Asymptotics of Laplace Eigenfunctions on the Disc

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Abstract

The main goal of this thesis is to study particular sequences of eigenfunction of the Laplacian on the disc which concentrate on annuli of varying width in the large eigenvalue limit. The method involves constructing a Hilbert basis of eigenfunctions using the method of separation of variables in polar variables. One is faced with solving two ordinary differential equations: the one in the angular coordinates is trivial but the one in the radial variable involves studying asymptotics of the Bessel functions of the first and second kind. We show that only the Bessel solutions of the first kind satisfy the requisite boundary conditions and then in Theorem 1 using the method of steepest descent, we show that these subsequences concentrate in the desired annuli in the large eigenvalue limit.

Résumé

Le but principal de ce mémoire de maîtrise est la construction de séquences de fonctions propres de l'opérateur laplacien sur le disque qui se concentrent sur des anneaux variant en largeur dans la limite de grandes valeurs propres. Nous établissons une base d'Hilbert pour les fonctions propres en utilisant une technique appelée la séparation de variables en coordonnées polaires. Par cette méthode, nous résolvons deux équations différentielles: celle angulaire est triviale, alors que l'équation radialle demande l'étude des fonctions de Bessel de la première et deuxième espèce. Utilisant les conditions limites, nous démontrons dans le théorème 1 qu'uniquement les fonctions de Bessel de la première espèce seront utiles. Avec des méthodes asymptotiques, incluant la méthode du point col, nous prouvons que les séquences choisies se concentrent sur les anneaux désirés dans la limite des valeurs propres.

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1. Introduction

Let $\mathbf{D} = \{(x,y) \in \mathbb{R}^2; x^2+y^2=1\}$ be the unit disc with exterior unit normal $\nu = (x,y)$ and consider the Dirichelt (or Neumann) problem on $\mathbf{D}$ given by

$$\begin{cases} -\Delta u = \mu\, u \text{ in } \mathbf{D}, \\ \\ u|_{\partial \mathbf{D}} = 0 \text{ (Dirichlet)}, \quad \partial_\nu u|_{\partial \mathbf{D}} = 0 \text{ (Neumann)}. \end{cases} \tag{1.1}$$

In the following, $\Delta = \partial_{xx} + \partial_{yy}$ is the standard planar Laplacian on $\mathbb{R}^2$. A standard computation in polar variables (see section Laplacian) shows that

$$[-\Delta, D_\theta] = 0, \tag{1.2}$$

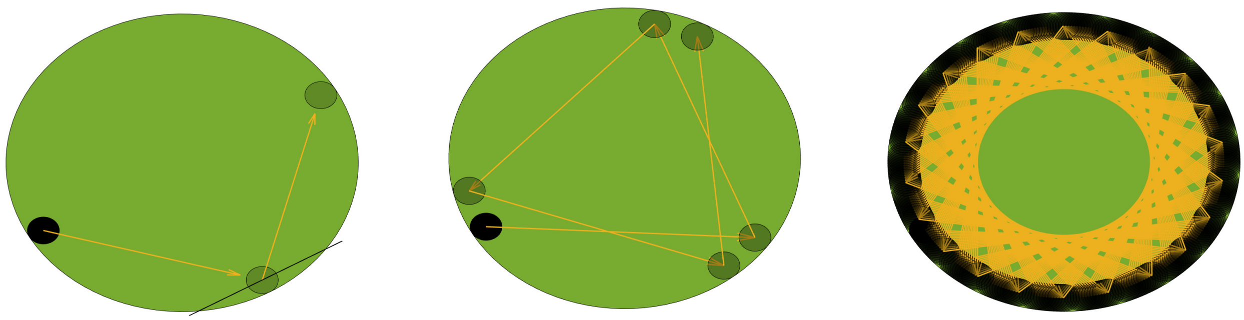
where $D_\theta = x\partial_y - y\partial_x$ is angular differentiation in the plane. Since $-\Delta$ is elliptic, it is well-known [4] that there exists a Hilbert basis $\{u_{\mu_j}\}_{j=1}^{\infty}$ of Dirichlet (or Neumann) eigenfunctions that are simultaneously *joint* eigenfunctions of Δ and D_θ. In the terminology of Toth and Zeldtich [1, 2] the operators $-\Delta$ and D_θ generate a *quantum completely integrable (QCI)* system on the disc. The main point of this thesis is to study the large-eigenvalue asymptotics of the joint eigenfunctions. In the following we denote the joint eigenvalues of D_θ and $-\Delta$ as follows

$$\operatorname{Spec}(D_\theta, -\Delta) = \begin{cases} \{(n, j_{k,n}^2); n \in \mathbb{N}_0, k \in \mathbb{N}\} & \text{(Dirichlet)} \\ \\ \{(n, (j'_{k,n})^2); n \in \mathbb{N}_0, n \in \mathbb{N}\} & \text{(Neumann)}. \end{cases}$$

As we will show in section 6, $j_{k,n}$ is the k-th zero of the Bessel function J_n and $j'_{k,n}$ is the k-th zero of the derivative of J_n. We will denote the corresponding L^2-normalized joint eigenfunctions by $\tilde{u}_{k,n}, \tilde{v}_{k,n}; k \in \mathbb{N},\ n \in \mathbb{N}_0$ for the Dirichlet and Neumann case respectively. As we show in sections 5 and 6, these joint eigenfunctions have a fairly simple form in polar coordinates; specifically,

$$\begin{aligned} \tilde{u}_{k,n}(r,\theta) &= c_{k,n}^{-1} e^{in\theta}\, J_n(j_{k,n} r) && \text{(Dirichlet)}, \\ \tilde{v}_{k,n}(r,\theta) &= d_{k,n}^{-1} e^{in\theta} J_n(j'_{k,n} r) && \text{(Neumann)}\ . \end{aligned} \tag{1.3}$$

In (1.3), $c_{k,n}$ and $d_{k,n}$ are L^2-normalization constants. Before stating the main result in Theorem 1 below, we give some background on the relationship between the above QCI system on the billiard dynamics in the disc. The general Bohr-correspondence principle asserts that eigenfunctions should "detect" the underlying billiard dynamics of a free particle travelling in the disc satisfying an equal angle Snell-type reflection principle at the boundary. It follows from the classical Appolonius principle that any billiard trajectory is tangential to an inner boundary of the annulus $A(\beta) = \{(x,y) \in D, \beta^2 \leq x^2+y^2 \leq 1\}$ after one link, remain constrained to $A(\beta)$ after arbitrary number of bounces, always intersecting the inner boundary circle $x^2 + y^2 = \beta^2$ tangentially (see Figure 1). The Bohr heuristics suggest that there should be a corresponding subsequence of joint eigenfunctions of $(D_\theta, -\Delta)$ that concentrates in such annulus $A(\beta)$. The main result of the thesis makes the correspondence principle precise for sequences of joint eigenfunction in the disc.



(A) Reflection at equal angle   (B) 5 iterations   (C) 250 iterations

FIGURE 1. Billiard reflections

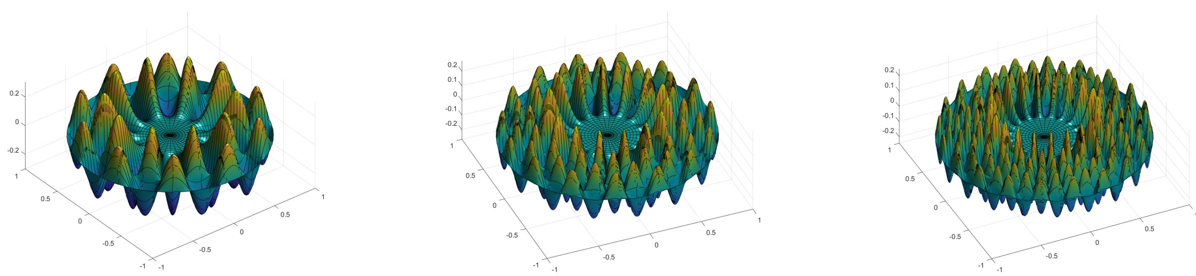


FIGURE 2. Dirichlet eigenfunctions

To identify the particular subsequences of interest, fix $0 < \beta < 1$. For concreteness we discuss the Dirichlet case, but the Neumann case follows in a similar fashion. The joint trace for commuting operators [1], [2] asserts that for any $\rho \in \mathbb{S}(\mathbb{R})$ with $\rho > 0$ and $\int_{\mathbb{R}} \rho = 1$,

$$\lim_{n\to\infty} \sum_{k=1}^{\infty} \rho(\beta\, j_{k,n} - n) = c(\beta). \tag{1.4}$$

Here $c(\beta) > 0$ is a geometric constant depending on β. There is a direct analogue of (1.4) in the Neumann case with $j'_{k,n}$ replacing $j_{k,n}$ and $c'(\beta) > 0$ on the RHS.

From (1.4) it follows that there exists a subsequence $(j_{k(n),n})_{n\in\mathbb{N}}$ such that

$$j_{k(n),n} = \beta^{-1} \cdot n + O(1) \qquad \text{as } n \to \infty \tag{1.5}$$

We continue to denote the associated subsequence of eigenfunctions by $(u_{k(n),n})_{n\in\mathbb{N}}$ The main result of this thesis is the following *correspondence principle:*

THEOREM **1.** *Fix* $0 < \beta < 1$ *and consider the corresponding annulus* $A(\beta) = \{(x,y) \in \mathbf{D}; \beta < |(x,y)| < 1\}$ *inside the unit disc. Then, there exist a subsequence* $(u_{k(n),n})_{n\in\mathbb{N}}$ *of joint Neumann or Dirichlet eigenfunctions with the property that for any* $\varepsilon > 0$,

$$(i)\ u_{k(n),n}(x,y) \sim \sum_{\pm} e^{\pm inf(x,y)} \sum_{\ell=1}^{\infty} a_\ell^\pm(x,y) n^{-m}, \quad (x,y) \in A(\beta+\varepsilon),$$

where $, a_\ell^\pm \in C^\infty(A(\beta+\varepsilon),\mathbb{C}); \ell \in \mathbb{N}$ *and* $f \in C^\infty(A(\beta+\varepsilon),\mathbb{R})$.

Moreover,

$$(ii) \sup_{(x,y)\in\mathbf{D}\setminus A(\beta-\varepsilon)} |u_{k(n),n}(x,y)| = O(e^{-C(\varepsilon)n}), \quad C(\varepsilon) > 0.$$

As a direct corollary of Theorem 1 it follows that

$$\int_{A(\beta-\varepsilon)} |u_{k(n),n}|^2\, dxdy \sim_{n\to\infty} 1. \tag{1.6}$$

The point of this thesis is to give a self-contained, direct proof of the correspondence principle in Theorem 1 using elementary methods: separation of variable in polar coordinates together with a detailed analysis of the large-n asymptotics of the relevant solutions of the radial Bessel equation using the method of steepest descent. We carry out the argument in several steps: The justification of the formula (1.3) in terms of polar variables is carried out in sections 1-6. The proof of Theorem 1 using the method of steepest descent is carried out in section 7.

The figures of the eigenfunctions in this thesis were made of MATLAB. I used Carey Smith's MATLAB function [11] to find zeros of the Bessel functions of the first kind as well as Greg Von Winckel's MATLAB function [12] for the zeros of the derivative of the Bessel functions of the first kind.

