TWO-DIMENSIONAL DILATON BLACK HOLES

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Abstract

In this thesis, we study toy models of two-dimensional gravity. We first review two known models: the classical and quantum corrected CGHS models and the quantum corrected model of RST. These two models have black holes solutions with curvature singularities, similar to the Schwarzschild black hole. This singularity becomes naked in the RST model at a certain event during the evaporation. In the third chapter, we build a more general version with new quantum corrections beyond those presented in the RST model, which enable us to find a model without curvature singularities. We will also see that these new quantum corrections can affect the rate of Hawking radiation flowing from the black hole.

Résumé

Dans cette thèse, nous étudions des modèles de gravité dilatonique bi-dimensionnelle. Nous allons premièrement aborder deux modèles existant: les modèles classique et quantique de CGHS, ainsi que le modèle de RST. Ces deux modèles possèdent des solutions de types trous noirs ayant une singularité semblable à celle de la solution de Schwarzschild. Dans le cas du modèle de RST, cette singularité évolue jusqu'à être nue à un certain événement lors de l'évaporation de Hawking. Dans le troisième chapitre, nous construisons une version généralisée des corrections quantiques introduites précédemment, nous permettant ainsi de construire un modèle ne contenant pas les singularités rencontrées dans les modèles précédents. De plus, ces corrections quantiques peuvent affecter le flux de la radiation de Hawking émise par le trou noir.

Preface

This thesis is the first brick in the wall of an old dream. The most important contributor to this thesis is surely my supervisor Professor R.C. Myers who always encouraged and supported me during the evolution of this work. I would also like to thank my lovely girlfriend, Marie-Hélène, for her love and support during some difficult moments of my work. I also have to thank my father Claude for the careful readings and remarks on the numerous drafts of this thesis. There is also my whole family for their support, even if a black hole is really strange, but maybe poetic for them. Finally, I thank the NSERC for its financial support.

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Introduction

The Universe. The large scale world where mankind is living has always been the subject of our admiration for its mysterious beauty. However, Man always wants more than just observation and has the desire to understand the laws driving our Universe. Many good scientists have spent their lives to improve our understanding of the Universe. One of the most famous is surely Sir Isaac Newton. He was the first physicist to give a good physical and mathematical description of the force that rules the large scale world, gravity.

His work remained the Bible of physicists for more than two hundred years. Then, Albert Einstein came at the dawn of this century with his theory of Special Relativity (1905) and one decade later with a modern theory of gravity (1914), the General Relativity. This beautiful theory gives a different explanation of phenomena already described by Newton's laws, but it also goes beyond that. It predicted new amazing phenomena such as deflection of light, gravitational red-shift, gravitational lensing and, one of the most famous, black hole. The latter will be the main subject of this thesis, so it is necessary to give a brief outlook of its origin.

When a huge cloud of dust and particles collapses, it heats until nuclear reactions start in its core. The energy radiated away by these nuclear reactions stops the collapse and we obtain a stable star such our Sun. After the star has burned most of its nuclear fuel, the collapse starts again and the future of the star depends on its mass. For stars like the Sun, *i.e.* around one solar mass ($M_\odot$), the stellar evolution model predicts that the star will end its life as a white dwarf. For masses around $2M_\odot$, the remnant will be a neutron star. In each of these two cases, it is the quantum Pauli exclusion principle that stops the collapse, leaving a stable remnant that will cool down with time.

For bigger stars of masses above 3 or $4M_\odot$, there are no known processes that can stop the collapse, and the star will undergo complete gravitational collapse to form a black hole. A black hole is a very spectacular object: below a certain limit, called the horizon, nothing can escape the black hole. Even light is trapped by the gravitational field. This is why they are called *black* holes.

These stellar monsters were not present in the gravitational theory of Newton; they are children of Einstein's General Relativity. The first appearance[1] of black holes as a solution of Einstein's equation was found by Schwarzschild in 1916 [3] and it describes spherically symmetric, non-rotating and uncharged black holes. Note that the interpretation of this solution as black hole was not yet understood until a few years later. After this first appearance in physicists' world, new type of black holes were discovered: axially symmetric, rotating and charged.

As described above, all black holes were thought to be completely black, and completely invisible from direct observations. In 1975, S. W. Hawking [4] shocked the world of physics when he showed that black holes are not entirely black: they radiate away energy. This spectacular result was obtained by using the tools of the other important branch of modern physics: quantum field theory. He proved by a semiclassical argument that black holes radiate energy when one includes quantum mechanical effects in the classical theory of General Relativity. Here, the word semiclassical, means that we keep the spacetime fixed.

Usually, we should expect the black hole space-time to be modified by the Hawking evaporation since if it radiates away energy, its mass should decrease. Because the spacetime curvature is mass dependent, we expect the spacetime to be modified. Let us point out that as far as we know, there is no satisfactory scenario for the end point of Hawking radiation (for an hypothesis, see [5]). Physicists hope that a complete theory of quantum gravity will eventually emerges and provide an answer to this question.

Such a theory would describe gravity in a complete quantum formalism. Unfortunately, we do not have this theory yet. So, how could we take backreaction into account? One approach is to look for toy models in which we can compute backreaction in an exact solution. The use of these toy models will, we hope, help us to understand how to build a quantum theory in a realistic case.

[1] However, during the 18th century, Laplace studied heavy objects from which light cannot escape, within the context of Newton's gravity [1]. See also appendix A of [2].

Usually these toy models are formulated in $1 + 1$ dimensions where quantum effects are easier to understand. One such model was proposed in 1992 by C.G. Callan, S.B. Giddings, J.A. Harvey and A. Strominger (CGHS) [6] in which they used a string-inspired action. Their model has classical black hole solutions and they have been able to include quantum corrections. A lot of work has been done on this model and the related ones (see for example [7, 8, 9, 10]) and there is surely more to be done. Especially, there are some models like the one of J.G. Russo, L. Susskind and L. Thorlacius (RST) [8] that can be solved exactly, including backreaction. This thesis is based on a model of this form. These models add counterterms to the one-loop corrected CGHS model to make it exactly solvable, but we will see that these new counterterms will affect the rate of Hawking radiation.

We will first review the two-dimensional model of CGHS, because it forms the basis of two-dimensional dilaton gravity models. In the second chapter, we will focus on the model of RST where a new counterterm is added to the quantum corrected CGHS model. The next chapter is the main part of the thesis and is a generalization of the RST model, which has strong effects on the rate of Hawking radiation and the creation of spacetime singularities. Finally, we will conclude by emphasizing on the results of the generalized model. Note that a short discussion of the Bondi mass for black hole's physics in two dimensions is presented in appendix A.

Chapter 1

The CGHS Model

In this chapter we will review the famous work of C.G. Callan, S.B. Giddings, J.A. Harvey and A. Strominger [6]. This model was developed for the study of black holes evaporation in two dimensions and it is known as a two-dimensional dilaton gravity theory[1]. They used a "classical" model containing black hole solutions and then they attempted to quantize it, taking advantage of the simplicity of the two-dimensional character of the model.

The classical action used by CGHS was already present in string theory as an effective action describing the radial modes of extremal dilatonic black holes in four or higher dimensions [6]. Apart from the origin of the action, it has an interesting feature: it is renormalizable. This is a great advantage for this toy model because as far as we know, all the theories of fundamental interactions are renormalizable and it is conjectured that the quantum theory of gravity should also be renormalizable. Some previous models of quantum gravity were not renormalizable and it was shown that they faced serious flaws.

This chapter will begin with the description of the classical solutions obtained from the classical action. In the following section, we will see how CGHS included some quantum effects in the theory, enabling them to describe Hawking radiation in a fixed background.

[1]Two-dimensional models are not recent in gravity physics and have been studied even before the discovery of Hawking [11]. Since Hawking's paper, a lot of work has been done on the evaporation of four-dimensional spherically symmetric black holes, which are effectively described by the two-dimensional metric of the $r-t$ coordinates. See for example, the work of Unruh [12] and Hajicek [13] and references therein.

1.1 Classical Solutions

In this section, we will look at the classical solutions derived from the classical action studied by CGHS. We will focus our attention on the solutions describing black holes, *i.e.* solutions with an event horizon and a physical singularity. Such solutions are similar to the radial part of the four-dimensional Schwarzschild black hole. The classical action describes dilaton gravity coupled to N massless scalar fields:

$$S = \frac{1}{2\pi}\int d^2x\sqrt{-g}\left[e^{-2\phi}[R+4(\nabla\phi)^2+4\lambda^2] - \frac{1}{2}\sum_{i=1}^{N}(\nabla f_i)^2\right] . \tag{1.1}$$

In this equation, g, ϕ and f_i represent the metric, dilaton and matter fields respectively. The constant λ is part of a dilaton dependent cosmological constant $e^{-2\phi}\lambda^2$. As it is common in classical physics, the equations of motion for the various fields will be derived from the minimum action principle, which means that $\delta S = 0$. The coefficients of $\delta\phi$ will give the equation of motion for the dilatonic field and similarly for the matter fields f_i. The terms proportional to δg_{ab}, the variation of the metric, will give us a set (three) of equations of motion for the metric. By functional differentiation, we obtain the covariant equations of motion of the dilaton and the matter fields:

$$\begin{aligned}
\frac{2\pi}{\sqrt{-g}}\frac{\delta S}{\delta\phi} &= 8g^{ab}\nabla_a\phi\nabla_b\phi - \left[R+4(\nabla\phi)^2+4\lambda^2\right] - 4\nabla^2\phi = 0 &(1.2)\\
\frac{2\pi}{\sqrt{-g}}\frac{\delta S}{\delta f_i} &= \nabla^2 f_i = 0 &(1.3)
\end{aligned}$$

and the metric covariant equations of motion (stress-energy tensor) are

$$\begin{aligned}
T_{ab} = \frac{2\pi}{\sqrt{-g}}\frac{\delta S}{\delta g^{ab}} &= \frac{1}{2}g_{ab}\left\{e^{-2\phi}\left[R+4(\nabla\phi)^2+4\lambda^2\right] - \frac{1}{2}\sum_{i=1}^{N}(\nabla f_i)^2\right\}\\
&\quad + e^{-2\phi}\left\{-2\nabla_a\nabla_b\phi - g_{ab}g^{cd}(4\partial_c\phi\partial_d\phi - 2\partial_c\partial_d\phi) - R_{ab}\right\}\\
&\quad + \frac{1}{2}\sum_{i=1}^{N}\nabla_a f_i\nabla_b f_i = 0 . &(1.4)
\end{aligned}$$

Covariance means that tensors transform according to simple rules under coordinate transformations (diffeomorphisms) [3].

Now, we would like to remove diffeomorphism invariance by imposing a particular form to the metric. However, this will not take rid of all coordinates invariance since a residual freedom will remain in the theory (see discussion before (1.19) below). It is common, and useful, in this simple model to work with the conformal gauge in which the two-dimensional metric takes the form:

$$g_{+-} = -\frac{1}{2}e^{2\rho} \ , \ g_{--} = g_{++} = 0 \tag{1.5}$$

where the null coordinates are defined by $x^{\pm} = (x^0 \pm x^1)$. With this choice of the metric, we have for the metric-related objects:

$$\begin{aligned} R_{+-} &= -2\partial_+\partial_-\rho \quad , \quad R = 8e^{-2\rho}\partial_+\partial_-\rho & (1.6)\\ (\nabla\phi)^2 &= -4e^{-2\rho}\partial_+\phi\partial_-\phi & (1.7)\\ \Gamma^{\pm}_{\pm\pm} &= 2\partial_\pm\rho & (1.8)\\ \nabla^2\phi &= -4e^{-2\rho}\partial_+\partial_-\phi & (1.9) \end{aligned}$$

where all other components of the Christoffel's symbol and the Ricci tensor are vanishing. Now all the information about the spacetime is encoded in the conformal field ρ. From the three metric equations, it turns out that one of them is the equation of motion (T_{+-}) and the two others are constraints ($T_{\pm\pm}$). The latter are called contraints since they are obtained from the functional differentiation of the action with respect to metric's components which are set to zero in 1.5. The components of T_{ab} are:

$$\begin{aligned} T_{+-} &= e^{-2\phi}\left(2\partial_+\partial_-\phi - 4\partial_+\phi\partial_-\phi - \lambda^2 e^{2\rho}\right) = 0 & (1.10)\\ T_{\pm\pm} &= e^{-2\phi}\left(4\partial_\pm\rho\partial_\pm\phi - 2\partial_\pm^2\phi\right) + \frac{1}{2}\sum_{i=1}^{N}\partial_\pm f_i\partial_\pm f_i = 0 \ , & (1.11) \end{aligned}$$

while the dilaton and matter equations of motion will be respectively given by:

$$\begin{aligned} e^{-2\phi}\left(-4\partial_+\partial_-\phi + 4\partial_+\phi\partial_-\phi + 2\partial_+\partial_-\rho + \lambda^2 e^{2\rho}\right) &= 0 & (1.12)\\ \partial_+\partial_- f_i &= 0 & (1.13) \end{aligned}$$

for all i. Let us notice that we have $N+2$ functions to solve, *i.e.* f_i, ϕ and ρ, and have $N+4$ equations for them. However, these equations are not all independent by virture of conservation of the stress-energy tensor $\nabla^a T_{ab} = 0$, which is a reflection of the covariance of the theory. This system of $N+4$ differential equations has a current equation that will be really helpful all along this work. This current is given by adding T_{+-} to the dilaton equation of motion:

$$\partial_+\partial_-(\rho - \phi) = 0 \ . \tag{1.14}$$

This enables us to write down a simple relation between ρ and ϕ:

$$\rho - \phi = \frac{1}{2}\left(w_+(x^+) + w_-(x^-)\right) \tag{1.15}$$

where $w_\pm(x^\pm)$ are called gauge functions for reasons that will become clear in a moment. A general solution of the equations of motion (1.10) and (1.12) is given by:

$$\begin{aligned} e^{-2\phi} &= u_+ + u_- - \lambda^2 \int^{x^+} dy^+ e^{w_+} \int^{x^-} dy^- e^{w_-} && (1.16) \\ e^{-2\rho} &= e^{-(w_+ + w_-)}\left[u_+ + u_- - \lambda^2 \int^{x^+} dy^+ e^{w_+} \int^{x^-} dy^- e^{w_-}\right] . && (1.17) \end{aligned}$$

Now, we have to solve for the free fields $u_\pm$ and $w_\pm$. We can solve for $u_\pm$ by substituting our solution for ϕ and ρ in the two constraints $T_{\pm\pm} = 0$. This procedure gives us a solution in term of $w_\pm$:

$$u_\pm(x^\pm) = \frac{M}{2\lambda} - \frac{1}{2}\int^{x^\pm} dy^\pm \left\{ e^{w_\pm} \int^{y^\pm} dz^\pm e^{-w_\pm} \sum_{i=1}^{N} \partial_\pm f_i \partial_\pm f_i \right\} \tag{1.18}$$

where M is an integration constant. And what about $w_\pm$? None of our equations of motion can make it explicit. However, the choice of the conformal gauge that we made before leaves a subgroup of diffeomorphism unfixed. This can be seen if we look at a coordinate transformation $\{x^\pm\} \to \{\sigma^\pm\}$ of the form:

$$x^\pm = h^\pm(\sigma^\pm) . \tag{1.19}$$

The metric will transform as a tensor by the rule:

$$\tilde{g}_{a'b'} = \Lambda^a{}_{a'}\Lambda^b{}_{b'} g_{ab} \tag{1.20}$$

where the transformation factor is given by $\Lambda^a{}_b = \frac{\partial h^a}{\partial \sigma^b}$ with a, b taking values $\pm$. For the conformal metric, we obtain:

$$\begin{aligned} \tilde{g}_{\pm\pm} &= 0 \\ \tilde{g}_{+-} &= -\frac{1}{2}\partial_{\sigma^+} h^+ \partial_{\sigma^-} h^- e^{2\rho} \\ &= -\frac{1}{2} e^{2\tilde{\rho}} \end{aligned} \tag{1.21}$$

where we defined a new conformal field $\tilde{\rho}$:

$$\tilde{\rho} = \rho + \frac{1}{2}\left[\ln\left[\partial_{\sigma^+} h^+\right] + \ln\left[\partial_{\sigma^-} h^-\right]\right] . \tag{1.22}$$

Thus, one sees that we recover a conformal metric by a coordinate transformation described by $h^{\pm}(\sigma^{\pm})$ which means that there is a subgroup of diffeomorphism that preserves the conformal gauge. From the simple current equation, we can relate the functions $h^{\pm}$ to the gauge functions $w_{\pm}$. If one starts with a system where $w_{\pm} = 0$, so that $\rho = \phi$ and makes a coordinate transformation to another system of coordinates $\{\sigma^{\pm}\}$, we will have the relation

$$\tilde{\rho} = \phi + \frac{1}{2}\ln\left[\partial_{\sigma^+}h^+\partial_{\sigma^-}h^-\right] \ . \tag{1.23}$$

Thus, from the simple current relation (1.15) we can write:

$$w_{\pm} = \ln\left[\partial_{\sigma^{\pm}}h^{\pm}(\sigma^{\pm}(x^{\pm}))\right] \ . \tag{1.24}$$

In other words, a particular choice for the gauge functions $w_{\pm}$ is simply a choice of the set of coordinates we will use. Thus, we will choose the simplest expression for the gauge functions, namely:

$$w_{\pm}(x^{\pm}) = 0 \tag{1.25}$$

which is called the Kruskal gauge. This choice simplifies the solutions and we obtain:

$$\begin{aligned} e^{-2\phi} = e^{-2\rho} \ &= \ \frac{M}{\lambda} - \lambda^2 x^+ x^- \\ &\quad - \frac{1}{2}\left[x^+\int^{x^+} dy^+ \sum_{i=1}^{N} \partial_+ f_i \partial_+ f_i + x^- \int^{x^-} dy^- \sum_{i=1}^{N} \partial_- f_i \partial_- f_i\right] . \end{aligned} \tag{1.26}$$

We still have to solve for the matter fields f_i, but the matter equation of motion (1.13) can easily be integrated out to yield the general solution:

$$f_i = f_{i+}(x^+) + f_{i-}(x^-) \tag{1.27}$$

and the special solutions will depend on the matter distribution we want to study. This completes the derivation of the general solution of the classical action (1.1).

