Particle dynamics in Kerr–Newman–de Sitter spacetimes

Steven Rayan

Department of Mathematics and Statistics,

McGill University, Montréal

Québec, Canada

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Equations are just the boring part of mathematics. I attempt to see things in terms of geometry.

Stephen Hawking

Abstract

As solutions of the Einstein-Maxwell field equations, the Kerr-Newman-de Sitter geometries are spacetimes that model the outer geometry of a charged, rotating black hole. We re-derive, using a lemma of Brandon Carter, first-order conserved quantities for the motion of zero rest mass and massive test particles in the Kerr-Newman-de Sitter spacetimes. The Liouville-Arnol'd integrability theory allows us to use first integral equations in place of second-order equations in the dynamical analysis. In examining the effects of first integral data on the equations, we expose some differences between particle dynamics in the electrically-neutral, asymptotically-flat Kerr geometries and those in the charged, de Sitter geometries.

Résumé

Les géométries de Kerr-Newman-de Sitter sont des solutions des équations d'Einstein-Maxwell avec constante cosmologique modélisant la partie extérieure de l'espace-temps au voisinage d'une configuration d'équilibre d'un trou noir en rotation. Nous calculons a l'aide d'un lemme de Brandon Carter les quantités conservées du premier ordre associées au mouvement de particules d'épreuve de masse nulle et non-nulle dans la métrique de Kerr-Newman-de Sitter. Le théorème d'intégrabilité de Liouville-Arnol'd nous permet d'utiliser ces quantités conservées pour analyser la dynamique. Ceci nous permet de mettre en evidence les différences entre la dynamique des particules selon que l'on introduit ou non un champ électromagnétique et une constante cosmologique.

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Introduction

0.1 Motivation

The large-scale structure of space and time, to some extent, can be predicted by means of the general theory of relativity. Ultimately, these predictions must be weighed against experimental evidence concerning both the local and asymptotic behaviour of the actual universe surrounding us. The search for exact solutions to the Einstein field equations has produced a plethora of models: the Schwarzschild universe; the de Sitter universe and its anti-de Sitter counterpart; the Gödel universe; and so on. Many more are found within [13]. In choosing a model to fit the actual universe, two factors to consider are (a) the existence of black holes, and (b) the magnitude of the cosmological constant Λ.

Our own galaxy contains an extremely dense star cluster, in which lies a compact radio source known as Sagittarius A*. Current observations made via high-resolution 10-metre telescope imaging suggest “the presence of a three million solar mass black hole in Sagittarius A* beyond any reasonable doubt. The Galactic Center thus constitutes the best astrophysical evidence for the existence of black holes which have long been postulated...” [7]. At the other extreme, “Recent experimental data indicate that the cosmological constant is most likely positive, suggesting (assuming a spacetime of constant curvature, at least in first approximation) a de Sitter universe at cosmological distances" [14].

The Kerr-Newman-de Sitter geometries comprise, collectively, the unique pseudo-stationary axisymmetric solution of the source-free Einstein-Maxwell equations with positive cosmological constant ([3]). Each such spacetime not only houses a black hole inside its horizons, but also has constant scalar curvature. While these special features capture some of the essential properties of our universe (as observed within the limits of modern experiments), the Kerr-Newman-de Sitter spacetimes are by no means models of the universe on a large scale. Rather, they are good models for the outer geometry of the equilibrium state of a rotating black hole viewed as an isolated system.

Regardless of how cosmologically suitable our model is, one question we may ask is "How do photons and massive test particles move in it?" While the motivation is physical, the problem itself is entirely mathematical. Particles transporting a charge q under the gravitational and electromagnetic influence of an empty spacetime are constrained along precisely those curves minimizing the action of a Lagrangian $\mathbf{L}_q$ determined by the metric and the Maxwell potential. When the charge q is set to zero, the components of the potential no longer contribute to the Lagrangian $\mathbf{L}_0$, whose action-minimizers are the geodesic curves in the spacetime. In the case of a Kerr-de Sitter spacetime, the Maxwell potential is zero, and so the charge on a particle plays no role in its dynamics. In particular, the only orbits assumed by particles are the geodesics in the manifold. B. O'Neill submits, in [10], a partial answer to the geodesic classification problem in the asymptotically-flat Kerr geometry. The treatment is global, as the trajectories are studied in the maximally-extended spacetime.

The variational orbit problem in Kerr-Newman-de Sitter has an equivalent Hamiltonian formulation, which we exploit in extracting the constants of motion. With sufficiently-many independent integrals in hand, we discuss the various trajectories determined by the initial conditions prescribed to a particle, charged or uncharged. While the treatment here concerns local dynamics only, the analysis, to some extent, mimics that performed by O'Neill.

The outline of the study is as follows:

1. **Properties of Kerr-Newman-de Sitter spacetimes.** In this chapter, we
   - introduce the metric formally;
   - discuss how the choice of a positive cosmological constant affects the number and type of horizons;
   - find sufficient conditions on the spacetime parameters such that the number of horizons is maximal;
   - define the Boyer-Lindquist and de Sitter blocks.
2. **Integrability of the equations of motion.** In this chapter, we
   - introduce the Hamiltonian formulation of the orbit problem;
   - extract the integrals of motion (with particular attention paid to the so-called *Carter's constant*);
   - use the first integrals to write down first-order differential equations for the components of a particle's velocity.
3. **Dynamics of charged particles.** In this chapter, we
   - analyze the first-order evolution equations for charged particle trajectories.

Before we begin, we fix the notation and conventions to be used in the thesis.

0.2 Notation, conventions, and preliminary definitions

Definition 0.1. A **Lorentz vector space** is a pair $(V, \langle -, - \rangle)$ in which V is a vector space of dimension $\dim_{\mathbb{R}}(V) \geq 2$ and $\langle -, - \rangle : V \times V \to \mathbb{R}$ is a scalar product of index $ind(\langle -, - \rangle) = 1$. When the scalar product is understood, we write V for $(V, \langle -, - \rangle)$. We say that each vector $v \in V$ has a **causal character**, described as

- **timelike** if $\langle v, v \rangle < 0$;
- **null** or **lightlike** if $\langle v, v \rangle = 0$;
- or **spacelike** if $\langle v, v \rangle > 0$.

(The competing convention is to require the index to be $n - 1$, under which $\langle v, v \rangle > 0$ defines *timelike* and $\langle v, v \rangle < 0$ defines *spacelike*.)

By a *semi-Riemannian manifold* $(\mathbb{M}, g)$, we mean a Hausdorff topological space $\mathbb{M}$ endowed with a smooth, real differentiable structure and a metric tensor, that is, a smooth covariant field g of nondegenerate scalar products $T_p\mathbb{M} \times T_p\mathbb{M} \to \mathbb{R}$ of fixed index. For economy, when the metric is understood, we write $\mathbb{M}$ for $(\mathbb{M}, g)$. Throughout, symbols in the style $\mathbb{M}$ or $\mathbb{X}$ will denote semi-Riemannian manifolds and hypersurfaces in them, save for $\mathbb{R}$, $\mathbb{Q}$, and $\mathbb{C}$ which of course denote number fields. Of particular concern in the forthcoming development are *Lorentz manifolds*.

Definition 0.2. A **Lorentz manifold** $\mathbb{M}$ is a semi-Riemannian manifold whose metric tensor g endows each fibre $V_p = \pi^{-1}(p)$ of the tangent bundle $T\mathbb{M} \xrightarrow{\pi} \mathbb{M}$ with a scalar product g_p such that (V_p, g_p) is a Lorentz vector space. (A Lorentz manifold is also referred to as a **spacetime**.)

As its tangent spaces are Lorentz spaces, a Lorentz manifold is necessarily of dimension at least 2.

A function $h : \mathbb{M} \to \mathbb{R}$ may be restricted to a curve $\gamma : I \to \mathbb{M}$, where $I \subset \mathbb{R}$, by the composition $h \circ \gamma : I \to \mathbb{R}$. Abusing notation, we denote the value taken by this composition at $s \in I$ by $h(s)$. We denote the rate of change of h with respect to the parameter s interchangeably by dots and primes, that is,

$$\frac{d(h \circ \gamma)}{ds} \equiv \dot{h}(s) \equiv h'(s).$$

Where explicit use of the summation symbol $\sum$ becomes cumbersome, we use in its place the Einstein summation convention, in which an index appearing once in subscript and once in superscript is summed over the range $1, \ldots, n$ (where n is the dimension of the manifold on which the summed quantities are relevant). For economy, when the trigonometric functions $\sin\theta$ and $\cos\theta$ appear with high frequency we write S and C in lieu of them.

Finally, geometrized units will be used throughout. In particular, the speed of light c and the gravitational constant G are unity.

Chapter 1

Properties of Kerr–Newman–de Sitter spacetimes

1.1 The metric

A *Kerr-Newman-(anti) de Sitter spacetime* $\mathbb{K}$ is a four-dimensional Lorentz manifold whose metric tensor $g = g_{ab}dx^a \otimes dx^b$, when cast in local coordinates (t, r, θ, φ), has the components

$$g_{tt} = \frac{a^2 \sin^2\theta \Delta_\theta - \Delta_r}{\lambda^2 \rho^2} \tag{1.1}$$

$$g_{rr} = \frac{\rho^2}{\Delta_r} \tag{1.2}$$

$$g_{\theta\theta} = \frac{\rho^2}{\Delta_\theta} \tag{1.3}$$

$$g_{\varphi\varphi} = \frac{\sin^2\theta}{\lambda^2\rho^2}\left[(r^2+a^2)^2\Delta_\theta - a^2\sin^2\theta\Delta_r\right] \tag{1.4}$$

$$g_{t\varphi} = g_{\varphi t} = \frac{a\sin^2\theta}{\lambda^2\rho^2}\left[\Delta_r - (r^2+a^2)\Delta_\theta\right] \tag{1.5}$$

$$g_{ij} = g_{ji} = 0 \;\text{ for all other components.}$$

Within these components are the functions

$$\rho^2 = r^2 + a^2 \cos^2\theta \tag{1.6}$$

$$\Delta_r = \left(1 - \frac{1}{3}\Lambda r^2\right)\left(r^2 + a^2\right) - 2Mr + e^2 \tag{1.7}$$

$$\Delta_\theta = 1 + \frac{1}{3}\Lambda a^2 \cos^2\theta \tag{1.8}$$

$$\lambda = 1 + \frac{1}{3}\Lambda a^2. \tag{1.9}$$

The coordinates (t, r, θ, φ) are referred to as *Boyer-Lindquist coordinates*[1]. The time coordinate t and the radial coordinate r are allowed to range over $\mathbb{R}$, so long as the coordinate functions remain non-singular at the values they assume. The colatitude θ takes values in the range $[0, \pi]$ while the longitude φ takes values in $[0, 2\pi)$. Points in $\mathbb{K}$ with $\theta = 0$ or π are called *poles*, and the set of all poles in $\mathbb{K}$ is the *axis*, denoted $\mathbb{A}$. Physically, $\mathbb{A}$ is the axis about which the black hole rotates. As the coordinates consist of a spherical coordinate system (θ, φ) taken at various radii $r \in \mathbb{R}$ and at various coordinate times $t \in \mathbb{R}$, the Boyer-Lindquist charts induce on $\mathbb{K}$ the topology of $\mathbb{R}^2 \times S^2$. We refer to the functions Δ_r and Δ_θ, respectively, as the *radial horizon* and *colatitudinal horizon* functions. The coefficients of dr^2 and $d\theta^2$ in the line element are not well-behaved at the roots of Δ_r and Δ_θ. Furthermore, spherical coordinates (θ, φ) fail at the poles of the sphere, since the longitudes φ converge at the poles and become indeterminate. Therefore, Boyer-Lindquist coordinates are only valid away from roots of Δ_r and Δ_θ, and away from points with $\theta = 0$ or π.

Respectively, the real parameters M, a, and e are called the *mass*, *angular momentum per unit mass*, and *charge per unit mass* of the spacetime. Although Ma and Me are the respective total angular momentum and total charge, for convenience

[1] The metric, in the coordinates displayed here, may be obtained from solution class [A] of [2] by the transformation $\Lambda \mapsto -\Lambda$, $\mu \mapsto a\cos\theta$, $p \mapsto a^2$, $q \mapsto 0$, $h \mapsto 1 - a^2\Lambda/3$, $\chi \mapsto t/\lambda - a^2\Psi$, and $\Psi \mapsto \varphi/(a\lambda)$.

we refer to a and e simply as the "angular momentum" and "charge" of $\mathbb{K}$. Normally, we take M and a in $[0, \infty)$ and—neglecting the quantization of charge—e in $(-\infty, \infty)$. Like the charge, the real parameter Λ can be negative or positive, and is referred to as a *cosmological constant*. If this constant is positive, $\mathbb{K}$ is referred to as a Kerr-Newman-de Sitter spacetime; if it is negative, then $\mathbb{K}$ is a Kerr-Newman-*anti* de Sitter spacetime[2].

Whenever we write $\mathbb{K}$ to refer to spacetime, we invariantly take the metric to be in its most general form, with all four parameters a, e, M, Λ possibly nonzero. To emphasize a particular choice of the parameters, we sometimes write $\mathbb{K}(a, e, M, \Lambda)$ for $\mathbb{K}$. Whenever we omit a symbol from the parameter list, we understand that the corresponding parameter is set to zero. For instance, $\mathbb{K}(a, e, M)$ has $\Lambda = 0$. When all parameters are zero, in which case we are left with empty Minkowski spacetime, we denote the manifold by $\mathbb{K}(0)$. For reference, we collect the traditional names of the families corresponding to the possible parameter lists. (We use $\simeq$ to indicate when two spacetimes are in fact isometric.)

Definition 1.1. Families of $\mathbb{K}$ spacetimes:

- $\mathbb{K} = \mathbb{K}(a, e, M, \Lambda)$ — *Kerr-Newman-(anti) de Sitter*

- $\mathbb{K}(a, M, \Lambda)$ — *Kerr-(anti) de Sitter*

[2]The metric admits yet a fifth parameter, p, for the magnetic monopole moment of the spacetime; see [12]. However, we take $p = 0$ throughout.

- $\mathbb{K}(e, M, \Lambda)$ — *Reissner-Nordström-(anti) de Sitter*

- $\mathbb{K}(M, \Lambda)$ — *Schwarzschild-(anti) de Sitter*

- $\mathbb{K}(a, e, \Lambda) \simeq \mathbb{K}(a, \Lambda) \simeq \mathbb{K}(e, \Lambda) \simeq \mathbb{K}(\Lambda)$ — *(anti) de Sitter*

- $\mathbb{K}(a, e, M)$ — *Kerr-Newman*

- $\mathbb{K}(a, M)$ — *Kerr*

- $\mathbb{K}(e, M)$ — *Reissner-Nordström*

- $\mathbb{K}(M)$ — *Schwarzschild*

- $\mathbb{K}(a, e) \simeq \mathbb{K}(a) \simeq \mathbb{K}(e) \simeq \mathbb{K}(0)$ —*Minkowski.*

Apart from naming the spacetimes, the list implies that, whenever we let the mass of the spacetime tend to zero, the parameters a and e lose their physical and/or geometric significance. This fact is consistent with their interpretations as quantities *per unit mass.* We easily see, for instance, that if all the parameters in the components (1.1)-(1.5) vanish save for a, then we are left with

$$g_{tt} = -1, \quad g_{rr} = 1 - a^2 \sin^2\theta/(r^2 + a^2), \quad g_{\theta\theta} = r^2 + a^2 \cos^2\theta, \quad g_{\varphi\varphi} = (r^2 + a^2)\sin^2\theta,$$

with all other components (in particular $g_{t\varphi} = g_{\varphi t}$) vanishing. The four-dimensional Lorentz spacetime with metric $g_{ab}dx^a \otimes dx^b$ taking these components is the well-known Minkowski spacetime, cast in an oblate spheroidal coordinate system. Explicitly, these coordinates are obtained by applying the transformation $r = \sqrt{\overline{r}^2 + a^2}\sin\theta$, $z = \overline{r}\cos\theta$ to

$$\eta = -dt^2 + d\overline{r}^2 + \overline{r}^2 d\varphi^2 + dz^2, \tag{1.10}$$

which is the usual form of the Minkowski metric in cylindrical coordinates.