2. THE LAPLACIAN OPERATOR

Let us define the Laplacian as

$$\triangle := \partial_{xx} + \partial_{xx}$$

on the space of twice continuously differentiable functions on the disc $\mathbf{D} = \{(x,y) \in \mathbb{R}^2 \mid x^2 + y^2 < 1\}$. It will be useful to express this operator in polar coordinates;

Lemma **2.1.** *For $r \geq 0$ and $0 \leq \theta < 2\pi$ defined by the equations $x = r\cos\theta$ and $y = r\sin\theta$ we have the following equality away from zero;*

$$\triangle = \partial_{rr} + \frac{1}{r}\partial_r + \frac{1}{r^2}\partial_{\theta\theta}$$

Proof.

$$\begin{aligned} x = r\cos\theta \implies \quad & \frac{\partial x}{\partial r} = \cos\theta \qquad & \frac{\partial x}{\partial \theta} = -r\sin\theta \\ y = r\sin\theta \implies \quad & \frac{\partial y}{\partial r} = \sin\theta \qquad & \frac{\partial y}{\partial \theta} = r\cos\theta \end{aligned}$$

Let u be a twice continuously differentiable function. We compute

$$\begin{aligned} \frac{\partial u}{\partial r} =& u_x\frac{\partial x}{\partial r} + u_y\frac{\partial y}{\partial r} = u_x\cos\theta + u_y\sin\theta \\ \frac{\partial^2 u}{\partial r^2} =& \frac{\partial}{\partial r}\left(u_x\cos\theta + u_y\sin\theta\right) = \frac{\partial u_x}{\partial r}\cos\theta + \frac{\partial u_x}{\partial r}\sin\theta \\ \frac{\partial u}{\partial \theta} =& u_x\frac{\partial^2 x}{\partial \theta^2} + u_y\frac{\partial^2 y}{\partial \theta^2} = -ru_x\sin\theta + ru_y\cos\theta \end{aligned}$$

and finally

$$\begin{aligned} \frac{\partial^2 u}{\partial \theta^2} =& \frac{\partial}{\partial \theta}(-ru_x\sin\theta + ru_y\cos\theta) \\ =& r\left[-\frac{\partial \sin\theta}{\partial\theta}u_x - \sin\theta\frac{\partial u_x}{\partial \theta} + \frac{\partial\cos\theta}{\partial\theta}u_y + \cos\theta\frac{\partial u_y}{\partial\theta}\right] \\ =& r\big[-\cos\theta u_x - \sin\theta\left(u_{xx}\frac{\partial^2 x}{\partial\theta^2} + u_{xy}\frac{\partial y}{\partial\theta}\right) \\ & \qquad - \sin\theta u_y + \cos\theta\left(u_{xy}\frac{\partial x}{\partial\theta} + u_{yy}\frac{\partial y}{\partial\theta}\right)\big] \\ =& r\left[-\cos\theta u_x - \sin\theta\left(u_{xx}(-r\sin\theta) + u_{xy}(r\cos\theta)\right) - \sin\theta u_y + \cos\theta\left(u_{xy}(-r\sin\theta) + u_{yy}r\cos\theta\right)\right] \\ =& -r\left(u_x\cos\theta + u_y\sin\theta\right) + r^2\left(u_{xx}\sin^2\theta - 2u_{xy}\sin\theta\cos\theta + u_{yy}\cos^2\theta\right) \end{aligned}$$

Which implies

$$\frac{1}{r^2}\frac{\partial^2 u}{\partial\theta^2} = -\frac{1}{r}\left(u_x\cos\theta + u_y\sin\theta\right) + u_{xx}\sin^2\theta - 2u_{xy}\sin\theta\cos\theta + u_{yy}\cos^2\theta$$

Summing up the u_{rr} and $r^{-2}u_{\theta\theta}$ equations, we get

$$\begin{aligned} u_{rr} + \frac{1}{r^2}u_{\theta\theta} =& -\frac{1}{r}\left(u_x\cos\theta + u_y\sin\theta\right) + u_{xx} + u_{yy} \\ =& -\frac{1}{r}u_r + u_{xx} + u_{yy} \end{aligned}$$

where at the last equality we use the u_r equation. Adding $\frac{1}{r}u_r$ on both sides finishes the proof.

$\square$

We also note for future reference, that by the above computations, in terms of polar variables in the disc $\mathbf{D}$,

$$L^2(dxdy)|_{\mathbf{D}} = L^2(rdrd\theta); \quad (r,\theta) \in (0,1] \times [0,2\pi]. \tag{2.1}$$

3. The method of separation of variables for the Laplacian

3.1. **Non-negativity of eigenvalues.**

We would like to find a rich family of eigenfunctions of the Laplacian on the disc;

$$\triangle u = \mu u \qquad \text{on } \mathbf{D} \tag{3.1}$$

There are two boundary conditions we may apply;

$$u\Big|_{\partial\mathbf{D}} \equiv 0 \qquad\qquad \text{Dirichlet}$$

$$\frac{\partial u}{\partial \nu}\Big|_{\partial\mathbf{D}} \equiv 0 \qquad\qquad \text{Neumann}$$

where ν is the outward pointing normal to the boundary. In other words, if u is expressed in polar coordinates (r,θ) Neumann's condition is the same as

$$\frac{\partial u}{\partial r}(1,\theta) = 0 \qquad\qquad \text{for all } \theta$$

We first make an important remark

Theorem **2.** *Let u be twice continuously differentiable and assume $-\triangle u = \mu u$*

- *if u satisfies the Dirichlet boundary condition, then $\mu > 0$*
- *if u satisfies the Neumann boundary condition, then $\mu \geq 0$*

Proof. Assume u satisfies the pde (3.1). Then

$$\begin{aligned}
\int_{\mathbf{D}} \mu u^2 &= -\int_{\mathbf{D}} \nu \triangle \nu && \text{using the pde} \\
&= \int_{\mathbf{D}} \|\nabla u\|^2 \, dx - \int_{\partial\mathbf{D}} u \frac{\partial u}{\partial n} dS(x) && \text{Green's identity} \\
&= \int_{\mathbf{D}} \|\nabla u\|^2 \geq 0. && \text{Boundary condition}
\end{aligned}$$

Moreover, if

$$0 = \int_{\mathbf{D}} \|\nabla u\|^2$$

then $\|\nabla u\| \equiv 0$ which then implies $u \equiv C$ for some constant C.

Now, if u satisfies the Dirichlet condition, then $C = 0$ by continuity up to the boundary. However, $u \equiv 0$ is not an eigenfunction. Hence, in the latter case $\int_{\mathbf{D}} \|\nabla u\|^2 > 0$ and so,

$$\mu \int_{\mathbf{D}} u^2 > 0. \tag{3.2}$$

Consequently, in this case, $\mu > 0$.

When u satisfies the Neumann condition, the integral

$$\int_{\mathbf{D}} \|\nabla u\|^2 \tag{3.3}$$

is allowed to be zero since a non-zero constant function is a Neumann eigenfunction and so, in the latter case, $\mu \geq 0$. $\square$

3.2. **Separation of variables.** Since the eigenvalues are non-negative, from now on we will write the eigenfunction equation in the form

$$-\triangle u = \lambda^2 u, \quad \text{on } \mathbf{D}. \tag{3.4}$$

This notation will make the writing somewhat more convenient later on and serves as a reminder that eigenvalues are non-negative.

In order to find solutions to 3.4 satisfying either Dirichlet or Neumann condition we will use *the method of separation of variables.* Namely, we look for formal particular solutions of the form $u(r, \theta) = R(r)\Phi(\theta)$ where Φ is 2π-periodic. We summarize the separation of variables argument in the following:

THEOREM **3.** *Let u be a eigenfunction of the Laplacian on $\mathbf{D}$ with eigenvalue as in λ^2 (3.4) and assume that u can be written as*

$$u(r, \theta) = R(r)\Phi(\theta) \tag{3.5}$$

Then there exists $a, b \in \mathbb{C}$ and $n \in \mathbb{N}_0$ such that

$$\Phi(\theta) = a\cos(n\theta) + b\cos(n\theta)$$

and furthermore the function $J : [0, \lambda] \to \mathbb{R}$ given by $J(\lambda r) = R(r)$ satisfies the Bessel equation of order n;

$$0 = x^2 J'' + xJ' + (x^2 - n^2)J \qquad x \in [0, \lambda]. \tag{3.6}$$

The relevant solutions $J = J_n$ to (3.6) are the n-th order Bessel functions of the first kind which have entire extensions to $\mathbb{C}$ for all $n = 1, 2, 3, \ldots$. In addition,

- *if u satisfies the Dirichlet condition, then $J(\lambda) = R(1) = 0$*
- *if u satisfies the Neumann condition, then $J'(\lambda) = R'(1) = 0$*

Consequently, in the Dirichlet case, $\lambda \in \{j_{k,n}\}_{k=1}^{\infty}$ where $j_{k,n}$ is the k-th (simple) zero of J_n. In the Neumann case $\lambda \in \{j'_{k,n}\}_{k=1}^{\infty}$ are the simple zeros of the derivative J'_n.

The proof of this theorem will be carried in several steps. Let $u(r, \theta) = R(r)\Phi(\theta)$ satisfy (3.4). Then, using the polar form of the Laplacian in Theorem 2.1,

$$\begin{aligned}
-\lambda^2 R\Phi &= -\lambda^2 u \\
&= \triangle u \\
&= \left(\partial_{rr} + \frac{1}{r}\partial_r + \frac{1}{r^2}\partial_{\theta\theta} \right) R\Phi \\
&= R''\Phi + \frac{1}{r}R'\Phi + \frac{1}{r^2}R\Phi''
\end{aligned}$$

By considering our function u where at a point (r, θ) where $r > 0$ and $\Phi \neq 0 \neq R$, we can rearrange the previous equation to get

$$-\frac{r^2R'' + rR' + r^2\lambda^2 R}{R} = \frac{\Phi''}{\Phi} \tag{3.7}$$

In (3.7), the right hand side only depends only on θ whereas the left hand side only depends on r. It follows that the only possiblity is that both sides of (3.7) are equal to a constant C; that is,

$$\frac{r^2R'' + rR' + r^2\lambda^2 R}{R} = -\frac{\Phi''}{\Phi} = C \tag{3.8}$$

3.3. **Analysis of Φ.** The second part of this triple equality (3.8) is

$$\Phi'' + C\Phi = 0 \tag{3.9}$$

where Φ is 2π-periodic. This equation is a constant coefficient linear homogeneous second order ODE with the characteristic equation

$$0 = \rho^2 + C$$

Three cases are now possible.
Case 1: $C < 0$
The characteristic equation yields the solutions $\rho_1 = -\sqrt{-C}$ and $\rho_2 = \sqrt{-C}$. Therefore the solutions are

$$\Phi(\theta) = ae^{\rho_1 \theta} + be^{\rho_2 \theta}$$

$\forall a, b \in \mathbb{R}$.
None of these solutions are admissible since Φ is required to be 2π-periodic.
Hence, we must reject the case $C < 0$.
Case 2: C=0
In this case, one gets the solutions

$$\Phi(\theta) = a + b\theta$$

It follows that that b must be 0 since Φ is required to be 2π-periodic.
Case 3: $C > 0$
The equation $0 = \rho^2 + C$ still has two solutions; $\rho_1 = i\sqrt{C}$ and $\rho_2 = -i\sqrt{C}$. Which gives rise the following two independent solutions

$$\begin{cases} \tilde{\Phi}_1(\theta) = e^{i\sqrt{C}\theta} \\ \tilde{\Phi}_2(\theta) = e^{-i\sqrt{C}\theta} \end{cases}$$

Again periodicity requires us to restrict the value of C to only $C \in \{n^2 \mid n \in \mathbb{N}\}$. In other words, $\tilde{\Phi}_1(\theta) = e^{in\theta}$ and $\tilde{\Phi}_2(\theta) = e^{-in\theta}$. But, as a basis for span$(\tilde{\Phi}_1, \tilde{\Phi}_2)$, we prefer to take

$$\Phi_1(\theta) := \sin(n\theta);$$
$$\Phi_2(\theta) := \cos(n\theta)$$

To resume, if $u = R\Phi$ solves (3.4), then there exists $n \in \mathbb{N}_0, a, b \in \mathbb{C}$ such that $\Phi(\theta) = a\cos(n\theta) + b\sin(n\theta)$. Moreover, the value of C in (3.8) is n^2.

3.4. **Analysis of R.** Coming back to our triple equality; equation (3.8), we have on the other hand that

$$\frac{r^2R'' + rR' + r^2\lambda^2 R}{R} = n^2 \qquad \text{for } r \in (0,1)$$

We multiply by R to get

$$0 = r^2R'' + rR' + \left(r^2\lambda^2 - n^2\right)R \qquad \text{for } r \in (0,1) \tag{3.10}$$

To simplify this ode, we define a new function J on $(0, \lambda)$ by

$$J(\lambda r) := R(r) \qquad r \in (0,1) \tag{3.11}$$

which by taking derivatives implies that

$$\lambda J'(\lambda r) = R'(r)$$
$$\lambda^2 J''(\lambda r) = R''(r)$$

and the equation (3.10) becomes

$$0 = \lambda^2 r^2 J''(\lambda r) + \lambda r J'(\lambda r) + (\lambda^2 r^2 - n^2) J(\lambda r) \qquad r \in (0, 1)$$

Calling $x = \lambda r$; the argument of J, gets us

$$0 = x^2 J'' + x J' + (x^2 - n^2) J \qquad x \in (0, \lambda) \tag{3.12}$$

We note that this last equation (3.12) is *independent* of the eigenvalue parameter λ and is well-known as the Bessel ODE of order n.

Since, in particular, the function $u = \Phi R$ is required to be continuous at the center of the disc $\mathbf{D}$, we require $\lim_{x \to 0^+} J(x)$ to exists.

