Space-time duality, superduality, and effective actions on anti-de-Sitter space-time

by

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Abstract

In part I of the thesis, a new class of duality symmetries amongst quantum field theories is introduced. The new class is based upon global *spacetime* symmetries, such as Poincaré invariance and supersymmetry, in the same way as the existing duality transformations are based on global *internal* symmetries. An interesting feature of the new duality transformations is that they can lead to fermions as the new dual variables. Illustrations of the new duality transformations offered in the case of scalar and spin-half field theories in 1+1 space-time dimensions, as well as $(1,1)$ and $(2,2)$ supersymmetric models. For $(2,2)$ models the new duality transformations can change whether a chiral multiplet is twisted or not.

In part II of the thesis, closed forms are derived for the effective actions for free, massive fields in anti-de-Sitter space-times in arbitrary dimensions. The results have simple expressions in terms of elementary functions (for odd dimensions) or multiple Gamma functions (for even dimensions). In the case of scalar fields in two dimensional anti de-Sitter space-time, the effective action is used to argue against the quantum validity of a recently proposed classical duality relating such theories with differing masses.

Résumé

Dans la première partie de cette thèse, est introduite une nouvelle classe de dualité entre les théories quantiques de champs. Cette nouvelle classe est basée sur des symétries globales spaciales, comme invariance de Poincaré et supersymétrie, de la même façon que présentement sont basés les dualités sur les symétries globales internes. Un aspect intéressant de la nouvelle dualité est que la transformation peut avoir des fermions comme nouvelles variables duelles. Les exemples de la nouvelle transformation sont presentés pour les cas de champ scalar et fermion en l'espace de (1+1) dimensions, ainsi que pour les modèles de supersymétrie (1,1) et (2,2). Dans le cas de modèles de type (2,2), la nouvelle dualité implique l'échange entre les modèles chiral qui sont et qui ne sont pas "twisted".

Dans la deuxiéme partie de la thèse, on calcule des formes exactes pour des actions effectives de champs scalars, massives et libres, dans l'espace anti-de-Sitter en dimensions générales. Dans les dimensions impaires, les résultats sont exprimés en fonctions élémentaires, et dans les dimensions paires, en fonctions G_n, dit "multiple gamma". Dans le cas de champ scalar en espace anti-de-Sitter de deux dimensions, on utilise l'action effective pour plaider contre la validité, dans le domaine quantique, d'une dualité entre les champs classiques proposés récemment.

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Finally, I would like to pay respects to the memory of Prof. Bernard Margolis, who accepted me as his student when I first arrived at McGill and with whom I began the Ph.D., and whose sudden departure was sadly premature.

Statement of Original Contribution

The calculations and results in the first part of the thesis, space-time duality and superduality, were carried out in collaboration with, Cliff Burgess, Marc Grisaru, Marcia Knutt, Philippe Page, Fernando Quevedo and Muhammad Zaberjad [1]. The author of the thesis is responsible for deriving constraint actions, and in the case of superduality, for the use of a supersymmetric multiplet of gauging fields.

The second part of the thesis is based on work done in collaboration with Cliff Burgess [2]. Here the author derived new relations among multiple Gamma functions, and used them in the evaluations of effective actions.

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Chapter 1

Introduction

This thesis is comprised of two parts. In the first part of the thesis a new duality formalism is presented, which generalizes useful ideas of duality based on global symmetries. The second part of the thesis involves the calculation of effective actions on anti-de-Sitter (AdS) space-times. These are a priori distinct subjects, which nonetheless find common ground as applications of quantum field theory (QFT) to curved space-times.

Duality is an invertible map among QFT's which proves their equivalence. As such, it is a useful piece of information, which may extend the applicability of usual calculational techniques to previously inaccesible domains. For instance, suppose a duality mapping associates the weak coupling regime ($g \ll 1$) of one QFT with the strong coupling regime ($e \gg 1$) of the dual by an inverse relation ($\frac{1}{g} = e$). One may perform well controlled approximations in the first theory, via usual perturbation theory and, using the duality, map the results onto the other formulation. This is effectively a well controlled analytical (rather than numerical) approximation in the non-perturbative sector of the latter theory.

Beyond its usefulness as a calculational tool, duality sheds light on the meaning of the QFT's it relates. The strong-weak coupling duality points to the following observation: that if thinking about a real physical system is best approached from perturbing a non-interacting idealization, then natural field(s) for specifying a QFT may differ from one coupling regime to another. The actual mapping may be of interest in itself, for instance relating the electric charge in one formulation to a solitonic charge in the other. It may also turn out that the map is non-local, and that the excitation of some state in one formulation contains an indefinite number of excitations in terms of the dual field.

Although examples of duality maps exist, including the particular strong-weak coupling case whose properties are listed above, there are only a few algorithms for generating dual pairs of QFT's. It may be argued that a successful algorithm should have the following properties: i) it should relate *local* field theories, and ii) it should be relatively simple to implement, at least within certain approximations.

One presently available duality construction is based on the presence of an internal rigid symmetry in the QFT. This algorithm is called "gauge duality" in this thesis. Examples of gauge duality in the literature are so-called T-duality and p-form duality. In gauge duality, the global symmetry is made local (gauged) by introducing a gauge field with the appropriate symmetry transformation properties. In the path integral, this is accomplished by integrating over the appropriately gauge-fixed new field. However, in order to maintain equivalence with the original formulation, the gauge field must be constrained away. One may enforce the constraint with a Lagrange multiplier(s). The dual formulation is reached by evaluating all path integrals except for the Lagrange multiplier(s), which then play the role of the new dynamic fields.

A few points bear notice with respect to the procedure just outlined. Duality, since it is expressed in the path integral language, is inherently a quantum mechanical mapping. In particular, one should make a distinction between duality and a classical canonical transformation relating two systems. Although there are dual systems whose classical limits are related by a canonical transformation (for instance for Maxwell equations in vacuum, swapping electric for magnetic field), this scenario is not the rule. An example of a canonical mapping failing as a quantum map is presented in Chapter 5.

The generalization of gauge duality presented in this thesis is rather straightforward. It is achieved by replacing a global internal symmetry with a space-time symmetry in the duality transformation, whence "space-time" duality. The new duality prescription may be applied to systems which lack an internal symmetry. Further, nothing prevents applying space-time duality to systems with gauge freedom, and at least in principle, there is a possibility of discovering a more interesting pattern of equivalence.

The second part of the thesis deals with the contribution of quantum matter to the effective actions in AdS space-time. An effective action describes gravitational dynamics in the presence of quantum matter. This is interesting, because evaluating path integrals in non-trivial gravitational backgrounds potentially offers insight into the interrelation of the quantum world and gravitation. From a more practical perspective, calculating effective actions on AdS may serve as the starting approximation for more realistic calculations, where the background is less symmetric.

The main result of the second part of this thesis is of technical interest: it is the observation that effective actions on AdS may be expressed simply in terms of multiple Gamma functions, which have known integral and infinite product representations, as well as asymptotic expansion. A new integral representation of the multiple Gamma functions and some integral relations are derived as well.

The outline of this thesis is as follows. The remainder of the Introduction is a presentation of some aspects of quantum field theory in curved space-times.

In Chapter 2, the reader is reminded of the gauge-duality transformation. The generation of dual pairs based on an internal rigid symmetry is summarized using a step-by-step prescription, and an example is worked out to demonstrate the procedure.

Chapter 3 is devoted to the presentation of space-time duality. Here, instead of an internal symmetry, one gauges Lorentz invariance. Several examples are presented, including the "incomplete" bosonization of the Majorana fermion, a result which is not obvious, albeit expected, from gauge-duality.

In Chapter 4, space-time duality is extended to the case of supersymmetry, whence "superduality". A brief review of supersymmetry and supergravity is given, together with an introduction to the corresponding superspace formulation. Three calculations are presented, including the duality of chiral and twisted-chiral matter in (2,2) supergravity, which is perhaps the most interesting result in this thesis.

Chapter 5 contains a discussion of maximally symmetric spaces (MSS's) of which AdS space-time is an example. This is followed by a brief review of the path integral formulation of effective action calculations in curved backgrounds. The evaluation of the effective action is then specified to the scalar field living on AdS in arbitrary dimensions. Results for effective actions are compared with previous evaluations where available. Further, results for higher-spin fields are presented. In the case of a scalar field in two-dimensional AdS space-time, it is shown that a symmetry proposed recently for the classical action is not realized once quantum mechanical effects are considered.

The main results are recapped in the chapter entitled Summary and Outlook, which also includes an outline of future work stemming from the ideas in this thesis.

A summary of notation is presented in Appendix A, which although different from most other notations (an unfortunate standard in the field!), should make clear to the reader the material in the body of the thesis.

The formulation of a well defined path integral is of particular importance in light of subsequent chapters. Its construction is included in Appendix B, together with anomaly calculations.

Finally in Appendix C, key definitions and properties pertaining to the multiple Gamma functions are presented, together with a derivation of a new integral representation.

1.1 Quantum Fields in Curved Space time

The setting for calculations in this thesis is QFT in the presence of background gravitational field. In this section a brief review of selected aspects of QFT is offered; for a more detailed account the reader is referred to standard textbooks on field theory in curved spaces [3]. The present discussion begins with standard Minkowski space QFT, which is then generalized to curved space-times.

1.1.1 QFT on Minkowski Space-time

QFT on Minkowski space-time is based on the symmetry principle that physical observables are invariant under Poincaré transformations. QFT may be described by its action on a Hilbert space of states. Particle states are unitary representations of the Poincaré symmetry, where one physical particle state corresponds to an infinite collection of Hilbert space states related by the action of Poincaré symmetry.

Interactions among particles are dictated by details of the QFT, and must preserve Poincaré symmetry. A Lorentz invariant Lagrangian may be written in terms of fields, which when expanded in terms of the momentum modes describes the interaction among the particle states. Unlike particle representations of Poincaré symmetry, which are unitary and infinite dimensional, field representations are finite dimensional but not unitary.

In addition to Poincaré symmetry, other symmetry requirements may be placed on a QFT. The construction of a QFT invariant under several symmetries may be carried out by looking for multiplets of Lorentz invariant fields (and their actions) which represent the additional symmetry. Representations of supersymmetry in Chapter 4 are presented from this point of view.

1.1.2 QFT on a Curved Space

In this subsection Poincaré invariant QFT is generalized to curved backgrounds. In the path integral framework, going from Poincaré invariant to general coordinate transformation (GCT) invariant QFT requires the following steps.

1. In the Minkowski space-time scalar Lagrangian density $\mathcal{L}$, space-time derivatives are replaced by covariant derivatives, thereby making $\mathcal{L}$ a covariantly scalar quantity.

2. The integration measure for the action is multiplied by the determinant of the metric factor $\sqrt{-\det[g_{mn}]}$, to make it invariant under GCT's.

3. The path integral measure $[\mathcal{D}\phi]$ (for the generic field ϕ) is replaced with a GCT invariant measure $[\mathcal{D}\phi]_g$, which depends on the metric.

The first two steps ensure that the action is invariant under GCT's. Construction of a GCT invariant path integral measure is delegated to Appendix B. One important point regarding the path integral measure is the following. Besides GCT invariance, an action may contain additional symmetries, but it is not always possible to find a measure which preserves all the symmetries. In particular, the fact that conformal invariance of the action is anomalous, for the GCT invariant measure, is used in the duality calculations of Chapters 3 and 4.

Chapter 2

Gauge Duality

Gauge duality is a duality prescription based on an internal rigid symmetry, whose generalization to space-time duality and superduality are the main results of this thesis. This chapter contains a review of selected aspects of gauge duality, taken from the point of view most useful for its subsequent generalization.

Gauge duality is introduced by Buscher in reference [5]. He first considers the classical limit of a QFT model, i.e., he looks at the *action* which contains both dynamical fields (whose time evolution is of interest) as well as external fields, which couple to the dynamic sector. He proceeds to show that due to a global symmetry present in the model, the (classically) dynamic system may be parameterized by a different set of fields — where the difference is made explicit by the coupling to the external fields. This is a warm-up to the more interesting case in QFT. He considers next the quantum mechanical analogue of the same system by placing the action in a path integral, and by integrating over the dynamical sector while keeping the external fields fixed. It turns out that, quantum mechanically, the same physics may be rewritten in terms of a new set of dynamic fields, with *new* signifying a different coupling to the background. The new dynamic-external field coupling is also different from the classical prelude to the problem.

The essential ingredient needed for Buscher's result is the presence of an internal, rigid symmetry. It is the replacement of this internal symmetry with a space-time symmetry that is the subject of the subsequent chapters of this thesis. In the remainder of Chapter 2 a concrete example of Buscher's duality is presented, based on the point of view advocated in ref. [6].

2.1 Path Integral Derivation of Gauge Duality

The aim of this section is to clarify the procedure for generating a dual theory. For this purpose it is sufficient to consider the simple case of a massless scalar field in (1+1) dimensions. The path integral formulation of QFT is useful for the present derivation of duality.

Consider a system of a free scalar field, ϕ, in flat space-time, with the quantum dynamics captured by the following path integral:

$$Z[a_m] = \int [\mathcal{D}\phi] \exp\left\{-\frac{i}{2}\int d^2x\ (\partial_m\phi - a_m)(\partial^m\phi - a^m)\right\} \quad . \tag{2.1}$$

In addition to the dynamical scalar field, a coupling to an external background (non-integrated) gauge field, a_m, is included here because it facilitates the generalization to the case of space-time duality in Chapter 3. Additionally, the result after performing the path integral has a dependence on the a_m field. This is important, because otherwise the path integral would yield a number, and overall constants may be ignored when performing functional integrals, since they drop out in the normalization. Physically, (logarithmic) derivatives of $Z[a_m]$ with respect to a_m are the T^*-ordered, connected correlation functions for the operator $\partial_m\phi$ (as discussed later in this chapter).

When following the dependence of background fields, such as a_m, it is important to keep in mind that some dependence may appear implicitly in the definition of the measure of the functional integration. This is so, because the measure is chosen to preserve a set of symmetries, whose transformation rules may involve the background fields as in the case of (super)gravitational fields in subsequent chapters. Fortunately this complication does not arise for the simple system considered here.

The key observation, from the point of view of duality, is the presence of a *rigid* internal invariance in the path integral (2.1): the global symmetry, $\phi \to \phi - \omega$, for arbitrary constant ω. *Rigid* refers here to the fact that having chosen the amount by which field ϕ transforms at one point in space-time, the transformation is fixed for all other points.

The chosen transformation does not exhaust all symmetries of the system — in fact there is a related *local* gauge invariance, whereby $\phi \to \phi - u(x)$ together with $a_m \to a_m + \partial_m u(x)$. This is a transformation on the background, which is irrelevant for the purpose of demonstrating duality.

The following steps outline the duality procedure for the theory (2.1):

1. The global symmetry $\phi \to \phi + \omega$ is gauged by letting ω be an arbitrary function of space-time, and by introducing a compensating dynamical gauge field A_m over which a functional integral is to be performed.

2. To be well defined, the evaluation of the path integral over A_m requires a gauge-fixing condition, $\mathcal{F}(A_m) = 0$, together with the corresponding Fadeev–Popov–deWitt determinant, J_{FPW}.

3. Next, a gauge-covariant constraint is imposed, which makes A_m a pure gauge. This constraint, together with the gauge condition just described, is designed to ensure that the path integral over A_m is equivalent to evaluating the integrand at the configuration $A_m = 0$, and therefore to maintain equality with (2.1).

4. The constraint of Item 3 is expressed by introducing a Lagrange multiplier (Λ) whose path integration imposes this constraint. Integrating over Λ, and then over A_m, therefore, reproduces the original theory, (2.1).

5. Finally, performing the path integral in a different order: integrating first over ϕ and A_m and leaving the integral over Λ unperformed, results in the dual formulation of the same physical theory, with Λ playing the role of the new field variable.

Following the steps just outlined, the partition function (2.1) may be rewritten as follows:

$$
\begin{aligned}
Z[a_m] \;&=\; \int [\mathcal{D}\phi][\mathcal{D}A_m]\Delta_{LM}[A_m + a_m, a_m]\Delta[\mathcal{F}(A_m)]J_{FPW} \times \\
&\qquad \exp \times \left\{ -\frac{i}{2} \int d^2x \; (\partial_m \phi - a_m - A_m)\,(\partial^m \phi - a^m - A^m) \right\} \;, \qquad (2.2)
\end{aligned}
$$

where Δ stands for a functional δ function. $[\mathcal{D}A_m]$ refers here to summing over all possible values of the field A_m, including the configurations related by a gauge transformation. To have a well defined integration, gauge "overcounting" is eliminated by choosing a functional gauge-fixing condition $\mathcal{F}(A_m) = 0$ (functional, since there are infinitely many gauge related configurations). The choice of a particular gauge condition is rendered arbitrary with the Fadeev–Popov–deWitt determinant, J_{FPW}, defined as the functional determinant of the variation of the gauge-fixing condition with respect to a gauge transformation. It may be shown that the gauge field measure is invariant under a gauge transformation, by an exact cancellation between $\Delta[\mathcal{F}(A_m)]$ and J_{FPW} [7].

The explicit expression for the Lagrange multiplier constraint may be derived from the requirement that eq. (2.2) reduces to eq. (2.1). It is expressed as a path integral over a scalar field Λ,

$$
\Delta_{LM}[A_m + a_m, a_m] = \int [\mathcal{D}\Lambda] \exp \left\{ -i \int d^2x \, \Lambda \, \epsilon^{mn} \partial_m A_n \right\} \quad , \qquad (2.3)
$$

so that vector field A_m is constrained to have a vanishing field strength. This allows the rewriting of (2.2) as follows:

$$
\begin{aligned}
Z[a_m] \;&=\; \int [\mathcal{D}\phi][\mathcal{D}A_m]\Delta[\mathcal{F}(A_m)] \; J_{FPW}[\mathcal{D}\Lambda] \times \\
&\quad \exp \left\{ -\frac{i}{2} \int d^2x \; \left((\partial_m \phi + a_m + A_m)\,(\partial^m \phi + a^m + A^m) + 2\Lambda \epsilon^{mn} \partial_m A_n \right) \right\} \,. \\
&\hfill (2.4)
\end{aligned}
$$

In order to check that the Lagrange multiplier action (2.3) is correct, consider eq. (2.4) in detail. The gauge field may be expanded as follows: $A_m = \partial_m Y + \epsilon_{mn} \partial^n Z$, where $Y = \frac{1}{\Box} \partial^m A_m$ and $Z = \frac{1}{\Box} \partial^m \epsilon_{mn} A^n$. Although the gauge field expansion is non-local, it is still well defined in the path integral context, where it is used as a change of integration variables. The gauge field measure may be expressed in terms of Y and Z as follows: $[\mathcal{D}A_m] = [\mathcal{D}Y][\mathcal{D}Z]\det[-\Box]$, where the determinant term is the Jacobian. Taking Lorentz gauge, $\mathcal{F} = \partial^m A_m$, implies the gauge-fixing condition $\Delta[\Box Y] = (\det[-\Box])^{-1}\Delta[Y]$. The Fadeev–Popov–deWitt factor, J_{FPW}, is equal to $\det[-\Box]$, and the Lagrange multiplier path integral enforces the condition $\Delta[\Box Z] = (\det[-\Box])^{-1}\Delta[Z]$. Combining all these factors in (2.4), one sees that the powers of $\det[-\Box]$ cancel, and that one is left with the original action (2.1).

For the dual formulation the order of integration is exchanged, integrating first A_m and ϕ. As before, the gauge choice $\mathcal{F} = \partial^m A_m = \Box Y$ is made, with $J_{FPW} = \det[-\Box]$. The integral over A_m is Gaussian, so completing the square in the action (2.4),

$$\frac{1}{2}\left(a_m + A_m\right)\left(a^m + A^m\right) + \epsilon^{mn}\Lambda\partial_m A_n \tag{2.5}$$

and performing the A_m integral leads to the following expression,

$$Z[a_m] = \int [\mathcal{D}\Lambda] \exp\left\{-\frac{i}{2}\int d^2x \;\left(\partial_m\Lambda\partial^m\Lambda - 2\epsilon^{mn}a_m\partial_n\Lambda\right)\right\} \quad . \tag{2.6}$$

This is then the gauge-dual of the original partition function. The significance of the present dual formulation rests in the observation that the coupling to a_m differs in eq. (2.6) from that of eq. (2.1). In particular, the difference in the linear term in the respective actions indicates that the field operators dualize according to the standard relation (Poincaré duality):

$$\partial_m\phi \leftrightarrow \epsilon_{mn}\,\partial^n\Lambda. \tag{2.7}$$

In addition, notice that the action for ϕ in eq. (2.1) contains the quadratic term, $a_m a^m$, but no such term appears in the dual action for Λ in eq. (2.6). This also has physical implications, since twice differentiating eqs. (2.1) and (2.6) implies:

$$(-i)^2 \left.\frac{\delta^2 Z}{\delta a^m(x)\,\delta a^n(y)}\right|_{a_m\to 0} = \langle\epsilon_{mp}\partial^p\Lambda(x)\epsilon_{nq}\partial^q\Lambda(y)\rangle = \langle\partial_m\phi(x)\partial_n\phi(y)\rangle + i\eta_{mn}\delta^2(x-y), \tag{2.8}$$

where η is the usual Minkowski-space metric. $\langle\cdots\rangle$ here indicates the covariant T^* product, which is related to the usual time-ordered (T) product by, for example,

$$\left\langle 0 \left| T^* \left[\frac{\partial}{\partial x^m}\phi(x)\frac{\partial}{\partial y^n}\phi(y)\right]\right| 0 \right\rangle \equiv \frac{\partial}{\partial x^m}\frac{\partial}{\partial y^n}\langle 0|T[\phi(x)\phi(y)]|0\rangle \quad . \tag{2.9}$$

In a Lorentz-covariant QFT the naive time ordering of operators, inside the expectation value, is not satisfactory for derivatives of the fields. This stems from the fact that time-ordering (T) is not a symmetric operation between space- and time-derivatives. The time ordering is "improved" to maintain Lorentz-covariance, by adding sufficient terms to the T prescription so to effectively pull derivatives of operators outside the expectation value. This T^* prescription is manifestly Lorentz-covariant, and it is in fact the expectation value that comes from the path-integral formulation of QFT.

In the T^* prescription for operator ordering, the δ-function contact term in the last of the equalities in (2.8) is just what is required for this equation to make sense. The correspondence (2.7) implies that time derivatives, $\partial_t\phi$, dualize to space derivatives, $\partial_x\Lambda$, and while time derivatives get δ-function contributions when the derivatives hit the time ordering ($\partial_t\Theta(t) = \delta(t)$) space derivatives do not. Field commutation relations can be used to show that the contact term is just what is required to make both sides of eq. (2.8) agree.

Chapter 3

Space-time Duality

In this chapter a generalization of the prescription for dualizing a QFT is presented. Following some general considerations, the new duality algorithm is summarized with a step-by-step prescription, and three examples are worked out to argue in favour of the proposed method. The examples of space-time duality are the self-duality of a massless scalar field, bosonization of a Dirac fermion and the incomplete bosonization of a Majorana spinor.

In Chapter 2, the key step in the gauge-duality prescription is the gauging of an internal rigid symmetry. It is interesting to consider whether or not other symmetries can serve as a mechanism for (some other) duality transformations. In this chapter some first steps are taken in this direction and a new duality formalism based on Lorentz invariance is presented, which is called here space-time duality.

In the example of the scalar invariant under a constant shift (2.1), the dual field is the Lagrange multiplier from eq. (2.3) which eliminates the field strength of the dynamic gauge field. Therefore, the Lagrange multiplier contains the same number of degrees of freedom as the field strength. An equivalent point of view for the same example is the following: the Lagrange multiplier eliminates the degrees of freedom remaining after the Fadeev–Popov–deWitt gauge fixing of the dynamic gauge field. For the gauge vector field A_m in $(1+1)$ dimensions there is one non-gauge degree of freedom, which implies that one degree of freedom is needed in the Lagrange multiplier. Therefore the dual theory has one dynamic degree of freedom, a scalar field, with coupling to the background different from the original.

For space-time duality in this chapter and for superduality in the next, constraining the field strength is not equivalent to constraining the non-gauge degrees of freedom. Constraints on the gauge fields are more convenient than constraining field strengths, since they can be expressed as actions linear in the gauge field which is being integrated in the path integral.

The purpose of this chapter is to demonstrate that a duality prescription is available for the Lorentz symmetry, and the subsequent chapter is devoted to establishing the validity of superduality, or the duality based on supersymmetry. This is a novel idea, because Lorentz invariance is fundamentally different from a gauge symmetry as it acts on space-time coordinates in addition to the internal field degrees of freedom. Supersymmetry is a hybrid space-time/gauge symmetry, in the sense that supersymmetry acts on both space-time coordinates and on internal field degrees of freedom.

3.1 General Algorithm

The steps outlining the procedure of gauge duality are presented in Section 2.1. In this section, the gauge duality procedure is naively modified to the case of Lorentz symmetry. The remainder of Chapter 3 is devoted to demonstrating the validity of the proposed procedure.

1. The global Lorentz symmetry is made local (gauged) by introducing a dynamical metric field (g_{mn}), which is integrated in the generating functional.

2. As usual in theories with local freedom, one must divide out the volume of the gauge group in the path integral measure, which in this case is the group of diffeomorphisms (i.e. local coordinate transformations); equivalently a gauge fixing condition $\mathcal{F}(g_{mn})$ is introduced together with the corresponding Fadeev–Popov–deWitt determinant.

3. A constraint is imposed stating that the dynamic metric may be chosen to be a pure diffeomorphism of the background, by introducing the necessary Lagrange

multiplier(s).

4. Performing the path integrals over the Lagrange multipliers has to reproduce the original theory. This is a check on the Lagrange multiplier action.

5. Integrating out the dynamic metric gives the dual theory, with the Lagrange multipliers acting as the new fundamental fields.

Immediately it may be deduced that in (1+1) dimensions the dual system contains one degree of freedom. There are three degrees of freedom in the (symmetric) metric tensor, and the local symmetry of diffeomorphism invariance can be parametrized with a two component vector field. This leaves one degree of freedom which remains after gauge fixing, and this degree of freedom needs to be constrained with a Lagrange multiplier which becomes the dynamic field in the dual theory.

In three dimensional space-time, the dual fields contain, in total, three degrees of freedom coming from six independent components of the metric less three degrees of freedom for diffeomorphism invariance. A similar argument yields six degrees of freedom for the dual theory in four dimensional space-time.