The various $\mathbb{K}$ spacetimes arise in the general theory of relativity as solutions to field equations for gravity coupled to electromagnetism. We define a one-form A that vanishes when the charge e of the spacetime tends to zero:

$$A = \frac{er}{\lambda\rho^2}\left[dt - a\sin^2\theta d\varphi\right]. \tag{1.11}$$

With g the metric reconstructed from the line element ds^2, and with A the choice of electromagnetic potential[3], the couple (g, A) is a solution of the source-free Einstein-Maxwell field equations with contribution from a nonzero cosmological constant Λ:

$$G_{ab} - 8\pi T_{ab} + \Lambda g_{ab} = 0. \tag{1.12}$$

In (1.12), the Einstein tensor G_{ab} is $R_{ab} - \frac{1}{2}Rg_{ab}$ while the stress-energy tensor T_{ab} is $\frac{1}{4\pi}\left(\sum_{c=1}^{4} F_{ac}F^{c}{}_{b} - \frac{Ig_{ab}}{4}\right)$. The symbols R_{ab} and R denote, respectively, the Ricci tensor and the scalar curvature, while $F_{ab} = 2(dA)_{ab}$ is the antisymmetric Maxwell field tensor and $I = \sum_{1\leq a,b\leq 4} F_{ab}F^{ab}$ is a scalar invariant.

[3]This potential is but one example of an entire family $A(\alpha)$ of one-forms with which g is a solution of the Einstein-Maxwell equations. The parameter α determines the "complexion" of the electromagnetic field, as Carter explains in [2]. Our choice of potential is equivalent to setting $\alpha = 0$.

We note that the cosmological constant Λ makes no entry into the differential forms satisfying Maxwell's equations; however, the rotation a does. This observation is consistent with the fact that the electromagnetic potential A, as defined above, is precisely the one coupled to the Kerr-Newman metric (in which $\Lambda = 0$ but $a \neq 0$).

So long as the rotation parameter a is nonzero, the Killing vector fields ∂_t and ∂_φ remain non-orthogonal, which is indicative of the loss of full spherical symmetry in favour of axial symmetry. In terms of isometries, while the map $t \mapsto -t,\ \varphi \mapsto -\varphi$ is an isometry of $\mathbb{K}$, the reversal of only one of the two signs is not. The physical interpretation is simple: when we run time backwards, the rotation reverses as well. When we let $a \to 0$, the component $g_{t\varphi}$ also $\to 0$, and so spherical symmetry is regained by the non-rotating sub-family $\mathbb{K}(e, M, \Lambda)$. In this case, the four maps $t \mapsto \pm t,\ \varphi \mapsto \pm\varphi$ are all spacetime isometries.

It is a simple observation that the Boyer-Lindquist components (1.1), (1.4), and (1.5) for the Killing fields satisfy the following identities.

Lemma 1.1. *Using the symbol S in lieu of $\sin\theta$, we have*

$$
\begin{aligned}
g_{\varphi\varphi} + aS^2 g_{t\varphi} &= \langle W, \partial_\varphi\rangle = \frac{(r^2+a^2)S^2\Delta_\theta}{\lambda^2} && (1.13)\\
g_{t\varphi} + aS^2 g_{tt} &= \langle W, \partial_t\rangle = -\frac{aS^2\Delta_\theta}{\lambda^2} && (1.14)\\
ag_{\varphi\varphi} + (r^2+a^2) g_{t\varphi} &= \langle V, \partial_\varphi\rangle = \frac{aS^2\Delta_r}{\lambda^2} && (1.15)\\
ag_{t\varphi} + (r^2+a^2) g_{tt} &= \langle V, \partial_t\rangle = -\frac{\Delta_r}{\lambda^2}, && (1.16)
\end{aligned}
$$

where $V := (r^2+a^2)\partial_t + a\partial\varphi$ and where $W := \partial_\varphi + aS^2\partial_t$.

As the vector fields V and W pervade the study of Kerr-type spacetimes as rotational versions of ∂_t and ∂_φ, we make a formal

Definition 1.2. The **canonical Kerr vector fields** ([10]) are

$$V = (r^2 + a^2)\partial_t + a\partial_\varphi \tag{1.17}$$

and

$$W = \partial_\varphi + aS^2\partial_t, \tag{1.18}$$

where S again stands in place of $\sin\theta$.

It is clear that $\langle \partial_r, V\rangle = \langle \partial_r, W\rangle = \langle \partial_\theta, V\rangle = \langle \partial_\theta, W\rangle = 0$. Moreover:

Lemma 1.2. *$\langle V, V\rangle = -\dfrac{\rho^2\Delta_r}{\lambda^2}$, $\langle W, W\rangle = \dfrac{\rho^2 S^2\Delta_\theta}{\lambda^2}$, and $\langle V, W\rangle = \langle W, V\rangle = 0$.*

Proof. By Lemma 1.1,

$$\begin{aligned}
\langle V, V\rangle &= (r^2 + a^2)\langle V, \partial_t\rangle + a\langle V, \partial_\varphi\rangle \\
&= -(r^2 + a^2)\frac{\Delta_r}{\lambda^2} + a\frac{(aS^2\Delta_r)}{\lambda^2} \\
&= -\frac{\rho^2\Delta_r}{\lambda^2} &(1.19)\\
\langle W, W\rangle &= \langle W, \partial_\varphi\rangle + aS^2\langle W, \partial_t\rangle \\
&= \frac{(r^2 + a^2)S^2\Delta_\theta}{\lambda^2} - aS^2\frac{(aS^2\Delta_\theta)}{\lambda^2} \\
&= \frac{\rho^2 S^2\Delta_\theta}{\lambda^2} &(1.20)\\
\langle V, W\rangle &= (r^2 + a^2)\langle \partial_t, W\rangle + a\langle \partial_\varphi, W\rangle \\
&= -(r^2 + a^2)\frac{aS^2\Delta_\theta}{\lambda^2} + a\frac{(r^2 + a^2)S^2\Delta_\theta}{\lambda^2} \\
&= 0. &(1.21)
\end{aligned}$$

□

With our verification of the orthogonality of V and W, we see that the set $\{\partial_r, \partial_\theta, V, W\}$ is an orthogonal basis for the fibres of the bundle $T(\mathbb{K})$. This fact will be put to use time and time again in chapters 2 and 3.

When the cosmological constant is zero, the Boyer-Lindquist components of the metric tend to Minkowskian components (in spherical coordinates) as $r \to 0$. For this reason, we say that spacetimes $\mathbb{K}(a, e, M)$ are *asymptotically flat.*

The more general spacetimes $\mathbb{K}(a, e, M, \Lambda)$ with $\Lambda \neq 0$ are not asymptotically flat, but rather are characterized by constant nonzero scalar curvature. To compute the trace of the Ricci tensor, we apply the contravariant metric $g^{-1} = g^{ab}\partial_a \otimes \partial_b$ to each side of (1.12). The result is

$$R - 2R + 4\Lambda \quad = \quad 8\pi T_a^a.$$

However, we have $8\pi T_a^a = 2\left(F_c^b F_b^c - I g^{ab} g_{ab}/4\right) = 0$, since $I := F_{cd}F^{cd}$. Therefore, we have that our solution is one of constant scalar curvature, namely

$$R \quad = \quad 4\Lambda,$$

When $\Lambda > 0$, the spacetime has constant positive curvature (de Sitter), and when $\Lambda < 0$, the spacetime has constant negative curvature (anti-de Sitter). A characteristic feature of Lorentz spacetimes with constant positive curvature is the existence of a unique hypersurface partitioning the spacetime into two *causal components*—that is, a boundary across which no massless or massive particle may travel ([8]). To discover where this feature arises in Kerr-Newman-de Sitter geometry, we must investigate the various regions of coordinate failure.

1.2 Singularities and horizons

Boyer-Lindquist coordinates fail in several regions of $\mathbb{R}^2 \times S^2$. Some of the failures are more serious than others. The spherical coordinates (θ, φ) on S^2 invariably fail at the poles, where the longitudinal curves φ converge. However, we will see in Chapter 2 that, while Boyer-Lindquist coordinates fail at the poles, the first-order differential equations for a particle's Boyer-Lindquist coordinates generally do not. For our purposes, the failure is harmless.

The most serious failure occurs at the *ring singularity*: the set of zeros of the function $\rho^2 = r^2 + a^2 \cos^2\theta$, which appears in the denominators of g_{tt}, $g_{t\varphi}$, and $g_{\varphi\varphi}$. In the general case $a > 0$, the function ρ^2 is zero if and only if both $r = 0$ and $\cos\theta = 0$, that is, if and only if $r = 0$ and $\theta = \pi/2$. The set Σ of all spacetime events (t, r, θ, φ) with $r = 0$ and $\theta = \pi/2$ is a circle in the equatorial plane, persisting through time: $\Sigma \cong \mathbb{R}^1 \times S^1$. Although the radius of this ring is $r = 0$, we must remember that, as r may assume negative values, the region of the equatorial plane $\mathbb{E} := \{(t, r, \pi/2, \varphi)\}$ that is bounded inside the ring is just as expansive as the region outside the ring. (It makes more sense to draw the "radius" of the ring, or that of any circle concentric with it, by taking the center at $r = -\infty$ rather than $r = 0$.)

As Σ is "a circle of infinite gravitational forces" [10], it is the site of the black hole contained in $\mathbb{K}$. The coordinate failure along Σ is most serious because Σ is the site of an actual *metric singularity*, rather than a singularity owing to poor coordinates. That the Riemann curvature of $\mathbb{K}$ is singular at Σ is observed directly in the Kerr curvature computations performed in the second chapter of [10]. We will not repeat these here. For comparison with the Reissner-Nordström-de Sitter family $\mathbb{K}(e, M, \Lambda)$ (which further reduces to the Schwarzschild geometry as $e, \Lambda \to 0$), the function ρ^2 is simply r^2, and so the singularity collapses to a single point of degenerate curvature. Therefore, the ring singularity owes its existence exclusively to the rotation of the black hole.

Further regions of coordinate failure are the *horizons*, which are the hypersurfaces $\{(t, r_0, \theta, \varphi)\}$ with r_0 a root of Δ_r, and $\{(t, r, \theta_0, \varphi)\}$ with θ_0 a root of Δ_θ. We denote the (set-theoretic) union of these hypersurfaces by $\mathbb{H}$. These failures vary in severity. Some horizons may be *event horizons*, which act as one-way membranes through which particles may pass but never return. Others may be *cosmological horizons*, across which no physical information may be sent or received. The latter type partitions an otherwise topologically path-connected spacetime into two spacetimes $\mathbb{S}_1$ and $\mathbb{S}_2$, causally-disconnected from one another. (Here, *causal disconnection* means there exists no smooth geodesic curve γ, with connected domain $I \subset \mathbb{R}$, such that $\gamma(s_1) \in \mathbb{S}_1$ for at least one number $s_1 \in I$, and $\gamma(s_2) \in \mathbb{S}_2$ for at least one number $s_2 \in I$.) Before we discuss the types of horizons manifesting themselves in $\mathbb{K}$, we determine conditions on the parameters a, e, M, Λ that control the number of horizons and their locations.

We may write $\mathbb{H}$ as the union of two subsets, $\mathbb{H}_r$ and $\mathbb{H}_\theta$, where $\mathbb{H}_r$ consists of those points whose r coordinates are roots of Δ_r, and where $\mathbb{H}_\theta$ consists of those points whose θ coordinates are roots of Δ_θ. (As we will see in the case of $\Lambda < 0$, these sets are not necessarily disjoint.) Due to the current opinions in cosmology cited in the introduction, we choose to investigate de Sitter spacetimes rather than anti-de Sitter ones. In choosing $\Lambda > 0$, we are eliminating the existence of θ-horizons, for

$$\Delta_\theta \;=\; 1 + \frac{\Lambda}{3} a^2 \cos^2\theta \;>\; 0.$$

Consequently, $\mathbb{H} = \mathbb{H}_r$ when $\Lambda > 0$.

The remaining possibilities for horizons are those corresponding to roots of the quartic polynomial

$$\Delta_r(r) = -Lr^4 + (1 - La^2)r^2 - 2Mr + a^2 + e^2 \tag{1.22}$$

with $L := \Lambda/3 > 0$. The polynomial $\Delta_r(r)$ has exactly four roots, though these roots are generally complex. If a complex root $a + ib$ has nonzero imaginary part b, then the root is one of a conjugate pair $(a \pm ib)$. Such roots may be discarded, as they do not manifest themselves physically. Subsequently, there are three possibilities: no real roots; two real roots and one complex conjugate pair; or four real roots.

In the case of no real roots, $\mathbb{H}$ is empty and the curvature singularity has no horizons to conceal it. To use the standard terminology, Σ is "naked." An observer in the vicinity of such a black hole would be able to see light entering the singularity itself, as there are no points between the observer and the singularity where the coordinates (his or her way of looking at the world) tend to fail. The imperative to study this case is subject to some debate. On the one hand, the "cosmic censorship" hypothesis of R. Penrose asserts that stellar collapse does not admit the formation of naked singularities ([4]). On the other hand, computer simulations by S. Shapiro and S. Teukolsky ([11]) have shown that, given assumptions on the compactness of the material undergoing gravitational collapse, physics may in fact admit naked singularities. Still, D. Christodoulou has rigorously proved that, while there exist initial data with which the spherically-symmetric Einstein equations for gravity coupled to a massless scalar field yield naked singularities (see [5]), he also shows that these singularities are unstable (see [6]).

In the two-parameter Kerr geometry $\mathbb{K}(a, M)$, the horizon function reduces to the quadratic polynomial

$$\delta(r) \quad = \quad r^2 - 2Mr + a^2,$$

which gives rise to a naked singularity whenever $a > M$. When $a < M$, two distinct horizons emerge. In the three-parameter geometry $\mathbb{K}(a, e, M)$, the horizon function changes only minimally to become

$$\delta(r) \quad = \quad r^2 - 2Mr + a^2 + e^2.$$

Nakedness occurs when $a^2+e^2 > M^2$, and distinct horizons appear when $a^2+e^2 < M^2$. We must note, however, that the nakedness condition $a^2 + e^2 > M^2$ is highly unphysical, given the expectedly large mass of a star collapsing to a black hole.

Considering the nonexistent observational evidence for naked black holes, and the instability proofs of Christodoulou (at least in the spherically-symmetric case), we keep our investigation conservative and consider only those Kerr-Newman-de Sitter spacetimes with real r-horizons. To ensure that the scenario with $\Lambda > 0$ is maximally different from that of the asymptotically-flat Kerr spacetimes, we seek the maximal number of horizons. For the quartic horizon function of the general spacetime $\mathbb{K} = \mathbb{K}(a, e, M, \Lambda)$, we make the following claim, which generalizes the conditions for the maximal number of horizons in the Kerr-Newman geometry.

Proposition 1.1. *There exists no anti-de Sitter Kerr spacetime* $(L < 0)$ *with more than two radial horizons. If* L *is positive,* $L^2 << 1$*,* $La^2 << 1$*, and* $a^2 + e^2 << M^2 << L^{-1}$*, then* Δ_r *has four distinct real roots.*

The proof is given in Appendix A.

Before leaving behind the anti-de Sitter spacetimes entirely, we note in passing that even in the case where no r-horizons are present, there always exists a pair of θ-horizons, as the solutions of the equation

$$1 - \frac{|\Lambda|}{3} a^2 \cos^2\theta \quad = \quad 0.$$

The horizons manifest themselves as three-dimensional hyperplanes placed symmetrically to the north and south of the equatorial plane $\mathbb{E}$. (The equatorial plane itself can never be a horizon.) In particular, when $\Lambda = -3/a^2$ these horizons occur at the poles $\theta = 0, \pi$, and so the entire axis $\mathbb{A}$ is the θ-horizon. When we introduce r-horizons (of which there can be at most two), the two types of horizons have nonempty intersections, which are homeomorphic to circles $S^1 \times \{t\}$. A detailed study of the implications of the existence of θ-horizons, and of their intersections with r-horizons, is left for future study.