3.4.1. *Dirichlet boundary condition.* We additionally assume that

$$u\Big|_{\partial \mathbf{D}} \equiv 0$$

Since $u = \Phi R$ where Φ is not always 0, we conclude that

$$0 = R(1)$$

Which in turn gives us a boundary condition on (3.12);

$$0 = J(\lambda)$$

3.4.2. *Neumann boundary condition.* Another usual assumption which we will consider is

$$\frac{\partial u}{\partial r}\Big|_{\partial \mathbf{D}} \equiv 0$$

Again using that $u = \Phi R$ where Φ is not always 0, we get

$$0 = R'(1)$$

Or in terms of the function J;

$$0 = J'(\lambda)$$

4. Solving Bessel equation

In view of the previous section, we turn to the analysis of the solutions of (3.12).

Theorem **4.** *Given $n \in \mathbb{N}_0$, the Bessel ODE of order n given by*

$$x^2 J'' + xJ' + (x^2 - n^2)J = 0, \quad x \in (0, \infty) \tag{4.1}$$

has two linearly independent solutions J_n and K_n. J_n is given by

$$J_n(x) = \sum_{\ell=0}^{\infty} \frac{(-1)^\ell}{\ell!(n+\ell)!} \left(\frac{x}{2}\right)^{2\ell+n} \qquad \text{for } n \geq 0$$

and extends to an entire function. On the other hand, the function K_n explodes at $x = 0$; $\lim_{x\to 0^+} |K_n(x)| = \infty$.

The proof of this result is deferred to section 4.2. We first recall the techniques used to solved this equation in section 4.1

4.1. **The Frobenius method.**

Definition **5.** *A second order differential equation with regular singular point at $x_0 = 0$ is of the form*

$$L[\varphi] := x^2\varphi'' + a(x)x\varphi' + b(x)\varphi = 0 \tag{4.2}$$

where a and b are both analytic functions with power series

$$a(x) = \textstyle\sum_{n=0}^{\infty} \beta_n x^n, \qquad b(x) = \sum_{n=0}^{\infty} \beta_n x^n$$

which are convergent for $x \in (0, M_0)$ for some $M_0 > 0$.

Definition **6.** *The indicial polynomial of* (4.2) *is*

$$q(s) = s(s-1) + a(0)s + b(0) \tag{4.3}$$

Theorem **7.** *For $0 < x < M_0$ there is a solution φ_1 to* (4.2) *the form*

$$\varphi_1(x) = x^{s_1} \sum_{k=0}^{\infty} c_k x^k, \quad (c_0 = 1)$$

For the second solution, let s_2, s_1 denote the roots of the indicial polynomial $q(s)$. If $s_2 \leq s_1$ and

- *$s_2 + m \neq s_1$ for every $m \in \mathbb{N}_0$, then there is a second solution of the form*

$$\varphi_2(x) = x^{s_2} \sum_{k=0}^{\infty} \tilde{c}_k x^k, \quad (\tilde{c_0} = 1)$$

- *$s_2 + 0 = s_1$, then the second solution is*

$$\varphi_2(x) = x^{s_1+1}\sigma_2(x) + \ln(x)\varphi_1(x)$$

where σ_2 is a power series centered at $x_0 = 0$.

- *$s_2 + m = s_1$ for some $m \in \mathbb{N}$, the second solution is of the form*

$$\varphi_2(x) = x^{s_2}\sigma_2(x) + c\ln(x)\varphi_1(x),$$

where σ_2 is a power series centered at $x_0 = 0$ and c is a constant.

In all cases the series converge for $x \in (0, M_0)$ and φ_1, φ_2 are both solutions to (4.2) on $(0, M_0)$

Proof. For a detailed proof, we refer to Chapter 4 of [8]. Here, we provide a brief sketch of the ideas involved in the proof. Since q is a quadratic polynomial, q has two roots s_1, s_2. We assume here that they are real and $s_2 \le s_1$.

Given $s \in \mathbb{R}$ and $c_0 \in \mathbb{R}$, we define by recursion

$$c_n(c_0, s) := \frac{-1}{q(n+s)} \sum_{k=0}^{n-1} c_k(c_0, s)\left[(k+s)\alpha_{n-k} + \beta_{n-k}\right] \tag{4.4}$$

where we have abused notation slightly by writing $c_0 = c_0(c_0, s)$ in (4.4). Note that this definition fails if $s \in \mathbb{R}$ is such that $q(n+s) = 0$ for some $n \in \mathbb{N}$. Hence, we let $E(s_2, s_1) := \mathbb{R} \setminus (\{s_1 - 1, s_1 - 2, s_1 - 3, \ldots\} \cup \{s_2 - 1, s_2 - 2, \ldots\})$; for every s in that set, the definition above makes sense.

Finally, we let

$$\Phi(x; c_0, s) := \sum_{k=0}^{\infty} c_k(c_0, s) x^{k+s} \tag{4.5}$$

for the permitted values of s.

Assume that the series $\Phi(x; c_0, s)$ converges for $|x| < M_0$ for a fixed c_0 and $s \in E(s_1, s_2)$. By differentiating inside the summation sign and carrying out some algebra, we get that

$$\begin{aligned}
L[\Phi(x; c_0, s)] &= x^2 \frac{\partial^2}{\partial x^2}\Phi(x; c_0, s) + a(x)x\frac{\partial}{\partial x}\Phi(x; c_0, s) + b(x)\Phi(x; c_0, s) && (4.6)\\
&= x^s\left[q(s)c_0 + \sum_{n=1}^{\infty}\left[q(n+s)c_n + \sum_{k=0}^{n-1} c_k\left[(k+s)\alpha_{n-k} + \beta_{n-k}\right]\right]\right] && (4.7)\\
&= x^s\left[q(s)c_0 + \sum_{n=1}^{\infty} 0\right] && (4.8)\\
&= x^s q(s) c_0 && (4.9)
\end{aligned}$$

where the dramatic simplification from line (4.7) to (4.8) is due to our definition (4.4). We note that s_1; the largest root of q, is an element of $E(s_1, s_2)$ and so

$$\begin{aligned} L[\Phi(x; c_0, s_1)] &= x^{s_1} q(s_1) c_0 \\ &= 0 \qquad\qquad\qquad\qquad (s_1 \text{ is a root of } q) \end{aligned}$$

Therefore, we let

$$\varphi_1(x) := \Phi(x; c_0, s_1) \tag{4.10}$$

be our first solution.

To get a second solution, there are several different cases to consider.

- $s_1 - s_2 \neq 0, 1, 2, \dots$
  Then the second solution is given in the same way as φ_1.
- $s_2 = s_1$.
  In this case, we remark that $q(s_1) = 0$ and $q'(s_1) = 0$ since $q(s) = (s - s_1)^2$. Differentiaion of (4.9) in s gives

$$\begin{aligned} 0 &= \frac{\partial}{\partial s} \left[x^s q(s) c_0 \right] |_{s=s_1} \\ &= \frac{\partial}{\partial s} \left[L[\Phi(x; c_0, s)] \right] |_{s=s_1} \\ &= L[\frac{\partial}{\partial s} \Phi(x; c_0, s) |_{s=s_1}] \end{aligned}$$

  where at the last line we assume that Φ is C^2 in x and s, so that one dan commute the operators ∂_x and ∂_s. As a result, a second solution is given by

$$\begin{aligned} \varphi_2(x) :&= \frac{\partial}{\partial s} \Phi(x; c_0, s) |_{s=s_1} \\ &= \frac{\partial}{\partial s} \left[x^s \sum_{k=0}^{\infty} c_k(c_0, s) x^k \right] |_{s=s_1} \\ &= \left[\ln(x) x^s \sum_{k=0}^{\infty} c_k(c_0, s) x^k + x^s \sum_{k=0}^{\infty} \frac{\partial}{\partial s} \left[c_k(c_0, s) \right] x^k \right] |_{s=s_1} \\ &= \ln(x) x^{s_1} \sum_{k=0}^{\infty} c_k(c_0, s_1) x^k + x^{s_1} \sum_{k=0}^{\infty} \frac{\partial}{\partial s} \left[c_k(c_0, s) \right] |_{s=s_1} x^k \\ &= \ln(x) \Phi(x; c_0, s_1) + x^{s_1} \sum_{k=0}^{\infty} \frac{\partial}{\partial s} \left[c_k(c_0, s) \right] |_{s=s_1} x^k \end{aligned}$$

- $s_2 + m = s_1$.

  Take s close to s_2. Note that both $q(s)$ and $(s - s_2)$ vanish at $s = s_2$ and so,

$$\begin{aligned} 0 &= \frac{\partial}{\partial s}\left[x^s q(s)(s - s_2)\right] |_{s=s_2} \\ &= \frac{\partial}{\partial s}\left[L[\Phi(x; s - s_2, s)]\right] |_{s=s_2} \\ &= L[\frac{\partial}{\partial s}\Phi(x; s - s_2, s) |_{s=s_2}] \end{aligned}$$

  As a result, in this case the second solution is

$$\begin{aligned} \varphi_2(x) :&= \frac{\partial}{\partial s}\Phi(x; s - s_2, s) |_{s=s_2} \\ &= \frac{\partial}{\partial s}\left[x^s \sum_{k=0}^{\infty} c_k(s - s_2, s)x^k\right] |_{s=s_2} \\ &= \left[\ln(x)x^s \sum_{k=0}^{\infty} c_k(s - s_2, s)x^k + x^s \sum_{k=0}^{\infty} \frac{\partial}{\partial s} c_k(s - s_2, s)x^k\right] |_{s=s_2} \\ &= \ln(x)x^{s_2} \sum_{k=0}^{\infty} \lim_{s \to s_2} \left[c_k(s - s_2, s)\right] x^k + x^{s_2} \sum_{k=0}^{\infty} \frac{\partial}{\partial s}\left[c_k(s - s_2, s)\right] |_{s=s_2} x^k \end{aligned}$$

  We note that since $c_0 = s - s_2$ here, its limit as $s \to s_2$ is 0. Also, by definition,

$$c_n(c_0, s) := \frac{-1}{q(n + s)} \sum_{k=0}^{n-1} c_k(c_0, s)\left[(k + s)\alpha_{n-k} + \beta_{n-k}\right] \tag{4.11}$$

  and using induction, we can show that the coefficients $c_1(s-s_2, s), \ldots, c_{m-1}(s-s_2, s)$ all have the factor $s - s_2$. We conclude that their limit are also 0. But $c_m(c_0, m)$ on the other hand might not be 0. All in all we have,

$$\begin{aligned} x^{s_2} \sum_{k=0}^{\infty} \lim_{s \to s_2} \left[c_k(s - s_2, s)\right] x^k &= x^{s_2} \sum_{k=m}^{\infty} \lim_{s \to s_2} \left[c_k(s - s_2, s)\right] x^k \\ &= x^{s_2+m} \sum_{k=0}^{\infty} \lim_{s \to s_2} \left[c_{k+m}(s - s_2, s)\right] x^k \\ &= x^{s_1} \sum_{k=0}^{\infty} \lim_{s \to s_2} \left[c_{k+m}(s - s_2, s)\right] x^k \end{aligned}$$

  It is not difficult to see that the last function is a multiple of φ_1

$\square$

4.2. **Solving Bessel equation using Frobenius method.** In the case of the Bessel ODE of order n

$$0 = x^2 J''(x) + xJ'(x) + (x^2 - n^2)J(x) \tag{4.12}$$
$$= x^2 J''(x) + xa(x)J'(x) + b(x)J(x) \tag{4.13}$$

(4.12) does have a regular singular point $x_0 = 0$ since

$$a(x) \equiv 1$$
$$b(x) = x^2 - n^2$$

are both analytic near zero. Note that $\alpha_0 = 1$ and $\beta_0 = -n^2$. Hence, the indicial polynomial

$$\begin{aligned} q(s) &= s(s-1) + r\alpha_0 + \beta_0 \\ &= s^2 - n^2 \end{aligned}$$

has roots

$$s_2 = -n \qquad s_1 = n.$$

4.2.1. *Finding a particular solution to the Bessel ODE.* Using Frobenius, we choose $c_0 \in \mathbb{R}$ and set

$$c_m = \frac{-1}{q(m+s_1)} \sum_{k=0}^{m-1} c_k \left((k+s_1)\alpha_{n-k} + \beta_{n-k}\right) \tag{4.14}$$

Since $q(s) = s^2 - n^2$ and $s_1 = n$, it follows that in (4.14)

$$\alpha_n = \begin{cases} 1 & n = 0 \\ 0 & \text{else} \end{cases}$$
$$\beta_n = \begin{cases} -n^2 & n = 0 \\ 1 & n = 2 \\ 0 & \text{else.} \end{cases}$$

Thus, equation (4.14) simplifies to

$$\begin{aligned} c_1 &= 0 \\ c_m &= \left(\frac{-1}{m^2 + 2nm}\right) c_{m-2} \qquad && m \geq 2 \end{aligned}$$

It follows by induction that

$$
\begin{aligned}
c_{2\ell+1} &= 0 \\
c_{2\ell} &= \frac{(-1)^\ell c_0}{\ell!(n+1)(n+2)\dots(n+\ell)2^{2\ell}} && \forall \ell \geq 0
\end{aligned}
$$

where $(n+1)(n+2)\dots(n+\ell)$ is understood to be 1 when $\ell = 0$.
Substituting this in the original power series gives

$$
\begin{aligned}
J(x) &= \sum_{k=0}^{\infty} c_k x^{k+n} \\
&= \sum_{\ell=0}^{\infty} c_{2\ell} x^{2\ell+n} \\
&= \sum_{\ell=0}^{\infty} \frac{(-1)^\ell c_0}{\ell!(n+1)(n+2)\dots(n+\ell)2^{2\ell}} x^{2\ell+n}
\end{aligned}
$$

We take $c_0 := 2^n/(n!)$.

