varphi^2} \cdot \frac{\partial^2 f}{\partial \varphi'^2} - \left(\frac{\partial^2 f}{\partial \varphi \, \partial \varphi'}\right)^2 \geq 0 \tag{2.46}$$

In conclusion, it may be stated that if the strong minimum condition is satisfied and strict inequality of (2.38) holds, then the solution to the differential equation is unique. On the other hand, if only the weak minimum condition is satisfied, then the solution obtained by solving the field problem, according to Ladyzhenskaya and Ural'tseva [2.24, p. 61] is not necessarily unique. If, however, a minimising sequence of trial functions is set up for minimising the functional, known as 'the direct method' of solving the variational problem, a unique solution will be obtained, even though the functional attains only a weak minimum [2.24, p. 61].

All such functions are called generalized solutions of the variational problem. The existence of such functions is assured by simple assumptions of boundedness of $\mathfrak{F}$ from below (true for a minimising sequence which has a limit, see Reference [2.24, pp. 59-62])

and the convexity of $\mathfrak{F}$ with respect to p_k, which are defined by equations (6) and (7) of Reference [2.24, p.22]. However, to answer the question whether the trial functions are sufficiently smooth to ensure the minimization of the nonlinear functional, further discussion is necessary.

2.5 The Energy Functional for the Nonlinear Poisson's Equation

In nearly every engineering application, there exists a vari ional formulation corresponding to the partial differential equations of the field problem ; that is to say there exist certain scalar quantities, e.g. energy which must be minimised if a given field is to exist, and the field differential equations are the conditions for minimisation.

The process of obtaining such an energy functional can be easily comprehended for a linear case, and, therefore, the variational formulation of the linear electromagnetic field problem will be considered first and will be later extended to the nonlinear case.

Let us consider the linear Poisson's equation obtained from Equation(2·3) of Section 2.2, by assuming the reluctivity to be a constant single-valued quantity, so that

$$\nu \nabla^2 \varphi_o = - J_o \tag{2.47}$$

satisfying mixed homogeneous boundary conditions on some closed surface or set of surfaces.

Supposing an approximate solution to the problem is given by some function φ, we may differentiate this approximate solution and construct a corresponding source function J so as to arrive at a consistent solution to the boundary value problem

$$\nu \nabla^2 \varphi = - J \tag{2.48}$$

This, of course, is not the true solution since $J \neq J_o$, unless the approximate solution just happens to be the right one. Taking the difference between the two equations (2.47) and (2.48), it is seen that the error in the magnetic vector potential satisfies the Poisson's equation whose source function is the error in the current density distribution.

$$\nu \nabla^2 (\varphi - \varphi_o) = - (J - J_o) \tag{2.49}$$

The electromagnetic energy associated with the potential and current density errors is given by [Reference 2.25]

$$\overline{F} = \frac{1}{2} \int (\varphi - \varphi_o) (J - J_o) \, dU \tag{2.50}$$

This integral is obviously zero if the correct solution has been found. Also its value must be positive, since in physical terms it equals the stored energy for a null solution,

$$\varphi_o = J_o = 0 \tag{2.51}$$

This latter observation implies that the zero value associated with the correct solution is not

merely a zero, but is also a true minimum value. Therefore, of any two given approximate solutions, that which yields a lower value of $\bar{F}$ is always the better one.

It is not possible to form $\bar{F}$ unless the correct answer is already known. However, the requirement that $\bar{F}$ be a minimum may be used to choose the best from among several trial solutions even if the actual values of $\bar{F}$ cannot be evaluated, since it is sufficient to know its relative magnitude for the several approximations. Rewriting $\bar{F}$ and expanding in detail,

$$\bar{F} = \frac{1}{2} \int \varphi \, J \, dU - \frac{1}{2} \int \varphi \, J_o \, dU - \frac{1}{2} \int \varphi_o \, J \, dU + \frac{1}{2} \int \varphi_o \, J_o \, dU \tag{2.52}$$

Since J_o appears as the source term in the Poisson's equation, and using suitable vector identities [2.26, p.804] one obtains

$$\begin{aligned} -\int \varphi \, J_o \, dU &= \nu \int \varphi \, \nabla^2 \varphi_o \, dU \\ &= \nu \int \text{div} \, (\varphi \, \text{grad} \, \varphi_o) \, dU - \nu \int \text{grad} \, \varphi \, \text{grad} \, \varphi_o \, dU \\ &= \nu \oint \varphi \, \text{grad} \, \varphi_o \, \overline{ds} - \nu \int \text{grad} \, \varphi \cdot \text{grad} \, \varphi_o \, dU \end{aligned} \tag{2.53}$$

For homogeneous boundary conditions, the surface integral vanishes, leaving

$$\begin{aligned} -\int \varphi \, J_o \, dU &= -\nu \int \text{grad} \, \varphi \cdot \text{grad} \, \varphi_o \, dU \\ &= -\int \varphi_o \, J \, dU \end{aligned} \tag{2.54}$$

Hence

$$\bar{F} = \frac{1}{2} \int \varphi \, J \, dU - \int \varphi J_o \, dU + \frac{1}{2} \int \varphi_o \, J_o \, dU \qquad (2.55)$$

The third term represents the energy associated with the correct solution and is independent of the choice of approximate solution to be tried. This cannot be evaluated unless the exact solution is already known, but since it is a fixed quantity, we may define

$$\mathfrak{F} = \bar{F} - \frac{1}{2} \int \varphi_o \, J_o \, dU \qquad (2.56)$$

and seek the minimum of $\mathfrak{F}$. The minimum value will now not be zero, but it is clear that the minimum value of $\mathfrak{F}$ (equal to the stored energy associated with the correct solution) will correspond to $\varphi = \varphi_o$. Also from (2.55) and (2.56) we have

$$\mathfrak{F} = \frac{1}{2} \int \varphi \, J \, dU - \int \varphi \, J_o \, dU \qquad (2.57)$$

This expression can be made useful for solving a variety of electromagnetic field problems, by re-writing the first term as

$$\frac{1}{2} \int \varphi \, J \, dU = -\frac{\nu}{2} \int \varphi \, \nabla^2 \varphi \, dU \qquad (2.58)$$

$$= -\frac{\nu}{2} \oint \varphi \, \text{grad} \, \varphi \cdot \overline{ds} + \frac{\nu}{2} \int |\text{grad} \, \varphi|^2 \, dU$$

and since the boundary conditions are homogeneous, this reduces to

$$\frac{1}{2} \int \varphi \, J \, dU = \frac{\nu}{2} \int | \text{ grad } \varphi |^2 \, dU \tag{2.59}$$

There results, finally,

$$\mathfrak{F} = \frac{\nu}{2} \int | \text{ grad } \varphi |^2 \, dU - \int \varphi \, J_o \, dU \tag{2.60}$$

For a two-dimensional field problem, since $\varphi = \varphi_z (x, y)$, it can be shown that

$$B = |\text{curl } \overline{\varphi}| = | \text{ grad } \varphi | \tag{2.61}$$

so that the first term of Equation (2.59) can be re-written as

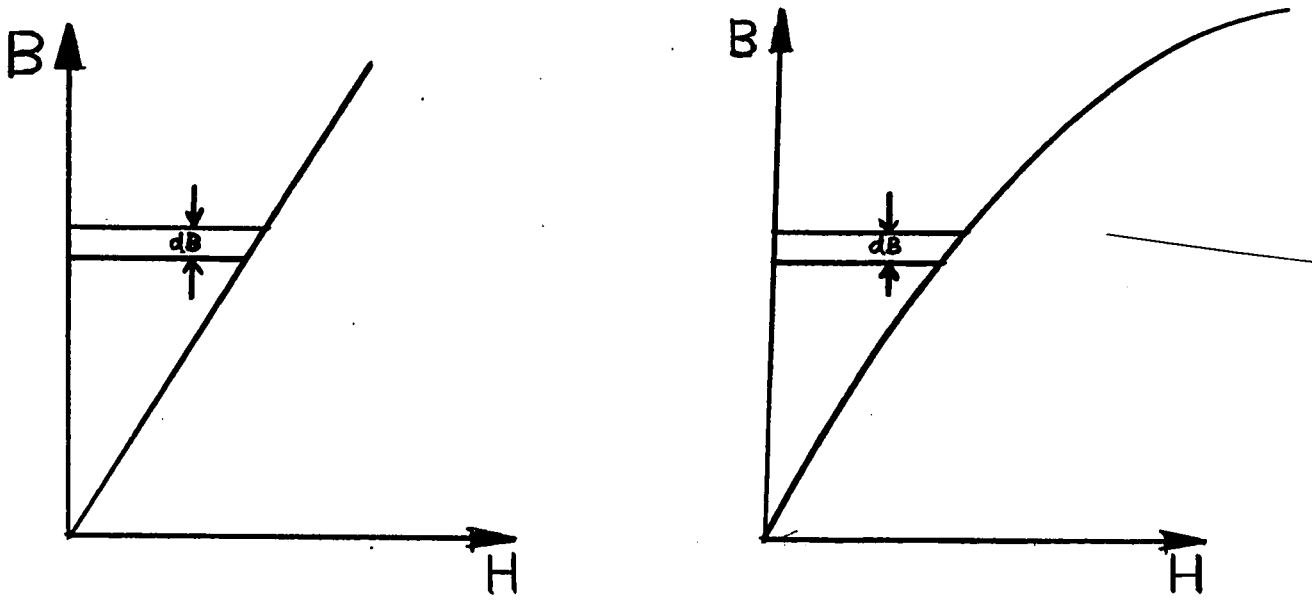
$$\frac{\nu}{2} \int | \text{ grad } \varphi |^2 \, dU = \frac{1}{2} \int \nu B^2 \, dU = \int [\int_o^B H \, db] \, dU \tag{2.62}$$

which is the stored energy in the magnetic field under linear conditions (Figure 2.4a). The formulation $\mathfrak{F}$ which is an energy expression for the given source function and must be minimised for obtaining the true solution of the field problem is termed the <u>energy functional</u>.

For the nonlinear case, the energy stored in the magnetic field is obtained by reference to Figure 2.4b as

$$W = \int [\int_o^B H \, db] \, dU = \int [\int_o^B \nu \, b \, db] \, dU \tag{2.63}$$

since $\overline{H} = \nu \overline{B}$



(a) Linear Conditions.

(b) Nonlinear Conditions.

FIGURE 2.4. ENERGY STORED IN THE MAGNETIC FIELD.

Therefore, the energy functional to be used for the nonlinear field problem can, by a similar analysis to the linear case, be expressed as

$$\mathfrak{F} = \int \left[\int_0^B \nu b \, db \right] dU - \int \varphi J_o \, dU \tag{2.64}$$

For a two-dimensional bounded region, this expression reduces to the form

$$\mathfrak{F} = \int_R \int \left[\int_0^B \nu b \, db \right] dx \, dy - \int_R \int \varphi J_o \, dx \, dy \tag{2.65}$$

where φ and J_o have components only in the Z direction and ν is a single-valued function of B. It is shown in the appendix of Reference [2.27] that Equation (9) is the Euler equation of (2.65), which is a necessary condition for $\mathfrak{F}$ to have an extremum.

For a local minimum, by Legendre transformation,

$$\delta^2 \mathfrak{F} = \frac{\partial^2 f}{\partial \varphi'^2} = \int \int \frac{\partial}{\partial \varphi'} \left[\frac{\partial}{\partial \varphi'} \int_0^B \nu b \, db - J\varphi \right] dx \, dy \tag{2.66}$$

$$= \nu + \varphi' \cdot \frac{\partial \nu}{\partial \varphi'} \tag{2.67}$$

Since $B = | \nabla \varphi |$, Equation (2.66) can be re-written as

$$\delta^2 \mathfrak{F} = \nu + B \frac{\partial \nu}{\partial B} = \frac{\partial}{\partial B} (\nu B) = \frac{\partial H}{\partial B} \tag{2.68}$$

It is readily seen that its value will be equal to ν_{inc}, the incremental permeability (slope of the B-H curve) which is always positive, since the

magnetisation characteristic is monotonically increasing.

Let us now examine if a strong minimum exists. Evaluating Equation (2.38) for the functional of (2.65), one obtains by using the expression of (2.46)

$$\frac{\partial^2 f}{\partial \varphi'^2} \cdot \frac{\partial^2 f}{\partial \varphi^2} - \left(\frac{\partial^2 f}{\partial \varphi \, \partial \varphi'}\right)^2 = 0 \qquad (2.69)$$

since

$$\frac{\partial^2 f}{\partial \varphi^2} = 0 \, , \quad \frac{\partial f}{\partial \varphi} = -J \, , \quad \frac{\partial^2 f}{\partial \varphi \, \partial \varphi'} = 0$$

This indicates that the second variation of the functional is zero and the functional is a minimum ; but since strict inequality of (2.46) does not hold in (2.69), it cannot be asserted that the solution obtained is unique. Having thus proved conclusively that the nonlinear functional formulation in (2.64) attains a local as well as a general minimum, it only remains for us to construct a minimising sequence of trial functions which will ensure a unique convergent solution.

Mikhlin [2.28] has shown that a Rayleigh-Ritz approximation to the solution of the nonlinear functional

$$\mathfrak{F}(\varphi) = \int_S \int \sum_{i=1}^{K} \int_0^{\tau_i^2(\varphi)} \rho_i(\xi) \, d\xi \, dx \, dy - \langle f, \varphi \rangle \qquad (2.70)$$

can be constructed, which will minimise the functional and attain a limit if the number of terms of the sequence tend to infinity. By a simple substitution for ξ, $\rho(\xi)$ and f such that

$$\lambda^2 = \xi , \qquad J_o = f \text{ and}$$

$$\rho(\xi) = \frac{1}{2} \psi(\sqrt{\xi}) \tag{2.71}$$

it can be shown that Equations (2.70) and (2.65) are identical to one another and therefore, one may conclude that the R - R type approximation would ensure a unique solution of (2.64). The trial functions in this case will be of the form

$$\varphi_{n\,j} = \sum_{k=1}^{n} a_k \theta_k \tag{2.72}$$

from which the a's are evaluated by setting

$$\frac{\partial \mathcal{I}}{\partial a_i} = 0 \tag{2.73}$$

REFERENCES

[2.1] W.F. Ames, Nonlinear Partial Differential Equations in Engineering, Academic Press, New York, 1965.

[2.2] L. Bolzmann, "Ann. Physik", 1894, Vol. 53, p. 959 .

[2.3] O. Benedikt, The nomographic computation of complicated and highly saturated Magnetic circuits, Pergamon Press, London 1964.

[2.4] K.J. Binns, "Predetermination of the no-load magnetisation characteristics of large Turbogenerators", Proceedings IEE (GB), Vol. 112, No. 4, pp. 720 - 30, April 1965.

[2.5] C.F. Gauss, "Brief an Gerling", Werke, Vol. 9, pp. 278 - 81, Dec. 1823, (Translated by G.E. Forsythe, Math. Tab. Wash., Vol. 5, pp. 255 - 8, 1951).

[2.6] L. Seidel, "Über ein Verfahren die Gleichungen, auf welche die Methode der kleinsten Quadrate führt, Sowie lineare Gleichungen überhaupt durch successive Annäherung aufzulosen, "ABHANDLUNGEN DER BAYERISCHEN AKADEMIE, Vol. 11, pp. 81 - 108, 1873.

[2.7] R.V. Southwell, "Stress calculation in frameworks by the method of 'Systematic Relaxation of Constraints,' " Proc. Roy. Soc., Vol. 151, A, p. 56, 1935.

[2.8] B. Friedman, Principles and Techniques of Applied Mathematics, New York, John Wiley, 1960.

[2.9] E.A. Erdélyi and S.V. Ahmed, "Flux distribution in Saturated DC Machines", IEEE TRANS. POWER APPARATUS AND SYSTEMS, Vol. 84, No. 5, pp. 375 - 381, May 1965.

[2.10] E.A. Erdélyi, M.S. Sarma and S.S. Coleman, "Magnetic fields in nonlinear salient pole alternators", IEEE TRANS, POWER APPARATUS AND SYSTEMS, Vol. 87, No. 10, October 1968.

[2.11] F. de la Vallée Poussin and A. Lion, "Iterative calculation of magnetic induction in electrical machines", REVUE GÉNÉRALE DE L'ÉLECTRICITÉ', Vol. 76, pp. 731 - 739, April 1967.

[2.12] A.M. Winslow, "Magnetic field calculation in an irregular triangle mesh", Proceedings of the International Symposium on Magnet Technology, Stanford, pp. 170 - 181, October 1965.

[2.13] P. Concus, "On the calculation of Nonlinear Magnetostatic Fields", Proceedings of the International Symposium on Magnet Technology, Stanford, pp. 164 - 169, October 1965.

[2.14] H.M. Lieberstein, "Overrelaxation for Non-linear Elliptic Partial Differential Problems", MRC Tech. Summary Report No. 80, University of Wisconsin, 1959.

[2.15] P. Rosen, "Use of Restricted Variational Principles for the solution of Differential Equations", Journal of Applied Physics, Vol. 25, p. 336, 1954.

[2.16] I. Stakgold, Boundary Value Problems in Mathematical Physics, Macmillan series in Advanced Mathematics and Theoretical Physics, Vol. II, pp. 337 - 40.

[2.17] P. W. Berg, "Calculus of Variations", Engineering Mechanics Handbook, ed. W. Flugge, McGraw-Hill, Chapter 16, 1962.

[2.18] F.B. Hildebrand, Methods of Applied Mathematics, Prentice Hall, Englewood Cliffs, New Jersey, p. 136, Equation 135, 1965.

[2.19] F.B. Hildebrand, Advanced calculus for Applications, Prentice Hall, Englewood Cliffs, New Jersey, p. 297, 1962.

[2.20] M. Becker, The Principles and Applications of Variational Methods, Research Monograph No. 27, The M.I.T. Press, Cambridge, Massachusetts, Chapter 2, p. 7, Section 2.3.

[2.21] O. Bolza, Lectures on the calculus of Variations, Dover Publications Inc., New York, p. 46, Chapter 2, Section 11.b.

[2.22] Legendre, "Memoire sur la manière de distinguer les maxima des minima dans le calcul des variations", Memoires de l'Academie des Sciences, 1786.

[2.23] Lagrange, "Oeuvres", Vol. IX, p. 303, 1797.

[2.24] O.A. Ladyzhenskaya and N.N. Ural'tseva, "Quasi-linear Elliptic Equations and Variational Problems with many independent variables", Russian Mathematical Surveys, (translated from USPEKHI MATHEMATICHES - KIKH NAUK), Cleaver-Hume Press, London, Vol. XVI, Section 8, pp. 59 - 62, 1961.

[2.25] P. Silvester, Modern Electromagnetic Fields, Prentice Hall, Englewood Cliffs, New Jersey 1968, Chapter 5, Section 5, pp. 151 - 153.

[2.26] Morse-Feshbach, Methods of Theoretical Physics, McGraw-Hill, New York 1953, Part I, Chapter 7, pp. 804 - 805.

[2.27] P. Silvester and M.V.K. Chari, "Finite Element Solution of Saturable Magnetic Field Problems", IEEE PAPER 70 TP 185 - PWR, January 1970.

[2.28] S.G. Mikhlin and K.L. Smolitskiy, Approximate Methods for solution of Differential and Integral Equations, Modern Analytical and computational methods in Science and Mathematics, Elsevier, New York, Vol. 5, Section 10.4, p. 258, Equation (2.230), 1967.

CHAPTER III

FINITE ELEMENT FORMULATION

3.1 Description of Direct Methods

If a differential equation is found to be the Euler equation of some functional, and if a direct method is used to establish that the functional has an extremum in a class of functions differentiable a sufficient number of times, then the differential equation has a solution for the specified boundary conditions. A direct method, as described in Reference [2.24] consists of constructing a sequence of functions that converges to the desired solution function. Thus, the method establishes the existence of a solution and provides a way of constructing an approximate solution. The principal constituents of a direct method are (see Reference [3.1, pp. 130-131]) .

(a) The construction of a minimising sequence.

(b) Proof of the existence of a limit for this sequence.

(c) Proof of the semi-continuity [3.1, p. 130] of the functional at the limit.

The Rayleigh-Ritz method is shown to belong to this class of direct methods in References [2.28, pp. 256-57, Theorem 32] and [3.2], satisfying the requirements above. De Arantes [3.3, p. 942] has shown that the Finite Element method becomes a special case of the R-R method, provided certain conformity and completeness conditions are satisfied by the elements and their interfaces. Some of these aspects will be discussed in this chapter to demonstrate the validity of the technique as a direct method of functional

minimisation, and its applicability to the solution of two-dimensional nonlinear elliptic boundary value problems. For detailed theoretical analysis, however, the reader is referred to Pian and Pin Tong [3.4] and de Arantes [3.3].

3.2.1 Finite Element Method

The finite element method is a general technique of numerical analysis which belongs to the class of direct methods and provides an approximate solution for the field problem. In this method, the continuous region R is subdivided into a finite number of sub-regions, in each of which families of functions having different analytical expressions are defined.

A finite element may be described as a closed sub-region with a family of functions prescribed within it. This family is a linear combination of the prescribed values of the field at discrete points called nodes on the boundary and in the interior of the element. The type of an element refers to its general shape, nodal point specification and to the functions analytically defined in it in terms of their nodal values and the co-ordinate system chosen.

In the present analysis, the two-dimensional region R bounded by S, is represented by a finite number of triangles and the field inside each triangle is specified as a function of its nodal values. The nodal vector potentials thus defined will constitute a matrix, the element functions of which will be assumed to be continuous and to have continuous principal derivatives of order $(p_i - 1)$ or less in the closed sub-region of the

finite element. Here p_i is used to denote the order of the differential operator of the field problem.

Having thus defined the approximate function in each element in terms of the nodal values, the problem now reduces to one of minimising the functional of the variational problem with respect to each of the nodal values of the field. This results in a set of nonlinear algebraic equations which are solved iteratively to obtain the appropriate solution to the problem.

3.2.2 Discrete Representation of the Nonlinear Energy Functional by Finite Elements and Derivation of the Coefficient Matrix

The method used in this analysis is based on the higher order finite elements described by Silvester [3.5], and it is specialised to the first order element for the minimisation of the nonlinear energy functional defined by Equation (2.65) of Chapter II, Section 2.5.

The two dimensional region R is subdivided into triangles in an arbitrary manner as shown in Figure 3.1, ensuring only that material interfaces and other physical boundaries coincide with triangle edges. The vector potential φ in each triangular element will be assumed to be a complete polynomial in x and y, so that the orientation of the element is of no importance [3.6]. The number of terms in a complete polynomial of degree N is given by

$$n = (N + 1) \cdot (N + 2) / 2 \qquad (3.1)$$

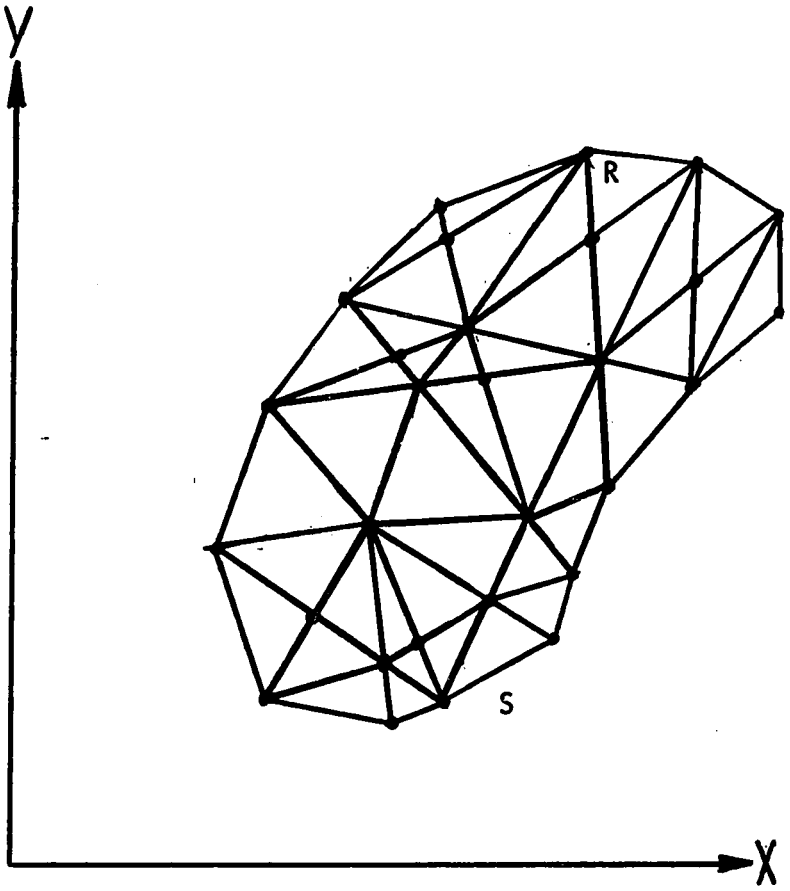


FIGURE 3.1. CONTINUOUS REGION R DISCRETISED BY FINITE ELEMENTS.

and, therefore, each element must have n degrees of freedom, i.e., n independently specified parameters. Further, if the functional $\mathfrak{F}$ is to be evaluated without additional assumptions, the first derivatives of φ must be finite everywhere. Consequently φ must be continuous and this requirement is met by defining φ along any triangle edge as a polynomial function of order N along the edge. The number of coefficients of the polynomial must depend on the values of φ along the edge, if continuity with the adjoining triangle is to be assured. In Reference [3.5] it is shown that it is necessary and sufficient to specify φ at the triangle vertices and N - 1 other points along each edge.

In order to define a complete polynomial expression of order greater than 2, more information is necessary, such as for example specification of interior points.

In order to facilitate analysis, the so-called area co-ordinates (vide Appendix 1) will be used. In this system, the position of any point within the triangle is given by the distance measured along the perpendicular to each of the sides passing through the point, distances being expressed as fractions of the triangle altitude. The numerical values of the area co-ordinates range from 0 to 1 in every triangle. The area co-ordinates are conveniently specified as ζ_1, ζ_2, ζ_3, the lines of constant ζ_n being parallel to side n of the triangle. Only two of these co-ordinates are independent, since they are related by

$$\zeta_1 + \zeta_2 + \zeta_3 = 1 \tag{3.2}$$

A regularly spaced set of points P_{stu} may be defined in a triangle by the area co-ordinate values

$$\left(\frac{s}{N}, \frac{t}{N}, \frac{u}{N}\right), \quad 0 \leq s, t, u \leq N, \tag{3.3}$$

where s, t, u are nonnegative integers satisfying the relation $s+t+u=N$. Such a set of points is shown in Figure 3.2 for a first order element and it is evident that the number of points specified is

$$\frac{1}{2}(N+1) \cdot (N+2) = 3 \tag{3.4}$$

which is the total number of independent coefficients of the polynomial expression of order N. There are no intermediate points along each edge, since (N - 1) = 0.

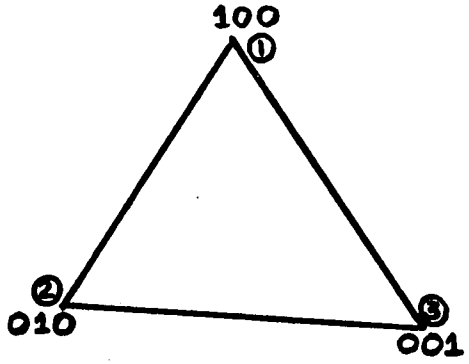


FIGURE 3.2. TRIPLE AND SINGLE SUBSCRIPT NOTATION OF FINITE ELEMENTS.

Polynomial Approximation to Functional Minimisation

We shall now specify the potential at each of these points as $\varphi_{s,t,u}$, so that the function φ will be defined throughout the triangular element by the polynomial

$$\varphi(\xi_1, \xi_2, \xi_3) = \sum_{s=0}^{N} \sum_{t=0}^{N-s} \alpha_{s,t,u}(\xi_1, \xi_2, \xi_3)\, \varphi_{s,t,u} \qquad (3.5)$$

where $\alpha_{s,t,u}$ are polynomials of order N in ξ_1, ξ_2, ξ_3, which are independent

of φ and triangle size and shape. The polynomials defined by Equation (3.5) are complete and satisfy the continuity requirements. Since u depends on s and t, the summation over u is not required.

The forcing function J is not expressed as a polynomial, since it is constant over each triangle.

For the two dimensional case, where the dimension along the Z direction is large and the vector potential is defined by

$$\varphi_z = \varphi(x, y),$$

the magnetic induction B is given by

$$|B| = |\text{curl } \overline{\varphi}| = |\text{grad } \varphi_z| = \sqrt{\left(\frac{\partial \varphi_z}{\partial x}\right)^2 + \left(\frac{\partial \varphi_z}{\partial y}\right)^2} \qquad (3.6)$$

The necessary condition for minimising the functional defined by Equation (2.65) of Chapter II, Section 2.5 is that its first variation with respect to each of the point values of potential in each triangle must be set to zero, so that

$$\frac{\partial \mathfrak{F}}{\partial \varphi_k} = 0 \quad \text{for all } k \qquad (3.7)$$

To perform this minimisation, and thereby obtain an approximate solution for φ everywhere, it is convenient to rewrite the surface integrals of Equation (2.65) Chapter II, in such a way as to permit evaluation over one triangle at a time, so that

$$\mathfrak{F} = \sum \left(\int \int \left[\int_o^B \nu b \, db \right] dx \, dy - \int \int J \varphi \, dx \, dy \right) \qquad (3.8)$$

The typical term of the above series, say the t'th term will be

$$\mathfrak{F}_t = \int \int_t \left[\int_o^B \nu b \, db - J \varphi \right] dx \, dy \qquad (3.9)$$

Equation (3.7) may now be written as

$$\frac{\partial \mathfrak{F}}{\partial \varphi_k} = \sum_{\text{triangles}} \frac{\partial \mathfrak{F}_t}{\partial \varphi_k} = 0 \qquad (3.10)$$

where the summation is carried over all triangles in the sub-division. Substituting for $\mathfrak{F}$ in Equation (3.10) from Equation (3.8) and differentiating one obtains

$$\frac{\partial \mathfrak{F}}{\partial \varphi_k} = \int_R \int \frac{\partial}{\partial \varphi_k} \left[\int_o^B \nu b \, db \right] dx \, dy - \int_R \int \frac{\partial}{\partial \varphi_k} [J \varphi] \, dx \, dy \qquad (3.11)$$

$$= \int_R \int \left[\nu B \frac{\partial B}{\partial \varphi_k} \right] dx \, dy - \int \int J \left(\frac{\partial \varphi}{\partial \varphi_k} \right) dx \, dy = 0 \qquad (3.12)$$

where the index k ranges over the points defined by (s, t, u) in the triangle.

Differentiation with respect to φ_k clearly produces a zero, unless k is one of the points defined on the edges of the triangle, as appropriate to the order of the element chosen. Therefore, Equation (3.5) is an expression in a number of independent variables φ_k, the number being defined by Equation (3.1). For the first order case, the above differentiation yields an equation in three variables only.

Substituting for B from Equation (3.6) in the first term of the energy functional expression above, Equation (3.12) reduces to the form

$$\int_R \int \nu \left[\frac{\partial \varphi}{\partial x} \cdot \frac{\partial}{\partial \varphi_x} \left(\frac{\partial \varphi}{\partial x} \right) + \frac{\partial \varphi}{\partial y} \cdot \frac{\partial}{\partial \varphi_y} \left(\frac{\partial \varphi}{\partial y} \right) \right] dx\, dy = \int \int J \left(\frac{\partial \varphi}{\partial \varphi_k} \right) dx\, dy \tag{3.13}$$

Substituting for φ from Equation (3.5) in Equation (3.13), one obtains the result, after repeated differentiation

$$\int_R \int \sum_{q=1}^{n} \nu \left[\frac{\partial \alpha_p}{\partial x} \cdot \frac{\partial \alpha_q}{\partial x} + \frac{\partial \alpha_p}{\partial y} \cdot \frac{\partial \alpha_q}{\partial y} \right] \varphi_q = \int_R \int \sum_{k=1}^{n} J \alpha_q \, dx\, dy \tag{3.14}$$

To convert to triangle area coordinates, the relations of Equation (A.1.3) of Appendix 1 are used, namely

$$\zeta_i = \frac{(a_i + b_i x + c_i y)}{2 \Delta}$$

$$a_i = -(x_m y_j - x_j y_m)$$

where $b_i = y_j - y_m$ and so on

$$c_i = x_m - x_j$$

and Δ represents the triangle area. The index i ranges over the triangle vertices and not the points $P_{s,t,u}$. The derivatives in Equation (3.14) can then be transformed in

terms of area coordinates as follows. For a typical partial derivative

$$\frac{\partial \alpha_p}{\partial x} = \sum_i \frac{\partial \alpha_p}{\partial \zeta_i} \cdot \frac{\partial \zeta_i}{\partial x} \tag{3.15}$$

and using Equation (A.1.3), this reduces to the form

$$\frac{\partial \alpha_p}{\partial x} = \sum_i \frac{\partial \alpha_p}{\partial \zeta_i} \cdot \frac{b_i}{2\Delta} \tag{3.16}$$

Similarly the other partial derivatives can be evaluated and substituting these values in Equation (3.14) above, there is obtained

$$\int\int \frac{\nu}{4\Delta^2} \sum_{q=1}^{n} \left[\sum_i \sum_j (b_i b_j + c_i c_j) \frac{\partial \alpha_p}{\partial \zeta_i} \frac{\partial \alpha_q}{\partial \zeta_j} \right] \varphi_q \, ds = \int \frac{1}{2\Delta} \sum_{q=1}^{n} J \alpha_q \, ds \tag{3.17}$$

Here the indices i, j, m range over the triangle vertices, while indices p, q assume values 1, 2, ...n corresponding to the points P_{stu}. It is worth noting that the bracketed expression is symmetric in p and q.

From the geometric properties of a triangle, it can be shown as in Appendix II that

$$\begin{aligned} b_i b_j + c_i c_j &= -2\Delta \operatorname{Cot} \theta_k \qquad i \neq j \\ b_i^2 + c_i^2 &= 2\Delta (\operatorname{Cot} \theta_i + \operatorname{Cot} \theta_k) \end{aligned} \tag{3.18}$$

where θ_k denotes the included angle at vertex k . Substituting equations (3.18) into (3.17), expanding and collecting the respective terms, Equation (3.17) is transformed as

$$\int\int \frac{\nu}{2\Delta} \sum_{q=1}^{n} \left[\sum_{i}^{m} \left(\frac{\partial \alpha_p}{\partial \zeta_i} - \frac{\partial \alpha_p}{\partial \zeta_k} \right) \cdot \left(\frac{\partial \alpha_q}{\partial \zeta_i} - \frac{\partial \alpha_q}{\partial \zeta_k} \right) \operatorname{Cot} \theta_i \right] \varphi_q \, ds$$

$$= \int_R \int \frac{1}{2\Delta} \sum_{q=1}^{n} J \alpha_q \, ds \qquad (3.19)$$

where i, j, k assume values 1, 2, 3 cyclically.

If we define the typical elements of the above Equation (3.19) so that

$$S_{mq} = \frac{1}{2\Delta} \sum_{i}^{m} \operatorname{Cot} \theta_i \int \nu \left(\frac{\partial \alpha_p}{\partial \zeta_i} - \frac{\partial \alpha_p}{\partial \zeta_k} \right) \left(\frac{\partial \alpha_q}{\partial \zeta_i} - \frac{\partial \alpha_q}{\partial \zeta_k} \right) ds \qquad (3.20)$$

and $$R_{mq} = \frac{1}{2\Delta} \sum_{q=1}^{n} \int J \alpha_q \, ds \qquad (3.21)$$

then (3.19) may be written in matrix form as

$$S \varphi = R \qquad (3.22)$$

The matrix R is seen to be independent of the triangle shape. S may be written as

$$S = \sum_{i}^{m} Q_i \operatorname{Cot} \theta_i \qquad (3.23)$$

where Q_i is independent of the shape of the triangle, but is dependent on the reluctivity function. We shall now derive the element coefficient matrices.

Evaluation of the Polynomial Coefficients

In order to evaluate the matrices R and Q, above, the polynomials α_{stu} of Equation (3.3) are required. It is, therefore, necessary to define an auxiliary function $P_m(z)$ such that

$$P_m(z) = \prod_{i=1}^{m} \left(\frac{Nz - i + 1}{i}\right), \quad m \geq 1 \qquad (3.24)$$
$$= 1 \qquad m = 0$$

where $P_m(z)$ is a polynomial of order m in z. Let a polynomial α_{stu} of order N be defined over the triangle such that

$$\alpha_{stu} = P_s(\zeta_1)\, P_t(\zeta_2)\, P_u(\zeta_3), \quad s+t+u = N \qquad (3.25)$$

These polynomials satisfy all the conditions necessary to qualify as the required polynomials of Equation (3.5) ; and since there are as many distinct linearly independent polynomials α_{stu} as there are terms in a complete polynomial of order N, the use of these expressions makes the polynomial of Equation (3.5) complete. Let us consider the values of the area coordinates at one of the points given by (3.3), i.e. at the typical point $(p/N,\ q/N,\ r/N)$. From Equation (3.24), one finds that for s, p integers

$$P_s\left(\frac{p}{N}\right) = 0 \qquad s > p$$
$$= 1, \qquad s = p \tag{3.26}$$

Therefore, at the typical point all α_{stu} given by (3.25) are identically zero, except for α_{pqr}, which assumes the value unity. Equations (3.5) becomes a simple identity at the points where φ are specified.