1.3 Boyer-Lindquist blocks

As the de Sitter geometry—to which we now confine ourselves—is devoid of θ-horizons, the term "horizon" will henceforth refer only to radial ones. We note that the sufficient conditions presented in Proposition 1.1 for the existence of a maximal number of horizons are physically reasonable. As we expect the mass of the rotating star collapsing to the ring singularity to be quite large, the condition $M^2 << \Lambda^{-1}$ places a very small upper bound on the value of the cosmological constant, which is consistent with what experiments lead us to believe ([14]). Also, the condition $a^2 + e^2 << M^2$ mimics the analogous condition for a maximal number of horizons in the asymptotically-flat Kerr-Newman geometry.

Let us label the four distinct roots as r_{--}, r_-, r_+, and r_{++} in such a way that

$$r_{--} \quad < \quad r_- \quad < \quad r_+ \quad < \quad r_{++}.$$

We may characterize fully the signs of these roots.

Corollary 1.1. *Suppose that Δ_r has four distinct real roots. If $a^2 + e^2 \neq 0$, then they are all positive save for r_{--}, which is negative.*

Proof. When $a^2 + e^2 \neq 0$, no root can be zero. Since $L > 0$, we can rearrange $\Delta_r = 0$ into an equation in which each side is strictly positive for $r < 0$:

$$Lr^4 + La^2r^2 \quad = \quad r^2 - 2Mr + a^2 + e^2. \tag{1.23}$$

If we take the derivatives with respect to r of each side, we get

$$4Lr^3 + 2La^2r \quad = \quad 2r - 2M.$$

Each side of the differentiated equation is strictly negative on $r < 0$. Hence, each side of (1.23) is strictly decreasing on $r < 0$. For large negative r, the left side is greater than the right side. But while the left side of (1.23) is zero at $r = 0$, the right side takes the positive value $a^2 + e^2$ there. Hence, the two curves must cross at a unique point r less than zero. The crossing point is r_{--}. The remaining roots, distinct from r_{--}, are therefore positive.

□

It is well-known that while de Sitter manifolds are geodesically complete, there are points in them which cannot be connected by geodesics. The causal disconnection is such that there exist two maximal causal components, separated by a three-dimensional r-hypersurface ([3] and [8]). In other words, de Sitter spacetime possesses a (unique) cosmological horizon. The massless de Sitter manifold, $\mathbb{K}(\Lambda)$, has the horizon function

$$\Delta_r = -Lr^4 + r^2,$$

which has nonzero roots at $\pm\sqrt{1/L}$ and a coincident pair of roots at $r = 0$. The roots at $r = 0$ are artificial, however, since $\rho^2 = r^2$ for $\mathbb{K}(\Lambda)$, and hence $g_{rr} = r^2/(-Lr^4 + r^2) = (-Lr^2 + 1)^{-1}$. Only the roots $\pm\sqrt{1/L}$ persist. In can be shown that, in the maximally-extended de Sitter spacetime, the horizons associated to each of these roots are identified, and hence form the unique cosmological horizon (see [3]).

In Schwarzschild-de Sitter $\mathbb{K}(M, \Lambda)$, the horizon function is $-Lr^4 + r^2 - 2Mr$, and so the radial Boyer-Lindquist component is $g_{rr} = r/(-Lr^3 + r - 2M)$. In the event that the denominator has three roots, one is negative and two are positive. The smaller of the two positive roots is the generalization of the Schwarzschild radius from the asymptotically-flat Schwarzschild geometry. As is commonly known, the Schwarzschild radius ($= 2M$ when $\Lambda = 0$) is the event horizon, for geodesics may cross it only in the ingoing direction, towards the singularity at $r = 0$, and may never return. Null geodesic coordinate systems that pass smoothly over this horizon can be constructed (e.g. Kruskal coordinates, see [8]), and so the metric failure is artificial. As in de Sitter spacetime, the larger positive root identifies with the negative one to form the cosmological barrier. With respect our labelling of zeros of Δ_r, these roots corresponds to r_{--}, r_+, and r_{++}, with r_+ as the Schwarzschild radius. The horizon structure for Reissner-Nordström-de Sitter is no different (although the cosmological horizon applies more generally to also halt charged particle trajectories, which are extremal with respect to some action but not geodesic).

Introducing $a > 0$, we regain the third (and smallest) positive root r_-, assuming the parameter choices allow four real roots to exist. The Kerr-specific hypersurface $\{(t, r_-, \theta, \varphi)\}$ is a *Cauchy surface*, that is, a surface penetrable by a particle at most once, but not necessarily in an ingoing direction (see [8] and [10]). In [10], the Boyer-Lindquist coordinates are replaced by null geodesic *Kerr-star coordinates* that pass smoothly over both $r_\pm$, showing that these are coordinate singularities but not metric singularities. In concordance with the non-rotating case, the r_+ hypersurface supplies the event horizon of the rotating black hole, while the identified roots r_{--} and r_{++} supply the cosmological horizon. When r_{--} and r_{++} are left unidentified, we can imagine the r_{++} surface dividing $\mathbb{K}(a, e, M, \Lambda)$ roughly into a "Kerr" half (for $r < r_{++}$) and a "de Sitter" half (for $r > r_{++}$).

The following definitions (which generalize those made in [10]) help us to classify the location of a spacetime event relative to the horizons.

Definition 1.3. The set of all spacetime events (t, r, φ, θ) with

- $r > r_{++}$ is the *de Sitter block*;
- $r_+ < r < r_{++}$ is the *Boyer-Lindquist block I*;
- $r_- < r < r_+$ is the *Boyer-Lindquist block II*;
- $r < r_-$ and $r \neq r_{--}$ is the *Boyer-Lindquist block III.*

The four blocks are disjoint, and no horizon is contained in a block. By definition, block III skips over the negative horizon, but includes the curvature singularity Σ.

As r_{--} and r_{++} are identified in the maximally-extended spacetime, the region of block III with $r < r_{--}$ is actually contained in block I, while the de Sitter block is contained in the region of block III with $r > r_{--}$. In this way, referring to the blocks collectively as the *Boyer-Lindquist blocks* does not exclude the de Sitter block. For convenience, we will often write **dS** for the de Sitter block, and **I**, **II**, and **III** for the others.

In Chapter 3 especially, the definitions of the blocks will help us to organize our study of particle orbits.

Chapter 2

Integrability of the equations of motion

In the relativistic theory of mechanics, particles moving under the gravitational and electromagnetic influence of a spacetime are constrained to nonspacelike curves minimizing the action of the Lagrangian determined by the electromagnetic and metric fields. We call these minimizers *spacetime orbits* or *trajectories*. Referring to such a curve as timelike, null, or spacelike is well-defined, since as we shall see, the tangent vectors along an action-minimizer are of invariant causal character. (Therefore, a spacelike minimizer is wholly unphysical, for all of its velocity vectors lie outside the nullcones of the tangent spaces to which they belong.)

The Lagrangian for this variational problem (see [12]) is given by

$$\mathbf{L}_q(s, x^j, \dot{x}^j) \;=\; \frac{1}{2} g_{ab}\dot{x}^a\dot{x}^b - qA_a\dot{x}^a, \tag{2.1}$$

where A is a choice of potential whereby (g, A) is a solution of the source-free Einstein-Maxwell equations (1.12). Such a choice is made in (1.11). The number q is a real parameter fixing the charge of the geodesics. (We must not confuse this number with the charge e per unit mass of the spacetime itself.) For a particle of charge q, we obtain its possible trajectories as solutions of the second-order system of Euler-Lagrange equations arising from (2.1). In the special case $q = 0$ corresponding to photon and neutral-particle orbits, the Euler-Lagrange equations are the familiar geodesic equations

$$\left(x^k\right)'' + \Gamma^k_{ij} \left(x^i\right)' \left(x^j\right)' = 0, \tag{2.2}$$

where the derivatives $' \equiv \dfrac{\partial}{\partial s}$ are taken with respect to an affine parameter s, and where the Γ^k_{ij} are the Christoffel symbols coupled to the metric. We are right, however, to be wary of the seemingly simplistic appearance of these equations. They are complicated enough to rarely yield to exact solubility, and this poses an obstacle to our study of test matter dynamics.

Alternatively, it is possible to construct a system of first-order differential equations amongst whose families of solutions we find all those solutions of the second-order geodesic equations. Such a system arises whenever we have access to sufficiently-many so-called *first integrals*—quantities that are necessarily conserved along action-minimizing curves.

2.1 First integrals

Consider an action-extremizing problem on a manifold $\mathbb{M}$, that is, the problem of finding extremal curves $(\gamma(s), \gamma'(s)) \subset T(\mathbb{M})$ of the action

$$\mathbf{A}[\gamma(s)] = \int_{s_0}^{s_1} \mathbf{L}(s, \gamma(s), \gamma'(s)) ds$$

for a given Lagrangian $\mathbf{L}$.

Definition. A *first integral* for the solutions of an action-extremizing problem on a manifold $\mathbb{M}$ is a smooth function $f : T(\mathbb{M}) \to \mathbb{R}$ such that for each solution $\gamma : I_\gamma \to \mathbb{M}$ ($I_\gamma \subset \mathbb{R}$), the function $s \mapsto f(\gamma(s), \gamma'(s))$ is a constant, denoted by f_γ or $f(\gamma)$.

The smooth assignment of a constant $k = f(\gamma)$ to each solution γ determines a first integral $T(\mathbb{M}) \to \mathbb{R}$, $v \mapsto f(\gamma_v)$, where γ_v is the extremizing curve with tangent vector $v \in T_{\gamma(0)}\mathbb{M}$. In this way, a first integral f becomes a first-order differential equation for the coordinates of γ, with additive constant $k = f(\gamma)$.

We consider coordinates $(x^1, \dots, x^n, x^{1'}, \dots, x^{n'})$ on the tangent bundle $T(\mathbb{M})$ in which the x^k are functions of the parameter s and the $x^{k'}$ are the derivatives of the x^k with respect to s. We introduce canonical coordinates $(x^1, \dots, x^n, p_1, \dots, p_n)$ on the cotangent bundle $T^*(\mathbb{M})$ by means of a Legendre transformation, defined fibre-wise by

$$x^{k'} \quad \mapsto \quad \frac{\partial \mathbf{L}}{\partial x^{k'}} \quad =: \quad p_k(x^{k'}). \tag{2.3}$$

In this way, the *conjugate momenta* p_k in $T^*(\mathbb{M})$ depend upon the *generalized velocities* $x^{k'}$ in $T(\mathbb{M})$. The inverse Legendre transformation is

$$p_k \quad \mapsto \quad \frac{\partial \mathbf{H}}{\partial p_k} \quad = \quad x^{k'}(p_k),$$

where $\mathbf{H} : T^*(\mathbb{M}) \to \mathbb{R}$ is the Hamiltonian derived from the Lagrangian. From these transformations, we obtain Hamilton's equations: $x^{k'} = \partial\mathbf{H}/\partial p_k$ and $p_k' = -\partial\mathbf{H}/\partial x^k$.

If a smooth, real-valued function f is defined on $T^*(\mathbb{M})$, then we may view it as a function of the $x^{k'}$ by the dependence in (2.3). Subsequently, we may ask how f varies with the parameter s of an extremizing curve. If $\dfrac{d}{ds} f(x^k, p_k) = 0$ along each curve, then f is a first integral for the variational problem.

Exploiting the cotangent bundle representation of first integrals, we have at our disposal an extremely useful tool for identifying first integrals:

Lemma 2.1. *A function $f : T^*(\mathbb{M}) \to \mathbb{R}$ is a first integral if and only if it is in involution with the Hamiltonian* $\mathbf{H}$*, that is, if and only if H and f Poisson commute.*

Proof. Applying Hamilton's equations $x^{k\prime} = \partial \mathbf{H}/\partial p_k$ and $p_k' = -\partial \mathbf{H}/\partial x^k$, we have

$$\frac{d}{ds} f(x^j, p_j) = \sum_{k=1}^{n} \left(\frac{\partial f}{\partial p_k} \frac{dp_k}{ds} + \frac{\partial f}{\partial x^k} \frac{dx^k}{ds} \right) = \sum_{k=1}^{n} \left(\frac{\partial f}{\partial x^k} \frac{\partial \mathbf{H}}{\partial p_k} - \frac{\partial f}{\partial p_k} \frac{\partial \mathbf{H}}{\partial x^k} \right) = \{f, \mathbf{H}\}.$$

□

An immediate corollary of this lemma is that, since the Poisson bracket $\{-,-\}$ is skew-symmetric, $\{\mathbf{H}, \mathbf{H}\} = 0$ and so the Hamiltonian itself is a first integral. While we may proceed to collect further first integrals, we should discuss the actual advantages of possessing them in the first place. To this end, we cite Liouville's theorem as found in [1]:

Theorem 2.1. (Liouville-Arnol'd) *Suppose that $f_1, \ldots, f_n$ are first integrals for a Hamiltonian $\mathbf{H} : T^*(\mathbb{M}) \to \mathbb{R}$, where $\dim T^*(\mathbb{M}) = 2n$. Consider a level set $\mathbb{M}_c = \{x \in T^*(\mathbb{M}) \, : \, f_i(x) = c_i, \, i = 1, \ldots n\}$, such that $df_1 \wedge \cdots \wedge df_n \neq 0$ at every point of $\mathbb{M}_c$ (so that the f_i are functionally independent on $\mathbb{M}_c$). If the f_i pairwise Poisson commute, then the Hamiltonian flow on $\mathbb{M}_c$ is completely integrable (by quadratures).*

The theorem can be interpreted as saying that, for a $2n$-dimensional Hamiltonian system with n Poisson-commuting first integrals, those solutions $\gamma(s)$ whose constants $f_1(\gamma) = c_1, \ldots, f_n(\gamma) = c_n$ determine a level set $\mathbb{M}_c$ on which the f_i are functionally independent are solutions of the first-order system constructed from the first integrals. (In other words, these solutions are not lost in considering the first-order system in lieu of the higher-order Euler-Lagrange system.)

We apply the integrability theory for Hamiltonian systems to the problem of determining particle orbits in the vicinity of a Kerr-Newman-de Sitter singularity. Stated beforehand in (2.1), the Lagrangian for particle orbits in $\mathbb{K}$ is

$$\mathbf{L}_q(s, x^j, \dot{x}^j) \;=\; \frac{1}{2} g_{ab}\dot{x}^a\dot{x}^b - qA_a\dot{x}^a, \tag{2.4}$$

which we differentiate by $\dot{x}^a$ to obtain the conjugate momenta $p_a = g_{ab}\dot{x}^b - qA_a$. (We recall the convention that derivatives with respect to the parameter s are denoted equivalently by dots and primes.) Furthermore, the Hamiltonian takes the form

$$\mathbf{H}(x^j, p_j) \;=\; \frac{1}{2} g^{ab}(p_a + qA_a)(p_b + qA_b) \tag{2.5}$$

in terms of the components of the contravariant metric $g^{-1} = g^{ab}\partial_a \otimes \partial_b$ and of the potential one-form $A = A_a dx^a$. When we transform the momenta p_a to functions $g_{ab}\dot{x}^b - qA_a$ on the tangent spaces, we are left with

$$\mathbf{H} \;=\; \frac{1}{2} g^{ab} g_{ab} g_{ab} \dot{x}^a\dot{x}^b \;=\; \frac{1}{2}\delta^a_a g_{ab}\dot{x}^a\dot{x}^b \;=\; \frac{1}{2} g_{ab}\dot{x}^a\dot{x}^b \;=\; \frac{1}{2}\langle \gamma', \gamma' \rangle .$$

Thus, the scalar product of a minimizer's tangent vector with itself remains constant along the curve, for $\langle \gamma', \gamma' \rangle$ and the first integral $\mathbf{H}$ are scalar multiples of each other. In particular, a minimizer has a well-defined causal character. If γ' is timelike (or null or spacelike) at a point then it is timelike (or null or spacelike) at every point $p = \gamma(s)$. We refer to the first integral $\mathfrak{q} := \langle \gamma', \gamma' \rangle = 2\mathbf{H}$ as the *rest mass* of the particle travelling along the curve. The constancy of $\mathfrak{q}$ therefore represents the conservation of the particle's mass. (We set $m^2 := -\mathfrak{q}$ so that timelike curves correspond to particles of real mass, lightlike curves to massless particles, and spacelike curves to particles of imaginary mass.)