Definition 8. *We define the n^{th} Bessel function of the first kind as*

$$J_n(x) = \sum_{\ell=0}^{\infty} \frac{(-1)^\ell}{\ell!(n+\ell)!} \left(\frac{x}{2}\right)^{2\ell+n} \qquad \textit{for } n \geq 0, x \in \mathbb{C} \tag{4.15}$$

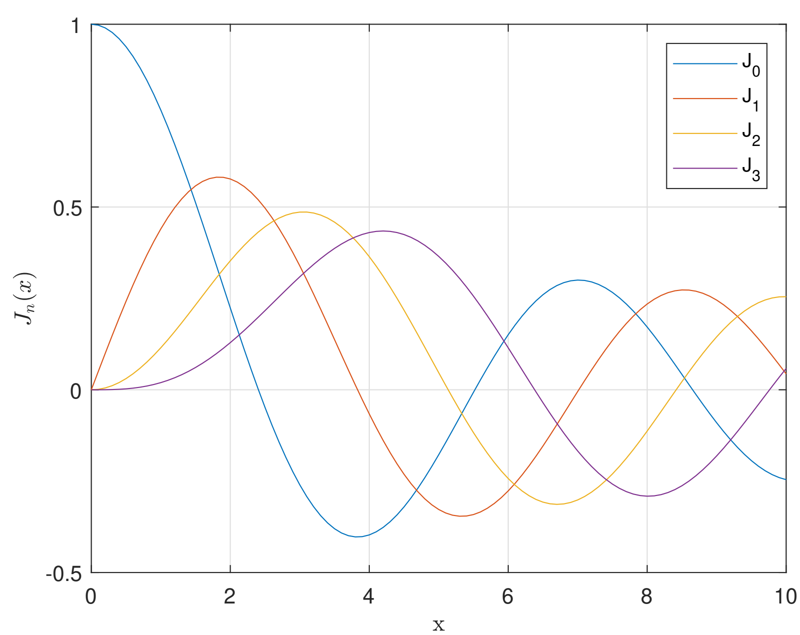


Figure 3. Graph of J_n on $[0, 10] \subset \mathbb{R}$

Claim: For $n = 0, 1, \ldots$ the functions $J_n : \mathbb{C} \to \mathbb{C}$ is well-defined, entire, and a solution to Bessel ode on $\mathbb{C}$.

Proof. To see that J_n has an entire extension and well-defined, we recall that the coefficients of

$$J_n(x) = \sum_{k=0}^{\infty} c_k x^{k+n}$$

are

$$c_k = \begin{cases} 0 & k \text{ is odd} \\ \frac{(-1)^\ell}{\ell!(n+\ell)!2^{2\ell+n}} & k = 2\ell \end{cases}$$

Since

$$\limsup_{\ell\to\infty} \left|\frac{c_{2\ell+2}}{c_{2\ell}}\right| \leq \limsup_{\ell\to\infty} \left|\frac{1}{(\ell+1)(\ell+n+1)}\right| = 0,$$

it follows by the ratio test that

$$J_n(x) = x^n \sum_{k=0}^{\infty} c_k x^k \tag{4.16}$$

converges absolutely for all $x \in \mathbb{R}$ and uniformly on any interval. It is then clear that J_n has an entire extension.

It is easy to verify that (4.16) satisfies the Bessel equation; Indeed,

$$J_n' = \sum_{k=0}^{\infty} c_k (k+n) x^{k+n-1}$$
$$J_n'' = \sum_{k=0}^{\infty} c_k (k+n)(k+n-1) x^{k+n-2}$$

We evaluate

$$x^2 J_n'' + x J_n' + (x^2 - n^2) J_n \tag{4.17}$$
$$= \left(n^2 - n^2\right) c_0 x^n + \left((1+n)^2 - n^2\right) c_1 x^{n+1}$$
$$+ \sum_{k=2}^{\infty} \left(\left[(k+n)^2 - n^2\right] c_k + c_{k-2}\right) x^{k+n} \tag{4.18}$$

equals 0 iff $c_1 = 0$ and $\left(\left[(k+n)^2 - n^2\right] c_k + c_{k-2}\right) = 0 \quad \forall\ k \geq 2$ which is obviously satisfied by the coefficients c_k that we previously chose.

$\square$

4.2.2. *A second independent solution of the Bessel equation of order 0.* In the case of order 0;

$$0 = x^2 J''(x) + xJ'(x) + x^2 J(x)$$

with indicial polynomial $q(s) = s^2$ with repeated roots $s_1 = s_2 = 0$. By theorem (7), our second solution has the form

$$\varphi_2(x) = \sum_{k=0}^{\infty} c_k x^k + \ln(x)\varphi_1(x)$$

for some coefficients c_k to be determined. We have

$$\varphi_2'(x) = \sum_{k=1}^{\infty} k c_k x^{k-1} + \frac{\varphi_1}{x} + \ln(x)\varphi_1(x)$$

$$\varphi_2''(x) = \sum_{k=2}^{\infty} k(k-1) c_k x^{k-2} - \frac{\varphi_1}{x^2} + \frac{2}{x}\varphi_1'(x) + \ln(x)\varphi_1''(x)$$

Therefore,

$$\begin{aligned} L[\varphi_2](x) &= x^2\varphi_2''(x) + x\varphi_2'(x) + x^2\varphi_2(x) \\ &= c_1 x + 4c_2 x^2 + \sum_{k=3}^{\infty}(k^2 c_k + c_{k-2})x^k + 2x\varphi_1'(x) \end{aligned}$$

where we used $\ln(x)L[\varphi_1](x) = 0$ in our simplifications. Since $L[\varphi_2] = 0$ we get that

$$\begin{aligned} c_1 x + 4c_2 x + \sum_{k=3}^{\infty}(k^2 c_k + c_{k-2})x^k &= -2x\varphi_1' \\ &= -2\sum_{m=1}^{\infty} \frac{(-1)^m 2m}{2^{2m}(m!)^2} x^{2m} \end{aligned}$$

Comparing the coefficients and using induction, we conclude that

$$\begin{cases} c_{2m-1} = 0 \\ c_{2m} = \frac{(-1)^{m-1}}{2^{2m}(m!)^2} H_m \quad m = 1, 2, \ldots \end{cases}$$

where $H_m = 1/1 + 1/2 + \ldots + 1/m$ as usual.

Definition **9.** *We define the Bessel function of the 0th order of the second kind to be*

$$K_0(x) = \sum_{m=1}^{\infty} \frac{(-1)^{m+1} H_m}{(m!)^2} \left(\frac{x}{2}\right)^{2m} + \ln(x)J_0(x) \tag{4.19}$$

4.2.3. *A second independent solution of Bessel equation for $n \in \mathbb{N}$.* Let $n \in \{1, 2 \ldots\}$ be the order of the Bessel equation. Recall that the roots of the indicial polynomial $q(r) =$

$(r-n)(r+n)$; $+n$ and $-n$ are separated by an integer. Thus, we seek a solution of the form

$$\varphi_2(x) = \ln(x)cJ_n(x) + x^{r_2}\sum_{k=0}^{\infty} c_k'(r_2)x^k \tag{4.20}$$

We compute

$$\begin{aligned} L[\varphi_2] = 0c_0x^{-n} + [1-2n]c_1x^{1-n} + x^{-n}\sum_{k=2}^{\infty}\left[\left[k^2-2kn\right]c_k + c_{k-2}\right]x^k \\ + 2cxJ_n'(x) \end{aligned}$$

where we used $L[\varphi_1] = 0$ in our simplifications. Therefore, we want

$$0c_0x^{-n} + [1-2n]c_1x^{1-n} + x^{-n}\sum_{k=2}^{\infty}\left[\left[k^2-2kn\right]c_k + c_{k-2}\right]x^k = -2cxJ_n'(x)$$

By multiplying by x^n and recalling our definition of J_n we can rewrite this as

$$[1-2n]c_1x + \sum_{k=2}^{\infty}\left[\left[k^2-2kn\right]c_k + c_{k-2}\right]x^k = \sum_{m=0}^{\infty}\frac{-2c(2m+n)(-1)^{-m}}{2^{2m+n}m!(m+n)!}x^{2m+2n} \tag{4.21}$$

the coefficients of the odd powers on the right hand side are all zeros, hence $0 = c_1 = c_3 = c_5 \ldots$. Also the series on the right starts only at the power $2n$, therefore we require the coefficients of $x^0, x^2, x^4, \ldots, x^{2n-2}$ to be 0 on the left. So, using induction and equation (4.21);

$$\begin{aligned} &c_0 \in \mathbb{R} \\ &c_{2j} = \frac{c_0}{2^{2j}j!(n-1)\ldots(n-j)} \qquad j = 1,2,\ldots,n-1 \end{aligned} \tag{4.22}$$

Now looking at the coefficient of x^{2n} we have

$$[[(2n)^2 - 2(2n)n]c_{2n} + c_{2n-2} = c_{2n-2} = \frac{-c}{2^{n-2}(n-1)} \tag{4.23}$$

Also from (4.22);

$$c_{2n-2} = \frac{c_0}{2^{2n-2}(n-1)!(n-1)!}$$

Hence,

$$c = \frac{-c_0}{2^{n-1}(n-1)!} \tag{4.24}$$

The coefficient c_{2n} is for the moment allowed to be a free parameter. It remains to deal with the coefficients of $x^{2+2n}, x^{4+2n}, \ldots$. Looking at the coefficients of x^{2n+2m} on both sides of

(4.21), we find

$$2m[2n+2m]c_{2n+2m} + c_{2n+2m-2} = \frac{-2c(2m+n)(-1)^{-m}}{2^{2m+n}m!(m+n)!} \quad m = 1, 2, \ldots \tag{4.25}$$

By choosing

$$c_{2n} = \frac{-cH_n}{2^{n+1}n!}$$

one can prove by induction that

$$c_{2n+2m} = \frac{-c(-1)^{-1}[H_m + H_{m+n}]}{2^{2m+n+1}m!(m+n)!}$$

We also end up choosing that value $c = 1$ which implies $c_0 = -2^{n-1}(n-1)!$. And so *the Bessel function of order n of the second kind* is

$$\begin{aligned} K_n(x) = &\frac{-1}{2}\left(\frac{x}{2}\right)^{-n} \sum_{j=0}^{n-1} \frac{(n-j-1)!}{j!}\left(\frac{x}{2}\right)^{2j} - \frac{1}{2}\frac{1}{n!}H_n\left(\frac{x}{2}\right)^n \\ &- \frac{1}{2}\left(\frac{x}{2}\right)^n \sum_{m=1}^{\infty} \frac{(-1)^m}{m!(m+n)!}\left[H_m + H_{m+n}\right]\left(\frac{x}{2}\right)^{2m} + \ln(x)J_n(x) \end{aligned}$$

where as usual $H_n = 1/1 + \ldots + 1/n$ for $n \in \{1, 2, 3, \ldots\}$.

Remark. It is worth noting that the formula remains valid for $n = 0$.

