ABSTRACT

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This thesis deals primarily with matrix norms. The preliminary general treatment is largely confined to non-Archimedian norms, i.e., norms which satisfy the strong triangle inequality. It includes the establishment of a correspondence between a class of such norms and certain submodules of a vector space over a non-Archimedian field as well as a discussion of the properties of bounds (special norms on spaces of linear transformations) and of unit spheres in normed spaces and algebras. Finite dimensional vector spaces over an arbitrary valuated field are considered next. Duality is discussed and a new necessary and sufficient condition for a matrix norm to be a bound established as well as a relation between non-Archimedian matrix norms and the "natural" bound. There follow results concerning "$\mathcal{S}$-unitary" matrix groups, spectral radii, convergence, and methods of successive approximation over the field of p-adic numbers.

NON-ARCHIMEDIAN NORMS AND BOUNDS

by

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§ 0. Introduction

Non-Archimedian normed spaces have been studied by I.S. Cohen, I. Fleischer, A.F. Monna, J.-P. Serre, T.A. Springer and others. The concept of K-convexity which leads to the investigation of R-submodules of a vector space over a non-Archimedian valuated field is due to A.F. Monna ([18]). This concept is employed by J. van Tiel for the study of certain topological spaces in a dissertation which also contains a summary of previous work ([29]).

The study of matrix norms has a fairly long history but non-Archimedian matrix norms do not appear to have been considered. The association of norms with convex bodies has been systematically exploited by A.S. Householder (see [9]). Ju.I. Ljubič in [13] established a necessary and sufficient condition for a matrix norm to be a bound. The first systematic treatment of the use of norms in convergence proofs of numerical analysis was given by V.N. Fadeeva in [7].

Chapter I of the present thesis contains a general treatment of norms, mainly non-Archimedian. In §1 we prove the existence of a cannonical pseudonorm on a quotient module of a pseudonormed module over a valuated ring. In §2 with the help of an algebraic definition of boundedness we establish an association between a class of non-Archimedian norms on a vector space E over a non-Archimedian field and certain R-submodules of E. In §3 we prove that the open unit sphere of a complete non-Archimedian algebra A is an ideal contained in the Jacobson radical of the closed unit sphere of A with resultant effects on convergence and invertibility.

Chapter 2 deals with finite dimensional vector spaces over a valuated field K and particularly with K^m and K_m. In §4 we show that, although

duality in such a space need not be an involutory relation, the dual norm has useful properties. In particular, we obtain a new proof of Ljubič's theorem. We investigate the properties of the natural norm and bound for a non-Archimedian K. We prove an inequality satisfied by non-Archimedian matrix norms and establish a necessary and sufficient condition for such a norm being a natural bound. In § 5 we consider duality for norms determined by R-submodules of K^m and prove a new necessary and sufficient condition for a matrix norm over a valuated field to be a bound. We also investigate the relation between non-Archimedian bounds and spectral radii and use the concept of an $\mathcal{S}$ - unitary matrix to establish the non-singularity of a class of matrices over a non-Archimedian field as well as certain properties of their inverses and spectra. § 6 contains estimates for the rate of convergence of certain iterative processes over a non-Archimedian field and examples of the use of methods of successive approximation for the solution of systems of linear equations and for the inversion of matrices over the field of p-adic numbers.

We shall use the following notation:

Implication will be denoted by $\Rightarrow$. $\emptyset$ will stand for the empty set. $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Z}^+$, $\mathbb{R}$, $\mathbb{R}^+$, and $\mathbb{C}$ will denote the natural numbers, the integers, non-negative integers, the reals, non-negative reals, and the complex numbers respectively. For sets A and B, $A \subset B$ will mean $x \in A \Rightarrow x \in B$.

All rings and algebras will contain the identity usually denoted by 1 and all modules will be unital. If R is a ring $\oplus R = \{(a_i) \mid a_i \in R\}$ will stand for the restricted external direct sum

of a number of copies of R; when this number is countable $\oplus R$ will be the R-module of sequences in R with a finite number of non-zero terms.

K^m and K_m will denote respectively the space of m-dimensional column vectors and the algebra of $m \times m$ matrices over the field K. If $x, y \in K^m$, $A \in K_m$ then

$$x = \begin{pmatrix} \xi_1 \\ \vdots \\ \xi_m \end{pmatrix}, \quad y = \begin{pmatrix} \eta_1 \\ \vdots \\ \eta_m \end{pmatrix} \quad \text{and} \quad A = (\alpha_{ij}) = \begin{pmatrix} \alpha_{11} & \cdots & \alpha_{1m} \\ \vdots & & \vdots \\ \alpha_{m1} & & \alpha_{mm} \end{pmatrix}$$

The transpose of a vector x or a matrix A will be denoted by x' and A' respectively. Thus $x'y$ will be the scalar product of x and y while xy' will be a matrix of rank $\leqslant 1$. I will stand for the identity matrix and e_i for the i'th column of I.

CHAPTER I

Non-Archimedian Normed Spaces

§ 1. Normed and Pseudonormed Modules.

A valuation on a ring D is a function $a \to |a|$ from D to $\mathbb{R}^+$ which satisfies the following conditions:

(1) $|a| = 0 \iff a = 0$,

(2) $|ab| = |a|\,|b|$ for all $a, b \in D$,

(3) $|a + b| \leq |a| + |b|$ for all $a, b \in D$.

A valuation defines a topology on D under which D is a topological ring ([11], p.329). The couple $(D, |\ |)$ is called a valuated ring. Since $\mathbb{R}$ is a field, a valuated ring cannot have any divisors of zero (ibid, p.294).

A valuation is said to be non-Archimedian if, instead of the triangle inequality (3), it satisfies the stronger condition:

(3′) $|a + b| \leq \max(|a|, |b|)$ for all $a, b \in D$.

If D is a non-Archimedian valuated ring, then

$$R = \{a \in D \mid |a| \leq 1\}$$

is a subring of D called the valuation ring of D. The set

$$P = \{a \in D \mid |a| < 1\}$$

is a prime ideal of R called the valuation ideal.

If D is a valuated ring and M a torsion-free D-module, a norm may be defined on D entirely analogous to a norm on a vector space over a valuated field (see [4], p.65). The resultant definition of a module norm must be distinguished from the one used by Monna and Springer (see [22], p.613).

Definition

Let D be a valuated ring and M a torsion-free D-module. A norm on M is a function $x \to \|x\|$ from M to $\mathbb{R}^+$ which satisfies the following conditions:

(1) $\|x\| = 0 \iff x = 0$,

(2) $\|\lambda x\| = |\lambda| \, \|x\|$ for all $\lambda \in D$, $x \in M$,

(3) $\|x + y\| \leq \|x\| + \|y\|$ for all $x, y \in M$.

The couple $(M, \| \ \|)$ is a normed module which is said so be non-Archimedian if $\| \ \|$ satisfies the strong triangle inequality:

(3′) $\|x + y\| \leq \max (\|x\|, \|y\|)$ for all $x, y \in M$.

A valuated ring D may be regarded as a normed D-module. If M is a normed D-module and C is a subring of D then every submodule of M is a normed C-module ([2], p.49) under the norm defined on M.

Since for $\lambda, \mu \in D$, $0 \neq x \in M$, $|\lambda| \leq |\mu| \iff \|\lambda x\| \leq \|\mu x\|$, a non-Archimedian norm on M implies a non-Archimedian valuation on D (cf [15], p.1046), but the converse statement is false. In this connection Monna considered the spaces of sequences (λ_i), $\lambda \in K$, K a non-Archimedian valuated field, for which the series $\sum_{i=1}^{\infty} |\lambda_i|^p$ is convergent (see [16], p.480). We prove a general result:

Theorem 1.1

If D is a valuated ring and M a free D-module, then each monotonic norm on $\oplus \mathbb{R}$ determines a norm on M.

Proof:

Note first that M is torsion-free ([12], p.134).

Establish a partial order on $\oplus \mathbb{R}$ by setting $x \leq y$ if $\xi_i \leq \eta_i$ for all i, where $x = (\xi_i)$, $y = (\eta_i) \in \oplus \mathbb{R}$.

A norm on $\oplus \mathbb{R}$ is monotonic if $|x| \leqslant |y| \Rightarrow \|x\| \leqslant \|y\|$, where $|x| = (|\xi_i|)$, $|\xi_i|$ denoting the ordinary absolute value on $\mathbb{R}$.

If $\{u_i\}$ is a basis for M define a map

$$M \to \oplus \mathbb{R}$$

by $x = \sum \lambda_i u_i \to |x| = (|\lambda_i|)$, $\lambda_i \in D$.

We have $|x| = 0 \iff x = 0$,

$|\lambda x| = |\lambda|\,|x|$ for all $\lambda \in D$, $x \in M$,

$|x + y| \leqslant |x| + |y|$ for all $x, y \in M$.

Define $\|x\| = \| |x| \|$.

Then $\| \ \|$ is a norm on M.

Corollary 1.

If D is a valuated ring, then for $1 \leqslant p \leqslant \infty$

$$\| (\lambda_i) \|_p = \left(\sum |\lambda_i|^p \right)^{\frac{1}{p}}$$

is a norm on $\oplus D$, which may be called a Hölder norm.

Corollary 2.

A Hölder norm on a free D-module whose dimension is greater than 1 is non-Archimedian if and only if the valuation on D is non-Archimedian and $p = \infty$.

Remarks

1. More generally, if M is a free D-module with basis $\{u_i\}$ then $\| \sum \lambda_i u_i \| = \max |\lambda_i| \, \|u_i\|$, where $\{\|u_i\|\}$ is an arbitrary set of positive reals, is a norm on M, which is non-Archimedian if and only if the valuation on D is non-Archimedian.

2. The proof of theorem 1.1 will apply to the subspaces of K^∞ discussed by Monna.

3. If $\oplus \mathbb{R}$ is ordered lexicographically, the function $x \to |x|$ becomes a non-real-valued norm on the free D-module M. This norm will be non-Archimedian if the valuation on D is non-Archimedian.

The requirement that a normed module be torsion-free ceases to be necessary if condition (2) in the definition of a norm is replaced by a suitable weaker condition. In particular, we have the following (cf [22], p.613):

Definition

Let D be a valuated ring and M an arbitrary D-module. A non-Archimedian pseudonorm on D is a function $x \to \|x\|$ from D to $\mathbb{R}^+$ which satisfies the following conditions:

(1) $\|x\| = 0 \iff x = 0$,

(2′) $\|\lambda x\| \leq |\lambda| \, \|x\|$ for all $\lambda \in D$, $x \in M$,

(3′) $\|x + y\| \leq \max(\|x\|, \|y\|)$ for all $x, y \in M$.

We observe that a non-Archimedian pseudonorm on M does not imply a non-Archimedian valuation on D.

A pseudonorm on a vector space E over a field with a non-trivial valuation (see § 2) is Köthe's "F - norm" (see [10], p.167) under which E is a topological vector space (cf [20], p.357). In the general case, a non-Archimedian pseudonorm on M defines a non-Archimedian metric $d(x,y) = \|x - y\|$ under which M is at least a topological group.

Certain properties of non-Archimedian valuated fields and normed vector spaces are the direct consequences of the strong triangle inequality and the resulting non-Archimedian metric. These will hold for non-Archimedian pseudonormed D-modules (see [24], p.76; [20], p.353). In the list below we assume in item 7 that the valuation on D is non-Archimedian.

1. It follows from the strong triangle inequality that $x, y \in M$ with $\|x\| > \|y\| \Rightarrow \|x \pm y\| = \|x\|$ ([24], p.73; [20], p.353).