Calculation of the Element Matrices for First Order Elements

The triangular element is subscripted as shown in Figure 3.2. Since there are no intermediate points along the edges of the triangle, one obtains by using Equation (3.24)

$$P_s(\zeta_i) = \frac{\zeta_{i-1+1}}{1} = \zeta_i$$
$$P_t(\zeta_j) = 1 \qquad \text{and so on for other vertices} \tag{3.27}$$
$$P_u(\zeta_k) = 1$$

Hence α_{stu} for the vertex $i = \zeta_i$

Similarly

$$\alpha_{stu} \text{ for the vertex } j = \zeta_j \tag{3.28}$$
$$\alpha_{stu} \text{ for the vertex } k = \zeta_k$$

Substituting the values of a_{stu} from Equations (3.28) in the integral of Equation (3.20), its value is obtained as

$$\left(\frac{\partial a_p}{\partial \zeta_i} - \frac{\partial a_p}{\partial \zeta_k}\right)\cdot\left(\frac{\partial a_q}{\partial \zeta_i} - \frac{\partial a_q}{\partial \zeta_k}\right) = \left(\frac{\partial(\zeta_i)}{\partial \zeta_i} - \frac{\partial(\zeta_j)}{\partial \zeta_k}\right)\cdot\left(\frac{\partial(\zeta_k)}{\partial \zeta_i} - \frac{\partial(\zeta_k)}{\partial \zeta_k}\right) = -1 \tag{3.29}$$

Using Equation (3.29) above and the value of $\cot\theta_i$ from Equation (3.18) in Equation (3.20), there is

$$S_{mq} = \frac{1}{4\Delta^2} \cdot (b_i b_j + c_i c_j) \int\int \nu \, ds \tag{3.30}$$

The value of $\int\int ds$ is Δ, and therefore, matrix S reduces to

$$S = \sum_{i}^{j}\sum_{i}^{k} \nu \frac{(b_i b_j + c_i c_j)}{4\Delta} \varphi_i \tag{3.31}$$

Now for a first order element, using Equations (3.5) and (3.28)

$$\varphi = a_{stu}\, \varphi_q = (\zeta_i \varphi_i + \zeta_j \varphi_j + \zeta_K \varphi_k) \tag{3.32}$$

Differentiating the above expression for φ with respect to x and y partially and substituting the results in Equation (3.6), the magnetic induction B is evaluated as

$$B = \frac{1}{2\Delta}\sqrt{(b_i \varphi_i + b_j \varphi_j + b_k \varphi_k)^2 + (c_i \varphi_i + c_j \varphi_j + c_K \varphi_K)^2} \tag{3.33}$$

The value of B is seen to be independent of the coordinate system, and is therefore, a constant in each triangle. The reluctivity, ν, which is a function of B is likewise a constant in each triangle and can therefore be taken outside of Equation (3.31). Thus the final expression for the coefficient matrix is given by

$$S = \frac{\nu}{4\Delta} \sum_{i}^{j} \sum_{i}^{k} (b_i b_j + c_i c_j) \varphi_i \qquad (3.34)$$

By a procedure similar to the derivation of the matrix S, the right hand side of Equation (3.19) is evaluated as (See Appendix I)

$$R = \int_R \int \frac{1}{2\Delta} \sum_{q=1}^{n} J a_q \, ds = 2\Delta . J \int \int \zeta_i \, d\zeta_i \, d\zeta_j \qquad (3.35)$$

since J, the current density is constant in each triangle.

As shown in Appendix I, the value of this integral will be equal to $J\Delta/3$, so that the final field equation in discretised variational form can be written as

$$\frac{\nu}{4\Delta} \cdot \sum_{i}^{k} (b_i b_j + c_i c_j) \varphi_i = J\Delta/3 \qquad (3.36)$$

Expanding this expression in detail, the complete matrix equation representing the non-linear field problem is obtained as

$$\frac{\nu}{4\Delta}\cdot\begin{vmatrix} (b_i\,b_i + c_i\,c_i) & (b_i\,b_j + c_i\,c_j) & (b_i\,b_k + c_i\,c_k) \\ (b_j\,b_i + c_j\,c_i) & (b_j\,b_j + c_j\,c_j) & (b_j\,b_k + c_j\,c_k) \\ (b_k\,b_i + c_k\,c_i) & (b_k\,b_j + c_k\,c_j) & (b_k\,b_k + c_k\,c_k) \end{vmatrix} \begin{bmatrix} \varphi_i \\ \varphi_j \\ \varphi_k \end{bmatrix} = J\Delta\cdot\begin{bmatrix} 1/3 \\ 1/3 \\ 1/3 \end{bmatrix} \tag{3.37}$$

For only one triangle in the field region, a suitable discrete representation of the energy functional (3.8) is given by the above matrix Equation (3.37). To obtain a corresponding discrete representation for the entire problem region, it is only necessary to write one equation similar to (3.37) for each and every triangle in the sub-division, just as it is necessary in finite difference methods to write the finite difference expressions anew at each mesh point. It should be noted, however, that the variational expression (3.8) need not be modified for triangles near the boundaries, since natural boundary conditions (Dirichlet or homogeneous Neumann boundary) are implicit in the functional formulation.

When an equation corresponding to (3.37) has been written for each of the triangles in the sub-division, the resulting assembly of algebraic equations may be combined immediately into a single matrix equation given by

$$[S]\,.\,[\varphi] = 1/3\,.\,[R] \tag{3.38}$$

where φ is the column vector of vertex values of φ, R the corresponding column vector of currents and S is the matrix obtained by adding the individual element matrices. It is

at once clear that this problem is in fact a nonlinear algebraic one, since S depends not only on the shape and size of each triangle, but also on the reluctivities, which depend on φ. The nonlinear continuum problem of Equation (2.65) of Chapter II, Section 2.5 has thus been represented by a set of nonlinear algebraic equations. Matrix S, it should be noted, is of order equal to the number of vertex potentials, and it corresponds to the stiffness matrices encountered in structural finite element analysis.

3.3 Boundary Conditions and their Effect on the Variational Formulation

The energy functional defined by Equation (2.65), Chapter II yields the nonlinear Poisson's equation as its Euler equation satisfying Dirichlet and natural Neumann type boundary conditions. Therefore, homogeneous boundary conditions are implicit in the variational formulation. In such cases the functional does not require any modifications, since it is self-sufficient. In the case of loaded boundaries, however, expressed by the relation (see [3.7], p. 149, Equation 10.3) on the boundary by

$$\nu \frac{\partial \varphi}{\partial x} l_x + \nu \frac{\partial \varphi}{\partial y} l_y + q + \alpha \varphi = 0 \qquad (3.39)$$

wher l_x, l_y are the direction cosines of the outward normal to the boundary surface, further consideration of the variational formulation is necessary.

In an electrical machine, usually the external boundaries of the region coincide with symmetry lines and, therefore, the quantity q which represents net flow of flux across the boundary will be zero. The term $\alpha \varphi$ in Equation (3.39) represents power loss

per unit area across the boundary. This is a more complex type of boundary condition than the natural boundary conditions defined earlier. While it is quite feasible to satisfy the same by directly imposing constraints on the boundary elements, such an approach is inelegant and suffers from the draw-back that the physical problem becomes obscured. Consequently it is advantageous to modify the variational problem so that the value of φ on the boundary can take up any value without constraint. This is easily accomplished by adding appropriate terms to the functional which is to be minimised, and the resulting expression for the energy functional becomes

$$\mathfrak{F} = \int\int \left[\int_{o}^{B} \nu b \, db - J\varphi \right] dS + \int_{c} q\varphi \, dS + \int \frac{1}{2} \alpha \varphi^2 \, dS \qquad (3.40)$$

In Equation (3.40), the last two integrals are taken along the boundary subject to the boundary conditions of Equation (3.39) and along which φ is not constrained.

The minimisation of the energy functional will now result in adding the derivatives of the last two terms with respect to the nodal values of φ, to the Equation (3.11) of Section 3.2.2. Such derivatives will exist only for elements which actually form the boundary, as for example the element r, s, k illustrated in Figure 3.3. The values of these additional terms will now be evaluated. If φ_r and φ_s are the potentials at the vertices r and s respectively and L is the length of the face r - s, then the expression for a minimum value of the functional can be shown, by using Equations (A.3.3) & (A.3.5) of Appendix III, as

$$\left[\frac{\partial \mathfrak{F}}{\partial \varphi_r}\right] = 0 = [S] \cdot [\varphi] - [T] + \sum \left[\frac{qL}{2} + \frac{\alpha L}{3} \left(\varphi_r - \frac{1}{2} \varphi_s \right) \right] \qquad (3.41)$$

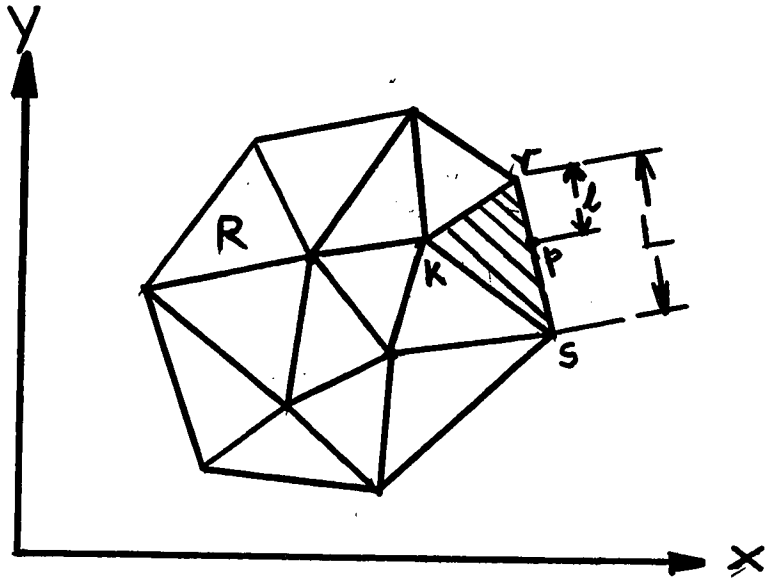


FIGURE 3.3. DETAILS OF FINITE TRIANGULAR ELEMENT ON THE BOUNDARY OF THE FIELD REGION R.

For an unloaded boundary $q = \alpha = 0$, so that the additional terms vanish. As already described, symmetry lines fall into this category. If neither the potential φ nor its normal derivative is prescribed on the boundary, the variational formulation will imply homogeneous Neumann or Dirichlet boundaries automatically.

3.4 Continuity of the Magnetic Field Across the Interface Between Finite Elements

For the present two-dimensional continuum problem, the magnetic vector potential has only a component along the Z direction and is invariant in that direction, so that the magnetic induction B is given by

$$| B | = | \nabla \times \bar{\varphi} | = | \nabla \varphi_z | \qquad (3.42)$$

In Cartesian co-ordinates, the following relations hold

$$\vec{\nabla} \times \vec{\varphi} = \frac{\partial \vec{\varphi}}{\partial y} \hat{i} - \frac{\partial \vec{\varphi}}{\partial x} \hat{j}$$

$$\vec{\nabla} \varphi_z = \frac{\partial \varphi_z}{\partial x} \hat{i} + \frac{\partial \varphi_z}{\partial y} \hat{j} \qquad (3.43)$$

Since φ has only a component φ_z, it is seen that the curl of the vector potential and the gradient of φ_z have the same magnitude, but are orthogonal to each other. Hence, the tangential component of the magnetising force H can be expressed as

$$\vec{H}_t = \nu \vec{B}_t = \nu (\vec{\nabla} \times \vec{\varphi})_t \qquad (3.44)$$

and from Equation (3.43), this relation reduces to

$$\vec{H}_t = \nu (\vec{\nabla} \varphi)_n \qquad (3.45)$$

Using Equations (3.43), the normal component of the magnetic induction B is obtained as

$$\vec{B}_n = (\vec{\nabla} \times \vec{\varphi})_n = (\vec{\nabla} \varphi)_t \qquad (3.46)$$

Since the vector potential is continuous, the tangential component of its derivatives will also be continuous, resulting in the continuity of B_n. As regards the tangential component of H, continuity will be ensured so long as the product $\nu (\vec{\nabla} \varphi)_n$

is continuous. This requirement is met by defining the reluctivity function in such a way that in association with the normal derivative of φ, it will result in the continuity of the tangential magnetising force H, although ν and φ may not necessarily be continuous across the interfaces of every element.

3.5 Minimisation of the Functional in the Discretised Region and Uniqueness of the Approximate Solution

The criterion for a strong minimum of the functional is defined by the differential equation (2.37), Chapter II, Section 2.4.3. We shall now determine the value of this expression using the approximate trial functions defined in the finite element discretisation by Equation (3.32), Section 3.2.2.

Since grad φ_z is non-zero in each triangular element and the reluctivity ν is differentiable with respect to the magnetic induction B, by substituting for f in Equation (2.46), Chapter II, Section 2.2.4 for a strong minimum, one obtains for the second variation of the nonlinear functional, the value

$$\frac{\partial^2 f}{\partial \varphi'^2} \cdot \frac{\partial^2 f}{\partial \varphi^2} - \left(\frac{\partial^2 f}{\partial \varphi \, \partial \varphi'}\right)^2 = \left\{\frac{\partial \nu}{\partial \varphi'} \cdot \varphi' + \nu\right\} \times 0 - 0 = 0 \qquad (3.47)$$

Obviously the energy functional attains a minimum value in the discretised case just as it attains a minimum in the continuum problem. Nevertheless, as before, strict inequality of (2.38), Chapter II, Section 2.4.3 does not hold and hence the solution to the field problem cannot be asserted as being unique. Further the contributions, if any,

of the element boundaries will have to be considered. The uniqueness of the solution can only be ensured by appeal to a minimising sequence such as the R - R method. It was shown by de Arantes [3.3] that the finite element method is a specialisation of the R - R method provided, of course, conformity and completeness conditions are satisfied. Conformity is said to obtain, if the piecewise defined fields in the elements have continuity of the field and its principal derivatives across the element boundaries. The completeness criterion, however, is satisfied if the minimising sequence has a limit and is bounded (see [3.3], pp. 944 - 45, Equations 81 and 82), although the principal derivatives of the functions may not be continuous. Further, the completeness requirement overrides the conformity condition and the derivatives across the element boundaries need not be continuous in order to ensure that the finite element method belongs to the R - R class of a minimising sequence.

As an alternative method of proving that the functional attains a minimum in the bounded region R, we may for the present neglect the element boundaries. Then we can show that the minimisation process is satisfied in each element by recourse to Equation (3.47). It is then only required to show that the interfaces do not affect the problem in any way for the type of element chosen and the conditions of the problem. It has already been shown in Section 3.4 of this chapter that the tangential component of the magnetising force H and the normal component of the magnetic induction B are continuous at element interfaces. Therefore, if we take a line integral around the element boundary as shown in Figure 3.4, its value will be zero, indicating that the interface makes no contribution to the functional in any way. Thus the minimisation of the functional is valid whether or not the interfaces are considered.

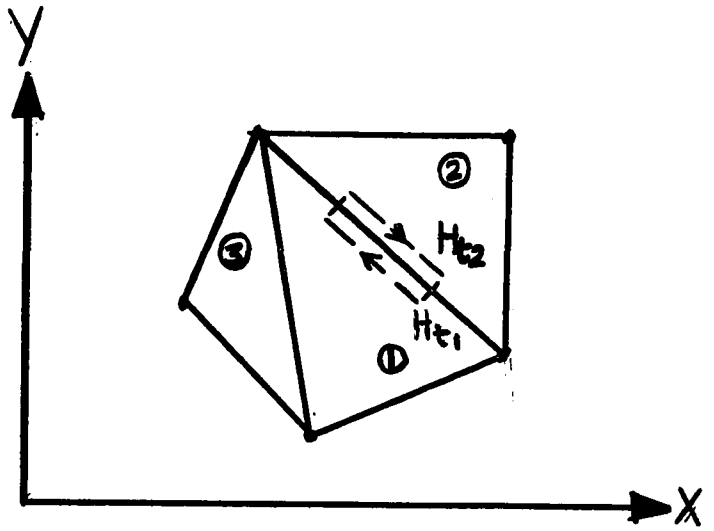


FIGURE 3.4. LINE INTEGRAL OF THE TANGENTIAL COMPONENT OF MAGNETISING FORCE AT THE ELEMENT BOUNDARY.

Having thus satisfied the conditions for the second variation of the functional and having set up a sequence by the method of finite elements, it is necessary to show that the sequence is a minimising one. In other words the set of nonlinear algebraic equations defined by (3.38), Section 3.2.2.

$$[S] \cdot [\varphi] = 1/3 \cdot [R]$$

must yield a solution which converges to a limit. This is best achieved by solving the given set of equations iteratively, starting from an initial guess, such that the sequence of approximate solutions in successive iterates satisfies the Lipschitz condition

$$|| G(x) - G(x') || \leq \lambda || x - x' || \qquad (3.48)$$

for all values of x, x' in the closed interval $[x_o - p, x_o + p]$ where the Lipschitz

constant has a value $0 \leq \lambda < 1$, and the sequence converges to an acceptable unique solution. The advantages of this method are

(a) that the nonlinear equations are recursively solved by a set of linear equations, and

(b) if the process converges, it will do so rapidly to the unique solution of the original equations.

However, since the principle of superposition has to be used in setting up the recursive relation (see [3.8, p.2]), ill conditioning of the linear algebraic equations may present convergence difficulties. Nevertheless, such quasilinearisation techniques have been effectively used resulting in rapid convergence of the iteration process. Two such methods, the chord method and the Newton-Raphson iterative scheme, have been successfully employed for solving field problems in the present analysis. The chord method provides linear convergence, while the generalised Newton-Raphson method yields a quadratically convergent sequence to the solution of the problem. In both these cases, the approximate solution obtained is unique. These techniques will be discussed in greater detail in the following chapter along with other schemes for solving nonlinear algebraic equations.

REFERENCES

[3.1] N.I. Akhiezer, The Calculus of Variations, Blaisdell Publishing Company, New York, Chapter 4, pp. 130-131, 1962.

[3.2] R.B. Simpson, "The Rayleigh-Ritz Process for the Simplest Problem in the Calculus of Variations", SIAM Journal, Numerical Analysis, Vol. 6, No. 2, June 1969.

[3.3] E.R. de Arantes, "Theoretical Foundations of the Finite Element Method", International Journal of Solids and Structures, Vol. 4, pp. 929-952, 1968.

[3.4] T.H.H. Pian and Pin Tong, "Basis of Finite Element Methods for Solid Continua", International Journal. Numerical Methods in Engineering, Vol. 1, pp. 3 - 28, 1969.

[3.5] P. Silvester, "High-Order Polynomial Triangular Finite Elements for Potential Problems", International Journal. Engineering Science, Vol. 7, pp. 849 - 861, 1969.

[3.6] P.C. Dunne, Journal of Royal Aeronautical Society, Vol. 72, p. 245, 1968.

[3.7] O.C. Zienkiewicz and Y.K. Cheung, The Finite Element Method in Structural and Continuum Mechanics, Chapter 10, McGraw-Hill, 1967.

[3.8] E.S. Lee, Quasilinearization and Invariant Imbedding, Chapter 1, Academic Press, 1968.

CHAPTER IV

ITERATIVE SOLUTION OF NONLINEAR ALGEBRAIC EQUATIONS

4.1 Introduction

In the preceding chapters, the nonlinear field problem was formulated in terms of variational calculus and the resulting continuum problem was discretised by the method of finite elements for obtaining a unique approximate solution. Other numerical methods discussed were finite difference schemes which were either based on a divided difference approach or on a restricted functional formulation. Reference was also made to the Rayleigh-Ritz method as a technique of functional minimisation. A factor common to all these methods is that the continuous nonlinear problem is finally transformed into a set of nonlinear algebraic equations. Therefore, the solution of the field problem reduces to one of obtaining a solution to the set of nonlinear algebraic equations, represented in matrix form or in any other manner.

Several methods currently in use for solving nonlinear equations will be discussed in this chapter, in particular the chord method and the Newton-Raphson iteration scheme, and proof of existence, convergence and uniqueness of the solutions obtained will be furnished wherever applicable. The generalised Newton-Raphson method will be presented in a multi-dimensional form and applied to the system of matrix equations (3.38), Chapter III, Section 3.2.2 representing the nonlinear field problem.

Various methods of solving the resulting set of linear equations are also discussed, such as Gaussian elimination and other techniques and a

brief reference is made to the condition of the matrices, accuracy of solutions obtained and the computational work involved in employing these methods.

4.1.1 Functional Iteration Method

The methods of solving nonlinear algebraic equations discussed in the following sections belong to the group of methods called functional iteration, as distinct from direct methods, for example the method of successive bisection, inverse linear interpolation etc. [4.1, Chap.3, pp. 72-82].

Let us suppose that the system of equations to be solved can be expressed as

$$f(x) = 0 \tag{4.1}$$

where f and x are vectors of the same dimension k. When $k = 1$, we have a single equation and if $k = n$, we have a system of n equations. If we now define a new function $G(x)$ such that

$$G(x) = x - f(x) \tag{4.2}$$

then most of the iteration methods can be written in the form

$$x_{n+1} = G(x_n) \tag{4.3}$$

for some suitable function G and initial approximation x_o. The convergence of the iteration process is assured if the mapping $G(x)$ carries a closed and bounded set

$S \in C_k$ into itself and if the mapping is contracting [4.2, p.85], i.e. if

$$|| G(x) - G(y) || \leq M || x - y || \tag{4.4}$$

for some norm, for all x, y in S and for $M < 1$, known as the Lipschitz constant. Such an iteration scheme is sometimes called the Picard method. As an example, a first order unaccelerated scheme is illustrated by the following flow chart (Figure 4.1)

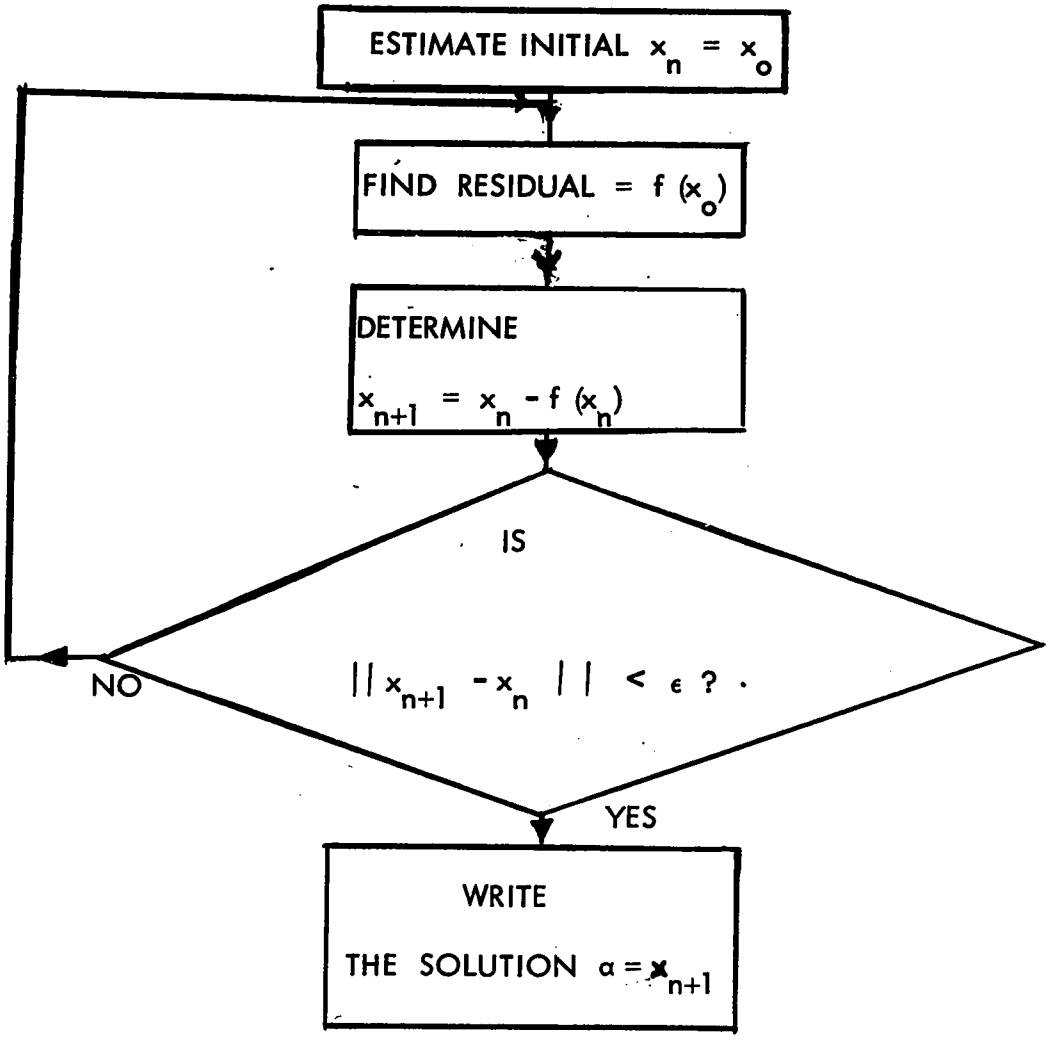


FIGURE 4.1. DESCRIPTION OF FUNCTIONAL ITERATIONS.

Here $\epsilon = M \, || \, x_n - x_{n-1} \, ||$ ≤ a small number

and $M < 1$

Isaacson and Keller [4.2, pp. 86 - 88] state the criteria for convergence of the iteration method and the existence of a unique solution in the form of the following theorems.

Theorem 1

Let G (x) defined by Equation (4.2) satisfy the Lipschitz condition

$$|| \, G(x) - G(x') \, || \leq \lambda \, || \, x - x' \, || \tag{4.5}$$

for all values of x, x' in the closed interval $[x_o - \xi, \; x_o + \xi]$, where the Lipschitz constant satisfies $0 \leq \lambda < 1$.

If the initial estimate x_o is such that

$$|| \, x_o - G(x_o) \, || \leq (1 - \lambda) \, \xi \tag{4.6}$$

then

(i) all the iterates x_n defined by the foregoing iteration sequence lie within the interval

$$x_o - \xi \leq x_n \leq x_o + \xi \tag{4.7}$$

(ii) (existence), the iterates converge to some point say a,

$$\lim_{n\to\infty} x_n = \alpha \quad (\text{in fact, } \| x_n - \alpha \| \leq \lambda^n \rho) \tag{4.8}$$

which is a root of the equation

$$x - G(x) = 0 \tag{4.9}$$

and

(iii) (uniqueness) α is the only root in the interval $[x_o - \xi, x_o + \xi]$.

The above theorem holds equally well for a multi-dimensional case, provided vector norms are used in place of absolute values.

Theorem 2

If $x = G(x)$ has a root α and if the components $G_i(x)$ have first partial derivatives and satisfy

$$\| \frac{\partial G_i(x)}{\partial x_j} \| \leq \frac{\lambda}{n}, \quad \lambda < 1 \tag{4.10}$$

for all x in $\| x - \alpha \|_\infty \leq \rho$ (4.11)

then

i. for any $x^{(o)}$ satisfying Equation (4.11), all the iterates $x^{(n)}$ of the sequence (4.3) also satisfy (4.11),

ii. for any $x^{(o)}$ satisfying (4.11), the iterates (4.3) converge to the root α of (4.9) which is unique in the interval (4.11).

Proofs of these theorems are given in Reference [4.2, Chapter 3, pp. 110-111]

4.2 Explicit Iteration Scheme or the Chord Method

Let us consider the general case of a system of equations of the form

$$f(x) = 0 \tag{4.12}$$

where

$$f(x) = [f_1(x), f_2(x), f_3(x) \ldots f_n(x)]^T \tag{4.13}$$

is an n - component vector. Such a system can be written in the form $x = G(x)$ in a variety of ways. We can examine the choice

$$G(x) = x - A(x) \cdot f(x) \tag{4.14}$$

where $A(x)$ is an nth order matrix with components $a_{ij}(x)$. Equations (4.9) and (4.12) will have the same set of solutions if $A(x)$ is non-singular (since in that case $A(x) \cdot f(x) = 0$ implies $f(x) = 0$).

The simplest choice for $A(x)$ is

$$A(x) = A \tag{4.15}$$

a constant non-singular matrix. If we introduce a matrix

$$J(x) = \left(\frac{\partial f_i(x)}{\partial x_i} \right) \tag{4.16}$$

whose determinant is the Jacobian of the functions $f_i(x)$, then from Equations (4.14) and (4.16), we have by differentiation and substitution

$$F(x) = \left(\frac{\partial G_i(x)}{\partial x_i}\right) = I - A J(x) \tag{4.17}$$

By Theorem 2, the iteration determined by using

$$x^{n+1} = x^n - A f(x^n) \tag{4.18}$$

will converge, for $x^{(0)}$ sufficiently close to a, if the elements of the matrix (4.15) are sufficiently small, for example, as in the case that $J(\alpha)$ is non-singular and A is approximately the inverse of $J(\alpha)$. This procedure is the n-dimensional analogue of the chord method. This iteration scheme has a geometric meaning as shown in Figure 4.2 for the one dimensional case in which the value x_{n+1} is the x intercept of the line with slope $1/m$ through $(x_n, f(x_n))$. The inequality implies that this slope should be between ∞ and $\frac{1}{2}f'(\alpha)$, i.e., half the slope of the tangent of the curve $y = f(x)$ at the root. Hence the name 'chord method'.

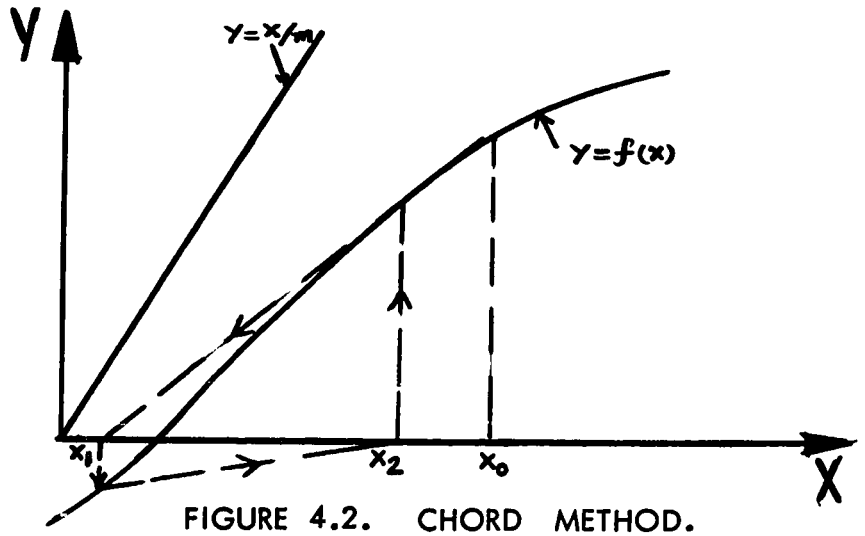


FIGURE 4.2. CHORD METHOD.

4.3 Second Order Methods [Reference 4.2, pp. 112-113]

If the function $G(x)$ is such that at a root the matrix

$$G_{ij}(x) = \frac{\partial G_i(x)}{\partial x_j} = 0 \ , \quad i, j = 1, 2, \ldots, n \tag{4.19}$$

and these derivatives are continuous near the root, then

$$\text{(a)} \quad \| \frac{\partial G_i(x)}{\partial x_j} \| < \frac{\lambda}{n} \quad \text{for } \lambda < 1 \tag{4.20}$$

$$\text{(b)} \quad \max_i \sum_{j=1}^{n} \| G_{ij}(x) \| \leq \lambda < 1 \tag{4.21}$$

$$\text{for all} \quad \| x - \alpha \|_\infty < \rho \tag{4.22}$$

will be satisfied for some $\rho > 0$. If in addition the second derivatives $\frac{\partial^2 G_i(x)}{\partial x_j \partial x_k}$ all exist in a neighbourhood of the root, then it can be shown as in Reference [4.2, pp. 112] that

$$\| x^{(n)} - \alpha \|_\infty \leq M \| x^{(n-1)} - \alpha \|_\infty^2 \tag{4.23}$$

where M is such that

$$\max_{i,j,k} \| \frac{\partial^2 G_i(x)}{\partial x_j \partial x_k} \| \leq \frac{2M}{n^2} \ .$$

This shows that quadratic convergence can occur in solving a system of equations by iteration, i.e., the error in any iterate is proportional to the square of the previous error and hence if

$G''(\alpha) \neq 0$, this procedure will be called a second order method. Illustrative second order schemes are described in the following sections.

4.3.1 The Generalised Newton-Raphson Method

The generalised Newton-Raphson method also known as the Newton method for functional iteration is a quasi-linearisation technique for solving nonlinear algebraic equations. This is also called the method of tangents, since in this scheme the function $f(x)$, whose solution is required, is approximated by the tangents drawn at the respective points to the function curve. The intersection of the tangents with the axis successively leads to the true solution as illustrated in Figure 4.3 for the one dimensional case. Here the function considered is convex and the root α is simple. Also it is assumed that

$$f'(x) < 0 \tag{4.24}$$

Suppose we are given an estimate x_i of a real root of the equation

$$f(x) = 0 \tag{4.25}$$

the equation of the tangent to $f(x)$ at $x = x_i$ can be expressed as

$$y(x) = f(x_i) + f'(x_i) \cdot (x - x_i) \tag{4.26}$$

Let $(x_{i+1}, 0)$ denote the intersection of this tangent line with the x axis. This point is found by setting $y(x)$ equal to zero in Equation (4.26). Then this

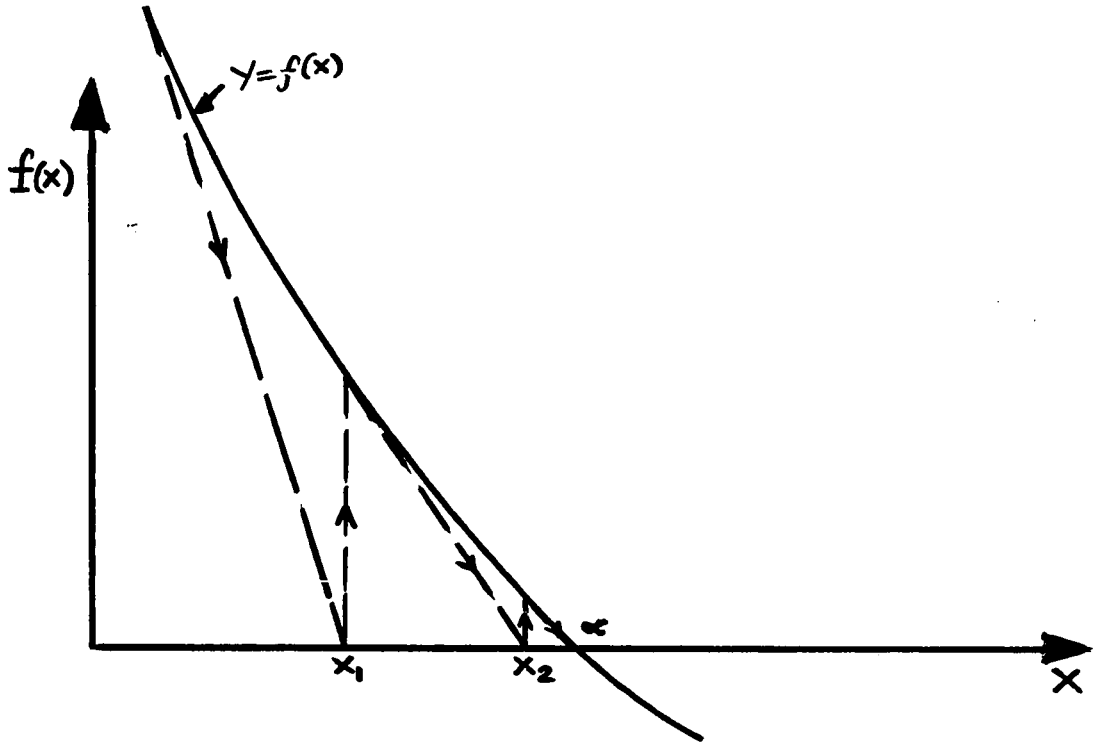


FIGURE 4.3. NEWTON -RAPHSON METHOD.

equation reduces to

$$0 = f(x_i) + f'(x_i) \cdot (x_{i+1} - x_i) \tag{4.27}$$

Solving the above Equation (4.27) for x_{i+1}, we obtain

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)} \tag{4.28}$$

which is the classical Newton-Raphson iterative scheme. In effect, we are obtaining a refined approximation x_{i+1} of a root α of $f(x) = 0$ by approximating the graph of $f(x)$ by the line tangent of $f(x)$ at $x = x_i$.