3.2 2D Massless Scalar

In order to test the idea of dualizing on a space-time symmetry, it is useful to consider a simple system, such as a real massless scalar field in a background gravitational field. The coupling to gravity serves as a particular choice of source terms on which the generating functional depends.

The action for the scalar field ϕ coupled to the background metric h_{mn}, in $(1+1)$ dimensions, takes the following form,

$$S[h,\phi] = -\frac{1}{2}\int d^2x\sqrt{-h}h^{mn}\partial_m\phi\partial_n\phi = \frac{1}{2}\int d^2x\sqrt{-h}\phi\Box_h\phi \quad , \tag{3.1}$$

where the Laplacian $\Box_h$ is defined as $\frac{1}{\sqrt{-h}}\partial_m\sqrt{-h}h^{mn}\partial_n$, and the subscript h indicates the explicit metric dependence of the Laplacian (only the flat space Laplacian, $\partial_m\partial_n\eta^{mn}$, is denoted by $\Box$ in this thesis).

For the following applications it is useful to consider symmetries of the action (3.1). A Weyl transformation takes the metric $h_{mn}(x)$ to the rescaled value $e^{\sigma(x)}h_{mn}(x)$ under which the scalar action $S[h, \phi]$ is invariant (note that in higher dimensions additional curvature-scalar terms are required to maintain the scale invariance of the scalar action). In addition, use of covariant derivatives and the integration measure $d^2x\sqrt{-h}$ ensure the invariance of the action under GCT's, and under Lorentz transformations.

The partition function that depends on the background metric h_{mn} may be written as follows:

$$Z[h_{mn}] = \int [\mathcal{D}\phi]_h \, e^{iS[h,\phi]} \quad . \tag{3.2}$$

In order that eq. (3.2) is well defined, one must specify the measure of path integration $[\mathcal{D}\phi]_h$. This measure should preserve the symmetries of the action. Unlike the example of gauging an internal symmetry in Chapter 2, here the measure depends non-trivially on the background field, the metric h_{mn}. The explicit construction of possible path integral measures is relegated to Appendix B, from which it is evident that one cannot preserve all symmetries of the action in the measure. A choice of a preferred symmetry should be made from one fewer degrees of freedom than are present in the symmetries of the action. For the remainder of this thesis, the *choice* of the GCT invariant measure is made, a measure which is not invariant under Weyl scaling of the metric. Scaling of the metric may still be performed resulting in a non-trivial Jacobian, i.e. the Weyl anomaly, that can be exponentiated and written in terms of the Liouville action for the conformal degree of freedom. For $g_{mn} = e^{\sigma}h_{mn}$, the Liouville action, S_L, is the (integrated) anomaly contribution given by:

$$S_L[h,\sigma] \equiv -\frac{1}{48\pi}\int d^2x\sqrt{-h}\left(-\frac{1}{2}h^{mn}\partial_m\sigma\partial_n\sigma + \mathcal{R}_h\sigma + \mu^2 e^{\sigma}\right) \tag{3.3}$$

$$= -\frac{1}{96\pi}\left\{\int d^2x\sqrt{-g}\left(\mathcal{R}_g\frac{1}{\Box_g}\mathcal{R}_g + \mu^2\right) - \int d^2x\sqrt{-h}\left(\mathcal{R}_h\frac{1}{\Box_h}\mathcal{R}_h\right)\right\} \tag{3.4}$$

Here, $\mathcal{R}_g$ is the Ricci curvature scalar defined from the metric g_{mn} (and likewise $\mathcal{R}_h$ from the conformally related h_{mn}), and a regularization dependent scale, μ^2, renormalizes the cosmological constant. It is assumed that an appropriate bare cosmological constant is present in the initial action in eq. (3.2) to ensure a precise cancellation of the μ^2 term, therefore any term of the form, $\int\sqrt{-g}\mu^2$ is to be omitted from now on. $\Box_h^{-1}\mathcal{R}_h$ denotes the convolution $\int d^2y\sqrt{-h}G_h(x,y)\mathcal{R}_h(y)$, of $\mathcal{R}_h$ with the Feynman Green's function for $\Box_h$, defined by $\Box_h G_h(x,y) = \delta^2(x-y)/\sqrt{-h}$. Notice also, that the sign of the conformal factor (σ) kinetic term in (3.3) is opposite to the usual scalar field action. Although the sign of the Weyl anomaly action is not crucial in space-time duality, the relative signs of various anomaly terms play an important role in superduality.

To proceed with the duality program, one observes that the scalar field model (3.2) is invariant under global Lorentz transformations. Gauging this global symmetry is equivalent to coupling the system to a dynamical, i.e. functionally integrated, gravitational field g_{mn}. One also needs to include a generally-covariant constraint $\Delta_{LM}(g_{mn}, h_{mn})$ which forces the dynamical field, g_{mn} to be gauge-equivalent to the background metric, h_{mn}, thereby making the gauged system identical to the original, (3.2). Duality is then achieved by interchanging the order of functional integrations.

The gauged functional integral takes the following form:

$$Z[h_{mn}] = \int [\mathcal{D}\phi]_g[\mathcal{D}g_{mn}]_g\,\Delta_{LM}(g_{mn}, h_{mn})\,e^{iS[g,\phi]} \quad , \tag{3.5}$$

where the construction of $\Delta_{LM}(g_{mn}, h_{mn})$ is discussed below. To define the measure $[\mathcal{D}g_{mn}]_g$ it is useful to parameterize the dynamical metric as a Weyl (scale) transformation of the background metric h_{pq}, followed by a coordinate transformation,

$$g_{mn}(\xi) = e^{\sigma(\xi(x))}h_{pq}(\xi(x))\,\frac{\partial\xi^p}{\partial x^m}\frac{\partial\xi^q}{\partial x^n} \quad . \tag{3.6}$$

The measure $[\mathcal{D}g_{mn}]_g$ is written as $[\mathcal{D}\sigma]_g[\mathcal{D}\xi^m]_g\Delta[\mathcal{F}^m]\,[J_{FPW}]_g$. Here $\mathcal{F}^m(g_{mn}) = 0$ denotes the coordinate condition which fixes ξ^m, which is chosen to be $\mathcal{F}^m = \xi^m - x^m$. With this choice the Fadeev–Popov–deWitt determinant becomes $[J_{FPW}]_g = \det[-\Box_g^v + \frac{1}{2}\mathcal{R}_g]^{1/2}$ [8], where $\Box_g^v$ is the Laplacian operating on vector fields, $\left(\Box_g^v A\right)_p = g^{mn}D_mD_nA_p$.

There is a conformal anomaly associated with each of the GCT invariant measure factors. For D scalar fields $\{\phi^i\}$,

$$\int [\mathcal{D}\phi^i]_{e^\sigma h}\, e^{iS[e^\sigma h_{mn},\phi^i]} = e^{D\, iS_L[h_{mn},\sigma]} \int [\mathcal{D}\phi^i]_h\, e^{iS[h_{mn},\phi^i]} \quad , \tag{3.7}$$

where use is made of the fact that the massless scalar action is Weyl invariant, $S[e^\sigma h_{mn}, \phi] = S[h_{mn}, \phi]$. The path integral for the metric degrees of freedom contains the integration over the conformal factor σ and the Fadeev–Popov–deWitt determinant, J_{FPW}, which satisfy [8, 10, 11, 12], :

$$[\mathcal{D}\sigma]_{e^\sigma h} = [\mathcal{D}\sigma]_h\, e^{iS_L[h_{mn},\sigma]} \ , \quad [J_{FPW}]_{e^\sigma h} = [J_{FPW}]_h\, e^{-26 iS_L[h_{mn},\sigma]} \quad . \tag{3.8}$$

Combining the factors of the Liouville action, permits eq. (3.5) to be written in the following way,

$$\begin{aligned} Z[h_{mn}] &= \int [\mathcal{D}\phi]_{e^\sigma h}\, [\mathcal{D}\sigma]_{e^\sigma h}\, [J_{FPW}]_{e^\sigma h}\ \Delta_{LM}(e^\sigma h_{mn}, h_{mn})\ e^{iS[e^\sigma h_{mn},\phi]} \\ &= \int [\mathcal{D}\phi]_h\, [\mathcal{D}\sigma]_h\, [J_{FPW}]_h\ \Delta_{LM}(e^\sigma h_{mn}, h_{mn})\ e^{iS[h_{mn},\phi]-24iS_L[h_{mn},\sigma]} \ . \end{aligned} \tag{3.9}$$

At this point a suitable constraint term $\Delta_{LM}(g_{mn}, h_{mn})$ remains to be defined. The guide in this construction consists of two requirements. First, Δ_{LM} must be proportional to $\Delta[\sigma]$ in the second equality of (3.9), in order to remove the integration over σ by setting $\sigma = 0$. Second, it must remove the σ-independent factor, $[J_{FPW}]_h$ in the same equation, since such a term does not appear in the original expression (3.2) for $Z[h_{mn}]$. These two conditions do not suffice to fix Δ_{LM} completely, since they leave the freedom to multiply by an arbitrary function which approaches unity as $\sigma \to 0$. Employing this freedom to combine as many factors of h_{mn} and σ together into g_{mn}'s as possible, leads to the following choice for the constraint:

$$\begin{aligned} \Delta_{LM}[g_{mn}, h_{mn}] &= \int [\mathcal{D}\Lambda]_g \exp\left\{-i\int d^2x \left(\sqrt{-g}\,\mathcal{R}_g - \sqrt{-h}\,\mathcal{R}_h\right)\Lambda\right\} \frac{\det[-\Box_g]}{[J_{FPW}]_g} \\ &= \int [\mathcal{D}\Lambda]_g \exp\left\{-i\int d^2x \left(\sqrt{-g}\,\mathcal{R}_g - \sqrt{-h}\,\mathcal{R}_h\right)\Lambda \right. \\ &\qquad\qquad \left. + \frac{i}{4\pi}\int d^2x \sqrt{-g}\left(\mathcal{R}_g \frac{-1}{\Box_g} \mathcal{R}_g\right)\right\} \end{aligned} \tag{3.10}$$

For conformally related (1+1) dimensional metrics, $g_{mn} = e^{\sigma} h_{mn}$, the Ricci curvature scalars are related by $\sqrt{-g}\mathcal{R}_g = \sqrt{-h}\mathcal{R}_h + \sqrt{-h}\Box_h\sigma$. The Lagrange multiplier constraint may be expressed as,

$$\begin{aligned}\Delta_{LM}[e^{\sigma}h_{mn}, h_{mn}] &= \int[\mathcal{D}\Lambda]_h \exp\left\{-i\int d^2x\sqrt{-h}\,(\Lambda\,\Box_h\sigma)\right\}\frac{\det[-\Box_h]}{[J_{FPW}]_h}\,e^{25iS_L[h_{mn},\sigma]} \\ &= \frac{\Delta[\sigma]}{[J_{FPW}]_h}\,e^{25iS_L[h_{mn},\sigma]} \quad . \end{aligned} \tag{3.11}$$

As a check, expression (3.11) is inserted into eq. (3.9), the functional delta function, $\Delta[\sigma]$, is used to perform the σ path integral. The original expression (3.2) for $Z[h_{mn}]$ is recovered following the identification $S_L[h_{mn}, \sigma = 0] = 0$.

In order to obtain the dual version of the theory path integrals over ϕ and σ are performed in eq. (3.9), leaving Λ as the new dynamic variable.

$$\begin{aligned} Z[h_{mn}] &= \det[-\Box_h]\int[\mathcal{D}\phi]_h\,[\mathcal{D}\sigma]_h\,[\mathcal{D}\Lambda]_h \exp\left\{i\int d^2x\sqrt{-h}\left[\frac{1}{2}\phi\Box_h\phi\right.\right. \\ &\qquad \left.\left. -\frac{1}{48\pi}\left(\frac{1}{2}\sigma\Box_h\sigma + \mathcal{R}_h\,\sigma\right) - \Lambda\,\Box_h\,\sigma\right]\right\} \\ &= \det[-\Box_h]\int[\mathcal{D}\phi]_h\,[\mathcal{D}\sigma]_h\,[\mathcal{D}\Lambda]_h \exp\left\{\frac{i}{2}\int d^2x\sqrt{-h}\left(\phi\Box_h\phi\right.\right. \\ &\qquad -\frac{1}{48\pi}\left[\left(\sigma - \frac{1}{\Box_h}\mathcal{R}_h + 48\pi\,\Lambda\right)\Box_h\left(\sigma - \frac{1}{\Box_h}\mathcal{R}_h + 48\pi\,\Lambda\right)\right. \\ &\qquad \left.\left.\left. +\frac{1}{48\pi}\left(\frac{1}{\Box_h}\mathcal{R}_h - 48\pi\,\Lambda\right)\Box_h\left(\frac{1}{\Box_h}\mathcal{R}_h - 48\pi\,\Lambda\right)\right]\right)\right\} \quad , \end{aligned} \tag{3.12}$$

where the second equality is obtained from the first by completing the square on σ. The field σ is rescaled to $\sqrt{48\pi}\,\sigma$ and Λ to $\Lambda/\sqrt{48\pi}$. The remaining two Gaussian path integrations over ϕ and over σ are performed producing factors of $(\det[-\Box_h])^{-1/2} \times (\det[-\Box_h])^{-1/2}$, which cancel $\det[-\Box_h]$ appearing in eq. (3.12). Therefore, the dual expression for $Z[h_{mn}]$ takes the following form:

$$Z[h_{mn}] = \int[\mathcal{D}\Lambda]_h \exp\left\{\frac{i}{2}\int d^2x\sqrt{-h}\left(\Lambda - \frac{1}{\sqrt{48\pi}}\frac{1}{\Box_h}\mathcal{R}_h\right)\Box_h\left(\Lambda - \frac{1}{\sqrt{48\pi}}\frac{1}{\Box_h}\mathcal{R}_h\right)\right\} . \tag{3.13}$$

In a conformally-flat background, $h_{mn} = e^{\varphi}\eta_{mn}$, the dual theory (3.13) simplifies to:

$$Z[e^{\varphi}\eta_{mn}] = \int[\mathcal{D}\Lambda]_\eta \exp\left\{\frac{i}{2}\int d^2x\left(\Lambda - \frac{1}{\sqrt{48\pi}}\varphi\right)\Box_\eta\left(\Lambda - \frac{1}{\sqrt{48\pi}}\varphi\right) + iS_L[\eta_{mn}, \varphi]\right\} . \tag{3.14}$$

After shifting Λ by the factor of $\frac{1}{\sqrt{48\pi}}\frac{1}{\Box_h}\mathcal{R}_h$ in (3.13) — or, equivalently by $\frac{1}{\sqrt{48\pi}}\varphi$ in (3.14) — the original massless scalar theory is recovered.

The example of the massless scalar field in (1, 1) dimensions demonstrates that the procedure of space-time duality is consistent, in the sense that evaluating the remaining path integral in (3.13) one recovers the same result as with the original theory (3.2). Unfortunately, the scalar coupled to background gravitation is too simple to (space-time) dualize to a different field theory. Still, the fact that a local expression is obtained is suggestive of possible usefulness of the procedure.

3.3 Dirac Fermion

A less trivial example of space-time duality occurs in the case of fermions coupled to a curved background, which is the topic of present section. Spinor notation conventions may be found in Appendix A.

In order to describe spinors in the context of a curved space-time it is necessary to invoke the principle that a curved manifold looks flat when restricted to a small enough region. This concept is captured by the construction of the tangent space, a linear space spanned by the (curved) coordinate derivative basis (∂_m). One can define a flat, non-coordinate basis of tangent space, (∂_a) related to the coordinate basis with frame vector fields e_a^m (called zweibein in 2 dimensions, or vielbein in general) by $\partial_a = e_a^m\partial_m$. Note that (m) is the vector component label, where as (a) labels N such vectors in N dimensional space-time. The inverse vielbein e_m^a are related to the curved metric by $h_{mn} = e_m^a e_n^b \eta_{ab}$, therefore $\sqrt{-\det[h_{mn}]} = \det[e_m^a]$, which is denoted by $[e^{-1}]$.

Spinors are defined locally on the flat tangent space, hence frame vectors are needed in the curved space generalization of the Dirac action:

$$Z[e_m^a] = \int [\mathcal{D}\overline{\chi}]_e[\mathcal{D}\chi]_e \exp\left\{i\int d^2x[e^{-1}]\,\frac{i}{2}\overline{\chi}\gamma^a e_a^m D_m\chi\right\} \tag{3.15}$$

where D_m is the GCT covariant derivative,

$$D_m = \partial_m + \frac{1}{2}\omega_m\gamma_5 = \partial_m + \frac{1}{2}\epsilon^{ab}e_a^n\partial_n e_{b,m}\gamma_5 \tag{3.16}$$

To carry out the duality procedure, dynamic gravitational degrees of freedom are introduced into the path integral. Because spinors are involved, instead of considering a dynamic metric one makes use of a dynamic zweibein: $f^a_m = \left(e^{\sigma/2} e^b_m \theta^a_b(\alpha)\right)^\xi$, where $\theta^a_b(\alpha)$ is a local Lorentz transformation (LLT) on the locally defined Minkowski frame, namely a boost in the (0-1) plane, and vector ξ_m parameterizes diffeomorphisms. The dynamic metric g_{mn} is defined by $f^a_m f^b_n \eta_{ab}$, where σ and ξ parameterize the same degrees of freedom as in the previous section (3.6). Therefore the partition function (3.15) is rewritten as follows:

$$Z[e^a_m] = \int [\mathcal{D}\overline{\chi}]_f [\mathcal{D}\chi]_f \left\{ ([\mathcal{D} f^a_m])_f \, \Delta_{LM}[f^a_m, e^a_m] \right\} \exp\left\{ iS[f^a_m, \overline{\chi}, \chi] \right\} \quad . \tag{3.17}$$

Well defined integration over the gravitational degrees of freedom requires that the volume of the diffeomorphism group is factored out. The measure for the zweibein may be expanded as follows:

$$[\mathcal{D} f^a_m]_f \to [\mathcal{D}\sigma]_f [\mathcal{D}\xi_m]_f [\mathcal{D}\alpha]_f \Delta[\mathcal{F}_{GCT}] \Delta[\mathcal{F}_{LLT}][J_{FPW}]_f \tag{3.18}$$

Choosing the gauge $\mathcal{F}_{GCT} = \xi(x) - x$ and $\mathcal{F}_{LLT} = \alpha$ (i.e., no tangent space boosts from the reference zweibein e^a_m) the path integral (3.17) may be expressed in terms of the gauge fixed zweibein $f^a_m \to e^{\sigma/2} e^a_m$ as follows,

$$Z[e^a_m] = \int [\mathcal{D}\overline{\chi}]_{e^{\sigma/2}e} [\mathcal{D}\chi]_{e^{\sigma/2}e} [\mathcal{D}\sigma]_{e^{\sigma/2}e} [J_{FPW}]_{e^{\sigma/2}e} \Delta_{LM}[e^{\sigma/2} e^a_m, e^a_m] \exp\left\{ iS[e^{\sigma/2} e^a_m, \overline{\chi}, \chi] \right\} \tag{3.19}$$

The Fadeev–Popov–deWitt determinant, J_{FPW}, now ensures gauge invariance with respect to GCT and LLT, but since LLT gauge fixing is algebraic and the corresponding Jacobian a number, $\Delta_{LM}[e^{\sigma/2} e^a_m, e^a_m]$ takes the same form as in the scalar case, namely $\Delta_{LM}[e^\sigma h_{mn}, h_{mn}]$ in eq. (3.11).

As in the case of the scalar field measure, the Weyl anomaly can be explicitly calculated for the Dirac fermion [13],

$$[\mathcal{D}\overline{\chi}]_{e^{\sigma/2}e} = [\mathcal{D}\overline{\chi}]_e \, e^{\frac{i}{2} S_L[h_{mn},\sigma]} \qquad [\mathcal{D}\chi]_{e^\sigma e} = [\mathcal{D}\chi]_e \, e^{\frac{i}{2} S_L[h_{mn},\sigma]} \quad . \tag{3.20}$$

Transforming all measures to depend only on the original background metric induces the following anomaly terms in the action,

$$\begin{aligned} Z[e^a_m] &= \det[-\Box_h] \int [\mathcal{D}\overline{\chi}]_e[\mathcal{D}\chi]_e[\mathcal{D}\sigma]_e[\mathcal{D}\Lambda]_e \exp\Big\{ iS[e^a_m, \overline{\chi}, \chi] + iS_L[h_{mn}, \sigma] \\ &\quad + i\int d^2x\sqrt{-h}\sigma\Box_h\Lambda\Big\} \quad . \end{aligned} \tag{3.21}$$

At this point, equation (3.21) may be compared with eq. (3.12), and the subsequent steps following it. Evaluating the Gaussian path integrals $[\mathcal{D}\overline{\chi}]_e[\mathcal{D}\chi]_e[\mathcal{D}\sigma]_e$ the dual theory takes the form,

$$\begin{aligned} Z[e^a_m] &= \det[i\not{D}]_e(\det[-\Box_e])^{1/2} \int [\mathcal{D}\Lambda]_e \times \\ &\quad \exp\left\{\frac{i}{2}\int d^2x\sqrt{-h}\left(\Lambda + \frac{1}{\sqrt{48\pi}}\frac{1}{\Box_h}\mathcal{R}_h\right)\Box_h\left(\Lambda + \frac{1}{\sqrt{48\pi}}\frac{1}{\Box_h}\mathcal{R}_h\right)\right\} \quad . \end{aligned} \tag{3.22}$$

In principle, $\det[i\not{D}]_e$ may contain a contribution from the Lorentz anomaly in addition to the conformal anomaly [13]. However, in the case of Dirac (or Majorana) fermions, where the number of left- and right-moving components are matched, the Lorentz anomaly drops out. The remaining dependence on the background cancels between $\det[i\not{D}]_e$ and $(\det[-\Box_e])^{1/2}$ as can be seen by expressing the determinants in flat background with the Liouville actions containing all dependence on e^a_m. The two determinants combine to $\det[-\Box]$ which does not depend on the background.

Therefore, the final form of the dual theory to the QFT in eq. (3.15) is

$$Z[e^a_m] = \det[-\Box] \int [\mathcal{D}\Lambda]_e \exp\left\{\frac{i}{2}\int d^2x\sqrt{-h}\Lambda\Box_h\Lambda\right\} \quad . \tag{3.23}$$

This is a local bosonic theory, and it demonstrates an alternate derivation of "Bosonization as duality" to reference [6].

3.4 Majorana Fermion

Consider next the case of a Majorana spinor χ, which satisfies the self charge conjugate condition (A.18), (A.19).

$$\begin{aligned} \chi^c_\alpha &= \chi_\alpha \\ \chi^c_\alpha &= \left[C\,(\overline{\chi})^T\right]_\alpha = \left[\gamma^1\left(\chi^\dagger\gamma^1\right)^T\right]_\alpha = -\chi^*_\alpha \quad , \end{aligned} \tag{3.24}$$

where $C = \gamma^1$ is the charge conjugation matrix and superscript T denotes the transpose in the Dirac matrix space. From condition (3.24), it can be seen that the Majorana spinor χ is anti-Hermitian.

In the path integral one integrates over half as many degrees of freedom as in the Dirac fermion case,

$$Z[e^a_m] = \int [\mathcal{D}\chi]_e \exp\{iS[e^a_m, \overline{\chi}, \chi]\} \quad . \tag{3.25}$$

The duality procedure follows analogous steps as before, except that the the conformal anomaly contribution is half as big.

$$\begin{aligned} Z[e^a_m] &= \int [\mathcal{D}\chi]_f \{[\mathcal{D}f^a_m]_f \Delta_{LM}[f^a_m, e^a_m]\} \exp\{iS[f^a_m, \overline{\chi}, \chi]\} \\ &= \int [\mathcal{D}\chi]_e [\mathcal{D}\sigma]_e [\mathcal{D}(\Box_e \Lambda)]_e \exp\Big\{iS[e^a_m, \overline{\chi}, \chi] + \frac{i}{2} S_L[e^a_m, \sigma] \\ &\qquad + i \int d^2x [e^{-1}] \sigma \Box_e \Lambda\Big\} \end{aligned} \tag{3.26}$$

Completing the square and evaluating the $[\mathcal{D}\chi]_e[\mathcal{D}\sigma]_e$ path integrals, the following dual theory is obtained,

$$\begin{aligned} Z[e^a_m] &= (\det[i\not{D}]_e)^{1/2} (\det[-\Box_e])^{1/2} \int [\mathcal{D}\Lambda]_e \times \\ &\quad \exp\left\{-\frac{i}{2}\int d^2x \sqrt{-h} (\Lambda - \sqrt{\frac{1}{96\pi}} \frac{1}{\Box_h} \mathcal{R}_h) \Box_h (\Lambda - \sqrt{\frac{1}{96\pi}} \frac{1}{\Box_h} \mathcal{R}_h)\right\} \quad . \end{aligned} \tag{3.27}$$

Like in the Dirac fermion case, the number of left- and right-handed components of χ are equal, which implies that the functional determinant $(\det[i\not{D}]_e)^{1/2}$ does not have a Lorentz anomaly contribution. However, unlike the Dirac case, the metric dependence of $(\det[i\not{D}]_e)^{1/2}(\det[-\Box_e])^{1/2}$ does not cancel. This metric dependence from the determinants may be exponentiated with a Liouville action,

$$\begin{aligned} Z[e^a_m] &= (\det[i\not{\partial}])^{1/2} (\det[-\Box])^{1/2} \int [\mathcal{D}\Lambda]_e \times \\ &\quad \exp\left\{\frac{i}{2}\int d^2x [e^{-1}] \Lambda \Box_e \Lambda + \frac{i}{192\pi} \int d^2x \sqrt{-h} \mathcal{R}_h \frac{1}{\Box_h} \mathcal{R}_h\right\} \quad . \end{aligned} \tag{3.28}$$

Because the Liouville action is non-local, the Majorana spinor is inequivalent to a local bosonic theory. This result confirms the fact that the degrees of freedom are well matched only in the Dirac spinor – scalar duality, and that this matching is far from arbitrary. In addition, the fact that in (1+1) dimensions Majorana spinors do not couple to gauge fields implies that a similar conclusion cannot be reached from the gauge duality prescription.