2. Hence, $x_n \to x \neq 0 \Rightarrow \exists\, n_o \in \mathbb{N}$ such that $\|x_n\| = \|x\|$ for $n > n_o$ ([15], p.1045; [20], p.353).

3. If $\sum_{n=1}^{\infty} x_n$ converges, then $\| \sum_{n=1}^{\infty} x_n \| \leq \max_n \|x_n\|$ ([8], p.165).

4. For every M there exists a completion $\hat{M}$ ([4], p.68) such that $W_M = W_{\hat{M}}$, where $W_M = \{ \|x\| \mid 0 \neq x \in M \}$ and $W_{\hat{M}}$ has a similar meaning for the extension of $\| \ \|$ to $\hat{M}$ ([14], p.81).

5. If M is complete, then the series $\sum_{n=1}^{\infty} x_n$ converges if and only if $\lim_{n \to \infty} x_n = 0$ ([24], p.75).

6. For $a \in M$ and real $\rho > 0$ the sets $\{x \in M \mid \|x - a\| \leq \rho\}$, $\{x \in M \mid \|x - a\| < \rho\}$ are called respectively the <u>closed</u> and <u>open spheres</u> with center a and radius ρ. However, all spheres are both open and closed in the topology determined by the pseudonorm. Further, each point of a sphere is its center and two spheres are either disjoint or one is contained in the other ([24], p.74).

7. If R is the valuation ring of D, a sphere $S \subset M$ is an R-submodule of M if and only if $0 \in S$ (cf [18], p.532).

<u>Remark</u>

The R-submodules of a vector space E over a non-Archimedian valuated field K are precisely those non-empty sets in E which Monna described as having property C and von Tiel calls K-convex (see [29], p.253).

The "K-convex null of a set $S \subset E$" (p.254) is simply the R-submodule of E generated by S. Monna called K-convex sets of the form $x_o + S$, where S has property C and x_o is a fixed vector in E ([18], p.532).

If N is a closed subspace of a vector space E over the real or complex field it is known that an F-norm of E determines an F-norm on the quotient space E/N (see [10], p.167). We prove an analogous result for non-Archimedian pseudonormed modules.

Theorem 1.2

Let D be a valuated ring, M a non-Archimedian pseudonormed D-module, and N a closed submodule of M. Then

$$\| x + N\| = \inf_{n \in N} \|x + n\|$$

defines a non-Archimedian pseudonorm on the quotient module M/N. If M is complete so is M/N.

Proof:

We verify the three properties of a non-Archimedian pseudonorm.

(1) follows from the fact that N is closed (see [27], p.213).

(2′) We note that for all $\lambda \in D$ $\lambda N \subset N$. Hence

$$\| \lambda(x + N) \| = \| \lambda x + N \| = \inf_{n \in N} \| \lambda x + n \| \leqslant \inf_{n \in N} \| \lambda x + \lambda n \| =$$

$$|\lambda| \|x + N\|$$

(3′) Let $\|x + N\| \geqslant \|y + N\|$.

Then $\forall\, n \in N \;\; \exists \;\; n' \in N$ such that $\|x + n\| \geqslant \|y + n'\|$.

Hence, $\|x + n + y + n'\| \leqslant \|x + n\|$ so that

$$\|(x + N) + (y + N)\| = \|x + y + N\| = \inf_{n'' \in N} \|x + y + n''\| =$$

$$\inf_{n \in N} \|x + n + y + n'\| \leqslant \inf_{n \in N} \|x + n\| = \|x + N\|.$$

For proof of the last assertion see [27].

Corollary 1.

If M is a non-Archimedian pseudonormed D-module, R the valuation ring of D, and S a sphere in M, $0 \subset S$, then

$$\|S\| = 0, \quad \|x + S\| = \|x\|, \quad x \notin S,$$

is a non-Archimedian pseudonorm on the R-module M/S.

Corollary 2.

If E is a non-Archimedian normed vector space over a valuated field K and N is a closed subspace of E, then

$$\|x + N\| = \inf_{n \in N} \|x + n\|$$

defines a non-Archimedian norm on the quotient space E/N.

Remarks

1. An Archimedian pseudonorm on M also induces a pseudonorm on M/N, which may be non-Archimedian.

2. Corollary 2 is false for normed modules.

Example

Consider $\mathbb{Z}$ as a normed $\mathbb{Z}$-module under ordinary absolute value or diadic valuation.

$2\mathbb{Z}$ is a closed submodule of $\mathbb{Z}$. In fact, in the latter case $2\mathbb{Z}$ is a sphere in $\mathbb{Z}$ centered at the origin. In either case the induced pseudonorm on $\mathbb{Z}/2\mathbb{Z}$ is given by

$$\| \bar{0} \| = 0, \quad \| \bar{1} \| = 1.$$

$\| \ \|$ is a non-Archimedian valuation on the field $K = \mathbb{Z}/2\mathbb{Z}$ and hence a non-Archimedian norm on the vector space K over K. But $\| \ \|$ is not a norm on the $\mathbb{Z}$-module $\mathbb{Z}/2\mathbb{Z}$, for

$$\|2 \cdot \bar{1}\| = \|\bar{0}\| = 0 < |2| \ \|\bar{1}\| \neq 0.$$

§ 2. Non-Archimedian Norms and R-submodules of a Vector Space.

Let K be a non-Archimedian valuated field. Then K is the field of quotients of its valuation ring $R = \{\alpha \in K \mid |\alpha| \leq 1\}$, while the valuation ideal $P = \{\alpha \in K \mid |\alpha| < 1\}$ is a unique maximal ideal of R. The set

$$U = \{\alpha \in K \mid |\alpha| = 1\}$$

is the group of units of R, usually called the group of units of K, and

$$\bar{K} = R/P$$

is a field, called the residue field of K ([8], p.81).

The set

$$W_K = \{|\alpha| \mid 0 \neq \alpha \in K\}$$

is a multiplicative group, called the value group of K. Moreover, for real $e > 1$ the map $|\alpha| \to -\log_e |\alpha|$ is an isomorphism of W_K onto a subgroup of $(\mathbb{R}, +)$. Accordingly, the valuation on K is called trivial if $W_K = \{1\}$, dense if W_K is everywhere dense in $\mathbb{R}^+$, and discrete in the remaining case, i.e., if W_K is discrete in $\mathbb{R}^+ - \{0\}$ or, equivalently, if 0 is the only limit point of W_K. The valuation on K is discrete if and only if there exists $\pi \in K$ such that $|\pi| = \max_{\alpha \in P} |\alpha|$ ([25], p.9). Then W_K is the infinite cyclic group $\{|\pi|^i \mid i \in \mathbb{Z}\}$. In fact, if $0 \neq \alpha \in K$, then $\alpha = \eta \pi^i$, $\eta \in U$, $i \in \mathbb{Z}$, so that π is a prime element of R ([14], p.107).

The topology of K is discrete if and only if its valuation is trivial ([4], p.62). For a field with a non-trivial valuation to be locally compact it is necessary and sufficient that it be complete with respect to a discrete valuation and have a finite residue field ([29,p.251).

Let E be a non-Archimedian normed vector space over K. The unit sphere in E is the R-module

$$S = \{x \in E \mid \|x\| \leqslant 1\}.$$

$$W_E = \{\|x\| \mid 0 \neq x \in E\}$$

will be called the value set of the norm on E.

It may happen that $W_E \subset W_K$ and even $W_E = W_K$. This is the case, for example, with the norm $\| \ \|_\infty$ on $\oplus K$ defined by

$$\|(\alpha_i)\|_\infty = \max_i |\alpha_i| , \quad (\alpha_i) \in \oplus K.$$

If $W_E \subset W_K$ then $\forall\, x \in E \ \ x = \alpha x_0, \ \alpha \in K, \ \|x_0\| = 1$, so that the norm is completely determined by the set $\{x \in E \mid \|x\| = 1\}$ (cf [15], p.1045).

On the other hand, every non-Archimedian normed space contains a set of non-zero vectors $\{x_i\}$ such that $W_K\|x_i\| \cap W_K\|x_j\| = \emptyset$ for $i \neq j$. The strong triangle inequality gives

$$\| \sum_{i=1}^{n} \xi_i x_i \| = \max_{1 \leqslant i \leqslant n} \| \xi_i x_i \|, \quad \xi_i \in K,$$

so that $\{x_i\}$ is an orthogonal family of vectors (see [22], p.603). It is clear that every orthogonal family is linearly independent. If such a family forms a basis for E, then $W_E = \bigcup_i W_K\|x_i\|$. In particular, if the valuation on K is trivial, then $\|u_i\| \neq \|u_j\|$ for $i \neq j$ on a basis $\{u_i\}$ for E implies that

$$\| \sum_{i=1}^{n} \alpha_i u_i \| = \max_{1 \leqslant i \leqslant n} \|u_i\|, \quad \alpha_i \in K.$$

W_E does not possess properties comparable to the trichotomy for W_K. We shall say that a norm on E is dense or discrete if W_E is respectively dense in $\mathbb{R}^+$ or discrete in $\mathbb{R}^+ - \{0\}$ (Monna and Springer

in [22], p.606, adopt a more general definition of a discrete norm). A dense valuation on K implies a dense norm on E, so that a discrete norm on E implies a discrete valuation on K whenever it is known that the latter is non-trivial ([15], p.1049). The converse statements hold if E is finite dimensional.

When E is finite dimensional, E is complete with respect to its norm if K is complete with respect to its valuation ([1], p.98) but this implication is not true in general. Nor does completeness of E imply completeness of K. However, when E is complete the scalar multiplication in E over K may be extended to scalar multiplication in the normed space E over $\hat{K}$, where $\hat{K}$ is the completion of K ([15], p.1054). If both K and E are complete, then E is known as a non-Archimedian Banach space.

When the valuation on K is non-trivial, the unit sphere in E is compact or, equivalently, E is locally compact if and only if K is locally compact and E is finite dimensional ([15], pp.1048-1053; [5], p.695), implying a complete and discrete valuation on K.

Let E be a vector space over a field K with a non-trivial non-Archimedian valuation. Monna ([18]) investigated the relationship between K-convex subsets of E and non-Archimedian seminorms on E, i.e., real valued functions p such that $p(\lambda x) = |\lambda| p(x)$ and $p(x + y) \leq \max (p(x), p(y))$ for all $x, y \in E$, $\lambda \in K$. For this purpose he chose a definition of a K-convex body in E independent of the topology on E (see [20], p.357). He also proved Kolmogorov's Criterion for non-Archimedian normed spaces (ibid, p.360). Here, however, he adopted

a topological definition of boundedness (p.353). We shall establish a relation between non-Archimedian norms and R-submodules of E using algebraic concepts only.

For the remainder of this section (with the exception of the last two paragraphs) K will denote a field with a non-trivial non-Archimedian valuation and E a vector space over K which may or may not be normed. We shall need the following result of Monna's:

Lemma 2.1

Let S be an R-submodule of E. Then for $\alpha, \beta \in K$ with $|\alpha| \geq |\beta|$ αS is an R-submodule of E and $\alpha S = \alpha S + \beta S$. Conversely, $0 \in S$ and $\alpha S = \alpha S + \beta S$ for $\alpha, \beta \in K$ with $|\alpha| \geq |\beta|$ imply that S is an R-submodule of E.

Proof:

See [20], p. 354.

Definition

A set $S \subset E$ is said to be equilibrated if $\lambda x \in S$ for all $x \in S$, $\lambda \in R$, i.e., if $\lambda \in R \Rightarrow \lambda S \subset S$.