Newton-Raphson Method for a System of Equations

The method described above for a single equation can be extended to a system of equations as follows. Let us consider the following equations in the two independent variables x and y.

$$\begin{aligned} f(x, y) &= 0 \\ g(x, y) &= 0 \end{aligned} \tag{4.29}$$

If an initial estimate of the solution namely (x_o, y_o) is available and that this is incremented by changes δx, δy, then the functions can be expanded by Taylor's theorem as

$$f(x_o + \delta x, y_o + \delta y) = f(x_o, y_o) + f_x(x_o, y_o) \cdot \delta x + f_y(x_o, y_o) \cdot \delta y + (\text{higher order terms}) \tag{4.30}$$

$$g(x_o + \delta x, y_o + \delta y) = g(x_o, y_o) + g_x(x_o, y_o) \cdot \delta x + g_y(x_o, y_o) \cdot \delta y + (\text{higher order terms}) \tag{4.31}$$

where the subscripts x, y denote partial derivatives of the functions.

If we now truncate the series after terms of the first degree, we will obtain first order approximations of the resulting changes in f (x, y) and g (x, y) as the total differentials

$$\delta f = f_x (x_o , y_o) \cdot \delta x + f_y (x_o , y_o) \delta y \tag{4.32}$$

$$\delta g = g_x (x_o , y_o) \cdot \delta x + g_y (x_o , y_o) \delta y \tag{4.33}$$

A solution of system (4.29) can be obtained by determining δx, δy such that the total differentials δf, δg satisfy the constraints

$$\begin{aligned} \delta f &= -f (x_o , y_o) \\ \delta g &= -g (x_o , y_o) \end{aligned} \tag{4.34}$$

Substituting the values of these constraints in (4.32) and solving the resulting set of linear equations namely

$$\begin{aligned} -f (x_o , y_o) &= f_x (x_o , y_o) \cdot \delta x + f_y (x_o , y_o) \cdot \delta y \\ -g (x_o , y_o) &= g_x (x_o , y_o) \cdot \delta x + g_y (x_o , y_o) \cdot \delta y \end{aligned} \tag{4.35}$$

the values of δx and δy are determined.

From Equations (4.30) and (4.31), it is evident that if f and g are evaluated at $(x_o + \delta x , y_o + \delta y)$ and expressed in a Taylor expansion truncated after the first-order terms, the following relations hold.

$$\begin{aligned} f (x_o + \delta x, y_o + \delta y) &= f (x_o , y_o) + f_x (x_o , y_o) \cdot \delta x + f_y (x_o , y_o) \cdot \delta y = 0 \\ g (x_o + \delta x, y_o + \delta y) &= g (x_o , y_o) + g_x (x_o , y_o) \cdot \delta x + g_y (x_o , y_o) \cdot \delta y = 0 \end{aligned} \tag{4.36}$$

If these linear expansions are sufficiently accurate, then $(x_o + \delta x, y_o + \delta y)$ are fairly good approximations of the solution of Equation (4.29). If $|\delta x| > \epsilon$ or if $|\delta y| > \epsilon$, where ϵ is a small positive quantity, it is necessary to replace x_o by $x_o + \delta x$ and y_o by $y_o + \delta y$ and repeat the entire process. Usually a few iterates of the process will produce accurate values of the root, provided that the original estimates (x_o, y_o) are sufficiently close to the true solution.

Existence and Convergence

The Newton-Raphson method has two important properties namely monotone convergence and quadratic convergence. From Figure 4.3 for the one dimensional case, the intercepts on the axis increase successively from an initial approximation x_o towards the root α such that

$$x_o < x_1 < x_2 \ldots\ldots < \alpha \qquad (4.37)$$

The property expressed by Equation (4.37) is known as monotone convergence, by virtue of which the value of the sequence x_n increases monotonically to the root α. This property provides an upper or lower bound for the convergent interval (see [4.3, Chapter 1, Section 7, p. 20]), and ensures automatic improvement of the initial approximation after each iteration. The monotone convergence property is assured for the Newton-Raphson iteration formula only if the function $f(x)$ is a monotone decreasing or monotone increasing function and it is strictly convex or concave. In the present analysis, since the magnetisation characteristic of the iron is monotonic, the

solution sequence of the resulting matrix equation

$$[\nu S]\,[\phi] = [R] \tag{4.38}$$

will be monotonic and therefore monotone convergence of the Newton-Raphson method is assured for this application.

In general, the Newton method always yields quadratic convergence, even in such cases when the function is not monotonic. The quadratic convergence is a consequence of using the first and second terms in the Taylor series expansion. One other important fact about the Newton-Raphson formula is that the resulting equations are always linear although the original function may be nonlinear.

For the present application of this technique to the solution of the above matrix Equation (4.38), existence of a solution is assured since the iteration sequence

$$G(\phi) = [\nu S]\,[\phi] - [R] \tag{4.39}$$

is monotonic. The convergence and uniqueness of the solution obtained for a single equation is described by McCalla [4.1, pp. 84-86] and for a system of equations is concisely presented by Henrici [4.4, p. 106].

Rate of Convergence of the Newton Method

Let the error term of the ith iterate be expressed as

$$\delta_i = x_i - \alpha \tag{4.40}$$

By finding a relation between δ_{i+1} and δ_i, we can estimate how rapidly (or how slowly), the algorithm converges to a root α of the equation $f(x) = 0$, provided the algorithm converges. Such a relation can be determined by expanding the iteration function $G(x)$ in a Taylor series about $x = \alpha$ so that

$$G(x) = G(\alpha) + G'(\alpha) \cdot (x - \alpha) + \frac{G''(\alpha) \cdot (x - \alpha)^2}{2} \cdots\cdots \quad (4.41)$$

For the Newton-Raphson algorithm, the iteration function $G(x)$ and its first two derivatives are

$$G(x) = x - f(x) / f'(x) \quad (4.42)$$

$$G'(x) = \frac{f(x) \cdot f''(x)}{f'(x)^2} \quad (4.43)$$

$$G''(x) = \frac{(f'(x))^2 \cdot (f.f'' + f' \cdot f'') - (f.f'') \cdot 2f' \cdot f''}{(f'(x))^4} \quad (4.44)$$

and $G(\alpha) = \alpha$, $G'(\alpha) = 0$, $G''(\alpha) = f''(\alpha) / f'(\alpha)$ (4.45)

Substituting the values of (4.45) into (4.41) and truncating the series after terms of the 2nd degree and evaluating at x_i we obtain

$$G(x_i) = \alpha + \frac{f''(\alpha) \cdot (x_i - \alpha)^2}{2 f'(\alpha)} \quad (4.46)$$

Using the relation $x_{i+1} = G(x_i)$ and the definition (4.40), we find that

$$\delta_{i+1} = \frac{f''(\alpha) \cdot \delta_i^2}{2 f'(\alpha)} \quad (4.47)$$

From the above relation (4.47), the Newton-Raphson method is said to yield quadratic convergence [4.1, p. 87].

4.3.2 Some Modifications of the Newton Method

The evaluation of the derivative G' of the given iteration function G may not be a trivial problem in many practical situations especially if G is itself the result of a complicated computation. Hence a variety of methods have been devised by different authors to obviate the need for calculating G'. Some of the methods in common use are presented below.

i. Whittaker's Method [4.5]

In this method, the derivative $G'(x_n)$ is simply replaced by a constant and the resulting iteration formula given by

$$x_{n+1} = x_n - G(x_n)/m \tag{4.48}$$

defines for a certain range of values of m, a linearly converging sequence, unless we happen to pick $m = G'(\alpha)$. If $m = 1$, the simple unaccelerated functional iteration results. If the estimate of m is good, convergence will be rapid. Further in the initial stages of the Newton's process, it is usually not necessary to recompute G' at each step.

ii. Regula Falsi [4.4, p.87]

Here the value of the derivative $f'(x_n)$ is approximated by the difference

quotient

$$\frac{f(x_n) - f(x_{n-1})}{x_n - x_{n-1}} \tag{4.49}$$

formed with two preceding approximations. The resulting iteration formula is given as

$$x_{n+1} = x_n - \frac{(x_n - x_{n-1}) \cdot f(x_n)}{f(x_n) - f(x_{n-1})} \tag{4.50}$$

The algorithm suggested above is termed regula falsi, defined by a difference equation of order 2. Ostrowski [4.6] has shown that the degree of convergence of the method lies somewhere between that of Newton's and the ordinary functional iteration.

iii. Muller's Method

Regula Falsi can be obtained by approximating the graph of the function f by the straight line passing through the points $(x_{n-1}, G(x_{n-1}))$ and $(x_n, G(x_n))$. The point of intersection of this line with the x - axis defines the new approximation x_{n+1}. Instead of approximating f by a linear function, more rapid convergence can be obtained by approximating f by a polynomial p of degree $k \geq 1$ coinciding with f at the points $x_n, x_{n-1}, \ldots\ldots\ldots x_{n-k}$ and determining x_{n+1} as one of the zero's of p. Muller's study [4.7] reveals that the choice of $k = 2$ yields very satisfactory results. Since the construction of p depends on the theory of interpolation polynomials, the matter will not be pursued any further herein.

iv. Accelerated Iteration Method

This modification of Newton's method is the nonlinear analogue of the linear over-relaxation scheme [4.8, p. 102] and can be described as follows. Consider the system of equations

$$f_i (x_1 , x_2 , x_3 \ldots x_n) = 0, \quad i = 1, 2, \ldots n \tag{4.51}$$

Let

$$f_{ij} = \partial f_i / \partial x_j \tag{4.52}$$

and assume that

$$f_{ii} \neq 0 , \quad i = 1, 2, \ldots n \tag{4.53}$$

Then for m a fixed non-zero real constant, and for initial vector,

$$x^{(o)} = (x_1^{(o)} , x_2^{(o)} , \ldots . x_n^{(o)}) \tag{4.54}$$

a sequence of vectors

$$x^k = (x_1^{(k)} , x_2^{(k)} \ldots x_n^{(k)}), \quad k = 1, 2 \ldots n \tag{4.55}$$

is defined by the following iteration process.

$$x_1^{(k+1)} = x_1^{(k)} - m \frac{f_1 (x_1^{(k)} , x_2^{(k)} , \ldots x_n^{(k)})}{f_{11} (x_1^{(k)} , x_2^{(k)} \ldots x_n^{(k)}} \tag{4.56}$$

$$\vdots$$

$$x_n^{(k+1)} = x_n^{(k)} - \frac{m\, f_n\left(x_1^{(k+1)},\ x_2^{(k+1)} \ \ldots\ x_n^{(k+1)},\ x_n^{(k)}\right)}{f_{nn}\left(x_1^{(k+1)},\ x_2^{(k+1)} \ \ldots\ x_{n-1}^{(k+1)},\ x_n^{(k)}\right)} \qquad (4.57)$$

Where the double subscript notation denotes partial derivatives, m is the over-relaxation factor, and $k = 1, 2, \ldots\ldots n$. Then $x^{(k)}$ is a solution of (4.57) if and only if $x^{(k+1)} = x^{(k)}$. If the system of Equations (4.57) is linear, the method described above will become identical to the linear over-relaxation process. 'm' is chosen to lie somewhere between 1 and 2 [4.9, p. 234].

4.3.3 Application of the Generalised Newton-Raphson Method to the Multi-Dimensional Nonlinear Field Problem Using First Order Finite Elements

Of the various methods available, the generalised Newton-Raphson method was found to be most suitable for solving the system of nonlinear algebraic equations resulting from the variational analysis of the field problem. However, since the functions approximating the solutions are defined in each sub-region, the matrix Equation (3.38) of Chapter III, Section 3.2.2 must be recast for obtaining a recursion formula for the iterative scheme.

In the finite element method, the vector potentials, reluctivity and magnetic induction are defined piecewise in each triangular element and, therefore, it is sufficient to construct the Newton-Raphson iteration formula in each triangle separately and assemble the overall coefficient matrix with like elements for each triangle. Thus, the multi-

dimensional problem, in effect, reduces to one in three independent variables only namely the vertex values of potential for each triangle.

For the nonlinear field problem discretised by finite elements, the matrix equation can be re-written as three separate iteration functions for each triangle as follows.

$$\begin{aligned} f(x, y, z) &= 0 \\ g(x, y, z) &= 0 \\ h(x, y, z) &= 0 \end{aligned} \tag{4.58}$$

If (x_o, y_o, z_o) are the initial approximate values, the method seeks to obtain a correction Δx, Δy, Δz on x_o, y_o, z_o so that the corrected values will be

$$\begin{aligned} x &= x_o + \Delta x \\ y &= y_o + \Delta y \\ z &= z_o + \Delta z \end{aligned} \tag{4.59}$$

for which

$$\begin{aligned} f(x_o + \Delta x, \; y_o + \Delta y, \; z_o + \Delta z) &= 0 \\ g(x_o + \Delta x, \; y_o + \Delta y, \; z_o + \Delta z) &= 0 \\ h(x_o + \Delta x, \; y_o + \Delta y, \; z_o + \Delta z) &= 0 \end{aligned} \tag{4.60}$$

Expanding the above equations by Taylor's theorem for a function of three variables, one obtains

$$f(x+\Delta x, y+\Delta y, z+\Delta z) = f(x_o, y_o z_o) + f_x(x_o, y_o, z_o) \cdot \Delta x$$

$$+ f_y(x_o, y_o z_o) \cdot \Delta y + f_z(x_o, y_o, z_o) \cdot \Delta z + \text{higher order terms} = 0$$

$$g(x+\Delta x, y+\Delta y, z+\Delta z) = g(x_o, y_o, z_o) + g_x(x_o, y_o, z_o)\,\Delta x$$

$$+ g_y(x_o, y_o, z_o)\,\Delta y + g_z(x_o, y_o, z_o) \cdot \Delta z + \text{higher order terms} = 0 \qquad (4.61)$$

$$h(x+\Delta x, y+\Delta y, z+\Delta z) = h(x_o, y_o, z_o) + h_x(x_o, y_o, z_o)\,\Delta x$$

$$+ h_y(x_o, y_o, z_o)\,\Delta y + h_z(x_o, y_o, z_o) + \text{higher order terms} = 0$$

where the suffixes x, y, z denote the partial derivatives of the functions f, g and h. If we ignore all terms of order higher than the first, we are left with a system of three linear equations, in the three unknowns Δx, Δy, Δz, i.e.,

$$f_x(x_o, y_o, z_o)\,\Delta x + f_y(x_o, y_o, z_o)\,\Delta y + f_z(x_o, y_o, z_o)\,\Delta z = -f(x_o, y_o, z_o)$$

$$g_x(x_o, y_o, z_o)\,\Delta x + g_y(x_o, y_o, z_o)\,\Delta y + g_z(x_o, y_o, z_o)\,\Delta z = -g(x_o, y_o, z_o) \qquad (4.62)$$

$$h_x(x_o, y_o, z_o)\,\Delta x + h_y(x_o, y_o, z_o)\,\Delta y + h_z(x_o, y_o, z_o)\,\Delta z = -h(x_o, y_o, z_o)$$

The process may now be repeated by using Equation (4.59) and the true solution obtained.

Let us, now, consider the matrix Equation (3.37) of Chapter III for the nonlinear field problem represented by finite elements in each triangle of the region.

$$\nu \begin{bmatrix} B_{ii} & B_{ij} & B_{im} \\ B_{ji} & B_{ii} & B_{jm} \\ B_{mi} & B_{mj} & B_{mm} \end{bmatrix} \cdot \begin{bmatrix} \varphi_i \\ \varphi_j \\ \varphi_m \end{bmatrix} = \frac{\Delta}{3} \cdot \begin{bmatrix} J_i \\ J_j \\ J_m \end{bmatrix} \tag{4.63}$$

Differentiating the above Equation (4.63) partially with respect to φ_i , φ_j , φ_m etc., and substituting the result in Equation (4.62), one obtains the following iteration formula (vide Appendix IV) .

$$\frac{\nu}{4\Delta} \cdot \begin{bmatrix} B_{ii} & B_{ij} & B_{im} \\ B_{ji} & B_{ii} & B_{jm} \\ B_{mi} & B_{mj} & B_{mm} \end{bmatrix} \cdot \begin{bmatrix} \delta\varphi_i \\ \delta\varphi_j \\ \delta\varphi_m \end{bmatrix} + \frac{1}{4\Delta} \begin{bmatrix} \Sigma B_{ii}\varphi_i & \Sigma B_{ii}\varphi_i & \Sigma B_{ii}\varphi_i \\ \Sigma B_{ii}\varphi_i & \Sigma B_{ii}\varphi_i & \Sigma B_{ii}\varphi_i \\ \Sigma B_{mm}\varphi_i & \Sigma B_{mm}\varphi_i & \Sigma B_{mm}\varphi_i \end{bmatrix} \begin{bmatrix} \frac{\partial\nu}{\partial\varphi_i} & 0 & 0 \\ 0 & \frac{\partial\nu}{\partial\varphi_j} & 0 \\ 0 & 0 & \frac{\partial\nu}{\partial\varphi_m} \end{bmatrix} \cdot \begin{bmatrix} \delta\varphi_i \\ \delta\varphi_j \\ \delta\varphi_m \end{bmatrix}$$

$$= -\frac{\nu}{4\Delta} \cdot \begin{bmatrix} B_{ii} & B_{ij} & B_{im} \\ B_{ji} & B_{ii} & B_{jm} \\ B_{mi} & B_{mj} & B_{mm} \end{bmatrix} \cdot \begin{bmatrix} \varphi_i \\ \varphi_j \\ \varphi_m \end{bmatrix} + \frac{\Delta}{3} \cdot \begin{bmatrix} J_i \\ J_j \\ J_m \end{bmatrix} \tag{4.64}$$

where B_{ii} , B_{ij} , B_{im} etc. are defined in terms of the geometric constants b_i , b_j, b_m, c_i, c_j, c_m as

$$
\begin{aligned}
B_{ii} &= b_i b_i + c_i c_i & \Sigma B_{ii} &= b_i b_i \varphi_i + b_i b_j \varphi_j + b_i b_m \varphi_m + \ldots \\
B_{ij} &= b_i b_j + c_i c_j & \Sigma B_{ij} &= b_j b_i \varphi_i + b_j b_j \varphi_j + b_j b_m \varphi_m + \ldots \\
B_{im} &= b_i b_m + c_i c_m & \Sigma B_{mm} &= b_m b_i \varphi_i + b_m b_j \varphi_j + b_j b_m \varphi_m + \ldots \\
B_{mm} &= b_m b_m + c_m c_m
\end{aligned}
\qquad (4.65)
$$

An equation similar to the above is derived for each triangle and the overall Newton-Raphson iteration formula is constructed by adding the corresponding elements of each of the triangles. The set of linear equations is then solved and the approximation to the solution is improved by using Equation (4.59), so that

$$
\begin{aligned}
\varphi_i &= \varphi_{io} + \Delta \varphi_i \\
\varphi_j &= \varphi_{jo} + \Delta \varphi_j \\
\varphi_m &= \varphi_{mo} + \Delta \varphi_m
\end{aligned}
\qquad (4.66)
$$

4.4 Solution of Linear Algebraic Equations

Solutions of linear equations fall into two principal categories, (i) direct methods [13] and (ii) iterative techniques [10, 11, 12]. The linear algebraic equations in this present analysis are solved conveniently by Gaussian elimination, since it is a simple technique and requires less computational work than any other direct method and is free of the instability and errors that occur in linear iterative schemes.

The Gaussian method is fully described in mathematical literature [4.14] and consists of successive elimination of the elements of the lower triangle of the coefficient matrix, with the result the latter is completely transformed into an upper triangular one. This process is called triangular decomposition. The transformation of the right hand vector is termed forward modification. From the decomposed matrix and the modified forcing function, the solution vector is obtained by a process of back substitution.

The solution of the linear equations obtained by the above method is applicable to symmetric complete matrices as well as sparse banded matrices. In the case of band-matrices, however, only the lower or the upper half of the original coefficient matrix along with the principal diagonal elements need to be known for obtaining the solution by this method. The method uses the diagonal terms as the pivots in the elimination process, and therefore it is necessary to ensure that they are non-zero and preferably large. It is shown in Reference [4.13, p. 80] that if the coefficient matrix is positive definite, then the solution is assured, although certain semi-definite or negative definite matrices may well yield good results by this method.

If for any reason the diagonal terms are found to be zero, then the method has to be modified by pivotal condensation or other techniques, with the result the band property of the coefficient matrix may be entirely lost and the computational time enhanced considerably.

4.4.1 Conditioning of the Matrix and Accuracy of the Solutions Obtained

The Gaussian method described above is a direct method which gives explicit solutions to the simultaneous equations. Provided the pivots are non-zero and are not very

small, the accuracy of the method is assured subject to the conditions

i. the data of the problem is accurate,

ii. there are no round off errors.

However neither of the above conditions can be fully met in practice with the result that there will always be some errors included in the solutions.

The conditioning of the matrix has a great bearing on the accuracy of the solutions obtained and the subject is fully discussed in Reference [4.13, pp. 136-174]. If the coefficient matrix is an ill-conditioned one or, in other words, it is nearly singular, an iterative improvement of the solution obtained by the direct methods is warranted. Such a scheme is fully described by Fox [4.13, pp. 143 - 145]. For the present problem of solving the set of nonlinear algebraic equations defined by Equation (4.63), numerical experiments with Gaussian elimination have indicated that no such iterative improvement is really necessary, although the coefficient matrix is an ill-conditioned one.

REFERENCES

[4.1] T.R. McCalla, Numerical Methods and Fortran Programming, Chapter 3, John Wiley, New York, 1967.

[4.2] E. Isaacson and Keller, Analysis of Numerical Methods, Chapter 3, John Wiley, New York, 1966.

[4.3] E.S. Lee, Quasilinearization and Invariant Imbedding, Academic Press, New York, 1968, p.20.

[4.4] P. Henrici, Elements of Numerical Analysis, John Wiley, New York, 1964.

[4.5] E.T. Whittaker and G. Robinson, The Calculus of Observations, Blackie, London, 1924.

[4.6] A. Ostrowski, Solution of Equations and Systems of Equations, Academic Press, New York, 1960.

[4.7] D.E. Muller, "A Method for Solving Algebraic Equations Using an Automatic Computer", Math. Tables, Aids Comput., Vol. 10, pp. 208-215, 1956.

[4.8] D. Greenspan, "On Approximating Extremals of Functionals", Bulletin of International Computer Centre, Vol. 4, p.102, 1965.

[4.9] G.E. Forsythe and C.B. Moler, Computer Solution of Linear Algebraic Systems, p. 234, Prentice Hall, Englewood Cliffs, New Jersey, 1967.

[4.10] D.W. Martin and G.J. Tee, "Iterative Methods for Linear Equations with Symmetric Positive Definite Matrix", Computer Journal, Vol. 4, 1961, pp. 242 - 248.

[4.11] I.M. Khabaza, "An Iterative Least Square Method Suitable for Solving Large Sparse Matrices", The Computer Journal, Vol. 6, 1963, pp. 202 - 206.

[4.12] R.P. Tewarson, "Projection Methods for Solving Sparse Linear Systems", Computer Journal, Vol. 12. No. 1, February 1969, pp. 77 - 80.

[4.13] L. Fox, An Introduction to Numerical Analysis, Clarendon Press, England, 1964.

[4.14] F.B. Hilderbrand, "Modern Mathematics for Engineers and Physicists", Prentice Hall, Englewood Cliffs, New Jersey, 1962.

CHAPTER V

PRACTICAL COMPUTATIONAL ASPECTS OF THE FINITE ELEMENT METHOD

5.1 Introduction

Solution of the nonlinear Poisson's equation by the finite element scheme is best accomplished in three stages :

(i) Generation of the matrices of geometrical coefficients and the formation of the current vector contributions for each triangular element.

(ii) Calculation of the reluctivities from the potential vector and construction of the total coefficient matrix and the total current vector.

(iii) Solution of the matrix equation.

The steps in computation are illustrated by the flow diagram of Figure 5.1. For an efficient and economical program, all the geometrical coefficients, material characteristics and current vectors which are repeatedly used in the iterative scheme, are best evaluated in the beginning and stored for subsequent use. The method of evaluating these parameters and the indexing algorithm for their compact storage will be discussed in the following sections.

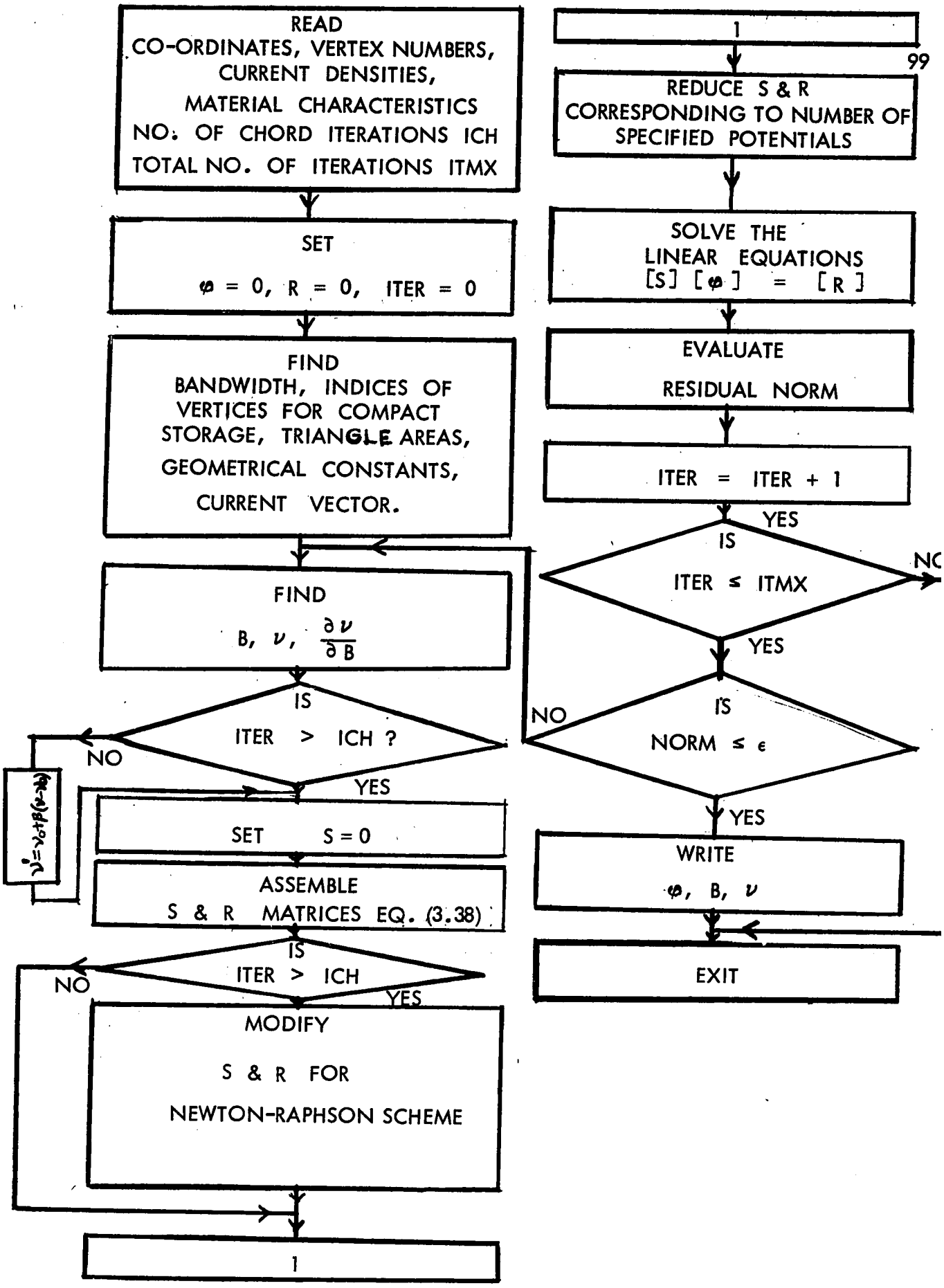


FIGURE 5.1. FLOW-CHART FOR THE NONLINEAR FIELD SOLUTION.

5.2.1 Calculation of the Geometrical Coefficients and Current Vectors

The triangle co-ordinates, vertex numbers, current densities, material characteristics for the iron and non-iron regions, the number of specified potentials and the iterations to be carried out form the input data for the program. The following geometrical constants and other parameters are then obtained :

(a) the area of each triangle,

(b) the coefficients defined by

$$B_{ii} = b_i b_i + c_i c_i \qquad B_{ii} = b_i b_i + c_i c_i$$

$$B_{ji} = b_j b_i + c_j c_i \qquad B_{jm} = b_j b_m + c_j c_m$$

$$B_{mi} = b_m b_i + c_m c_i \qquad B_{mm} = b_m b_m + c_m c_m$$

where i, j, m are the vertices of the individual triangles and

$$b_i = y_j - y_m$$

and so on

$$c_i = x_m - x_j$$

(c) the current vector contributions $J \cdot \Delta/3$, where J is the the current density and Δ the triangle area.

The magnetic characteristics of the iron parts are stored as point values of the magnetic intensity vector H corresponding to successive values of the magnetic in-

duction B. A linear interpolation routine was developed for determining the reluctivity and its derivative with respect to the magnetic induction B in each triangular element.

5.2.2 Special Indexing Routine

In the triangular sub-division of the field region as described in earlier chapters, it is seen that no one node is connected to all of the other nodes. Consequently the total coefficient matrix assembled from the elements for the individual triangles is always sparse, but symmetric and can be arranged to have a band structure. It would be not only uneconomical, but well nigh impossible to store all the elements of a large matrix arising in practical machine problems. However, it is only necessary to store the lower or the upper triangular matrix along with those on the principal diagonal. The elements of the coefficient matrix are best stored as a vector, consisting of either the row elements in succession or the column elements. Figures 5.2 and 5.3 illustrate two alternative schemes of matrix storage, both of which use a single subscript indexing procedure, since double subscripting would require a large memory.

Any of the above methods would facilitate compact storage of large sparse matrices. Since the number of multiplications involved in the solution of such matrix equations increases with the band-width, it would be ideal if one could somehow compact the matrix such that only non-zero elements populate the band area. However, much economy can still be achieved even if the band area does include some zero elements provided, of course, the band-width itself is sufficiently small. A skillful numbering

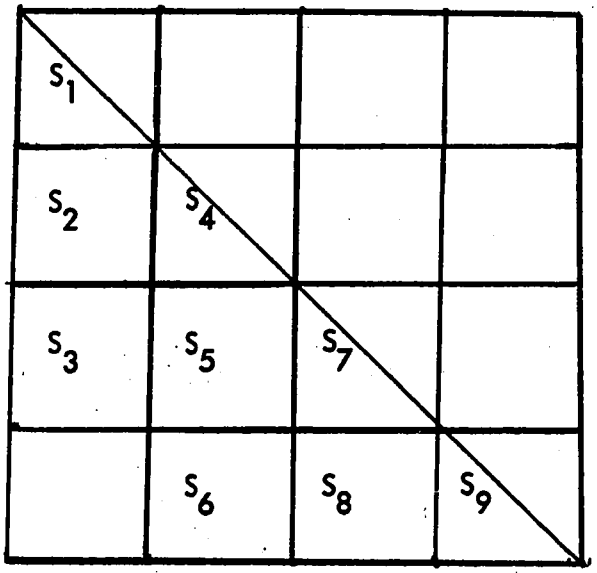


FIGURE 5.2. LOWER TRIANGULAR MATRIX ELEMENTS STORED COLUMN WISE.

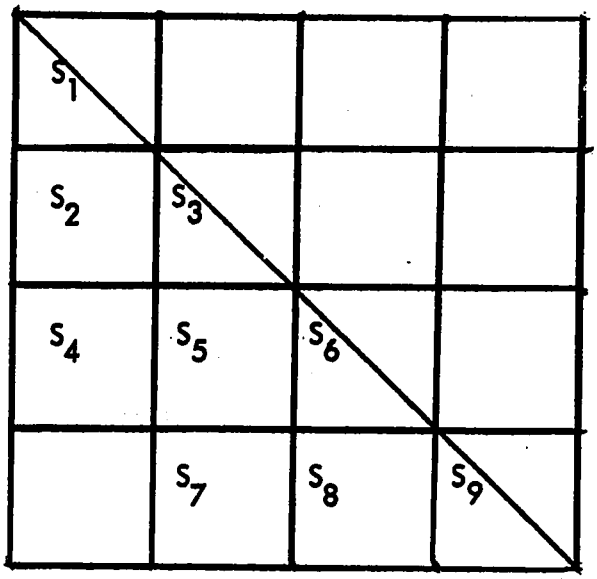


FIGURE 5.3. LOWER TRIANGULAR MATRIX ELEMENTS STORED ROW WISE.

system of the triangle vertices does lead to a small band-width and for the present, the task of achieving an optimum band-width in the finite element method rests largely on the diligence and ingenuity of the engineer.

For an economical storage, therefore, the vertices of the triangles must be transformed to a single index notation in order to locate the respective matrix elements of individual triangles in the over-all coefficient matrix. This is best illustrated by considering two finite triangular elements as shown in Figure 5.4 (a). The double index scheme for the total coefficient matrix and its single index equivalent are illustrated in Figure 5.4(b). The indexing for the constituent matrices is shown in Figure 5.4 (c).

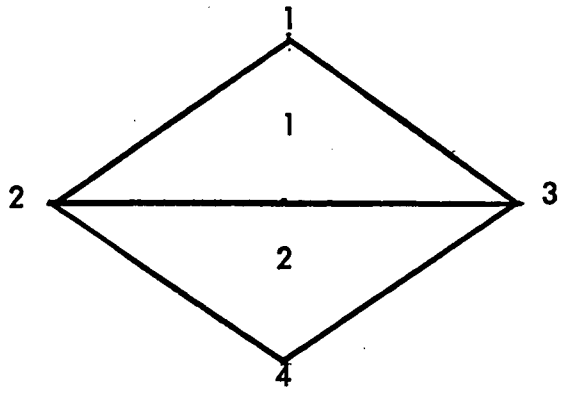


FIGURE 5.4 (a). TRIANGULAR FINITE ELEMENTS.

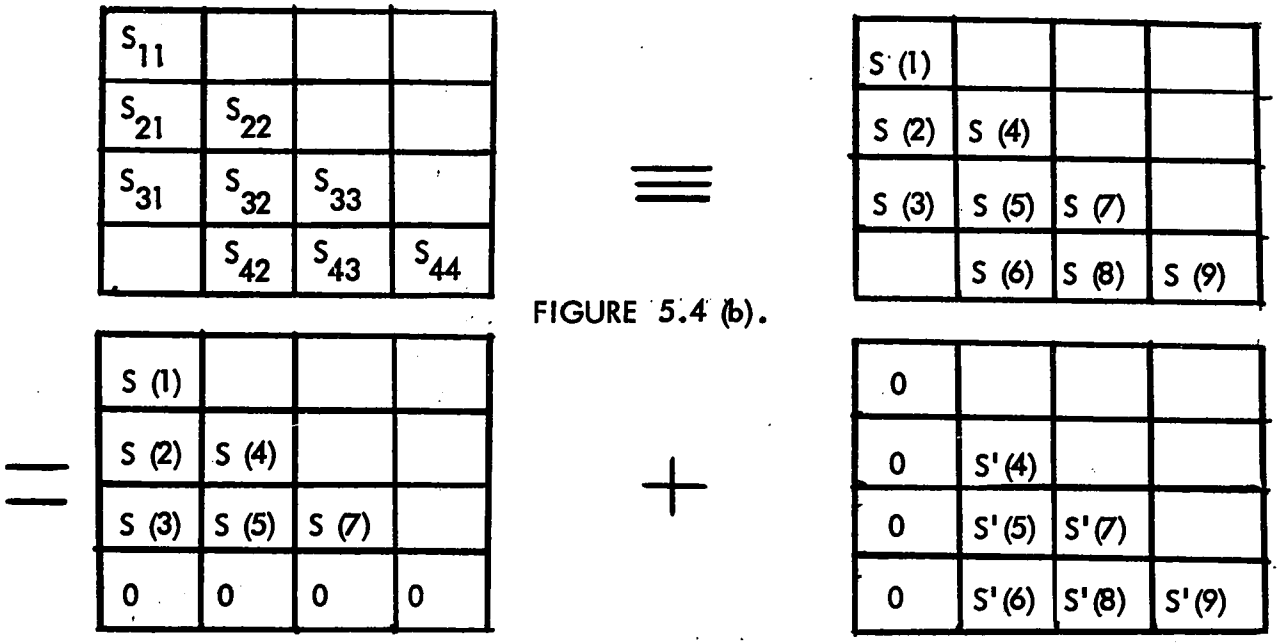


FIGURE 5.4 (b).

