ROBUST CONTROL OF UNCERTAIN TIME-DELAY SYSTEMS

Ammar Haurani

Department of Electrical and Computer Engineering

McGill University, Montreal

April 2004

A thesis submitted to McGill University

in partial fulfillment of the requirements for the degree of

Doctor of Philosophy

Your file Votre référence
ISBN: 0-612-98270-X
Our file Notre référence
ISBN: 0-612-98270-X

Keywords: time-delay, robust stabilization, H_∞ control, finite-horizon, infinite-horizon, Lyapunov-Krasovskii theorem, disturbance attenuation, norm-bounded parametric uncertainties, linear matrix inequalities, Schur complement, time-invariant, time-varying, relaxation techniques, control saturation, difference inclusions, polytopic sets.

In the name of Allah the all merciful, the all bountiful

Gratefully dedicated to my mother, my father and my wife

Abstract

This work addresses the problem of robust stabilization and robust H_∞ control of uncertain time-delay systems. The time-delays are considered to be present in the states and/or the outputs, and the uncertainties in the system representation are of the parametric norm-bounded type. Both cases of actuators, with and without saturation are studied, and the state-feedback and output-feedback control designs are presented. Two methods for analysis and synthesis of controllers are used: The first is based on the transfer function, and the second on the use of functionals.

In the context of the design method based on transfer functions, the problem of H_∞ output feedback design for a class of uncertain linear continuous-time or discrete-time systems, with delayed states and/or outputs (only for the continuous-time case), and norm-bounded parametric uncertainties is considered. The objective is to design a linear output feedback controller such that, for the unknown state and output time-delays and all admissible norm-bounded parameter uncertainties, the feedback system remains robustly stable and the transfer function from the exogenous disturbances to the state-error outputs meets the prescribed H_∞ norm upper-bound constraint. The output feedback structure does not depend on the time-delay. The conditions for the existence of the desired robust H_∞ output feedback and the analytical expression of these controllers, are then characterized in terms of matrix Riccati-type inequalities. In the continuous-time context, both the time-invariant and the time-varying cases are treated. Finally, examples are presented to demonstrate the validity and the solvability of the proposed design methods.

In the context of the design method based on the use of functionals, the state-feedback H_∞ control problem is first presented for continuous-time, finite-horizon, time-varying linear neutral systems with parametric uncertainties entering all the matrices of the system representation. The controller is given as the solution to

a set of differential matrix inequalities by employing a descriptor model transformation of the system and a least conservative Lyapunov-Krasovskii functional. The conditions that guarantee robust H_∞ control are dependent on the value of the time-delay and its rate of change. In the infinite-horizon case, the solution of the robust H_∞ control problem is obtained in terms of linear matrix inequalities. Numerical examples are presented and illustrate the effectiveness of the proposed approach and the reduction in conservatism as compared to previous results in the literature.

Still in the same context, the state-feedback robust stabilization problem for neutral systems with time-varying delays and saturating actuators is addressed. The systems considered are continuous-time, with parametric uncertainties entering all the matrices in the system representation. The model used for the representation of actuator saturations is that of differential inclusions. A saturating control law is designed and a region of initial conditions is specified within which local asymptotic stability of the closed-loop system is ensured. The least conservative approach, which employs the Lyapunov-Krasovskii functional, is adopted to ensure stabilization. The controller is dependent on the time-delay and its rate of change. It is constructed in terms of the solution to a set of matrix inequalities. Numerical examples illustrate the increase in the ball of initial conditions over previous results in the literature.

Finally, the robust output-feedback stabilization problem for state-delayed systems with time-varying delays and saturating actuators is addressed. The systems considered are again continuous-time, with parametric uncertainties entering all the matrices in the system representation. Two models are used for the representation of actuator saturations: sector modeling and differential inclusions. Saturating control laws are designed, and in the case of differential inclusions, a region of initial conditions is specified within which local asymptotic stability of the closed-loop system is ensured. The designed controllers are dependent on the time-delay and its rate of change. The controllers are constructed in terms of the solution to a set of matrix inequalities. Numerical examples are presented and illustrate the effectiveness of the proposed designs, and in the case of differential inclusions, the increase in the ball of initial conditions over previous results in the literature.

Résumé

Cette thèse aborde le problème de la stabilisation robuste et de la commande robuste en H_∞ des systèmes incertains à retard. Les retards sont présents dans les états et/ou dans les sorties, et les incertitudes dans la représentation du système sont du type à norme limitée. Les deux cas d'actionneurs avec ou sans saturation sont traités, et les conceptions pour l'asservissement d'états ou de sorties sont présentés. Deux méthodes pour l'analyse et la synthèse des contrôleurs sont utilisées : La première méthode est basée sur la fonction de transfert, et la seconde est basée sur l'utilisation des fonctionnelles.

Dans le contexte de la méthode de conception basée sur la fonction de transfert, le problème de conception de l'asservissement de sortie en H_∞ est présenté pour une classe de systèmes incertains continus ou discrets, avec des retards dans les états et/ou dans les sorties (juste dans le cas continu), et avec des incertitudes paramétriques à norme limitée. L'objectif est de concevoir un contrôleur à asservissement de sorties de façon que, pour les retards inconnus dans l'état et la sortie, et pour toutes les incertitudes paramétriques admissibles à norme limitée, le système asservi reste stable et robuste, et la fonction de transfert liant le bruit extérieur à l'erreur dans l'état répond à une limite supérieure choisie dans le sens de la norme H_∞. La structure de l'asservissement de sortie ne dépend pas du retard. Les conditions pour l'existence de l'asservissement robuste de sortie en H_∞ et l'expression analytique de ces contrôleurs, sont caractérisées en fonction d'inégalités matricielles du type Riccati. Dans le contexte du temps continu, les cas d'invariance et de variance dans le temps sont traités. Finalement, des exemples sont présentés pour démontrer la validité et la solvabilité des méthodes de conceptions proposées.

Dans le contexte de la méthode de conception basée sur l'utilisation des fonctionnelles, le problème de commande en H_∞ par asservissement d'état est d'abord présenté pour les systèmes neutres à temps

continu, horizon fini, et variance dans le temps, avec des incertitudes dans toutes les matrices du système. Le contrôleur est donné comme la solution d'inégalité matricielle linéaire, en utilisent une transformation du système de modèle descripteur, et la fonctionnelle la moins conservatrice de type Lyapunov-Krasovskii. Les conditions garantissant la commande robuste en H_∞ dépendent de la valeur du retard et de sa vitesse de changement. Dans le cas de l'horizon infini, la solution du problème de commande robuste en H_∞ est obtenue en fonction d'inégalités matricielles linéaires. Des exemples numériques sont présentés, et illustre l'efficacité de l'approche proposée, et la réduction du conservatisme en comparaison avec les résultats antérieurs dans la littérature.

Toujours dans le même contexte, le problème de stabilisation robuste en H_∞ pour les systèmes neutres avec des retards variables dans le temps et des saturations dans les actionneurs est étudié. Les systèmes considérés sont continus, avec des incertitudes paramétriques incluses dans toutes les matrices du système. Le modèle utilisé pour la représentation des saturations des actionneurs est celui des inclusions différentielles. Une loi de contrôle à saturation est conçue, et une région de conditions initiales est spécifiée dans laquelle la stabilité locale asymptotique de la boucle fermée est assurée. La moins conservatrice des approches, employant la fonctionnelle de type Lyapunov-Krasovskii, est adoptée pour assurer la stabilisation. Le contrôleur est dépendant du retard et de sa vitesse de changement. Il est construit en fonction de la solution d'un groupe d'inégalités matricielles. Des exemples numériques illustrent l'élargissement de la boule de conditions initiales en comparaison avec les résultats antérieurs dans la littérature.

Finalement, le problème de stabilisation robuste en asservissement de sortie pour des systèmes retardés avec des retards variables dans le temps, et avec des saturations dans les actuateurs est traité. Les systèmes considérés sont encore continus, avec des incertitudes dans toutes les matrices du système. Deux modèles sont utilisés pour la représentation des saturations dans les actuateurs: le modèle sectoriel et les inclusions différentielles. Des lois de contrôle saturées sont conçues, et dans le cas des inclusions différentielles une région de conditions initiales est spécifiée dans laquelle la stabilité locale asymptotique du système en boucle est assurée. Les contrôleurs sont dépendants du retard et sa vitesse de changement. Ils sont construits en fonction de la solution à un groupe d'inégalités matricielles. Des exemples numériques sont présentés, et illustrent l'efficacité de l'approche proposée, et dans le cas des inclusions différentielles, démontrent l'élargissement de la boule conditions initiales en comparaison avec les résultats ultérieures dans la literature.

Claims of Originality

The following novel contributions are made in this dissertation:

- The development of a delay-independent solution for robust H_∞ output feedback control of linear uncertain retarded systems. Both the continuous and discrete-time [P2] cases are treated . The structure of the observer is such that it allows for the separation of the controller and observer design. The value of the controller gain is chosen by solving a Riccati type inequality independent of the observer gain. The observer gain is then obtained from a second Riccati inequality in which the a priori selected value of the controller gain is used.
- The development of a delay-dependent finite-horizon time-varying solution for robust H_∞ state feedback control of linear uncertain neutral systems with time-varying delays, using the least conservative model transformation and Lyapunov functional [P3]. The design is presented as the solution to Linear Matrix Inequalities (LMI) even for the finite horizon time-varying case, which is not found in the literature of previous work.
- The development of a delay-dependent solution for robust H_∞ state feedback control of uncertain neutral systems with time-varying delays and actuators saturations, using the least conservative model transformation and Lyapunov functional and saturation model (differential inclusions) [P4, P5]. Previous work had only dealt with the nominal case (no uncertainties) and constant time-delays. The solution presented as compared to previous work for the nominal case achieves a larger set of initial conditions guaranteeing local asymptotic stability.
- The development of a delay-dependent solution for robust H_∞ output feedback control of uncertain retarded systems with time-varying delays and actuators saturations, using the least

conservative model transformation and Lyapunov functional. The model used for actuator saturation is that of sector modeling. The proposed design is less conservative than the only previous paper in the literature dealing with the same problem **[72]**. Also the system treated in the present thesis includes uncertainties in the control and in the output, and an extra uncertain feedforward term in the output as compared to **[72]**.

- The development of a delay-dependent solution for robust H_∞ output feedback control of uncertain retarded systems with time-varying delays and actuators saturations, using the least conservative model transformation and Lyapunov functional and saturation model (differential inclusions) [P6, P7]. Previous work using differential inclusions had only dealt with delay-free systems. The solution presented as compared to previous work for the delay-free case achieves a larger set of initial conditions guaranteeing local asymptotic stability.

This research work has been partially reported in the following publications:

Publications

[P1] A. Haurani, Othman Taha, H. Michalska, B. Boulet. Multivariable Control of a Paper Coloring Process: a Case Study. *Proc. of the 2001 American Control Conference,* Arlington, Virginia, U.S.A., 25-27 June 2001, pp. 2210-2215.

[P2] A. Haurani, H. Michalska, B. Boulet. Discrete-Time Robust H_∞ Output Feedback Control of State Delayed Systems. *Proc. of the 15th IFAC Congress,* Barcelona, Spain, 21-26 July 2002, 6 Pages.

[P3] A. Haurani, H. Michalska, B. Boulet. Delay-dependent finite-horizon time-varying bounded real lemma for uncertain linear neutral systems. *Proc. of the 2003 American Control Conference,* Denver, Colorado, U.S.A., 4-6 June 2003, pp. 1512-1517.

[P4] A. Haurani, H. Michalska, B. Boulet. Delay-dependent robust stabilization of uncertain neutral systems with saturating actuators. *Proc. of the 2003 American Control Conference,* Denver, Colorado, U.S.A., 4-6 June 2003, pp. 509-514.

[P5] A. Haurani, H. Michalska, B. Boulet. Delay-dependent robust stabilization of uncertain neutral systems with time-varying delays and saturating actuators. Submitted to *International Journal of Robust and Nonlinear Control,* John Wiley & Sons, July 2003.

[P6] A. Haurani, H. Michalska, B. Boulet. Robust output feedback stabilization of uncertain time-varying state-delayed systems with saturating actuators. *International Journal of Control,* 10 March 2004, Vol.77 (4), pp. 399-414.

[P7] A. Haurani, H. Michalska, B. Boulet. Delay-dependent robust output feedback stabilization of uncertain state-delayed systems with saturating actuators. Accepted in the *2004 American Control Conference,* Boston, MA, U.S.A., June 2004.

Acknowledgements

Many persons have given me help, guidance, support and inspiration during my years at McGill University. To all of them I am very grateful.

First and foremost, I wish to express my special thanks to my thesis supervisor, Professor Hannah Michalska, for her guidance and encouragement without which my research could not have been accomplished. I am deeply thankful to Professor Michalska for her generous support and advice beyond the duties of any supervisor, and for her contribution to the different papers that were worked jointly. I thank her for always providing a friendly environment.

I am very grateful to Professor Benoit Boulet for his moral, technical and financial support, his contribution to the papers that were worked jointly, and for having always been responsive to my needs.

I am also very thankful to Dr. Othman Taha of Nexfor Technology, Noranda for having put his trust and confidence in me. His moral support and the funding from Noranda were very important in helping me go through my studies.

I thank Professor Peter Caines for the special attention he had given me when I audited his Nonlinear Control class, especially the interesting papers he used to leave for me on my desk.

I am very grateful to my friend and officemate Shahid Shaikh for having taught me Linear Systems in the fall of 1999, for his constant support and valuable advices on variables subjects, and for enjoying his company and friendship.

I wish to thank the staffs of CIM department. In particular, many thanks to Cynthia Davidson, Jan Binder, and Danny Chouinard for their good-natured advice and help through various administrative and operational hurdles.

I am grateful to all the friends at CIM, especially Dr. Nabil Aouf for his pleasant company and for the computer he left me, Dr. Miguel Torres-Torriti for all precious information and help he used to give me, and Dr. Minyi Huang for his short, yet precious remarks.

Finally, but most important, to my mother and father who made me what I am today, and my wife who supported me at every moment and at every step and my children who were very patient in not being able to see me more often. Unfortunately, there are no words to adequately express my gratitude. In the absence of words, or their insufficiency to express my love, I dedicate this dissertation to them.

Ammar Haurani

Montreal, Quebec, 30 September 2003

TABLE OF CONTENTS

LIST OF FIGURES

LIST OF TABLES

CHAPTER 1

Introduction

In recent years, much work has been devoted to the analysis and synthesis of controllers for state-delayed systems with or without parametric uncertainties **[9, 10, 12-14, 22, 37, 49, 50, 69]**. This interest is strongly motivated by the fact that delays and uncertainties are the two most important causes of instability. Furthermore, both delays and uncertainties occur frequently in the chemical and in the process industries, which provides another reason for the study of new and less conservative stability conditions and the synthesis of high performance controllers.

Many methods exist for the stabilization and control of time-delay systems: $\mathbb{C}$-plane roots, Matrix pencils, Norm measure, 1st Lyapunov, Krasovskii, Razumikhin, Comparison techniques, LaSalle Invariance, Smith Predictor, Sliding Mode control, Linear Parameter Variation, μ-synthesis, etc. (see **[65]** for a recent review). In this thesis, two major methods are used for the analysis and synthesis of control laws for time-delay systems: The first is based on the bounding of the H_∞-norm of the closed-loop transfer function, and the second is based on the use of a cost function and of functionals.

In the context of the design method based on transfer functions used in the first part (Chapters 2 and 3) of this thesis, most work has been directed towards the study of state feedback controller design **[21, 23, 25, 26, 91, 92]**, and state observer design **[15, 48, 63, 64, 80, 85, 86, 92]**, as separate issues. Very little effort has so far been put into the design and analysis of systems using output-feedback **[66]**. The latter is however of greatest practical relevance as usually the states of the system are not directly available for measurement. Also, the output feedback control case needs special attention when uncertainties are present. As an example, for the general form of uncertainties, the separation principle does not hold, and the observer design is no longer the dual of the controller design. Furthermore, most of the previous work involving output feedback is concerned with continuous-time systems. Very little attention has been given to the discrete-time case.

In **[86]**, a continuous-time observer for state-delayed systems with parametric uncertainties has been developed. In this thesis, a robust H_∞ output feedback controller which complements the observer design of **[86]** is proposed in order to maintain robust stability of the combined system. The design procedure and the resulting closed loop system properties are delay-independent. Although more conservative than delay-dependent methods, especially for small time-delays, the conservatism in the proposed procedure can be considerably reduced by using optimization algorithms as in **[89]**. The time-delays are included in the state and in the output equations, and uncertainties are added to all the state matrices, making the design very general and applicable to a very large class of problems. Both the time-invariant and time-varying cases are presented.

Similarly, a discrete-time observer for state-delayed systems with parametric uncertainties has been developed in **[85]**. In this context, the contribution of this thesis is the development of a discrete-time robust H_∞ output feedback controller which complements the observer design of **[85]**, in such a way as to maintain robust stability of the combined system. Furthermore, uncertainty in the delayed state matrix is taken into account as an improvement over the observer design in **[85]**.

More specifically, using the methodology based on the transfer function, the aim is to design observer and controller gains for both cases of continuous and discrete-time systems such that, for all admissible parameter uncertainties, the output feedback system remains robustly stable and the transfer function from the exogenous disturbances to the state error output meets a prescribed H_∞-norm upper bound constraint, independently of the time delay. The parameter uncertainties are norm-bounded and appear in the state, the output and the control input matrices. A penalty is considered on both the state output error and the control input. A simple algebraic parameterized approach is exploited, which enables us to derive the existence conditions for the observer and controller gains and to characterize the set of robust H_∞ output feedback controllers in terms of several free design parameters. These free parameters, that appear in the observer and controller gains, offer additional design freedom and can be utilized to account for additional performance constraints.

The design formulation of our H_∞ output feedback control problem requires the solution of two Riccati matrix inequalities which is not a difficult task. The computational feasibility of our approach is demonstrated by examples.

As seen above, the second major method for the design of controllers for time-delay systems is based on the use of functionals, which is the method used in the second part (Chapters 4-7) of this thesis.

In this context, most of the previous results related to robust control design for time-delayed systems refer to simple retarded type systems. Delay-independent stability conditions for such systems were obtained in terms of linear matrix inequalities (LMIs) or Riccati equations **[53, 83]**, while delay-dependent stability conditions were derived in **[19, 36]**.

The control design for the general case of neutral (descriptor) systems, in which the time-delays can appear both in the state and its derivative, has so far obtained relatively little attention **[54, 67]**. Unlike simple retarded systems, neutral systems are particularly sensitive to delays and can be easily destabilized **[29, 52]**.

Most H_∞ design approaches for retarded systems refer to the infinite-horizon case while, to our knowledge, the more challenging finite-horizon time-varying case has only been discussed in **[67]**.

It is the choice of an appropriate Lyapunov-Krasovskii functional, needed in the derivation of the bounded real criterion (see **[19, 24, 42]** for examples of such functionals), that is the deciding factor in the particular type of time-delay dependency of the resulting control law. The most general form of this functional leads to a complicated system of Riccati type partial differential equations **[1]** or inequalities **[22]**. Some special forms of Lyapunov-Krasovskii functionals lead to simpler, delay-independent control laws **[59, 60]**, while other forms lead to less conservative, delay-dependent control laws **[12, 24]**.

Some authors **[12]**, construct control laws which are dependent on the actual value of the time-delay which is thus assumed known, while others **[40]**, construct laws which are dependent on the rate of the time-delay. Both, the value and the rate of the time-delay, are used in **[39]**.

The choice of the descriptor model transformation used and the choice of the Lyapunov-Krasovskii functional should be such that the resulting control law is least conservative. The descriptor model transformation employed in **[12, 36, 67]** does not result in full equivalence of the transformed and the original systems (see **[8]** for a proof of this fact), and some of the bounds resulting from the choice of the

Lyapunov-Krasovskii functional, are redundant. Recently a better descriptor model transformation was introduced in **[19]** and was applied in **[24]** to significantly reduce the conservatism of the design. The bounds used in the bounded real lemma were further tightened in **[20]** using a recent idea introduced in **[62]**. However, the result in **[20]** applies only to the infinite-horizon time-invariant case without norm bounded parametric uncertainties.

Concerning the subject of constrained control of linear systems, a great effort has been made during the last decade to take into account saturating controls in linear systems control design. In fact, this is an important practical constraint usually disregarded in classical control design methods, despite the fact that no practical system can deliver unlimited control input as can be occasionally requested by the controller; see, for example, the two special issues **[4]**, **[71]** and the references therein for an overview on this subject.

In **[27]**, the authors compared the different control saturation models used in the literature, concluding that the differential inclusions model leads to the least conservative design.

In the context of continuous-time systems with both time-delays and saturating controls, some delay-independent results addressing local as well as global stabilization via memoryless feedback control laws have been proposed **[61, 76]**.

To the author's best knowledge, stabilization of neutral systems with actuator saturation has only been studied in **[77]**. However, the controller designed in **[77]** is delay-independent (which is relatively conservative), and the system representation includes no uncertainties, while the delays are considered known and time-invariant.

The majority of the design procedures known to date and applicable to time-delayed systems assume perfect availability of the state measurement. However, it is well known that such an assumption is rarely realistic in practical situations, be it only for the reason of excessive cost of full state measurements. Output feedback then imposes itself as the most practical solution for achieving a realistic implementation of a control design. The reason for which the state feedback has so far been treated more extensively is that it lends itself to simpler analysis as compared to the output feedback case. To the author's best knowledge, the issue of output feedback stabilization of uncertain state-delayed systems with saturating control was only addressed in **[72]**. The control design presented in **[72]** is delay-independent and so relatively conservative. Uncertainties are only included in the state and delayed-state system matrices. The remaining, control and output system matrices are assumed to be known exactly. Furthermore, the time-delay is considered time-invariant. As shown in **[27]**, the control saturation model used in **[72]** is more conservative than the differential inclusions model used in the present thesis. Furthermore, the design procedure proposed by **[72]** is also characterized by overly strong conservatism of the design introduced by the specific bounding techniques used.

Having enumerated some of the issues that are still not treated in the subject of robust stabilization and control of time-delay systems with the use of functionals, the objective in this thesis is to present solutions to some of these problems. More specifically, the problems treated here are:

- Finite-horizon time-varying robust H_∞ state-feedback control of parametric norm-bounded uncertain neutral systems with time-varying delays.
- Robust state-feedback stabilization of parametric norm-bounded uncertain neutral systems with time-varying delays, and actuator saturations represented by differential inclusions.
- Robust output feedback stabilization of parametric norm-bounded uncertain retarded systems with time-varying delays, and actuators saturations represented by sector modeling.
- Robust output feedback stabilization of parametric norm-bounded uncertain retarded systems with time-varying delays, and actuators saturations represented by differential inclusions.

1.1. Notation

The following notation is adopted. For any matrix A, the expressions A^T and $diag\{A\}$ denote the transpose of A and the diagonal of A, respectively. The notation $col\{v_1, v_2, ..., v_n\}$ is used for the column vector formed by stacking column vectors v_1, $v_2, ..., v_n$. $\mathbb{R}^n$ is the n dimensional Euclidean space with vector norm $\| \,.\, \|$, $\mathbb{R}^{n\times m}$ is the set of all $n\times m$ real matrices, and for any matrix $P\in \mathbb{R}^{n\times n}$, the inequality $P>0$, signifies that P is positive definite. The symbols $\lambda_{\max}(P)$ and $\lambda_{\min}(P)$ denote the maximal and minimal eigenvalue of a matrix P, respectively. The standard notation of $L_2^q[a,b]$ is adopted for the space of all functions $f:\mathbb{R}\rightarrow\mathbb{R}^q$ which are Lebesque integrable in the square over the interval $[a,b]$, with the standard norm $\| \cdot \|_{L_2}$. The symbol $C_{d,n} = C([-d,0],\mathbb{R}^n)$ denotes the Banach space of continuous vector functions mapping the interval $[-d,0]$ into $\mathbb{R}^n$ with the topology of uniform convergence; i.e. $\|\phi\|_C \triangleq \sup_{-d\le t\le 0}\|\phi(t)\|$ is defined as the norm for $\phi\in C_{d,n}$. The subset $C^w_{d,n}\subset C_{d,n}$ is defined by $C^w_{d,n}\triangleq\{\phi\in C_{d,n};\ \|\phi\|_C<w\}$, where w is a positive real number. For any two vectors $v\triangleq[v_1 ... v_m]^T\in\mathbb{R}^m$ and $\bar{u}\triangleq[\bar{u}_1 ... \bar{u}_m]^T\in\mathbb{R}^m$ such that $\bar{u}_i>0$, $i=1,...,m$, the saturation function sat is defined by: $sat(v)\triangleq[sat(v_1)...sat(v_m)]^T$, where $sat(v_i)\triangleq sign(v_i)\min(\bar{u}_i,|v_i|)$. For any $t\ge t_0$, let $x_t:[-d,0]\to\mathbb{R}^n$ denote the function defined by: $x_t(\psi)=x(t+\psi)$, for all $\psi\in[-d,0]$.

1.2. System Description

This work addresses the problem of robust H_∞ control and stabilizing feedback designs for uncertain neutral and retarded time-delay systems. The state space representation for a neutral time-delay system (a retarded time-delay system being a special case) is:

$$\dot{x}(t)-\bar{A}_g(t)\dot{x}(t-g(t))=\bar{A}(t)x(t)+\bar{A}_h(t)x(t-h(t))+B_1(t)w(t)+\bar{B}(t)u(t) \tag{1.1}$$

$$y(t)=\bar{C}(t)x(t)+\bar{C}_h(t)x(t-h(t))+B_2(t)w(t)+\bar{D}(t)u(t). \tag{1.2}$$

The initial condition for system (1.1)-(1.2) is:

$$x(t_0+\psi)=\phi(\psi),\ \forall\psi\in[-d_{\max},0],\ (t_0,\phi)\in\mathbb{R}^+\times C^w_{d_{\max},n}. \tag{1.3}$$

In system (1.1)-(1.3), $x(t)\in\mathbb{R}^n$ is the system state vector, $w\in L_2^q[0,T]$ is the exogenous disturbance signal with $T\in\mathbb{R}^+$ denoting the control time horizon, $u(t)\in\mathbb{R}^m$ is the control input, and $y(t)\in\mathbb{R}^p$ is the system output. The system delays $h(t)>0$ and $g(t)>0$ are assumed to be some unknown functions of time. Delay $d_{\max}$ is defined as $d_{\max}\triangleq\max(g(t),h(t))$, for all $t\in[0,T]$. The initial value $\phi(t)$ for $x(t)$ is a smooth vector-valued continuous function defined in the Banach space $\mathbb{C}^n[-d_{\max},0]$ of smooth functions ψ: $[-d_{\max},0]\to\mathbb{R}^n$ with $\|\psi\|_\infty\triangleq\sup_{-d_{\max}\le\tau\le 0}\|\psi(\tau)\|$.

Matrices $B_1(t)\in\mathbb{R}^{n\times q}$ and $B_2(t)\in\mathbb{R}^{p\times q}$ are exactly known, while matrices $\bar{A}_g(t)\in\mathbb{R}^{n\times n}$, $\bar{A}(t)\in\mathbb{R}^{n\times n}$, $\bar{A}_h(t)\in\mathbb{R}^{n\times n}$, $\bar{B}(t)\in\mathbb{R}^{n\times m}$, $\bar{C}(t)\in\mathbb{R}^{p\times n}$, $\bar{C}_h(t)\in\mathbb{R}^{p\times n}$ and $\bar{D}(t)\in\mathbb{R}^{p\times m}$ are assumed to be uncertain. The uncertainties are represented by the widely used **[12, 49, 69]** norm-bounded parametric uncertainty model, as follows:

$$\bar{A}_g(t)=A_g(t)+\Delta A_g(t),\ \bar{A}(t)=A(t)+\Delta A(t),\ \bar{A}_h(t)=A_h(t)+\Delta A_h(t),\ \bar{B}(t)=B(t)+\Delta B(t),$$

$$\bar{C}(t)=C(t)+\Delta C(t),\ \bar{C}_h(t)=C_h(t)+\Delta C_h(t),\ \bar{D}(t)=D(t)+\Delta D(t) \tag{1.4}$$

The matrices $A_g(t)$, $A(t)$, $A_h(t)$, $B(t)$, $B_1(t)$, $B_2(t)$, $C(t)$, $C_h(t)$ and $D(t)$ are bounded, real, and time-varying, with continuous entries over $[0,T]$, and are assumed to be known exactly. The matrices $\Delta A_g(t)$, $\Delta A(t)$, $\Delta A_h(t)$, $\Delta B(t)$, $\Delta C(t)$, $\Delta C_h(t)$ and $\Delta D(t)$ are real-valued, represent the norm-bounded parameter uncertainties, and are assumed to be of the following form:

$$\Delta A_g(t)=H_g(t)F_g(t)E_g(t),\qquad \Delta A(t)=H_A(t)F_A(t)E_A(t),\qquad \Delta A_h(t)=H_h(t)F_h(t)E_h(t),$$

$$\Delta B(t)=H_B(t)F_B(t)E_B(t),\qquad \Delta C(t)=H_C(t)F_C(t)E_C(t),\qquad \Delta C_h(t)=H_{C_h}(t)F_{C_h}(t)E_{C_h}(t),$$

$$\Delta D(t)=H_D(t)F_D(t)E_D(t) \tag{1.5}$$

where $F_g(t)\in\mathbb{R}^{i_g\times j_g}$, $F_A(t)\in\mathbb{R}^{i_A\times j_A}$, $F_h(t)\in\mathbb{R}^{i_h\times j_h}$, $F_B(t)\in\mathbb{R}^{i_B\times j_B}$, $F_C(t)\in\mathbb{R}^{i_C\times j_C}$, $F_{C_h}(t)\in\mathbb{R}^{i_{C_h}\times j_{C_h}}$ and $F_D(t)\in\mathbb{R}^{i_D\times j_D}$ are real, uncertain, time-varying matrices with Lebesgue measurable entries which, additionally, meet the following requirements: $F_g(t)F_g^T(t)\le I$, $F_A(t)F_A^T(t)\le I$, $F_h(t)F_h^T(t)\le I$, $F_B(t)F_B^T(t)\le I$, $F_C(t)F_C^T(t)\le I$, $F_{C_h}(t)F_{C_h}^T(t)\le I$ and $F_D(t)F_D^T(t)\le I$. The matrices $H_g(t)$, $H_A(t)$, $H_h(t)$, $H_B(t)$, $H_C(t)$, $H_{C_h}(t)$, $H_D(t)$, $E_g(t)$, $E_A(t)$, $E_h(t)$, $E_B(t)$, $E_C(t)$, $E_{C_h}(t)$ and $E_D(t)$ are known, real, time-varying, with piece-wise continuous entries over $[0,T]$, and characterize the way in which the uncertain parameters of $F_g(t)$, $F_A(t)$, $F_h(t)$, $F_B(t)$, $F_C(t)$, $F_{C_h}(t)$ and $F_D(t)$ enter the nominal matrices $A_g(t)$, $A(t)$, $A_h(t)$, $B(t)$, $C(t)$, $C_h(t)$ and $D(t)$.

1.3. Motivation

The motivation to this research is provided by the following:

- The existence of very few studies treating the output feedback control of time delay systems.
- The state feedback control of time-delay systems presented in the literature deals very often with the uncertainty-free case.
- There is work to do in terms of decreasing the conservatism of the controller design presented in the literature, especially for the robust treatment, where uncertainties are present in the system.
- The rate of change of the time-delay is rarely taken into account in the design of controllers for time-delay systems, thus rendering the design often unrealistic, and that is because if the time-delay is varying in time in the actual plant, any design made with the assumption that the time-delay is time-invariant might lead to an unstable closed-loop system when actually implemented.
- The case of state-feedback control of time-delay systems with actuator saturation is almost always treated for the nominal system. Thus a robust treatment where uncertainties in the system are considered is needed.
- The robust output feedback control of time-delay systems with actuator saturation is treated only in **[72]** in a very conservative approach.

1.4. Functional Differential Equations

Prior to giving real examples about time-delay models, it is necessary to put time-delay systems into their theoretical context. Time-delay systems are part of what is known as systems represented by functional differential equations (FDEs) **[41]**, which makes it necessary at this point to have an overview of this major mathematical class.

As is well known, an ordinary differential equation (ODE) is an equation connecting the values of an unknown function and some of its derivatives for one and the same argument value. For example, the equation $f\left(t,x,dx/dt,d^2x/dt^2\right)=0$ may be written as $f\left(t,x(t),\dot{x}(t),\ddot{x}(t)\right)=0$, where dots indicate derivatives: $\dot{x}(t)=dx/dt$.

A functional equation (FE) is an equation involving an unknown function for different argument values. The equations $x(2t)+2x(3t)=1$, $x(t)=t^2x(t+1)-\left[x(t-2)\right]^2$, $x(x(t))=x(t)+1$, etc. are examples of FEs. The differences between the argument values of an unknown function and t in an FE are called argument deviations. If all argument deviations are constant (as in the second example above), then the FE is called a difference equation.

Combining the notions of differential and functional equations, we obtain the notion of functional differential equation (FDE), or, equivalently, differential equation with deviating argument. Thus, this is an equation connecting the unknown function and some of its derivatives for, in general, different argument values. Here also the argument values can be discrete, continuous or mixed. Correspondingly one introduces the notions of differential-difference equation (DDE), integro-differential equation (IDE), etc. A FDE is called autonomous if it is invariant under the change $t \mapsto t+T$ for all $T \in \mathbb{R}$.

The order of a FDE is the order of the highest derivative of the unknown function entering in the equation. So, a FE may be regarded as a FDE of order zero. Hence the notion of FDE generalizes all equations of mathematical analysis for function of a continuous argument. A similar assertion holds for functions depending on several arguments. Therefore the creation of a sufficiently substantial theory of FDEs is possible only for certain reasonably restricted classes of FDEs. Most of these classes are chosen guided by applications.

As will be seen in the examples below, FDEs with aftereffect arise when modeling biological, physical, etc., processes whose rate of change of state at any moment of time t is determined not only by the present state, but also by past states.

First we restrict ourselves to the case when there are finitely many discrete argument deviations whose dependence on t is known. Then we obtain the equation of general form

$$x^{(m)}(t)=f\left(t,x^{(m_1)}\left(t-h_1(t)\right),\ldots,x^{(m_k)}\left(t-h_k(t)\right)\right). \tag{1.6}$$

Here $x(t)\in\mathbb{R}^n$, all $m_i \geq 0$ and represent the order of the derivative, $h_i(t)\geq 0$, i.e. all argument deviations are nonnegative. In (1.6) the function f and delays h_i are given, and x is the unknown function of t.

According to a now universally accepted proposal of G. Kamenskii, (1.6) is called a functional differential equation of retarded type, or retarded functional differential equation (RDE) if $\max\{m_1,...,m_k\} < m$; a functional differential equation of neutral type (NDE) if $\max\{m_1,...,m_k\} = m$; and a functional differential equation of advanced type (ADE) if $\max\{m_1,...,m_k\} > m$.

Experience in mathematical modeling has shown that the evolution equations of actual processes with aftereffect are almost exclusively RDEs and NDEs. On the other hand, the investigation of various problems for these equations has revealed that RDEs and NDEs have many 'nice' mathematical properties. As for ODEs, we can transform the equation (1.6) to a first order equation by taking as new unknown functions the lower derivatives of x. Preserving the notation x for the new unknown function and f for the new right-hand side, we can write an RDE as

$$\dot{x}(t) = f\left(t, x\left(t-h_1(t)\right),...,x\left(t-h_k(t)\right)\right), \tag{1.7}$$

and an NDE as

$$\dot{x}(t) = f\left(t, x\left(t-h_1(t)\right),...,x\left(t-h_k(t)\right), \dot{x}\left(t-g_1(t)\right),...,\dot{x}\left(t-g_l(t)\right)\right). \tag{1.8}$$

Note that any FDE is equivalent to a hybrid system of ODEs and functional equations, in particular, difference equations. For example (1.8) is equivalent to the following hybrid system:

$$\left.\begin{array}{c}\dot{x}(t) = y(t) \\ y(t) = f\left(t, x\left(t-h_1(t)\right),...,x\left(t-h_k(t)\right), y\left(t-g_1(t)\right),...,y\left(t-g_l(t)\right)\right).\end{array}\right\}$$

Equation (1.7) is the most widely used type of nonlinear RDEs, which can be written as a general RDE in the form

$$\dot{x}(t) = F(t, x_t). \tag{1.9}$$

Here $x(t) \in \mathbb{R}^n$ and x_t (for a given t) is the function defined by

$$x_t(\theta) = x(t+\theta), \qquad \theta \in J_t \subseteq (-\infty, 0],$$

where J_t is a given interval $\left[-h(t), -g(t)\right]$ or $\left(-\infty, -g(t)\right]$. The transition from x to x_t for $J = [-h, 0]$ is shown in Figure 1.1. Note that x_t may be treated as the fragment of the function x at the left of the point

t, observed from this point. The right-hand side of (1.9) is a function of t, and a functional of x_t, i.e. to any t and any function $\psi : J_t \to R^n$ in some class of functions corresponds a vector $f(t,\psi) \in \mathbb{R}^n$. It is to note that for FDEs, the function x_t is the actual state of the system. Vector $x(t)$ is the solution at time t. However, it is the custom in the literature to use the word "state" for vector $x(t)$, which is also adopted in this thesis.

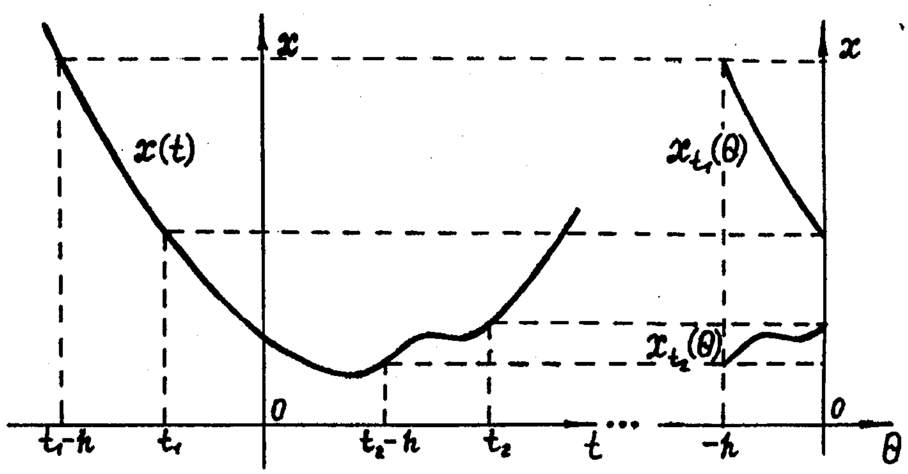

Figure 1.1. Geometrical interpretation of the transition from x to x_t

Similarly, the general NDE can be written as

$$\dot{x}(t) = F(t, x_t, \dot{x}_t). \tag{1.10}$$

Note that if J_t does not reduce to a point, then formally the right-hand side of (1.10) can be written as $\tilde{F}(t, x_t)$, because by giving a function we also give its derivative. Of course, this does not mean that there are no principal distinctions between RDEs and NDEs, since the conditions natural for RDEs are usually not satisfied by $\tilde{F}$. Roughly speaking, for variable t the right-hand side of an RDE must define a bounded operator on an appropriate space, and the right-hand side of a 'true' NDE, an unbounded operator.

In recent years certain authors preferred another general form of NDEs:

$$\frac{d}{dt}\left[x(t) - G(t, x_t)\right] = F(t, x_t). \tag{1.11}$$

In general, (1.10) and (1.11) cannot be reduced to each other. However, the main distinctions between these forms of NDEs usually vanish in simple cases, provided the solution concept is reasonably modified.

1.5. Examples of Delay Systems

Delay systems find their way into the representation of many real world applications. The examples seem to touch almost every field of science. The list of examples given in the following is far from being exhaustive but it gives a good idea about the importance of delays. For a more thorough coverage of delay examples in all fields of sciences, one can refer to **[41]**.

1.5.1. Nuclear Reactors

FDEs are widely used to model the dynamics of nuclear reactors. The physical reasons for the appearance of delays are various: transportation delays caused by finiteness of time of heat transport along different elements of the circulation contours (can be represented by corresponding ODEs or PDEs); warming up time of the reactor; snapping time of the control system; etc.

In **[28]** the following model was used:

$$\left.\begin{aligned} \dot{x}(t) &= \left[ax(t)+by(t-h)\right]\left[1+x(t)\right], \\ \dot{y}(t) &= x(t)-y(t); \end{aligned}\right\}$$

$$\left.\begin{aligned} \dot{x}(t) &= \left[\phi\left(x(t-h_1)\right)+\psi\left(y(t-h_2)\right)\right]\left[1+x(t)\right], \\ \dot{y}(t) &= x(t)-y(t); \end{aligned}\right\}$$

$$\left.\begin{aligned} \dot{x}(t) &= \left[a_1\theta_1(t)+a_2\theta_2(t)\right]\left[1+x(t)\right]-a_3\left[x(t)-y(t)\right], \\ \dot{y}(t) &= a_4\left[x(t)-y(t)\right], \\ \gamma\dot{\theta}_1(t) &= (1-a)x(t)-b\left[\theta_1(t)-\theta_2(t)\right], \\ \dot{\theta}_2(t) &= \alpha\theta_2(t-h)-\theta_2(t)+ax(t)+b\left[\theta_1(t)-\theta_2(t)\right]. \end{aligned}\right\} \qquad (1.12)$$

Here, $x(t)$ is the relative change of neutron density, $y(t)$, $\theta_1(t)$, $\theta_2(t)$ are proportional to the relative change in temperature of the reactor, fuel and de-acceleration device, respectively. The first two models do not take into account the delayed neutrons, but (1.12) does. In (1.12), the delay h is the time of liquid fuel transportation along a circular contour.

1.5.2. Models of Lasers

FDEs are widely used to model the dynamics properties of lasers. E.g., the following equations were introduced in **[70]**:

$$\left.\begin{aligned}\dot{x}_1(t) &= v x_1(t)\left[x_2(t) - 1 - m - \alpha m x_1(t-h)\right] + vU_0,\\ \dot{x}_2(t) &= K_0 - K(t)\left[x_1(t) + 1\right],\end{aligned}\right\}$$

where $x_1(t)$ is the radiation density and $x_2(t)$ the amplification coefficient. The other parameters are constants depending on the properties of the laser.

1.5.3. Combustion in Rocket Motor Chambers [88]

We consider a liquid monopropellant rocket motor with a pressure feeding system. Assuming nonsteady flow and taking non-uniform lag into account, a linearized model of the feeding system and the combustion chamber equations has been obtained by **[11, 18, 94]**. Their model is:

$$\dot{x}(t) = A_0 x(t) + A_1 x(t-h) + \int_{-d}^{0} A_d x(t+s)\,ds$$

where the entries $x_i(t)$, $i = 1,\ldots,3$ of the state vector are, respectively, the relative deviations of the instantaneous combustion chamber pressure, the instantaneous mass flow upstream of the capacitance and the instantaneous mass rate of the injected propellant from their steady values, and $x_4(t)$ is the ratio between the deviation of the instantaneous pressure in a special place in the feeding line from its value in steady operation and twice the injector pressure drop in steady operation.

The model matrices are:

$$A_0 = \begin{bmatrix} \rho - 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \dfrac{-1}{\varsigma J} \\ -0.5\dfrac{p}{(1-\varsigma)J} & 0 & -\dfrac{1}{(1-\varsigma)J} & \dfrac{1}{(1-\varsigma)J} \\ 0 & \dfrac{1}{E_e} & -\dfrac{1}{E_e} & 0 \end{bmatrix}, \qquad A_1 = 0, \qquad A_d = \begin{bmatrix} -\dfrac{\rho}{d} & 0 & \dfrac{1}{d} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

where ς is the fractional length for pressure supply, J is the line inertia, E_e is the line elasticity parameter, p is the ratio of steady-state pressure and steady-state injector pressure drop and ρ is the pressure of the combustion process.

1.5.4. Paper Coloring Process [33]

The paper coloring process uses three colour dyes. Desired paper colour shade is obtained as a mixture of the three basic dye concentrations *[ounces of dye/ton of fiber]*, which are used as inputs to the process and are denoted by u_1, u_2 and u_3. The basic dyes are black, and two other dyes chosen from the set of yellow, blue, and red. The basic dye concentrations are initially injected in the so called "wet-end process" which can be modeled as a first order low pass filter with a transport delay. The three wet basic dye concentrations are then mixed to produce a desired shade of a given colour which is measured by a Xenon-based spectrophotometer as the reflectance spectrum data vector, denoted by $V = [X \quad Y \quad Z]^T$, where X, Y and Z are normalized by their maximum values according to the CIELAB standard. The measured reflectance spectrum data vector $V = [X \quad Y \quad Z]^T$, is output by the spectrometer in the form of "colour space values" L^*, a^*, and b^*, (called the CIELAB coordinates) calculated using the following non-linear transformations:

$$L^* = 116Y^{1/3} - 16$$

$$a^* = 500(X^{1/3} - Y^{1/3})$$

$$b^* = 200(Y^{1/3} - Z^{1/3})$$

The CIELAB coordinates are familiar to the process operators and end-users and are the colour process output variables to be controlled.

Assuming that the three dyes are of the same type (acid-type or base-type) with similar fixing dynamics, or that the dyes fix to the pulp fibers so rapidly that the dye dynamics are the same as that of the pulp (the color literature consistently makes this assumption **[5]**), then the model of the coloring process is captured by the following equations:

$$\Delta\dot{X}(t) = -\frac{1}{\tau}\left[\Delta X(t) + k_{11}u_1(t-h) + k_{12}u_2(t-h) + k_{13}u_3(t-h)\right]$$

$$\Delta\dot{Y}(t) = -\frac{1}{\tau}\left[\Delta Y(t) + k_{21}u_1(t-h) + k_{22}u_2(t-h) + k_{23}u_3(t-h)\right]$$

$$\Delta\dot{Z}(t) = -\frac{1}{\tau}\left[\Delta Z(t) + k_{31}u_1(t-h) + k_{32}u_2(t-h) + k_{33}u_3(t-h)\right]$$

where the k_{ij} 's represent DC gains which depend on the selected combination of the 3 basic dyes, and can be determined either by identification using step responses or analytically using the reflectances of the added dyes, h is the plant delay, τ is the time constant, $\Delta V \triangleq V - V_0 = [\Delta X \quad \Delta Y \quad \Delta Z]$, with V_0 being a constant initial value for V giving the initial reflectances of the un-dyed paper measured by the sensor.

1.5.5. Infeed Grinding and Cutting

To describe processes of infeed grinding and cutting, many mathematical models with delay were proposed **[17, 44-46, 57, 68, 81]**.

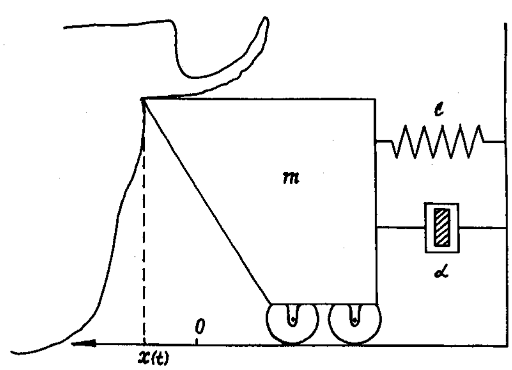

Figure 1.2. A simplified cutting scheme

Consider the simplified cutting scheme in Figure 1.2, where m is the mass of the cutter, α the viscosity coefficient, and c the elasticity coefficient. Let v_0 be the velocity of shaving relative to the cutter, $f(v_0)$ the friction coefficient, d_0 the desirable cutting depth, and $x(t)$ the position of the cutter. For the deviation

$z(t)=x(t)-x_0$ from the static rest position $x_0=f(v_0)qd_0c^{-1}$, where the coefficient q depends on the geometry of the cutter and the width of the layer under consideration, we obtain

$$m\ddot{z}(t)+\alpha\dot{z}(t)+cz(t)+f(v_0)q\left[z(t)-z(t-h)\right]=-q\left[f(v_0+\dot{z}(t))-f(v_0)\right]\left[d_0+z(t)-z(t-h)\right].$$

In this model, the time delay h arises due to the dependence of the cutting process on the surface state at the previous rotation. A detailed stability analysis of this model was given in **[81]**.

One of the most important problems connected with cutting is oscillation of the cutter. Such oscillation can be described by an RDE with delays depending on the unknown solution **[81]**:

$$\left.\begin{aligned}\ddot{x}(t)+\alpha_1\dot{x}_1(t)+\alpha_2x(t)+\alpha_3vx(t-h_1(\dot{x},\dot{y},v))=0,\\ \ddot{y}(t)+\beta_1\dot{y}(t)+\beta_2y(t)+\beta_3x(t-h_2(\dot{y},v))=0.\end{aligned}\right\}$$

Here α_i and β_i are constants, v is the rate of cutting, and the delays h_i are given by

$$h_1(\dot{x},\dot{y},v)=\frac{\alpha_4}{v+\dot{y}}+\frac{\alpha_5}{v+\dot{x}+\dot{y}},\qquad h_2(\dot{y},v)=\frac{\alpha_4}{v+\dot{y}}.$$

1.5.6. Technological Delay

Physical and chemical processes in reactors are characterized by their complexity. Changes in the amount of liquid entering the system happen to cause a change in the amount of liquid leaving the system only during a time h. In reality there is also certain time needed to mix the liquids in the vessel, for chemical reaction, and for transportation of liquid from one part of the reactor to another.

The feedback control loops in integrated communication and control systems are subject to network-induced delays, in addition to the delays incurred in digital sampling and data processing **[51]**.

Such type of delay is called technological delay, and occurs if it is necessary to take into account the finiteness of the time needed to complete a technological process.

E.g., processes in an absorbing column with recycling have been described by the following equation **[56]** (after transition to dimensionless variables and substitution of concrete numerical values for the parameters):

$$\dot{x}(t)=3.2\left[-x(t)+x(t-h)\right]+u(t-0.625h),$$

where $x(t)$ is a value of circulating mixture and $u(t)$ a control.

Another type of delay is connected with energy or signal transmission. Hereditary phenomena are especially important when controlling objects with high velocity (e.g., airplanes, rockets), and creating long distance control devices. Delays may occur in the automatic regulation landing system of an airplane, because of the finiteness of time of the propulsion reaction on a deviation of the control lever of the engine. It is also essential to take into account delays when treating the control aerodynamic rudder servomechanisms.

1.5.7. Car Chasing [3, 7, 16, 32]

Consider $n \geq 2$ identical cars, one following the other without passing possibility. It is assumed that acceleration at time t of the second car is proportional to the relative velocity of the two cars at time $t-h$, and inversely proportional to the distance between them at this moment $t-h$, where h is the driver reaction time. Let m be the mass of a car, $x_i(t)(i=1,\dots,n)$ the position of car i at time $t\left(t \mapsto x_n(t) \text{ is a given function}\right)$. Then the system of dynamic equations of motion is:

$$m\ddot{x}(t)\left[x_i(t-h)-x_{i+1}(t-h)\right]=\alpha\left[\dot{x}(t-h)-\dot{x}_{i+1}(t-h)\right] \qquad (i=1,\dots,n-1),$$

where α is the constant sensitivity coefficient of the driver.

1.5.8. Control Problems in Microbiology

Certain delay models are used to control processes of microbiological growth of cells and production of a useful product. We will consider one of them, describing the continuous reproduction of micro-organisms, the production of ferments, the degradation of wastes, etc. The process here discussed is as follows (Figure 1.3).

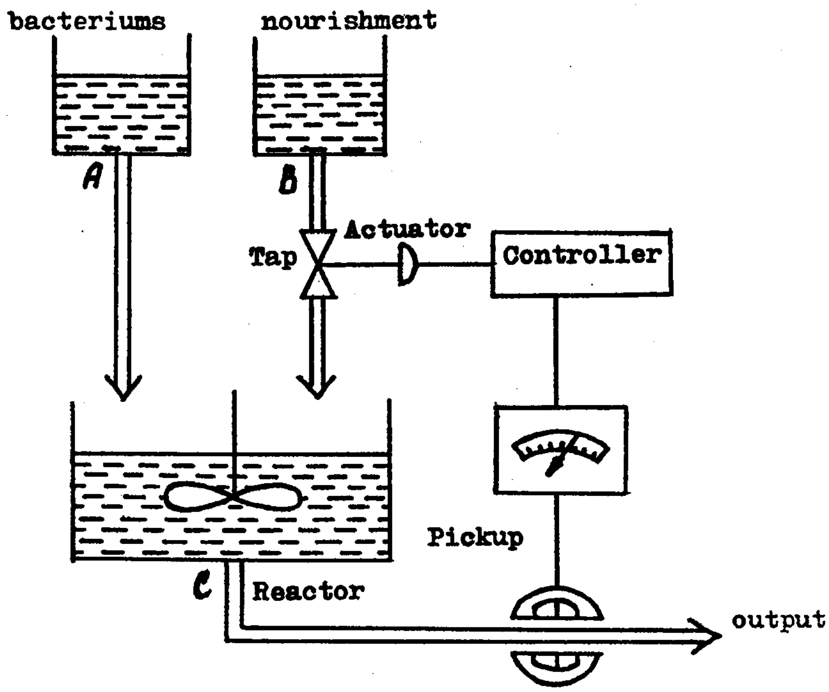

Figure 1.3. Biological reactor

Bacteria are introduced into a vessel with an entrance for nourishing substances and an entrance for extraction of resulting products. The bacteria consume the nourishing substances , re-product, and produce at a certain moment of time some quantity of the resulting product.

This process can be described by the bilinear delay model **[43]**

$$\left.\begin{aligned} \dot{x}_1(t) &= \gamma(t)x_1(t) - u(t)x_1(t) - \beta x_1(t-h), \\ \dot{x}_2(t) &= \gamma(t)\alpha^{-1}x_1(t) - u(t)x_2(t) + bu(t). \end{aligned}\right\} \tag{1.13}$$

The first equation is the balance equation of biological substrate, the second equation characterizes the production of resulting mass by the bacteria. Here,

$x_1(t)$ is the volume of microbiological substrate;

$x_2(t)$ is the volume of the resulting product;

$u(t)$ is the volume of nourishing environment in the vessel;

$\gamma(t)$ is the rate of biological growth;

$x_1(t-h)$ accounts for the loss of bacteria of great vitality during a finite time h ;

β and b are constants in the model;

α is the rate of growth of the useful product.

Models like (1.13) can be used in the process of biological clearance of sewage, when dirty water goes into a vessel with active substances that come to react with the contaminator. The rates of supply of dirty water and active substances are controlled. As a result of the reaction we obtain clean water, and a residue of biopolymers. The values of the delays in this model depends on the rate of mixing, temperature, density, etc.

1.6. Stability of FDE

One of the most important aspects in control is stability, which is the first issue to look at before thinking of the performance of the control design. Many methods were used in the literature for the analysis of time-delay systems **[65]** as shown in Table 1.1.

Methods	LTI, const. delay	Nonlin.	Varying delays	Neutral
$\mathbb{C}$-plane, roots	Low dimension	No	No	Yes
Matrix pencils	Low dimension	No	No	?
Norm, measure	Yes	Yes	Yes	Yes
1st Lyapunov	Obvious	Yes	Yes	?
Krasovskii	Yes	Yes	$\frac{dh}{dt} < 1$	Yes
Razumikhin	Yes	Yes	$h < \infty$	Yes
Comparison tech.	Yes	Yes	Yes	Yes
LaSalle invar.	Yes	Yes	?	?

Table 1.1. *Different methods for stability analysis of time-delay systems*

In this section we present the stability theorems that were developed using Lyapunov-Krasovskii functionals. As shown in Section 1.4, functional differential equations are classified into two major types: retarded and neutral. The stability theorems will be presented for both types.

1.6.1. Retarded Type Functional Differential Equations (RDE) [31]

We consider the stability problem for RDEs of the form:

$$\dot{x}(t) = f(t, x_t), \qquad t \geq 0 \tag{1.14}$$

$$x_{t_0}(\theta) = \phi(\theta), \qquad \forall \theta \in [-\tau, 0]$$

Assume that $f : \mathbb{R} \times C_{d,n} \to \mathbb{R}^n$ is continuous and Lipschitzian (Appendix A) in the second variable, and $f(t,0) = 0$, $\forall t$, then we have the following theorem:

Theorem 1.1. [31] *Suppose $f : \mathbb{R} \times C_{d,n} \to \mathbb{R}^n$ takes $\mathbb{R} \times$ (bounded sets of $C_{d,n}$) into bounded sets of $\mathbb{R}^n$, and u, v, $w : \mathbb{R}^+ \to \mathbb{R}^+$ are continuous non-decreasing functions, $u(s)$ and $v(s)$ are positive for $s > 0$, and $u(0) = v(0) = 0$. If there is a continuous functional $V : \mathbb{R} \times C_{d,n} \to \mathbb{R}$ such that*

$$u(\|\phi(0)\|) \leq V(t,\phi) \leq v(\|\phi\|_C), \qquad \forall \theta \in [-\tau, 0]$$

$$\dot{V}(t, x_t) \leq -w(\|x(t)\|)$$

then the solution $x = 0$ of equation (1.14) is uniformly stable. If $w(s) > 0$ for $s > 0$, the solution $x = 0$ is uniformly asymptotically stable.

1.6.2. Neutral Type Functional Differential Equations (NDE) [41]

1.6.2.1. Degenerate Lyapunov Functionals

We consider the stability problem for NDEs of the form:

$$\frac{d}{dt}\left[x(t) - G(t, x_t)\right] = F(t, x_t), \qquad t \geq t_0, \tag{1.15}$$

$$x_{t_0}(\theta) = \phi(\theta), \qquad \forall \theta \in [-\tau, 0]. \tag{1.16}$$

Here $F,G:[t_0,\infty)\times C_{d,n}^{w}\to\mathbb{R}^n$ are continuous maps satisfying

$$F(t,0)=G(t,0)=0, \qquad t\ge t_0. \qquad (1.17)$$

The method given below for studying stability is based on the use of positive semi-definite functionals and the study of the stability of the following type of functional inequalities:

$$\|z(t,y_t)\|\le f(t), \quad t\ge t_0; \qquad y_{t_0}=\phi, \qquad (1.18)$$

where $z(t,y_t)\triangleq y(t)-G(t,y_t)$, and $f:[t_0,\infty)\to[0,\infty)$ is a continuous function.

Then we have the following:

Theorem 1.2. [41] *Suppose that the trivial solution of* (1.18) *is stable and that there exists a continuous functional* $V:[t_0,\infty)\times C_{d,n}^{w}\times\mathbb{R}^n\to\mathbb{R}$ *such that*

$$u(\|z(t,\psi)\|)\le V(t,\psi,z(t,\psi))\le v(\|\psi\|_C), \qquad (1.19)$$

$$\dot V\triangleq\frac{d}{dt}V(t,x_t,z(t,x_t))\le 0 \qquad (1.20)$$

for all solutions x *of equation* (1.15), *where* $u(s)$ *and* $v(s)$ *are continuous, nonnegative, and nondecreasing with* $u(s),v(s)>0$ *for* $s\ne0$, *and* $u(s)=v(s)=0$. *Then the trivial solution of* (1.15) *is stable. If* V *satisfies* (1.19), *and*

$$\dot V\le -w(\|z(t,x_t)\|),$$

where $w(s)$ *is continuous, nonnegative, and nondecreasing with* $w(s)>0$ *for* $s\ne0$, *and* $w(0)=0$, *then the trivial solution of* (1.15) *is asymptotically stable.*

1.6.2.2. The Use of Functionals Depending on Derivatives

We consider the stability problem for NDEs of the form

$$\dot x(t)=f(t,x_t,\dot x_t), \qquad \left.\begin{array}{ll} x:[t_0-\tau,\infty)\to\mathbb{R}^n, & t\ge t_0 \\ x_{t_0}(\theta)=\phi(\theta), & \forall\theta\in[-\tau,0] \end{array}\right\} \qquad (1.21)$$

Let W be the Banach space of absolutely continuous functions $[-\tau,0]\to R^n$ with square-integrable derivative, with form

$$\|\psi\|_W = \left[\left\|\psi(0)^2\right\| + \int_{-h}^{0}\|\dot{\psi}(\theta)\|^2 \, d\theta\right]^{1/2}.$$

Let $Q_H \subset W$ be a ball, $Q_H \triangleq \{\psi: \|\psi\|_W \le H\}$, $H > 0$. Let $f:[t_0,\infty)\times Q_H \times L_2[-\tau,0] \to \mathbb{R}^n$ be a continuous function which satisfies a Lipschitz condition (Appendix A) in the second and third arguments, with Lipschitz constant for the third argument less than 1, and $f(t,0,0)=0$, the we have the following:

Theorem 1.3. [41] *If there is a continuous functional* $V:[t_0,\infty)\times Q_H \times L_2[-\tau,0] \to \mathbb{R}$ *satisfying the inequalities*

$$\left.\begin{array}{c} u(\|\psi(0)\|) \le V(t,\psi,\chi) \le v(\|\psi\|_W), \\ \dot{V} \le 0, \end{array}\right\}$$

then the solution $x(t)=0$ *is stable. If, moreover, we have* $\dot{V} \le -w(\|x(t)\|)$, *then the trivial solution of* (1.21) *is asymptotically stable. Here* $u(s)$, $v(s)$ *and* $w(s)$ *are as defined in* Theorem 1.2 *above.*

1.7. Approaches Relevant to the Control of Time-Delay Systems

As stated before, two methods are used in this thesis for the design of controllers for time-delay systems: the transfer function method and the functionals method.