Lemma 2.2

$S \subset E$ is equilibrated if and only if for $\alpha, \beta \in K$ with $|\alpha| \geq |\beta|$, $(\alpha + \beta)S \subset \alpha S$.

Proof:

Let S be equilibrated and $x \in S$.

Then $(\alpha + \beta)x = \alpha(1 + \alpha^{-1}\beta)x$ with $|1 + \alpha^{-1}\beta| \leq \max(1, |\alpha^{-1}\beta|) = 1$.

Hence $(1 + \alpha^{-1}\beta)x \in S$ and inclusion follows.

Conversely, let $\lambda \in R$.

Put $\mu = \lambda - 1$ so that $|\mu| \leq |1|$.

Therefore $\lambda S = (1 + \mu)S \subset 1.S = S$.

Definition

A set $S \subset E$ is absorbant if for each $x \in E$ there exists a real $\alpha > 0$ such that $x \in \lambda S$ for all $\lambda \in K$ with $|\lambda| \geq \alpha$.

Lemma 2.3

If S is an R-submodule of E, then the following statements are equivalent:

(a) S is absorbant.

(b) For each $x \in E$ there exists $0 \neq \mu \in K$ such that $x \in \mu S$.

(c) S contains a basis for E.

Proof:

(a) $\Leftrightarrow$ (b). See [3], p.6, remembering that S is an equilibrated set.

(a) $\Rightarrow$ (c) because an absorbant set in E generates E (ibid., p.7).

(c) $\Rightarrow$ (b). Let $\{u_i\} \subset S$ be a basis for E, $x = \sum_{i=1}^{n} \alpha_i u_i \in E$, and $|\alpha_k| = \max_{1 \leq i \leq n} |\alpha_i|$.

If $|\alpha_k| = 0$, then $\alpha_i = 0$ for all i, and $x \in S = 1.S$.

If $|\alpha_k| \neq 0$, then $\alpha_k \neq 0$ and $x = \alpha_k\left(\sum_{i=1}^{n} \alpha_k^{-1}\alpha_i u_i\right)$. We have $|\alpha_k^{-1}\alpha_i| \leq 1$ for all i, $\Sigma\alpha_k^{-1}\alpha_i u_i \in S$, and $x \in \alpha_k S$.

Corollary

An R-submodule of E which contains an absorbant submodule is itself absorbant.

Definition

We shall say that a set $S \subset E$ is bounded if for each $0 \neq x \in E$

there exists a real $\alpha > 0$ such that $\lambda x \notin S$ for all $\lambda \in K$ with $|\lambda| > \alpha$.

Clearly every subset of a bounded set is bounded. Further, boundedness with respect to any norm on E (possibly Archimedian) implies boundedness in the above sense (but not conversely; see example on p.14).

Lemma 2.4

Let E be a non-Archimedian normed space over K and S a sphere in E centered at the origin. Then S is an absorbant and bounded R-submodule of E.

Proof:

Clear.

Corollary

Every R-submodule of E which is contained in a sphere and contains a sphere centered at the origin is absorbant and bounded.

Remark

When a normed space E is infinite dimensional an absorbant and bounded R-submodule of E need not be contained in a sphere, i.e., it may be unbounded with respect to the norm on E. Nor does it have to contain a sphere. In fact, its interior may be empty.

Example (cf [29], p.254).

Let $E = \oplus K$ be the space of sequences in K with a finite number of non-zero terms. If the valuation on K is dense choose $\rho \in K$ such that $|\rho| > 1$; if the valuation is discrete put $\rho = \pi^{-1}$. Then

$$S_1 = \left\{(\alpha_i) \in E \mid |\alpha_i| < |\rho|^i, \quad i \in \mathbb{N}\right\} \text{ and}$$

$$S_2 = \left\{(\alpha_i) \in E \mid |\alpha_i| < |\rho|^{-i}, \quad i \in \mathbb{N}\right\}$$

are absorbant and bounded R-submodules of E. But S_1 is unbounded and S_2 has

an empty interior with respect to the topology defined on E by the non-Archimedian norm $\|(\alpha_i)\|_\infty = \max_i |\alpha_i|$.

Theorem 2.1

Let E be a vector space over K and S an absorbant and bounded R-submodule of E. Define $\| \ \|_S$ on E by $\|x\|_S = \inf_{x \in \lambda S} |\lambda|$ for all $x \in E$. Then:

(a) $\| \ \|_S$ is a non-Archimedian norm on E.

(b) $\{x \in E \mid \|x\|_S < 1\} \subset S \subset \{x \in E \mid \|x\|_S \leqslant 1\}$.

(c) If the valuation on K is discrete, then $W_E \subset W_K$ and $S = \{x \in E \mid \|x\|_S \leqslant 1\}$.

Proof:

(a) The first norm property (§1, p.2) is an immediate consequence of the boundedness of S. The second property is evident. It remains to prove the strong triangle inequality.

Let $x, y \in E$ with $\|x\|_S \geqslant \|y\|_S$.

Then for all $\lambda \in K$ with $x \in \lambda S$ there exists $\mu \in K$ with $y \in \mu S$ and $|\lambda| \geqslant |\mu|$.

Now, $x + y \in \lambda S + \mu S = \lambda S$ by lemma 2.1.

So $\|x + y\|_S \leqslant \|x\|_S = \max(\|x\|_S, \|y\|_S)$ (cf [18], p 534).

(b) Since $\| \ \|_S$ is a seminorm, see [18], p 535.

(c) The first result follows from the properties of W_K, for the second, see [29], p 256.

Remark

An Archimedian norm on E cannot be defined in the above manner by any subset of E. For, if S is to define a norm on E, we should have at least:

(i) S is equilibrated.

(ii) $(\alpha + \beta)S = \alpha S + \beta S$ for all $\alpha, \beta \in K$.

The first condition gives for $|\alpha| \geq |\beta|$

$$(\alpha + \beta)S \subset \alpha S \subset \alpha S + \beta S,$$

while the second forces equality. Hence S is an R-module in E so that any norm defined by S will be non-Archimedian.

Every non-Archimedian norm on a vector space E over K determines a unit sphere which is an absorbant and bounded R-submodule of E. Conversely, every absorbant and bounded R-submodule S of E defines a non-Archimedian norm $\| \ \|_S$ on E. Further, if the valuation on K is discrete, then S is the unit sphere for $\| \ \|_S$. It does not follow, however, that if S is the unit sphere for a non-Archimedian norm $\| \ \|$ on E then $\|x\|_S = \|x\|$ for all $x \in E$. In fact, we already know that this cannot happen in the discrete case unless the value set of the norm on E is contained in the value group of K. The actual state of affairs resembles closely the situation described by Monna for seminorms (see [18], p.531).

Lemma 2.5

Let $\| \ \|$ be a non-Archimedian norm on E, S the corresponding unit sphere, and $\| \ \|_S$ the norm on E determined by S. Then:

(a) If the valuation of K is dense, $\|x\| = \|x\|_S \quad \forall\, x \in E$.

(b) If the valuation of K is discrete, $|\pi| \, \|x\|_S < \|x\| \leq \|x\|_S, \quad x \in E$.

(c) In the discrete case $\|x\| = \|x\|_S \quad \forall\, x \in E$ if and only if the value set of $\| \ \|$ is contained in the value group of K.

Proof:

We have for $x \in E$, $\lambda \in K$

$$\|x\| \leq |\lambda| \iff x \in \lambda S \text{ so that } \|x\| \leq \inf_{x \in \lambda S} |\lambda| = \|x\|_S .$$

If the valuation of K is dense,

$$\|x\| < \|x\|_S \Rightarrow \exists\, \mu \in K \text{ such that } \|x\| < |\mu| < \inf_{x \in \lambda S} |\lambda|.$$

Then $x \in \mu S$ and $x \notin \mu S$! Hence (a).

If the valuation of K is discrete,

$$\inf_{x \in \lambda S} |\lambda| = |\pi|^n \text{ for some } n \in \mathbb{Z}.$$

If $\|x\| \leq |\pi|^{n+1}$, then $x \in \pi^{n+1} S$ with $|\pi^{n+1}| < \inf_{x \in \lambda S} |\lambda|$,

which is a contradiction. Thus

$$|\pi|^{n+1} < \|x\| \leq |\pi|^n$$

and both (b) and (c) follow immediately.

Remark

Instead of the unit sphere the open sphere $S_o = \{x \in E \mid \|x\| < 1\}$ may be used to establish the norm $\| \ \|_{S_o}$ on E (cf [18], p.530). We then have:

(a′) $\|x\| = \|x\|_S \ \forall\, x \in E$, if the valuation of K is dense.

(b′) $|\pi| \ \|x\|_{S_o} \leq \|x\| < \|x\|_{S_o}$, if the valuation of K is discrete.

(c′) In the latter case, $W_E \subset W_K \Leftrightarrow |\pi| \ \|x\|_{S_o} = \|x\|$.

We recall that two norms $\| \ \|_1$ and $\| \ \|_2$ on a vector space E are said to be equivalent if there exist two positive reals a and b such that

$$a\|x\|_1 \leq \|x\|_2 \leq b\|x\|_1 \quad \forall\, x \in E.$$

Lemma 2.6

Let $\| \ \|_1$ and $\| \ \|_2$ be a norm on E determined by the absorbant and bounded R-submodules S_1 and S_2 of E such that $S_1 \subset \alpha S_2$, $\alpha \in K$. Then $\|x\|_2 \leq |\alpha| \ \|x\|_1$, $\forall\, x \in E$.

Proof:

We have

$$x \in \lambda S_1 \Rightarrow x \in \lambda\alpha S_2, \quad \lambda \in K.$$

Hence

$$\|x\|_2 = \inf_{x \in \mu S_2} |\mu| \leq \inf_{x \in \lambda\alpha S_2} |\lambda\alpha| = |\alpha| \inf_{x \in \lambda S_1} |\lambda| = |\alpha| \, \|x\|_1.$$

Corollary

$S_1 \subset \alpha S_2 \subset \beta S_1$, $\beta \in K \Rightarrow \| \ \|_1$ and $\| \ \|_2$ are equivalent.

Theorem 2.2

Let $\| \ \|$ be a non-Archimedian norm on E and M an R-submodule of E such that $\{x \in E \mid \|x\| < 1\} \subset M \subset \{x \in E \mid \|x\| \leq 1\}$. Then M determines a non-Archimedian norm on E equivalent to $\| \ \|$. In fact, if the valuation on K is dense, $\|x\|_M = \|x\|$ for all $x \in E$; if the valuation of K is discrete, then $\forall x \in E \quad |\pi| \, \|x\|_M \leq \|x\| \leq \|x\|_M$.

Proof:

The given inclusion relation implies that M is absorbant and bounded so that $\| \ \|_M$ is a non-Archimedian norm on E.

Thus the equality in the dense case and the inequality in the discrete case follows from Monna's results (see [18], p 537).

Remarks

1. Equivalence in both cases may also be proved as follows:

With the notation of lemma 2.5 and the remark which follows it, we have $\mu S \subset S_0$, $|\mu| < 1$, in the case of discrete valuation and $\pi S = S_0$ when the valuation on K is discrete. Thus, the given inclusion and the corollary to lemma 2.6 imply that $\| \ \|_M$ is equivalent to $\| \ \|_S$ which is equivalent to $\| \ \|$ by lemma 2.5.