Chapter 4

Superduality

In this chapter the formalism of space-time duality is extend to the case of supersymmetry. The key idea in space-time duality is turning the global Lorentz symmetry into a local symmetry, by introducing dynamical gravitational degrees of freedom. Similarly, in superduality, one makes a global supersymmetry present in a theory into a local symmetry, by introducing supergravity (SUGRA) degrees of freedom. The duality prescription presented in this chapter results in a supersymmetric multiplet of dual fields.

The outline of this chapter is as follows. (1+1) dimensional supersymmetry is introduced from the point of view of enlarging the Poincaré symmetry. The supersymmetry considered first is generated by one Majorana generator (so-called (1,1) supersymmetry), and a field representation of the supersymmetry is presented. Taking the locally supersymmetric version of the same model requires (1,1) SUGRA fields. Having both global and local versions of (1,1)-supersymmetry, one is able to dualize a free, massless, scalar supermultiplet model. This model turns out to be self-dual under superduality, similar to the case of a (1+1) dimensional, free, massless scalar field under space-time duality. A less trivial example of superduality is considered next. A model with a richer (2,2) supersymmetry structure is presented, together with the irreducible versions of the corresponding SUGRA. To facilitate keeping track of supersymmetry, both global and local, the device of superspace is introduced. It is in the superspace formulation of supersymmetry that the main result of this chapter is presented, namely, the pattern of duality between chiral and twisted-chiral matter in irreducible (2,2) SUGRA. But first to the preliminaries.

4.1 Introducing...Supersymmetry

In this section supersymmetry is introduced as an extension of Poincaré symmetry. Following the discussion of global supersymmetry and its field representation for (1,1) supersymmetric model, SUGRA extension of the same model is considered.

4.1.1 (1+1) Dimensional Supersymmetry Algebra

Recall, that in (3+1) dimensions, Poincaré symmetry is generated by four translations $\{P_a\}$, and six Lorentz transformations: three rotations $\{J_{ij}\}$ and three boosts $\{J_{0m}\}$.

$$\begin{aligned} [P_a, P_b] &= 0 \quad , \qquad [P_a, J_{bc}] = \eta_{ab} P_c - \eta_{ac} P_b \quad , \\ [J_{ab}, J_{cd}] &= -\left(\eta_{ac} J_{bd} + \eta_{bd} J_{ac} - \eta_{ad} J_{bc} - \eta_{bc} J_{ad}\right) \quad . \end{aligned} \tag{4.1}$$

The above Poincaré algebra may be specified to (1+1) dimensions. Since there is only one possible Lorentz transformation, denoted here by $\mathcal{M}$, the last relation in (4.1) is satisfied identically. Requiring a QFT to satisfy Poincaré symmetry implies: i) that the fields should furnish a representation of the symmetry, and ii) that the generating functional should be invariant under the symmetry transformation.

Supersymmetry algebra is generated by the supersymmetry generator, Q, which is a fermionic, hence anticommuting, charge in the spinor represention of the Poincaré symmetry. Further, the fermionic property of the generator implies that two such charges, if moved past one another, result in the change of the overall sign.

The Poincaré algebra (4.1) is complemented by the following relations containing the supersymmetry generator:

$$\begin{aligned} &\{Q_\alpha, \overline{Q}^\beta\} = -(\gamma^a)_\alpha^\beta P_a \qquad \{Q_\alpha, Q_\beta\} = \{\overline{Q}^\alpha, \overline{Q}^\beta\} = [Q_\alpha, P_a] = 0 \\ &[\mathcal{M}, Q_\alpha] = \frac{1}{2}(\gamma_5)_\alpha^\beta Q_\beta \quad , \end{aligned} \tag{4.2}$$

where the curly brackets stand for *anticommutation*, and the Dirac conjugate for charge Q is defined analogously to the Dirac conjugate of a spinor. An important property of the algebra is that the product of two fermionic generators is bosonic, and the product of a fermionic generator with a bosonic one is a fermionic generator.

This is a convenient point to note some general features of the supersymmetry algebra. First, as already mentioned, supersymmetry and Poincaré symmetry intertwine, and in this sense supersymmetry may be considered an extension of Poincaré symmetry. This situation is unlike the direct product structure of taking gauge and Poincaré symmetries together. Second, the fact that the generator Q is fermionic implies that supersymmetry relates bosonic and fermionic states, both of which are required to furnish a particle representation of supersymmetry. Similarly, at the level of fields, one expects that a multiplet of fields of both statistics is needed to furnish a representation of supersymmetry. This is the task of the next section. Finally, from the observation that matter fields are fermionic and force fields bosonic, supersymmetry may be interpreted as relating forces to matter, where as usual symmetries relate matter to matter, and forces to forces [14].

So far the supersymmetry generator Q is taken to be a complex charge in a spinor representation. A convenient form for labeling possible supersymmetries makes use of the decomposition of spinors in (1+1) dimensions. It is possible to impose restrictions on Q, taking it to be anti-Hermitian[1]. Besides anti-Hermiticity, one has the choice of imposing the chirality (Weyl) condition with the matrix $\gamma_5 = \gamma^0\gamma^1$:

$$Q_+ = +\gamma_5 Q_+ \qquad \text{or} \qquad Q_- = -\gamma_5 Q_- \quad . \tag{4.3}$$

A feature of (1+1) dimensions is the possibility of imposing, simultaneously, the anti-Hermiticity and chirality conditions. The resulting spinor charges are anti-Hermitian and of positive or negative chirality. It is customary to label (1+1)-dimensional supersymmetric theories as (N,M), corresponding to N positive- and M negative chirality, anti-Hermitian components of the supersymmetry algebra.

[1]the following discussion is generally unaltered if charge Q is taken to be Hermitian

4.1.2 Field Representation of (1,1) Supersymmetry

The goal of this section is to present a model which represents (1,1) supersymmetry and Poincaré symmetry. Poincaré symmetry is represented by covariant fields which appear in a scalar action. (1,1) supersymmetry, which is generated by anti-Hermitian Weyl charges Q_α satisfying relations (4.2), imposes the additional condition that covariant fields come in multiplets. The basic model (Wess-Zumino (WZ) model) contains the multiplet of fields $\{\phi, \chi, N\}$, where ϕ is a real scalar, χ a spinor which satisfies the Majorana constraint, and finally N is a real auxiliary scalar field whose role is explained below.

The simplest (1,1)-supersymmetric invariant action takes the following form,

$$S_0 = \frac{1}{2}\int d^2x \left(-(\partial_a\phi)^2 + i\overline{\chi}\partial\!\!\!/\chi - N^2\right) \quad . \tag{4.4}$$

The action of the supersymmetry generator Q on the multiplet fields may be stated in terms of a constant Majorana parameter ε:

$$\delta\phi = \overline{\varepsilon}\chi \qquad \delta\chi = i\partial\!\!\!/\phi\varepsilon + N\varepsilon \qquad \delta N = i\overline{\varepsilon}\partial\!\!\!/\chi \tag{4.5}$$

Invariance of the action (4.4) under supersymmetry (4.5) may be anticipated from equal number of bosonic and fermionic degrees of freedom. With the equations of motion satisfied, there is only one bosonic degree of freedom (ϕ). The auxiliary field (N) does not propagate because its equation of motion is a constraint. The spinor equation of motion relates the Majorana (χ) to its conjugate, thereby reducing the number of fermionic degrees of freedom to one. The purpose of the auxiliary field is to guarantee invariance under supersymmetry even when the equations of motion are not satisfied. In this case, one has two bosonic degrees of freedom (ϕ, N) and two fermionic degrees of freedom, (χ) unrestricted by its equation of motion.

It is important to stress that the invariance of the action (4.4) under supersymmetry transformations (4.5) is not sufficient to prove that the multiplet of fields represents supersymmetry (4.2). One must show, in addition, that the commutation rules of the algebra (4.2) are represented by the fields, which is in deed the case for the proposed representation.

4.1.3 (1,1) Supergravity

In order to obtain a locally supersymmetric version of the theory (4.4), gravitational degrees of freedom (with superpartners) need to be introduced. This may be anticipated from the intertwining of the supersymmetry and the Lorentz algebra in (4.2). Since there are spinors present, gravitational degrees of freedom are parameterized by a zweibein, as discussed in Section 3.3. Supersymmetry further requires that gravitational fields come in supersymmetric multiplets, therefore a spin $\frac{3}{2}$ Majorana fermion ψ_m, the gravitino, is present in the theory. In addition, one requires a real auxiliary scalar field S.

In (1+1) dimensions, the Einstein gravity action $\int d^2x\sqrt{-g}\,\mathcal{R}$ is a topological invariant proportional to the Euler characteristic [15], a quantity which depends on global properties of the space on which the fields rest. The gravitino action,

$$S_{\psi_m} = \int d^2x\sqrt{-g}\,\overline{\psi}_m\gamma^{[m}\gamma^n\gamma^{\rho]}\partial_n\psi_\rho \tag{4.6}$$

vanishes due to the antisymmetrization of three γ matrices, where in (1+1) dimensions there are only two distinct such matrices, γ^0 and γ^1. Therefore, neither the vielbein nor the gravitino contain propagating degrees of freedom, and their contribution to the dynamics of the system is indirect. The graviton and gravitino equations of motion impose constraints on the other fields.

Locally supersymmetric extension of the action (4.4) takes the following form:

$$S_{(1,1)} = \frac{1}{2}\int d^2x[e^{-1}]\left\{\phi\Box_e\phi + i\overline{\chi}\not{D}\chi - N^2 - \kappa\overline{\psi}_a\not{\partial}A\gamma^a\chi + \frac{\kappa^2}{8}\overline{\psi}^B_n\gamma^m\gamma^n\psi^B_m\overline{\chi}\chi\right\} \quad . \tag{4.7}$$

where κ is the SUGRA coupling constant and $[e^{-1}]$ stands for the determinant of the inverse zweibein, $\det[e^a_m]$. Action $S_{(1,1)}$ is invariant under *local* supersymmetry (i.e. SUGRA) which is parametrized by a space-time dependent, fermionic, Majorana parameter $\varepsilon(x)$. The individual fields transform as follows:

$$\begin{aligned}
\delta\phi &= \bar{\varepsilon}\chi \qquad \delta\chi = (i\hat{\not{\mathbf{D}}}\phi + N)\varepsilon \qquad \delta N = i\bar{\varepsilon}\hat{\not{\mathbf{D}}}\chi \\
\delta e_m^a &= i\kappa\bar{\varepsilon}\gamma^a\psi_m \qquad \delta\psi_m = \frac{-2}{\kappa}\mathbf{D}_m\varepsilon - S\gamma_m\varepsilon + \frac{i\kappa}{2}\overline{\psi}_a\eta^{ab}\psi_b\gamma_m\varepsilon + i\kappa\left(\overline{\psi}_b\gamma_5\gamma^b\varepsilon\right)\eta_{ad}\epsilon^{de}\psi_e \\
\delta S &= \frac{i}{2}\bar{\varepsilon}\left(-2\epsilon^{ab}\gamma_5\mathbf{D}_a\psi_b + (i\phi + \kappa^2\overline{\psi}_a\gamma^b\gamma^a\psi_b)\gamma^c\psi_c\right) \qquad (4.8)
\end{aligned}$$

where the SUGRA covariant derivatives, denoted by boldface ($\mathbf{D}_a$), contain dependence on the gravitino in addition to the zweibein. In particular,

$$\begin{aligned}
\hat{\mathbf{D}}_m\phi &= \partial_m\phi + \frac{\kappa}{2}\overline{\psi}_m\chi \quad , \\
\hat{\mathbf{D}}_m\chi &= \mathbf{D}_m\chi + \frac{\kappa}{2}(i\hat{\not{\mathbf{D}}}\phi + N)\psi_m \quad , \\
\mathbf{D}_m\chi &= \left(\partial_m + \frac{1}{2}\omega_m\gamma_5\right)\chi \quad , \qquad (4.9)
\end{aligned}$$

where the spin connection ω_m is the sum of usual zweibein term and an additional gravitino term (evaluated in Appendix A):

$$\omega_m = \epsilon^{ab}e_a^n\partial_n e_{b,m} - \kappa^2\overline{\psi}_m\gamma_5\gamma^b\psi_b \quad . \qquad (4.10)$$

In addition to local supersymmetry, the action (4.7) is left unchanged by both general coordinate and local Lorentz transformations, which originates from the fact that the action is constructed to be a coordinate and Lorentz scalar (with the help of the zweibein).

There are three further *local* symmetries which play an important role in the evaluation of the generating functional. The first one is Weyl scaling, which is parameterized infinitesimally by a scalar $\sigma(x)$,

$$\begin{aligned}
\delta e_m^a &= \sigma e_m^a \qquad \delta\psi_m = \frac{1}{2}\sigma\psi_m \qquad \delta S = 0 \\
\delta\phi &= 0 \qquad \delta\chi = -\frac{1}{2}\sigma\chi \qquad \delta N = -\sigma N \qquad (4.11)
\end{aligned}$$

The other two local symmetries are related to the above Weyl (scale) transformation. S-supersymmetry is parameterized by a Majorana spinor and the "auxiliary" symmetry [16] by a mass dimension one real scalar, both of which appear in a $(1,1)$ supersymmetric multiplet with the scalar σ. For the present purpose it is not necessary to list these transformations; still, the fact that the anomalies come in a supersymmetric multiplet implies that the the anomaly action (corresponding to Liouville action in Chapter 3) is supersymmetric.

4.2 (1,1) Superduality

In this section, superduality is applied to the case of (1,1) supersymmetric massless WZ matter. Superduality maps the theory back onto itself, reminiscent of the space-time duality action on the massless scalar field. In spite of the trivial nature of superduality in this section, the example does demonstrate that the path integrals involved are under control, and that the duality map takes a local QFT to a local QFT.

The superduality prescription is a straightforward generalization of space-time duality of Chapter 3. The initial model is a QFT with a global (1,1) supersymmetry. The external source is a background (1,1) SUGRA multiplet, and as in the case of space-time duality, special attention should be paid to the path integral measure, which here depends on the SUGRA fields. Global supersymmetry is made local by introducing "fake" SUGRA which is a perturbation about the background. A gauge fixing condition with appropriate Fadeev–Popov–deWitt determinant is needed to have a well defined QFT. Finally, in order to maintain explicit equality with the original theory, a Lagrange multiplier (multiplet) eliminates the "fake" SUGRA degrees of freedom that still persist after gauge fixing. Evaluating the Lagrange multiplier path integral one must recover the initial configuration, which is a condition used to deduce the form of the Lagrange multiplier action. To reach the dual formulation, the original matter and dynamic SUGRA fields are integrated out. One leaves the path integral, which still depends on the background SUGRA, in terms of the Lagrange multipliers which play the role of the dual fields.

To begin, recall from the action for the (1,1)-supersymmetric WZ scalar multiplet in flat Minkowski space (4.4), $S_0 = \frac{1}{2}\int d^2x\,(\phi\Box\phi - i\overline{\chi}\gamma^r\partial_r\chi + N^2)$. It is useful to introduce a four-dimensional matrix notation for the field multiplet and the field operator, as in reference [16], which is inspired by the superspace formulation of supersymmetry. This notation helps to keep track of the various multiplets in the problem, and makes more evident invariance properties under supersymmetry.

$$\rho := \begin{pmatrix} \phi \\ N \\ \chi \end{pmatrix} \qquad \mathbf{T} := \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ C & 0 & 0 \end{pmatrix} \qquad \mathbf{\Theta_0} := \begin{pmatrix} 0 & 0 & i\partial\!\!\!/ \\ 0 & -1 & 0 \\ \Box & 0 & 0 \end{pmatrix} \quad , \tag{4.12}$$

where $C = \gamma^1$ is the charge conjugation matrix. With this notation, the action (4.4) can be rewritten in the compact form,

$$S_0 = \frac{1}{2}\int d^2x \bar{\rho}\mathbf{\Theta_0}\rho \quad , \tag{4.13}$$

where the conjugate $\bar{\rho}$ is defined as $\rho^\dagger \mathbf{T}$.

The background SUGRA is a multiplet of fields $\mathbf{B} = (e^a_m, S^B, \psi^B_m)$, where e^a_m is the inverse zweibein, ψ^B_m a majorana gravitino, and S^B is a real auxiliary scalar. In background SUGRA, the invariant extension of the free action (4.4) takes the form of the action (4.7). Since there are no kinetic terms for either the zweibein or the gravitino, the scalar multiplet action in background SUGRA can be written in the matrix notation, as was done for the flat-space multiplet (4.13) above,

$$S_1 = \frac{1}{2}\int d^2x \bar{\rho}\mathbf{\Theta_B}\rho \quad . \tag{4.14}$$

The SUGRA dependent field operator now takes the form:

$$\mathbf{\Theta_B} := [e^{-1}] \begin{pmatrix} \frac{-\kappa}{2}\gamma^m\gamma^n\psi^B_m\partial_n & 0 & i D\!\!\!\!/ + \frac{\kappa^2}{8}\overline{\psi}^B_m\gamma^n\gamma^m\psi^B_n \\ 0 & -1 & 0 \\ \Box_e & 0 & -\frac{\kappa}{2}[e]\partial_m[e^{-1}]\overline{\psi}^B_n\gamma^m\gamma^n \end{pmatrix} \tag{4.15}$$

with the derivatives in $\mathbf{\Theta_B}$ acting on everything that stands to the right. In order to have the form (4.15), the fourth term in the SUGRA action (4.7) has been symmetrized, in the sense of factoring out a total derivative.

The quantum system of interest is the dynamic (integrated) (1,1) WZ multiplet, with background SUGRA playing the role of the external source on which the generating functional depends. For the duality procedure one may consider the following path integral:

$$Z[\mathbf{B}] = \int [\mathcal{D}\rho]_{\mathbf{B}} \exp\{iS_1[\rho, \mathbf{B}]\} \quad . \tag{4.16}$$

A SUGRA invariant path integral measure is chosen for (4.16). It may be constructed implicitly, in analogy with covariant measure as in Appendix B, or explicitly, as is done in ref. [16].

In order to simplify the evaluation of functional integrals, the background is specified in the superconformal gauge; however, the duality program presented here should be valid in any gauge. In particular, the choice of background gauge does not carry any physical meaning, so long as SUGRA gauge independent questions are asked of the field theory.

Using general coordinate transformations, local Lorentz transformations and local supersymmetry, background SUGRA fields may be specified in the following gauge,

$$e^a_m \to e^{\sigma^B}\delta^a_m \qquad \psi^B_m \to -i\gamma_m\psi^B \qquad S^B \to S^B \tag{4.17}$$

Note that with foresight, the conformal factor σ^B – parameterizing the zweibein – is chosen to be half of the factor φ in Chapter 3. This turns out to be convenient in order to display supersymmetry properties of the dual action.

In the superconformal gauge (4.17), the SUGRA multiplet has the field content identical to that of a scalar (1,1) multiplet of global supersymmetry, namely a scalar, a Majorana spinor and a scalar auxiliary field. The background, superconformal gauge, SUGRA fields may be grouped into a multiplet $\mathcal{B} = (\sigma^B, S^B, \psi^B)$, in analogy with the matter multiplet (ρ).

In the superconformal gauge, the action (4.14) takes the form: $\frac{1}{2}\int d^2x\bar{\rho}\mathbf{\Theta}^{\mathbf{SC}}_{\mathcal{B}}\rho$, with $\mathbf{\Theta}^{\mathbf{SC}}_{\mathcal{B}}$ the superconformal limit of the field operator (4.15),

$$\mathbf{\Theta}^{\mathbf{SC}}_{\mathcal{B}} = \begin{pmatrix} 0 & 0 & ie^{\sigma^B}\partial\!\!\!/ \\ 0 & -e^{2\sigma^B} & 0 \\ \square & 0 & 0 \end{pmatrix} . \tag{4.18}$$

Looking at the gauge fixed field operator (4.18) it seems that the path integral no longer has a dependence on the gravitino ψ^B. Still, through the matter fields' path integral measure, chosen to be SUGRA invariant, dependence on all components of the background $(\mathcal{B})$ does remain [16].

The starting point for the duality transformation is the superconformal gauge limit of the path integral (4.16)

$$Z[\mathcal{B}] = \int [\mathcal{D}\rho]_{\mathcal{B}} \exp\{iS_{\text{l}}[\rho, \mathcal{B}]\} \tag{4.19}$$

Proceeding with the duality algorithm, fluctuations about background SUGRA are introduced via a dynamic (integrated) SUGRA multiplet $\mathbf{T} = (f^a_m, S^T, \psi^T_m)$.

$$Z[\mathcal{B}] = \int [\mathcal{D}\rho]_{\mathbf{T}} [\mathcal{D}\mathbf{T}]_{\mathbf{T}} \Delta_{LM}[\mathbf{T}, \mathcal{B}] \exp\{iS_{\text{l}}[\rho, T]\} \tag{4.20}$$

All path integral measures are now invariant under local supersymmetry. The integral over SUGRA degrees of freedom, $[\mathcal{D}\mathbf{T}]_{\mathbf{T}}$, requires gauge fixing which is discussed shortly. The functional constraint $\Delta_{LM}[\mathbf{T}, \mathcal{B}]$ is constructed to enforce the equivalence of the total and background SUGRA multiplets, $\mathbf{T} = \mathcal{B}$, thereby maintaining equality with the original generating functional (4.19).

It is convenient to impose the superconformal gauge on the dynamic SUGRA degrees of freedom using the non-anomalous symmetries of the action, namely GCT, LLT and local supersymmetry invaraince,

$$\begin{aligned} f^a_m &= e^{\sigma^Q} e^a_m = e^{\sigma^Q + \sigma^B} \delta^a_m \\ \psi^T_m &= -i f^a_m \gamma_a (\psi^Q + \psi^B) \\ S^T &= S^Q + S^B \quad . \end{aligned} \tag{4.21}$$

The measure for the supergravity multiplet may be expanded as follows,

$$[\mathcal{D}\mathbf{T}]_{\mathbf{T}} = [\mathcal{D}f^a_m]_{\mathbf{T}} [\mathcal{D}\psi^T_m]_{\mathbf{T}} [\mathcal{D}S^T]_{\mathbf{T}} \Delta[f^a_m - \frac{1}{2}\delta^a_m (f^b_p \delta^p_b)] \Delta[\gamma^l \gamma_m \psi^T_l][J_{FPW}]_{\mathbf{T}} \quad , \tag{4.22}$$

where the functional delta functions enforce the superconformal gauge (4.21). The SUGRA measure in eq. (4.22) is well defined because the gauge fixing condition takes care of GCT, LLT, and local supersymmetry invariance. The remaining symmetries of the action (4.7), Weyl scaling and S-supersymmetry, are anomalous, and therefore do not require a gauge fixing. To obtain the Fadeev–Popov–deWitt Jacobian [9], the gauge fixing conditions in (4.22) are varied under the non-anomalous symmetries, and the determinant of the resulting operator enters in the generating functional. Since the explicit form of the Jacobian is not needed, it is left unevaluated here. Note, also, that in the superconformal gauge, the dynamic SUGRA degrees of freedom may be conveniently grouped into a (1, 1) scalar multiplet: $\mathcal{Q} = (\sigma^Q, S^Q, \psi^Q)$.

In order to find the appropriate constraint $\Delta_{LM}[\mathbf{T}, \mathcal{B}]$ it is useful to recall the corresponding procedure in space-time duality. There, the form of the constraint is determined by two conditions: i) that it eliminates the dynamic, gauge fixed gravity degrees of freedom, and ii) that it removes from the path integral any additional factors, which do not depend on dynamical gravity, to maintain equality with the original generating functional. The same two conditions, with SUGRA replacing "gravity" are used presently for superduality.

Before determining the precise form of the constraint, it is convenient to evaluate the gauge fixing conditions in the path integral.

$$Z[\mathcal{B}] = \int [\mathcal{D}\rho]_{\mathcal{B}+\mathcal{Q}}[\mathcal{D}\mathcal{Q}]_{\mathcal{B}+\mathcal{Q}}[J_{FPW}]_{\mathcal{B}+\mathcal{Q}}\Delta_{LM}[\mathcal{B}+\mathcal{Q},\mathcal{B}]\exp\{iS_1[\rho,\mathcal{B}+\mathcal{Q}]\} \quad . \quad (4.23)$$

Defining a Lagrange multiplier scalar multiplet $\Lambda = (L, F, \eta)$, the following choice is made for the constraint:

$$\Delta_{LM}[\mathcal{B}+\mathcal{Q},\mathcal{B}] = \int [\mathcal{D}\Lambda]_{\mathcal{B}+\mathcal{Q}}[J_{FPW}]^{-1}_{\mathcal{B}+\mathcal{Q}}\,\mathrm{sdet}[\mathbf{\Theta}^{\mathbf{SC}}_{\mathcal{B}+\mathcal{Q}}]\exp\left\{-i\int d^2x\overline{\mathcal{Q}}\mathbf{\Theta}^{\mathbf{SC}}_{\mathcal{B}}\Lambda\right\} \quad (4.24)$$

In equation (4.24) there appears a superdeterminant of the field operator $\mathrm{sdet}[\mathbf{\Theta}^{\mathbf{SC}}_{\mathcal{B}+\mathcal{Q}}]$. Superdeterminant is defined to keep track of the statistics of the multiplet fields which sandwich the operator. For example, consider a square matrix M

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \quad , \quad (4.25)$$

where block A relates bosonic variables to bosonic, block B relates fermionic variables to bosonic, C bosonic to fermionic, and D fermionic to fermionic. The superdeterminant of M is defined as

$$\mathrm{sdet}[M] = \det[A]\det[D - CA^{-1}B]^{-1} \quad , \quad (4.26)$$

and it is related to the supertrace,

$$\mathrm{sdet}[M] = \exp\{\mathrm{sTr}[\ln M]\} \quad , \tag{4.27}$$

defined as

$$\mathrm{sTr}[M] := \sum_j A_{jj} - \sum_i D_{ii} \quad . \tag{4.28}$$

Using the expression for the Δ_{LM} (4.24), the path integral (4.23) simplifies to the following form,

$$\begin{aligned} Z[\mathcal{B}] &= \int [\mathcal{D}\rho]_{\mathcal{B}+\mathcal{Q}} [\mathcal{D}\mathcal{Q}]_{\mathcal{B}+\mathcal{Q}} [\mathcal{D}\Lambda]_{\mathcal{B}+\mathcal{Q}} \,\mathrm{sdet}[\boldsymbol{\Theta}^{\mathbf{SC}}_{\mathcal{B}+\mathcal{Q}}] \\ &\qquad \times \exp\left\{ iS_1[\rho, \mathcal{B}+\mathcal{Q}] - i \int d^2x\, \overline{\mathcal{Q}} \boldsymbol{\Theta}^{\mathbf{SC}}_{\mathcal{B}}\, \Lambda \right\} \end{aligned} \tag{4.29}$$

To check that eq. (4.24) is the correct choice for Δ_{LM}, the integral $[\mathcal{D}\Lambda]$ is evaluated in eq. (4.29). One first makes use of the superconformal invariance of the action to rewrite the second term in (4.29) as $-i\int d^2x \overline{\mathcal{Q}} \boldsymbol{\Theta}^{\mathbf{SC}}_{\mathcal{B}+\mathcal{Q}}\Lambda$. Next, performing the $[\mathcal{D}\Lambda]_{\mathcal{B}+\mathcal{Q}}$ path integral introduces $\Delta[\boldsymbol{\Theta}^{\mathbf{SC}}_{\mathcal{B}+\mathcal{Q}}\mathcal{Q}] = (\mathrm{sdet}[\boldsymbol{\Theta}^{\mathbf{SC}}_{\mathcal{B}+\mathcal{Q}}])^{-1}\Delta[\mathcal{Q}]_{\mathcal{B}+\mathcal{Q}}$. Finally, using $\Delta[\mathcal{Q}]_{\mathcal{B}+\mathcal{Q}}$, the integration over $\mathcal{Q}$ is performed, removing all dependence on the dynamic multiplet. The original formulation (4.16) is recovered, thereby validating the choice of the constraint (4.24).