The existence of Killing vector fields for $\mathbb{K}$ gives rise to two other independent first integrals. The Poisson bracket of a conjugate momentum p_k with the Hamiltonian is

$$\{p_k, \mathbf{H}\} \;=\; \frac{\partial \mathbf{H}}{\partial x^k}.$$

Since the components of both the metric and the potential one-form are independent of the coordinate functions t and φ, we have $\dfrac{\partial \mathbf{H}}{\partial t} = \dfrac{\partial \mathbf{H}}{\partial \varphi} = 0$. It follows from Lemma 2.1 that the momenta p_t and p_φ are first integrals for the minimizers of the action $\displaystyle\int \mathbf{L}\, ds$. Furthermore, as momenta Poisson-commute amongst themselves, and since they are functionally independent as coordinates on $T^*(\mathbb{K})$, p_t and p_φ are independent in the sense required by Liouville's theorem. We call the first integral $p_t(\gamma)$ the *energy* of the particle travelling along γ, and write $\mathcal{E} := -p_t$. We call $p_\varphi(\gamma)$ the *angular momentum* of the particle—which is not to be confused with the angular momentum a of the spacetime itself. We write $\mathcal{L} := p_\varphi$. The constancy of these first integrals reflects the conservation of energy and of (axial) angular momentum of a particle moving under the gravitational and electromagnetic influence of spacetime.

Thus far, we have a set $\{\mathfrak{q} = 2\mathbf{H},\ \mathcal{E} = -p_t,\ \mathcal{L} = p_\varphi\}$ of three first integrals in involution. To achieve the Liouville-Arnol'd integrability of the Hamiltonian system, we need to identify at least one more first integral, in involution with the others. The existence of this fourth first integral, referred to in the literature ([10], [12], etc.) as *Carter's constant*, is not so obvious. We can, however, motivate its discovery by expanding the first integral $\mathfrak{q} = \langle \gamma', \gamma' \rangle$ and writing it in terms of the canonical vector fields V and W.

2.2 Carter's constant and first-order equations for r and θ

Our quest for a fourth first integral begins by writing the invariant mass $\mathfrak{q} = \langle \gamma', \gamma' \rangle$ in terms of the Boyer-Lindquist functions $r(s)$ and $\theta(s)$, and their first derivatives. First, we examine the linear combination

$$\Psi \quad := \quad \frac{\langle \gamma', V \rangle}{\langle V, V \rangle} V + \frac{\langle \gamma', W \rangle}{\langle W, W \rangle} W. \tag{2.6}$$

We use the identities in lemmas 1.1 and 1.2, as well as our knowledge that ∂_r, ∂_θ, V, and W are pairwise orthogonal, to expand (2.6) as

$$\begin{aligned}
\Psi &= \frac{t' \langle \partial_t, V \rangle + \varphi' \langle \partial_\varphi, V \rangle}{-\rho^2 \Delta_r / \lambda^2} V + \frac{t' \langle \partial_t, W \rangle + \varphi' \langle \partial_\varphi, W \rangle}{\rho^2 S^2 \Delta_\theta / \lambda^2} W \\
&= \frac{-t' \Delta_r / \lambda^2 + \varphi' a S^2 \Delta_r / \lambda^2}{-\rho^2 \Delta_r / \lambda^2} V + \frac{-t' a S^2 \Delta_\theta / \lambda^2 + \varphi' (r^2 + a^2) S^2 \Delta_\theta / \lambda^2}{\rho^2 S^2 \Delta_\theta / \lambda^2} W \\
&= \frac{-t' \Delta_r / \lambda^2 + \varphi' a S^2 \Delta_r / \lambda^2}{-\rho^2 \Delta_r / \lambda^2} V + \frac{-t' a S^2 \Delta_\theta / \lambda^2 + \varphi' (r^2 + a^2) S^2 \Delta_\theta / \lambda^2}{\rho^2 S^2 \Delta_\theta / \lambda^2} W \\
&= \frac{t' - a S^2 \varphi'}{\rho^2} V + \frac{(r^2 + a^2) \varphi' - a t'}{\rho^2} W \\
&= \frac{(V - aW) t' + ((r^2 + a^2) W - a S^2 V) \varphi'}{\rho^2}.
\end{aligned}$$

But $V - aW = \rho^2 \partial_t$ and $(r^2 + a^2) W - a S^2 V = \rho^2 \partial \varphi$, and so

$$\frac{\langle \gamma', V \rangle}{\langle V, V \rangle} V + \frac{\langle \gamma', W \rangle}{\langle W, W \rangle} W \quad = \quad t' \partial_t + \varphi' \partial_\varphi.$$

Therefore, the velocity vector field of a curve, in Boyer-Lindquist coordinates, is

$$\gamma' = r' \partial_r + \theta' \partial_\theta + \frac{\langle \gamma', V \rangle}{\langle V, V \rangle} V + \frac{\langle \gamma', W \rangle}{\langle W, W \rangle} W.$$

It follows that

$$\langle \gamma', \gamma' \rangle \quad = \quad (r')^2 g_{rr} + (\theta')^2 g_{\theta\theta} + \frac{\langle \gamma', V \rangle^2}{\langle V, V \rangle^2} \langle V, V \rangle + \frac{\langle \gamma', W \rangle^2}{\langle W, W \rangle^2} \langle W, W \rangle.$$

Inserting the definitions of g_{rr} and $g_{\theta\theta}$, we have

$$\begin{aligned} \langle \gamma', \gamma' \rangle &= (r')^2 \frac{\rho^2}{\Delta_r} + (\theta')^2 \frac{\rho^2}{\Delta_\theta} + \frac{\langle \gamma', V \rangle^2}{\langle V, V \rangle} + \frac{\langle \gamma', W \rangle^2}{\langle W, W \rangle} \\ &= (r')^2 \frac{\rho^2}{\Delta_r} + (\theta')^2 \frac{\rho^2}{\Delta_\theta} - \frac{\langle \gamma', V \rangle^2}{\rho^2 \Delta_r / \lambda^2} + \frac{\langle \gamma', W \rangle^2}{\rho^2 S^2 \Delta_\theta / \lambda^2}, \end{aligned}$$

and so we may write

$$\mathfrak{q}\,\rho^2 = \frac{\rho^4 r'^2 - \lambda^2 \langle \gamma', V \rangle^2}{\Delta_r} + \frac{\rho^4 \theta'^2 + \lambda^2 \langle \gamma', W \rangle^2 / S^2}{\Delta_\theta}. \tag{2.7}$$

As $\mathfrak{q}$ assigns a (finite) constant to each action-minimizing curve, and since $\rho^2 = r^2 + a^2 \cos^2 \theta$ is defined everywhere in spacetime, it follows that the right-hand side of (2.7) must also be defined everywhere in $\mathbb{K}$, even on horizons and the axis. Of great interest to us is the minimal coupling of r and θ in this equation. Upon our writing out the left-hand side as $\mathfrak{q}\,r^2 + \mathfrak{q}\,a^2 C^2$, a slight rearrangement of (2.7) gives

$$\mathfrak{q}\,r^2 - \frac{\rho^4 r'^2 - \lambda^2 \langle \gamma', V \rangle^2}{\Delta_r} = -\mathfrak{q}\,a^2 C^2 + \frac{\rho^4 \theta'^2 + \lambda^2 \langle \gamma', W \rangle^2 / S^2}{\Delta_\theta}. \tag{2.8}$$

On the left-hand side, Δ_r is a function only of r alone, whereas on the right-hand side, Δ_θ, S^2, and C^2 are functions of θ alone. (We remind ourselves that the symbol λ, which appears on both sides, is just the positive constant $1 + 1/3\Lambda a^2$.) If (2.8) were fully separated, with the left side depending only on r and the right side only on θ, then each side would necessarily be constant. As was shown by B. Carter ([2]), this is in fact the case.

Theorem 2.2. (Carter) *For each $\mathbb{K}$-curve γ minimizing the action of* (2.1)*, there exists a constant $\mathcal{K} = \mathcal{K}_\gamma$ such that, if $\mathfrak{q}$ is the rest mass of the curve, then*

$$\mathcal{K} = \mathfrak{q}\,r^2 - \frac{\rho^4 r'^2 - \lambda^2 \langle \gamma', V \rangle^2}{\Delta_r} = -\mathfrak{q}\,a^2 C^2 + \frac{\rho^4 \theta'^2 + \lambda^2 \langle \gamma', W \rangle^2 / S^2}{\Delta_\theta}. \tag{2.9}$$

For the case of the electrically-neutral, asymptotically-flat Kerr spacetimes $\mathbb{K}(a, M)$, a surprisingly simple proof of this fact is due to B. O'Neill ([10]). In this case, $A_t = A_\varphi = 0$ and the Lagrangian becomes

$$\mathbf{L}(s, x^j, \dot{x}^j) \;=\; \frac{1}{2}\left(\lim_{e,\Lambda\to 0} g_{ab}\right) \dot{x}^a\, \dot{x}^b.$$

Applying the argument of O'Neill, we may integrate directly the corresponding Euler-Lagrange equations for θ' and r' to find $\mathcal{K}$ as a constant of integration. This method does not generalize to $\mathbb{K}(a, e, \Lambda, M)$ with nonzero charge e (and q) and nonzero cosmological constant Λ. Instead, we determine $\mathcal{K}$ by studying the Hamiltonian

$$\mathbf{H}(x^j, p_j) \;=\; \frac{1}{2} g^{ab}(p_a + qA_a)(p_b + qA_b).$$

We suppose for the moment that the Hamiltonian can be partially separated in the following manner:

$$\mathbf{H} \;=\; \frac{1}{2}\frac{H_r + H_\theta}{U_r + U_\theta}, \tag{2.10}$$

where U_r and U_θ are single-variable functions of r and θ, respectively, and where

- H_r is independent of p_θ and of all Boyer-Lindquist coordinates save for r;
- H_θ is independent of p_r and of all Boyer-Lindquist coordinates save for θ.

Lemma 2.2. *If the Hamiltonian* $\boldsymbol{H}$ *partially separates in the manner described above, then the function* $\mathcal{K} : T^*(\mathbb{K}) \to \mathbb{R}$ *defined by*

$$\mathcal{K} \;=\; \frac{U_r H_\theta - U_\theta H_r}{U_r + U_\theta} \tag{2.11}$$

is in involution with $\boldsymbol{H}$.

The proof of this lemma can be found in [3].

Assuming that $\mathbf{H}$ separates as desired, we would have $\mathcal{K}$ as a fourth first integral for the geodesics in $\mathbb{K}$. We now complete the proof of Carter's theorem by showing that $\mathbf{H}$ does indeed separate as desired.

Proof. **(Theorem 2.2)**

To determine whether the Hamiltonian admits the special form (2.10), we must compute the contravariant metric $g^{-1} : T^*(\mathbb{K})^2 \to \mathcal{C}^\infty(\mathbb{K})$. As the matrix for g, in Boyer-Lindquist coordinates, is of the form

$$g_{ab} = \begin{bmatrix} \alpha & 0 & 0 & 0 \\ 0 & \beta & 0 & 0 \\ 0 & 0 & \eta & \sigma \\ 0 & 0 & \sigma & \psi \end{bmatrix},$$

it has an inverse of the equally sparse block matrix form

$$g^{ab} = \begin{bmatrix} 1/\alpha & 0 & 0 & 0 \\ 0 & 1/\beta & 0 & 0 \\ 0 & 0 & \psi/d & -\sigma/d \\ 0 & 0 & -\sigma/d & \eta/d \end{bmatrix},$$

where $d := \psi\eta - \sigma^2$. As $\eta = g_{tt}$, $\psi = g_{\varphi\varphi}$ and $\sigma = g_{t\varphi}$, the determinant d can be readily calculated and simplified as

$$d = -\frac{\Delta_r \Delta_\theta S^2}{\lambda^4}, \tag{2.12}$$

which is only zero on horizons or poles (where Boyer-Lindquist coordinates fail in any case).

Referring to definition (1.11) in the introduction, we recall the particular choice of potential with which g solves the source-free Einstein-Maxwell equations:

$$A = \frac{er}{\lambda\rho^2}\left[dt - aS^2 d\varphi\right].$$

Choosing $\alpha = g_{rr}$, $\beta = g_{\theta\theta}$, and η, ψ, and σ as above, we may write down the Hamiltonian (in terms of the covariant metric components) as

$$\mathbf{H} = \frac{1}{2}\left[\frac{p_r^2}{g_{rr}} + \frac{p_\theta^2}{g_{\theta\theta}} + \frac{g_{\varphi\varphi}(p_t + qA_t)^2}{d} + \frac{g_{tt}(p_\varphi + qA_\varphi)^2}{d}\right] - \frac{g_{t\varphi}(p_t + qA_t)(p_\varphi + qA_\varphi)}{d}.$$

Substituting A_t, A_φ, and (1.1)-(1.5) into this expression, we obtain after simplification

$$\begin{aligned} 2\rho^2\mathbf{H} &= \Delta_r p_r^2 + \Delta_\theta p_\theta^2 \\ &\quad + \lambda^2\left(\frac{a^2S^2}{\Delta_\theta} - \frac{(r^2+a^2)^2}{\Delta_r}\right)\left(p_t + q\frac{er}{\lambda\rho^2}\right)^2 \\ &\quad + \lambda^2\left(\frac{1}{S^2\Delta_\theta} - \frac{a^2}{\Delta_r}\right)\left(p_\varphi - q\frac{eraS^2}{\lambda\rho^2}\right)^2 \\ &\quad + 2\lambda^2\left(\frac{a}{\Delta_\theta} - \frac{a(r^2+a^2)}{\Delta_r}\right)\left(p_t + q\frac{er}{\lambda\rho^2}\right)\left(p_\varphi - q\frac{eraS^2}{\lambda\rho^2}\right). \end{aligned} \tag{2.13}$$

Inspecting the terms in (2.13), we find that the desired near-separability comes to fruition as

$$H_r(r, p_r, p_t, p_\varphi) = \Delta_r p_r^2 - \frac{\lambda^2}{\Delta_r}\left[(r^2+a^2)p_t + ap_\varphi + \frac{qer}{\lambda}\right]^2 \tag{2.14}$$

$$H_\theta(\theta, p_\theta, p_t, p_\varphi) = \Delta_\theta p_\theta^2 + \frac{\lambda^2}{\Delta_\theta S^2}\left[aS^2 p_t + p_\varphi\right]^2 \tag{2.15}$$

$$U_r(r) = r^2 \tag{2.16}$$

$$U_\theta(\theta) = a^2C^2, \tag{2.17}$$

so that $U_r + U_\theta = \rho^2$ and $H_r + H_u = 2\rho^2\mathbf{H}$. Subsequently, the function

$$\mathcal{K} = \frac{U_r H_\theta - U_\theta H_r}{U_r + U_\theta}$$

is a constant of motion, as per Lemma 2.2. We may rewrite this expression as

$$\begin{aligned}\mathcal{K} &= \frac{U_r H_\theta - U_\theta H_r + (U_r H_r - U_r H_r)}{U_r + U_\theta} \\ &= \frac{U_r(H_r + H_\theta) - H_r(U_r + U_\theta)}{U_r + U_\theta} \\ &= U_r(2\mathbf{H}) - H_r \\ &= \mathfrak{q}\, U_r - H_r. \end{aligned} \tag{2.18}$$

With the definition

$$\mathcal{P}(r) \;:=\; -\left((r^2 + a^2)p_t + ap_\varphi + \frac{qer}{\lambda}\right), \tag{2.19}$$

we see from (2.18) and (2.14) that

$$\mathcal{K} \;=\; \mathfrak{q}\, r^2 - \Delta_r p_r^2 + \frac{\lambda^2 \mathcal{P}^2}{\Delta_r}.$$