FIGURE 5.4 (c). INDEXING OF THE TOTAL COEFFICIENT MATRIX FOR TWO FINITE ELEMENTS AND THE CONSTITUENT MATRICES.

The following algorithm describes one method of achieving the required index transformation. If i, j, m are the vertices of a triangular element, II, IJ, IM, JJ, JM, MM are the single subscript indices for the matrix elements $S_{i,i}$, $S_{i,j}$, $S_{i,m}$, $S_{j,j}$, $S_{j,m}$ and $S_{m,m}$ then the following relations hold.

Defining a term X called pivot,

$$X = N - B + 2 \qquad (5.1)$$

where N is the order of the total coefficient matrix, and B = (Bandwidth + 1) / 2.

The index of the last term will be

$$Y = (2BN - B^2 + B)/2 \tag{5.2}$$

The following further relations hold

$$II = \begin{cases} i + (B-1)(i-1), & i < X \\ i + [2(B-1)(X-1) + (i-X)(2B+X-i-3)]/2, & i > X \end{cases} \tag{5.3}$$

$$IJ = \begin{cases} j + (B-1)(i-1), & i < x, \text{ or } j < x \\ i + (B-1)(j-1), & i > j \text{ and } i < x \text{ or } j < x \\ j + [2(B-1)(x-1) + (i-x)(2B+x-i-3)]/2 & \text{for } i > x \text{ and } j > x \text{ and } i < j \\ i + [2(B-1)(x-1) + (j-x)(2B+x-j-3)]/2 & \text{for } i > x \text{ and } j > x \text{ and } i > j \end{cases} \tag{5.4}$$

$$JJ = \begin{cases} j + (B-1)(j-1), & j < x \\ j + [2(B-1)(x-1) + (j-x)(2B+x-j-3)]/2 & j > x \end{cases} \tag{5.5}$$

$$JM = m + (B-1)(j-1), \quad j < x \text{ or } m < x \text{ and } j < m$$

$$= j + (B-1)(m-1), \quad j < x \text{ or } m < x \text{ and } j > m$$

$$= m + [2(B-1)(x-1) + (j-x)(2B+x-j-3)]/2 \quad \text{for } j > x \text{ and } m > x \text{ and } j < m \qquad (5.6)$$

$$= j + [2(B-1)(x-1) + (m-x)(2B+x-m-3)]/2 \quad \text{for } j > x \text{ and } m > x \text{ and } j > m$$

$$MM = m + (B-1)(m-1), \quad m < x \qquad (5.7)$$

$$= m + (2(B-1)(x-1) + (m-x)(2B+x-m-3))/2$$

$$IM = i + (B-1)(m-1), \quad m < x \text{ or } i < x \text{ and } m < i$$

$$= m + (B-1)(i-1), \quad m < x \text{ or } i < x \text{ and } m > i \qquad (5.8)$$

$$= i + (2(B-1)(x-1) + (m-x)(2B+x-m-3))/2 \quad m > x \text{ and } i > x \text{ and } m < i$$

$$= m + (2(B-1)(x-1) + (i-x)(2B+x-i-3))/2$$

5.3.1 Assembly of the Total Coefficient Matrix and the Current Vector

Having thus evaluated all the geometrical coefficients for each individual triangle and the current vectors and having transformed the indices of the element vertices, we proceed to assemble the total coefficient matrix S and the current vector R. Assuming an initial value of the vector potential φ, normally zero value for most purposes, the magnetic induction B is evaluated from the relation

$$B = \frac{1}{2\Delta} \sqrt{(b_i \varphi_i + b_j \varphi_j + b_m \varphi_m)^2 + (c_i \varphi_i + c_j \varphi_j + c_m \varphi_m)^2} \qquad (5.8)$$

The corresponding reluctivities and their derivatives with respect to the magnetic induction are obtained from the B-H characteristic for the iron parts, while for other regions ν has a constant value equal to the reciprocal permeability of free space. The S and R matrices are now easily assembled from their respective component values for each of the triangular elements as

$$\begin{aligned} \nu S (I J) \Big|_{i,j,m} &= \frac{1}{4\Delta} \sum [\nu B_{ij}] \Big|_{i,j,m} \\ R (I) \Big|_{i,j,m} &= \sum \left[\frac{R_i}{3}\right] \Big|_{i,j,m} \end{aligned} \qquad (5.9)$$

where $[R_i]$ is the current vector.

5.3.2 Chord and Newton-Raphson Iteration Schemes

For the first few iterations, the reluctivities are transformed by the chord method defined by the relation

$$\nu_{i+1}^{new} = \nu_i^{old} + \beta\,(\nu_{i+1} - \nu_i^{old}) \tag{5.10}$$

Almost similar results were obtained by transforming the vector potentials instead of the reluctivities and, therefore, either alternative is considered satisfactory. The value of the constant β was chosen as 0.1, since it gave good linear convergence without causing instability of the iteration cycle. A lower value is also found to be satisfactory, but requires more iterations, since the step size is smaller. However, since the solution obtained by the chord method is only being used as an initial estimate for the quadratically convergent Newton-Raphson scheme, no further experiments for optimising the value of β are considered necessary.

As all the parameters for each triangle have already been calculated and stored, the implementation of the Newton-Raphson algorithm becomes an easy matter. This can be expressed as

$$\underset{\text{(over all triangles)}}{\frac{1}{4\Delta}\sum \nu B_{ij}}\Big|_{i,j,m} + \underset{\text{(over all triangles)}}{\frac{1}{16\Delta^2}\sum (B_{ij}\varphi_j)(B_{im}\varphi_m)}\Big|_{i,j,m} \cdot \left(\frac{1}{B\cdot\Delta}\frac{\partial\nu}{\partial B}\right) \tag{5.11}$$

$$= -\underset{\text{(over all triangles)}}{\frac{1}{4\Delta}\sum (B_{ij}\varphi_j)} \cdot \nu + \underset{\text{(over all triangles)}}{\sum \frac{R_i}{3}}\Big|_{i,j,m}$$

5.3.3 Matrix Reduction

It is well known that Laplace's and Poisson's equations have unique solutions for Dirichlet or mixed boundary conditions, but the solutions for the Neumann problem differ by an arbitrary constant. Since Neumann boundary conditions are implicit in the variational formulation, at least one potential needs to be specified for obtaining a unique solution. This is conveniently carried out by specifying a flux line boundary wherever convenient, or by some other means. Since the number of unknown variables of the potential vector is reduced by the number of specified potentials, the coefficient matrix and the current vector must be correspondingly transformed. This is accomplished by either of the following methods :

(a) By a row and column reduction of the coefficient matrix corresponding to the potentials specified and a row reduction of the current vector.

(b) By introducing zeroes in the rows and columns, and a value of unity in the diagonal terms of the coefficient matrix corresponding to the potentials specified as constants in the right hand current vector (see Reference [5.1]). These schemes are illustrated by Figures 5.5, 5.6(a) and 5.6(b).

Alternative (b) is well suited for matrices with a band structure, since the transformation preserves the band property of the coefficient matrix, thereby permitting the use of an equation solving routine suitable for band matrices.

$$\begin{bmatrix} S_1 & & & \\ S_2 & S_4 & & \\ S_3 & S_5 & S_7 & \\ & S_6 & S_8 & S_9 \end{bmatrix} \begin{bmatrix} \varphi_1 \\ \varphi_2 \\ \varphi_3 \\ \varphi_4 \end{bmatrix} = \begin{bmatrix} R_1 \\ R_2 \\ R_3 \\ R_4 \end{bmatrix}$$

FIGURE 5.5. ORIGINAL MATRIX EQUATION.

$$\begin{bmatrix} S_1 & & \\ S_2 & S_4 & \\ & S_6 & S_9 \end{bmatrix} \begin{bmatrix} \varphi_1 \\ \varphi_2 \\ \varphi_4 \end{bmatrix} = \begin{bmatrix} R_1 \\ R_2 \\ R_4 \end{bmatrix} - \alpha \begin{bmatrix} S_3 \\ S_5 \\ S_8 \end{bmatrix}$$

FIGURE 5.6 (a). REDUCED MATRIX EQUATION (Alternative (a)).

$$\begin{bmatrix} S_1 & & & \\ S_2 & S_4 & & \\ 0 & 0 & 1 & \\ & S_6 & 0 & S_9 \end{bmatrix} \begin{bmatrix} \varphi_1 \\ \varphi_2 \\ \varphi_3 \\ \varphi_4 \end{bmatrix} = \begin{bmatrix} R_1 \\ R_2 \\ \alpha \\ R_4 \end{bmatrix} - \alpha \begin{bmatrix} S_3 \\ S_5 \\ 0 \\ S_8 \end{bmatrix}$$

FIGURE 5.6 (b). TRANSFORMED MATRIX EQUATION (Alternative (b)).

5.3.4 Solution of Linear Equations for a Coefficient Matrix with a Band Structure

The two principal objectives in developing an algorithm for solving linear equations are that it must ensure compact and economical storage by a suitable indexing system and must facilitate fast computation by reducing the number of arithmetic operations to a minimum.

The first of these objectives is accomplished by using an indexing routine described in Section 5.2.2, which leads to a numbering scheme of the coefficient matrix as shown in Figure 5.4(b). Further indexing is, however necessary, in order to execute the linear equation solving routine, which in the present case is the Gaussian elimination process. Referring to Figure 5.4(b), the lower triangle of this band-structured matrix can be conveniently divided up into a parallelogram and a triangle as shown in Figure 5.7. It

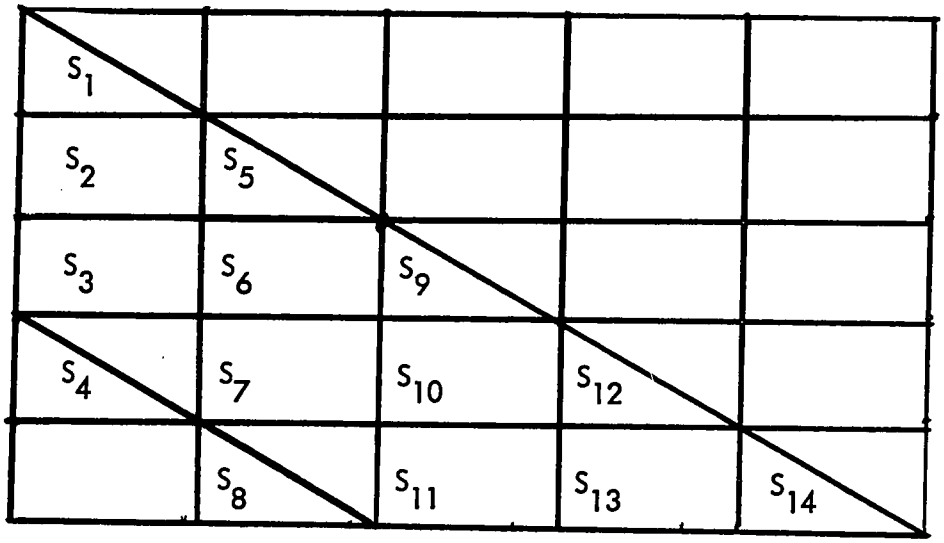


FIGURE 5.7. SUBDIVISION OF COEFFICIENT MATRIX INTO A TRIANGLE AND A PARALLELOGRAM.

is observed that the number of columns containing M elements below the principal diagonal is only two (where M is $\frac{\text{band-width} + 1}{2}$). The remaining three columns consist of elements whose indices decrease successively from M - 1 to 1. The determination of the indices of the elements S_5 and S_9, therefore, is of vital significance. Denoting the index of S_5 by IX, it can be shown that IX = (M * N - M^2 + 1), where N is the order of the matrix, and M is defined as before. Once the indices of these two elements are determined, the indexing of the rest of the elements of the respective columns is a simple matter, since they increase by 1 at a time. Also the indices of the diagonal terms increase by M at a time from 1 until the $S_{IX + M}$ element is reached. Thereafter the diagonal indices increase successively by M - 1, M - 2, etc.

Using this special indexing scheme, the elements of the lower triangular band-structured matrix are modified by Gaussian elimination. This is accomplished in the innermost DO loop of the program. The multiplying factors required for the modification of the matrix elements are determined in the outer loop. The forward modification of the right hand vector is likewise carried out in the outer DO loop. The allied indexing schemes and arithmetic such as additions and subtractions are relegated to the outermost DO loop to keep the computation to a minimum. The details of this scheme are shown in the flow chart of Figure 5.8.

5.3.5 Formula for the Total Number of Arithmetic Operations Required

If N is the order of the matrix and 2M - 1 the band-width, then the following arithmetic operations are required. All the operations are evaluated in terms of multiplications for estimating the work to be done.

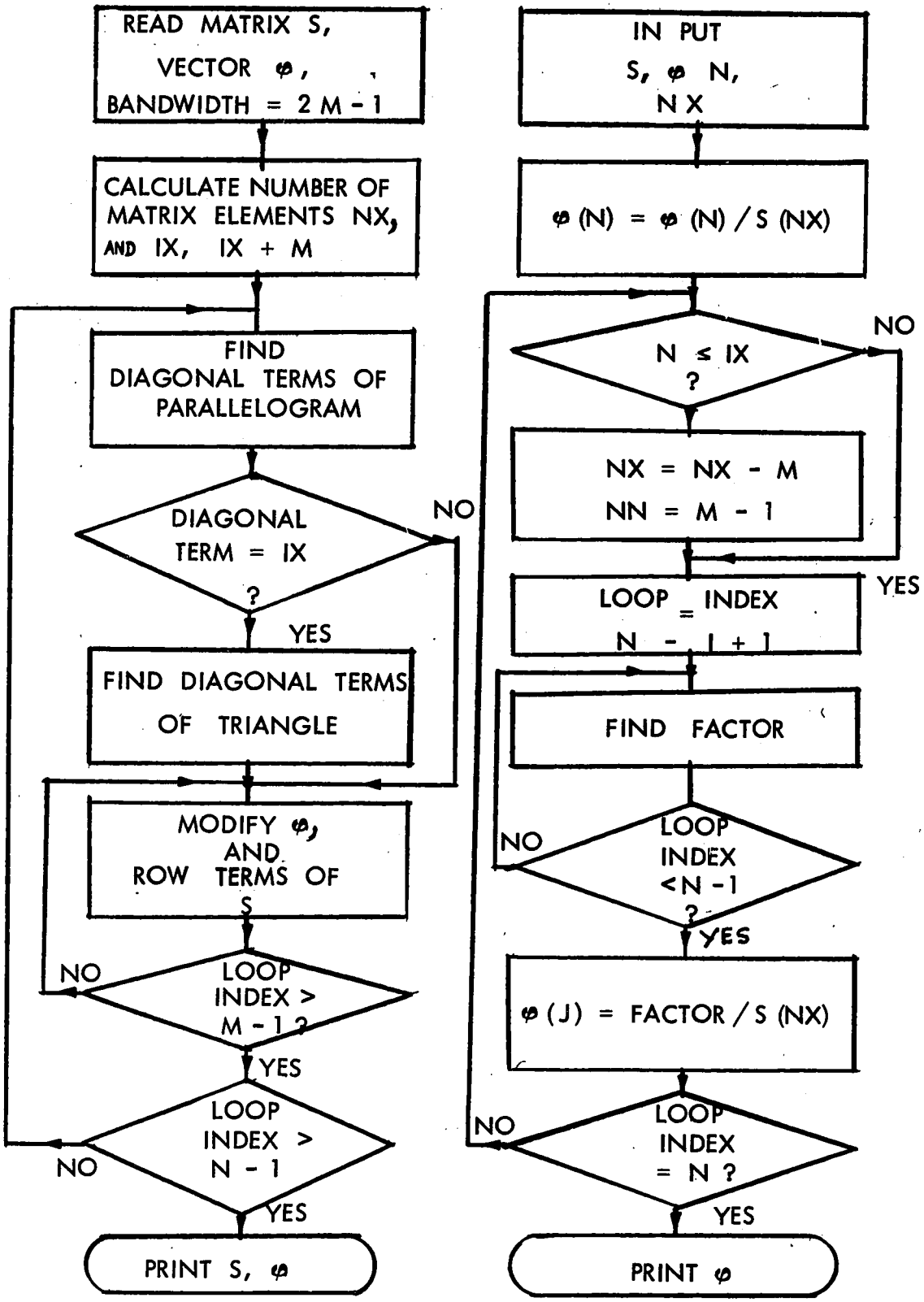


FIGURE 5.8. FLOW-CHART FOR SOLUTION OF EQUATIONS WITH BAND-STRUCTURED MATRICES.

Indexing

Number of multiplications and additions = $3.15 + 0.3\,(N - 1)$

Forward Modification

Number of multiplications and divisions = $\left[(N - M + 1)(M - 1) + \sum_{2}^{M-1} (M - K \right] . 2$

= $(2N - M)(M - 1)$

Number of additions = $0.15\,(2N - M)(M - 1)$

Hence, the total number of multiplications, additions, etc. = $1.15\,(2N - M)(M - 1)$

Triangular Decomposition

Number of multiplications = $(M - 1)^2 (N - M + 1) + \frac{(M-2)(M-1)(2M-3)}{6}$

Back Substitution

Number of multiplications = $(M - 1)(N - M + 1) + \sum_{k=1}^{M-1} (M - K) + 0.25\,N$

Therefore, the total number of Arithmetic operations required will be

$\approx \frac{(M - 1)}{6} . (7N - 5M + 6MN - 4M^2 + 6)$

$\approx \frac{M^2 (3N - 2M)}{3}$ . . . (multiplications)

5.3.6 Comparison of the Arithmetic Operations Required by the Triangular Decomposition Method and by an Inverse Routine for a Band Matrix and also for a Full Matrix

Arithmetic Required to Solve N Simultaneous Equations

Method	Band-Matrix	Full-Matrix
Simultaneous Equation Solution	$\frac{M^2 (3N - 2M)}{3}$	$\frac{N^3}{3} + N^2$
Inverse and Post Multiplications	$\frac{3}{2} \cdot MN^2$	N^3

5.4.1 Convergence Criteria and Rate of Convergence

The computational work increases with the number of iterations of the matrix equation solution, and therefore, some criteria must be applied for terminating the iterative sequence, in order to achieve efficiency and economy of programming. An error norm would be the ideal basis for determining whether or not the number of iterations are adequate for obtaining a solution of acceptable accuracy. Since there is no easy way of determining the true error for a nonlinear problem discretised by finite elements, it is necessary to adopt other norms as the bases of terminating the iteration process. Following are some of the norms used in the present analysis :

(a) infinity norm of the displacement vector,

(b) Euclidean norm of the displacement vector,

(c) infinity norm of the residual vector,

(d) Euclidean norm of the residual vector.

All these norms are evaluated by the computer program, and a value of 10^{-6} or less for the Euclidean norm of the residual vector is used as the basis for stopping the iterative sequence.

The convergence rates for the chord and Newton-Raphson iterations are shown in Figure 5.9 for a transformer problem. In this case, however, an error norm was used as the criterion, assuming the solution achieved after 25 iterations to be the true solution of the problem. It is evident that between six and eight iterations are adequate for achieving an acceptable minimum value of the norm, and this has been further confirmed by later numerical experiments on larger matrices. In all these cases of field analysis the initial approximation to the solution vector was taken to be zero and convergence to the required solution was achieved in a few iterations, owing to the quadratic-convergence property of the Newton-Raphson method. If more than one evaluation of the field solution is required for a given problem,for example, if the applied current density is varied in steps then the computation could be considerably reduced by using the vector potential solution corresponding to one value of current density as the initial value for the next. One such scheme is used in the latest versions of the computer program.

5.5 Flux Plotting Routine

An automatic flux plotting routine has been developed, which in conjunction with a digital X - Y plotter, produces flux plots directly from the input data of co-ordinate values, vertex numbers of triangles and the potential values obtained from the finite element analysis. The steps in the plotting routine are as follows :

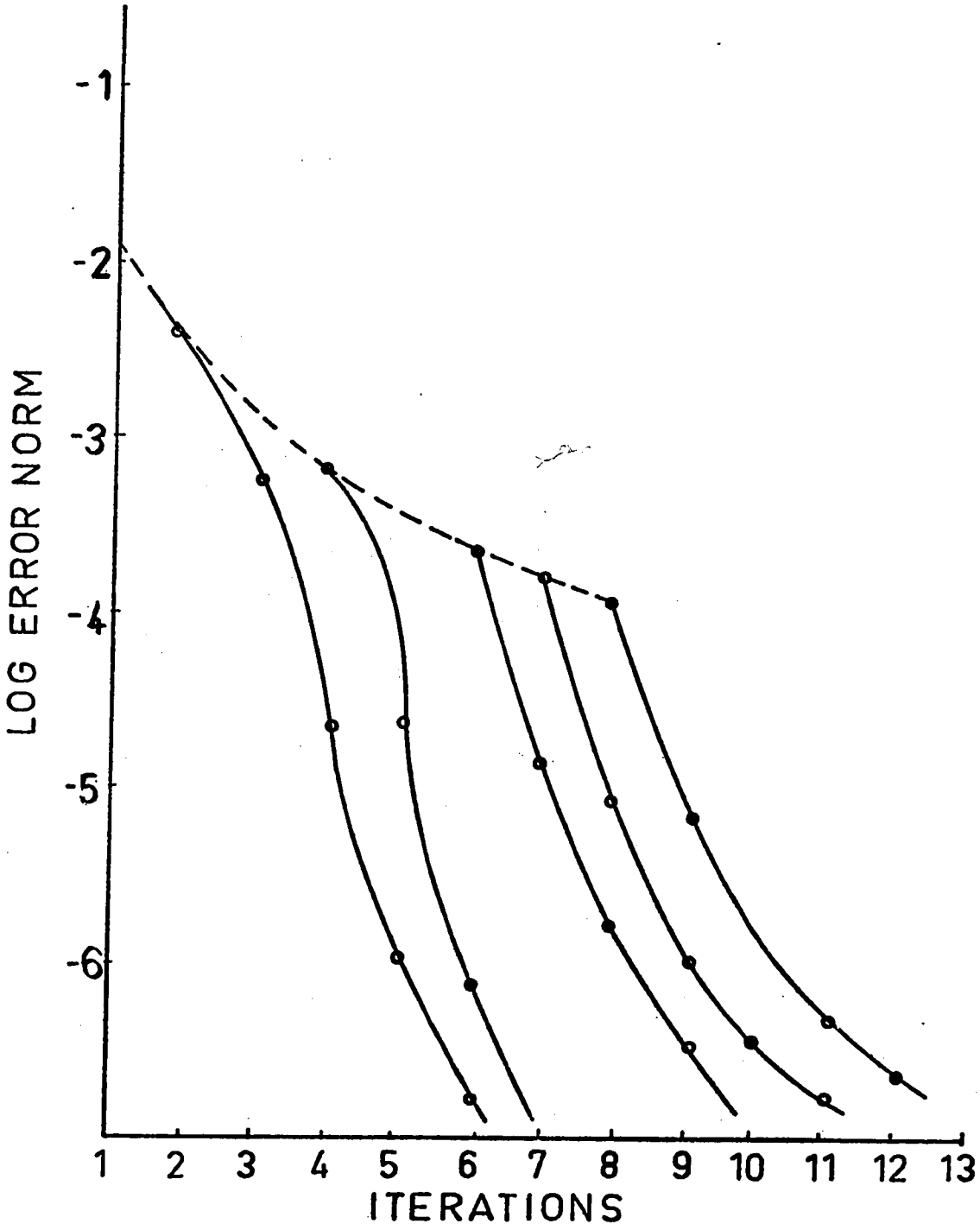


FIGURE 5.9. VARIATION OF THE EUCLIDEAN ERROR NORM OF THE COMPUTED MAGNETIC-VECTOR POTENTIAL, WITH THE NUMBER OF ITERATIONS.

---------- REDUCTION IN THE NORM OBTAINED BY THE 'CHORD METHOD'.

———————— REDUCTION IN THE NORM OBTAINED BY THE 'NEWTON-RAPHSON METHOD'.

(i) Determination of the maximum and minimum values of vertex potentials, and a suitable sub-division of their range.

(ii) Scanning of the triangle edges to check whether each of the sub-divisions of potential as described in (i) lies along the sides.

(iii) Evaluation of the X and Y co-ordinates of the points corresponding to the potential sub-division, by linear interpolation.

(iv) Drawing the flux lines for each potential sub-division by joining the co-ordinates obtained from (iii), by a line plotting routine.

The flux plots executed by the above routine are sectionally straight and truly represent the field solution obtained by first order finite elements. Examples of such plots will be found in Chapter VI.

REFERENCES

[5.1] W.Y.J. Shieh, S.L. Lee and R.A. Parmalee, "Analysis of Plate bending by triangular elements", Journal of the Engineering Mechanics Division, Proc. ASCE, EM5, p. 1095, October 1968.

CHAPTER VI

APPLICATION OF THE METHOD TO ELECTRICAL MACHINES

In order to ascertain the efficacy of the finite element method for solving practical nonlinear field problems occurring in electrical machinery, three distinct applications were considered, namely

(a) a transformer,

(b) a large turbo-generator and

(c) a direct current generator.

These machines are representative of the wide range of electrical machines in present day use and possess all the complexity of geometry, material characteristics and mode of energy conversion. The diversity that these devices present in respect of constructional details and electrical performance poses an interesting challenge to currently available field solution techniques and, therefore, any new method that endeavours to solve the field problem efficiently should be of general application and yet be flexible enough to suit particular needs. One of the considerations for the choice of these applications besides the above, is that the computed field solutions could be verified by laboratory experiments or by factory test results that are readily available.

6.1.1 The Transformer Problem

A three limbed transformer core was assembled out of transformer steel and a small winding was placed on the central limb as shown in Figure 6.1. A number of

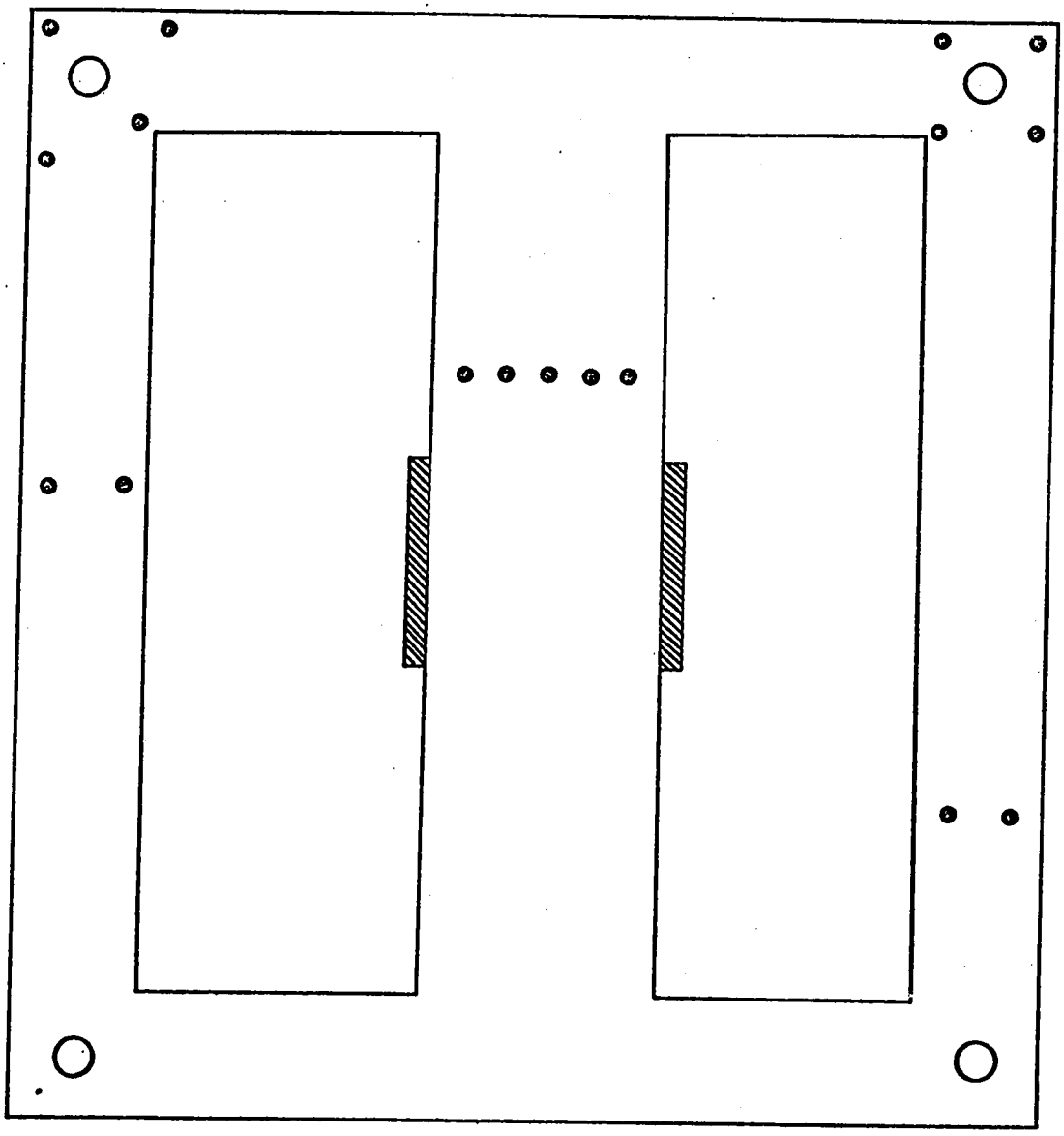


FIGURE 6.1. LOCATION OF SEARCH COILS IN THE TRANSFORMER CORE, USED FOR FLUX MEASUREMENTS. THE MAXIMUM DIMENSION OF THE CORE IS 28 CM ; THE OTHER DIMENSIONS ARE TO SCALE.

small holes were drilled through the core in order to permit insertion of search coils, so that the flux distribution in the core could be measured directly. The mean magnetisation curve for this steel was determined by punching a ring sample out of randomly selected stampings. Measurements were made with a fluxmeter, eliminating any possible eddy-current effects by avoidance of alternating current testing. The measured magnetisation curve is shown in Figure 6.2.

For the finite element analysis, the core and surrounding air space, were subdivided into triangles in a variety of ways. Figure 6.3 shows one of the subdivisions used, wherein the iron and the immediately surrounding air are modelled, with large triangles in the latter where the solution is expected to have little importance. A flux-line boundary is assumed around the outer edge of the air-space modelled. It has been found, in fact, that leakage flux at reasonable distances is small enough that such detailed representation, in the majority of cases, adds little to the analysis, and can occasionally be ignored altogether without loss of accuracy. Figure 6.4 shows a predicted flux distribution in the transformer, the external air space not being drawn in. A noteworthy point is that the flux lines obtained by this method are sectionally straight. As described earlier, this is a consequence of the linear interpolation of the potential in each triangle in terms of its vertex values. The flux plot is deliberately shown without any smoothing so as to illustrate the essential characteristics of the first order finite element method. Only one such distribution is shown, since it is more or less the same for other current values. It might be added that the leakage flux lines shown crossing the transformer window are not equally spaced with the lines shown in the iron, but are additional lines drawn in for explanatory purposes.

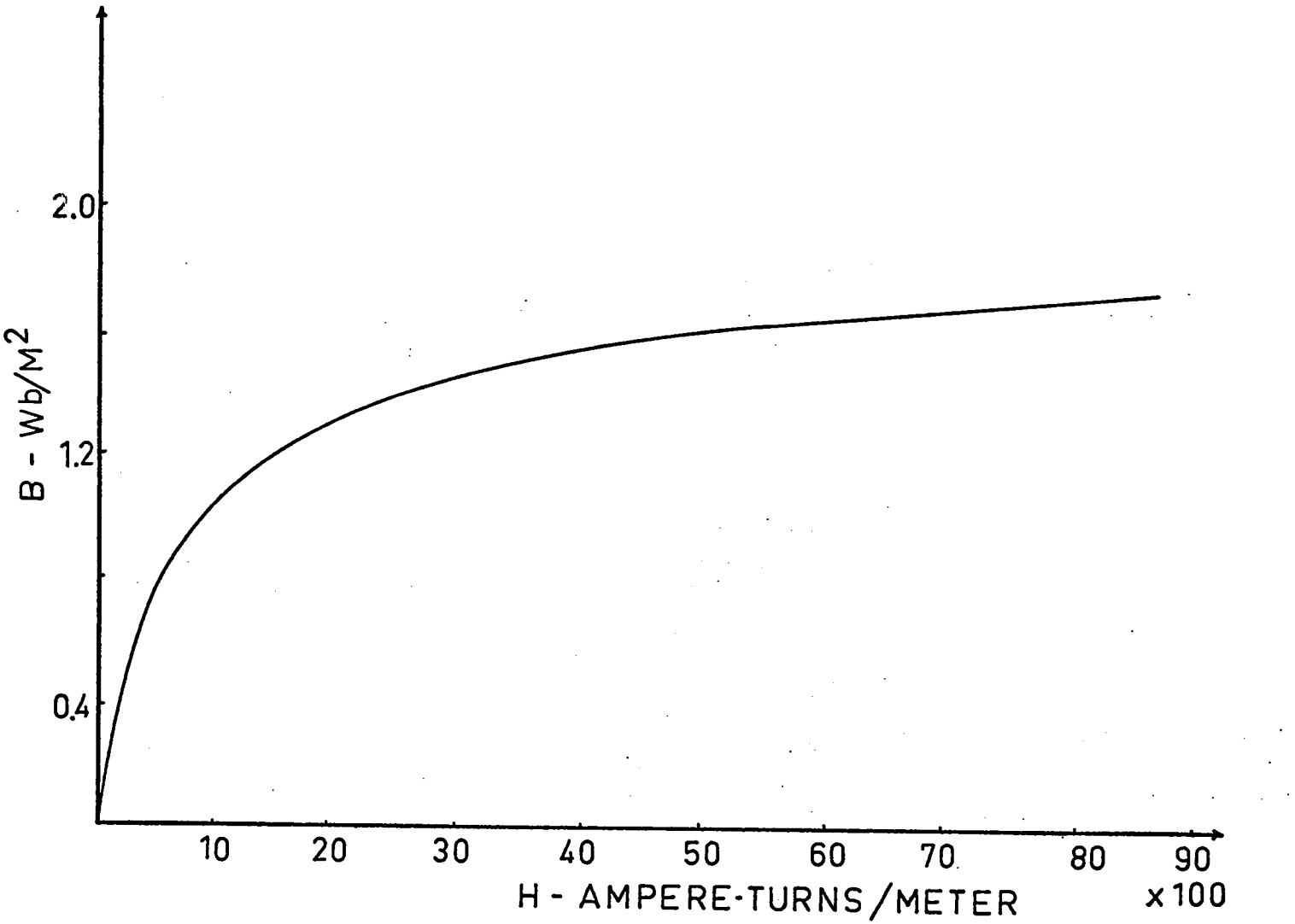


FIGURE 6.2. B - H CHARACTERISTIC OF RING SAMPLE OF THE TRANSFORMER CORE MATERIAL.