Now we will look at some special cases of the general solution (1.26). We first look at the system where there are no matter fields: $f_i = 0$ for all i. Thus, the solution (1.26) becomes:

$$e^{-2\phi} = e^{-2\rho} = \frac{M}{\lambda} - \lambda^2 x^+ x^- \ . \tag{1.28}$$

The case $M = 0$ is simply the linear dilaton vacuum as it appears in higher-dimensional dilaton gravity. It has this name because this vacuum state is a linear function of the coordinates, in the Minkowskian vacuum ($\phi = \frac{\lambda}{2}(\sigma^- - \sigma^+)$) as we will see later on. When the constant M is different than zero, we have a black hole of mass M. The solution is similar to the $r-t$ plane of the (static) Schwarzschild black hole, even if the metric does not have exactly the same form. Here the line element is written as:

$$\begin{aligned} ds^2 &= -\frac{1}{2}e^{2\rho}dx^+dx^- \\ &= -\frac{1}{2}\frac{dx^+dx^-}{M/\lambda - \lambda^2 x^+x^-} . \end{aligned} \tag{1.29}$$

This spacetime has a physical singularity (*i.e.* $R \to \infty$) at $\lambda^2 x^+ x^- = M/\lambda$.

In order to well understand the geometry of this spacetime, we can construct its Penrose (conformal) diagram. A Penrose diagram enables us to describe the infinite spacetime within a diagram of finite dimensions. This is done by performing the following conformal transformation of the coordinates:

$$x^{\pm} = \sqrt{\frac{M}{\lambda^3}}\tan(q^{\pm}) \tag{1.30}$$

where

$$-\frac{\pi}{2} < q^{\pm} < \frac{\pi}{2} . \tag{1.31}$$

According to (1.22), the conformal factor of the classical black hole (1.28) becomes:

$$-2g_{+-} = e^{2\tilde{\rho}} = \left[\cos q^+ \cos q^- \cos(q^+ + q^-)\right]^{-1} . \tag{1.32}$$

The Penrose diagram constructed from this metric is depicted in figure 1.1. The points i^0 corresponds to the spatial infinity where $x \to \pm\infty$ at time $t = 0$. The two other points i^- and i^+ are the past and future timelike infinities, respectively, for $x = 0$. The four lines (called null infinities) $\mathcal{J}^{\pm}_{R,L}$ represents the regions at infinity joining the four points described above. The physical singularities are obtained from the Ricci scalar

$$\begin{aligned} R &= 8e^{-2\tilde{\rho}}\partial_{q^+}\partial_{q^-}\tilde{\rho} \\ &= 4\cos q^+ \cos q^- \left[\cos(q^+ + q^-) + \sin(q^+ + q^-)\tan(q^+ + q^-)\right] \end{aligned} \tag{1.33}$$

which diverges for $q^+ + q^- = \pm\frac{\pi}{2}$. We readily see that this Penrose diagram is identical to the diagram of the extended Schwarzschild spacetime, built from the well known metric [3]:

$$ds^2 = -\left(1 - \frac{2M}{r}\right)dt^2 + \left(1 - \frac{2M}{r}\right)^{-1}dr^2 + r^2 d\Omega^2 . \tag{1.34}$$

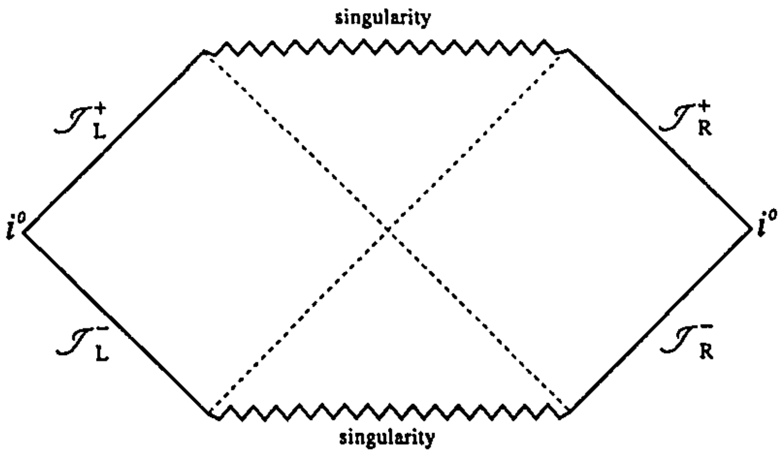

Figure 1.1: Penrose diagram of the extended Schwarzschild black hole.

This shows that the classical black hole described by (1.28) has the same causal structure as the r-t part of the four-dimensional Schwarzschild metric.

The fact that the constant M is the mass of the black hole is not obvious *a priori*, but it can be seen by a computation of the Bondi mass, as done in Appendix A. So, we have obtained a solution describing a static and uncharged black hole in vacuum. Apart from this solution, we could imagine a solution describing the collapse of matter, creating a black hole. One possible choice for such a collapsing configuration is given by the stress-energy tensor:

$$\begin{aligned} T^f_{++} &= \frac{1}{2}\sum_{i=1}^{N}\partial_+ f_i \partial_+ f_i = m\delta(x^+ - x_0^+) \\ T^f_{--} &= \frac{1}{2}\sum_{i=1}^{N}\partial_- f_i \partial_- f_i = 0 \end{aligned} \tag{1.35}$$

which represents an infalling shock wave with amplitude m. This stress-energy tensor can be obtained from the singular limit of a gaussian wavepacket. By inserting these relations in the equations for the functions $u_\pm$ in Kruskal gauge, we perform the integral and we obtain the solution for a collapsing matter wave:

$$e^{-2\phi} = e^{-2\rho} = -m(x^+ - x_0^+)\theta(x^+ - x_0^+) - \lambda^2 x^+ x^- . \tag{1.36}$$

This spacetime is depicted in figure 1.2.

For $x^+ < x_0^+$ this solution represents the linear dilaton vacuum discussed above. So,

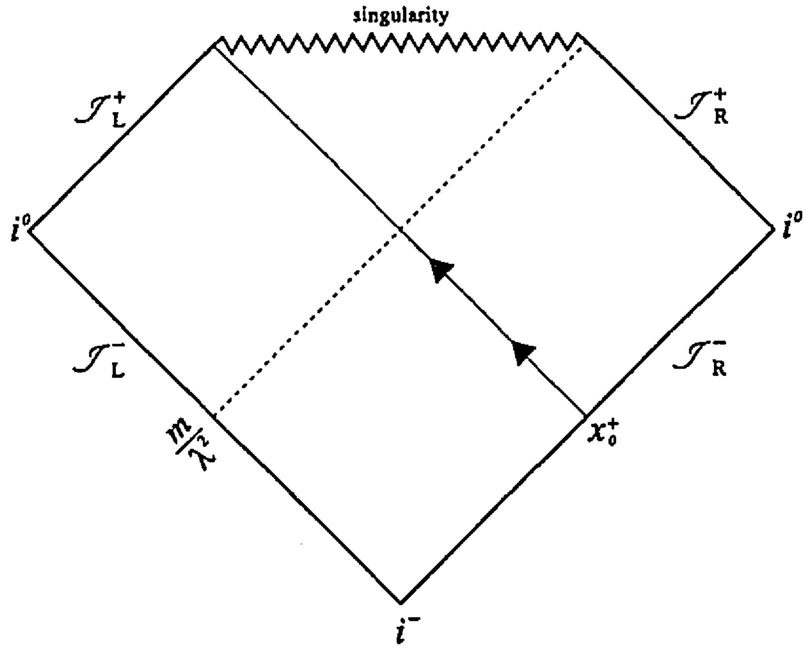

Figure 1.2: Penrose diagram of infalling matter creating a classical black hole.

observers in this region of spacetime do not see any black hole. For $x^+ > x_0^+$, we can shift x^- by m/λ^2 and we obtain a classical black hole of mass $mx_0^+\lambda$, by comparison with the solution (1.28). Again the mass can be computed with the Bondi mass method (see Appendix A).

In this section, we have only solved classical equations; there was no quantum effects on our system. We can conjecture that the inclusion of such effects in the theory would make the black hole emit energy, according to the famous conclusion of S.W. Hawking: black holes evaporate [4]. We will see in the next section how to take some quantum effects into account in this model.

1.2 One-Loop Corrections

In this section, we will show how to include quantum effects in the classical theory presented in the previous section. Obviously, our goal is to study Hawking radiation emanating from black holes. In quantum mechanics, there are two principal formalisms for the quantization of a system: the canonical and the path integral formalism. The latter is today's most popular and most efficient method in field theory and we will use it for the quantization of the model of dilaton-gravity coupled to matter (1.1).

Following the usual procedure of path integral, we will look at the functional integral:

$$Z = \int \mathcal{D}(g, \phi, f_i) e^{iS_0 + iS_M} \tag{1.37}$$

where S_0 is the dilaton gravity action and S_M is the matter action, both in (1.1), and $\mathcal{D}(g, \phi, f_i)$ represents the measures for the metric, dilaton and matter fields. It can be found in the literature how to integrate the matter functional integral at first order in the loop expansion [14], and it gives the familiar Polyakov-Liouville action:

$$\int \mathcal{D}(f_i) e^{iS_M} = e^{i\kappa S_{PL}} \tag{1.38}$$

with

$$S_{PL} = -\frac{1}{8\pi} \int d^2x \sqrt{-g} R \frac{1}{\nabla^2} R \tag{1.39}$$

where we defined $\kappa = N/12$ and $x = x^{\pm}$. The Green's function $\frac{1}{\nabla^2} = G(x,y)$ of the d'Alembertian satisfies $\nabla_x^2 G(x,y) = \delta^2(x-y)$. Thus, our system is now described by the path integral:

$$Z = \int \mathcal{D}(g, \phi) e^{iS_0 + i\kappa S_{PL}} \tag{1.40}$$

and then we have to solve the equations of motion for this one-loop corrected action. Let us note something about the matter functional integral. The integration leading to the Polyakov-Liouville action is, in some sense, arbitrary. This means that the functional integral over the matter fields f_i permits the addition of local, covariant counterterms to S_{PL}. In this chapter, we will not add such arbitrary terms and we will only keep the Polyakov-Liouville action. However, in the next chapters, we will see how the addition of such counterterms will modify the theory and how we can make a solvable quantum-corrected theory.

We do not want to solve the full quantum theory, which is beyond the scope of this thesis. We only want to solve the semi-classical system, using the minimum action principle, as done in the classical case. We will also work in the large N limit, where the contributions from the ghosts, dilaton and conformal measure to the effective action are negligible. This requirement will hold for the CGHS model, but it will not be used in the discussion of the generalized model in the third chapter of this thesis. So, we can derive the stress-energy tensor's components T_{ab} from the effective action $S = S_0 + \kappa S_{\text{PL}}$ and we obtain:

$$T_{ab} = T_{ab}^{\rm CGHS} - \frac{\kappa}{4} T_{ab}^{\rm quant} = 0 \tag{1.41}$$

where

$$\begin{aligned} T_{ab}^{\rm quant} &= 4 g_{ab} g^{cd} \partial_c \partial_d \rho_0 + 4 \Gamma^c_{ab} \partial_c \rho_0 - 4 \partial_a \partial_b \rho_0 - 2 g_{ab} \rho_0 R + 4 \rho_0 R_{ab} \\ &\quad + 2 g_{ab} g^{cd} \partial_c \rho_0 \partial_d \rho_0 - 4 \partial_a \rho_0 \partial_b \rho_0 \, . \end{aligned} \tag{1.42}$$

In this stress-energy tensor, the conformal field ρ_0 comes from the application of the Green's function $\frac{1}{\nabla^2}$ on the Ricci scalar R:

$$\frac{1}{\nabla^2} R = \int d^2 y \; G(x,y) \; R(y) = -2\rho_0 \, . \tag{1.43}$$

We are denoting the conformal field arising from the Green's function differently because we could want to use a different reference vacuum for the propagator of the massless scalar field and then relate it to ρ by a coordinate transformation[2]. On the other hand, the dilaton equation of motion (1.12) remains unchanged by the addition of quantum corrections, since (1.39) is dilaton independent. Thus, we now have a quantum correction to the stress-energy tensor, whose components are:

$$\begin{aligned} T_{\pm\pm}^{\rm quant} &= 8 \partial_\pm \rho \partial_\pm \rho_0 - 4 \partial_\pm \rho_0 \partial_\pm \rho_0 - 4 \partial_\pm^2 \rho_0 && (1.44) \\ T_{+-}^{\rm quant} &= 4 \partial_+ \partial_- \rho_0 \, . && (1.45) \end{aligned}$$

We can relate the conformal field ρ_0 to the field ρ by a relation $\rho_0 = \rho + v_0$ where $v_0 = v_0(x^+, x^-)$ is an arbitrary function. We can constrain the form of the function v_0 by applying the operator ∇_x^2 on the left-hand side of equation (1.43):

$$\int d^2 y \nabla_x^2 G(x,y) \; R(y) = R(x) = 8 e^{-2\rho} \partial_+ \partial_- \rho \tag{1.46}$$

where we used $\nabla_x^2 G(x,y) = \delta^2(x-y)$. For the right-hand side, we have:

$$-2\nabla_x^2 (\rho + v_0) = 8 e^{-2\rho} \partial_+ \partial_- \rho + 8 e^{-2\rho} \partial_+ \partial_- v_0 \, . \tag{1.47}$$

Thus, the equality will hold if $\nabla_x^2 v_0 = 0$ *i.e.* if $v_0 = \frac{1}{2}(v_+(x^+) + v_-(x^-))$, yielding:

$$\rho_0 = \rho + \frac{1}{2} \left(v_+(x^+) + v_-(x^-) \right) \, . \tag{1.48}$$

[2]For the CGHS model, we will take $\rho_0 = \rho$, so that the vacuum of the propagator is defined using the present coordinates where $g_{+-} = -\frac{1}{2} e^{2\rho}$. However, this equality will not hold in the next chapter where we study the RST model.

In this case, the quantum stress-energy tensor's components will be given by

$$T^{\text{quant}}_{\pm\pm} = -4\left(\partial^2_\pm \rho - \partial_\pm \rho \partial_\pm \rho\right) - \frac{4}{\kappa} t_\pm(x^\pm) \quad (1.49)$$

$$T^{\text{quant}}_{+-} = 4\partial_+\partial_- \rho \quad (1.50)$$

where we defined

$$t_\pm(x^\pm) = \frac{\kappa}{4}\left[2\partial^2_\pm v_\pm + \partial_\pm v_\pm \partial_\pm v_\pm\right] \ . \quad (1.51)$$

Thus we see that a complete quantum stress-energy tensor must include the functions $t_\pm(x^\pm)$. These functions are related to the zero-modes ambiguity of the Green's function $G(x,y) = \frac{1}{\nabla^2}$ as it can be seen in the following derivation [15]. Let us look at the eigenfunctions of the d'Alembertian ∇^2:

$$\nabla^2_x \omega_i(x) = \lambda_i \omega_i(x) \quad (1.52)$$

where λ_i is the eigenvalue corresponding to the eigenfunction $\omega_i(x)$ and $x = x^\pm$. We can use these eigenfunctions as a basis for function, say $\psi(x)$, so that we can decompose them on this basis:

$$\psi(x) = \sum_i a_i \omega_i(x) \ . \quad (1.53)$$

These eigenfunctions also satisfy an orthogonality relation:

$$\int d^2x \ \omega_i(x)\omega_j(x) = \delta_{ij} \ . \quad (1.54)$$

One would like to write the Green's function $G(x,y)$ in the basis $\{\omega_i\}$, *i.e.* we must determine the constants a_i for this function. We will show that the correct decomposition for the Green's function would be:

$$G(x,y) = \sum_i \frac{1}{\lambda_i}\omega_i(x)\omega_i(y) \ . \quad (1.55)$$

To check this expression, we will use the definition of the Green's function $\nabla^2_x G(x,y) = \delta^2(x-y)$ and the integral of the Dirac delta function:

$$\int d^2y \ \delta^2(x-y)\psi(y) = \psi(x) \ . \quad (1.56)$$

Using (1.52) and (1.55) we obtain:

$$\nabla^2_x G(x,y) = \sum_i \omega_i(x)\omega_i(y) \quad (1.57)$$

and we have to check if it is equal to the Dirac delta-function $\delta^2(x-y)$. In fact, we can show that the RHS of (1.57) behaves like a Dirac delta function:

$$\begin{aligned} \int d^2y \sum_i \omega_i(x)\omega_i(y) \sum_k a_k\omega_k(y) &= \sum_{i,k} a_k\omega_i(x)\delta_{ik} \\ &= \sum_i a_i\omega_i(x) \\ &= \psi(x) \end{aligned} \tag{1.58}$$

where we used the orthogonality relation (1.54). Thus, we see that the decomposition (1.55) is appropriate for the Green's function $G(x,y)$. On the other hand, this decomposition of the Green's function is obviously undetermined for the zero mode $\lambda_i = 0$. For these modes, we have $\nabla_x^2\omega_0(x) = 0$ which has the same form as the condition on the function $v_0(x)$ that generated the functions $t_\pm(x^\pm)$ discussed above. This establishes the relation between the zero modes ambiguity of the Green's function and the functions $t_\pm(x^\pm)$.

The explicit form of these functions will be determined in the next section, where we compute Hawking radiation in a fixed background. In the next chapter, we will see another way of determining these functions, which is more convenient in some generalizations beyond the CGHS model. In the third chapter, we will see that the two methods used lead to different Hawking radiation rates and we will discuss this discrepancy.