Transforming to tangent space coordinates, that is, replacing p_r with $g_{rr}r' - qA_r = \frac{\rho^2}{\Delta_r} r'$, we have

$$\mathcal{K} \;=\; \mathfrak{q}\, r^2 - \frac{\rho^4 r'^2 - \lambda^2 \mathcal{P}^2}{\Delta_r}.$$

Now, we examine $-\mathcal{P}(r)$ in the tangent space coordinates:

$$\begin{aligned} -\mathcal{P} &= (r^2 + a^2)(g_{tt}t' + g_{t\varphi}\varphi' - qA_t) + a(g_{\varphi\varphi}\varphi' + g_{\varphi t}t' - qA_\varphi) + \frac{qer}{\lambda} \\ &= \left\langle t'\partial_t + \varphi'\partial_\varphi, (r^2 + a^2)\partial_t \right\rangle + \langle t'\partial_t + \varphi'\partial_\varphi, a\partial_\varphi \rangle + q\left(-(r^2 + a^2)A_t - aA_\varphi + \frac{er}{\lambda}\right) \\ &= \langle t'\partial_t + \varphi'\partial_\varphi, V \rangle + q\left(-(r^2 + a^2)\frac{er}{\lambda\rho^2} + a^2 S^2 \frac{er}{\lambda\rho^2} + \frac{er}{\lambda}\right) \\ &= \langle t'\partial_t + \varphi'\partial_\varphi, V \rangle + q\left(-\frac{er}{\lambda} + \frac{er}{\lambda}\right), \end{aligned}$$

so that any terms associated with the electromagnetic potential vanish. Since V, ∂_r, and ∂_θ are mutually orthogonal (with respect to the metric), we may write

$$-\mathcal{P} \;=\; \langle r'\partial_r + \theta'\partial_\theta + t'\partial_t + \varphi'\partial_\varphi, V \rangle \;=\; \langle \gamma', V \rangle\,.$$

We conclude that

$$\mathcal{K} \;=\; \mathfrak{q}\, r^2 - \frac{\rho^4 r'^2 - \lambda^2 \langle \gamma', V \rangle^2}{\Delta_r},$$

as desired. Alternatively, we define the θ-dependent function

$$\mathcal{D}(\theta) \;:=\; p_\varphi + p_t a S^2. \tag{2.20}$$

Since $\mathcal{D}(\theta) = \langle \gamma', W \rangle$, we may use $\mathcal{K} = H_\theta - U_\theta \mathfrak{q}$ to write

$$\mathcal{K} \;=\; -\mathfrak{q}\, a^2 C^2 + \frac{\rho^4 \theta'^2 + \lambda^2 \langle \gamma', W \rangle^2 / S^2}{\Delta_\theta},$$

thereby completing the proof.

□

Since $\mathcal{K}$ is a function neither of coordinate time t nor of the Boyer-Lindquist longitude φ, we see that it Poisson commutes with $\mathcal{E} = -p_t$ and $\mathcal{L} = p_\varphi$ for the same reason that $\mathbf{H}$ commutes with them. It follows that $\{\mathfrak{q}, \mathcal{E}, \mathcal{L}, \mathcal{K}\}$ is a complete set of four first integrals in mutual involution. Upon obtaining first-order equations from them, we will remark upon their functional independence (on level sets), as required by Theorem 2.1.

Instead of writing $\mathcal{K} = \mathfrak{q}\, r^2 - \dfrac{\rho^4 r'^2 - \lambda^2 \langle \gamma', V \rangle^2}{\Delta_r}$, we prefer to write

$$\mathcal{K} \;=\; \mathfrak{q}\, r^2 - \frac{\rho^4 r'^2 - \lambda^2 \mathcal{P}^2}{\Delta_r}, \tag{2.21}$$

because the function $\mathcal{P}(r) = (r^2 + a^2)\mathcal{E} - a\mathcal{L} - qer/\lambda$ involves the energy and the angular momentum of the particle, and so using equation (2.21), we may calculate $\mathcal{K}$ in terms of the other first integrals $\mathcal{E}$, $\mathcal{L}$, and $\mathfrak{q}$. Similarly, we write

$$\mathcal{K} \;=\; -\mathfrak{q}\, a^2 C^2 + \frac{\rho^4 \theta'^2 + \lambda^2 \mathcal{D}^2 / S^2}{\Delta_\theta} \tag{2.22}$$

for $\mathcal{D}(\theta) = \mathcal{L} - \mathcal{E} a S^2$. A slight rearrangement of these equations yields the following

Corollary 2.1. *Coupled to each particle trajectory* γ *in Kerr-Newman-de Sitter spacetime* $\mathbb{K}$ *is a constant* $\mathcal{K} = \mathcal{K}_\gamma$ *which, together with the coordinates* r *and* θ *of* γ*, satisfies the equations*

$$R(r) \;:=\; \rho^4 r'^2 \;=\; \Delta_r\left(\mathfrak{q}\, r^2 - \mathcal{K}\right) + \lambda^2\mathcal{P}^2 \tag{2.23}$$

$$\Theta(\theta) \;=\; \rho^4\theta'^2 \;=\; \Delta_\theta\left(\mathfrak{q}\, a^2C^2 + \mathcal{K}\right) - \lambda^2\mathcal{D}^2/S^2. \tag{2.24}$$

Appropriately, we call these equations the *first integral equations for (the Boyer-Lindquist coordinates)* r *and* θ. They are also known in the literature (e.g. [10]) as the *radial equation* and *colatitude equation*, respectively.

2.3 First-order equations for t and φ

Now that we have first-order equations for $r(s)$ and $\theta(s)$, we wish to use the pair $\mathcal{E}$, $\mathcal{L}$ to construct a system of first-order differential equations for the coordinates $t(s)$ and $\varphi(s)$ of an arbitrary orbit $\gamma(s)$. The definitions of these integrals are themselves a pair of first-order equations for t and φ, but the equations are coupled: each equation involves both t and φ explicitly. The first-order coupled equations are

$$-\mathcal{E} + qA_t \;=\; t'g_{tt} + \varphi' g_{t\varphi} \tag{2.25}$$

$$\mathcal{L} + qA_\varphi \;=\; t'g_{t\varphi} + \varphi' g_{\varphi\varphi}. \tag{2.26}$$

Since the determinant of the matrix of coefficients,

$$g_{tt}g_{\varphi\varphi} - g_{t\varphi}^2 \;=\; -\frac{\Delta_r\Delta_\theta S^2}{\lambda^4},$$

is nonzero away from coordinate singularities, the system can be solved exactly for t' and φ'. To find the solution for t', we subtract $g_{t\varphi}$ times equation (2.26) from $g_{\varphi\varphi}$ times (2.25). We are left with

$$t'(g_{tt}g_{\varphi\varphi} - g_{t\varphi}^2) \;=\; -g_{\varphi\varphi}\mathcal{E} - g_{t\varphi}\mathcal{L} + q(g_{\varphi\varphi}A_t - g_{t\varphi}A_\varphi),$$

or

$$t' \frac{\Delta_r \Delta_\theta S^2}{\lambda^4} \quad = \quad g_{\varphi\varphi}\mathcal{E} + g_{t\varphi}\mathcal{L} + q(g_{t\varphi}A_\varphi - g_{\varphi\varphi}A_t).$$

Inserting the actual Boyer-Lindquist components and performing a preliminary simplification, we obtain

$$t' \frac{\Delta_r \Delta_\theta S^2}{\lambda^4} \quad = \quad \frac{(r^2+a^2)\Delta_\theta S^2}{\lambda^2 \rho^2}\left((r^2+a^2)\mathcal{E} - a\mathcal{L} - \frac{qer}{\lambda}\right) + \frac{a\Delta_r S^2}{\lambda^2\rho^2}\left(\mathcal{L} - \mathcal{E}aS^2\right),$$

that is,

$$t' \frac{\Delta_r \Delta_\theta S^2}{\lambda^4} \quad = \quad \frac{(r^2+a^2)\Delta_\theta S^2}{\lambda^2\rho^2}\mathcal{P} + \frac{a\Delta_r S^2}{\lambda^2\rho^2}\mathcal{D}.$$

Rearranging this for t', we get

$$t' \quad = \quad \frac{\lambda^2}{\rho^2}\left(\frac{(r^2+a^2)\mathcal{P}}{\Delta_r} + \frac{a\mathcal{D}}{\Delta_\theta}\right).$$

To find an equation for φ' that is independent of t', we mimic this procedure and subtract g_{tt} times equation (2.26) from $g_{t\varphi}$ times (2.25). After the necessary simplifications, we arrive at

$$\varphi' \quad = \quad \frac{\lambda^2}{\rho^2}\left(\frac{\mathcal{D}}{S^2\Delta_\theta} + \frac{\mathcal{P}}{\Delta_r}\right). \tag{2.27}$$

We refer to the decoupled system of equations

$$t' \quad = \quad \frac{\lambda^2}{\rho^2}\left(\frac{(r^2+a^2)\mathcal{P}}{\Delta_r} + \frac{a\mathcal{D}}{\Delta_\theta}\right) \tag{2.28}$$

$$\varphi' \quad = \quad \frac{\lambda^2}{\rho^2}\left(\frac{\mathcal{D}}{S^2\Delta_\theta} + \frac{\mathcal{P}}{\Delta_r}\right) \tag{2.29}$$

as the *first integrals equations for t and φ*, respectively.

Equations (2.28), (2.29), (2.23), and (2.24) comprise a system of first-order differential equations whose solutions are minimizers of the action of (2.1). Amongst the solutions of these equations, do we find *all* the minimizers? In other words, have we lost any solutions of the *second-order* Euler-Lagrange equations? By Theorem (2.1), we necessarily retain those solutions $\gamma(s)$ whose constants of motion correspond to level sets $\mathbb{K}_c = \{x \in T^*(\mathbb{K}) : \mathfrak{q}\,(x) = c_{\mathfrak{q}}\,, \mathcal{E}(x) = c_{\mathcal{E}}, \mathcal{L}(x) = c_{\mathcal{L}}, \mathcal{K}(x) = c_{\mathcal{K}}\}$ on which the first integrals are functionally independent. So far, we only know that $\mathcal{E}$ and $\mathcal{L}$ are independent in this sense, owing to the fact that they are momenta. Our goal in this chapter, however, was not only to prove the existence of four Poisson-commuting first integrals for our Hamiltonian system, but also to construct them explicitly. Having actually constructed them, we may answer the question of "lost solutions" in a different way: by differentiating the four first-order equations to check that we retrieve the Euler-Lagrange equations themselves, which are

$$\frac{d}{ds}\left(g_{kb}\dot{x}^b - qA_k\right) = \frac{1}{2}g_{ab,k}\dot{x}^a\dot{x}^b - qA_{a,k}\dot{x}^a, \quad \text{for } k = 1, \ldots, 4.$$

The Euler-Lagrange equations corresponding to t or φ have right-hand sides identically zero, since the metric components and the potential components are t-independent and φ-independent, and so we see immediately that (2.25) and (2.26) are integrated Euler-Lagrange equations with additive constants $\mathcal{E}$ and $\mathcal{L}$. The radial and colatitude equations do not yield to inspection so easily, but with effort they can be shown to be integrated Euler-Lagrange equations for r and θ with constants of integration $\mathfrak{q}$ and $\mathcal{K}$. Having done this, we can be confident that no orbits have been lost along the way. (Knowing that the first integral equations determine all the minimizers, we may conclude from the form of these equations that the $\mathbb{K}_c \subset T^*(\mathbb{K})$ on which the first integrals are functionally independent are precisely those level sets determined by numbers $(c_{\mathfrak{q}}\,, c_{\mathcal{E}}, c_{\mathcal{L}}, c_{\mathcal{K}})$ for which $R(r)|_{\mathbb{K}_c}, \Theta(\theta)|_{\mathbb{K}_c} \geq 0$.)

While each minimizer of (2.1) is a solution of the first integral equations (2.28), (2.29), (2.23), and (2.24) for some four-tuple $(\mathfrak{q}\,, \mathcal{E}, \mathcal{L}, \mathcal{K})$, is the converse true? For any such four-tuple is there a minimizer whose conserved quantities are these numbers? A minimizer is uniquely determined by its starting position and velocity. When a starting position $(t_0, r_0, \theta_0, \varphi)$ and constants $\mathcal{E}$ and $\mathcal{L}$ are supplied, the system of equations (2.25) and (2.26) for t' and φ' is linear and invertible wherever Boyer-Lindquist coordinates are valid. Hence, at each starting position $(t_0, r_0, \theta_0, \varphi_0)$ in $\mathbb{K}$, the pair $(\mathcal{E}, \mathcal{L})$ determines uniquely the initial data $t'(0)$ and $\varphi'(0)$ of a minimizing curve and vice-versa. The starting position and the pair $(\mathcal{E}, \mathcal{L})$ also determine $\mathcal{P}(r_0)$ and $\mathcal{D}(\theta_0)$ in the radial and colatitude equations, so that equations (2.23) and (2.24) become linear equations for r'^2 and θ'^2 in $\mathfrak{q}$ and $\mathcal{K}$. Given numbers $\mathfrak{q}, \mathcal{K}$ such that $R(r_0), \Theta(\theta_0) \geq 0$, we may then solve for $r'(0)$ and $\theta'(0)$—but not uniquely, since (2.23) and (2.24) are equations for the squares of r' and θ'. Hence, for each eight-tuple $(t_0, r_0, \theta_0, \varphi_0, \mathfrak{q}, \mathcal{E}, \mathcal{L}, \mathcal{K})$ such that $R(r_0), \Theta(\theta_0) \geq 0$, there are four minimizing curves departing from $(t_0, r_0, \theta_0, \varphi_0)$, depending on the signs of $r'(0)$ and $\theta'(0)$.

Remark 2.1. Are the right-hand sides of (2.24) and (2.29) singular at $\theta = 0, \pi$? While we might suspect so, the fact is that the equations do not fail on the axis. Since $\mathcal{L} = g_{\varphi\varphi}\varphi' + g_{\varphi t}t' - qA_\varphi$, and since $g_{t\varphi}$, $g_{\varphi\varphi}$, and A_φ are all proportional to S^2, we have that $\mathcal{D}(\theta) = \mathcal{L} - \mathcal{E}aS^2$ is proportional to S^2. Therefore, $\mathcal{D}/S^2$ is a quotient whose numerator is equal to S^2 times a function $\mathcal{D}^*(\theta)$ which does not blow up as $S^2 \to 0$. Upon cancelling S^2 from the top and bottom, the quotient is just $\mathcal{D}^*(\theta)$.

The verification of the well-behavedness of (2.29) on $\mathbb{A}$ reveals a fact about the motion of test particles in $\mathbb{K}$. Since $\mathcal{L}$ is written as a combination of functions proportional S^2, it must be zero on the axis. (This agrees with our intuition, as *axial* angular momentum is taken about $\mathbb{A}$.) Since $\mathcal{L}$ is fixed along every action-minimizing trajectory, we are forced to conclude that every particle crossing the axis must have zero angular momentum. This already places a severe restriction on the types of orbits that may pass through the axis, and we will apply this fact in the next chapter, where we shed light on the dynamics implied by the first-order equations.

Chapter 3

Dynamics of charged particles

In this chapter, we examine the dynamical implications of the first-order equations (2.28), (2.29), (2.23), and (2.24) derived in the previous chapter. Our goal is not to completely exhaust all the possible orbits that light and charged test matter can achieve under the gravitational and electromagnetic impetus of a Kerr-Newman-de Sitter-type spacetime. Rather, we strive to emphasize the differences between orbits in the general Kerr-Newman-de Sitter setting and those in electrically-neutral, asymptotically-flat Kerr spacetimes, which are treated almost exhaustively in [10]. Glimpses of the differences arise naturally when we seek to understand the geometric and physical meanings of Carter's constant $\mathcal{K}$.

3.1 Principal orbits

In the previous chapter, we emphasized the physical interpretations of $\mathfrak{q}$ $(= -m^2)$ as mass, $\mathcal{E}$ as energy, and $\mathcal{L}$ as (axial) angular momentum. These beg us, in turn, to find meaning for Carter's constant—geometric, physical, or both. We start with a geometric one.