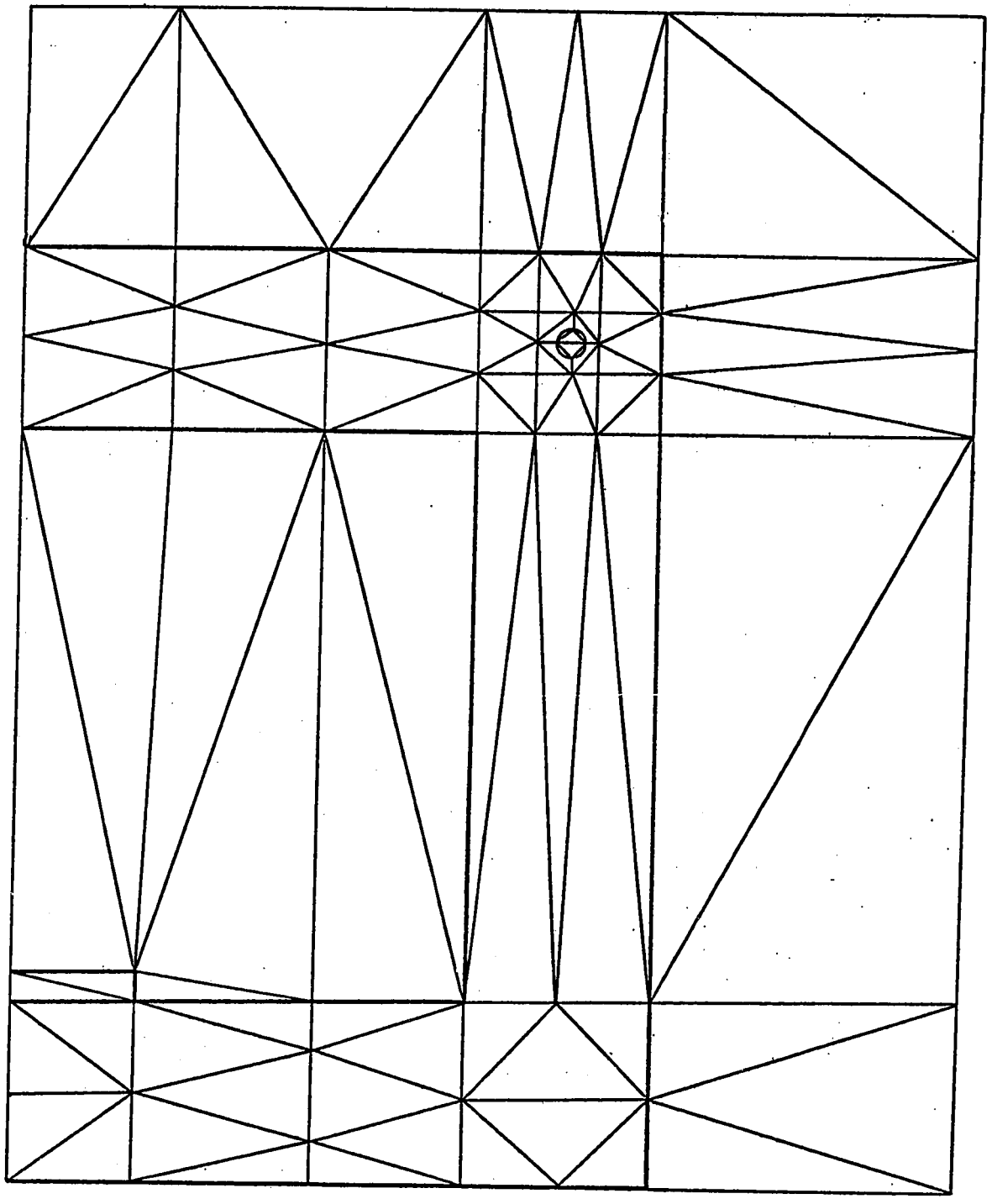


FIGURE 6.3. SECTIONAL VIEW OF A TRANSFORMER SHOWING IRREGULAR TRIANGULAR SUBDIVISIONS.

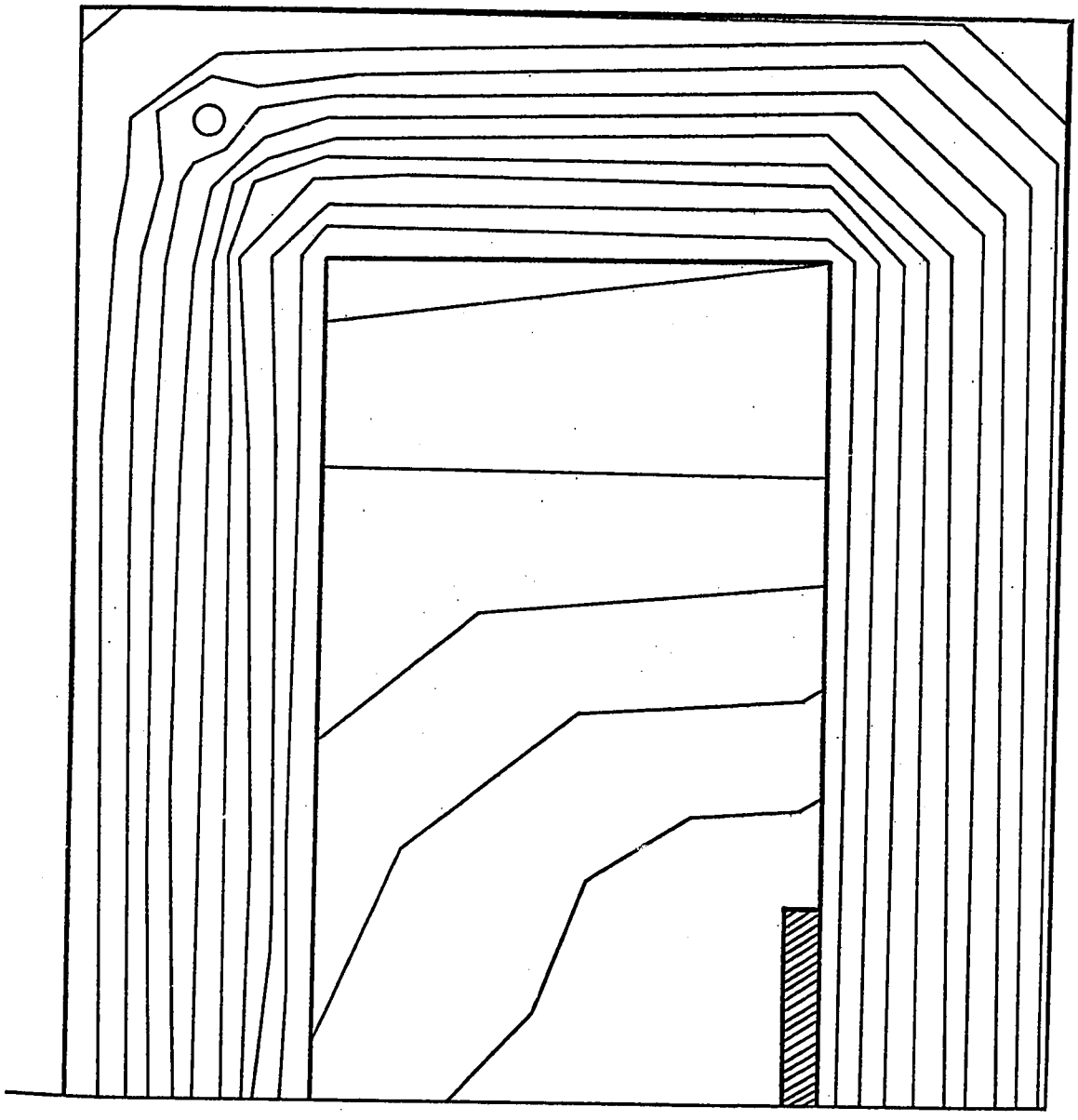


FIGURE 6.4. PLOT OF THE PREDICTED FLUX DISTRIBUTION IN THE TRANSFORMER WITH A MAXIMUM FLUX DENSITY OF 1.5 Wb/m^2 IN THE CORE AND 2.1 Wb/m^2 AT THE CORNERS.

A direct comparison of experimental and predicted results is shown in Figure 6.5, which shows the fluxes linked by search coils placed in the central limb of the transformer. These coils were placed in small drilled holes located as indicated by the heavy dots in Figure 6.6. The agreement is extremely good, particularly in view of the very considerable degree of saturation at the higher current levels. Figure 6.7 similarly exhibits the predicted and measured fluxes for the search coils in the immediate vicinity of the corner region of the core, where high local flux densities and rapid variation in flux density are to be anticipated. Again, the experiment amply justifies the theoretical treatment.

6.1.2 Determination of the Magnetising Current

In order to indicate the manner in which the calculated flux distribution might be employed directly for the solution of practical problems, prediction of the magnetising current waveform for the core at hand was attempted. Although approximate calculations of this quantity are easily made, accurate prediction requires considerable care. For this analysis, the following method was used.

Let us suppose that at a particular instant the current density in the transfromer winding has a certain known value J. Using the finite-element technique, the corresponding magnetic vector potential ϕ may be determined everywhere in and around the transformer, and the total instantaneous stored energy W calculated by using the expression [2.25, p. 152] Equation 5.37.

$$W = \frac{1}{2} \iiint \bar{\phi} \cdot \bar{J} \, dx \, dy \, dz \qquad (6.1)$$

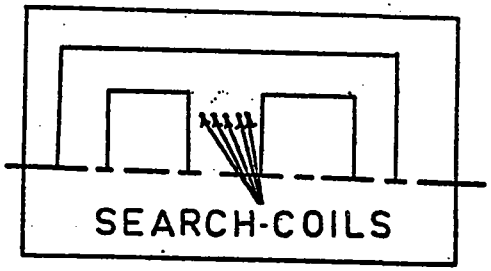


FIGURE 6.6. SEARCH COIL POSITION.

FIGURE 6.5. COMPARISON OF MEASURED TRANSFORMER CORE FLUX WITH ESTIMATED VALUES FROM THEORY, AT 1.08 Wb / m^2 AND 1.75 Wb / m^2 PEAK FLUX DENSITY.

———— Experimental Results.
---------- Computed Values.

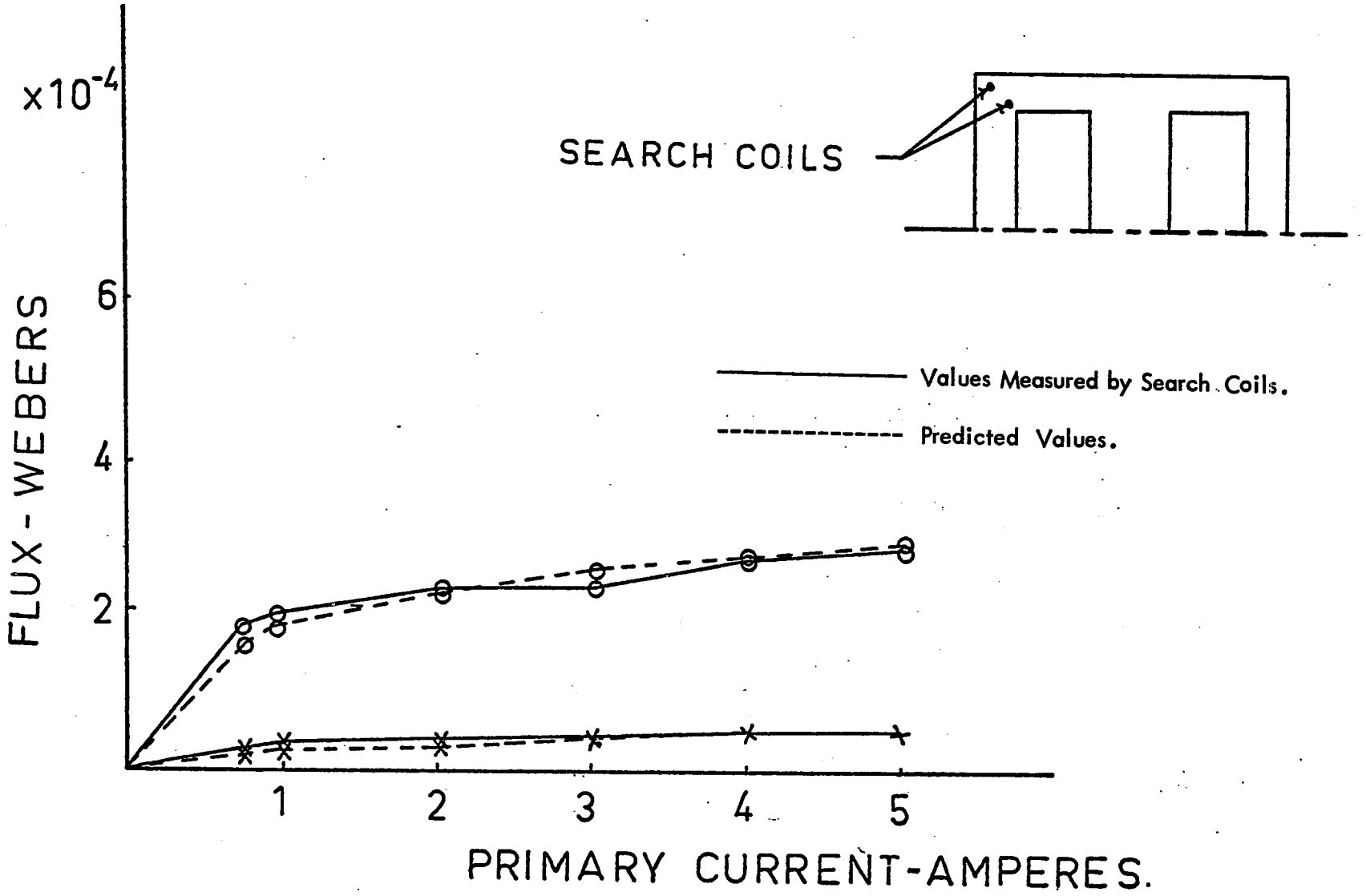


FIGURE 6.7. VARIATION OF PEAK FLUX IN THE CORNER REGION OF THE TRANSFORMER WITH PRIMARY CURRENT. TWO SETS OF CURVES ARE SHOWN CORRESPONDING TO PEAK FLUX DENSITIES OF 0.8 Wb / m^2 AND 2.15 Wb / m^2 RESPECTIVELY.

It will be assumed that the winding terminal e.m.f. is sinusoidal expressed by the relation

$$e = E \cos \omega t \tag{6.2}$$

The total flux linkages λ for any coil are given by

$$\lambda = \int_0^t e \, dt \tag{6.3}$$

and the coil stored energy may be stated in terms of flux linkages and coil current i as

$$W = \frac{1}{2} \lambda i \tag{6.4}$$

Combining Equations (6.1) to (6.4), one obtains

$$\frac{1}{\omega} E \sin \omega t = \frac{\int\int\int \bar{\varphi} \cdot \bar{J} \, dx \, dy \, dz}{\int\int J \, dx \, dy} \tag{6.5}$$

where the current i is expressed as

$$i = \int\int J \, dx \, dy \tag{6.6}$$

In this formulation, it is not required to associate any particular flux lines, nor any particular fraction of the total flux, with all or part of the total winding. This point is essential, for as the iron saturates, the flux distribution in the core alters considerably -- for example, the relative amount of leakage flux rises rapidly -- so that one cannot easily specify what part of the total flux is linked by any one turn. However, the

integrals in (6.5) are easily evaluated numerically as part of the computer program. By repeating the solution for various current values, it is possible to find the corresponding instantaneous voltages. In this way, the magnetizing current waveform for a given sinusoidal terminal voltage is constructed easily.

The computed magnetizing current waveshape for one value of terminal voltage is compared with the characteristic determined experimentally as shown in Figure 6.8. During the tests, care was taken to ensure that no external resistances were included in the primary circuit of the transformer, lest the input voltage waveform should be distorted and differ appreciably from a sinusoid. A slight discrepancy, most probably arising from the mean magnetisation curve rather than a true hysteresis loop, is discernible. However, the correlation of experiment with prediction is very satisfactory. In this analysis of the transformer, 64 triangles with 51 vertex potentials were used to represent a quarter section of the transformer, and the initial computer program developed could produce a complete solution for one value of current density at a cost of $6.00 on an IBM 360/75 computer. The later programs using an improved algorithm for a band-structured coefficient matrix solve the same problem under two dollars per current value.

6.2.1 Evaluation of Turbogenerator Performance

A 30 MW, 2 pole, 3000 r.p.m. turbogenerator operating at 11 KV, 0.8 p.f. was chosen for predicting its performance under various conditions by field analysis as described in earlier chapters, since the computed values could be compared with available factory test results. As shown in Figure 6.9, the rotor and stator slots are fully

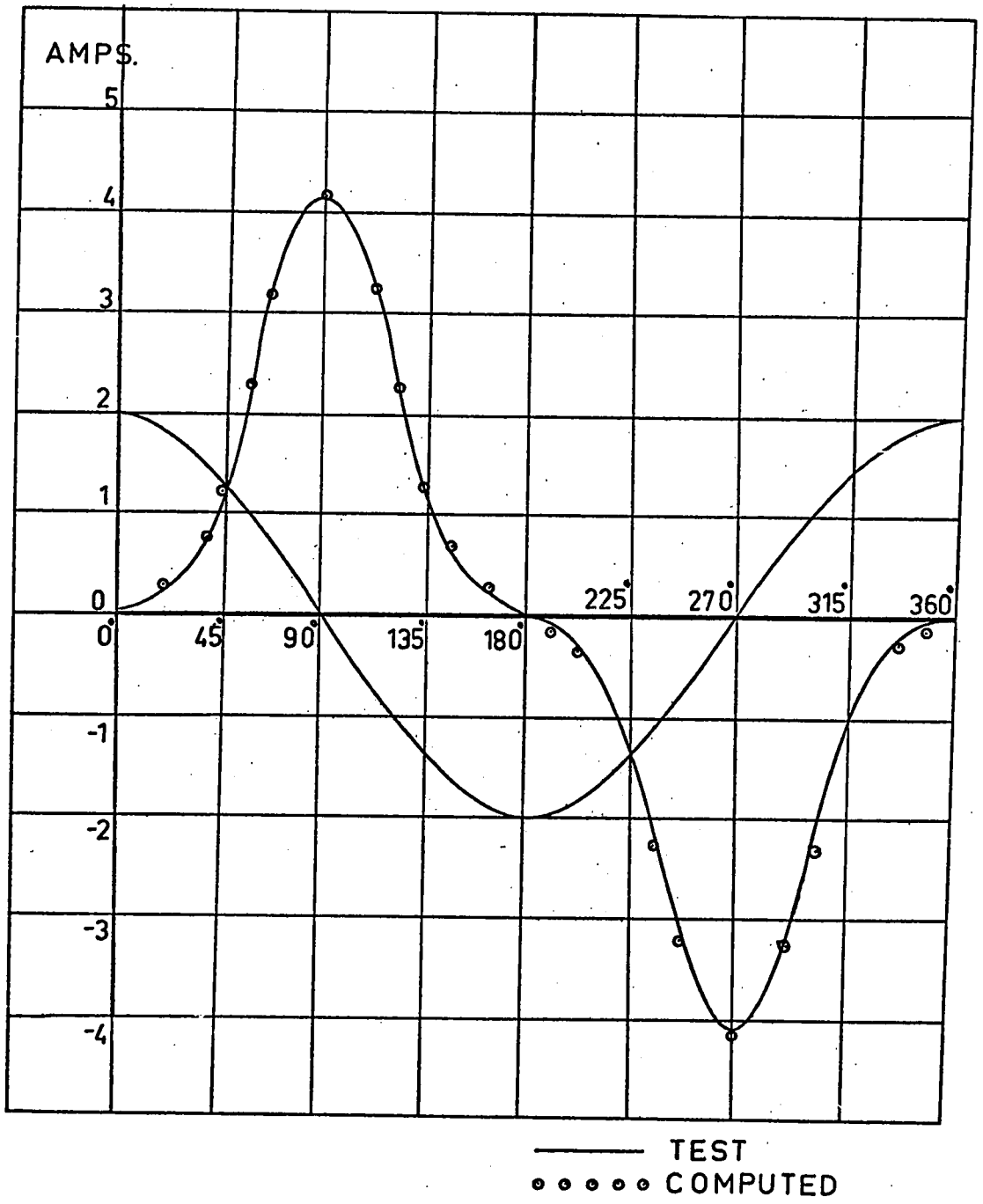


FIGURE 6.8. MAGNETIZING CURRENT WAVE-FORM OF THE TRANSFORMER.

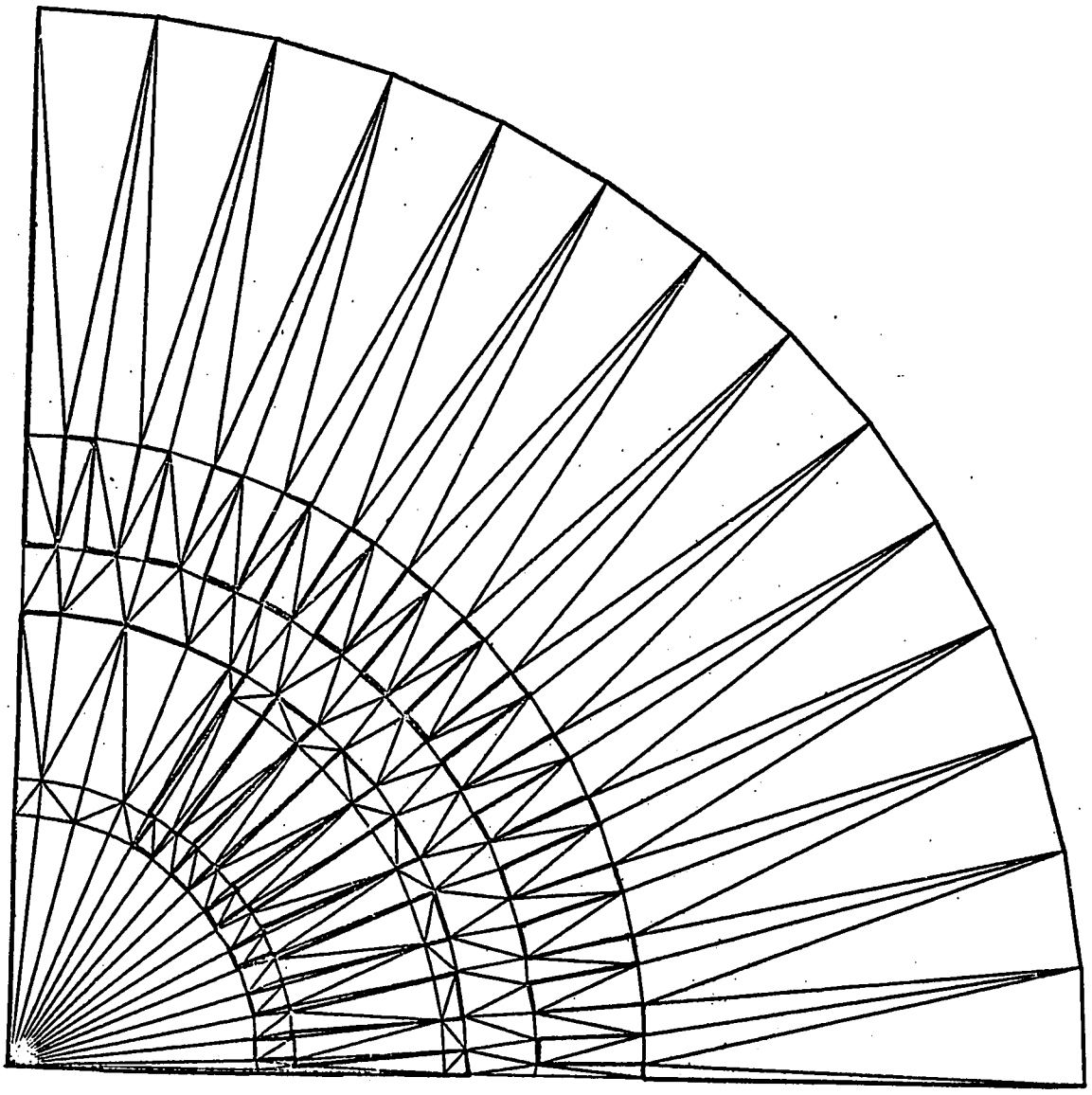


FIGURE 6.9. SUBDIVISION OF A QUARTER MACHINE CROSS-SECTION INTO TRIANGLES.

represented, except for wedge grooves, which could easily be accommodated in the program if they are especially required. Only a quarter of the machine is considered owing to symmetry. The B-H characteristics for the stator and rotor iron are shown in Figure 6.10.

In determining the magnetic field distribution in the machine by finite elements, the following assumptions are made in addition to those already stated in Chapters III and IV for the two-dimensional field problem.

(1) The individual currents in the straps forming the stator and rotor conductors are replaced by a uniform current density field over the cross-section of the armature and field coils.

(2) The magnetic field outside the machine is assumed negligible and the machine contour is treated as a line of constant vector potential (a flux line).

(3) At no-load and purely reactive loads, the direct axis is also a line of constant vector potential.

The flux distribution in the turbogenerator is obtained by solving Equation (3.38) of Chapter III, satisfying the aforesaid conditions and the natural boundary conditions implicit in the variational formulation. The result, of course, appears merely as a listing of the values of vector potential at the various triangle vertices. For further computations, this is entirely sufficient. However, the flux distribution pattern itself is often thought to be informative for the designer, and it is, therefore, on occasion desirable to plot it in detail. Figures 6.11 and 6.12 show typical flux plots obtained. It will be

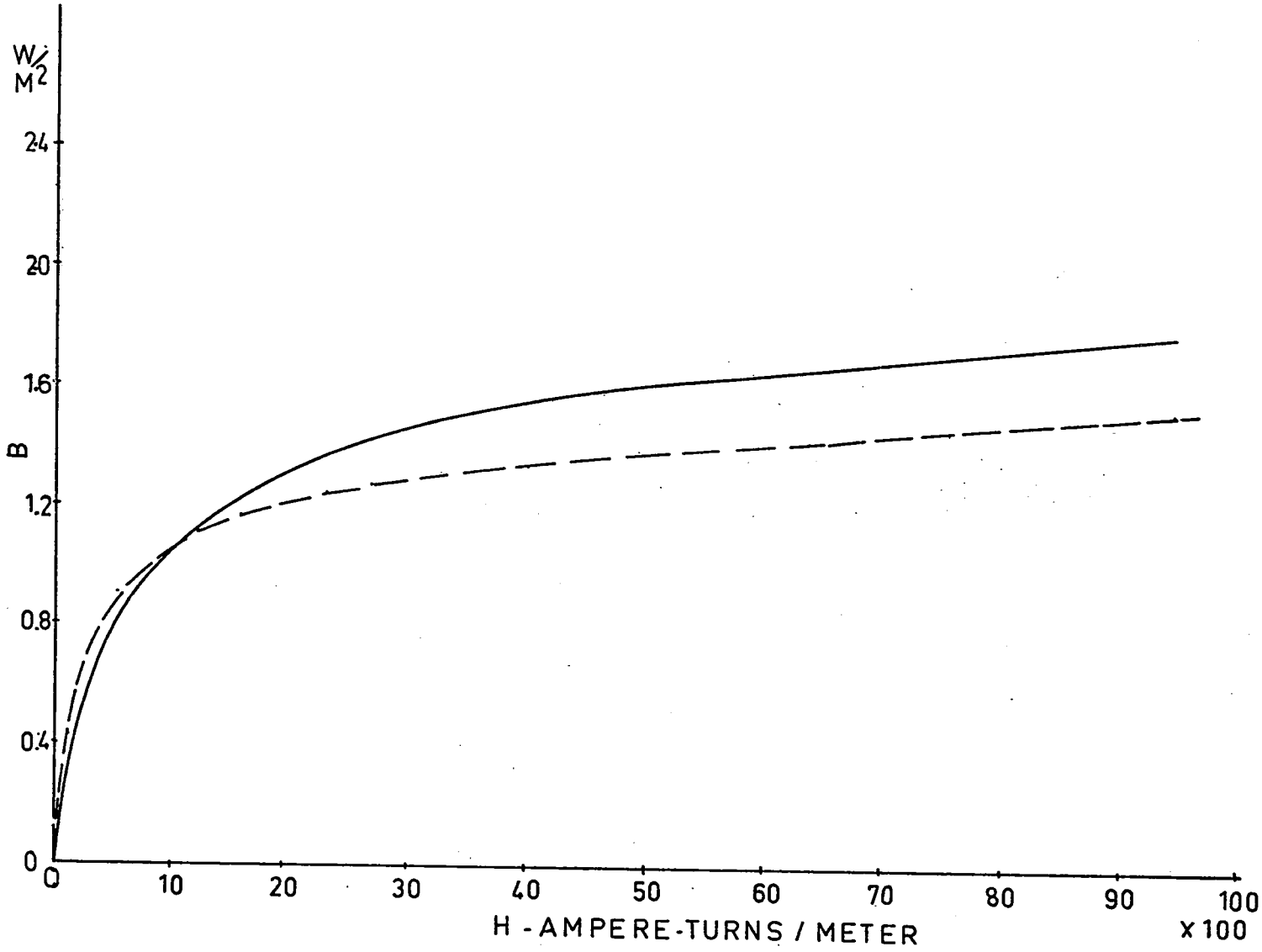


FIGURE 6.10. MEAN MAGNETIZATION CURVES OF ROTOR AND STATOR IRON : SOLID LINE REPRESENTS ROTOR, BROKEN CURVE FOR STATOR.

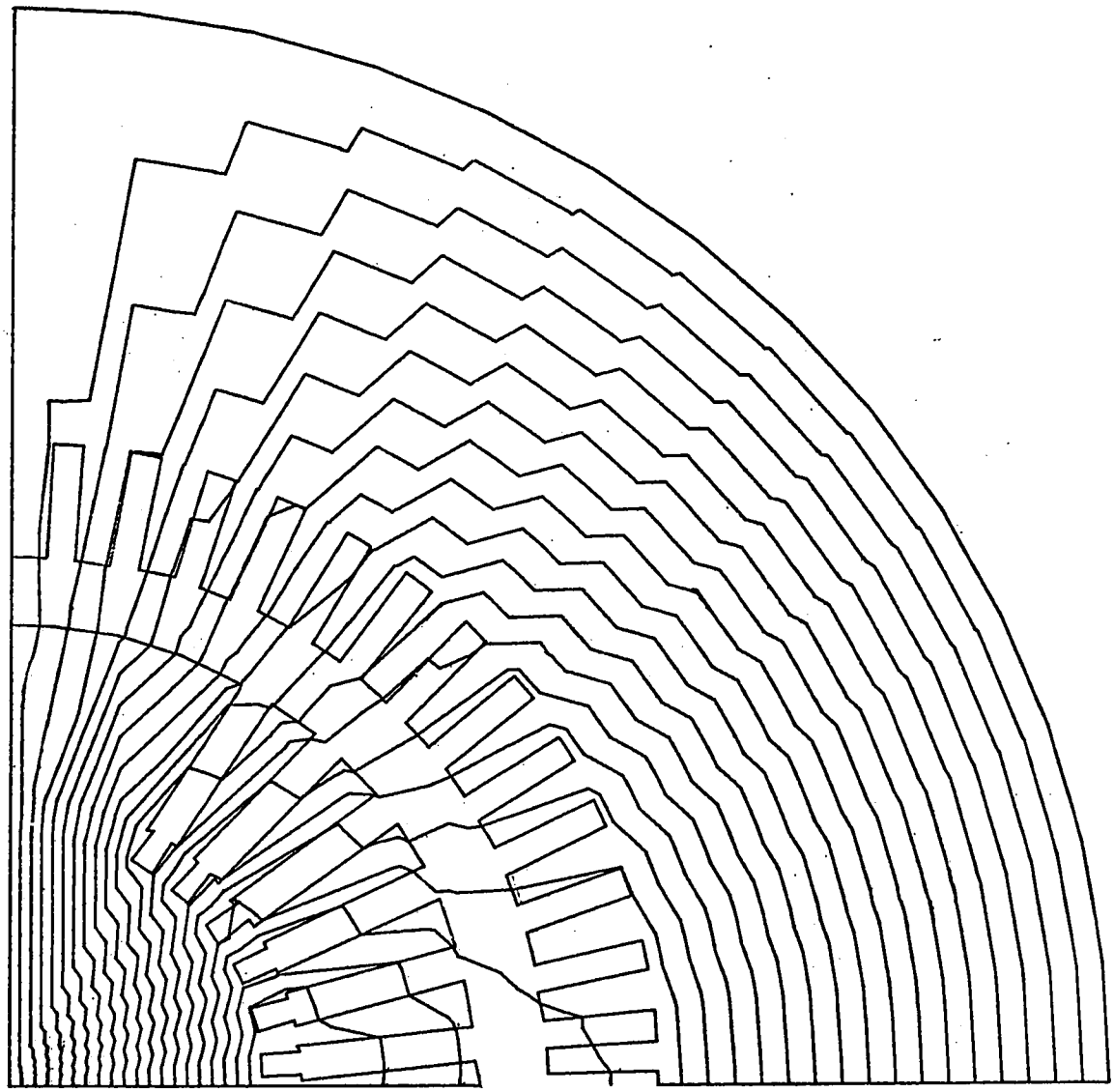


FIGURE 6.11. FLUX DISTRIBUTION IN TURBOALTERNATOR AT ZERO POWER FACTOR.

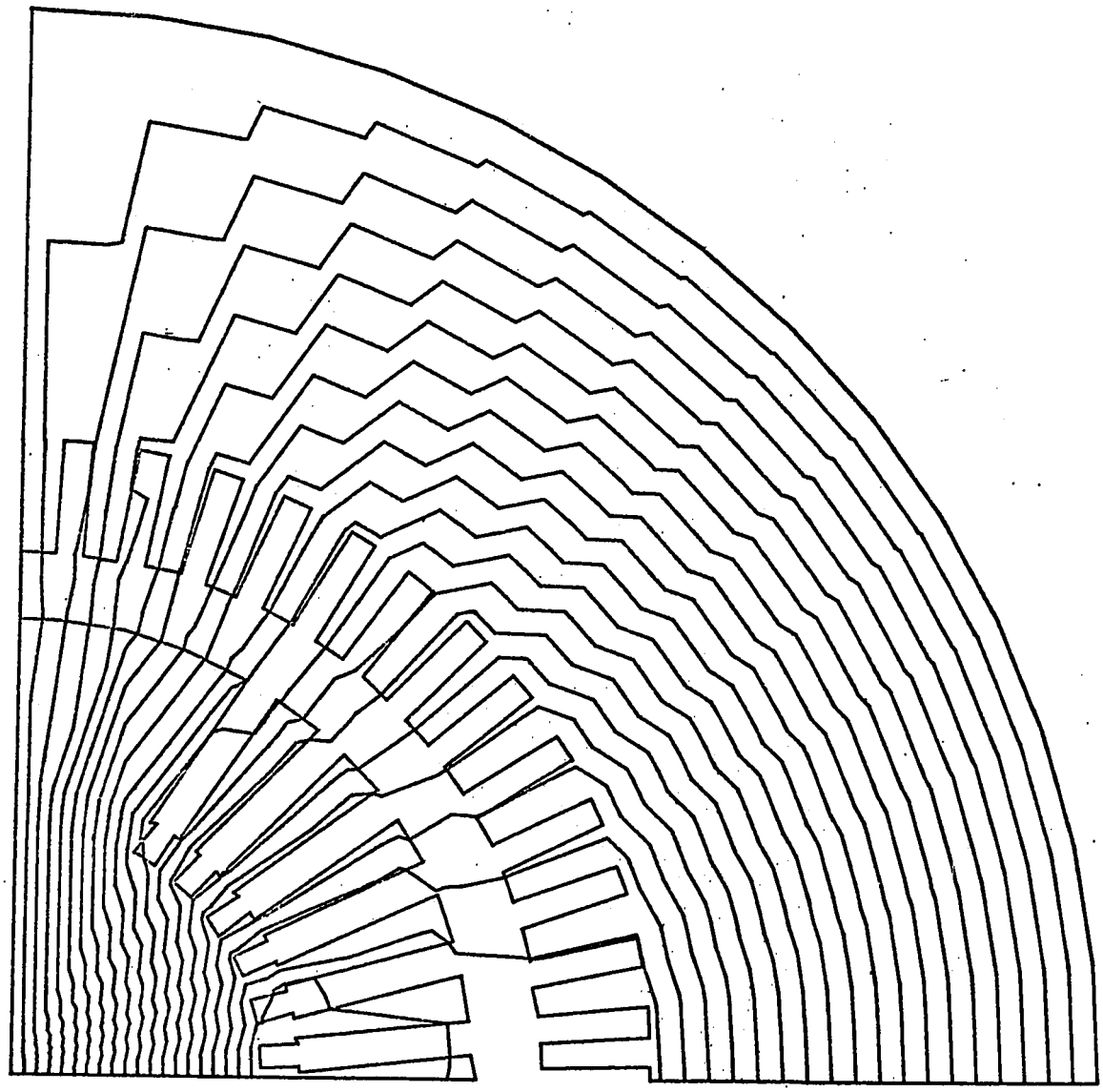


FIGURE .6.12. FLUX DISTRIBUTION IN TURBOALTERNATOR AT NO LOAD.

noted that the total lack of restriction on triangle size and shape has permitted using a crude approximation in regions where low accuracy is acceptable, while in the region of the slots and air-gap, where improved accuracy is desired, much smaller triangles have been used.

6.2.2 No-Load Saturation Curve and Iron Losses

Different values of current density were used in the rotor slots and the vector potentials were evaluated by the field solution method described above. The flux linkages ψ of the different stator conductors were calculated using the following relations [2.25, p. 165, Equation 5-10]

$$\psi = \oint \varphi \cdot dz \tag{6.7}$$

$$\psi = (\varphi_i - \varphi_j) \cdot l \cdot N \tag{6.8}$$

where i and j are the vertices bounding the individual coils, l the axial length of the coil and N is the number of turns. The computed and test results of the no load voltage for different excitation currents are found to agree exceedingly well as shown in Figure 6.13. The iron losses were computed from manufacturer's curves of magnetic induction B vs. watts loss / kg of the silicon iron stampings and vent plates used for the stator core. Even here, the agreement between test and computed values is good, as in Figure 6.14.