1.7.1. The Transfer Function Method

1.7.1.1. The Continuous-Time Case

This method is a frequency domain methodology, where the H_∞-norm of the transfer function of the system is required to be less than a prescribed positive scalar. The system is assumed time-invariant including the uncertainty matrices F, since no transfer function is defined for the time-varying case. More specifically, let a continuous-time retarded system have the following state equations:

$$\dot{x}(t) = (A+\Delta A)x(t) + (A_h + \Delta A_h)x(t-h) + Bw(t) \tag{1.22}$$

$$z(t) = Cx(t) \tag{1.23}$$

with the corresponding transfer function,

$$T_{zw}(s) = C\left[sI - (A + \Delta A) - (A_h + \Delta A_h)e^{-sh}\right]^{-1} B$$

then the following lemma was presented in **[47]**:

Lemma 1.1. *For a given constant* $\gamma > 0$ *and a symmetric matrix* $Q > 0$*, if there exists a symmetric matrix* $P > 0$ *satisfying the inequality*

$$(A + \Delta A)^T P + P(A + \Delta A) + P(A_h + \Delta A_h)Q^{-1}(A_h + \Delta A_h)^T P + Q + C^T C + \gamma^{-2} PBB^T P < 0 \quad (1.24)$$

for all admissible parameter uncertainties ΔA *and* ΔA_h*, then the system* (1.22) *and* (1.23) *is robustly asymptotically stable and* $\| T_{zw}(s) \|_\infty \le \gamma$*, independently of the time-delay* h.

1.7.1.2. The Discrete-Time Case

As for the continuous-time case, this method is a frequency domain methodology, where the H_∞-norm of the transfer function of the system is required to be less than a prescribed positive scalar. The system is time-invariant including the uncertainty matrices F. More specifically, let a discrete-time retarded system have the following state equations:

$$x(k+1) = (A + \Delta A)x(k) + (A_h + \Delta A_h)x(k-h) + Bw(k) \quad (1.25)$$

$$z(k) = Cx(k) \quad (1.26)$$

$$\Delta A = HFE, \qquad \Delta A_h = H_h F E_h \quad (1.27)$$

with the corresponding transfer function,

$$T_{zw}(z) = C\left[zI - (A + \Delta A) - (A_h + \Delta A_h)z^{-h}\right]^{-1} B \quad (1.28)$$

then we have the following lemma **[69]**:

Lemma 1.2. *If there exist a positive-definite matrix* P *and positive scalars* $\varepsilon_1 > 0$, $\varepsilon_2 > 0$ *and* $\varepsilon_3 > 0$ *such that the following inequalities hold*

$$P^{-1} - \gamma^{-2} C^T C - \varepsilon_1 A_h^T A_h - \varepsilon_2 E^T E - \varepsilon_3 E_h^T E_h > 0 \quad (1.29)$$

$$A\left(P^{-1} - \gamma^{-2} C^T C - \varepsilon_1 A_h^T A_h - \varepsilon_2 E^T E - \varepsilon_3 E_h^T E_h\right)^{-1} A^T - P + BB^T + \varepsilon_1^{-1} I + \varepsilon_2^{-1} HH^T + \varepsilon_3^{-1} H_h H_h^T < 0 \quad (1.30)$$

then the augmented system (1.25) *and* (1.26) *is asymptotically stable and meets the specified* H_∞ *norm upper-bound constraint* $\left\|T_{zw}(z)\right\|_\infty \le \gamma$, *independent of the positive integer state time-delay* h.

1.7.2. The Functionals Method

Consider a simple nominal retarded system:

$$\dot{x}(t) = Ax(t) + A_h x(t-h) + B_1 w(t) \tag{1.31}$$

$$z(t) = Cx(t) + B_2 w(t). \tag{1.32}$$

Using the Liebniz-Newton formula $x(t-h) = x(t) - \int_{t-h}^{t} \dot{x}(s) ds$, the state equation (1.31) can be transformed to the equivalent descriptor form in one of two ways:

- $$\dot{x}(t) = (A + A_h) x(t) - A_h \int_{t-h}^{t} \dot{x}(s) ds + B_1 w(t) \tag{1.33}$$
- $$y(t) = \dot{x}(t), \quad y(t) = (A + A_h) x(t) - A_h \int_{t-h}^{t} y(s) ds + B_1 w(t). \tag{1.34}$$

As noticed, transformation (1.34) augments the order of the system to twice the order of the original system (1.31), and so makes the treatment more difficult, especially when uncertainties are added. Despite this fact, transformation (1.34) is more advantageous than transformation (1.33), as shown in **[8]**, because it preserves the dynamics of the original system (1.31) in the sense that the transfer functions for systems (1.31) and (1.34) are identical. This is not the case for transformation (1.33), since additional poles **[8]** are introduced to the original system (1.31). In terms of control this means an increase in conservatism, since the controller designed by the use of transformation (1.33) controls more dynamics than the original system (1.33). In this thesis, transformation (1.34) will be used, since one of the main objectives is to decrease conservatism. This choice will be at the expense of more difficult treatment.

Once the transformation is chosen, the methodology using functionals for the design of an H_∞ controller for time-delay system is based on the use of a cost function of the form,

$$J = \int_0^\infty \left[z^T(t) z(t) - \gamma^2 w^T(t) w(t) + \dot{V}(t) \right] dt \tag{1.35}$$

with V being a functional to be chosen.

Requiring $J < 0$ and $\dot{V} < 0$ for all time t and for a certain prescribed positive scalar γ guarantees the asymptotic stability (see Section 1.6) of the system in question and the attenuation of the disturbances w entering the system. The choice of the functional V determines the type of controller thus designed (delay-independent/dependent). Some examples of functionals V are:

- $V(t) = x^T(t)Px(t)$ (1.36)

- $V(t) = \int_{t-h}^{t} x^T(\tau)Sx(\tau)d\tau$ (1.37)

- $V(t) = \int_{-h_{\max}}^{0} \int_{t+\theta}^{t} x^T(s)A^T R_3 Ax(s)dsd\theta$. (1.38)

Functional (1.36) corresponds to the descriptor system **[55, 74]**, and functional (1.37) corresponds to the delay-independent conditions with respect to delay h, while functional (1.38) corresponds to the delay-dependent conditions with respect to delay h. In this thesis, a general functional formed by the sum of all three functionals will be used. This is again to decrease conservatism, but at the expense of more difficult treatment and complexity of solution.

1.8. Research Objective

The objective of this research is to further the treatment of time-delay systems in the following:

- Present the delay-independent and delay-dependent robust output feedback controllers for time-delay systems with and without actuator saturation.
- Present the time-varying finite horizon treatment of the robust state-feedback control of time-delay systems.
- Decrease conservatism in the control design for time-delay systems.
- Include the rate of change of the time-delay in the proposed design.
- Obtain as much as possible solutions in terms of Linear matrix inequalities (LMI: see Appendix B) or Bilinear matrix inequalities, which are easily solved using the methods proposed in **[6]**.

1.9. Contributions of this Thesis

In relation to the research objectives stated above, the contributions of this dissertation are summarized as follows:

1.9.1. The Delay-Independent Designs using Transfer Function Methodology

- The delay-independent robust H_∞ output-feedback control of uncertain continuous retarded systems is presented in full generality, with delays and disturbances in the state and the output equations. The uncertainties are present in all the state equation matrices. The discrete-time case is also treated.

1.9.2. The Delay-Dependent Designs using Functionals Methodology

- The finite-horizon time-varying robust H_∞ control problem is addressed here for linear neutral systems in full generality, with norm-bounded parametric uncertainties entering all the matrices in the system representation.
- The least conservative approach to the derivation of the bounded real lemma, as proposed in **[19, 20]** for the uncertainty-free infinite-horizon problem, is used in this thesis. However, the approach of **[19, 20]** is modified and adapted here to suit the finite-horizon time-varying case with parametric uncertainties. Conservatism is reduced by the use of the most efficient descriptor model transformation and Lyapunov-Krasovskii functional. A more conservative functional was used in **[77, 79]**.
- When solved numerically, the solution to the finite-horizon time-varying case is shown to result in a set of linear matrix inequalities (LMI) at every discretized time. The latter can easily be solved using recently developed algorithms **[6]**.
- The solution to the infinite horizon case which incorporates all the parametric uncertainties is also presented and leads to a design in terms of a set of linear matrix inequalities.
- A delay-dependent robust state-feedback stabilization problem is addressed for neutral systems in full generality, including actuators constraints and norm-bounded parametric uncertainties entering all the matrices in the system representation. Previous results **[77]**, address only the nominal case.

- The least conservative model for the actuator saturation **[27]**, as delivered by the differential inclusions approach is employed here.
- The value of the time-delay as well as its rate of change are taken into account in the design methods presented and further permits to reduce the conservatism of the approach. This delivers a feedback controller that, unlike the ones presented in the previous literature, is both dependent on the value of the time delay as well as its rate of change.
- The method developed here is applied to example systems which were previously used in **[73, 78]**, and, shows that the new design yields less conservative results, in that stabilization is ensured for a wider range of the time-delays and a larger set of initial conditions.
- A delay-dependent robust output feedback stabilization problem is addressed for state-delayed systems in full generality, including actuators constraints and norm-bounded parametric uncertainties entering all the matrices in the system representation. Also, an additional feed-forward term is included in the output system equation. The saturation is modeled using both sector modeling and differential inclusions. The main results deliver computationally verifiable criteria guaranteeing asymptotic robust stabilization under control saturation. In the case of differential inclusions a larger region of local asymptotic stability as compared to previous results **[35, 75]** is also given.
- Employing relaxation techniques, as suggested in **[77]**, the stabilization problem with control saturation considered here is reduced to a set of LMIs (see **[6]** for efficient numerical algorithms). Despite the fact that the main results of the thesis provide seemingly complicated sufficient conditions for stability of the closed loop system, the last are surprisingly easy to satisfy. The LMI based robust stability verification procedure is very efficient as demonstrated by way of many examples.

1.10. Thesis Outline

In an attempt to make the presentation of this thesis as readable and clear as possible, the many different time-delay control problems that are treated are put in separate chapters, which are self contained. The chapters can be read without any necessary order. Any lemmas or parts of proofs that are common to some chapters were repeated in each of them for the sake of clarity. With this in mind, the thesis is organized as follows:

- **Chapter 2: Continuous-Time Delay-Independent Robust H_∞ Output Feedback Control of Uncertain Retarded Systems with Time-Delays in States and Outputs.**

  This chapter presents the design for the robust H_∞ output feedback control of uncertain retarded continuous systems using the method of transfer function. The time-delays are present in the state and in the output, and the parametric norm bounded uncertainties are present in all the state matrices. Disturbances are included in the state and the output equations. The observer is presented with an additional parameter allowing the separation in the design of the controller and the observer. The augmented system is written and the corresponding transfer function is formulated. The H_∞ norm of the transfer function is then required to be less than a prescribed positive scalar. The solution thus obtained is delay-independent and is given in terms of two Riccati-type inequalities, and a numerical example is given to show the efficiency of the design.

- **Chapter 3: Discrete-Time Delay-Independent Robust H_∞ Output Feedback Control of Uncertain Retarded Systems.**

  This chapter presents the discrete equivalent of the robust H_∞ output feedback control of uncertain retarded continuous systems presented in the previous chapter. The method used for solving the problem is that of the transfer function. The time-delay is present in the state, and the parametric norm bounded uncertainties are present in all the state matrices except the control matrix. Disturbances are included in the state and the output equations. The observer is presented with an additional parameter allowing the separation in the design of the controller and the observer. The augmented system is written and the corresponding transfer function is formulated. The H_∞ norm of the transfer function is then required to be less than a prescribed positive scalar.

The solution thus obtained is delay-independent and is given in terms of two Riccati-type inequalities, and a numerical example is given to show the efficiency of the design.

- **Chapter 4: Finite-Horizon Time-Varying H_∞ State-Feedback Control of Linear Neutral Systems with Parametric Uncertainties.**

  This chapter presents the finite-horizon time-varying H_∞ state-feedback control of uncertain linear neutral systems using the method of functionals. Time-delays which are assumed time-varying are present in the state and in its time-derivative, and the parametric norm bounded uncertainties are present in all the state matrices. Starting with the control free case, the H_∞ control problem is formulated in terms of an integral cost function relating a weighted output to the disturbances to the system, and a Lyapunov-Krasovskii type functional is presented to solve the problem. In the finite horizon case, the solution thus obtained is delay-dependent and is presented in terms of a linear differential matrix inequality, in which the derivative is approximated by a difference scheme thus reducing the solution to a linear matrix inequality solved at every iteration time, once the time step is chosen. In the infinite horizon case, the time derivative of the Lyapunov-Krasovskii is made to be strictly negative thus ensuring the asymptotic stability of the origin. The solution is thus presented in terms of a linear matrix inequality and numerical examples are presented to show the decrease in conservatism as compared to previous results in the literature.

  Next, control is introduced to solve the state feedback control problem using the Bounded Real Lemma presented for the control free case. The solution thus obtained is nonlinear, but the use of a certain parametrization reduces it to a linear matrix inequality. Finally, a numerical example is presented to show the efficiency of the design.

- **Chapter 5: Delay-Dependent State-Feedback Robust Stabilization of Uncertain Neutral Systems with Saturating Actuators: The Differential Inclusions Model.**

This chapter presents the robust state-feedback stabilization of uncertain neutral systems with actuator saturations, using the method of functionals. Time-delays which are assumed time-varying are present in the state and in its time-derivative, and the parametric norm bounded uncertainties are present in all the state matrices. The control saturation is modeled using differential inclusions, and a Lyapunov-Krasovskii type functional is presented to solve the problem. The time derivative of the Lyapunov-Krasovskii is made to be strictly negative thus ensuring the asymptotic stability of the origin for a convex set of initial conditions. The solution thus obtained is delay-dependent and is presented in terms of a set of bilinear matrix inequalities. Using relaxation and optimization techniques, the solution is reduced to a set of linear matrix inequalities and the set of initial conditions guaranteeing asymptotic stability is maximized. Numerical examples are presented to show the increase in the set of initial conditions and thus a decrease in conservatism as compared to previous results in the literature.

- **Chapter 6: Delay-Dependent Robust Output Feedback Stabilization of Uncertain State Delayed Systems with Time-Varying Delays and Saturating Actuators: The Sector Modeling Model.**

  This chapter presents the robust output-feedback stabilization of uncertain retarded systems with actuator saturations, using the method of functionals. The time-delays which is assumed time-varying is present in the state, and the parametric norm bounded uncertainties are present in all the state matrices. The control saturation is modeled using sector modeling, and a Lyapunov-Krasovskii type functional is presented to solve the problem. The time derivative of the Lyapunov-Krasovskii is made to be strictly negative thus ensuring the asymptotic stability of the origin. The solution thus obtained is delay-dependent and is presented in terms of a set of bilinear matrix inequalities. Using relaxation techniques, the solution is reduced to a set of linear matrix inequalities. A numerical example is presented to show the effectiveness of the design as compared to previous results in the literature.

- **Chapter 7: Robust Output Feedback Stabilization of Uncertain Time-Varying State Delayed Systems with Saturating Actuators: The Differential Inclusions Method.**

  This chapter presents the robust output-feedback stabilization of uncertain retarded systems with actuator saturations, using the method of functionals. The time-delay which is assumed time-varying is present in the state, and the parametric norm bounded uncertainties are present in all the state matrices. The observer-controller design is presented and the augmented system is written. The control saturation is modeled using differential inclusions, and a Lyapunov-Krasovskii type functional is presented to solve the problem. The time derivative of the Lyapunov-Krasovskii is made to be strictly negative thus ensuring the asymptotic stability of the origin for a convex set of initial conditions. The solution thus obtained is delay-dependent and is presented in terms of a set of bilinear matrix inequalities. Using relaxation and optimization techniques, the solution is reduced to a set of linear matrix inequalities and the set of initial conditions guaranteeing asymptotic stability is maximized. Numerical examples are presented to show the increase in the set of initial conditions and thus a decrease in conservatism as compared to previous results in the literature.

- **Chapter 8: Conclusions and Future Research.**

  The last chapter concludes the thesis with a brief review of the main contributions of the research presented in the preceding chapters. Some general remarks concerning the advantages and potential of the proposed approaches are presented. Suggestions on issues for future research are given.

Reference material which will be cited when needed, is included in several appendices for the reader's convenience:

- Appendix A: Useful Theorems and Other Results.
- Appendix B: Linear Matrix Inequality.

1.11. Originality of the Research Contributions

The proposed approaches constitute an original contribution to the control and stabilization of time-delay systems in that:

1.11.1. The Delay-Independent Designs using Transfer Function Methodology

- The robust output feedback control is presented, whereas in the literature the cases that are mostly presented are that of state feedback control and of observer designs.
- The problem solved for the continuous-time case is of the most general treatment, since delays are included in both the state and the output, and the uncertainties are included in the matrices of the system representation.
- For the discrete-time case, the treatment is one of the few found in literature due to the relative negligence of the discrete-time treatment in the literature, partially due to the difficulty of treatment and to the complexity of the equations.
- The structure of the observer allows for the separation in the design of the controller and the observer gains. This is not possible in general for the output feedback control with uncertainties in the system, since previous designs lead generally to coupled design between the control and the observer parts.

1.11.2. The Delay-Dependent Designs using Functionals Methodology

- The finite-horizon time-varying robust H_∞ state-feedback control of linear uncertain neutral systems is presented. The only paper found in the literature treating the finite-horizon case is **[67]**, where the solution is given in terms of nonlinear differential Riccati type inequalities. In this thesis the solution is given in the form of linear differential matrix inequalities, which are reduced to an LMI **[6]** easily solved at every iteration time.
- The robust state-feedback stabilization of uncertain neutral systems with actuator saturation is presented. The only previous study **[77]** treating neutral systems is restricted to the nominal case (uncertainty free). The design presented in this thesis is shown to be less conservative in that it achieves a larger set of initial conditions guaranteeing asymptotic stability.

- The robust output feedback stabilization of uncertain retarded systems with actuator saturation is presented. Two models for saturations are used: sector modeling and differential inclusions model. In the case of sector modeling, the design presented here is less conservative than the only previous study found in the literature **[72]**, while for the case of differential inclusions the design presented in this thesis is the first in the literature.

CHAPTER 2

Continuous-Time Delay-Independent Robust H_∞ Output Feedback Control of Uncertain Retarded Systems with Time-Delays in States and Outputs

2.1. System Description

Consider the retarded version of system (1.1)-(1.5) taken in the infinite-horizon time-invariant context, where the system matrices are all time-invariant. The uncertainty matrices F_i are assumed to be equal (assumption frequently made in the literature and is not restrictive in practice) and also time-invariant. The time-delay h is assumed to be time-invariant and exactly known (see Remark 2.1 below). More specifically, the system under consideration is:

$$\dot{x}(t) = \bar{A}x(t) + \bar{A}_h x(t-h) + B_1 w(t) + \bar{B}u(t) \tag{2.1}$$

$$y(t) = \bar{C}x(t) + \bar{C}_h x(t-h) + B_2 w(t) \tag{2.2}$$

$$x(t) = \phi(t), \qquad t \in [-h, 0] \tag{2.3}$$

$$\bar{A} = A + \Delta A\,,\ \bar{A}_h = A_h + \Delta A_h\,,\ \bar{B} = B + \Delta B\,,\ \bar{C} = C + \Delta C\,,\ \bar{C}_h = C_h + \Delta C_h \tag{2.4}$$

$$\begin{bmatrix} \Delta A \\ \Delta C \end{bmatrix} = \begin{bmatrix} H_A \\ H_C \end{bmatrix} F E_A, \qquad \begin{bmatrix} \Delta A_h \\ \Delta B \end{bmatrix} = H_A F \begin{bmatrix} E_h \\ E_B \end{bmatrix}, \qquad \Delta C_h = H_{C_h} F E_{C_h}\ . \tag{2.5}$$

2.2. Assumptions

The following assumption is needed for the subsequent development:

Assumption 2.1. Either one of the matrices H_C, H_{C_h}, or B_2 is of full rank.

2.3. Full Order Observer

The proposed full-order linear and delay independent state observer with the associated feedback controller proposed in this chapter are assumed to take the following form:

$$\dot{\hat{x}}(t) = G\hat{x}(t) + Ly(t) + Bu(t) \tag{2.6}$$

$$u(t) = K\hat{x}(t) \tag{2.7}$$

where G and L denote the observer gains, K denotes the controller gain, and all three gains are design parameters.

Remark 2.1. Although the design presented in this chapter is delay-independent, knowledge of the value of the delay is still needed because as seen in (2.2) and (2.6), the observer chosen for our design depends on the delay h through the output $y(t)$, for the state estimate to be constructed. In case the value of the delay is not available, then the observer (2.6) can be chosen without delay in the state estimate $\hat{x}(t)$, but at the expense of a less accurate design.

Defining the error state,

$$e(t) \triangleq x(t) - \hat{x}(t) \tag{2.8}$$

it then follows from (2.1), (2.2), (2.6) and (2.7) that,

$$\dot{x}(t) = \left(A + \Delta A + (B + \Delta B)K\right)x(t) - (B + \Delta B)Ke(t) + \left(A_h + \Delta A_h\right)x(t-h) + B_1 w(t) \tag{2.9}$$

and

$$\begin{aligned} \dot{e}(t) = {} & \left(A + \Delta A - G - L(C + \Delta C) + \Delta BK\right)x(t) + (G - \Delta BK)e(t) \\ & + \left(A_h + \Delta A_h - L(C_h + \Delta C_h)\right)x(t-h) + \left(B_1 - LB_2\right)w(t) \end{aligned} \tag{2.10}$$

Let $z(t) \in \mathbb{R}^t$ denote the state-error output which is assumed to be given by:

$$z(t) = Me(t) + Nu(t) \tag{2.11}$$

where $M \in \mathbb{R}^{t \times n}$ and $N \in \mathbb{R}^{t \times q}$ are given constant matrices.

Using (2.7) and then (2.2), (2.11) becomes,

$$z(t) = NKx(t) + (M - NK)e(t) \tag{2.12}$$

Defining

$$\xi(t) \triangleq \begin{bmatrix} x(t) \\ e(t) \end{bmatrix}, \tag{2.13}$$

$$\hat{A} \triangleq \begin{bmatrix} A+BK & -BK \\ A-LC-G & G \end{bmatrix}, \quad \hat{A}_h \triangleq \begin{bmatrix} A_h & 0 \\ A_h - LC_h & 0 \end{bmatrix}, \tag{2.14}$$

$$\hat{H}_1 \triangleq \begin{bmatrix} H_A \\ H_A - LH_C \end{bmatrix}, \qquad \hat{E}_1 \triangleq [E_A \quad 0], \tag{2.15}$$

$$\hat{H}_2 \triangleq \begin{bmatrix} H_A \\ H_A \end{bmatrix}, \qquad \hat{E}_2 \triangleq [E_B K \quad -E_B K] \tag{2.16}$$

$$\hat{H}_h \triangleq \begin{bmatrix} H_A \\ H_A - LH_{C_h} \end{bmatrix}, \qquad \hat{E}_h \triangleq [E_h \quad 0], \tag{2.17}$$

$$\Delta\hat{A} \triangleq \hat{H}_1 F \hat{E}_1 + \hat{H}_2 F \hat{E}_2, \qquad \Delta\hat{A}_h \triangleq \hat{H}_h F \hat{E}_h, \tag{2.18}$$

$$\hat{C} \triangleq [NK \quad M - NK], \qquad \hat{B} = \begin{bmatrix} B_1 \\ B_1 - LB_2 \end{bmatrix} \tag{2.19}$$

and combining (2.9), (2.10) and (2.12), the following augmented system is easily obtained:

$$\dot{\xi}(t) = (\hat{A} + \Delta\hat{A})\xi(t) + (\hat{A}_h + \Delta\hat{A}_h)\xi(t-h) + \hat{B}w(t) \tag{2.20}$$

$$z(t) = \hat{C}\xi(t). \tag{2.21}$$

The transfer function from the disturbance $w(t)$ to the state-error output $z(t)$ is thus given by

$$T_{zw}(s) = \hat{C}\left[sI - (\hat{A} + \Delta\hat{A}) - (\hat{A}_h + \Delta\hat{A}_h)e^{-sh}\right]^{-1}\hat{B}. \tag{2.22}$$

2.4. Problem Statement

In the above context, the objective is to design for the parameter values: G, L and K, in such a way that for all admissible parameters uncertainties ΔA, ΔA_h, ΔB, ΔC and ΔC_h, the augmented system (2.20) and (2.21) is asymptotically stable and the following upper bound constraint on the H_∞-norm of $T_{zw}(s)$ is simultaneously guaranteed:

$$\|T_{zw}(s)\|_{\infty} \le \gamma \tag{2.23}$$

for all time-delay values $h \in \mathbb{R}^{+}$ and all uncertainties of type (2.5), where $\|T_{zw}(s)\|_{\infty} = Sup_{W \in R}\sigma_{\max}\left[T_{zw}(jW)\right]$ with W being the frequency, $\sigma_{\max}[T]$ denotes the largest singular value of T, and $\gamma < 1$ is a given positive constant.

2.5. Main Results

2.5.1. Time-Invariant Case

The following lemmas, proved in **[84]** will be useful in the design of the robust H_{∞} output feedback controller for the uncertain linear continuous-time state delayed system (2.20) and (2.21).

Lemma 2.1. [84] *For an arbitrary scalar $\varepsilon_1 > 0$ and a symmetric matrix $P > 0$, the following matrix inequality is valid*

$$\left(\Delta\hat{A}\right)^{T} P + P\left(\Delta\hat{A}\right) \le \varepsilon_1 P\hat{H}\hat{H}^{T}P + \varepsilon_1^{-1}\hat{E}^{T}\hat{E} \tag{2.24}$$

where $\hat{H}$ can take the value of $\hat{H}_1$ or $\hat{H}_2$, and $\hat{E}$ can take the value of $\hat{E}_1$ or $\hat{E}_2$, respectively.

Lemma 2.2. [84] *Let a scalar $\varepsilon_2 > 0$ and $Q > 0$ be such that $\hat{E}_h Q^{-1}\hat{E}_h^{T} < \varepsilon_2 I$, then*

$$\left(\hat{A}_h + \Delta\hat{A}_h\right)Q^{-1}\left(\hat{A}_h + \Delta\hat{A}_h\right)^{T} \le \hat{A}_h\left(Q - \varepsilon_2^{-1}\hat{E}_h^{T}\hat{E}_h\right)^{-1}\hat{A}_h^{T} + \varepsilon_2\hat{H}_h\hat{H}_h^{T} \tag{2.25}$$

Next lemma is easy to see.

Lemma 2.3. *For a given negative definite matrix $\Upsilon < 0$ $\left(\Upsilon \in \mathbb{R}^{n\times n}\right)$, there always exists a matrix* $E_o \in \mathbb{R}^{n\times p}$ $(p \le n)$ such that $\Upsilon + E_o E_o^{T} < 0$.

The last lemma, found in **[47]**, will be essential to our result.

Lemma 2.4. [47] *For a given constant $\gamma > 0$ and a symmetric matrix $Q > 0$, if there exists a symmetric matrix $P > 0$ satisfying the inequality*

$$\left(\hat{A} + \Delta\hat{A}\right)^{T} P + P\left(\hat{A} + \Delta\hat{A}\right) + P\left(\hat{A}_h + \Delta\hat{A}_h\right)Q^{-1}\left(\hat{A}_h + \Delta\hat{A}_h\right)^{T} P + Q + \hat{C}^{T}\hat{C} + \gamma^{-2}P\hat{B}\hat{B}^{T}P < 0 \tag{2.26}$$

for all admissible parameter uncertainties ΔA, ΔA_h, ΔB, ΔC *and* ΔC_h *given in* (2.5), *then the system* (2.20) *and* (2.21) *is robustly asymptotically stable and* $\| T_{zw}(s) \|_\infty \le \gamma$, *independently of the time-delay* h.

For the sake of simplicity, we introduce the following definitions prior to stating the main results of the chapter:

$$\Gamma \triangleq \left(Q_1 - \varepsilon_2^{-1} E_h^T E_h\right) \tag{2.27}$$

$$\Phi \triangleq A_h \Gamma^{-1} A_h^T + \varepsilon_2 H_A H_A^T \tag{2.28}$$

$$\tilde{A} \triangleq A + (\varepsilon_1 + \varepsilon_3) H_A H_A^T P_1 + \Phi P_1 + \gamma^{-2} B_1 B_1^T P_1 \tag{2.29}$$

$$\hat{A} \triangleq \tilde{A} - P_2^{-1}(BK)^T P_1 - \varepsilon_3^{-1} P_2^{-1} (E_B K)^T E_B K + P_2^{-1}(M - NK)^T NK \tag{2.30}$$

$$V \triangleq \varepsilon_1 H_C H_A^T + \varepsilon_2 H_{C_h} H_A^T + \gamma^{-2} B_2 B_1^T + C_h \Gamma^{-1} A_h^T \tag{2.31}$$

$$\tilde{C} \triangleq C + V P_1 \tag{2.32}$$

$$R \triangleq \varepsilon_1 H_C H_C^T + \varepsilon_2 H_{C_h} H_{C_h}^T + \gamma^{-2} B_2 B_2^T + C_h \Gamma^{-1} C_h^T \tag{2.33}$$

$$S \triangleq \tilde{C} + V P_2 \tag{2.34}$$

The following theorem offers the theoretical basis for achieving the desired design goal.

Theorem 2.1. *Let* σ *be a sufficiently small positive constant,* Q_1 *be a positive-definite matrix, and let the matrices* Φ, $\tilde{A}$, $\hat{A}$, $\tilde{C}$, R *and* S *be defined as in* (2.27)-(2.34).

Let the controller gain matrix $K \in \mathbb{R}^{q \times n}$ *be chosen such that,*

$$A + BK \text{ is stable.} \tag{2.35}$$

Suppose there exist positive scalars ε_1, ε_2 *and* ε_3 *such that the following two quadratic matrix inequalities have positive-definite solutions* P_1 *and* P_2, *respectively,*

$$(A+BK)^T P_1 + P_1(A+BK) + P_1\left((\varepsilon_1+\varepsilon_3) H_A H_A^T + \gamma^{-2} B_1 B_1^T + \Phi\right) P_1$$
$$+ \varepsilon_1^{-1} E_A^T E_A + \varepsilon_3^{-1}(E_B K)^T E_B K + (NK)^T NK + Q_1 < 0 \tag{2.36}$$

$$\Upsilon \triangleq \left(\tilde{A} - V^T R^{-1} \tilde{C}\right)^T P_2 + P_2\left(\tilde{A} - V^T R^{-1}\tilde{C}\right) + P_2\left((\varepsilon_1+\varepsilon_3) H_A H_A^T + \gamma^{-2} B_1 B_1^T + \Phi - V^T R^{-1} V\right) P_2$$
$$- P_1 BK - (BK)^T P_1 - \varepsilon_3^{-1}(E_B K)^T E_B K + (M - NK)^T M + (NK)^T (M - NK)$$

$$-\tilde{C}^T R^{-1}\tilde{C}+\sigma I<0 \tag{2.37}$$

with $E_h Q_1^{-1} E_h^T < \varepsilon_2 I$. (2.38)

Under these conditions, if G *and* L *are the observer gain matrices which for some chosen orthogonal matrix* $U \in \mathbb{R}^{p\times p}$ $\left(UU^T = I\right)$, *satisfy:*

$$L = P_2^{-1}\left(S^T R^{-1} + E_o U R^{-1/2}\right) \tag{2.39}$$

$$G = \hat{A} - L\tilde{C} \tag{2.40}$$

where $E_o \in \mathbb{R}^{n\times p}$ *is an arbitrary matrix meeting the condition that,*

$$\Upsilon + E_o E_o^T < 0 \tag{2.41}$$

and Υ *is defined by* (2.37), *then the resulting output feedback system using* G, L *and* K *will be such that, for all admissible parameter uncertainties* ΔA, ΔA_h, ΔB, ΔC *and* ΔC_h, *and for all delay values* $h \in \mathbb{R}^+$:

(1) the augmented state-delayed system (2.20) *and* (2.21) *is asymptotically stable.*

(2) $\left\|T_{zw}(s)\right\|_\infty \le \gamma$.

Proof. By Assumption 2.1 and equation (2.33), the matrix B_2 is of full rank, so R^{-1} exists. By virtue of Lemma 2.1 and Lemma 2.2, and definitions (2.13)-(2.19), (2.28)-(2.34),

$$\begin{aligned}
&\left(\hat{A}+\Delta\hat{A}\right)^T P + P\left(\hat{A}+\Delta\hat{A}\right) + P\left(\hat{A}_h+\Delta\hat{A}_h\right)Q^{-1}\left(\hat{A}_h+\Delta\hat{A}_h\right)^T P \\
&\qquad \le \hat{A}^T P + P\hat{A} + \varepsilon_1 P\hat{H}_1\hat{H}_1^T P + \varepsilon_1^{-1}\hat{E}_1^T\hat{E}_1 + \varepsilon_3 P\hat{H}_2\hat{H}_2^T P + \varepsilon_3^{-1}\hat{E}_2^T\hat{E}_2 \\
&\qquad\qquad + P\left[\hat{A}_h\left(Q-\varepsilon_2^{-1}\hat{E}_h^T\hat{E}_h\right)^{-1}\hat{A}_h^T + \varepsilon_2\hat{H}_h\hat{H}_h^T\right]P
\end{aligned} \tag{2.42}$$

Adopting the following definitions,

$$P \triangleq \begin{bmatrix} P_1 & 0 \\ 0 & P_2 \end{bmatrix} > 0, \quad Q \triangleq \begin{bmatrix} Q_1 & 0 \\ 0 & \sigma I \end{bmatrix} > 0 \tag{2.43}$$

$$\Sigma \triangleq \begin{bmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{12}^T & \Sigma_{22} \end{bmatrix} = \hat{A}^T P + P\hat{A} + \varepsilon_1 P\hat{H}_1\hat{H}_1^T P + \varepsilon_1^{-1}\hat{E}_1^T\hat{E}_1 + \varepsilon_3 P\hat{H}_2\hat{H}_2^T P + \varepsilon_3^{-1}\hat{E}_2^T\hat{E}_2$$

$$+P\left[\hat{A}_h\left(Q-\varepsilon_2^{-1}\hat{E}_h^T\hat{E}_h\right)^{-1}\hat{A}_h^T+\varepsilon_2\hat{H}_h\hat{H}_h^T\right]P+Q+\hat{C}^T\hat{C}+\gamma^{-2}P\hat{B}\hat{B}^TP \tag{2.44}$$

the block matrices of Σ are easily calculated as:

$$\Sigma_{11}=(A+BK)^TP_1+P_1(A+BK)+P_1\left((\varepsilon_1+\varepsilon_3)H_AH_A^T+\gamma^{-2}B_1B_1^T+\Phi\right)P_1$$

$$+\varepsilon_1^{-1}E_A^TE_A+\varepsilon_3^{-1}(E_BK)^TE_BK+(NK)^TNK+Q_1 \tag{2.45}$$

$$\Sigma_{12}=(A-LC-G)^TP_2-P_1BK+\varepsilon_1P_1H_A(H_A-LH_C)^TP_2+\varepsilon_3P_1H_AH_A^TP_2$$

$$-\varepsilon_3^{-1}(E_BK)^TE_BK+P_1\Phi P_2-P_1A_h\Gamma^{-1}(LC_h)^TP_2-\varepsilon_2P_1H_A\left(LH_{C_h}\right)^TP_2$$

$$+(NK)^T(M-NK)+\gamma^{-2}P_1B_1(B_1-LB_2)^TP_2 \tag{2.46}$$

$$\Sigma_{22}=G^TP_2+P_2G+\varepsilon_1P_2(H_A-LH_C)(H_A-LH_C)^TP_2+\varepsilon_3P_2H_AH_A^TP_2+\varepsilon_3^{-1}(E_BK)^TE_BK$$

$$+P_2\Phi P_2-P_2LC_h\Gamma^{-1}A_h^TP_2-P_2A_h\Gamma^{-1}(LC_h)^TP_2+P_2LC_h\Gamma^{-1}(LC_h)^TP_2$$

$$-\varepsilon_2P_2LH_{C_h}H_A^TP_2-\varepsilon_2P_2H_A\left(LH_{C_h}\right)^TP_2+\varepsilon_2P_2LH_{C_h}\left(LH_{C_h}\right)^TP_2$$

$$+\gamma^{-2}P_2(B_1-LB_2)(B_1-LB_2)^TP_2+(M-NK)^T(M-NK)+\sigma I. \tag{2.47}$$

From (2.45) and (2.36), it then follows that $\Sigma_{11}<0$.

Using (2.47), and (2.40),

$$\Sigma_{22}=\left(\hat{A}-L\tilde{C}\right)^TP_2+P_2\left(\hat{A}-L\tilde{C}\right)+\varepsilon_1P_2(H_A-LH_C)(H_A-LH_C)^TP_2+\varepsilon_3P_2H_AH_A^TP_2$$

$$+P_2\Phi P_2-P_2LC_h\Gamma^{-1}A_h^TP_2-P_2A_h\Gamma^{-1}(LC_h)^TP_2+P_2LC_h\Gamma^{-1}(LC_h)^TP_2$$

$$-\varepsilon_2P_2LH_{C_h}H_A^TP_2-\varepsilon_2P_2H_A\left(LH_{C_h}\right)^TP_2+\varepsilon_2P_2LH_{C_h}\left(LH_{C_h}\right)^TP_2$$

$$+\gamma^{-2}P_2(B_1-LB_2)(B_1-LB_2)^TP_2+\varepsilon_3^{-1}(E_BK)^TE_BK+(M-NK)^T(M-NK)+\sigma I. \tag{2.48}$$

Re-grouping the terms yields

$$\Sigma_{22}=\hat{A}^TP_2+P_2\hat{A}+P_2\left((\varepsilon_1+\varepsilon_3)H_AH_A^T+\gamma^{-2}B_1B_1^T+\Phi\right)P_2+\varepsilon_3^{-1}(E_BK)^TE_BK+\sigma I$$

$$+(M-NK)^T(M-NK)-P_2L\left(\tilde{C}+\varepsilon_1H_CH_A^TP_2+\varepsilon_2H_{C_h}H_A^TP_2+\gamma^{-2}B_2B_1^TP_2+C_h\Gamma^{-1}A_h^TP_2\right)$$

$$-\left(\tilde{C}+\varepsilon_1H_CH_A^TP_2+\varepsilon_2H_{C_h}H_A^TP_2+\gamma^{-2}B_2B_1^TP_2+C_h\Gamma^{-1}A_h^TP_2\right)^T(P_2L)^T$$

$$+P_2L\left(\varepsilon_1 H_C H_C^T+\varepsilon_2 H_{C_h} H_{C_h}^T+\gamma^{-2} B_2 B_2^T+C_h \Gamma^{-1} C_h^T\right)\left(P_2 L\right)^T .$$

Substituting the definitions of R and S, (2.33) and (2.34),

$$\Sigma_{22}=\hat{A}^T P_2+P_2 \hat{A}+P_2\left(\left(\varepsilon_1+\varepsilon_3\right) H_A H_A^T+\gamma^{-2} B_1 B_1^T+\Phi\right) P_2+\varepsilon_3^{-1}\left(E_B K\right)^T E_B K+\sigma I$$

$$+(M-N K)^T(M-N K)-S^T R^{-1} S+\left(P_2 L R^{1/2}-S^T R^{-1/2}\right)\left(P_2 L R^{1/2}-S^T R^{-1/2}\right)^T$$

By (2.39), $\left(P_2 L R^{1/2}-S^T R^{-1/2}\right)\left(P_2 L R^{1/2}-S^T R^{-1/2}\right)^T=E_o E_o^T$, so that

$$\Sigma_{22}=\hat{A}^T P_2+P_2 \hat{A}+P_2\left(\left(\varepsilon_1+\varepsilon_3\right) H_A H_A^T+\gamma^{-2} B_1 B_1^T+\Phi\right) P_2$$

$$+\varepsilon_3^{-1}\left(E_B K\right)^T E_B K+(M-N K)^T(M-N K)+\sigma I-S^T R^{-1} S+E_o E_o^T$$

Finally, using (2.30), (2.34) and Lemma 2.3, gives

$$\Sigma_{22}=\left(\tilde{A}-V^T R^{-1} \tilde{C}\right)^T P_2+P_2\left(\tilde{A}-V^T R^{-1} \tilde{C}\right)+P_2\left(\left(\varepsilon_1+\varepsilon_3\right) H_A H_A^T+\gamma^{-2} B_1 B_1^T+\Phi-V^T R^{-1} V\right) P_2$$

$$-P_1 B K-(B K)^T P_1-\varepsilon_3^{-1}\left(E_B K\right)^T E_B K+(M-N K)^T M+(N K)^T(M-N K)+\sigma I$$

$$-\tilde{C}^T R^{-1} \tilde{C}$$

which, by virtue of (2.37), clearly implies that $\Sigma_{22}<0$.

Moreover, substituting (2.40) into (2.46) we get:

$$\Sigma_{12}=\left(A-\hat{A}-L(C-\tilde{C})\right)^T P_2-P_1 B K+\varepsilon_1 P_1 H_A\left(H_A-L H_C\right)^T P_2+\varepsilon_3 P_1 H_A H_A^T P_2$$

$$-\varepsilon_3^{-1}\left(E_B K\right)^T E_B K+P_1 \Phi P_2-P_1 A_h \Gamma^{-1}\left(L C_h\right)^T P_2-\varepsilon_2 P_1 H_A\left(L H_{C_h}\right)^T P_2$$

$$+(N K)^T(M-N K)+\gamma^{-2} P_1 B_1\left(B_1-L B_2\right)^T P_2$$

Again, re-grouping the terms and using (2.29)-(2.32), yields $\Sigma_{12}=0$. Hence, $\Sigma=\begin{bmatrix}\Sigma_{11} & 0 \\ 0 & \Sigma_{22}\end{bmatrix}<0$, as $\Sigma_{11}<0$ and $\Sigma_{22}<0$, as demonstrated above. By virtue of Lemma 2.4 and (2.44), system (2.20) and (2.21) is therefore robustly asymptotically stable and $\left\|H_{zw}(s)\right\|_{\infty} \leq \gamma$, for all values of time-delay $h \in \mathbb{R}^{+}$ and for all uncertainties (2.5). QED

Remark 2.2. The result of Theorem 2.1 may be conservative because of inequalities in Lemma 2.1, Lemma 2.2 and Lemma 2.4. However, the conservatism can be significantly reduced by the proper selection of ε_1, ε_2 and ε_3 in a matrix norm sense. The relevant discussion and corresponding optimization algorithm can be found in [89] and references therein.

Remark 2.3. In the case where $\sigma > 0$, $Q_1 > 0$, $\varepsilon_1 > 0$, $\varepsilon_2 > 0$ and $\varepsilon_3 > 0$ are fixed, the two quadratic matrix inequalities (2.36) and (2.37) can further be converted into two linear matrix inequalities using the well-known results on Shur complements of partitioned symmetric matrices [6]. In this way, the computational complexity of the design would be further simplified.

Remark 2.4. The presented robust H_∞ output feedback control design procedure still offers much additional design freedom. This freedom is reflected by the arbitrary choice of the positive-definite matrix $Q_1 > 0$, the free gain parameter $E_o \in \mathbb{R}^{n\times p}$ $\left(\Upsilon + E_o E_o^T < 0\right)$, and the orthogonal matrix $U \in \mathbb{R}^{p\times p}$. Introducing additional performance constraints into the problem formulation *(1)* and *(2)* of Theorem 2.1, which would exploit this design freedom is currently under investigation.

2.5.2. Existence of a Positive Definite Solution

In this section, we discuss the existence of a positive-definite solution to inequalities (2.36) and (2.37). Starting with inequality (2.36), it is obvious that we can re-write it as an equality by adding a positive definite matrix δ_1 to obtain,

$$(A+BK)^T P_1 + P_1(A+BK) + P_1\left((\varepsilon_1+\varepsilon_3)H_A H_A^T + \gamma^{-2}B_1B_1^T + \Phi\right)P_1$$

$$+\varepsilon_1^{-1}E_A^T E_A + \varepsilon_3^{-1}(E_B K)^T E_B K + (NK)^T NK + Q_1 + \delta_1 = 0 \qquad (2.49)$$

Let Θ be defined as,

$$\Theta \triangleq \left((\varepsilon_1+\varepsilon_3)H_A H_A^T + \gamma^{-2}B_1B_1^T + \Phi\right) \qquad (2.50)$$

From (2.27) and (2.28), it is easy to see (Appendix A) that Θ can only verify $\Theta \geq 0$ since Γ must be positive by assumption (2.38) in Theorem 2.1. In this case, since $A+BK$ is stable as required by Theorem 2.1, we then have the following result which can be found in **[38]**.

Corollary 2.1. *For $A+BK$ stable and $\Theta \geq 0$ for some positive scalars ε_1 and ε_3, and a positive definite matrix Q_1, then there exists a positive-definite solution P_1 to the inequality* (2.36) *if and only if*

$$\left\| \left(\varepsilon_1^{-1} E_A^T E_A + \varepsilon_3^{-1} (E_B K)^T E_B K + (NK)^T NK + Q_1\right)^{1/2} (sI - A - BK)\Theta^{1/2} \right\|_\infty < 1. \tag{2.51}$$

Finally, if $\Theta > 0$, we give the following necessary conditions for the existence of a positive definite solution to such a parameter-dependent Riccati equation.

Corollary 2.2. *Consider the algebraic matrix equation* (2.49) *with $\Theta > 0$. If there exists a positive definite solution $P_1 > 0$ to equation* (2.49), *then*

$$(A+BK)^T \Theta^{-1} (A+BK) - \left(\varepsilon_1^{-1} E_A^T E_A + \varepsilon_3^{-1} (E_B K)^T E_B K + (NK)^T NK + Q_1 + \delta_1\right) \geq 0. \tag{2.52}$$

Proof. Equation (2.49) can be rearranged as

$$\left[P_1 \Theta^{1/2} + (A+BK)^T \Theta^{-1/2}\right]\left[P_1 \Theta^{1/2} + (A+BK)^T \Theta^{-1/2}\right]^T$$

$$= (A+BK)^T \Theta^{-1} (A+BK) - \left(\varepsilon_1^{-1} E_A^T E_A + \varepsilon_3^{-1} (E_B K)^T E_B K + (NK)^T NK + Q_1 + \delta_1\right). \tag{2.53}$$

Clearly, the right-hand side of equation (2.53) must be non-negative. QED

Remark 2.5. Corollary 2.1 and Corollary 2.2, and Remark 2.2 provide some guides for the selection of proper design parameters ε_1, ε_3 and Q_1 in order to ensure the effectiveness of the proposed design procedure.

A similar analysis could be done for inequality (2.37).

2.5.3. Numerical Example

Example 2.1. In terms of an example, consider the linear continuous uncertain time-delay system (2.1) and (2.2) with system matrices given by

$$A = \begin{bmatrix} -2 & -0.5 \\ 0.5 & -3 \end{bmatrix}, \quad A_h = \begin{bmatrix} 0.5 & 0 \\ 0 & 0.5 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 \end{bmatrix}, \quad C_h = \begin{bmatrix} 1 & 1 \end{bmatrix},$$

$$B_1 = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.1 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0.5 & 0.8 \end{bmatrix}, \quad M = \begin{bmatrix} 0.1 & 0.1 \end{bmatrix}, \quad N = 0.1,$$

$$H_A = \begin{bmatrix} 0.1 & 0.05 \\ -0.02 & 0.1 \end{bmatrix}, \qquad H_C = \begin{bmatrix} -0.2 & 0.8 \end{bmatrix}, \qquad H_{C_h} = \begin{bmatrix} 0.3 & -0.5 \end{bmatrix},$$

$$E_A = \begin{bmatrix} 0.8 & 0 \\ 0 & 0.8 \end{bmatrix}, \qquad E_h = \begin{bmatrix} 0.1 & 0.01 \\ 0.02 & 0.5 \end{bmatrix}, \qquad E_B = \begin{bmatrix} 0.1 \\ 0.1 \end{bmatrix}, \qquad E_{C_h} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

In what follows, the H_∞ output feedback design procedure will aim at the satisfaction of $\|T_{zw}(s)\|_\infty \le \gamma$ with $\gamma = 0.8$.

To satisfy the constraint $E_h Q_1^{-1} E_h^T < \varepsilon_2 I$, we choose $\varepsilon_2 = 0.5$ and $Q_1 = I$. Also, we select $\varepsilon_1 = 0.1$ to yield from (2.33), $R = 4.74$. Furthermore, we choose $\varepsilon_3 = 0.1$ and $\sigma = 1$, and the controller gain K is selected (with $A + BK$ stable) as $K = \begin{bmatrix} -0.35 & 0.35 \end{bmatrix}$, to yield a positive-definite solution to (2.36),

$$P_1 = \begin{bmatrix} 3 & -1 \\ -1 & 5 \end{bmatrix}$$

The matrix $\tilde{C}$ is then computed from (2.32) as, $\tilde{C} = \begin{bmatrix} 1.72 & 5.03 \end{bmatrix}$. Finally, the matrix $\tilde{A}$ in (2.29) and the positive-definite solution to (2.37), are further obtained as,

$$\tilde{A} = \begin{bmatrix} -1.17 & -0.71 \\ 0.02 & -0.40 \end{bmatrix}, \qquad P_2 = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}$$

which by (2.30) and (2.34), results in

$$\hat{A} = \begin{bmatrix} -1.18 & -0.71 \\ -0.67 & -1.81 \end{bmatrix}, \qquad S = \begin{bmatrix} 1.82 & 5.54 \end{bmatrix}$$

The matrix E_o which needs to satisfy $\Upsilon + E_o E_o^T < 0$, is selected as,

$$E_o = \begin{bmatrix} -0.7 \\ 0.2 \end{bmatrix}$$

Finally, since the dimension of the measured output is $p = 1$, the arbitrary orthogonal matrix U can be either 1 or -1. Corresponding to these two cases, the desired observer gains L_1 and G_1 (for $U = 1$), and L_2 and G_2 (for $U = -1$) can be obtained from (2.39) and (2.40), respectively:

$$L_1 = \begin{bmatrix} 1.32 \\ 2.58 \end{bmatrix}, \qquad G_1 = \begin{bmatrix} -3.46 & -7.36 \\ -5.11 & -14.81 \end{bmatrix} \tag{2.54}$$

$$L_2 = \begin{bmatrix} 1.78 \\ 2.86 \end{bmatrix}, \qquad G_2 = \begin{bmatrix} -4.25 & -9.67 \\ -5.58 & -16.19 \end{bmatrix} \tag{2.55}$$

Using the numerical values in this example and taking $s = jW$ in (2.22), the expression of the transfer function which is in this case a vector of two entries, is obtained. The norms of both entries of the transfer function are then plotted in Figure 2.1 for the values (2.54) of the gains. The delay is chosen as $h = 1$.

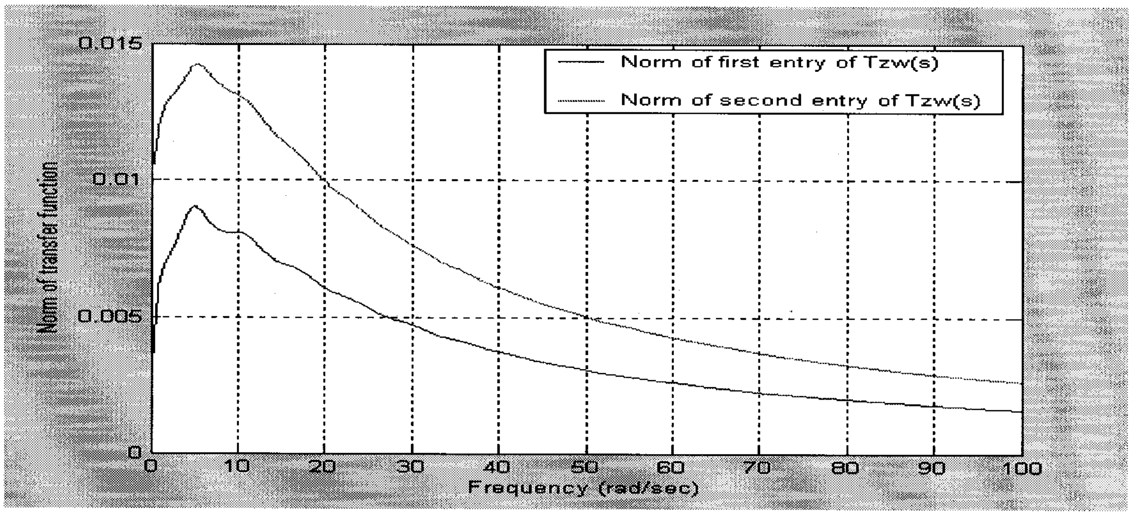

Figure 2.1. *Norm of the transfer function of the closed loop system vs the frequency for gain values* L_1 *and* G_1

In Figure 2.2, the plot is repeated for the values (2.55) of the gains.

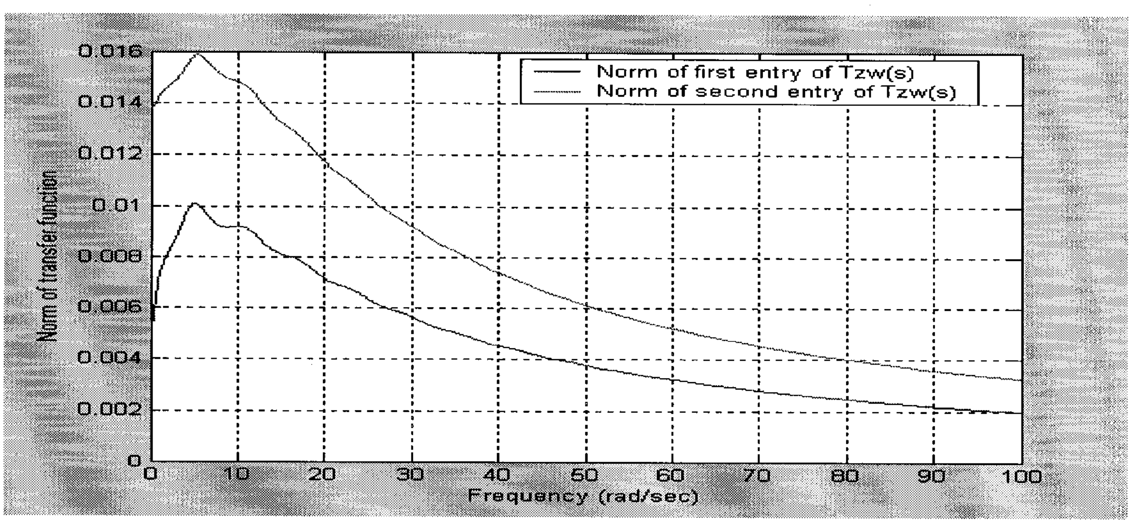

Figure 2.2. *Norm of the transfer function of the closed loop system vs the frequency for gain values* L_2 *and* G_2

We see that the condition $\|T_{zw}(s)\|_\infty \le \gamma$ is fulfilled as in this example $\gamma = 0.8$. Also the maximum amplitude of the norm will not be affected by varying the delay h. The plots above will only exhibit more oscillations, as the delay is found in the exponential term in the transfer function.

Also by imposing some other performance restrictions (rise time, settling time, etc.), the condition $\|T_{zw}(s)\|_\infty \le \gamma$ can be met more tightly.

2.5.4. Asymptotic Stability for the Time-Varying Case

Consider the following linear continuous uncertain time-varying counterpart of system (2.1)-(2.5):

$$\dot{x}(t) = [A(t) + \Delta A(t)]x(t) + [A_h(t) + \Delta A_h(t)]x(t-h) + B_1(t)w(t) + [B(t) + \Delta B(t)]u(t) \tag{2.56}$$

$$y(t) = [C(t) + \Delta C(t)]x(t) + [C_h(t) + \Delta C_h(t)]x(t-h) + B_2(t)w(t) \tag{2.57}$$

$$z(t) = M(t)e(t) + N(t)u(t) \tag{2.58}$$

$$\begin{bmatrix} \Delta A(t) \\ \Delta C(t) \end{bmatrix} = \begin{bmatrix} H_A(t) \\ H_C(t) \end{bmatrix} F(t) E_A(t), \quad \begin{bmatrix} \Delta A_h(t) \\ \Delta B(t) \end{bmatrix} = H_A(t) F(t) \begin{bmatrix} E_h(t) \\ E_B(t) \end{bmatrix}, \quad \Delta C_h(t) = H_{C_h}(t) F(t) E_{C_h}(t) \tag{2.59}$$

where $F(t) \in \mathbb{R}^{i \times j}$ is a real uncertain time-varying matrix with Lebesgue measurable elements which meets the requirement that $F(t)F^T(t) \le I$.

The full-order linear time-varying and delay independent state observer with the associated feedback controller proposed in this chapter are assumed to take the following form:

$$\dot{\hat{x}}(t) = G(t)\hat{x}(t) + L(t)y(t) + B(t)u(t) \tag{2.60}$$

$$u(t) = K(t)\hat{x}(t) \tag{2.61}$$

where $G(t)$ and $L(t)$ denote the time-varying observer gains, $K(t)$ denotes the time-varying controller gain, and all three gains are design parameters.

Using the same definitions for $e(t)$ and $\xi(t)$ as in (2.8) and (2.13) respectively, the following augmented system is easily obtained:

$$\dot{\xi}(t) = [\hat{A}(t) + \Delta\hat{A}(t)]\xi(t) + [\hat{A}_h(t) + \Delta\hat{A}_h(t)]\xi(t-h) + \hat{B}(t)w(t) \tag{2.62}$$

$$z(t) = \hat{C}(t)\xi(t) \tag{2.63}$$

where $\hat{A}(t)$, $\Delta\hat{A}(t)$, $\hat{A}_h(t)$, $\Delta\hat{A}_h(t)$, $\hat{C}(t)$ and $\hat{B}(t)$ are defined as in (2.14)-(2.19), with all quantities being time-varying.

In the above context, the objective is to design for the parameter values: $G(t)$, $L(t)$ and $K(t)$, in such a way that for all admissible parameters uncertainties $\Delta A(t)$, $\Delta A_h(t)$, $\Delta B(t)$, $\Delta C(t)$ and $\Delta C_h(t)$, and for all time-delay values $h \in \mathbb{R}^+$, the augmented system (2.62)-(2.63) is asymptotically stable.

Theorem 2.2. [87] *Given a constant positive definite matrix $Q > 0$. If the following differential Riccati inequality*

$$\frac{d}{dt}P(t)+\left[\hat{A}(t)+\Delta\hat{A}(t)\right]^T P(t)+P(t)\left[\hat{A}(t)+\Delta\hat{A}(t)\right]$$

$$+P(t)\left[\hat{A}_h(t)+\Delta\hat{A}_h(t)\right]Q^{-1}\left[\hat{A}_h(t)+\Delta\hat{A}_h(t)\right]^T P+Q<0 \tag{2.64}$$

has positive definite solution $P(t)$ for all admissible uncertainties, then the system (2.62)-(2.63) *is asymptotically stable.*

For the sake of simplicity, we introduce the following definitions:

$$\Gamma(t) \triangleq \left[Q_1 - \varepsilon_2^{-1}E_h^T(t)E_h(t)\right] \tag{2.65}$$

$$\Phi(t) \triangleq A_h(t)\Gamma^{-1}(t)A_h^T(t)+\varepsilon_2 H_A(t)H_A^T(t) \tag{2.66}$$

$$\tilde{A}(t) \triangleq A(t)+(\varepsilon_1+\varepsilon_3)H_A(t)H_A^T(t)P_1(t)+\Phi(t)P_1(t) \tag{2.67}$$

$$\hat{A}(t) \triangleq \tilde{A}(t)-P_2^{-1}(t)\left[B(t)K(t)\right]^T P_1(t)-\varepsilon_3^{-1}P_2^{-1}(t)\left[E_B(t)K(t)\right]^T E_B(t)K(t) \tag{2.68}$$

$$V(t) \triangleq \varepsilon_1 H_C(t)H_A^T(t)+\varepsilon_2 H_{C_h}(t)H_A^T(t)+C_h(t)\Gamma^{-1}(t)A_h^T(t) \tag{2.69}$$

$$\tilde{C}(t) \triangleq C(t)+V(t)P_1(t) \tag{2.70}$$

$$R(t) \triangleq \varepsilon_1 H_C(t)H_C^T(t)+\varepsilon_2 H_{C_h}(t)H_{C_h}^T(t)+C_h(t)\Gamma^{-1}(t)C_h^T(t) \tag{2.71}$$

$$S(t) \triangleq \tilde{C}(t)+V(t)P_2(t) \tag{2.72}$$

Following the same line of the proof of Theorem 2.1, we obtain the following result for the robust output feedback stabilization problem in the time-varying case..

Theorem 2.3. *Let σ be a sufficiently small positive constant, Q_1 be a positive-definite matrix, and let the matrices $\Phi(t)$, $\tilde{A}(t)$, $\hat{A}(t)$, $\tilde{C}(t)$, $R(t)$ and $S(t)$ be defined as in* (2.65)-(2.72).