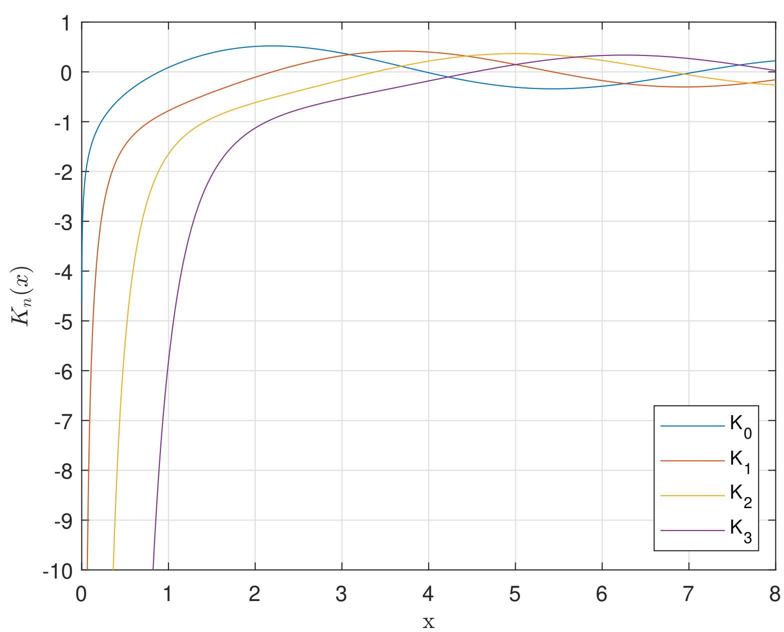


FIGURE 4. Graph of K_n on $[0, 8] \subset \mathbb{R}$

It is useful to summarize the above results in the following

LEMMA **4.1.** *The Bessel functions of the first kind is analytic on $\mathbb{C}$; $J_n \in C^\omega(\mathbb{C})$, $\forall n \geq 0$. The Bessel functions of the second kind, K_n, are singular at $x = 0$. In particular,*

- $\lim_{x\to 0^+} J_0(x) = 1$, $\lim_{x\to 0^+} J_n(x) = 0$, $n \geq 1$,
- $\lim_{x\to 0^+} K_n(x) = -\infty$, $n \geq 0$.

Proof. The fact that $J_n \in C^\omega(\mathbb{C})$, $n \geq 0$ is an immediate consequence of (4.16) since the radius of convergence for the series representing J_n is infinite. Indeed, as we have already pointed out, the Bessel functions of the first kind have entire extensions $J_n : \mathbb{C} \to \mathbb{C}$. As for the computation of the limit at $x = 0$,

$$\begin{aligned}\lim_{x\to 0^+} J_n(x) &= \lim_{x\to 0^+} \sum_{\ell=0}^{\infty} \frac{(-1)^\ell}{\ell!(n+\ell)!}\left(\frac{x}{2}\right)^{2\ell+n} \\ &= \sum_{\ell=0}^{\infty} \lim_{x\to 0^+} \frac{(-1)^\ell}{\ell!(n+\ell)!}\left(\frac{x}{2}\right)^{2\ell+n}\end{aligned} \tag{4.26}$$

The interchange of the summation and the limit is allowed as the series converges absolutely and uniformly on compact intervals. It follows directly from (4.26) that $\lim_{x\to 0^+} J_0(x) = 1$ and $\lim_{x\to 0^+} J_n(x) = 0 \quad n \geq 1$.

As for the Bessel functions of the second kind, from subsection 4.2.3

$$\begin{aligned}\limsup_{x\to 0^+} K_n(x) = &\limsup_{x\to 0^+} \Big[\frac{-1}{2}\left(\frac{x}{2}\right)^{-n} \sum_{j=0}^{n-1} \frac{(n-j-1)!}{j!}\left(\frac{x}{2}\right)^{2j} - \frac{1}{2}\frac{1}{n!}H_n\left(\frac{x}{2}\right)^n \\ &- \frac{1}{2}\left(\frac{x}{2}\right)^n \sum_{m=1}^{\infty} \frac{(-1)^m}{m!(m+n)!}\left[H_m + H_{m+n}\right]\left(\frac{x}{2}\right)^{2m} + \ln(x)J_n(x)\Big]\end{aligned}$$

We will investigate the convergence of each of these four terms. To begin, we note that second term converges to a finite number. Same applies to the third term by switching the limit and the summation sign. We now look at

$$\limsup_{x\to 0^+}[\ln(x)J_n(x)] = \limsup_{x\to 0^+}\left[\ln(x)\sum_{\ell=0}^{\infty} \frac{(-1)^\ell}{\ell!(n+\ell)!}\left(\frac{x}{2}\right)^{2\ell+n}\right]$$

J_n is of the form $J_n(x) = \sum_0^\infty (-1)^\ell b_\ell(x)$ where $b_\ell(x) \geq 0$ has limit 0 as $\ell \to \infty$ and $b_0(x) \geq b_1(x) \geq b_2(x) \ldots$ for all $0 < x < 2$.

By using the estimate of alternative series, we note that for all $0 < x < 1$, $J_n(x) \geq 0$, from which we conclude that

$$\limsup_{x \to 0^+} \ln(x) J_n(x) \leq 0$$

Lastly, we take a look at the first term

$$\begin{aligned}
&\limsup_{x \to 0^+} \frac{-1}{2} \left(\frac{x}{2}\right)^{-n} \sum_{j=0}^{n-1} \frac{(n-j-1)!}{j!} \left(\frac{x}{2}\right)^{2j} \\
&\qquad = \frac{-1}{2} \limsup_{x \to 0^+} \left((n-1)! \left(\frac{x}{2}\right)^{-n} + \sum_{j=1}^{n-1} \frac{(n-j-1)!}{j!} \left(\frac{x}{2}\right)^{2j-n} \right) \\
&\qquad = -\infty
\end{aligned}$$

if $n \geq 1$. Combining these four facts, we conclude that

$$\lim_{x \to 0} K_n(x) = -\infty \quad n = 1, 2, \ldots \tag{4.27}$$

The case $n = 0$ is easier;

$$K_0(x) = \sum_{m=1}^{\infty} \frac{(-1)^{m+1} H_m}{(m!)^2} \left(\frac{x}{2}\right)^{2m} + \ln(x) J_0(x)$$

The first term remains finite as $x \searrow 0$. Also, and $\lim_{x \searrow 0} J_0(x) = 1$. The behavior of ln near zero then implies our result. This concludes the proof of the lemma. $\square$

5. Proof of Theorem 3

Proof. We are now in a position to complete the proof of Theorem 3. So far, we have shown that joint eigenfunctions of $(-\Delta, D_\theta)$ on the disc must be of the form

$$u_{\lambda,n}(r, \theta) = J(\lambda r)\, e^{in\theta}, \; n \in \mathbb{Z},$$

where J solves the Bessel equation of order n. Since the eigenfunctions u must be smooth (in fact, analytic) at $r = 0$, it follows that $J = J_n$, a Bessel function of order n of the first kind. Moreover, in the Dirichlet case, the boundary condition $u_{\lambda,n}(1, \theta) = 0$ is equivalent to

$$J_n(\lambda) = 0,$$

whereas, in the Neumann case,

$$J_n'(\lambda) = 0.$$

$\square$

In view of Theorem 3, it follows that the eigenvalues λ are precisely the zeros of J_n (resp. J_n') in the Dirichlet (resp. Neumann) case.

We will need the following basic results regarding zeros of Bessel functions of the first kind along with their derivatives.

LEMMA **5.1.** *The zeros J_n and J_n' are discrete, real-valued, countably infinite and simple.*

Proof. For the detailed proofs of these facts, we refer the reader[10](Chapter 17). We indicate here only the main ideas involved. First, we note that from the series formula (4.16) it is clear that if $J_n(z_0)=0$ then $J_n(-z_0)=0$ and if $J_n(z_0)=0$ then $J_n(\bar{z}_0)=0$ since the series coefficients in (4.16) are real-valued.

The simplicity of zeros follows directly from the Bessel ODE. To see this one argues by contradiction: if x_0 is not simple, then $J_n(x_0)=J_n'(x_0)=0$, but then from the ODE it follows that $J_n''(x_0)=0$. Differentiation of the ODE and iteration of this argument then implies that $J_n^{(k)}(x_0)=0, \quad k\in\mathbb{Z}^+$. Since J_n is real-analytic, this would inturn imply that $J_n\equiv 0$ which is a contradiction. The set of zeros $\{z_k\}$ are discrete since if there would exists a finite limit point (ie. not equal to $\pm\infty$), by unique continuation, $J_n\equiv 0$, which is absurd.

To show that zeros of J_n are real, it is convenient to use the integral formula [9]

$$\begin{aligned}&(a^2-b^2)\int_0^x tJ_n(at)J_n(bt)dt\\&=x\big[bJ_n(ax)J_n'(bx)-aJ_n'(ax)J_n(bx)\big],\ \ a,b\in\mathbb{C},\ \ n=0,1,2,3,....\end{aligned}\tag{5.1}$$

which can be derived fairly easily from the Bessel ODE itself. Let $z_0\in\mathbb{C}$ be a Bessel zero. Then, subsitution of $x=1,a=z_0$ and $b=\bar{z}_0$ in (5.2) gives

$$0=(z_0^2-\bar{z}_0{}^2)\int_0^1 t|J_n(z_0t)|^2dt,$$

which implies that $z_0^2=\bar{z}_0{}^2$. There are two possibilites: either $\operatorname{Im} z_0=0$ or $\operatorname{Re} z_0=0$. In the latter case, subsitution of $z_0=iy$, $y\neq 0$ in the series formula for J_n implies that $J_n(iy)>0$, which is a contradiction.

Finally, to show that there are infinitely-many zeros, one uses the well-known integral formula

$$J_n(x)=\frac{1}{2\pi}\int_{-\pi}^{\pi}e^{i(x\sin\tau-n\tau)}\,d\tau=\frac{1}{\pi}\int_0^{\pi}\cos(n\tau-x\sin\tau)\,d\tau.\tag{5.2}$$

An application of stationary phase [6] in (5.2) gives

$$J_n(x)=\sqrt{\frac{2}{\pi x}}\cos\left(x-\frac{2n+1}{4}\pi\right)+O\left(\frac{1}{x^{3/2}}\right)\quad(x\to\infty)\tag{5.3}$$

and similarily for the derivative,

$$J_n'(x) = -\sqrt{\frac{2}{\pi x}}\sin\left(x - \frac{2n+1}{4}\pi\right) + O\left(\frac{1}{x^{3/2}}\right) \quad (x \to \infty) \tag{5.4}$$

where $|x_k| \geq R$ where R sufficiently large with $x_k = \frac{2n+1}{4}\pi + \frac{2k+1}{2}\pi$, so that the first cosine term on the RHS of (5.3) vanishes and consider the interval $I_k := [x_k - R^{-1/2}, x_k + R^{-1/2}]$. It then follows from (5.3) that J_n must change sign on I_k and so, by the intermediate value theorem, J_n must have a zero in I_k. The same agument holds for all $k \in \mathbb{N}$ with $|x_k| \geq R$, so J_n has infinitely many zeros.

The proof of these facts for the derivative J_n' follows in a same way as for J_n. $\square$

It follows from Lemma 5.1 that one can list the positive zeros of J_n in the form $0 < j_{1,n} < j_{2,n} < \dots$ and the roots of the derivative of J_n in the form $0 \leq j'_{1,n} < j'_{2,n} < \dots$. Since zeros occur in $\pm$-pairs, there are corresponding negative eigenvalues in both case.

Consequently, the Laplace eigenvalues consist of the sequence

$$0 < j_{1,n}^2 < j_{2,n}^2 < j_{3,n}^2 < \cdots \; (Dirichlet),$$

and

$$0 \leq (j'_{1,n})^2 < (j'_{2,n})^2 < (j'_{3,n})^2 < \cdots \; (Neumann).$$

The following result is a consequence of Theorem 3 and Lemma 5.1.

Theorem 10. *The sequence of functions*

$$(i)\; u_{k,n}(r,\theta) := J_n(j_{k,n}r)e^{\pm in\theta};\; n \in \{0,1,2\dots\} \text{ and } k \in \{1,2\dots\}$$

is an orthogonal sequence of Dirichlet eigenfunctions on the disc $\mathbf{D}$ *with eigenvalue* $j_{k,n}^2$. *Similarly, the sequence of functions*

$$(ii)\; v_{k,n}(r,\theta) := J_n(j'_{k,n}r)e^{\pm in\theta};\; n \in \{0,1,2\dots\} \text{ and } k \in \{1,2\dots\}$$

is an orthonormal sequence of Neumann eigenfunctions on the Laplacian on the disc $\mathbf{D}$ *with eigenvalue* $(j'_{k,n})^2$. *In both cases (i) and (i), the eigenfunction sequences are complete in* $L^2(\mathbf{D})$.

Proof. The fact that the $u_{k,n}$ (resp. $v_{k,n}$) form a sequence of Dirichlet (resp. Neumann) eigenfunctions on $\mathbf{D}$ follows directly from Lemma 5.1 and Theorem 3.

In the Dirichlet case, the orthogonality can be proved as follows.

$$\int_{\mathbf{D}} u_{k,n}\overline{u_{k',n'}}rdrd\theta = \Big(\int_0^{2\pi} e^{i(n-n')\theta}d\theta\Big)\cdot\Big(\int_0^1 J_n(j_{k,n}r)\,J_{n'}(j_{k',n'}r)\,rdr\Big). \tag{5.5}$$

Since $\int_0^{2\pi} e^{i(n-n')\theta}d\theta = 2\pi\delta_{n,n'}$, it follows that for $n \neq n'$ the eigenfunctions are orthogonal.