To reach the dual formulation, all integrals except the Lagrange multiplier multiplet are performed. In eq. (4.29) the matter multiplet $[\mathcal{D}\rho]_{\mathcal{B}+\mathcal{Q}}$ integrates out to give $(\mathrm{sdet}[\boldsymbol{\Theta}^{\mathbf{SC}}_{\mathcal{B}+\mathcal{Q}}])^{-1/2}$, which combines with a similar superdeterminant factor in the path integral. In order to perform the integration over $\mathcal{Q}$, the dependence of the measure on $\mathcal{Q}$ itself should be eliminated. For this purpose, it is useful to express the path integral measures in the flat background at the penalty of introducing a Jacobian, which may be expressed as a local action:

$$[\mathcal{D}\mathcal{Q}]_{\mathcal{B}+\mathcal{Q}} [\mathcal{D}\Lambda]_{\mathcal{B}+\mathcal{Q}} (\mathrm{sdet}[\boldsymbol{\Theta}^{\mathbf{SC}}_{\mathcal{B}+\mathcal{Q}}])^{1/2} = [\mathcal{D}\mathcal{Q}]_0 [\mathcal{D}\Lambda]_0 \,\mathrm{sdet}[\boldsymbol{\Theta}_0]^{1/2} \exp\left[iS_{SL}(\mathcal{B}+\mathcal{Q})\right] \quad . \tag{4.30}$$

In eq. (4.30), the subscript “0” indicates that there is no SUGRA dependence in the measure. The exponentiated Jacobian determinant is a supersymmetric generalization of the Liouville action [16]:

$$\begin{aligned} S_{SL}(\mathcal{B}) &= -\frac{1}{8\pi}\int d^2x\,\left(\frac{1}{2}\sigma\,\Box\,\sigma + \frac{i}{2}\overline{\psi}\gamma^m\partial_m\psi - \frac{1}{2}\,S^2\right) \\ &= -\frac{1}{16\pi}\int d^2x\,\overline{\mathcal{B}}\Theta_0\,\mathcal{B} \quad , \end{aligned} \tag{4.31}$$

where $\mathcal{B}$ may be replaced by the relevant multiplet. The outline of the anomaly calculation is found in Appendix B.

The generating functional takes the following form:

$$\begin{aligned} Z[\mathcal{B}] &= \int[\mathcal{D}\mathcal{Q}]_0[\mathcal{D}\Lambda]_0\,(\mathrm{sdet}[\Theta_0])^{1/2}\,\exp\Big\{-i\int d^2x\,\overline{\mathcal{Q}}\Theta_0\Lambda - \\ &\qquad \frac{i}{16\pi}\int d^2x\,(\overline{\mathcal{B}}+\overline{\mathcal{Q}})\Theta_0(\mathcal{B}+\mathcal{Q})\Big\} \quad . \end{aligned} \tag{4.32}$$

The action may be expressed in a suggestive form as

$$\begin{aligned} Z[\mathcal{B}] &= \int[\mathcal{D}\mathcal{Q}]_0[\mathcal{D}\Lambda]_0\,\mathrm{sdet}[\Theta_0]^{1/2}\,\exp\Big\{\frac{i}{16\pi}\int d^2x\,\left(8\pi\,\overline{\Lambda}+\overline{\mathcal{B}}\right)\Theta_0\,(8\pi\,\Lambda+\mathcal{B}) \\ &\quad -\frac{i}{16\pi}\int d^2x\,\left(\overline{\mathcal{Q}}+\overline{\mathcal{B}}+8\pi\,\overline{\Lambda}\right)\Theta_0\,(\mathcal{Q}+\mathcal{B}+8\pi\,\Lambda) - \frac{i}{16\pi}\int d^2x\,\overline{\mathcal{B}}\,\Theta_0\,\mathcal{B}\Big\} \,. \end{aligned} \tag{4.33}$$

Multiplets may be rescaled as follows: $\mathcal{Q}\to\sqrt{8\pi}\mathcal{Q}$ and $\Lambda\to\Lambda/\sqrt{8\pi}$ with the corresponding Jacobians canceling. Performing the $[\mathcal{D}\mathcal{Q}]_0$ integral cancels with the superdeterminant in the path integral. Therefore,

$$Z[\mathcal{B}] = \int[\mathcal{D}\Lambda]_0\,\exp\left\{\frac{i}{2}\int d^2x\,\left(\overline{\Lambda}+\frac{\overline{\mathcal{B}}}{\sqrt{8\pi}}\right)\Theta_0\left(\Lambda+\frac{\mathcal{B}}{\sqrt{8\pi}}\right) - \frac{i}{16\pi}\int d^2x\,\overline{\mathcal{B}}\,\Theta_0\,\mathcal{B}\right\} , \tag{4.34}$$

Finally, shifting the Λ multiplet by a background dependent term and recognizing the last term in (4.34) as a super-Liouville action allows the rewriting of the Λ measure in the SUGRA background $\mathcal{B}$. The dual formulation takes the following form:

$$Z[\mathcal{B}] = \int[\mathcal{D}\Lambda]_{\mathcal{B}}\,\exp\left\{\frac{i}{2}\int d^2x\,\overline{\Lambda}\Theta^{\mathrm{SC}}_{\mathcal{B}}\Lambda\right\} \quad . \tag{4.35}$$

Therefore, just as in the massless scalar under space-time duality, the super dual formulation (4.35) is identical to the original formulation (4.19). The trivial mapping of non-interacting, massless (1,1) matter under superduality does demonstrate some level of consistency of the procedure, mapping a local QFT to a local QFT. One could consider massive and interacting matter in (1,1) SUGRA, where superduality may be less trivial. Instead, it is technically more feasible to consider massless supersymmetric matter in the context of (2,2) SUGRA, which is the topic of the next section.

4.3 (2,2) Superduality In Components

In this section the procedure of superduality is applied to free massless (2,2) supersymmetric matter in background (2,2) SUGRA. The duality procedure follows the corresponding steps of the previous section.

After a brief introduction to (2,2) supersymmetry and to (2,2) supergravity, superduality is applied to the simplest (2,2) action, in background SUGRA. Although superficially it seems that, just as in the case of (1,1) matter, one obtains a trivial mapping under superduality, a more careful investigation leads one to conclude otherwise.

In the next section, superspace formulation of global and local symmetry is introduced, where the non-trivial nature of superduality is most easily evident. Superspace is a construct useful for keeping track of various supersymmetric multiplets, where local coordinate transformations correspond to ordinary local coordinate transformations *and* to local supersymmetric transformations. Therefore to be covariant in superspace implies covariance under both general coordinate transformations and local supersymmetry. In superspace analysis it becomes evident that there are four "simplest" models of (2,2) supersymmetry in background SUGRA. In absence of masses or interactions the four models reduce to the same field content, and are indistinguishable in terms of component actions. Rather than introducing masses or interactions, performing superduality in superspace makes it possible to keep track of the different minimal models. Superduality is shown to have a non-trivial action, mapping so-called twisted-chiral multiplet to a chiral one.

In the remainder of this section the component (2,2) superduality is presented.

Although superficially, component superduality is trivial, it is presented here in order to introduce the field content of (2,2) supersymmetry and supergravity, which the superspace formulation must reproduce. In addition, the component calculation demonstrates that superduality maps a local action into another local action, by keeping explicit track of the cancellation of all nonlocal functional determinants.

4.3.1 (2,2) Supersymmetry

(2,2) supersymmetry refers to the symmetry generated by two left-moving and two right-moving anti-Hermitian–Weyl charges. Equivalently, two anti-Hermitian charges, Q^1 and Q^2, can generate the (2,2) algebra. It is customary to combine two Majorana charges into a single complex generator $Q = \frac{1}{\sqrt{2}}(Q^1 - iQ^2)$, so that the (anti)commutation relations (4.2) are now satisfied by the complex generator Q.

The simplest model which represents the (2,2) algebra has the field content of a complex scalar, a complex auxiliary field, and a Dirac spinor (ϕ, N, χ). Supersymmetry transformations of the fields may be parametrized in terms of a constant complex spinor ε, with ε^c referring to the charge conjugate as defined in (3.24), and $\overline{\varepsilon}^c$ refers to the Dirac conjugate of the charge conjugate.

$$\delta\phi = \overline{\varepsilon}\psi \ , \qquad \delta\chi = i\partial\!\!\!/\phi\varepsilon + N\varepsilon^c \ , \qquad \delta N = i\overline{\varepsilon}^c\partial\!\!\!/\chi \quad . \tag{4.36}$$

The supersymmetric action takes the form:

$$S = \frac{1}{2}\int d^2x \left(-|\partial_a\phi|^2 + i\overline{\chi}\partial\!\!\!/\chi - N^*N\right) \quad . \tag{4.37}$$

As in the (1,1) case, invariance under supersymmetry of the proposed model may be anticipated from counting the degrees of freedom. With or without equations of motion, the number of bosonic and fermionic components match in the supersymmetric multiplet. Also, as in the (1,1) case, invariance of the action (4.37) under (4.36) is not sufficient to guarantee a representation of supersymmetry. It is important that in addition to the invariance of the action, the supersymmetry transformations on the fields (4.36) represent the algebra (4.2), which can be shown to be true just as in the (1,1) case.

4.3.2 (2,2) Supergravity

Brink and Schwarz [17] have derived an action with a local (2,2) supersymmetry. In addition to (2,2) matter multiplet, (ϕ, N, χ), the $(2,2)$ SUGRA theory contains a Dirac gravitino (ψ_m), a gauge field (A_m), an inverse zweibein (e^a_m) as well as a complex auxiliary field $(\mathcal{K})$. The SUGRA auxiliary field appears only in the SUGRA transformation rules [18, 19] and in the path integral measure. The appearance of the gauge fields follows from the fact that the (2,2) supersymmetric model (4.37) contains Dirac spinors, which are invariant under a global gauge transformation. In fact, in the local version of (2,2) supersymmetry one may choose to gauge either the vector like or the axial-vector like symmetry, with the resulting SUGRA labeled as $U_V(1)$ and $U_A(1)$, respectively [20].

The component SUGRA action [17, 19, 21], takes the following form:

$$\begin{aligned} S \;=\; & \frac{1}{2}\int d^2x[e^{-1}]\Big\{-g^{mn}\partial_m\phi\partial_n\phi^* - i\overline{\chi}\gamma^m\mathbf{D}_m\chi + \frac{1}{2}A_m\overline{\chi}\gamma_5\gamma^m\chi + N^*N \\ & - \left(\partial_m\phi^* + \frac{1}{2}\overline{\psi}_a\chi\right)\overline{\chi}\gamma^b\gamma^a\psi_b - \left(\partial_a\phi + \frac{1}{2}\overline{\chi}\psi_a\right)\overline{\psi}_b\gamma^a\gamma^b\chi\Big\} \,. \end{aligned} \tag{4.38}$$

This action is correct for both $U_V(1)$ and $U_A(1)$ types of SUGRA, since a field redefinition takes one from $U_V(1)$ to $U_A(1)$ theory [21].

For superduality applications it is useful to list the local symmetries of the SUGRA action (4.38). In addition to local supersymmetry, coordinate and Lorentz invariance, one has invariance under local gauge transformations. Further, there is invariance under local Weyl, chiral gauge, and (complex) S-supersymmetry, which are anomalous for the SUGRA invariant path integral measure.

4.3.3 Superduality in Components

The purpose of this section is to demonstrate that the duality procedure yields a local, well defined result. Superduality is applied to a $(2,2)$ matter multiplet in $U_A(1)$ SUGRA. As in the section on $(1,1)$ superduality, the starting point is $(2,2)$ matter $M = (\phi, N, \chi)$ coupled to a background SUGRA multiplet $\mathbf{B} = (e^a_m, \psi_m, A_m, \mathcal{K})$. It is useful to define an (8 dimensional) matrix notation, similarly as in the (1, 1) case,

$$M = \begin{pmatrix} \phi \\ \phi^* \\ F \\ F^* \\ \chi_+ \\ \chi_- \\ -\chi_+^* \\ -\chi_-^* \end{pmatrix} \qquad T = \begin{pmatrix} 0 & 0 & 0 & 1_2 \\ 0 & 0 & 1_2 & 0 \\ 0 & C & 0 & 0 \\ C & 0 & 0 & 0 \end{pmatrix} \quad , \tag{4.39}$$

with $\overline{M} = M^\dagger T$, so that the generating functional may be written as

$$Z[\mathbf{B}, \overline{\mathbf{B}}] = \int \left[\mathcal{D}(M, \overline{M})\right]_{\mathbf{B}} \exp\left\{ i \int d^2x \overline{M} \Theta_{\mathbf{B}} M \right\} \quad . \tag{4.40}$$

The field operator $\Theta_{\mathbf{B}}$ depends on the background supergravity multiplet $\mathbf{B}$ and may be extracted from (4.38). For the present purposes, it is sufficient to consider the superconformal gauge limit of background SUGRA, by imposing the following conditions:

$$e_m^a - \frac{1}{2}\delta_m^a \left(e_l^b \delta_b^l\right) = 0 \tag{4.41}$$

$$\gamma^l \gamma_m \psi_l = 0 \tag{4.42}$$

$$\mathbf{D}_m \Box_e^{-1} \mathbf{D}_l A^l = 0 \tag{4.43}$$

Superconformal gauge conditions result in the SUGRA multiplet reducing to the conformal factor (σ^B) (4.41), a Dirac spinor (λ^B) (4.42), the transverse component of the gauge field (ρ^B) coming from $A_a = \epsilon_{ab}\partial^b \rho^B$ (4.43), and the auxiliary field ($\mathcal{K}^B$) remains as before. It should be emphasized that both $U_A(1)$ and $U_V(1)$ SUGRA models have the same field content, and are indistinguishable in absence of masses or self-interactions. The fields σ^B and ρ^B can be combined into a single complex scalar field, $\sigma^B + i\rho^B$. Therefore, collectively, the superconformal gauge fields may be denoted by $\mathcal{B} = (\sigma^B + i\rho^B, \mathcal{K}^B, \lambda^B)$. Notice, also, that the SUGRA field content in superconformal gauge is the same as for the (2,2) matter multiplet.

In superconformal gauge the field operator is

$$\boldsymbol{\Theta}_{\mathcal{B}}^{\mathbf{SC}} = \frac{1}{4}\begin{pmatrix} 0 & 0 & 0 & ie^{\sigma^B/2-i\rho^B/2}\not{\partial}e^{\sigma^B/2+i\rho^B/2} \\ 0 & 0 & ie^{\sigma^B/2+i\rho^B/2}\not{\partial}e^{\sigma^B/2-i\rho^B/2} & 0 \\ 0 & e^{2\sigma^B}1_2 & 0 & 0 \\ \Box 1_2 & 0 & 0 & 0 \end{pmatrix} \tag{4.44}$$

and the generating functional (4.40) becomes

$$Z[\mathcal{B},\overline{\mathcal{B}}] = \int \left[\mathcal{D}(M,\overline{M})\right]_{\mathcal{B}} \exp\left\{i\int d^2x\overline{M}\boldsymbol{\Theta}_{\mathcal{B}}^{\mathbf{SC}}M\right\} . \tag{4.45}$$

The $\mathcal{B}$-dependence of the path integral measure in 4.45 is chosen to ensure invariance with respect to local (2,2) SUGRA. The choice of this preferred symmetry implies the existence of a superconformal anomaly for this model [19]: $\left[\mathcal{D}(M,\overline{M})\right]_{\mathcal{B}} = \left[\mathcal{D}(M,\overline{M})\right]_0 \exp\{iS_{SL}(\mathcal{B})\}$, where

$$\begin{aligned} S_{SL}(\mathcal{B}) &= -\frac{1}{4\pi}\int d^2x \left(-\frac{1}{2}\partial_a\sigma^B\,\partial^a\sigma^B - \frac{1}{2}\partial_a\rho^B\,\partial^a\rho^B + \frac{i}{2}\overline{\lambda}^B\gamma^a\partial_a\lambda^B - \frac{1}{2}\mathcal{K}^{B*}\mathcal{K}^B\right) \\ &= -\frac{1}{4\pi}\int d^2x\,\overline{\mathcal{B}}\,\boldsymbol{\Theta}_0\,\mathcal{B}, \end{aligned} \tag{4.46}$$

In eq. (4.46), $\boldsymbol{\Theta}_0$ refers to the flat background limit of the field operator (4.44), i.e. $\sigma^B, \rho^B \to 0$. The anomaly action is proportional to the kinetic action for (2,2) matter multiplet (4.37).

Note that the expression for the anomaly as derived by [19, 13] differs from the expression (4.46) by the opposite sign for the ρ term. The discrepancy is explained in Appendix B.

Superduality proceeds by introducing fluctuations about background SUGRA fields. It is convenient to impose the superconformal gauge on the dynamic SUGRA, and the corresponding degrees of freedom may be grouped into a scalar multiplet as was done for the background: $\mathcal{Q} = (\sigma^Q + i\rho^Q, \mathcal{K}^Q, \lambda^Q)$. The total SUGRA multiplet becomes $\mathcal{B} + \mathcal{Q}$.

The $\mathcal{Q}$ degrees of freedom must be constrained away to ensure equality with the original path integral (4.45). This is accomplished using a multiplet of Lagrange multipliers, denoted by $\Lambda = (L_1 + iL_2, \mathcal{G}, \eta)$. Note that Λ has the same field content as both the superconformal supergravity multiplets, $\mathcal{B}$, $\mathcal{Q}$, and the matter multiplet, M.

Taking care of the Fadeev–Popov–deWitt Jacobian and choosing the constraint to maintain equality with (4.45), the generating functional takes the following form:

$$
\begin{aligned}
Z[\mathcal{B},\overline{\mathcal{B}}] &= \int \left[\mathcal{D}(M,\overline{M})\right]_{\mathcal{B}+\mathcal{Q}} \left[\mathcal{D}(\mathcal{Q},\overline{\mathcal{Q}})\right]_{\mathcal{B}+\mathcal{Q}} \left[\mathcal{D}(\Lambda,\overline{\Lambda})\right]_{\mathcal{B}+\mathcal{Q}} \mathrm{sdet}[\boldsymbol{\Theta}^{\mathbf{SC}}_{\mathcal{B}+\mathcal{Q}}] \\
&\quad \times \exp\left\{+i\int d^2x \overline{M}\boldsymbol{\Theta}^{\mathbf{SC}}_{\mathcal{B}+\mathcal{Q}} M + i\int d^2x \left(\overline{\mathcal{Q}}\boldsymbol{\Theta}^{\mathbf{SC}}_{\mathcal{B}}\Lambda + \overline{\Lambda}\boldsymbol{\Theta}^{\mathbf{SC}}_{\mathcal{B}}\mathcal{Q}\right)\right\} \quad .
\end{aligned}
\tag{4.47}
$$

Note, that the power of sdet$[\boldsymbol{\Theta}^{\mathbf{SC}}_{\mathcal{B}+\mathcal{Q}}]$ in eq.(4.47) is the same as in the corresponding step in (1,1) superduality (4.29). This is because the field operator is eight dimensional here, where as it is four dimensional in the (1,1) case.

In order to perform the relevant path integrals, it is useful to make all SUGRA dependence explicit by performing a superconformal transformation. When this is done for eq. (4.47), the transformation of the measures for $\mathcal{Q}$ and Λ cancels the transformation of the superdeterminant, leaving only a single factor of the anomaly action coming from $[\mathcal{D}(M,\overline{M})]_{\mathcal{B}+\mathcal{Q}}$. Therefore, eq. (4.47) simplifies to:

$$
\begin{aligned}
Z[\mathcal{B},\overline{\mathcal{B}}] &= \int \left[\mathcal{D}(M,\overline{M})\right]_0 \left[\mathcal{D}(\mathcal{Q},\overline{\mathcal{Q}})\right]_0 \left[\mathcal{D}(\Lambda,\overline{\Lambda})\right]_0 \mathrm{sdet}[\boldsymbol{\Theta}_0] \\
&\quad \times \exp\left\{i\int d^2x \overline{M}\boldsymbol{\Theta}_0 M + i\int d^2x \left(\overline{\mathcal{Q}}\boldsymbol{\Theta}_0\Lambda + \overline{\Lambda}\boldsymbol{\Theta}_0\mathcal{Q}\right) + iS_{SL}(\mathcal{B}+\mathcal{Q})\right\} .
\end{aligned}
\tag{4.48}
$$

One should first verify the equivalence of eq.(4.48) with the starting expression, eq. (4.45). To do so the variable Λ is changed to $\boldsymbol{\Theta}_0\Lambda$, and the integration over $\boldsymbol{\Theta}_0\Lambda$ is performed to obtain the functional delta function $\Delta[\mathcal{Q}]$. Using this to perform the $\mathcal{Q}$ integration leaves:

$$
Z[\mathcal{B},\overline{\mathcal{B}}] = \int \left[\mathcal{D}(M,\overline{M})\right]_0 \exp\left\{i\int d^2x \overline{M}\boldsymbol{\Theta}_0 M + iS_{SL}(\mathcal{B})\right\}
\tag{4.49}
$$

which reproduces eq. (4.43) once the path integral measure is rescaled to depend on the background multiplet, and the (superconformally invariant) matter action is rewritten as $\int \overline{M}\boldsymbol{\Theta}^{\mathbf{SC}}_{\mathcal{B}} M$.

To obtain the dual, the path integral over M is evaluated to obtain $(\mathrm{sdet}[\Theta_0])^{-1/2}$, which partly cancels the explicit superdeterminant which is already present in eq. (4.48). This leaves

$$\begin{aligned} Z[\mathcal{B},\overline{\mathcal{B}}] &= \int \left[\mathcal{D}(\mathcal{Q},\overline{\mathcal{Q}})\right]_0 \left[\mathcal{D}(\Lambda,\overline{\Lambda})\right]_0 (\mathrm{sdet}[\Theta_0])^{1/2} \\ &\times \exp\left\{i\int d^2x\left(\overline{\mathcal{Q}}\Theta_0\Lambda+\overline{\Lambda}\Theta_0\mathcal{Q}\right)-\frac{i}{4\pi}\int d^2x(\overline{\mathcal{B}}+\overline{\mathcal{Q}})\Theta_0(\mathcal{B}+\mathcal{Q})\right\}. \end{aligned} \tag{4.50}$$

Completing the squares in the exponent allows it to be rewritten as

$$\begin{aligned} S &= -\frac{1}{4\pi}\int d^2x\left(\overline{\mathcal{Q}}+\overline{\mathcal{B}}-4\pi\,\overline{\Lambda}\right)\Theta_0\left(\mathcal{Q}+\mathcal{B}-4\pi\,\Lambda\right) \\ &+ \frac{1}{4\pi}\int d^2x\left(\mathcal{B}-4\pi\,\overline{\Lambda}\right)\Theta_0\left(\mathcal{B}-4\pi\,\Lambda\right)-\frac{1}{4\pi}\int d^2x\overline{\mathcal{B}}\,\Theta_0\,\mathcal{B} \end{aligned} \tag{4.51}$$

Rescaling $\Lambda\to\Lambda/\sqrt{4\pi}$, and performing the $\mathcal{Q}$ path integration,

$$\begin{aligned} Z[\mathcal{B},\overline{\mathcal{B}}] &= \int \left[\mathcal{D}(\Lambda,\overline{\Lambda})\right]_0 \exp\left\{-\frac{i}{4\pi}\int d^2x\overline{\mathcal{B}}\Theta_0\mathcal{B}\right\} \\ &\times \exp\left\{i\int d^2x\left(\overline{\Lambda}-\frac{1}{\sqrt{4\pi}}\overline{\mathcal{B}}\right)\Theta_0\left(\Lambda-\frac{1}{\sqrt{4\pi}}\mathcal{B}\right)\right\}, \end{aligned} \tag{4.52}$$

which, after shifting the dual multiplet Λ to absorb $\mathcal{B}/\sqrt{4\pi}$ and rescaling $\mathcal{B}$ back into the measure $[\mathcal{D}(\Lambda,\overline{\Lambda})]_0$, is recognized as the action for a massless (2,2)-supersymmetric WZ multiplet:

$$Z[\mathcal{B},\overline{\mathcal{B}}] = \int \left[\mathcal{D}(\Lambda,\overline{\Lambda})\right]_{\mathcal{B}} \exp\left\{i\int d^2x\;\overline{\Lambda}\Theta_{\mathcal{B}}^{\mathrm{SC}}\Lambda\right\}\quad . \tag{4.53}$$

Superficially, it appears that the dual theory (4.53) is identical to the starting action, eq. (4.45). One cannot conclude, however, that the (2,2) case is trivially self dual, because a component action does not distinguish between two possible (2,2) matter multiplets: chiral and twisted-chiral. The distinction does become visible if one were to use more complicated actions. Alternatively, one can examine the situation in superspace, as is done next.