Let the controller gain matrix $K(t) \in \mathbb{R}^{q \times n}$ be chosen such that,

$$A(t)+B(t)K(t) \text{ is stable.} \tag{2.73}$$

Suppose there exist positive scalars ε_1, ε_2 and ε_3 such that the following two differential quadratic matrix inequalities have positive-definite solutions $P_1(t)$ and $P_2(t)$, respectively,

$$\begin{aligned}&\frac{d}{dt}P_1(t)+\left[A(t)+B(t)K(t)\right]^T P_1(t)+P_1(t)\left[A(t)+B(t)K(t)\right]\\&\qquad+P_1(t)\left[(\varepsilon_1+\varepsilon_3)H_A(t)H_A^T(t)+\Phi(t)\right]P_1(t)\\&\qquad+\varepsilon_1^{-1}E_A^T(t)E_A(t)+\varepsilon_3^{-1}\left[E_B(t)K(t)\right]^T E_B(t)K(t)+Q_1<0\end{aligned} \tag{2.74}$$

$$\begin{aligned}\Upsilon(t)\triangleq&\frac{d}{dt}P_2(t)+\left[\tilde{A}(t)-V^T(t)R^{-1}(t)\tilde{C}(t)\right]^T P_2(t)+P_2(t)\left[\tilde{A}(t)-V^T(t)R^{-1}(t)\tilde{C}(t)\right]\\&\qquad+P_2(t)\left[(\varepsilon_1+\varepsilon_3)H_A(t)H_A^T(t)+\Phi(t)-V^T(t)R^{-1}(t)V(t)\right]P_2(t)\\&\qquad-P_1(t)B(t)K(t)-\left[B(t)K(t)\right]^T P_1(t)-\varepsilon_3^{-1}\left[E_B(t)K(t)\right]^T E_B(t)K(t)\\&\qquad-\tilde{C}^T(t)R^{-1}(t)\tilde{C}(t)+\sigma I<0\end{aligned} \tag{2.75}$$

with $E_h(t)Q_1^{-1}E_h^T(t)<\varepsilon_2 I$. (2.76)

Under these conditions, if $G(t)$ and $L(t)$ are the observer gain matrices which for some chosen orthogonal matrix $U \in \mathbb{R}^{p\times p}$ $\left(UU^T=I\right)$, satisfy:

$$L(t)=P_2^{-1}(t)\left(S^T(t)R^{-1}(t)+E_o U R^{-1/2}(t)\right) \tag{2.77}$$

$$G(t)=\hat{A}(t)-L(t)\tilde{C}(t) \tag{2.78}$$

where $E_o \in \mathbb{R}^{n\times p}$ is an arbitrary matrix meeting the condition that,

$$\Upsilon(t)+E_o E_o^T<0 \tag{2.79}$$

and $\Upsilon(t)$ *is defined by* (2.75), *then the resulting output feedback system using* $G(t)$, $L(t)$ *and* $K(t)$ *will be such that, for all admissible parameter uncertainties* $\Delta A(t)$, $\Delta A_h(t)$, $\Delta B(t)$, $\Delta C(t)$ *and* $\Delta C_h(t)$, *and for all time-delay values* $h \in \mathbb{R}^+$, *the augmented state-delayed system* (2.62)-(2.63) *is asymptotically stable.*

CHAPTER 3

Discrete-Time Delay-Independent Robust H_∞ Output Feedback Control of Uncertain Retarded Systems

3.1. System Description

Consider the retarded version of system (1.1)-(1.5) taken in the infinite-horizon, time-invariant and discrete-time context, where the system matrices are all time-invariant. The uncertainty matrices F_i are assumed to be equal (assumption frequently made in the literature and is not restrictive in practice) and also time-invariant. The constant time-delay h is an integer, and is assumed to be known exactly (see Remark 3.1 below). More specifically, the system under consideration is:

$$x(k+1) = \bar{A}x(k) + \bar{A}_h x(k-h) + B_1 w(k) + Bu(k) \tag{3.1}$$

$$y(k) = \bar{C}x(k) + B_2 w(k) \tag{3.2}$$

$$x(k) = \phi(k), \qquad k \in [-h, 0] \tag{3.3}$$

$$\bar{A} = A + \Delta A\,,\ \bar{A}_h = A_h + \Delta A_h\,,\ \bar{C} = C + \Delta C \tag{3.4}$$

$$\begin{bmatrix} \Delta A \\ \Delta C \end{bmatrix} = \begin{bmatrix} H_A \\ H_C \end{bmatrix} F E_A\,, \qquad \Delta A_h = H_A F E_h \tag{3.5}$$

3.2. Assumptions

The following assumption is needed for the subsequent development:

Assumption 3.1. The matrix B_2 or H_C is of full rank.

3.3. Full Order Observer

In the discrete-time case, the full order linear state observer, as proposed in **[85]**, is of the form:

$$\hat{x}(k+1) = G\hat{x}(k) + A_h\hat{x}(k-h) + Ly(k) + Bu(k) \tag{3.6}$$

Remark 3.1. Although the design presented in this chapter is delay-independent, knowledge of the value of the delay is still needed because as seen in (3.6), the observer chosen for our design depends on the delay h, for the state estimate to be constructed. In case the value of the delay is not available, then the observer (3.6) can be chosen without delay in the state estimate $\hat{x}(t)$, but at the expense of a less accurate design.

The controller to be designed will be assumed linear, delay-free and of the form:

$$u(k) = K\hat{x}(k) \tag{3.7}$$

where G and L are the observer gains and K is the controller gain to be determined.

Defining the state error $e(k) \triangleq x(k) - \hat{x}(k)$, it then follows from (3.1), (3.2), (3.6) and (3.7) that

$$e(k+1) = Ge(k) + (A + \Delta A - L(C + \Delta C) - G)x(k) + A_h e(k-h) + \Delta A_h x(k-h) + (B_1 - LB_2)w(k) \tag{3.8}$$

Let $z(k)$ then be the state-error output, which is assumed to be given by:

$$z(k) = Me(k) \tag{3.9}$$

where $M \in \mathbb{R}^{m \times n}$ is a given constant matrix. Defining

$$\xi(k) \triangleq \begin{bmatrix} x(k) \\ e(k) \end{bmatrix}, \tag{3.10}$$

$$\hat{A} \triangleq \begin{bmatrix} A + BK & -BK \\ A - LC - G & G \end{bmatrix}, \hat{A}_h \triangleq \begin{bmatrix} A_h & 0 \\ 0 & A_h \end{bmatrix} \tag{3.11}$$

$$\hat{H} \triangleq \begin{bmatrix} H_A \\ H_A - LH_C \end{bmatrix}, \qquad \hat{E} \triangleq [E_A \quad 0] \tag{3.12}$$

$$\hat{H}_h \triangleq \begin{bmatrix} H_A \\ H_A \end{bmatrix}, \qquad \hat{E}_h \triangleq [E_h \quad 0] \tag{3.13}$$

$$\Delta\hat{A} \triangleq \hat{H}F\hat{E}, \qquad \Delta\hat{A}_h \triangleq \hat{H}_h F\hat{E}_h \tag{3.14}$$

$$\hat{B} \triangleq \begin{bmatrix} B_1 \\ B_1 - LB_2 \end{bmatrix}, \qquad \hat{C} \triangleq [0 \quad M] \tag{3.15}$$

and combining (3.1), (3.2), (3.5), (3.7) and (3.8), the following augmented system is easily obtained:

$$\xi(k+1)=\left(\hat{A}+\Delta\hat{A}\right)\xi(k)+\left(\hat{A}_h+\Delta\hat{A}_h\right)\xi(k-h)+\hat{B}w(k) \tag{3.16}$$

$$z(k)=\hat{C}\xi(k) \tag{3.17}$$

The transfer function from the disturbance $w(k)$ to the state-error output $z(k)$ is thus given by:

$$T_{zw}(z)=\hat{C}\left[zI-\left(\hat{A}+\Delta\hat{A}\right)-\left(\hat{A}_h+\Delta\hat{A}_h\right)z^{-h}\right]^{-1}\hat{B} \tag{3.18}$$

3.4. Problem Statement

The objective is to design the parameters G, L and K, such that for all admissible parameter uncertainties ΔA, ΔA_h and ΔC, the augmented system (3.16) and (3.17) is asymptotically stable and the following upper-bound constraint on the H_∞ norm of $T_{zw}(z)$ is simultaneously guaranteed:

$$\left\|T_{zw}(z)\right\|_\infty \le \gamma, \tag{3.19}$$

for all positive integer time-delay values $h\in\mathbb{N}^+$, and all uncertainties (3.5), where $\left\|T_{zw}(z)\right\|_\infty = \max_{\theta\in[0,2\pi]}\sigma_{\max}\left[T_{zw}\left(e^{j\theta}\right)\right]$ and $\sigma_{\max}[T]$ denotes the largest singular value of T, and $\gamma<1$ is a given positive constant.

3.5. Main Result

The following lemma will play a key role in designing the robust H_∞ output feedback controller for the uncertain linear discrete-time state delayed system (3.1) and (3.2).

Lemma 3.1. *If there exist a positive-definite matrix P and positive scalars $\varepsilon_1>0$, $\varepsilon_2>0$ and $\varepsilon_3>0$ such that the following inequalities hold*

$$P^{-1}-\gamma^{-2}\hat{C}^T\hat{C}-\varepsilon_1\hat{A}_h^T\hat{A}_h-\varepsilon_2\hat{E}^T\hat{E}-\varepsilon_3\hat{E}_h^T\hat{E}_h>0 \tag{3.20}$$

$$\hat{A}\left(P^{-1}-\gamma^{-2}\hat{C}^T\hat{C}-\varepsilon_1\hat{A}_h^T\hat{A}_h-\varepsilon_2\hat{E}^T\hat{E}-\varepsilon_3\hat{E}_h^T\hat{E}_h\right)^{-1}\hat{A}^T-P+\hat{B}\hat{B}^T+\varepsilon_1^{-1}I+\varepsilon_2^{-1}\hat{H}\hat{H}^T+\varepsilon_3^{-1}\hat{H}_h\hat{H}_h^T<0 \tag{3.21}$$

then the augmented system (3.16) *and* (3.17) *is asymptotically stable and meets the specified* H_∞ *norm upper-bound constraint* $\|T_{zw}(z)\|_\infty \le \gamma$, *independent of the positive integer state time-delay* h.

Proof. It is easy to see that the system (3.16), (3.17) is asymptotically stable if and only if the following auxiliary system is asymptotically stable:

$$y(k+1)=\left(\hat{A}+\Delta\hat{A}\right)^T y(k)+\left(\hat{A}_h+\Delta\hat{A}_h\right)y(k-h)+\hat{C}^T w_1(k) \tag{3.22}$$

$$z_1(k)=\hat{B}^T y(k) \tag{3.23}$$

where the state $y(k)\in\mathbb{R}^{2n}$, the disturbance input $w_1(k)\in\mathbb{R}^m$, the system output $z_1(k)\in\mathbb{R}^r$, and the transfer functions of the systems (3.16)-(3.17) and (3.22)-(3.23) have the same H_∞-norm values. Then, based on the auxiliary system (3.22)-(3.23), the proof of this lemma is completely similar to that of Theorem 2 in **[69]** and is thus omitted. QED

For the sake of simplicity, the following definitions are introduced prior to stating the main results of the chapter:

$$\Phi_1 \triangleq \left(P_1^{-1}-\varepsilon_1 A_h^T A_h-\varepsilon_2 E_A^T E_A-\varepsilon_3 E_h^T E_h\right)^{-1} \tag{3.24}$$

$$\Phi_2 \triangleq \left(P_2^{-1}-\gamma^{-2}M^T M-\varepsilon_1 A_h^T A_h\right)^{-1} \tag{3.25}$$

$$\Gamma_1 \triangleq -A\Phi_2\left(\Phi_1+\Phi_2\right)^{-1}+\left(A\left(\Phi_1^{-1}+\Phi_2^{-1}\right)^{-1}A^T+B_1B_1^T+\left(\varepsilon_2^{-1}+\varepsilon_3^{-1}\right)H_AH_A^T\right)\left(E_cU_c\right)^{-T}\left(\Phi_1+\Phi_2\right)^{-1/2} \tag{3.26}$$

where $E_c\in\mathbb{R}^{n\times n}$ is an invertible matrix and $U_c\in\mathbb{R}^{n\times n}$ is an arbitrary chosen orthogonal matrix $\left(U_cU_c^T=I\right)$.

$$\tilde{A}\triangleq A+\Gamma_1 \tag{3.27}$$

$$\Gamma_2 \triangleq -C\Phi_2\left(\Phi_1+\Phi_2\right)^{-1}+\left(C\left(\Phi_1^{-1}+\Phi_2^{-1}\right)^{-1}A^T+B_2B_1^T+\varepsilon_2^{-1}H_CH_A^T\right)\left(E_cU_c\right)^{-T}\left(\Phi_1+\Phi_2\right)^{-1/2} \tag{3.28}$$

$$\tilde{C}\triangleq C+\Gamma_2 \tag{3.29}$$

$$\Theta_1 \triangleq \varepsilon_1 A_h^T A_h+\varepsilon_2 E_A^T E_A+\varepsilon_3 E_h^T E_h,\qquad \Theta_2 \triangleq \gamma^{-2}M^T M+\varepsilon_1 A_h^T A_h \tag{3.30}$$

$$R_c \triangleq \Phi_1 + \Phi_2, \qquad S_c \triangleq -\Phi_1 A^T \tag{3.31}$$

$$R_o \triangleq \Gamma_2 \Phi_1 \Gamma_2^T + \tilde{C}\Phi_2 \tilde{C}^T + B_2 B_2^T + \varepsilon_2^{-1} H_C H_C^T \tag{3.32}$$

$$S_o \triangleq \Gamma_2 \Phi_1 \Gamma_1^T + \tilde{C}\Phi_2 \tilde{A}^T + B_2 B_1^T + \varepsilon_2^{-1} H_C H_A^T \tag{3.33}$$

$$\Omega_1 \triangleq AP_1A^T + AP_1\Theta_1^{1/2}\left(I - \Theta_1^{1/2}P_1\Theta_1^{1/2}\right)^{-1}\Theta_1^{1/2}P_1A^T - P_1 + B_1B_1^T + \left(\varepsilon_2^{-1} + \varepsilon_3^{-1}\right)H_AH_A^T - S_c^TR_c^{-1}S_c + \varepsilon_1^{-1}I \tag{3.34}$$

$$\Omega_2 \triangleq \tilde{A}P_2\tilde{A}^T + \tilde{A}P_2\Theta_2^{1/2}\left(I - \Theta_1^{1/2}P_2\Theta_2^{1/2}\right)^{-1}\Theta_2^{1/2}P_2\tilde{A}^T - P_2 + \Gamma_1\Phi_1\Gamma_1^T + B_1B_1^T + \left(\varepsilon_2^{-1} + \varepsilon_3^{-1}\right)H_AH_A^T$$
$$-S_o^TR_o^{-1}S_o + \varepsilon_1^{-1}I \tag{3.35}$$

The following theorem provides the theoretical basis for achieving the desired design goal.

Theorem 3.1. *Let the matrices* Φ_1, $\tilde{A}$, Θ_1, Θ_2, Γ_1, S_c, R_c, S_o *and* R_o *be defined as in* (3.24)-(3.33). *Suppose there exist positive scalars* ε_1, ε_2 *and* ε_3, *an invertible matrix* $E_c \in \mathbb{R}^{n\times n}$, *and a matrix* $E_o \in \mathbb{R}^{n\times p}$ *such that the following Riccati-type matrix inequalities*

$$AP_1A^T - P_1 + AP_1\Theta_1^{1/2}\left(I - \Theta_1^{1/2}P_1\Theta_1^{1/2}\right)^{-1}\Theta_1^{1/2}P_1A^T + B_1B_1^T + \left(\varepsilon_2^{-1} + \varepsilon_3^{-1}\right)H_AH_A^T - S_c^TR_c^{-1}S_c$$
$$+E_cE_c^T + \varepsilon_1^{-1}I < 0 \tag{3.36}$$

$$\tilde{A}P_2\tilde{A}^T - P_2 + \tilde{A}P_2\Theta_2^{1/2}\left(I - \Theta_1^{1/2}P_2\Theta_2^{1/2}\right)^{-1}\Theta_2^{1/2}P_2\tilde{A}^T + \Gamma_1\Phi_1\Gamma_1^T + B_1B_1^T + \left(\varepsilon_2^{-1} + \varepsilon_3^{-1}\right)H_AH_A^T - S_o^TR_o^{-1}S_o$$
$$+E_oE_o^T + \varepsilon_1^{-1}I < 0 \tag{3.37}$$

along side with the corresponding matrix inequality constraints

$$P_1^{-1} - \varepsilon_1 A_h^T A_h - \varepsilon_2 E_A^T E_A - \varepsilon_3 E_h^T E_h > 0 \tag{3.38}$$

$$P_2^{-1} - \gamma^{-2} M^T M - \varepsilon_1 A_h^T A_h > 0 \tag{3.39}$$

have symmetric positive-definite solutions P_1 *and* P_2 *respectively.*

Under these conditions, if G, L *and* K *are gain matrices which for some chosen orthogonal matrices* $U_c \in \mathbb{R}^{n\times n}$ *(* $U_cU_c^T = I$ *) and* $U_o \in \mathbb{R}^{p\times p}$ *(* $U_oU_o^T = I$ *), satisfy:*

$$BK = S_c^T R_c^{-1} + E_c U_c R_c^{-1/2} \tag{3.40}$$

$$L = S_o^T R_o^{-1} + E_o U_o R_o^{-1/2}, \tag{3.41}$$

$$G = \tilde{A} - L\tilde{C} \tag{3.42}$$

then the resulting output feedback system using G, L and K will be such that, for all admissible parameter uncertainties ΔA, ΔA_h and ΔC, and for all positive integer time-delay values h,

(1) the augmented state-delayed system (3.16) *and* (3.17) *is asymptotically stable.*

(2) $\left\|T_{zw}(z)\right\|_\infty \le \gamma$.

Proof. By virtue of Lemma 3.1, the validity of (3.20) and (3.21) needs to be shown. To this end, defining,

$$P \triangleq \begin{bmatrix} P_1 & 0 \\ 0 & P_2 \end{bmatrix} > 0 \tag{3.43}$$

and considering the definitions (3.11)-(3.15) and (3.24)-(3.33), it is easy to see that inequality (3.20) follows from inequalities (3.38) and (3.39). Also, for simplicity of notation, define the left-hand side of (3.21) by Σ, where

$$\Sigma \triangleq \begin{bmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{12}^T & \Sigma_{22} \end{bmatrix} \tag{3.44}$$

Substituting (3.43) into (3.21) yields:

$$\Sigma_{11} = (A+BK)\Phi_1(A+BK)^T + BK\Phi_2(BK)^T - P_1 + B_1B_1^T + \varepsilon_1^{-1}I + \left(\varepsilon_2^{-1}+\varepsilon_3^{-1}\right)H_AH_A^T \tag{3.45}$$

$$\Sigma_{12} = (A+BK)\Phi_1(A-LC-G)^T - BK\Phi_2G^T + B_1(B_1-LB_2)^T + \varepsilon_2^{-1}H_A(H_A-LH_C)^T + \varepsilon_3^{-1}H_AH_A^T \tag{3.46}$$

$$\begin{aligned}\Sigma_{22} = {} & (A-LC-G)\Phi_1(A-LC-G)^T + G\Phi_2G^T - P_2 + (B_1-LB_2)(B_1-LB_2)^T \\ & + \varepsilon_2^{-1}(H_A-LH_C)(H_A-LH_C)^T + \varepsilon_3^{-1}H_AH_A^T + \varepsilon_1^{-1}I \end{aligned} \tag{3.47}$$

It follows from the matrix inversion Lemma,

$$\left(A_{11} - A_{12}A_{22}^{-1}A_{21}\right)^{-1} = A_{11}^{-1} + A_{11}^{-1}A_{12}\left(A_{22} - A_{21}A_{11}^{-1}A_{12}\right)^{-1}A_{21}A_{11}^{-1}$$

and the definitions of Θ_1 and Θ_2 given in (3.30), that

$$\Phi_1 = P_1 + P_1\Theta_1^{1/2}\left(I - \Theta_1^{1/2}P_1\Theta_1^{1/2}\right)^{-1}\Theta_1^{1/2}P_1 \tag{3.48}$$

$$\Phi_2 = P_2 + P_2\Theta_2^{1/2}\left(I - \Theta_2^{1/2}P_2\Theta_2^{1/2}\right)^{-1}\Theta_2^{1/2}P_2 \tag{3.49}$$

Re-writing Σ_{11} as,

$\Sigma_{11} =$

$$BK(\Phi_1+\Phi_2)(BK)^T + BK(\Phi_1 A^T) + A\Phi_1(BK)^T + A\Phi_1 A^T - P_1 + B_1 B_1^T + \varepsilon_1^{-1} I + (\varepsilon_2^{-1}+\varepsilon_3^{-1}) H_A H_A^T \quad (3.50)$$

and using the definitions of R_c and S_c in (3.31), while noting that R_c is invertible because Φ_1 and Φ_2 are positive-definite (due to (3.38) and (3.39)),

$$\Sigma_{11} = (BKR_c^{1/2} - S_c^T R_c^{-1/2})(BKR_c^{1/2} - S_c^T R_c^{-1/2})^T - S_c^T R_c^{-1} S_c + A\Phi_1 A^T - P_1 + B_1 B_1^T + \varepsilon_1^{-1} I + (\varepsilon_2^{-1}+\varepsilon_3^{-1}) H_A H_A^T$$

Using the definition of BK in (3.40),

$$\Sigma_{11} = A\Phi_1 A^T - P_1 + B_1 B_1^T + \varepsilon_1^{-1} I + (\varepsilon_2^{-1}+\varepsilon_3^{-1}) H_A H_A^T - S_c^T R_c^{-1} S_c + E_c E_c^T$$

so that by (3.48),

$$\Sigma_{11} = AP_1 A^T + AP_1\Theta_1^{1/2}(I - \Theta_1^{1/2} P_1 \Theta_1^{1/2})^{-1}\Theta_1^{1/2} P_1 A^T - P_1 + B_1 B_1^T + \varepsilon_1^{-1} I + (\varepsilon_2^{-1}+\varepsilon_3^{-1}) H_A H_A^T - S_c^T R_c^{-1} S_c + E_c E_c^T$$

From (3.36), $\Sigma_{11} < 0$.

Similarly, Σ_{22} of (3.47) can be re-written as,

$$\Sigma_{22} = (A - LC - \tilde{A} + L\tilde{C})\Phi_1(A - LC - \tilde{A} + L\tilde{C})^T + (\tilde{A} - L\tilde{C})\Phi_2(\tilde{A} - L\tilde{C})^T + (B_1 - LB_2)(B_1 - LB_2)^T$$

$$-P_2 + \varepsilon_1^{-1} I + \varepsilon_2^{-1}(H_A - LH_C)(H_A - LH_C)^T + \varepsilon_3^{-1} H_A H_A^T$$

where G has been replaced by its expression (3.42).

Grouping the terms with respect to L,

$$\Sigma_{22} = L\left[\Gamma_2\Phi_1\Gamma_2^T + \tilde{C}\Phi_2\tilde{C}^T + B_2 B_2^T + \varepsilon_2^{-1} H_C H_C^T\right]L^T - L\left[\Gamma_2\Phi_1\Gamma_1^T + \tilde{C}\Phi_2\tilde{A}^T + B_2 B_1^T + \varepsilon_2^{-1} H_C H_A^T\right]$$

$$-\left[\Gamma_2\Phi_1\Gamma_1^T + \tilde{C}\Phi_2\tilde{A}^T + B_2 B_1^T + \varepsilon_2^{-1} H_C H_A^T\right]^T L^T + \Gamma_1\Phi_1\Gamma_1^T + \tilde{A}\Phi_2\tilde{A}^T - P_2 + B_1 B_1^T + (\varepsilon_2^{-1}+\varepsilon_3^{-1}) H_A H_A^T + \varepsilon_1^{-1} I$$

From (3.32) and (3.33),

$$\Sigma_{22} = LR_o L^T - LS_o - S_o^T L^T + \Gamma_1\Phi_1\Gamma_1^T + \tilde{A}\Phi_2\tilde{A}^T - P_2 + B_1 B_1^T + (\varepsilon_2^{-1}+\varepsilon_3^{-1}) H_A H_A^T + \varepsilon_1^{-1} I$$

Assumption 3.1 implies that the matrix R_o defined in (3.32) is positive-definite and hence invertible, thus

$$\Sigma_{22} = (LR_o^{1/2} - S_o^T R_o^{-1/2})(LR_o^{1/2} - S_o^T R_o^{-1/2})^T - S_o^T R_o^{-1} S_o + \Gamma_1\Phi_1\Gamma_1^T + \tilde{A}\Phi_2\tilde{A}^T - P_2 + B_1 B_1^T$$

$$+(\varepsilon_2^{-1}+\varepsilon_3^{-1}) H_A H_A^T + \varepsilon_1^{-1} I$$

From (3.41) for L,

$$\Sigma_{22} = \tilde{A}\Phi_2\tilde{A}^T - P_2 + \Gamma_1\Phi_1\Gamma_1^T + B_1B_1^T + \left(\varepsilon_2^{-1} + \varepsilon_3^{-1}\right)H_AH_A^T - S_o^TR^{-1}S_o + E_oE_o^T + \varepsilon_1^{-1}I$$

and by (3.49),

$$\Sigma_{22} = \tilde{A}P_2\tilde{A}^T + \tilde{A}P_2\Theta_2^{1/2}\left(I - \Theta_1^{1/2}P_2\Theta_2^{1/2}\right)^{-1}\Theta_2^{1/2}P_2\tilde{A}^T - P_2 + \Gamma_1\Phi_1\Gamma_1^T + B_1B_1^T + \left(\varepsilon_2^{-1} + \varepsilon_3^{-1}\right)H_AH_A^T$$

$$-S_o^TR^{-1}S_o + E_oE_o^T + \varepsilon_1^{-1}I$$

From (3.37), $\Sigma_{22} < 0$.

Finally,

$$\Sigma_{12} = (A + BK)\Phi_1(A - LC - G)^T - BK\Phi_2G^T + B_1(B_1 - LB_2)^T + \varepsilon_2^{-1}H_A(H_A - LH_C)^T + \varepsilon_3^{-1}H_AH_A^T$$

Grouping the terms with respect to G^T,

$$\Sigma_{12} = -(A\Phi_1 + BK\Phi_1 + BK\Phi_2)G^T + A\Phi_1(A - LC)^T + BK\Phi_1(A - LC)^T$$

$$+B_1(B_1 - LB_2)^T + \varepsilon_2^{-1}H_A(H_A - LH_C)^T + \varepsilon_3^{-1}H_AH_A^T$$

Replacing BK by it expression (3.40) and grouping the terms with respect to L,

$$\Sigma_{12} = -E_cU_c(\Phi_1 + \Phi_2)^{1/2}G^T + \Big[A\Phi_1(\Phi_1 + \Phi_2)^{-1}\Phi_1C^T - A\Phi_1C^T$$

$$-E_cU_c(\Phi_1 + \Phi_2)^{-1/2}\Phi_1C^T - B_1B_2^T - \varepsilon_2^{-1}H_AH_C^T\Big]L^T + A\Phi_1A^T - A\Phi_1(\Phi_1 + \Phi_2)^{-1}\Phi_1A^T$$

$$+E_cU_c(\Phi_1 + \Phi_2)^{-1/2}\Phi_1A^T + B_1B_1^T + \left(\varepsilon_2^{-1} + \varepsilon_3^{-1}\right)H_AH_A^T$$

Noting that $(\Phi_1 + \Phi_2)^{-1} = \Phi_2^{-1}\left(\Phi_1^{-1} + \Phi_2^{-1}\right)^{-1}\Phi_1^{-1} = \Phi_1^{-1} - \Phi_1^{-1}\left(\Phi_1^{-1} + \Phi_2^{-1}\right)^{-1}\Phi_1^{-1}$, the following is obtained:

$$\Sigma_{12} = -E_cU_c(\Phi_1 + \Phi_2)^{1/2}G^T - \left[A\left(\Phi_1^{-1} + \Phi_2^{-1}\right)^{-1}C^T + E_cU_c(\Phi_1 + \Phi_2)^{-1/2}\Phi_1C^T + B_1B_2^T + \varepsilon_2^{-1}H_AH_C^T\right]L^T$$

$$+A\left(\Phi_1^{-1} + \Phi_2^{-1}\right)^{-1}A^T + E_cU_c(\Phi_1 + \Phi_2)^{-1/2}\Phi_1A^T + B_1B_1^T + \left(\varepsilon_2^{-1} + \varepsilon_3^{-1}\right)H_AH_A^T$$

Replacing G, $\tilde{A}$ and $\tilde{C}$ by their expressions in (3.42), (3.27) and (3.29) respectively and subsequently Γ_1 and Γ_2 by their expressions in (3.26) and (3.28) respectively, it then implies that $\Sigma_{12} = 0$ and so,

$$\Sigma = \begin{bmatrix} \Sigma_{11} & 0 \\ 0 & \Sigma_{22} \end{bmatrix} < 0, \tag{3.51}$$

as $\Sigma_{11} < 0$ and $\Sigma_{22} < 0$ as shown above. By virtue of Lemma 3.1, the output feedback system (3.16) and (3.17) is robustly asymptotically stable and $\left\| T_{zw}(z) \right\|_{\infty} \le \gamma$ for all values of positive integer time-delay h, and all uncertainties (3.5). QED

Remark 3.2. The invertibility of A required in **[85]** is not necessary in this chapter.

Remark 3.3. The numerical solutions to (3.36) and (3.37) are easily obtained by solving two auxiliary Riccati-type inequalities as explained in [34].

Remark 3.4. The presented robust H_{∞} output feedback control design procedure still offers much additional design freedom. This freedom is reflected by the arbitrary choice of the free gains parameters $E_c \in \mathbb{R}^{n \times n}$ and $E_o \in \mathbb{R}^{n \times p}$, and the orthogonal matrices $U_c \in \mathbb{R}^{n \times n}$ and $U_o \in \mathbb{R}^{p \times p}$. Introducing additional performance constraints into the problem formulation (1) and (2) of Theorem 3.1, which would exploit this design freedom is currently under investigation.

3.6. Numerical Example

Example 3.1. In terms of an example, consider the linear discrete uncertain time-delay system (3.1) and (3.2) with system matrices given by

$$A = \begin{bmatrix} 0.5 & 0.01 \\ 0.01 & -0.5 \end{bmatrix}, \qquad A_h = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.1 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \qquad C = \begin{bmatrix} 1 & 0 \end{bmatrix},$$

$$B_1 = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.1 \end{bmatrix}, \qquad B_2 = \begin{bmatrix} 0.5 & 0.8 \end{bmatrix}, \qquad M = \begin{bmatrix} 0.5 & 0 \\ 0 & 0.5 \end{bmatrix},$$

$$H_A = \begin{bmatrix} 0.1 & 0.05 \\ -0.02 & 0.1 \end{bmatrix}, \quad H_C = \begin{bmatrix} -0.2 & 0.8 \end{bmatrix}, \quad E_A = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.1 \end{bmatrix}, \quad E_h = \begin{bmatrix} 0.1 & 0.01 \\ 0.02 & 0.5 \end{bmatrix}$$

In what follows, the H_{∞} output feedback design procedure will aim at the satisfaction of $\left\| T_{zw}(z) \right\|_{\infty} \le \gamma$ with $\gamma = 0.8$.

To satisfy constraints (3.38) and (3.39), we choose

$$\varepsilon_1 = 5, \qquad \varepsilon_2 = 5, \qquad \varepsilon_3 = 1.$$

Taking $U_c = -1$ and invertible matrix $E_c = \begin{bmatrix} 10 & 0 \\ 0 & 1 \end{bmatrix}$, and initial values for P_1 and P_2, we compute (3.24)-(3.35) and (3.36)-(3.37). We then iterate in P_1 and P_2 until (3.36) and (3.37) are met while satisfying constraints (3.38)-(3.39). This yields the following values:

$$P_1 = \begin{bmatrix} 1.643 & 0.006 \\ 0.006 & 1.132 \end{bmatrix}, \qquad P_2 = \begin{bmatrix} 0.877 & 0.006 \\ 0.006 & 0.662 \end{bmatrix}$$

$$\tilde{A} = \begin{bmatrix} 0.280 & 0.009 \\ 0.006 & 0.440 \end{bmatrix}, \qquad \tilde{C} = \begin{bmatrix} 0.559 & -0.054 \end{bmatrix},$$

$$R_o = 1.873, \qquad S_o = \begin{bmatrix} 0.473 & 0.116 \end{bmatrix},$$

$$R_c = \begin{bmatrix} 3.438 & 0.067 \\ 0.067 & 2.811 \end{bmatrix}, \qquad S_c = \begin{bmatrix} -1.005 & 0.007 \\ -0.046 & 0.938 \end{bmatrix}$$

Finally, taking $E_o = \begin{bmatrix} 0.5 \\ 0.4 \end{bmatrix}$, we obtain the following values for our design parameters:

$$K = \begin{bmatrix} -1.1366 & -0.0947 \end{bmatrix}, \qquad L = \begin{bmatrix} 0.618 \\ 0.354 \end{bmatrix}, \qquad G = \begin{bmatrix} -0.066 & 0.042 \\ -0.193 & -0.421 \end{bmatrix}$$

This process can be done for other combinations of U_c and U_o like for instance 1 and 1. Also, we notice that $A + BK$ is stable with poles –0.224 and –1.102.

CHAPTER 4

Finite-Horizon Time-Varying H_∞ State-Feedback Control of Linear Neutral Systems with Parametric Uncertainties

4.1. System Description

Consider the neutral system (1.1) taken in the finite-horizon time-varying context, where the system matrices are all time-varying. More specifically, the system under consideration is:

$$\dot{x}(t)-\bar{A}_g(t)\dot{x}(t-g(t))=\bar{A}(t)x(t)+\bar{A}_h(t)x(t-h(t))+B_1(t)w(t)+\bar{B}(t)u(t) \tag{4.1}$$

with initial condition,

$$x(t)=0, \quad \forall t\leq 0 \tag{4.2}$$

Also, we define an auxiliary output $z(t)\in\mathbb{R}^p$ to be attenuated,

$$z(t)=col\{\bar{C}_0(t)x(t)+B_2(t)w(t),\bar{C}_1(t-h(t))x(t-h(t)),\bar{C}_2(t-g(t))x(t-g(t)),D(t)u(t)\} \tag{4.3}$$

with,

$$\bar{C}_0(t)=C_0(t)+\Delta C_0(t),\quad \bar{C}_1(t)=C_1(t)+\Delta C_1(t),\quad \bar{C}_2(t)=C_2(t)+\Delta C_2(t) \tag{4.4}$$

The uncertainties are:

$$\Delta A_g(t)=H_g(t)F_g(t)E_g(t),\qquad \begin{bmatrix}\Delta A(t)\\ \Delta B(t)\end{bmatrix}=H_A(t)F_A(t)\begin{bmatrix}E_A(t)\\ E_B(t)\end{bmatrix},\qquad \Delta A_h(t)=H_h(t)F_h(t)E_h(t),$$

$$\begin{bmatrix}\Delta C_0(t)\\ \Delta C_1(t)\\ \Delta C_2(t)\end{bmatrix}=\begin{bmatrix}H_{C_0}(t)\\ H_{C_1}(t)\\ H_{C_2}(t)\end{bmatrix}F_C(t)E_C(t) \tag{4.5}$$

4.2. Assumptions

The delays h and g in the system are functions of time and are assumed to be continuously differentiable, with their respective amplitudes and rates of change bounded as follows:

$$0 \le h(t) \le h_{\max}, \quad 0 \le g(t) < \infty, \qquad \text{for } t \in [0,T] \tag{4.6}$$

$$0 \le \dot{h}(t) \le \alpha < 1, \quad 0 \le \dot{g}(t) \le \beta < 1, \qquad \text{for } t \in [0,T] \tag{4.7}$$

where $h_{\max}$, α and β are given positive constants.

Also, $\bar{A}_h(t)$ is bounded as follows (see Remark 4.1 below):

$$\bar{A}_h^T(t)\bar{A}_h(t) \le \bar{A}_{h,\max}^T(t)\bar{A}_{h,\max}(t), \quad \text{for all } t \in [0,T] \tag{4.8}$$

where $\bar{A}_{h,\max}(t)$ is a continuously differentiable given matrix for all $t \in [0,T]$.

4.3. Problem Statement

The robust H_∞ control problem (RCP):

For any scalar $\gamma > 0$, let the following performance index be defined:

$$J \triangleq x^T(T)P_{1T}x(T) + \int_0^T \left[z^T(t)z(t) - \gamma^2 w^T(t)w(t)\right]dt \tag{4.9}$$

where P_{1T} is a given weighting matrix for the terminal state $x(T)$.

The robust H_∞ control design problem with disturbance attenuation level γ can now be translated into finding a stabilizing control law for system (4.1) and (4.3) which yields $J < 0$, for all disturbances $w \in L_2^q[0,T]$, subject to the usual assumption that $x(t) = 0$, for all $t \le 0$, and $w(t) = 0$, for all $t < 0$.

4.4. Preliminaries

The following lemmas will prove helpful in the sequel:

Lemma 4.1. [39] *Let the function* $v(t) = \int_{a(t)}^{b(t)} \int_{t-\theta}^{t} f(s)\,ds\,d\theta$. *Then* $v(t)$ *is a solution of the differential equation,*

$$\frac{dv(t)}{dt} = \left(b(t) - a(t)\right)f(t) - \left(1 - \dot{b}(t)\right)\int_{t-b(t)}^{t-a(t)} f(s)\,ds + \left(\dot{b}(t) - \dot{a}(t)\right)\int_{t-a(t)}^{t} f(s)\,ds.$$

Lemma 4.2. [12] *Let* A, L, E *and* F *be real matrices (possibly time-varying) of appropriate dimensions, with* F *satisfying* $FF^T \le I$. *Then the following holds:*

1- For any scalar $\varepsilon > 0$, $P^T LFE + E^T F^T L^T P \le \varepsilon^{-1} P^T LL^T P + \varepsilon E^T E$

2- For any matrix $P > 0$ *and any scalar* $\varepsilon > 0$ *such that* $\varepsilon I - EPE^T > 0$,

$$(A+LFE)P(A+LFE)^T \le APA^T + APE^T\left(\varepsilon I - EPE^T\right)^{-1} EPA^T + \varepsilon LL^T$$

3- For any matrix $P > 0$ *and any scalar* $\varepsilon > 0$ *such that* $P - \varepsilon LL^T > 0$,

$$(A+LFE)^T P^{-1}(A+LFE) \le A^T\left(P - \varepsilon LL^T\right)^{-1} A + \varepsilon^{-1} E^T E$$

Remark 4.1. Statement 3 in Lemma 4.2 as applied to $\bar{A}_h$ can be used to choose bound $\bar{A}_{h,\max}$ of (4.8).

Lemma 4.3. [62] *Assume that* $a(s) \in R^n$ *and* $b(s) \in R^m$ *are integrable over* $s \in \Omega$. *Then, for any positive definite matrix* $R \in R^{n\times m}$ *and any matrix* $M \in R^{m\times m}$, *the following holds:*

$$-2\int_\Omega b^T(s)a(s)ds \le \int_\Omega \begin{bmatrix} a(s) \\ b(s) \end{bmatrix}^T \begin{bmatrix} R & RM \\ M^T R & \Gamma \end{bmatrix} \begin{bmatrix} a(s) \\ b(s) \end{bmatrix} ds \tag{4.10}$$

where $\Gamma = \left(M^T R + I\right) R^{-1} (MR + I)$.

4.5. Main Results

It is the objective of this section to derive a stabilizing control law for our robust H_∞ control problem which depends on the time-delay h, the rates of change $\dot{h}$ and $\dot{g}$, but not on g itself. This is to ensure that, any variations in g, over an infinite-time horizon, do not destabilize the system; see e.g. **[36]**, and **[53]**.

The following will present the main results of this chapter.

4.5.1. Finite-Horizon Case

4.5.1.1. The L_2-gain Finite-Horizon Analysis of Linear Neutral Systems (the Bounded Real Lemma)

Following **[20]**, the unforced system ($u(t)=0$) is first considered, and equation (4.1) is written in its equivalent descriptor form:

$$\dot{x}(t)=y(t),\quad y(t)=\bar{A}_g(t)y(t-g(t))+\bar{A}(t)x(t)+\bar{A}_h(t)x(t-h(t))+B_1(t)w(t) \tag{4.11}$$

Using the Liebniz-Newton formula $x(t-h(t))=x(t)-\int_{t-h(t)}^{t}\dot{x}(s)ds$ permits to re-write (4.11) yet in a more tractable form. Introduction of the augmented state as in (4.11) and the use of the Liebnitz-Newton formula allows to avoid the introduction of any additional dynamics, so that the transfer function of the system obtained by freezing the time-variable in the system matrices does not exhibit any additional poles; see **[8]**. This way of transforming system (4.11) is particularly useful as it allows to avoid unnecessary conservatism in the design that follows.

The last transformation of (4.11) yields:

$$\dot{x}(t)=y(t),\quad 0=-y(t)+\bar{A}_g(t)y(t-g(t))+\left[\bar{A}(t)+\bar{A}_h(t)\right]x(t)-\bar{A}_h(t)\int_{t-h(t)}^{t}y(s)ds+B_1(t)w(t) \tag{4.12}$$

so that for $E=\begin{bmatrix} I_n & 0 \\ 0 & 0 \end{bmatrix}$, the augmented system is:

$$E\begin{bmatrix}\dot{x}(t)\\ \dot{y}(t)\end{bmatrix}=\begin{bmatrix}0 & I\\ \bar{A}(t)+\bar{A}_h(t) & -I\end{bmatrix}\begin{bmatrix}x(t)\\ y(t)\end{bmatrix}+\begin{bmatrix}0\\ \bar{A}_g(t)\end{bmatrix}y(t-g(t))-\begin{bmatrix}0\\ \bar{A}_h(t)\end{bmatrix}\int_{t-h(t)}^{t}y(s)ds+\begin{bmatrix}0\\ B_1(t)\end{bmatrix}w(t) \tag{4.13}$$

A time-varying generalization of the Lyapunov-Krasovskii functional, introduced in **[20]**, will be used here:

$$V(t)=V_0(t)+V_1(t)+V_2(t)+V_3(t) \tag{4.14}$$

where, $V_0(t)=\begin{bmatrix}x^T(t) & y^T(t)\end{bmatrix}EP(t)\begin{bmatrix}x(t)\\ y(t)\end{bmatrix}=x^T(t)P_1(t)x(t)$ (4.15)

$$V_1(t)=\int_{t-h(t)}^{t}x^T(\tau)S(t)x(\tau)d\tau \tag{4.16}$$

$$V_2(t)=\int_{t-g(t)}^{t}y^T(\tau)U(t)y(\tau)d\tau \tag{4.17}$$

$$V_3(t)=\int_{-h_{\max}}^{0}\int_{t+\theta}^{t}y^T(s)\bar{A}_{h,\max}^T(t)R_3(t)\bar{A}_{h,\max}(t)y(s)dsd\theta \tag{4.18}$$

with, $P(t)=\begin{bmatrix}P_1(t) & 0\\ P_2(t) & P_3(t)\end{bmatrix}$, $P_1(t)=P_1^T(t)>0$, $U(t)=U^T(t)>0$, $S(t)=S^T(t)>0$,

$$R_3(t) = R_3^T(t) > 0 \ . \tag{4.19}$$

The following theorem delivers the main result of this section; for simplicity of notation the time parameter t is omitted in the entries of all matrices.

Theorem 4.1. (Bounded Real Lemma) *Consider the system* (4.1)-(4.3). *For a given* $\gamma > 0$ *and a given symmetric, positive-definite matrix* P_{1T}, *suppose that there exist* $n \times n$ *-matrices:* $P_1(t) = P_1^T(t) > 0$ *such that* $P_1(T) = P_{1T}$, $P_2(t)$, $P_3(t)$, $S(t) = S^T(t) > 0$ *with* $\dot{S}(t) \le 0$, $U(t) = U^T(t) > 0$ *with* $\dot{U}(t) \le 0$, $W_1(t)$, $W_2(t)$, $W_3(t)$, $W_4(t)$, *and* $R_1(t) = R_1^T(t)$, $R_2(t)$, $R_3(t) = R_3^T(t) > 0$, $t \in [0,T]$, *and positive scalars* $\varepsilon_i(t)$; $i = 1 \ldots 8$, *which satisfy the following matrix inequality:*

$$\begin{bmatrix} \Omega_1 & \Omega_2 \\ \Omega_2^T & \Omega_3 \end{bmatrix} < 0 \tag{4.20}$$

with,

$$\Omega_1 = \begin{bmatrix} \Psi_1 & \Psi_2 & P_2^T B_1 & h_{\max}\Phi_1 & -W_3^T A_h - \varepsilon_4 E_h^T E_h & P_2^T A_g & \tilde{C}^T & C_0^T \\ \Psi_2^T & \Psi_3 & P_3^T B_1 & h_{\max}\Phi_2 & -W_4^T A_h & P_3^T A_g & 0 & 0 \\ B_1^T P_2 & B_1^T P_3 & -\gamma^2 I & 0 & 0 & 0 & 0 & B_2^T \\ h_{\max}\Phi_1^T & h_{\max}\Phi_2^T & 0 & -h_{\max} R & 0 & 0 & 0 & 0 \\ -A_h^T W_3 - \varepsilon_4 E_h^T E_h & -A_h^T W_4 & 0 & 0 & -(1-\alpha)S + \varepsilon_4 E_h^T E_h & 0 & 0 & 0 \\ A_g^T P_2 & A_g^T P_3 & 0 & 0 & 0 & -(1-\beta)U + \varepsilon_3 E_g^T E_g & 0 & 0 \\ \tilde{C} & 0 & 0 & 0 & 0 & 0 & -I & 0 \\ C_0 & 0 & B_2 & 0 & 0 & 0 & 0 & -I_p \end{bmatrix} \tag{4.21}$$

$$\Omega_2 = \begin{bmatrix} P_2^T H_A & P_2^T H_h & P_2^T H_g & W_3^T H_h & 0 & C_0^T H_{C_0} & C_1^T H_{C_1} & C_2^T H_{C_2} \\ P_3^T H_A & P_3^T H_h & P_3^T H_g & W_4^T H_h & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & B_2^T H_{C_0} & 0 & 0 & 0 \\ & & & & 0 & & & \end{bmatrix} \tag{4.22}$$

$$\Omega_3 = diag\left\{-\varepsilon_1 I, -\varepsilon_2 I, -\varepsilon_3 I, -\varepsilon_4 I, -\varepsilon_5 I, -\left(\varepsilon_6 I - H_{C_0}^T H_{C_0}\right), -(1-\alpha)\left(\varepsilon_7 I - H_{C_1}^T H_{C_1}\right), -(1-\beta)\left(\varepsilon_8 I - H_{C_2}^T H_{C_2}\right)\right\} \tag{4.23}$$

where,

$$\Psi_1 = \dot{P}_1^T + \left(A^T + A_h^T\right)P_2 + P_2^T\left(A + A_h\right) + W_3^T A_h + A_h^T W_3 + S + \varepsilon_1 E_A^T E_A + \left(\varepsilon_2 + \varepsilon_4\right)E_h^T E_h + \left(\varepsilon_5 + \varepsilon_6 + \varepsilon_7 \frac{1}{1-\alpha} + \varepsilon_8 \frac{1}{1-\beta}\right)E_C^T E_C$$

$$\Psi_2 = P_1^T - P_2^T + \left(A^T + A_h^T\right)P_3 + A_h^T W_4 , \qquad \Psi_3 = -P_3 - P_3^T + U + h_{\max}\overline{A}_{h,\max}^T R_3 \overline{A}_{h,\max}$$

$$\Phi_1 = \begin{bmatrix} W_1^T + P_1^T & W_3^T + P_2^T \end{bmatrix}, \qquad \Phi_2 = \begin{bmatrix} W_2^T & W_4^T + P_3^T \end{bmatrix}$$

$$R = R^T = \begin{bmatrix} R_1 & R_2 \\ R_2^T & R_3 \end{bmatrix} > 0, \qquad \tilde{C}^T \tilde{C} = C_0^T C_0 + \frac{1}{1-\alpha} C_1^T C_1 + \frac{1}{1-\beta} C_2^T C_2$$

Also suppose that $R_3(t)$ *satisfies the following matrix inequality,*

$$\dot{\bar{A}}_{h,\max}^T R_3 \bar{A}_{h,\max} + \bar{A}_{h,\max}^T R_3 \dot{\bar{A}}_{h,\max} + \bar{A}_{h,\max}^T \dot{R}_3 \bar{A}_{h,\max} < 0 \tag{4.24}$$

Under these conditions, the cost function (4.9) *satisfies* $J < 0$ *for all nonzero* $w \in L_2^q[0,T]$, *and for any value of the positive delay* $g(t)$, $t \in [0,T]$

Proof. The performance index J of (4.9) can be written as,

$$J = x^T(T) P_{1T} x(T) + \int_0^T \left[z^T(t) z(t) - \gamma^2 w^T(t) w(t) + \dot{V}(t) \right] dt + V(0) - V_0(T) - V_1(T) - V_2(T) - V_3(T) \tag{4.25}$$

Since $V(0) = 0$, $V_0(T) = x^T(T) P_{1T} x(T)$, and $V_i(T) \geq 0$; $i = 1...3$, then $J < 0$ is guaranteed if,

$$\bar{J} = \int_0^T \left[z^T(t) z(t) - \gamma^2 w^T(t) w(t) + \dot{V}(t) \right] dt < 0 \tag{4.26}$$

So, the need is to compute the expression for $\bar{J}$ or more precisely an upper bound $\bar{\bar{J}}$ on $\bar{J}$ which then can be required to verify $\bar{J} < \bar{\bar{J}} < 0$.

In doing so, we start first by evaluating $\dot{V}(t) = \dfrac{dV(t)}{dt}$. Differentiating (4.15) and using (4.13), yields,

$$\frac{dV_0}{dt} = 2x^T(t) P_1(t) \dot{x}(t) = 2\begin{bmatrix} x^T(t) & y^T(t) \end{bmatrix} P^T(t) \begin{bmatrix} \dot{x}(t) \\ 0 \end{bmatrix}$$

$$= 2\begin{bmatrix} x^T(t) & y^T(t) \end{bmatrix} P^T(t) \begin{bmatrix} 0 & I \\ \bar{A}(t) + \bar{A}_h(t) & -I \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} + 2\begin{bmatrix} x^T(t) & y^T(t) \end{bmatrix} P^T(t) \begin{bmatrix} 0 \\ \bar{A}_g(t) \end{bmatrix} y(t - g(t))$$

$$+ 2\begin{bmatrix} x^T(t) & y^T(t) \end{bmatrix} P^T(t) \begin{bmatrix} 0 \\ B_1(t) \end{bmatrix} w(t) + \begin{bmatrix} x^T(t) & y^T(t) \end{bmatrix} \dot{P}^T(t) \begin{bmatrix} x(t) \\ 0 \end{bmatrix}$$

$$-2\int_{t-h(t)}^{t} \begin{bmatrix} x^T(t) & y^T(t) \end{bmatrix} P^T(t) \begin{bmatrix} 0 \\ \bar{A}_h(t) \end{bmatrix} y(s) ds \tag{4.27}$$

A bound for the last term of (4.27) will be derived as follows:

Define $\eta(t) = -2\int_{t-h(t)}^{t} \begin{bmatrix} x^T(t) & y^T(t) \end{bmatrix} P^T(t) \begin{bmatrix} 0 \\ \bar{A}_h(t) \end{bmatrix} y(s)\, ds$ (4.28)

Using Lemma 4.3 with $a(s) = \begin{bmatrix} 0 \\ \bar{A}_h(t) \end{bmatrix} y(s)$, $b(s) = P(t)\begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$ and with $\Omega = \left[t-h(t), t\right]$, gives:

$$\eta(t) \le \int_{t-h(t)}^{t} \begin{bmatrix} x^T(t) & y^T(t) \end{bmatrix} P^T(t)\left(M^T(t)R(t)+I\right)R^{-1}(t)\left(M(t)R(t)+I\right)P(t)\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} ds$$

$$+2\int_{t-h(t)}^{t} y^T(s)\,ds \begin{bmatrix} 0 & \bar{A}_h^T(t) \end{bmatrix} R(t) M(t) P(t) \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} + \int_{t-h(t)}^{t} y^T(s) \begin{bmatrix} 0 & \bar{A}_h^T(t) \end{bmatrix} R(t) \begin{bmatrix} 0 \\ \bar{A}_h(t) \end{bmatrix} y(s)\, ds$$

$$\le h_{\max} \begin{bmatrix} x^T(t) & y^T(t) \end{bmatrix} P^T(t)\left(M^T(t)R(t)+I\right)R^{-1}(t)\left(M(t)R(t)+I\right)P(t)\begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$$

$$+2\left[x^T(t) - x^T(t-h(t))\right]\begin{bmatrix} 0 & \bar{A}_h^T(t) \end{bmatrix} R(t) M(t) P(t) \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$$

$$+\int_{t-h_{\max}}^{t} y^T(s) \begin{bmatrix} 0 & \bar{A}_h^T(t) \end{bmatrix} R(t) \begin{bmatrix} 0 \\ \bar{A}_h(t) \end{bmatrix} y(s)\, ds \tag{4.29}$$

as $R(t) > 0$.

Differentiating $V_1(t)$ and $V_2(t)$ of (4.16) and (4.17),

$$\frac{dV_1(t)}{dt} = x^T(t)S(t)x(t) - \left(1-\dot{h}(t)\right)x^T(t-h(t))S(t)x(t-h(t)) + \int_{t-h(t)}^{t} x^T(\tau)\dot{S}(t)x(\tau)\,d\tau$$

$$\frac{dV_2(t)}{dt} = y^T(t)U(t)y(t) - \left(1-\dot{g}(t)\right)y^T(t-g(t))U(t)y(t-g(t)) + \int_{t-g(t)}^{t} y^T(\tau)\dot{U}(t)y(\tau)\,d\tau$$

and using the assumptions $S(t) > 0$, $\dot{S}(t) \le 0$, $U(t) > 0$ and $\dot{U}(t) \le 0$, and the bounds (4.7),

$$\frac{dV_1(t)}{dt} \le x^T(t)S(t)x(t) - (1-\alpha)x^T(t-h(t))S(t)x(t-h(t)) \tag{4.30}$$

$$\frac{dV_2(t)}{dt} \le y^T(t)U(t)y(t) - (1-\beta)y^T(t-g(t))U(t)y(t-g(t)) \tag{4.31}$$

Applying Lemma 4.1 to $V_3(t)$,

$$\frac{dV_3(t)}{dt} = h_{\max} y^T(t)\bar{A}_{h,\max}^T(t)R_3(t)\bar{A}_{h,\max}(t)y(t) - \int_{t-h_{\max}}^{t} y^T(s)\bar{A}_{h,\max}^T(t)R_3(t)\bar{A}_{h,\max}(t)y(s)\,ds$$

$$+\int_{-h_{\max}}^{0}\int_{t+\theta}^{t} y^T(s)\frac{d}{dt}\left[\bar{A}_{h,\max}^T(t)R_3(t)\bar{A}_{h,\max}(t)\right]y(s)\,ds\,d\theta \tag{4.32}$$

If,

$$\frac{d}{dt}\left[\bar{A}_{1,\max}^T(t)R_3(t)\bar{A}_{1,\max}(t)\right]=$$

$$\dot{\bar{A}}_{1,\max}^T(t)R_3(t)\bar{A}_{1,\max}(t)+\bar{A}_{1,\max}^T(t)R_3(t)\dot{\bar{A}}_{1,\max}(t)+\bar{A}_{1,\max}^T(t)\dot{R}_3(t)\bar{A}_{1,\max}(t)<0 \tag{4.33}$$

then,

$$\frac{dV_3(t)}{dt}<h_{\max}y^T(t)\bar{A}_{h,\max}^T(t)R_3(t)\bar{A}_{h,\max}(t)y(t)-\int_{t-h_{\max}}^{t}y^T(s)\bar{A}_{h,\max}^T(t)R_3(t)\bar{A}_{h,\max}(t)y(s)\,ds \tag{4.34}$$

Employing (4.27), (4.29), (4.30), (4.31) and (4.34), the following upper bound on $\dot{V}(t)$ is obtained:

$$\begin{aligned}
\dot{V}(t)<{}&2\begin{bmatrix}x^T(t) & y^T(t)\end{bmatrix}P^T(t)\begin{bmatrix}0 & I\\ \bar{A}(t)+\bar{A}_h(t) & -I\end{bmatrix}\begin{bmatrix}x(t)\\ y(t)\end{bmatrix}+2\begin{bmatrix}x^T(t) & y^T(t)\end{bmatrix}P^T(t)\begin{bmatrix}0\\ \bar{A}_g(t)\end{bmatrix}y(t-g(t))\\
&+2\begin{bmatrix}x^T(t) & y^T(t)\end{bmatrix}P^T(t)\begin{bmatrix}0\\ B_1(t)\end{bmatrix}w(t)+\begin{bmatrix}x^T(t) & y^T(t)\end{bmatrix}\dot{P}^T(t)\begin{bmatrix}x(t)\\ 0\end{bmatrix}\\
&+h_{\max}\begin{bmatrix}x^T(t) & y^T(t)\end{bmatrix}P^T(t)\left(M^T(t)R(t)+I\right)R^{-1}(t)\left(M(t)R(t)+I\right)P(t)\begin{bmatrix}x(t)\\ y(t)\end{bmatrix}\\
&+2\left[x^T(t)-x^T(t-h(t))\right]\begin{bmatrix}0 & \bar{A}_h^T(t)\end{bmatrix}R(t)M(t)P(t)\begin{bmatrix}x(t)\\ y(t)\end{bmatrix}\\
&+\int_{t-h_{\max}}^{t}y^T(s)\begin{bmatrix}0 & \bar{A}_h^T(t)\end{bmatrix}R(t)\begin{bmatrix}0\\ \bar{A}_h(t)\end{bmatrix}y(s)\,ds\\
&+x^T(t)S(t)x(t)-(1-\alpha)x^T(t-h(t))S(t)x(t-h(t))\\
&+y^T(t)U(t)y(t)-(1-\beta)y^T(t-g(t))U(t)y(t-g(t))\\
&+h_{\max}y^T(t)\bar{A}_{h,\max}^T(t)R_3(t)\bar{A}_{h,\max}(t)y(t)-\int_{t-h_{\max}}^{t}y^T(s)\bar{A}_{h,\max}^T(t)R_3(t)\bar{A}_{h,\max}(t)y(s)\,ds
\end{aligned} \tag{4.35}$$

Since $R_3(t)>0$ and using bound (4.8), we see that the two remaining integrals in (4.35) verify,

$$\int_{t-h_{\max}}^{t}y^T(s)\begin{bmatrix}0 & \bar{A}_h^T(t)\end{bmatrix}R(t)\begin{bmatrix}0\\ \bar{A}_h(t)\end{bmatrix}y(s)\,ds-\int_{t-h_{\max}}^{t}y^T(s)\bar{A}_{h,\max}^T(t)R_3(t)\bar{A}_{h,\max}(t)y(s)\,ds$$

$$=\int_{t-h_{\max}}^{t} y^T(s)\left[\bar{A}_h^T(t)R_3(t)\bar{A}_h(t)-\bar{A}_{h,\max}^T(t)R_3(t)\bar{A}_{h,\max}(t)\right]y(s)ds\leq 0$$

thus reducing (4.35) to,

$$\dot{V}(t)<2\begin{bmatrix}x^T(t) & y^T(t)\end{bmatrix}P^T(t)\begin{bmatrix}0 & I\\ \bar{A}(t)+\bar{A}_h(t) & -I\end{bmatrix}\begin{bmatrix}x(t)\\ y(t)\end{bmatrix}+2\begin{bmatrix}x^T(t) & y^T(t)\end{bmatrix}P^T(t)\begin{bmatrix}0\\ \bar{A}_g(t)\end{bmatrix}y(t-g(t))$$

$$+2\begin{bmatrix}x^T(t) & y^T(t)\end{bmatrix}P^T(t)\begin{bmatrix}0\\ B_1(t)\end{bmatrix}w(t)+\begin{bmatrix}x^T(t) & y^T(t)\end{bmatrix}\dot{P}^T(t)\begin{bmatrix}x(t)\\ 0\end{bmatrix}$$

$$+h_{\max}\begin{bmatrix}x^T(t) & y^T(t)\end{bmatrix}P^T(t)\left(M^T(t)R(t)+I\right)R^{-1}(t)\left(M(t)R(t)+I\right)P(t)\begin{bmatrix}x(t)\\ y(t)\end{bmatrix}$$

$$+2\left[x^T(t)-x^T(t-h(t))\right]\begin{bmatrix}0 & \bar{A}_h^T(t)\end{bmatrix}R(t)M(t)P(t)\begin{bmatrix}x(t)\\ y(t)\end{bmatrix}$$

$$+x^T(t)S(t)x(t)-(1-\alpha)x^T(t-h(t))S(t)x(t-h(t))$$

$$+y^T(t)U(t)y(t)-(1-\beta)y^T(t-g(t))U(t)y(t-g(t))$$

$$+h_{\max}y^T(t)\bar{A}_{h,\max}^T(t)R_3(t)\bar{A}_{h,\max}(t)y(t) \qquad (4.36)$$

Next,

$$\int_0^T z^T(t)z(t)dt=\int_0^T\left[\bar{C}_0(t)x(t)+B_2(t)w(t)\right]^T\left[\bar{C}_0(t)x(t)+B_2(t)w(t)\right]dt$$

$$+\int_0^T\left[\bar{C}_1(t-h(t))x(t-h(t))\right]^T\left[\bar{C}_1(t-h(t))x(t-h(t))\right]dt$$

$$+\int_0^T\left[\bar{C}_2(t-g(t))x(t-g(t))\right]^T\left[\bar{C}_2(t-g(t))x(t-g(t))\right]dt. \qquad (4.37)$$

Making the change of variables $\tau=t-h(t)$ and $\theta=t-g(t)$, we can write (4.37) as,

$$\int_0^T z^T(t)z(t)dt=\int_0^T\left[\bar{C}_0(t)x(t)+B_2(t)w(t)\right]^T\left[\bar{C}_0(t)x(t)+B_2(t)w(t)\right]dt$$

$$+\int_{-h(0)}^{T-h(T)}\left[\bar{C}_1(\tau)x(\tau)\right]^T\left[\bar{C}_1(\tau)x(\tau)\right]\frac{1}{1-\dot{h}(t)}d\tau$$

$$+\int_{-g(0)}^{T-g(T)}\left[\bar{C}_2(\theta)x(\theta)\right]^T\left[\bar{C}_2(\theta)x(\theta)\right]\frac{1}{1-\dot{g}(t)}d\theta \qquad (4.38)$$

Since $x(t)=0$ for all $t\leq 0$, (4.38) is written,

$$\int_0^T z^T(t)z(t)dt = \int_0^T \left[\bar{C}_0(t)x(t)+B_2(t)w(t)\right]^T \left[\bar{C}_0(t)x(t)+B_2(t)w(t)\right]dt$$

$$+\int_0^{T-h(T)} \left[\bar{C}_1(\tau)x(\tau)\right]^T \left[\bar{C}_1(\tau)x(\tau)\right]\frac{1}{1-\dot{h}(t)}d\tau$$

$$+\int_0^{T-g(T)} \left[\bar{C}_2(\theta)x(\theta)\right]^T \left[\bar{C}_2(\theta)x(\theta)\right]\frac{1}{1-\dot{g}(t)}d\theta \tag{4.39}$$