Now, we have a semiclassical theory of two-dimensional gravity which has black hole solutions. As shown by S.W. Hawking [4], including quantum corrections in a classical black hole solution will force the black hole to evaporate. In the next section, we perform a first computation of Hawking radiation in a classically fixed background.

1.3 Hawking Radiation in a Fixed Background

Now we will make a first calculation of Hawking radiation. The following calculation does not include the backreaction effect on the spacetime caused by the decreasing of the mass of the evaporating black hole. The computation will be done for the collapsing matter wave solution (1.36), which will be our fixed spacetime. Of course, this method of computation is only an approximation because in reality the spacetime will be modified during the evaporation, as a consequence of the decrease of black hole's mass. Anyway, it is very instructive to see some qualitative features of the Hawking radiation for this system.

Before going into direct computation, we will perform a coordinate transformation on the metric (1.36). The new coordinate system will often be referred to as asymptotically Minkowskian, because it is a manifestly Minkowski flat spacetime "far" from the origin (asymptotic region) and will be much more natural for the description of the black hole evaporation. We use it because in Minkowskian coordinates, the notion of particles is well defined in field theory, while it is not the case in a general curved spacetime, where curvature can "create" particles, leading to a problem in the definition of particles [16]. The coordinate transformation $\{x^\pm\} \to \{\sigma^\pm\}$, for (1.36) with $x^+ > x_0^+$, is defined as:

$$x^+ = \frac{1}{\lambda} e^{\lambda\sigma^+} \tag{1.59}$$

and

$$x^- = -\frac{1}{\lambda} e^{-\lambda\sigma^-} - \frac{m}{\lambda^2}\,. \tag{1.60}$$

For the region below the infall line, the line element in σ-coordinates is flat Minkowskian:

$$ds^2 = -\frac{1}{2} d\sigma^+ d\sigma^- \tag{1.61}$$

and it is asymptotically flat on $\mathcal{J}_R^+$, *i.e.* in the limit $\sigma^+ \to +\infty$. This latter limit corresponds to the limit $x^+ \to +\infty$ in the x-coordinates. Note that this coordinate system describes the spacetime only above the classical event horizon $x^- = \frac{m}{\lambda^2}$. According to equation (1.22) and following, the conformal field does not transform as a scalar. Thus, using the transformations (1.59) and (1.60), the spacetime (1.36) for the collapsing matter will now be written as:

$$\begin{aligned} e^{2\tilde{\rho}} &= \left[1 + \left(\tfrac{m}{\lambda}\right) e^{\lambda\sigma^-}\right]^{-1} && \text{for } \sigma^+ < \sigma_0^+ \\ &= \left[1 + \left(\tfrac{m}{\lambda}\right) e^{\lambda(\sigma^- - \sigma^+ + \sigma_0^+)}\right]^{-1} && \text{for } \sigma^+ > \sigma_0^+ \end{aligned} \tag{1.62}$$

and the components of the quantum stress-energy tensor become simply:

$$\tilde{T}^{\text{quant}}_{\pm\pm} = 4[\partial_\pm\tilde{\rho}\partial_\pm\tilde{\rho} - \partial_\pm^2\tilde{\rho}] - \frac{4}{\kappa}\tilde{t}_\pm(\sigma^\pm) \tag{1.63}$$

$$\tilde{T}^{\text{quant}}_{+-} = \partial_+\partial_-\tilde{\rho} \tag{1.64}$$

where $\partial_\pm$ now represents a derivative with respect to $\sigma^\pm$ and $\tilde{t}_\pm(\sigma^\pm)$ is defined in terms of $t_\pm(x^\pm)$ by:

$$\tilde{t}_\pm(\sigma^\pm) = e^{\pm\lambda\sigma^\pm} t_\pm(x^\pm(\sigma^\pm)) + \frac{\kappa\lambda^2}{4} \tag{1.65}$$

for both regions $x^+ < x_0^+$ and $x^+ > x_0^+$.

Because this fixed spacetime satisfies the classical equations of motion, so that $T_{ab}^{\rm CGHS} = 0$, the quantum expectation value of the stress-energy tensor's components will only be the quantum corrections:

$$\langle \tilde{T}_{ab} \rangle = -\frac{\kappa}{4} \tilde{T}_{ab}^{\rm quant} \tag{1.66}$$

We still have to find the correct expression for the functions $\tilde{t}_\pm$. For this, we apply boundary conditions on the quantum expectation values, and the most natural one in this case, is to require that the quantum expectations values vanish in the linear dilaton vacuum, which is present at $\sigma^+ < \sigma_0^+$. As a result of this condition , the complete equation of motion $\tilde{T}_{ab}^{\rm CGHS} - \frac{\kappa}{4} \tilde{T}_{ab}^{\rm quant} = 0$ will be satisfied. This requirement ($\langle \tilde{T}_{ab} \rangle = 0$) forces the functions $\tilde{t}_\pm(x^\pm)$ to be:

$$\begin{aligned} \tilde{t}_+(\sigma^+) &= 0 \\ \tilde{t}_-(\sigma^-) &= -\frac{\kappa}{4}\lambda^2 \left[1 - \left(1 + \frac{m}{\lambda} e^{\lambda\sigma^-}\right)^{-2}\right] . \end{aligned} \tag{1.67}$$

This completes the expression for the stress-energy tensor in the spacetime. Now let us evaluate this solution at the future null infinity $\mathcal{J}_R^+$, far from the black hole, *i.e.* in the limit $\sigma^+ \to \infty$. In this limit we recover a Minkowski metric since $\tilde{\rho} \to 0$, leaving vanishing values for $\langle T_{++} \rangle$ and $\langle T_{+-} \rangle$, but:

$$\langle T_{--} \rangle \to \frac{\kappa\lambda^2}{4} \left[1 - (1 + \frac{m}{\lambda} e^{\lambda\sigma^-})^{-2}\right] \tag{1.68}$$

thus, far observers will detect energy coming from the black hole. This represents a flux of f-matter particles reaching the future null infinity $\mathcal{J}_R^+$ and it is interpreted as Hawking radiation flowing from an evaporating black hole, since $\langle \tilde{T}_{ab} \rangle_{\rm LDV} = 0$. There are some interesting features about this radiation. First of all, it tends to zero at the spacelike infinity i^0 ($\sigma^- \to -\infty$). This result corresponds to the intuitive fact that there is no Hawking radiation when observers are unaware of the formation of a black hole. However, it is not surprising that at the future timelike infinity i^+, Hawking radiation does **not** stop. This is a consequence of the use of a fixed background; because the black hole geometry does not change, its mass remains the same and it can evaporate forever. This non-stopping radiation has a constant flux that asymptotically ($\sigma^- \to \infty$) tends to $\frac{N\lambda^2}{48}$, which is curiously mass-independent. This is a characteristic feature of two-dimensional gravity [6], while in four dimensions it is mass-dependent [4]. Moreover, this unending flux is unreasonable since as radiation is flowing out, the mass should decrease and reach an endpoint where radiation will stop. So, because this computation is somehow incomplete, we would like to compute Hawking radiation by including the backreaction on the geometry. This might be done by a computation of the Bondi mass (see Appendix A) for the one-loop corrected solutions. Unfortunately, it cannot be done exactly for this model because the quantum corrected equations of motions are not exactly solvable. We will see in the next chapters how to modify the theory so that we obtain a solvable quantum corrected theory.

Chapter 2

The RST Model

In this chapter we will examine an interesting variation of the CGHS model, studied in the previous chapter. The work presented here was done by J.G. Russo, L. Susskind and L. Thorlacius (RST) [8] and was an attempt to make the semiclassical CGHS theory exactly solvable at the one-loop level. They have essentially added a new local, covariant counterterm and made a field redefinition, which leads to a solvable Liouville theory. It is not the only solvable model obtained from a variation of the CGHS model. A. Bilal and C. Callan [7] studied one of these CGHS inspired model. They essentially modified the cosmological constant to make the model solvable. The RST model has the advantage of having exact classical solutions, but, as we will see, the new fields do not span the whole real axis giving rise to a spacetime singularity. We will first see how they modified the CGHS one-loop corrected model by adding another counterterm. In the next section, we will solve the semiclassical equations using a fields redefinition. At the end, we will consider some features of the solutions.

2.1 New Counterterm

The important point of this section, is that the matter path integral (1.39) may be changed by the addition of local covariant counterterms since, in the functional integration, the measure is ill-defined and can be modified by such counterterms. The addition of these new counterterms may have strong effects on the theory. For example, in the present chapter, the added counterterm will produce a solvable semiclassical theory as we will see. Russo *et al.* investigated the modified action $S = S_0 + S_1 + S_2$ where S_0 is the CGHS action and

$$S_1 = \kappa S_{\rm PL} = -\frac{\kappa}{8\pi}\int d^2x\sqrt{-g}\;\; R\frac{1}{\nabla^2}R \tag{2.1}$$

is the Polyakov-Liouville action derived in the CGHS model, with $\kappa = \frac{N}{12}$. The last term

$$S_2 = -\frac{\kappa}{8\pi}\int d^2x\sqrt{-g}\;\; 2\phi R \tag{2.2}$$

is the new local counterterm they added to the theory. Again, we will use the conformal gauge (1.5) in which the action takes the form:

$$\begin{aligned} S &= \frac{1}{\pi}\int d^2x\Big\{e^{-2\phi}(2\partial_+\partial_-\rho - 4\partial_+\phi\partial_-\phi + \lambda^2e^{2\rho}) + \frac{1}{2}\sum_{i=1}^{N}\partial_+f_i\partial_-f_i \\ &\quad - \kappa(\partial_+\rho\partial_-\rho + \phi\partial_+\partial_-\rho)\Big\}\,. \end{aligned} \tag{2.3}$$

As before, we use the minimum action principle to derive the equations of motion for the metric, matter and dilaton fields. This procedure is correct for a semiclassical theory, which might differ from the full quantum theory. These equations will essentially be the same as in the previous chapter, apart from some extra terms coming from the new counterterm. We obtain for the stress-energy tensor's components:

$$T_{ab} = T_{ab}^{(0)} - \frac{\kappa}{4}\left(T_{ab}^{(1)} + T_{ab}^{(2)}\right) = 0 \tag{2.4}$$

where $T_{ab}^{(0)} = T_{ab}^{\rm CGHS}$ is given by (1.4) and:

$$\begin{aligned} T_{ab}^{(1)} &= -2g_{ab}\rho R - 4\partial_a\partial_b\rho + 4\Gamma^c_{ab}\partial_c\rho + 4g_{ab}g^{cd}\partial_c\partial_d\rho \\ &\quad + 4\rho R_{ab} + 2g_{ab}g^{cd}\partial_c\rho\partial_d\rho - 4\partial_a\rho\partial_b\rho \\ T_{ab}^{(2)} &= g_{ab}\phi R + 2\partial_a\partial_b\phi - 2\Gamma^c_{ab}\partial_c\phi - 2g_{ab}\partial_c\partial_d\phi - 2\phi R_{ab}\,. \end{aligned} \tag{2.5}$$

Note that everything is expressed in terms of the conformal field ρ and there is no ρ_0. The contributions of the latter are incorporated in the functions $t_\pm(x^\pm)$, which are part of $T_{\pm\pm}$, and remain to be determined (see discussion following equation (1.45)). Explicitly, we obtain one equation of motion and two constraints for the metric:

$$\begin{aligned} T_{\pm\pm} &= \left(e^{-2\phi} + \frac{\kappa}{4}\right)\left(4\partial_\pm\rho\partial_\pm\phi - 2\partial_\pm^2\phi\right) + \frac{1}{2}\sum_{i=1}^{N}\partial_\pm f_i\partial_\pm f_i \\ &\quad + \kappa\left(\partial_\pm^2\rho - \partial_\pm\rho\partial_\pm\rho\right) + t_\pm(x^\pm) \\ T_{+-} &= e^{-2\phi}\left(2\partial_+\partial_-\phi - 4\partial_+\phi\partial_-\phi - \lambda^2e^{2\rho}\right) \\ &\quad + \frac{\kappa}{2}\left(\partial_+\partial_-\phi - 2\partial_+\partial_-\rho\right)\,. \end{aligned} \tag{2.6}$$

Of course, the N matter equations of motion survive in the same form:

$$\partial_+\partial_- f_i = 0 \ . \tag{2.7}$$

Finally, the dilaton equation of motion is modified to:

$$e^{-2\phi}\left[4\partial_+\partial_-\phi - 2\partial_+\partial_-\rho - 4\partial_+\phi\partial_-\phi - \lambda^2 e^{2\rho}\right] - \frac{\kappa}{2}\partial_+\partial_-\rho = 0 \ . \tag{2.8}$$

All these equations of motion and constraints contain the one-loop corrections and so, they have evaporating black hole solutions. The addition of the counterterm in the one-loop corrected action has the advantage of restoring the classical current $\partial_+\partial_-(\rho-\phi)=0$ at the one loop-level as can be seen by adding the dilaton equation of motion to $T_{+-}=0$. This current was essential for the solvability of the classical CGHS theory and it was destroyed in the one-loop corrected CGHS model [6]. It has been shown recently by Y. Kazama *et al.* that this condition is essential for the solvability of two-dimensional dilaton models [17]. We will see later that this characteristic greatly simplifies the solutions of the RST model.

2.2 Liouville-Like Theory

Now we want to solve the equations of motion derived in the previous section. In their given form, the solutions are not obvious. If we make a field redefinition, we can transform to a simpler solvable theory. So, let us define [8]:

$$\begin{aligned} \Omega &= \frac{1}{2}\phi + \frac{1}{\kappa}e^{-2\phi} \\ X &= \rho - \frac{1}{2}\phi + \frac{1}{\kappa}e^{-2\phi} \ . \end{aligned} \tag{2.9}$$

These definitions change the action to a Liouville-like action:

$$S = \frac{1}{\pi}\int d^2x \left\{\kappa\left[-\partial_+X\partial_-X + \partial_+\Omega\partial_-\Omega + \lambda^2 e^{2(X-\Omega)}\right] + \frac{1}{2}\sum_{i=1}^{N}\partial_+f_i\partial_-f_i\right\} \ . \tag{2.10}$$

Applying the transformations (2.9) to the stress-energy tensor (2.6), we obtain:

$$\begin{aligned} T_{\pm\pm} &= \kappa\left[-\partial_\pm X\partial_\pm X + \partial_\pm\Omega\partial_\pm\Omega + \partial_\pm^2 X\right] \\ &\quad + \frac{1}{2}\sum_{i=1}^{N}\partial_\pm f_i\partial_\pm f_i + t_\pm(x^\pm) = 0 \end{aligned} \tag{2.11}$$

$$T_{+-} = -\kappa\partial_+\partial_- X - \lambda^2 e^{2(X-\Omega)} = 0 \tag{2.12}$$

and the simple current equation may be written as:

$$\partial_+\partial_-(X - \Omega) = 0 \ . \tag{2.13}$$

This theory can be solved exactly. We will derive the general solution of the Liouville theory in the next chapter where we will look at a more general model of dilaton gravity. For now, we want to point out that the new field Ω does not range over the whole real values as we might expect; the function $\Omega(\phi)$ has a minimum at a critical value $\phi = \phi_{\rm crit}$ and this point turns out to be a spacetime singularity.

2.2.1 Curvature singularity

The fact that the field Ω is bound from below gives rise to a curvature singularity. This can be seen by looking at the Ricci scalar (R) of g_{ab}:

$$R = 8e^{-2\rho}\partial_+\partial_-\rho \tag{2.14}$$

in the conformal metric. From the definitions of the Liouville fields, the Laplacian of the conformal field ρ in the Ricci scalar may be rewritten as:

$$\partial_+\partial_-\rho = \frac{1}{\Omega'}\left[\partial_+\partial_- X - \frac{\Omega''}{\Omega'^2}\partial_+\Omega\partial_-\Omega\right] . \tag{2.15}$$

where the prime ($'$) denotes a derivative with respect to the field ϕ. A curvature singularity will arise at $\Omega' \to 0$, *i.e.* when

$$\phi = \phi_{\rm crit} = -\frac{1}{2}\ln\left[\frac{\kappa}{4}\right] . \tag{2.16}$$

This critical value of the dilaton field will define a curve in spacetime, whose characteristic will depend on the particular solution we want to investigate. We will have to be very careful when the dilaton field reaches the critical value on account of the fact that the solutions will not be properly defined on this curve, since the Ricci scalar blows up. One usually wants to impose boundary conditions on the fields on this critical curve to get a well behaved theory. Many authors attempted to find the right boundary conditions and it is beyond the scope of this thesis to go into the details of such problems. We refer the reader to [8, 18] and references therein for such an analysis.

We will see in the next chapter how we can remove this singularity, by taking advantage of the addition of new counterterms. For now, we will look at some solutions of the equations of motions. There are two main types of solutions: eternal and evaporating black holes. The difference between the two types comes from the different choices of the functions $t_\pm(x^\pm)$, which correspond to different choices of reference vacuum for the matter propagator. For eternal black hole, these functions are set to zero, while for the evaporating black holes they are non-vanishing.