4.4 (2,2) Superduality in Superspace

In order to clarify the distinction between (2,2) matter multiplets it is convenient to work in superspace. Superspace is an enlarged space, which treats general coordinate and local supersymmetry invariance at par. The goal of the present section is to introduce enough machinery to be able to carry out the superduality procedure in superspace. Discussion of global and local (2,2) superspace follows closely the corresponding construction for (1,1) superspace, which is included in Appendix A. Also in Appendix A the reader may find a summation of notation conventions, which for the most part are left out from the chapters. For a more complete description of superspace the reader is referred to textbooks on the subject, for instance references [23, 22, 24].

The presentation of superspace begins with the description of rigid (or global) (2,2) superspace. It is shown that there are in fact two types of scalar supersymmetric matter: chiral and twisted-chiral. The subsequent description of local (2,2) superspace confirms the fact that there are two types of (2,2) SUGRA. The result of the present discussion is the realization that there are four minimal (2,2) matter–SUGRA systems, depending on whether one chooses chiral or twisted-chiral matter and whether one chooses $U_A(1)$ or $U_V(1)$ SUGRA. With theses preliminaries at hand, superduality is shown to map among the matter–SUGRA systems in the last subsection of this chapter.

4.4.1 Rigid (2,2) Superspace

In order to derive superspace, it is convenient to work in the light-cone basis defined by the following relations:

$$x^{\ddagger} = \frac{1}{2}(x^0 + x^1) \qquad x^{=} = \frac{1}{2}(-x^0 + x^1) \quad . \tag{4.54}$$

The corresponding derivatives are defined so that $[\partial_m, x^n] = \delta_m^n$. Therefore,

$$\partial_{\ddagger} = \partial_0 + \partial_1 \qquad \partial_{=} = -\partial_0 + \partial_1 \quad . \tag{4.55}$$

Lightcone translation operators take the following form,

$$P_{\ddagger} = P_0 + P_1 \qquad P_{=} = -P_0 + P_1 \quad , \tag{4.56}$$

so that supersymmetry anticommutation rules (4.2) may be expressed in terms of the one dimensional (Weyl) component generators, as follows,

$$\{Q_+, Q_{\dot{+}}\} = -(P_0 + P_1) = -P_{\ddagger} \qquad \{Q_-, Q_{\dot{-}}\} = -(-P_0 + P_1) = -P_{=} \quad . \tag{4.57}$$

Dotting the lower (upper) spinor index is equivalent to taking the negative (positive) Hermitian conjugate of the Weyl component (as explained in Appendix A). The (2,2) algebra is generated by $\{P_{\ddagger}, P_{=}, Q_+, Q_-, Q_{\dot{+}}, Q_{\dot{-}}, \mathcal{M}, \mathcal{Y}, \mathcal{Y}'\}$, where $\mathcal{M}$ is the Lorentz generator J_{01}, and $\mathcal{Y}, \mathcal{Y}'$ are internal $U_V(1), U_A(1)$ generators, respectively. For completeness, the non-zero commutators (in addition to the relations (4.57)) are stated here:

$$\begin{aligned}
[\mathcal{M}, Q_\pm] &= \pm\frac{1}{2}Q_\pm \,, \qquad [\mathcal{M}, Q_{\dot{\pm}}] = \pm\frac{1}{2}Q_{\dot{\pm}} \,, \\
[\mathcal{Y}, Q_\pm] &= -\frac{i}{2}Q_\pm \,, \qquad \left[\mathcal{Y}, Q_{\dot{\pm}}\right] = +\frac{i}{2}Q_{\dot{\pm}} \,, \\
[\mathcal{Y}', Q_\pm] &= \mp\frac{i}{2}Q_\pm \,, \qquad \left[\mathcal{Y}', Q_{\dot{\pm}}\right] = \pm\frac{i}{2}Q_{\dot{\pm}} \,, \\
\left[P_{\ddagger}, \mathcal{M}\right] &= -P_{\ddagger} \,, \qquad [P_{=}, \mathcal{M}] = P_{=} \quad .
\end{aligned} \tag{4.58}$$

The basic idea of superspace is to have supersymmetry algebra represented by derivatives, just as translational generators are represented by derivatives in Minkowski space-time: $P_{\ddagger} = i\partial_{\ddagger}$ and $P_{=} = i\partial_{=}$. Since there are two independent, complex, one dimensional supersymmetry generators, (2,2) superspace is parametrized by the coordinates: $\{x^M\} = \{x^{\ddagger}, x^{=}, \theta^+, \theta^-, \theta^{\dot{+}}, \theta^{\dot{-}}\}$, where $\theta^{\dot{\pm}} = (\theta^\pm)^*$. The coordinates θ are complex and anticommuting, to reflect the nature of the generators Q which are to be represented by derivatives.

Derivatives on superspace are defined analogously with the lightcone derivatives, $[\partial_M, x^N\} = \delta_M^N$, where the (anti-)communtation refers to (fermionic) bosonic coordinates. Therefore there are the following six "naive" derivatives on superspace: $\{\partial_{\ddagger}, \partial_{=}, \partial_+, \partial_-, \partial_{\dot{+}}, \partial_{\dot{-}}\}$. Note that the (lower index) spinor derivatives are Hermitian, where as space-time derivatives are anti-Hermitian, which may be seen from the above defining relation. It is straightforward to show that the following operators (linear combinations of derivatives on superspace) represent the supercharges:

$$
\begin{aligned}
Q_{+} &= \partial_{+} - \frac{i}{2}\theta^{\dot{+}}\partial_{\ddagger} & Q_{\dot{+}} &= \partial_{\dot{+}} - \frac{i}{2}\theta^{+}\partial_{\ddagger} \\
Q_{-} &= \partial_{-} - \frac{i}{2}\theta^{\dot{-}}\partial_{=} & Q_{\dot{-}} &= \partial_{\dot{-}} - \frac{i}{2}\theta^{-}\partial_{=} \quad ,
\end{aligned} \tag{4.59}
$$

where $Q_{\pm}$ and $Q_{\dot{\pm}}$ are Hermitian conjugates.

Having defined the action of supersymmetry generators on superspace, the supersymmetry transformation may be parametrized infinitesimally with fermionic parameters $\{\varepsilon^{\pm}, \varepsilon^{\dot{\pm}}\}$ as follows, $\varepsilon^{+}Q_{+} + \varepsilon^{-}Q_{-} + \varepsilon^{\dot{+}}Q_{\dot{+}} + \varepsilon^{\dot{-}}Q_{\dot{-}}$. Acting on superspace coordinates, one obtains the following transformation rules:

$$
\begin{aligned}
\delta\theta^{+} &= \varepsilon^{+}, \quad \delta\theta^{-} = \varepsilon^{-}, & \delta\theta^{\dot{+}} &= \varepsilon^{\dot{+}}, \quad \delta\theta^{\dot{-}} = \varepsilon^{\dot{-}} \\
\delta x^{\ddagger} &= -\frac{i}{2}\varepsilon^{+}\theta^{\dot{+}} - \frac{i}{2}\varepsilon^{\dot{+}}\theta^{+}, & \delta x^{=} &= -\frac{i}{2}\varepsilon^{-}\theta^{\dot{-}} - \frac{i}{2}\varepsilon^{\dot{-}}\theta^{-} \quad .
\end{aligned} \tag{4.60}
$$

One can also define six supersymmetry covariant derivatives, $\{D_{\ddagger},\ D_{=},\ D_{\pm},\ D_{\dot{\pm}}\}$, by the requirement that they commute with the supersymmetry generators Q. One finds explicitly,

$$
\begin{aligned}
D_{\ddagger} &= \partial_{\ddagger}, & D_{=} &= \partial_{=} \\
D_{+} &= \partial_{+} + \frac{i}{2}\theta^{\dot{+}}\partial_{\ddagger}, & D_{\dot{+}} &= \partial_{\dot{+}} + \frac{i}{2}\theta^{+}\partial_{\ddagger} \\
D_{-} &= \partial_{-} + \frac{i}{2}\theta^{\dot{-}}\partial_{=}, & D_{\dot{-}} &= \partial_{\dot{-}} + \frac{i}{2}\theta^{-}\partial_{=} \quad .
\end{aligned} \tag{4.61}
$$

In Minkowski space-time one defines fields which form representations of Poincaré symmetry. Similarly, in superspace, covariant fields form representations of supersymmetry. The scalar superfield $\Phi(x,\theta)$ is a function of superspace coordinates. Since $(2,2)$ superspace is a complex space, so then, in general, are the superfields which live on it. In order to make contact with the usual Minkowski field representations of supersymmetry, the superfield is Taylor expanded in anticommuting coordinates. This expansion terminates, because the anticommuting nature of the superspace coordinates restricts terms with more than four of them.

$$
\Phi = \phi(x) + \theta^{+}\chi_{+}(x) + \theta^{-}\chi_{-}(x) + \theta^{\dot{+}}\eta_{\dot{+}}(x) + \theta^{\dot{-}}\eta_{\dot{-}}(x)
$$

$$
\begin{aligned}
&+ \ \{6 \text{ terms containing } 2\ \theta\text{'s}\} \\
&+ \ \{4 \text{ terms containing } 3\ \theta\text{'s}\} + \theta^+\theta^-\theta^{\dot{+}}\theta^{\dot{-}}D(x)
\end{aligned}
\tag{4.62}
$$

The coefficients of the expansion (4.62) are complex valued fields in Minkowski space. Therefore taking the Hermitian conjugate of the scalar superfield one obtains:

$$
\begin{aligned}
\overline{\Phi} &= \phi^*(x) + \theta^{\dot{+}}\chi_{\dot{+}}(x) + \theta^{\dot{-}}\chi_{\dot{-}}(x) + \theta^+\eta_+(x) + \theta^-\eta_-(x) \\
&+ \ \{6 \text{ terms containing } 2\ \theta\text{'s}\} \\
&+ \ \{4 \text{ terms containing } 3\ \theta\text{'s}\} + \theta^+\theta^-\theta^{\dot{+}}\theta^{\dot{-}}D^*(x)
\end{aligned}
\tag{4.63}
$$

Associating the leading terms in the superfield expansion, $\{\phi,\ \phi^*\}$, with the scalar component of the $(2,2)$ supersymmetry multiplet (4.37), one finds that "one θ" terms in eqs. (4.62) and (4.63) contain too many fermionic degrees of freedom for a minimal representation of supersymmetry. Therefore two fermionic constraints should be imposed on the superfield Φ (together with the Hermitian conjugate conditions) to match the Minkowski space field content to (4.37). The key point in the present discussion is the following: that there are two ways of restricting fermionic components of the superfield. A *chiral* superfield is defined by

$$
D_+\overline{\Phi} = D_-\overline{\Phi} = D_{\dot{+}}\Phi = D_{\dot{-}}\Phi = 0 \quad , \tag{4.64}
$$

and a *twisted chiral* superfield by

$$
D_+\overline{\Phi} = D_{\dot{-}}\overline{\Phi} = D_{\dot{+}}\Phi = D_-\Phi = 0 \quad . \tag{4.65}
$$

With either set of constraints, the number and type of component fields in the superfield is reduced to match with the (4.37).

There is an alternate way to define the Minkowski field components from the superspace expansion (4.62). Using supersymmetry covariant derivatives and projections one defines the chiral superfield components by

$$
\begin{array}{rclrcl}
\Phi| &=& \phi & \overline{\Phi}| &=& \phi^* \\
D_+\Phi| &=& \chi_+ & D_{\dot{+}}\overline{\Phi}| &=& \chi_{\dot{+}} \\
D_-\Phi| &=& \chi_- & D_{\dot{-}}\overline{\Phi}| &=& \chi_{\dot{-}} \\
\frac{i}{2}[D_+, D_-]\Phi| &=& N & \frac{i}{2}[D_{\dot{+}}, D_{\dot{-}}]\overline{\Phi}| &=& N^*
\end{array}
\tag{4.66}
$$

where "|" denotes the projection: $\theta^{\pm}, \theta^{\dot{\pm}} \to 0$. Similarly, the twisted-chiral multiplet component fields are defined by the following projections,

$$\begin{aligned} \Phi\,| &= \phi & \overline{\Phi}\,| &= \phi^* \\ D_+\Phi\,| &= \chi_+ & D_{\dot{+}}\overline{\Phi}\,| &= \chi_{\dot{+}} \\ D_{\dot{-}}\Phi\,| &= \eta_{\dot{-}} & D_-\overline{\Phi}\,| &= \eta_- \\ \tfrac{i}{2}[D_+, D_{\dot{-}}]\Phi\,| &= G & \tfrac{i}{2}[D_{\dot{+}}, D_-]\overline{\Phi}\,| &= G^* \end{aligned} \tag{4.67}$$

The use of supersymmetry covariant derivatives in equations (4.66) and (4.67) turns out to be convenient for deriving component actions from superspace as well as for supersymmetry transformations.

For the chiral scalar superfield one may define the action on $(2,2)$ superspace as follows:

$$S = \int d^2x d\theta^+ d\theta^- d\theta^{\dot{+}} d\theta^{\dot{-}} \overline{\Phi}\Phi = \int d^2x D_+ D_- D_{\dot{+}} D_{\dot{-}} \overline{\Phi}\Phi\,| \tag{4.68}$$

The twisted chiral scalar action takes the same form, except for an overall minus sign.

For concreteness, taking the definition of component fields from eq. (4.66), and expanding the action (4.68), one obtains

$$S = \int dx^{\ddagger} dx^{=} \left(\phi^* \partial_{\ddagger} \partial_{=} \phi - i\chi_{\dot{+}} \partial_{=} \chi_+ - i\chi_{\dot{-}} \partial_{\ddagger} \chi_- - N^* N\right) \tag{4.69}$$

which may be rewritten in the usual form,

$$S = \frac{1}{2} \int dx^0 dx^1 \left(-\partial_m \phi^* \partial^m \phi + i\overline{\chi}\partial\!\!\!/ \chi - N^* N\right) \tag{4.70}$$

It is useful to introduce the notion of tangent space to superspace. Although arguably such a construct is not necessary for doing calculations in rigid superspace, it is essential in the case of local superspace. The present discussion should facilitate the introduction of local superspace as a generalization of geometry of rigid superspace.

One may define the tangent space as the space spanned by the "naive" coordinate derivatives $\{\partial_{\ddagger}, \partial_{=}, \partial_{\pm}, \partial_{\dot{\pm}}\}$. This is the so-called coordinate basis of the tangent space. It is possible to consider a non-coordinate basis of derivatives which is related by a general linear transformation to the coordinate basis. In particular, one may choose a basis which (anti-)commutes with supersymmetry charges, i.e., the basis of covariant derivatives, as is done for $(1,1)$ superspace in Appendix A. Covariant derivatives are related to the coordinate basis as follows:

$$D_A = \hat{E}_A^M \partial_M\,, \qquad \partial_M = \hat{E}_M^A D_A \quad . \tag{4.71}$$

The above relation defines the vielbein and its inverse, respectively, which in the context of local superspace in the next subsection become space-time dependent superfields. The dual (in the linear algebra sense) coordinate and non-coordinate line elements are related similarly, as follows:

$$dx^{\mathrm{M}} = \hat{E}_A^{\mathrm{M}} dx^A\,, \qquad dx^A = \hat{E}_{\mathrm{M}}^A dx^{\mathrm{M}} \quad . \tag{4.72}$$

The invariant integration measure for an action on superspace is defined in terms of non-coordinate line elements (which are invariant under supersymmetry), and the definition can be cast in terms of the coordinate basis as follows:

$$\prod_A dx^A = \prod_{\mathrm{M}} dx^{\mathrm{M}} \left[\hat{E}^{-1}\right] \tag{4.73}$$

where $\left[\hat{E}^{-1}\right] = \mathrm{sdet}\left[\hat{E}_{\mathrm{M}}^A\right]$.

Having the covariant derivatives D_{A} (4.61), it is straightforward to reconstruct the flat superspace vielbein,

$$\hat{E}_A^M = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ \frac{i}{2}\theta^{\dot{+}} & 0 & 1 & 0 & 0 & 0 \\ 0 & \frac{i}{2}\theta^{\dot{-}} & 0 & 1 & 0 & 0 \\ \frac{i}{2}\theta^{+} & 0 & 0 & 0 & 1 & 0 \\ 0 & \frac{i}{2}\theta^{-} & 0 & 0 & 0 & 1 \end{pmatrix} , \tag{4.74}$$

and its inverse,

$$\hat{E}_M^A = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ -\frac{i}{2}\theta^{\dot{+}} & 0 & 1 & 0 & 0 & 0 \\ 0 & -\frac{i}{2}\theta^{\dot{-}} & 0 & 1 & 0 & 0 \\ -\frac{i}{2}\theta^{+} & 0 & 0 & 0 & 1 & 0 \\ 0 & -\frac{i}{2}\theta^{-} & 0 & 0 & 0 & 1 \end{pmatrix} . \tag{4.75}$$

With the explicit form for the supervielbein, one observes that sdet $\left[\hat{E}_{\mathrm{M}}^{A}\right] = 1$. Therefore, the coordinate and non-coordinate measures are related trivially, whence the previous definition of the superspace action (4.68) is consistent with (4.73).

Before proceeding to a discussion of local superspace, it is useful to point out that there are non-trivial consequences for the geometry of rigid superspace. In general relativity, one captures the non-trivial geometry of space-time with a non-zero curvature tensor, defined as the commutator of covariant derivatives. Minkowski or Euclidean spaces have trivial geometry since there one sees that (covariant) derivatives commute.

Considering the tensor structure, one sees that in addition to curvature (R) terms in the commutation relation for covariant derivatives, one also has the possibility of torsion (T) terms, as follows,

$$[D_A, D_B] = T_{AB}^{C} D_C + R_{AB}^{CD} J_{CD} \quad . \tag{4.76}$$

Torsion describes the lack of closure of a parallelogram of infinitesimal displacements generated by covariant derivatives. The second term in (4.76) is the usual Riemann tensor of general relativity. Although both torsion and curvature tensors are indexed with non-coordinate basis indices, one is free to convert to coordinate indices with the help of the vielbein.

In principle, if the covariant derivatives are also covariant under local internal symmetries, one has additional terms in the relation (4.76). These are field strengths for the corresponding gauge fields (just as the Riemann tensor is the field strength for the metric), and they enter in expression (4.76) multiplied by the generators of the local symmetry, in a representation appropriate to the field on which the covariant derivatives act.

Having the global superspace covariant derivatives (4.71), one can explicitly evaluate the commutation relation to confirm that rigid superspace is indeed flat (i.e., the Riemann tensor vanishes) but that it has non-zero torsion. This suggests that rigid (2, 2) superspace has non-trivial geometry.

4.4.2 Local (2,2) Superspace

In order to represent SUGRA one needs to introduce metric degrees of freedom. This means that $(2,2)$ local superspace is a curved manifold, and its construction follows, for the most part, the construction of $(1,1)$ local superspace in Appendix A. The present description of $(2,2)$ SUGRA comes from reference [21].

Local superspace is parametrized by six coordinates with world indices $\{x^M\} = \{x^m, \theta^\mu\} = \{x^{\ddagger}, x^{\approx}, \theta^{+}, \theta^{\sim}, \theta^{\dot{+}}, \theta^{\dot{\sim}}\}$. At every point on the manifold, one may define the tangent space to be a linear space spanned by the derivatives which form the (coordinate) basis for vector components $\{\, \partial_{\ddagger}\, , \partial_{\approx}\, , \partial_{+}\, , \partial_{\sim}\, , \partial_{\dot{+}}\, , \partial_{\dot{\sim}}\, \}$. Note, that unlike the $(1,1)$ case in Appendix A, and in order to agree with reference [21], here naive derivatives are chosen for the coordinate basis of the tangent space.

Locally, one may define a flat, non-coordinate basis of the tangent space: $\{\partial_A\} = \{\partial_{\ddagger}, \partial_{=}, \partial_{+}, \partial_{-}, \partial_{\dot{+}}, \partial_{\dot{-}}\}$, related to the coordinate basis with a locally defined vielbein superfield,

$$\partial_A \;=\; E_A^M \partial_M \quad . \tag{4.77}$$

The vielbein contains SUGRA degrees of freedom.

Covariant derivatives of $(2,2)$ SUGRA are defined as follows,

$$\hat{\nabla}_A \;=\; \partial_A + \Phi_A \mathcal{M} + \Gamma_A \mathcal{Y} + \Gamma'_A \mathcal{Y}' \quad , \tag{4.78}$$

where Φ_A is the connection for local Lorentz symmetry, Γ is the gauge field for local $U_V(1)$ symmetry and Γ' is the gauge field for the local $U_A(1)$ symmetry. Therefore, $\hat{\nabla}_A$ derivatives define a SUGRA theory with both $U_V(1)$ and $U_A(1)$ invariance.

As in the case of $(1,1)$ SUGRA in superspace, the vielbein, connection and gauge superfields contain too many degrees of freedom for the minimal $(2,2)$ SUGRA model in the previous section (4.38). The vielbein now contains $6 \times 6 = 36$ superfield degrees of freedom, and the three connection/gauge fields contain 18 superfield degrees of freedom. Using local coordinate freedom on superspace, as well as local Lorentz transformations and local $U(1)$ symmetries reduces the number of non-gauge superfield degrees of freedom to 45. Therefore just as in the case $(1,1)$ case, the superspace

SUGRA formulation is highly redundant, and covariant constraints need to be placed on field strengths (torsion, curvature, and gauge field strengths), in order to remove unwanted degrees of freedom. As in the $(1,1)$ case, primary constraints are stated in terms of spinor covariant derivatives,

$$\begin{aligned}
\{\hat{\nabla}_+, \hat{\nabla}_+\} &= 0\,, & \{\hat{\nabla}_-, \hat{\nabla}_-\} &= 0\,, & \{\hat{\nabla}_+, \hat{\nabla}_{\dot{-}}\} &= -\overline{F}\,\overline{N}\,, \\
\{\hat{\nabla}_+, \hat{\nabla}_-\} &= -\overline{R}\,\overline{M} & \{\hat{\nabla}_+, \hat{\nabla}_{\dot{+}}\} &= i\hat{\nabla}_{\mp}\,, & \{\hat{\nabla}_-, \hat{\nabla}_{\dot{-}}\} &= i\hat{\nabla}_{=}
\end{aligned} \tag{4.79}$$

where

$$\begin{aligned}
M &= \frac{1}{2}(\mathcal{M} + i\mathcal{Y}')\,, & \overline{M} &= \frac{1}{2}(\mathcal{M} - i\mathcal{Y}')\,, \\
N &= \frac{1}{2}(\mathcal{M} + i\mathcal{Y})\,, & \overline{N} &= \frac{1}{2}(\mathcal{M} - i\mathcal{Y}) \quad .
\end{aligned} \tag{4.80}$$

Using generalized Bianchi identities (A.67) consequences of the constraints may be derived as further commutators of covariant derivatives. In this way, all torsion, curvature and gauge field strengths may be expressed in terms of two superfields F and R.

In the component version of $(2,2)$ SUGRA (4.38), only one gauge field appears. This suggests that the SUGRA theory defined by the covariant derivatives (4.78) is further reducible to one containing either $U_V(1)$ or $U_A(1)$ symmetry.

$U_V(1)$ theory is defined by setting the superfield $R = 0$. The covariant derivatives (now without "hats") take the following form,

$$\nabla_A = \partial_A + \Phi_A\mathcal{M} + \Gamma_A\mathcal{Y} \quad , \tag{4.81}$$

and the SUGRA constraint relations simplify to

$$\begin{aligned}
\{\nabla_+, \nabla_+\} &= 0 & \{\nabla_-, \nabla_-\} &= 0 & \{\nabla_+, \nabla_{\dot{-}}\} &= -\overline{F}\,\overline{N} \\
\{\nabla_+, \nabla_-\} &= 0 & \{\nabla_+, \nabla_{\dot{+}}\} &= i\nabla_{\mp} & \{\nabla_-, \nabla_{\dot{-}}\} &= i\nabla_{=} \quad .
\end{aligned} \tag{4.82}$$

To obtain the SUGRA theory with local axial-vector symmetry, one sets the superfield $F = 0$. The $U_A(1)$ covariant derivatives are:

$$\nabla_A = \partial_A + \Phi_A\mathcal{M} + \Gamma'_A\mathcal{Y}' \quad , \tag{4.83}$$

and the corresponding constraints on covariant derivatives are as follows:

$$
\begin{array}{llllll}
\{\nabla_+,\nabla_+\} & =0 & \{\nabla_-,\nabla_-\} & =0 & \{\nabla_+,\nabla_{\dot{-}}\}=0 & \\
\{\nabla_+,\nabla_-\} & =-\overline{R}\,\overline{M} & \{\nabla_+,\nabla_{\dot{+}}\} & =i\nabla_{\ddagger} & \{\nabla_-,\nabla_{\dot{-}}\}=i\nabla_{=} & . \quad (4.84)
\end{array}
$$

Further constraints are generated by applying Bianchi identities (A.67) to the above relations.

The SUGRA constraints may be expressed as conditions on the vielbein and connection superfields. In reference [25] the vielbein and connections are solved in terms of unconstrained superfields (called prepotentials), a real vector superfield prepotential (H^m) and either a chiral or twisted-chiral scalar superfield, Σ, for $U_A(1)$ and $U_V(1)$ respectively.

For both $U_A(1)$ and $U_V(1)$ supergravities the Ricci scalar evaluated from the vielbein, $\mathcal{R}$, can be placed within a scalar superfield, R and F respectively. A key distinction between the two is that for $U_A(1)$ supergravity R is chiral, while for $U_V(1)$ supergravity F is twisted-chiral.

SUGRA covariant chiral superfields are defined analogously to the global superspace definition in eq. (4.64),

$$
\nabla_+\overline{\Phi}\ ,=\ \nabla_-\overline{\Phi}\ =\ \nabla_{\dot{+}}\Phi\ =\ \nabla_{\dot{-}}\Phi\ =0\quad , \tag{4.85}
$$

similarly, twisted-chiral, SUGRA covariant, scalar superfields are defined as follows,

$$
\nabla_+\overline{\Phi}\ ,=\ \nabla_{\dot{-}}\overline{\Phi}\ =\ \nabla_{\dot{+}}\Phi\ =\ \nabla_-\Phi\ =0\quad . \tag{4.86}
$$

In principle there are four (2,2)-supersymmetric WZ models to consider, depending on whether the matter supermultiplet is chiral (C) or twisted-chiral (T), and on whether supergravity is $U_A(1)$ or $U_V(1)$. For brevity of reference, the four systems are denoted by: CA, CV, TA and TV. Note, that in the absence of couplings to other fields, the field redefinitions $\chi_- \leftrightarrow \chi_{\dot{-}}$ and $\psi_m^- \leftrightarrow \psi_m^{\dot{-}}$ simultaneously interchanges $T \leftrightarrow C$ and $A \leftrightarrow V$. Therefore, at the classical level (where no distinction is made between background and dynamic fields) all four of these Wess-Zumino models are not independent, $TA \equiv CV$ and $CA \equiv TV$.