Also using bounds (4.7) and the fact that the integrands are positive, (4.39) is bounded as,

$$\int_0^T z^T(t)z(t)dt \le$$

$$\int_0^T \left[\bar{C}_0(t)x(t)+B_2(t)w(t)\right]^T \left[\bar{C}_0(t)x(t)+B_2(t)w(t)\right]dt + \int_0^T \left[\bar{C}_1(t)x(t)\right]^T \left[\bar{C}_1(t)x(t)\right]\frac{1}{1-\alpha}dt$$

$$+\int_0^T \left[\bar{C}_2(t)x(t)\right]^T \left[\bar{C}_2(t)x(t)\right]\frac{1}{1-\beta}dt\,. \tag{4.40}$$

and finally integrating (4.36) from time 0 to T, we obtain,

$$\int_0^T \left[z^T(t)z(t)-\gamma^2 w^T(t)w(t)+\dot{V}(t)\right]dt <$$

$$\int_0^T \Bigg\{2\left[x^T(t) \quad y^T(t)\right]P^T(t)\begin{bmatrix} 0 & I \\ \bar{A}(t)+\bar{A}_h(t) & -I \end{bmatrix}\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} + 2\left[x^T(t) \quad y^T(t)\right]P^T(t)\begin{bmatrix} 0 \\ \bar{A}_g(t) \end{bmatrix}y(t-g(t))$$

$$+2\left[x^T(t) \quad y^T(t)\right]P^T(t)\begin{bmatrix} 0 \\ B_1(t) \end{bmatrix}w(t)+\left[x^T(t) \quad y^T(t)\right]\dot{P}^T(t)\begin{bmatrix} x(t) \\ 0 \end{bmatrix}$$

$$+h_{\max}\left[x^T(t) \quad y^T(t)\right]P^T(t)\left(M^T(t)R(t)+I\right)R^{-1}(t)\left(M(t)R(t)+I\right)P(t)\begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$$

$$+2\left[x^T(t)-x^T(t-h(t))\right]\left[0 \quad \bar{A}_h^T(t)\right]R(t)M(t)P(t)\begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$$

$$+x^T(t)S(t)x(t)-(1-\alpha)x^T(t-h(t))S(t)x(t-h(t))$$

$$+y^T(t)U(t)y(t)-(1-\beta)y^T(t-g(t))U(t)y(t-g(t))$$

$$+h_{\max}y^T(t)\bar{A}_{h,\max}^T(t)R_3(t)\bar{A}_{h,\max}(t)y(t)$$

$$+\left[\bar{C}_0(t)x(t)+B_2(t)w(t)\right]^T\left[\bar{C}_0(t)x(t)+B_2(t)w(t)\right]$$

$$+\frac{1}{1-\alpha}\left[\bar{C}_1(t)x(t)\right]^T\left[\bar{C}_1(t)x(t)\right]+\frac{1}{1-\beta}\left[\bar{C}_2(t)x(t)\right]^T\left[\bar{C}_2(t)x(t)\right]-\gamma^2 w^T(t)w(t)\Big\}dt \tag{4.41}$$

Next applying Lemma 4.2 to (4.41) results in a bound for $\bar{J}$ which has the following form:

$$\int_0^T\left[z^T(t)z(t)-\gamma^2 w^T(t)w(t)+\dot{V}(t)\right]dt<$$

$$\int_0^T\Big\{\left[x^T(t)\quad y^T(t)\right]P^T(t)\begin{bmatrix}0 & I\\ A(t)+A_h(t) & -I\end{bmatrix}\begin{bmatrix}x(t)\\ y(t)\end{bmatrix}+\left[x^T(t)\quad y^T(t)\right]\begin{bmatrix}0 & A^T(t)+A_h^T(t)\\ I & -I\end{bmatrix}P(t)\begin{bmatrix}x(t)\\ y(t)\end{bmatrix}$$

$$+\left[x^T(t)\quad y^T(t)\right]P^T(t)\begin{bmatrix}0\\ A_g(t)\end{bmatrix}y(t-g(t))+y^T(t-g(t))\left[0\quad A_g^T(t)\right]P(t)\begin{bmatrix}x(t)\\ y(t)\end{bmatrix}$$

$$+\left[x^T(t)\quad y^T(t)\right]P^T(t)\begin{bmatrix}0\\ B_1(t)\end{bmatrix}w(t)+w^T(t)\left[0\quad B_1^T(t)\right]P(t)\begin{bmatrix}x(t)\\ y(t)\end{bmatrix}+\left[x^T(t)\quad y^T(t)\right]\dot{P}^T(t)\begin{bmatrix}x(t)\\ 0\end{bmatrix}$$

$$+h_{\max}\left[x^T(t)\quad y^T(t)\right]P^T(t)\left(M^T R(t)+I\right)R^{-1}(t)\left(MR(t)+I\right)P(t)\begin{bmatrix}x(t)\\ y(t)\end{bmatrix}$$

$$+\left[x^T(t)\quad y^T(t)\right]P^T(t)M^T(t)R(t)\begin{bmatrix}0\\ A_h(t)\end{bmatrix}\left[x(t)-x(t-h(t))\right]$$

$$+x^T(t)S(t)x(t)-(1-\alpha)x^T(t-h(t))S(t-h(t))x(t-h(t))$$

$$+y^T(t)U(t)y(t)-(1-\beta)y^T(t-g(t))U(t-g(t))y(t-g(t))$$

$$+h_{\max}y^T(t)\bar{A}_{h,\max}^T(t)R_3(t)\bar{A}_{h,\max}(t)y(t)$$

$$+x^T(t)C_0^T(t)B_2(t)w(t)+w^T(t)B_2^T(t)C_0(t)x(t)+w^T(t)B_2^T(t)B_2(t)w(t)$$

$$-\gamma^2 w^T(t)w(t)$$

$$+w^T(t)\varepsilon_6^{-1}B_2^T(t)H_{C_0}H_{C_0}^T B_2(t)w(t)+x^T(t)\varepsilon_6 E_C^T E_C x(t)$$

$$+x^T(t)C_0^T(t)C_0(t)x(t)+x^T(t)C_0^T(t)H_{C_0}\left(\varepsilon_6 I-H_{C_0}^T H_{C_0}\right)H_{C_0}^T C_0(t)x(t)+\varepsilon_6 x^T(t)E_C^T E_C x(t)$$

$$+\frac{1}{1-\alpha}x^T(t)C_1^T(t)C_1(t)x(t)+x^T(t)C_1^T(t)H_{C_1}\left(\varepsilon_7 I-H_{C_1}^T H_{C_1}\right)H_{C_1}^T C_1(t)x(t)+\varepsilon_7\frac{1}{1-\alpha}x^T(t)E_C^T E_C x(t)$$

$$+\frac{1}{1-\beta}x^T(t)C_2^T(t)C_2(t)x(t)+x^T(t)C_2^T(t)H_{C_2}\left(\varepsilon_8 I-H_{C_2}^T H_{C_2}\right)H_{C_2}^T C_2(t)x(t)+\varepsilon_8\frac{1}{1-\beta}x^T(t)E_C^T E_C x(t)$$

$$+\varepsilon_1^{-1}x^T(t)P_2^T(t)H_AH_A^TP_2(t)x(t)+\varepsilon_1^{-1}y^T(t)P_3^T(t)H_AH_A^TP_2(t)x(t)+\varepsilon_1^{-1}x^T(t)P_2^T(t)H_AH_A^TP_3(t)y(t)$$

$$+\varepsilon_1^{-1}y^T(t)P_3^T(t)H_AH_A^TP_3(t)y(t)+\varepsilon_1x^T(t)E_A^TE_Ax(t)$$

$$+\varepsilon_2^{-1}x^T(t)P_2^T(t)H_hH_h^TP_2(t)x(t)+\varepsilon_2^{-1}y^T(t)P_3^T(t)H_hH_h^TP_2(t)x(t)+\varepsilon_2^{-1}x^T(t)P_2^T(t)H_hH_h^TP_3(t)y(t)$$

$$+\varepsilon_2^{-1}y^T(t)P_3^T(t)H_hH_h^TP_3(t)y(t)+\varepsilon_2x^T(t)E_h^TE_hx(t)$$

$$+\varepsilon_3^{-1}x^T(t)P_2^T(t)H_gH_g^TP_2(t)x(t)+\varepsilon_3^{-1}y^T(t)P_3^T(t)H_gH_g^TP_2(t)x(t)+\varepsilon_3^{-1}x^T(t)P_2^T(t)H_gH_g^TP_3(t)y(t)$$

$$+\varepsilon_3^{-1}y^T(t)P_3^T(t)H_gH_g^TP_3(t)y(t)+\varepsilon_3y^T(t-g(t))E_g^TE_gy(t-g(t))$$

$$+\varepsilon_4^{-1}x^T(t)W_3^T(t)H_hH_h^TW_3(t)x(t)+\varepsilon_4^{-1}y^T(t)W_4^T(t)H_hH_h^TW_3(t)x(t)+\varepsilon_4^{-1}x^T(t)W_3^T(t)H_hH_h^TW_4(t)y(t)$$

$$+\varepsilon_4^{-1}y^T(t)W_4^T(t)H_hH_h^TW_4(t)y(t)+\varepsilon_4\left[x^T(t)-x^T(t-h(t))\right]E_h^TE_h\left[x(t)-x(t-h(t))\right]\Big\}dt \qquad (4.42)$$

Defining the aggregated state $\xi(t)$ as:

$$\xi(t)=\begin{bmatrix} x(t) \\ y(t) \\ w(t) \\ (M+R)^{-1}P\begin{bmatrix} x(t) \\ y(t)\end{bmatrix} \\ x(t-h(t)) \\ y(t-g(t)) \\ \tilde{C}(t)x(t) \\ B_2(t)w(t) \end{bmatrix}$$

then bound (4.42) can be written as:

$$\bar{J}\le\int_0^T\xi^T(t)\left[\Omega_1(t)-\Omega_2(t)\Omega_3^{-1}(t)\Omega_2^T(t)\right]\xi(t)\,dt$$

where $\Omega_1(t)$, $\Omega_2(t)$ and $\Omega_3(t)$ are as in (4.21), (4.22) and (4.23), respectively.

Sufficient conditions for guaranteeing that $\bar{J}<0$ (and so $J<0$) can thus be stated as follows:

1.) $\Omega_1(t)-\Omega_2(t)\Omega_3^{-1}(t)\Omega_2^T(t)<0$.

2.) $\Omega_3(t)<0$ (assumptions of Lemma 4.2).

3.) $\dot{\bar{A}}_{h,\max}^{T}(t)R_3(t)\bar{A}_{h,\max}(t)+\bar{A}_{h,\max}^{T}(t)R_3(t)\dot{\bar{A}}_{h,\max}(t)+\bar{A}_{h,\max}^{T}(t)\dot{R}_3(t)\bar{A}_{h,\max}(t)<0$ (used for obtaining (4.34)).

By the Schur complements lemma (see Appendix A), conditions 1.) and 2.) are equivalent to assumption (4.20) of the Theorem, and condition 3.) is assumption (4.24) of the Theorem. QED

Remark 4.2. Concerning a practical numerical procedure for finding the solutions to the above inequalities, the time derivatives of $P_1(t)$, $S(t)$, $U(t)$ and $R_3(t)$ can be approximated using a backward difference scheme as follows:

$$\dot{P}_1(t)\approx\frac{P_1(k)-P_1(k-1)}{T_s},\quad \dot{S}(t)\approx\frac{S(k)-S(k-1)}{T_s},\quad \dot{U}(t)\approx\frac{U(k)-U(k-1)}{T_s},\quad \dot{R}_3(t)\approx\frac{R_3(k)-R_3(k-1)}{T_s}$$

where T_s is the discretization step selected based on numerical considerations, and k is the discretization index. Once T_s is selected, inequalities (4.20) and (4.24) reduce to a set of LMIs **[6]** to be solved backward at each discretized time with terminal conditions $P_1(T)=P_{1T}$, $S(T)=S_T$, $U(T)=U_T$ and $R_3(T)=R_{3T}$. Here, the choice of S_T, U_T and R_{3T} is arbitrary provided that the conditions of the Theorem are satisfied. Also notice that the variables $\varepsilon_i(t)$ being time-varying does not make the solution more difficult, since the LMI (see Appendix B) solved at each discretized time gives as solution, the value of all the variables listed in the Theorem (P_i, S, U, W_i, R_i and ε_i). Taking $\varepsilon_i(t)$ as time-varying allows the LMIs to be satisfied more easily and less conservatively at every discretized time.

4.5.1.2. Finite-Horizon State-Feedback Control of Linear Time-Varying Neutral Systems

In this section, a memoryless state-feedback gain matrix $K(t)\in\mathbb{R}^{l\times n}$ will be constructed such that,

$$u(t)=K(t)x(t) \tag{4.43}$$

guarantees that $J<0$ for all nonzero $w(t)\in L_2^q[0,T]$.

Substituting (4.43) into the system equations (4.1) and (4.3), yields,

$$\dot{x}(t)-\overline{A}_g(t)\dot{x}(t-g(t))=\hat{A}(t)x(t)+\overline{A}_h(t)x(t-h(t))+B_1(t)w(t) \tag{4.44}$$

$$z(t)=col\left\{\hat{C}_0(t)x(t)+\hat{B}_2(t)w(t),\overline{C}_1(t-h(t))x(t-h(t)),\overline{C}_2(t-g(t))x(t-g(t))\right\} \tag{4.45}$$

with,

$$\hat{A}(t)=\overline{A}(t)+\overline{B}(t)K(t),\quad \hat{C}_0(t)=\begin{bmatrix}\overline{C}_0(t)\\ D(t)K(t)\end{bmatrix}\quad \text{and}\quad \hat{B}_2(t)=\begin{bmatrix}B_2(t)\\ 0\end{bmatrix} \tag{4.46}$$

Direct application of the BRL derived in the previous section to the above system leads to a nonlinear matrix inequality which involves terms $P_2^T(t)\overline{B}(t)K(t)$ and $K^T(t)\overline{B}^T(t)P_3(t)$. Instead, a different form of (4.20) is used,

$$\begin{bmatrix}\overline{\Omega}_1 & \overline{\Omega}_2\\ \overline{\Omega}_2^T & \overline{\Omega}_3\end{bmatrix}<0 \tag{4.47}$$

with,

$$\overline{\Omega}_1=\begin{bmatrix}\overline{\Psi} & P^T\begin{bmatrix}0\\ B_1\end{bmatrix} & h_{\max}\Phi & -W^T\begin{bmatrix}0\\ A_h\end{bmatrix}-\begin{bmatrix}\varepsilon_4 E_h^T E_h\\ 0\end{bmatrix} & P^T\begin{bmatrix}0\\ A_g\end{bmatrix} & \begin{bmatrix}C_0^T\\ 0\end{bmatrix}\\ \begin{bmatrix}0 & B_1^T\end{bmatrix}P & -\gamma^2 I & 0 & 0 & 0 & B_2^T\\ h_{\max}\Phi^T & 0 & -h_{\max}R & 0 & 0 & 0\\ -\begin{bmatrix}0 & A_h^T\end{bmatrix}W-\begin{bmatrix}\varepsilon_4 E_h^T E_h & 0\end{bmatrix} & 0 & 0 & -(1-\alpha)S+\varepsilon_4 E_h^T E_h & 0 & 0\\ \begin{bmatrix}0 & A_g^T\end{bmatrix}P & 0 & 0 & 0 & -(1-\beta)U+\varepsilon_3 E_g^T E_g & 0\\ \begin{bmatrix}C_0 & 0\end{bmatrix} & B_2 & 0 & 0 & 0 & -I_p\end{bmatrix} \tag{4.48}$$

$$\overline{\Omega}_2=\begin{bmatrix}P^T\begin{bmatrix}0\\ H_A\end{bmatrix} & P^T\begin{bmatrix}0\\ H_h\end{bmatrix} & P^T\begin{bmatrix}0\\ H_g\end{bmatrix} & W^T\begin{bmatrix}0\\ H_h\end{bmatrix} & \begin{bmatrix}0\\ 0\end{bmatrix} & \begin{bmatrix}C_0^T\\ 0\end{bmatrix}H_{C_0} & \begin{bmatrix}C_1^T\\ 0\end{bmatrix}H_{C_1} & \begin{bmatrix}C_2^T\\ 0\end{bmatrix}H_{C_2}\\ 0 & 0 & 0 & 0 & B_2^T H_{C_0} & 0 & 0 & 0\\ & & & & 0 & & & \\ & & & & & & & \end{bmatrix} \tag{4.49}$$

$$\overline{\Omega}_3=diag\left\{-\varepsilon_1 I,-\varepsilon_2 I,-\varepsilon_3 I,-\varepsilon_4 I,-\varepsilon_5 I,-\left(\varepsilon_6 I-H_{C_0}^T H_{C_0}\right),-(1-\alpha)\left(\varepsilon_7 I-H_{C_1}^T H_{C_1}\right),-(1-\beta)\left(\varepsilon_8 I-H_{C_2}^T H_{C_2}\right)\right\} \tag{4.50}$$

where,

$$\bar{\Psi} = \Psi + \begin{bmatrix} C_0^T C_0 + \frac{1}{1-\alpha} C_1^T C_1 + \frac{1}{1-\beta} C_2^T C_2 & 0 \\ 0 & 0 \end{bmatrix} + W^T \begin{bmatrix} 0 & 0 \\ A_h & 0 \end{bmatrix} + \begin{bmatrix} 0 & A_h^T \\ 0 & 0 \end{bmatrix} W$$

$$+ \begin{bmatrix} \varepsilon_1 E_A^T E_A + (\varepsilon_2 + \varepsilon_4) E_h^T E_h + \left(\varepsilon_5 + \varepsilon_6 + \varepsilon_7 \frac{1}{1-\alpha} + \varepsilon_8 \frac{1}{1-\beta} \right) E_c^T E_c & 0 \\ 0 & 0 \end{bmatrix} \tag{4.51}$$

$$\Psi = \dot{P}^T E + P^T \begin{bmatrix} 0 & I \\ A + A_h & -I \end{bmatrix} + \begin{bmatrix} 0 & A^T + A_h^T \\ I & -I \end{bmatrix} P + \begin{bmatrix} S & 0 \\ 0 & U + h_{\max} \bar{A}_{h,\max}^T R_3 \bar{A}_{h,\max} \end{bmatrix} \tag{4.52}$$

$$W = RMP, \quad \Phi = W^T + P^T, \quad W = \begin{bmatrix} W_1 & W_2 \\ W_3 & W_4 \end{bmatrix}, \quad \Phi = \begin{bmatrix} \Phi_1 \\ \Phi_2 \end{bmatrix}, \quad R = R^T = \begin{bmatrix} R_1 & R_2 \\ R_2^T & R_3 \end{bmatrix}$$

We notice that (4.20) results from (4.47) by expansion of the matrix blocks in (4.47).

Next, (4.46) is first substituted into (4.47), and then Lemma 4.2 with a scalar parameter ε_9 is used to yield,

$$\begin{bmatrix} \hat{\Omega}_1 & \hat{\Omega}_2 \\ \hat{\Omega}_2^T & \hat{\Omega}_3 \end{bmatrix} < 0 \tag{4.53}$$

with,

$$\hat{\Omega}_1 = \begin{bmatrix} \tilde{\Psi} & P^T \begin{bmatrix} 0 \\ B_1 \end{bmatrix} & h_{\max} \Phi & -W^T \begin{bmatrix} 0 \\ A_h \end{bmatrix} - \varepsilon_4 \begin{bmatrix} E_h^T E_h \\ 0 \end{bmatrix} & P^T \begin{bmatrix} 0 \\ A_g \end{bmatrix} & \begin{bmatrix} C_0^T \\ 0 \end{bmatrix} \\ \begin{bmatrix} 0 & B_1^T \end{bmatrix} P & -\gamma^2 I & 0 & 0 & 0 & B_2^T \\ h_{\max} \Phi^T & 0 & -h_{\max} R & 0 & 0 & 0 \\ -\begin{bmatrix} 0 & A_h^T \end{bmatrix} W - \varepsilon_4 \begin{bmatrix} E_h^T E_h & 0 \end{bmatrix} & 0 & 0 & -(1-\alpha) S + \varepsilon_4 E_h^T E_h & 0 & 0 \\ \begin{bmatrix} 0 & A_g^T \end{bmatrix} P & 0 & 0 & 0 & -(1-\beta) U + \varepsilon_3 E_g^T E_g & 0 \\ \begin{bmatrix} C_0 & 0 \end{bmatrix} & B_2 & 0 & 0 & 0 & -I_p \end{bmatrix}$$

$$\hat{\Omega}_2 = \begin{bmatrix} P^T \begin{bmatrix} 0 \\ H_A \end{bmatrix} & P^T \begin{bmatrix} 0 \\ H_h \end{bmatrix} & P^T \begin{bmatrix} 0 \\ H_g \end{bmatrix} & W^T \begin{bmatrix} 0 \\ H_h \end{bmatrix} & \begin{bmatrix} 0 \\ 0 \end{bmatrix} & \begin{bmatrix} C_0^T \\ 0 \end{bmatrix} H_{C_0} & \begin{bmatrix} C_1^T \\ 0 \end{bmatrix} H_{C_1} & \begin{bmatrix} C_2^T \\ 0 \end{bmatrix} H_{C_2} & P^T \begin{bmatrix} 0 \\ H_A \end{bmatrix} \\ 0 & 0 & 0 & 0 & B_2^T H_{C_0} & 0 & 0 & 0 & 0 \\ & & & & 0 & & & & \end{bmatrix}$$

$$\hat{\Omega}_3 = diag\left\{ -\varepsilon_1 I, -\varepsilon_2 I, -\varepsilon_3 I, -\varepsilon_4 I, -\varepsilon_5 I, -\left(\varepsilon_6 I - H_{C_0}^T H_{C_0} \right), -(1-\alpha)\left(\varepsilon_7 I - H_{C_1}^T H_{C_1} \right), -(1-\beta)\left(\varepsilon_8 I - H_{C_2}^T H_{C_2} \right), -\varepsilon_9 I \right\}$$

$$\tilde{\Psi} = \hat{\Psi} + \begin{bmatrix} C_0^T C_0 + \frac{1}{1-\alpha} C_1^T C_1 + \frac{1}{1-\beta} C_2^T C_2 & 0 \\ 0 & 0 \end{bmatrix} + W^T \begin{bmatrix} 0 & 0 \\ A_h & 0 \end{bmatrix} + \begin{bmatrix} 0 & A_h^T \\ 0 & 0 \end{bmatrix} W$$

$$+\begin{bmatrix} \varepsilon_1 E_A^T E_A + (\varepsilon_2+\varepsilon_4) E_h^T E_h + \left(\varepsilon_5+\varepsilon_6+\varepsilon_7\frac{1}{1-\alpha}+\varepsilon_8\frac{1}{1-\beta}\right) E_C^T E_C & 0 \\ 0 & 0 \end{bmatrix}$$

$$\hat{\Psi} = \dot{P}^T E + P^T \begin{bmatrix} 0 & I \\ A+BK+A_h & -I \end{bmatrix} + \begin{bmatrix} 0 & A^T+K^T B^T + A_h^T \\ I & -I \end{bmatrix} P$$

$$+\begin{bmatrix} S & 0 \\ 0 & U + h_{\max}\bar{A}_{h,\max}^T R_3 \bar{A}_{h,\max} \end{bmatrix} + \begin{bmatrix} K^T D^T D K + \varepsilon_9 K^T E_B^T E_B K & 0 \\ 0 & 0 \end{bmatrix}$$

Next, a new time-varying parameters $\delta(t) \in \mathbb{R}$ is introduced such that $W(t) = \delta(t) P(t)$. By Schur complements (see Appendix A) as applied to (4.20),

$$\begin{bmatrix} -P_3(t) - P_3^T(t) + U(t) & P_3^T(t) G(t) \\ G^T(t) P_3(t) & -U(t) \end{bmatrix} < 0$$

which in turn is equivalent to,

$$-U(t) < 0 \tag{4.54}$$

$$-P_3(t) - P_3^T(t) + U(t) + P_3^T(t) G(t) U^{-1}(t) G^T(t) P_3(t) < 0 . \tag{4.55}$$

Since by the assumption in the BRL, $U(t) > 0$, (4.54) is verified and (4.55) implies that $-P_3(t) - P_3^T(t) < 0$ and so $P_3(t)$ is nonsingular. Now recall that $P(t) = \begin{bmatrix} P_1(t) & 0 \\ P_2(t) & P_3(t) \end{bmatrix}$ and that $P_1(t)$ is nonsingular, hence $P(t)$ is nonsingular. This permits to define:

$$Q(t) \triangleq \begin{bmatrix} Q_1(t) & 0 \\ Q_2(t) & Q_3(t) \end{bmatrix} \triangleq P^{-1}(t) = \begin{bmatrix} P_1^{-1}(t) & 0 \\ -P_3^{-1}(t) P_2(t) P_1^{-1}(t) & P_3^{-1}(t) \end{bmatrix} \text{ and } \Delta \triangleq diag\{Q, I_{q+p+27n+3i_c}\} \tag{4.56}$$

and to parametrize $K(t)$ so that

$$K(t) = Y(t) Q_1^{-1}(t) . \tag{4.57}$$

Multiplication of (4.53) by Δ^T and Δ, on the left and the right, respectively and the application of the Schur lemma to the quadratic term in Q, yields the following,

$$\begin{bmatrix} \Pi_1 & \Pi_2 \\ \Pi_2^T & \Pi_3 \end{bmatrix} < 0 \tag{4.58}$$

with,

$$\Pi_1 = \begin{bmatrix} \Lambda_1 & \Lambda_2 & \Lambda_4 \\ \Lambda_2^T & \Lambda_3 & 0 \\ \Lambda_4^T & 0 & \Lambda_5 \end{bmatrix}$$

$$\Lambda_1 = \begin{bmatrix} \Theta & \begin{bmatrix} 0 \\ B_1 \end{bmatrix} & h_{\max}(\delta I + I) & -\delta \begin{bmatrix} 0 \\ A_h \end{bmatrix} - \varepsilon_4 Q^T \begin{bmatrix} E_h^T E_h \\ 0 \end{bmatrix} & \begin{bmatrix} 0 \\ A_g \end{bmatrix} & Q^T \begin{bmatrix} C_0^T \\ 0 \end{bmatrix} \\ \begin{bmatrix} 0 & B_1^T \end{bmatrix} & -\gamma^2 I & 0 & 0 & 0 & B_2^T \\ h_{\max}(\delta I + I) & 0 & -h_{\max} R & 0 & 0 & 0 \\ -\delta \begin{bmatrix} 0 & A_h^T \end{bmatrix} - \varepsilon_4 \begin{bmatrix} E_h^T E_h & 0 \end{bmatrix} Q & 0 & 0 & -(1-\alpha) S + \varepsilon_4 E_h^T E_h & 0 & 0 \\ \begin{bmatrix} 0 & A_g^T \end{bmatrix} & 0 & 0 & 0 & -(1-\beta) U + \varepsilon_3 E_g^T E_g & 0 \\ \begin{bmatrix} C_0 & 0 \end{bmatrix} Q & B_2 & 0 & 0 & 0 & -I_p \end{bmatrix}$$

$$\Lambda_2 = \begin{bmatrix} Q^T \begin{bmatrix} I_n \\ 0 \end{bmatrix} & Q^T \begin{bmatrix} 0 \\ I_n \end{bmatrix} & \begin{bmatrix} Y^T & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} D^T \\ 0 \end{bmatrix} & h_{\max} Q^T \begin{bmatrix} 0 & 0 \\ 0 & \bar{A}_{h,\max}^T \end{bmatrix} & Q^T \begin{bmatrix} I_n \\ 0 \end{bmatrix} & Q^T \begin{bmatrix} C_0^T \\ 0 \end{bmatrix} & Q^T \begin{bmatrix} C_1^T \\ 0 \end{bmatrix} & Q^T \begin{bmatrix} C_2^T \\ 0 \end{bmatrix} \\ & & & 0 & & & & \end{bmatrix}$$

$$\Lambda_3 = diag\left\{-S^{-1}, -U^{-1}, -I, -h_{\max} R^{-1}, -\dot{Q}_1^T, -I, -(1-\alpha) I, -(1-\beta) I\right\}$$

$$\Lambda_4 = \begin{bmatrix} \varepsilon_1 Q^T \begin{bmatrix} E_A^T \\ 0 \end{bmatrix} & \varepsilon_2 Q^T \begin{bmatrix} E_h^T \\ 0 \end{bmatrix} & \varepsilon_4 Q^T \begin{bmatrix} E_h^T \\ 0 \end{bmatrix} & \varepsilon_5 Q^T \begin{bmatrix} E_C^T \\ 0 \end{bmatrix} & \varepsilon_6 Q^T \begin{bmatrix} E_C^T \\ 0 \end{bmatrix} & \varepsilon_7 Q^T \begin{bmatrix} E_C^T \\ 0 \end{bmatrix} & \varepsilon_8 Q^T \begin{bmatrix} E_C^T \\ 0 \end{bmatrix} & \varepsilon_9 \begin{bmatrix} Y^T & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} E_B^T \\ 0 \end{bmatrix} \\ & & & & 0 & & & \end{bmatrix}$$

$$\Lambda_5 = diag\left\{-\varepsilon_1 I, -\varepsilon_2 I, -\varepsilon_4 I, -\varepsilon_5 I, -\varepsilon_6 I, -\varepsilon_7 (1-\alpha) I, -\varepsilon_8 (1-\beta) I, -\varepsilon_9 I\right\}$$

$$\Pi_2 = \begin{bmatrix} \begin{bmatrix} 0 \\ H_A \end{bmatrix} & \begin{bmatrix} 0 \\ H_h \end{bmatrix} & \begin{bmatrix} 0 \\ H_g \end{bmatrix} & \delta \begin{bmatrix} 0 \\ H_h \end{bmatrix} & \begin{bmatrix} 0 \\ 0 \end{bmatrix} & Q^T \begin{bmatrix} C_0^T \\ 0 \end{bmatrix} H_{C_0} & Q^T \begin{bmatrix} C_1^T \\ 0 \end{bmatrix} H_{C_1} & Q^T \begin{bmatrix} C_2^T \\ 0 \end{bmatrix} H_{C_2} & \begin{bmatrix} 0 \\ H_A \end{bmatrix} \\ 0 & 0 & 0 & 0 & B_2^T H_{C_0} & 0 & 0 & 0 & 0 \\ & & & & & 0 & & & \end{bmatrix}$$

$$\Pi_3 = diag\left\{-\varepsilon_1 I, -\varepsilon_2 I, -\varepsilon_3 I, -\varepsilon_4 I, -\varepsilon_5 I, -\left(\varepsilon_6 I - H_{C_0}^T H_{C_0}\right), -(1-\alpha)\left(\varepsilon_7 I - H_{C_1}^T H_{C_1}\right), -(1-\beta)\left(\varepsilon_8 I - H_{C_2}^T H_{C_2}\right), -\varepsilon_9 I\right\}$$

$$\Theta = \begin{bmatrix} 0 & I \\ A + A_h & -I \end{bmatrix} Q + Q^T \begin{bmatrix} 0 & A^T + A_h^T \\ I & -I \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ BY & 0 \end{bmatrix} + \begin{bmatrix} 0 & Y^T B^T \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ \delta A_h & 0 \end{bmatrix} Q + Q^T \begin{bmatrix} 0 & \delta A_h^T \\ 0 & 0 \end{bmatrix}$$

Finally defining, $\bar{S}(t) = S^{-1}(t)$, $\bar{U}(t) = U^{-1}(t)$ and $\bar{R}(t) = R^{-1}(t)$, and expanding the block matrices in (4.58), leads to the following version of the BRL for the closed-loop system.

Theorem 4.2. *Consider the system* (4.1)-(4.3) *with a performance index* (4.9). *Given* $\gamma > 0$ *and a symmetric, positive-definite matrix* P_{1T}, *suppose that for some chosen time-varying parameter* $\delta(t) \in R$, *there exist* $n \times n$*-matrices:* $Q_1(t) = Q_1^T(t) > 0$ *with* $Q_1(T) = P_{1T}^{-1}$, $Q_2(t)$, $Q_3(t)$, $S(t) = S^T(t) > 0$ *with* $\dot{S}(t) \le 0$, $U(t) = U^T(t) > 0$ *with* $\dot{U}(t) \le 0$, $R_1(t) = R_1^T(t)$, $R_2(t)$, $R_3(t) = R_3^T(t) > 0$ *and* $Y(t) \in R^{l \times n}$, *and positive scalars* $\varepsilon_i(t)$, $i = 1,\ldots,9$, *which satisfy the following inequality,*

$$\begin{bmatrix} \overline{\Pi}_1 & \overline{\Pi}_2 \\ \overline{\Pi}_2^T & \overline{\Pi}_3 \end{bmatrix} < 0 \tag{4.59}$$

with,

$$\overline{\Pi}_1 = \begin{bmatrix} \overline{\Lambda}_1 & \overline{\Lambda}_2 & \overline{\Lambda}_4 \\ \overline{\Lambda}_2^T & \overline{\Lambda}_3 & 0 \\ \overline{\Lambda}_4^T & 0 & \overline{\Lambda}_5 \end{bmatrix}$$

$$\overline{\Lambda}_1 = \begin{bmatrix} Q_2 + Q_2^T & \Theta_1 & 0 & h_{\max}(1+\delta)\overline{R}_1 & h_{\max}(1+\delta)\overline{R}_2 & -\varepsilon_4 Q_1^T E_h^T E_h & 0 & Q_1^T C_0^T \\ \Theta_1^T & -Q_3 - Q_3^T & B_1 & h_{\max}(1+\delta)\overline{R}_2^T & h_{\max}(1+\delta)\overline{R}_3 & -\delta A_h & A_g & 0 \\ 0 & B_1^T & -\gamma^2 I & 0 & 0 & 0 & 0 & B_2^T \\ h_{\max}(1+\delta)\overline{R}_1^T & h_{\max}(1+\delta)\overline{R}_2 & 0 & -h_{\max}\overline{R}_1 & -h_{\max}\overline{R}_2 & 0 & 0 & 0 \\ h_{\max}(1+\delta)\overline{R}_2^T & h_{\max}(1+\delta)\overline{R}_3^T & 0 & -h_{\max}\overline{R}_2^T & -h_{\max}\overline{R}_3 & 0 & 0 & 0 \\ -\varepsilon_4 E_h^T E_h Q_1 & -\delta A_h^T & 0 & 0 & 0 & -(1-\alpha)S + \varepsilon_4 E_h^T E_h & 0 & 0 \\ 0 & A_g^T & 0 & 0 & 0 & 0 & -(1-\beta)U + \varepsilon_3 E_g^T E_g & 0 \\ C_0 Q_1 & 0 & B_2 & 0 & 0 & 0 & 0 & -I_p \end{bmatrix}$$

$$\overline{\Lambda}_2 = \begin{bmatrix} Q_1^T & Q_2^T & Y^T B_2^T & 0 & h_{\max} Q_2^T \overline{A}_{h,\max}^T & Q_1^T & Q_1^T C_0^T & Q_1^T C_1^T & Q_1^T C_2^T \\ 0 & Q_3^T & 0 & 0 & h_{\max} Q_3^T \overline{A}_{h,\max}^T & 0 & 0 & 0 & 0 \\ & & & & 0 & & & & \end{bmatrix}$$

$$\overline{\Lambda}_3 = diag\left\{-\overline{S}, -\overline{U}, -I, -h_{\max}\begin{bmatrix} \overline{R}_1 & \overline{R}_2 \\ \overline{R}_2^T & \overline{R}_3 \end{bmatrix}, -\dot{Q}_1^T, -I, -(1-\alpha)I, -(1-\beta)I\right\}$$

$$\overline{\Lambda}_4 = \begin{bmatrix} \varepsilon_1 Q_1^T E_A^T & \varepsilon_2 Q_1^T E_h^T & \varepsilon_4 Q_1^T E_h^T & \varepsilon_5 Q_1^T E_C^T & \varepsilon_6 Q_1^T E_C^T & \varepsilon_7 Q_1^T E_C^T & \varepsilon_8 Q_1^T E_C^T & \varepsilon_9 Y^T E_B^T \\ & & & 0 & & & & \end{bmatrix}$$

$$\overline{\Lambda}_5 = diag\left\{-\varepsilon_1 I, -\varepsilon_2 I, -\varepsilon_4 I, -\varepsilon_5 I, -\varepsilon_6 I, -\varepsilon_7(1-\alpha)I, -\varepsilon_8(1-\beta)I, -\varepsilon_9 I\right\}$$

$$\bar{\Pi}_2 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & Q_1^T C_0^T H_{C_0} & Q_1^T C_1^T H_{C_1} & Q_2^T C_2^T H_{C_2} & 0 \\ H_A & H_h & H_g & \delta H_h & 0 & 0 & 0 & 0 & H_A \\ 0 & 0 & 0 & 0 & B_2^T H_{C_0} & 0 & 0 & 0 & 0 \\ & & & & & 0 & & & \end{bmatrix}$$

$$\bar{\Pi}_3 = diag\left\{-\varepsilon_1 I, -\varepsilon_2 I, -\varepsilon_3 I, -\varepsilon_4 I, -\varepsilon_5 I, -\left(\varepsilon_6 I - H_{C_0}^T H_{C_0}\right), -(1-\alpha)\left(\varepsilon_7 I - H_{C_1}^T H_{C_1}\right), -(1-\beta)\left(\varepsilon_8 I - H_{C_2}^T H_{C_2}\right), -\varepsilon_9 I\right\}$$

where, $\bar{S}(t) = S^{-1}(t)$, $\bar{U}(t) = U^{-1}(t)$, *and* $\bar{R}_1(t)$, $\bar{R}_2(t)$ *and* $\bar{R}_3(t)$ *are the* $(1,1)$, $(1,2)$ *and* $(2,2)$ *blocks of* $\bar{R}(t) = \begin{bmatrix} R_1(t) & R_2(t) \\ R_2^T(t) & R_3(t) \end{bmatrix}^{-1} > 0$ *respectively, and,*

$$\Theta_1 = Q_3 - Q_2^T + Q_1^T\left(A^T + A_h^T + \delta A_h^T\right) + Y^T B^T.$$

Also suppose that $R_3(t)$ *satisfies the following matrix inequality,*

$$\dot{\bar{A}}_{1,\max}^T R_3 \bar{A}_{1,\max} + \bar{A}_{1,\max}^T R_3 \dot{\bar{A}}_{1,\max} + \bar{A}_{1,\max}^T \dot{R}_3 \bar{A}_{1,\max} < 0 \qquad (4.60)$$

Under these conditions, the state-feedback law given below:

$$u(t) = K(t)x(t) = Y(t)Q_1^{-1}(t)x(t) \qquad (4.61)$$

guarantees that the cost function (4.9) *satisfies* $J < 0$ *for all nonzero* $w \in L_2^q[0,T]$, *and for any value of the positive delay* $g(t)$, $t \in [0,T]$.

4.5.2. Infinite Horizon Case

It is useful to interpret the results derived in the previous sections as they refer to systems on an infinite-time horizon. It is not surprising that the solution to the infinite-horizon problem can be shown to translate into linear matrix inequalities and hence can be solved using efficient algorithms **[6]**, which do not require parameter tuning. To demonstrate this in some detail, it is assumed that all the system matrices are constant except $F_g(t)$, $F(t)$, $F_h(t)$ and $F_C(t)$, while the time-delays g and h are still considered time-varying, unknown, and bounded along with their rates of change as in (4.6)-(4.7).

The following assumption **[30]** is needed to enable the application of Lyapunov's second method for stability of neutral systems in the infinite-horizon case :

Assumption 4.1. The difference operator D: $C_n(-\infty,0]\times\mathbb{R}\rightarrow\mathbb{R}^n$, given by $D(x_t,t)=x(t)-\bar{A}_g(t)x(t-g(t))$, is delay-independently stable (i.e., the homogeneous difference equation $Dx_t=0$ is asymptotically stable irrespective of the delay g).

For any input signal $w\in L_2^q[0,\infty)$ the performance index J is now written as,

$$J_0=\int_0^\infty\left[z^T(t)z(t)-\gamma^2w^T(t)w(t)\right]dt \tag{4.62}$$

The theorem found below is an analog of the BRL of Theorem 4.1 which additionally provides conditions under which the system (4.1) is globally uniformly asymptotically stable (g.u.a.s), and under which $J_0<0$, for all $w\in L_2^q[0,\infty)$.

Theorem 4.3. (bounded real lemma for the infinite-horizon case) *Consider the time-invariant version of system* (4.1)-(4.3) *as defined above. For a given* $\gamma>0$, *suppose there exist* $n\times n$ *-matrices:* $P_1>0$, P_2, P_3, $S=S^T>0$, $U=U^T>0$, W_1, W_2, W_3, W_4 *and* $R_1=R_1^T$, R_2, $R_3=R_3^T>0$, *and positive scalars* ε_i; $i=1...8$, *which satisfy the following linear matrix inequality:*

$$\begin{bmatrix}\tilde{\Omega}_1 & \tilde{\Omega}_2\\ \tilde{\Omega}_2^T & \tilde{\Omega}_3\end{bmatrix}<0 \tag{4.63}$$

where, (4.63) *has the same expression as* (4.20) *in Theorem 4.1, except that all entries are time-invariant. Under these conditions, system* (4.1) *is globally uniformly asymptotically stable (g.u.a.s), and the cost function* (4.62) *satisfies* $J_0<0$, *for all nonzero* $w\in L_2^q[0,\infty)$, *and for any value of the positive delay* g.

Proof. By the application of the Schur complements lemma (see Appendix A) to (4.63), it is easy to show that $\dot{V}<0$, which along with Assumption 4.1 guarantees the global uniform asymptotic stability of system (4.1), and also implies that $z\in L_2^p[0,\infty)$ and that J_0 is well defined. Close inspection of (4.24) further implies that this condition is redundant. The proof of $J_0<0$ then follows exactly like that of Theorem 4.1.

QED

The next result is the equivalent of Theorem 4.2.

Theorem 4.4. (Infinite-Horizon Case) *Consider the time-invariant version of system* (4.1)-(4.3) *with a performance index* (4.62). *Given* $\gamma>0$, *suppose that for some chosen parameter* $\delta\in R$, *there exist* $n\times n$ -

matrices: $Q_1 = Q_1^T > 0$, Q_2, Q_3, $S = S^T > 0$, $U = U^T > 0$, $R_1 = R_1^T$, R_2, $R_3 = R_3^T > 0$ *and* $Y \in \mathbb{R}^{l \times n}$, *and positive scalars* ε_i, $i = 1 \ldots 9$, *which satisfy the following inequality,*

$$\begin{bmatrix} \hat{\Pi}_1 & \hat{\Pi}_2 \\ \hat{\Pi}_2^T & \hat{\Pi}_3 \end{bmatrix} < 0 \tag{4.64}$$

with

$$\hat{\Pi}_1 = \begin{bmatrix} \hat{\Lambda}_1 & \hat{\Lambda}_2 & \hat{\Lambda}_4 \\ \hat{\Lambda}_2^T & \hat{\Lambda}_3 & 0 \\ \hat{\Lambda}_4^T & 0 & \hat{\Lambda}_5 \end{bmatrix}$$

$$\hat{\Lambda}_1 = \begin{bmatrix} Q_2 + Q_2^T & \hat{\Theta}_1 & 0 & h_{\max}(1+\delta)\bar{R}_1 & h_{\max}(1+\delta)\bar{R}_2 & -\varepsilon_4 Q_1^T E_h^T E_h & 0 & Q_1^T C_0^T \\ \hat{\Theta}_1^T & -Q_3 - Q_3^T & B_1 & h_{\max}(1+\delta)\bar{R}_2^T & h_{\max}(1+\delta)\bar{R}_3 & -\delta A_h & A_g & 0 \\ 0 & B_1^T & -\gamma^2 I & 0 & 0 & 0 & 0 & B_2^T \\ h_{\max}(1+\delta)\bar{R}_1^T & h_{\max}(1+\delta)\bar{R}_2 & 0 & -h_{\max}\bar{R}_1 & -h_{\max}\bar{R}_2 & 0 & 0 & 0 \\ h_{\max}(1+\delta)\bar{R}_2^T & h_{\max}(1+\delta)\bar{R}_3^T & 0 & -h_{\max}\bar{R}_2^T & -h_{\max}\bar{R}_3 & 0 & 0 & 0 \\ -\varepsilon_4 E_h^T E_h Q_1 & -\delta A_h^T & 0 & 0 & 0 & -(1-\alpha)S + \varepsilon_4 E_h^T E_h & 0 & 0 \\ 0 & A_g^T & 0 & 0 & 0 & 0 & -(1-\beta)U + \varepsilon_3 E_g^T E_g & 0 \\ C_0 Q_1 & 0 & B_2 & 0 & 0 & 0 & 0 & -I_p \end{bmatrix}$$

$$\hat{\Lambda}_2 = \begin{bmatrix} Q_1^T & Q_2^T & Y^T D^T & 0 & h_{\max} Q_2^T \bar{A}_{h,\max}^T & Q_1^T C_0^T & Q_1^T C_1^T & Q_1^T C_2^T \\ 0 & Q_3^T & 0 & 0 & h_{\max} Q_3^T \bar{A}_{h,\max}^T & 0 & 0 & 0 \\ & & & & 0 & & & \end{bmatrix}$$

$$\hat{\Lambda}_3 = diag\left\{ -\bar{S}, -\bar{U}, -I, -h_{\max} \begin{bmatrix} \bar{R}_1 & \bar{R}_2 \\ \bar{R}_2^T & \bar{R}_3 \end{bmatrix}, -I, -(1-\alpha)I, -(1-\beta)I \right\}$$

$$\hat{\Lambda}_4 = \begin{bmatrix} \varepsilon_1 Q_1^T E_A^T & \varepsilon_2 Q_1^T E_h^T & \varepsilon_4 Q_1^T E_h^T & \varepsilon_5 Q_1^T E_C^T & \varepsilon_6 Q_1^T E_C^T & \varepsilon_7 Q_1^T E_C^T & \varepsilon_8 Q_1^T E_C^T & \varepsilon_9 Y^T E_B^T \\ & & & 0 & & & & \end{bmatrix}$$

$$\hat{\Lambda}_5 = diag\left\{ -\varepsilon_1 I, -\varepsilon_2 I, -\varepsilon_4 I, -\varepsilon_5 I, -\varepsilon_6 I, -\varepsilon_7 (1-\alpha) I, -\varepsilon_8 (1-\beta) I, -\varepsilon_9 I \right\}$$

$$\hat{\Pi}_2 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & Q_1^T C_0^T H_{C_0} & Q_1^T C_1^T H_{C_1} & Q_2^T C_2^T H_{C_2} & 0 \\ H_A & H_h & H_g & \delta H_h & 0 & 0 & 0 & 0 & H_A \\ 0 & 0 & 0 & 0 & B_2^T H_{C_0} & 0 & 0 & 0 & 0 \\ & & & & & 0 & & & \end{bmatrix}$$

$$\hat{\Pi}_3 = diag\left\{ -\varepsilon_1 I, -\varepsilon_2 I, -\varepsilon_3 I, -\varepsilon_4 I, -\varepsilon_5 I, -\left(\varepsilon_6 I - H_{C_0}^T H_{C_0}\right), -(1-\alpha)\left(\varepsilon_7 I - H_{C_1}^T H_{C_1}\right), -(1-\beta)\left(\varepsilon_8 I - H_{C_2}^T H_{C_2}\right), -\varepsilon_9 I \right\}$$

where, $\bar{S} = S^{-1}$, $\bar{U} = U^{-1}$, *and* $\bar{R}_1$, $\bar{R}_2$ *and* $\bar{R}_3$ *are the* $(1,1)$, $(1,2)$ *and* $(2,2)$ *blocks of*

$\bar{R} = \begin{bmatrix} R_1 & R_2 \\ R_2^T & R_3 \end{bmatrix}^{-1} > 0$ *respectively, and,*

$$\hat{\Theta}_1 = Q_3 - Q_2^T + Q_1^T\left(A^T + A_h^T + \delta_1 A_h^T\right) + Y^T B^T$$

Under these conditions, the state-feedback law given below:

$$u(t) = Kx(t) = YQ_1^{-1}x(t) \tag{4.65}$$

guarantees that the system (4.1) *is globally uniformly asymptotically stable (g.u.a.s), and the cost function* (4.62) *satisfies* $J_0 < 0$, *for all nonzero* $w \in L_2^q[0,\infty)$, *and for any value of the positive delay* g.

Remark 4.3. When uncertainties enter only matrices A_g, A and B, then $\bar{A}_{h,\max} = A_h$ and inequality (4.64) reduces to the following LMI:

$$\begin{bmatrix} \tilde{\Pi}_1 & \tilde{\Pi}_2 \\ \tilde{\Pi}_2^T & \tilde{\Pi}_3 \end{bmatrix} < 0 \tag{4.66}$$

$$\tilde{\Pi}_1 = \begin{bmatrix} \tilde{\Lambda}_1 & \tilde{\Lambda}_2 & \tilde{\Lambda}_4 \\ \tilde{\Lambda}_2^T & \tilde{\Lambda}_3 & 0 \\ \tilde{\Lambda}_4^T & 0 & \tilde{\Lambda}_5 \end{bmatrix}$$

$$\tilde{\Lambda}_1 = \begin{bmatrix} Q_2 + Q_2^T & \tilde{\Theta}_1 & 0 & h_{\max}(1+\delta)\bar{R}_1 & h_{\max}(1+\delta)\bar{R}_2 & 0 & 0 & Q_1^T C_0^T \\ \tilde{\Theta}_1^T & -Q_3 - Q_3^T & B_1 & h_{\max}(1+\delta)\bar{R}_2^T & h_{\max}(1+\delta)\bar{R}_3 & -\delta A_h \bar{S} & A_g \bar{U} & 0 \\ 0 & B_1^T & -\gamma^2 I & 0 & 0 & 0 & 0 & B_2^T \\ h_{\max}(1+\delta)\bar{R}_1^T & h_{\max}(1+\delta)\bar{R}_2 & 0 & -h_{\max}\bar{R}_1 & -h_{\max}\bar{R}_2 & 0 & 0 & 0 \\ h_{\max}(1+\delta)\bar{R}_2^T & h_{\max}(1+\delta)\bar{R}_3^T & 0 & -h_{\max}\bar{R}_2^T & -h_{\max}\bar{R}_3 & 0 & 0 & 0 \\ 0 & -\delta\bar{S}A_h^T & 0 & 0 & 0 & -(1-\alpha)\bar{S} & 0 & 0 \\ 0 & \bar{U}A_g^T & 0 & 0 & 0 & 0 & -(1-\beta)\bar{U} & 0 \\ C_0 Q_1 & 0 & B_2 & 0 & 0 & 0 & 0 & -I_p \end{bmatrix}$$

$$\tilde{\Lambda}_2 = \begin{bmatrix} Q_1^T & Q_2^T & Y^T D^T & 0 & h_{\max} Q_2^T A_h^T & Q_1^T C_0^T & Q_1^T C_1^T & Q_1^T C_2^T \\ 0 & Q_3^T & 0 & 0 & h_{\max} Q_3^T A_h^T & 0 & 0 & 0 \\ & & & & 0 & & & \end{bmatrix}$$

$$\tilde{\Lambda}_3 = diag\left\{-\bar{S}, -\bar{U}, -I, -h_{\max}\begin{bmatrix} \bar{R}_1 & \bar{R}_2 \\ \bar{R}_2^T & \bar{R}_3 \end{bmatrix}, -I, -(1-\alpha)I, -(1-\beta)I\right\}$$

$$\tilde{\Lambda}_4 = \begin{bmatrix} Q_1^T E_A^T & Y^T E_B^T \\ 0_{7\times 1} & 0_{7\times 1} \end{bmatrix}$$

$\tilde{\Lambda}_5 = diag\{-\bar{\varepsilon}_1 I, -\bar{\varepsilon}_9 I\}$

$$\tilde{\Pi}_2 = \begin{bmatrix} \bar{\varepsilon}_1 H_A & 0 & \bar{\varepsilon}_9 H_A \\ & 0_{4\times3} & \\ 0 & E_g^T & 0 \\ & 0_{11\times3} & \end{bmatrix}$$

$\tilde{\Pi}_3 = diag\{-\bar{\varepsilon}_1 I, -\bar{\varepsilon}_3 I, -\bar{\varepsilon}_9 I\}$

with,

$\tilde{\Theta}_1 = Q_3 - Q_2^T + Q_1^T\left(A^T + A_h^T + \delta A_h^T\right) + Y^T B^T$, $\quad \bar{\varepsilon}_1 = \varepsilon_1^{-1}$, $\bar{\varepsilon}_3 = \varepsilon_3^{-1}$ and $\bar{\varepsilon}_9 = \varepsilon_9^{-1}$.

For the infinite-horizon time-invariant case with no uncertainties, the results of **[20]** are recovered.

4.5.3. Numerical Examples

The following examples were run on LMI-LAB Toolbox in MATLAB. Bound $\bar{A}_{h,\max}$ is chosen using Lemma 4.2 (see Remark 4.1). Parameter ε in Lemma 4.2 is adjusted until the resulting bound $\bar{A}_{h,\max}$ results in the best stability limit for $h_{\max}$ when the LMI of Theorem 4.3 is solved. This tuning of parameter ε is actually relatively easy because of the bell-like relation between $h_{\max}$ and ε. The results presented below are more competitive than any other result presented in the literature thus far.

Example 4.1. Consider the uncertain time-delay system (4.1)-(4.3) with:

$A = \begin{bmatrix} -2 & 0 \\ 0 & -0.9 \end{bmatrix}$, $\quad A_h = \begin{bmatrix} -1 & 0 \\ -1 & -1 \end{bmatrix}$, $\quad B_1 = \begin{bmatrix} -0.5 \\ 1 \end{bmatrix}$, $\quad C_0 = \begin{bmatrix} 1 & 0 \end{bmatrix}$, $\quad H_A = H_h = diag\{0.2, 0.2\}$ and

$E_A = E_h = diag\{1,1\}$

and assume that all the other matrices in (4.1)-(4.3) are zero. The same example was used in **[12]** to compare the method developed there with previous results while using $h_{\max} = 0.4437$.

The application of Theorem 4.3 with $\alpha = 0$ and $\beta = 0$ while varying parameter ε used in the selection of $\bar{A}_{h,\max}$ reveals a bell-like relation between $h_{\max}$ and ε as shown in Figure 4.1. The maximum delay for which robust stability is guaranteed is $h_{\max} = 1.4657$, proving that the approach adopted here is much less conservative than **[12]** despite the presence of parametric uncertainties. The corresponding ε and $\bar{A}_{h,\max}$ are

$$\varepsilon = 6.2\text{, and } \bar{A}_{h,\max} = \begin{bmatrix} -1.6072 & -0.4877 \\ -0.4877 & -1.1195 \end{bmatrix}.$$

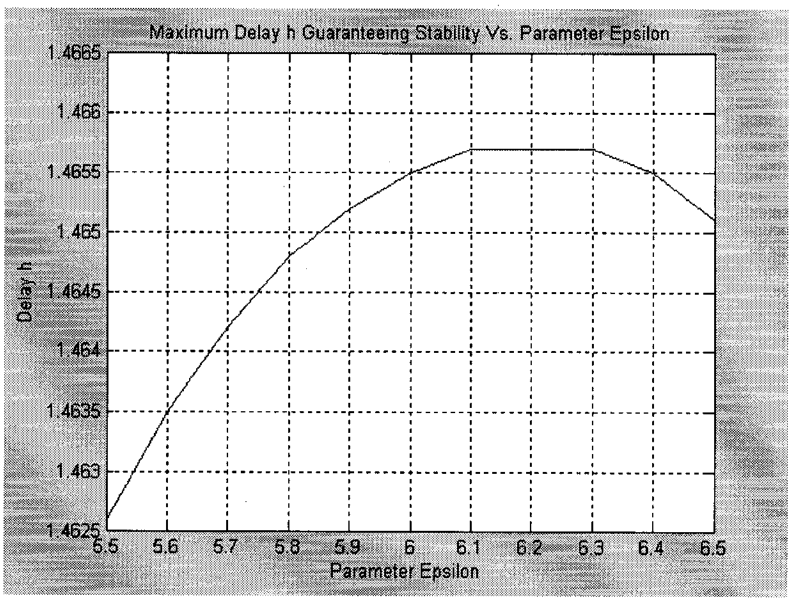

Figure 4.1. *Maximum delay* $h_{\max}$ *guaranteeing robust stability vs tuning parameter* ε *:* Example 4.1

Finally, for a delay $h = 0.7$, $\alpha = 0$ and $\beta = 0$, Figure 4.2 shows the relation between the achieved disturbances attenuation γ and tuning parameter ε.

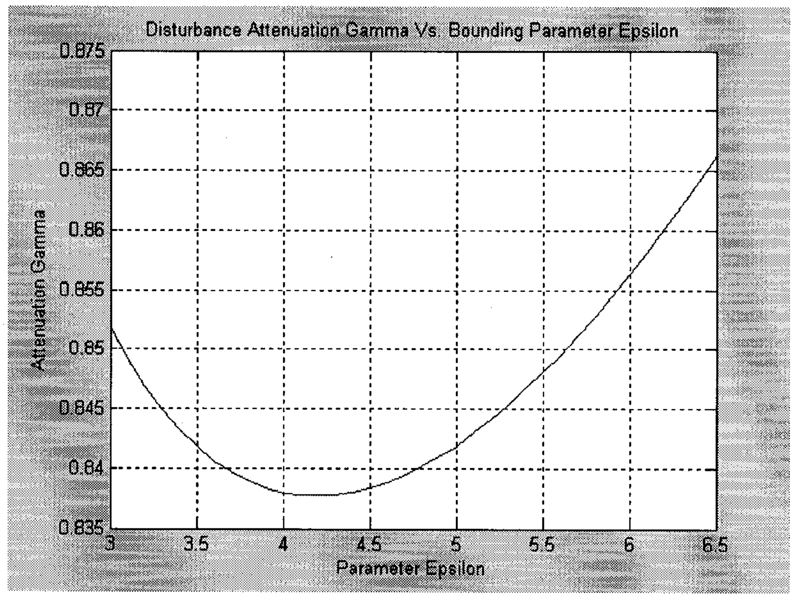

Figure 4.2. *Minimum disturbance attenuation γ vs tuning parameter ε :* Example 4.1

It is seen that the best disturbances attenuation is $\gamma = 0.8377$ and is achieved for $\varepsilon = 4.1$ and $\bar{A}_{h,\max} = \begin{bmatrix} -1.5608 & -0.4473 \\ -0.4473 & -1.1136 \end{bmatrix}$. It is noticed that the highest $h_{\max}$ and the lowest γ are not achieved for the same value of parameter ε, although from Figure 4.2, it is seen that the attenuation γ achieved for $\varepsilon = 6.2$ (corresponding to the highest $h_{\max}$, see above) is $\gamma = 0.8601$, which is very close to the lowest value, namely $\gamma = 0.8377$ (achieved for $\varepsilon = 4.1$).

Example 4.2. Consider the uncertain time-delay system (4.1)-(4.3) with:

$$A = \begin{bmatrix} -2 & 0 \\ 0 & -1 \end{bmatrix}, \quad A_h = \begin{bmatrix} -1 & 0 \\ -1 & -1 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad C_0 = \begin{bmatrix} 1 & 0 \end{bmatrix}, \quad H_A = E_A = diag\left\{\sqrt{1.6}, \sqrt{0.05}\right\} \quad \text{and}$$

$$H_h = E_h = diag\left\{\sqrt{0.1}, \sqrt{0.3}\right\}$$

and assume that all the other matrices in (4.1)-(4.3) are zero. The same example was used in **[39]** to compare the method developed there with previous results while using $h_{\max} = 0.2412$.

The application of Theorem 4.3 with $\alpha = 0$ and $\beta = 0$ while varying parameter ε used in the selection of $\bar{A}_{h,\max}$ reveals a bell-like relation between $h_{\max}$ and ε as shown in Figure 4.3. The maximum delay for which robust stability is guaranteed is $h_{max} = 0.6562$, proving that the approach adopted here is much less conservative than **[39]** despite the presence of parametric uncertainties. The corresponding ε and $\overline{A}_{h,max}$ are $\varepsilon = 0.6$, and $\overline{A}_{h,max} = \begin{bmatrix} -1.5003 & -0.4462 \\ -0.4462 & -1.2331 \end{bmatrix}$.

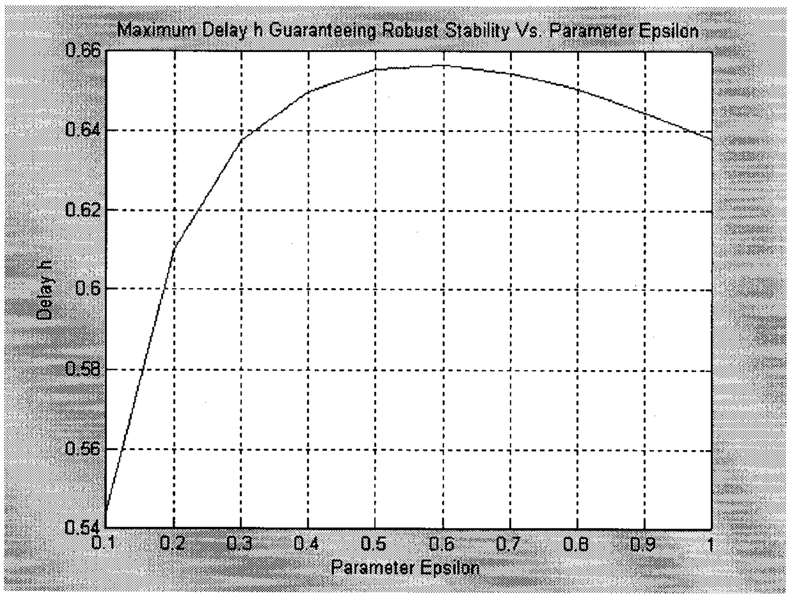

Figure 4.3. *Maximum delay* h_{max} *guaranteeing robust stability vs tuning parameter* ε *:* Example 4.2

In **[93]**, for a rate of change $\alpha = 0.9$, robust stability is guaranteed with $h_{max} = 0.1561$. Using Theorem 4.3 with $\alpha = 0.9$, robust stability is found to be guaranteed for $h_{max} = 0.4964$. Figure 4.4 gives the values of the maximum time-delay h_{max} guaranteeing robust stability in function of the rate of change α of time-delay h.