2. The equivalence of $\| \ \|$ and $\| \ \|_S$ in the case of discrete valuation

(as well as the equality of S and the unit shere of $\| \ \|_S$) has been mentioned by Fleischer ([8], p 168). Serre's equivalent norm for the same case defined by $\|x\|' = \inf \{|\lambda| \in W_K \mid \|x\| \leq |\lambda|\}$ ([26], p 69) is our $\| \ \|_S$.

We conclude this section with some remarks concerning the case of trivial valuation on the field K. Obviously the definitions of absorbant and bounded sets may be extended to a vector space E over such a field. However, since now the R-submodules of E are precisely the subspaces of E, the only absorbant R-submodule of E is E itself. Since E is bounded it defines a non-Archimedian norm $\| \ \|_E$ whose value set is clearly {1}. Conversely, if E is a normed space with $W_E = \{1\}$, then E is its own unit shere and $\|x\| = \|x\|_E$ for all $x \in E$.

With the help of a concept of Serre's (ibid.) these observations may be combined with previous results (lemma 2.5) to obtain the following theorem:

Theorem 2.3

If E is a non-Archimedian normed space over a valuated field K and S is the unit shere in E, then $\|x\| = \|x\|_S$ for all $x \in E$ if and only if $W_E \subset \overline{W}_K$, where $\overline{W}_K$ is the closure of W_K.

§3. Linear Transformations and Normed Algebras.

Let K be a non-Archimedian valuated field, E_1 and E_2 vector spaces over K, and T a linear transformation from E_1 to E_2. Since a linear transformation is an R-module homomorphism, the image $T(S_1)$ of an R-module $S_1 \subset E_1$ and the inverse image of an R-module $S_2 \subset E_2$ are also R-modules (cf [18], p 533; [20], p 355).

Let norms $\| \ \|_1$ and $\| \ \|_2$ be defined on E_1 and E_2 respectively. T is said to be bounded if there exists a real $c > 0$ such that $\|T(x)\|_2 \leq c\|x\|_1$ for every $x \in E_1$. The following theorem is an exact counterpart of a well known result for normed spaces over $\mathbb{R}$ or $\mathbb{C}$ (see [27], p.219):

Theorem 3.1

Let E_1 be a non-Archimedian normed space over a field K with a non-trivial valuation. Let E_2 be an arbitrary normed space over K, T a linear transformation from E_1 to E_2, and S the unit sphere in E_1. Then the following statements are equivalent:

(a) T is bounded.

(b) T is continuous.

(c) T(S) is bounded with respect to the norm on E_2.

Proof:

(a) $\Rightarrow$ (b) and (c).

Clear.

(b) $\Rightarrow$ (c).

Since the valuation on K is non-trivial, there exists $\rho \in K$ with $|\rho| > 1$.

If T(S) is unbounded, then for each $n \in \mathbb{N}$ there exists $x_n \in S$ with $\|T(x_n)\|_2 > |\rho|^n$.

Let $y_n = \rho^{-1}x_n$. Then $\|y_n\|_1 = \frac{\|x_n\|_1}{|\rho|^n} \leq \frac{1}{|\rho|^n}$, so that $y_n \to 0$ as $n \to \infty$.

But $\|T(y_n)\|_2 = \frac{\|T(x_n)\|_2}{|\rho|^n} > 1$ for all $n \in \mathbb{N}$.

Thus $T(y_n) \not\to 0$, contradicting the continuity of T.

(c) $\Rightarrow$ (a).

Suppose $T(S)$ is contained in a closed sphere centered at the origin with radius k.

Clearly the result holds if $x = 0$.

For $0 \neq x \in E$, $x \in \lambda S$, $\lambda \in K \Rightarrow \|x\|_1 \leq |\lambda|$, so that $\lambda^{-1}x \in S$.

Hence, $\dfrac{\|T(x)\|_2}{|\lambda|} \leq k$ and $\|T(x)\|_2 \leq k|\lambda|$.

Therefore, $\|T(x)\|_2 \leq k \inf\limits_{x \in \lambda S} |\lambda| = k \|x\|_S$

The required result follows from the equivalence of $\| \ \|_1$ and $\| \ \|_S$.

In fact, $\|T(x)\|_2 \leq c \|x\|_1$ for all $x \in E$ where $c = k$ if $W_{E_1} \subset \overline{W}_K$ and $c < \dfrac{k}{|\pi|}$ in the remaining case.

Remark

The equivalence of (a) and (b) was proved by Monna without the assumption that $\| \ \|_1$ is non-Archimedian ([15], pp 1134-1135). The equivalence of (b) and (c) is essentially a special case of a theorem proved by van Tiel for certain classes of K-convex topological spaces (see [29], p.269).

Let $\mathfrak{B}(E_1, E_2)$ be the space of bounded linear transformations from E_1 to E_2, where for the moment we permit the valuation on K to be trivial. Then

$$\|T\|_{1,2} = \sup_{0 \neq x \in E_1} \frac{\|T(x)\|_2}{\|x\|_1} = \inf_{x \in E_1} \{c \mid \|T(x)\|_2 \leq c \|x\|_1\}, \ T \in \mathfrak{B},$$

defines a norm on $\mathfrak{B}(E_1, E_2)$, called the bound of T, such that

$$\|T(x)\|_2 \leq \|T\|_{1,2} \|x\|_1 \quad \text{for all } x \in E_1.$$

If $W_{E_1} \subset W_K$, then for all $T \in \mathfrak{B}$

$$\|T\|_{1,2} = \sup_{\|x\|_1 = 1} \|T(x)\|_2 .$$

Another norm on $\mathcal{B}$ is given by

$$\|T\|_S = \sup_{x \in S} \|T(x)\|_2 = \inf_{x \in S} \{k \mid \|T(x)\|_2 \leq k\}, \quad T \in \mathcal{B}.$$ We have

$$\|T\|_S \leq \|T\|_{1,2} \quad \text{for all} \quad T \in \mathcal{B}.$$

If the valuation on K is non-trivial, then $\mathcal{B}(E_1, E_2)$ is the space of continuous linear transformations from E_1 to E_2 and the two norms on $\mathcal{B}$ are equivalent. In fact,

$W_{E_1} \subset \overline{W}_K \Rightarrow \|T\|_S = \|T\|_{1,2}$ for all $T \in \mathcal{B}$ (cf [26], p.71), so that

$$\|T(x)\|_2 \leq \|T\|_S \, \|x\|_1 \quad \text{for all} \quad x \in E_1 \text{ (cf [21], p.124).}$$

Both norms on $\mathcal{B}$ will be non-Archimedian if $\| \ \|_2$ is non-Archimedian. If the valuation on K is non-trivial and $W_{E_1} \subset \overline{W}_K$, $\mathcal{B}$ will be a complete normed space whenever E_2 is complete, for the proof in [27], p.221, will hold in this case (cf [26]). Finally, if the norm on E_2 is discrete, then there exists $0 \neq x \in S$ such that $\|T(x)\|_2 = \|T\|_S$.

Since a discrete norm on E_2 implies $W_K = \overline{W}_K$, the above assertions yield the following result (cf [21], [26]):

Theorem 3.2

Let E_1 and E_2 be non-Archimedian normed spaces over a field K with non-trivial valuation such that $W_{E_1} \subset \overline{W}_K$.

Let $\mathcal{B}(E_1, E_2)$ be the space of continuous linear transformations from E_1 to E_2. Then

$$\|T\| = \sup_{\|x\|_1 \leq 1} \|T(x)\|_2, \quad T \in \mathcal{B},$$

defines a non-Archimedian norm on $\mathcal{B}(E_1, E_2)$ such that

$$\|T(x)\|_2 \leq \|T\| \, \|x\|_1 \text{ for all } x \in E_1.$$

If E_2 is a Banach space, so is $\mathcal{B}(E_1, E_2)$. If the norm on E_2 is discrete,

then $\|T\| = \sup_{\|x\|_1 = 1} \|T(x)\|_2$, $T \in \mathfrak{B}$,

and there exists $x_0 \in E_1$ with $\|x_0\| = 1$ such that

$$\|T(x_0)\|_2 = \|T\| .$$

Remark

Since Theorem 3.1 is trivially true if $W_{E_1} = W_{E_2} = \{1\}$, Theorem 3.2 will also hold in this case.

Corollary 1.

Let K be a field with a non-dense valuation, E a non-Archimedian normed space over K such that $W_E \subset W_K$, and f a continuous functional on E. Then

$$\|f\| = \sup_{\|x\| = 1} |f(x)|$$

is a non-Archimedian bound for f so that

$$|f(x)| \leq \|f\| \, \|x\| \text{ for all } x \in E.$$

Further, there exists $x_0 \in E$ with $\|x_0\| = 1$ such that $|f(x_0)| = \|f\|$.

Definition

Let A be an algebra over K. A non-Archimedian vector norm $\| \ \|$ on A will be called a non-Archimedian algebra norm if it satisfies the additional conditions:

(4) $\|xy\| \leq \|x\| \, \|y\|$ for all $x, y \in A$,

(5) $\|1\| = 1$.

A is known as a non-Archimedian Banach algebra if it is a non-Archimedian Banach space.

We note that property (5) implies $W_K \subset W_A$.

If $\mathfrak{B}(E)$ is the algebra of bounded linear operators on a non-Archimedian normed space E, then

$$\|T\|_{1,1} = \sup_{0 \neq x \in E} \frac{\|T(x)\|}{\|x\|}, \quad T \in \mathfrak{B},$$

defines a non-Archimedian algebra norm on $\mathfrak{B}(E)$ (for proof of (4) see [23], p.76). We thus have an additional corollary to theorem 3.2 :

Corollary 2.

Let E be a non-Archimedian vector space over a field K such that $W_E \subset \overline{W}_K$. Let $\mathfrak{B}(E)$ be the algebra of continuous linear operators on E. Then

$$\|T\| = \sup_{\|x\| \leq 1} \|T(x)\|, \quad T \in \mathfrak{B},$$

defines a non-Archimedian algebra norm on $\mathfrak{B}(E)$ such that $W_{\mathfrak{B}} \subset \overline{W}_K$ and

$$\|T(x)\| \leq \|T\| \, \|x\| \quad \text{for all } x \in E.$$

$\mathfrak{B}(E)$ is a non-Archimedian Banach algebra if E is a non-Archimedian Banach space. If the norm on E is discrete, then

$$\|T\| = \sup_{\|x\| = 1} \|T(x)\|, \quad T \in \mathfrak{B},$$

and there exists $x_o \in E$ with $\|x_o\| = 1$ such that

$$\|T(x_o)\| = \|T\|.$$

In this case, $W_{\mathfrak{B}} = W_K$.

We shall see that an algebra over a non-Archimedian valuated field may well have an Archimedian norm. This cannot happen, however, if the algebra is commutative and its norm is multiplicative. In fact, we have a somewhat stronger result:

Theorem 3.3

Let A be a commutative algebra over a non-Archimedian valuated field and $\| \ \|$ a norm on A such that $\|x^2\| = \|x\|^2$ for all $x \in A$. Then $\| \ \|$ is non-Archimedian.

Proof:

By induction, $\|x^{2^n}\| = \|x\|^{2^n}$ for every $n \in \mathbb{N}$.