2.3 Evaporating black hole

Since the Liouville theory obtained is classically solvable, we can now analyze the different solutions. Before solving for the fields, we must find the right expressions for the functions $t_\pm(x^\pm)$, which arise from the zero-modes ambiguity of the Green's function $\frac{1}{\nabla^2}$ (see chapter 1). In the previous chapter, we determined these functions by requiring the quantum expectation values of the stress-energy tensor to vanish in the linear dilaton vacuum. In the present chapter, we will use a different method, which, in fact, yields the same results. We will define the vacuum of the Green's function $\frac{1}{\nabla^2}$ in the Minkowskian coordinates $\{\sigma^\pm\}$ and then transform to the $\{x^\pm\}$ coordinates system. We use the stress-energy tensor (1.45) derived from the Liouville-Polyakov action, which depends on both ρ_0 and ρ. We already know that the conformal field does not transform as a scalar, but rather transforms according to equation (1.23):

$$\rho_0 = \rho + \frac{1}{2}\left(w_+(x^+) + w_-(x^-)\right) . \tag{2.17}$$

This transformation has the same form as (1.48) if we substitute $v_\pm(x^\pm) \to w_\pm(x^\pm)$. This now makes a relation between the quantum functions $t_\pm(x^\pm)$ and the coordinate transformation which is, from (1.51):

$$t_\pm(x^\pm) = \frac{\kappa}{4}\left[2\partial_\pm^2 w_\pm + \partial_\pm w_\pm \partial_\pm w_\pm\right] . \tag{2.18}$$

Now, we must determine the functions $w_\pm(x^\pm)$ which correspond to a transformation to Minkowskian vacuum. We already know the correct form from the computation of Hawking radiation in a fixed background:

$$w_\pm(x^\pm) = Q \ln\left[\pm\lambda x^\pm\right] , \tag{2.19}$$

where Q is a constant, and it yields to the functions $t_\pm(x^\pm)$:

$$t_\pm(x^\pm) = -\frac{\kappa}{4}\frac{P}{(x^\pm)^2} \tag{2.20}$$

with the redefinition $P = Q(Q-2)$. The constant P labels different Minkowskian vacua with line element given by:

$$ds^2 = -\frac{1}{2}P^2 d\sigma^+ d\sigma^- \ . \tag{2.21}$$

We can solve the equations of motion with these functions in the static vacuum case $T^f_{\pm\pm} = 0$, we obtain the following solution, in the Kruskal gauge, for asymptotically flat spacetime:

$$\kappa\Omega = \kappa X = -\lambda^2 x^+ x^- - \frac{P}{4}\kappa \ln\left[-\lambda^2 x^+ x^-\right] + \frac{M}{\lambda} \tag{2.22}$$

where the constant M labels different solutions. The solution for $P = 0$ is called eternal black hole solution because its mass would not decrease when it evolves. This might seem surprising since we are including quantum effects in these solutions and we would expect that, for this reason, black hole solutions would always evaporate. This can be remedied if one considers the black hole as being in thermal equilibrium with an heat bath at infinity [19]. Note that the fact that the black hole stays in thermal equilibrium indicates that the Hawking temperature of these two-dimensional black hole is independent of the mass [4].

The case where $P \neq 0$, is more interesting because it corresponds to an evaporating black hole. First of all, we examine the limit $M = 0$. By comparison with the transformations (2.9) in Kruskal gauge, we see that it corresponds to the linear dilaton vacuum if $P = 1$. Since this solution is a vacuum solution, the Bondi mass of the system must vanish. This would be the case only if the constant $P = 1$. In the other cases where $P \neq 0$ and $P \neq 1$, the Bondi mass on $\mathcal{J}^+_R$ diverges, and it has no good solutions for the description of black holes physics. Thus one can claim that the most natural choice is $P = 1$ for black holes physics.

A physical stress-energy tensor for the matter is the collapsing shock wave tensor. Now let us look at the solution obtained from the collapse of a matter shock wave as we did for the CGHS model:

$$T^f_{++} = m\delta(x^+ - x^+_0) \tag{2.23}$$

$$T^f_{--} = 0 \ . \tag{2.24}$$

The solution of the equation of motion and constraints is then, in the Kruskal gauge:

$$\kappa\Omega = \kappa X = -\lambda^2 x^+ x^- - \frac{\kappa}{4}\ln\left[\lambda^2 x^+ x^-\right] - m(x^+ - x^+_0)\theta(x^+ - x^+_0) \tag{2.25}$$

where m is the amplitude of the incoming matter wave. First of all, this black hole has an apparent horizon defined by the curve $\partial_+\phi = 0$ [20]. In the Liouville fields, this definition translates to $\partial_+\Omega = 0$, since we can write:

$$\partial_+\Omega = \Omega'\partial_+\phi = 0 \tag{2.26}$$

as long as the function $\Omega'(\phi)$ is well behaved (which is not the case at the spacetime singularity $\Omega' = 0$), the equation defining the apparent horizon may be derived from the above equation. This equation defines a curve (x_H^+, x_H^-) given by the substitution of solution (2.25) above the infall line ($x^+ > x_0^+$) in definition (2.26):

$$x_H^+ = -\frac{\kappa}{4}\frac{1}{\lambda^2 x_H^- + m}\,. \tag{2.27}$$

The singularity also defines a curve (x_s^+, x_s^-) in spacetime, given by the substitution of $\phi_{\rm crit}$ in solution (2.9):

$$1 - \ln\left[\frac{\kappa}{4}\right] = -\frac{4\lambda^2}{\kappa}x_s^+x_s^- - \ln\left[-\lambda^2 x_s^+ x_s^-\right] - \frac{4}{\kappa}m(x_s^+ - x_0^+)\,. \tag{2.28}$$

The rate at which the apparent horizon recedes agrees with the semiclassical calculations of Hawking radiation performed by CGHS [8].

At a certain event in spacetime, the apparent horizon and the singularity will meet each other. The intersection event $(x_H^+, x_H^-) = (x_s^+, x_s^-) = (x_i^+, x_i^-)$, which is above the infall line is:

$$\begin{aligned} x_i^- &= -\frac{m}{\lambda^2}\left(1 - e^{-4mx_0^+/\kappa}\right)^{-1} \\ x_i^+ &= \frac{\kappa}{4m}\left(1 - e^{4mx_0^+/\kappa}\right)\,. \end{aligned} \tag{2.29}$$

At this event, the singularity goes from being spacelike to timelike, giving rise to a naked singularity. Therefore, we cannot determine uniquely the future evolution of the black hole without making any assumption on the boundary conditions on this singularity. What physically happens when a singularity becomes naked is not well understood and Hawking has speculated that the formation of a naked singularity would produce a cataclysm, called a thunderbolt, which would propagate outward at the speed of light [5].

However, the singularity is still present in the vacuum since the critical value $\phi_{\rm crit}$ does not depend on the particular solution we are looking at. So, even in the linear dilaton vacuum we have the problem of a naked singularity and one must also solve this question in order to find out what happen with a naked singularity. As we pointed out in the discussion of curvature singularity, the hope is to apply a proper set of boundary conditions on this singularity and it is still an open question.

We now turn our attention to a generalization of the two dilatonic models studied above. In the next chapter, we build a generalized model for two-dimensional dilaton gravity which will have some interesting features about the rate of Hawking radiation and the presence of the singularity in the quantum corrected model.

Chapter 3

Generalized Model

In this chapter, we will apply the ideas of the previous chapter to a more general one-loop correction to the two-dimensional dilaton gravity of CGHS. We will first introduce our new counterterms, which are a generalization of the RST counterterm. We will see that the rate of Hawking radiation can be affected by the choice of these counterterms. We will impose conditions on the coefficients of these counterterms such that we recover the same current as in the classical CGHS and RST models, namely:

$$\partial_+\partial_-(\rho - \phi) = 0 \; . \tag{3.1}$$

This condition, which makes the theory solvable, will be referred to as the simple current condition.

In the two previous models, the quantization assumed that the contributions from the ghosts was negligible because the models were designed to work in the large N limit, but we will not make this assumption in the present chapter. We will include the ghosts' contribution to the action in an attempt to construct a more complete quantum theory at the one-loop level. This introduction of the ghosts in the theory will give us a conformal field theory as it will be checked by solving the β-functions. After this, we will try to compute Hawking radiation in a fixed background, but the computation will be altered by a non-vanishing vacuum's contribution to the flux. Finally, we will compute the Bondi mass of our evaporating black hole solution.

3.1 One-loop Corrected Action

As in the RST model, we will add counterterms to the usual one-loop anomaly term. Our choice for these counterterms is based on the conditions that they must be local, covariant and have the proper mass dimension, which restrict us to terms with two derivatives of the dilaton. The action we will study will be written as $S = S_0 + S_1 + S_2 + S_3$ where S_0 is the classical CGHS action (1.1) and the other terms are:

$$
\begin{aligned}
S_1 &= -\frac{\kappa}{8\pi}\int d^2x\sqrt{-g}\; R\frac{1}{\nabla^2}R && (3.2)\\
S_2 &= -\frac{\kappa}{8\pi}\int d^2x\sqrt{-g}\;\left[\alpha\phi R + \beta(\nabla\phi)^2\right] && (3.3)\\
S_3 &= -\frac{\kappa}{8\pi}\int d^2x\sqrt{-g}\;\sum_{n=2}^{K}\left[a_n\phi^n R + b_n\phi^{n-1}(\nabla\phi)^2\right] && (3.4)
\end{aligned}
$$

where α, β, a_n and b_n are constants parameters, K is an integer and $\kappa = N/12$. We have separated the $n = 1$ term (*i.e.* S_2) of the sum for simplicity of derivation in the future. The counterterms S_2 and S_3 are new to this thesis, even though some special cases have been studied in the literature. For example, the case $\alpha = 2$ and $\beta = 0$ corresponds to the RST model studied in the previous chapter. Also, the case $\alpha = 4$ and $\beta = -4$ yields a model studied by Bose *et al.* [21]. If we were to proceed as before we would derive the equation of motions and then transform to a Liouville-like theory. This procedure would be right if we assumed that the number of matter fields is large, so that the ghosts' contribution is negligible. However in this chapter, we will include the ghosts' contribution into the equations of motion.

3.1.1 Contribution from the ghosts

The work presented here is based mainly on the work of A. Strominger [22] (see also [23]), in which he describes an original procedure to include the ghost effects in the one-loop corrected two-dimensional dilaton gravity. This procedure will immediately lead us to a conformal field theory.

We first rewrite the action S in term of a fixed background metric $\bar{g}$ defined by the relation:

$$g_{ab} = e^{2\rho}\bar{g}_{ab}\,. \tag{3.5}$$

By substituting this metric in the definition for the Ricci tensor and all metric-related objects used in General Relativity, we can derive the following relations between the fiducial (fixed background) objects and the original one. We have, in two dimensions:

$$\begin{aligned}
\nabla^2 &= e^{-2\rho}\overline{\nabla}^2 \quad , \quad \sqrt{-g} = e^{2\rho}\sqrt{-\bar{g}} \\
R_{ab} &= \bar{R}_{ab} - \bar{g}_{ab}\overline{\nabla}^2\rho \\
R &= e^{-2\rho}\left(\bar{R} - 2\overline{\nabla}^2\rho\right) \\
(\nabla\phi)^2 &= e^{-2\rho}\overline{\nabla}\phi\bar{\cdot}\overline{\nabla}\phi \,.
\end{aligned} \tag{3.6}$$

The symbol $\bar{\cdot}$ means that the dot product is taken with respect to the fiducial metric and similarly for $\overline{\nabla}$. These relations, will transform the action into the following form:

$$\begin{aligned}
S_0 &= \frac{1}{2\pi}\int d^2x\sqrt{-\bar{g}}\left[e^{-2\phi}\left(\bar{R} - 4\overline{\nabla}\phi\bar{\cdot}\overline{\nabla}\rho + 4\overline{\nabla}\phi\bar{\cdot}\overline{\nabla}\phi + 4\lambda^2 e^{2\rho}\right) - \frac{1}{2}\sum_{i=1}^{N}(\overline{\nabla}f_i)^2\right] \\
S_1 &= -\frac{\kappa}{8\pi}\int d^2x\sqrt{-\bar{g}}\left[\bar{R}\frac{1}{\overline{\nabla}^2}\bar{R} - 4\rho\bar{R} - 4\overline{\nabla}\rho\bar{\cdot}\overline{\nabla}\rho\right] \\
S_2 &= -\frac{\kappa}{8\pi}\int d^2x\sqrt{-\bar{g}}\left[\alpha\phi\bar{R} + 2\alpha\phi\overline{\nabla}\phi\bar{\cdot}\overline{\nabla}\rho + \beta\overline{\nabla}\phi\bar{\cdot}\overline{\nabla}\phi\right] \\
S_3 &= -\frac{\kappa}{8\pi}\int d^2x\sqrt{-\bar{g}}\sum_{n=2}^{K}\left[a_n\phi^n\bar{R} + 2a_n n\phi^{n-1}\overline{\nabla}\phi\bar{\cdot}\overline{\nabla}\rho + b_n\phi^{n-1}\overline{\nabla}\phi\bar{\cdot}\overline{\nabla}\phi\right] \,.
\end{aligned} \tag{3.7}$$

This action includes the one-loop contribution of the matter only. However, as it is well known in quantum field theory, a more complete quantum theory also includes terms arising from the measure of others fields present in the theory and also from the ghosts' fields. Previous attempts to build a correct action including all measure terms, simply shifted the constant κ from $\frac{N}{12}$ to $\frac{N-24}{12}$. Unfortunately, this shift brings some problematic results when we look at the Hawking radiation of the black hole.

Usually, the purpose of the introduction of the ghosts in quantum field theory, is to simplify the computations of Feynman diagrams. It is only a mathematical trick and the ghosts cannot be considered as real particles and they should never appear as free particles in any process. The problem here is that the simple shift mentioned before makes black holes to Hawking radiate ghosts. In fact, the black hole will gain mass by Hawking radiating negative-energy ghosts, which is certainly unphysical, when we perform a fixed background computation of Hawking radiation. Thus, we must find a different method to include ghosts in the quantum corrected two-dimensional dilaton gravity.

A. Strominger proposed a solution to this problem in [22]. Mainly, his proposition was to use a different metric to define the measure of the dilaton's and metric's path integrals. Instead of using the standard metric $g_{ab} = e^{2\rho}\bar{g}_{ab}$, we should use the "shifted" metric $e^{-2\phi}g_{ab} = e^{2(\rho-\phi)}\bar{g}_{ab}$. This kind of shift is permitted in the theory, because of the freedom we have to add local, covariant counterterms at the one-loop level. Of course, other choices for the ghosts metric are permitted by this freedom, but it is easier to use the simplest one.

So, the ghosts will now contribute to the action by shifting the action:

$$N\, S^{\text{quant}}(\bar{g},\rho) \rightarrow N\, S^{\text{quant}}(\bar{g},\rho) - 24\, S^{\text{quant}}(\bar{g},\rho-\phi) \tag{3.8}$$

where S^{quant} is the complete one-loop correction $(1/N)(S_1 + S_2 + S_3)$. By adding this term to the action derived earlier, we obtain the complete action:

$$\begin{aligned} S = \frac{1}{2\pi}\int d^2x \quad \sqrt{-\bar{g}} \quad &\Big\{\Big(4e^{-2\phi} - \beta\gamma' - \gamma' b_n\phi^n - \alpha - a_n n\phi^{n-1} - 2\Big)\,\overline{\nabla}\phi\bar{\cdot}\overline{\nabla}\phi \\ &+ \Big(-4e^{-2\phi} - 2\alpha\gamma' - 2a_n n\gamma'\phi^{n-1} + 4\Big)\,\overline{\nabla}\phi\bar{\cdot}\overline{\nabla}\rho \\ &+ 4\gamma'\overline{\nabla}\rho\bar{\cdot}\overline{\nabla}\rho + 4\lambda^2 e^{2(\rho-\phi)} - \frac{1}{2}\sum_{i=1}^{N}\overline{\nabla}f_i\bar{\cdot}\overline{\nabla}f_i \\ &+ \Big[e^{-2\phi} + 2\phi + \gamma'\Big(4\rho - \bar{R}\frac{1}{\overline{\overline{\nabla}}} - \alpha\phi - a_n\phi^n\Big)\Big]\,\bar{R}\Big\} \end{aligned} \tag{3.9}$$

where we defined

$$\gamma' = \frac{1}{4}\gamma = \frac{1}{4}(\kappa - 2) = \frac{1}{4}\frac{(N-24)}{12}\,. \tag{3.10}$$

For simplicity, we dropped the summation sign over n but the reader must keep in mind it is still present. Now we will analyse the action (3.9) in a similar way as done before. However, let first look at the conformal character of the theory.

3.1.2 Conformal invariance

A theory will be called conformally invariant if it is invariant under conformal transformations like:

$$g \rightarrow g' = \Omega g \tag{3.11}$$

where Ω is some conformal factor. Usually, quantum gravity theories are conformally invariant and we will look if it is the case for our model. For more information, see [16, 23, 24, 25].