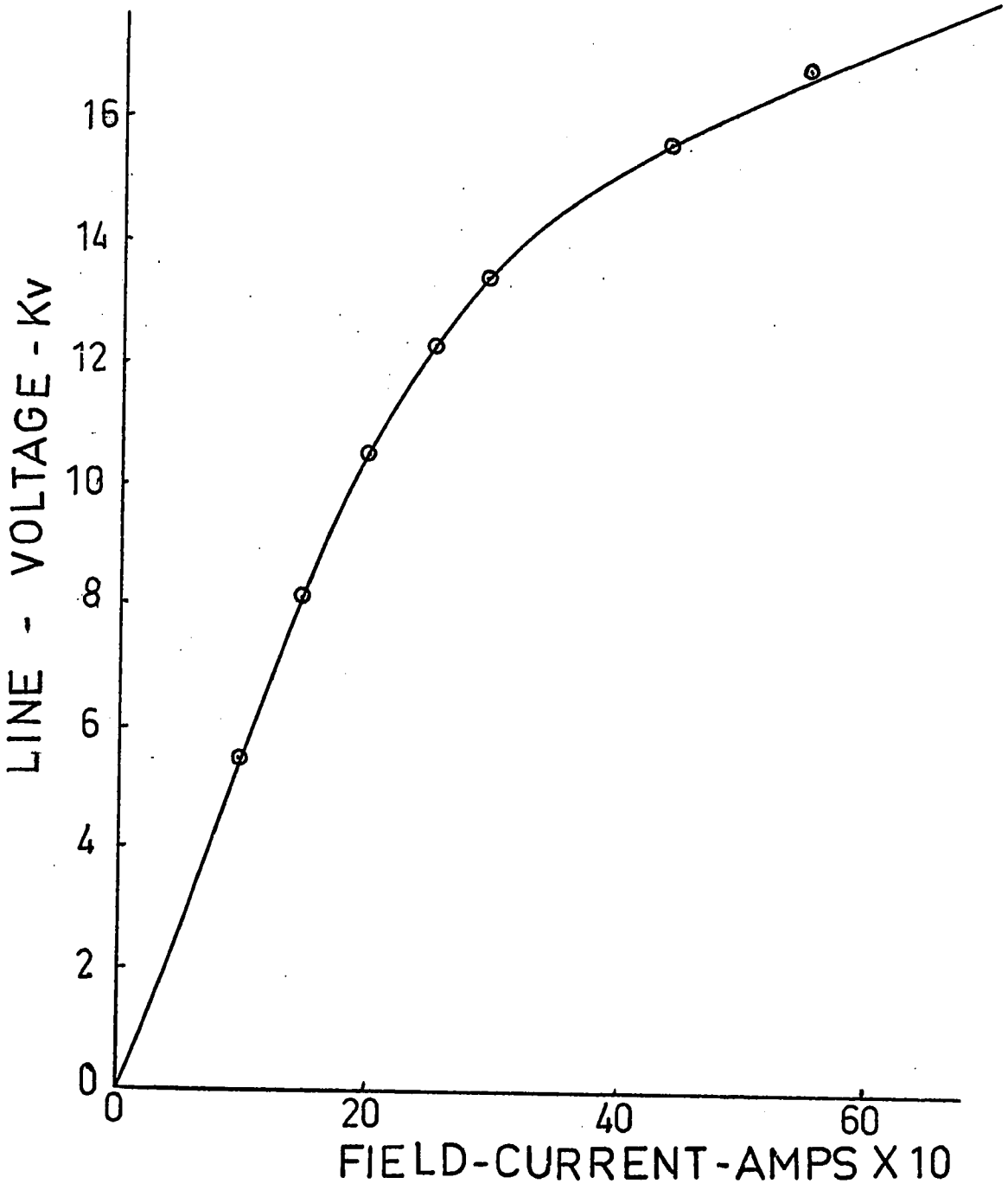


FIGURE 6.13. OPEN-CIRCUIT CHARACTERISTIC OF TURBOALTERNATOR. SOLID CURVE EXPERIMENTAL, CIRCLED POINTS PREDICTED BY FINITE-ELEMENT ANALYSIS.

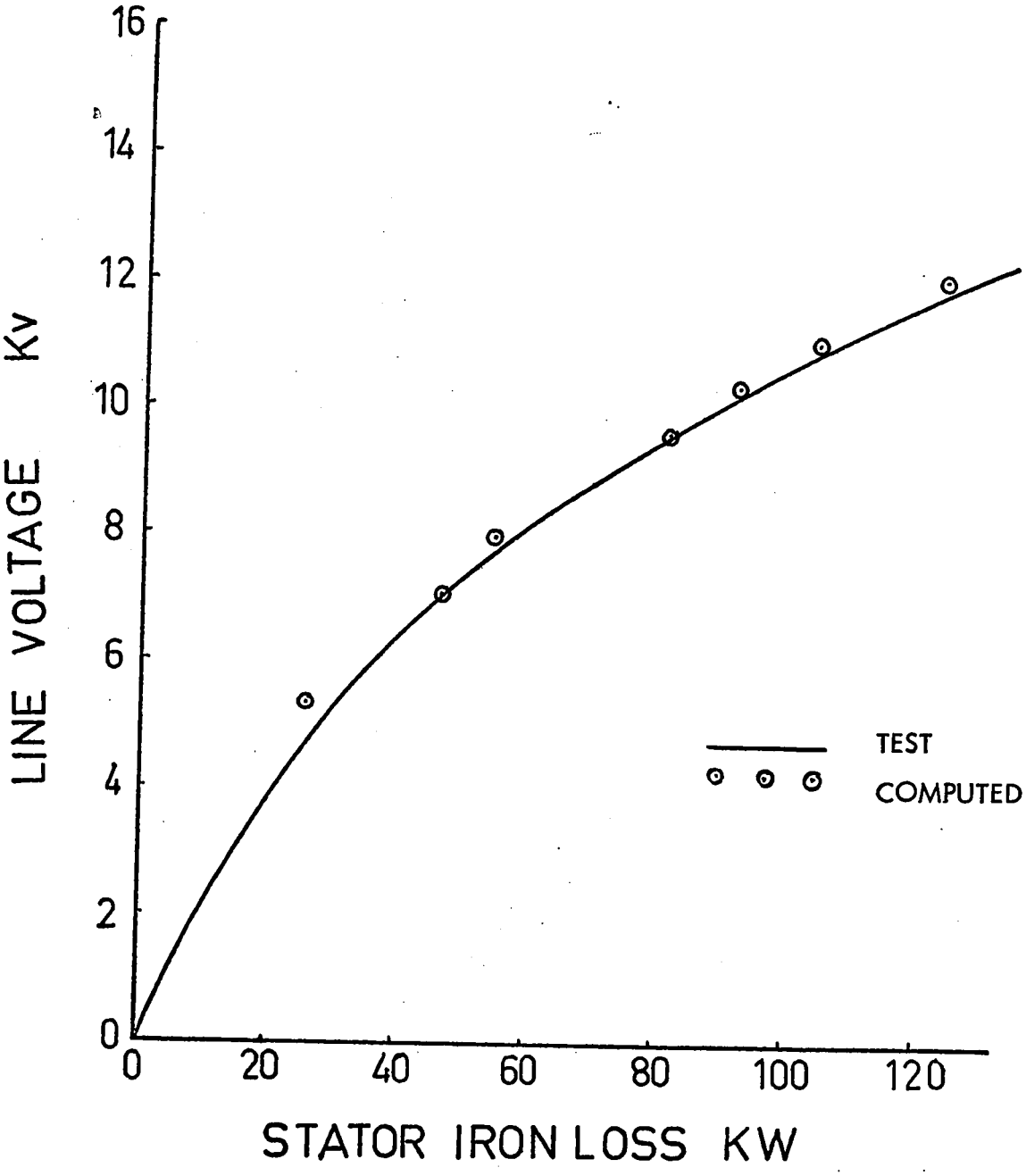


FIGURE 6.14. PREDICTED AND MEASURED IRON LOSSES AT NO LOAD.

6.2.3 Short-circuit and Zero Power Factor Characteristics

Under these conditions, both armature and field are magnetized along the direct or pole axis. Further, the armature reaction is entirely demagnetizing, since the effect of the resistance of its windings is negligibly small for generators such as the 30 MW unit being analyzed.

The process of computation here is similar to the open-circuit case, except for the fact that armature as well as the field coils carry current. The computed values compare well with test results, as shown in Figures 6.15 and 6.16.

The short circuit ratio (SCR) is a parameter often required in the stability analysis of machines and power systems, and is used in estimating the reactive volt-ampere capability of the generator. It is defined as

$$\text{SCR} = \frac{\text{Field current for rated open-circuit voltage}}{\text{Field current for rated short-circuit current}}$$

The value of SCR obtained from the computed open and short-circuit curves is 0.829 which agrees well with the test value of 0.827.

6.2.4 Waveform Analysis of the No-Load Voltage

The wave-form of the flux linking the stator slots, Figure 6.17, was analyzed and the harmonic coefficients were computed by a suitable Fourier analysis program. It was noted that all even harmonics were practically nonexistent and also the odd harmonics were

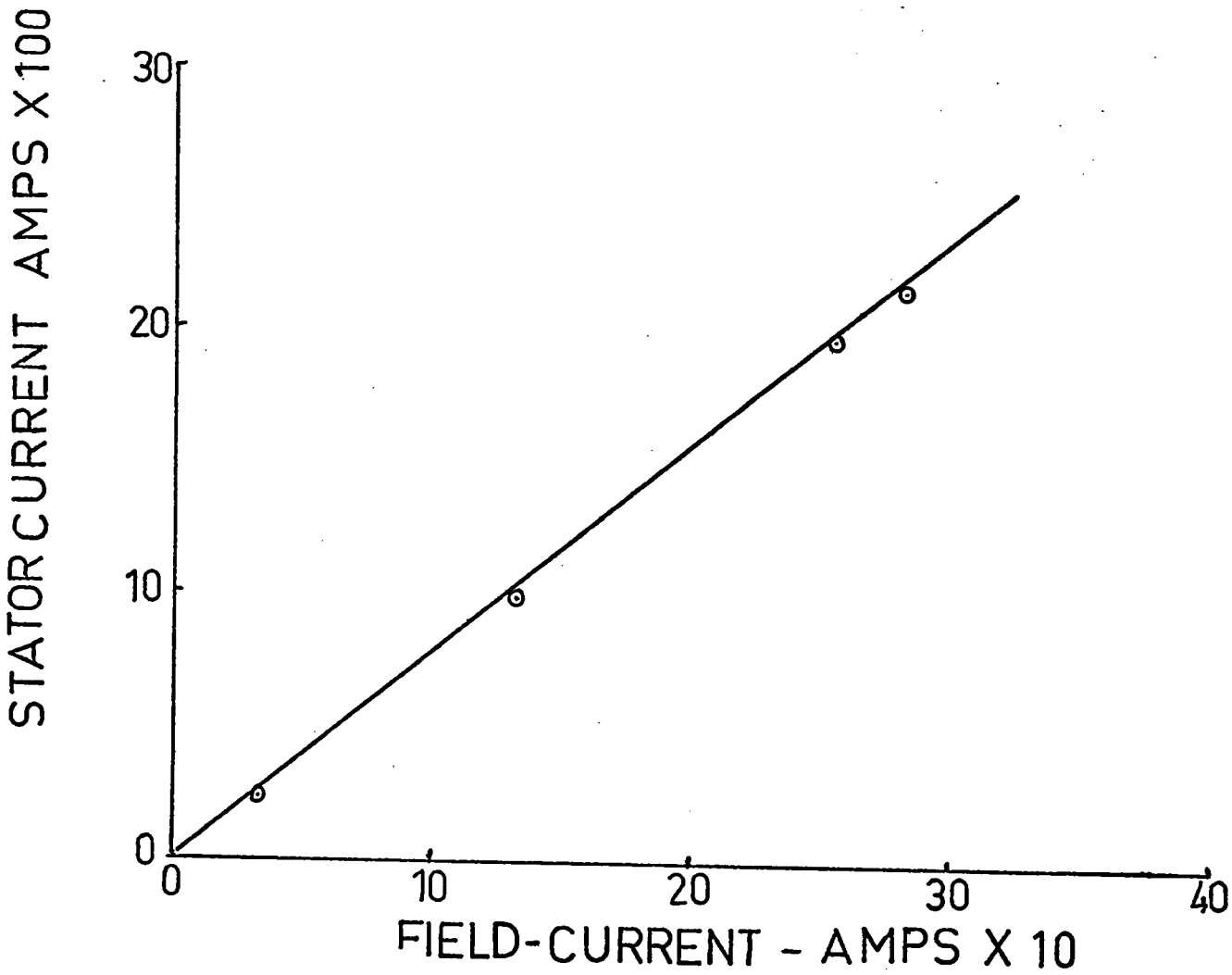


FIGURE 6.15. SHORT-CIRCUIT CHARACTERISTIC. SOLID LINE REPRESENTS EXPERIMENTAL RESULTS, CIRCLED POINTS PREDICTED FROM FINITE - ELEMENT ANALYSIS.

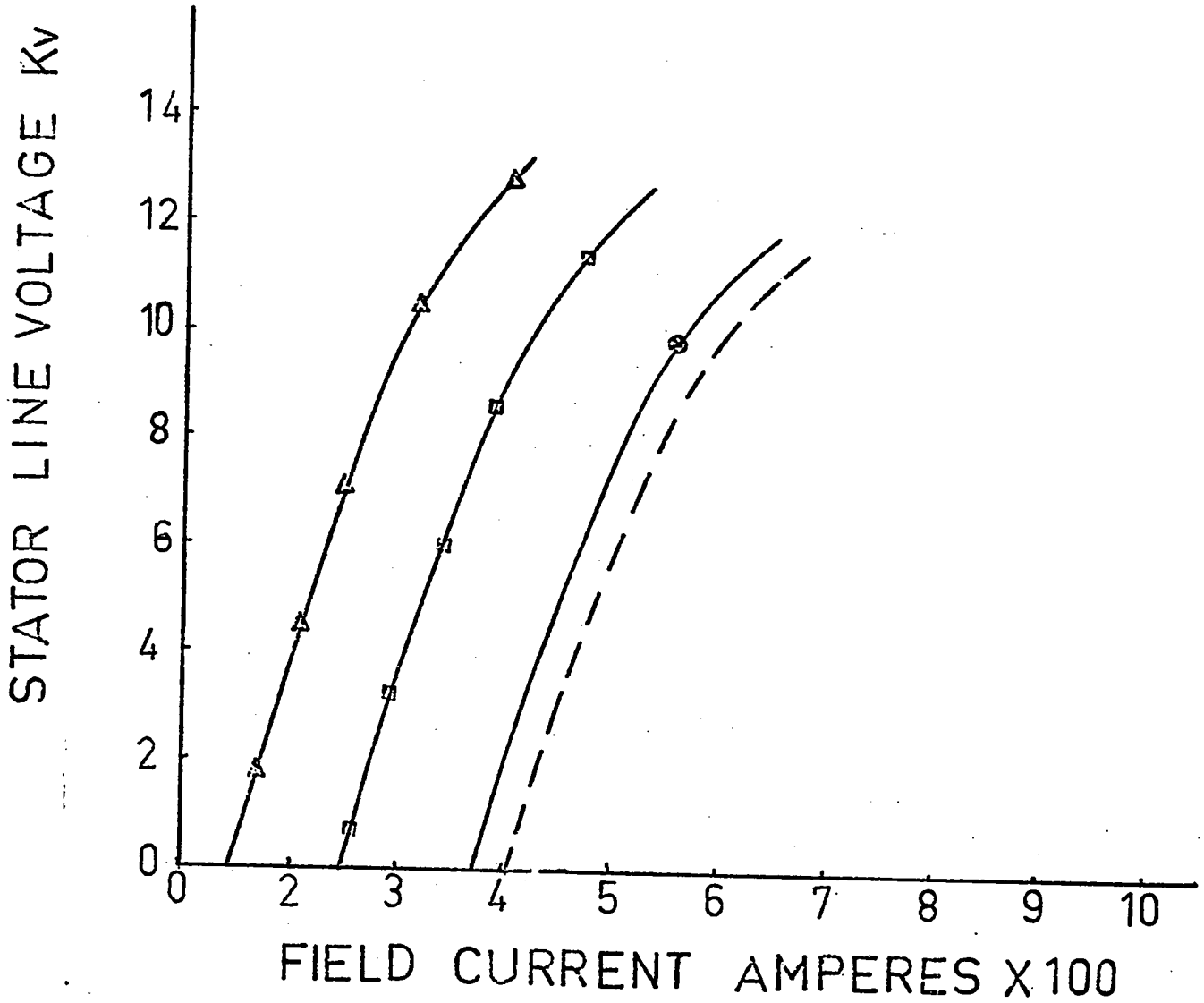


FIGURE 6.16. ZERO POWER FACTOR TESTS COMPUTED AT CURRENT VALUES OF 250, 1000, 1968 AND 2196 AMPERES. THE CIRCLED POINT AT 1968 AMPERES HAS BEEN VERIFIED EXPERIMENTALLY.

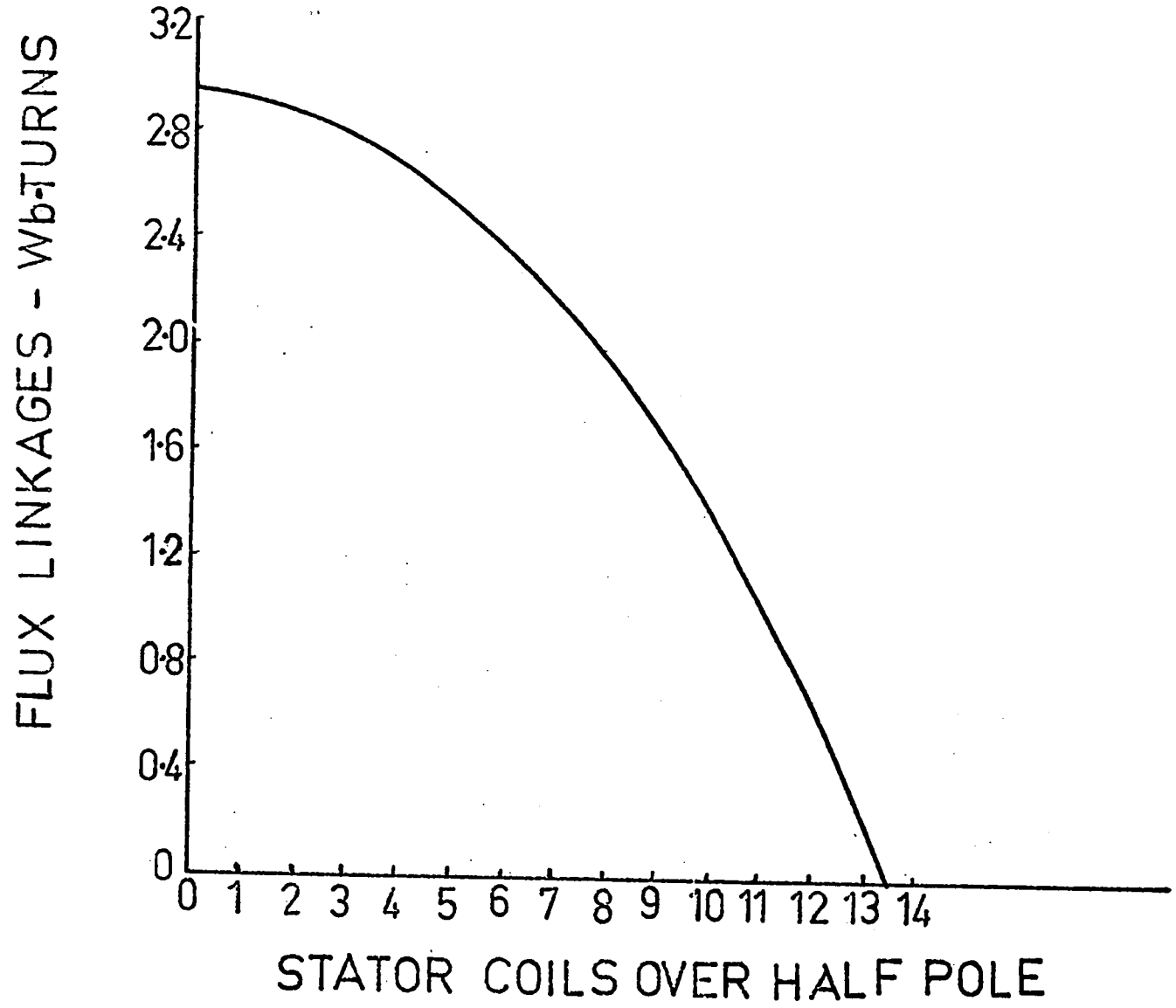


FIGURE 6.17. GENERATED VOLTAGE WAVEFORM ON OPEN CIRCUIT.

of small magnitude. A comparison of the computed values and test results of wave-form analysis of the phase-to-neutral voltage of the turboalternator is shown in Table II.

6.2.5 Evaluation of Sequence Reactances

If the generator is run at rated speed with the field excited to circulate rated current in the Y - B phases, according to the circuit diagram of Figure 6.18 [see 6.1, pp. 169 - 170], the negative sequence reactance will be proportional to the ratio of the line voltage to the current circulating in the short-circuit phases (see Appendix V), so that

$$X_2 = E_3 / \sqrt{3}.1 \text{ ohms} \tag{6.9}$$

To achieve this condition in the computer program, only the Y - B phase armature coils are made to carry current and the resulting voltage is computed for any fixed value of field current. From a plot of armature current against line voltage, the value of voltage corresponding to rated current is obtained. Then the negative sequence reactance is given by

$$X_2 \text{ (p. u)} = \frac{\text{Line voltage}}{\text{Rated voltage}} \tag{6.10}$$

The zero power factor test is carried out as described in Reference [6.1, pp. 170 - 173] with the terminal connections of Figure 6.19. The zero sequence reactance is then obtained, as shown in the Appendix, from the relation

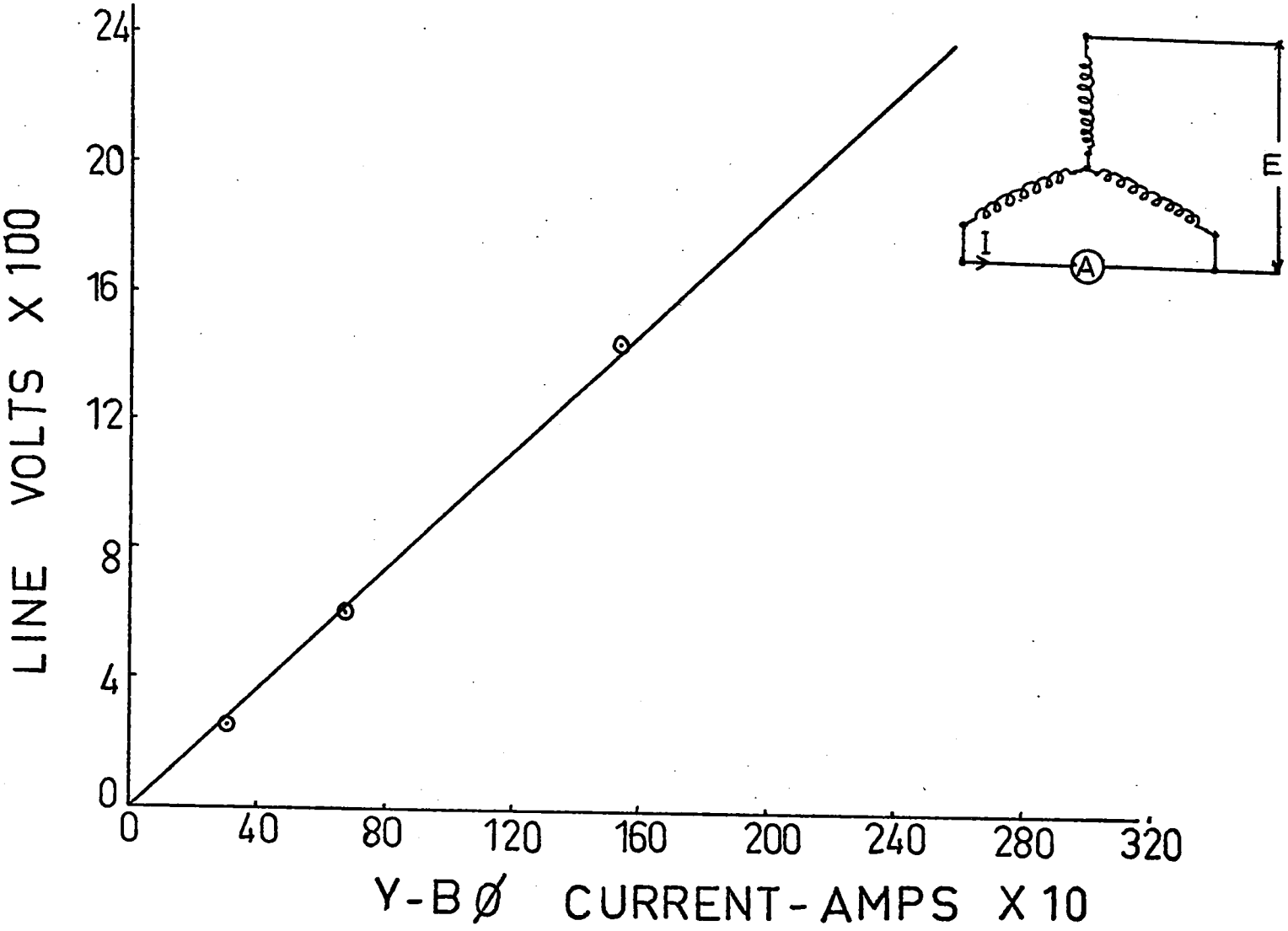


FIGURE 6.18. NEGATIVE - SEQUENCE TEST. SOLID CURVE MEASURED, CIRCLES PREDICTED.

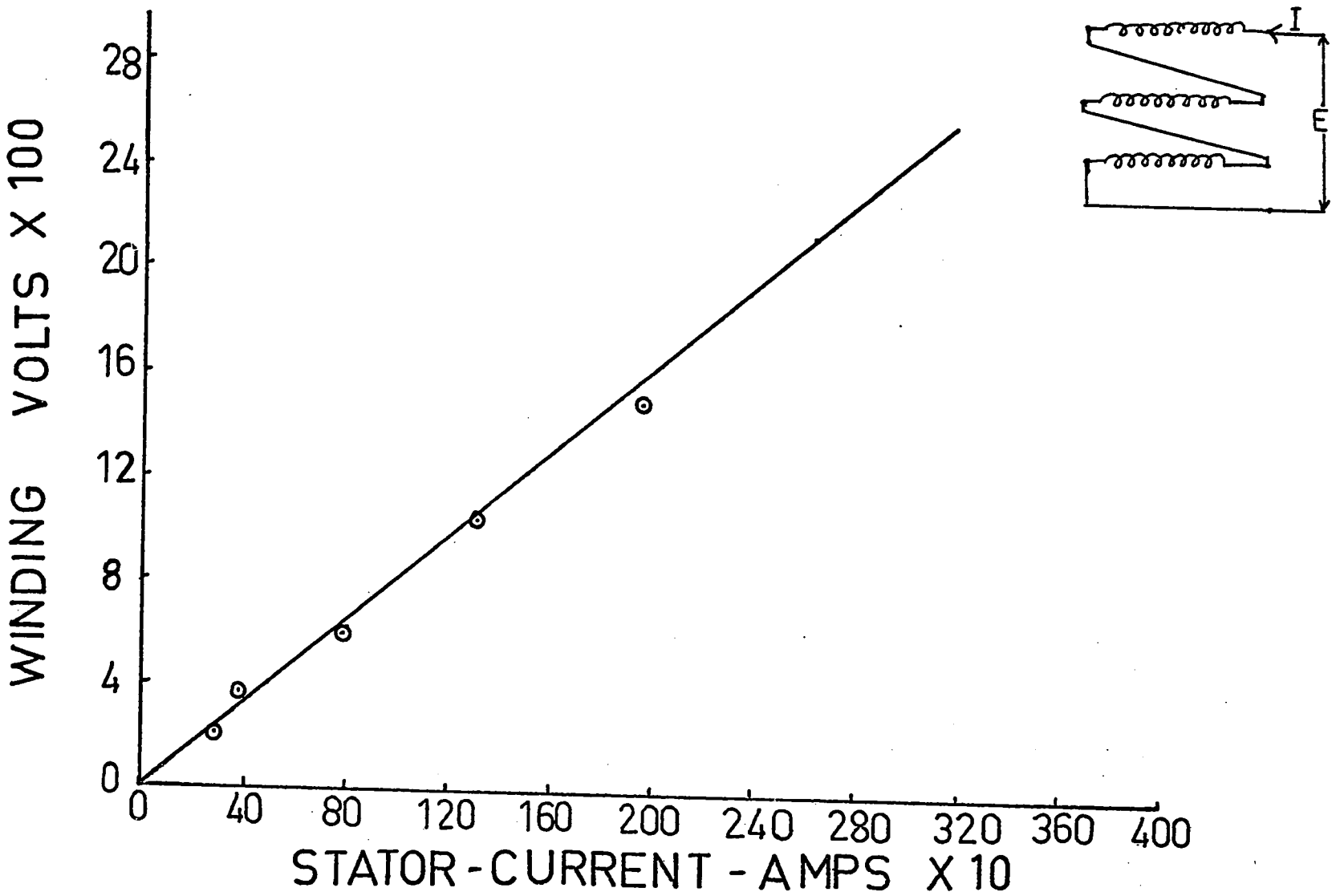


FIGURE 6.19. ZERO-SEQUENCE TEST. SOLID CURVE REPRESENTS MEASURED, CIRCLES PREDICTED VALUES.

$$Z_o = E / 3.1 \text{ ohms} \qquad (6.11)$$

its per unit value being

$$Z_o = E / \sqrt{3} . \text{ Rated volts} \qquad (6.12)$$

A comparison of test results and computed values is shown in Table II, which is good for most practical purposes at the design stage.

In the finite element analysis of the turbogenerator field problem, 273 triangles with 151 values of potentials were used to represent a quarter section of the machine. The cost of programming of the entire series of tests was around $60, with a computation time of less than 10 minutes on an IBM 360/75 computer.

6.3.1 Determination of the Performance Characteristics of a D.C. Generator

For determining the no-load and on-load characteristics by this method, a 5 KW, 4 pole, 1750 r.p.m., 200 - 220 volts , separately excited DC generator with interpoles and compensating windings was used.

As before, the currents in the individual straps of the main field, interpole compensating and armature windings are replaced by a uniform current density, and the outside of the machine yoke is considered a flux line boundary. Since the load characteristics are required as well as the open-circuit performance, it is necessary to represent the field region over one pole pitch for obtaining a general computer program, although

TABLE I.

COMPARISON OF ESTIMATED NEGATIVE AND ZERO SEQUENCE PERFORMANCE WITH TEST RESULTS

Type of Test	Stator Volts Test	Stator Volts Computed	Stator Amps. Test	Stator Amps. Computed	P. U. Reactance Test	P. U. Reactance Computed
Negative Sequence Test	1760	1740	1968	1968	0.160	0.158
Zero Sequence	1550	1500	1968	1968	0.0788	0.081

TABLE II.

HARMONIC ANALYSIS OF THE OPEN - CIRCUIT VOLTAGE WAVE - FORM

Order of the Harmonic Component	Computed Values	Test Results
Fundamental	1.03	1.0
Third	0.001595	0.001649
Fifth	0.000268	0.0000535

for determining the no-load characteristic of the machine, the region over half a pole pitch would suffice. The triangular sub-divisions used in this analysis are shown in Figure 6.20. Two different B - H characteristics are used ; one for the main-pole and armature laminations and another for the remaining iron parts of the machine.

6.3.2 Open-Circuit Characteristics of the DC Generator

The analysis for the no-load case of the DC machine is identical to that of the turbogenerator, and from the listing of the nodal potentials obtained, the air-gap flux is evaluated and the armature voltage is determined using the relation

$$E = \phi Z N \text{ volts} \qquad (6.13)$$

where ϕ is the flux per pole in webers, Z the number of armature conductors and N the speed in r.p.s. In the alternative, the method described by Equations (6.7) and (6.8) of Section 6.2 are used to calculate the flux linkages and the resulting induced voltage in the machine. A comparison of the predicted values and test results of the no-load characterisitcs are shown in Figure 6.21, and the correlation is obviously very good considering the fact that a two dimensional analysis has been used for a finite-length machine with a L / D ratio of less than unity and the B - H characteristics have been considered as single-valued, ignoring hysteresis effects.

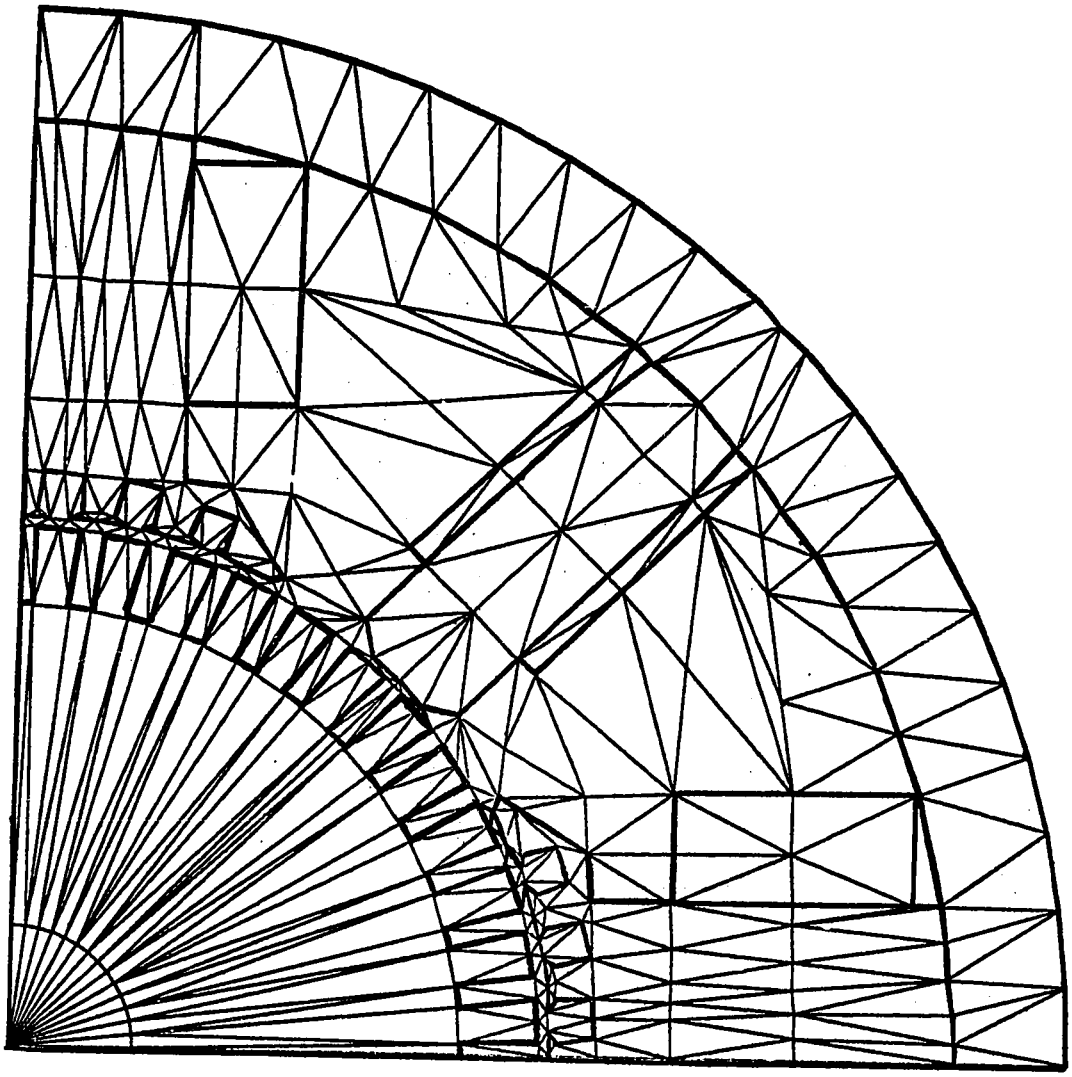


FIGURE 6.20. SUBDIVISION OF D C GENERATOR OVER A POLE PITCH INTO TRIANGLES.