As a preliminary step, we notice that $g_{\theta\theta} = \rho^2/\Delta_\theta$ is everywhere positive—save for at the ring, where it vanishes. Similarly, $\langle W, W\rangle = \rho^2 S^2 \Delta_\theta/\lambda^2$ is positive away from the ring and axis. Let us consider a trajectory $\gamma : I \to \mathbb{K} - (\mathbb{H} \cup \Sigma \cup \mathbb{A})$, that is, an orbit contained in a Boyer-Lindquist block that steers clear of poles and the ring. At every point of $\mathbb{K} - (\mathbb{H} \cup \Sigma \cup \mathbb{A})$, $g_{\theta\theta}$ and W are positive, and so at each point p visited by γ, the two-dimensional vector subspace $\mathrm{Span}\{\partial_\theta, W\}$ of $T_p\mathbb{K}$ is spacelike. The orthogonal plane $\Pi := \mathrm{Span}\,\{\partial_r, V\}$ at each point is necessarily timelike. We call the timelike planes Π the *principal planes* along γ. In the special case where $\gamma' \in \Pi$ at each point of γ, we refer to γ a *principal orbit.*[1]

We fix the symbols γ'_Π and $\gamma'_\perp$ for the components of γ' in Π and $\Pi^\perp = \mathrm{Span}\,\{\partial_\theta, W\}$, respectively. To emphasize that the scalar product on $\Pi^\perp$ is positive definite, we write the line element $\langle v, v\rangle$ as $|v|^2$ when $v \in \Pi^\perp$. We now reveal a geometric meaning of $\mathcal{K}$, as a measure of the relation of γ' to the principal planes Π along γ.

Lemma 3.1. *If γ is the orbit of a particle with mass $\mathfrak{q}$ and Carter's constant $\mathcal{K}$, then*

$$\mathcal{K} \;=\; \rho^2\,|\gamma'_\perp|^2 - \mathfrak{q}\,a^2C^2 \;=\; \mathfrak{q}\,r^2 - \rho^2\,\langle\gamma'_\Pi, \gamma'_\Pi\rangle\,. \tag{3.1}$$

Proof. We consider the formula for γ' derived at the beginning of Section 2.2, namely

$$\gamma' = r'\partial_r + \theta'\partial_\theta + \frac{\langle\gamma', V\rangle}{\langle V, V\rangle}V + \frac{\langle\gamma', W\rangle}{\langle W, W\rangle}W.$$

Accordingly, the component of γ' in $\Pi^\perp = \mathrm{Span}\,\{\partial_\theta, W\}$ is

$$\gamma'_\perp \;=\; \theta'\partial_r + \frac{\langle\gamma', W\rangle}{\langle W, W\rangle}W,$$

[1] We borrow this terminology from [10], where the analogous definition is made in the special case of geodesic orbits.

so that

$$\begin{aligned} |\gamma'_\perp|^2 &= \langle \gamma'_\perp, \gamma'\perp \rangle = \theta'^2 g_{\theta\theta} + \frac{\mathcal{D}^2}{\langle W, W \rangle} \\ &= \theta'^2 \frac{\rho^2}{\Delta_\theta} + \frac{\mathcal{D}^2 \lambda^2}{\rho^2 S^2 \Delta_\theta}. \end{aligned}$$

We may solve for the square of θ':

$$\theta'^2 = \frac{\Delta_\theta}{\rho^2} \left(|\gamma'_\perp|^2 - \frac{\mathcal{D}^2 \lambda^2}{\rho^2 S^2 \Delta_\theta} \right).$$

Inserting this expression into the colatitude equation (2.24) gives us

$$\begin{aligned} \mathcal{K} &= -\mathfrak{q}\, a^2 C^2 + \Delta_\theta^{-1} \left[\Delta_\theta \rho^2 \left(|\gamma'_\perp|^2 - \frac{\mathcal{D}^2 \lambda^2}{\rho^2 S^2 \Delta_\theta} \right) + \frac{\mathcal{D}^2 \lambda^2}{S^2} \right] \\ &= -\mathfrak{q}\, a^2 C^2 + \Delta_\theta^{-1} \left[|\gamma'_\perp|^2 \Delta_\theta \rho^2 - \frac{\mathcal{D}^2 \lambda^2}{S^2} + \frac{\mathcal{D}^2 \lambda^2}{S^2} \right] \\ &= -\mathfrak{q}\, a^2 C^2 + \rho^2 |\gamma'_\perp|^2 . \end{aligned}$$

Since $T(\mathbb{K}) = \Pi \oplus \Pi^\perp$, the invariant mass of γ decomposes as

$$\mathfrak{q} = \langle \gamma'_\Pi, \gamma'_\Pi \rangle + |\gamma'_\perp|^2 .$$

Applying this to the new formula for $\mathcal{K}$, we arrive at

$$\begin{aligned} \mathcal{K} &= \rho^2 |\gamma'_\perp|^2 - \left(\langle \gamma'_\Pi, \gamma'_\Pi \rangle + |\gamma'_\perp|^2 \right) a^2 C^2 \\ &= (r^2 + a^2 C^2) |\gamma'_\perp|^2 - \left(\langle \gamma'_\Pi, \gamma'_\Pi \rangle + |\gamma'_\perp|^2 \right) a^2 C^2 \\ &= r^2 |\gamma'_\perp|^2 - \langle \gamma'_\Pi, \gamma'_\Pi \rangle a^2 C^2 \\ &= r^2 |\gamma'_\perp|^2 + \left(r^2 \langle \gamma'_\Pi, \gamma'_\Pi \rangle - r^2 \langle \gamma'_\Pi, \gamma'_\Pi \rangle \right) - \langle \gamma'_\Pi, \gamma'_\Pi \rangle a^2 C^2 \\ &= r^2 \left(|\gamma'_\perp|^2 + \langle \gamma'_\Pi, \gamma'_\Pi \rangle \right) - (r^2 + a^2 C^2) \langle \gamma'_\Pi, \gamma'_\Pi \rangle \\ &= r^2 \mathfrak{q} - \rho^2 \langle \gamma'_\Pi, \gamma'_\Pi \rangle . \end{aligned}$$

□

As r and θ are global functions, so too are these expressions for $\mathcal{K}$. Since $\mathcal{K}$ appears here as the difference of two quadratic forms ($\mathfrak{q}$ and $\langle v_\Pi, v_\Pi \rangle$), $\mathcal{K}$ itself can be regarded as a quadratic form on each fibre of $T(\mathbb{K})$. Moreover, this form is positive definite, as $\langle v_\Pi, v_\Pi \rangle$ is a negative definite inner product on the timelike plane Π.

These expressions for Carter's constant also bring to light the following facts concerning $\mathbb{K}$-orbits of the three causal types:

Corollary 3.1. *Let γ be an orbit in $\mathbb{K}$.*

1. *If γ is timelike, then $\mathcal{K} \geq 0$; $\quad \mathcal{K} = 0 \Leftrightarrow \gamma$ is a principal equatorial orbit.*

2. *If γ is lightlike, then $\mathcal{K} \geq 0$; $\quad \mathcal{K} = 0 \Leftrightarrow \gamma$ is principal.*

3. *If γ is spacelike, then $\mathcal{K} \geq -\mathfrak{q}\, a^2$; $\quad \mathcal{K} = -\mathfrak{q}\, a^2 \Leftrightarrow \gamma$ is a principal orbit constrained to the axis.*

Proof.

- In the case that $\mathfrak{q} < 0$, $\mathcal{K} = \rho^2 \left|\gamma'_\perp\right|^2 - \mathfrak{q}\, a^2 C^2$ is nonnegative, and is equal to zero if and only if both $\left|\gamma'_\perp\right|^2$ and $\cos^2\theta$ are equal to zero.

- In the case that $\mathfrak{q} = 0$, $\mathcal{K} = \rho^2 \left|\gamma'_\perp\right|^2$ is nonnegative, and vanishes if and only if $\left|\gamma'_\perp\right|^2$ is zero.

- In the case that $\mathfrak{q} > 0$, $\mathcal{K} = \rho^2 \left|\gamma'_\perp\right|^2 - \mathfrak{q}\, a^2 C^2 \geq -\mathfrak{q}a^2$, and is equal to $-\mathfrak{q}a^2$ whenever $\left|\gamma'_\perp\right|^2 = 0$ and $\cos^2\theta = 1$.

□

While these results are not different from the analogous results in the $\mathbb{K}(a, M)$ spacetimes (see [10]), we nevertheless use the corollary to expose crucial differences, particularly in the way a particle may approach the ring singularity, Σ.

But in the meantime, we discuss the ways in which the t and φ coordinates of a particle may evolve given initial data $\mathcal{E}$ and $\mathcal{L}$.

3.2 Evolution of the time and longitude coordinates

We remind ourselves of the first-order equations for t and φ:

$$\begin{aligned} t' &= \frac{\lambda^2}{\rho^2}\left(\frac{(r^2+a^2)\mathcal{P}}{\Delta_r} + \frac{a\mathcal{D}}{\Delta_\theta}\right) \\ \varphi' &= \frac{\lambda^2}{\rho^2}\left(\frac{\mathcal{D}}{S^2\Delta_\theta} + \frac{\mathcal{P}}{\Delta_r}\right). \end{aligned}$$

As the simplest dynamics to understand are those of what we call “lazy” orbits—particles with $\mathcal{E} = \mathcal{L} = 0$—we pay particular attention to them, and ask where the differences arise when their motion is compared to that in $\mathbb{K}(a, M)$.

If a neutral particle (such as a photon) is devoid of energy or axial angular momentum, then $\mathcal{P} = \mathcal{D} = 0$ along its orbit. Consequently, $t' = \varphi' = 0$ by equations (2.28) and (2.29). Such a particle remains frozen in time—*coordinate* time, not proper time—and in its longitudinal position. Radial and colatitudinal motion may still be possible, as equations (2.23) and (2.24) imply, at least when $\mathfrak{q}$ and $\mathcal{K}$ are nonzero.

Thus, to a distant observer who views the world through Boyer-Lindquist coordinates, a particle with the attributes $\mathcal{E} = \mathcal{L} = 0$ may instantaneously vanish from his or her line of sight, for any change of the particle's latitude or radial position would occur without the elapse of Boyer-Lindquist time. However, the geodesics traversed by lazy uncharged particles are necessarily trapped in special submanifolds of spacetime called *closed, totally-geodesic submanifolds.* If a particle starts in such a submanifold, with an initial velocity tangent to it, then its orbit remains constrained to the submanifold indefinitely. (More details on these submanifolds, including a proof of this fact, can be found in [9].) It is also true that the set of fixed points of a spacetime isometry, as well as the set of fixed points of a Killing vector field, are closed, totally-geodesic submanifolds. This makes both the axis $\mathbb{A}$ and the equatorial plane $\mathbb{E}$ closed, totally-geodesic submanifolds. Furthermore, polar planes (hypersurfaces of fixed $t = t_0$ and $\varphi = \varphi_0$) are closed and totally-geodesic, as well as the Kerr and de Sitter horizons.

We suppose that a lazy, neutral geodesic is not trapped in $\mathbb{H}$ or $\mathbb{A}$. It follows that Boyer-Lindquist coordinates are valid, and so the first integral equations imply $\mathcal{L} = \mathcal{E} = 0 \Leftrightarrow \mathcal{P} = \mathcal{D} = 0 \Leftrightarrow t' = \varphi' = 0$. The velocity vector γ' is tangent to a polar plane, and is therefore doomed to remain in it. The fates of the lazy neutral geodesics are sealed in this characterization.

In Kerr(-de Sitter), these remarks are the end of the story for particles with no energy or angular momentum. A key difference, however, arises in Kerr-Newman(-de Sitter) geometry: when $q \neq 0$, $\mathcal{L} = \mathcal{E} = 0$ does not imply $\mathcal{P} = 0$. In $\mathcal{P}$, a linear function of r, proportional to the charges q and e, persists even after $\mathcal{L}$ and $\mathcal{E}$ vanish. Motion in the t and φ directions is still possible, and the first-order equations take the simple forms

$$t' \;=\; -(r^2+a^2)\frac{qer\lambda}{\rho^2\Delta_r}$$

and

$$\varphi' \;=\; -\frac{qer\lambda}{\rho^2\Delta_r}$$

in which the rates of progression along the t and φ axes are proportional to the charge q. In particular, t' and φ' have the same sign. Based on the signs of Δ_r in the Boyer-Lindquist blocks, we can construct the following table for the sign of t' and φ' along orbits with $\mathcal{E} = \mathcal{L} = 0$:

Table 3.1: *Directions of travel along the t and φ axes for $\mathcal{E}(\gamma) = \mathcal{L}(\gamma) = 0$.*

	$qe < 0$	$qe > 0$
dS	-	+
I	+	-
II	-	+
$(r > 0)$ III	+	-
$(0 > r > r_{--})$ III	-	+
$(r < r_{--})$ III	+	-

For instance, a lazy particle with initial position in the de Sitter block and a charge opposite in sign to the charge of the black hole will invariably move backwards in Boyer-Lindquist time, and rotate in a direction opposite to that of the rotation of the black hole. In agreement with the identification of r_{++} with r_{--}, the behaviours exhibited in I and in the $r < r_{--}$ region of III are the same. Furthermore, the behaviours exhibited in the de Sitter block and in the $0 > r > r_{--}$ region of III are the same.

While much more can be said about the evolution of t and φ, particularly for the much larger class of particles with energy and angular momentum, we instead focus on unravelling the more complicated behaviour of the θ coordinate.

3.3 Evolution of the colatitude coordinate

While the first integral equations for the t and φ coordinates of a particle depend only on the values of $\mathcal{E}$ and $\mathcal{L}$, the radial and colatitude equations depend on all four first integrals. However, information about the local evolution of $r(s)$ and $\theta(s)$ may be gleaned from one overriding fact about $R(r)$ and $\Theta(\theta)$: neither can be negative. An immediate dynamical consequence of this is as follows:

Theorem 3.1. *If a trajectory γ approaches the ring Σ, then it must have $\mathcal{K} = 0$ and an angular momentum proportional to its energy (specifically $\mathcal{L} = a\mathcal{E}$).*

Proof. We define a first integral $\mathcal{Q}$, dependent on the others, by

$$\mathcal{K} = \mathcal{Q} + \lambda^2(\mathcal{L} - a\mathcal{E})^2. \tag{3.2}$$

While $\mathcal{Q}$, at first glance, may only be a shifted variant of Carter's constant, the usefulness of this alternate form will become quite clear. Without confusion, we apply the name "Carter's constant" to $\mathcal{Q}$, and let the symbols $\mathcal{K}$ and $\mathcal{Q}$ indicate the form being used.

Now, if $\gamma(s) \to \Sigma$ as $s \to s^*$ (where $s^* \leq \infty$), then necessarily $\lim_{s\to s^*} r(s) = 0$ and $\lim_{s\to s^*} \theta(s) = \pi/2$. Since $R(r(s)), \Theta(\theta(s)) \geq 0$ for all $s \in \mathbb{R}$, it follows by the continuity of the functions R and Θ as defined in Corollary 2.1 that $R(0), \Theta(\pi/2) \geq 0$. But

$$\begin{aligned}
R(0) &= \Delta_r(0)(-\mathcal{K}) + \lambda^2\mathcal{P}^2 \\
&= -(a^2+e^2)(\mathcal{Q} + \lambda^2(\mathcal{L} - a\mathcal{E})^2) + \lambda^2(a^2\mathcal{E} - \mathcal{L}a)^2 \\
&= -(a^2+e^2)\mathcal{Q} - e^2\lambda^2(\mathcal{L} - a\mathcal{E})^2 - a^2\lambda^2(\mathcal{L} - a\mathcal{E})^2 - a^2 + a^2\lambda^2(a\mathcal{E} - \mathcal{L})^2 \\
&= -(a^2+e^2)\mathcal{Q} - e^2\lambda^2(\mathcal{L} - a\mathcal{E})^2.
\end{aligned}$$

If we enforce the nonnegativity of this expression, then

$$\mathcal{Q} \leq -\frac{e^2\lambda^2(\mathcal{L} - a\mathcal{E})^2}{a^2+e^2},$$

which forces $\mathcal{Q}$ to be negative whenever $\mathcal{L}$ is different from $a\mathcal{E}$. Examining Θ at $\theta = \pi/2$, we find the inequality

$$\begin{aligned}
\Theta(\pi/2) &= \Delta_\theta(\pi/2)(\mathfrak{q}\, a^2C^2(\pi/2) + \mathcal{K}) - \frac{\lambda^2\mathcal{D}^2(\pi/2)}{S^2(\pi/2)} \\
&= 1 \cdot (0 + \mathcal{Q} + \lambda^2(\mathcal{L} - a\mathcal{E})^2) - \lambda^2(\mathcal{L} - a\mathcal{E}S^2(\pi/2))/1^2 \\
&= \mathcal{Q} \geq 0.
\end{aligned}$$

The two inequalities for $\mathcal{Q}$ are non-contradictory only when $\mathcal{Q}$ is zero, which occurs only when $\mathcal{L} = a\mathcal{E}$. Subsequently, Carter's constant $\mathcal{K}$ for this orbit is $\mathcal{K} = \mathcal{Q} + \lambda^2(\mathcal{L} - a\mathcal{E})^2 = 0$.