We will check if our model is conformally invariant, which is a consequence of the covariance of General Relativity. A conformally invariant theory may be written in the form of a σ-model theory. Note that string theory is such a theory:

$$S(g,X) = -\frac{1}{2\pi}\int d^2x\sqrt{-\bar{g}}\,\left[G_{\mu\nu}(X^\lambda)\overline{\nabla}X^\mu\cdot\overline{\nabla}X^\nu + \frac{1}{2}\Phi(X^\lambda)\bar{R} + T(X^\lambda)\right] \qquad (3.12)$$

where g is the fiducial metric described before and $X^\lambda = (\phi, \rho, f_i)$. The quantum theory described by this action will be conformally invariant if it satisfies three sets of equations called β-functions. These functions are usually very difficult to solve exactly and one must look at a small parameter (usually $\hbar$) expansion of the theory to solve these functions. For dilaton gravity, such a parameter would obviously be $e^{2\phi}$. The reason for this is that the Lagrangian in the classical action (1.1) is proportional to $e^{-2\phi}$ and when we built the one-loop correction we assumed $e^{-2\phi} \gg 1$ for the perturbation expansion. Then the first order β-functions are:

$$\begin{aligned}
\beta^T &= \frac{1}{2}\nabla_\mu\Phi\nabla^\mu T - 2T - \frac{1}{4}\nabla^2 T + \ldots = 0 \\
\beta^0_{\mu\nu} &= \nabla_\mu\nabla_\nu\Phi + \frac{1}{2}\mathcal{R}_{\mu\nu} + \ldots = 0 \\
\beta^\Phi &= \frac{1}{2}(\nabla\Phi)^2 - \frac{1}{4}\nabla^2\Phi + 2\Upsilon + \mathcal{R} + \ldots = 0
\end{aligned} \qquad (3.13)$$

where the covariant derivatives are taken with respect to the metric $G_{\mu\nu}$, and $\mathcal{R}_{\mu\nu}$ is the Ricci tensor of $G_{\mu\nu}$. The dots $(\ldots)$ stand for the terms of higher order in the expansion. If we compare the action (3.12) with our action (3.9), we can write down the following expression for $G_{\mu\nu}$, Φ and T:

$$\begin{aligned}
G_{\phi\phi} &= -4e^{-2\phi} + \beta\Upsilon' + b_n\Upsilon'\phi^{n-1} + \alpha + a_n n\phi^{n-1} + 2 \\
G_{\phi\rho} &= 2e^{-2\phi} + \alpha\Upsilon' + a_n\Upsilon' n\phi^{n-1} - 2 \\
G_{\rho\rho} &= -4\Upsilon' \\
G_{f_i f_i} &= \frac{1}{2} \\
\Phi &= -2e^{-2\phi} - 8\Upsilon'\rho + 2\alpha\Upsilon'\phi + 2a_n\Upsilon'\phi^n - 4\phi + 2\Upsilon'\bar{R}\frac{1}{\overline{\nabla}^2} \\
T &= -4\lambda^2 e^{2(\rho-\phi)}\,.
\end{aligned} \qquad (3.14)$$

Substituting these equations in the β-functions (3.14), we can check that β^Φ and $\beta^0_{\mu\nu}$ are exactly satisfied at first order in the parameter expansion. For the tachyon β-function, β^T, the first order terms give an expression proportional to $e^{2\phi}$. However, we must remember that our expansion parameter in the β-functions is $e^{2\phi}$, hence $e^{2\phi} \ll 1$. Thus, all β-functions are satisfied at first order and we can conclude that we have a conformal field theory, at least to leading order in $e^{2\phi}$.

Now the task is to solve the model developed so far, which includes ghosts and is conformally invariant. We will solve it from a minimum action principle (*i.e.* by using $\delta S = 0$), as we did for the two previous models.

3.2 Equations of Motion

As done before, we have to derive the stress-energy tensor components, the dilaton and the matter equations of motion. For this purpose, we need to rewrite the action (3.9) in terms of the metric g instead of the fiducial metric $\bar{g}$. In order to recover the conformal metric (1.5), we require the fiducial metric to be flat:

$$\bar{g}_{\pm\pm} = 0 \ , \ \bar{g}_{+-} = -\frac{1}{2} \ . \tag{3.15}$$

Substituting this metric in the action (3.9) and after some algebra, we can write the action in the covariant form:

$$\begin{aligned} S = \frac{1}{\pi}\int d^2x \quad \sqrt{-g} \quad & \left\{ \frac{1}{2}e^{-2\phi}\left[R + 4(\nabla\phi)^2 + 4\lambda^2\right] + \frac{1}{2}\sum_{i=1}^{N}(\nabla f_i)^2 \right. \\ & - \frac{\gamma}{8}R\frac{1}{\nabla^2}R - \frac{\gamma}{8}\left[\alpha\phi R + \beta(\nabla\phi)^2\right] \\ & - \frac{\gamma}{8}\sum_{n=2}^{K}\left[a_n\phi^n R + b_n\phi^{n-1}(\nabla\phi)^2\right] \\ & \left. - \frac{1}{2}\left[\alpha + 2 + \sum_{n=2}^{K} a_n n\phi^{n-1}\right](\nabla\phi)^2 + \phi R \right\} \end{aligned} \tag{3.16}$$

where we have reinstated the sum over n. This is essentially the same action as (3.4) except that κ has been replaced by γ and we gained some extra terms from the introduction of the ghosts, dilaton and metric measures. This action shows explicitly that the prescription of A. Strominger for the contribution from the ghosts is different from the shift $\kappa \to \kappa - 2$ used before. We can simplify the above action by redefining the constant parameters:

$$\begin{aligned} \alpha &\rightarrow \hat{\alpha} = \alpha - \frac{8}{\gamma} \\ \beta &\rightarrow \hat{\beta} = \beta + \frac{4}{\gamma}\alpha + \frac{8}{\gamma} \\ b_n &\rightarrow \hat{b}_n = b_n + \frac{4}{\gamma}a_n n \ . \end{aligned} \tag{3.17}$$

The action then becomes:

$$S = \frac{1}{\pi}\int d^2x \quad \sqrt{-g} \quad \left\{\frac{1}{2}e^{-2\phi}\left[R + 4(\nabla\phi)^2 + 4\lambda^2\right] + \frac{1}{2}\sum_{i=1}^{N}(\nabla f_i)^2 \right.$$
$$- \frac{\gamma}{8}R\frac{1}{\nabla^2}R - \frac{\gamma}{8}\left[\hat{\alpha}\phi R + \hat{\beta}(\nabla\phi)^2\right]$$
$$\left. -\frac{\gamma}{8}\sum_{n=2}^{K}\left[a_n\phi^n R + \hat{b}_n\phi^{n-1}(\nabla\phi)^2\right]\right\} . \tag{3.18}$$

This rewriting has the advantage of collecting all the quantum correction into terms proportional to γ, so that the classical limit is obtained by $\gamma \to 0$.

We can derive the stress-energy tensor components by functional differentiation as before. The complete expression is $T_{ab} = T^{(0)}_{ab} + \frac{\gamma}{4}T^{(1)}_{ab} + \frac{\gamma}{4}T^{(2)}_{ab} + \frac{\gamma}{4}T^{(3)}_{ab} = 0$. The components are:

$$\begin{aligned}
T^{(0)}_{\pm\pm} &= e^{-2\phi}\left(4\partial_\pm\rho\partial_\pm\phi - 2\partial^2_\pm\phi\right) + \frac{1}{2}\sum_{i=1}^{N}\partial_\pm f_i\partial_\pm f_i \\
T^{(1)}_{\pm\pm} &= 4\partial^2_\pm\rho - 4\partial_\pm\rho\partial_\pm\rho + \frac{4}{\gamma}t_\pm(x^\pm) \\
T^{(2)}_{\pm\pm} &= 2\hat{\alpha}\partial_\pm\rho\partial_\pm\phi - \hat{\alpha}\partial^2_\pm\phi + \hat{\beta}\partial_\pm\phi\partial_\pm\phi \\
T^{(3)}_{\pm\pm} &= \sum_{n=2}^{K}\left\{a_n n\left[2\phi^{n-1}\partial_\pm\rho\partial_\pm\phi - \phi^{n-1}\partial^2_\pm\phi - (n-1)\phi^{n-2}\partial_\pm\phi\partial_\pm\phi\right]\right. \\
&\quad \left. + \hat{b}_n\phi^{n-1}\partial_\pm\phi\partial_\pm\phi\right\}
\end{aligned} \tag{3.19}$$

where the functions $t_\pm(x^\pm)$ appear because of the zero modes ambiguity of the Green's function $\frac{1}{\nabla^2}$, as discussed in chapter 1. Moreover, the off-diagonal components are:

$$\begin{aligned}
T^{(0)}_{+-} &= e^{-2\phi}(2\partial_+\partial_-\phi - 4\partial_+\phi\partial_-\phi - \lambda^2 e^{2\rho}) \\
T^{(1)}_{+-} &= -4\partial_+\partial_-\rho \\
T^{(2)}_{+-} &= \hat{\alpha}\partial_+\partial_-\phi \\
T^{(3)}_{+-} &= \sum_{n=2}^{K}a_n n\left(\phi^{n-1}\partial_+\partial_-\phi + (n-1)\phi^{n-2}\partial_+\phi\partial_-\phi\right) .
\end{aligned} \tag{3.20}$$

For the dilaton equation of motion we obtain:

$$\begin{aligned}
D^{(0)} &= e^{-2\phi}\left(4\partial_+\partial_-\phi - 2\partial_+\partial_-\rho - 4\partial_+\phi\partial_-\phi - \lambda^2 e^{2\rho}\right) \\
D^{(1)} &= 0 \\
D^{(2)} &= -\hat{\alpha}\partial_+\partial_-\rho - \hat{\beta}\partial_+\partial_-\phi \\
D^{(3)} &= \sum_{n=2}^{K}\left[-a_n n\phi^{n-1}\partial_+\partial_-\rho - \hat{b}_n\phi^{n-1}\partial_+\partial_-\phi + \frac{1}{2}\hat{b}_n(n-1)\phi^{n-2}\partial_+\phi\partial_-\phi\right]
\end{aligned} \tag{3.21}$$

with the dilaton equation of motion:

$$D^{(0)} + \frac{\gamma}{4} D^{(1)} + \frac{\gamma}{4} D^{(2)} + \frac{\gamma}{4} D^{(3)} = 0 \,. \tag{3.22}$$

We remember that in the classical CGHS and RST model, we got a simple current $\partial_+\partial_-(\rho - \phi) = 0$ by subtracting the dilaton equation of motion from the metric equation of motion $T_{+-} = 0$. With this current, we have the freedom to conformally transform to the Kruskal gauge $\phi = \rho$. We will demand that this current equation arises in this model also, so that we will get relations between the constants parameters in our counterterms. The subtraction gives:

$$\begin{aligned} 0 &= 2e^{-2\phi}\partial_+\partial_-(\rho - \phi) \\ &\quad + \frac{\gamma}{4}\left[(\hat{\alpha} - 4)\partial_+\partial_-\rho + (\hat{\alpha} + \hat{\beta})\partial_+\partial_-\phi\right] \\ &\quad + \phi^{n-1}\frac{\gamma}{4}\left[(a_n n + \hat{b}_n)\partial_+\partial_-\phi - a_n n\partial_+\partial_-\rho\right] \\ &\quad + \phi^{n-2}\,\frac{\gamma}{8}(n-1)(\hat{b}_n + 2a_n n)\partial_+\phi\partial_-\phi \,. \end{aligned} \tag{3.23}$$

The simple current must be satisfied for all powers of ϕ. For ϕ^0, we obtain the following relation between the constants $\hat{\alpha}$ and $\hat{\beta}$:

$$\begin{aligned} &(\hat{\alpha} - 4) = -(\hat{\alpha} + \hat{\beta}) \\ &\Rightarrow \quad 2\hat{\alpha} = 4 - \hat{\beta} \,. \end{aligned} \tag{3.24}$$

For ϕ^{n-1}, the simple current condition leads to the constraint:

$$\begin{aligned} -a_n n &= (a_n n + \hat{b}_n) \\ \Rightarrow \quad a_n &= -\frac{1}{2n}\hat{b}_n \,. \end{aligned} \tag{3.25}$$

For ϕ^{n-2}, we only have $\partial_+\phi\partial_-\phi$ and it must vanish if we want to have the simple current in our theory. The vanishing of this coefficient reproduces the same constraints (3.25) as before. It is attractive to obtain the same condition in the last two cases, since it means that the simple current is satisfied for all powers of n.

All these conditions come from the arbitrary imposition of a simple current condition. We must remember that we made this choice to produce a solvable set of equations of motion. It could look strange that the spectrum, N, of massless scalar fields appears in our conditions, through the definitions of $\hat{\alpha}$, $\hat{\beta}$ and $\hat{b}_n$, but for us it is only a mathematical trick to make the theory solvable. Whether or not Nature arranges itself to be solvable is another question.

Since we have obtained a conformal theory including one-loop quantum effects, the natural question to ask is whether or not black holes will evaporate in this theory. The natural way to answer this question would be to perform a semiclassical computation of Hawking radiation in a fixed classical background. We will see that such a computation is really problematic for the generalized model since the linear dilaton vacuum is not a solution of the equations of motion $T_{+-} = 0$ and $D = 0$, and the constraints $T_{\pm\pm} = 0$.

3.3 Problems with Hawking Radiation in a Fixed Background

In this section, we will show that the introduction of the new counterterms makes the computation of Hawking radiation in a fixed background impossible to perform in the simple way introduced by CGHS. The reason for this behavior of our generalized model comes from the fact that the linear dilaton vacuum (LDV) is no longer a solution of the complete theory, which includes classical and quantum terms in the action. As a consequence, energy will be created from the vacuum suggesting that the LDV is not the proper vacuum for the theory. To see where the problems come from, we will perform the calculations as far as we can with the method of CGHS.

As in the CGHS model, we will use the classical solution of a collapsing matter shock wave producing a vacuum region ($x^+ < x_0^+$) and a black hole region ($x^+ > x_0^+$). The metric is given by (1.36), in the Kruskal gauge $\rho = \phi$:

$$e^{-2\phi} = e^{-2\rho} = -m(x^+ - x_0^+)\theta(x^+ - x_0^+) - \lambda^2 x^+ x^- \tag{3.26}$$

where m is the amplitude of the matter wave (1.35). We will perform a coordinate transformation to the asymptotically Minkowskian coordinate system $\{\sigma^\pm\}$ defined by a general transformation $x^\pm = h^\pm(\sigma^\pm)$. Usually, the functions $h^\pm(\sigma^\pm)$ are given by (1.59) and (1.60) for the collapsing matter solution, but we will keep them undetermined for the moment. This system is the most natural because, for this Minkowskian system, we have a good definition for particles in the asymptotic (flat) region; for others, the concept of particles is not well defined [16]. This coordinate transformation preserves the conformal gauge and the new conformal field is given by (see chapter 1):

$$\tilde{\rho} = \rho + \frac{1}{2}\left[\ln[\partial_{\sigma^+} h^+] + \ln[\partial_{\sigma^-} h^-]\right] . \tag{3.27}$$

Under this coordinate transformation, the stress-energy tensor components will transform in the usual way:

$$\tilde{T}_{a'b'} = \Lambda^a{}_{a'}\Lambda^b{}_{b'}T_{ab} . \tag{3.28}$$

This transformation leads to the following tensor's components in the σ-coordinates (using (3.21) and (3.20)):

$$\begin{aligned}
\tilde{T}_{\pm\pm} &= \left[e^{-2\phi} + F(\phi)\right](4\partial_\pm\tilde{\rho}\partial_\pm\phi - 2\partial_\pm^2\phi) + G(\phi)\partial_\pm\phi\partial_\pm\phi \\
&\quad + \gamma\left(\partial_\pm^2\tilde{\rho} - \partial_\pm\tilde{\rho}\partial_\pm\tilde{\rho}\right) + \tilde{T}^f_{\pm\pm} + \tilde{t}_\pm(\sigma^\pm) \qquad (3.29) \\
\tilde{T}_{+-} &= e^{-2\phi}(2\partial_+\partial_-\phi - 4\partial_+\phi\partial_-\phi - \lambda^2 e^{2\tilde{\rho}}) - \gamma\partial_+\partial_-\tilde{\rho} \\
&\quad + \frac{\gamma}{4}\left[\hat{\alpha} + \sum_{n=2}^{K} a_n n\phi^{n-1}\right]\partial_+\partial_-\phi - \frac{\gamma}{4}\sum_{n=2}^{K} a_n n(n-1)\phi^{n-2}\partial_+\phi\partial_-\phi \qquad (3.30)
\end{aligned}$$

where $\tilde{}$ denotes the σ-coordinates and we redefined $\partial_\pm = \frac{\partial}{\partial\sigma^\pm}$. We have also defined the functions:

$$\begin{aligned}
F(\phi) &= \frac{\gamma}{8}\left[\hat{\alpha} + \sum_{n=2}^{K} a_n n\phi^{n-1}\right] \qquad (3.31) \\
G(\phi) &= \frac{\gamma}{4}\left[\hat{\beta} + \sum_{n=2}^{K}\left(\hat{b}_n\phi^{n-1} - a_n n(n-1)\phi^{n-2}\right)\right] \qquad (3.32) \\
\tilde{t}_\pm(\sigma^\pm) &= \left(\partial_\pm h^\pm\right)^2 t_\pm(x^\pm(\sigma^\pm)) - \frac{\gamma}{2}\mathrm{D}_\pm h^\pm \qquad (3.33)
\end{aligned}$$

the Schwarz derivative being defined by:

$$\mathrm{D}_\pm h^\pm = \frac{\partial_\pm^3 h^\pm}{\partial_\pm h^\pm} - \frac{3}{2}\left(\frac{\partial_\pm^2 h^\pm}{\partial_\pm h^\pm}\right)^2 . \tag{3.34}$$

The dilaton equation of motion (3.22) keeps the same form except that $\rho \to \tilde{\rho}$ and $\partial_{x^\pm} \to \partial_{\sigma^\pm}$.

From these expressions, we derive the quantum expectation values for the stress-energy tensor's components, which are only the quantum corrections to the classical stress-energy tensor, because $\tilde{T}^{(0)}_{ab} = 0$ since the fixed metric is classical. So,

$$\begin{aligned}
\langle\tilde{T}_{\pm\pm}\rangle &= F(\phi)(4\partial_\pm\tilde{\rho}\partial_\pm\phi - 2\partial_\pm^2\phi) \\
&\quad + \gamma(\partial_\pm^2\tilde{\rho} - \partial_\pm\tilde{\rho}\partial_\pm\tilde{\rho}) + G(\phi)\partial_\pm\phi\partial_\pm\phi + \tilde{t}_\pm(\sigma^\pm) \qquad (3.35)
\end{aligned}$$

$$
\begin{aligned}
\langle \tilde{T}_{+-} \rangle &= \frac{\gamma}{4}\left\{(\hat{\alpha}+\sum_{n=2}^{K} a_n n \phi^{n-1})\partial_+\partial_-\phi\right\} \\
&+ \frac{\gamma}{4}\left\{\sum_{n=2}^{K} a_n n(n-1)\phi^{n-2}\partial_+\phi\partial_-\phi - 4\partial_+\partial_-\tilde{\rho}\right\} . \qquad (3.36)
\end{aligned}
$$

Now, we have to find the functions $\tilde{t}_\pm(\sigma^\pm)$. In this section, we will use the method of CGHS, which requires that the quantum expectation values of the stress-energy tensor vanish in the linear dilaton vacuum. However, we will see that this requirement is unfortunately impossible for the generalized model.