The invariant action for a free generic scalar superfield Φ with SUGRA takes the following general form:

$$S = \int d^2x d^4\theta [E^{-1}] \overline{\Phi}\Phi \quad , \tag{4.87}$$

where the determinant of the inverse vielbein ensures invariance under local supersymmetry transformations.

For $U_A(1)$ supergravity, if the superfield Φ, and hence the Lagrangian density, are covariantly chiral, the action simplifies to

$$S_C = \int d^2x d\theta^+ d\theta^- [\mathcal{E}^{-1}] \nabla_{\dot{+}} \nabla_{\dot{-}} \overline{\Phi}\Phi \quad . \tag{4.88}$$

The measure is given by

$$[\mathcal{E}^{-1}] = e^{-2\Sigma}(1 \cdot e^{\overleftarrow{H}}) \tag{4.89}$$

with $U_A(1)$ SUGRA prepotentials, chiral superfield Σ and the vector superfield $\overleftarrow{H} = iH^m \overleftarrow{\partial}_m$. A similar simplification exists for the covariantly twisted-chiral Lagrangian in $U_V(1)$ supergravity, but now a twisted-chiral scalar superfield takes the place of the chiral Σ. For chiral matter coupling to $U_V(1)$ supergravity, or for twisted-chiral matter in $U_A(1)$, no such simplifications are known, and one must work with the generic action (4.87). The difficulty in finding a reduced action measure for twisted-chiral scalar matter in $U_A(1)$ supergravity can be anticipated from the fact that chirality conditions, which define the matter multiplets, would be mixed if the chiral action measure were used.

4.4.3 Superduality in Superspace

The key observation which makes the (2,2) example more interesting than its (1,1) counterpart, is that (2,2) chiral supermultiplets come in more than one type (B.19). The existence of these two kinds of multiplets is important because superduality can map one type of multiplet into another in the following way:

$$\begin{array}{ll} CA \to CA & \qquad TA \to CA \\ CV \to TV & \qquad TV \to TV \end{array} \tag{4.90}$$

In order to follow the supersymmetry-transformation properties of the dual model, and to see the transformation patterns of eq. (4.90) emerge, it is necessary to keep manifest the full symmetries of the problem.

Although superconformal gauge is used extensively in this superspace calculation, the duality procedure may nonetheless follow the supersymmetry-transformation properties of the various multiplets. The crucial point is that the superconformal gauge used in this section does not also impose the non-supersymmetric Wess-Zumino gauge conditions. The Wess-Zumino gauge [21] is needed for obtaining component SUGRA fields from superspace, and it is defined analogously with the (1,1) case (A.69) by the projections

$$\begin{aligned} \nabla_\alpha| &= \partial_\alpha \qquad \text{and} \\ \nabla_a| &= \mathbf{D}_a + \psi_a^+\partial_+ + \psi_a^-\partial_- + \psi_a^{\dot+}\partial_{\dot+} + \psi_a^{\dot-}\partial_{\dot-} \quad . \end{aligned} \tag{4.91}$$

A quick way to see how transformations (4.90) arise is to examine the (2,2)-invariant extension of the Lagrange multiplier Lagrangian used in space-time duality (3.10), which involved terms of the form $\int \sqrt{-g}\,\Lambda\,\mathcal{R}_g$. In superspace formulation of SUGRA, $\mathcal{R}_g$ lives in a chiral scalar superfield, R, for $U_A(1)$ SUGRA, and in twisted-chiral superfield F in $U_V(1)$ SUGRA. The (2,2)-supersymmetric generalization of the condition that imposes $\sqrt{-g}\mathcal{R}_g - \sqrt{-h}\mathcal{R}_h = 0$ necessarily involves a Lagrange-multiplier superfield with the same chirality properties as either R or F. Taking $U_A(1)$ background SUGRA, and additionally dynamic $U_A(1)$ SUGRA in superconformal gauge (parameterized by the chiral superfield $\mathcal{Q}$), the full superspace vielbein may be written as $(E^T)_\alpha^M = e^{\mathcal{Q}}(E^B)_\alpha^M$. Thus, the constraint $[\mathcal{E}_T^{-1}]R_T - [\mathcal{E}_B^{-1}]R_B = [\mathcal{E}_B^{-1}]\bar\nabla_B^2\bar{\mathcal{Q}} = 0$ is enforced by a chiral integral

$$\int d^2x d^2\theta\, [\mathcal{E}_B^{-1}]\,\Lambda\,\bar\nabla_B^2\bar{\mathcal{Q}} = \int d^2x d^4\theta\,[E^{-1}]\Lambda\bar{\mathcal{Q}} \quad , \tag{4.92}$$

where Λ is a chiral superfield and $[E^{-1}]$ is the (nonchiral) superdeterminant of the vielbein for the full superspace integral. Correspondingly in $U_V(1)$ theory, F, the Lagrange multiplier Λ, and the density are all twisted chiral. Because the Lagrange multipliers end up being the dual fields, superduality takes both C and T multiplets into C multiplets in $U_A(1)$ supergravity. Conversely in $U_V(1)$ supergravity, superduality always produces a T multiplet as in eq. (4.90).

Consider, for concreteness, a twisted-chiral superfield, $\mathcal{X}$, in $U_A(1)$ supergravity. The invariant generating functional is defined as:

$$Z[E_A^M] = \int \left[\mathcal{D}(\mathcal{X}, \bar{\mathcal{X}})\right]_E \exp\left\{-i\int d^2x\, d^4\theta\, [E^{-1}]\, \bar{\mathcal{X}}\mathcal{X}\right\} \tag{4.93}$$

Here $[E^{-1}]$ is the (nonchiral) superdeterminant of the vielbein which guarantees that the action is locally supersymmetric.

In superspace contact is made with superconformal gauge by expressing $[E^{-1}] = 1$ [25] and $[\mathcal{E}^{-1}] = [\mathcal{E}^{-1}(\Sigma)]$ as functions of a the compensator superfield Σ, with the vector prepotential H set to zero. For $U_A(1)$ supergravity, both Σ and $[\mathcal{E}^{-1}(\Sigma)] = e^{-2\Sigma}$ are chiral, while both are twisted-chiral for $U_V(1)$ supergravity.

Performing the functional integral over $\mathcal{X}$ involves subtleties related to maintaining consistent chirality conditions, and are not addressed here (see however [26]). One finds [21]:

$$\begin{aligned} Z[E_A^M] &= (\det[\Box_+]_E)^{-1/2} \\ &= (\det[\Box_+]_0)^{-1/2} \exp\left[\frac{i}{8\pi}\int d^2x\, d^4\theta\, [E^{-1}] \left(\overline{R}\frac{1}{\Box_+}R\right)\right] \\ &\to (\det[\Box_+]_0)^{-1/2} \exp\left[-\frac{2i}{\pi}\int d^2x\, d^4\theta\, \overline{\Sigma}\Sigma\right], \qquad \text{(in superconformal gauge)} \end{aligned} \tag{4.94}$$

Here $\Box_+ = \overline{\nabla}^2\nabla^2$ is the superspace d'Alembertian acting on a chiral superfield, and subscript "0" indicates decoupling from the background. In superconformal gauge, $R = -4\overline{\nabla}^2\overline{\Sigma}$ and $\overline{R} = 4\nabla^2\Sigma$, where Σ is the chiral compensator field.

To start dualizing, one needs to choose a gauge for dynamical supergravity such that dynamical vielbein $(E^T)_\alpha^M$ equals to the scaling of the background $e^{\mathcal{Q}}(E^B)_\alpha^M$ and in addition the background SUGRA is put in conformal gauge $(E^B)_\alpha^M = e^{\mathcal{B}}\widehat{E}_\alpha^M$ where $\widehat{E}_\alpha^M$ is the vielbein for flat superspace (4.74). The point of this parameterization is to ensure a linear background/quantum splitting $\mathcal{B} + \mathcal{Q}$ in terms of chiral superfields, just as in the conformal factor in space-time duality. As discussed above, a Lagrange-multiplier chiral superfield Λ is introduced, as in eq. (4.92), Next, the following gauged version of eq. (4.93) is chosen:

$$Z[\mathcal{B},\bar{\mathcal{B}}] = \int \left[\mathcal{D}(\mathcal{X},\bar{\mathcal{X}})\right]_{\mathcal{B}+\mathcal{Q}} \left[\mathcal{D}(\mathcal{Q},\bar{\mathcal{Q}})\right]_{\mathcal{B}+\mathcal{Q}} \left[\mathcal{D}(\Lambda,\bar{\Lambda})\right]_{\mathcal{B}+\mathcal{Q}} \det[\Box_{+}^{\mathcal{B}+\mathcal{Q}}]$$
$$\times \exp\left\{-i\int d^2x\, d^4\theta\, \bar{\mathcal{X}}\mathcal{X} + i\int d^2x\, d^4\theta\, (\bar{\mathcal{Q}}\Lambda + \bar{\Lambda}\mathcal{Q})\right\} \tag{4.95}$$

where the determinant is chosen by the requirement that eq. (4.95) reduce to eq. (4.93), once the Λ and $\mathcal{Q}$ are functionally integrated. To see this, one rewrites the Lagrange multiplier term as

$$\int d^2x\, d^4\theta\, (\bar{\mathcal{Q}}\Lambda + \bar{\Lambda}\mathcal{Q}) = \int d^2x d^2\theta\, [\mathcal{E}_{\bar{B}}^{-1}]\, \mathcal{Q}\bar{\nabla}^2\bar{\Lambda} + \int d^2x d^2\bar{\theta}\, [\bar{\mathcal{E}}_{B}^{-1}]\, \bar{\mathcal{Q}}\nabla^2\Lambda \tag{4.96}$$

and makes the change of variables $\Lambda \to \bar{\nabla}^2\bar{\Lambda}'$, $\bar{\Lambda} \to \nabla^2\Lambda'$ with Jacobian $(\det[\bar{\nabla}]^2 \times \det[\nabla^2])^{-1} = (\det[\Box_+])^{-1}$. This cancels the determinant factor already present and produces delta-functions that set $\mathcal{Q} = 0$, which guarantees the required equivalence of eqs. (4.95) and (4.93).

To reach the dual formulation superfields $\mathcal{X}$ and $\mathcal{Q}$ are integrated, leaving only Λ as the dual field. As in eq. (4.94), the $\mathcal{X}$ integral gives a factor of $(\det[\Box_+]_{\mathcal{B}+\mathcal{Q}})^{-\frac{1}{2}}$ and Weyl rescaling the $\mathcal{Q}$ and Λ measures gives

$$\left[\mathcal{D}(\mathcal{Q},\bar{\mathcal{Q}})\right]_{\mathcal{B}+\mathcal{Q}} \left[\mathcal{D}(\Lambda,\bar{\Lambda})\right]_{\mathcal{B}+\mathcal{Q}} = \left[\mathcal{D}(\mathcal{Q},\bar{\mathcal{Q}})\right]_0 \left[\mathcal{D}(\Lambda,\bar{\Lambda})\right]_0 (\det[\Box_+]_{\mathcal{B}+\mathcal{Q}})^{-1} \quad . \tag{4.97}$$

Therefore,

$$Z[\mathcal{B},\bar{\mathcal{B}}] = \int \left[\mathcal{D}(\mathcal{Q},\bar{\mathcal{Q}})\right]_0 \left[\mathcal{D}(\Lambda,\bar{\Lambda})\right]_0 \exp\left\{i\int d^2x\, d^4\theta\, (\bar{\mathcal{Q}}\Lambda + \bar{\Lambda}\mathcal{Q})\right\}$$
$$\times \exp\left\{-\frac{2i}{\pi}\int d^2x\, d^4\theta\, (\bar{\mathcal{Q}} + \bar{\mathcal{B}})(\mathcal{Q} + \mathcal{B})\right\} \tag{4.98}$$

where the last exponential is a representation of $(\det[\Box_+]_{\mathcal{B}+\mathcal{Q}})^{-\frac{1}{2}}$, as in eq. (4.94).

Suitably completing squares, performing the Gaussian $\mathcal{Q}$ integral and rescaling Λ finally gives:

$$Z[\mathcal{B},\bar{\mathcal{B}}] = \int \left[\mathcal{D}(\Lambda,\bar{\Lambda})\right]_{\mathcal{B}} \exp\left\{i\int d^2x d^4\theta \left(\bar{\Lambda} - \frac{2\bar{\mathcal{B}}}{\pi}\right)\left(\Lambda - \frac{2\mathcal{B}}{\pi}\right)\right\}, \tag{4.99}$$

which is recognized, after a shift, as the superspace generating functional for the massless chiral multiplet Λ.

This section demonstrates that super-duality takes the twisted-chiral multiplet, $\mathcal{X}$, into a chiral one. As is clear from the derivation, so long as background $U_A(1)$ SUGRA is considered, the dual variable Λ is chiral. Similar manipulations for $U_V(1)$ supergravity fill out the rest of the relationships of eq.(4.90).

Chapter 5

Effective Actions on Anti-de-Sitter Space-time

The focus of this chapter is the evaluation of effective actions for scalar fields in anti-de-Sitter (AdS) space-time, which is an example of a maximally symmetric space (MSS).

There are several reasons which make the study of effective actions in curved background interesting, besides their intrinsic interest as exact expressions for quantum systems in non-trivial gravitational backgrounds. AdS space-time is the background for extended SUGRA theories, in the sense that the extended SUGRA field equations, in the constant curvature limit, choose AdS as a solution over Minkowski space-time. Another interest in AdS stems from the conjecture by Maldacena [27] which relates supersymmetric nonabelian gauge theory in large-N limit to AdS SUGRA. In addition, having effective actions on MSS may serve as a starting point for building approximate effective gravitational actions in less symmetric spaces.

The outline of this chapter is as follows. The present discussion begins with the construction of MSS's. A brief review of select aspects of QFT on generic curved spaces is offered, and then specified to AdS space-time. Next, effective actions for scalar and spinor fields are evaluated on AdS of various dimensions. In passing, it is shown that a proposed classical duality on $(1+1)$ dimensional AdS is not maintained at the quantum level. Metric, curvature and other conventions used in this chapter are listed in Appendix A.

5.1 Maximally Symmetric Spaces

MMS are space-times with non-zero curvature, but which nonetheless contain the same number of symmetries as the Poincaré group for Minkowski space-time. Because the symmetry requirement restricts MMS's to take particular forms, their structure has been studied, and a well posed QFT on these spaces is available. In order to demonstrate some symmetry properties of MMS's, their construction in general is presented in this section.

The GCT of the metric tensor takes the following form [28],

$$g_{mn}(x) = \frac{\partial x'^p}{\partial x^m}\frac{\partial x'^s}{\partial x^n}\, g'_{ps}(x') \quad . \tag{5.1}$$

One is often interested in space-time isometries, a restricted class of GCT's which preserve the form of the metric as follows,

$$g_{mn}(x) = \frac{\partial x'^p}{\partial x^m}\frac{\partial x'^s}{\partial x^n}\, g_{ps}(x') \quad . \tag{5.2}$$

Locally, a GCT is generated by a vector parameter (ξ^m), so that $x'^m = x^m + \epsilon\xi^m(x)$ with ϵ infinitesimal. Substituting this relation into eq. (5.2) results in the Killing condition for the generator of an isometry (Killing vector),

$$\xi_{s;p} + \xi_{p;s} = 0 \quad , \tag{5.3}$$

where semicolon denotes the covariant derivative.

Combining eq. (5.3) with the expression for the commutator of covariant derivatives acting on a vector:

$$\xi_{s;p;m} - \xi_{s;m;p} = -\mathcal{R}^l_{spm}\xi_l \quad , \tag{5.4}$$

gives an identity which relates the second covariant derivative of the Killing vector to itself,

$$\xi_{m;p;s} = -\mathcal{R}^l_{spm}\xi_l \quad . \tag{5.5}$$

Suppose the value of a Killing vector and its first covariant derivative is known at some point (x_0). Constructing a Taylor series and relating higher to lower derivatives with eq. (5.5) allows the specification of the Killing vector, in the neighbourhood of (x_0), in terms of the first two terms of the Taylor expansion,

$$\xi_m(x) = A^n_m(x, x_0)\xi_n(x_0) + B^{np}_m(x, x_0)\xi_{n;p}(x_0) \quad . \tag{5.6}$$

From eq. (5.6) the greatest number of possible killing vectors may be deduced. Counting the degrees of freedom in (ξ_m) and $(\xi_{m;n})$, and remembering antisymmetry implied by eq. (5.3), one finds at most $N + \frac{1}{2}N(N-1) = \frac{1}{2}N(N+1)$ linearly independent Killing vectors in an N dimensional space. A MSS is defined to contain all $\frac{1}{2}N(N+1)$ independent Killing vectors.

MSS's are uniquely specified by i) the Ricci scalar $\mathcal{R}$ and by ii) the signature of the metric tensor [28]. The curvature tensor and the Ricci scalar may be parametrized as follows (with K a real constant),

$$\mathcal{R}_{lpsn} = K(g_{sp}g_{ln} - g_{np}g_{ls}) \quad , \quad \mathcal{R} = -N(N-1)K \quad . \tag{5.7}$$

N-dimensional MSS's may be constructed by restricting the coordinates of an $N+1$ dimensional flat space to rest on a hyperboloid. In this way, the generalized rotations on the hyperboloid, for instance the $\frac{1}{2}N(N+1)$ rotations of the N-dimensional sphere, correspond to the maximal number of Killing vectors. The generalized rotations play an equivalent role to Poincaré symmetry in Minkowski space-time, and are needed in the corresponding construction of QFT.

Riemannian and pseudo-Riemannian MSS's can be summarized by the table below. dS stands for de Sitter and AdS for Anti-de-Sitter space-time. For dS and the sphere, $K > 0$, where as for the hyperbolic space and AdS space-time $K < 0$.

Space **Topology**	**Signature**	**Curvature**	**$N+1$ Dimensional Metric** **Restriction: equation for hypersurface**
Euclidean	$(+, ..., +)$	0	
Minkowski	$(-, +, ..., +)$	0	
sphere S^N	$(+, ..., +)$	$R < 0$	$ds^2 = dx_1^2 + ... + dx_N^2 + dz^2$ $x_1^2 + ... + x_N^2 + z^2 = K^{-1}$
dS $R^1 \times S^{N-1}$	$(-, +, ..., +)$	$R < 0$	$ds^2 = -dx_0^2 + dx_1^2 ... + dx_{N-1}^2 + dz^2$ $-x_0^2 + x_1^2 + ... + x_{N-1}^2 + z^2 = K^{-1}$
hyperbolic H^N	$(+, ..., +)$	$R > 0$	$ds^2 = dx_1^2 + ... + dx_N^2 - dz^2$ $x_1^2 + ... + x_N^2 - z^2 = K^{-1}$
AdS $S^1 \times R^{N-1}$	$(-, +, ..., +)$	$R > 0$	$ds^2 = -dx_0^2 + dx_1^2 ... + dx_{N-1}^2 - dz^2$ $-x_0^2 + x_1^2 + ... + x_{N-1}^2 - z^2 = K^{-1}$

Table 1: Maximally Symmetric Spaces

5.2 QFT on Curved Spaces

QFT on curved spaces has been considered in references [29, 30], among others. The goal of their analysis is to determine the contribution to the effective action (Σ) which arises from the inclusion of quantum mechanical matter in background gravity. The effective action contains the classical gravity action (S) in addition to (Σ), and together represents a modified gravitational dynamics in the presence of quantum fields.

Consider for simplicity a massive real scalar field with the partition function written as follows,

$$Z[g_{mn}, m^2] = \int [\mathcal{D}\phi]_g \exp\left\{\frac{i}{2}\int d^N x\sqrt{-g}\phi(-\Box_g + m^2)\phi\right\} = e^{i\Sigma(g,m^2)} \tag{5.8}$$

where $\Box_g := \frac{1}{\sqrt{g}}\partial_m g^{mn}\sqrt{g}\partial_n$ is the Laplacian, and $\Sigma(g, m^2)$ is the contribution to the effective action which is to be evaluated.

The QFT represented by the path integral (5.8) is constructed to preserve a set of symmetries, which in the case at hand is GCT invariance. To maintain GCT invariance of the path integral, not only the action but also the path integral measure need to satisfy the symmetry. De Witt [29], and later Polyakov [8] have devised an implicit definition of the GCT invariant measure. The presentation here is based on a later, explicit construction by Fujikawa [31] (which is presented in detail in Appendix B).

Consider the scalar action in (5.8) and the quadratic field operator Δ defined as the variation of the action with respect to the field ϕ: $\delta S/\delta\phi =: \Delta\phi$. The field operator defines the eigenvalue problem,

$$(-\Box_g + m^2)\phi_n = \lambda_n \phi_n \quad , \tag{5.9}$$

which allows the scalar field ϕ to be expanded in the orthonormal modes (ϕ_n) with coefficients (a_n), i.e., $\phi = \sum_n a_n \phi_n$. The consistent and covariant path integral measure is constructed as the product of one dimensional integrals of the coefficients in the orthogonal mode expansion (a_n). In the normal mode basis the quadratic action is diagonalized, and one obtains the following expression for the path integral,

$$\begin{aligned} Z[g_{mn}] &= \prod_n{}' \int_{-\infty}^{\infty} da_n e^{-\frac{i}{2}\sum_n \lambda_n a_n^2} = \prod_n{}' (\lambda_n)^{-1/2} = \det{}'[-\Box_g + m^2]^{-1/2} \\ &= e^{-\frac{1}{2}\mathrm{Tr}' \ln[-\Box_g + m^2]} = e^{i\Sigma(g,m^2)} \quad , \end{aligned} \tag{5.10}$$

where prime indicates omission of the zero modes and where constant factors such as $2, \pi$ and -1 are absorbed in the measure. Since taking the trace (or determinant) is invariant under a change of basis, the contribution to the effective action $\Sigma(g, m^2)$ may be evaluated in the position basis. To avoid dealing with the logarithm, $\Sigma(g, m^2)$ is differentiated with respect to the mass parameter m^2,

$$\frac{\partial}{\partial m^2}\Sigma(g, m^2) = \frac{i}{2}\mathrm{Tr}'\left(\frac{1}{-\Box_g + m^2}\right) = \frac{i}{2}\int d^N x \sqrt{-g} \lim_{x' \to x} G(x, x'), \tag{5.11}$$

thereby introducing the inverse of the field operator (Green's function) into the problem,

$$(-\Box_g + m^2)_x G(x, x') = \frac{\delta^N(x, x')}{\sqrt{g}} \quad . \tag{5.12}$$

As is the case in Minkowski field theory the inverse of the Laplacian is not unique and a generalization of the flat-space Feynman propagator is needed. Fortunately, for MSS's these results are available, for dS [33, 34], and for AdS [35, 36, 37] space-times. This is the topic of the next section.

To obtain the contribution $\Sigma(g, m^2)$ to the effective action, eq. (5.11) is integrated with respect to m^2:

$$\Sigma(g, m^2) - \Sigma(g, m_0^2) = \int_{m_0^2}^{m^2} dm^2 \left\{ \frac{i}{2} \int d^N x \sqrt{-g} G(x, x) \right\} \quad , \tag{5.13}$$

and the functional determinant at m_0 should be evaluated using other means. For this purpose $m_0 = 0$ or $m_0 = \infty$ may be useful limits to consider.

Another method for obtaining a regularized effective action is the ζ-function technique [38] . Here, a generalized ζ function is defined as

$$\zeta(s) := \sum_n \lambda_n^{-s} \quad , \tag{5.14}$$

where $\{\lambda_n\}$ are the eigenvalues of the operator whose determinant is to be evaluated. Formally, the determinant equals $\exp(-d\zeta/ds)|_{s=0}$, and the effective action (say for a scalar field) may be *defined* as $-\frac{1}{2}\zeta'(0) - \frac{1}{2}\zeta(0)\left[\frac{1}{s} + \ln(\nu^2)\right]$, where ν is an arbitrary regularization scale coming from the path integral measure.

5.3 QFT on AdS

N dimensional AdS may be thought of as a hyperboloid in $N + 1$ dimensional flat space with signature $(-, +, ...+, -)$, as in **Table 1**. The metric on N-dimensional AdS space-time, in intrinsic coordinates, may be written as follows [37],

$$ds^2 = \frac{-K}{(\cos\rho)^2}\left[-dt^2 + d\rho^2 + \sin(\rho)^2 d^{N-2}\Omega\right] \tag{5.15}$$

where $0 \leq \rho \leq \frac{\pi}{2}$ with $\frac{\pi}{2}$ corresponding to spatial infinity, $-\pi \leq t < \pi$ and where $d^{N-2}\Omega$ is the usual metric for $(N-2)$ dimensional sphere.

Quantization of scalar field theory on AdS space-time involves additional complications over those which arise for flat Minkowski space. These can be summarized in general terms as follows. AdS space-time is not simply connected and it possesses closed time-like paths. In addition, AdS space-time is not globally hyperbolic [39].

To see how these issues arise in practice, consider the metric (5.15). The time coordinate t has a compact domain, with $-\pi$ and π identified. In order not to lose causality, QFT on AdS is actually defined on a related space, called the covering space (CAdS), with the same metric as before but with the time coordinate now spanning $-\infty \leq t < \infty$. The topology of AdS ($S^1 \times R^{N-1}$) is now replaced with (R^N) for CAdS.

The more curious feature of AdS is tied up with lack of hyperbolicity. Recall, that in the Minkowski field theory, specifying initial data on a constant time hypersurface is a sufficient input for an unambiguous time evolution of a dynamic system. The same is not the case for CAdS, namely, in order for the scalar-field equations to formulate a well posed boundary-value problem, boundary information is required on a time-like surface at spatial infinity in addition to the usual initial conditions which are sufficient in Minkowski space-time. The absence of a global Cauchy surface may be seen by looking at the equation for radial null geodesics from metric (5.15), namely, $\rho = t$. It follows that information travels from the origin to spatial infinity in a finite time, which allows the possibility of information loss from the dynamical system of interest. Conversely, information may leak from infinity to the physical system at the origin in a finite time, thereby affecting its evolution.