□

Taken together, the theorem and Corollary 3.1 indicate that the only material (timelike) particles that can approach the ring are those that do so following principal trajectories in the equator. Similarly, the only photons that can approach Σ are those that are principal. (The same corollary seems to suggest that a hypothetical spacelike *tachyon* with $\mathcal{K} = 0$ need not follow a special path to meet its destruction at the ring.)

It is in these results that we begin to see characteristic differences between the behaviour of action-minimizing curves in Kerr-Newman-de Sitter spacetime and, say, that of geodesics in Kerr-de Sitter spacetime. In Kerr-de Sitter (or simply Kerr), the upper bound on $\mathcal{Q}$ obtained in the preceding proof is zero since $e^2 = 0$. Thus, we may have $\mathcal{Q} = 0$ without requiring $\mathcal{L} = a\mathcal{E}$, and consequently $\mathcal{K}$ may be greater than zero, in which case a greater variety of timelike and null geodesics may be permitted into the ring. As is remarked in [10] for curves in Kerr spacetime, "it is rare for a Kerr geodesic to hit the ring singularity. By contrast, in Schwarzschild spacetime every particle falling through the horizon inexorably meets the central singularity (unless it perishes earlier)." The results above indicate that it is even less likely for a Kerr-Newman(-de Sitter) orbit to enter the ring. Should the charge per unit mass of the spacetime be even slightly different from zero, then the upper bound on $\mathcal{Q}$ is negative, such that the only matter or light that can meet the ring must do so via the special paths mentioned above. Furthermore, these restrictions come into effect *regardless* of whether the particle is charged or not. Remarkably, the electromagnetic field seems to affect all particles, repelling away from Σ all those with angular momentum $\mathcal{L}$ different from $a\mathcal{E}$. This strange phenomenon is an effect of the combined presence of the black-hole rotation with the Maxwell field, for if a were zero, the e^2 terms in the upper bound on $\mathcal{Q}$ would necessarily cancel.

If we consider the vanishing of rotation (but not necessarily of the cosmological constant), then the upper bound on $\mathcal{Q}$ becomes $-\lambda^2\mathcal{L}^2$. Consequently, the only geodesics in Reissner-Nordström(-de Sitter) that meet the singularity—no longer a ring but a point—are those with zero angular momentum. In particular, there are no apparent energy restrictions on particles.

While the previous theorem provides us with a lot information concerning particle behaviour, we can extract even more by writing the constant $\mathcal{Q}$ independently of $\mathcal{K}$. To achieve this, we make note of the following identity, which follows immediately from the definition of Δ_θ.

Lemma 3.2.

$$\Delta_\theta(\mathcal{L} - a\mathcal{E})^2 - \frac{(\mathcal{L} - a\mathcal{E}S^2)^2}{S^2} = a^2C^2\left(L(\mathcal{L} - a\mathcal{E})^2 + \mathcal{E}^2 - \frac{\mathcal{L}^2}{a^2S^2}\right),$$

where $L := \Lambda/3$.

Now, using the θ-formulation of $\mathcal{K}$, we write

$$\begin{aligned} \mathcal{Q} &= \mathcal{K} - \lambda^2(\mathcal{L} - a\mathcal{E})^2 \\ &= -\mathfrak{q}\, a^2C^2 + \frac{1}{\Delta_\theta}\left[\rho^4\theta'^2 + \lambda^2\frac{\mathcal{D}^2}{S^2}\right] - \lambda^2\frac{\Delta_\theta}{\Delta_\theta}(\mathcal{L} - a\mathcal{E})^2 \\ &= -\mathfrak{q}\, a^2C^2 + \frac{1}{\Delta_\theta}\left[\rho^4\theta'^2 + \lambda^2\left(\frac{(\mathcal{L} - a\mathcal{E}S^2)^2}{S^2} - \Delta_\theta(\mathcal{L} - a\mathcal{E})^2\right)\right]. \end{aligned}$$

By Lemma 3.2 we have

$$\mathcal{Q} = -\mathfrak{q}\, a^2C^2 + \frac{1}{\Delta_\theta}\left[\rho^4\theta'^2 - \lambda^2a^2C^2\left(L(\mathcal{L} - a\mathcal{E})^2 + \mathcal{E}^2 - \frac{\mathcal{L}^2}{a^2S^2}\right)\right],$$

which we rearrange as

$$\mathcal{Q} = T + V \tag{3.3}$$

for

$$T\left((r,\theta),\theta'\right) \;:=\; \frac{\rho^4}{\Delta_\theta}\theta'^2 \tag{3.4}$$

and

$$V(\theta) \;:=\; -a^2C^2\left[\mathfrak{q}+\frac{\lambda^2}{\Delta_\theta}\left(L(\mathcal{L}-a\mathcal{E})^2+\mathcal{E}^2-\frac{\mathcal{L}^2}{a^2S^2}\right)\right]. \tag{3.5}$$

As the expression $L(\mathcal{L}-a\mathcal{E})^2+\mathcal{E}^2$ is a positive constant, we use a single symbol, $\mathcal{N}$, to denote it. Under this convention, (3.5) becomes

$$V(\theta) \;:=\; -a^2C^2\left[\mathfrak{q}+\frac{\lambda^2}{\Delta_\theta}\left(\mathcal{N}-\frac{\mathcal{L}^2}{a^2S^2}\right)\right]. \tag{3.6}$$

The decomposition of $\mathcal{Q}$ as $T+V$ answers the question of the physical meaning of Carter's constant. The functions T and V can be interpreted as *rotational kinetic* and *rotational potential* energies, respectively, of a particle moving along the θ-axis. As $T \geq 0$ and $T = 0 \;\Leftrightarrow\; \theta' = 0$, the function T exhibits the positive definiteness that we would expect of a kinetic energy. (We assume that the radial positions adopted during the motion of the particle are bounded away from the singularity Σ, such that ρ^2 is always strictly greater than zero.) Subsequently, the constancy of $T+V$ can be interpreted as the conservation of rotational mechanical energy of the particle.

We may acquire a deeper understanding of the evolution of the colatitudinal coordinate $\theta(s)$ by examining the potential function $V(\theta)$. Suppose that we were to draw a graph of V versus $\theta \in (0,\pi)$. At any height $V = \mathcal{Q}$, we draw a horizontal line through the graph. At the points where the line intersects the graph, the kinetic energy T is zero. Since $T \geq 0$, the θ coordinate may only take on values in $(0,\pi)$ at which $V(\theta)$ lies below or on the line $\mathcal{Q}$. Any other values of θ are forbidden.

We list some important properties of V for $\theta \in (0,\pi)$:

- $V(\theta) \to +\infty$ as $\theta \to 0, \pi$: as the particle moves within reach of the poles, its potential energy blows up. Such unabated growth in V would necessarily require a large Carter constant $\mathcal{Q}$, but even then, $\mathcal{Q}$ is a ceiling that the values of V cannot exceed. Hence, the particle can never actually reach the poles. (The range $\theta \in (0, \pi)$ excludes the event that γ actually starts at a pole. In this case, the term $\mathcal{L}^2/(a^2S^2)$ vanishes, as $\mathcal{L}^2$ can be written as S^4 times a function that is non-singular on the axis. The particle completes its journey with the simpler potential

$$u(\theta) \quad = \quad -a^2C^2 \left[\mathfrak{q} + \frac{\lambda^3\mathcal{E}^2}{\Delta_\theta} \right]. \tag{3.7}$$

For a photon ($\mathfrak{q} = 0$) starting at a pole, the function u is always negative, and so the kinetic energy has lower bound $\mathcal{Q} + |u|$.)

- $V(\theta)$ has an obvious repeated root: $\theta = \pi/2$. In other words, when the particle meets the equatorial plane, its kinetic energy is maximal (and is equal to $\mathcal{Q}$). For some particles, this is the only root of V; photons emitted from the axis are one example. A sufficient condition for further roots to exist, distinct from $\pi/2$, is the satisfaction of the inequality

$$0 \quad < \quad \frac{\mathcal{L}^2}{a^2}\left(\frac{\mathfrak{q}}{\lambda} + \mathcal{N}\right) \quad < \quad 1 \tag{3.8}$$

by the parameters of the trajectory. Any additional roots of V are zeros of the equation

$$S^2 \quad = \quad \frac{\mathcal{L}^2}{a^2}\left(\frac{\Delta_\theta \mathfrak{q}}{\lambda^2} + \mathcal{N}\right),$$

and the right-hand side is bounded above by $\frac{\mathcal{L}^2}{a^2}\left(\frac{\mathfrak{q}}{\lambda} + \mathcal{N}\right)$. Therefore, when the inequality holds, $S^2 < 1$ has two distinct roots $\theta_\pm$ different from $\pi/2$ and symmetric about the equator.

The existence of these roots, combined with (a) the continuity of V; (b) the limits $V \to +\infty$ as $\theta \to 0, \pi$; and (c) the third root $V(\pi/2) = 0$, imply that there exist minima at points θ_-^* and θ_+^* such that

$$0 \; < \; \theta_- \; < \; \theta_-^* \; < \; \pi/2 \; < \; \theta_+^* \; < \; \theta_+ \; < \; \pi.$$

By examining $V'(\theta)$ directly, we may discover other sufficient conditions for the existence of critical points of V.

- The slope of V is

$$\frac{dV}{d\theta} \;=\; 2a^2SC\left[\mathfrak{q} + \frac{\lambda^2}{\Delta_\theta^2}\left(\mathcal{N} - L\frac{\mathcal{L}^2C^2}{S^2}\right) - \frac{\lambda^2\mathcal{L}^2}{a^2\Delta_\theta S^4}\right]. \tag{3.9}$$

It follows that $V'(\theta) \to -\infty$ as $\theta \to 0$ and that $V'(\theta) \to +\infty$ as $\theta \to \pi$. This information describes the way in which V asymptotically approaches the potential barriers at $\theta = 0, \pi$.

- The function V' has an obvious root at $\theta = \pi/2$. If there are no other roots, then $\pi/2$ is the site of a global minimum: $V(\pi/2) = 0$. In this case, $\mathcal{Q} \geq 0$.

Above, we discovered a sufficient condition for the existence of further roots of V'. We see now that these critical points must lie symmetrically to each side of $\theta = \pi/2$, since all the sine and cosine functions in V' appear to even powers (save for the SC out in front). The sufficient conditions (3.8) necessarily require $\mathcal{L} \neq 0$. Alternatively, we may investigate the possibility of sufficient conditions for V-minima in $\mathcal{L} = 0$ orbits. Subject to this restriction, V' reduces to

$$\frac{dV}{d\theta} = 2a^2 SC \left[\mathfrak{q} + \frac{\lambda^3 \mathcal{E}^2}{\Delta_\theta^2} \right].$$

If $\mathfrak{q} = 0$, then $\theta = \pi/2$ is the only critical point in $0 < \theta < \pi$. For material particles ($\mathfrak{q} < 0$), there exist two additional critical points θ_-^* and θ_+^*, symmetric about $\pi/2$, if and only if

$$\mathcal{E} > (|\mathfrak{q}\,| / \lambda^3)^{1/2} \quad \text{and} \quad \left(\lambda \mathcal{E} \sqrt{\frac{\lambda}{|\mathfrak{q}\,|}} - 1 \right) < La^2. \tag{3.10}$$

Now, when $\mathcal{L} = 0$, the potential takes the form

$$V(\theta) = -a^2 C^2 \left[\mathfrak{q} + \frac{\lambda^3 \mathcal{E}^2}{\Delta_\theta} \right].$$

When the two additional roots of V' exist, we have $\Delta_\theta(\theta_\pm^*) = \lambda \mathcal{E} \sqrt{\lambda / |\mathfrak{q}\,|}$, and so

$$V(\theta_\pm^*) = -a^2 C^2 (\mathfrak{q} + \mathcal{E} \sqrt{|\mathfrak{q}\,| \lambda^3}),$$

which is strictly negative when $\theta \neq \pi/2$, owing to the lower bound on $\mathcal{E}$. Since $V(\pi/2) = 0 > V(\theta_\pm^*)$, $V(\pi/2)$ is a local maximum while $V(\theta_\pm^*)$ are minima. In particular, $\mathcal{Q} \geq V(\theta_\pm^*)$, i.e. $\mathcal{Q}$ may assume a negative value. When $\mathcal{Q} < 0$, we call the orbit *vortical*, a term coined by de Felice ([10]).

Hence, we have two sets of sufficient conditions, (3.8) and (3.10), such that symmetric minima of V exist—one set of conditions for $\mathcal{L} \neq 0$ and one for $\mathcal{L} = 0$. (The $\mathcal{L} = 0$ conditions are not only sufficient but also necessary.) Our collection of facts concerning the potential $V(\theta)$ lead us immediately to the following dynamical conclusions. The first concerns the $\mathcal{L} \neq 0$ case.

Corollary 3.2. *(See Fig. 3.2.) If γ is an orbit for which* (3.8) *holds, then*

1. $\mathcal{Q} > 0 \Rightarrow \theta$ *oscillates symmetrically about* $\pi/2$;

2. $\mathcal{Q} = 0 \Rightarrow \theta$ *lies unstably at* $\pi/2$, *or approaches the equator asymptotically;*

3. $\mathcal{Q}_{min} < \mathcal{Q} < 0 \Rightarrow \theta$ *oscillates between* η_1 *and* η_2, *where*

$$\begin{array}{ccccccccccc} & \theta_- & < & \eta_1 & < & \theta_-^* & < & \eta_2 & < & \pi/2 & \\ \textit{or} & \pi/2 & < & \eta_1 & < & \theta_+^* & < & \eta_2 & < & \theta_+ \,;. \end{array}$$

4. $\mathcal{Q} = \mathcal{Q}_{min} \Rightarrow \theta$ *is lies stably at* θ_-^* *or* θ_+^*.

Remark 3.1. In Case 2, if the particle does not already lie at the equator, then because $d\Theta/d\theta|_{\theta=\pi/2} = 0$, $\theta(s)$ can only approach $\pi/2$ asymptotically. It may also approach the ring singularity $\Sigma \subset \mathbb{E}$, but only if its angular momentum is equal to $a\mathcal{E}$, as demonstrated earlier. In Case 3, particles trapped in the minima of $V(\theta)$ find themselves caged either in the northern hemisphere or the southern hemisphere of spacetime, with no possibility of crossing the equator.

As the conditions (3.8) are sufficient but not necessary, Corollary 3.2 is the most we can say about orbits with $\mathcal{L} \neq 0$. However, the $\mathcal{L} = 0$ conditions for symmetric minima are necessary and sufficient, and so we may make the following

Corollary 3.3. *If γ is a massless particle with $\mathcal{L} = 0$, or a massive particle with $\mathcal{L} = 0$ that fails to meet condition* (3.10), *then $\mathcal{Q} > 0 \Longrightarrow \theta$ oscillates symmetrically about $\pi/2$; $\mathcal{Q} = 0 \Longrightarrow \pi/2$ is a node at which γ is confined stably (see Fig. 3.1). If γ is a massive particle with $\mathcal{L} = 0$ that meets condition* (3.10), *then the θ-behaviour is precisely that described in Corollary 3.2 for particles with axial angular momentum.*

Figure 3.1: *Potential versus colatitude,* $\theta \in (0, \pi)$*, when* $\mathcal{Q}_{min} = 0$.