3.3.1 Linear dilaton vacuum region

The classical spacetime (3.26) has a linear dilaton vacuum region ($x^+ < x_0^+$) which is described in the σ-coordinates by:

$$
\begin{aligned}
\tilde{\rho} &= \frac{1}{2}\left[\ln[\partial_+ h^+ \partial_- h^-] - \ln[-\lambda^2 h^+ h^-]\right] \qquad (3.37)\\
\phi &= -\frac{1}{2}\ln[-\lambda^2 h^+ h^-] \qquad (3.38)
\end{aligned}
$$

where the relation for $\tilde{\rho}$ comes from (3.27) with $\rho = \phi$. By substituting these relations into the quantum expectation values, we obtain what we call $\langle \tilde{T}_{ab} \rangle_{\rm LDV}$ whose components are:

$$
\begin{aligned}
\langle \tilde{T}_{+-} \rangle_{\rm LDV} &= \frac{A(\phi)}{4}\frac{\partial_+ h^+ \partial_- h^-}{h^+ h^-} \qquad (3.39)\\
\langle \tilde{T}_{\pm\pm} \rangle_{\rm LDV} &= \frac{B(\phi)}{4}\left[\frac{\partial_\pm h^\pm}{h^\pm}\right]^2 + \frac{\gamma}{2}\left\{ \mathrm{D}_\pm h^\pm + \frac{1}{2}\left[\frac{\partial_\pm h^\pm}{h^\pm}\right]^2\right\} + \tilde{t}_\pm(\sigma^\pm) \qquad (3.40)
\end{aligned}
$$

where

$$
\begin{aligned}
A(\phi) = A(h^+, h^-) &= \frac{\gamma}{4}\sum_{n=2}^{K} a_n n(n-1)\left[-\frac{1}{2}\ln[-\lambda^2 h^+ h^-]\right]^{n-2} \qquad (3.41)\\
B(\phi) = B(h^+, h^-) &= \frac{\gamma}{4}\hat{\beta} + \frac{\gamma}{4}\sum_{n=2}^{K} \hat{b}_n \left[-\frac{1}{2}\ln[-\lambda^2 h^+ h^-]\right]^{n-1} \\
&- \frac{\gamma}{4}\sum_{n=2}^{K} a_n n(n-1)\left[-\frac{1}{2}\ln[-\lambda^2 h^+ h^-]\right]^{n-2} . \qquad (3.42)
\end{aligned}
$$

We must remember that in the CGHS model, we get $\langle \tilde{T}_{+-} \rangle_{\rm LDV} = 0$ independently of the particular transformation relations that we have, because of the special form of the stress-energy tensor. This allowed us to set all components to zero in the LDV, which means that there were no incoming particles from the vacuum region except for the collapsing matter. For the generalized model, it is obvious from (3.39) and (3.40) that we cannot demand $\langle \tilde{T}_{ab} \rangle_{\rm LDV} = 0$ in the whole vacuum region $\sigma^+ < \sigma_0^+$. Also, demanding $\langle \tilde{T}_{\pm\pm} \rangle_{\rm LDV} = 0$ will require the functions $\tilde{t}_\pm$ to depend on both σ^+ and σ^-, which is not a permitted solution. This dependence is forbidden by the vanishing divergence of the stress-energy tensor: $\nabla_a T^{ab} = 0$.

Since we cannot set $\langle \tilde{T}_{ab} \rangle_{\rm LDV} = 0$ in the whole LDV, we can look at a specific region where it is zero. From (3.39), we see that $\langle \tilde{T}_{+-} \rangle_{\rm LDV}$ will vanish at the zeros of $A(h^+, h^-)$, which are at $-\lambda^2 h^+ h^- = 1$. From now on, we will use the asymptotically Minkowskian coordinates defined by:

$$h^+ = \frac{1}{\lambda} e^{\lambda\sigma^+} \tag{3.43}$$

$$h^- = -\frac{1}{\lambda} e^{-\lambda\sigma^-} - \frac{m}{\lambda^2}\,. \tag{3.44}$$

Using these relations, the zeros of $A(h^+, h^-)$ will then lie on a curve defined by:

$$\sigma^+ = -\frac{1}{\lambda} \ln\left[\frac{m}{\lambda} + e^{-\lambda\sigma^-}\right]. \tag{3.45}$$

The interesting behavior of this curve is that it brings the zeros to the asymptotic region $\sigma^+ \to -\infty$ if $\sigma^- \to -\infty$. So, we could be tempted to impose the condition of vanishing quantum expectation values at the past timelike infinity i^-. At other events in the LDV region, the vacuum will create particles since the quantum expectation values are non-zero. Imposing the weaker condition $\langle \tilde{T}_{ab} \rangle_{\rm LDV}^{i^-} = 0$, we can derive the expressions for the functions $\tilde{t}_\pm(\sigma^\pm)$:

$$\tilde{t}_+(\sigma^+) = -\frac{\gamma\lambda^2}{16}\hat{\beta} \tag{3.46}$$

$$\tilde{t}_-(\sigma^-) = \frac{\gamma\lambda^2}{16}\left[4 - (\hat{\beta}+4)\left(1 + \frac{m}{\lambda} e^{\lambda\sigma^-}\right)^{-2}\right]. \tag{3.47}$$

We have to make two remarks on these functions. First of all, we would have obtained the same results if one considers the case $a_n = \hat{b}_n = 0$, where we do not encounter the problems of a non-vanishing $\langle \tilde{T}_{+-} \rangle_{\rm LDV}$. This special case will be discussed later. Secondly, the function $A(h^+, h^-)$ has no zeros when $n = 2$ and there is no asymptotic past region where we can find a vanishing $\langle \tilde{T}_{+-} \rangle_{\rm LDV}$.

The quantum expectation values are now completely determined and we can study the black hole region. However, since (3.39) and (3.40) are not vanishing, particles creation will take place in the vacuum and we should expect a vacuum contribution to the radiation flowing out to $\mathcal{J}_R^+$. As a consequence, Hawking radiation should be obtained after the subtraction of this vacuum contribution from the total radiation. In order to determine the vacuum contribution, we can consider the black hole as a "perturbation" of the linear dilaton vacuum. Thus, the radiation flowing out to infinity will have a part coming from the vacuum energy flux at $\mathcal{J}_R^+$. So, we will have to substract the quantum expectation values $\langle\tilde{T}_{ab}\rangle_{\text{LDV}}^{\mathcal{J}_R^+}$ from the one obtained in the black hole region. We have:

$$\begin{aligned}
\langle\tilde{T}_{+-}\rangle_{\text{LDV}}^{\mathcal{J}_R^+} &= -\frac{\lambda^2}{16}\gamma\sum_{n=2}^{K} a_n n(n-1)\left(1+\frac{m}{\lambda}e^{\lambda\sigma^-}\right)^{-1} \\
&\quad\times\lim_{\sigma^+\to\infty}\left[-\frac{\lambda}{2}(\sigma^+-\sigma^-)-\frac{1}{2}\ln\left[1+\frac{m}{\lambda}e^{\lambda\sigma^-}\right]\right]^{n-2} \\
\langle\tilde{T}_{++}\rangle_{\text{LDV}}^{\mathcal{J}_R^+} &= \frac{\gamma\lambda^2}{16}\lim_{\sigma^+\to\infty}\left\{\sum_{n=2}^{K}\hat{b}_n\left[-\frac{\lambda}{2}(\sigma^+-\sigma^-)-\frac{1}{2}\ln\left[1+\frac{m}{\lambda}e^{\lambda\sigma^-}\right]\right]^{n-1}\right. \\
&\quad\left.-\sum_{n=2}^{K}a_n n(n-1)\left[-\frac{\lambda}{2}(\sigma^+-\sigma^-)-\frac{1}{2}\ln\left[1+\frac{m}{\lambda}e^{\lambda\sigma^-}\right]\right]^{n-2}\right\} \qquad (3.48) \\
\langle\tilde{T}_{--}\rangle_{\text{LDV}}^{\mathcal{J}_R^+} &= \left(1+\frac{m}{\lambda}e^{\lambda\sigma^-}\right)^{-2}\langle\tilde{T}_{++}\rangle_{\text{LDV}}^{\mathcal{J}_R^+}\,.
\end{aligned}$$

We see that these expressions are divergent in the limit $\sigma^+\to\infty$, and one might hope that they will extract the possible divergences in some radiation flowing out from the black hole region. Now we turn our attention to this black hole region $\sigma^+>\sigma_0^+$.

3.3.2 Black hole region

Now, we can compute the Hawking radiation flowing from the black hole using the CGHS method described in chapter 1. We will use the asymptotically Minkowskian coordinate system (3.44) and we will evaluate (3.35) and (3.36) in this region ($\sigma^+>\sigma_0^+$). The dilaton and conformal fields, in the black hole region, are given by:

$$\tilde{\rho} = -\frac{1}{2}\ln\left[1+\frac{m}{\lambda}e^{(\sigma^--\sigma^++\sigma_0^+)}\right] \qquad (3.49)$$

$$\phi = -\frac{1}{2}\ln\left[e^{(\sigma^+-\sigma^-)}+\frac{m}{\lambda}e^{\sigma_0^+}\right]\,. \qquad (3.50)$$

With these fields, we compute the quantum expectation values and we evaluate them at the future null infinity $\mathcal{J}_R^+$. Let us first evaluate the dilaton on $\mathcal{J}_R^+$:

$$\phi|_{\mathcal{J}_R^+} = \lim_{\sigma^+\to\infty}-\frac{1}{2}\ln\left[e^{\lambda(\sigma^+-\sigma^-)}+\frac{m}{\lambda}e^{\lambda\sigma_0^+}\right]$$

$$= \lim_{\sigma^+ \to \infty} -\frac{1}{2}\left\{\lambda(\sigma^+ - \sigma^-) + \ln\left[1 + \frac{m}{\lambda}e^{\lambda(\sigma^- - \sigma^+ + \sigma_0^+)}\right]\right\}$$

$$= \lim_{\sigma^+ \to \infty} -\frac{\lambda}{2}(\sigma^+ - \sigma^-) . \qquad (3.51)$$

The quantum expectation values become:

$$\langle \tilde{T}_{++}\rangle_{\mathrm{BH}}^{\mathcal{J}_R^+} = \frac{\lambda^2}{16}\gamma \lim_{\sigma^+ \to \infty} \sum_{n=2}^{K} \left\{ \hat{b}_n \left[-\frac{\lambda}{2}(\sigma^+ - \sigma^-)\right]^{n-1} - a_n n(n-1)\left[-\frac{\lambda}{2}(\sigma^+ - \sigma^-)\right]^{n-2} \right\} \qquad (3.52)$$

$$\langle \tilde{T}_{--}\rangle_{\mathrm{BH}}^{\mathcal{J}_R^+} = \frac{\lambda^2}{16}\gamma(\hat{\beta} + 4)\left\{1 - \left(1 + \frac{m}{\lambda}e^{\lambda\sigma^-}\right)^{-2}\right\} + \frac{\gamma\lambda^2}{16} \lim_{\sigma^+ \to \infty} \sum_{n=2}^{K} \left\{ \hat{b}_n \left[-\frac{\lambda}{2}(\sigma^+ - \sigma^-)\right]^{n-1} - a_n n(n-1)\left[-\frac{\lambda}{2}(\sigma^+ - \sigma^-)\right]^{n-2} \right\} \qquad (3.53)$$

$$\langle \tilde{T}_{+-}\rangle_{\mathrm{BH}}^{\mathcal{J}_R^+} = -\frac{\lambda^2}{16}\gamma \lim_{\sigma^+ \to \infty} \sum_{n=2}^{K} a_n n(n-1)\left[-\frac{\lambda}{2}(\sigma^+ - \sigma^-)\right]^{n-2} . \qquad (3.54)$$

All these expressions are diverging, but we would be able to extract the contribution of the vacuum (3.48) and then obtain the Hawking radiation of the black hole. When we look in the far past ($\sigma^- \to -\infty$), we can subtract the vacuum contributions (3.48) and we get the Hawking radiation there:

$$\lim_{\sigma^- \to -\infty} \langle \tilde{T}_{+-}\rangle_{\mathrm{BH}}^{\mathcal{J}_R^+} \to 0$$

$$\lim_{\sigma^- \to -\infty} \langle \tilde{T}_{++}\rangle_{\mathrm{BH}}^{\mathcal{J}_R^+} \to 0 \qquad (3.55)$$

$$\lim_{\sigma^- \to -\infty} \langle \tilde{T}_{--}\rangle_{\mathrm{BH}}^{\mathcal{J}_R^+} \to 0 \qquad (3.56)$$

which agrees with the computations of CGHS, namely that there is no radiation if we do not observe a black hole in spacetime. When we look in the far future ($\sigma^- \to \infty$), we cannot explicitly extract the vacuum contributions and the quantum expectation values remain divergent.

There are some possible answers for the failure of the computation of Hawking radiation in a fixed background. One possibility is to argue that the one-loop corrections are so strong that any semiclassical approximation is wrong. If this is the case, we must perform a computation of Hawking radiation that takes care of the backreaction of the metric and this will be done later when we will compute the Bondi mass of the generalized model. It is also possible that the method used to extract the diverging vacuum contribution to the radiation at $\mathcal{J}_R^+$ was wrong. In this case, interpreting the black hole as a perturbation of the vacuum would be wrong and we should find another way for the extraction of the vacuum contribution. It is also possible that the condition $\langle\tilde{T}_{ab}\rangle_{\rm LDV} = 0$ is not the right condition to impose, but we do not know what should be the right one.

We noticed in the subsection on the linear dilaton vacuum region that there was a special case where everything was similar to the CGHS model. We will discuss this case shortly.

3.3.3 Special case

We can easily see that there is no problem at all for the case $a_n = \hat{b}_n = 0$ because (3.39) and (3.40) are similar to the CGHS model:

$$
\begin{aligned}
\langle\tilde{T}_{+-}\rangle_{\rm LDV} &= 0 && (3.57)\\
\langle\tilde{T}_{\pm\pm}\rangle_{\rm LDV} &= \frac{\gamma}{16}\hat{\beta}\left[\frac{\partial_\pm h^\pm}{h^\pm}\right]^2 + \frac{\gamma}{2}\left\{{\rm D}_\pm h^\pm + \frac{1}{2}\left[\frac{\partial_\pm h^\pm}{h^\pm}\right]^2\right\} + \tilde{t}_\pm(\sigma^\pm)\,. && (3.58)
\end{aligned}
$$

Since $\langle\tilde{T}_{+-}\rangle_{\rm LDV}$ is vanishing, we can perform the semiclassical computation as in the CGHS model. Using the usual asymptotically Minkowskian coordinates, we obtain for $\tilde{t}_\pm(\sigma^\pm)$:

$$
\begin{aligned}
\tilde{t}_+(\sigma^+) &= -\frac{\gamma}{4}\lambda^2\hat{\beta} && (3.59)\\
\tilde{t}_-(\sigma^-) &= \frac{\gamma}{4}\lambda^2\left[1 - \frac{\hat{\beta}+4}{4}\left(1+\frac{m}{\lambda}e^{\lambda\sigma^-}\right)^{-2}\right]. && (3.60)
\end{aligned}
$$

Now, we turn out to the black hole region. We can evaluate the quantum expectation values on the future null infinity $\mathcal{J}_R^+$ and we obtain:

$$
\begin{aligned}
\langle\tilde{T}_{+-}\rangle_{\rm BH}^{\mathcal{J}_R^+} &= 0 && (3.61)\\
\langle\tilde{T}_{++}\rangle_{\rm BH}^{\mathcal{J}_R^+} &= 0 && (3.62)\\
\langle\tilde{T}_{--}\rangle_{\rm BH}^{\mathcal{J}_R^+} &= \frac{\gamma}{16}\lambda^2(\hat{\beta}+4)\left[1-\left(1+\frac{m}{\lambda}e^{\lambda\sigma^-}\right)^{-2}\right]. && (3.63)
\end{aligned}
$$

The last of these three equations is the Hawking radiation flowing out on $\mathcal{J}_R^+$. It has the same behavior as the expression derived by CGHS, which can be recovered in the limit

$\hat{\beta} \to 0$ and $\gamma \to N/12$. We also recover the RST model in this limit, which implies that both models have the same Hawking radiation flux. The flux of Hawking radiation goes to a constant at the future timelike infinity i^+ ($\sigma^- \to \infty$):

$$\langle \hat{T}_{--} \rangle_{\rm BH}^{i^+} = \frac{\gamma}{16} \left(\hat{\beta} + 4 \right) . \tag{3.64}$$

We also notice that the flux of radiation depends on the parameter $\hat{\beta}$. For the special case where $\hat{\beta} = -4$, the Hawking radiation is zero. So, for these two-dimensional models, we see that the rate of Hawking radiation is sensitive to the choice of the parameter for the counterterms. Then, it is possible to find quantum black holes that are not radiating, as in classical General Relativity. In the other hand, we cannot remove the spacetime singularity as we will do for the general case where $a_n \neq 0$ and $\hat{b}_n \neq 0$, as we will see in the following sections.

Let us emphasize again on the assumptions of the calculations done before. These calculations use a fixed classical background, but since energy is emitted by the black hole, its mass should decrease and when the mass is changing, the geometry should also change. Thus, it would be very interesting to include this backreaction in our calculations and see how the black hole evolves. To include the backreaction, we will use a method inspired from the field redefinition performed in the RST model.

3.4 Liouville Theory

Since we are not able to compute Hawking radiation in a semiclassical way, we would like to be able to compute it with a method that includes the backreaction of the metric. In chapter 2, we showed that Russo *et al.* performed a field redefinition that led them to a Liouville-like theory. Their new fields were bound from below: they did not cover the complete range of values.