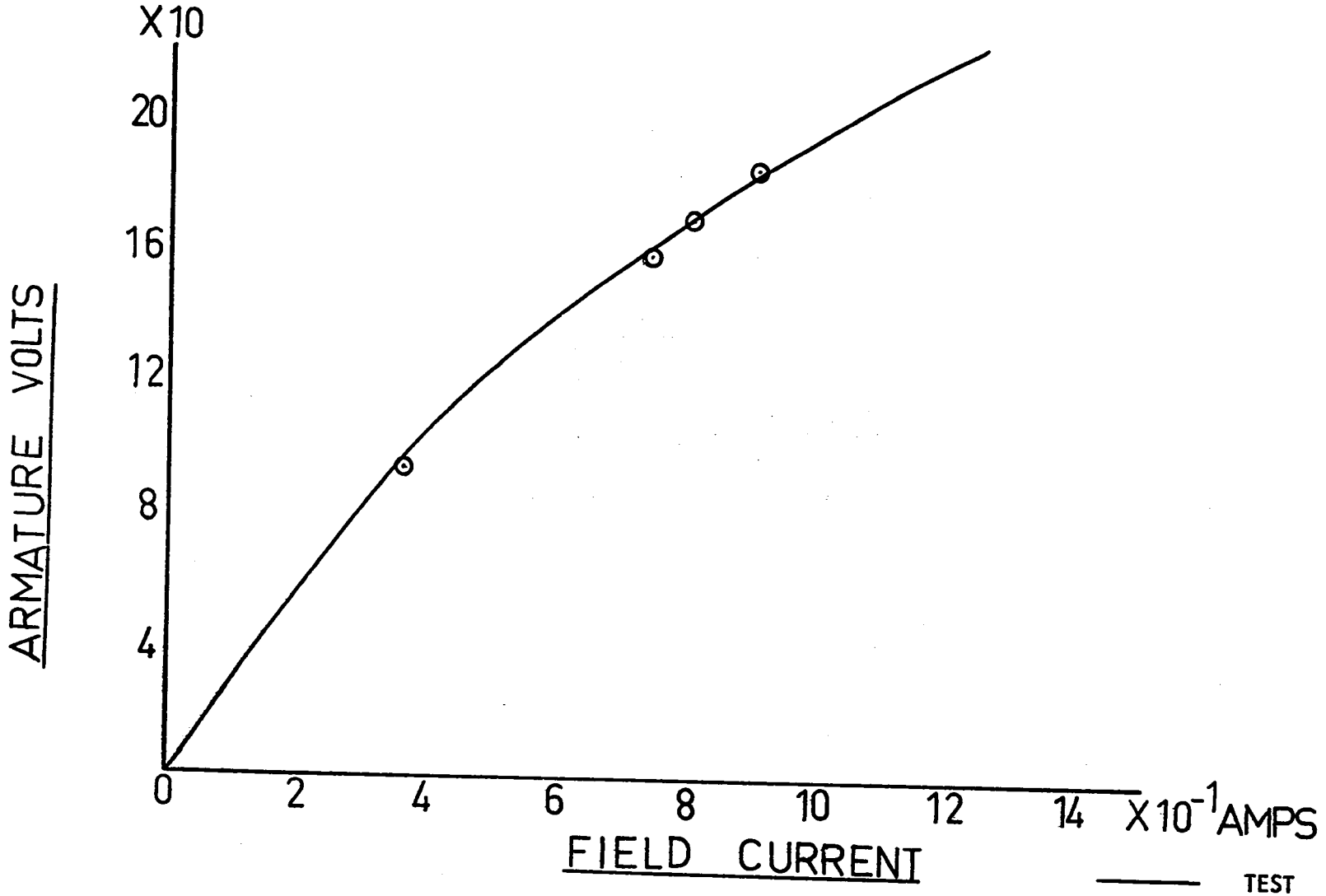


FIGURE 6.21. NO - LOAD CHARACTERISTIC OF D C MACHINE.

6.3.3 Voltage Regulation On-Load

When a rotating electrical machine is delivering power to a load, the axes of symmetry of the magnetic field depart considerably from the polar or interpolar axes as shown in Figure 6.22, and the latter cannot serve as a flux line boundary for the field problem. However, for the field pattern shown, it will be noted that the potential at any point is related to that of the corresponding point one pole-pitch away, so that the two potentials are equal in magnitude but of opposite sign. This is commonly known as the periodicity condition. The potentials on either side of the origin along the yy' axis of Figure 6.22 are likewise subject to this additional boundary condition. Using the periodicity condition, the coefficient matrix is modified and the resulting set of equations similar to (3.37) of Chapter III, are solved to obtain the field solution. The modification of the coefficient matrix and the forcing function are accomplished by the use of a special connection matrix as shown in Appendix VI.

From the vector potential solutions obtained for three different values of exciting current and different armature currents, the induced e.m.f. in the armature is evaluated by either of two methods described for the no-load case. Figure 6.23 shows the comparison of the voltage regulation obtained by computation with experimental results on the 5 Kw D.C. generator with the main field winding separately excited. Even here, the correlation is very good, indicating the validity of the finite element method for solving practical nonlinear field problems.

In this analysis, the field region is subdivided into 490 triangles and 266 nodes and the flux plots obtained for the no-load and full-load conditions are illustrated in Figures 6.24, 6.25 and 6.26. The use of the periodicity condition has

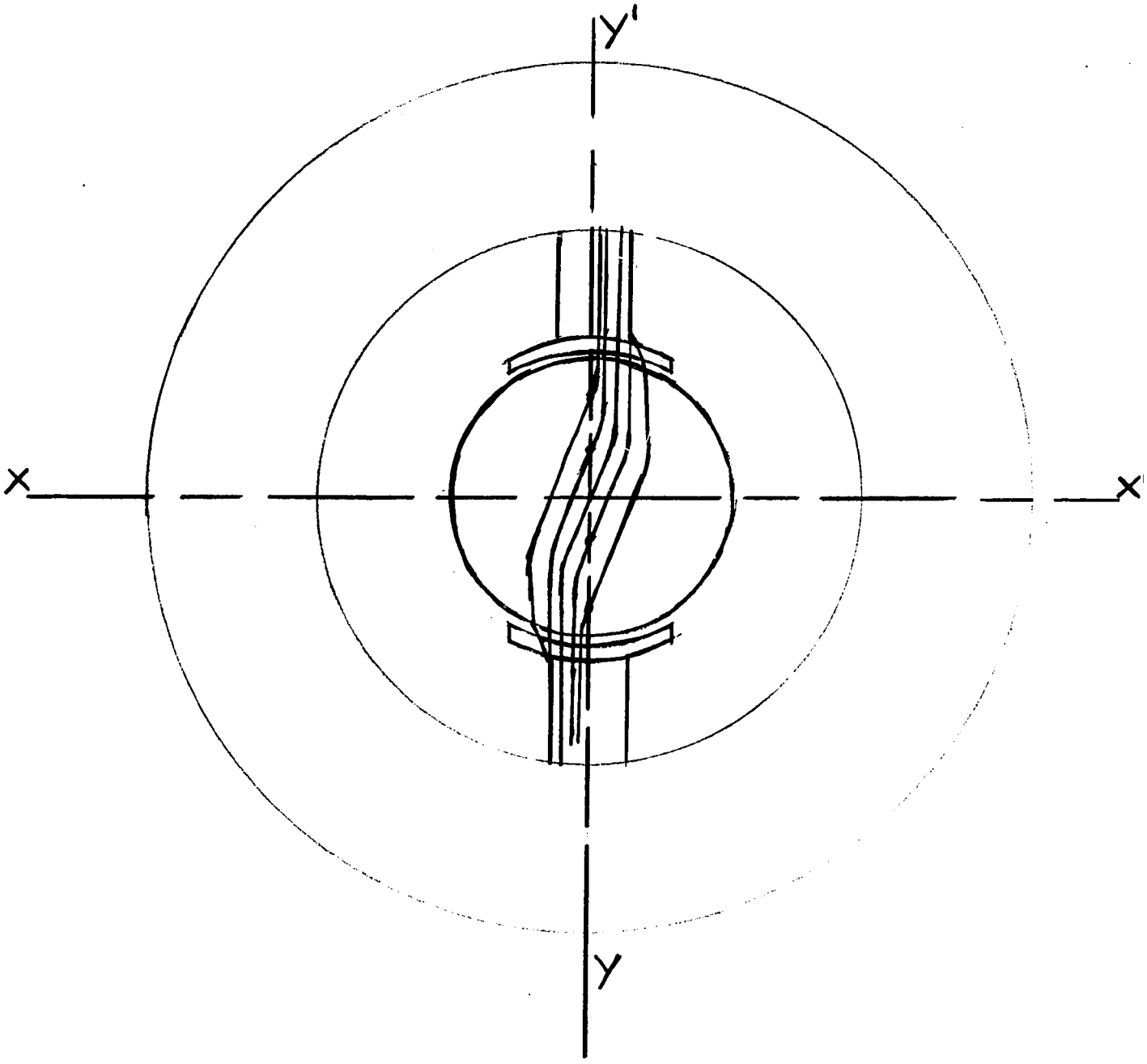


FIGURE 6.22. LOADED D C MACHINE.

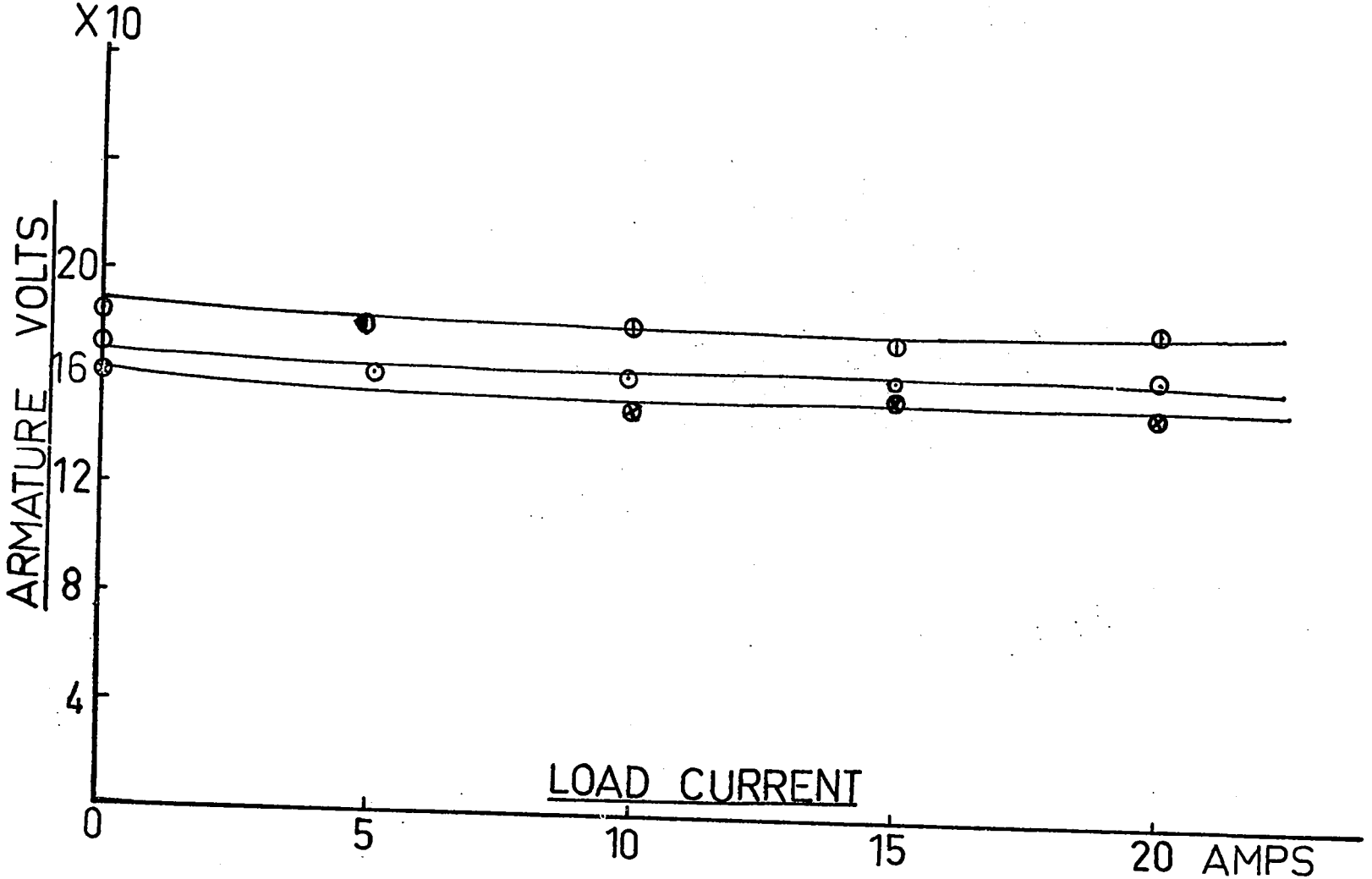


FIGURE 6.23. LOAD CHARACTERISTIC OF DC GENERATOR.

TEST

COMPUTED

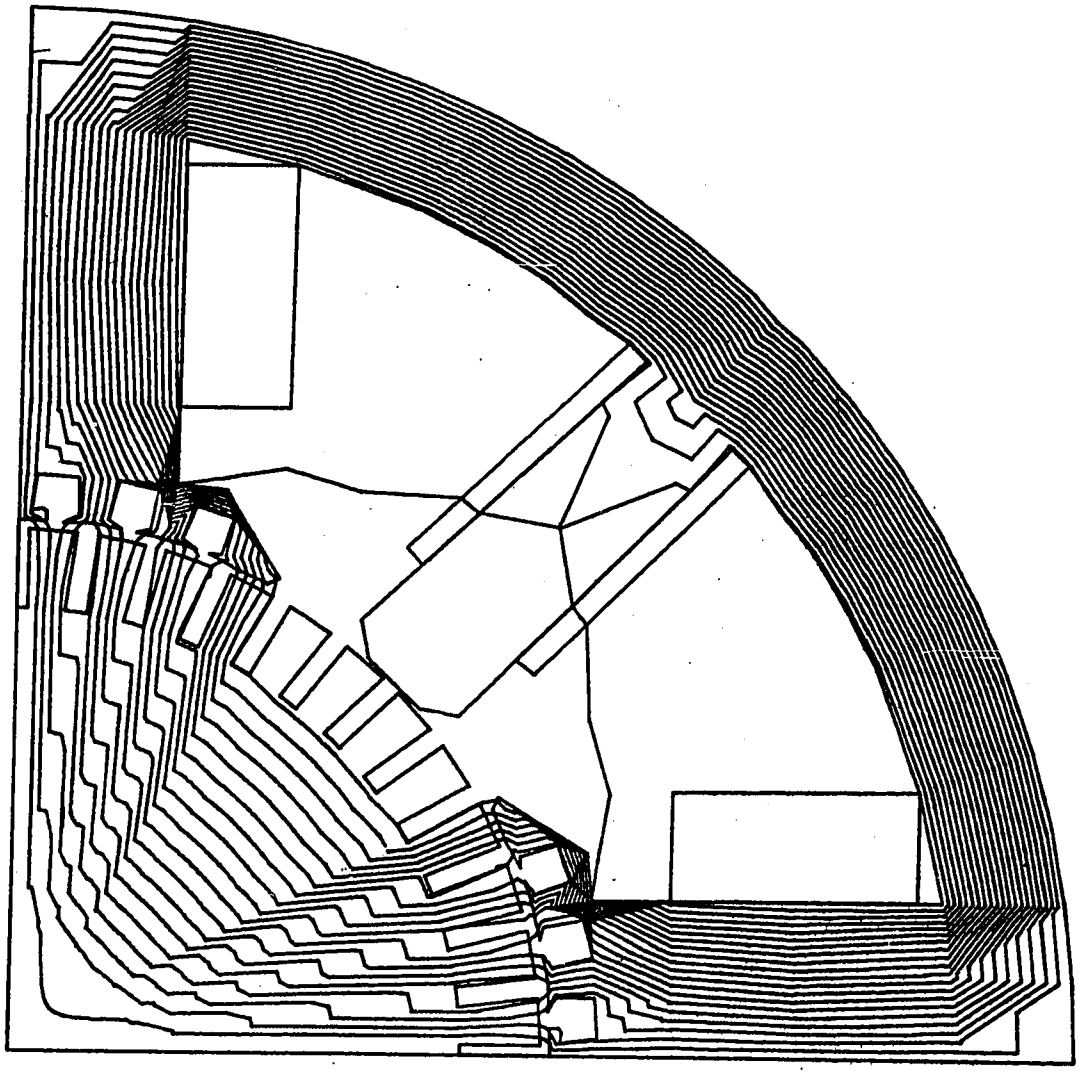


FIGURE 6.24. FLUX DISTRIBUTION OF D C MACHINE ON NO-LOAD.

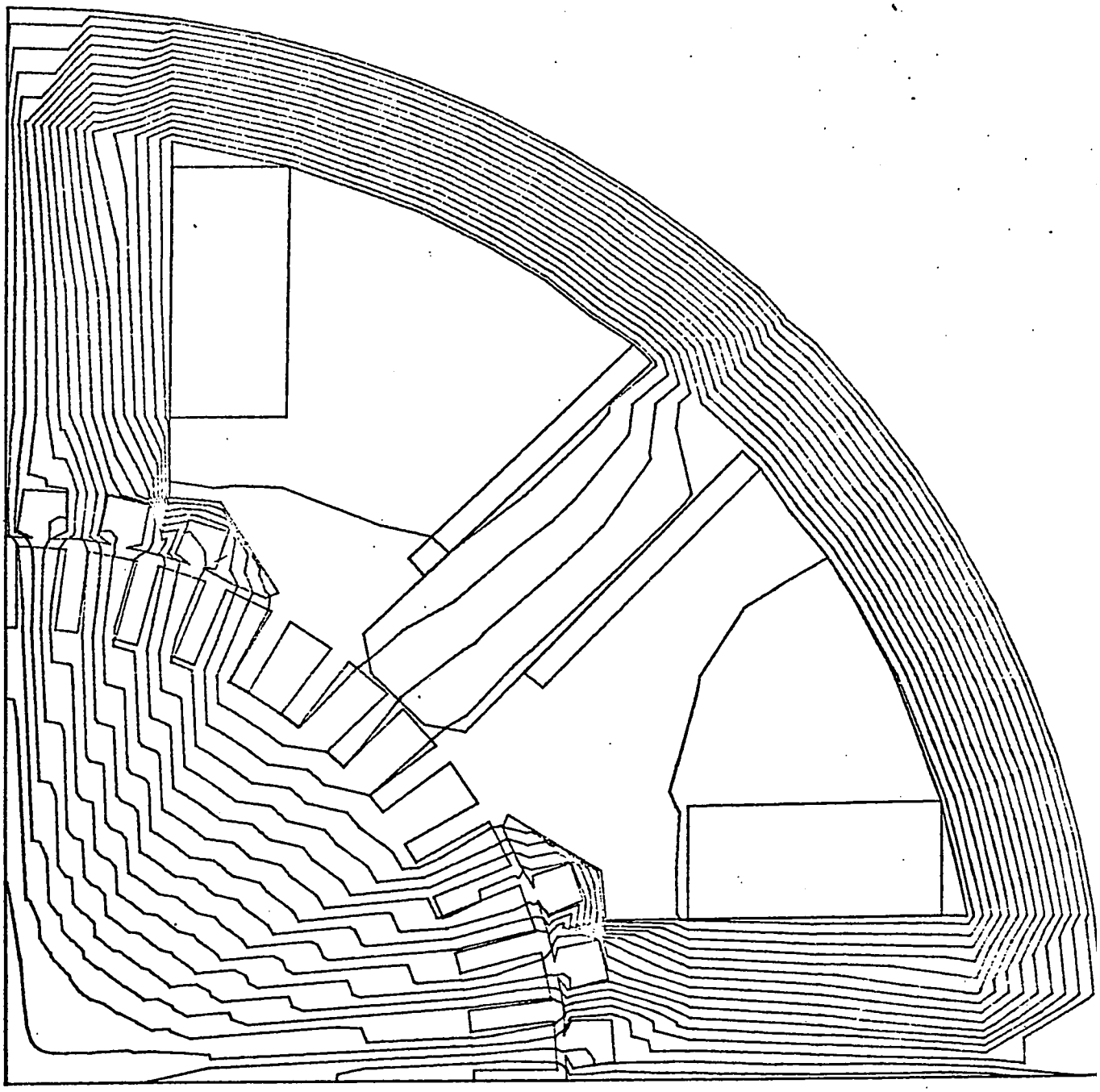


FIGURE 6.25. FLUX DISTRIBUTION IN D C MACHINE ON FULL LOAD (FULLY COMPENSATED).

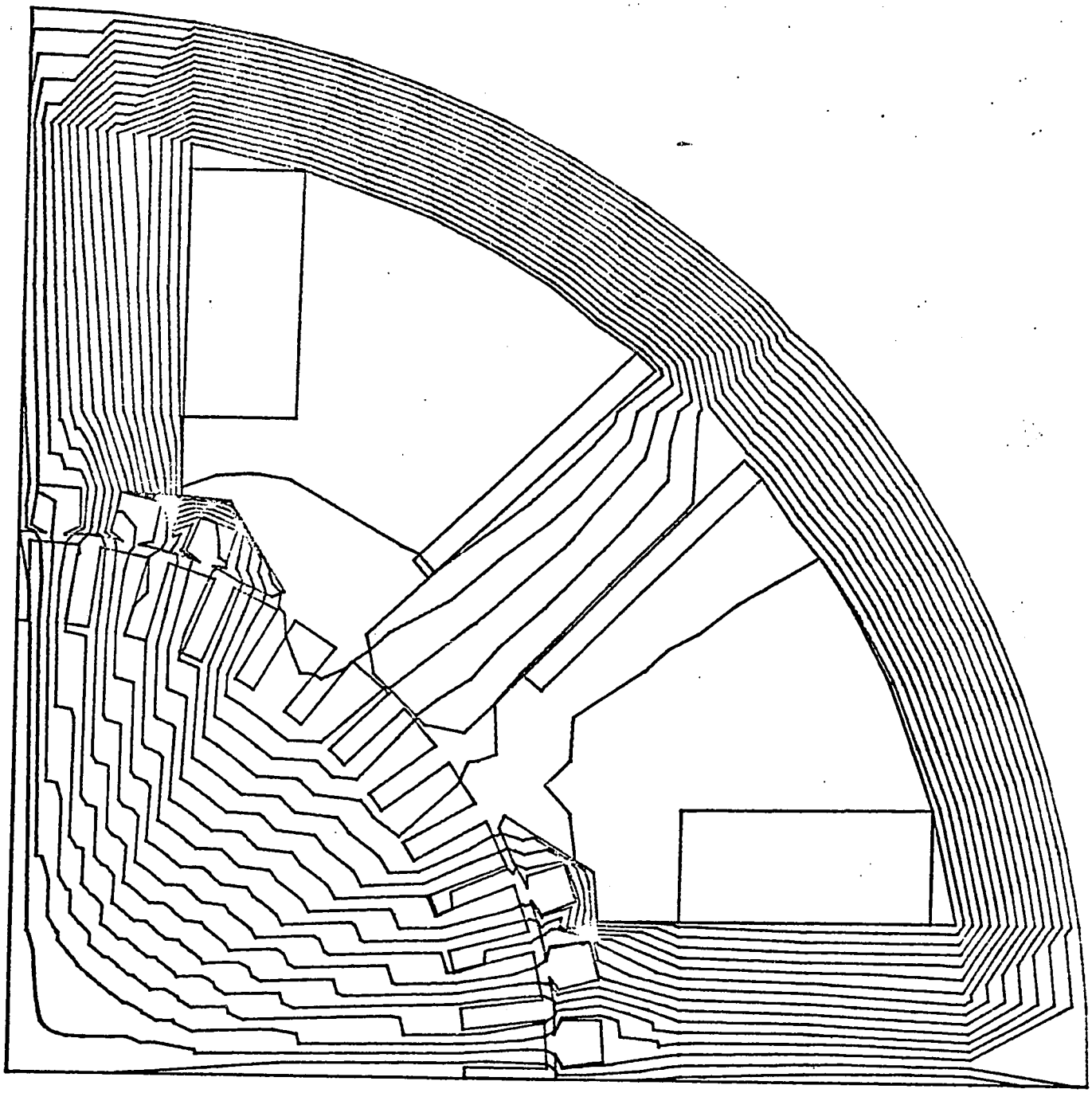


FIGURE 6.26. FLUX DISTRIBUTION IN THE D.C. MACHINE AT FULL LOAD. (UNCOMPENSATED).

made a detailed representation of the DC generator and its solution under load conditions possible, which could not be accomplished otherwise in view of the limitation on storage of large matrices in the computer. However, the band-width of the modified coefficient matrix is larger than for the no-load case, since nodes a pole-pitch away are connected to each other electrically. A careful re-numbering of the triangle vertices in the first instance leads to a small band-width and economy in storage. This is not always possible and, perhaps, in future work on the subject a method for compacting band-structured matrices may be developed as a separate algorithm for effecting further economy and efficiency in programming. However, despite this limitation in the present analysis, the cost per solution for each field and armature current value is still far more economical than any field solution obtained on a similar problem by currently available finite difference schemes and others.

6.3.4. Flux Distribution

From the values of flux densities predicted, it is observed that a much higher degree of saturation occurs in the rotor body of the turbo-generator and the field of the D.C. machine than has been assumed in conventional design practice. Consequently the design constants which are based on an average value of flux density require considerable modification.

CHAPTER VII

CONCLUSIONS

A general variational approach has been presented in this thesis for solving two-dimensional nonlinear electromagnetic field problems. The finite element method has been applied for the first time to electric machines for evaluating their performance under conditions of magnetic saturation.

The method developed in this analysis consists of deriving the true energy functional for nonlinear conditions and replacing the continuum problem by a set of finite triangular elements which represent the geometry and material characteristics of the medium and define the approximation to the magnetic field in the region. Minimisation of the energy functional by a set of approximate functions thus defined yields the required field solution. This process results in a set of nonlinear algebraic equations, which are solved by a rapidly convergent iterative scheme.

The following conclusions are drawn from the foregoing analysis.

(1) The variational formulation of the field problem yields a general nonlinear energy functional which satisfies Dirichlet and homogeneous Neumann boundary conditions. Since the natural boundary conditions are implicit in the functional formulation, the complexity of boundary specifications encountered in divided difference schemes is entirely avoided.

(2) The energy functional is not restrcited in any way since the reciprocal permeability of the medium which causes the non-linearity is not held fixed, although it is assumed single valued by neglecting hysteresis effects. Therefore, the Euler equation is found to be the partial differential equation of the original field problem. As a result, no approximation is made in formulating the variational expression or in its minimisation.

(3) The finite element representation of the field region including current sources, inhomogeneities and nonlinearities leads to far fewer nodes and equations than the finite difference schemes. Further, the triangular elements can be of arbitrary shape, number and unrestricted topology.

(4) The set of nonlinear algebraic equations resulting from setting the first variation of the functional to zero is solved by the generalised Newton-Raphson scheme, which assures nearly quadratic convergence of the iteration process, starting from an initial estimate of the solution obtained by the first order chord method. In this scheme, the convergence of the iterations is not seriously affected by an arbitrary choice of the initial value of the potentials or reluctivities. From the potential solution obtained, the magnetic induction is evaluated and from the B - H characteristic of the medium, the reluctivity is determined. The set of nonlinear equations is thereby modified in each iterative pass. The linear

equations in every iteration are solved by direct Gaussian elimination thus avoiding the slow convergence and instability encountered in iterative solution methods common to most finite difference schemes.

(5) The correct choice of an under-relaxation factor for modifying the reluctivities or the potentials is not a critical requirement for the iteration algorithm developed in this analysis. Since the B - H characteristic is monotonic, a solution is always assured even starting from an initial potential value of zero.

(6) The use of the periodicity condition in addition to the natural boundary conditions for evaluating the load characteristics of rotating electrical machinery, and the necessary connection matrix derived for the purpose have permitted the field representation over one pole pitch only instead of the entire region, with the result that large practical magnetic field problems can be solved without exceeding the limits of computer memory.

(7) A comparison of the computed values and test results of the steady state characteristics of a transformer, a turbogenerator and a D.C. generator amply demonstrates the efficacy of the finite element method and its practical applicability to the determination of field distribution in electric machines in the presence of magnetic saturation.

(8) The programs developed to date yield results which are accurate within limits of experimental errors and are computationally cheaper than any of the finite difference schemes of nonlinear field analysis.

The Contributions of this Thesis can be summarised as follows :

(a) The derivation of a general unrestricted nonlinear energy functional by variational methods for the two dimensional field problem in electric machines and proof of its minimality at the solution point.

(b) The application of the first order finite elements for the discrete representation of electric machines including complex geometrical shapes of the different regions, current sources and widely differing material characteristics.

(c) The problem is formulated in such a way as to permit the application of the generalised Newton-Raphson method to first order triangular finite elements.

(d) Derivation of a suitable connection matrix for the periodicity condition used in the evaluation of load performance of rotating electrical machinery.

(e) Prediction of the flux distribution and steady state performance characteristics of a transformer, turbogenerator and a direct current generator by computation, and proof of their accuracy by comparison with test results.

(f) Development of a general computer algorithm for electric machine application capable of yielding results of the same accuracy as earlier numerical methods, but at a much reduced cost.

APPENDIX I

SURFACE INTEGRALS IN AREA CO-ORDINATES

We can evaluate surface integrals conveniently in terms of area co-ordinates (Reference [3.7, pp. 99 - 100]), which are presented here for completeness.

Consider the triangle ABC divided up into three distinct areas as shown in Figure A.1.1

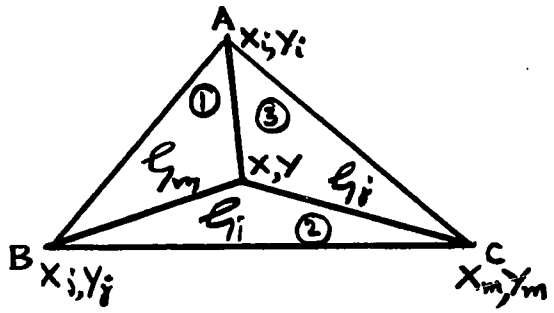


FIGURE A.1.1. SUBDIVISIONS OF TRIANGLE.

$$\text{Area of Triangle (1)} = \frac{1}{2} \cdot \begin{vmatrix} 1 & x_i & y_i \\ 1 & x_j & y_j \\ 1 & x & y \end{vmatrix} = \Delta_1$$

$$\text{Area of Triangle (2)} = \frac{1}{2} \cdot \begin{vmatrix} 1 & x & y \\ 1 & x_j & y_j \\ 1 & x_m & y_m \end{vmatrix} = \Delta_2$$

$$\text{Area of Triangle (3)} = \frac{1}{2} \cdot \begin{vmatrix} 1 & x_i & y_i \\ 1 & x & y \\ 1 & x_m & y_m \end{vmatrix} = \Delta_3$$

$$\text{so that } \Delta = (\Delta_1 + \Delta_2 + \Delta_3) = \frac{1}{2} \cdot \begin{vmatrix} 1 & x_i & y_i \\ 1 & x_j & y_j \\ 1 & x_m & y_m \end{vmatrix}$$

We shall now define certain ratios called 'area co-ordinates', ζ_i , ζ_j , ζ_m as

$$\begin{aligned} \zeta_i &= \frac{\Delta_2}{\Delta} \\ \zeta_j &= \frac{\Delta_3}{\Delta} \\ \zeta_m &= \frac{\Delta_1}{\Delta} \end{aligned} \qquad \text{(A.1.1)}$$

Expanding the second of the Equations (A.1.1) and re-arranging one obtains

$$\zeta_i = \frac{\Delta_2}{\Delta} = \frac{(x_j y_m - y_j x_m) + (y_j - y_i) x + (x_m - x_j) y}{2 \Delta} \qquad \text{(A,1.2)}$$

Reducing the above equation in terms of the geometrical constants

$$\begin{aligned} a_i &= x_j y_m - x_m y_j \\ b_i &= y_j - y_m \\ c_i &= x_m - x_j \end{aligned}$$

and so on, one obtains

$$\xi_i = \frac{(a_i + b_i x + c_i y)}{2\Delta}$$

$$\xi_j = \frac{(a_j + b_j x + c_j y)}{2\Delta} \qquad \text{(A.1.3)}$$

$$\xi_m = \frac{(a_m + b_m x + c_m y)}{2\Delta}$$

The magnitude of these area co-ordinates varies from zero at the sides to a maximum of unity at the opposite triangle vertices along the altitudes, and they satisfy the relation

$$\xi_i + \xi_j + \xi_m = 1 \qquad \text{(A.1.4)}$$

It is, therefore, only necessary to consider two of the area co-ordinates for any given problem. We shall now evaluate the surface integral of Equation (3.19), Chapter III, Section 3.2.2, given by

$$\int_R \int \frac{1}{2\Delta} \sum_{i=1}^{n} J \alpha q \, ds = \frac{1}{2\Delta} \int_R \int J (a_i + b_i x + c_i y) \, ds \qquad \text{(A.1.5)}$$

Consider an area element defined by the small parallelogram shown shaded in Figure A.1.2.

The parallelogram elementary area is given by

$$ds = a_2 \, d\xi_2 \, dl \qquad \text{(A.1.6)}$$

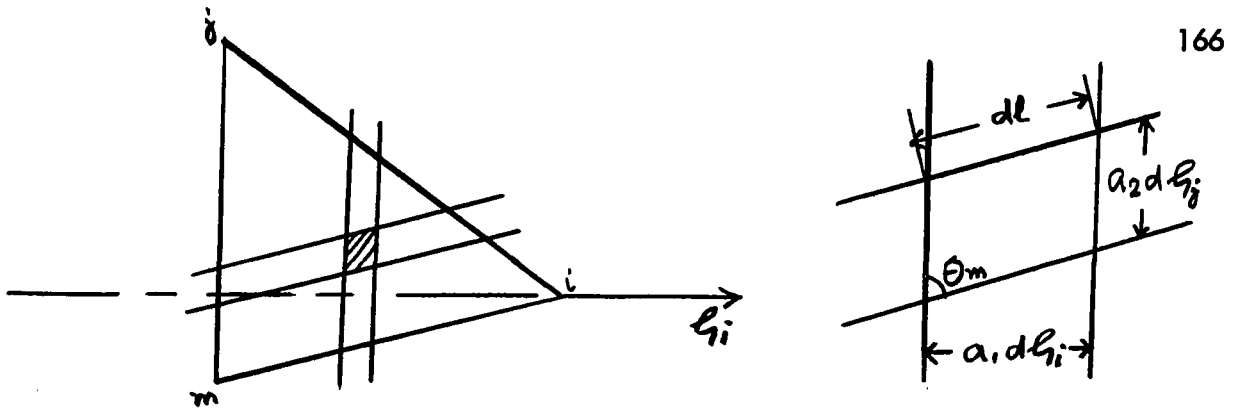


FIGURE A.1.2. ELEMENTARY AREA PARALLELOGRAM.

From trigonometry, it is evident that

$$\sin \varphi_m = \frac{a_1 \, d \, \xi_i}{d \, l} \, . \tag{A.1.7}$$

so that the area element is expressed as

$$d s = a_1 \, a_2 \, \text{cosec} \, \varphi_m \, d \, \xi_i \, d \, \xi_j \tag{A.1.8}$$

The sides of the elementary area parallelogram run parallel to two of the sides of the triangle and, therefore, it is similar to the parallelogram constructed by the corresponding sides of the entire triangle. Also if we bisect the elementary area, it would yield a small triangle similar to the large one. Thus we can derive the area of the large triangle as

$$\Delta = \frac{1}{2} \, a_1 \, a_2 \, \text{cosec} \, \varphi_m \tag{A.1.9}$$

since $d \, \xi_i$ and $d \, \xi_j$ attain a value of unity.

The area of the elementary parallelogram can now be written as

$$d s = 2 \Delta \, d \zeta_i \, d \zeta_j \qquad \text{(A.1.10)}$$

Substituting for $d s$ from Equation (A.1.10) in Equation (A.1.5) and using the value of ζ_i from Equation (A.1.3), the surface integral of Equation (A.1.5) becomes

$$\frac{1}{2\Delta} \iint_P J (a_i + b_i x + c_i y) \, d s = 2 \Delta \int_0^1 \int_0^{1 - \zeta_j} J \zeta_i \, d \zeta_i \, d \zeta_j \qquad \text{(A.1.11)}$$

where the limits of integration range from zero to a maximum value of $(1 - \zeta_j)$ or unity as shown.

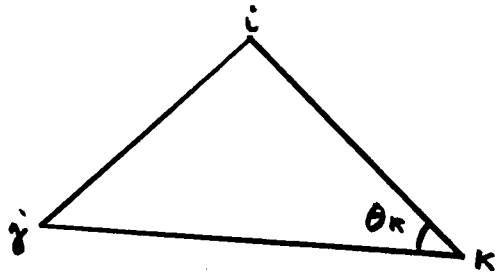
After performing the double integration in the usual manner, the surface integral reduces to the value

$$\frac{1}{2\Delta} \iint_R J (a_i + b_i x + c_i y) \, d s = \frac{J \Delta}{3} \qquad \text{(A.1.12)}$$

APPENDIX II

COTANGENT IDENTITY FOR TRIANGULAR FINITE ELEMENTS

Let us consider the triangle of Figure A.2·1, whose area is denoted by Δ and the vertices i, j, k .