A well defined dynamics is obtained by imposing boundary conditions for fields at infinity [35, 36]. This is done by recognizing that the square bracket in the metric (5.15) describes half of the Einstein static universe (ESU) [39] (half because the coordinate ρ does not extend over the full spherical domain $0 \leq \rho \leq \pi$). In other words, a conformal scaling of the CAdS metric is related to half of ESU. One pertinent property of ESU is that it is globally hyperbolic, therefore QFT constructed on ESU may lead to a consistent construction for CAdS. This is indeed the case, and the subtle step is in the resolution of the question "what to do with the other "half" of the ESU?". The solution is to impose boundary conditions at the equator of ESU (i.e.

at $\rho = \frac{1}{2}\pi$), and to truncate the available field modes appropriately. In particular, one may constrain the scalar field or its derivative to vanish, given by Dirichlet and Neumann conditions. It is interesting to point out that the corresponding two sets of field modes are both complete separately . An analogy can be made here with Fourier decomposition [36]. One needs both $\sin(nx)$ and $\cos(nx)$ functions to parameterize an arbitrary function in a symmetric interval about the origin, but either set is sufficient if only the positive half of the interval is considered (resulting in either odd or even extension of the function). The fact that more than one set of modes is available leads to the existence of more than one Fock vacuum for the CAdS QFT. As a consequence, different physical situations can lead to different boundary conditions, and so to different QFT's.

5.4 Calculation of Effective Actions in Various Dimensions

As discussed in Section 5.2, having an explicit expression for the Feynman propagator allows the evaluation of the corresponding field's contribution to the effective action. In this section the propagator which satisfies the energy-conserving boundary conditions on anti-de Sitter space [37] is used for effective action calculation. Following a general exposition for calculating AdS scalar effective action in arbitrary dimensions, specific results are reported for 2, 4, and odd dimensions. Finally, higher spin fields in AdS are considered.

5.4.1 General Considerations

The AdS Feynman propagator, which satisfies the energy-conserving boundary conditions, is given in terms of standard hypergeometric functions, $F(a, b; c; x)$ [40], by:

$$-\frac{i}{2}\,G_F(z) = \frac{C_{F,N}}{2\,z^{\beta}}\,F\left(\frac{\beta}{2}\,,\;\frac{\beta+1}{2}\;;\;\beta-\frac{N}{2}+\frac{3}{2}\;;\;z^{-2}\right), \tag{5.16}$$

where $z = 1 + |K|\sigma$, and σ is the the square of the geodesic distance [1] between the points x and x', and β denotes the expression

$$\beta = \frac{N-1}{2} \pm \sqrt{\frac{(N-1)^2}{4} + \frac{m^2}{|K|}} \quad . \tag{5.17}$$

Finally, the coefficient $C_{F,N}$ is a known constant, defined in equation (9) of ref. [37]:

$$C_{F,N} = \frac{|K|^{(N-2)/2}\,\Gamma(\beta)}{2^{\beta+1}\,\pi^{N/2-1/2}\,\Gamma\left(\beta - \frac{1}{2}(N-3)\right)} \quad . \tag{5.18}$$

Taking the coincidence limit ($\sigma \to 0$) is accomplished by taking $z \to 1$ in eq. (5.16). Using the corresponding limit for the hypergeometric function:

$$F([a,b],[c],1) = \frac{\Gamma(c)\Gamma(c-b-a)}{\Gamma(c-b)\Gamma(c-a)} \quad , \tag{5.19}$$

and simplifying further the Γ functions in the denominator the, propagator takes the form:

$$-\frac{i}{2}G_F(1) = \frac{C_{F,N}\,2^{\beta-N}\,\Gamma(\beta-\frac{n}{2}+\frac{3}{2})\,\Gamma(1-\frac{N}{2})}{\sqrt{\pi}\Gamma(\beta-N+2)} \quad . \tag{5.20}$$

When N is a positive, even integer, expression (5.20) suffers from the usual divergences that are associated with the coincidence limit of the Feynman propagator. These divergences are regularized by temporarily imagining the spacetime dimension, N, to be complex n, with n taken to the physical dimension of space-time only at the end of the calculation.

For n dimensional spacetime, the following result is obtained: [2]

$$-\frac{i}{2}\,G_F(1) = \frac{\Gamma\left(\frac{n}{2}-\frac{1}{2}+\sqrt{\frac{(n-1)^2}{4}+\frac{m^2}{|K|}}\right)\,\Gamma(1-\frac{n}{2})\,|K|^{(n-2)/2}}{2^{n+1}\,\pi^{n/2}\,\Gamma\left(-\frac{n}{2}+\frac{3}{2}+\sqrt{\frac{(n-1)^2}{4}+\frac{m^2}{|K|}}\right)}. \tag{5.21}$$

To proceed, eq. (5.21) is integrated with respect to the mass parameter m^2. The limit $n \to N$ of eq. (5.21), when N is an odd integer, is well-defined and so may be taken directly, and the result integrated with respect to m^2. When N is even, however,

[1] Recall from **Table 1**, that for AdS $K < 0$, therefore absolute value for K appears in most sections in this chapter

[2] A correction is made here to a typo in the coincidence limit of ref. [37].

the pole from the Γ-function in the numerator gives a divergent result, which may be isolated by performing a Laurent series in powers of $(n - N)$. It is generally useful to perform this expansion first, and reserving until last the integration over m^2.

5.4.2 $N = 2$

Specializing to $N = 2$, the Laurent expansion of the scalar propagator becomes (neglecting terms which are $O(n-2)$):

$$\frac{i}{2}\, G_f(1) = \frac{1}{4\pi\,(n-2)} - \frac{1}{8\pi}\left[\ln\left(\frac{4\pi\,\Lambda^2}{|K|}\right) - \gamma - 2\,\Psi\left(\frac{1}{2} + \frac{1}{2}\sqrt{1 + \frac{4\,m^2}{|K|}}\right)\right] , \quad (5.22)$$

where $\Psi(x) := d\ln\Gamma(x)/dx$ and Λ is the usual arbitrary scale which enters when dimensions are continued to complex values.

Integrating the expression in eq. (5.22) with respect to mass, the effective action is expressed as the integral over an effective potential,

$$\Sigma = -\int d^2x\, \sqrt{-g}\, V_{eff}(g, m^2) \quad , \quad (5.23)$$

with

$$\begin{aligned} V_{eff}(g,m^2) \;=\; & V_{eff}(g,0) - \left[-\frac{1}{4\pi\,(n-2)} + \frac{1}{8\pi}\left(-\gamma + \ln\left(\frac{4\pi\,\Lambda^2}{|K|}\right) - 2\right)\right] m^2 \\ & + \frac{|K|}{8\pi}\left[2\,\ln G_1\left(\frac{1}{2}\sqrt{1+\frac{4m^2}{|K|}} + \frac{1}{2}\right) + 4\,\ln G_2\left(\frac{1}{2}\sqrt{1+\frac{4m^2}{|K|}} + \frac{1}{2}\right)\right. \\ & \left. + \left(1 - \sqrt{1+\frac{4m^2}{|K|}}\right)\,\ln(2\pi)\right] \quad . \end{aligned} \quad (5.24)$$

Here $G_n(x)$ denote the multiple Gamma functions [41], which are defined in Appendix C. In two dimensions, the massless reference point is useful because the functional integral for massless scalars is known to give the Liouville action (3.3).

Using the asymptotic expansions of the functions G_n, given in Appendix C, the small curvature limit ($|K| \ll m^2$) of eq. (5.24) is found to be:

$$\begin{aligned} V_{eff}(g,m^2) \;\sim\; & V_{eff}(g,0) - \frac{m^2}{8\pi}\left[\frac{2}{(n-2)} - \ln\left(\frac{4\pi\,\Lambda^2}{m^2}\right) + \gamma - 1\right] \\ & - \frac{|K|}{24\,\pi}\left[\ln\left(\frac{|K|}{8\pi^3\,m^2}\right) + \frac{3}{2} - 12\,\zeta'(1)\right] + \frac{|K|^2}{120\,\pi\,m^2} + O\left(|K|^3\right) \quad , \end{aligned} \quad (5.25)$$

where $\zeta(x)$ denotes the usual Reimann zeta function.

Expression (5.25) is consistent with the derivative expansion evaluated directly from eq. (5.13). This is done by discretizing the mass integral $\int dm^2$, with the trace of the Feynman propagator evaluated in the position basis as in ref. [37].

5.4.3 $N = 4$

Evaluating eq. (5.21) for $n \to N = 4$, permits a comparison of this expression with previous work. The following result is obtained for the coincidence limit of the propagator,

$$\begin{aligned} \frac{i}{2} G_f(1) &= -\frac{2\,|K| + m^2}{16\pi^2\,(n-4)} + \frac{m^2}{32\pi^2} \\ &\quad + \left(\frac{2\,|K| + m^2}{32\pi^2}\right)\left[\ln\left(\frac{4\pi\,\Lambda^2}{|K|}\right) - \gamma - 2\,\Psi\left(\frac{1}{2} + \sqrt{\frac{9}{4} + \frac{m^2}{|K|}}\right)\right] + O(n-4)\,. \end{aligned} \tag{5.26}$$

Integrating with respect to mass (with the lower limit set to zero) the effective potential is obtained:

$$\begin{aligned} V_{eff}(g, m^2) &= V_{eff}(g, 0) + \\ &\quad -\frac{|K|^2}{64\pi^2}\left\{\left(-\frac{2}{n-4} + \ln\left(\frac{4\pi\,\Lambda^2}{|K|}\right) - \gamma + \frac{1}{3}\right)\left(b^2 - \frac{9}{4}\right)\left(b^2 + \frac{7}{4}\right)\right. \\ &\quad + \left[(6 + 8\,C_2)\left(\frac{1}{2} + b\right) - 9 + 24\,C_3 + 8\,C_2\right]\left(b^2 - \frac{9}{4}\right) \\ &\quad + (24\,C_2 + 11 + 48\,C_3 + 48\,C_4)\left(-\frac{3}{2} + b\right) \\ &\quad \left. -72\,\ln G_3\left(\frac{1}{2} + b\right) - 24\,\ln G_2\left(\frac{1}{2} + b\right) - 48\,\ln G_4\left(\frac{1}{2} + b\right)\right\}\,, \end{aligned} \tag{5.27}$$

where $b^2 := \frac{9}{4} + \frac{m^2}{|K|}$, and the constants C_n are defined in the Appendix C. Setting mass m to zero ($b = \frac{3}{2}$) in eq. (5.27), the effective potential vanishes as expected.

This expression can be compared with earlier calculations. These have been computed in terms of the integral over m^2 in ref. [37] (using the same methods as used here) and in ref. [42] (using ζ-function methods). The result of ref. [42] is:

$$\begin{aligned} V_{eff} = -\mathcal{L}_{eff} &= -\frac{|K|^2}{64\pi^2}\left[\left(b^4 - \frac{1}{2}b^2 - \frac{17}{240}\right)\ln\left(\frac{\nu^2}{|K|}\right) + b^4 + \frac{1}{6}b^2 + 8\,c\right] \\ &\quad + \frac{|K|^2}{16\pi^2}\int_{1/2}^{1/2+b} x\,(x-1)\,(2\,x-1)\,\Psi(x)\,dx \end{aligned} \tag{5.28}$$

where ν is the arbitrary scale which arises in ζ-function regularization, and the constant c is given by [3] [43]:

$$\begin{aligned} c &= \int_0^\infty \frac{2\lambda\,(\lambda^2+1/4)\ln\lambda}{e^{2\pi\lambda}+1}\,d\lambda \\ &= -\frac{\ln 2}{160} - \frac{17}{960}\ln\pi + \frac{137}{5760} - \frac{17}{960}\gamma + \frac{21}{32}\frac{\zeta'(4)}{\pi^4} + \frac{1}{16}\frac{\zeta'(2)}{\pi^2} \\ &= -0.01744158583\ldots \end{aligned} \tag{5.29}$$

If the integrals in eq. (5.28) are evaluated in terms of the multiple Gamma functions, and the result for $m = 0$ limit is subtracted, agreement is found with eq. (5.27) provided that the arbitrary scales ν and Λ are related in the following way:

$$\Lambda = \nu\,\exp\left[\frac{(12b^2+21)(\gamma - \ln(4\pi)) + 56}{6(4b^2+7)}\right]\quad . \tag{5.30}$$

5.4.4 Scalar Fields in Odd Dimensions

In this section the effective actions for massive scalar fields in odd-dimensional AdS space-times are evaluated. As is usually the case for dimensionally-regularized quantities, the resulting expressions are easier to evaluate due to the absence in odd dimensions of logarithmic divergences at one loop.

For brevity, the final results for the effective potentials for the lowest odd dimensions are quoted here.

$N = 3$

For 3-dimensional AdS spacetimes the massive scalar effective lagrangian density becomes:

$$V_{eff}(K,m) - V_{eff}(K,0) = -\frac{|K|^{3/2}}{12\pi}\left[\left(\frac{|K|+m^2}{|K|}\right)^{3/2} - 1\right]. \tag{5.31}$$

[3] A correction is made here to a typo in ref. [42], where the value for the constant c is incorrect by the factor of $-137/360$

$N = 5$

The corresponding result for 5-dimensional AdS spacetimes is:

$$V_{eff}(K,m) - V_{eff}(K,0) = \frac{|K|^{5/2}}{360\,\pi^2}\left[\left(\frac{4\,|K| + m^2}{|K|}\right)^{3/2}\left(\frac{7\,|K| + 3\,m^2}{|K|}\right) - 56\right]. \quad (5.32)$$

$N = 7$

$$\begin{aligned} V_{eff}(K,m) \quad - \quad V_{eff}(K,0) &= -\frac{|K|^{7/2}}{5,040\,\pi^3}\left[\left(\frac{9\,|K| + m^2}{|K|}\right)^{3/2}\right. \\ &\times \left.\left(\frac{82\,|K|^2 + 33\,|K|m^2 + 3\,m^4}{|K|^2}\right) - 2,214\right]. \end{aligned} \quad (5.33)$$

$N = 9$

$$\begin{aligned} V_{eff}(K,m) \quad - \quad V_{eff}(K,0) &= \frac{|K|^{9/2}}{151,200\,\pi^4}\left[\left(\frac{16\,|K| + m^2}{|K|}\right)^{3/2}\right. \\ &\times \left.\left(\frac{3,956\,|K|^3 + 1401\,|K|^2m^2 + 150\,|K|m^4 + 5\,m^6}{|K|^3}\right) - 253,184\right]. \end{aligned} \quad (5.34)$$

$N = 11$

Finally, the 11-dimensional expression is:

$$\begin{aligned} V_{eff}(K,m) - V_{eff}(K,0) \;&=\; -\;\frac{|K|^{11/2}}{1,995,840\,\pi^5}\left[-\,16,067,000\right. \\ &+ \left(\frac{128,536\,|K|^4 + 40,188\,|K|^3m^2 + 4,287\,|K|^2m^4 + 190\,|K|m^6 + 3\,m^8}{|K|^4}\right) \times \\ &\times \left.\left(\frac{25\,|K| + m^2}{|K|}\right)^{3/2}\right]. \end{aligned} \quad (5.35)$$

5.4.5 Higher Spins for $N = 4$ Anti-de Sitter Space

Some results are also available in four dimensions for higher-spin particles. It is often possible to express the one-loop functional determinants for higher-spin (s) fields in the form

$$\det\left(-\Box_s + X\right) \quad , \tag{5.36}$$

where $\Box_s$ is the Laplacian operator acting on various constrained tensor or spinor fields. Constrained, for spin-1 particles for instance, means that the relevant field is a divergenceless vector field. The functional determinants for these fields have been evaluated for dS space-times in ref. [44], and for AdS space-times in ref. [45], using ζ-function regularization. Following these references, various fields are labeled here by the corresponding spin. For tensor fields ($s =$ integer) on AdS with $N = 4$ ref. [45] gives the following result (with the overall sign chosen for bose statistics):

$$\begin{aligned} V^s_{eff} &= -\,g(s)\,\frac{|K|^2}{64\pi^2}\left\{\left[b^4 - \left(s+\frac{1}{2}\right)^2\left(2\,b^2+\frac{1}{6}\right) - \frac{7}{240}\right]\ln\left(\frac{\nu^2}{|K|}\right) + b^4 \right. \\ &\qquad \left. +\frac{1}{6}\,b^2 + 8\,c_+\right\} \\ &\quad -g(s)\,\frac{|K|^2}{8\pi^2}\int_0^b\left[\left(s+\frac{1}{2}\right)^2 - t^2\right]\,\Psi\left(t+\frac{1}{2}\right)\,t\,dt \end{aligned} \tag{5.37}$$

with $g(s) = 2s+1$. The quantity b is given in refs. [45] and [44], and depends on both $m^2/|K|$ and s. For the special case $s = 0$, $b^2 = \frac{9}{4} + \frac{m^2}{|K|}$, while for $s = 1$, $b^2 = \frac{1}{4} + \frac{m^2}{|K|}$. The constant c_+ is given by [43],

$$\begin{aligned} c_+ &= \int_0^\infty \frac{2\lambda\left[\lambda^2 + \left(s+\frac{1}{2}\right)^2\right]\ln\lambda}{e^{2\pi\lambda}+1}\,d\lambda \\ &= \frac{s(s+1)}{24}\left(-\ln\pi + 1 - \gamma + \frac{6\,\zeta'(2)}{\pi^2}\right) - \frac{\ln 2}{160} - \frac{17\,\ln\pi}{960} \\ &\quad +\frac{137}{5760} - \frac{17\,\gamma}{960} + \frac{21\,\zeta'(4)}{32\pi^4} + \frac{\zeta'(2)}{16\pi^2} \end{aligned} \tag{5.38}$$

Evaluating the integrals in eq. (5.37), the effective Lagrangian produced by (constrained) tensor fields on AdS can also be expressed in terms of the multiple Gamma functions as follows:

$$\begin{aligned} V_{eff} &= g(s)\,\frac{|K|^2}{64\pi^2}\left\{\left[\ln\left(\frac{|K|}{\nu^2}\right) - \frac{1}{3}\right]b^4 - (8\,C_2+6)\,b^3 \right. \\ &\quad + \left[-2s(s+1)\left(1+\ln\left(\frac{|K|}{\nu^2}\right)\right) - 24\,C_3 + \frac{3}{2} - 12\,C_2 - \frac{1}{2}\ln\left(\frac{|K|}{\nu^2}\right)\right]b^2 \end{aligned}$$

$$
\begin{aligned}
&+\left[2s(s+1)\,(4\,C_2+1)-48\,C_3+\frac{5}{2}-48\,C_4-6\,C_2\right]\,b\\
&+\left[-\frac{1}{6}\,\ln\left(\frac{|K|}{\nu^2}\right)+4\,\ln G_1\left(\frac{1}{2}\right)+8\,\ln G_2\left(\frac{1}{2}\right)-4\,\ln G_1\left(\frac{1}{2}+b\right)\right.\\
&\left.-8\,\ln G_2\left(\frac{1}{2}+b\right)\right]\,s(s+1)+24\,\ln G_2\left(\frac{1}{2}+b\right)+72\,\ln G_3\left(\frac{1}{2}+b\right)-8\,c_+\\
&-24\,\ln G_2\left(\frac{1}{2}\right)-\frac{17}{240}\,\ln\left(\frac{|K|}{\nu^2}\right)-48\,\ln G_4\left(\frac{1}{2}\right)-72\,\ln G_3\left(\frac{1}{2}\right)\\
&+48\,\ln G_4\left(\frac{1}{2}+b\right)\Big\}\qquad\qquad \text{for } s=\text{integer}\,. \qquad (5.39)
\end{aligned}
$$

A similar result may be derived for (constrained) spinor fields. Ref. [45] gives the following expression (assuming fermi statistics):

$$
\begin{aligned}
V^s_{eff} &= g(s)\,\frac{|K|^2}{64\pi^2}\left\{\left[b^4-\left(s+\frac{1}{2}\right)^2\left(2\,b^2-\frac{1}{3}\right)+\frac{1}{30}\right]\ln\left(\frac{\nu^2}{|K|}\right)\right.\\
&\qquad\left.+b^4-\frac{4\,b^3}{3}-\frac{b^2}{3}+4\left(s+\frac{1}{2}\right)^2 b-8\,c_-\right\}\\
&\qquad+g(s)\,\frac{|K|^2}{8\pi^2}\int_0^b\left[\left(s+\frac{1}{2}\right)^2-t^2\right]\,\Psi(t)\,t\,dt\quad , \qquad (5.40)
\end{aligned}
$$

where b is again spin dependent, equal to $b^2=\frac{m^2}{|K|}$ for $s=\frac{1}{2}$, and the constant c_- is [43]:

$$
\begin{aligned}
c_- &= \int_0^\infty \frac{2\lambda\left[\lambda^2+\left(s+\frac{1}{2}\right)^2\right]\ln\lambda}{e^{2\pi\lambda}-1}\,d\lambda\\
&= -\frac{7\ln 2}{240}-\frac{7\ln\pi}{240}+\frac{13}{360}-\frac{7\gamma}{240}+\frac{3\,\zeta'(4)}{4\pi^4}\\
&\qquad+\frac{s(s+1)}{12}\left[-\ln(2\pi)+1-\gamma+\frac{6\,\zeta'(2)}{\pi^2}\right]+\frac{1}{8}\,\frac{\zeta'(2)}{\pi^2}\,. \qquad (5.41)
\end{aligned}
$$

Combining expressions the following form is obtained for the spinor effective potential on AdS:

$$
\begin{aligned}
V^s_{eff} &= g(s)\,\frac{|K|^2}{64\pi^2}\left\{\left[-\ln\left(\frac{|K|}{\nu^2}\right)-\frac{13}{3}\right]\,b^4+\left(64\,C_2+\frac{124}{3}\right)\,b^3\right.\\
&\quad+\left[\left(16+2\,\ln\left(\frac{|K|}{\nu^2}\right)\right)s(s+1)+\frac{1}{2}\,\ln\left(\frac{|K|}{\nu^2}\right)-\frac{101}{3}+96\,C_2+192\,C_3\right]\,b^2\\
&\qquad+\left[\,(-64\,C_2-28)s(s+1)-39+384\,C_3+384\,C_4+48\,C_2\right]\,b
\end{aligned}
$$

$$
\begin{aligned}
&+\left[64\,\ln G_2(b) - \frac{1}{3}\ln\left(\frac{|K|}{\nu^2}\right) + 64\,\ln G_1(b)\right] s(s+1) \\
&-768\,\ln G_3(b) - \frac{7}{60}\ln\left(\frac{|K|}{\nu^2}\right) - 8\,c_- - 432\,\ln G_2(b) \\
&-48\quad \ln G_1(b) - 384\,\ln G_4(b)\Big\} \qquad \text{for} \qquad s = \text{half-integer}\,.
\end{aligned}
\tag{5.42}
$$

The following technical point bears notice. When evaluated for massless, spin $\frac{1}{2}$ fermions ($b = 0$), eq.(5.42) superficially appears to be ill-defined, due to the appearance of the divergent quantities $\ln G_2(0)$, $\ln G_3(0)$ and $\ln G_4(0)$. It happens that these divergences cancel in eq.(5.42), leaving a well-defined massless limit.

5.5 2D AdS Canonical Equivalence Broken by Quantum Effects

In ref. [46], Cruz proposes the classical equivalence of two types of free scalar fields in 2 dimensional AdS having Ricci scalar $\mathcal{R}$. The proposed equivalence relates a massless, minimally-coupled scalar with massive scalar having mass $m^2 = \mathcal{R} = 2|K|$, for which he argues by constructing a time-dependent canonical transformation which maps one system into the other.

The argument may be summarized as follows. Consider the scalar field denoted by ϕ and the Lagrangian density:

$$
\mathcal{L}_\phi \;=\; (\partial_m\phi)^2 + |K|\phi^2 \tag{5.43}
$$

and denote the conjugate momentum by π_ϕ. Then both quantities may be expanded in terms of functions of definite chirality, $a_{(\ddagger)}(x^{\ddagger}), a_{(=)}(x^{=}), a'_{(\ddagger)}(x^{\ddagger}) = \partial_{\ddagger}a_{(\ddagger)}(x^{\ddagger}), a''_{(\ddagger)}(x^{\ddagger}) = \partial_{\ddagger}\partial_{\ddagger}a_{(\ddagger)}(x^{\ddagger})$, etc., as follows:

$$
\begin{aligned}
\phi &= -\frac{1}{4}\left(a'_{(\ddagger)} - a'_{(=)}\right) - \frac{|K|}{8}\frac{a_{(\ddagger)}x^{=} + a_{(=)}x^{\ddagger}}{1 - \frac{|K|}{2}x^{\ddagger}x^{=}} \\
\pi_\phi &= -\frac{1}{8}(a''_{(\ddagger)} - a''_{(=)}) - \frac{|K|}{8}\frac{a'_{(\ddagger)}x^{=} + a'_{(=)}x^{\ddagger}}{1 - \frac{|K|}{2}x^{\ddagger}x^{=}}
\end{aligned}
$$

$$+\frac{|K|}{8}\frac{a_{(\mp)}-a_{(=)}}{1-\frac{|K|}{2}x^{\mp}x^{=}}+\frac{|K|^2}{16}\frac{(x^{\mp}-x^{=})(a_{(\mp)}x^{=}+a_{(=)}x^{\mp})}{(1-\frac{|K|}{2}x^{\mp}x^{=})^2} \quad . \tag{5.44}$$

Cruz shows that defining new phase space variables as:

$$\psi=-\frac{1}{4}\left(a'_{(\mp)}-a'_{(=)}\right), \qquad \pi_{\psi}=-\frac{1}{8}(a''_{(\mp)}-a''_{(=)}) \quad , \tag{5.45}$$

is canonical in the sense that the Poisson bracket takes the same form in the new variables as it did in terms of the old, and the dual Lagrangian density is given by

$$\mathcal{L}_{\psi} = (\partial_m\psi)^2 \quad . \tag{5.46}$$

In this section arguments are presented against the survival of this equivalence at the quantum level. This failure to survive promotion to the quantum theory is similar to what happens for the Liouville action, which is canonically equivalent to a free field theory — and so is integrable [47] — but is nonetheless quantum mechanically distinct from it (see, ref. [12], and references therein).