Let $a, b \in A$ with $\|a\| \geq \|b\|$. Then

$$\|a + b\|^{2^n} = \|(a + b)^{2^n}\| = \| \sum_{i=0}^{n} \binom{2^n}{i} a^{2^n - i} b^i \|$$

$$\leq \sum_{i=0}^{n} \| \binom{2^n}{i} a^{2^n - i} b^i \|, \quad \text{by property (3)},$$

$$= \sum | \binom{2^n}{i} | \, \| a^{2^n - i} b^i \|, \quad \text{by property (2)},$$

$$\leq \sum \| a^{2^n - i} b^i \|, \quad \text{since } | \ | \text{ is non-Archimedian},$$

$$\leq \sum_{i=0}^{n} \|a\|^{2^n - i} \|b\|^i, \quad \text{by property (4)},$$

$$\leq (2^n + 1)\|a\|^{2^n}.$$

Hence, $\|a + b\| \leq (2^n + 1)^{1/2^n} \|a\|$ for every $n \in \mathbb{N}$.

Letting $n \to \infty$ we get

$$\|a + b\| \leq \|a\| = \max (\|a\|, \|b\|).$$

Lemma 3.1

If A is a non-Archimedian normed K-algebra and S is the unit sphere in A, then S is an R-subalgebra of A, where R is the valuation ring of K. The set

$$Q = \{x \in A \mid \|x\| < 1\}$$

is an algebra ideal of S.

Proof:

Clear.

Theorem 3.4

With the notation of lemma 3.1, let $\mathfrak{R}(S)$ be the Jacobson radical of S. Then:

(a) If $x \in A$ and $\sum_{n=0}^{\infty} x^n$ is convergent, then $\sum_{n=0}^{\infty} x^n$ is the inverse of $1 - x$ and $\|(1 - x)^{-1}\| \leq \max_n \|x^n\|$.

(b) If $x \in Q \cap \mathfrak{R}(S)$, then $\sum_{n=0}^{\infty} x^n$ converges to $(1 - x)^{-1} \in S$. In fact, $\|1 - x\| = \|(1 - x)^{-1}\| = 1$.

(c) If A is complete, then $Q \subset \mathfrak{R}(S)$.

Proof:

All the assertions are immediate consequences of lemma 3.1, item 3 of §1, and the following known results:

1. If A is a normed algebra over a valuated field, $x \in A$, and $\sum_{n=0}^{\infty} x^n$ converges, then $1 - x$ is invertible and $(1 - x)^{-1} = \sum_{n=0}^{\infty} x^n$. Conversely, if $\|x\| < 1$ and $1 - x$ is invertible, then $\sum_{n=0}^{\infty} x^n$ converges to $(1 - x)^{-1}$. If A is complete and $\|x\| < 1$, then $1 - x$ has an inverse in A ([4], pp 75-76).

2. The radical of a ring S is the largest ideal I such that, for all $x \in I$, $1 - x$ is a unit ([12], p 57).

CHAPTER 2

Norms and Bounds on Finite Dimensional Spaces

§ 4. Norms on K^m. Duality.

Let K be a valuated field and E a finite dimensional vector space over K. If K is complete, then all norms on E are equivalent ([1], p.95). If in addition the valuation of K is non-trivial, then all m-diemnsional normed spaces over K are topologically isomorphic ([3], p.27; for non-Archimedian spaces see [5], p.695). For each is topologically isomorphic to K^m normed with $\| \ \|_\infty$ and hence to K^m with the product topology ([4], pp 67, 69).

On the other hand, without special assumptions about K, all the Hölder norms on E are equivalent to $\| \ \|_\infty$. So is any non-Archimedian norm for which E has an orthogonal basis $\{u_i\}_{1 \leq i \leq m}$ ([17], p.464). For then

$$x = \sum_{i=1}^{m} \alpha_i u_i \ \Rightarrow \ \|x\| = \max_i \|\alpha_i u_i\| \ .$$

If $\|x\|_\infty = \max_i |\alpha_i| = |\alpha_k|$, we have

$$\|x\|_\infty \min_i \|u_i\| \leq |\alpha_k| \ \|u_k\| \leq \max_i \|\alpha_i u_i\| \leq \|x\|_\infty \max_i \|u_i\| .$$

Again, when E is finite diemnsional over a valuated field K, convergence to zero with respect to any norm on E is equivalent to coordinatewise convergence, for the proof referred to in [28], p.253, will hold here. Hence all linear transformations from E to an arbitrary normed space over K are continuous and bounded. The boundedness is obvious when the valuation on K is trivial and follows from the remark after theorem 3.1 in the non-trivial case. We calculate the bound of a linear

transformation between finite dimensioral non-Archimedian normed spaces with orthogonal bases.

Theorem 4.1

Let E_1 and E_2 be finite dimensional non-Archimedian normed spaces over K with orthogonal bases $\{u_j\}_{1 \leq j \leq n}$ and $\{v_i\}_{1 \leq i \leq m}$. Let T be a linear transformation from E_1 to E_2. Then the bound of T is given by

$$\|T\|_{1,2} = \max_{i,j} \frac{|\alpha_{ij}| \, \|v_i\|_2}{\|u_j\|_1}$$

where (α_{ij}) is the matrix of T with respect to the given bases.

Proof:

$$\text{Let } \quad 0 \neq x = \sum_j \xi_j u_j \ , \quad \|x\|_1 = \max_j \|\xi_j u_j\| = \|\xi_k u_k\| .$$

$$\text{Then } \quad T(x) = \sum_{i,j} \alpha_{ij} \xi_j v_i \qquad \text{and}$$

$$\|T(x)\|_2 = \max_{i,j} \|\alpha_{ij} \xi_j v_i\| = |\alpha_{pq}| \, |\xi_q| \, \|v_p\|_2 \ , \text{ say.}$$

We have

$$\frac{\|T(x)\|_2}{\|x\|_1} = \frac{|\alpha_{pq}| \, |\xi_q| \, \|v_p\|_2}{\|\xi_k u_k\|_1}$$

$$= |\alpha_{pq}| \, \frac{\|\xi_q u_q\|_1}{\|\xi_k u_k\|_1} \cdot \frac{\|v_p\|_2}{\|u_q\|_1}$$

$$\leq |\alpha_{pq}| \, \frac{\|v_p\|_2}{\|u_q\|_1} \ .$$

Thus

$$\sup_{0 \neq x \in E_1} \frac{\|T(x)\|_2}{\|x\|_1} \leq \max_{i,j} \frac{|\alpha_{ij}| \|v_i\|_2}{\|u_j\|_1} = \frac{|\alpha_{rs}| \, \|v_r\|_2}{\|u_s\|_1} \ , \text{ say.}$$

Now put $x = u_s$.

Then,

$$\frac{\|T(u_s)\|_2}{\|u_s\|_1} = \frac{\|\sum_i \alpha_{is} v_i\|_2}{\|u_s\|_1} = \frac{\max_i \|\alpha_{is} v_i\|_2}{\|u_s\|_1} = \frac{|\alpha_{rs}|\ \|v_r\|_2}{\|u_s\|_1}$$

So,

$$\|T\|_{1,2} = \frac{|\alpha_{rs}|\ \|v_r\|_2}{\|u_s\|_1} = \max_{i,j} \frac{|\alpha_{ij}|\ \|v_i\|_2}{\|u_j\|_1}$$

Corollary

Let E be a non-Archimedian normed vector space over K with an orthogonal basis $\{u_i\}_{1 \leq i \leq m}$. Let T be a linear operator on E. Then the bound of T is given by

$$\|T\|_{1,1} = \max_{i,j} \frac{|\alpha_{ij}|\ \|u_i\|}{\|u_j\|}$$

where (α_{ij}) is the matrix of T with respect to the given basis. If also $W_E \subset \bar{W}_K$, then

$$\|T\|_S = \max_{i,j} \frac{|\alpha_{ij}|\ \|u_i\|}{\|u_j\|}$$

Let K be a valuated field. An algebra norm on K_m will be called a matrix norm. We note that this definition implies that $\|I\| = 1$. Since K_m may be identified with the algebra of linear operators on K^m, every vector norm on K^m in determining an operator bound also determines a matrix norm

$$\text{lub}\ (A) = \sup_{x \neq 0} \frac{\|Ax\|}{\|x\|}, \quad A \in K_m,$$

which may be called a matrix bound.

From now on we take $E = K^m$. Lower case Latin letters will denote column vectors while ordinary capitals will stand for matrices.

If a vector and a matrix norm satisfy the condition

$$\|Ax\| \leqslant \|A\| \, \|x\|$$

for every x and A, then the two norms are said to be consistent. A bound is of course consistent with the corresponding vector norm. The following results known for $\mathbb{R}$ and $\mathbb{C}$ remain valid for K :

For any matrix norm and for the bound associated with a consistent vector norm

$$r(A) \leqslant \text{lub } (A) \leqslant \|A\|,$$

where r(A) is the spectral radius of the matrix A ([9], p.45). If y is a fixed non-zero vector, then for any matrix norm

$$\|x\| = \|xy'\|$$

defines a consistent vector norm ([13]; in [9] the Hermitian transpose is used throughout). If follows that for a matrix norm to be a bound it is sufficient that the norm be minimal (cf [13]). The converse has been proved by Ljubić for real matrices (ibid). The following theorem is weaker than the corresponding results in $\mathbb{R}^m$.

Theorem 4.2

Let K be a valuated field and $\| \ \|$ a norm on K^m. For all $x \in K^m$ let

$$\|x\|^{\star} = \sup_{y \neq 0} \frac{|x'y|}{\|y\|} \ .$$

Then:

(a) $\| \ \|^{\star}$ is a norm on K^m .

(b) $\| \ \|^{\star}$ is non-Archimedian if and only if the valuation of K is non-Archimedian.

(c) $|x'y| \leqslant \|x\|^{\star} \, \|y\|$

Proof:

All the assertions follow from the fact that $\| \ \|^*$ is the bound of a linear functional generated by x.

Corollary 1. (see [13]).

For the bound associated with $\| \ \|$

$$\text{lub } (xy') = \|x\| \ \|y\|^*.$$

Proof: (cf [13]):

$$\text{lub } (xy') = \sup_{z \neq 0} \frac{\|xy'z\|}{\|z\|} = \|x\| \sup_{z \neq 0} \frac{|y'z|}{\|z\|} = \|x\| \ \|y\|^* .$$

Corollary 2. (cf [9], p.43).

For all $x, y \in K^m$, $A \in K_m$

$$|x'Ay| \leq \|x\|^* \|y\| \text{ lub } (A) = \text{lub } (yx') \text{ lub } (A).$$

Definition

Since $\| \ \|^*$ is a norm on the dual space of K^m it will be called the dual norm of $\| \ \|$.

Remarks

1. As distinct from the situation in $\mathbb{R}^m$ in general duality is not an involutory relation. To see this we need only consider an Archimedian norm on K^m when K is non-Archimedian. For then $(\| \ \|^*)^* = \| \ \|$ contradicts (b).

2. It follows from (c) that $\|e_i\| \ \|e_i\|^* \geq 1$. Thus, if S and S^* are the unit spheres for $\| \ \|$ and $\| \ \|^*$ respectively,

$$e_i \in S \cap S^* \iff \|e_i\| = \|e_i\|^* = 1.$$

If this condition is fulfilled for all i, then for all i, j

$$\text{lub } (e_i e_j') = 1 \quad \text{and} \quad |\alpha_{ij}| \leq \text{lub } (A).$$

Lemma 4.1 (cf [13])

Let K be a valuated field. If norms $\| \ \|_1$ and $\| \ \|_2$ are defined on K^m, then for the corresponding bounds

$$\text{lub}_1(A) = \text{lub}_2(A) \iff \|x\|_1 = a\|x\|_2 \text{ for some } a > 0.$$

Proof:

One implication is obvious and the other follows from corollary 1 to theorem 4.2.