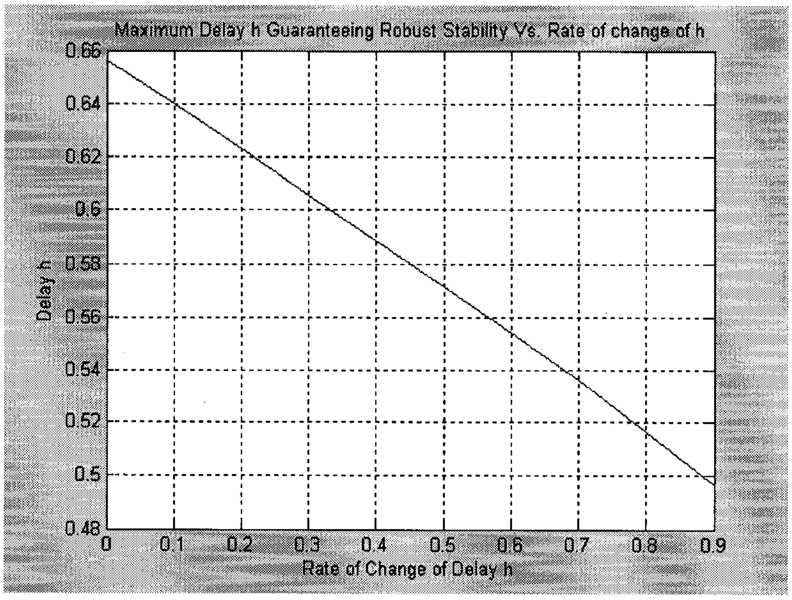

Figure 4.4. *Maximum delay h_{max} guaranteeing robust stability Vs. rate of change of h :* Example 4.2

Finally, for $h = 0.4$, $\alpha = 0$, $\beta = 0$ and $\varepsilon = 0.6$, a minimum value of disturbance attenuation $\gamma = 0.6064$ is achieved.

Example 4.3. Consider the uncertain time-delay system (4.1)-(4.3) with:

$$A = \begin{bmatrix} -2 & 0 \\ 1 & -3 \end{bmatrix}, \quad A_h = \begin{bmatrix} -1 & 0 \\ -0.8 & -1 \end{bmatrix}, \quad B_1 = \begin{bmatrix} -0.5 \\ 1 \end{bmatrix}, \quad C_0 = \begin{bmatrix} 1 & 0 \end{bmatrix},$$

$H_A = E_A = diag\{\sqrt{0.2}, \sqrt{0.2}\}$ and $H_h = E_h = diag\{\sqrt{0.2}, \sqrt{0.2}\}$

and assume that all the other matrices in (4.1)-(4.3) are zero. The same example was used in **[39]** to compare the method developed there with previous results while using $h_{max} = 0.5351$.

The application of Theorem 4.3 with $\alpha = 0$ and $\beta = 0$ while varying parameter ε used in the selection of $\bar{A}_{h,\max}$ reveals a bell-like relation between h_{max} and ε as shown in Figure 4.5. The maximum delay for which robust stability is guaranteed is $h_{max} = 3.4092$, proving that the approach adopted here is much less conservative than **[39]** despite the presence of parametric uncertainties. The corresponding ε and $\bar{A}_{h,\max}$ are $\varepsilon = 0.9$, and $\bar{A}_{h,\max} = \begin{bmatrix} -1.4420 & -0.3779 \\ -0.3779 & -1.1397 \end{bmatrix}$.

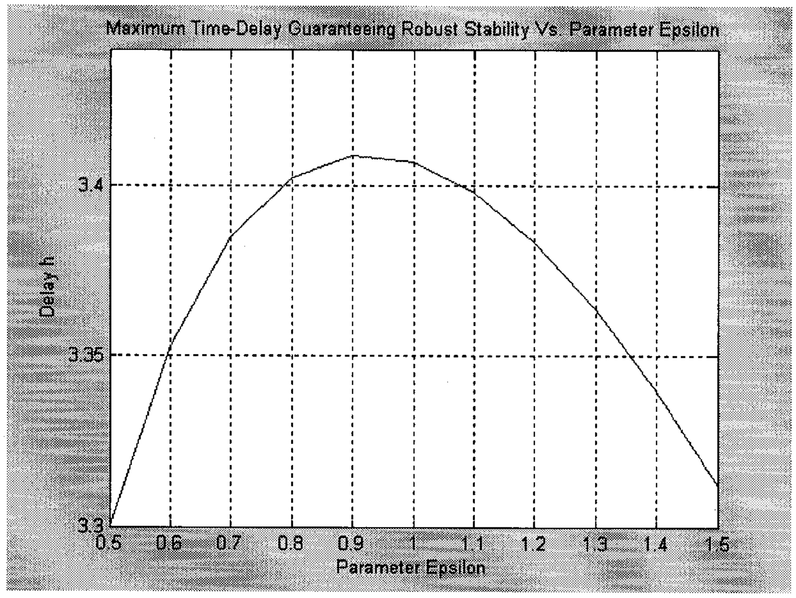

Figure 4.5. *Maximum delay* $h_{\max}$ *guaranteeing robust stability vs tuning parameter* ε *:* Example 4.3

In **[93]**, for a rate of change $\alpha = 0.9$, robust stability is guaranteed for $h_{\max} = 0.3151$. Using Theorem 4.3 with $\alpha = 0.9$, robust stability is found to be guaranteed for $h_{\max} = 0.8773$. Figure 4.6 gives the values of the maximum time-delay $h_{\max}$ guaranteeing robust stability in function of the rate of change α of time-delay h.

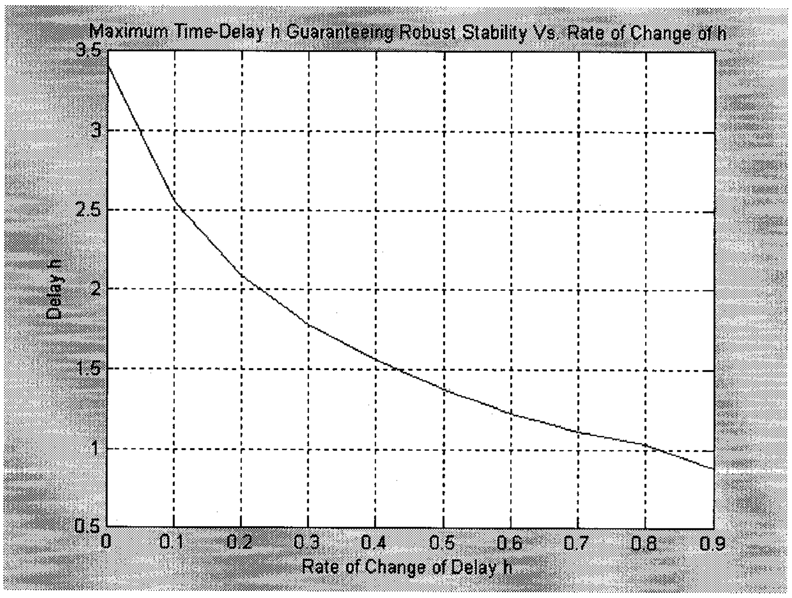

Figure 4.6. *Maximum delay* $h_{\max}$ *guaranteeing robust stability vs rate of change of* h *:* Example 4.3

Finally, for $\alpha = 0$ and $\beta = 0$, and assuming the following value for $A_g = \begin{bmatrix} -0.8 & 0 \\ 0.8 & -0.1 \end{bmatrix}$, the system is still stable for $h_{\max} = 0.2074$. For $h = 0.09$, $\alpha = 0$, $\beta = 0$ and $\varepsilon = 0.9$, a minimum value of $\gamma = 0.9272$ is achieved.

Example 4.4. Consider the uncertain time-delay system (4.1) and (4.3) with:

$$A = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix},\ A_h = \begin{bmatrix} -1 & -1 \\ 0 & -0.9 \end{bmatrix},\ B_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix},\ B = \begin{bmatrix} 0 \\ 1 \end{bmatrix},\ C_0 = \begin{bmatrix} 0 & 1 \end{bmatrix},\ D = 0.1$$

$$H_A = diag\{0.2, 0.2\},\ E_A = diag\{1,1\} \text{ and } E_B = \begin{bmatrix} 0.2 \\ 0.2 \end{bmatrix}$$

and assume that all the other matrices in (4.1)-(4.3) are zero.

Applying the LMI (4.66) with the parameter $\delta_1 = -0.3$, the closed-loop system is still stable for $h_{\max} = 0.9297$. The corresponding controller gain matrix is $K = \begin{bmatrix} 47.5 & -22478 \end{bmatrix}$.

For $h = 0.8$, the minimum value of attenuation achieved is $\gamma = 0.5373$. The corresponding controller gain matrix is then $K = \begin{bmatrix} -0.2215 & -327.3820 \end{bmatrix}$.

When the neutral delay term is added with $A_g = diag\{-0.1, -0.2\}$, the maximum h for which a state-feedback controller stabilizes the system is $h_{\max} = 0.7998$. The corresponding controller gain matrix is $K = \begin{bmatrix} 322.3 & -7193.3 \end{bmatrix}$.

For $h = 0.75$, the minimum value of attenuation is $\gamma = 0.4236$. The corresponding controller gain matrix is $K = \begin{bmatrix} -0.2620 & -89.8172 \end{bmatrix}$.

CHAPTER 5

Delay-Dependent State-Feedback Robust Stabilization of Uncertain Neutral Systems with Saturating Actuators: The Differential Inclusions Model

5.1. System Description

Consider the neutral system (1.1) and (1.3)-(1.5) taken in the infinite-horizon context, where all the system matrices are time-invariant except the uncertainty matrices F_i. More specifically, the system under consideration is:

$$\dot{x}(t)-A_g(t)\dot{x}(t-g(t))=A(t)x(t)+A_h(t)x(t-h(t))+B(t)u(t) \tag{5.1}$$

$$x(t_0+\psi)=\phi(\psi),\ \forall\psi\in[-d_{\max},0],\ (t_0,\phi)\in\mathbb{R}^+\times C^w_{d_{\max},n} \tag{5.2}$$

$$A_g(t)=A_g+\Delta A_g(t),\quad A(t)=A+\Delta A(t),\quad A_h(t)=A_h+\Delta A_h(t),\quad B(t)=B+\Delta B(t) \tag{5.3}$$

$$\Delta A_g(t)=H_gF_g(t)E_g,\quad \begin{bmatrix}\Delta A(t)\\ \Delta B(t)\end{bmatrix}=H_AF(t)\begin{bmatrix}E_A\\ E_B\end{bmatrix},\qquad \Delta A_h(t)=H_hF_h(t)E_h \tag{5.4}$$

5.2. Assumptions

The delays h and g in the system are functions of time and are assumed to be continuously differentiable, with their respective amplitudes and rates of change bounded as follows:

$$0\le h(t)\le h_{\max},\qquad 0\le g(t)<\infty,\qquad \text{for all } t\ge 0 \tag{5.5}$$

$$0\le \dot{h}(t)\le\alpha<1,\quad 0\le\dot{g}(t)\le\beta<1,\qquad \text{for all } t\ge 0 \tag{5.6}$$

where $h_{\max}$, α and β are given positive constants.

Also, $A_h(t)$ is assumed to be bounded as follows (see Remark 5.2 below):

$$A_h^T(t)A_h(t) \le A_{h,\max}^T A_{h,\max} \tag{5.7}$$

where matrix $A_{h,\max}$ is constant and known.

Finally, the following is assumed to hold:

Assumption 5.1. $(A + A_h, B)$ is stabilizable.

Assumption 5.2. The input vector is subject to amplitude constraints, i.e. $u \in U_0 \subset \mathbb{R}^m$, with

$$U_0 \triangleq \{u \in R^m \ ; \ -\overline{u}_i \le u_i \le \overline{u}_i \ , \ \ i = 1...m\} \tag{5.8}$$

where vector $\overline{u} \triangleq [\overline{u}_1,...,\overline{u}_m]^T$ has strictly positive entries and is given.

The following additional assumption **[30]**, is needed to enable the application of Lyapunov's second method for the stability of neutral systems :

Assumption 5.3. The difference operator $D: C((-\infty,0],\mathbb{R}^n)\times\mathbb{R} \to \mathbb{R}^n$, given by $D(x_t,t) = x(t) - A_g(t)x(t-g(t))$, is delay-independently asymptotically stable (i.e., the homogeneous difference equation $Dx_t = 0$ is asymptotically stable irrespective of the delay g).

5.3. Problem Statement

The robust stabilization problem with saturating actuators:

Find a matrix $K \in \mathbb{R}^{m\times n}$ and a set of initial conditions $S_0 \subset \mathbb{R}^n$ such that the closed-loop system:

$$\dot{x}(t) - A_g(t)\dot{x}(t-g(t)) = A(t)x(t) + A_h(t)x(t-h(t)) + B(t)u(t)$$

with $u(t) = sat(Kx(t))$ (5.9)

is asymptotically stable.

Remark 5.1. Generally, global asymptotic stability for an open-loop unstable system with bounded controls cannot be achieved so only local asymptotic stability will be sought.

It is the objective of this chapter to derive a stabilizing control law which depends on the bound h_{max} of the time-delay h, the bounds β_h and β_g on the rates of change $\dot{h}$ and $\dot{g}$, but not on the time-delay g itself. This is to ensure that, any variation in g does not destabilize the system; see **[29, 52]**.

5.4. Preliminaries

The following lemmas will prove helpful in the sequel:

Lemma 5.1. [39] *Let $a(t): R^+ \to R$, $b(t): R^+ \to R$ and $f(s): R \to R$ be continuously differentiable functions.*

Let the function $z(t) = \int_{a(t)}^{b(t)} \int_{t-\theta}^{t} f(s) ds d\theta$. Then $z(t)$ is a solution of the differential equation,

$$\frac{dz(t)}{dt} = (b(t) - a(t)) f(t) - (1 - \dot{b}(t)) \int_{t-b(t)}^{t-a(t)} f(s) ds + (\dot{b}(t) - \dot{a}(t)) \int_{t-a(t)}^{t} f(s) ds .$$

Lemma 5.2. [12] *Let A, L, E and F be real matrices (possibly time-varying) of appropriate dimensions, with F satisfying $FF^T \le I$. Then the following holds:*

1- For any scalar $\varepsilon > 0$ and any matrix P,

$P^T LFE + E^T F^T L^T P \le \varepsilon^{-1} P^T L L^T P + \varepsilon E^T E$

2- For any matrix $P > 0$ and any scalar $\varepsilon > 0$ such that $\varepsilon I - EPE^T > 0$,

$(A + LFE) P (A + LFE)^T \le APA^T + APE^T (\varepsilon I - EPE^T)^{-1} EPA^T + \varepsilon L L^T$

3- For any matrix $P > 0$ and any scalar $\varepsilon > 0$ such that $P - \varepsilon L L^T > 0$,

$(A + LFE)^T P^{-1} (A + LFE) \le A^T (P - \varepsilon L L^T)^{-1} A + \varepsilon^{-1} E^T E$

Remark 5.2. Statement 3 in Lemma 5.2 as applied to $A_h(t)$ can be used to choose the bound $A_{h,\max}$ in (5.7), as follows:

$$A_h^T(t) A_h(t) = (A_h + H_h F_h(t) E_h)^T (A_h + H_h F_h(t) E_h)$$

$$\le A_h^T (I - \zeta H_h H_h^T)^{-1} A_h + \lambda^{-1} E_h^T E_h = A_{h,\max}^T A_{h,\max} \qquad (5.10)$$

with,

$\zeta > 0$ a positive scalar chosen such that, $\left(I - \zeta H_h H_h^T\right) > 0$.

Lemma 5.3. [62] *Assume that* $a : \Omega \rightarrow \mathbb{R}^n$ *and* $b : \Omega \rightarrow \mathbb{R}^n$, $\Omega \subset \mathbb{R}$ *are integrable functions over their common domain* Ω. *Then, for any positive definite matrix* $R \in \mathbb{R}^{n\times m}$ *and any matrix* $M \in \mathbb{R}^{m\times m}$, *the following inequality holds:*

$$-2\int_{\Omega} b^T(s)a(s)ds \leq \int_{\Omega} \begin{bmatrix} a(s) \\ b(s) \end{bmatrix}^T \begin{bmatrix} R & RM \\ M^T R & \Upsilon \end{bmatrix} \begin{bmatrix} a(s) \\ b(s) \end{bmatrix} ds \tag{5.11}$$

where $\Upsilon = \left(M^T R + I\right) R^{-1} \left(RM + I\right)$.

5.5. Main Result

A locally equivalent polytopic representation for the closed loop nonlinear system (5.9) based on the concept of differential inclusions **[58]** is used here and leads to,

$$\dot{x}(t) - A_g(t)\dot{x}(t-g(t)) = \left(A(t) + B(t)\Gamma(\alpha(x))K\right)x(t) + A_h(t)x(t-h(t)) \tag{5.12}$$

where $\Gamma(\alpha(x)) \triangleq diag\{\alpha_i(x);\ i = 1,\ldots,m\}$ is a diagonal matrix whose diagonal elements are defined by :

$$\alpha_i(x) \triangleq \begin{cases} \dfrac{\bar{u}_i}{K_{(i)}x} & \text{if} \quad K_{(i)}x > \bar{u}_i \\ 1 & \text{if} \quad -\bar{u}_i \leq K_{(i)}x \leq \bar{u}_i \qquad i = 1,\ldots,m \\ -\dfrac{\bar{u}_i}{K_{(i)}x} & \text{if} \quad K_{(i)}x < -\bar{u}_i \end{cases} \tag{5.13}$$

where $K_{(i)}$ is the i-th row of matrix K. Clearly $0 < \alpha_i(x) \leq 1,\ i = 1,\ldots,m,\ \forall x \in \mathbb{R}^n$. The value $\alpha_i(x)$ can be interpreted as an indicator of the saturation degree of the control law. The smaller is α_i, the farther is x from the region of linearity of the control u, $S(\bar{u}, 1_m)$,

$$S(\bar{u}, 1_m) \triangleq \left\{x \in \mathbb{R}^n : \left|K_{(i)}x\right| \leq \bar{u}_i,\ i = 1,\ldots,m\right\} \tag{5.14}$$

In an effort to estimate the size of the region of attraction for the local stabilization of the constructed robust controller, the following lower bound for $\alpha_i(x)$ is introduced to correspond to any compact set $S_c \subset \mathbb{R}^n$,

$$\underline{\alpha}_i \triangleq \min\{\alpha_i(x): x \in S_c\}, \quad i = 1,\dots,m \tag{5.15}$$

so that,

$$0 < \underline{\alpha}_i \leq \alpha_i(x) \leq 1, \quad \forall x \in S_c, \qquad i = 1,\dots,m \tag{5.16}$$

For a fixed vector $\underline{\alpha} \triangleq [\underline{\alpha}_1 \dots \underline{\alpha}_m]^T$, define the following vertex matrices:

$$A_j(t) \triangleq A(t) + B(t)\Gamma_j(\underline{\alpha})K, \qquad j = 1,\dots,2^m \tag{5.17}$$

where $\Gamma_j(\underline{\alpha})$ is a diagonal matrix whose diagonal elements take the values 1 (no saturation) or $\underline{\alpha}_i$, $i = 1,\dots,m$ (saturation). Hence, if $x \in S_c$ then the velocity $\dot{x}$ must satisfy the following equation (see Lemma 1 in [79]):

$$\dot{x}(t) - A_g(t)\dot{x}(t - g(t)) = \sum_{j=1}^{2^m} \lambda_j A_j(t)x(t) + A_h(t)x(t - h(t)) \tag{5.18}$$

with $\sum_{j=1}^{2^m} \lambda_j = 1, \ \lambda_j \geq 0$.

Furthermore, it is important to note that the vector $\underline{\alpha}$ allows to define a polyhedral set:

$$S(\bar{u}, \underline{\alpha}) \triangleq \left\{ x \in \mathbb{R}^n : -\left|K_{(i)}x\right| \leq \frac{\bar{u}_i}{\underline{\alpha}_i}, \ i = 1,\dots,m \right\} \tag{5.19}$$

The set $S(\bar{u}, \underline{\alpha})$ contains S_c and corresponds to the maximal set in which (5.18) equivalently represents system (5.9).

In the context of the above, let S_c be a closed ellipsoid defined by a symmetric positive definite matrix $P_1 > 0$,

$$S_c \triangleq \{x \in \mathbb{R}^n : x^T P_1 x \leq \gamma^{-1}\} \tag{5.20}$$

where γ is a positive scalar.

The following theorem delivers the main result of this section:

Theorem 5.1. *Consider the closed-loop system* (5.9)*. Suppose that there exist* $n \times n$ *-matrices:* $P_1 = P_1^T > 0$, P_2, P_3, $S = S^T > 0$, $U = U^T > 0$, W_1, W_2, W_3, W_4, *and* $R_1 = R_1^T$, R_2, $R_3 = R_3^T > 0$, *a matrix* $K \in \mathbb{R}^{m \times n}$, *a* *vector $\underline{\alpha} \in \mathbb{R}^m$, positive scalars ε_i; $i = 1,...,4$, and a positive scalar γ, which satisfy the following matrix inequalities:*

$$\begin{bmatrix} \Omega_{1j} & \Omega_2 \\ \Omega_2^T & \Omega_3 \end{bmatrix} < 0, \qquad \forall j = 1,...,2^m \tag{5.21}$$

$$\begin{bmatrix} P_1 & \underline{\alpha}_i K_{(i)}^T \\ \underline{\alpha}_i K_{(i)} & \gamma \bar{u}_i^2 \end{bmatrix} \geq 0, \qquad \forall i = 1,...,m \tag{5.22}$$

$$\underline{\alpha}_i \in (0,1], \qquad \forall i = 1,...,m \tag{5.23}$$

with,

$K = \left[K_{(1)}^T,...,K_{(m)}^T\right]^T$, $K_{(i)}^T \in R^n$ *(i.e. $K_{(i)}$ is the $i-th$ row of K).*

$$\Omega_{1j} \triangleq \begin{bmatrix} \Psi_{1j} & \Psi_{2j} & h_{\max}\Phi_1 & -W_3^T A_h - \varepsilon_4 E_h^T E_h & P_2^T A_g \\ \Psi_{2j}^T & \Psi_3 & h_{\max}\Phi_2 & -W_4^T A_h & P_3^T A_g \\ h_{\max}\Phi_1^T & h_{\max}\Phi_2^T & -h_{\max}R & 0 & 0 \\ -A_h^T W_3 - \varepsilon_4 E_h^T E_h & -A_h^T W_4 & 0 & -(1-\beta_h)S + \varepsilon_4 E_h^T E_h & 0 \\ A_g^T P_2 & A_g^T P_3 & 0 & 0 & -(1-\beta_g)U + \varepsilon_3 E_g^T E_g \end{bmatrix} \tag{5.24}$$

$$\Omega_2 \triangleq \begin{bmatrix} P_2^T H_A & P_2^T H_h & P_2^T H_g & W_3^T H_h \\ P_3^T H_A & P_3^T H_h & P_3^T H_g & W_4^T H_h \\ & 0_{3\times 4} & & \end{bmatrix} \tag{5.25}$$

$$\Omega_3 \triangleq diag\{-\varepsilon_1 I, -\varepsilon_2 I, -\varepsilon_3 I, -\varepsilon_4 I\} \tag{5.26}$$

where,

$$\Psi_{1j} \triangleq \left(A + A_h + B\Gamma_j(\underline{\alpha})K\right)^T P_2 + P_2^T\left(A + A_h + B\Gamma_j(\underline{\alpha})K\right) + W_3^T A_h + A_h^T W_3 + S$$
$$+\varepsilon_1\left(E_A + E_B\Gamma_j(\underline{\alpha})K\right)^T\left(E_A + E_B\Gamma_j(\underline{\alpha})K\right) + (\varepsilon_2 + \varepsilon_4)E_h^T E_h$$

$$\Psi_{2j} \triangleq P_1^T - P_2^T + \left(A + A_h + B\Gamma_j(\underline{\alpha})K\right)^T P_3 + A_h^T W_4, \qquad \Psi_3 \triangleq -P_3 - P_3^T + U + h_{\max} A_{h,\max}^T R_3 A_{h,\max}$$

$$\Phi_1 \triangleq \begin{bmatrix} W_1^T + P_1^T & W_3^T + P_2^T \end{bmatrix}, \qquad \Phi_2 \triangleq \begin{bmatrix} W_2^T & W_4^T + P_3^T \end{bmatrix}$$

$$R = R^T = \begin{bmatrix} R_1 & R_2 \\ R_2^T & R_3 \end{bmatrix} > 0$$

Under these conditions, system (5.9) *is locally asymptotically stable for any initial condition* $\phi(\sigma)$ *in the ball,*

$$\Phi(\sigma)=\left\{\phi\in C^{w}_{d_{\max},n};\ \|\phi\|_C^2\le\sigma\right\} \tag{5.27}$$

with,

$$\sigma=\frac{1}{\gamma\pi_2} \tag{5.28}$$

where,

$$\pi_2=\max\left\{\lambda_{\max}(P_1)+2\frac{h_{\max}}{(1-\beta_h)}\lambda_{\max}(S),\ 2\frac{h_{\max}^2}{(1-\beta_h)}\lambda_{\max}(S)+\frac{1}{(1-\beta_g)}\lambda_{\max}(U)+h_{\max}\lambda_{\max}\left(A_{h,\max}^T R_3 A_{h,\max}\right)\right\} \tag{5.29}$$

Remark 5.3. In **[27]**, a comparison between the different saturation models used in the literature was made, concluding that the differential inclusion method used in this chapter leads to the least conservative design.

Remark 5.4. In **[77]**, the proposed design is delay-independent in both the neutral and the retarded delays, while in Theorem 5.1, the presented controller is delay-independent in the neutral delay and delay-dependent in the retarded delay. This leads to a less conservative design than that of **[77]**. The reason for choosing a delay-independent design for the neutral delay is that, unlike simple retarded systems, neutral systems are particularly sensitive to delays and can be easily destabilized; see **[29, 52]**. Also, in **[77]**, only a design for the nominal system (no uncertainties) is considered and the delays are assumed known and time-invariant, which makes the above theorem more general.

Remark 5.5. In **[79]**, a delay-dependent design for retarded systems was presented. However, the descriptor system transformation used there introduces additional dynamics to the original system **[8]**, and thus makes the design more conservative than the one proposed in the present thesis. Also, the bounding technique (see Lemma 5.3) employed in the proof of Theorem 5.1, further reduces the conservatism of the presented approach as compared with **[79]**.

Remark 5.6. The main difficulty in the application of the design procedure as stated in terms of Theorem 5.1 is that the inequalities (5.21) and (5.22) are nonlinear in the parameters P_2, P_3, $\underline{\alpha}$ and K. This difficulty can be overcome by employing relaxation techniques, as suggested in **[77, 78]**. A suitable relaxation technique in this case is to choose $\underline{\alpha}$ and K and solve for P_2, P_3, and then re-iterate the choice of $\underline{\alpha}$ and K. In this way inequalities (5.21) and (5.22) then reduce to linear matrix inequalities (LMIs) in the variables P_2, P_3. The latter are easily solved using the algorithms of **[6]**. Also, more sophisticated optimization techniques could be used to maximize the set of initial conditions $\Phi(\sigma)$; see **[77]** for examples of such techniques.

Proof. By virtue of condition (5.22) of the Theorem the ellipsoid defined by (5.20) is included in the set $S(\overline{u},\underline{\alpha})$ as defined by (5.19), where the vector $\underline{\alpha}$ verifies (5.23). Therefore, $\dot{x}(t)$ satisfies the polytopic system equation (5.18).

The last can be further written in its equivalent descriptor form:

$$\dot{x}(t)=y(t), \quad y(t)=A_g(t)y(t-g(t))+\sum_{j=1}^{2^m}\lambda_j A_j(t)x(t)+A_h(t)x(t-h(t)) \tag{5.30}$$

Using the Liebnitz-Newton formula $x(t-h(t))=x(t)-\int_{t-h(t)}^{t}\dot{x}(s)ds$ permits to re-write (5.30), yet in a more tractable form. Introduction of the augmented state as in (5.30) and the use of the Liebnitz-Newton formula allows to avoid the introduction of any additional dynamics, so that the transfer function of the system obtained by freezing the time-variable in the system matrices does not exhibit any additional poles; see **[8]**. This way of transforming system (5.30) is particularly useful as it allows to avoid unnecessary conservatism in the design that follows.

The last transformation of (5.30) yields:

$$\dot{x}(t)=y(t), \quad 0=-y(t)+A_g(t)y(t-g(t))+\left[\sum_{j=1}^{2^m}\lambda_j A_j(t)+A_h(t)\right]x(t)-A_h(t)\int_{t-h(t)}^{t}y(s)ds \tag{5.31}$$

so that if $E\triangleq\begin{bmatrix} I_n & 0 \\ 0 & 0\end{bmatrix}$, then the augmented system is given by :

$$E\begin{bmatrix}\dot{x}(t)\\ \dot{y}(t)\end{bmatrix}=\begin{bmatrix} 0 & I \\ \sum_{j=1}^{2^m}\lambda_j A_j(t)+A_h(t) & -I\end{bmatrix}\begin{bmatrix}x(t)\\ y(t)\end{bmatrix}+\begin{bmatrix}0\\ A_g(t)\end{bmatrix}y(t-g(t))-\begin{bmatrix}0\\ A_h(t)\end{bmatrix}\int_{t-h(t)}^{t}y(s)ds \tag{5.32}$$

The following Lyapunov-Krasovskii functional is used here:

$$V(t)=V_0(t)+V_1(t)+V_2(t)+V_3(t) \tag{5.33}$$

where,

$$V_0(t)=\begin{bmatrix}x^T(t) & y^T(t)\end{bmatrix}EP\begin{bmatrix}x(t)\\ y(t)\end{bmatrix}=x^T(t)P_1x(t) \tag{5.34}$$

$$V_1(t)=\int_{t-h(t)}^{t}x^T(\tau)Sx(\tau)d\tau \tag{5.35}$$

$$V_2(t)=\int_{t-g(t)}^{t}y^T(\tau)Uy(\tau)d\tau \tag{5.36}$$

$$V_3(t)=\int_{-h_{\max}}^{0}\int_{t+\theta}^{t}y^T(s)A_{h,\max}^TR_3A_{h,\max}y(s)dsd\theta=\int_{-h_{\max}}^{0}\int_{t+\theta}^{t}y^T(s)\begin{bmatrix}0 & A_{h,\max}^T\end{bmatrix}R\begin{bmatrix}0\\ A_{h,\max}\end{bmatrix}y(s)dsd\theta \tag{5.37}$$

with,

$$P=\begin{bmatrix}P_1 & 0\\ P_2 & P_3\end{bmatrix},\ P_1=P_1^T>0,\ U=U^T>0,\ S=S^T>0,\ R_3=R_3^T>0,\ R=R^T=\begin{bmatrix}R_1 & R_2\\ R_2^T & R_3\end{bmatrix}>0,\ R_1=R_1^T \tag{5.38}$$

Differentiating (5.34) and using (5.32), yields,

$$\frac{dV_0(t)}{dt}=2x^T(t)P_1\dot{x}(t)=2\begin{bmatrix}x^T(t) & y^T(t)\end{bmatrix}P^T\begin{bmatrix}\dot{x}(t)\\ 0\end{bmatrix}$$

$$=2\begin{bmatrix}x^T(t) & y^T(t)\end{bmatrix}P^T\begin{bmatrix}0 & I\\ \sum_{j=1}^{2^m}\lambda_jA_j(t)+A_h(t) & -I\end{bmatrix}\begin{bmatrix}x(t)\\ y(t)\end{bmatrix}+2\begin{bmatrix}x^T(t) & y^T(t)\end{bmatrix}P^T\begin{bmatrix}0\\ A_g(t)\end{bmatrix}y(t-g(t))$$

$$-2\int_{t-h(t)}^{t}\begin{bmatrix}x^T(t) & y^T(t)\end{bmatrix}P^T\begin{bmatrix}0\\ A_h(t)\end{bmatrix}y(s)ds \tag{5.39}$$

A bound for the last term of (5.39) is derived as follows:

Define $\eta(t)\triangleq-2\int_{t-h(t)}^{t}\begin{bmatrix}x^T(t) & y^T(t)\end{bmatrix}P^T\begin{bmatrix}0\\ A_h(t)\end{bmatrix}y(s)ds$ (5.40)

Using Lemma 5.3 with $a(s)=\begin{bmatrix}0\\ A_h(t)\end{bmatrix}y(s)$, $b(s)=P\begin{bmatrix}x(t)\\ y(t)\end{bmatrix}$ and with $\Omega=\begin{bmatrix}t-h(t),t\end{bmatrix}$, gives:

$$\eta(t)\le\int_{t-h(t)}^{t}\begin{bmatrix}x^T(t) & y^T(t)\end{bmatrix}P^T\left(M^TR+I\right)R^{-1}\left(RM+I\right)P\begin{bmatrix}x(t)\\ y(t)\end{bmatrix}ds$$

$$+2\int_{t-h(t)}^{t} y^T(s)ds\begin{bmatrix}0 & A_h^T(t)\end{bmatrix}RMP\begin{bmatrix}x(t)\\ y(t)\end{bmatrix}+\int_{t-h(t)}^{t} y^T(s)\begin{bmatrix}0 & A_h^T(t)\end{bmatrix}R\begin{bmatrix}0\\ A_h(t)\end{bmatrix}y(s)ds$$

$$\le h_{\max}\begin{bmatrix}x^T(t) & y^T(t)\end{bmatrix}P^T\left(M^TR+I\right)R^{-1}(RM+I)P\begin{bmatrix}x(t)\\ y(t)\end{bmatrix}$$

$$+2\left[x^T(t)-x^T(t-h(t))\right]\begin{bmatrix}0 & A_h^T(t)\end{bmatrix}RMP\begin{bmatrix}x(t)\\ y(t)\end{bmatrix}+\int_{t-h_{\max}}^{t} y^T(s)\begin{bmatrix}0 & A_h^T(t)\end{bmatrix}R\begin{bmatrix}0\\ A_h(t)\end{bmatrix}y(s)ds \quad (5.41)$$

as $R>0$.

Differentiating $V_1(t)$ and $V_2(t)$ of (5.35) and (5.36), yields,

$$\frac{dV_1(t)}{dt}=x^T(t)Sx(t)-\left(1-\dot{h}(t)\right)x^T(t-h(t))Sx(t-h(t))$$

$$\frac{dV_2(t)}{dt}=y^T(t)Uy(t)-\left(1-\dot{g}(t)\right)y^T(t-g(t))Uy(t-g(t)).$$

Using the assumptions $S>0$, $U>0$, and the bounds specified by (5.6),

$$\frac{dV_1(t)}{dt}\le x^T(t)Sx(t)-\left(1-\beta_h\right)x^T(t-h(t))Sx(t-h(t)) \quad (5.42)$$

$$\frac{dV_2(t)}{dt}\le y^T(t)Uy(t)-\left(1-\beta_g\right)y^T(t-g(t))Uy(t-g(t)). \quad (5.43)$$

Applying Lemma 5.1 to $V_3(t)$,

$$\frac{dV_3(t)}{dt}=h_{\max}y^T(t)A_{h,\max}^TR_3A_{h,\max}y(t)-\int_{t-h_{\max}}^{t} y^T(s)A_{h,\max}^TR_3A_{h,\max}y(s)ds \quad (5.44)$$

Employing (5.33), (5.39), (5.41), (5.42), (5.43) and (5.44), the following upper bound for $\dot{V}(t)$ is obtained:

$$\dot{V}(t)\le 2\begin{bmatrix}x^T(t) & y^T(t)\end{bmatrix}P^T(t)\begin{bmatrix}0 & I\\ \sum_{j=1}^{2^m}\lambda_jA_j(t)+A_h(t) & -I\end{bmatrix}\begin{bmatrix}x(t)\\ y(t)\end{bmatrix}+2\begin{bmatrix}x^T(t) & y^T(t)\end{bmatrix}P^T(t)\begin{bmatrix}0\\ A_g(t)\end{bmatrix}y(t-g(t))$$

$$+h_{\max}\begin{bmatrix}x^T(t) & y^T(t)\end{bmatrix}P^T\left(M^TR+I\right)R^{-1}(RM+I)P\begin{bmatrix}x(t)\\ y(t)\end{bmatrix}$$

$$+2\left[x^T(t)-x^T(t-h(t))\right]\begin{bmatrix}0 & A_h^T(t)\end{bmatrix}RMP\begin{bmatrix}x(t)\\ y(t)\end{bmatrix}$$

$$+\int_{t-h_{\max}}^{t} y^T(s)\begin{bmatrix}0 & A_h^T(t)\end{bmatrix}R\begin{bmatrix}0\\ A_h(t)\end{bmatrix}y(s)ds$$

$$+x^T(t)Sx(t)-(1-\beta_h)x^T(t-h(t))Sx(t-h(t))$$

$$+y^T(t)Uy(t)-(1-\beta_g)y^T(t-g(t))Uy(t-g(t))$$

$$+h_{\max}y^T(t)A_{h,\max}^T R_3 A_{h,\max}y(t)-\int_{t-h_{\max}}^{t} y^T(s)A_{h,\max}^T R_3 A_{h,\max}y(s)ds \tag{5.45}$$

Since $R_3>0$, then, by virtue of the bound given by (5.7), the two remaining integrals in (5.45) satisfy the following inequality :

$$\int_{t-h_{\max}}^{t} y^T(s)\begin{bmatrix}0 & A_h^T(t)\end{bmatrix}R\begin{bmatrix}0\\ A_h(t)\end{bmatrix}y(s)ds-\int_{t-h_{\max}}^{t} y^T(s)A_{h,\max}^T R_3 A_{h,\max}y(s)ds$$

$$=\int_{t-h_{\max}}^{t} y^T(s)\left[A_h^T(t)R_3A_h(t)-A_{h,\max}^T R_3 A_{h,\max}\right]y(s)ds\le 0$$

Now, it is possible to reduce (5.45) to,

$$\dot V(t)\le 2\begin{bmatrix}x^T(t) & y^T(t)\end{bmatrix}P^T\begin{bmatrix}0 & I\\ \sum_{j=1}^{2^m}\lambda_jA_j(t)+A_h(t) & -I\end{bmatrix}\begin{bmatrix}x(t)\\ y(t)\end{bmatrix}+2\begin{bmatrix}x^T(t) & y^T(t)\end{bmatrix}P^T\begin{bmatrix}0\\ A_g(t)\end{bmatrix}y(t-g(t))$$

$$+h_{\max}\begin{bmatrix}x^T(t) & y^T(t)\end{bmatrix}P^T\left(M^TR+I\right)R^{-1}(RM+I)P\begin{bmatrix}x(t)\\ y(t)\end{bmatrix}$$

$$+2\left[x^T(t)-x^T(t-h(t))\right]\begin{bmatrix}0 & A_h^T(t)\end{bmatrix}RMP\begin{bmatrix}x(t)\\ y(t)\end{bmatrix}$$

$$+x^T(t)Sx(t)-(1-\beta_h)x^T(t-h(t))Sx(t-h(t))$$

$$+y^T(t)Uy(t)-(1-\beta_g)y^T(t-g(t))Uy(t-g(t))$$

$$+h_{\max}y^T(t)A_{h,\max}^T R_3 A_{h,\max}y(t) \tag{5.46}$$

From (5.30) and the fact that x is square integrable on $[0,\infty)$, it follows that $D(y_t)\in L_2^n[0,\infty)$. Under Assumption 5.3, the latter implies that $y_t\in L_2^n[0,\infty)$.

Next, the application of Lemma 5.2 to the uncertainty terms in $A_g(t)$, $A_j(t)$, $A_h(t)$, and $B(t)$, results in the following bound for $\dot V(t)$:

$$\dot{V}(t) \le \begin{bmatrix} x^T(t) & y^T(t) \end{bmatrix} P^T \begin{bmatrix} 0 & I \\ \sum_{j=1}^{2^m} \lambda_j \left(A + B\Gamma_j(\underline{\alpha})K\right) + A_h & -I \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$$

$$+ \begin{bmatrix} x^T(t) & y^T(t) \end{bmatrix} \begin{bmatrix} 0 & \sum_{j=1}^{2^m} \lambda_j \left(A + B\Gamma_j(\underline{\alpha})K\right)^T + A_h^T \\ I & -I \end{bmatrix} P \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$$

$$+ \begin{bmatrix} x^T(t) & y^T(t) \end{bmatrix} P^T \begin{bmatrix} 0 \\ A_g \end{bmatrix} y(t-g(t)) + y^T(t-g(t)) \begin{bmatrix} 0 & A_g^T \end{bmatrix} P \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$$

$$+ h_{\max} \begin{bmatrix} x^T(t) & y^T(t) \end{bmatrix} P^T \left(M^T R + I\right) R^{-1} (RM + I) P \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$$

$$+ \begin{bmatrix} x^T(t) & y^T(t) \end{bmatrix} P^T M^T R \begin{bmatrix} 0 \\ A_h \end{bmatrix} \left[x(t) - x(t-h(t))\right] + \left[x^T(t) - x^T(t-h(t))\right] \begin{bmatrix} 0 & A_h^T \end{bmatrix} RMP \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$$

$$+ x^T(t) S x(t) - (1-\beta_h) x^T(t-h(t)) S x(t-h(t))$$

$$+ y^T(t) U y(t) - (1-\beta_g) y^T(t-g(t)) U y(t-g(t))$$

$$+ h_{\max} y^T(t) A_{h,\max}^T R_3 A_{h,\max} y(t)$$

$$+ \varepsilon_1^{-1} x^T(t) \sum_{j=1}^{2^m} \lambda_j P_2^T H_A H_A^T P_2 x(t) + \varepsilon_1^{-1} y^T(t) \sum_{j=1}^{2^m} \lambda_j P_3^T H_A H_A^T P_2 x(t) + \varepsilon_1^{-1} x^T(t) \sum_{j=1}^{2^m} \lambda_j P_2^T H_A H_A^T P_3 y(t)$$

$$+ \varepsilon_1^{-1} y^T(t) \sum_{j=1}^{2^m} \lambda_j P_3^T H_A H_A^T P_3 y(t) + \varepsilon_1 x^T(t) \sum_{j=1}^{2^m} \lambda_j \left(E_A + E_B \Gamma_j(\underline{\alpha}) K\right)^T \left(E_A + E_B \Gamma_j(\underline{\alpha}) K\right) x(t)$$

$$+ \varepsilon_2^{-1} x^T(t) P_2^T H_h H_h^T P_2 x(t) + \varepsilon_2^{-1} y^T(t) P_3^T H_h H_h^T P_2 x(t) + \varepsilon_2^{-1} x^T(t) P_2^T H_h H_h^T P_3 y(t)$$

$$+ \varepsilon_2^{-1} y^T(t) P_3^T H_h H_h^T P_3 y(t) + \varepsilon_2 x^T(t) E_h^T E_h x(t)$$

$$+ \varepsilon_3^{-1} x^T(t) P_2^T H_g H_g^T P_2 x(t) + \varepsilon_3^{-1} y^T(t) P_3^T H_g H_g^T P_2 x(t) + \varepsilon_3^{-1} x^T(t) P_2^T H_g H_g^T P_3 y(t)$$

$$+ \varepsilon_3^{-1} y^T(t) P_3^T H_g H_g^T P_3 y(t) + \varepsilon_3 y^T(t-g(t)) E_g^T E_g y(t-g(t))$$

$$+ \varepsilon_4^{-1} x^T(t) W_3^T H_h H_h^T W_3 x(t) + \varepsilon_4^{-1} y^T(t) W_4^T H_h H_h^T W_3 x(t) + \varepsilon_4^{-1} x^T(t) W_3^T H_h H_h^T W_4 y(t)$$

$$+ \varepsilon_4^{-1} y^T(t) W_4^T H_h H_h^T W_4 y(t) + \varepsilon_4 \left[x^T(t) - x^T(t-h(t))\right] E_h^T E_h \left[x(t) - x(t-h(t))\right] \tag{5.47}$$

with,

$$\varepsilon_i > 0,\ i = 1,\dots,4 \tag{5.48}$$

Defining $W \triangleq \begin{bmatrix} W_1 & W_2 \\ W_3 & W_4 \end{bmatrix} \triangleq RMP$, and the aggregated state $\xi(t)$ by,

$$\xi(t) \triangleq \begin{bmatrix} x(t) \\ y(t) \\ (M + R^{-1})P\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} \\ x^T(t - h(t)) \\ y^T(t - g(t)) \end{bmatrix}$$

the inequality of (5.47) can now simply be written as :

$$\dot{V}(t) \le \xi^T(t)\left[\Omega_1 - \Omega_2\Omega_3^{-1}\Omega_2^T\right]\xi(t) \triangleq \xi^T(t)\Theta\xi(t) \tag{5.49}$$

where the matrix Θ involves the summation of terms of the form $\sum_{j=1}^{2^m} \lambda_j(\dots)$; see the inequality (5.47).

To guarantee that $\dot{V}(t) < 0$, for all $t \ge 0$, it is sufficient that $\Theta < 0$. By convexity, $\Theta < 0$ is guaranteed if $\Omega_{1j} - \Omega_2\Omega_3^{-1}\Omega_2^T < 0$, for $j = 1,\dots,2^m$, where Ω_{1j}, Ω_2 and Ω_3 are given by (5.24), (5.25) and (5.26), respectively.

Sufficient conditions for $\dot{V}(t) < 0$ can thus be summarized as follows :

1.) $\Omega_{1j} - \Omega_2\Omega_3^{-1}\Omega_2^T < 0$, $\qquad j = 1,\dots,2^m$

2.) $\Omega_3 < 0$ (assumptions (5.48) resulting from the application of Lemma 5.2 to obtain inequality (5.47)).

By the Schur Complements Lemma, conditions 1.) and 2.) above are equivalent to assumption (5.21) of the Theorem.

Thus, there exists $\pi_3 > 0$ such that $\dot{V}(x_t(\psi) = x(t+\psi)) \le -\pi_3\|x(t)\|^2$, so that $V(x_t) \le V(x_{t_0})$ provided that the model (5.18) is valid, i.e., for any time $t \ge t_0$ such that $x(t) \in S(\overline{u}, \underline{\alpha})$.

Furthermore, following **[90]**, the Lyapunov functional defined in (5.33) can be shown to satisfy,

$$w_1(\|\phi(0)\|) \le V(\phi, t) \le w_2(\|\phi\|_W) \tag{5.50}$$

where,

$$\|\phi\|_W = \left[\|\phi(0)\|^2 + \int_{-d_{\max}}^{0} \|\dot{\phi}(\theta)\|^2 \, d\theta\right]^{1/2} \tag{5.51}$$

$$w_1(\|\phi(0)\|) = \pi_1 \|\phi(0)\|^2 \tag{5.52}$$

$$w_2(\|\phi\|_W) = \pi_2 \|\phi\|_W^2 \tag{5.53}$$

with $\pi_1 \triangleq \lambda_{\min}(P_1)$ and

$$\pi_2 \triangleq \max\left\{\lambda_{\max}(P_1) + 2\frac{h_{\max}}{(1-\beta_h)}\lambda_{\max}(S),\ 2\frac{h_{\max}^2}{(1-\beta_h)}\lambda_{\max}(S) + \frac{1}{(1-\beta_g)}\lambda_{\max}(U) + h_{\max}\lambda_{\max}\left(A_{h,\max}^T R_3 A_{h,\max}\right)\right\} \tag{5.54}$$

Inequality (5.50) can be shown by first noting that,

$$V(\phi,t) \geq \lambda_{\min}(P_1)\|\phi(0)\|^2$$

Therefore, we choose,

$$w_1(\|\phi(0)\|) = \lambda_{\min}(P_1)\|\phi(0)\|^2 = \pi_1 \|\phi(0)\|^2$$

Next, recalling that $d_{\max} = \max\{h(t), g(t)\}$, and noting that for $\theta \leq 0$,

$$\begin{aligned}
\|\phi(\theta)\| &= \left\|\phi(0) - \int_\theta^0 \dot{\phi}(u)\,du\right\| \\
&\leq \|\phi(0)\| + \left\|\int_\theta^0 \dot{\phi}(u)\,du\right\| \\
&\leq \|\phi(0)\| + \int_\theta^0 \|\dot{\phi}(u)\|\,du
\end{aligned}$$

then we have that for $\theta \leq 0$,

$$\begin{aligned}
\|\phi(\theta)\|^2 &\leq \left(\|\phi(0)\| + \int_\theta^0 \|\dot{\phi}(u)\|\,du\right)^2 \\
&\leq 2\|\phi(0)\|^2 + 2\left(\int_\theta^0 \|\dot{\phi}(u)\|\,du\right)^2 \\
&\leq 2\|\phi(0)\|^2 + 2\int_\theta^0 \|\dot{\phi}(u)\|^2\,du \int_\theta^0 1^2\,du \\
&= 2\|\phi(0)\|^2 - 2\theta\int_\theta^0 \|\dot{\phi}(u)\|^2\,du
\end{aligned}$$

where the Cauchy-Schwarz inequality is used. It then follows that,

$$\int_{-h(0)}^{0}\|\phi(\theta)\|^{2}d\theta \le \int_{-h(0)}^{0}\left(2\|\phi(0)\|^{2}-2\theta\int_{\theta}^{0}\|\dot{\phi}(u)\|^{2}du\right)d\theta$$

$$\le 2h(0)\|\phi(0)\|^{2}+2h(0)\int_{-h(0)}^{0}\int_{\theta}^{0}\|\dot{\phi}(u)\|^{2}dud\theta$$

$$=2h(0)\|\phi(0)\|^{2}+2h(0)\int_{-h(0)}^{0}\int_{-h(0)}^{u}\|\dot{\phi}(u)\|^{2}d\theta du$$

$$=2h(0)\|\phi(0)\|^{2}+2h(0)\int_{-h(0)}^{0}\left(u+h(0)\right)\|\dot{\phi}(u)\|^{2}du$$

$$\le 2h(0)\|\phi(0)\|^{2}+2h^{2}(0)\int_{-h(0)}^{0}\|\dot{\phi}(u)\|^{2}du$$

From this, we obtain,

$$V(\phi,0)=\phi^{T}(0)P_{1}\phi(0)+\int_{-h(0)}^{0}\phi^{T}(\tau)S\phi(\tau)d\tau+\int_{-g(0)}^{0}\dot{\phi}^{T}(\tau)U\dot{\phi}(\tau)d\tau$$

$$+\int_{-h_{\max}}^{0}\int_{\theta}^{0}\dot{\phi}^{T}(s)\bar{A}_{1,\max}^{T}R_{3}\bar{A}_{1,\max}\dot{\phi}(s)dsd\theta$$

$$\le \lambda_{\max}(P_{1})\|\phi(0)\|^{2}+\frac{1}{(1-\beta_{h})}\lambda_{\max}(S)\int_{-h(0)}^{0}\|\phi(\tau)\|^{2}d\tau+\frac{1}{(1-\beta_{g})}\lambda_{\max}(U)\int_{-g(0)}^{0}\|\dot{\phi}(\tau)\|^{2}d\tau$$

$$+\int_{-h_{\max}}^{0}\int_{\theta}^{0}\dot{\phi}^{T}(s)\bar{A}_{1,\max}^{T}R_{3}\bar{A}_{1,\max}\dot{\phi}(s)dsd\theta$$

Also,

$$\int_{-h_{\max}}^{0}\int_{\theta}^{0}\dot{\phi}^{T}(s)\bar{A}_{1,\max}^{T}R_{3}\bar{A}_{1,\max}\dot{\phi}(s)dsd\theta=\int_{-h_{\max}}^{0}\int_{-h_{\max}}^{s}\dot{\phi}^{T}(s)\bar{A}_{1,\max}^{T}R_{3}\bar{A}_{1,\max}\dot{\phi}(s)d\theta ds$$

$$=\int_{-h_{\max}}^{0}(s+h_{\max})\dot{\phi}^{T}(s)\bar{A}_{1,\max}^{T}R_{3}\bar{A}_{1,\max}\dot{\phi}(s)ds$$

$$\le h_{\max}\int_{-h_{\max}}^{0}\dot{\phi}^{T}(s)\bar{A}_{1,\max}^{T}R_{3}\bar{A}_{1,\max}\dot{\phi}(s)ds$$

and so,

$$V(\phi,t)\le \lambda_{\max}(P_{1})\|\phi(0)\|^{2}+\frac{1}{(1-\beta_{h})}\lambda_{\max}(S)\int_{-h(0)}^{0}\|\phi(\tau)\|^{2}d\tau+\frac{1}{(1-\beta_{g})}\lambda_{\max}(U)\int_{-g(0)}^{0}\|\dot{\phi}(\tau)\|^{2}d\tau$$

$$+h_{\max}\lambda_{\max}\left(\bar{A}_{1,\max}^{T}R_{3}\bar{A}_{1,\max}\right)\int_{-h_{\max}}^{0}\|\dot{\phi}(s)\|^{2}ds$$

$$\leq \lambda_{\max}(P_1)\|\phi(0)\|^2 + \frac{1}{(1-\beta_h)}\lambda_{\max}(S)\left(2h(0)\|\phi(0)\|^2 + 2h^2(0)\int_{-h(0)}^{0}\|\dot{\phi}(u)\|^2 du\right)$$

$$+\frac{1}{(1-\beta_g)}\lambda_{\max}(U)\int_{-g(0)}^{0}\|\dot{\phi}(\tau)\|^2 d\tau + h_{\max}\lambda_{\max}\left(\bar{A}_{1,\max}^T R_3 \bar{A}_{1,\max}\right)\int_{-h_{\max}}^{0}\|\dot{\phi}(s)\|^2 ds$$

$$\leq \left(\lambda_{\max}(P_1) + 2\frac{h_{\max}}{(1-\beta_h)}\lambda_{\max}(S)\right)\|\phi(0)\|^2$$

$$+\left(2\frac{h_{\max}^2}{(1-\beta_h)}\lambda_{\max}(S) + h_{\max}\lambda_{\max}\left(\bar{A}_{1,\max}^T R_3 \bar{A}_{1,\max}\right)\right)\int_{-h_{\max}}^{0}\|\dot{\phi}(u)\|^2 du$$

$$+\frac{1}{(1-\beta_g)}\lambda_{\max}(U)\int_{-g_{\max}}^{0}\|\dot{\phi}(\tau)\|^2 d\tau$$

$$\leq \left(\lambda_{\max}(P_1) + 2\frac{h_{\max}}{(1-\beta_h)}\lambda_{\max}(S)\right)\|\phi(0)\|^2$$

$$+\left(2\frac{h_{\max}^2}{(1-\beta_h)}\lambda_{\max}(S) + \frac{1}{(1-\beta_g)}\lambda_{\max}(U) + h_{\max}\lambda_{\max}\left(\bar{A}_{1,\max}^T R_3 \bar{A}_{1,\max}\right)\right)\int_{-d_{\max}}^{0}\|\dot{\phi}(\tau)\|^2 d\tau$$

$$\leq \pi_2\left[\|\phi(0)\|^2 + \int_{-d_{\max}}^{0}\|\dot{\phi}(\tau)\|^2 d\tau\right]$$

where π_2 is given by (5.54).

Hence, for all $\phi(\psi)\in\Phi$, $\psi\in[-d_{\max},0]$,

$$x^T(t)P_1x(t)\leq V(x_t)\leq V(x_{t_0})\leq\gamma^{-1},\ \forall t\geq t_0. \tag{5.55}$$

Therefore, for any initial condition ϕ in the ball $\Phi(\sigma)$ defined by (5.27), the system (5.9) verifies the conditions of the Lyapunov-Krasovskii Theorem **[41]** and $V(x_t)$ is a local strictly decreasing Lyapunov function. Therefore the asymptotic stability of system (5.9) is ensured. QED

5.6. Numerical Examples

The following examples were solved using the LMI (see Appendix B) tool in Matlab.

Example 5.1. Consider the time-delay system (5.9) with no uncertainties, and with the following system matrices:

$$A = \begin{bmatrix} 1 & 1.5 \\ 0.3 & -2 \end{bmatrix}, \qquad A_h = \begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} 10 \\ 1 \end{bmatrix} \tag{5.56}$$

and with all the other matrices in (5.9) set to zero. The retarded time-delay is chosen to be equal to $h = 1$, and the saturation limit is imposed to be $\bar{u} = 15$. The coefficient of tolerance for control saturation is chosen to be $\underline{\alpha} = 0.5$. The same example was used in **[78]**, where the authors achieved a ball of admissible initial conditions $\Phi(\sigma)$ characterized by the square radius of $\sigma = 1791.9$. Now, employing the results of Theorem 5.1, the ball of admissible initial conditions $\Phi(\sigma)$ is calculated as follows. For a fixed controller gain vector K, inequalities (5.21)-(5.22) are first solved using the standard LMI package to yield a corresponding value of the parameter π_2 of (5.29). The value of the parameter γ is next decreased so that the inequality (5.22) is satisfied sharply. Repeating this procedure while iterating with respect to the controller gain vector K allows us to achieve a (generally local) minimum for the product $\gamma\pi_2$. In this way, a ball of admissible initial conditions $\Phi(\sigma)$ characterized by $\sigma = 2183.7$ was achieved for the above example, and is then 21.87 % larger than the one previously achieved in **[78]**.

The corresponding solution matrices pertinent to the maximal ball $\Phi(\sigma)$ are :

$$P_1 = \begin{bmatrix} 0.8940 & -0.0167 \\ -0.0167 & 2.3860 \end{bmatrix}, \ S = \begin{bmatrix} 0.2763 & -0.0666 \\ -0.0666 & 1.1200 \end{bmatrix}, \ R_3 = \begin{bmatrix} 1.4594 & 0 \\ 0 & 0.7186 \end{bmatrix}, \ U = \begin{bmatrix} 0.5135 & 0.0295 \\ 0.0295 & 0.7201 \end{bmatrix}$$

and the controller gain is :

$$K = \begin{bmatrix} -0.2770 & -0.0800 \end{bmatrix}$$

Finally, the response of the closed loop system to example initial conditions is shown in Figure 5.1 and Figure 5.2. Figure 5.1 shows the state response and the corresponding control for the case without control saturation, while Figure 5.2 shows the state response and control for the case with control saturation.

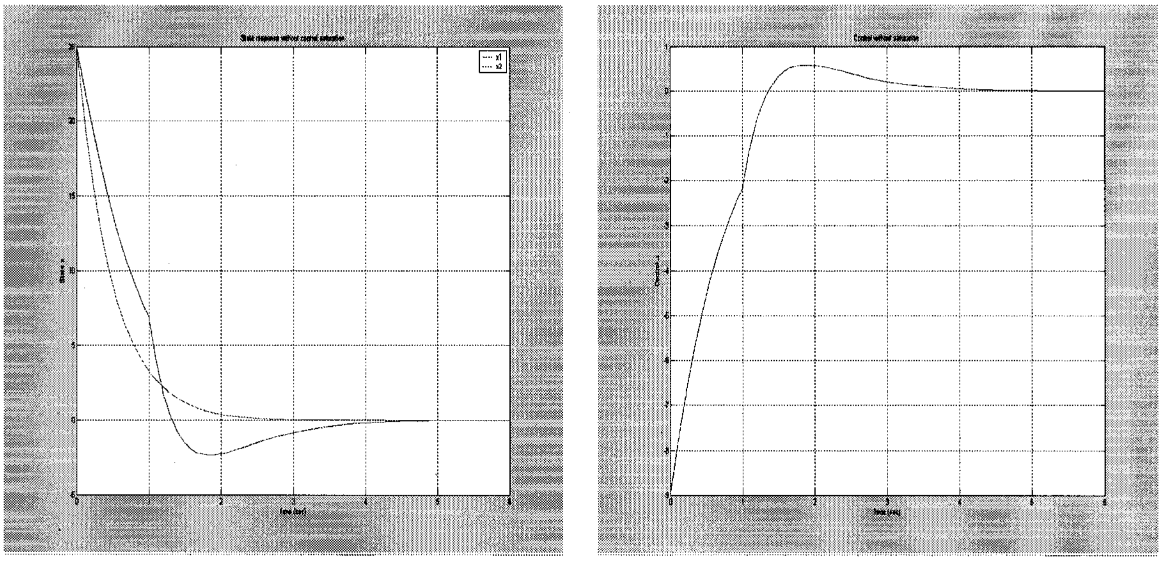

Figure 5.1. *Response of closed-loop system to initial condition* $x_0 = (25, 25)$ *: no control saturation*

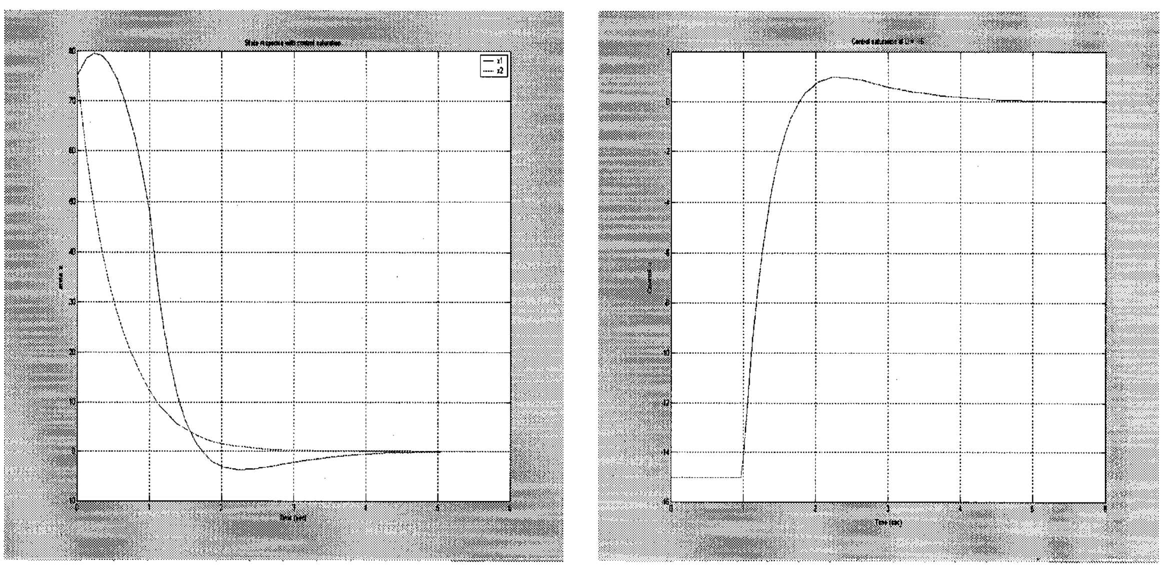

Figure 5.2. *Response of closed-loop system to initial condition* $x_0 = (75, 75)$ *: with control saturation*

Simulations demonstrate that the closed system stability is ensured for some initial states outside the computed ball $\Phi(\sigma)$. The initial conditions used in Figure 5.2 are such an example, which implies that the stabilizable set of initial conditions may have a different shape than a ball and that $\Phi(\sigma)$ is only an interior approximation of this stabilizable set. This was to be expected as stabilizable sets seldom have the exact shape of a ball.