FIG· A·2·1

It is evident that

$$\theta_k = \tan^{-1} \frac{\bar{y}_j}{\bar{x}_j} - \tan^{-1} \frac{\bar{y}_i}{\bar{x}_i} \qquad (A.2.1)$$

where

$$\bar{y}_i = (y_i - y_k) \ , \quad \bar{x}_i = (x_i - x_k) \qquad (A.2.2)$$

Hence

$$\cot \theta_k = \cot \left(\tan^{-1} \frac{\bar{y}_j}{\bar{x}_j} - \tan^{-1} \frac{\bar{y}_i}{\bar{x}_i}\right)$$

$$= \frac{\cot \tan^{-1} \frac{\bar{y}_j}{\bar{x}_j} \cot \tan^{-1} \frac{\bar{y}_i}{\bar{x}_i} + 1}{\cot \tan^{-1} \frac{\bar{y}_j}{\bar{x}_j} - \cot \tan^{-1} \frac{\bar{y}_i}{\bar{x}_i}} \qquad (A.2.3)$$

But $\cot \tan^{-1} \psi = \frac{1}{\psi}$, so that

$$\cot \theta_k = \frac{\frac{x_j}{\bar{y}_j} \cdot \frac{x_i}{\bar{y}_i} + 1}{\frac{\bar{x}_j}{\bar{y}_j} - \frac{\bar{x}_i}{\bar{y}_i}} \qquad \text{(A.2.4)}$$

$$= \frac{\bar{x}_i \bar{x}_j + \bar{y}_i \bar{y}_j}{\bar{x}_j \bar{y}_i - \bar{x}_i \bar{y}_j} \qquad \text{(A.2.5)}$$

which finally becomes after substituting (A.2.2) in Equation (A.2.5) .

$$\cot \theta_k = \frac{(x_j - x_k)(x_i - x_k) + (y_j - y_k)(y_i - y_k)}{-2\Delta} \qquad \text{(A.2.6)}$$

We shall now consider the factor $(\bar{b}_i \bar{b}_j + \bar{c}_i \bar{c}_j)$ which can be expressed as

$$(\bar{b}_i \bar{b}_j + \bar{c}_i \bar{c}_j) = (y_i - y_k)(y_j - y_k) + (x_k - x_i)(x_k - x_j) \qquad \text{(A.2.7)}$$

since $\bar{b}_i = y_j - y_k$

and so on

$\bar{c}_i = x_k - x_j$

From Equations (A.2.6) and (A.2.7) one obtains

$$(\bar{b}_i \bar{b}_j + \bar{c}_i \bar{c}_j) = -2\Delta \cot \theta_k , \quad i \neq j \qquad \text{(A.2.8)}$$

$$= l_{jk}^2 , \quad j = i \quad (\text{i.e. } \bar{b}_i^2 + \bar{c}_i^2 = l_{jk}^2) \qquad \text{(A.2.9)}$$

Here, as previously, l_{jk} denotes the lengths of the side spanning nodes $j - k$. The following further relation can be obtained easily

$$l_{jk}^2 = \frac{2\Delta \cdot l_{jk}^2}{2\Delta} = 2\Delta \cdot \frac{l_{jk}^2}{h_i \, l_{jk}}$$

$$= 2\Delta \cdot \frac{l_{jk}}{h_i}$$

$$= 2\Delta (\cot \theta_j + \cot \theta_k) \qquad \text{(A.2.10)}$$

where h_i is the triangle altitude through vertex i. Thus we finally have

$$\bar{b}_i \bar{b}_j + \bar{c}_i \bar{c}_j = -2\Delta \cot \theta_k \,, \qquad i \neq j$$

$$\bar{b}_i^2 + \bar{c}_i^2 = 2\Delta (\cot \theta_j + \cot \theta_k) \,. \qquad \text{(A.2.11)}$$

APPENDIX III

EVALUATION OF THE CHANGES TO THE DIFFERENTIAL OF THE FUNCTIONAL DUE TO A LOADED BOUNDARY

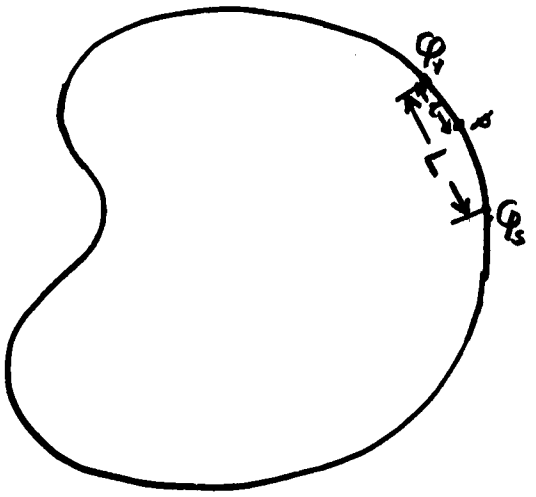


FIGURE A.3.1. ELEMENT OF A LOADED BOUNDARY.

At a point p as shown in Figure A.3.1, the vector potential is assumed to be a linear interpolate of φ_r and φ_s so that

$$\varphi_p = \varphi_s + \frac{(\varphi_r - \varphi_s)}{L} \cdot l \qquad \text{(A.3.1)}$$

Hence

$$\int_c q \, \varphi \, ds = \int_0^L q \left[\varphi_s + (\varphi_r - \varphi_s) \cdot \frac{l}{L} \right] dl$$

$$= \left[\varphi_s \left(l - \frac{l^2}{2L} \right) + \frac{\varphi_r l^2}{2L} \right]_0^L \qquad \text{(A.3.2)}$$

We can, therefore, evaluate the derivative as

$$\frac{\partial}{\partial \varphi_r}\left[\int q\, \varphi\, ds \right] = \frac{qL}{2} \tag{A.3.3}$$

Similarly $$\int \frac{1}{2}\, \alpha\, \varphi^2\, ds = \int_0^L \frac{1}{2}\, \alpha \left[\varphi_s + \overline{\varphi_r - \varphi_s}\, \frac{l}{L} \right]^2 dl \tag{A.3.4}$$

which after some algebra

$$= \frac{\alpha}{2}\left[\varphi_s^2\, L + \varphi_s\, L\, (\varphi_r - \varphi_s) + (\varphi_r - \varphi_s)^2\, \frac{L}{3} \right]$$

The differential of Equation (A.3.4) is then obtained as

$$\frac{\partial}{\partial \varphi_r}\left[\int \frac{1}{2}\, \alpha\, \varphi^2\, ds \right] = \frac{\alpha L}{3}\left[\varphi_r - \frac{\varphi_s}{2} \right] \tag{A.3.5}$$

APPENDIX IV

THE NEWTON-RAPHSON FORMULATION FOR FIRST ORDER TRIANGULAR FINITE ELEMENTS

Let us define functions f, g, h such that

$$f = \frac{\nu}{4\Delta}\left[B_{ii}\,\varphi_i + B_{ij}\,\varphi_j + B_{im}\,\varphi_m\right] - \frac{J\Delta}{3}$$

$$g = \frac{\nu}{4\Delta}\left[B_{ji}\,\varphi_i + B_{jj}\,\varphi_j + B_{jm}\,\varphi_m\right] - \frac{J\Delta}{3} \qquad \text{(A.4.1)}$$

$$h = \frac{\nu}{4\Delta}\left[B_{mi}\,\varphi_i + B_{mj}\,\varphi_j + B_{mm}\,\varphi_m\right] - \frac{J\Delta}{3}$$

The partial derivatives of f, g, h are given by

$$f_i = \frac{\nu B_{ii}}{4\Delta} + \frac{B_{ii}\,\varphi_i}{4\Delta}\cdot\frac{\partial\nu}{\partial\varphi_i} + \frac{B_{ij}\,\varphi_j}{4\Delta}\cdot\frac{\partial\nu}{\partial\varphi_i} + \frac{B_{im}\,\varphi_m}{4\Delta}\cdot\frac{\partial\nu}{\partial\varphi_i}$$

$$f_j = \frac{B_{ji}\,\varphi_i}{4\Delta}\cdot\frac{\partial\nu}{\partial\varphi_j} + \frac{\nu B_{ij}}{4\Delta} + \frac{B_{jj}\,\varphi_j}{4\Delta}\cdot\frac{\partial\nu}{\partial\varphi_j} + \frac{B_{jm}\,\varphi_m}{4\Delta}\cdot\frac{\partial\nu}{\partial\varphi_j}$$

$$f_m = \frac{B_{mi}\,\varphi_i}{4\Delta}\frac{\partial\nu}{\partial\varphi_m} + \frac{B_{mj}}{4\Delta}\varphi_j\frac{\partial\nu}{\partial\varphi_m} + \frac{\nu B_{jm}}{4\Delta} + \frac{B_{jm}\,\varphi_m}{4\Delta}\cdot\frac{\partial\nu}{\partial\varphi_m}$$

$$g_i = \frac{\nu B_{ji}}{4\Delta} + \frac{B_{ji}\,\varphi_i}{4\Delta}\cdot\frac{\partial\nu}{\partial\varphi_i} + \frac{B_{jj}\,\varphi_j}{4\Delta}\cdot\frac{\partial\nu}{\partial\varphi_i} + \frac{B_{jm}\,\varphi_m}{4\Delta}\cdot\frac{\partial\nu}{\partial\varphi_i} \qquad \text{(A.4.2)}$$

$$g_i = \frac{B_{ii}\,\varphi_i}{4\Delta}\,\frac{\partial \nu}{\partial \varphi_i} + \frac{\nu\, B_{ii}}{4\Delta} + \frac{B_{ij}\,\varphi_j}{4\Delta} \cdot \frac{\partial \nu}{\partial \varphi_i} + \frac{B_{im}\,\varphi_m}{4\Delta} \cdot \frac{\partial \nu}{\partial \varphi_i}$$

$$g_m = \frac{B_{ji}\,\varphi_i}{4\Delta} \cdot \frac{\partial \nu}{\partial \varphi_m} + \frac{B_{jj}\,\varphi_j}{4\Delta} \cdot \frac{\partial \nu}{\partial \varphi_m} + \frac{\nu\, B_{jm}}{4\Delta} + \frac{B_{jm}\,\varphi_m}{4\Delta} \cdot \frac{\partial \nu}{\partial \varphi_m}$$

$$h_i = \frac{\nu\, B_{mi}}{4\Delta} + \frac{B_{mi}\,\varphi_i}{4\Delta}\,\frac{\partial \nu}{\partial \varphi_i} + \frac{B_{mj}\,\varphi_j}{4\Delta} \cdot \frac{\partial \nu}{\partial \varphi_i} + \frac{B_{mm}}{4\Delta}\,\varphi_m \cdot \frac{\partial \nu}{\partial \varphi_i}$$

$$h_j = \frac{B_{mi}\,\varphi_i}{4\Delta} \cdot \frac{\partial \nu}{\partial \varphi_j} + \frac{\nu\, B_{mj}}{4\Delta} + \frac{B_{mj}\,\varphi_j}{4\Delta} \cdot \frac{\partial \nu}{\partial \varphi_j} + \frac{B_{mm}}{4\Delta}\,\varphi_m \cdot \frac{\partial \nu}{\partial \varphi_j}$$

$$h_m = \frac{B_{mi}\,\varphi_i}{4\Delta} \cdot \frac{\partial \nu}{\partial \varphi_m} + \frac{B_{mj}\,\varphi_j}{4\Delta} \cdot \frac{\partial \nu}{\partial \varphi_m} + \frac{\nu\, B_{mm}}{4\Delta} + \frac{B_{mm}\,\varphi_m}{4\Delta} \cdot \frac{\partial \nu}{\partial \varphi_m}$$

Hence $f_i\,\Delta\varphi_i + f_j\,\Delta\varphi_j + f_m\,\Delta\varphi_m = -f\,(\varphi_i, \varphi_j, \varphi_m)$ can be evaluated as

$$= \left[\frac{\nu\, B_{ii}}{4\Delta} \cdot \Delta\varphi_i + \frac{\nu\, B_{ij}}{4\Delta} \cdot \Delta\varphi_j + \frac{\nu\, B_{im}}{4\Delta} \cdot \Delta\varphi_m\right]$$

$$+ \frac{1}{4\Delta}\left[B_{ii}\,\varphi_i + B_{ij}\,\varphi_j + B_{im}\,\varphi_m\right] \cdot \left[\frac{\partial \nu}{\partial \varphi_i} \cdot \Delta\varphi_i\right.$$

$$\left. + \frac{\partial \nu}{\partial \varphi_j} \cdot \Delta\varphi_j + \frac{\partial \nu}{\partial \varphi_m} \cdot \Delta\varphi_m\right] \qquad \text{(A.4.3)}$$

The above equation can be recast in the form

$$\sum f_i \Delta\varphi_i = \frac{\nu}{4\Delta}\left[B_{ii},\ B_{ij},\ B_{im}\right] \cdot \begin{bmatrix} \Delta\varphi_i \\ \Delta\varphi_j \\ \Delta\varphi_m \end{bmatrix}$$

$$+ \frac{1}{4\Delta} \cdot \left[\sum B_{ii}\,\varphi_i \frac{\partial\nu}{\partial\varphi_i},\ \sum B_{ii}\,\varphi_i \frac{\partial\nu}{\partial\varphi_j},\ \sum B_{ii}\,\varphi_i \frac{\partial\nu}{\partial\varphi_m}\right] \cdot \begin{bmatrix} \Delta\varphi_i \\ \Delta\varphi_j \\ \Delta\varphi_m \end{bmatrix}$$

$$= -\frac{\nu}{4\Delta}\left[B_{ii},\ B_{ij},\ B_{im}\right] \begin{bmatrix} \varphi_i \\ \varphi_j \\ \varphi_m \end{bmatrix} + \frac{J\Delta}{3} \qquad \text{(A.4.4)}$$

Similarly

$$\sum g_i \Delta\varphi_i = \frac{\nu}{4\Delta}\left[B_{ji},\ B_{jj},\ B_{jm}\right] \begin{bmatrix} \Delta\varphi_i \\ \Delta\varphi_j \\ \Delta\varphi_m \end{bmatrix} +$$

$$\frac{1}{4\Delta} \cdot \left[\sum B_{ji}\,\varphi_i \cdot \frac{\partial\nu}{\partial\varphi_i},\ \sum B_{ji}\,\varphi_i \cdot \frac{\partial\nu}{\partial\varphi_j},\ \sum B_{ji}\,\varphi_i \frac{\partial\nu}{\partial\varphi_m}\right] \cdot \begin{bmatrix} \Delta\varphi_i \\ \Delta\varphi_j \\ \Delta\varphi_m \end{bmatrix}$$

$$= - \frac{\nu}{4\Delta} [\, B_{ii} \,,\; B_{ij} \,,\; B_{im} \,] \cdot \begin{bmatrix} \varphi_i \\ \varphi_j \\ \varphi_m \end{bmatrix} + \frac{J\,\Delta}{3} \qquad \text{(A.4.5)}$$

$$\sum h_i \, \Delta \varphi_i = \frac{\nu}{4\Delta} [\, B_{mi} \,,\; B_{mj} \,,\; B_{mm} \,] \cdot \begin{bmatrix} \Delta \varphi_i \\ \Delta \varphi_j \\ \Delta \varphi_m \end{bmatrix}$$

$$+ \frac{1}{4\Delta} \cdot \left[\sum B_{mi} \, \varphi_i \, \frac{\partial \nu}{\partial \varphi_i} \,,\; \sum B_{mi} \, \varphi_i \, \frac{\partial \nu}{\partial \varphi_j} \,,\; \sum B_{mi} \, \varphi_i \, \frac{\partial \nu}{\partial \varphi_m} \right]$$

$$= - \frac{\nu}{4\Delta} [\, B_{mi} \,,\; B_{mj} \,,\; B_{mm} \,] \cdot \begin{bmatrix} \varphi_i \\ \varphi_j \\ \varphi_m \end{bmatrix} + \frac{J\,\Delta}{3} \qquad \text{(A.4.6)}$$

Adding the like elements of the above matrix equations one obtains the complete matrix equation (4.64) of Chapter IV.

We shall now evaluate the Second Set of matrices on the L.H.S. of Equation (4.64) of Chapter IV.

$$\sum B_{ii} \, \varphi_i \cdot \frac{\partial \nu}{\partial \varphi_i} = \sum B_{ii} \, \varphi_i \cdot \frac{\partial \nu}{\partial B} \cdot \frac{\partial B}{\partial \varphi_i} \qquad \text{(A.4.7)}$$

Substituting for B in terms of the derivatives of φ and after some algebra

$$\frac{\partial B}{\partial \varphi_i} = \frac{1}{2\Delta} \cdot \frac{\sum B_{ii} \, \varphi_i}{\sqrt{\left(\sum b_i \, \varphi_i \right)^2 + \left(\sum c_i \, \varphi_i \right)^2}} \qquad \text{(A.4.8)}$$

Similarly

$$\frac{\partial B}{\partial \varphi_j} = \frac{1}{2\Delta} \cdot \frac{\sum B_{ij}\ \varphi_i}{\sqrt{(\sum b_i\ \varphi_i)^2 + (\sum c_i\ \varphi_i)^2}} \tag{A.4.9}$$

$$\frac{\partial B}{\partial \varphi_m} = \frac{1}{2\Delta} \cdot \frac{\sum B_{im}\ \varphi_i}{\sqrt{(\sum b_i\ \varphi_i)^2 + (\sum c_i\ \varphi_i)^2}} \tag{A.4.10}$$

Therefore

$$\sum B_{ii}\ \varphi_i \frac{\partial \nu}{\partial \varphi_j} = \frac{1}{2\Delta} \cdot \sum B_{ii}\ \varphi_i \frac{\partial \nu}{\partial B} \cdot \frac{\sum B_{ij}\ \varphi_i}{\sqrt{(\sum b_i\ \varphi_i)^2 + (\sum c_i\ \varphi_i)^2}} \tag{A.4.11}$$

$$\sum B_{ii}\ \varphi_i \frac{\partial \nu}{\partial \varphi_i} = \frac{1}{2\Delta} \sum B_{ii}\ \varphi_i \frac{\partial \nu}{\partial B} \cdot \frac{\sum B_{ii}\ \varphi_i}{\sqrt{(\sum b_i\ \varphi_i)^2 + (\sum c_i\ \varphi_i)^2}} \tag{A.4.12}$$

By the above procedure all the elements of the second term of Equation (4.64) can be obtained from which it is seen that

$$\sum B_{ii}\ \varphi_i \frac{\partial \nu}{\partial \varphi_j} = \sum B_{ji}\ \varphi_i \frac{\partial \nu}{\partial \varphi_i}$$

$$\sum B_{ii}\ \varphi_i \frac{\partial \nu}{\partial \varphi_m} = \sum B_{mi}\ \varphi_i \frac{\partial \nu}{\partial \varphi_i} \tag{A.4.13}$$

APPENDIX V

EXPRESSIONS FOR THE NEGATIVE AND ZERO SEQUENCE REACTANCES OF A TURBOGENERATOR

Zero Sequence Reactance

If the three phase windings of a generator are connected in series and a single phase voltage E is impressed across the windings, as in Figure 6.18, while the rotor is unexcited and stationary, then the following relations hold :

$$E_1 = I_1 Z_1 + I_2 Z_2 + I_0 Z_0 \qquad \text{(A.5.1)}$$

$$E_2 = a^2 I_1 Z_1 + a I_2 Z_2 + I_0 Z_0 \qquad \text{(A.5.2)}$$

$$E_3 = a I_1 Z_1 + a^2 I_2 Z_2 + I_0 Z_0 \qquad \text{(A.5.3)}$$

where I_1 , I_2 and I_0 are sequence components of current and Z_1 , Z_2 and Z_0 the sequence reactances ; $a = -0.5 + j\ 0.366$

Adding (A.5.1), (A.5.2) and (A.5.3), there is obtained

$$E_1 + E_2 + E_3 = E = 3\ I_0\ Z_0 \qquad \text{(A.5.4)}$$

Also

$$I_R = I_1 + I_2 + I_0$$

$$I_Y = a^2 I_1 + a I_2 + I_0 \qquad \text{(A.5.5)}$$

$$I_B = a I_1 + a^2 I_2 + I_0$$

Since $I_R = I_Y = I_B$, there results from (A.5.5)

$$I_0 = \frac{I_R + I_Y + I_B}{3} = I \qquad \text{(A.5.6)}$$

Substituting for I_0 from (A.5.6) in (A.5.4) one obtains

$$Z_0 = E / 3.I \qquad \text{(A.5.7)}$$

In order to perform the field analysis for predicting the results of the above test, it is necessary to convert all electrical quantities of the circuit of Figure 6.1E based on a voltage source to that of an equivalent current source. The criterion of equivalence of the above two circuits is that the voltage must be the same in both cases. It is evident, therefore, that

$$E = I' Z / 3 \qquad \text{(A.5.8)}$$

$$= 3 I Z \qquad \text{(A.5.9)}$$

From equations (A.5.8) and (A.5.9), the two currents are related by

$$I = I'/9 \qquad \text{(A.5.10)}$$

If the new current is impressed on the stator in the present field analysis, then the correct flux will be obtained from which the required voltage can be evaluated. Using Equation (A.5.7), one can then compute Z_0.

Negative Sequence Reactance

Using Equations (A.5.1) to (A.5.3) and the additional relations applicable to the circuit of Figure 6.17,

$$I_Y = -I_B \qquad \text{(A.5.11)}$$

$$i_0 = 0 \qquad \text{(A.5.12)}$$

one obtains the expression for the negative sequence reactance, after some algebra, as

$$X_2 = E/\sqrt{3.1} \qquad \text{(A.5.13)}$$

If now the circuit with the voltage source is converted to that of a current source as before for the field problem, it is found that the equivalent current to be impressed

is given by

$$I = 2I'/9 \qquad \text{(A.5.14)}$$

The negative sequence reactance can then be computed by using Equation (A.5.13).

APPENDIX VI

PERIODICITY CONDITION FOR ROTATING ELECTRICAL MACHINERY

1. Invariance of Energy and Formulation of the Connection Matrix for Symmetry Conditions

Let us for example consider a 2 pole DC machine as in Figure A.6.1(a) operating on no-load.

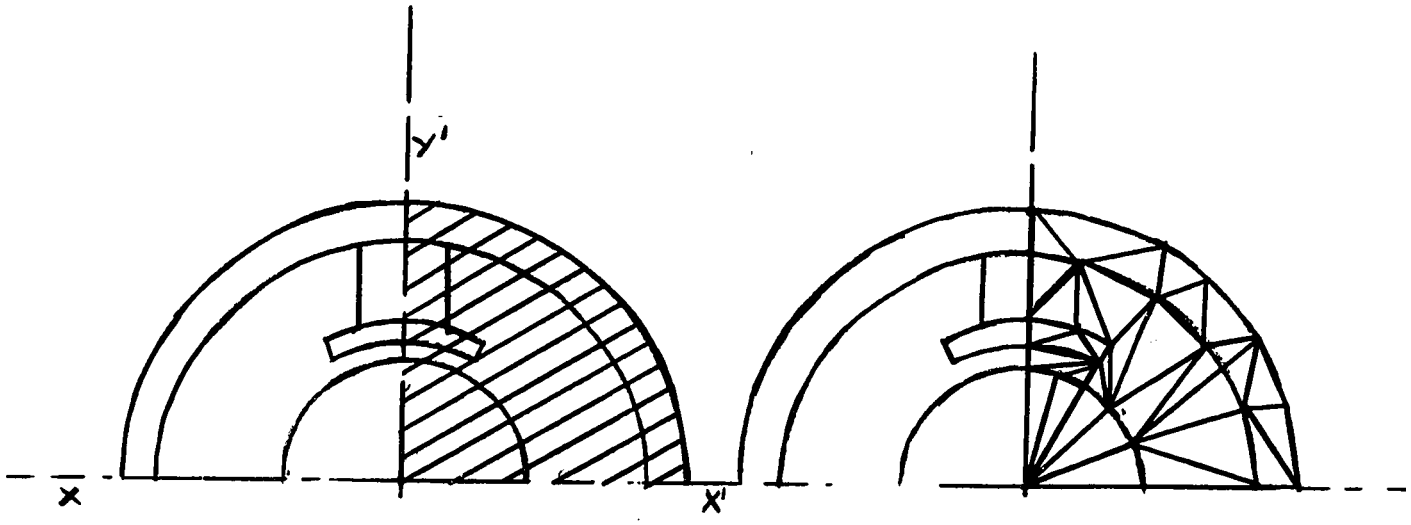


FIGURE A.6.1 (a)

FIGURE A.6.1 (b)

The shaded region alone need be considered owing to the symmetry of the problem, and in view of the fact that no even harmonics are generated by rotating machines. This region is now divided up into a number of triangles as shown in Figure A.6.1(b). For simplicity we shall consider only two triangles as in Figure A.6.2 .

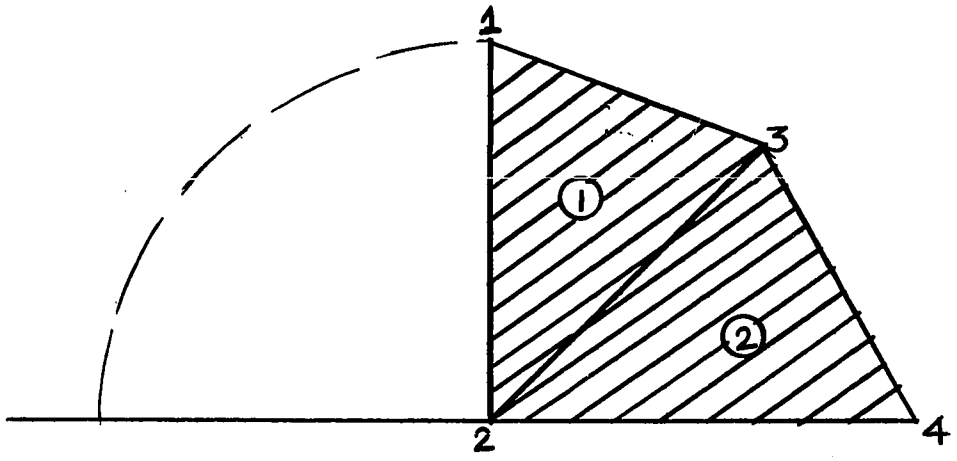


FIGURE A.6.2.

The individual triangular coefficient matrices will be

$$\begin{bmatrix} S_{11} & S_{12} & S_{13} \\ S_{21} & S_{22} & S_{23} \\ S_{31} & S_{32} & S_{33} \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} S'_{22} & S'_{23} & S'_{24} \\ S'_{32} & S'_{33} & S'_{34} \\ S'_{42} & S'_{43} & S'_{44} \end{bmatrix} \qquad \text{(A.6.1)}$$

Also the potential matrices and forcing functions will be respectively

$$\begin{bmatrix} \varphi_1 \\ \varphi_2 \\ \varphi_3 \end{bmatrix} ; \begin{bmatrix} \varphi_2 \\ \varphi_3 \\ \varphi_4 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} J_1 & \Delta/3 \\ J_2 & \Delta/3 \\ J_3 & \Delta/3 \end{bmatrix} ; \begin{bmatrix} J_2 & \Delta/3 \\ J_3 & \Delta/3 \\ J_4 & \Delta/3 \end{bmatrix} \qquad \text{(A.6.2)}$$

It is now required to form the total S matrix and the vectors of potentials and forcing functions. If the total potential vector for the connected triangles is represented by φ' and the sum of the individual potential vectors for the unconnected triangles by φ, then the following relations hold, where C is the connection matrix [6.2, p. 206, Eq. (8)].

$$\varphi = C \cdot \varphi' \tag{A.6.3}$$

or

$$\begin{bmatrix} \varphi_1 \\ \varphi_2 \\ \varphi_3 \\ \varphi_2 \\ \varphi_3 \\ \varphi_4 \end{bmatrix} = \begin{array}{c|cccc} & 1 & 2 & 3 & 4 \\ \hline 1 & 1 & 0 & 0 & 0 \\ 2 & 0 & 1 & 0 & 0 \\ 3 & 0 & 0 & 1 & 0 \\ 2 & 0 & 1 & 0 & 0 \\ 3 & 0 & 0 & 1 & 0 \\ 4 & 0 & 0 & 0 & 1 \end{array} \cdot \begin{bmatrix} \varphi_1 \\ \varphi_2 \\ \varphi_3 \\ \varphi_4 \end{bmatrix} \tag{A.6.4}$$

Note that the connection matrix has entries 1 or 0 depending on whether or not a node is connected to another.

Similarly the forcing functions are also related. If F" and F correspond to the total connected and the sum of the individual unconnected vectors respectively, then

$$F'' = C_T \cdot F \tag{A.6.5}$$

or

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} F_1 \\ F_2 \\ F_3 \\ F'_2 \\ F'_3 \\ F'_4 \end{bmatrix} = \begin{bmatrix} F_1 \\ F_2 + F'_2 \\ F_3 + F'_3 \\ F_4 \end{bmatrix} \qquad \text{(A.6.6)}$$

Now since the energy in the magnetic field or in a conservatice system is invariant with respect to the method of evaluating the energy [6.3, pp. 30 - 34], the sum of the energies calculated one triangle at a time will be the same as the energy of the entire field calculated all at once. Therefore, if we define the sum of the individual energy densities as $J \cdot \varphi$ and the total energy density as $J'' \cdot \varphi'$, we have the following relation,

$$\iiint J \cdot \varphi \; dU = \iiint J'' \cdot \varphi' \; dU,$$ so that the integrands are equal. Therefore

$$J \cdot \varphi = J'' \cdot \varphi' \qquad \text{(A.6.7)}$$

or

$$F \cdot \varphi = F'' \cdot \varphi' \qquad \text{(A.6.8)}$$

The left hand side of Equation (A.6.6) can be rewritten as

$$F \cdot \varphi = F \cdot C \cdot \varphi' = F'' \cdot \varphi'$$

or $F'' = C_T \cdot F$ (as already stated in (A.6.5))

Further $F = S \cdot \varphi$ (A.6.9)

$F'' = S'' \cdot \varphi'$ (A.6.10)

Substituting for φ from Equation (A.6.3) in Equation (A.6.8), there is

$$S \cdot \varphi = S \cdot C \cdot \varphi' = F \quad \text{(A.6.11)}$$

From Equation (A.6.5) we have

$$C_T \cdot F = F'' = [C_T \cdot S \cdot C.] \cdot \varphi' = S'' \cdot \varphi' \quad \text{(A.6.12)}$$

Here S is the block diagonal matrix

S_{11}	S_{12}	S_{13}			
S_{21}	S_{22}	S_{23}			
S_{31}	S_{32}	S_{33}			
			S'_{22}	S'_{23}	S'_{24}
			S'_{32}	S'_{33}	S'_{34}
			S'_{42}	S'_{43}	S'_{44}

2. Load Analysis and Periodicity Condition

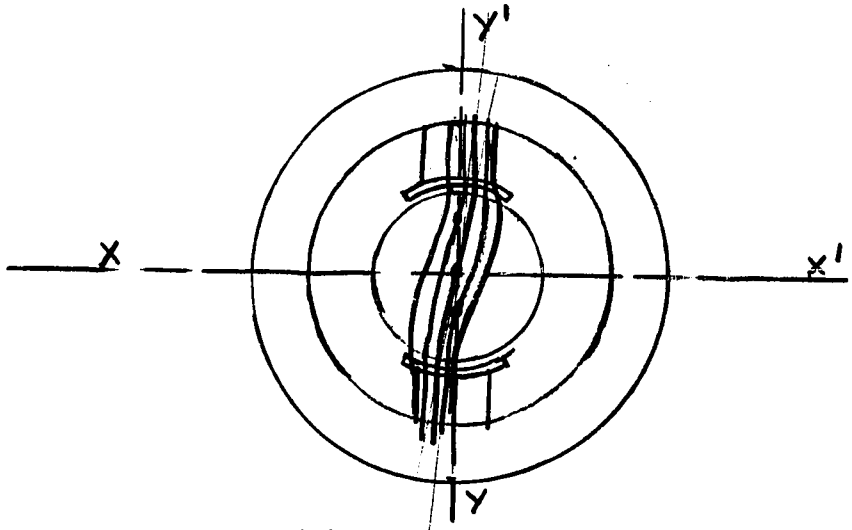


FIGURE A.6.3. LOADED DC MACHINE.

For the on load condition of the DC generator, the field pattern will be as shown in Figure A.6.3. It is seen that the pole axis is no longer the axis of symmetry and cannot serve as a flux line boundary. Therefore the periodicity condition will have to be used in solving the problem.

If we now consider the 2 triangle problem in the illustration as in Figure (A.6.2) and assume that the nodes 1 and 4 lie on either side of the origin along yy', i.e., they are 180^{o} (electrical degrees) away, the potential vector can be written as

$$\phi' = \begin{bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{bmatrix} \qquad \text{(A.6.14)}$$

which is obtained from the vector of the primitives based on the relation

$$\varphi = C \cdot \varphi' \quad \text{or}$$

$$\begin{bmatrix} \varphi_1 \\ \varphi_2 \\ \varphi_3 \\ \varphi_2 \\ \varphi_3 \\ -\varphi_1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} \varphi_1 \\ \varphi_2 \\ \varphi_3 \end{bmatrix} \qquad \text{(A.6.15)}$$

It may be noted that in Equation (A.6.15) above, the connection matrix C has been altered so that the problem now reduces to one of determining only φ_1, φ_2, φ_3 which are the three independent unknown variables.

Using the relation of Equation (A.6.15) and the new connection matrix, we have

$$S'' = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} S_{11} & S_{12} & S_{13} & & & \\ S_{21} & S_{22} & S_{23} & & & \\ S_{31} & S_{32} & S_{33} & & & \\ & & & S'_{22} & S'_{23} & S'_{24} \\ & & & S'_{32} & S'_{33} & S'_{34} \\ & & & S'_{42} & S'_{43} & S'_{44} \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & 0 & 0 \end{bmatrix} \qquad \text{(A.6.16)}$$

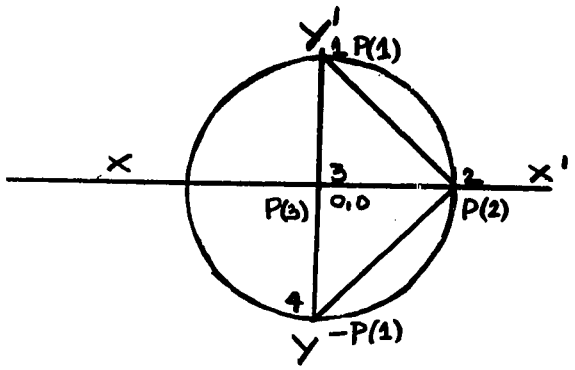


FIGURE A.6.4.

which reduces to

$$S'' = \begin{array}{|c|c|c|c|c|c|} \hline 1 & 0 & 0 & 0 & 0 & -1 \\ \hline 0 & 1 & 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 1 & 0 & 1 & 0 \\ \hline \end{array} \cdot \begin{array}{|c|c|c|} \hline S_{11} & S_{12} & S_{13} \\ \hline S_{21} & S_{22} & S_{23} \\ \hline S_{31} & S_{32} & S_{33} \\ \hline -S'_{24} & S'_{22} & S'_{23} \\ \hline -S'_{34} & S'_{32} & S'_{33} \\ \hline -S'_{44} & S'_{42} & S'_{43} \\ \hline \end{array} \qquad \text{(A.6.1}$$

$$= \begin{array}{|c|c|c|} \hline S_{11} + S'_{44} & S_{12} - S'_{42} & S_{13} - S'_{43} \\ \hline S_{21} - S'_{24} & S_{22} + S'_{22} & S_{23} + S'_{23} \\ \hline S_{31} - S'_{34} & S_{32} + S'_{32} & S_{33} + S'_{33} \\ \hline \end{array} \qquad \text{(A.6.18}$$

which can be split up as

$$S'' = \begin{bmatrix} S_{11} & S_{12} & S_{13} \\ S_{21} & S_{22} & S_{23} \\ S_{31} & S_{32} & S_{33} \end{bmatrix} + \begin{bmatrix} S'_{44} & -S'_{42} & -S'_{43} \\ -S'_{24} & S'_{22} & S'_{23} \\ -S'_{34} & S'_{32} & S'_{33} \end{bmatrix} \qquad \text{(A.6.19)}$$

It is apparent that the sub-matrix for triangle 2 is modified such that the terms of the rows and columns involving the point for which the periodicity condition applies, are changed in sign excepting the diagonal term.

The forcing function has to be likewise modified so that

$$\begin{bmatrix} F''_1 \\ F''_2 \\ F''_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} F_1 \\ F_2 \\ F_3 \\ F'_2 \\ F'_3 \\ F'_4 \end{bmatrix} = \begin{bmatrix} F_1 - F'_4 \\ F_2 + F'_2 \\ F_3 + F'_3 \end{bmatrix} \qquad \text{(A.6.20)}$$

REFERENCES

[6.1] L.H.A. Carr, Testing of Electrical Machines, Macdonald, London, 1960, pp. 169 - 173.

[6.2] P. Silvester, "A general High-Order Finite Element Wave Guide Analysis program", IEEE Trans. MTT - Vol. 17, pp. 204 - 210, April 1969.

[6.3] L.V. Bewley, Tensor Analysis of Electric Circuits and Machines, NewYork, Ronald Press, 1961, Chapter 3, pp. 30 - 34.