Lemma 4.2

Let K be a valuated field. Let $\| \ \|$ be a norm on K_m and suppose that for each matrix A there exist vectors x and y with $\|yx'\| = 1$ such that $|x'Ay| = \|A\|$. Then $\| \ \|$ is minimal. In fact, $\| \ \|$ is a bound.

Proof:

Let $\| \ \|_1$ be a matrix norm such that $\|A\|_1 \leqslant \|A\|$ for all A. Then for the bound associated with a vector norm consistent with $\| \ \|_1$ we have

$$\text{lub}(yx') \leqslant \|yx'\|_1 \leqslant \|yx'\| = 1.$$

Hence by corollary 2 to theorem 4.2,

$$|x'Ay| \leqslant \text{lub}\,(yx')\ \text{lub}\,(A) \leqslant \text{lub}\,(A) \leqslant \|A\|_1 \leqslant \|A\|$$

and the given condition forces

$$\text{lub}\,(A) = \|A\|_1 = \|A\| \quad \text{for all A.}$$

It is known that when $K = \mathbb{R}$ or $K = \mathbb{C}$ non-zero vectors may be found for which equality holds in corollary 2 to theorem 4.2 (see [9]). If these are suitably normalized, then the conditions of lemma 4.2 are satisfied and we obtain an alternative proof of Ljubič's theorem:

Theorem 4.3

When $K = \mathbb{R}$ or $K = \mathbb{C}$, a matrix norm on K_m is a bound if and only if it is minimal.

It is known that every Archimedian valuated field is topologically isomorphic to a subfield of $\mathbb{C}$ valuated with the ordinary absolute value ([28], p.246). Accordingly in this case it is customary to take $K = \mathbb{R}$ or $K = \mathbb{C}$. Further, every norm on $\mathbb{R}^m$ or $\mathbb{C}^m$ is determined by its unit sphere which is an equilibrated convex body ([6], p.108; [9], p.41). Theorem 2.3 shows that this statement remains true for certain norms on K^m when K is non-Archimedian, where however S is an absorbant and bounded R-module. Writing $\| \ \| = \| \ \|_S$ we have in all cases

$$\text{lub } (A) = \sup_{x \in S} \|Ax\|_S = \text{lub}_S (A) .$$

We also have (cf [9]):

Lemma 4.3

Let K be a non-Archimedian valuated field, $K = \mathbb{R}$ or $K = \mathbb{C}$. Let $S_1 \subset K^m$ be an absorbant and bounded R-module if K is non-Archimedian and an equilibrated convex body in the other cases. If the matrix P is non-singular, then for $x \in K^m$, $A \in K_m$

$$S_2 = PS_1 \Rightarrow \|x\|_{S_2} = \|P^{-1}x\|_{S_1} \text{ and } \text{lub}_{S_2}(A) = \text{lub}_{S_1}(P^{-1}AP).$$

Proof:

We may assume that K is non-Archimedian. Then S_2 is an absorbant and bounded R-submodule of K^m. Further,

$$\forall \lambda \in K, x \in \lambda S_2 \Leftrightarrow P^{-1}x \in \lambda S_1 .$$

Hence, $$\inf_{x \in \lambda S_2} |\lambda| = \inf_{P^{-1}x \in \lambda S_1} |\lambda| .$$

Also, $$\sup_{x\neq 0} \frac{\|Ax\|_{S_2}}{\|x\|_{S_2}} = \sup_{x\neq 0} \frac{\|P^{-1}Ax\|_{S_1}}{\|P^{-1}x\|_{S_1}} = \sup_{y\neq 0} \frac{\|P^{-1}APy\|_{S_1}}{\|y\|_{S_1}} .$$

The required results are now obtained by a reference to definitions.

Corollary

$$S_2 = \alpha S_1 , \alpha \in K \Rightarrow \|x\|_{S_1} = |\alpha| \, \|x\|_{S_2} \text{ and } \mathrm{lub}_{S_1}(A) = \mathrm{lub}_{S_2}(A).$$

Let K be a non-Archimedian valuated field. Consider the non-Archimedian norm on K^m defined by

$$\|x\|_\infty = \max_i |\xi_i| .$$

The unit sphere of $\| \ \|_\infty$ is the cube $N = \{x \in K^m \mid |\xi_i| \leqslant 1, \ 1 \leqslant i \leqslant m\}$.

We have $W_{K^m} = W_K$ so that by theorem 2.3

$$\|x\|_\infty = \|x\|_N \text{ for all } x \in K^m .$$

It follows from § 3 that

$$\mathrm{lub}_N(A) = \sup_{\|x\|_N = 1} \|Ax\|_N ,$$

while theorem 4.1 gives

$$\mathrm{lub}_N(A) = \max_{i,j} |\alpha_{ij}| .$$

Definition

When the valuation of K is non-Archimedian we shall describe as natural the norm and bound defined above.

The natural norm is self-dual. For, if $\|y\|_N = 1$, then

$$|x'y| \leqslant \max_i |\xi_i \eta_i| \leqslant \max |\xi_i| = |\xi_k| \text{ say,}$$

and equality is attained for $y = e_k$. Since

$$W_{K^m} = W_K \Rightarrow \|x\|^* = \sup_{\|y\|_N = 1} |x'y| ,$$

we have

$$\|x\|^* = |\xi_k| = \|x\|_N .$$

Further,

$$\|x\|^{\star} = |e'_k x| = |x' e_k| = \|x\|_N$$

indicates that for each x and for each y there exists a non-zero vector such that equality holds in (c) of theorem 4.2.

Again, if $\max_{i,j} |\alpha_{ij}| = |\alpha_{rs}|$, then

$$|e'_r A e_s| = |\alpha_{rs}| = \text{lub}_N(A)$$

so that by lemma 4.2 the natural bound is minimal.

Finally, $\| \ \|_N$ is the only non-Archimedian norm on K^m for which (with the notation of the remark after theorem 4.2) $e_i \in S \cap S^{\star}$ for all i. For, if $|\eta_\ell| = \max_i |\eta_i|$,

$$|\eta_\ell| = |e'_\ell y| \leqslant \|e_\ell\|^{\star} \|y\| = \|y\| \leqslant \max_i \|\eta_i e_i\| = \max_i |\eta_i|.$$

Hence $\|y\| = \max |\eta_i| = \|y\|_N$ for all $y \in K^m$.

If $G = \text{diag}(\gamma_1, \gamma_2, \ldots, \gamma_m)$, $\gamma_i \neq 0$, then

$$GN = \{x \in K^m \mid |\xi_i| \leqslant |\gamma_i|, \ 1 \leqslant i \leqslant m\}$$

also defines a norm on K^m and a corresponding bound on K_m, which may be called a g-norm and a g-bound respectively (see [9], p 45). By lemma 4.3 we have

$$\|x\|_{GN} = \max_i |\gamma_i^{-1} \xi_i| \quad \text{and} \quad \text{lub}_{GN}(A) = \max_{i,j} |\gamma_i^{-1} \alpha_{ij} \gamma_j|.$$

It may be shown that the dual of $\| \ \|_{GN}$ is given by

$$\|x\|_{G^{-1}N} = \max_i |\gamma_i \xi_i|.$$

Remark

The non-Archimedian bound which corresponds to the natural norm should be compared with the cubic bound for real matrices (see [6], p 108). On the other hand, a slight modification of the calculation in [6] shows that, even when K is non-Archimedian, the bound associated with the Archimedian norm $\|x\| = \sum_{i=1}^{m} |\xi_i|$ on K^m is given by the same formula as in the real case. For its dual, however, we have

$$\|x\|^* = \max_i |\xi_i| = \|x\|_N,$$

verifying that duality need not be involutory.

Let K be a valuated field. If $\| \ \|$ is a matrix norm, then for the bound associated with a consistent vector norm we have

$$1 \leqslant \text{lub}\,(e_i e_i') \leqslant \|e_i e_i'\| \quad \text{for all } i,$$

so that

$$1 \leqslant \max_{i,j} \|e_i e_j'\| = n(J) \quad \text{say (theorem 4.2).}$$

Further, for all i, j

$$|\alpha_{ij}| \leqslant \text{lub}\,(e_j e_i')\ \text{lub}\,(A) \leqslant \|e_j e_i'\|\ \|A\| \leqslant n(J)\ \|A\|.$$

Therefore,

$$\frac{\max_{i,j} |\alpha_{ij}|}{n(J)} \leqslant \|A\|.$$

If $\| \ \|$ is non-Archimedian, then

$$\|A\| \leqslant \max_{i,j} \|\alpha_{ij} e_i e_j'\| \leqslant n(J)\ \text{lub}_N(A)$$

We have proved the following theorem:

Theorem 4.4

If $\| \ \|$ is a non-Archimedian matrix norm, then

$$\frac{\text{lub}_N(A)}{n(J)} \leqslant \|A\| \leqslant n(J)\ \text{lub}_N(A)$$

where $n(J) = \max_{i,j} \|e_i e'_j\|$.

Corollary

A non-Archimedian matrix norm is the natural bound if and only if $\max_{i,j} \|e_i e'_j\| = 1$.

§ 5. The Polar. Norms and Bounds on K_m.

Let K be a valuated field. The polar of a set $S \subset K^m$ is defined by

$$S' = \{x \in K^m \mid u \in S \Rightarrow |x'u| \leqslant 1\}.$$

The following lemma is analogous to a result in [9], p.42.

Lemma 5.1

If the valuation of K is non-Archimedian and S is an absorbant set in K^m, then S' is an absorbant and bounded R-submodule of K^m.

Proof:

Let $u \in S$.

Then for $x, y \in S'$ and $\lambda, \mu \in R$

$$|(\lambda x + \mu y)'u| \leqslant \max(|\lambda|\ |x'u|, |\mu|\ |y'u|) \leqslant 1$$

so that $\lambda x + \mu y \in S'$ and S' is an R-submodule of K^m.

To show that S' is absorbant, let $x \in K^m$ and $x'u = \mu \in K$. If $\mu = 0$, $x \in S' = 1 \cdot S'$, otherwise $x \in \mu S'$ (see lemma 2.3).

Again, let $0 \neq x \in K^m$. Since S is absorbant there exists $u \in S$ such that $x'u \neq 0$. Let $\alpha = \frac{1}{|x'u|}$. Then for all $\lambda \in K$ with $|\lambda| > \alpha$

$$|(\lambda x)'u| = |\lambda| \, |x'u| = \frac{|\lambda|}{\alpha} > 1$$

so that $\lambda x \notin S'$ and S' is bounded.

Now suppose that S itself is an absorbant and bounded R-module and that a non-Archimedian norm $\| \ \| = \| \ \|_S$ is defined on K^m. Then $W_{K^m} \subset W_K$ and

$$\|x\|^* = \sup_{y \in S} |x'y|, \ x \in K^m.$$

Moreover, S' is the unit sphere for $\| \ \|^*$ and the set of values of $\| \ \|^*$ is contained in $\bar{W}_K$. Hence

$$\|x\|^* = \|x\|_{S'} \text{ for all } x \in K^m .$$

If the valuation of K is non-dense, then by corollary 1 to theorem 3.2

$$\|x\|_{S'} = \sup_{\|y\|_S = 1} |x'y|$$

and for each $x \in K^m$ there exists a vector y_o with $\|y_o\|_S = 1$ such that

$$|x'y_o| = \|x\|_{S'} \|y_o\|_S .$$

In this case duality is an involutory relation, for we also have

$$\|y\|_S = \sup_{\|x\|_{S'} = 1} |x'y| .$$

To see this put $\|y\|^* = \sup_{\|x\|_{S'} = 1} |y'x| = \sup_{\|x\|_{S'} = 1} |x''y|$. Then by (c) of theorem 4.2 $\|y\|^* \leq \|y\|_S$. Further, there exists $x_o \in K^m$ with $\|x_o\|_{S'} = 1$ such that $\|y\|^* = |x_o'y|$. Therefore,

$$\|x_o\|_{S'}\|y\|^{\star} = |x_o'y| \leq \|x_o\|_{S'}\|y\|_S$$

and $\|y\|^{\star} = \|y\|_S$.