Example 5.2. Consider the uncertain time-delay system (5.9) with the following system matrices:

$$A=\begin{bmatrix}-2 & 0\\ 1 & -3\end{bmatrix},\quad A_h=\begin{bmatrix}-1 & 0\\ -0.8 & -1\end{bmatrix},\quad B=\begin{bmatrix}0\\ 1\end{bmatrix},\quad H_A=H_h=\begin{bmatrix}0.2 & 0\\ 0 & 0.2\end{bmatrix},\quad E_A=E_h=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} \tag{5.57}$$

with all the other matrices in (5.9) set to zero. Suppose that the saturation limit for the scalar control is $\bar{u}=1$.

Using (5.10), the matrix $A_{h,\max}$ in the bound (5.7) is chosen as:

$$A_{h,\max}=\begin{bmatrix}-1.4362 & -0.3618\\ -0.3618 & -1.1468\end{bmatrix} \text{ for } \zeta=3.6.$$

The choice of ζ and thus of $A_{h,\max}$ is not arbitrary. The value of the maximum time-delay $h_{\max}$ for guaranteeing asymptotic stability varies in function of parameter ζ in a parabolic behavior, and so ζ is chosen at the highest $h_{\max}$.

This example was used in **[73]**, where for a level of saturation $\underline{\alpha}=0.3$, system (5.9) was found to be robustly stable with $h_{\max}=4.4206$. Since the example of **[73]** does not take into account the rate of change of the time-delay h, for the purpose of fair comparison, it is assumed here that h does not vary with time, and so $\beta_h=0$.

The application of the design procedure implied by Theorem 5.1 leads to a better stabilizing controller. For the same saturation level of $\underline{\alpha}=0.3$, the solution of (5.21)-(5.22) with controller gain vector $K=[-0.65 \quad -0.65]$, provides a controller which guarantees robust stability up to $h_{\max}=6.9152$. The increase in the width of the stability margin is related to the decrease in the conservatism of the present approach as compared to that of **[73]**.

As the approach developed here allows us to take account of a time-varying time-delay h in the system, further computations with the same example demonstrated how the upper limit $h_{\max}$ on the time-delay changes with the rate $\frac{dh(t)}{dt}$. The curve in Figure 5.3 illustrates that robust stability margin as expressed by

$h_{\max}$ decreases very rapidly with the increase in the rate of change of the time-delay. It further amplifies the importance of considering the rate of change of the delay as a factor in the design, as any incorrectness in the assumption about this rate of change may lead to instabilities in the real closed-loop system. Figure 5.3 also shows that for the rate of change β_h as high as 0.9 robust stability of the closed-loop system is still ensured with delays not exceeding $h_{\max} = 0.7015$.

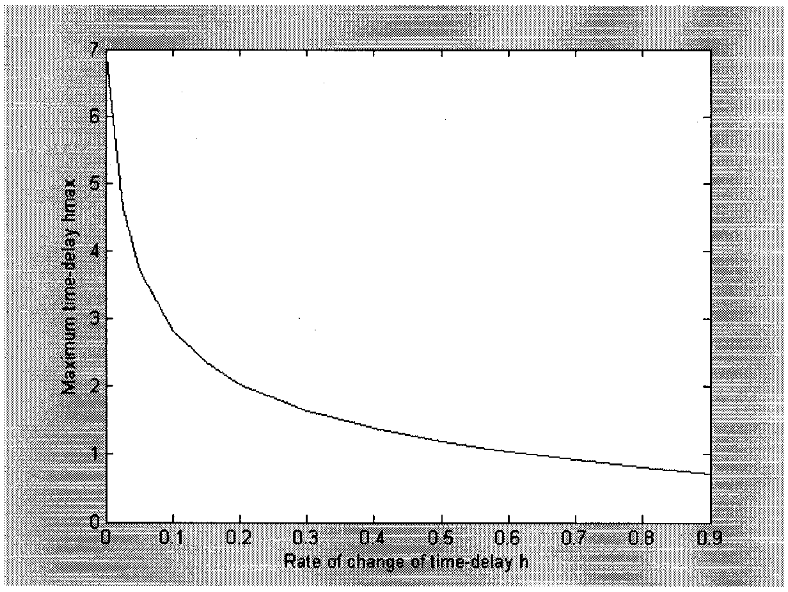

Figure 5.3. *Maximum time-delay h for guaranteed stability vs. rate of change of h*

A neutral system version of (5.57) with $A_g = \begin{bmatrix} -0.3 & 0 \\ 0.3 & -0.1 \end{bmatrix}$, $H_g = \begin{bmatrix} 0.2 & 0 \\ 0 & 0.2 \end{bmatrix}$, $E_g = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $\beta_h = \beta_g = 0$, is also considered. In this case, robust stability is ensured for all delays smaller than $h_{\max} = 0.8026$. As before, Figure 5.4 shows the dependence between $h_{\max}$ and the rate of change of neutral time-delay $\dfrac{dg(t)}{dt}$. Not surprisingly, the stability of the closed-loop system is very sensitive with respect to rapid time-variation in the neutral time-delay.

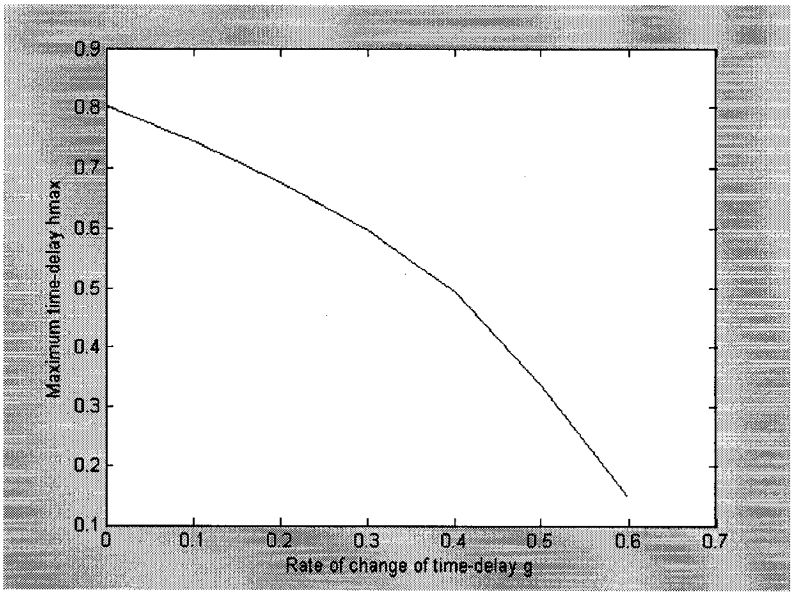

Figure 5.4. *Maximum time-delay h for guaranteed stability vs. rate of change of g*

CHAPTER 6

Delay-Dependent Robust Output Feedback Stabilization of Uncertain State-Delayed Systems with Time-Varying Delays and Saturating Actuators:

The Sector Modeling Model

6.1. System Description

Consider the retarded system (1.1)-(1.5) taken in the infinite-horizon context, where all the system matrices are time-invariant except the uncertainty matrices F_i. More specifically the system under consideration is:

$$\dot{x}(t) = A(t)x(t) + A_h(t)x(t-h(t)) + B(t)u_{sat}(t)$$

$$y(t) = C(t)x(t) + D(t)u_{sat}(t)$$

$$u_{sat}(t) = sat(u(t)),\ sat(u(t)) = \left[sat(u_1(t))\quad sat(u_2(t))\quad \ldots\quad sat(u_m(t))\right]$$

$$x(t) = \phi(t),\ t \in [-h_{\max}, 0] \tag{6.1}$$

$$A(t) = A + \Delta A(t),\ A_h(t) = A_h + \Delta A_h(t),\ B(t) = B + \Delta B(t),\ C(t) = C + \Delta C(t),\ D(t) = D + \Delta D(t) \tag{6.2}$$

$$\begin{bmatrix} \Delta A(t) & \Delta B(t) \\ \Delta C(t) & \Delta D(t) \end{bmatrix} = \begin{bmatrix} H_A \\ H_C \end{bmatrix} F(t)[E_A\quad E_B],\ \Delta A_h(t) = H_h F_h(t) E_h \tag{6.3}$$

6.2. Assumptions

The time-delay h is a function of time and is assumed to be continuously differentiable, with its amplitude and rate of change bounded as follows:

$$0 \le h(t) \le h_{\max}, \qquad 0 \le \dot{h}(t) \le \alpha < 1, \qquad \text{for all } t \ge 0 \tag{6.4}$$

where $h_{\max}$ and α are given positive constants.

Finally, the following is assumed to hold:

Assumption 6.1. $(A + A_h, B)$ is stabilizable.

Assumption 6.2. (C, A) is detectable.

6.3. Preliminaries

The following lemmas will prove helpful in the sequel:

Lemma 6.1. [39] *Let* $a(t): \mathbb{R}^+ \to \mathbb{R}$, $b(t): \mathbb{R}^+ \to \mathbb{R}$ *and* $f(s): \mathbb{R} \to \mathbb{R}$ *be continuously differentiable functions.*

Let the function $z(t) = \int_{a(t)}^{b(t)} \int_{t-\theta}^{t} f(s) ds d\theta$. *Then* $z(t)$ *is a solution of the differential equation,*

$$\frac{dz(t)}{dt} = \left(b(t) - a(t)\right) f(t) - \left(1 - \dot{b}(t)\right) \int_{t-b(t)}^{t-a(t)} f(s) ds + \left(\dot{b}(t) - \dot{a}(t)\right) \int_{t-a(t)}^{t} f(s) ds .$$

Lemma 6.2. [12, 72] *Let* A, L, E *and* F *be real matrices (possibly time-varying) of appropriate dimensions, with* F *satisfying* $FF^T \le I$. *Then the following holds:*

1- For any scalar $\varepsilon > 0$ *and any matrix* P,

$$P^T LFE + E^T F^T L^T P \le \varepsilon^{-1} P^T LL^T P + \varepsilon E^T E$$

2- For any matrix $P > 0$ *and any scalar* $\varepsilon > 0$ *such that* $\varepsilon I - EPE^T > 0$,

$$(A + LFE) P (A + LFE)^T \le APA^T + APE^T \left(\varepsilon I - EPE^T\right)^{-1} EPA^T + \varepsilon LL^T$$

3- For any matrix $P > 0$ *and any scalar* $\varepsilon > 0$ *such that* $P - \varepsilon LL^T > 0$,

$$(A + LFE)^T P^{-1} (A + LFE) \le A^T \left(P - \varepsilon LL^T\right)^{-1} A + \varepsilon^{-1} E^T E$$

4 – For any scalar $\varepsilon > 0$ *and any vectors* X *and* Y,

$$2X^T FY \le \varepsilon X^T X + \varepsilon^{-1} Y^T Y$$

Lemma 6.3. [62] *Assume that* $a:\Upsilon \to \mathbb{R}^n$ *and* $b:\Upsilon \to \mathbb{R}^n$, $\Upsilon \subset \mathbb{R}$ *are integrable functions over their common domain* Υ. *Then, for any positive definite matrix* $R \in \mathbb{R}^{n\times m}$ *and any matrix* $M \in \mathbb{R}^{m\times m}$, *the following inequality holds:*

$$-2\int_{\Omega} b^T(s)a(s)ds \le \int_{\Omega} \begin{bmatrix} a(s) \\ b(s) \end{bmatrix}^T \begin{bmatrix} R & RM \\ M^T R & \Upsilon \end{bmatrix} \begin{bmatrix} a(s) \\ b(s) \end{bmatrix} ds \tag{6.5}$$

where $\Upsilon = (M^T R + I)R^{-1}(RM + I)$.

6.4. The Observer-Based Dynamic Output Feedback

The results to be presented are concerned with providing sufficient conditions for the design of an observer-based dynamic output feedback law for asymptotically stabilizing system (6.1). This law is assumed to take the following form:

$$u(t) = 2K\hat{x}(t) \tag{6.6}$$

$$\dot{\hat{x}}(t) = A\hat{x}(t) + Bu_{sat}(t) + L\big(y(t) - C\hat{x}(t) - Du_{sat}(t)\big) \tag{6.7}$$

where $\hat{x}(t) \in \mathbb{R}^n$ is the observer state vector, $K \in \mathbb{R}^{m\times n}$ and $L \in \mathbb{R}^{n\times r}$ are the constant controller and observer gains, respectively. The factor 2 in (6.6) is for convenience in order to facilitate algebraic manipulations.

Defining the usual observer error:

$$e(t) \triangleq x(t) - \hat{x}(t) \tag{6.8}$$

and using (6.1) and (6.7), permits to write,

$$\dot{e}(t) = (A - LC)e(t) + \big(\Delta A(t) - L\Delta C(t)\big)x(t) + \big(A_h + \Delta A_h(t)\big)x(t - h(t)) + \big(\Delta B(t) - L\Delta D(t)\big)u_{sat}(t) \tag{6.9}$$

Employing the particular form of the control law (6.6)-(6.7) and the using the equation for the observer error (6.9), the augmented system representation is:

$$\begin{bmatrix} \dot{x}(t) \\ \dot{e}(t) \end{bmatrix} = \left\{ \begin{bmatrix} A + BK & -BK \\ 0 & A - LC \end{bmatrix} + \begin{bmatrix} \Delta A(t) + \Delta B(t)K & -\Delta B(t)K \\ \Delta A(t) + \Delta B(t)K - L\Delta C(t) - L\Delta D(t)K & -\Delta B(t)K + L\Delta D(t)K \end{bmatrix} \right\} \begin{bmatrix} x(t) \\ e(t) \end{bmatrix}$$

$$+\left\{\begin{bmatrix} A_h & 0 \\ A_h & 0 \end{bmatrix}+\begin{bmatrix} \Delta A_h(t) & 0 \\ \Delta A_h(t) & 0 \end{bmatrix}\right\}\begin{bmatrix} x(t-h(t)) \\ e(t-h(t)) \end{bmatrix}+\left\{\begin{bmatrix} B \\ 0 \end{bmatrix}+\begin{bmatrix} \Delta B(t) \\ \Delta B(t)-L\Delta D(t) \end{bmatrix}\right\}\eta(t) \tag{6.10}$$

where,

$$\eta(t) \triangleq sat(2K\hat{x}(t))-K\hat{x}(t) \tag{6.11}$$

Using (6.3),

$$\begin{bmatrix} \dot{x}(t) \\ \dot{e}(t) \end{bmatrix}=\left\{\begin{bmatrix} A+BK & -BK \\ 0 & A-LC \end{bmatrix}+\begin{bmatrix} H_A \\ H_A-LH_C \end{bmatrix}F(t)[E_A+E_BK \quad -E_BK]\right\}\begin{bmatrix} x(t) \\ e(t) \end{bmatrix}$$

$$+\left\{\begin{bmatrix} A_h & 0 \\ A_h & 0 \end{bmatrix}+\begin{bmatrix} H_h \\ H_h \end{bmatrix}F_h(t)[E_h \quad 0]\right\}\begin{bmatrix} x(t-h(t)) \\ e(t-h(t)) \end{bmatrix}+\left\{\begin{bmatrix} B \\ 0 \end{bmatrix}+\begin{bmatrix} H_A \\ H_A-LH_C \end{bmatrix}F(t)E_B\right\}\eta(t) \tag{6.12}$$

Finally, introducing the following definitions,

$$\xi(t) \triangleq \begin{bmatrix} x(t) \\ e(t) \end{bmatrix},\ \hat{A} \triangleq \begin{bmatrix} A+BK & -BK \\ 0 & A-LC \end{bmatrix},\ \Delta\hat{A}(t) \triangleq \hat{H}F(t)\hat{E} \triangleq \begin{bmatrix} H_A \\ H_A-LH_C \end{bmatrix}F(t)[E_A+E_BK \quad -E_BK],$$

$$\hat{A}_h \triangleq \begin{bmatrix} A_h & 0 \\ A_h & 0 \end{bmatrix},\ \Delta\hat{A}_h(t) \triangleq \hat{H}_hF_h(t)\hat{E}_h \triangleq \begin{bmatrix} H_h \\ H_h \end{bmatrix}F_h(t)[E_h \quad 0],\ \hat{B} \triangleq \begin{bmatrix} B \\ 0 \end{bmatrix},$$

$$\Delta\hat{B}(t) \triangleq \hat{H}F(t)E_B \triangleq \begin{bmatrix} H_A \\ H_A-LH_C \end{bmatrix}F(t)E_B$$

$$\hat{A}(t) \triangleq \hat{A}+\Delta\hat{A}(t),\ \hat{A}_h(t) \triangleq \hat{A}_h+\Delta\hat{A}_h(t),\ \hat{B}(t) \triangleq \hat{B}+\Delta\hat{B}(t) \tag{6.13}$$

permits to re-write equation (6.12) in yet more aggregated compact form,

$$\dot{\xi}(t)=\hat{A}(t)\xi(t)+\hat{A}_h(t)\xi(t-h(t))+\hat{B}(t)\eta(t). \tag{6.14}$$

6.5. Problem Statement

The robust output feedback stabilization problem with saturating actuators:

Find a matrix $K \in \mathbb{R}^{m\times n}$, an a matrix $L \in \mathbb{R}^{n\times r}$ such that there exists a Lyapunov functional $V(\xi(t),t)$ corresponding to the closed-loop system (6.14), that satisfies,

$$\dot{V}(\xi(t),t) \le -\pi\|\xi(t)\|^2 \tag{6.15}$$

for some constant $\pi > 0$.

The above will ensure the quadratic stabilizability **[2]** of system (6.1) via dynamic output feedback control law (6.6)-(6.7), and so the asymptotic stability of the closed-loop system formed by system (6.1) and control law (6.6)-(6.7).

6.6. Main Result

The following theorem provides sufficient conditions for robust output feedback stabilization of the uncertain time-delay system (6.14) with control saturation.

Theorem 6.1. *Consider the closed-loop system consisting of* (6.1) *and controller-observer pair* (6.6)-(6.7). *Suppose that there exist,* $n\times n$ *-matrices:* P_{i11}, P_{i12}, P_{i21}, P_{i22}; $i=1,\ldots,3$, W_{i11}, W_{i12}, W_{i21}, W_{i22}; $i=1,\ldots,4$, R_{i11}, R_{i12}, R_{i21}, R_{i22}; $i=1,\ldots,3$, $S=S^T>0$, *an* $m\times n$ *matrix* K *and an* $n\times r$ *matrix* L, *which together with some suitable positive scalars* ε_i; $i=1,\ldots,5$, *satisfy the following matrix conditions:*

$$\begin{bmatrix} P_{111} & P_{112} \\ P_{121} & P_{122} \end{bmatrix} > 0,\ \textit{with}\ P_{111}=P_{111}^T,\ P_{112}=P_{121}^T,\ P_{122}=P_{122}^T \tag{6.16}$$

$$\begin{bmatrix} R_{311} & R_{312} \\ R_{321} & R_{322} \end{bmatrix} > 0,\ \textit{with}\ R_{311}=R_{311}^T,\ R_{312}=R_{321}^T,\ R_{322}=R_{322}^T \tag{6.17}$$

$$\begin{bmatrix} R_{111} & R_{112} & R_{211} & R_{212} \\ R_{121} & R_{122} & R_{221} & R_{222} \\ R_{211}^T & R_{221}^T & R_{311} & R_{312} \\ R_{212}^T & R_{222}^T & R_{321} & R_{322} \end{bmatrix} > 0,\ \textit{with}\ R_{111}=R_{111}^T,\ R_{112}=R_{121}^T,\ R_{122}=R_{122}^T \tag{6.18}$$

$$\begin{bmatrix} \Omega_1 & \Omega_2 \\ \Omega_2^T & \Omega_3 \end{bmatrix} < 0 \tag{6.19}$$

where,

$$\Omega_1 = \begin{bmatrix} \Psi_{111} & \Psi_{112} & \Psi_{211} & \Psi_{212} & h_{\max}\left(W_{111}^T + P_{111}^T\right) & h_{\max}\left(W_{121}^T + P_{121}^T\right) & h_{\max}\left(W_{311}^T + P_{211}^T\right) & h_{\max}\left(W_{321}^T + P_{221}^T\right) & -W_{311}^T A_h - W_{321}^T A_h - \varepsilon_4 E_h^T E_h \\ & \Psi_{122} & \Psi_{221} & \Psi_{222} & h_{\max}\left(W_{112}^T + P_{112}^T\right) & h_{\max}\left(W_{122}^T + P_{122}^T\right) & h_{\max}\left(W_{312}^T + P_{212}^T\right) & h_{\max}\left(W_{322}^T + P_{222}^T\right) & -W_{312}^T A_h - W_{322}^T A_h \\ & & \Psi_{311} & \Psi_{312} & h_{\max}\left(W_{211}^T\right) & h_{\max}\left(W_{221}^T\right) & h_{\max}\left(W_{411}^T + P_{311}^T\right) & h_{\max}\left(W_{421}^T + P_{321}^T\right) & -W_{411}^T A_h - W_{421}^T A_h \\ & & & \Psi_{322} & h_{\max}\left(W_{212}^T\right) & h_{\max}\left(W_{222}^T\right) & h_{\max}\left(W_{412}^T + P_{312}^T\right) & h_{\max}\left(W_{422}^T + P_{322}^T\right) & -W_{412}^T A_h - W_{422}^T A_h \\ & & & & -h_{\max} R_{111} & -h_{\max} R_{112} & -h_{\max} R_{211} & -h_{\max} R_{212} & 0 \\ & & & & & -h_{\max} R_{122} & -h_{\max} R_{221} & -h_{\max} R_{222} & 0 \\ & & & & & & -h_{\max} R_{311} & -h_{\max} R_{312} & 0 \\ & & & & & & & -h_{\max} R_{322} & 0 \\ & & & & & & & & -(1-\alpha)S + \varepsilon_4 E_h^T E_h \end{bmatrix}$$

(6.20)

$$\Omega_2 = \begin{bmatrix} P_{211}^T H_A + P_{221}^T (H_A - LH_C) & P_{211}^T H_h + P_{221}^T H_h & W_{311}^T H_h + W_{321}^T H_h & P_{211}^T B & P_{211}^T B E_B^T & P_{211}^T H_A + P_{221}^T (H_A - LH_C) \\ P_{212}^T H_A + P_{222}^T (H_A - LH_C) & P_{212}^T H_h + P_{222}^T H_h & W_{312}^T H_h + W_{322}^T H_h & P_{212}^T B & P_{212}^T B E_B^T & P_{212}^T H_A + P_{222}^T (H_A - LH_C) \\ P_{311}^T H_A + P_{321}^T (H_A - LH_C) & P_{311}^T H_h + P_{321}^T H_h & W_{411}^T H_h + W_{421}^T H_h & P_{311}^T B & P_{311}^T B E_B^T & P_{311}^T H_A + P_{321}^T (H_A - LH_C) \\ P_{312}^T H_A + P_{322}^T (H_A - LH_C) & P_{312}^T H_h + P_{322}^T H_h & W_{412}^T H_h + W_{422}^T H_h & P_{312}^T B & P_{312}^T B E_B^T & P_{312}^T H_A + P_{322}^T (H_A - LH_C) \end{bmatrix}$$

(6.21)

$$\Omega_3 = diag\left\{-\varepsilon_1 I, -\varepsilon_2 I, -\varepsilon_4 I, -\varepsilon_5 I, -\varepsilon_5\left(\varepsilon_3 I - E_B E_B^T\right), -\varepsilon_5 \varepsilon_3^{-1} I\right\} \tag{6.22}$$

and where,

$$\Psi_{111} = \left[A + BK + A_h\right]^T P_{211} + P_{211}^T\left[A + BK + A_h\right] + A_h^T P_{221} + P_{221}^T A_h + \left(W_{311}^T + W_{321}^T\right) A_h + A_h^T\left(W_{311} + W_{321}\right) + S$$

$$+\varepsilon_1\left(E_A + E_B K\right)^T\left(E_A + E_B K\right) + \left(\varepsilon_2 + \varepsilon_4\right) E_h^T E_h + 2\varepsilon_5 K^T K$$

$$\Psi_{112} = \left[A + BK + A_h\right]^T P_{212} + A_h^T P_{222} - P_{211}^T BK + P_{221}^T (A - LC) + A_h^T\left(W_{312} + W_{322}\right) - \varepsilon_1\left(E_A + E_B K\right)^T E_B K$$

$$\Psi_{122} = -(BK)^T P_{212} - P_{212}^T BK + (A - LC)^T P_{222} + P_{222}^T (A - LC) + \varepsilon_1\left(E_B K\right)^T E_B K + 2\varepsilon_5 K^T K$$

$$\Psi_{211} = P_{111}^T - P_{211}^T + \left(A + BK + A_h\right)^T P_{311} + A_h^T P_{321} + A_h^T\left(W_{411} + W_{421}\right)$$

$$\Psi_{212} = P_{121}^T - P_{221}^T + \left(A + BK + A_h\right)^T P_{312} + A_h^T P_{322} + A_h^T\left(W_{412} + W_{422}\right)$$

$$\Psi_{221} = P_{112}^T - P_{212}^T - (BK)^T P_{311} + (A - LC)^T P_{321}$$

$$\Psi_{222} = P_{122}^T - P_{222}^T - (BK)^T P_{312} + (A - LC)^T P_{322}$$

$$\Psi_{311} = -P_{311} - P_{311}^T + h_{\max} \hat{A}_{h11,\max}^T R_{311} \hat{A}_{h11,\max}$$

$$\Psi_{312} = -P_{312} - P_{321}^T$$

$$\Psi_{322} = -P_{322} - P_{322}^T$$

Under these conditions, the closed-loop system consisting of (6.1) *and controller-observer pair* (6.6)-(6.7) *is asymptotically stable.*

Proof. Following **[24]**, equation (6.14) is written in its equivalent descriptor form:

$$\dot{\xi}(t) = y(t), \quad y(t) = \hat{A}(t)\xi(t) + \hat{A}_h(t)\xi(t-h(t)) + \hat{B}(t)\eta(t) \tag{6.23}$$

Using the Liebnitz-Newton formula $\xi(t-h(t)) = \xi(t) - \int_{t-d(t)}^{t} \dot{\xi}(s)ds$ permits to re-write (6.23) yet in a more tractable form. Introduction of the augmented state as in (6.23) and the use of the Liebnitz-Newton formula allows to avoid the introduction of any additional dynamics, so that the transfer function of the system obtained by freezing the time-variable in the system matrices does not exhibit any additional poles **[8]**. This way of transforming system (6.23) is particularly useful as it allows to avoid unnecessary conservatism in the design that follows.

The last transformation of (6.23) yields:

$$\dot{\xi}(t) = y(t), \; 0 = -y(t) + \left[\hat{A}(t) + \hat{A}_h(t)\right]\xi(t) - \hat{A}_h(t)\int_{t-h(t)}^{t} y(s)\,ds + \hat{B}(t)\eta(t) \tag{6.24}$$

so that for $E = \begin{bmatrix} I_{2n} & 0 \\ 0 & 0 \end{bmatrix}$, the augmented system is:

$$E\begin{bmatrix} \dot{\xi}(t) \\ \dot{y}(t) \end{bmatrix} = \begin{bmatrix} \dot{\xi}(t) \\ 0 \end{bmatrix} = \begin{bmatrix} 0 & I \\ \hat{A}(t) + \hat{A}_h(t) & -I \end{bmatrix}\begin{bmatrix} \xi(t) \\ y(t) \end{bmatrix} - \begin{bmatrix} 0 \\ \hat{A}_h(t) \end{bmatrix}\int_{t-h(t)}^{t} y(s)\,ds + \begin{bmatrix} 0 \\ \hat{B}(t) \end{bmatrix}\eta(t) \tag{6.25}$$

Then the following Lyapunov-Krasovskii functional is used here:

$$V(t) = V_0(t) + V_1(t) + V_2(t) \tag{6.26}$$

where,

$$V_0(t) = \begin{bmatrix} \xi^T(t) & y^T(t) \end{bmatrix} EP \begin{bmatrix} \xi(t) \\ y(t) \end{bmatrix} = \xi^T(t) P_1 \xi(t) \tag{6.27}$$

$$V_1(t) = \int_{t-h(t)}^{t} x^T(\tau) S x(\tau)\,d\tau \tag{6.28}$$

$$V_2(t)=\int_{-h_{\max}}^{0}\int_{t+\theta}^{t} y^T(s)\hat{A}_{h,\max}^T R_3 \hat{A}_{h,\max} y(s)dsd\theta=\int_{-h_{\max}}^{0}\int_{t+\theta}^{t} y^T(s)\begin{bmatrix}0 & \hat{A}_{h,\max}^T\end{bmatrix} R \begin{bmatrix}0\\ \hat{A}_{h,\max}\end{bmatrix} y(s)dsd\theta \tag{6.29}$$

with $P=\begin{bmatrix}P_1 & 0\\ P_2 & P_3\end{bmatrix}$, $P_1=P_1^T>0$, $S=S^T>0$, $R_3=R_3^T>0$, $R=R^T=\begin{bmatrix}R_1 & R_2\\ R_2^T & R_3\end{bmatrix}>0$, $R_1=R_1^T$ (6.30)

and $\hat{A}_{h,\max}$ is such that,

$$\hat{A}_h^T(t)\hat{A}_h(t)\le \hat{A}_{h,\max}^T\hat{A}_{h,\max}, \qquad \text{for all } t\ge 0 \tag{6.31}$$

The bound in (6.31) can be evaluated using Lemma 6.2 as follows:

Let $\varepsilon_6>0$ be a scalar such that $\left(I-\varepsilon_6\hat{H}_h\hat{H}_h^T\right)>0$. Then

$$\hat{A}_h^T(t)\hat{A}_h(t)=\left(\hat{A}_h+\hat{H}_hF_h(t)\hat{E}_h\right)^T\left(\hat{A}_h+\hat{H}_hF_h(t)\hat{E}_h\right)\le \hat{A}_h^T\left(I-\varepsilon_6\hat{H}_h\hat{H}_h^T\right)^{-1}\hat{A}_h+\varepsilon_6^{-1}\hat{E}_h^T\hat{E}_h$$

$$=\begin{bmatrix}A_h^T & A_h^T\\ 0 & 0\end{bmatrix}\left(I-\varepsilon_6\begin{bmatrix}H_hH_h^T & H_hH_h^T\\ H_hH_h^T & H_hH_h^T\end{bmatrix}\right)^{-1}\begin{bmatrix}A_h & 0\\ A_h & 0\end{bmatrix}+\varepsilon_6^{-1}\begin{bmatrix}E_h^TE_h & 0\\ 0 & 0\end{bmatrix}=\hat{A}_{h,\max}^T\hat{A}_{h,\max} \tag{6.32}$$

Partitioning $\hat{A}_{h,\max}$ as follows,

$$\hat{A}_{h,\max}=\begin{bmatrix}\hat{A}_{h11,\max} & \hat{A}_{h12,\max}\\ \hat{A}_{h21,\max} & \hat{A}_{h22,\max}\end{bmatrix} \tag{6.33}$$

it is easy to see that,

$$\hat{A}_{h11,\max}=\hat{A}_{h11,\max}^T, \qquad \hat{A}_{h12,\max}=0, \qquad \hat{A}_{h22,\max}=0,$$

and thus (6.33) is reduced to,

$$\hat{A}_{h,\max}=\begin{bmatrix}\hat{A}_{h11,\max} & 0\\ 0 & 0\end{bmatrix}, \text{ with } \hat{A}_{h11,\max}=\hat{A}_{h11,\max}^T. \tag{6.34}$$

Differentiating (6.27) and using (6.25), yields,

$$\frac{dV_0(t)}{dt}=2\xi^T(t)P_1\dot{\xi}(t)=2\begin{bmatrix}\xi^T(t) & y^T(t)\end{bmatrix}P^T\begin{bmatrix}\dot{\xi}(t)\\ 0\end{bmatrix}$$

$$=2\begin{bmatrix}\xi^T(t) & y^T(t)\end{bmatrix}P^T\begin{bmatrix}0 & I\\ \hat{A}(t)+\hat{A}_h(t) & -I\end{bmatrix}\begin{bmatrix}\xi(t)\\ y(t)\end{bmatrix}+2\begin{bmatrix}\xi^T(t) & y^T(t)\end{bmatrix}P^T\begin{bmatrix}0\\ \hat{B}(t)\end{bmatrix}\eta(t)$$

$$-2\int_{t-h(t)}^{t}\begin{bmatrix}\xi^T(t) & y^T(t)\end{bmatrix}P^T\begin{bmatrix}0\\ \hat{A}_h(t)\end{bmatrix}y(s)ds \tag{6.35}$$

A bound for the last term of (6.35) is now derived as follows:

Define $\mu(t) \triangleq -2\int_{t-h(t)}^{t} \left[\xi^T(t) \quad y^T(t)\right] P^T \begin{bmatrix} 0 \\ \hat{A}_h(t) \end{bmatrix} y(s)\,ds$ (6.36)

Using Lemma 6.3 with:

$a(s) \triangleq \begin{bmatrix} 0 \\ \hat{A}_h(t) \end{bmatrix} y(s), b(s) \triangleq P \begin{bmatrix} \xi(t) \\ y(t) \end{bmatrix}, \Upsilon \triangleq [t-h(t), t]$ gives:

$$\mu(t) \le \int_{t-h(t)}^{t} \left[\xi^T(t) \quad y^T(t)\right] P^T \left(M^T R + I\right) R^{-1} (RM + I) P \begin{bmatrix} \xi(t) \\ y(t) \end{bmatrix} ds + 2\int_{t-h(t)}^{t} y^T(s)\,ds \left[0 \quad \hat{A}_h^T(t)\right] RMP \begin{bmatrix} \xi(t) \\ y(t) \end{bmatrix}$$
$$+\int_{t-h(t)}^{t} y^T(s) \left[0 \quad \hat{A}_h^T(t)\right] R \begin{bmatrix} 0 \\ \hat{A}_h \end{bmatrix} y(s)\,ds$$

$$\le h_{\max} \left[\xi^T(t) \quad y^T(t)\right] P^T \left(M^T R + I\right) R^{-1} (RM + I) P \begin{bmatrix} \xi(t) \\ y(t) \end{bmatrix} + 2\left[\xi^T(t) - \xi^T(t-h(t))\right] \left[0 \quad \hat{A}_h^T(t)\right] RMP \begin{bmatrix} \xi(t) \\ y(t) \end{bmatrix}$$
$$+\int_{t-h_{\max}}^{t} y^T(s) \left[0 \quad \hat{A}_h^T(t)\right] R \begin{bmatrix} 0 \\ \hat{A}_h(t) \end{bmatrix} y(s)\,ds \qquad (6.37)$$

as $R > 0$.

Differentiating $V_1(t)$ of (6.28),

$$\frac{dV_1(t)}{dt} = x^T(t) Sx(t) - \left(1 - \dot{h}(t)\right) x^T(t-h(t)) Sx(t-h(t))$$

and using the assumption $S > 0$, as well as the bounds specified by (6.4),

$$\frac{dV_1(t)}{dt} \le x^T(t) Sx(t) - (1-\alpha) x^T(t-h(t)) Sx(t-h(t)) \qquad (6.38)$$

Applying Lemma 6.1 to $V_3(t)$,

$$\frac{dV_2(t)}{dt} = h_{\max} y^T(t) \hat{A}_{h,\max}^T R_3 \hat{A}_{h,\max} y(t) - \int_{t-h_{\max}}^{t} y^T(s) \hat{A}_{h,\max}^T R_3 \hat{A}_{h,\max} y(s)\,ds \qquad (6.39)$$

Employing (6.26), (6.35), (6.37), (6.38) and (6.39), the following upper bound for $\dot{V}(t)$ is obtained :

$$\dot{V}(t) \le 2\left[\xi^T(t) \quad y^T(t)\right] P^T \begin{bmatrix} 0 & I \\ \hat{A}(t) + \hat{A}_h(t) & -I \end{bmatrix} \begin{bmatrix} \xi(t) \\ y(t) \end{bmatrix} + 2\left[\xi^T(t) \quad y^T(t)\right] P^T \begin{bmatrix} 0 \\ \hat{B}(t) \end{bmatrix} \eta(t)$$
$$+h_{\max} \left[\xi^T(t) \quad y^T(t)\right] P^T \left(M^T R + I\right) R^{-1} (RM + I) P \begin{bmatrix} \xi(t) \\ y(t) \end{bmatrix}$$

$$+2\left[\xi^T(t)-\xi^T(t-h(t))\right]\left[0 \quad \hat{A}_h^T(t)\right]RMP\begin{bmatrix}\xi(t)\\ y(t)\end{bmatrix}$$

$$+\int_{t-h_{\max}}^{t} y^T(s)\left[0 \quad \hat{A}_h^T(t)\right]R\begin{bmatrix}0\\ \hat{A}_h(t)\end{bmatrix}y(s)\,ds$$

$$+x^T(t)Sx(t)-(1-\alpha)x^T(t-h(t))Sx(t-h(t))$$

$$+h_{\max}y^T(t)\hat{A}_{h,\max}^T R_3\hat{A}_{h,\max}y(t)-\int_{t-h_{\max}}^{t} y^T(s)\hat{A}_{h,\max}^T R_3\hat{A}_{h,\max}y(s)\,ds \tag{6.40}$$

Since $R_3>0$, then, by virtue of the bound given by (6.31), the two remaining integrals in (6.40) satisfy the following inequality:

$$\int_{t-h_{\max}}^{t} y^T(s)\left[0 \quad \hat{A}_h^T(t)\right]R\begin{bmatrix}0\\ \hat{A}_h(t)\end{bmatrix}y(s)\,ds-\int_{t-h_{\max}}^{t} y^T(s)\hat{A}_{h,\max}^T R_3\hat{A}_{h,\max}y(s)\,ds$$

$$=\int_{t-h_{\max}}^{t} y^T(s)\left[\hat{A}_h^T(t)R_3\hat{A}_h(t)-\hat{A}_{h,\max}^T R_3\hat{A}_{h,\max}\right]y(s)\,ds\le 0$$

Now, it is possible to reduce (6.40) to:

$$\frac{dV(t)}{dt}\le 2\left[\xi^T(t) \quad y^T(t)\right]P^T\begin{bmatrix}0 & I\\ \hat{A}(t)+\hat{A}_h(t) & -I\end{bmatrix}\begin{bmatrix}\xi(t)\\ y(t)\end{bmatrix}+2\left[\xi^T(t) \quad y^T(t)\right]P^T\begin{bmatrix}0\\ \hat{B}(t)\end{bmatrix}\eta(t)$$

$$+h_{\max}\left[\xi^T(t) \quad y^T(t)\right]P^T\left(M^TR+I\right)R^{-1}(RM+I)P\begin{bmatrix}\xi(t)\\ y(t)\end{bmatrix}$$

$$+2\left[\xi^T(t)-\xi^T(t-h(t))\right]\left[0 \quad \hat{A}_h^T(t)\right]RMP\begin{bmatrix}\xi(t)\\ y(t)\end{bmatrix}$$

$$+x^T(t)Sx(t)-(1-\alpha)x^T(t-h(t))Sx(t-h(t))+h_{\max}y^T(t)\hat{A}_{h,\max}^T R_3\hat{A}_{h,\max}y(t) \tag{6.41}$$

Applying part 4 of Lemma 6.2 to the expression $2\left[\xi^T(t) \quad y^T(t)\right]P^T\begin{bmatrix}0\\ \hat{B}(t)\end{bmatrix}\eta(t)$, with $X=\left[0 \quad \hat{B}^T(t)\right]P\begin{bmatrix}\xi(t)\\ y(t)\end{bmatrix}$ and $Y=\eta(t)$, yields,

$$2\left[\xi^T(t) \quad y^T(t)\right]P^T\begin{bmatrix}0\\ \hat{B}(t)\end{bmatrix}\eta(t)\le\alpha_1\left[\xi^T(t) \quad y^T(t)\right]P^T\begin{bmatrix}0\\ \hat{B}(t)\end{bmatrix}\left[0 \quad \hat{B}^T(t)\right]P\begin{bmatrix}\xi(t)\\ y(t)\end{bmatrix}+\alpha_1^{-1}\eta^T(t)\eta(t) \tag{6.42}$$

It is easy to check that the vector $\eta(t) = sat(2K\hat{x}(t)) - K\hat{x}(t)$ has its norm bounded as follows,

$$\eta^T(t)\eta(t) \le 2\left[x^T(t)K^TKx(t) + e^T(t)K^TKe(t)\right] \tag{6.43}$$

Using (6.42) and (6.43) in (6.41), gives the following upper bound on $\dot{V}(t)$,

$$\begin{aligned}
\frac{dV(t)}{dt} \le\ & 2\begin{bmatrix}\xi^T(t) & y^T(t)\end{bmatrix}P^T\begin{bmatrix}0 & I\\ \hat{A}(t)+\hat{A}_h(t) & -I\end{bmatrix}\begin{bmatrix}\xi(t)\\ y(t)\end{bmatrix} + \alpha_1\begin{bmatrix}\xi^T(t) & y^T(t)\end{bmatrix}P^T\begin{bmatrix}0\\ \hat{B}(t)\end{bmatrix}\begin{bmatrix}0 & \hat{B}^T(t)\end{bmatrix}P\begin{bmatrix}\xi(t)\\ y(t)\end{bmatrix}\\
& + h_{\max}\begin{bmatrix}\xi^T(t) & y^T(t)\end{bmatrix}P^T\left(M^TR+I\right)R^{-1}(RM+I)P\begin{bmatrix}\xi(t)\\ y(t)\end{bmatrix}\\
& + 2\left[\xi^T(t) - \xi^T(t-h(t))\right]\begin{bmatrix}0 & \hat{A}_h^T(t)\end{bmatrix}RMP\begin{bmatrix}\xi(t)\\ y(t)\end{bmatrix}\\
& + x^T(t)Sx(t) - (1-\alpha)x^T(t-h(t))Sx(t-h(t)) + h_{\max}y^T(t)\hat{A}_{h,\max}^TR_3\hat{A}_{h,\max}y(t)\\
& + 2\alpha_1^{-1}\left[x^T(t)K^TKx(t) + e^T(t)K^TKe(t)\right]
\end{aligned}$$

Next, setting $\varepsilon_5 \triangleq \alpha_1^{-1}$ and applying Lemma 6.2 to the uncertainty terms of $\hat{A}(t)$, $\hat{A}_h(t)$ and $\hat{B}(t)$, results in the following upper bound for $\dot{V}(t)$:

$$\begin{aligned}
\frac{dV(t)}{dt} \le\ & \begin{bmatrix}\xi^T(t) & y^T(t)\end{bmatrix}P^T\begin{bmatrix}0 & I\\ \hat{A}+\hat{A}_h & -I\end{bmatrix}\begin{bmatrix}\xi(t)\\ y(t)\end{bmatrix}\\
& + \begin{bmatrix}\xi^T(t) & y^T(t)\end{bmatrix}\begin{bmatrix}0 & \hat{A}^T+\hat{A}_h^T\\ I & -I\end{bmatrix}P\begin{bmatrix}\xi(t)\\ y(t)\end{bmatrix}\\
& + \varepsilon_5^{-1}\begin{bmatrix}\xi^T(t) & y^T(t)\end{bmatrix}P^T\begin{bmatrix}0\\ \hat{B}\end{bmatrix}\begin{bmatrix}0 & \hat{B}^T\end{bmatrix}P\begin{bmatrix}\xi(t)\\ y(t)\end{bmatrix}\\
& + \varepsilon_5^{-1}\begin{bmatrix}\xi^T(t) & y^T(t)\end{bmatrix}P^T\begin{bmatrix}0\\ \hat{B}E_B^T\end{bmatrix}\left(\varepsilon_3I - E_BE_B^T\right)^{-1}\begin{bmatrix}0 & E_B\hat{B}^T\end{bmatrix}P\begin{bmatrix}\xi(t)\\ y(t)\end{bmatrix}\\
& + \varepsilon_3\varepsilon_5^{-1}\begin{bmatrix}\xi^T(t) & y^T(t)\end{bmatrix}P^T\begin{bmatrix}0\\ \hat{H}\end{bmatrix}\begin{bmatrix}0 & \hat{H}^T\end{bmatrix}P\begin{bmatrix}\xi(t)\\ y(t)\end{bmatrix}\\
& + h_{\max}\begin{bmatrix}\xi^T(t) & y^T(t)\end{bmatrix}P^T\left(M^TR+I\right)R^{-1}(RM+I)P\begin{bmatrix}\xi(t)\\ y(t)\end{bmatrix}\\
& + \begin{bmatrix}x^T(t) & y^T(t)\end{bmatrix}P^TM^TR\begin{bmatrix}0\\ \hat{A}_h\end{bmatrix}\left[x(t) - x(t-h(t))\right]
\end{aligned}$$

$$+\left[\xi^T(t)-\xi^T(t-h(t))\right]\left[0 \quad \hat{A}_h^T\right]RMP\begin{bmatrix}\xi(t)\\ y(t)\end{bmatrix}$$

$$+x^T(t)Sx(t)-(1-\alpha)x^T(t-h(t))Sx(t-h(t))+h_{\max}y^T(t)\hat{A}_{h,\max}^T R_3\hat{A}_{h,\max}y(t)$$

$$+2\varepsilon_5\left[x^T(t)K^TKx(t)+e^T(t)K^TKe(t)\right]$$

$$+\varepsilon_1^{-1}\xi^T(t)P_2^T\hat{H}\hat{H}^TP_2\xi(t)+\varepsilon_1^{-1}y^T(t)P_3^T\hat{H}\hat{H}^TP_2\xi(t)+\varepsilon_1^{-1}\xi^T(t)P_2^T\hat{H}\hat{H}^TP_3y(t)$$

$$+\varepsilon_1^{-1}y^T(t)P_3^T\hat{H}\hat{H}^TP_3y(t)+\varepsilon_1\xi^T(t)\hat{E}^T\hat{E}\xi(t)$$

$$+\varepsilon_2^{-1}\xi^T(t)P_2^T\hat{H}_h\hat{H}_h^TP_2\xi(t)+\varepsilon_2^{-1}y^T(t)P_3^T\hat{H}_h\hat{H}_h^TP_2\xi(t)+\varepsilon_2^{-1}\xi^T(t)P_2^T\hat{H}_h\hat{H}_h^TP_3y(t)$$

$$+\varepsilon_2^{-1}y^T(t)P_3^T\hat{H}_h\hat{H}_h^TP_3y(t)+\varepsilon_2\xi^T(t)\hat{E}_h^T\hat{E}_h\xi(t)$$

$$+\varepsilon_4^{-1}\xi^T(t)W_3^T\hat{H}_h\hat{H}_h^TW_3\xi(t)+\varepsilon_4^{-1}y^T(t)W_4^T\hat{H}_h\hat{H}_h^TW_3\xi(t)+\varepsilon_4^{-1}\xi^T(t)W_3^T\hat{H}_h\hat{H}_h^TW_4y(t)$$

$$+\varepsilon_4^{-1}y^T(t)W_4^T\hat{H}_h\hat{H}_h^TW_4y(t)+\varepsilon_4\left[\xi^T(t)-\xi^T(t-h(t))\right]\hat{E}_h^T\hat{E}_h\left[\xi(t)-\xi(t-h(t))\right] \tag{6.44}$$

with,

$$\varepsilon_i>0,\ i=1,\ldots,5. \tag{6.45}$$

Defining $W\triangleq\begin{bmatrix}W_1 & W_2\\ W_3 & W_4\end{bmatrix}\triangleq RMP$, and the aggregated state $\chi(t)$ by,

$$\chi(t)\triangleq\begin{bmatrix}\xi(t)\\ y(t)\\ \left(M+R^{-1}\right)P\begin{bmatrix}\xi(t)\\ y(t)\end{bmatrix}\\ \xi^T(t-h(t))\end{bmatrix}$$

inequality (6.44) can now simply be written in the form:

$$\dot{V}(t)\le\chi^T(t)\Theta\chi(t). \tag{6.46}$$

To guarantee that $\dot{V}(t)<0$, for all $t\ge0$, it is sufficient that,

$$\Theta<0. \tag{6.47}$$

Letting,

$$P_i\triangleq\begin{bmatrix}P_{i11} & P_{i12}\\ P_{i21} & P_{i22}\end{bmatrix}, \qquad i=1,\ldots,3$$

with, $P_{111} = P_{111}^T$, $\quad P_{112} = P_{121}^T$, $\quad P_{122} = P_{122}^T$

$$W_i \triangleq \begin{bmatrix} W_{i11} & W_{i12} \\ W_{i21} & W_{i22} \end{bmatrix}, \quad i = 1,\ldots,4$$

$$R_i \triangleq \begin{bmatrix} R_{i11} & R_{i12} \\ R_{i21} & R_{i22} \end{bmatrix}, \quad i = 1,\ldots,3$$

with, $R_{111} = R_{111}^T$, $\quad R_{112} = R_{121}^T$, $\quad R_{122} = R_{122}^T$, $\quad R_{311} = R_{311}^T$, $\quad R_{312} = R_{321}^T$, $\quad R_{322} = R_{322}^T$ (6.48)

and expanding all the matrices within Θ using the definitions (6.13) and (6.48), inequality (6.47) becomes $\Omega_1 - \Omega_2\Omega_3^{-1}\Omega_2^T < 0$ where Ω_1, Ω_2 and Ω_3 are given by (6.20), (6.21) and (6.22), respectively.

Sufficient conditions for $\dot{V}(t) < 0$ can thus be summarized as follows:

1.) $\Omega_1 - \Omega_2\Omega_3^{-1}\Omega_2^T < 0$

2.) $\Omega_3 < 0$ (assumptions (6.45) resulting from the application of Lemma 6.2 to obtain inequality (6.44)).

By the Schur Complements Lemma, conditions 1.) and 2.) above are equivalent to assumption (6.19) of the Theorem.

Thus, from $\dot{V}(t) \le \chi^T(t)\Theta\chi(t) < 0$ there exists $\pi > 0$ such that the quadratic stabilization condition **[2]** is satisfied as, $\dot{V}(\xi_t(\psi) = \xi(t+\psi)) \le -\pi\|\chi(t)\|^2 \le -\pi\|\xi(t)\|^2$. This implies that the closed-loop system formed by (6.1) and control law (6.6)-(6.7) is asymptotically stable. QED

6.7. Numerical Example

Example 6.1. The result presented within Theorem 6.1 involves the solution of several complicated matrix inequalities. To demonstrate its usefulness for control design processes and explain how it might be employed, an example is considered of an uncertain time-delay system of the type (6.1), in which the system matrices are given by:

$$A = \begin{bmatrix} -2 & 1 \\ 3 & -4 \end{bmatrix}, \quad A_h = \begin{bmatrix} 1 & 0 \\ -2 & 3 \end{bmatrix}, \quad B = \begin{bmatrix} 3 \\ 5 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 2 \end{bmatrix}$$

$$H_A = \begin{bmatrix} 1/10 \\ 0 \end{bmatrix}, \quad E_A = \begin{bmatrix} 1 & 1 \end{bmatrix}, \quad H_h = \begin{bmatrix} 1/10 \\ 1/10 \end{bmatrix}, \quad E_h = \begin{bmatrix} 0 & 1 \end{bmatrix}, \quad H_C = 0, \quad E_B = 0 \qquad (6.49)$$

The time-delay is considered time-invariant with $h = 0.3$, and the control saturation level is determined by $\bar{u} = 1$. The same example was earlier used by **[72]** where only the values of gains K and L were given without any further information about the response of the system.

With the above values for the state matrices, an observer gain matrix for the nominal system (in the absence of uncertainties and saturation) is chosen first using standard methods to minimize the state estimation error for the nominal system. The next step consists in the design of a nominal controller gain according to pre-selected performance criteria. With reference to the above example, the following values have so proved suitable:

$$L = \begin{bmatrix} 50 \\ 50 \end{bmatrix}, \quad K = \begin{bmatrix} -0.1 & -0.1 \end{bmatrix} \tag{6.50}$$

Theorem 6.1 was then applied to verify that (6.50) in fact yield a robust observer and controller design for the pre-specified uncertainties and the given actuator saturation level. This was done as follows. The parameters ε_3 and ε_6 were chosen first to be $\varepsilon_3 = \varepsilon_6 = 1$, and the matrix $\hat{A}_{h,11} = \begin{bmatrix} -2.5766 & 1.8446 \\ 1.8446 & -3.9965 \end{bmatrix}$ was computed using (6.32). It should be pointed out that the parameters ε_3 and ε_6 could be tuned further to reduce the overall conservatism of the design. This, however, has not been our purpose here.

With fixed design parameters K , L and ε_3, the matrix inequalities of Theorem 6.1 become linear in the remaining parameters and can be solved using the Matlab LMI toolbox, based on the methods developed by **[6]**. This was carried out to confirm that system (6.49) controlled by output feedback with observer and controller gains (6.50) is robustly stable with respect to the admissible uncertainties and in spite of the given control saturation. Thus, the control laws (6.6) and (6.7) constitute a robust output feedback controller for this time-delay system.

Figure 6.1 shows the open-loop state response for the nominal system with initial condition $x_0 = (2,1)$.

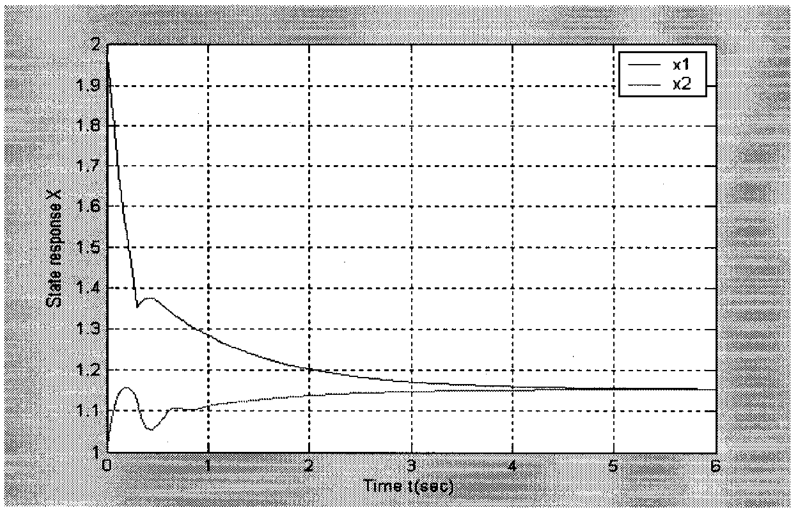

Figure 6.1. *Open-loop state response of the nominal system with initial condition X0=(2,1)*

Figure 6.2 shows the corresponding closed-loop state response of the nominal system with observer and controller gains as in (6.50). It is seen that the overshoot and the steady-state error were eliminated and that the settling time is smaller as compared with the open-loop system of Figure 6.1.

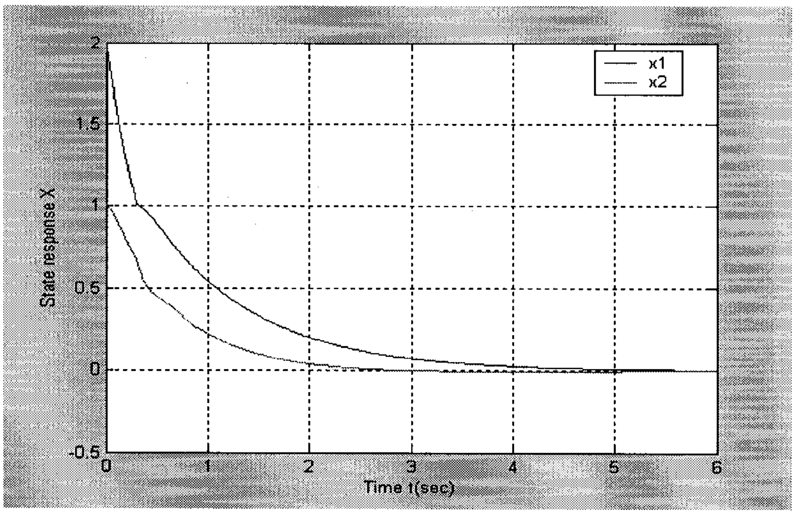

Figure 6.2. *Closed-loop state response of the nominal system with initial condition X0=(2,1)*

Figure 6.3 shows the control input to the plant corresponding to the response in Figure 6.2. No control saturation is seen.

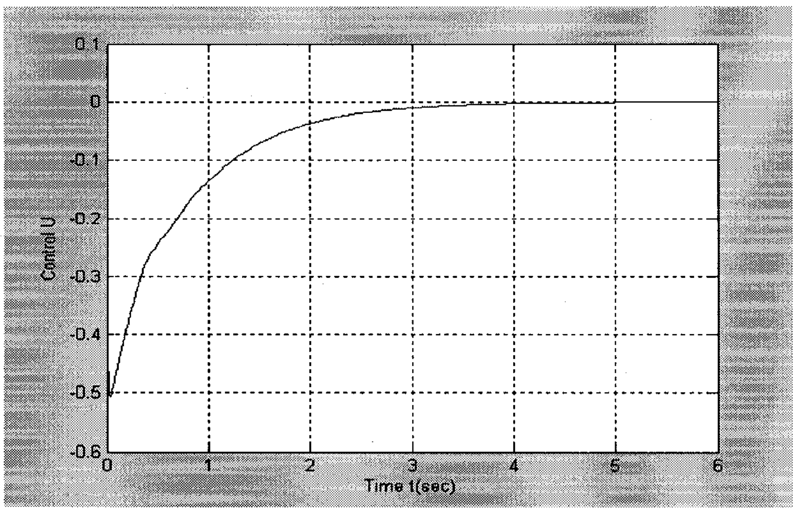

Figure 6.3. *Closed-loop control input to the plant in the nominal system with initial condition X0=(2,1)*

Finally, the controller and observer are tested using a remote initial condition $x_0 = (10,5)$; see Figure 6.4 for the closed-loop nominal system.

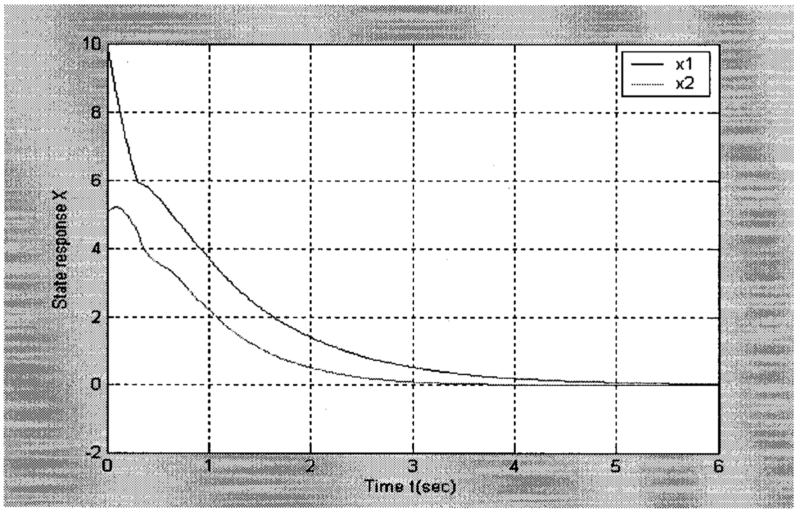

Figure 6.4. *Closed-loop state response of the nominal system with initial condition X0=(10,5)*

Figure 6.5 shows the corresponding control input to the plant where the actuator is clearly seen to saturate at the specified level of -1. Despite saturation, the responses of Figure 6.4 are still those of a stable system and the overall system performance has not been compromised excessively.

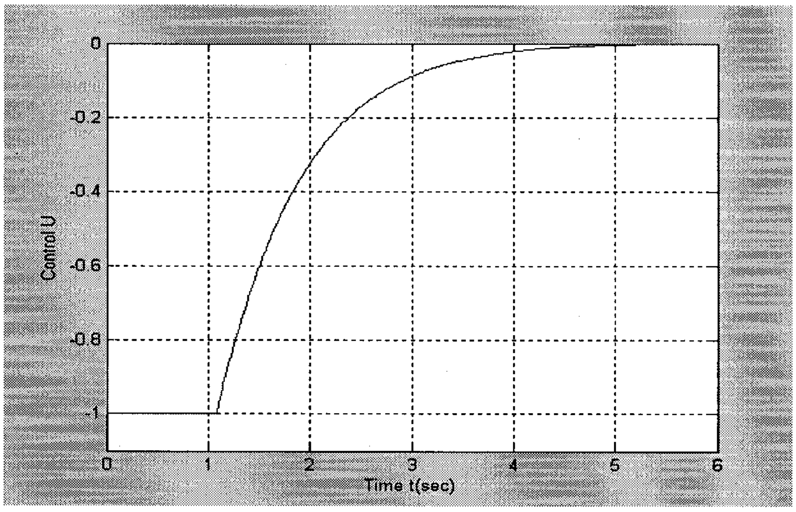

Figure 6.5. *Closed-loop input to the plant in the nominal system with initial condition X0=(10,5)*

Simulations were made for different constant values of the uncertainty matrices $F(t)$ and $F_h(t)$, with initial condition $x_0 = [10,5]$. As seen in Figure 6.6, despite uncertainties and control saturation, the closed loop system remains asymptotically stable.