We will also perform a similar fields redefinition, but we will see that we can obtain fields with more natural range, if we impose some constraints on the free parameters of the theory. The new fields will be defined by the following relations with the previous fields ρ and ϕ:

$$\chi = \rho - \frac{\hat{\alpha}}{4}\phi + \frac{1}{\gamma}e^{-2\phi} - \frac{1}{4}\sum_{n=2}^{K} a_n \phi^n$$

$$\Omega = \left(1-\frac{\hat{\alpha}}{4}\right)\phi + \frac{1}{\gamma}e^{-2\phi} - \frac{1}{4}\sum_{n=2}^{K} a_n\phi^n \,. \tag{3.65}$$

These definitions considerably simplify the expression for the action (3.18) and the stress-energy tensor's components (3.20) and (3.21), since we obtain a Liouville theory. We obtained similar expressions for the RST model, except that κ is changed to γ:

$$S = \frac{1}{\pi}\int d^2x \left\{-\gamma\partial_+ X\partial_- X + \gamma\partial_+\Omega\partial_-\Omega + \lambda^2 e^{2(X-\Omega)} + \frac{1}{2}\sum_{i=1}^{N}\partial_+ f_i\partial_- f_i\right\} \tag{3.66}$$

$$T_{\pm\pm} = -\gamma\partial_\pm X\partial_\pm X + \gamma\partial_\pm\Omega\partial_\pm\Omega + \gamma\partial_\pm^2 X + T^f_{\pm\pm} + t_\pm(x^\pm) \tag{3.67}$$

$$T_{+-} = -\gamma\partial_+\partial_- X - \lambda^2 e^{2(X-\Omega)} \tag{3.68}$$

We pointed out in the discussion of the RST model that Ω was bound from below and this turned out to coincide with a spacetime singularity, as it was seen from the Ricci scalar:

$$\begin{aligned} R &= 8e^{-2\rho}\partial_+\partial_-\rho \\ &= 8e^{-2\rho}\frac{1}{\Omega'}\left[\partial_+\partial_- X - \frac{\Omega''}{\Omega'^2}\partial_+\Omega\partial_-\Omega\right] \end{aligned} \tag{3.69}$$

where the prime designates a derivative with respect to ϕ. The curvature will blow up when Ω' will be zero, which is the extremum of the function $\Omega(\phi)$. So, could it be possible in a generalized model to prevent Ω from having an extremum? The answer is yes. However, the transcendentality of $\Omega(\phi)$ prevents us to find an analytic solution to the equation $\Omega' \neq 0$. On the other hand, it is still possible to find numerical solutions. Let us write $\Omega(\phi)$ in the form:

$$\Omega = A\phi + \sum_{n=2}^{K} B_n\phi^n + Ce^{-2\phi} \tag{3.70}$$

where

$$\begin{aligned} A &= 1-\frac{\hat{\alpha}}{4} \\ B_n &= -\frac{1}{4}a_n \\ C &= \frac{1}{\gamma}\,. \end{aligned} \tag{3.71}$$

Then, Ω' is given by:

$$\Omega'(\phi) = A + \sum_{n=2}^{K} nB_n\phi^{n-1} - 2Ce^{-2\phi}\,. \tag{3.72}$$

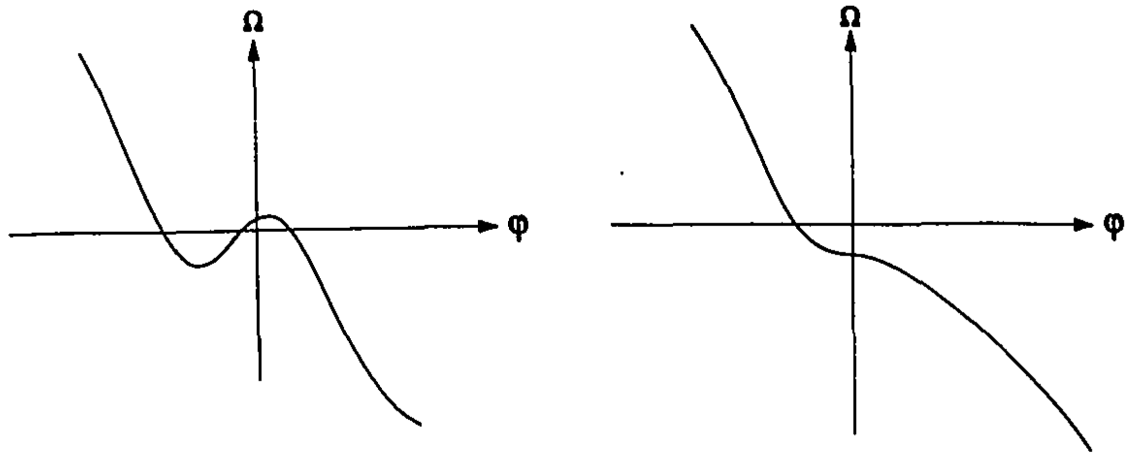

Figure 3.1: Sketch of the field dependence of a bound $\Omega(\phi)$ (left) and of an unbound $\Omega(\phi)$ (right).

From this, we immediately see that the function $\Omega(\phi)$ will be unbound if n takes only odd values and if the following conditions are satisfied

$$A > 0, \quad B_n > 0, \quad C < 0 \tag{3.73}$$

or

$$A < 0, \quad B_n < 0, \quad C > 0\,. \tag{3.74}$$

Because C changes sign when N crosses 24, the above conditions imply that A and B_n are also proportional to $N - 24$. For n even, the problem is more subtle and we must look at the function Ω numerically in order to see whether or not it has an extremum.

Let us look at special cases for n even, where $A = 0$, and see how the function $\Omega(\phi)$ behaves. There is a possibility of having no extrema only when B_n and C have opposite signs, depending on the particular relative values of these constants and K. This must be checked numerically because we could have two possibilities, shown in figure 3.1.

For example, the right graph of figure 3.1 could be obtained by setting $K = 2$ and $B_2 = -C$. It is not necessary that the equality holds exactly, but B_n and C must at least be very close to each other. Thus it is possible to build a theory without singularity or, equivalently, an unbound Ω. This feature would be very interesting if we could prove that the Hawking radiation of such models without singularity has a good behavior. This means that the Hawking radiation should be positive definite for any number of matter fields N and, if possible, goes to zero at future timelike infinity (i^+).

The latter can not be satisfied since most of the existing models of two-dimensional dilaton gravity have a problem with this point: the Hawking radiation goes to a constant, even when we include the backreaction. We will see that it is still a problem with the generalized model. For the positiveness, we have seen in the previous section that a semiclassical calculation did not solve this question; we have not been able to obtain a good description of Hawking radiation, except for the special case $a_n = \hat{b}_n = 0$. In this special case, $B_n = 0$ and it is not possible to build an unbound field $\Omega(\phi)$, *i.e.* we cannot remove the spacetime singularity.

Now, we have to solve the equations of motion and constraints of this Liouville theory.

3.4.1 Solutions

Since we transformed to a Liouville theory, the solutions will be similar to the ones of RST, but here, we will make a more detailed derivation of the various solutions of the Liouville theory. We will simplify the Liouville theory a bit further by making a field redefinition:

$$\begin{aligned} U &= \frac{\gamma}{2}(\Omega + X) \\ V &= X - \Omega\,. \end{aligned} \tag{3.75}$$

These simplify the action, equation of motion and constraints to:

$$S = \frac{1}{\pi}\int d^2x \left\{ -2\partial_+ U \partial_- V + \lambda^2 e^{2V} + \frac{1}{2}\sum_{i=1}^{N} \partial_+ f_i \partial_- f_i \right\} \tag{3.76}$$

$$T_{\pm\pm} = -2\partial_\pm U \partial_\pm V + \frac{\gamma}{2}\partial_\pm^2 V + \partial_\pm^2 U + T^f_{\pm\pm} + t_\pm(x^\pm) = 0 \tag{3.77}$$

$$T_{+-} = -\frac{\gamma}{2}\partial_+\partial_- V - \partial_+\partial_- U - \lambda^2 e^{2V} = 0 \tag{3.78}$$

$$\partial_+\partial_- V = 0 \tag{3.79}$$

where the last equation is the simple current obtained before from $T_{+-} - D = 0$. As a consequence of the latter, we assume that the simple current conditions (3.24) and (3.25) are satisfied. If not, the generalized theory will not be solvable. This action is essentially the classical CGHS action if we would have defined the fields U and V as:

$$\begin{aligned} U &= e^{-2\phi} \\ V &= \rho - \phi\,. \end{aligned} \tag{3.80}$$

From this similarity, we should expect that the two models will have solutions of the same form. In the other hand, we must notice that the constraints of the generalized model are different from that of the classical CGHS model since we have terms proportional to γ and $t_{\pm}(x^{\pm})$. This will bring down some difference in the general solutions. We first solve the simple current, which yields:

$$V = \frac{1}{2}(w_{+} + w_{-}) \tag{3.81}$$

where the functions $w_{\pm}(x^{\pm})$ are called gauge functions. Then we can substitute this solution in the equation of motion $T_{+-} = 0$, which gives us the solution:

$$U = u_{+} + u_{-} - \lambda^2 \int^{x^+} dy^+ e^{w_+} \int^{x^-} dy^- e^{w_-}. \tag{3.82}$$

The functions $u_{\pm}(x^{\pm})$ will be determined by the constraints $T_{\pm\pm} = 0$. Substituting the solution for U in them, we obtain the constraints:

$$T_{\pm\pm} = \partial_{\pm}^2 u_{\pm} - \partial_{\pm} u_{\pm} \partial_{\pm} w_{\pm} + \frac{\gamma}{4}\partial_{\pm}^2 w_{\pm} + T^f_{\pm\pm} + t_{\pm}(x^{\pm}) = 0 \tag{3.83}$$

and the solution of this equation may be written in the following form:

$$u_{\pm}(x^{\pm}) = \frac{\mu}{2\lambda} - \int^{x^{\pm}} dy^{\pm} \left\{ e^{w_{\pm}} \int^{y^{\pm}} dz^{\pm} e^{-w_{\pm}} \left[T^f_{\pm\pm} + \frac{\gamma}{4}\partial_{\pm}^2 w_{\pm} + t_{\pm} \right] \right\} \tag{3.84}$$

where μ is a constant of integration. The matter stress-energy tensor $T^f_{\pm\pm}$ depends on the particular distribution of matter in spacetime. In the other hand, we have already seen how to determine the two other functions $t_{\pm}(x^{\pm})$ in the first chapter (see discussion following equation (1.45)). These functions depends on the particular choice of the reference vacuum we are considering for the massless matter field propagator.

As a special case, the Kruskal gauge $w_{\pm}(x^{\pm}) = 0$ simplify the solutions, as we saw in the CGHS model. This choice for the gauge functions $w_{\pm}$ was allowed because of the invariance of the conformal metric under a subgroup of diffeomorphisms (see chapter 1). In this gauge the solutions are given by:

$$\begin{aligned} U = \gamma\Omega &= \gamma X \\ \gamma X &= \frac{\mu}{\lambda} - \lambda^2 x^+ x^- - \int^{x^+}\int^{y^+} dy^+ dz^+ \left[T^f_{++} + t_+\right] \\ &\quad - \int^{x^-}\int^{y^-} dy^- dz^- \left[T^f_{--} + t_-\right]. \end{aligned} \tag{3.85}$$

$$V = 0 \tag{3.86}$$

The solutions of the equations of motion (3.79) may also be written, in Kruskal gauge, as:

$$\gamma\Omega = \gamma\chi = \frac{\mu}{\lambda} - \lambda^2 x^+ x^- - \int^{x^+}\int^{y^+} dy^+ dz^+ t_+ - \int^{x^-}\int^{y^-} dy^- dz^- t_- \\ - x^+ P_+(x^+) + \Delta_+(x^+) - x^- P_-(x^-) + \Delta_-(x^-) \tag{3.87}$$

where we defined

$$P_\pm(x^\pm) = \int^{x^\pm} dy^\pm T^f_{\pm\pm}(y^\pm)$$
$$\Delta_\pm(x^\pm) = \int^{x^\pm} y^\pm T^f_{\pm\pm}(y^\pm)\,. \tag{3.88}$$

However, there is another useful gauge, which defines the asymptotically Minkowskian coordinate system. This gauge is often referred as the σ-gauge and is defined by $w_\pm(\sigma^\pm) = \pm\lambda\sigma^\pm$. We will denote by a "$\tilde{\ }$" the objects defined in this gauge, so that the stress-energy tensor components are:

$$\tilde{T}_{\pm\pm} = -2\partial_\pm\tilde{U}\partial_\pm\tilde{V} + \frac{\gamma}{2}\partial^2_\pm\tilde{V} + \partial^2_\pm\tilde{U} + \tilde{T}^f_{\pm\pm} + \tilde{t}_\pm(\sigma^\pm) \tag{3.89}$$
$$\tilde{T}_{+-} = -\partial_+\partial_-\tilde{U} - \lambda^2 e^{2\tilde{V}} \tag{3.90}$$

where we used $\partial_+\partial_-\tilde{V} = 0$. Then, the general solution is given by:

$$\tilde{U} = \tilde{u}_+ + \tilde{u}_- + e^{\lambda(\sigma^+ - \sigma^-)} \tag{3.91}$$
$$\tilde{V} = \frac{\lambda}{2}\left(\sigma^+ - \sigma^-\right) \tag{3.92}$$

where the functions $\tilde{u}_\pm(\sigma^\pm)$ are given by

$$\tilde{u}_\pm = \frac{\mu}{2\lambda} - \int^{\sigma^\pm} dy^\pm \left\{ e^{\pm\lambda y^\pm} \int^{y^\pm} dz^\pm e^{\mp\lambda z^\pm}\left[\tilde{T}^f_{\pm\pm} + \tilde{t}_\pm\right]\right\}\,. \tag{3.93}$$

Now, we have the general solutions expressed in two different gauges: the Kruskal gauge and the σ-gauge. The former is useful when we analyse the curves defined by the apparent horizon and the curvature singularity. The σ-gauge is useful when we compute the Bondi mass for a specific matter stress-energy tensor. One such stress-energy tensor describes the collapsing matter wave, and we will focus on this one in the next section.

3.4.2 Collapsing Matter

In this section, we will derive the solution for the collapsing matter stress-energy tensor in both the Kruskal and the σ-gauges. We first look at the Kruskal gauge and, as in the previous models, we will assume that all the matter fields f_i are vanishing, except for one of them, say f_1. Then the matter stress-energy tensor has the usual form:

$$T^f_{++} = m\delta(x^+ - x_0^+) \tag{3.94}$$

$$T^f_{--} = 0\,. \tag{3.95}$$

When we substitute them in the general solutions, we obtain after integration in the Kruskal gauge:

$$\begin{aligned}\gamma\Omega = \gamma X &= \frac{\mu}{\lambda} - \lambda^2 x^+ x^- - m(x^+ - x_0^+)\theta(x^+ - x_0^+) \\ &\quad - \int^{x^+}\int^{y^+} dy^+ dz^+ t_+ - \int^{x^-}\int^{y^-} dy^- dz^- t_- \,.\end{aligned} \tag{3.96}$$

Now, we have to integrate the functions $t_\pm(x^\pm)$. When we discussed Hawking radiation in a fixed background, we tried to derive these functions by requiring the quantum expectation values of the stress-energy tensor to vanish in the linear dilaton vacuum and we had some problems as discussed earlier. For the present calculation, we will derive these functions using the method developed in the RST model, *i.e.* by fixing the reference vacuum of the matter propagator to be Minkowskian. Since the addition of our new counterterms did not change the Green's function $\frac{1}{\nabla^2}$, we will obtain the same $t_\pm(x^\pm)$ as derived in the RST model. One may think that we should shift $\kappa \to \gamma$ since the ghosts contribute a non-local term like S_1 in (3.2) to the total action given in (3.17). So, we still use (2.20) with $P = 1$:

$$t_\pm(x^\pm) = -\frac{\kappa}{4}\frac{1}{(x^\pm)^2}\,. \tag{3.97}$$

Then, we can integrate and (3.96) becomes:

$$\gamma\Omega = \gamma X = \frac{\mu}{\lambda} - \lambda^2 x^+ x^- - m(x^+ - x_0^+)\theta(x^+ - x_0^+) - \frac{\kappa}{4}\ln\left[-\lambda^2 x^+ x^-\right]\,. \tag{3.98}$$

In the other hand, we will also need the solution in the asymptotically Minkowskian coordinates $\{\sigma^\pm\}$. These coordinates, for the black hole spacetime, are related to the Kruskal coordinates by the transformations:

$$x^+ = h^+(\sigma^+) = \frac{1}{\lambda}e^{\lambda\sigma^+} \tag{3.99}$$

$$x^- = h^-(\sigma^-) = -\frac{1}{\lambda}e^{-\lambda\sigma^-} - \frac{m}{\lambda^2}\,. \tag{3.100}$$

Also, the functions $t_\pm(x^\pm)$ will transform according to equation (3.33), with $h^\pm(\sigma^\pm)$ defined above. Using these transformations and the functions $t_\pm(x^\pm)$ in Kruskal gauge, we obtain:

$$\begin{aligned}\tilde{t}_+(\sigma^+) &= \frac{\lambda^2}{4}(\gamma-\kappa) \\ \tilde{t}_-(\sigma^-) &= \frac{\lambda^2}{4}\left[\gamma - \frac{\kappa}{\left(1+\frac{m}{\lambda}e^{\lambda\sigma^-}\right)^2}\right].\end{aligned} \tag{3.101}$$

The stress-energy tensor is also given by:

$$\begin{aligned}\tilde{T}^f_{++} &= m\delta(\sigma^+ - \sigma_0^+) \\ \tilde{T}^f_{--} &= 0\end{aligned} \tag{3.102}$$

Thus, performing the integrals in equation (3.93), we obtain the solutions:

$$\begin{aligned}\tilde{U} &= \frac{\mu}{\lambda} + e^{\lambda(\sigma^+-\sigma^-)} - \frac{m}{\lambda}\left(e^{\lambda(\sigma^+-\sigma_0^+)} - 1\right)\Theta(\sigma^+ - \sigma_0^+) \\ &\quad -\frac{\kappa}{4}\ln\left[1+\frac{m}{\lambda}e^{\lambda\sigma^-}\right] + \frac{\kappa\lambda}{4m}e^{-\lambda\sigma^-} + \frac{\lambda}{4}(\gamma-\kappa)\left(\sigma^+ - \sigma^-\right)\end{aligned} \tag{3.103}$$

$$\tilde{V} = \frac{\lambda}{2}\left(\sigma^+ - \sigma^-\right). \tag{3.104}$$

The solution in terms of the fields $\tilde{X}$ and $\tilde{\Omega}$ are easily obtained from the definitions of the fields $\tilde{U}$ and $\tilde{V}$, which are similar to the definitions (3.75) of the fields U and V. Now, we have to determine the Hawking evaporation rate of this black hole.