When $n = n'$, it follows from (5.5) that

$$\int_{\mathbf{D}} u_{k,n}\overline{u_{k',n}} r dr d\theta = 2\pi \int_0^1 J_n(j_{k,n}r)\, J_n(j_{k',n}r)\, r dr = 0, \;\; k \neq k'. \tag{5.6}$$

The last line follows from (5.2) and the fact that $j_{k,n} \neq j_{k',n}$ when $k \neq k'$.

A very similar argument to the one above can also be carried out for the Neumann eigenfunctions.

As for completeness of the above eigenfunction sequence, this follows from general completeness results for eigenfunctions of Sturm-Liouville boundary value problems [5] (Chapter 7). □

5.1. L^2 **-Normalization.** Finally, to L^2-normalize the eigenfunctions , we evaluate

$$\begin{aligned}\|\tilde{u}_{k,n}\|^2_{L^2(\mathbf{D})} &= \int_{\mathbf{D}} |u_{k,n}|^2 \\ &= \int_{[0,2\pi)}\int_{[0,1)} \left|J_n(j_{k,n}r)e^{in\theta}\right|^2 r dr d\theta \\ &= 2\pi \int_{[0,1)} J_n(j_{k,n}r)^2 r dr.\end{aligned}$$

We set

$$c_{k,n} := \left(2\pi \int_{[0,1)} J_n(j_{k,n}r)^2 r dr\right)^{1/2}.$$

and

$$d_{k,n} := \left(2\pi \int_{[0,1)} J_n(j'_{k,n}r)^2 r dr\right)^{1/2}$$

The L^2-normalized Dirichlet (resp. Neumann) eigenfunctions are then

$$\tilde{u}_{k,n} = \frac{u_{k,n}}{c_{k,n}}, \;\; \tilde{v}_{k,n} = \frac{v_{k,n}}{d_{k,n}}.$$

6. Asymptotics for the Neumann eigenfunctions: Proof of Theorem 1

We carry out the proof of Theorem 1 for Neumann eigenfunctions in several steps.

In the following we set $\alpha := \frac{1}{\beta}$ where $\beta \in (0,1)$ and so, $\alpha \in (1,\infty)$. We would like to study the behavior of $\tilde{u}_{k(n),n} = c^{-1}_{k(n),n} J_n(j'_{k(n),n}r)e^{in\theta}$ as $n \to \infty$ where $k(n)$ is chosen as in (1.5) with

$$j_{k(n),n} = \alpha n + O(1), \;\text{ as } n \to \infty.$$

Direct computation using the integral formula (5.2) gives

$$\begin{aligned} J_n(j'_{k(n),n}r) &= J_n((\alpha n + O(1))r) \\ &= \frac{1}{\pi}\int_0^\pi \cos\left(n\left[\alpha r\sin(x) - x\right]\right)\, a(x,r)\, dx \\ &= \frac{1}{\pi}\mathrm{Re}\left(\int_0^\pi e^{in[\alpha r\sin(x)-x]}\, a(x,r)\, dx\right). \end{aligned} \tag{6.1}$$

In (6.1), $a(x,r) = e^{iO(1)r\sin x} \in C^\infty$ so that $|a(x,r)| = 1$ and $\partial_x^\alpha a(x,r) = O_\alpha(1)$ uniformly for $n \geq 0$ and $x \in [0,\pi]$.

Consequently, our goal is to find the asymptotics of the integral:

$$\frac{1}{\pi}\mathrm{Re}\left(\int_0^\pi e^{in\varphi(x;\alpha r)}\, a(x,r) dx\right), \tag{6.2}$$

where

$$\varphi(x;\alpha r) := \alpha r\sin(x) - x. \tag{6.3}$$

We sometimes abuse notation and simply write $\varphi(x)$ for $\varphi(x,\alpha r)$.

6.0.1. *The method of steepest descent.* To compute the large n asymptotics of (6.2), we will apply the method of steepest descent [6]. Roughly speaking, this technique consist of intepreting (6.2) as a contour integral in the complex plane $\mathbb{C}$ and using Cauchy integral formula to modify the path of integration in order to pass through critical points of the phase function φ. Moreover, on this modified path, it is useful to require that $\mathrm{Re}\,(\varphi) = \textit{const.}$. since, If $\mathrm{Re}\,(\varphi)$ is constant, then the integrand $e^{in\varphi} = e^{-n\mathrm{Im}\,(\varphi)}e^{in\mathrm{Re}\,(\varphi)}$ consist of a real part $e^{-n\mathrm{Im}\,(\varphi)}$ being multiplied by a *constant* term $e^{in\mathrm{Re}\,(\varphi)}$ which can be removed from the integral. It is then possible to extract the asymptotics of the remaining real integral using Laplace's method [6].

6.0.2. *Finding the critical points.* Since

$$\begin{aligned} \varphi(x) &= \alpha r\sin x - x \\ \varphi'(x) &= \alpha r\cos x - 1 \\ \varphi''(x) &= -\alpha r\sin x, \end{aligned}$$

the non-degenerate critical points $x = c$ of φ are the solutions of the equation $\varphi'(x) = 0$ and $\varphi''(x) \neq 0$. In other words

$$\cos(c) = \frac{1}{\alpha r}. \tag{6.4}$$

For $1 > r > \frac{1}{\alpha}$, we have the following critical point for φ;

$$c = \arccos(1/\alpha r), \quad c \in (0,\pi). \tag{6.5}$$

We will deform the path so that it passes through c along a path of steepest descent.

6.0.3. *Finding the contour of steepest descent.* Now let us identify the complex contours on which path $\mathrm{Re}\,\varphi$ is constant. Using a trigonometric identity, we find

$$\begin{aligned}\varphi(x+iy) &= \alpha r \sin(x+iy) - (x+iy) \\ &= \alpha r\left[\sin(x)\cosh(y) + i\cos(x)\sinh(y)\right] - (x+iy).\end{aligned} \tag{6.6}$$

Hence,

$$\mathrm{Re}\left(\varphi(x+iy,r)\right) = \alpha r \sin(x)\cosh(y) - x$$

Consequently, the required contour will be of the form $\{(x,y)\in\mathbb{R}^2; \alpha r\sin(x)\cosh(y) - x = \textit{const.}\}$, but as we will see, this curve consists of several branches and we will have to choose the appropriate one.

First, since the required curve passes through the saddle point c, it will be convenient to define

$$\begin{aligned} k(\alpha r) :&= \mathrm{Re}\left(\varphi(c,r)\right) \\ &= \alpha r \sin(c)\cosh(0) - c \\ &= \alpha r \sin\left(\arccos(1/\alpha r)\right)\cosh(0) - \arccos(1/\alpha r) \\ &= \alpha r \frac{\sqrt{(\alpha r)^2-1}}{\alpha r} 1 - \arccos(1/\alpha r) \qquad (*) \\ &= \sqrt{(\alpha r)^2-1} - \arccos(1/\alpha r). \end{aligned}$$

Lemma 11. For every $\alpha > 1$ and $r \in (0,1)$ such that $r > \frac{1}{\alpha}$, we have $k(\alpha r) > 0$

Proof. We defer the proof to Lemma 13 in the Appendix. □

We consider the set

$$S = \{x+iy \in \mathbb{C} \mid \alpha r \sin(x)\cosh(y) - x = k(\alpha r)\} \tag{6.7}$$

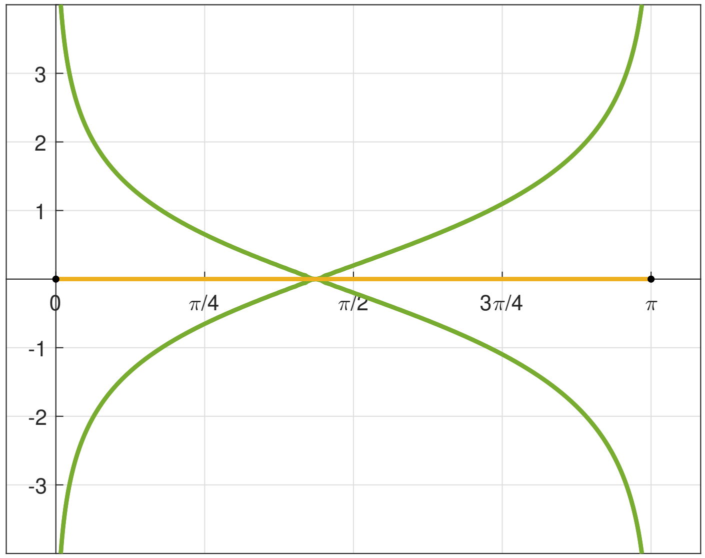


FIGURE 5. In green the set S, in yellow is the original path of integration

This set's shape is the same for every value of α and r such that $1 < \alpha r$. Indeed, fix $\alpha \in (1, \infty)$ and $r \in (0, 1)$ such that $1 < \alpha r$.

When $x + iy \in S$,

$$\sin(x)\cosh(y) = (k(\alpha r) + x)/\alpha r \tag{6.8}$$

where our constant $k(\alpha r) > 0$.

As $x \to 0^+$, the right hand side of (6.8) tends to a positive constant and $\sin(x) \to 0^+$, so we must have that $\cosh(y) \to \infty$. In other words,

$$x \to 0^+ \implies y \to \pm\infty \qquad \text{for } x + iy \in S$$

The same analysis can be made as x tends to π from the left. Thus, S on $0 < \mathrm{Re}\,(z) < \pi$ consists of two branches which intersect at c. Now in order to integrate from $(0, 0)$ to $(\pi, 0)$, there are multiple paths of integration possible. For example,

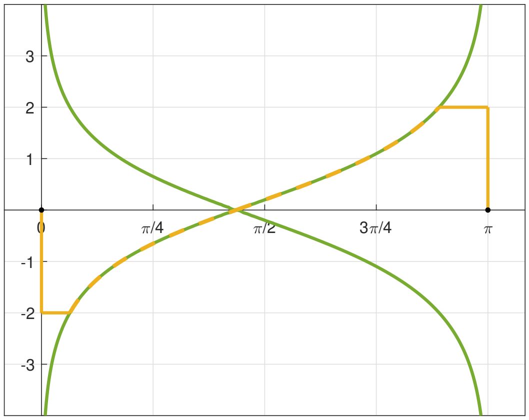


FIGURE 6. The lower potential path of integration

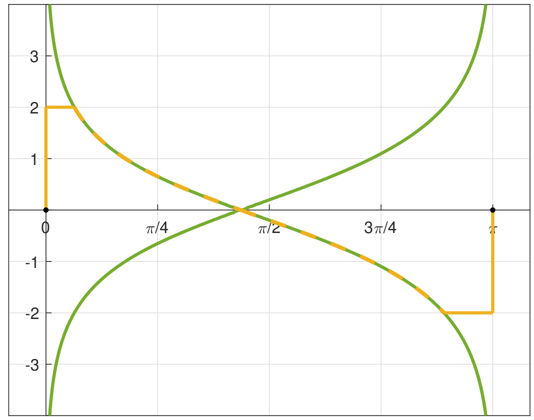


FIGURE 7. The higher potential path of integration

In order to decide which is best, we consider the imaginary part of the phase function (6.6) given by

$$\operatorname{Im}\varphi(x+iy;r) = \alpha r\cos(x)\sinh(y) - y.$$

When $x \in (0,\pi)$ is near 0, we have that

$$\operatorname{Im}\varphi(x+iy;r) \approx \alpha r\sinh(y) - y \tag{6.9}$$

We will eventually want to integrate the expression

$$\int e^{-n\operatorname{Im}(\varphi)}$$

over the chosen contour. For this integral to be well-behaved, we require that y to tend to ∞ (not $-\infty$) as x approaches 0 along the curve. In view of (6.9) this will ensure that along the relevant component of S,

$$\lim_{(x,y)\to(0^+,\infty)} \operatorname{Im}\varphi(x+iy,r) = \infty.$$

A similar analysis as $x \to \pi^-$ reveals that one should require y to tend towards $-\infty$ as x tends to π. We conclude that the path of integration shown in Figure 7 is the most adequate. In the next lemma, we rigorously prove the existence of this curve and its salient features.