We have proved the following theorem (cf [9]):

Theorem 5.1

Let K be a non-Archimedian valuated field and S an absorbant and bounded R-submodule of K^m. Let

$$\|x\|_{S'} = \sup_{y \in S} |x'y|, \quad x \in K^m.$$

Then:

(a) $\| \ \|_{S'}$ is a non-Archimedian norm on K^m such that

$$|x'y| \leq \|x\|_{S'}\|y\|_S \text{ for all } x, y \in K^m.$$

(b) If the valuation on K is non-dense, then

$$\|x\|_{S'} = \sup_{\|y\|_S=1} |x'y| \quad \text{and} \quad \|y\|_S = \sup_{\|x\|_{S'}=1} |x'y|.$$

Further, for each $x \in K^m$ there exists a vector y_o with $\|y_o\|_S = 1$ and for each $y \in K^m$ there exists a vector x_o with $\|x_o\|_{S'} = 1$ such that equality is attained in (a).

Corollary 1

If the valuation of K is non-dense, then the bound determined by S is minimal. In fact, for each matrix A there exist vectors x and y with $\text{lub}_S(yx') = 1$ such that

$$|x'Ay| = \text{lub}_S(A).$$

Proof:

By (b), for each $y \in K^m$ there exists a vector x_o with $\|x_o\|_{S'} = 1$ such that $|x_o'Ay| = \|Ay\|_S$. By corollary 1 to theorem 3.2 there exists y_o

with $\|y_o\|_S = 1$ such that $\|Ay_o\|_S = \text{lub}_S(A)$. Further, $\text{lub}_S(y_o x_o') = \|y_o\|_S \|x_o\|_{S'}$ by corollary 1 to theorem 4.2.

Corollary 2

If the valuation of K is non-dense, then the bound of A with respect to S is the bound of A′ with respect to S′.

Proof:

See [9].

We observe that it has been possible not only to extend the area of application of Ljubič's theorem but also to obtain different necessary and sufficient conditions. These may be summarized as follows:

Theorem 5.2

Let K be a valuated field. For a matrix norm on K_m to be a bound it is sufficient that for each $A \in K_m$ there exist vectors x and y with $\|yx'\| = 1$ such that

$$|x'Ay| = \|A\|.$$

This condition is necessary in the following cases: (a) $K = \mathbb{R}$ (b) $K = \mathbb{C}$, (c) K is non-Archimedian and the matrix bound is natural, (d) K is non-Archimedian with a non-dense valuation and the bound is associated with a non-Archimedian norm on K^m for which $W_{K^m} \subset W_K$.

Remark

For complex matrices $|x'Ay|$ may be replaced by $|x^HAy|$ or Re x^HAy while yx' becomes yx^H.

The next theorem should be compared with weaker results for complex matrices (see [9], p.46).

Theorem 5.3

Let K be a field with a non-trivial non-Archimedian valuation and let K contain all the characteristic roots of $A \in K_m$. Then there exists an absorbant and bounded R-submodule $S \subset K^m$ such that if $r(A) = 0$ $\text{lub}_S(A)$ is arbitrarily small and if $r(A) \neq 0$ $\text{lub}_S(A) = r(A)$.

Proof:

In view of lemma 4.3 we need consider only the Jordan connonical form of A and in fact only the Jordan block belonging to λ, where λ is a characteristic root of A such that $|\lambda| = r(A)$. Let $W(\varepsilon) = \text{diag}(1, \varepsilon, \varepsilon^2, \ldots)$, where $0 < |\varepsilon| < |\lambda|$ if $\lambda \neq 0$ and $|\varepsilon| > 0$ otherwise. Then

$$W(\varepsilon)\begin{pmatrix} \lambda & 0 & \ldots & 0 & 0 \\ 1 & \lambda & \ldots & 0 & 0 \\ \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & & \cdot & \cdot \\ 0 & 0 & \ldots & 1 & \lambda \end{pmatrix} \quad W^{-1}(\varepsilon) = \begin{pmatrix} \lambda & 0 & \ldots & 0 & 0 \\ \varepsilon & \lambda & \ldots & 0 & 0 \\ \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & & \cdot & \cdot \\ 0 & 0 & \ldots & \varepsilon & \lambda \end{pmatrix}$$

The natural bound of this matrix is equal to $|\varepsilon|$ if $\lambda = 0$ and to $|\lambda|$ if $\lambda \neq 0$.

Let K be a valuated field. To preserve uniformity of terminology we shall call

$$U = \{\alpha \in K \mid |\alpha| = |\alpha^{-1}| = 1\}$$

the group of units of K. For any norm on K_m the condition number of a non-singular matrix A will be defined by

$$c(A) = \|A\| \, \|A^{-1}\| .$$

Definition

Let $\mathcal{S}$ be the unit sphere of a norm on K_m. A matrix $B \in \mathcal{S}$ will be called $\mathcal{S}$-unitary if it has an inverse in $\mathcal{S}$.

It is clear that B is $\mathcal{S}$-unitary if and only if $\|B\| = \|B^{-1}\| = 1$ and that $\mathcal{S}$-unitary matrices form a multiplicative group $\mathcal{U}$ which may be

called the $\mathcal{S}$-unitary group of the norm on K_m. If $B \in \mathcal{U}$, then

$$c(B) = 1 \quad \text{and} \quad \|Bx\| = \|x\|$$

for every consistent vector norm on K^m. Conversely, if B is non-singular and $\|Bx\| = \|x\|$ for all x, then B belongs to the $\mathcal{S}$-unitary group of the associated bound.

Lemma 5.2

Let the matrix B be non-singular and have a non-empty spectrum. Then $B \in \mathcal{U}$ for a norm on K_m if and only if $c(B) = 1$ and the spectrum of B is contained in U.

Proof:

Let $\lambda \in K$ be a characteristic root of B.

$B \in \mathcal{U}$ implies that

$$1 = \frac{1}{\|B^{-1}\|} \leqslant |\lambda| \leqslant \|B\| = 1 \quad \text{so that } \lambda \in U.$$

Conversely we have

$$\|B\| \geqslant |\lambda| = 1 \quad \text{and} \quad \|B^{-1}\| \geqslant \frac{1}{|\lambda|} = 1$$

so that $c(B) = 1$ forces equality.

Lemma 5.3

Let C be a non-singular matrix with a non-empty spectrum. Then for any norm on K_m $c(C) = 1$ if and only if $C = \alpha B$ for some $0 \neq \alpha \in K$ and $B \in \mathcal{U}$.

Proof:

Let $\{\lambda_i\}_{1 \leqslant i \leqslant k}$ be the spectrum of C such that $|\lambda_k| \leqslant \ldots \leqslant |\lambda|$.

$c(C) = 1$ implies that

$$\|C\| = \frac{1}{\|C^{-1}\|} \leq |\lambda_k| \leq \ldots \leq |\lambda_1| \leq \|C\|$$

forcing equality.

Let $\alpha \in K$ such that $|\alpha| = |\lambda_i|$ for $1 \leq i \leq k$. Then by lemma 5.1

$$B = \alpha^{-1}C \in \mathcal{U}.$$

Since the converse is obvious, the result follows.

Theorem 5.4

Let K be a non-Archimedian valuated field, $K = \mathbb{R}$ or $K = \mathbb{C}$. Let $\| \ \|$ be a norm on K^m with the proviso in the first case that $\| \ \|$ be non-Archimedian with $W_{K^m} \subset W_K$. Then a non-singular matrix $C = \alpha B$, $0 \neq \alpha \in K$, $B \in \mathcal{U}$ for the associated bound on K_m if and only if $\text{lub}\,(C^{-1}AC) = \text{lub}\,(A)$ for all $A \in K_m$.

Proof:

Since the conclusion is obvious when C has the required form, it suffices to prove the converse.

Let S be the unit sphere in K^m. Then in all cases $\|x\| = \|x\|_S$ and $\text{lub}\,(A) = \text{lub}_S(A)$ for all $x \in K^m$, $A \in K_m$. By lemma 4.3 and similar results in the other cases, we have

$$\text{lub}_S(A) = \text{lub}_{CS}(A).$$

Hence by lemma 4.1

$$\|x\|_S = a\|x\|_{CS}, \quad a > 0.$$

Further, when K is non-Archimedian lemma 4.3 shows that the value set of $\| \ \|_{CS}$ is contained in W_K. Hence in all cases, $a = |\alpha|$ for some $\alpha \in K$.

Thus

$$CS = \alpha S.$$

It follows that

$$\text{lub}_S(C) = \sup_{x \in S} \|Cx\|_S = |\alpha| \sup_{x \in S} \|x\|_S = |\alpha|$$

Similarly $\text{lub}_B(C^{-1}) = \frac{1}{|\alpha|}$. Hence $c(C) = 1$ and the result follows from lemma 5.3 (it is easy to see that the constant involved is actually α).

Lemma 5.4

Let K be a complete non-Archimedian field, $\| \ \|$ a non-Archimedian norm on K_m and $\mathcal{Q} = \{A \in K_m \mid \|A\| < 1\}$. Then $\mathcal{G} = \{A \in K_m \mid A = I - H, H \in \mathcal{Q}\}$ is a normal subgroup of $\mathcal{U}$.

Proof:

K_m is complete and so $\mathcal{G} \subset \mathcal{U}$ by theorem 3.4.

Let $B_1, B_2 \in \mathcal{G}$. Then:

$$B_1 B_2 = (I - H_1)(I - H_2) = I - H_1 - H_2 + H_1 H_2$$

with $$\|H_1 + H_2 - H_1 H_2\| \leq \max(\|H_1\|, \|H_2\|) < 1.$$

Again, $$B_1^{-1} = I - C, \quad \text{where } C = I - (I - H_1)^{-1} \text{ and}$$

$$\|C\| = \Big\| \sum_{i=1}^{\infty} H_1^i \Big\| \leq \max_i \|H^i\| = \|H\| < 1 .$$

The normality of $\mathcal{G}$ is clear.

Remark

With an obvious extension of the definitions this lemma will hold in any complete non-Archimedian algebra.

Theorem 5.5

Let K be a complete non-Archimedian field with R, P and U as defined in § 2. Let A be a matrix in K_m and $I - A = (\beta_{ij})$ with $\beta_{ij} \in P$ for

all i, j. Then A is non-singular with a spectrum contained in U. Further, all the entries of A^{-1} are in R and $A^{-1} = I - C$ with the spectrum of C contained in P.

Proof:

If the natural bound $\mathrm{lub}_N((\alpha_{ij})) = \max_{i,j} |\alpha_{ij}|$ is defined on K_m, then $(\beta_{ij}) \in \mathcal{Q}$ and $A \in \mathcal{U}$ by the previous lemma.

This proves the assertion concerning the entries of A^{-1}, while an application of lemma 5.2 yields the spectral property of A (as well as of A^{-1}).

Again, $C \in \mathcal{Q}$ and so $r(C) \leqslant \mathrm{lub}_N(C) < 1$.

§ 6. p-Adic Numerical Analysis.