3.5 Bondi Mass of Evaporating Black Hole

In the present section, we will try to answer a very important question for the study of Hawking evaporation: what is the mass of the black hole. We are mainly interested in the evolution of the mass, which should decrease as the black hole evaporates. The method used is explained in Appendix A and is known as the Bondi mass.

Since a large class of two-dimensional dilaton gravity can be expressed as a Liouville theory by a proper field redefinition, even the classical theory of CGHS, we can derive a general expression for all these theories. We first need the linearizations $\delta\tilde{T}_{++}$ and $\delta\tilde{T}_{+-}$ of the stress-energy tensor's components, which are obtained from the variation of (3.89) and (3.90). Explicitly, we have:

$$\delta\tilde{T}_{++} = -2\partial_+\tilde{U}_0\partial_+\delta\tilde{V} - 2\partial_+\tilde{V}_0\partial_+\delta\tilde{U} + \partial_+^2\delta\tilde{U} \tag{3.105}$$

$$\delta\tilde{T}_{+-} = -\partial_+\partial_-\delta\tilde{U} - 2\lambda^2 e^{2\tilde{V}_0}\delta\tilde{V}\,. \tag{3.106}$$

The fields $\tilde{U}_0$ and $\tilde{V}_0$ are the reference solutions around which we are making the linearization. These reference fields are usually taken to be the solutions where there is no matter in the spacetime, which defines the vacuum state for a particular set of coordinates. From the derivation of equations (3.103) and (3.104), we see that the case where $m = 0$ leads to the reference solutions:

$$\tilde{U}_0 = \frac{\mu}{\lambda} + e^{\lambda(\sigma^+ - \sigma^-)} + \frac{\lambda}{4}(\gamma - \kappa)\left(\sigma^+ - \sigma^-\right) \tag{3.107}$$

$$\tilde{V}_0 = \frac{\lambda}{2}\left(\sigma^+ - \sigma^-\right)\,. \tag{3.108}$$

We note that this vacuum solution, does not reduce to the linear dilaton vacuum when we tranform back to the original fields $\tilde{\rho}$ and ϕ. This was the cause of the problems in the computation of Hawking radiation in a fixed background. However, it reduces to the LDV at the past timelike infinity i^-, so that we can argue that it is closely related to the LDV. To compute the Bondi mass of the black hole formed by the collapse of matter, we use the definition (A.1) stated in Appendix A, and after integrating it, we obtain:

$$\begin{aligned} M(\sigma^-) &= 2\lambda e^{\lambda(\sigma^+ - \sigma^-)}\delta\tilde{V} + \lambda\left[\frac{1}{2}(\gamma - \kappa)\delta\tilde{V} + \delta\tilde{U}\right] \\ &\quad - \frac{\gamma}{2}\left[\partial_+\delta\tilde{V} - \partial_-\delta\tilde{V}\right] + \partial_-\delta\tilde{U} - \partial_+\delta\tilde{U}\Big|_{\mathcal{I}_R^+}\,. \end{aligned} \tag{3.109}$$

The linear variations $\delta\tilde{U}$ and $\delta\tilde{V}$ will be obtained by writing the solutions (3.103) and (3.104) in the linear form:

$$\begin{aligned} \tilde{U} &= \tilde{U}_0 + \delta\tilde{U} \\ \tilde{V} &= \tilde{V}_0 + \delta\tilde{V}\,, \end{aligned} \tag{3.110}$$

which imply that we have:

$$\delta\tilde{U} = -\frac{m}{\lambda}\left(e^{\lambda(\sigma^+ - \sigma_0^+)} - 1\right)\Theta(\sigma^+ - \sigma_0^+) - \frac{\kappa}{4}\ln\left[1 + \frac{m}{\lambda}e^{\lambda\sigma^-}\right] + \frac{\kappa\lambda}{4m}e^{-\lambda\sigma^-} \tag{3.111}$$

$$\delta\tilde{V} = 0\,. \tag{3.112}$$

The Bondi mass (3.109) is then given by:

$$M(\sigma^-) = m - \frac{\lambda}{4}\kappa\left\{\ln\left[1+\frac{m}{\lambda}e^{\lambda\sigma^-}\right]+\frac{m}{\lambda e^{-\lambda\sigma^-}+m}\right\}\ . \tag{3.113}$$

A similar expression for the Bondi mass was derived in [18] for a conformally invariant model of two-dimensional dilaton gravity. This mass tends to the amplitude m of the incoming matter at the past infinity ($\sigma^- \to -\infty$). However, at the future infinity i^+ (the limit $\sigma^- \to \infty$), the Bondi mass goes to large negative values ($-\infty$). This is obviously nonsense, because the black hole cannot Hawking radiate more energy than its initial mass. This suggests that the vacuum is unstable; there is no stable ground state where the black hole could stop its evaporation. This diverging mass is present in most two-dimensional dilaton gravity models, but some models use new boundary conditions at $\Omega' = 0$ to stabilize the vacuum [18]. This problem can also be seen by looking at the Hawking radiation rate, defined as the variation of the Bondi mass with respect to σ^-, *i.e.*:

$$\frac{dM(\sigma^-)}{d\sigma^-} = \frac{\lambda^2}{4}\kappa\left[\frac{1}{\left(1+\frac{m}{\lambda}e^{\lambda\sigma^-}\right)^2}-1\right]\ . \tag{3.114}$$

We readily see that the radiation tends to a constant flux at the future infinity $\sigma^- \to \infty$, which means that the radiation never stops. This is a very important problem of these two-dimensional models, because this unending Hawking radiation occurs even when we are taking the backreaction of the metric into account. Previous attempts to find the right boundary condition on the singularity $\Omega' = 0$ were also designed to render a well-behaved vacuum state for the theory. It is also important to notice that the Hawking radiation rate is proportional to $\kappa = N/12$ and not to $\gamma = (N-24)/12$, so that the black hole does not Hawking radiate ghosts as some computations of Hawking radiation in a fixed background had shown for some early models.

On the other hand, for a proper choice of the parameters, we can now have a model without singularity, and this property enables us to avoid the problem of the possible event where the singularity becomes naked [5]. Thus in the generalized model we have developed so far, there is no point where the evolution of the spacetime becomes non-unique and where a cataclysm could happen. One would like to be able to modify the theory such that the Hawking radiation is well-behaved for the whole lifetime of the black hole, *i.e.* that it will stop somewhere, leaving either the vacuum or some sort of remnant. It could also be interesting to look at the possible implications of this model on the problem of the information loss in black holes physics. Interesting proposals on this subject can be found in [26].

Conclusion

In this thesis, we studied black holes solutions of two-dimensional dilaton gravity models. First, we saw in the CGHS [6] model that we can add quantum corrections to a classical model and make it Hawking evaporate. In this model, the quantum correction is simply the Polyakov anomaly term. We were able to compute the rate of Hawking radiation for a fiducial observer in a semiclassical approximation where the metric is kept fixed. The flux obtained has a good behavior in the far past, but tends to a constant in the far future. This unphysical constant rate was claimed to be caused by the approximation stated above. One would want to be able to include the dynamics of the metric in the computation of Hawking radiation, but the one-loop corrected model is not solvable. So, one would like to have a completely solvable model which will take care of the backreaction.

In the second chapter, we studied one solvable model: the RST model. In this model, a new counterterm has been added to the theory, making it solvable. Since the theory is completely solvable, we can take the backreaction into account in the Hawking evaporation. This model also has a curvature singularity defined by the equation

$$\frac{\delta\Omega}{\delta\phi} = 0 \ . \tag{3.115}$$

As the black hole evaporates, the apparent horizon recedes and the singularity will eventually become naked. A naked singularity is timelike and its evolution is not uniquely determined. So, in order to overcome this uniqueness problem, some authors tried to impose boundary conditions on the singularity, but nobody has found a satisfactory set of boundary conditions and research is still progressing on this subject [18, 26]. Another solution to the naked singularity problem is to build a model without singularity.

In the last chapter, we attempted to build such a model. We modified the quantum corrections to the classical model which gave what we called the generalized model action (3.4). We imposed a simple current relation, which enabled us to transform the theory to a solvable Liouville theory. The imposition of the simple current puts constraints on the parameters of the theory, which must satisfy the particular relations (3.24) and (3.25). We first tried to make a semiclassical computation of Hawking radiation, but it was not possible to perform it in a simple way. Then, we computed it using the Bondi mass method, but we obtained an unending Hawking radiation. This behavior is pathological to two-dimensional dilaton gravity models, and remained present in our generalized model, suggesting that the vacuum used in the dilaton gravity theories is not stable.

The important feature of the generalized theory is that we can build a theory without singularity by a proper choice of the parameters. Working with a theory without singularity removes the problem of the boundary conditions on a naked singularity. It is appealing that the quantum corrections to the classical CGHS model are able to remove the classical singularity and one might hope that such a behavior would appear to be true in higher dimensional black hole physics. This has to be checked, and there is no obvious way how to generalize the procedure developed in this thesis for higher dimensions. In higher dimensions, the dilaton field will not have a mass dimension of zero preventing us to add terms proportional to ϕ to the power of n. Also, the anomaly term would certainly be different from the simple Polyakov action arising from the matter path integral, since we cannot perform this path integral in four dimensions.

Finally, let us note that all these models are solved classically, *i.e.* using the minimal action principle $\delta S = 0$ to derive the equations of motion. This procedure leads to a deterministic solution which is opposite to the spirit of quantum mechanics. In quantum mechanics, we usually have a probabilistic evolution, which is absent from these two-dimensional models of dilaton gravity. So, one would like to bring this property in the models and one possible way to do it could be to use the influence functional method of Feynman and Vernon [27]. This still has to be done for the models studied in the present thesis.

Appendix A

Bondi Mass

It is often very useful to know the total energy (the mass) of a system when we are dealing with General Relativity. Since the discovery of Hawking evaporation, this question becomes more important if one wants to be able to follow the evolution of the black hole as it evaporates. Such a definition of the residual mass of a system is provided by the so-called Bondi mass (see [25] and [28] for application to black holes physics.). This appendix presents a brief description of this mass definition and we will compute a simple example.

A.1 Definition

First of all, the Bondi mass must be computed in an asymptotically Minkowskian coordinate system $\{\sigma^{\pm}\}$ because field theory is well understood in Minkowski spacetime. More precisely, the concept of particles is well defined [16]. Also, the computation is taken around a reference solution of the system for which we are computing the mass. Let us define δT_{ab} as the first variation of the stress-energy tensor's components around some reference solution, static or not. Then, the Bondi mass as measured at future null infinity $\mathcal{J}_R^+$ is defined by [25]:

$$M(\sigma^-) = \int^{\mathcal{J}_R^+} d\sigma^+ \delta\tilde{T}_+^0 = -\int^{\mathcal{J}_R^+} d\sigma^+ \left(\delta\tilde{T}_{++} + \delta\tilde{T}_{+-}\right) \ . \tag{A.1}$$

This is a simple expression; all we have to compute is the first variation δT_{ab} of the stress-energy tensor. From this definition, we can define in a natural way the Hawking radiation as the rate of decay of the Bondi mass with respect to σ^-, *i.e.* $\partial_{\sigma^-} M(\sigma^-)$. Now let us compute a simple example.

A.2 Classical CGHS Black Hole

We first have to look at the stress-energy tensor expressed in asymptotically Minkowskian coordinates

$$x^{\pm} = \pm\frac{1}{\lambda}e^{\pm\sigma^{\pm}}. \tag{A.2}$$

The stress-energy tensor's components (1.10) and (1.11) transform according to equations (3.30) and (3.29):

$$\begin{aligned}\tilde{T}_{\pm\pm} &= e^{-2\phi}\left(4\partial_{\pm}\tilde{\rho}\partial_{\pm}\phi - 2\partial_{\pm}^{2}\phi\right) & \text{(A.3)}\\ \tilde{T}_{+-} &= e^{-2\phi}\left(2\partial_{+}\partial_{-}\phi - 4\partial_{+}\phi\partial_{-}\phi - \lambda^{2}e^{2\tilde{\rho}}\right) & \text{(A.4)}\end{aligned}$$

where we redefined $\partial_{\pm} = \frac{\partial}{\partial\sigma^{\pm}}$. It is now simple to compute the linear variations of $\tilde{T}_{ab}$:

$$\begin{aligned}\delta\tilde{T}_{\pm\pm} &= -2e^{-2\phi_0}\left(4\partial_{\pm}\tilde{\rho}_0\partial_{\pm}\phi_0 - 2\partial_{\pm}^{2}\phi_0\right)\delta\phi\\ &\quad + e^{-2\phi_0}\left(4\partial_{\pm}\phi_0\partial_{\pm}\delta\tilde{\rho} + 4\partial_{\pm}\tilde{\rho}_0\partial_{\pm}\delta\phi - 2\partial^{2}\delta\phi\right) & \text{(A.5)}\\ \delta\tilde{T}_{+-} &= -2e^{-2\phi_0}\left(2\partial_{+}\partial_{-}\phi_0 - 4\partial_{+}\phi_0\partial_{-}\phi_0 - \lambda^{2}e^{2\tilde{\rho}_0}\right)\delta\phi\\ &\quad + e^{-2\phi_0}\left(2\partial_{+}\partial_{-}\delta\phi - 4\partial_{+}\delta\phi\partial_{-}\phi_0 - 4\partial_{+}\phi_0\partial_{-}\delta\phi - 2\lambda^{2}e^{2\tilde{\rho}_0}\delta\tilde{\rho}\right) & \text{(A.6)}\end{aligned}$$

where $\tilde{\rho}_0$ and ϕ_0 are the reference solutions around which we linearize and $\delta\phi$ and $\delta\tilde{\rho}$ are the linear variations of the fields from these reference solutions. The first part of each equation vanishes because the reference solution must satisfy the metric equation of motion and constraints $\tilde{T}_{ab} = 0$. Now we have to select a reference solution. The obvious choice here is the linear dilaton vacuum:

$$e^{-2\phi_0} = e^{-2\tilde{\rho}_0} = -\lambda^{2}x^{+}x^{-}\ . \tag{A.7}$$

However, in the σ-coordinates (A.2) defined above, this turns out to

$$\begin{aligned}\phi_0 &= -\frac{1}{2}\lambda(\sigma^{+} - \sigma^{-})\\ \tilde{\rho}_0 &= 0\ . & \text{(A.8)}\end{aligned}$$

Thus, the variations are given by:

$$\begin{aligned}\delta\tilde{T}_{\pm\pm} &= 2e^{\lambda(\sigma^{+}-\sigma^{-})}\left(-\lambda\partial_{\pm}\delta\tilde{\rho} - \partial_{\pm}^{2}\delta\phi\right) & \text{(A.9)}\\ \delta\tilde{T}_{+-} &= 2e^{\lambda(\sigma^{+}-\sigma^{-})}\left(\partial_{+}\partial_{-}\delta\phi + \lambda(\partial_{-} - \partial_{+})\delta\phi - \lambda^{2}\delta\tilde{\rho}\right)\ . & \text{(A.10)}\end{aligned}$$

Substituting these results in the definition of the Bondi mass, we have:

$$\begin{aligned} M(\sigma^-) &= \int^{\mathcal{J}_R^+} d\sigma^+ 2\partial_+ \left[\lambda e^{\lambda(\sigma^+-\sigma^-)}\delta\tilde{\rho} + e^{\lambda(\sigma^+-\sigma^-)}(\partial_+ - \partial_-)\delta\phi\right] \\ &= 2e^{\lambda(\sigma^+-\sigma^-)}\left(\lambda\delta\tilde{\rho} + \partial_+\delta\phi - \partial_-\delta\phi\right)\big|_{\mathcal{J}_R^+} . \end{aligned} \tag{A.11}$$

This is the result obtained by CGHS [6]. Note that this expression has to be evaluated at the future null infinity $\mathcal{J}_R^+$, which is the limit $\sigma^+ \to \infty$. The variations $\delta\phi$ and $\delta\tilde{\rho}$ depend on the particular solution we are looking at. For example, for the classical static black hole solution:

$$\begin{aligned} e^{-2\phi} &= \frac{M}{\lambda} + e^{\lambda(\sigma^+-\sigma^-)} \\ e^{-2\tilde{\rho}} &= 1 + \frac{M}{\lambda}e^{-\lambda(\sigma^+-\sigma^-)} , \end{aligned} \tag{A.12}$$

the variations are derived from the equations

$$\phi = \phi_0 + \delta\phi \tag{A.13}$$

$$\tilde{\rho} = \tilde{\rho}_0 + \delta\tilde{\rho} . \tag{A.14}$$

Using the linear dilaton vacuum (reference solution) $\tilde{\rho}_0$ and ϕ_0, we get:

$$\delta\phi = \delta\tilde{\rho} = -\frac{M}{2\lambda}e^{-\lambda(\sigma^+-\sigma^-)} . \tag{A.15}$$

Substituting this into the Bondi mass, we simply get the constant mass $M(\sigma^-) = M$. This proves that the constant of integration of the solution (1.28), suggestively named M, was in fact the mass of this classical black hole.

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