Figure 3.2: *Potential versus colatitude,* $\theta \in (0, \pi)$*, when* $\mathcal{Q}_{min} < 0$.

These corollaries capture a rough portrait of the θ-behaviour in Kerr-Newman-de Sitter geometry, with the primary difference between the motion in $\mathbb{K}(a, e, M, \Lambda)$ and motion in $\mathbb{K}(a, M)$ arising in the way that particles may approach the ring.

Finally, we turn to the radial coordinate, $r(s)$.

3.4 Evolution of the radial coordinate

Briefly, we consider some implications of the inequality $R(r) = \rho^4 r'^2 \geq 0$ on the motion of test particles. We pay particular attention to the existence of forbidden regions of radial motion, as they expose significant differences between the electrically-neutral and electrically-charged settings.

We consider a timelike trajectory γ, and its radial evolution equation

$$R(r) \quad = \quad \Delta_r(\mathfrak{q}\, r^2 - \mathcal{K}) + \lambda^2 \mathcal{P}^2. \tag{3.11}$$

To ensure that the right side of (3.11) is nonnegative, we require

$$\lambda^2 \mathcal{P}^2 \quad \geq \quad \Delta_r(\mathcal{K} - \mathfrak{q}\,). \tag{3.12}$$

Since $\mathcal{K} \geq 0$ (see Corollary 3.1) and $\mathfrak{q}\, < 0$, we have that $\mathcal{K} - \mathfrak{q}\,$ is strictly greater than zero. This means that whenever $\Delta_r > 0$, which occurs in block I and in part of block III, $\lambda^2 \mathcal{P}^2$ must be positive—and larger than $\Delta_r(\mathcal{K} - \mathfrak{q}\,)$. However, $\mathcal{P}^2$ is zero whenever

$$\mathcal{E} r^2 - \frac{qer}{\lambda} + a^2 \mathcal{E} - a\mathcal{L} \quad = \quad 0.$$

This quadratic equation has real roots only if and only if $q^2 e^2 \geq 4a\mathcal{E}\lambda^2(a\mathcal{E} - \mathcal{L})$. We suppose that the spacetime parameters and the first integral data of γ satisfy this condition. If either of the roots of $\mathcal{P}$ lies in the interval (r_+, r_{++}) or (r_{--}, r_-), then we will have both $\lambda^2 \mathcal{P}^2 = 0$ and $\Delta_r \mathcal{K} > 0$ at that root. In turn, R would be negative there. Consequently, by the continuity of $R(r)$, there would exist a connected interval (r^*_-, r^*_+), containing the root, on which $R(r) < 0$. Therefore, for each timelike orbit satisfying $q^2e^2 \geq 4a\mathcal{E}\lambda^2(a\mathcal{E} - \mathcal{L})$, there is the possible danger of forbidden zones in blocks I and III. (A particle travelling on an asymptotic equatorial orbit, however, might not consider these a danger—particularly if it finds itself in block III, where, depending on the location of the root(s) of $\mathcal{P}^2$, the barrier might offer safety from the ring at $r = 0$.)

Even if a particle's parameters satisfy $q^2e^2 \geq 4a\mathcal{E}\lambda^2(a\mathcal{E} - \mathcal{L})$, we cannot conclude based on this information alone that forbidden regions necessarily exist. The roots of $\mathcal{P}^2$ must be situated in blocks I and III (with $r > r_{--}$ in III) in order for the barriers to arise. While only the signs of the roots of Δ_r are known to us, we know that $\mathcal{P}^2$ is a parabola that opens up and has cusps at the roots, due to the squaring of $\mathcal{P}$. As $\mathcal{P} = (r^2 + a^2)\mathcal{E} - qer/\lambda - a\mathcal{L}$, we may adjust the position of the roots by changing the magnitude of $\mathcal{L}$. When we choose $\mathcal{L} > 0$ to be larger, $\mathcal{P}$ gets pulled vertically downwards, and so the roots of $\mathcal{P}^2$ occur wider apart. When $\mathcal{L}$ is chosen to be smaller, the roots get closer together. In terms of the condition $q^2e^2 \geq 4a\mathcal{E}\lambda^2(a\mathcal{E} - \mathcal{L})$, choosing $\mathcal{L}$ to be positive and large only makes it easier for the inequality to be satisfied. When we choose $\mathcal{L}$ small, we need only ensure that q^2e^2 is large (or to be more physically reasonable, that $\mathcal{E}$ is small). In this manner, we are able to construct timelike orbits with barriers in I or III by choosing $\mathcal{L}$ sufficiently large or small (so as to place at least one of the roots of $\mathcal{P}$ in the desired location).

The condition $q^2e^2 \geq 4a\mathcal{E}\lambda^2(a\mathcal{E} - \mathcal{L})$ highlights an important difference between the Kerr and Kerr-Newman spacetimes. When the charge of spacetime is zero, the condition reduces to $\mathcal{E}(a\mathcal{E} - \mathcal{L}) \leq 0$, which is satisfied if and only if $0 \leq a\mathcal{E} \leq \mathcal{L}$ or $\mathcal{L} \leq a\mathcal{E} \leq 0$. Thus, orbits may only encounter forbidden zones if the size of the orbit's angular momentum bounds that of its energy (times a). But as $q^2e^2 \geq 4a\mathcal{E}\lambda^2(a\mathcal{E}-\mathcal{L})$ shows, this is not a strict requirement in Kerr-Newman(-de Sitter).

Moreover, a nonzero cosmological constant introduces features of radial travel that do not exist in asymptotically-flat Kerr-Newman spacetimes. While $R(r)$ is a quartic polynomial in r in the asymptotically-flat Kerr geometry, the same function is sextic in r in Kerr-Newman-de Sitter, with dominant term $-\Lambda\mathfrak{q}\, r^6/3$. (We can see this by inserting the definitions of Δ_r and $\mathcal{P}(r)$ directly into the radial equation.) One geometric, though not necessarily physical, consequence of this fact is that no spacelike geodesics may be found at infinity in the de Sitter block.

As a final remark, we observe that $R(r)$ can be written in the form

$$R(r) \;\;=\;\; -\Delta_r\mathcal{Q} + R^*(r)$$

where $R^*(r)$ is $R(r)$ with $\mathcal{K} = \lambda^2(\mathcal{L}-a\mathcal{E})^2$. By the discussion in the previous section, this makes $R^*(r)$ the radial equation for particles moving in, or asymptotically to, the equatorial plane. If we let $\mathcal{Q}$ increase while the other first integrals are left fixed, we see that any forbidden regions will expand, consuming more and more once-navigable r values. In this way, Carter's constant finds another meaning.

Conclusion

In this thesis, we have outlined the basic geometric properties of the family of Kerr-Newman-de Sitter spacetimes, explained the physical significance of those properties, and made choices of parameters reflecting current observations in experimental physics. Using the Liouville-Arnol'd integrability theory, we extracted sufficiently-many independent first integrals for the spacetime motion of charged test particles. Amongst these first integrals is the so-called Carter's constant, whose discovery owes to the fundamental work in [2]. By attempting to interpret this constant physically and geometrically, we exposed several differences between orbits in Kerr-Newman-de Sitter and geodesics in Kerr. Some highlights are:

- Lazy particles, devoid of energy or angular momentum, are no longer constrained to closed, totally-geodesic submanifolds.

- Only particles with $\mathcal{K} = 0$ and $\mathcal{L} = a\mathcal{E}$ may approach the curvature singularity at $\Sigma = \{r = 0, \theta = \pi/2\}$, which is an effect of the electromagnetic field combined with the black-hole rotation. All particles—charged or uncharged—are subject to this restriction.

- The nonzero charge of the black hole makes it easier for forbidden regions of radial travel to arise. The size of a particle's forbidden region is intrinsically related to the size of Carter's constant $\mathcal{Q}$.

As for similarities between the Kerr and Kerr-Newman-de Sitter settings, the co-latitudinal behaviour of orbits in $\mathbb{K}(a, e, M, \Lambda)$ was found to be largely identical to that $\mathbb{K}(a, M)$, save for the much more complicated nature of the potential $V(\theta)$.

Future work can be carried out in several directions:

- the existence (particularly the number) of black-hole ergospheres, and the corresponding Penrose energy extraction process;

- the existence of closed timelike curves;

- the expansion of the study to anti-de Sitter cosmologies, and accordingly to the investigation of θ-horizons;

- the treatment of global dynamics in maximally-extended Kerr-Newman-de Sitter spacetimes.

Appendix A

Quartic polynomials

The details of Proposition 1.1, omitted from Chapter 1, are included below.

Proposition A.1. *There exists no anti-de Sitter Kerr spacetime* ($\Lambda < 0$) *with more than two radial horizons. If* $\Lambda > 0$ *is such that* $\Lambda^2 << 1$, $\Lambda a^2 << 1$ *and* $a^2 + e^2 << M^2 << \Lambda^{-1}$, *then* Δ_r *has four distinct real roots.*

Proof. The radial horizons of a Kerr-Newman-de Sitter spacetime $\mathbb{K}(a, e, M, \Lambda)$ are determined by the roots of the quartic polynomial

$$\Delta_r(r) \quad = \quad -Lr^4 + (1 - La^2)r^2 - 2Mr + a^2 + e^2,$$

where $L := \Lambda/3$.

If $p \in \mathbb{R}[x]$ is an n-th degree *parametric polynomial* (a polynomial whose coefficients are functions of one or more parameters), we can in principle extract necessary and sufficient conditions for the realization of a desired root structure—for instance, n real and distinct roots. These conditions normally take the form of a list of $d \leq n$ determinants, combined with lists of signs. Each sign list has length d and instructs us as to whether each determinant is to be positive, negative, identically zero, or any of the three. Associated to each possible root structure of p is at least one sign list. The realization of the signs in one of the lists coupled to a root structure is a sufficient condition for p to exhibit that structure. Conversely, if p has a given root structure, then one of the sign lists associated to it must be satisfied by the determinants.

For an arbitrary real parametric polynomial of degree four, a determinant list and sign lists are given in [15]. From this information, we distill the following information regarding the horizon function: *The polynomial Δ_r has four distinct real roots exist if and only if*

$$(D1) \quad \wedge \quad (D2) \quad \wedge \quad (D3),$$

where:

$$L - L^2a^2 \quad > \quad 0 \quad \text{(D1)}$$

$$-L \times (-1 + L^2a^4 - 4Le^2 - La^2 + L^3a^6 + 4e^2L^2a^2 + 18M^2L) \quad > \quad 0 \quad \text{(D2)}$$

$$\begin{aligned} -L \times (32L^2e^4a^2 + L^3a^6M^2 + 12L^3a^6e^2 - 33M^2La^2 + 22L^2a^4e^2 - 36Le^2M^2 \\ +12e^2La^2 + 8L^3a^4e^4 + L^4a^8e^2 + 33L^2a^4M^2 + 36L^2a^2e^2M^2 + a^2 + 4a^4L \\ -M^2 + e^2 + 6L^2a^6 + 8Le^4 + 4L^3a^8 + L^4a^{10} + 27LM^4 + 16L^2e^6) \quad > \quad 0 \quad \text{(D3)}. \end{aligned}$$

Since the coefficients of $\Delta_r(r)$ are polynomials in a, e, M, and L, the three determinants above are also polynomials in those parameters. For four real and distinct roots, there is a single sign list, demanding that each determinant be positive. We would like to show that the hypotheses of the Proposition are sufficient conditions for the sign list to be realized.

If $L < 0$, then condition (D1) is equivalent, after dividing both sides by L, to $La^2 > 1$. Since this can never hold for $L < 0$, the conditions can never be realized. Hence, there is no anti-de Sitter Kerr spacetime with more four horizons, and subsequently, there can be at most two. (Actually, as the condition (D1) reduces to $L > 0$ in the non-rotating de Sitter case, this is a property of all Lorentz manifolds of constant negative scalar curvature.)

Now, we consider the hypotheses on the parameters in the case that $L > 0$. In this case, (D1) reduces upon division by L to $La^2 < 1$, which is true by hypothesis. Upon our dividing both sides of condition (D2) by $-L$, the inequality to be satisfied becomes

$$P_2(a, e, M, L) := -1 + L^2a^4 - 4Le^2 - La^2 + L^3a^6 + 4e^2L^2a^2 + 18M^2L \quad < \quad 0. \quad \text{(D2)}'$$

Examining the terms above, we may write the function P_2 in a form that is far easier to inspect:

$$P_2(a, e, M, L) \quad = \quad (La^2 - 1)\left[1 + L(a^2 + 4e^2)\right] + 18M^2L.$$

Since $La^2 << 1$, we may safely take $La^2 - 1 \approx -1$, in order to write

$$P_2(a, e, M, L) \quad \approx \quad -1 - 4Le^2 + 18M^2L.$$

The condition to be satisfied reduces in turn to

$$4e^2 + M^2 < 1/L.$$

As $e^2 << M^2$ and $M^2 << 1/L$, the condition is realized.

Finally, in (D3), we safely ignore any terms involving $L^2 << 1$ or higher powers of L. If $P_3(a, e, M, L)$ is the polynomial on the left side of the inequality in (D3) (after division by $-L$), then

$$P_3(a,e,M,L) \approx -M^2 + L(-36e^2M^2 - 33M^2a^2 + 12e^2a^2 + 4a^4 + 8e^4 + 4a^4 + 27M^4)$$

is a valid approximation, given the assumptions made on the parameters. Let us put $P_{31} := P_3 + M^2$. The condition $P_3 < 0$ is then equivalent to $P_{31} < M^2$, or equivalently $P_{32} < 1$, with $P_{32} = M^{-2}P_{31}$. But

$$P_{32}(a,e,M,L) \approx 27LM^2.$$

By hypothesis, $M^2 << 1/L$, and so $P_{32} << L/L = 1$. Condition (D3) is therefore satisfied.

□

As a test, we compute horizons for given parameters, say $a = 0.01$, $e = 0.05$, $M = 2000$, and $3\Lambda = L = 10^{-9}$. The roots are:

$$\begin{aligned} r_{--} &\approx -33459.6 \\ r_{-} &\approx 6.5 \times 10^{-7} \\ r_{+} &\approx 4067.3 \\ r_{++} &\approx 29392.3\,. \end{aligned}$$

As we expect, there is a single negative horizon and three positive ones. It is interesting to note how near to the ring is the positive root r_-. While there are no physical units at play here, relative scale is meaningful: the other three horizons are astronomically farther from the ring. (In terms of our results in Chapter 3, these choices of parameters lower the likelihood for a equatorial particle in block III to have a wall of $R(r) < 0$ between it and the ring).

Suppose that we increase Λ by a single order of magnitude, and then repeat the calculation. The roots become

$$\begin{aligned} r_{--} &\approx -11597.0 \\ r_{-} &\approx 6.5 \times 10^{-7} \\ r_{+} &\approx 5798.5 + 932.0i \\ r_{++} &= \overline{r}_{+}\,. \end{aligned}$$

As we can see, the existence of a maximal number of roots is extremely sensitive to the size of Λ. If we now decrease M by an order of magnitude to compensate, we revive the four real roots, but find that r_- scales by an order of magnitude away from unity (whereas changing Λ had no appreciable effect on r_-). The change in M also brings each of r_+ and r_{++} an order of magnitude closer to unity. Hence, one root moves away from the ring (increasing the likelihood of block III barriers) while two move closer. In the large-scale picture, the "Kerr" half of spacetime loses volume to the "de Sitter" half. Letting $a, e, M \to 0$ and fixing $L = 10^{-8}$, we are left with three roots: a double root $r_- = r_+ = 0$, and the pair $r_{--}, r_{++} = \pm\sqrt{L^{-1}} = \pm 10000$, which overlap in the maximally-extended de Sitter spacetime to form a single cosmological horizon, as discussed in Chapter 1.

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