LEMMA **6.1.** *Fix $\alpha > 1$ and $\frac{1}{\alpha} < r < 1$. Then, there is a C^∞ function $y : (0,\pi) \to \mathbb{R}$ such that its graph*

$$\{(x, y(x)); x \in (0,\pi)\} \subset S \tag{6.10}$$

In addition, $\lim_{x\to 0^+} y(x) = \infty$*;* $\lim_{x\to\pi^-} y(x) = -\infty$*;* $y' < 0$ *on the interval* $(0,\pi)$*; and* $y(c) = y(\arccos(1/\alpha r)) = 0$.

Proof. Setting $u := \alpha r$, by the choice of α and r we have that $u > 1$. The equation of the set S

$$u\sin(x)\cosh(y) - x = k(u) \tag{6.11}$$

holds if and only if

$$\cosh(y) = \frac{k(u)+x}{u\sin(x)}, \quad x \in (0,\pi). \tag{6.12}$$

Thus, to solve for y, we need only show that for every $x \in (0,\pi)$ and $u > 1$,

$$h(x;u) := \frac{k(u)+x}{u\sin(x)} \geq 1 \tag{6.13}$$

where $k(u) = \sqrt{u^2-1} - \arccos(u^{-1})$. The proof of (6.13) is elementary but somewhat tedious, so we defer it to the Lemma 14 in the Appendix.
We want to isolate y in 6.12. Since $\cosh(x) = \frac{e^x+e^{-x}}{2}$ is even, it is not injective for $x \in \mathbb{R}$.

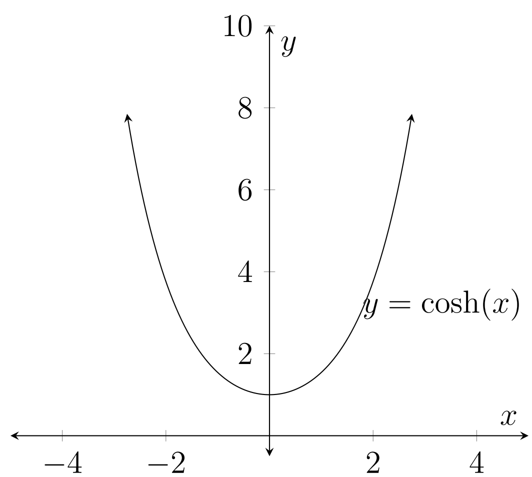


FIGURE 8. The graph of $\cosh(x)$

Nonetheless, when restricted to either $[0,\infty)$ or $(-\infty,0]$, cosh is a one-to-one function and can be inverted.

$$\left(\cosh\big|_{[0,\infty)}\right)^{-1}(x) = \ln(x+\sqrt{x^2-1})$$
$$\left(\cosh\big|_{(-\infty,0]}\right)^{-1}(x) = \ln(x-\sqrt{x^2-1})$$

for $x \in [1, \infty)$. Remembering the path of integration wanted in Figure 7, we define

$$y(x) = \begin{cases} \ln\left(\frac{k(\alpha r)+x}{\alpha r \sin(x)} + \sqrt{\left(\frac{k(\alpha r)+x}{\alpha r \sin(x)}\right)^2 - 1}\right) & \text{for } x \in (0, \arccos(1/\alpha r)] \\ \ln\left(\frac{k(\alpha r)+x)}{\alpha r \sin(x)} - \sqrt{\left(\frac{k(\alpha r)+x)}{\alpha r \sin(x)}\right)^2 - 1}\right) & \text{for } x \in [\arccos(1/\alpha r), \pi) \end{cases} \tag{6.14}$$

It is easily checked that the function $y \in C^\infty(0, \pi)$ and moreover by direct computation using (6.12) it follows that

$$y(c) = y(\arccos(1/\alpha r)) = 0,$$

with $\{(x, y(x)),\ x \in (0, \pi)\} \subset S$. The uniqueness is clear. □

Definition 12. *For the remaining of this thesis $y(x, \alpha r) = y(x)$ for $x \in (0, \pi)$ will refer to the function 6.14 defined in the proof of 6.1*

We now present the curve of steepest descent in more detail.

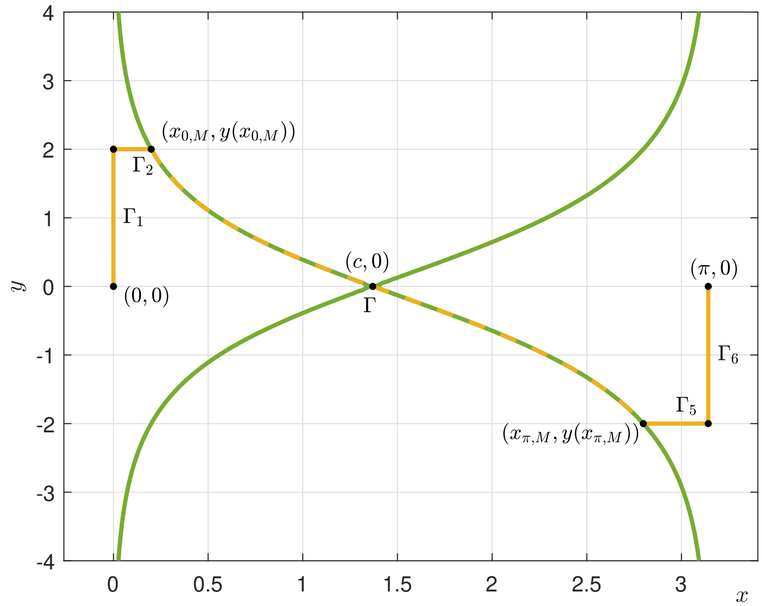


Figure 9. The path of integration

We first choose a large number $M >> 1$ and define

$$x_{0,M} = \frac{1}{M}, \quad x_{\pi,M} = \pi - \frac{1}{M}.$$

The values of $y(x_{0,M})$ and $y(x_{\pi,M})$ determine the "height" of Γ_1 and Γ_6 as pictured in Figure 9 It is easy to check from the formula in (6.12) and the fact that $k(u) > 0$ that

$$y(x_{0,M}) = y(1/M) \approx \log M, \quad y(x_{\pi,M}) = y(\pi - 1/M) \approx \log M.$$

To be more concrete, here are the detailed contours

$$\begin{aligned}\Gamma_1 &= \{0+it,\, 0\le t\le y(1/M)\},\\ \Gamma_2 &:= \{t+iy(1/M);\, 0\le t\le 1/M\},\\ \Gamma &:= \{t+iy(t);\; 1/M\le t\le \pi-1/M\},\\ \Gamma_5 &:= \{t+iy(\pi-1/M); \pi-1/M\le t\le\pi\},\\ \Gamma_6 &:= \{\pi+it; y(\pi-1/M)\le t\le 0\}.\end{aligned}$$

The contour of steepest descent is defined to be

$$\Gamma_{sd} := \Gamma_1\cup\Gamma_2\cup\Gamma\cup\Gamma_5\cup\Gamma_6.$$

6.0.4. *Evaluating the integral on* Γ_{sd}. By Cauchy's theorem, we have

$$\begin{aligned}\int_0^\pi e^{in\varphi(x;r)}a(x,r)dx &= \int_{\Gamma_{sd}} e^{in\varphi(z,r)}a(z,r)\,dz\\ &= (\int_{\Gamma_1}+\int_{\Gamma_2}+\int_{\Gamma}+\int_{\Gamma_5}+\int_{\Gamma_6})e^{in\varphi(z;r)}a(z,r)\,dz.\end{aligned}\tag{6.15}$$

The integral over Γ_1 and Γ_6:

We choose the parametrization $t\mapsto 0+it$ as t ranges from 0 to M:

$$\begin{aligned}\int_{\Gamma_1} e^{in\varphi(z;r)}a(z)dz &= \int_0^{y(1/M)}\exp\left[in\varphi(it;r)\right]a(it)\frac{dz(t)}{dt}dt\\ &= \int_0^{y(1/M)}\exp\left[in(\alpha r\sin(ti)-ti)\right]i\;a(it)dt\\ &= i\int_0^{y(1/M)}\exp\left[in(\alpha ri\sinh(t)-ti)a(it)\right]dt\\ &= i\int_0^{y(1/M)}\exp\left[-n(\alpha r\sinh(t)-t)\right]\;a(it)dt\end{aligned}$$

Since $a(it)=e^{iO(1)r\sin(it)}=e^{-O(1)r\sinh t}$ is real-valued, the integral over Γ_1 is purely imaginary. Since in (6.2), one takes real parts, the integral over Γ_1 does not contribute to the value of $J_n(\alpha nr)$.

A similar computation over the vertical contour Γ_6 gives

$$\mathrm{Re}\Big(\int_{\Gamma_6} e^{in\varphi(z)}\,a(z)\,dz\Big)=0.$$

Consequently, the integral over Γ_6 does not contribute either.

The integrals over Γ_2 and Γ_5:

For Γ_2, we consider $z(t) = t + y(1/M)i$ as t ranges from 0 to $x_{0,M}$.

$$\int_{\Gamma_2} e^{in\varphi(z;r)} a(z) dz = \int_0^{x_{0,M}} \exp\left[in\varphi(t + y(1/M)i; r)\right] a(z(t)) \frac{dz(t)}{dt} dt$$
$$= \int_0^{x_{0,M}} \exp\left[in(\alpha r \sin(t + y(1/M)i) - (t + y(1/M)i))\right] a(z(t)) dt$$
$$= \int_0^{x_{0,M}} \exp\left[in(\alpha r(\sin(t)\cosh(y(1/M)) + i\cos(t)\sinh(y(1/M))) - (t + y(1/M)i))\right] a(z(t)) dt$$
$$= \int_0^{x_{0,M}} \exp\left[-n(\cos(t)\sinh(y(1/M)) - y(1/M))\right] \exp\left[in(\alpha r \sin(t)\cosh(y(1/M)) - y(1/M))\right] a(z(t)) dt$$

Taking absolute value of both sides and recalling that $y(1/M) \approx \log M =: M'$ when $M \gg 1$ gives with some constants $C, C', C'' > 0$,

$$\left| \int_{\Gamma_2} e^{in\varphi(z;r)} dz \right| \leq C \int_0^{x_{0,M}} \left| \exp\left[-n(\cos(t)\sinh(M') - M')\right] \cdot \exp\left[in(\alpha r \sin(t)\cosh(M) - M)\right] \right| dt$$
$$\leq C' \int_0^{x_{0,M}} \exp\left[-n(\cos(t)\sinh(M') - M')\right] dt$$
$$\leq C'' x_{0,M} \max_{t \in [0, x_{0,M}]} \exp\left[-n(\cos(t)\sinh(M') - M')\right]$$

Since $M \gg 1$, we can assume without loss of generality that $\cos(t) \geq 1/2$ for all t in the range $[0, x_{0,M}] = [0, 1/M]$. The latter bound then gives

$$\left| \int_{\Gamma_2} e^{in\varphi(z;r)a(z,r)} dz \right| = O(M^{-1}) e^{n \log M} e^{-n \sinh(\log M)/2} = O(e^{-C_0(M)n}), \tag{6.16}$$

where $C_0(M) > 0$ provided we choose $M > 1$ large enough. A similar bound holds for the integral over Γ_5.

The integral over Γ:

The parametrization is given by

$$z(x) = x + iy(x, \gamma), \quad x \in [x_{0,M}, x_{\pi,M}] \tag{6.17}$$

where we defined $\gamma := \alpha r$. In particular, on $[x_{0,M}, c]$

$$z(x) = x + i \ln\left(\frac{k + x}{\gamma \sin(x)} + \sqrt{\left(\frac{k + x}{\gamma \sin(x)}\right)^2 - 1}\right)$$

using $k = k(\alpha r) = k(\gamma)$. We need $z'(x)$ to evaluate the integral:

$$\begin{aligned} z'(x) &= \frac{d}{dx}\left(x + iy(x)\right) \\ &= 1 + iy'(x) \end{aligned}$$

Where on $(x_{0,M}, c)$, we have

$$\begin{aligned} y'(x) &= \frac{d}{dx}\left(\ln\left(\frac{k+x}{\gamma\sin(x)} + \sqrt{\left(\frac{k+x}{\gamma\sin(x)}\right)^2 - 1}\right)\right) \\ &= \left(\frac{k+x}{\gamma\sin(x)} + \sqrt{\left(\frac{k+x}{\gamma\sin(x)}\right)^2 - 1}\right)^{-1} \frac{d}{dx}\left(\frac{k+x}{\gamma\sin(x)} + \sqrt{\left(\frac{k+x}{\gamma\sin(x)}\right)^2 - 1}\right) \\ &= \frac{\left(\frac{\sin(x) - (k+x)\cos(x)}{\gamma\sin^2(x)}\left(1 \;+\; \frac{k+x}{\sqrt{\gamma\sin(x)[k+x-\gamma\sin(x)]}}\right)\right)}{\frac{k+x}{\gamma\sin(x)} + \sqrt{\left(\frac{k+x}{\gamma\sin(x)}\right)^2 - 1}} \\ &= \frac{\left(\sin(x) - (k+x)\cos(x)\right)\left(1 \;+\; \frac{k+x}{\sqrt{\gamma\sin(x)[k+x-\gamma\sin(x)]}}\right)}{\sin(x)\left(k + x + \sqrt{(k+x)^2 - \gamma^2\sin^2(x)}\right)} \end{aligned}$$

We now compute the integral over Γ;

$$\begin{aligned} \int_\Gamma e^{in\varphi} a dz &= \int_\Gamma e^{in[\mathrm{Re}\,(\varphi) + i\mathrm{Im}\,(\varphi)]} a\, dz \\ &= \int_\Gamma e^{in\mathrm{Re}\,(\varphi)} e^{-n\mathrm{Im}\,(\varphi)} a\, dz \\ &= e^{ink(\alpha r)} \int_\Gamma e^{-n\mathrm{Im}\,(\varphi)} a\, dz, \end{aligned}$$

where in the last line we have used that $\mathrm{Re}\,(\varphi(z)) = k(\alpha r)$ when $z \in \Gamma$.