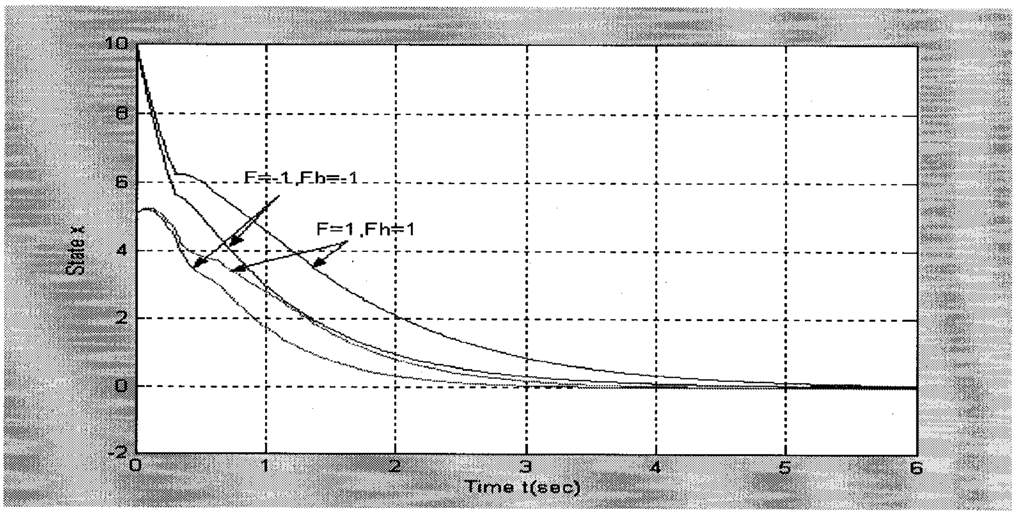

Figure 6.6. *Closed-loop state response for uncertainties* $(F=1, F_h=1)$ *and* $(F=-1, F_h=-1)$*, with initial condition X0=(10,5)*

CHAPTER 7

Robust Output Feedback Stabilization of Uncertain Time-Varying State-Delayed Systems with Saturating Actuators: The Differential Inclusions Method

7.1. System Description

Consider the retarded system (1.1)-(1.5) taken in the infinite-horizon context, where all the system matrices are time-invariant except the uncertainty matrices F_i. More specifically, the system under consideration is:

$$\dot{x}(t) = A(t)x(t) + A_h(t)x(t-h(t)) + B(t)u_{sat}(t)$$

$$y(t) = C(t)x(t) + D(t)u_{sat}(t)$$

$$u_{sat}(t) = sat(u(t)),\ sat(u(t)) = \left[sat(u_1(t)) \quad sat(u_2(t)) \quad \dots \quad sat(u_m(t))\right]$$

$$x(t_0+\psi) = \phi(\psi),\ \forall \psi \in [-h_{\max}, 0],\ (t_0, \phi) \in \mathbb{R}^+ \times C^{w}_{h_{\max}, n} \tag{7.1}$$

$$A(t) = A + \Delta A(t),\ A_h(t) = A_h + \Delta A_h(t),\ B(t) = B + \Delta B(t),\ C(t) = C + \Delta C(t),\ D(t) = D + \Delta D(t) \tag{7.2}$$

$$\begin{bmatrix} \Delta A(t) & \Delta B(t) \\ \Delta C(t) & \Delta D(t) \end{bmatrix} = \begin{bmatrix} H_A \\ H_C \end{bmatrix} F(t)[E_A \quad E_B],\ \Delta A_h(t) = H_h F_h(t) E_h \tag{7.3}$$

7.2. Assumptions

The time-delay h is a function of time and is assumed to be continuously differentiable, with its amplitude and rate of change bounded as follows:

$$0 \le h(t) \le h_{\max}, \qquad 0 \le \dot{h}(t) \le \beta < 1, \qquad \text{for all } t \ge 0 \tag{7.4}$$

where $h_{\max}$ and α are given positive constants.

Finally, the following is assumed to hold:

Assumption 7.1. $(A+A_h,B)$ is stabilizable.

Assumption 7.2. (C,A) is detectable.

Assumption 7.3. The input vector is subject to amplitude constraints, i.e., $u\in U_0\subset\mathbb{R}^m$, with

$$U_0\triangleq\{u\in\mathbb{R}^m\ ;\ -\bar{u}_i\le u_i\le\bar{u}_i\ ,\ \ i=1...m\} \tag{7.5}$$

where vector $\bar{u}\triangleq[\bar{u}_1,...,\bar{u}_m]^T$ has strictly positive entries and is given.

7.3. Preliminaries

The following lemmas will prove helpful in the sequel:

Lemma 7.1. [39] *Let $a(t):\mathbb{R}^+\to\mathbb{R}$, $b(t):\mathbb{R}^+\to\mathbb{R}$ and $f(s):\mathbb{R}\to\mathbb{R}$ be continuously differentiable functions.*

Let the function $z(t)=\int_{a(t)}^{b(t)}\int_{t-\theta}^{t}f(s)ds\,d\theta$. Then $z(t)$ is a solution of the differential equation,

$$\frac{dz(t)}{dt}=(b(t)-a(t))f(t)-(1-\dot{b}(t))\int_{t-b(t)}^{t-a(t)}f(s)ds+(\dot{b}(t)-\dot{a}(t))\int_{t-a(t)}^{t}f(s)ds\,.$$

Lemma 7.2. [12] *Let A, L, E and F be real matrices (possibly time-varying) of appropriate dimensions, with F satisfying $FF^T\le I$. Then the following holds:*

1- For any scalar $\varepsilon>0$ and any matrix P,

$$P^TLFE+E^TF^TL^TP\le\varepsilon^{-1}P^TLL^TP+\varepsilon E^TE$$

2- For any matrix $P>0$ and any scalar $\varepsilon>0$ such that $\varepsilon I-EPE^T>0$,

$$(A+LFE)P(A+LFE)^T\le APA^T+APE^T\left(\varepsilon I-EPE^T\right)^{-1}EPA^T+\varepsilon LL^T$$

3- For any matrix $P>0$ and any scalar $\varepsilon>0$ such that $P-\varepsilon LL^T>0$,

$$(A+LFE)^TP^{-1}(A+LFE)\le A^T\left(P-\varepsilon LL^T\right)^{-1}A+\varepsilon^{-1}E^TE$$

Lemma 7.3. [62] *Assume that* $a:\Upsilon \to \mathbb{R}^n$ *and* $b:\Upsilon \to \mathbb{R}^n$, $\Upsilon \subset \mathbb{R}$ *are integrable functions over their common domain* Υ. *Then, for any positive definite matrix* $R \in \mathbb{R}^{n\times m}$ *and any matrix* $M \in \mathbb{R}^{m\times m}$, *the following inequality holds:*

$$-2\int_{\Omega} b^T(s)a(s)ds \le \int_{\Omega} \begin{bmatrix} a(s) \\ b(s) \end{bmatrix}^T \begin{bmatrix} R & RM \\ M^T R & \Upsilon \end{bmatrix} \begin{bmatrix} a(s) \\ b(s) \end{bmatrix} ds \tag{7.6}$$

where $\Upsilon = \left(M^T R + I\right) R^{-1} \left(RM + I\right)$.

7.4. The Observer-Based Dynamic Output Feedback

The results to be presented are concerned with providing sufficient conditions for the design of an observer-based dynamic output feedback law for system (7.1). This law is assumed to take the following form:

$$u(t) = K\hat{x}(t) \tag{7.7}$$

$$\dot{\hat{x}}(t) = A\hat{x}(t) + Bu_{sat}(t) + L\left(y(t) - C\hat{x}(t) - Du_{sat}(t)\right) \tag{7.8}$$

where $\hat{x}(t) \in \mathbb{R}^n$ is the observer state vector, $K \in \mathbb{R}^{m\times n}$ and $L \in \mathbb{R}^{n\times r}$ are the constant controller and observer gains, respectively.

Using (7.1) and (7.8), permits to write,

$$\begin{bmatrix} \dot{x}(t) \\ \dot{\hat{x}}(t) \end{bmatrix} = \left\{ \begin{bmatrix} A & 0 \\ LC & A-LC \end{bmatrix} + \begin{bmatrix} \Delta A(t) & 0 \\ L\Delta C(t) & 0 \end{bmatrix} \right\} \begin{bmatrix} x(t) \\ \hat{x}(t) \end{bmatrix}$$

$$+\left\{ \begin{bmatrix} A_h & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} \Delta A_h(t) & 0 \\ 0 & 0 \end{bmatrix} \right\} \begin{bmatrix} x(t-h(t)) \\ \hat{x}(t-h(t)) \end{bmatrix} + \left\{ \begin{bmatrix} B \\ B \end{bmatrix} + \begin{bmatrix} \Delta B(t) \\ L\Delta D(t) \end{bmatrix} \right\} u_{sat}(t) \tag{7.9}$$

Using (7.3),

$$\begin{bmatrix} \dot{x}(t) \\ \dot{\hat{x}}(t) \end{bmatrix} = \left\{ \begin{bmatrix} A & 0 \\ LC & A-LC \end{bmatrix} + \begin{bmatrix} H_A \\ LH_C \end{bmatrix} F(t)[E_A \quad 0] \right\} \begin{bmatrix} x(t) \\ \hat{x}(t) \end{bmatrix}$$

$$+\left\{ \begin{bmatrix} A_h & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} H_h \\ 0 \end{bmatrix} F_h(t)[E_h \quad 0] \right\} \begin{bmatrix} x(t-h(t)) \\ \hat{x}(t-h(t)) \end{bmatrix} + \left\{ \begin{bmatrix} B \\ B \end{bmatrix} + \begin{bmatrix} H_A \\ LH_C \end{bmatrix} F(t) E_B \right\} u_{sat}(t) \tag{7.10}$$

Finally, introducing the following definitions,

$$\xi(t) \triangleq \begin{bmatrix} x(t) \\ \hat{x}(t) \end{bmatrix},\ \hat{A} \triangleq \begin{bmatrix} A & 0 \\ LC & A-LC \end{bmatrix},\ \Delta\hat{A}(t) \triangleq \hat{H}F(t)\hat{E} \triangleq \begin{bmatrix} H_A \\ LH_C \end{bmatrix} F(t)[E_A \quad 0],$$

$$\hat{A}_h \triangleq \begin{bmatrix} A_h & 0 \\ 0 & 0 \end{bmatrix}, \Delta\hat{A}_h(t) \triangleq \hat{H}_h F_h(t)\hat{E}_h \triangleq \begin{bmatrix} H_h \\ 0 \end{bmatrix} F_h(t)[E_h \quad 0],\ \hat{B} \triangleq \begin{bmatrix} B \\ B \end{bmatrix}, \Delta\hat{B}(t) \triangleq \hat{H}F(t)E_B \triangleq \begin{bmatrix} H_A \\ LH_C \end{bmatrix} F(t)E_B$$

$$\hat{A}(t) \triangleq \hat{A} + \Delta\hat{A}(t),\ \hat{A}_h(t) \triangleq \hat{A}_h + \Delta\hat{A}_h(t),\ \hat{B}(t) \triangleq \hat{B} + \Delta\hat{B}(t),\ \hat{K} = [0 \quad K] \tag{7.11}$$

permits to re-write equation (7.10) in a yet more compact form,

$$\dot{\xi}(t) = \hat{A}(t)\xi(t) + \hat{A}_h(t)\xi(t-h(t)) + \hat{B}(t)u_{sat}(t)\,. \tag{7.12}$$

with $u_{sat}(t) = sat\left(\hat{K}\xi(t)\right)$.

7.5. Problem Statement

The robust output feedback stabilization problem with saturating actuators:

Find a matrix $K \in \mathbb{R}^{m\times n}$, a matrix $L \in \mathbb{R}^{n\times r}$ and a set of initial conditions $S_0 \subset \mathbb{R}^{2n}$ such that the closed-loop system (7.12) is asymptotically stable.

Remark 7.1. Generally, global asymptotic stability for an open-loop unstable system with bounded controls cannot be achieved so only local asymptotic stability will be sought in this chapter since no assumption is made concerning the stability of the open-loop system.

7.6. Main Result

A locally equivalent polytopic representation for the closed loop nonlinear system (7.12) based on the concept of differential inclusions **[58]**, is used here and leads to,

$$\dot{\xi}(t) = \left(\hat{A}(t) + \hat{B}(t)\Gamma(\alpha(x))\hat{K}\right)\xi(t) + \hat{A}_h(t)\xi(t-h(t)) \tag{7.13}$$

where $\Gamma(\alpha(\xi)) \triangleq diag\{\alpha_i(\xi);\ i=1,\ldots,m\}$ is a diagonal matrix whose diagonal elements are defined by:

$$\alpha_i(\xi) \triangleq \begin{cases} \dfrac{\bar{u}_i}{\hat{K}_{(i)}\xi} & \text{if } \hat{K}_{(i)}\xi > \bar{u}_i \\ 1 & \text{if } -\bar{u}_i \le \hat{K}_{(i)}\xi \le \bar{u}_i \qquad i = 1,\dots,m \\ -\dfrac{\bar{u}_i}{\hat{K}_{(i)}\xi} & \text{if } \hat{K}_{(i)}\xi < -\bar{u}_i \end{cases} \tag{7.14}$$

where $\hat{K}_{(i)}$ is the i-th row of matrix $\hat{K}$. Clearly $0 < \alpha_i(\xi) \le 1,\ i = 1,\dots,m,\ \forall \xi \in \mathbb{R}^{2n}$. The value $\alpha_i(\xi)$ can be interpreted as an indicator of the saturation degree of the control law. The smaller is α_i, the farther is ξ from the region of linearity of the control u, $S(\bar{u}, 1_m)$,

$$S(\bar{u}, 1_m) \triangleq \left\{\xi \in \mathbb{R}^{2n} : \left|\hat{K}_{(i)}\xi\right| \le \bar{u}_i,\ i = 1,\dots,m\right\} \tag{7.15}$$

In an effort to estimate the size of the region of attraction for the local stabilization of the constructed robust controller, the following lower bound for $\alpha_i(\xi)$ is introduced to correspond to any compact set $S_c \subset \mathbb{R}^{2n}$,

$$\underline{\alpha}_i \triangleq \min\{\alpha_i(\xi) : \xi \in S_c\},\ i = 1,\dots,m \tag{7.16}$$

so that,

$$0 < \underline{\alpha}_i \le \alpha_i(\xi) \le 1,\ \forall \xi \in S_c, \quad i = 1,\dots,m \tag{7.17}$$

For a fixed vector $\underline{\alpha} \triangleq [\underline{\alpha}_1 \dots \underline{\alpha}_m]^T$, define the following vertex matrices:

$$\hat{A}_j(t) \triangleq \hat{A}(t) + \hat{B}(t)\Gamma_j(\underline{\alpha})\hat{K}, \qquad j = 1,\dots,2^m \tag{7.18}$$

where $\Gamma_j(\underline{\alpha})$ is a diagonal matrix whose diagonal elements take the values 1 (no saturation) or $\underline{\alpha}_i$, $i = 1,\dots,m$ (saturation). Hence, if $\xi \in S_c$ then the velocity $\dot{\xi}$ must satisfy the following equation (see Lemma 1 in [79]):

$$\dot{\xi}(t) = \sum_{j=1}^{2^m} \lambda_j \hat{A}_j(t)\xi(t) + \hat{A}_h(t)\xi(t - h(t)) \tag{7.19}$$

with $\sum_{j=1}^{2^m} \lambda_j = 1,\ \lambda_j \ge 0$.

Furthermore, it is important to note that the vector $\underline{\alpha}$ allows to define a polyhedral set:

$$S(\bar{u},\underline{\alpha}) \triangleq \left\{ \xi \in \mathbb{R}^{2n}: -\left|\hat{K}_{(i)}\xi\right| \le \frac{\bar{u}_i}{\underline{\alpha}_i},\ i=1,...,m \right\} \tag{7.20}$$

The set $S(\bar{u},\underline{\alpha})$ contains S_c and corresponds to the maximal set in which (7.19) equivalently represents system (7.12).

In the context of the above, let S_c be a closed ellipsoid defined by a symmetric positive definite matrix $P_1 > 0$,

$$S_c \triangleq \left\{ \xi \in R^{2n}: \xi^T P_1 \xi \le \gamma^{-1} \right\} \tag{7.21}$$

where γ is a positive scalar.

Remark 7.2. The asymptotic stability of the closed-loop system (7.19) guarantees that of the original closed-loop system (7.12). However, the reverse is not true; i.e. the stability of the original system does not guarantee that of system (7.19), and thus some unavoidable conservatism is introduced.

The following theorem provides sufficient conditions for robust output feedback stabilization of the uncertain time-delay system (7.12) with control saturation.

Theorem 7.1. *Consider the system* (7.12). *Suppose that there exist, $n\times n$-matrices: P_{i11}, P_{i12}, P_{i21}, P_{i22}; $i=1,...,3$, W_{i11}, W_{i12}, W_{i21}, W_{i22}; $i=1,...,4$, R_{i11}, R_{i12}, R_{i21}, R_{i22}; $i=1,...,3$, $S=S^T>0$, an $m\times n$ matrix K and an $n\times r$ matrix L, and a vector $\underline{\alpha}\in\mathbb{R}^m$, and a positive scalar γ, which together with some suitable positive scalars ε_i; $i=1,...,3$, satisfy the following matrix conditions:*

$$\begin{bmatrix} P_{111} & P_{112} \\ P_{121} & P_{122} \end{bmatrix} > 0,\ \textit{with}\ P_{111}=P_{111}^T,\ P_{112}=P_{121}^T,\ P_{122}=P_{122}^T \tag{7.22}$$

$$\begin{bmatrix} R_{311} & R_{312} \\ R_{321} & R_{322} \end{bmatrix} > 0,\ \textit{with}\ R_{311}=R_{311}^T,\ R_{312}=R_{321}^T,\ R_{322}=R_{322}^T \tag{7.23}$$

$$\begin{bmatrix} R_{111} & R_{112} & R_{211} & R_{212} \\ R_{121} & R_{122} & R_{221} & R_{222} \\ R_{211}^T & R_{221}^T & R_{311} & R_{312} \\ R_{212}^T & R_{222}^T & R_{321} & R_{322} \end{bmatrix} > 0,\ \textit{with}\ R_{111}=R_{111}^T,\ R_{112}=R_{121}^T,\ R_{122}=R_{122}^T \tag{7.24}$$

$$\begin{bmatrix} \Omega_{1,j} & \Omega_2 \\ \Omega_2^T & \Omega_3 \end{bmatrix} < 0, \qquad \forall j=1,...,2^m \tag{7.25}$$

$$
\begin{bmatrix} P_{111} & P_{112} & 0 \\ P_{121} & P_{122} & \underline{\alpha}_i K_{(i)}^T \\ 0 & \underline{\alpha}_i K_{(i)} & \gamma \bar{u}_i^2 \end{bmatrix} \geq 0, \quad \forall i = 1,\ldots,m \tag{7.26}
$$

$$
\underline{\alpha}_i \in (0,1], \qquad \forall i = 1,\ldots,m \tag{7.27}
$$

where,

$K_{(i)}$ *is the* $i-th$ *row of* K,

$$
\Omega_{1,j} = \begin{bmatrix}
\Psi_{111} & \Psi_{112,j} & \Psi_{211} & \Psi_{212} & h_{\max}\left(W_{111}^T + P_{111}^T\right) & h_{\max}\left(W_{121}^T + P_{121}^T\right) & h_{\max}\left(W_{311}^T + P_{211}^T\right) & h_{\max}\left(W_{321}^T + P_{221}^T\right) & -W_{311}^T A_h - \varepsilon_3 E_h^T E_h \\
 & \Psi_{122,j} & \Psi_{221,j} & \Psi_{222,j} & h_{\max}\left(W_{112}^T + P_{112}^T\right) & h_{\max}\left(W_{122}^T + P_{122}^T\right) & h_{\max}\left(W_{312}^T + P_{212}^T\right) & h_{\max}\left(W_{322}^T + P_{222}^T\right) & -W_{312}^T A_h \\
 & & \Psi_{311} & \Psi_{312} & h_{\max} W_{211}^T & h_{\max} W_{221}^T & h_{\max}\left(W_{411}^T + P_{311}^T\right) & h_{\max}\left(W_{421}^T + P_{321}^T\right) & -W_{411}^T A_h \\
 & & & \Psi_{322} & h_{\max} W_{212}^T & h_{\max} W_{222}^T & h_{\max}\left(W_{412}^T + P_{312}^T\right) & h_{\max}\left(W_{422}^T + P_{322}^T\right) & -W_{412}^T A_h \\
 & & & & -h_{\max} R_{111} & -h_{\max} R_{112} & -h_{\max} R_{211} & -h_{\max} R_{212} & 0 \\
 & & & & & -h_{\max} R_{122} & -h_{\max} R_{221} & -h_{\max} R_{222} & 0 \\
 & & & & & & -h_{\max} R_{311} & -h_{\max} R_{312} & 0 \\
 & & & & & & & -h_{\max} R_{322} & 0 \\
 & & & & & & & & -(1-\beta) S + \varepsilon_3 E_h^T E_h
\end{bmatrix}
$$

$$
\tag{7.28}
$$

$$
\Omega_2 = \begin{bmatrix} P_{211}^T H_A + P_{221}^T L H_C & P_{211}^T H_h & W_{311}^T H_h \\ P_{212}^T H_A + P_{222}^T L H_C & P_{212}^T H_h & W_{312}^T H_h \\ P_{311}^T H_A + P_{321}^T L H_C & P_{311}^T H_h & W_{411}^T H_h \\ P_{312}^T H_A + P_{322}^T L H_C & P_{312}^T H_h & W_{412}^T H_h \end{bmatrix} \tag{7.29}
$$

$$
\Omega_3 = diag\left\{-\varepsilon_1 I, -\varepsilon_2 I, -\varepsilon_3 I\right\} \tag{7.30}
$$

and where,

$$
\Psi_{111} = \left(A + A_h\right)^T P_{211} + P_{211}^T\left(A + A_h\right) + \left(LC\right)^T P_{221} + P_{221}^T LC + W_{311}^T A_h + A_h^T W_{311} + S + \varepsilon_1 E_A^T E_A + \left(\varepsilon_2 + \varepsilon_3\right) E_h^T E_h
$$

$$
\Psi_{112,j} = \left(A + A_h\right)^T P_{212} + \left(LC\right)^T P_{222} + P_{211}^T B\Gamma_j\left(\underline{\alpha}\right) K + P_{221}^T\left(A - LC + B\Gamma_j\left(\underline{\alpha}\right) K\right) + A_h^T W_{312} + \varepsilon_1 E_A^T E_B \Gamma_j\left(\underline{\alpha}\right) K
$$

$$
\Psi_{122,j} = \left(B\Gamma_j\left(\underline{\alpha}\right) K\right)^T P_{212} + P_{212}^T B\Gamma_j\left(\underline{\alpha}\right) K + \left(A - LC + B\Gamma_j\left(\underline{\alpha}\right) K\right)^T P_{222} + P_{222}^T\left(A - LC + B\Gamma_j\left(\underline{\alpha}\right) K\right)
$$

$$
+ \varepsilon_1\left(E_B \Gamma_j\left(\underline{\alpha}\right) K\right)^T E_B \Gamma_j\left(\underline{\alpha}\right) K
$$

$$
\Psi_{211} = P_{111}^T - P_{211}^T + \left(A + A_h\right)^T P_{311} + \left(LC\right)^T P_{321} + A_h^T W_{411}
$$

$$
\Psi_{212} = P_{121}^T - P_{221}^T + \left(A + A_h\right)^T P_{312} + \left(LC\right)^T P_{322} + A_h^T W_{412}
$$

$$\Psi_{221,j} = P_{112}^T - P_{212}^T + \left(B\Gamma_j(\underline{\alpha})K\right)^T P_{311} + \left(A - LC + B\Gamma_j(\underline{\alpha})K\right)^T P_{321}$$

$$\Psi_{222,j} = P_{122}^T - P_{222}^T + \left(B\Gamma_j(\underline{\alpha})K\right)^T P_{312} + \left(A - LC + B\Gamma_j(\underline{\alpha})K\right)^T P_{322}$$

$$\Psi_{311} = -P_{311} - P_{311}^T + h_{\max}\hat{A}_{h11,\max}^T R_{311}\hat{A}_{h11,\max}$$

$$\Psi_{312} = -P_{312} - P_{321}^T$$

$$\Psi_{322} = -P_{322} - P_{322}^T$$

Under these conditions, system (7.12) *is locally asymptotically stable for any initial condition* $\phi(\sigma)$ *in the ball*

$$\Phi(\sigma) = \left\{\phi \in C_{h_{\max},n}^{w};\ \|\phi\|_C^2 \le \sigma\right\} \tag{7.31}$$

with,

$$\sigma = \frac{1}{\gamma\pi_2} \tag{7.32}$$

where,

$$\pi_2 = \lambda_{\max}\left(\begin{bmatrix} P_{111} & P_{112} \\ P_{121} & P_{122} \end{bmatrix}\right) + \frac{h_{\max}}{(1-\beta)}\lambda_{\max}(S) + \frac{3}{2}h_{\max}^2\lambda_{\max}\left(\hat{A}_{h11,\max}^T R_{311}\hat{A}_{h11,\max}\right) \tag{7.33}$$

Remark 7.3. In **[27]**, a comparison between the different saturation models used in the literature was made, concluding that the differential inclusion method used in this chapter leads to the least conservative design.

Remark 7.4. In **[72]**, the proposed design is delay-independent, while in Theorem 7.1 the presented controller is delay-dependent. This leads to a less conservative design than that of **[72]**. Also, the bounding technique (see Lemma 7.3) employed in the proof of Theorem 7.1, further reduces the conservatism of the presented approach as compared with **[72]**. Finally, the method used in this chapter takes into account the rate of change of the time-delay along with its amplitude. This makes the design more realistic in that it can be implemented on the actual plant without exhibiting instability in the case the time-delay varies in time.

Remark 7.5. The main difficulty in the application of the design procedure as stated in terms of Theorem 7.1 is that the inequalities (7.25) and (7.26) are nonlinear in the parameters $\underline{\alpha}$, L, K and some of the P parameters (P_{i11}, P_{i12}, P_{i21}, P_{i22}, $i = 2,3$). This difficulty can be overcome by employing relaxation techniques, as suggested in **[78]**. A suitable relaxation technique in this case is to choose $\underline{\alpha}$, L and K and solve for the P 's, and then re-iterate the choice of $\underline{\alpha}$, L and K. In this way inequalities (7.25) and (7.26) then reduce to linear matrix inequalities (LMIs) in the P variables. The latter are easily solved using the algorithms of **[6]**. Also, more sophisticated optimization techniques could be used to maximize the set of initial conditions $\Phi(\sigma)$ (see **[78]** for examples of such techniques).

Proof. By virtue of condition (7.26) of the Theorem the ellipsoid defined by (7.21) is included in the set $S(\overline{u},\underline{\alpha})$ as defined by (7.20), where the vector $\underline{\alpha}$ verifies (7.27). Therefore, $\dot{\xi}(t)$ satisfies the polytopic system equation (7.19).

The last can be further written in its equivalent descriptor form

$$\dot{\xi}(t)=y(t),\quad y(t)=\sum_{j=1}^{2^m}\lambda_j\hat{A}_j(t)\xi(t)+\hat{A}_h(t)\xi(t-h(t)). \tag{7.34}$$

Using the Liebnitz-Newton formula $\xi(t-h(t))=\xi(t)-\int_{t-d(t)}^{t}\dot{\xi}(s)ds$ permits to re-write (7.34) yet in a more tractable form. Introduction of the augmented state as in (7.34) and the use of the Liebnitz-Newton formula allows to avoid the introduction of any additional dynamics, so that the transfer function of the system obtained by freezing the time-variable in the system matrices does not exhibit any additional poles **[8]**. This way of transforming system (7.34) is particularly useful as it allows to avoid unnecessary conservatism in the design that follows.

The last transformation of (7.34) yields:

$$\dot{\xi}(t)=y(t),\ 0=-y(t)+\left[\sum_{j=1}^{2^m}\lambda_j\hat{A}_j(t)+\hat{A}_h(t)\right]\xi(t)-\hat{A}_h(t)\int_{t-h(t)}^{t}y(s)ds \tag{7.35}$$

so that for $E=\begin{bmatrix} I_{2n} & 0\\ 0 & 0\end{bmatrix}$, the augmented system is:

$$E\begin{bmatrix}\dot{\xi}(t)\\ \dot{y}(t)\end{bmatrix}=\begin{bmatrix}\dot{\xi}(t)\\ 0\end{bmatrix}=\begin{bmatrix} 0 & I\\ \sum_{j=1}^{2^m}\lambda_j\hat{A}_j(t)+\hat{A}_h(t) & -I\end{bmatrix}\begin{bmatrix}\xi(t)\\ y(t)\end{bmatrix}-\begin{bmatrix}0\\ \hat{A}_h(t)\end{bmatrix}\int_{t-h(t)}^{t}y(s)ds \tag{7.36}$$

Then the following Lyapunov-Krasovskii functional is used here:

$$V(t)=V_0(t)+V_1(t)+V_2(t) \tag{7.37}$$

where,

$$V_0(t)=\begin{bmatrix}\xi^T(t) & y^T(t)\end{bmatrix}EP\begin{bmatrix}\xi(t)\\ y(t)\end{bmatrix}=\xi^T(t)P_1\xi(t) \tag{7.38}$$

$$V_1(t)=\int_{t-h(t)}^{t}x^T(\tau)Sx(\tau)d\tau \tag{7.39}$$

$$V_2(t)=\int_{-h_{\max}}^{0}\int_{t+\theta}^{t}y^T(s)\hat{A}_{h,\max}^T R_3\hat{A}_{h,\max}y(s)dsd\theta=\int_{-h_{\max}}^{0}\int_{t+\theta}^{t}y^T(s)\begin{bmatrix}0 & \hat{A}_{h,\max}^T\end{bmatrix}R\begin{bmatrix}0\\ \hat{A}_{h,\max}\end{bmatrix}y(s)dsd\theta \tag{7.40}$$

with $P=\begin{bmatrix}P_1 & 0\\ P_2 & P_3\end{bmatrix}$, $P_1=P_1^T>0$, $S=S^T>0$, $R_3=R_3^T>0$, $R=R^T=\begin{bmatrix}R_1 & R_2\\ R_2^T & R_3\end{bmatrix}>0$, $R_1=R_1^T$ (7.41)

and $\hat{A}_{h,\max}$ is such that,

$$\hat{A}_h^T(t)\hat{A}_h(t)\le\hat{A}_{h,\max}^T\hat{A}_{h,\max}, \qquad \text{for all } t\ge 0 \tag{7.42}$$

The bound in (7.42) can be evaluated using Lemma 7.2 as follows:

Let $\varepsilon_4>0$ be a scalar such that $\left(I-\varepsilon_4\hat{H}_h\hat{H}_h^T\right)>0$. Then,

$$\hat{A}_h^T(t)\hat{A}_h(t)=\left(\hat{A}_h+\hat{H}_hF_h(t)\hat{E}_h\right)^T\left(\hat{A}_h+\hat{H}_hF_h(t)\hat{E}_h\right)\le\hat{A}_h^T\left(I-\varepsilon_4\hat{H}_h\hat{H}_h^T\right)^{-1}\hat{A}_h+\varepsilon_4^{-1}\hat{E}_h^T\hat{E}_h$$

$$=\begin{bmatrix}A_h^T & 0\\ 0 & 0\end{bmatrix}\left(I-\varepsilon_4\begin{bmatrix}H_hH_h^T & 0\\ 0 & 0\end{bmatrix}\right)^{-1}\begin{bmatrix}A_h & 0\\ 0 & 0\end{bmatrix}+\varepsilon_4^{-1}\begin{bmatrix}E_h^TE_h & 0\\ 0 & 0\end{bmatrix}$$

$$=\begin{bmatrix}A_h^T\left(I-\varepsilon_4H_hH_h^T\right)^{-1}A_h+\varepsilon_4^{-1}E_h^TE_h & 0\\ 0 & 0\end{bmatrix}=\hat{A}_{h,\max}^T\hat{A}_{h,\max}\triangleq\begin{bmatrix}\hat{A}_{h11,\max} & 0\\ 0 & 0\end{bmatrix}^T\begin{bmatrix}\hat{A}_{h11,\max} & 0\\ 0 & 0\end{bmatrix} \tag{7.43}$$

Differentiating (7.38) and using (7.36), yields,

$$\frac{dV_0(t)}{dt}=2\xi^T(t)P_1\dot{\xi}(t)=2\begin{bmatrix}\xi^T(t) & y^T(t)\end{bmatrix}P^T\begin{bmatrix}\dot{\xi}(t)\\ 0\end{bmatrix}$$

$$=2\begin{bmatrix}\xi^T(t) & y^T(t)\end{bmatrix}P^T\begin{bmatrix}0 & I\\ \sum_{j=1}^{2^m}\lambda_j\hat{A}_j(t)+\hat{A}_h(t) & -I\end{bmatrix}\begin{bmatrix}\xi(t)\\ y(t)\end{bmatrix}-2\int_{t-h(t)}^{t}\begin{bmatrix}\xi^T(t) & y^T(t)\end{bmatrix}P^T\begin{bmatrix}0\\ \hat{A}_h(t)\end{bmatrix}y(s)ds \tag{7.44}$$

A bound for the last term of (7.44) is now derived as follows:

Define $\eta(t) \triangleq -2\int_{t-h(t)}^{t} \begin{bmatrix} \xi^T(t) & y^T(t) \end{bmatrix} P^T \begin{bmatrix} 0 \\ \hat{A}_h(t) \end{bmatrix} y(s)\,ds$ (7.45)

Using Lemma 7.3 with $a(s) \triangleq \begin{bmatrix} 0 \\ \hat{A}_h(t) \end{bmatrix} y(s), b(s) \triangleq P \begin{bmatrix} \xi(t) \\ y(t) \end{bmatrix}, \Upsilon \triangleq [t-h(t), t]$ gives:

$$\eta(t) \le \int_{t-h(t)}^{t} \begin{bmatrix} \xi^T(t) & y^T(t) \end{bmatrix} P^T (M^T R + I) R^{-1} (RM + I) P \begin{bmatrix} \xi(t) \\ y(t) \end{bmatrix} ds + 2\int_{t-h(t)}^{t} y^T(s)\,ds \begin{bmatrix} 0 & \hat{A}_h^T(t) \end{bmatrix} RMP \begin{bmatrix} \xi(t) \\ y(t) \end{bmatrix}$$

$$+\int_{t-h(t)}^{t} y^T(s) \begin{bmatrix} 0 & \hat{A}_h^T(t) \end{bmatrix} R \begin{bmatrix} 0 \\ \hat{A}_h \end{bmatrix} y(s)\,ds$$

$$\le h_{\max} \begin{bmatrix} \xi^T(t) & y^T(t) \end{bmatrix} P^T (M^T R + I) R^{-1} (RM + I) P \begin{bmatrix} \xi(t) \\ y(t) \end{bmatrix} + 2\begin{bmatrix} \xi^T(t) - \xi^T(t-h(t)) \end{bmatrix} \begin{bmatrix} 0 & \hat{A}_h^T(t) \end{bmatrix} RMP \begin{bmatrix} \xi(t) \\ y(t) \end{bmatrix}$$

$$+\int_{t-h_{\max}}^{t} y^T(s) \begin{bmatrix} 0 & \hat{A}_h^T(t) \end{bmatrix} R \begin{bmatrix} 0 \\ \hat{A}_h(t) \end{bmatrix} y(s)\,ds \qquad (7.46)$$

as $R > 0$.

Differentiating $V_1(t)$ of (7.39), yields,

$$\frac{dV_1(t)}{dt} = x^T(t) S x(t) - (1-\dot{h}(t)) x^T(t-h(t)) S x(t-h(t))$$

Using the assumption $S > 0$, as well as the rate bound specified by (7.4),

$$\frac{dV_1(t)}{dt} \le x^T(t) S x(t) - (1-\beta) x^T(t-h(t)) S x(t-h(t)) \qquad (7.47)$$

Applying Lemma 7.1 to $V_3(t)$,

$$\frac{dV_2(t)}{dt} = h_{\max} y^T(t) \hat{A}_{h,\max}^T R_3 \hat{A}_{h,\max} y(t) - \int_{t-h_{\max}}^{t} y^T(s) \hat{A}_{h,\max}^T R_3 \hat{A}_{h,\max} y(s)\,ds \qquad (7.48)$$

Employing (7.37), (7.44), (7.46), (7.47) and (7.48), the following upper bound for $\dot{V}(t)$ is obtained:

$$\dot{V}(t) \le 2\begin{bmatrix} \xi^T(t) & y^T(t) \end{bmatrix} P^T \begin{bmatrix} 0 & I \\ \sum_{j=1}^{2^m} \lambda_j \hat{A}_j(t) + \hat{A}_h(t) & -I \end{bmatrix} \begin{bmatrix} \xi(t) \\ y(t) \end{bmatrix}$$

$$+h_{\max}\left[\xi^T(t) \quad y^T(t)\right]P^T\left(M^TR+I\right)R^{-1}(RM+I)P\begin{bmatrix}\xi(t)\\ y(t)\end{bmatrix}$$

$$+2\left[\xi^T(t)-\xi^T(t-h(t))\right]\left[0 \quad \hat{A}_h^T(t)\right]RMP\begin{bmatrix}\xi(t)\\ y(t)\end{bmatrix}$$

$$+\int_{t-h_{\max}}^{t} y^T(s)\left[0 \quad \hat{A}_h^T(t)\right]R\begin{bmatrix}0\\ \hat{A}_h(t)\end{bmatrix}y(s)\,ds$$

$$+x^T(t)Sx(t)-(1-\beta)x^T(t-h(t))Sx(t-h(t))$$

$$+h_{\max}y^T(t)\hat{A}_{h,\max}^TR_3\hat{A}_{h,\max}y(t)-\int_{t-h_{\max}}^{t} y^T(s)\hat{A}_{h,\max}^TR_3\hat{A}_{h,\max}y(s)\,ds \tag{7.49}$$

Since $R_3>0$, then, by virtue of the bound given by (7.42), the two remaining integrals in (7.49) satisfy the following inequality:

$$\int_{t-h_{\max}}^{t} y^T(s)\left[0 \quad \hat{A}_h^T(t)\right]R\begin{bmatrix}0\\ \hat{A}_h(t)\end{bmatrix}y(s)\,ds-\int_{t-h_{\max}}^{t} y^T(s)\hat{A}_{h,\max}^TR_3\hat{A}_{h,\max}y(s)\,ds$$

$$=\int_{t-h_{\max}}^{t} y^T(s)\left[\hat{A}_h^T(t)R_3\hat{A}_h(t)-\hat{A}_{h,\max}^TR_3\hat{A}_{h,\max}\right]y(s)\,ds\le 0$$

Now, it is possible to reduce (7.49) to:

$$\frac{dV(t)}{dt}\le 2\left[\xi^T(t) \quad y^T(t)\right]P^T\begin{bmatrix}0 & I\\ \sum_{j=1}^{2^m}\lambda_j\hat{A}_j(t)+\hat{A}_h(t) & -I\end{bmatrix}\begin{bmatrix}\xi(t)\\ y(t)\end{bmatrix}$$

$$+h_{\max}\left[\xi^T(t) \quad y^T(t)\right]P^T\left(M^TR+I\right)R^{-1}(RM+I)P\begin{bmatrix}\xi(t)\\ y(t)\end{bmatrix}$$

$$+2\left[\xi^T(t)-\xi^T(t-h(t))\right]\left[0 \quad \hat{A}_h^T(t)\right]RMP\begin{bmatrix}\xi(t)\\ y(t)\end{bmatrix}$$

$$+x^T(t)Sx(t)-(1-\beta)x^T(t-h(t))Sx(t-h(t))+h_{\max}y^T(t)\hat{A}_{h,\max}^TR_3\hat{A}_{h,\max}y(t) \tag{7.50}$$

Next, the application of Lemma 7.2 to the uncertainty terms of $\hat{A}_j(t)$, $\hat{A}_h(t)$ and $\hat{B}(t)$, results in the following upper bound for $\dot{V}(t)$:

$$\frac{dV(t)}{dt} \le \left[\xi^T(t) \quad y^T(t)\right] P^T \begin{bmatrix} 0 & I \\ \sum_{j=1}^{2^m} \lambda_j \left(\hat{A} + \hat{B}\Gamma_j(\underline{\alpha})\hat{K}\right) + \hat{A}_h & -I \end{bmatrix} \begin{bmatrix} \xi(t) \\ y(t) \end{bmatrix}$$

$$+\left[\xi^T(t) \quad y^T(t)\right] \begin{bmatrix} 0 & \sum_{j=1}^{2^m} \lambda_j \left(\hat{A} + \hat{B}\Gamma_j(\underline{\alpha})\hat{K}\right)^T + \hat{A}_h^T \\ I & -I \end{bmatrix} P \begin{bmatrix} \xi(t) \\ y(t) \end{bmatrix}$$

$$+h_{\max}\left[\xi^T(t) \quad y^T(t)\right] P^T \left(M^T R + I\right) R^{-1} (RM + I) P \begin{bmatrix} \xi(t) \\ y(t) \end{bmatrix}$$

$$+\left[x^T(t) \quad y^T(t)\right] P^T M^T R \begin{bmatrix} 0 \\ \hat{A}_h \end{bmatrix} \left[x(t) - x(t-h(t))\right]$$

$$+\left[\xi^T(t) - \xi^T(t-h(t))\right]\left[0 \quad \hat{A}_h^T\right] RMP \begin{bmatrix} \xi(t) \\ y(t) \end{bmatrix}$$

$$+x^T(t) Sx(t) - (1-\beta) x^T(t-h(t)) Sx(t-h(t)) + h_{\max} y^T(t) \hat{A}_{h,\max}^T R_3 \hat{A}_{h,\max} y(t)$$

$$+\varepsilon_1^{-1}\xi^T(t)\sum_{j=1}^{2^m}\lambda_j P_2^T \hat{H}\hat{H}^T P_2 x(t) + \varepsilon_1^{-1} y^T(t)\sum_{j=1}^{2^m}\lambda_j P_3^T \hat{H}\hat{H}^T P_2 \xi(t) + \varepsilon_1^{-1}\xi^T(t)\sum_{j=1}^{2^m}\lambda_j P_2^T \hat{H}\hat{H}^T P_3 y(t)$$

$$+\varepsilon_1^{-1} y^T(t)\sum_{j=1}^{2^m}\lambda_j P_3^T \hat{H}\hat{H}^T P_3 y(t) + \varepsilon_1 \xi^T(t)\sum_{j=1}^{2^m}\lambda_j \left(\hat{E} + E_B\Gamma_j(\underline{\alpha})\hat{K}\right)^T \left(\hat{E} + E_B\Gamma_j(\underline{\alpha})\hat{K}\right)\xi(t)$$

$$+\varepsilon_2^{-1}\xi^T(t) P_2^T \hat{H}_h \hat{H}_h^T P_2 \xi(t) + \varepsilon_2^{-1} y^T(t) P_3^T \hat{H}_h \hat{H}_h^T P_2 x(t) + \varepsilon_2^{-1}\xi^T(t) P_2^T \hat{H}_h \hat{H}_h^T P_3 y(t)$$

$$+\varepsilon_2^{-1} y^T(t) P_3^T \hat{H}_h \hat{H}_h^T P_3 y(t) + \varepsilon_2 \xi^T(t) \hat{E}_h^T \hat{E}_h \xi(t)$$

$$+\varepsilon_3^{-1}\xi^T(t) W_3^T \hat{H}_h \hat{H}_h^T W_3 \xi(t) + \varepsilon_3^{-1} y^T(t) W_4^T \hat{H}_h \hat{H}_h^T W_3 \xi(t) + \varepsilon_3^{-1}\xi^T(t) W_3^T \hat{H}_h \hat{H}_h^T W_4 y(t)$$

$$+\varepsilon_3^{-1} y^T(t) W_4^T \hat{H}_h \hat{H}_h^T W_4 y(t) + \varepsilon_3\left[\xi^T(t) - \xi^T(t-h(t))\right]\hat{E}_h^T \hat{E}_h\left[\xi(t) - \xi(t-h(t))\right] \tag{7.51}$$

with,

$$\varepsilon_i > 0,\ i = 1,\ldots,3. \tag{7.52}$$

Defining $W \triangleq \begin{bmatrix} W_1 & W_2 \\ W_3 & W_4 \end{bmatrix} \triangleq RMP$, and the aggregated state $\chi(t)$ by,

$$\chi(t) \triangleq \begin{bmatrix} \xi(t) \\ y(t) \\ \left(M+R^{-1}\right)P\begin{bmatrix} \xi(t) \\ y(t) \end{bmatrix} \\ \xi^{T}\left(t-h(t)\right) \end{bmatrix}$$

inequality (7.51) can now simply be written in the form:

$$\dot{V}(t) \le \chi^{T}(t)\left[\Omega_1 - \Omega_2\Omega_3^{-1}\Omega_2^{T}\right]\chi(t) \triangleq \chi^{T}(t)\Theta\chi(t) \tag{7.53}$$

where the matrix Θ involves the summation of terms of the form $\sum_{j=1}^{2^m}\lambda_j(\ldots)$; see the inequality (7.51).

To guarantee that $\dot{V}(t)<0$, for all $t \ge 0$, it is sufficient that $\Theta<0$. By convexity, $\Theta<0$ is guaranteed if $\Theta_j<0$, for $j=1,\ldots,2^m$, in which Θ_j is obtained by dropping $\sum_{j=1}^{2^m}\lambda_j$ from Θ.

Next, letting,

$$P_i \triangleq \begin{bmatrix} P_{i11} & P_{i12} \\ P_{i21} & P_{i22} \end{bmatrix}, \qquad i=1,\ldots,3$$

with, $P_{111}=P_{111}^{T}$, $\quad P_{112}=P_{121}^{T}$, $\quad P_{122}=P_{122}^{T}$

$$W_i \triangleq \begin{bmatrix} W_{i11} & W_{i12} \\ W_{i21} & W_{i22} \end{bmatrix}, \qquad i=1,\ldots,4$$

$$R_i \triangleq \begin{bmatrix} R_{i11} & R_{i12} \\ R_{i21} & R_{i22} \end{bmatrix}, \qquad i=1,\ldots,3$$

with, $R_{111}=R_{111}^{T}$, $\quad R_{112}=R_{121}^{T}$, $\quad R_{122}=R_{122}^{T}$, $\quad R_{311}=R_{311}^{T}$, $\quad R_{312}=R_{321}^{T}$, $\quad R_{322}=R_{322}^{T}$ (7.54)

and expanding all the matrices within Θ_j using the definitions (7.11) and (7.54), the condition $\Theta_j<0$, for $j=1,\ldots,2^m$ becomes $\Omega_{1j}-\Omega_2\Omega_3^{-1}\Omega_2^{T}<0$, for $j=1,\ldots,2^m$, where Ω_{1j}, Ω_2 and Ω_3 are given by (7.28), (7.29) and (7.30), respectively.

Sufficient conditions for $\dot{V}(t)<0$ can thus be summarized as follows :

1.) $\Omega_{1j}-\Omega_2\Omega_3^{-1}\Omega_2^{T}<0$, $\qquad j=1,\ldots,2^m$

2.) $\Omega_3<0$ (assumptions (7.52) resulting from the application of Lemma 7.2 to obtain inequality (7.51)).

By the Schur Complements Lemma (see Appendix A), conditions 1.) and 2.) above are equivalent to assumption (7.25) of the Theorem.

Thus, from $\dot{V}(t) \leq \chi^T(t)\Theta\chi(t) < 0$ there exists $\pi_3 > 0$ such that the quadratic stabilization condition (see **[2]**) is satisfied as, $\dot{V}(\xi_t(\psi) = \xi(t+\psi)) \leq -\pi_3 \|\chi(t)\|^2 \leq -\pi_3 \|\xi(t)\|^2$, so that $V(\xi_t) \leq V(\xi_{t_0})$ provided that the model (7.19) is valid, i.e., for any time $t \geq t_0$ such that $\xi(t) \in S(\overline{u}, \underline{\alpha})$.

Furthermore, the Lyapunov functional defined in (7.37) can be shown to satisfy,

$$\pi_1 \|\xi(t)\|^2 \leq V(\xi_t) \leq \pi_2 \|\xi_t\|_C^2 \tag{7.55}$$

$$\text{with } \pi_1 \triangleq \lambda_{\min}(P_1) \text{ and } \pi_2 = \lambda_{\max}(P_1) + \frac{h_{\max}}{(1-\beta)}\lambda_{\max}(S) + \frac{3}{2}h_{\max}^2 \lambda_{\max}\left(\hat{A}_{h,\max}^T R_3 \hat{A}_{h,\max}\right) \tag{7.56}$$

Substituting P_1, $\hat{A}_{h,\max}$, and R_3 by their expressions found in (7.43) and (7.54),

$$\pi_1 \triangleq \lambda_{\min}\left(\begin{bmatrix} P_{111} & P_{112} \\ P_{121} & P_{122} \end{bmatrix}\right) \tag{7.57}$$

$$\pi_2 = \lambda_{\max}\left(\begin{bmatrix} P_{111} & P_{112} \\ P_{121} & P_{122} \end{bmatrix}\right) + \frac{h_{\max}}{(1-\beta)}\lambda_{\max}(S) + \frac{3}{2}h_{\max}^2 \lambda_{\max}\left(\hat{A}_{h11,\max}^T R_{311} \hat{A}_{h11,\max}\right) \tag{7.58}$$

Hence, for all $\phi(\psi) \in \Phi,\ \psi \in [-h_{\max}, 0]$,

$$\xi^T(t) P_1 \xi(t) \leq V(\xi_t) \leq V(\xi_{t_0}) \leq \gamma^{-1},\ \forall t \geq t_0. \tag{7.59}$$

Therefore, for any initial condition ϕ in the ball $\Phi(\sigma)$ defined by (7.31), the system (7.12) verifies the conditions of the Lyapunov-Krasovskii Theorem **[30, 31]** and $V(\xi_t)$ is a local strictly decreasing Lyapunov function. Therefore the asymptotic stability of system (7.12) is ensured. QED

7.7. Numerical Examples

The following examples were solved using the LMI (see Appendix B) toolbox in Matlab.

Example 7.1. Consider system (7.1) with the following state space matrices:

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \qquad C = \begin{bmatrix} 1 & 0 \end{bmatrix}, \qquad H_A = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \qquad E_A = \begin{bmatrix} 0 & 1 \end{bmatrix} \tag{7.60}$$

with all the other matrices being equal to zero. The time-delay in this example is assumed to be equal to zero. It is exactly the same example that was used previously by **[75]**. To the best knowledge of the author, the only paper that treats the robust output feedback stabilization of time-delay systems with control saturation is that of **[72]**, in which, as shown in **[27]**, the representation used for actuator saturation is more conservative than the differential inclusions modeling used in this chapter and in **[75]**. This motivates our choice of the example for the purpose of comparison. In **[75]**, the authors discussed this example in the context of robust output feedback stabilization with control saturation in both its amplitude and its rate. The following values of parameters were employed: $\bar{u} = 10$, $\bar{u}_r = 500$, $\underline{\alpha} = 0.9553$ and $\underline{\beta} = 0.9745$, corresponding to the saturation level for the control amplitude, the saturation level for the control rate, the lower bound imposed on the control amplitude saturation, and the lower bound imposed on the control rate saturation, respectively. With these parameters, the authors achieved a maximum volume, $Vol \triangleq \sqrt{\det\left(P_1^{-1}\gamma^{-1}\right)} = 1231.9$, for the region of the initial conditions for which asymptotic stability is guaranteed. It should be noted, that the above choice of parameter $\bar{u}_r = 500$ is a reasonable approximation of the situation when only amplitude saturation is present, justifying our comparison. Thus, using $\bar{u} = 10$, $\underline{\alpha} = 0.9553$, and selecting observer and controller gains $L = \begin{bmatrix} 1100 \\ 0.5 \end{bmatrix}$ and $K = \begin{bmatrix} -2.5 & -1.9 \end{bmatrix}$ using the relaxation techniques proposed in **[79]**, and assuming zero initial conditions for the state estimate $\hat{x}$, Theroem 1 of this chapter delivers a maximum volume of the region of the initial conditions $Vol = 3380$, which is much larger than that achieved by **[75]**. Hence it is just to claim that the result presented in Theorem 7.1 is less conservative.

Figure 7.1 presents the functional dependence of the volume Vol on the level of control saturation $\bar{u}$.

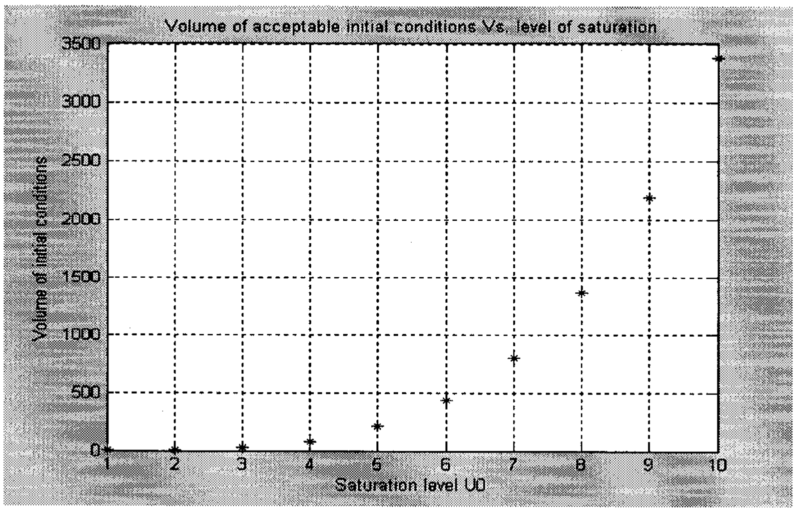

Figure 7.1. *Volume of initial conditions for which asymptotic stability is guaranteed as a function of the control amplitude saturation level.*

Example 7.2. Consider system (7.1) with the following state space matrices:

$$A = \begin{bmatrix} 0.1 & -0.1 \\ 0.1 & -3 \end{bmatrix}, \ B = \begin{bmatrix} 5 & 0 \\ 0 & 1 \end{bmatrix}, \ C = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \ H_A = \frac{1}{20}\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \ E_A = \begin{bmatrix} 1 & 1 \end{bmatrix}, \ E_B = \begin{bmatrix} 1 & 1 \end{bmatrix} \tag{7.61}$$

with all the other matrices being equal to zero. The time-delay in this example is assumed to be equal to zero. The actuators are constrained by saturation levels $\bar{u}_1 = 5$ and $\bar{u}_1 = 2$. It is exactly the same example that was used previously by **[35]**, where the authors using relaxation techniques achieved an initial state set (assuming zero initial conditions for the state estimate $\hat{x}$),

$$D_0 = \left\{ x: \ x^T Z x \le 1, \ Z = 10^{-6} \begin{bmatrix} 290.5 & -7.966 \\ -7.966 & 0.500 \end{bmatrix} \right\} \tag{7.62}$$

The corresponding volume of this set was computed as:

$$\log\left(Vol\left(D_0\right)\right) = \log\left(\sqrt{\det\left(Z\right)}\right) = 11.61 \tag{7.63}$$

In an attempt to achieve a larger set of initial conditions guaranteeing asymptotic stability of the closed-loop system, the following relaxation schemes were used in the present chapter:

LMIR 1: Given K, L, $\underline{\alpha}_1$ and $\underline{\alpha}_2$, solve for all the P matrices and γ, the problem $Min\left[w\text{Trace}(P_1)+\gamma\right]$, such that the LMIs of Theorem 7.1 are satisfied, where w is a column vector whose four entries are weights multiplying each diagonal entry of matrix P_1 to form its trace.

LMIR 2: Given all the P matrices, K, $\underline{\alpha}_1$ and $\underline{\alpha}_2$, solve for L and γ the problem $Min(\gamma)$, such that the LMIs of Theorem 7.1 are satisfied.

Using the above relaxation schemes, the achieved set of initial conditions (assuming zero initial conditions for the state estimate $\hat{x}$), is:

$$S_c = \left\{ x:\ x^T \gamma P_{111} x \le 1,\ \gamma P_{111} = 10^{-6} \begin{bmatrix} 183.3 & -4.396 \\ -4.396 & 0.355 \end{bmatrix} \right\} \tag{7.64}$$

and the corresponding computed volume is :

$$\log\left(Vol\left(S_c\right)\right) = \log\left(\sqrt{\det\left(\gamma P_{111}\right)}\right) = 11.90 \tag{7.65}$$

The corresponding observer and controller gains are:

$$L = \begin{bmatrix} 1400 & -33 \\ 1400 & -33 \end{bmatrix},\ K = \begin{bmatrix} -0.107 & 0.002 \\ -0.002 & 0.002 \end{bmatrix} \tag{7.66}$$

Figure 7.2 shows the sets of initial conditions achieved in **[35]** and in the present chapter, where it is seen that there is a substantial increase in the size of the set of initial conditions guaranteeing asymptotic stability.

In the use of scheme LMIR 1, the diagonal entries of matrix P_1 were weighted by w_1, w_2, w_3 and w_4. This means that $w\text{Trace}(P_1)$ is to be understood as $w_1 p_{1,11} + w_2 p_{1,22} + w_3 p_{1,33} + w_4 p_{1,44}$. The weights chosen for this example were: $w_1 = 1$, $w_2 = 2500$, $w_3 = 1$ and $w_4 = 1$. The objective function chosen for LMIR 1, namely $Min\left(w\text{Trace}(P_1)+\gamma\right)$ has proven to be an efficient method in maximizing the set of initial conditions with fewer iterations.

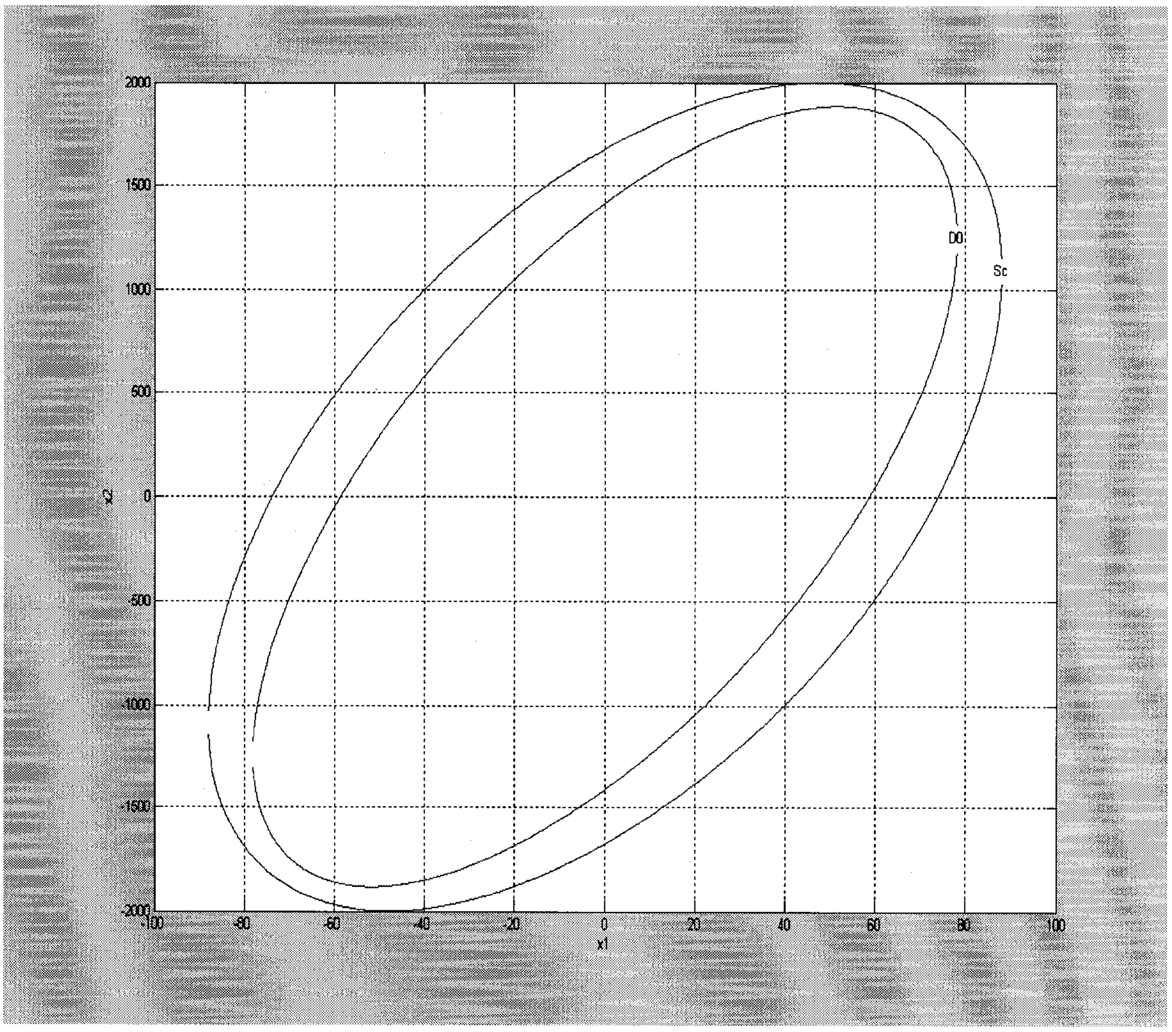

Figure 7.2. *Set S_c of initial conditions achieved in this chapter as compared to set D_0 of* **[35]**

Simulations were made for different constant values of the uncertainty matrix $F(t)$ and initial condition $x_0 = [60, -500]$. As seen in Figure 7.3, Figure 7.4 and Figure 7.5, despite uncertainties and control saturation, the closed loop system remains asymptotically stable.