Let the field K be complete with respect to a non-trivial non-Archimedian valuation and let A be a matrix over K. By § 1 the geometric series $\sum_{i=0}^{\infty} A^i$ converges with respect to a non-Archimedian norm if and only if $A^i \to 0$. Since convergence in K_m is equivalent to coordinate-wise convergence, the situation is exactly the same as for real matrices (see [6], p.113). Here too $A^i \to 0$ if and only if $r(A) < 1$ in the algebraic closure of K. Further, if $r(A) < 1$, then there exists a non-Archimedian norm on K_m such that $\|A\| < 1$ (theorem 5.3). In this event, by theorem 3.4 $I - A$ is non-singular and $(I - A)^{-1} = \sum_{i=0}^{\infty} A^i$. Put

$$S_n = \sum_{i=0}^{n} A^i ,$$

so that

$$(I - A)^{-1} - S_n = \sum_{i=n+1}^{\infty} A^i = (I - A)^{-1} A^{n+1} .$$

Since $\|A^{i+1}\| \leqslant \|A\|\ \|A^i\| < \|A^i\|$, $\left\| \sum_{i=n+1}^{\infty} A^i \right\| = \max_{i \geqslant n+1} \|A^i\| = \|A^{n+1}\|$.

In fact, we have obtained the following estimate for the rate of convergence of a geometric series:

Theorem 6.1

If $\|\ \|$ is a non-Archimedian matrix norm and $\|A\| < 1$, then $\|(I - A)^{-1} - \sum_{i=0}^{n} A^i\| = \|A^{n+1}\| \leqslant \|A\|^{n+1}$.

Remark

The above estimate is sharper than the corresponding estimate for real or complex matrices ([6]; [9], p.54). Actually we are dealing with a special case, for we also have $\|(I - A)^{-1} - S_n\| \leqslant \|(I - A)^{-1}\|\ \|A\|^{n+1}$ (see [9]). In our case, however, $\|(I - A)^{-1}\| = 1$. The same observation applies to the next theorem.

Theorem 6.2 (cf [9], p.55).

Let A be a non-singular matrix and C an approximation to A^{-1}. Let $H = I - AC$, $\|H\| < 1$, where $\|\ \|$ is non-Archimedian. Then:

(a) $\|A^{-1}\| = \|C\|$

(b) $\|A^{-1} - C\| \leqslant \|CH\|$

(c) $\|A - C^{-1}\| \leqslant \|HA\|$

Proof:

$I - H$ is non-singular and $\|I - H\| = \|(I - H)^{-1}\| = 1$.

Now, (a) follows from $A^{-1} = C(I - H)^{-1}$,

$C = A^{-1}(I - H)$;

(b) from $A^{-1} = C + CH(I - H)^{-1}$;

and (c) from $C^{-1} = A + (I - H)^{-1}HA$.

Methods of Successive Approximation

Let $x^{\star}$ be an approximate solution for a system of linear equations

$$Ax = h, \quad A \text{ non-singular.}$$

Define the error and residual vectors respectively by

$$s = x - x^{\star} \quad \text{and} \quad r = h - Ax^{\star} = As.$$

For any consistent non-Archimedian vector and matrix norms we have

$$\|s\| \leqslant \|C\| \, \|r\|,$$

where C is an approximation to A^{-1} such that $\|H\| < 1$.

In a method of successive approximation for the solution of such a system, given an initial approximation x_0, a sequence of approximating vectors is formed by the recursion formula

$$x_{n+1} = x_n + C_n r_n ,$$

where $C_0, C_1, \ldots$ is a certain sequence of matrices. We note that

$$x_{n+1} - x_n = s_n - s_{n+1} .$$

If

$$H_n = I - C_n A$$

then

$$s_{n+1} = H_n s_n , \quad r_{n+1} = A H_n A^{-1} r_n$$

Hence,

$$\|s_{n+1}\| \leqslant \|H_n\| \, \|s_n\|$$

and a sufficient condition for convergence is $\|H_n\| < 1 \quad \forall n$.

For non-Archimedian norms this condition implies that

$$\|x_{n+1} - x_n\| = \|s_n\|$$

so that

$$\|s_{n+1}\| \leq \|H_n\| \, \|x_{n+1} - x_n\| \quad \text{(cf [9], p.94).}$$

In the classical method of successive approximation

$$C_n = I, \quad H_n = I - A = H \quad \text{for all } n.$$

$$\begin{aligned} x_{n+1} &= Hx_n + h \\ &= H^{n+1}x_0 + \sum_{i=0}^{n} H^i h \, . \end{aligned}$$

If $\|H\| < 1$, we have in the non-Archimedian case

$$\begin{aligned} \|s_{n+1}\| &= \|(I - H)^{-1}h - \sum_{i=0}^{n} H^i h - H^{n+1}x_0\| \\ &= \|\sum_{i=n+1}^{\infty} H^i h - H^{n+1}x_0\| \\ &\leq \max\left(\|H\|^{n+1} \, \|h\|, \, \|H\|^{n+1} \, \|x_0\|\right) \quad \text{(cf [6], p.183).} \end{aligned}$$

If $x_0 = h$, then

$$x_n = \sum_{i=0}^{n} H^i h \quad \text{(p.185), and}$$

$$\|s_n\| = \|\sum_{i=n+1}^{\infty} H^i h\| = \|H^{n+1}h\| \leq \|H\|^{n+1} \, \|h\| \, .$$

Again, consider methods of successive approximation for finding the inverse of a non-singular matrix. If A is such a matrix and X_0 is an initial approximation to A^{-1}, error and residual matrices are defined by

$$S_n = A^{-1} - X_n \quad \text{and} \quad R_n = AS_n = I - AX_n \, .$$

All such methods depend on the possibility of finding a sequence of matrices C_n such that the sequence (X_n) tends to A^{-1} as a limit, where

$$X_{n+1} = X_n + C_n R_n .$$

If $C_n = X_n$, we have

$$X_{n+1} = X_n(I + R_n).$$

$$R_{n+1} = R_n^2 , \text{ and}$$

$$S_{n+1} = S_n A S_n .$$

Hence for any norm,

$$\|R_{n+1}\| \leq \|R_n\|^2 \quad \text{and} \quad \|S_{n+1}\| \leq \|A\| \, \|S_n\|^2,$$

and the process converges quadratically if either $\|R_n\| < 1$ or $\|S_n\| < 1$ $\forall n$ ([9], p.95).

In fact, since

$$R_n = R_o^{2^n} \quad \text{and} \quad S_n = A^{-1}R_n = X_o(I - R_o)^{-1}R_o^{2^n} , \quad \text{we have, if } \|R_o\| < 1$$

$$\|S_n\| = \|X_o R_o^{2^n}\| \leq \|X_o\| \, \|R_o\|^{2^n} \quad \text{(cf [6], p. 159).}$$

Let $K = Q_p$ be the field of p-adic numbers with the usual p-adic valuation. We shall express the elements of Q_p in decimal notation ([1],p.36) and note that for $a \in Q_p$ with $|a|_p < 1$ the number of zeros after the decimal point equals ord $a - 1$, where

$$\text{ord } a = -\log_p |a|_p$$

is the ordinal of a. As vector and matrix norms we shall use the natural norm $\|x\| = \max_i |\xi_i|_p$ and the corresponding natural bound $\|A\| = \max_{i,j} |\alpha_{ij}|_p$. We shall extend the concept of the ordinal function to these norms and observe that for any method of successive approximation

$$\text{ord } s_n - 1, \quad n > n_o ,$$

gives the number of correct decimal places in approximating vector x_n and that a similar statement applies to matrices. To fix our ideas we shall take $p = 5$.

Example 1

Consider the following system of 5-adic equations:

$$1.324444\,\xi_1 - .02\,\xi_2 - .12\,\xi_3 = .43$$
$$-.02\,\xi_1 + 1.404444\,\xi_2 - .04\,\xi_3 = .03$$
$$-.12\,\xi_1 - .04\,\xi_2 + 1.304444\,\xi_3 = 1.12$$

It will be seen that the coefficient matrix of the system satisfies the hypothesis of theorem 5.5 and is therefore non-singular. The table below presents the solution of this system by Gaussian elimination (see [6], p.130).

Single - Division Scheme

1.324444 -.02 -.12	-.02 1.404444 -.04	-.12 -.04 1.304444	.43 .03 1.12
1	-.024244 1.404014 -.042310	-.140024 -.042310 1.342032	.412310 .033341 1.110133
	1	-.041102 1.342421	.031231 1.110300
1	1	1	1.322413 .020241 .042020

Example 2

The equation in example 1 may be written in the following form:

$$\xi_1 = .22\xi_1 + .02\xi_2 + .12\xi_3 + .43$$

$$\xi_2 = .02\xi_1 + .14\xi_2 + .04\xi_3 + .03$$

$$\xi_3 = .12\xi_1 + .04\xi_2 + .24\xi_3 + 1.12 .$$

The solution by the classical method of successive approximation may be arranged as follows ([7], p.125):

Method of Successive Approximation

Formula: $x_{n+1} = Hx_n + h.$

H	.22 .02 .12	.02 .14 .04	.12 .04 .24
$x_0 = h$	.43	.03	1.12
x_1	.001040	.021401	1.321011
x_2	.044233	.020034	1.323244
x_3	.042013	.020201	1.322334
x_4	.042013	.020243	1.322441
x_5	.042021	.020241	1.322413

We note that

$$\text{ord } s_5 \geq -\log_5 \|H\|^6 \|h\| = 6$$

so that x_5 must be correct to 5 decimal places at least. A glance at the previous result shows that this is indeed the case. An alternative method of successive approximation (ibid, p.124) yields an exact measure of the accuracy of each approximating vector.

Method of Successive Approximation

Formula: $x_n = \sum_{i=0}^{n} H^i h$

h	.43	.03	1.12
Hh	.110040	.040401	.201011
H^2h	.043242	.004023	.002233
H^3h	.003224	.000220	.004044
H^4h	.000000	.000043	.000112
H^5h	.000013	.000003	.000021
H^6h	.000004	.000000	.000000
x_5	.042021	.020241	1.322413
x_6	.042020	.020241	1.322413

Here,

$$\text{ord } s_5 = \text{ord } H^6 h = 6,$$

while

$$\text{ord } s_6 \geqslant -\log_5 \|H\|^7 \|h\| = 7.$$

Thus x_5 and x_6 must be correct to 5 and 6 decimal places respectively. The latter is precisely the accuracy of example 1, for rounding errors do not occur in arithmetical operations with p-adic numbers.

Example 3

The coefficient matrix A of example 1 may be inverted by Gaussian elimination. To 6 decimals we get

$$A^{-1} = \begin{pmatrix} 1.222421 & .020320 & .113120 \\ .020320 & 1.100110 & .044002 \\ .113120 & .044002 & 1.240130 \end{pmatrix}$$

As the initial approximation for an iterative solution we shall take

$$X_0 = \begin{pmatrix} 1.22 & .02 & .11 \\ .02 & 1.10 & .04 \\ .11 & .04 & 1.24 \end{pmatrix}$$

so that

$$R_0 = -\begin{pmatrix} .003224 & .000124 & .002404 \\ .000214 & .000404 & .001323 \\ .002114 & .001204 & .000223 \end{pmatrix}$$

Thus

$$\text{ord } S_1 \geqslant -\log_5 \|X_0\| \|R_0\|^2 = 6 ,$$

and

$$X_1 = X_0 + X_0R_0$$

$$= \begin{pmatrix} 1.222423 & .020323 & .113124 \\ .020323 & 1.100114 & .044002 \\ .113124 & .044002 & 1.240130 \end{pmatrix}$$

is correct to 5 decimal places (cf [6], p.160).

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