Now using our parametrization we have

$$\int_\Gamma e^{-n\mathrm{Im}\,(\varphi)} dz = \int_{x_{0,M}}^{x_{\pi,M}} e^{-n\mathrm{Im}\,[\varphi(z(x))]} a(z(x)) \frac{dz(x)}{dx} dx \tag{6.18}$$

6.1. **Proof of Theorem 1 (i).**

Proof. To summarize, we have shown that with $j'_{k(n),n} = \alpha n + O(1)$, and with appropriate $C > 0$,,

$$J_n(j'_{k(n),n} r) = \mathrm{Re}\left[e^{ink(\alpha r)}\Big(\int_{x_{0,M}}^{x_{\pi,M}} e^{-n\mathrm{Im}\,[\varphi(z(t))]} a(z(t)) \frac{dz(t)}{dt} dt\Big)\right] + O(e^{-Cn}). \tag{6.19}$$

We compute the asymptotics of the RHS in (6.19) using the well-known asymptotic formula for Laplace integrals [6] (Section 2.4.)

$$\int_{x_{0,M}}^{x_{\pi,M}} e^{-n\mathrm{Im}\,[\varphi(z(t))]} a(z(t)) \frac{dz(t)}{dt} dt \sim_{n\to\infty} (2\pi n)^{-1/2} e^{-n\mathrm{Im}\,\varphi(c)} \sum_{k=0}^{\infty} a_k(r) n^{-k} \tag{6.20}$$

In (6.20), for any $\varepsilon > 0$ $a_k(r) \in C^\infty(\frac{1}{\alpha} + \varepsilon, 1)$ can be computed explicitly from the derivatives of the phase and amplitude and we have used that $\mathrm{Im}\,\varphi(c) = 0$. The leading coefficient in (6.20) is

$$a_0(r) = [\partial_t^2 h(c;r)]^{-1/2} a(c;r) \neq 0, \quad h(t) := \mathrm{Im}\,\varphi(z(t);r), \quad r \in (\frac{1}{\alpha} + \varepsilon, 1).$$

Dividing by $c_{k(n),n} = (\int_0^1 |J_n(j'_{k(n),n} r)|^2 r dr)^{1/2}$ in (6.20) and multiplying by $e^{in\theta}$ gives

$$\tilde{v}_{k(n),n}(r,\theta) \sim_{n\to\infty} \sum_{\pm} e^{\pm in[\theta + k(\alpha r)]} \big(\sum_{k=0}^{\infty} b_k^{\pm}(r) n^{-k} \big); \quad r \in (\frac{1}{\alpha} + \varepsilon, 1), \tag{6.21}$$

where $\int_0^1 |b_0^{\pm}(r)|^2 r dr = 1$. The proof in the Dirichlet case is essentially the same. This completes the proof of Theorem 1 (i).

$\square$

6.2. **Proof of Theorem 1 (ii).**

Proof. Fix $\varepsilon > 0$ arbitrarily small. and consider the complement of the annulus $A(\alpha)$ given by

$$\mathbf{D} \setminus A(\alpha) = \{z \in D; |z| \leq \frac{1}{\alpha} - \varepsilon\}.$$

To bound the eigenfunctions $\tilde{v}_{k(n),n}(z)$ when $z \in \mathbf{D} \setminus A(\alpha)$, we just note that whe $r \in [0, 1/(\alpha(1+\varepsilon_0))]$, the phase function

$$\varphi(x,r) = \alpha r \sin x - x$$

has no critical points for $x \in (0,\pi)$, since the putative critical point equation

$$\cos x = \frac{1}{\alpha r}$$

has no solution in $x \in (0,\pi)$ when $\alpha r < 1$.

We have the lower bound

$$|\varphi'(x;\alpha r)| > C(\varepsilon,\alpha) > 0 \qquad \forall\, x \in [-\pi,\pi],\ \forall\, r \in [0, \frac{1}{\alpha} - \varepsilon] \tag{6.22}$$

Fix $\delta > 0$ and consider the contour deformation

$$\Gamma : x \mapsto x + i\delta\overline{\varphi'(x;\alpha,r)}, \quad x \in (0,\pi).$$

Tnen, it follows by Cauchy integral formula that with $\delta > 0$ small,

$$J_n(j_{k(n),n} r) = \int_\Gamma e^{in\varphi(z;\alpha,r)}\, a(z;\alpha,r)\, dz + O(e^{-C_1 n}), \;\; C_1 > 0. \tag{6.23}$$

Since $\Gamma \ni z = x + i\delta\overline{\varphi'(x;\alpha,r)}$, and

$$\varphi(x + i\delta\overline{\varphi'(x)}) = \varphi(x) + i\delta|\varphi'(x)|^2 + O(\delta^2).$$

Consequently,

$$J_n(j'_{k(n),n} r) = \int_0^\pi e^{in[\varphi(x)+i\delta|\varphi'(x)|^2+O(\delta^2)]}\, a(x + i\delta\overline{\varphi'(x)};\alpha,r)\, dx + O(e^{-C_1 n}), \;\; C_1 > 0. \tag{6.24}$$

In view of (6.22) by choosing $\delta > 0$ sufficiently small, it follows that with an another constant $C_2 = C_2(\varepsilon,\delta,\alpha) > 0$,

$$\left| J_n(j'_{k(n),n} r) \right| \leq e^{-C_2 n}.$$

This completes the proof of Theorem 1 (ii). $\square$

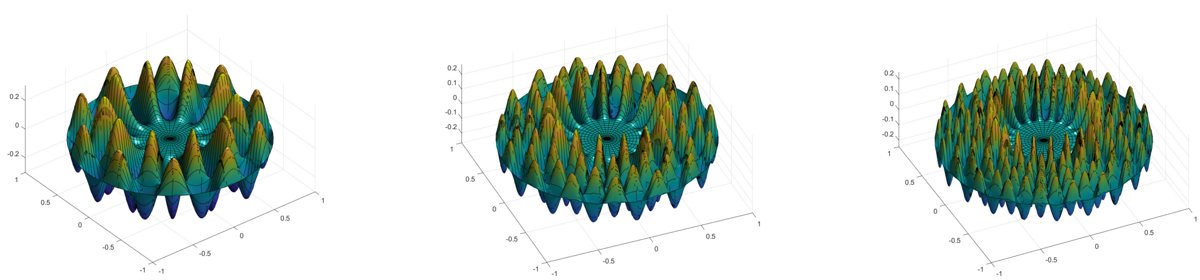


Figure 10. Sequence of eigenfunctions

7. The extreme case $\alpha = 1$

In this case, the degenerate annulus is the boundary circle

$$A(1) = \{r = 1\}$$

of the unit disc $\mathbf{D}$. It is well-known [3, 9] that as $n \to \infty$,

$$j'_{1,n} \sim n + o(n^{1/2})$$

The corresponding Neumann eigenfunctions $\tilde{v}(r,\theta) = d_{1,n}^{-1} J_n(j'_{1,n} r) e^{in\theta}$ are the *whispering gallery modes* and can be shown to decay exponentially in n outside any annular neighbourhood of the unit circle boundary $A(1)$.

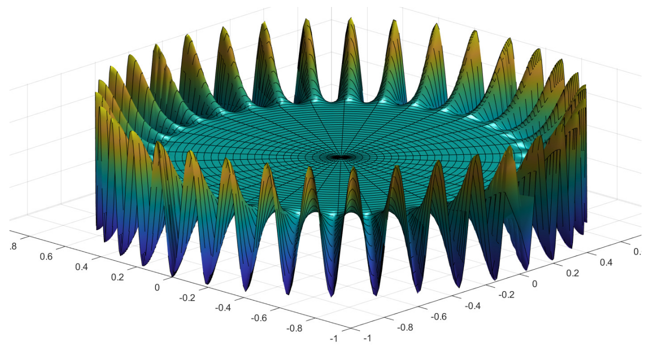


FIGURE 11. Whispering Gallery modes

8. Appendix

Lemma 13. For every $\alpha > 1$ and $r \in (0,1)$ such that $1/(\alpha r) < 1$ we have $k(\alpha r) > 0$

Proof. We need to show that on $(1, \infty)$ the function $f : x \mapsto \sqrt{x^2 - 1} - \arccos(x^{-1})$ is strickly positive. We note that the limit of $f(x)$ as $x \to 1^+$ is

$$-\sqrt{1^2 - 1} - \arccos(1^{-1}) = 0$$

Also for $x \in (1, \infty)$, the derivative

$$\frac{d}{dx}\left[\sqrt{x^2 - 1} - \arccos(x^{-1})\right] = \frac{x}{\sqrt{x^2 - 1}} + \frac{1}{x^2\sqrt{1 - x^{-2}}}$$

is strictly positive. $\square$

Lemma 14. On $(x, u) \in (0, \pi) \times (1, \infty)$

$$h(x; u) := \frac{k(u) + x}{u \sin(x)} \geq 1$$

where $k(u) = \sqrt{u^2 - 1} - \arccos(u^{-1})$.

Proof. For a fixed $u \in (1, \infty)$, since $x \mapsto f(x; u)$ is differentiable in x on $(0, \pi)$, we only need to show

- near $x = 0$ and $x = \pi$ we have $h(x; u) \geq 1$,
- There is only one critical point to $h(\cdot, u)$ and it is $x = \arccos(1/\alpha r)$, and
- At that that point, we have $h(\arccos(1/\alpha r)) = 1$

We easily see that h explodes to ∞ near 0 and π by the use of *lemma* 13 which says $k(u) > 0$. Thus, the first point is proven.

We now derive h

$$h'(x; u) = \frac{\sin(x) - (k(u) + x)\cos(x)}{u^2 \sin^2(x)}$$

Hence, $h'(x, u) = 0$ if and only if

$$\sin(x) - (k(u) + x)\cos(x) = 0$$

We now further define a function

$$g(x; u) := \sin(x) - (k(u) + x)\cos(x)$$

We note quickly that

- near $x = 0$, we have $g(x; u) \approx 0 - (k(u) + 0)1 = -k(u) < 0$,
- near $x = \pi$, we have $g(x; u) \approx 0 - (k(u) + \pi)(-1) = k(u) + \pi > 0$,

and also

$$\begin{aligned} g'(x;u) &= \cos(x) - \cos(x) - (k(u)+x)(-\sin(x)) \\ &= \sin(x)(k(u)+x) \end{aligned}$$

which is stricly positive on $(0,\pi)$. Therefore, there is at most 1 zero to $g(x;u)$ for every $u > 1$.

Thus, there is at most 1 critical point to the function f. And we note that

$$\begin{aligned} h'(\arccos(1/u)) &= \frac{\sin(\arccos(1/u)) - (k(u)+\arccos(1/u))\cos(\arccos(1/u))}{u^2\sin^2(\arccos(1/u))} \\ &= \frac{\sqrt{u^2-1}/u - \left(\sqrt{u^2-1} - \arccos(1/u) + \arccos(1/u)\right)1/u)}{u^2\sin^2(\arccos(1/u))} \\ &= 0 \end{aligned}$$

where we used the definition of $k(u)$ on the second to last line. Hence, $x = \arccos(1/u)$ is a critical point of h and lastly,

$$\begin{aligned} h(\arccos(1/u);u) &= \frac{k(u)+\arccos(1/u)}{u\sin(\arccos(1/u))} \\ &= \frac{\sqrt{u^2-1} - \arccos(1/u) + \arccos(1/u)}{u\sin(\arccos(1/u))} \\ &= 1 \end{aligned}$$

since $\sin(\arccos(1/u)) = \sqrt{u^2-1}/u$

$\square$

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