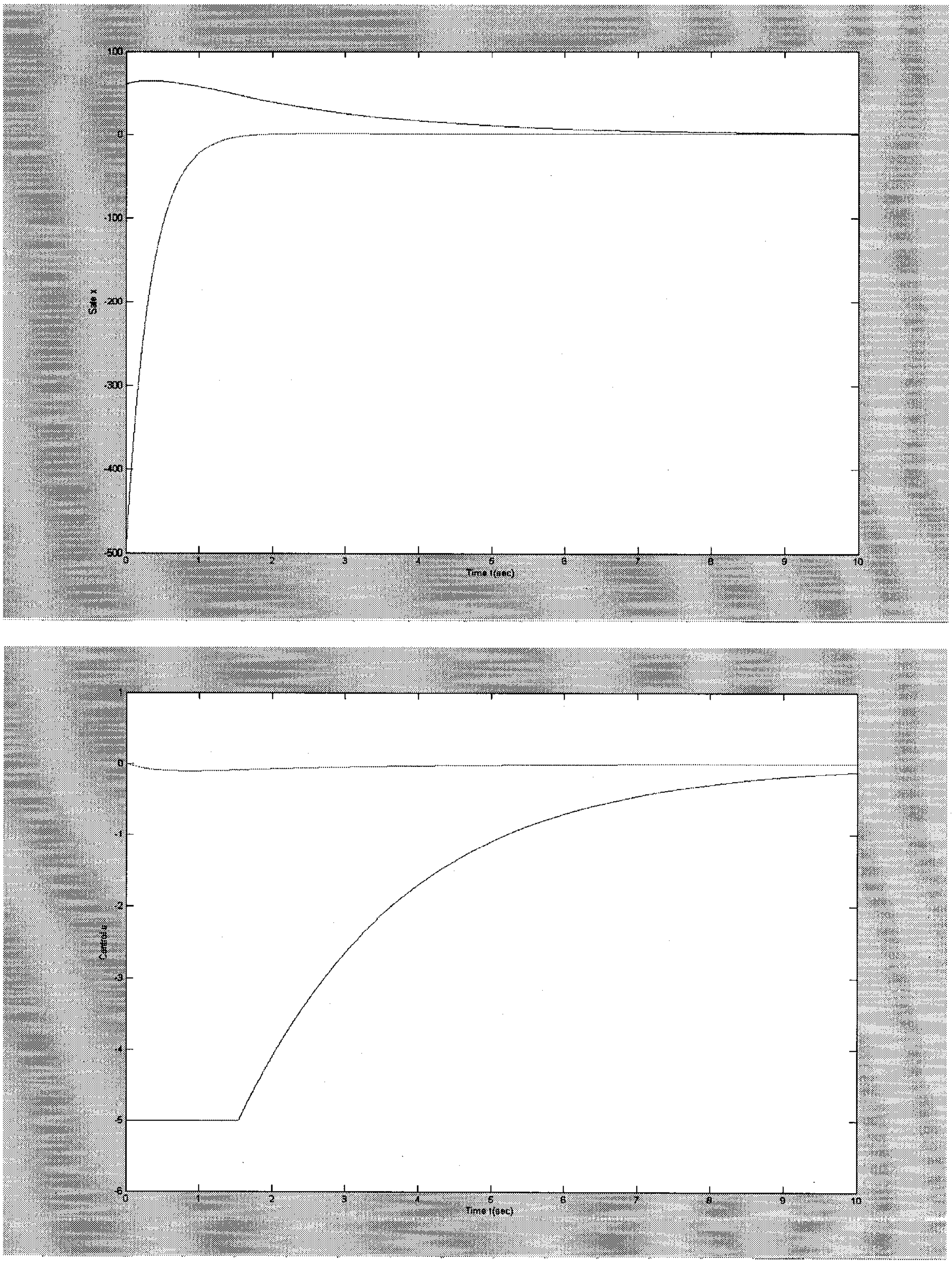

Figure 7.3. *State response (top) and corresponding control input (bottom) for uncertainty F=0*

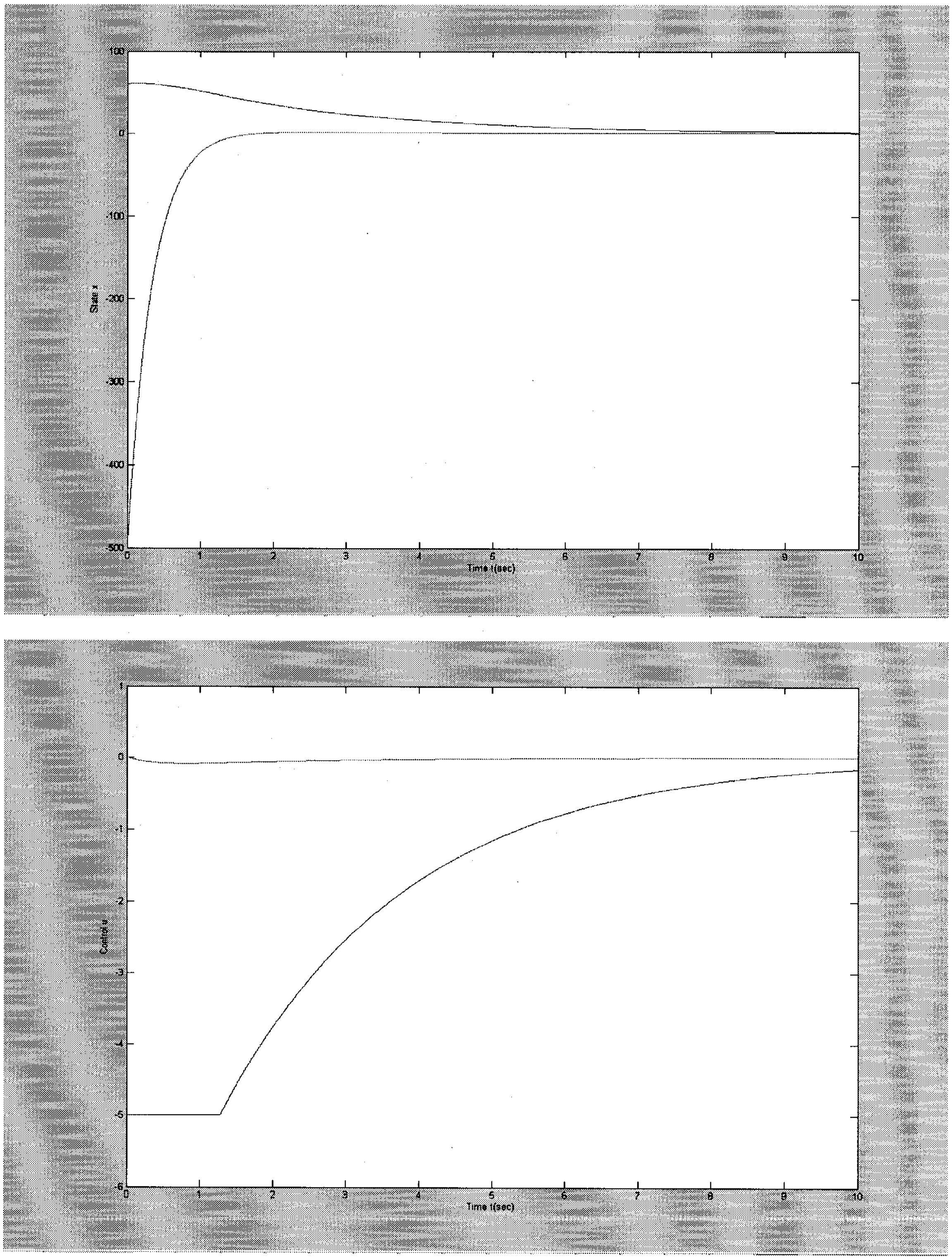

Figure 7.4. *State response (top) and corresponding control input (bottom) for uncertainty F=1*

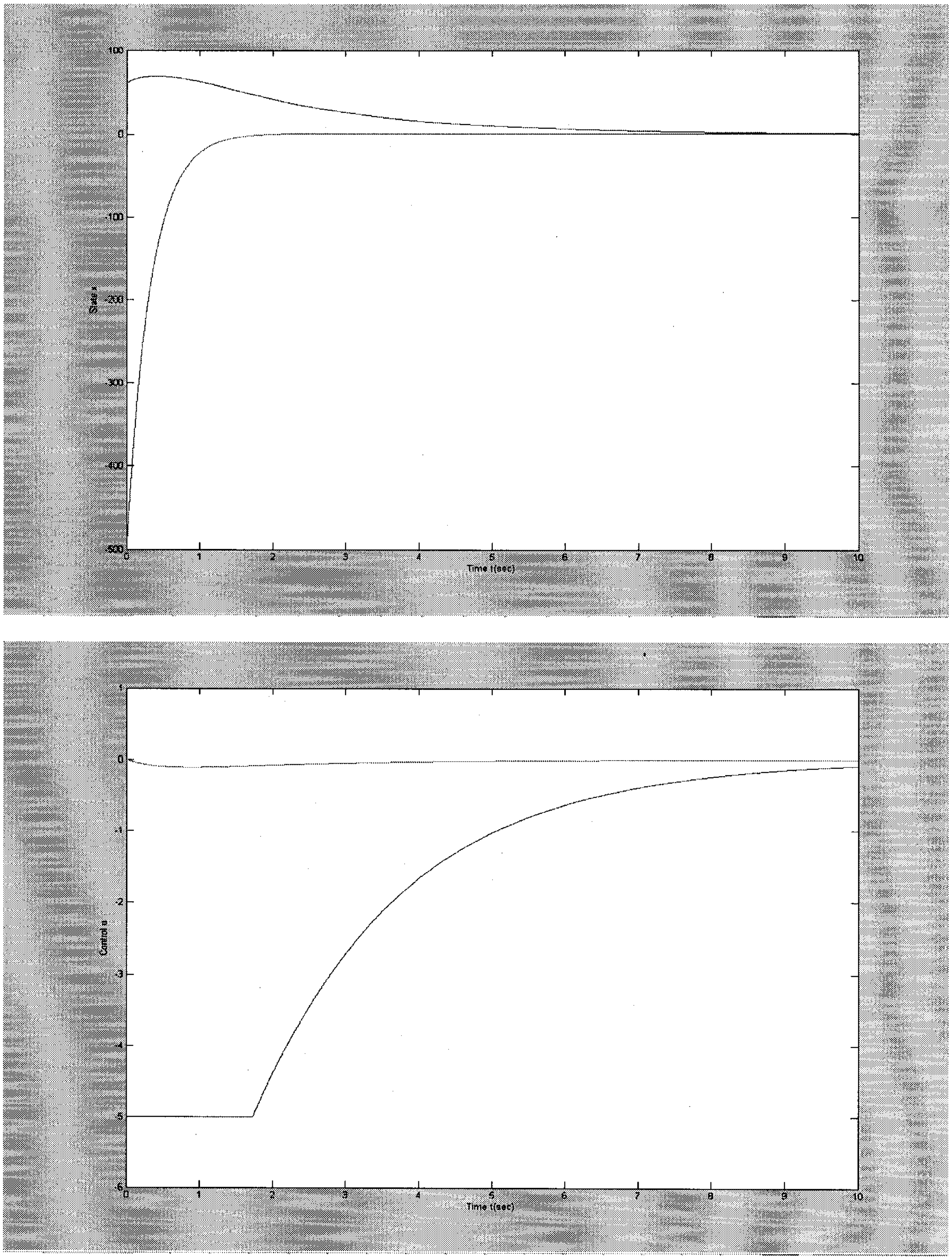

Figure 7.5. *State response (top) and corresponding control (bottom) input for uncertainty F=-1*

Example 7.3. The result presented within Theorem 7.1 involves the solution of several complicated matrix inequalities. To demonstrate its usefulness for control design processes and explain how it might be employed, an example is considered of an uncertain time-delay system of the type (7.1), in which the system matrices are given by:

$$A = \begin{bmatrix} -2 & 1 \\ 3 & -4 \end{bmatrix}, \quad A_h = \begin{bmatrix} 1 & 0 \\ -2 & 3 \end{bmatrix}, \quad B = \begin{bmatrix} 3 \\ 5 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 2 \end{bmatrix}$$

$$H_A = \begin{bmatrix} 1/10 \\ 0 \end{bmatrix}, \quad E_A = \begin{bmatrix} 1 & 1 \end{bmatrix}, \quad H_h = \begin{bmatrix} 1/10 \\ 1/10 \end{bmatrix}, \quad E_h = \begin{bmatrix} 0 & 1 \end{bmatrix}, \quad H_C = 0, \quad E_B = 0 \tag{7.67}$$

The time-delay is considered time-invariant with $h = 0.3$, and the control saturation level is determined by $\bar{u} = 1$. The same example was earlier used by **[72]** where only the values of gains K and L were given without any further information about the response of the system. With the above values for the state matrices, an observer gain matrix for the nominal system (in the absence of uncertainties and saturation) is chosen first using standard methods to minimize the state estimation error for the nominal system. The next step consists in the design of a nominal controller gain according to pre-selected performance criteria. With reference to the above example, the following values have so proved suitable:

$$L = \begin{bmatrix} 50 \\ 50 \end{bmatrix}, \quad K = \begin{bmatrix} -0.2 & -0.2 \end{bmatrix} \tag{7.68}$$

Theorem 7.1 was then applied to verify that (7.68) in fact yield a robust observer and controller design for the pre-specified uncertainties and the given actuator saturation level. This was done as follows. The parameter ε_4 was chosen first to be $\varepsilon_4 = 1$, and the matrix $\hat{A}_{h11,\max} = \begin{bmatrix} -1.8449 & 1.2675 \\ 1.2675 & -2.9129 \end{bmatrix}$ was computed using (7.43). It should be pointed out that the parameters ε_4 could be tuned further to reduce the overall conservatism of the design. This, however, has not been our purpose here. The control saturation lower bound was chosen as $\underline{\alpha} = 0.3$.

With fixed design parameters $\underline{\alpha}$, K and L, the matrix inequalities of Theorem 7.1 become linear in the remaining parameters and can be solved using the Matlab LMI toolbox, based on the methods developed in **[6]**. This was carried out to confirm that system (7.67) controlled by output feedback with observer and controller gains (7.68) is asymptotically robustly stable with respect to the admissible uncertainties and in spite of the given control saturation. Assuming zero initial conditions for the state estimate $\hat{x}$, the ball $\Phi(\sigma)$ of initial conditions for the state x guaranteeing asymptotic stability thus obtained is characterized by $\sigma = 17.5285$ as computed from equation (7.32). Thus the control laws (7.7) and (7.8) constitute a robust output feedback controller for this time-delay system.

Figure 7.6 shows the open-loop state response for the nominal system with initial condition $x_0 = (2,1)$.

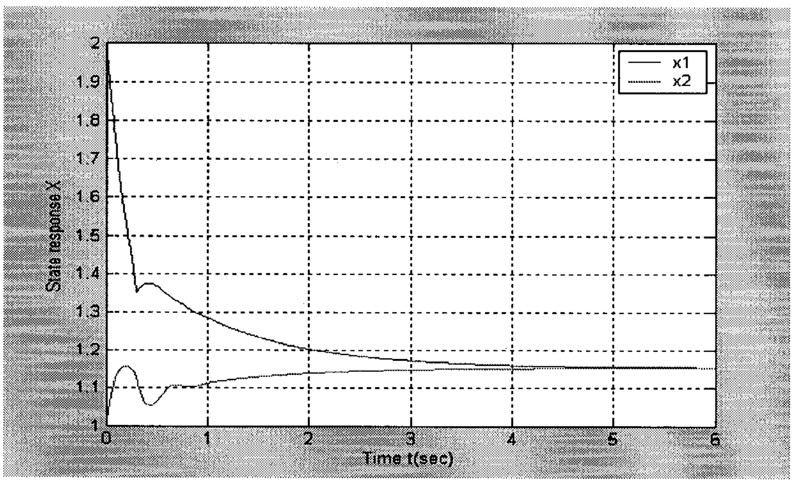

Figure 7.6. *Open-loop state response of the nominal system with initial condition X0=(2,1)*

Figure 7.7 shows the corresponding closed-loop state response of the nominal system with observer and controller gains as in (7.68). It is seen that the overshoot and the steady state error were eliminated and that the settling time is much smaller as compared with the open-loop system of Figure 7.6.

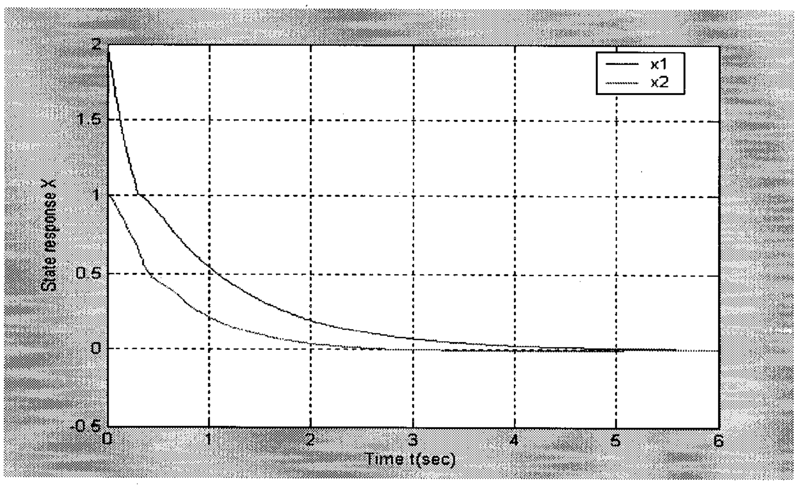

Figure 7.7. *Closed-loop state response of the nominal system with Initial condition X0=(2,1)*

Figure 7.8 shows the control input to the plant corresponding to the response in Figure 7.7. No control saturation is seen.

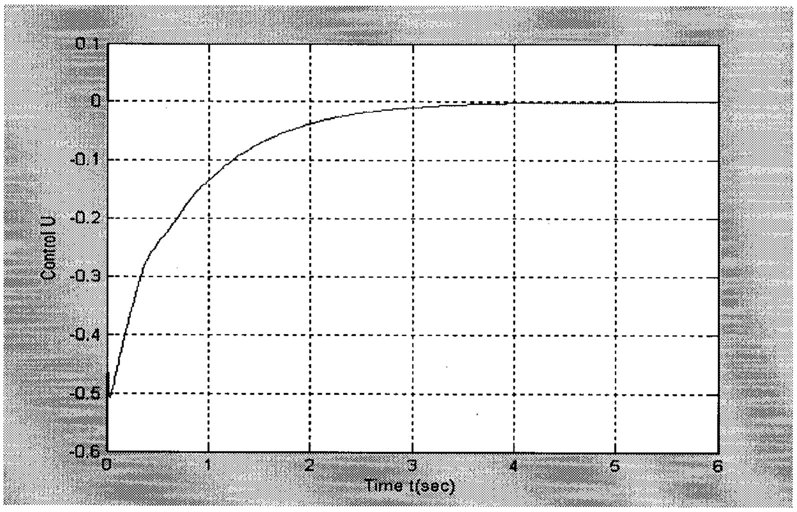

Figure 7.8. *Closed-loop control input to the plant in the nominal system with initial condition X0=(2,1)*

Finally, the controller and observer are tested using a remote initial condition $x_0 = (10,5)$; see Figure 7.9 for the closed-loop nominal system.

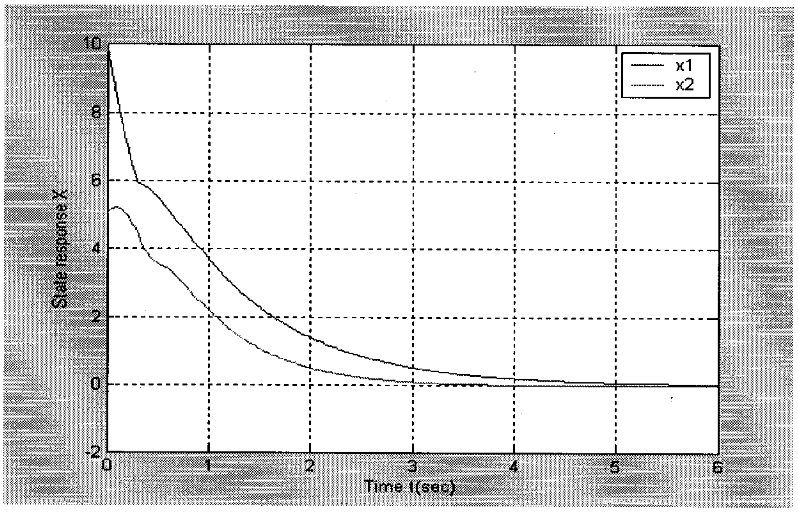

Figure 7.9. *Closed-loop state response of the nominal system with initial condition X0=(10,5)*

Figure 7.10 shows the corresponding control input to the plant where the actuator is clearly seen to saturate at the saturation level -1. Despite saturation, the responses of Figure 7.9 are still those of a stable system and the overall system performance has not been compromised excessively.

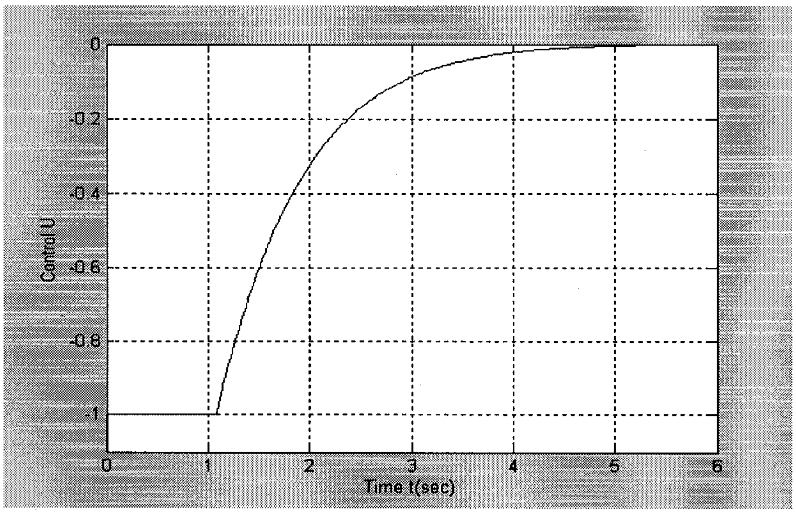

Figure 7.10. *Closed-loop input to the plant in the nominal system with initial condition X0=(10,5)*

Simulations demonstrate that the closed loop system stability is ensured for some initial states outside the computed ball $\Phi(\sigma)$. The initial conditions used in Figure 7.9 and Figure 7.10 are such an example, which implies that the stabilizable set of initial conditions may have a different shape than a ball and that $\Phi(\sigma)$ is only an interior approximation of this stabilizable set. This was to be expected as stabilizable sets seldom have the exact shape of a ball.

CHAPTER 8

Conclusion and Future Research

8.1. Conclusion

Using a methodology based on the bounding of the H_∞-norm of the closed-loop transfer function of the system, the first part (Chapters 2 and 3) of this thesis presents a robust, delay-independent, continuous-time H_∞ output feedback control design procedure for linear time-delay systems with parametric uncertainties. The design is quite general because of the inclusion of uncertainties in all matrices of the state equations and the delay is present in the state and output equations. The conditions for solvability of the robust H_∞ output feedback control problem is characterized in terms of the existence of solutions to two algebraic Riccati inequalities. The analytical expressions for the resulting observer and controller gains are given. Both time-invariant and time-varying cases are treated. In the time-varying case, only asymptotic stability is guaranteed.

Similarly, a delay-independent, discrete-time H_∞ output feedback control design procedure is presented. Specifically, the conditions for solvability of the robust H_∞ output feedback control problem is characterized in terms of the existence of solutions of two algebraic Riccati inequalities. The analytical expressions for the resulting observer and controller gains are given.

In the second part (Chapters 4-7) of this thesis, a methodology based on functionals, more specifically Lyapunov-Krasovskii functionals, is used to present delay-dependent robust stabilization and/or robust control designs for uncertain neutral or retarded time-delay systems, with or without actuators saturation.

First, the time-varying finite-horizon robust H_∞ control of uncertain neutral systems is presented. The problem formulation employed here is believed to be quite general with reference to the class of neutral systems. Such systems are highly relevant to applications in process industry and are likely to be employed there. As demonstrated, the design conditions derived in this paper are less restrictive as compared with previous works. A major innovation of the approach adopted here is the dependence of the feedback law on the value of the time-delay as well as on its rate of change.

Secondly, the robust stabilization of uncertain neutral systems with control saturation is presented. The saturations are modeled using differential inclusions. Again, the system representation is quite general in that uncertainties are included in the system matrices. A major innovation of the approach adopted here is that the stabilizing control design is made dependent on both the value of the time-delay as well as on its rate of change. It was shown through numerical examples that the presented method is less conservative than the most recent relevant designs found in the literature in that stabilization is ensured for a larger set of initial conditions. Further work is concerned with the extension of the present approach to a robust H_∞ control problem.

Finally, the robust output feedback stabilization of uncertain retarded systems with control saturation is presented. The problem formulation employed here is believed to be the first and the most general considered so far with reference to the delay-dependent robust output feedback stabilization of state-delayed systems with saturating actuators, using the Lypapunov-Krasovskii methodology. The problem is solved for both sector modeling and differential inclusions modeling for the actuator saturations. Again, a major innovation of the approach adopted here is that the stabilizing control design is made dependent on both the value of the time-delay as well as on its rate of change. For the case of sector modeling case, it was demonstrated by way of an example that the presented design provides an easily verifiable criterion for closed-loop robust stability of time-delayed systems with actuator saturation. For the differential inclusions model, it was demonstrated by way examples that the presented design provides an easily verifiable criterion for closed-loop robust stability of output feedback of time-delayed systems with actuator saturation, ensuring stability for larger sets of initial conditions than previous results obtained in the literature. Further work is concerned with the extension of the present approach to a robust H_∞ control problem.

8.2. Future Research Topics (see [65] for a very recent survey)

8.2.1. Extension to Robust H_∞ Control for the Case of Actuator Saturation

In Chapters 5-7, which dealt with time-delay systems with actuator saturation, the problem solved was that of robust stabilization. This work can be extended to the case of robust H_∞ control using the methodology presented in Chapter 4 for the case without actuator saturation. However, the presence of control constraints leads to a more complex problem. Specifically, the inclusion of time-varying disturbances necessitates a careful definition of the set of equilibria for the system (see **[79]**). A suitable modification of the S-procedure of **[6]** is expected to be useful in computing an approximation of the basin of attraction for such a set of equilibria.

8.2.2. Using the Delayed Inputs

Many results have been published about the control of systems with state delays, but without input or output delays. They lead to memory-less controllers, which means control laws of the form $u(t) = Kx(t)$, or to more general controllers with memory that include, nevertheless, an instantaneous feedback term (for example: $u(t) = Kx(t) + \sum_i K_i x(t - h_i)$).

But a more difficult and challenging question is to control a process without instantaneous measurement access to state variables, or via delayed actuators. For instance, it would be of theoretical and practical interest to consider systems such as:

$$\dot{x}(t) = Ax(t) + A_h x(t-h) + Bu(t) + B_h u(t-h)$$

for which the pairs (A, B) or $(A + A_h, B)$ are not controllable (for instance $B = 0$), which means one must use the $B_h u(t-h)$ term so as to obtain an efficient control.

8.2.3. Control Via the Delay Value

Another open problem is to control a process in which the input is the delay itself. For instance, the equation

$$\dot{y}(t) = g\left[y(t - lu(t))\right], \qquad u(t) \geq u_0 > 0 \tag{8.1}$$

corresponds to the crushing process depicted in Figure 8.1. Here, the recycled matter flow is supposed to be linear or nonlinear ratio $g(y)$ of the quantity of raw material y inside the crushing-mill, the output is the flow of processed material $h(y)$ (also depending on the filling level) and the rolling band with a variable speed u has a total path length l. Apparently the control of this kind of equation remains an open problem, as no theoretical grounding was stated.

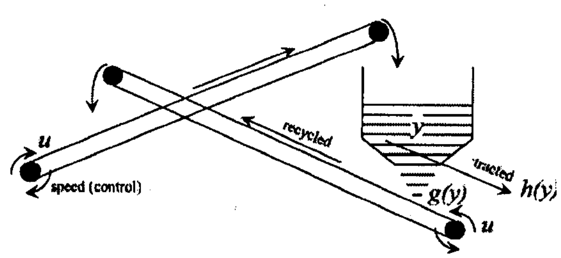

Figure 8.1. *Conveyor belts, speed* $u(t)$

Another example of such a control via the delay is given by a mixing tank with an impeller and a total recycle (Figure 8.2) in which a given quantity of salt is injected at the initial time. The salt concentration is measured by means of a conductivity probe which is placed at a different point. This corresponds to the following model:

$$T(u(t))\dot{y}(t) = y(t - h(u(t))) - y(t),$$

where $y(t) = z(t-h)$ is the conductivity measured at the probe position, $z(t)$ is the conductivity at the injection point, $u(t)$ is the rate, proportional to the rotation speed of the impeller, h is the time which the liquid, in total recycle, takes to flow from the injector to the probe (then, h is inversely proportional to u, and after rescaling: $h(u) = 1/u$), T is the mixing time constant, inversely proportional to the flow rate: $T(u) = 1/ku$.

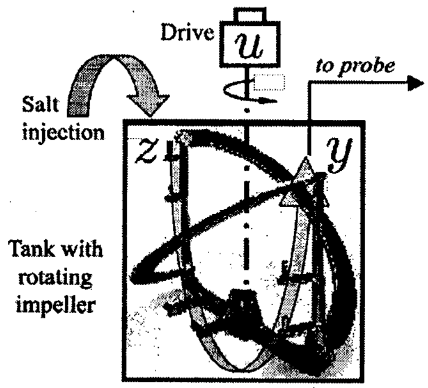

Figure 8.2. *Mixing tank with total recycle*

8.2.4. Collecting and Handling Information Relative to the Delay

Obviously, one can expect that the better the knowledge on the delay is, the higher the achievable control performances will be. For instance, in the case of a constant delay h, the simplest and best information is its value. If it is not available, then guaranteeing the robustness for $h \in [h_m, h_M]$ will be convenient. From this point of view, the poorest information corresponds to the most robust case: $h \geq 0$. Numerous authors, after proposing delay-independent stabilization results (assumption $h \geq 0$), concentrated on "delay-dependent" ones, as is the case in this thesis. The ideal would be to use the actual time-varying value of the delay, which calls more work into the identification of the delay.

8.2.4.1. Adaptive Identification of Delays

Even if several works considered the identification of either the delay or the parameters, the simultaneous identification remains to be done. Moreover, the real-time adaptive identification techniques of (varying) delays still need to improve. Works on identification of FDEs have shown the complexity of the question **[82]**. Identifying the delay is not an easy task for systems with both input and state delays, or when the delay is varying enough to require an adaptive identifier.

8.2.4.2. Using Stochastic Properties of the Delay

Until now the stability and stabilization of differential equations with stochastic delay were investigated under the assumption that, for each fixed value, the corresponding deterministic system is exponentially stable, uniformly with respect to all possible delay values. In other words, the stability conditions assume the delay to be known at each moment. More realistic stochastic models for time-delay need to be treated.

8.2.4.3. Delay Information for Observers

In the proposed observers in the literature the value of the delay (mainly constant) was involved in the realizations, which means that its measurement was assumed. In concrete applications, the delay invariance and delay knowledge remain assumptions coming from the identification and analysis limits than from technical facts. So, the robustness with regard to the delay estimation (and variation) should receive additional interest.

There are presently only few results in which the observer does not assume the delay knowledge. As far as is known, all of these few results are using delay-free observers, as is the case in Chapters 6 and 7 of this thesis. However, the stability conditions presented in Chapters 6 and 7 of this thesis are delay-dependent as compared to the delay-independent (more conservative) stability conditions presented in the literature and in Chapters 2 and 3 of this thesis.

REFERENCES

[1] T. Azuma, I. Ikeda, K. Uchida. Infinite-dimensional LMI approach to H_{Inf} control synthesis for linear systems with time delay. *5th European Control Conf.*, 1999.

[2] B. R. Barmish. Necessary and Sufficient Conditions for Quadratic Stabilizability of an Uncertain System. *Journal of Optimization Theory and Applications*, vol. 46: 4, pp. 399-408, 1985.

[3] E. A. Bencer, L. P. Neuwirth. Traffic flow: Laplace transform. *Amer. Math. Monthly*, vol. 80: 4, pp. 417-423, 1973.

[4] D. S. Bernstein, A. N. Michel(Eds). Special issue: Saturating Actuators. *International Journal of Robust and Nonlinear Control*, vol. 5: 5, pp. 375-540, 1995.

[5] T. Bond. Color Control in Dyed Paper, in *Electrical Engineering Department*: University of British Columbia, 1988.

[6] S. Boyd, L. E. Ghaoui, E. Feron, V. Balakrishnan. *Linear Matrix Inequalities in Systems and Control Theory*. Philadelphia, PA: SIAM, Studies in Applied Mathematics, 1994.

[7] B. Cahlon, J. Herband. A confidence headway for safe car following. *J. Math. Anal. & Appl.*, vol. 69: 2, pp. 511-530, 1979.

[8] V. Charitonov, D. Melchor-Aguilar. On delay-dependent stability conditions. *Systems&Control Letters*, vol. 40, pp. 71-76, 2000.

[9] H. H. Choi, M. J. Chung. An LMI approach to H_{Inf} controller design for linear time-delay systems. *Automatica*, vol. 33, pp. 737-739, 1997.

[10] A. Cohen, I. Yaesh, U. Shaked. H_{Inf} optimization of multivariable delayed measurements systems with an application to aircraft control. *Proceedings of Symposium on Large Scale Systems: theory and Applications*, pp. 613-618, 1999.

[11] L. Crocco. Aspects of combustion stability in liquid propellant rocket motors, part i: Fundamentals-Low frequency instability with monopropellants. *J. Amer. Rocket Society*, vol. 21, pp. 163-178, 1951.

[12] C. E. DeSouza, X. Li. Delay-dependent robust H_{Inf} control of uncertain linear state-delayed systems. *Automatica*, vol. 35, pp. 1313-1321, 1999.

[13] C. E. DeSouza, X. Li. LMI approach to decentralized stabilization of interconnected time-delay systems. *The 38th IEEE Conference on Decision and Control*, vol. 5, pp. 4700-4705, 1999.

[14] C. E. DeSouza, X. Li. LMI Approach to Delay-Dependent Robust Stability and Stabilization of Uncertain linear Delay Systems. *The 34th IEEE Conference on Decision and Control*, vol. 2, pp. 2023-2028, 1995.

[15] C. E. DeSouza, R. M. Palhares, P. L. D. Peres. Robust H_{Inf} filtering for uncertain linear systems with multiple time varying state delays; an LMI approach. *The 38th IEEE Conference on Decision and Control*, vol. 2, pp. 2023-2028, 1999.

[16] L. C. Edie. Car-following and steady state theory for noncongested traffic. *Oper. Res.*, vol. 9: 1, pp. 66-76, 1961.

[17] M. E. El'jasberg. About stability of metals cutting. *Izv. Akad. Nauk. SSSR, OTN*, vol. 9, 1958.

[18] Y. Fiagbedzi, A. Pearson. A multistage reduction technique for feedback stabilizing distributed time-lag systems. *Automatica*, vol. 23, pp. 311-326, 1987.

[19] E. Fridman. New Lyapunov-Krasovskii functionals for stability of linear retarded and neutral type systems. *Systems&Control Letters*, vol. 43: 4, pp. 309-319, 2001.

[20] E. Fridman, U. Shaked. A Descriptor System Approach to H_{Inf} Control of Linear Time-Delay Systems. *IEEE Transactions on Automatic Control*, vol. 47: 2, pp. 253-270, 2002.

[21] E. Fridman, U. Shaked. Finite horizon H_{Inf} state-feedback control of continuous-time systems with state delays. *IEEE Transactions on Automatic Control*, vol. 45, pp. 2406-2411, 2000.

[22] E. Fridman, U. Shaked. H_{Inf}-norm and invariant manifolds of systems with state delays. *Systems&Control Letters*, vol. 36: 2, pp. 157-165, 1999.

[23] E. Fridman, U. Shaked. H_{Inf}-state-feedback control of linear systems with small state delay. *Systems&Control Letters*, vol. 33, pp. 141-150, 1998.

[24] E. Fridman, U. Shaked. New bounded real lemma representations for time-delay systems and their applications. *IEEE Transactions on Automatic Control*, vol. 46, pp. 1973-1979, 2001.

[25] E. Fridman, U. Shaked. Sampled-data H_{Inf} state-feedback control of systems with state delays. *International Journal of Control*, vol. 73, pp. 1115-1128, 2000.

[26] J. Ge, P. M. Frank, C. F. Lin. Robust H_{Inf} state feedback control for linear systems with state delay and parameter uncertainty. *Automatica*, vol. 32, pp. 1183-1185, 1996.

[27] J. M. d. S. Gomes, S. Tarbouriech, R. Reginatto. Conservativity of ellipsoidal stability regions estimates for input saturated linear systems. *IFAC 2002*, 2002.

[28] V. D. Gorjachenko, S. L. Zolotarev, V. A. Kolchin. *Qualitative Methods in Nuclear Reactor Dynamics*. Moscow: Energoatomizdat, 1988.

[29] J. Hale, S. Lunel. *Effects of small delays on stability and control.* Amsterdam, the Netherlands, Rep. WS-528: Vrije University, 1999.

[30] J. K. Hale. Theory of functional differential equations. in *Applied Math. Sciences*, vol. 3. New York: Springer-Verlag, 1977.

[31] J. K. Hale, S. M. L. Verduyn. Introduction to functional differential equations. in *Applied Math. Sciences*, vol. 99. New York: Springer-Verlag, 1991.

[32] J. Harband. The existence of monotonic solutions of a nonlinear car-following equation. *J. Math. Anal. & Appl.*, vol. 57: 2, pp. 257-272, 1977.

[33] A. Haurani, H. Michalska, B. Benoit. Multivariable Control of a Paper Coloring Process: a Case Study. *Proc. of the 2001 American Control Conference*, pp. 2210-2215, 2001.

[34] A. Haurani, H. Michalska, B. Boulet. Discrete-time robust H_{Inf} output feedback controller design of linear state delayed systems with parametric uncertainties. *Report, McGill University*, 2001.

[35] D. Henrion, S. Tarbouriech, G. Garcia. Output Feedback Stabilization of Uncertain Linear Systems with Saturating Controls: an LMI Approach. *IEEE Transactions on Automatic Control*, vol. 44: 11, pp. 2230-2237, 1999.

[36] D. Ivanescu, J. M. Dion, L. Dugard, S. I. Niculescu. On delay-dependent stability of linear neutral systems. *Proc. 2nd IFAC Workshop Linear Time Delay Systems*, pp. 117-122, 2000.

[37] E. T. Jeung, D. C. Oh, J. H. Kim, H. B. Park. Robust controller design for uncertain systems with time delays: an LMI approach. *Automatica*, vol. 32, pp. 1229-1231, 1996.

[38] P. P. Khargonekar, I. R. Peterson, K. Zhou. Robust stabilization of uncertain linear systems: Quadratic stabilization and H_{Inf} control theory. *IEEE Trans. Automat. Control*, vol. 35, pp. 356-361, 1990.

[39] J. H. Kim. Delay and its Time-Derivative Dependent Robust Stability of Time-Delay Linear Systems with Uncertainty. *IEEE Transactions on Automatic Control*, vol. 46: 5, pp. 789-792, 2001.

[40] J. H. Kim. Robust stability of linear systems with delayed perturbations. *IEEE Transactions on Automatic Control*, vol. 41, pp. 1820-1822, 1996.

[41] V. Kolmanovskii, A. Myshkis. *Applied Theory of Functional Differential Equations.* Dordrecht, Boston: Kluwer Academic Publishers, 1992.

[42] V. Kolmanovskii, S. I. Niculescu, J. P. Richard. On the Lyapunov-Krasovskii functionals for stability analysis of linear delay systems. *Int. J. Control*, vol. 72, pp. 374-384, 1999.

[43] V. B. Kolmanovskii, N. I. Koroleva. Optimal control of bilinear hereditary systems and their applications. *Appl. Math. Letters*, vol. 4: 2, pp. 1-3, 1991.

[44] V. B. Kolmanovskii, V. R. Nosov. *Stability and Periodic modes of Regulated Systems with Delay.* Moscow: Nauka, 1981.

[45] V. B. Kolmanovskii, V. R. Nosov. *Stability of Functional Differential Equations.* New York-London: Acad. Press, 1986.

[46] V. B. Kolmanovskii, V. R. Nosov. Stability of neutral-type functional differential equations. *Nonlinear Anal. Theory, Methods & Appl.*, vol. 6: 9, pp. 873-910, 1982.

[47] J. H. Lee, S. W. Kim, W. H. Kwon. Memoryless H_{Inf} controllers for state delayed systems. *IEEE Transactions on Automatic Control*, vol. 39, pp. 159-162, 1994.

[48] H. Li, M. Fu. Linear matrix inequality approach to robust H_{Inf} filtering. *IEEE Transactions on Signal Processing*, vol. 45, pp. 2338-2350, 1997.

[49] X. Li, C. E. D. Souza. Criteria for robust stability and stabilization of uncertain linear systems with state delay. *Automatica*, vol. 33, pp. 1657-1662, 1997.

[50] X. Li, C. E. D. Souza. Delay-dependent robust stability and stabilization of uncertain linear delay systems: A linear matrix inequality approach. *IEEE Transactions on Automatic Control*, vol. 42, pp. 1144, 1997.

[51] L. W. Liou, A. Ray. Integrated communication and control systems. *J. Dynam. Systems, Measurement & Control*, vol. 112, pp. 357-364, 1990.

[52] H. Logemann, S. Townley. The effect of small delays in the feedback loop on the stability of the neutral systems. *Systems&Control Letters*, vol. 27, pp. 267-274, 1996.

[53] M. Mahmoud. *Robust control and Filtering for Time-Delay Systems*. New York: Marcel Decker, 2000.

[54] M. Mahmoud. Robust H_{Inf} control of linear neutral systems. *Automatica*, vol. 36, pp. 757-764, 2000.

[55] I. Masubuchu, Y. Kamitane, A. Ohara, N. Suda. H_{Inf} control for descriptor systems: A matrix inequalities approach. *Automatica*, vol. 33, pp. 669-673, 1997.

[56] V. M. Mazurov, D. I. Malov, V. I. Salomyrov. Systems of PH value control in absorbing column with recycle. *Himich. Promysh.*, vol. 4, pp. 63-65, 1974.

[57] V. N. Michelkevich, Y. A. Chabanov. Identification and synthesis of control system of infeed grinding process. *Elektromash. & Elektrooborudov*, vol. 31, pp. 42-52, 1980.

[58] A. P. Molchanov, E. S. Pyatniskii. Criteria of asymptotic stability of differential and difference inclusions encountered in control theory. *Systems & Control Letters*, vol. 13, pp. 59-64, 1989.

[59] S. I. Niculescu. H_{Inf} memoryless control with alpha-stability constraint for time-delay systems: An LMI approach. *IEEE Transactions on Automatic Control*, vol. 43, pp. 739-743, 1998.

[60] S. I. Niculescu, C. E. DeSouza, J. M. Dion, L. Dugard. Robust H_{Inf} memoryless control for uncertain linear systems with time-varying delay. *3rd European Control Conf.*, pp. 1814-1818, 1995.

[61] S. I. Niculescu, S. Tarbouriech, J. M. Dion, L. Dugard. Stability criteria for bilinear systems with delayed state and saturating controls. *Proceedings of the 34th IEEE Conference on Decision and Control*, pp. 2064-2069, 1995.

[62] P. Park. A delay-dependent stability criterion for systems with uncertain time-invariant delays. *IEEE Transactions on Automatic Control*, vol. 44, pp. 876-877, 1999.

[63] A. W. Pila, U. Shaked, C. E. D. Souza. H_{Inf} filtering for continuous-time linear systems with delay. *Proceedings of the 13th World Congress. Vol. J: Identification II, Discrete Event Systems*, pp. 49-54, 1997.

[64] A. W. Pila, U. Shaked, C. E. D. Souza. H_{Inf} Filtering for continuous-time linear systems with delay. *IEEE Transactions on Automatic Control*, vol. 44, pp. 1412-1417, 1999.

[65] J. P. Richard. Time-delay systems: an overview of some recent advances and open problems. *Automatica*, vol. 39, pp. 1667-1694, 2003.

[66] U. Shaked, I. Yaesh. H_{Inf} static output-feedback control of linear continuous-time systems with delay. *IEEE Transactions on Automatic Control*, vol. 43, pp. 1431-1436, 1998.

[67] U. Shaked, I. Yaesh, C. E. DeSouza. Bounded real Criteria for Linear Time-Delay Systems. *IEEE Transactions on Automatic Control*, vol. 43: 7, pp. 1016-1022, 1998.

[68] S. V. Shilman. *Method of Generating Functions in the Theory of Dynamical Systems*. Moscow: Nauka, 1978.

[69] S. H. Song, J. K. Kim. H_{Inf} control of discrete-time linear systems with norm bounded uncertainties and time delay in state. *Automatica*, vol. 34, pp. 137-139, 1998.

[70] H. Stats, G. A. d. Mars, D. T. Wilson, C. L. Tang. Problem of spike elimination in lasers. *J. Appl. Physics*, vol. 36: 1, pp. 1510, 1965.

[71] A. A. Stoorvogel, A. Saberi(Eds). Special Issue: Control Problems with Constraints. *International Journal of Robust and Nonlinear Control*, vol. 9: 10, 1995.

[72] H. Su, J. Lam, J. Hu, J. Chu. Output Feedback Stabilization of Uncertain Time-Delay Systems Containing Saturating Actuators. *Dev. Chem. Eng. Mineral Process*, vol. 9: 1/2, pp. 183-190, 2001.

[73] T. J. Su, C. Y. Lu, J. S. H. Tsai. Delay-dependent Robust Stabilization for Uncertain Time-delay Systems with Saturating Actuators-LMI Approach. *Proceedigns of the American Control Conference*, pp. 3357-3358, 2002.

[74] K. Takaba, N. Morihira, T. Katayama. A generalized Lyapunov theorem for descriptor system. *Systems & Control Letters*, vol. 24, pp. 49-51, 1995.

[75] S. Tarbouriech, G. Garcia. Output Feedback Compensators for Linear Systems with Position and Rate Bounded Actuators. in *Actuator Saturation Control*. New York: Marcel Dekker, 2002.

[76] S. Tarbouriech, G. Garcia. Robust stability of uncertain linear systems with saturating inputs: an LMI approach. *Proceedings of the IFAC Conference on System Structure and Control*, pp. 379-384, 1998.

[77] S. Tarbouriech, G. Garcia. Stabilization of Neutral Linear Time-Delay Systems with Saturating Actuators. *Proceedings of the 38th Conference on Decision & Control*, pp. 2011-2016, 1999.

[78] S. Tarbouriech, J. M. d. S. Gomes. Sysnthesis of controllers for continuous-time delay systems with saturating controls via LMIs. *IEEE Transactions on Automatic Control*, vol. 45: 1, pp. 105-111, 2000.

[79] S. Tarbouriech, P. Peres, G. Garcia, I. Queinnec. Delay-dependent stabilization of time-delay systems with saturating actuators. *Proceedings of the 39th Conference on Decision & Control*, pp. 3248-3253, 2000.

[80] H. Trinh, M. Aldeen. A memoryless state observer for discrete time-delay systems. *IEEE Trans. Automat. Control*, vol. 42, pp. 1572-1577, 1997.

[81] E. F. Tzar'kov. *Stochastic Disturbances of Functional Differential Equations*. Riga: Zinantne, 1989.

[82] S. M. Verdun-Lunel. Identification problems in functional differential equations. *36th IEEE CDC'97*, pp. 4409-4413, 1997.

[83] E. Verriest, S. I. Niculescu. Delay-independent stability of linear neutral systems: A Riccati equation approach. in *Stability and Control of Time-Delay Systems*, vol. 227, L. Dugard and E. Verriest, Eds. London, U.K.: Springer-Verlag, 1998, pp. 92-100.

[84] Y. Wang, L. Xie, C. E. DeSouza. Robust control of a class of uncertain nonlinear systems. *Systems&Control Letters*, vol. 19, pp. 139-149, 1992.

[85] Z. Wang, B. Huang, H. Unbehauen. Robust H_{Inf} observer design of linear state delayed systems with parametric uncertainty: the discrete-time case. *Automatica*, vol. 35, pp. 1161-1167, 1999.

[86] Z. Wang, B. Huang, H. Unbehauen. Robust H_{Inf} observer design of linear time-delay systems with parametric uncertainty. *Systems&Control Letters*, vol. 42, pp. 303-312, 2001.

[87] Z. Wang, F. Yang. Robust Uncertain Linear Systems with delayed States and Outputs. *IEEE Transactions on Circuits and Systems-I: Fundamental Theory and Applications*, vol. 49: 1, pp. 125-130, 2002.

[88] L. Xie, E. Fridman, U. Shaked. Robust H_{Inf} control of distributed delay systems with application to combustion control. *IEEE Transactions on Automatic Control*, vol. 46: 12, pp. 1930-1935, 2001.

[89] L. Xie, Y. C. Soh. Robust Kalman filtering for uncertain systems. *Systems&Control Letters*, vol. 22: 2, pp. 123-129, 1994.

[90] S. Xu, J. Lam, C. Yang, E. I. Verriest. An LMI approach to guaranteed cost control for uncertain linear neutral delay systems. *International Journal of Robust and Nonlinear control*, vol. 13, pp. 35-53, 2003.

[91] I. Yaesh, A. Cohen, U. Shaked. Delayed state-feedback H_{Inf} control. *Proceedings of IFAC Workshop on Linear Time Delay Systems*, pp. 51-56, 1999.

[92] Y. X. Yao, Y. M. Zhang, R. Kovacevic. Functional observer and state feedback for input time-delay systems. *International Journal of Control*, pp. 603-617, 1997.

[93] D. Yue, S. Won. An Improvement on 'Delay and Its Time-Derivative Dependent Robust Stability of Time-Delayed Linear Systems With Uncertainty'. *IEEE Transactions on Automatic Control*, vol. 47: 2, pp. 407-408, 2002.

[94] F. Zheng, P. M. Frank. Robust control of uncertain distributed delay systems with application to the stabilization of combustion in rocket motor chambers. *Automatica*, vol. 38: 3, pp. 487-497, 2002.

APPENDIX A

Useful Theorems and Other results

Some useful theorems and techniques referred to, but not original to this dissertation, are gathered in this particular appendix for the reader's convenience.

A.1. Matrix Inversion Formulas

Let A be a square matrix partitioned as follows:

$$A \triangleq \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$$

where A_{11} and A_{22} are also square matrices. Now suppose A_{11} is nonsingular; then A has the following decomposition:

$$\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} = \begin{bmatrix} I & 0 \\ A_{21}A_{11}^{-1} & I \end{bmatrix} \begin{bmatrix} A_{11} & 0 \\ 0 & \Delta \end{bmatrix} \begin{bmatrix} I & A_{11}^{-1}A_{12} \\ 0 & I \end{bmatrix}$$

with $\Delta \triangleq A_{22} - A_{21}A_{11}^{-1}A_{12}$, and A is nonsingular iff Δ in nonsingular. Dually, if A_{22} is nonsingular, then

$$\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} = \begin{bmatrix} I & A_{12}A_{22}^{-1} \\ 0 & I \end{bmatrix} \begin{bmatrix} \hat{\Delta} & 0 \\ 0 & A_{22} \end{bmatrix} \begin{bmatrix} I & 0 \\ A_{22}^{-1}A_{21} & I \end{bmatrix}$$

with $\hat{\Delta} \triangleq A_{11} - A_{12}A_{22}^{-1}A_{21}$, and A is nonsingular iff $\hat{\Delta}$ is nonsingular. The matrix Δ $\left(\hat{\Delta}\right)$ is called the Schur complement of A_{11} $\left(A_{22}\right)$ in A.

Moreover, if A is nonsingular, then

$$\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}^{-1} = \begin{bmatrix} A_{11}^{-1} + A_{11}^{-1}A_{12}\Delta^{-1}A_{21}A_{11}^{-1} & -A_{11}^{-1}A_{12}\Delta^{-1} \\ -\Delta^{-1}A_{21}A_{11}^{-1} & \Delta^{-1} \end{bmatrix}$$

and

$$\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}^{-1} = \begin{bmatrix} \hat{\Delta}^{-1} & -\hat{\Delta}^{-1} A_{12} A_{22}^{-1} \\ -A_{22}^{-1} A_{21} \hat{\Delta}^{-1} & A_{22}^{-1} + A_{22}^{-1} A_{21} \hat{\Delta}^{-1} A_{12} A_{22}^{-1} \end{bmatrix}.$$

The preceding matrix inversion formulas are particularly simple if A is block triangular:

$$\begin{bmatrix} A_{11} & 0 \\ A_{21} & A_{22} \end{bmatrix}^{-1} = \begin{bmatrix} A_{11}^{-1} & 0 \\ -A_{22}^{-1} A_{21} A_{11}^{-1} & A_{22}^{-1} \end{bmatrix}$$

$$\begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix}^{-1} = \begin{bmatrix} A_{11}^{-1} & -A_{11}^{-1} A_{12} A_{22}^{-1} \\ 0 & A_{22}^{-1} \end{bmatrix}.$$

The following identity is also very useful. Suppose A_{11} and A_{22} are both nonsingular matrices; then

$$\left(A_{11} - A_{12} A_{22}^{-1} A_{21}\right)^{-1} = A_{11}^{-1} + A_{11}^{-1} A_{12} \left(A_{22} - A_{21} A_{11}^{-1} A_{12}\right)^{-1} A_{21} A_{11}^{-1}.$$

As a consequence of the matrix decomposition formulas mentioned previously, we can calculate the determinant of a matrix by using its sub-matrices. Suppose A_{11} is nonsingular; then

$$\det A = \det A_{11} \det\left(A_{22} - A_{21} A_{11}^{-1} A_{12}\right).$$

On the other hand, if A_{22} is nonsingular, then

$$\det A = \det A_{22} \det\left(A_{11} - A_{12} A_{22}^{-1} A_{21}\right).$$

In particular, for any $B \in \mathbb{C}^{m\times n}$ and $C \in \mathbb{C}^{n\times m}$, we have

$$\det\begin{bmatrix} I_m & B \\ -C & I_n \end{bmatrix} = \det\left(I_n + CB\right) = \det\left(I_m + BC\right)$$

and for $x, y \in \mathbb{C}^n$

$$\det\left(I_n + xy^*\right) = 1 + y^* x.$$

A.2. The Lipschitz Condition

Let $f : \mathbb{R}^+ \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ be a functional defined an FDE

$$\dot{x}(t) = f\left(t, x_t, \dot{x}_t\right).$$

Functional f is then said to satisfy a Lipschitz condition in x_t if there is a piecewise continuous function $k(.): \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$\left\| f(t, x_t, \dot{x}_t) - f(t, y_t, \dot{x}_t) \right\| \le k(t) \left\| x_t - y_t \right\|$$

for all $t \in \mathbb{R}^+$ and for all $x_t, y_t \in \mathbb{R}^n$.

A.3. The Riccati Equation: Stability

A.3.1. The Continuous Time

We consider the asymptotic stability of

$$\dot{x}(t) = \left(A - BR^{-1}B^T P(\infty)\right)x(t) \tag{A.1}$$

where $P(\infty)$ is the maximal nonnegative definite solution of the ARE,

$$0 = A^T P(\infty) + P(\infty)A - P(\infty)BR^{-1}B^T P(\infty) + Q \tag{A.2}$$

Theorem A.3.1. *Consider the time-invariant linear vector differential equation* (A.1) *representing the closed loop of an infinite horizon LQ controlled system, where* $P(\infty)$ *is the maximal nonnegative definite solution,* $\overline{P}$ *of the ARE* (A.2).

Subject to the conditions:

- $[A, B]$ *is stabilizable,*
- $\left[A, Q^{1/2}\right]$ *is detectable,*
- $Q \ge 0$ *and* $R > 0$,

then (A.1) *is exponentially asymptotically stable.*

A.3.2. The Discrete Time

We consider the asymptotic stability of

$$x(k+1) = \left(A - B\left(B^t P(\infty)B + R\right)^{-1} B^T P(\infty)A\right)x(k) \tag{A.3}$$

where $P(\infty)$ is the maximal nonnegative definite solution of the ARE,

$$P(\infty) = A^T P(\infty) A - A^T P(\infty) B \left(B^T P(\infty) B + R\right)^{-1} B^T P(\infty) A + Q \tag{A.4}$$

Theorem A.3.2. *Consider the time-invariant linear vector difference equation* (A.3) *representing the closed loop of an infinite horizon LQ controlled system, where* $P(\infty)$ *is the maximal nonnegative definite solution,* $\overline{P}$ *of the ARE* (A.4).

Subject to the conditions:

- $[A, B]$ *is stabilizable,*
- $\left[A, Q^{1/2}\right]$ *is detectable,*
- $Q \geq 0$ *and* $R > 0$,

then (A.3) *is exponentially asymptotically stable.*

APPENDIX B

Linear Matrix Inequality

The goal of this appendix is to recall some important notions on linear matrix inequality (LMI). Mainly, we define the LMI problem and the related problems like the feasibility problem (FEAS), minimization of a linear objective under LMI constraints (MINCX), and the generalized eigenvalue minimization problem (GEVP).

B.1. LMI Functions

A linear matrix inequality (LMI) has the form

$$F(z) \triangleq F_0 + \sum_{i=1}^{m} z_i F_i < 0, \tag{B.1}$$

where $z = (z_1, \ldots, z_m) \in \mathbb{R}^m$ is the variable to be determined and the symmetric matrices $F_i \in \mathbb{R}^{n \times n}$, $0 \leq i \leq m$ are given. The inequality symbol in (B.1) means that $F(x)$ is negative-definite, i.e., $v^T F(x) v < 0$ for all nonzero $v \in \mathbb{R}^n$.

For example, a linear system with the following dynamic

$$\dot{x}(t) = Ax(t) \tag{B.2}$$

where $x(t) \in \mathbb{R}^2$, $A \in \mathbb{R}^{2\times 2}$, is stable if and only if there exists a symmetric and positive-definite matrix $P > 0$ such that

$$A^T P + PA < 0 \tag{B.3}$$

This problem. In fact, can be solved using the LMI toolbox. Let us now see how we can put this problem in the form of (B.1). For this purpose, let $A = \begin{bmatrix} a_1 & a_2 \\ a_2 & a_3 \end{bmatrix}$ and suppose that $P = \begin{bmatrix} z_1 & z_2 \\ z_2 & z_3 \end{bmatrix}$, where z_1, z_2 and z_3 are design parameters. Then,

$$A^T P + PA = z_1 \begin{bmatrix} 2a_1 & a_2 \\ a_2 & 0 \end{bmatrix} + z_2 \begin{bmatrix} 2a_2 & a_1 + a_3 \\ a_1 + a_3 & 2a_2 \end{bmatrix} + z_3 \begin{bmatrix} 0 & a_2 \\ a_2 & 2a_3 \end{bmatrix}.$$

Therefore, (B.3) is a standard LMI feasibility problem that can be solved using the LMI toolbox.

There are three kinds of generic LMI problems, which are shown next.

B.2. LMI Problems

As mentioned earlier, there exist three main LMI problems:

- The feasibility problem
- The linear optimization problem
- The generalized eigenvalue minimization problem.

B.2.1. Feasibility Problem

The LMI feasibility Problem (FEASP) consists of determining the variable $x \in \mathbb{R}^m$ such that $F(x) < 0$ holds. This problem can be solved using the function "feasp" of the LMI toolbox. A typical situation for the feasibility problem is the stability test for dynamical systems. In fact, based on control theory a system with the dynamic (B.2) is stable if and only if there exists a symmetric and positive-definite matrix $P > 0$ such that (B.3) is satisfied. The goal is then to find a matrix $P > 0$ such that the inequality (B.3) is satisfied.

Another example of the feasibility problem is by considering the stabilization problem of the system

$$\dot{x}(t) = Ax(t) + Bu(t) \tag{B.4}$$

where $u(t)$ is the control input. Using the stability result, a state feedback memoryless controller $u(t) = Kx(t)$ stabilizes system (B.4) if and only if there exists a symmetric and positive-definite matrix P such that

$$P(A+BK)+(A+BK)^T P<0. \tag{B.5}$$

PRE- and Post-multiplying both sides of (B.5) by $X=P^{-1}$ and letting $Y=KX$ yields that (B.5) is equivalent to

$$AX+BY+XA^T+Y^TB^T<0. \tag{B.6}$$

B.2.2. Minimization of a Linear Objective under LMI Constraints

Minimization of a linear objective under LMI constraints (MINCX) is another interesting problem that we have used extensively. The MINCX problem is stated as follows:

$$\min_{x\in\mathbb{R}^m} C^T x$$

$$\text{s.t. } F(x)<0$$

where $C\in\mathbb{R}^m$ is a given vector. This problem can be solved using "mincx" of LMI toolbox.

To provide an example using "mincx", let us consider an H_2 control problem. For this purpose let us assume the system dynamics are given by

$$\begin{cases}\dot{x}(t)=Ax(t)+Bw(t)\\ \quad y(t)=Cx(t)\end{cases} \tag{B.7}$$

where $w(t)$ is a white noise disturbance with unit covariance. Suppose that the H_2 performance is defined by

$$\|H\|_2^2=\lim_{t\to\infty}\mathbf{E}\left(\frac{1}{t}\int_0^t y^T(s)\,y(s)\,ds\right).$$

Then, it can be shown that the solution to this problem is given by:

$$\|H\|_2^2=\min\left\{tr\left(CPC^T\right):AP+PA^T+BB^T<0\right\}.$$

Obviously this optimization problem is equivalent to minimizing $tr(Q)$ subject to

$$AP+PA^T+BB^T<0 \tag{B.8}$$

$$CPC^T\le Q. \tag{B.9}$$

Using the Schur complement, (B.9) is equivalent to

$$\begin{bmatrix} -Q & CP \\ PC^T & -P \end{bmatrix} < 0 .$$

B.2.3. Generalized Eigenvalue Minimization Problem

The generalized eigenvalue minimization problem (GEVP) is the third interesting problem used extensively. This problem is stated as follows:

$$\min_{x \in \mathbb{R}^m} \lambda$$

$$\text{s.t. } F_1(x) < \lambda F_2(x)$$

where $F_1(x)$ and $F_2(x)$ are two matrices of form (B.1). The GEVP is quasi-convex with respect to the design parameters x and λ, which can be solved using "gevp" of the LMI toolbox.

The decay rate of system (B.2) is defined as the largest γ such that $\lim_{t \to \infty} e^{\gamma t} \|x(t)\| = 0$. Let us consider a Lyapunov function candidate $V(x(t)) = x^T(t) P x(t)$. If we can establish that

$$\frac{dV(x(t))}{dt} \le -2\gamma V(x(t)) \tag{B.10}$$

holds for all trajectories, then the decay rate of system (B.2) is at least γ.

Noting that (B.10) holds if and only if

$$A^T P + PA + 2\gamma P \le 0 , \tag{B.11}$$

we conclude that the largest lower bound on the decay rate can be found by solving the GEVP in P and γ

$$\max_{P>0} \gamma$$

$$\text{s.t. (B.11).}$$

To solve this optimization problem, let us rewrite (B.11) as

$$2P \le \frac{1}{\gamma}\left(-PA - A^T P\right) .$$