One expects that if duality is maintained at the quantum level to imply the equality of the effective actions Σ computed for the two types of scalars. This amounts to the vanishing of expression (5.24), which gives the difference between the massive and massless effective potentials. Since the arguments of ref. [46] apply for any $|K|>0$, eq.(5.24) should vanish for all such $|K|$ so long as the masses are at the dual points. Consider therefore the following expression,

$$\begin{aligned} V_{eff}(|K|,m^2) \quad - \quad & V_{eff}(|K|,0) = -\left[C+\frac{1}{8\pi}\ln\left(\frac{\Lambda^2}{|K|}\right)\right] m^2 \\ & +\frac{|K|}{8\pi}\left[\left(1-\sqrt{1+\frac{4m^2}{|K|}}\right)\ln(2\pi)+2\ln G_1\left(\frac{1}{2}\sqrt{1+\frac{4m^2}{|K|}}+\frac{1}{2}\right)\right. \\ & \left.+4\ln G_2\left(\frac{1}{2}\sqrt{1+\frac{4m^2}{|K|}}+\frac{1}{2}\right)\right] \quad , \end{aligned} \tag{5.47}$$

where C is the contribution of any counterterms. Besides cancelling the divergence of eq. (5.24) as $n\to 2$, these counterterms depend on Λ in just such a way as to ensure the Λ-independence of V_{eff}. Evaluating this expression for $m^2=2|K|$ the following result is obtained,

$$V_{eff}(|K|, m^2 = 2|K|) - V_{eff}(|K|, 0) = -\left[2\,C + \frac{1}{4\pi}\ln\left(\frac{2\pi\,\Lambda^2}{|K|}\right)\right]\,|K|, \qquad (5.48)$$

where the values $G_1(2) = G_2(2) = 1$ were used.

So long as C may depend arbitrarily on $|K|$ and m^2, one is always free to choose C to ensure the vanishing of eq. (5.48). C may certainly depend on $|K|$, since the counterterms can involve powers of the curvature, $\mathcal{R}$.

The reader might wonder why the possibility of curvature-dependent counterterms is entertained here, when for the non-interacting scalar on a fixed gravitational background under consideration it is seen that no $|K|$ dependence is required to cancel divergences in two dimensions. Background dependent counterterms are considered because more complicated counterterms *are* required once interactions are included, and if the gravitational field is also treated as a quantum field. Moreover, the possibility that duality at the quantum level may require special choices for finite counterterms, even if these are not required to cancel divergences, should be examined.

Now comes the main point. There are two ways to proceed, depending on how much $|K|$ dependence one is prepared to entertain.

Option 1: Arbitrary $|K|$ Dependence

One way to proceed is to permit C to depend arbitrarily on $|K|$. This might be reasonable if the metric is regarded strictly as a background field, and the addition to the classical action of an arbitrary metric-dependent functional which is independent of the scalar field, ϕ is permitted. In this case, in the interest of enforcing a quantum duality, the choice of C is made to ensure the vanishing of eq (5.48) for all $|K|$. With this choice, eq. (5.47) becomes:

$$\begin{aligned} V_{eff}(|K|, m^2) \quad - \quad V_{eff}(|K|, 0) = \frac{m^2}{8\pi}\ln(2\pi) + \frac{|K|}{8\pi}\left[\left(1 - \sqrt{1 + \frac{4m^2}{|K|}}\right)\ln(2\pi) + \right. \\ \left. + 2\ln G_1\left(\frac{1}{2}\sqrt{1 + \frac{4m^2}{|K|}} + \frac{1}{2}\right) + 4\ln G_2\left(\frac{1}{2}\sqrt{1 + \frac{4m^2}{|K|}} + \frac{1}{2}\right)\right]. \end{aligned}$$

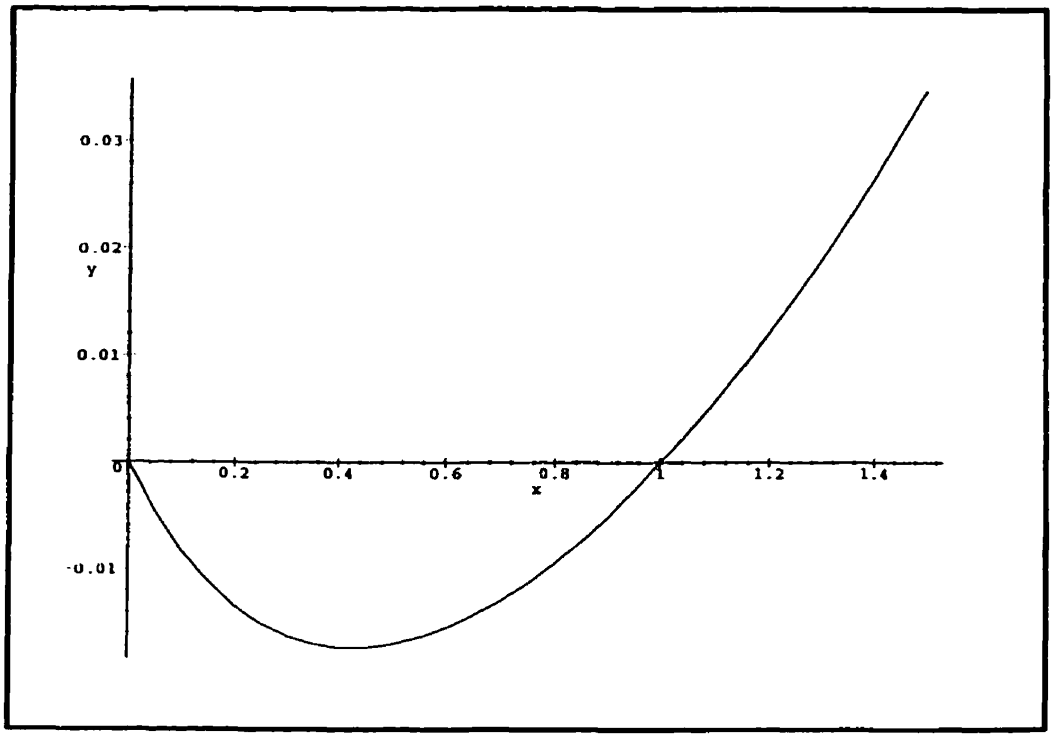

Figure 5.1: $y = [V_{eff}(|K|, m^2) - V_{eff}(|K|, 0)]/|K|$ vs $x = \frac{m^2}{2|K|}$

(5.49)

Eq. 5.49 is plotted in **Figure 5.1**, using the variables $y = [V_{eff}(|K|, m^2) - V_{eff}(|K|, 0)]/|K|$ vs $x = m^2/2|K|$. The following points emerge from an inspection of this plot.

1. By construction $y(0) = y(1) = 0$ indicating the equivalence of V_{eff} when evaluated at $m^2 = 0$ and $m^2 = 2|K|$. But the construction just given shows that there is nothing special about the choice $m^2 = 2|K|$, since $y = 0$ could have been renormalized to some other value of m^2.

2. Because $y(x)$ is not monotonically increasing or decreasing, there are many pairs $\{x_1, x_2\}$ which satisfy $y(x_1) = y(x_2)$, and so many pairs $\{m_1^2, m_2^2\}$ for which V_{eff} takes the same value.

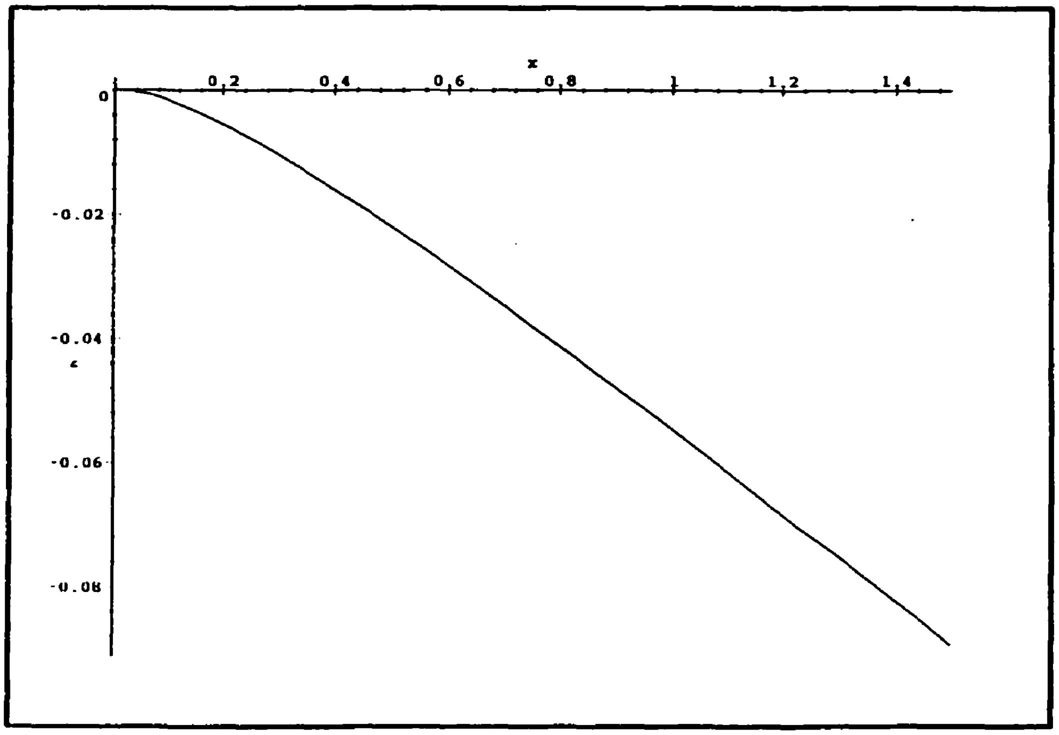

Figure 5.2: $z = \partial\left[V_{eff}(|K|, m^2) - V_{eff}(|K|, 0)\right]/\partial|K|$ vs. $x = \frac{m^2}{2|K|}$

3. What is less obvious from **Figure 5.1**, but demonstrated in **Figure 5.2** , is that the slope, $\partial V_{\rm eff}/\partial|K|$, is not the same for both members of these pairs. Since these slopes are related to the expectation $\langle T^p{}_p\rangle$ for the scalar field stress-energy tensor, this quantity must differ for m_1 and m_2 even though V_{eff} takes the same value for these two masses.

From this line of argument it is possible to conclude that duality is not a property of the quantum theory.

Option 2: Polynomial $|K|$ Dependence

A more reasonable requirement on C, is to require it to be at most a polynomial in $|K|$ (to any fixed order in perturbation theory). Physically, counterterms arise once higher-energy physics is integrated out, and so they should be interpreted in an effective-lagrangian sense. That is, they should be treated as perturbations in a low-energy derivative expansion. If so, to any fixed order in this expansion, they must be generally-covariant powers of the fields ϕ and g_{mn} and their derivatives, restricting C to be a polynomial in $|K|$.

If so, it is no longer possible to choose C to ensure the vanishing of eq. (5.48) because cancellation would require C to depend logarithmically on $|K|$. Once again the conclusion is reached that duality does not survive quantization.

Chapter 6

Summary and Outlook

A new duality formalism for quantum fields has been introduced in the first part of this thesis. The algorithm is based on the presence of a space-time symmetry in a QFT such as Lorentz invariance or supersymmetry.

As mentioned in the introduction, a successful duality algorithm should relate *local* field theories, and it should be relatively simple to implement, at least within certain approximations. With respect to these two criteria, the duality proposals in this thesis are only partially satisfactory.

One difficulty in the present procedure is the requirement of the evaluation of path integrals, which in general may be done only in an approximation or in some limit. Still, interesting examples exist where the steps involved are under control, as demonstrated with bosonization examples of Chapter 3 and with the pattern of (2,2) superduality for matter multiplets of definite chirality (4.90), in Chapter 4.

A more complicated issue of principle, than the technical ability of performing the duality transformation, is the question of locality of the dual action. As demonstrated with bosonization of Majorana spinor in Chapter 3, space-time duality (and hence also superduality) does not guarantee a local dual theory. A recovery of locality does exist, since it is possible to rewrite the non-local determinant in a local form, by introducing appropriate ghost fields as in ref. [49]. Still, it may be argued that with respect to ensuring locality of the dual theory, space-time and superduality are of limited success, and do not improve upon the gauge-duality program.

The main advantage of space-time duality over gauge duality is that most QFT's of interest (to high energy physics) are Lorentz invariant, which therefore makes the formalism widely applicable. For instance, in ref. [49], space-time duality is applied to a system without a global symmetry, to which therefore the gauge-duality prescription does not apply.

A further investigation of space-time duality may involve the application of the procedure within the effective Lagrangian content, for instance in the case of massive fields. Here, the AdS effective action, evaluated in Chapter 5, may serve as a useful first approximation in evaluating the relevant path integral. Further, it may be possible to extend space-time duality by introducing "fake torsion" in addition to "fake curvature" degrees of freedom.

It may be interesting to follow the consequences of superduality in the case of (4,4) supersymmetry in $(1 + 1)$ dimensions. One may also attempt to formulate superduality on BRST fermionic symmetry [48]. Here it should be interesting to see if superduality is equivalent to gauge duality, presented in Chapter 2.

In Chapter 5, effective actions are evaluated for various field content on AdS space-time. In even dimensions, these expressions may be stated simply in terms of multiple Gamma functions presented in Appendix C. One future application of the present results is the evaluation of the effective actions for supergravity theories in four dimensions. It is known that extended supergravity theories admit AdS space-time as the (lowest energy, constant) vacuum configuration. The particle content of extended SUGRA theories (i.e., spins, multiplicities, masses, statistics) is dictated by supersymmetry algebra on AdS. Combining results for various spins from Chapter 5, for appropriate multiplets, one may construct the effective potential for SUGRA theories. One expects, from ref. [50], that the divergent parts of the effective actions for $(3+1)$ SUGRA cancel (at one loop), provided the number of local supersymmetry parameters is greater than four. It should be interesting to check if simplifications also occur in the finite parts of the effective actions for these theories.

Appendix A

Notation

In this Appendix the notation used in the thesis is presented in detail.

A.1 Coordinate Labels

When discussing curved spaces, it is necessary to introduce two sets of indices. Coordinate basis indices are denoted by lower case Latin letters $\{m, n, p, ...\}$. Non-coordinate basis indices are denoted by lower case Latin letters from the beginning of the alphabet, $\{a, b, c, ...\}$.

For the discussion of superspace, one also needs to label spinor indices. Coordinate basis is denoted with Greek letters from the middle of the alphabet $\{\mu, \nu, \rho, ...\}$, and non coordinate basis with letters from the beginning of the alphabet, $\{\alpha, \beta, \gamma, ...\}$. Taking bosonic and spinor coordinates together, the coordinate basis is denoted with $\{M, N, P, ...\}$ and non-coordinate basis with $\{A, B, C, ...\}$.

Further coordinate labels are presented in the section on lightcone coordinates.

A.2 Curvature and Metric Conventions

There are two sign conventions for the curvature (Riemann) 4-tensor (and the related Ricci 2-tensor and scalar) which are found commonly in literature. It is customary to distinguish between them by the sign of the Ricci scalar for the sphere.

The curvature convention of this thesis is that of Weinberg [28], Equation (6.1.5), (as well as the built in routine in Maple [43]). Since a comparison in conventions is presented in this Appendix, Weinberg's curvature (Riemann) tensor is denoted with a superscript "minus" as follows:

$$\mathcal{R}^{-\,l}_{mnk} := \frac{\partial \Gamma^{l}_{\ mn}}{\partial x^k} - \frac{\partial \Gamma^{l}_{\ mk}}{\partial x^n} + \Gamma^{p}_{\ mn}\Gamma^{l}_{\ kp} - \Gamma^{p}_{\ mk}\Gamma^{l}_{\ np} \quad . \tag{A.1}$$

Ricci tensor and the curvature (Ricci) scalar are defined as

$$\mathcal{R}^{-}_{mk} = \mathcal{R}^{-\,l}_{mlk} \quad , \quad \mathcal{R}^{-} = g^{mn}\mathcal{R}_{mn} \quad . \tag{A.2}$$

In Weinberg's "minus" convention, Ricci scalar for the 2-sphere with the metric $\{1, \sin^2(\theta)\}$ is $\mathcal{R}^- = -2$.

The opposite convention, used in references [15, 36, 52, 51] for instance, is related to Weinberg's as follows. The Riemann tensor $\mathcal{R}^{+\,l}_{mnk}$ is defined by $\mathcal{R}^{+\,l}_{mnk} = -\mathcal{R}^{-\,l}_{mnk}$, which implies $\mathcal{R}^{+}_{mk} = -\mathcal{R}^{-}_{mk}$ and $\mathcal{R}^{+} = -\mathcal{R}^{-}$. Therefore, for the 2-sphere with the metric $\{1, \sin^2(\theta)\}$, $\mathcal{R}^+ = +2$.

Another notational headache is the signature of the Lorentzian metric. In this thesis God's metric [53] is used, namely $\{-,+,+,+\}$, which reflects the prejudice that a ruler ought be labeled with positive intervals. Still, the opposite signature (-2 instead of +2) is found abundantly in the literature. It may be useful to note that the Christoffel symbol (quadratic in the metric), the Riemann tensor (linear and quadratic in the Christoffel symbol), and the Ricci scalar (double contraction of the Riemann tensor), are independent of the choice of the signature of the metric. Ricci 2-tensor, however, as well as the Einstein tensor, and the stress-energy, do depend on the choice of signature.

A.3 Scalar Field Action and the Path Integral

In this section both Euclidean and Lorentzian actions are presented for the scalar field ϕ, and then mapped to one another by a Wick rotation.

A.3.1 Euclidean case

Let the Euclidean space metric be $\eta_{mn} = \mathrm{diag}(+, ..., +)$. A positive-definite scalar field action S_E^ϕ has the following form:

$$S_E^\phi = \frac{1}{2}\int d^N x\, \phi(-\Box + m^2)\phi \quad , \tag{A.3}$$

where $\Box$ denotes the negative-definite operator $\eta_{mn}\partial^m\partial^n$. Rewriting the action up to a total derivatives, its positive property is made manifest,

$$S_E^\phi = \frac{1}{2}\int d^N x\, \left((\partial_m\phi)^2 + m^2\phi^2\right) \quad . \tag{A.4}$$

The generating functional is defined as $Z_E = \int[\mathcal{D}\phi]_\eta \exp(-S_E^\phi)$. This definition ensures that the large and fast varying field configurations are suppressed.

A.3.2 Lorentzian Scalar Action

The Lorentzian action is defined so that the related Hamiltonian is positive. With the metric $\eta_{mn} = \mathrm{diag}(-, +, ..., +)$, the action takes the following form:

$$S_L^\phi = \frac{1}{2}\int dt d^{N-1}x\, \phi(\Box - m^2)\phi \quad , \tag{A.5}$$

where $\Box = \eta_{mn}\partial^m\partial^n$ is a negative definite operator following Wick rotation (to be defined in the next subsection). Upon rewriting the action, up to a total derivative, one has

$$S_L^\phi = \frac{1}{2}\int dt d^{N-1}x\, \left((\partial_0\phi)^2 - (\vec{\nabla}\phi)^2 - m^2\phi^2\right) \quad . \tag{A.6}$$

The conjugate momentum field to ϕ is $\Pi = \partial_0\mathcal{X}$, which implies that the Hamiltonian is

$$H_L^\phi = \frac{1}{2}\int d^{N-1}x\, \left(\Pi^2 + (\vec{\nabla}\phi)^2 + m^2\phi^2\right) \quad , \tag{A.7}$$

which is positive, as required.

The generating functional is defined as $Z_L = \int[\mathcal{D}\phi]_\eta \exp(iS_L^\phi)$

A.3.3 Wick rotation

The following prescription is used to map from Euclidean space to Lorentzian space-time. The Euclidean coordinate x^0 is mapped to Lorentzian $i\,t$. This mapping induces the following transformations:

1. Euclidean operator $\Box$ transforms to the Lorentzian operator $\Box$.

2. Euclidean measure $d^N x$ transforms to $i\,dt\,d^{N-1}x$.

3. The action and the generating functional transform as follows,

$$
\begin{aligned}
S_E^\phi &\overset{x^0 \to it}{\longrightarrow} -iS_L^\phi \quad , \\
Z_E = \int [\mathcal{D}\phi] \exp(-S_E^\phi) &\longrightarrow \quad Z_L = \int [\mathcal{D}\phi] \exp(iS_L^\phi) \quad .
\end{aligned} \tag{A.8}
$$

A.3.4 $i\epsilon$ prescription

In order to make the Lorentzian path integral well behaved for large ϕ (in the sense of absolute convergence), the mass term in the action S_L^ϕ is deformed as follows, $m^2 \to m^2(1 - i\epsilon)$ [29]. This change implies

$$
Z_L \to \int [\mathcal{D}\phi]_\eta \exp\left\{ iS_L^\phi - \frac{\epsilon}{2} \int dt\, d^{N-1}x\; m^2\phi^2 \right\} \quad , \tag{A.9}
$$

which damps contributions from large ϕ configurations.

A.4 Euler Characteristic

For a two-dimensional space, the Euler characteristic χ is a quantity which depends on global (topological) properties of the space. χ is defined as an integral of the Ricci scalar,

$$
\chi = -\frac{1}{8\pi} \int d^2x \mathcal{R}_h^- \;=\; \frac{1}{8\pi} \int d^2x \mathcal{R}_h^+ \quad . \tag{A.10}
$$

This definition can be tested on a two-sphere (S_2) with Ricci scalar equal to $\mathcal{R}^- = -2$ for the metric $\{1, \sin^2(\theta)\}$. Integrating, one obtains,

$$
\chi(S_2) \;=\; -\frac{1}{8\pi} \int \sin(\theta) d\theta\, d\phi\, (-2) = 1 \quad . \tag{A.11}
$$

An equivalent definition of the Euler characteristic is $\chi = 1 - g$, where for g stands for the number of handles. Taking the result of the equation (A.11) correctly verifies that the number of handles in a 2-sphere is $g = 0$.

A.5 Spinors in (1+1) Dimensions

Spinor notation in this thesis is chosen to be consistent with (2,2) superspace in reference [21]. Minkowski metric is the same as before: $\eta_{mn} = (-1, +1)$. In addition to the metric one has another invariant tensor density, the Levi-Civita antisymmetric symbol $\epsilon^{01} = \epsilon_{10} = +1$.

The following "Gamma" matrices are chosen,

$$\begin{aligned}\gamma^0 &= i\sigma_1 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \\ \gamma^1 &= -\sigma_2 = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} \\ \gamma_5 &= \gamma^0\gamma^1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad . \end{aligned} \tag{A.12}$$

The matrices (A.12) satisfy the following useful properties,

$$\gamma^a \gamma^b = \eta^{ab} + \gamma_5 \epsilon^{ab} \tag{A.13}$$

$$\gamma_a \gamma^b \gamma^a = 0. \quad . \tag{A.14}$$

The charge conjugation matrix is taken to be

$$C = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} = \gamma^1 \quad , \tag{A.15}$$

so that

$$C\gamma^a C^{-1} = -\gamma^0 \gamma^a \gamma^0 = -(\gamma^a)^T. \tag{A.16}$$

There are two ways of forming conjugate spinors. Dirac conjugate is defined as

$$\overline{\chi} = \chi^\dagger C = \chi^\dagger \gamma^1 = \begin{pmatrix} \chi_+ \\ \chi_- \end{pmatrix}^\dagger \gamma^1 = (-i\chi_-^*, i\chi_+^*) \quad , \tag{A.17}$$

where component spinor notation is used. Charge conjugate is defined as

$$\chi^c = C\overline{\chi}^T = \begin{pmatrix} -\chi_+^* \\ -\chi_-^* \end{pmatrix} \quad . \tag{A.18}$$

Further, one can form a Dirac conjugate (A.17) of the charge conjugate (A.18) spinor.

In general, (1+1) dimensional spinors are complex valued fields in a two-dimensional spinor space; they form a reducible representation of Lorentz symmetry called the Dirac representation. A representation may be reduced by imposing Lorentz invariant constraints on Dirac spinors. A Majorana spinor is defined to satisfy the following condition:

$$\chi^c = \chi \quad , \tag{A.19}$$

which implies that the spinor field is antihermitian, $\chi^* = -\chi$. Dirac spinor may be written in terms of two Majorana components: $\chi = \chi^1 + i\chi^2$. Weyl spinors, $\chi_\pm$, are defined as eigenstates of γ_5,

$$\chi_\pm = \pm\gamma_5\chi_\pm \quad . \tag{A.20}$$

Therefore, Dirac spinor may be decomposed into Weyl components by:

$$\chi = \begin{pmatrix} \chi_+ \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ \chi_- \end{pmatrix} = \begin{pmatrix} \chi_+ \\ \chi_- \end{pmatrix} \quad . \tag{A.21}$$

Finally, an irreducible spinor representation of Lorentz symmetry is obtained by imposing Majorana and Weyl constraints simultaneously. Dirac spinor may be decomposed into irreducible anti-Hermitian, one dimensional spinor components as follows,

$$\chi = \begin{pmatrix} \chi_+^1 + i\chi_+^2 \\ \chi_-^1 + i\chi_-^2 \end{pmatrix} \quad . \tag{A.22}$$

In order to facilitate discussion of spinors and superspace, the following notation is convenient. Spinors are denoted by one-dimensional Weyl spinor components, so that in general,

$$\chi = \begin{pmatrix} \chi_+ \\ \chi_- \end{pmatrix} \quad . \tag{A.23}$$

Charge conjugate Weyl spinor components are denoted by dotting the Weyl index,

$$\chi^c = \begin{pmatrix} \chi_{\dot{+}} \\ \chi_{\dot{-}} \end{pmatrix} \quad . \tag{A.24}$$

Upper spinor indices are defined by Dirac conjugate,

$$\overline{\chi} = (\chi^+, \chi^-) = (-i\chi_-^*, i\chi_+^*) = (i\chi_{\dot{-}}, -i\chi_{\dot{+}}) \quad , \tag{A.25}$$

and upper dotted indices are defied by the Dirac conjugate of a charge conjugate, as follows,

$$\overline{\chi}^c = (\chi^{\dot{+}}, \chi^{\dot{-}}) = (i\chi_-, -i\chi_+) \quad . \tag{A.26}$$

Note, that dotting a lower (upper) spinor field index is equivalent to taking a negative (positive) Hermitian conjugate of the spinor component.

In the case of an antihermitian Majorana spinor χ, taking the Dirac conjugate implies the following relation,

$$(\chi^+, \chi^-) = (i\,\chi_-, -i\,\chi_+). \tag{A.27}$$

Dirac action, in flat background, takes the following form,

$$S_\chi = \frac{1}{2}\int dx^0 dx^1\, i\overline{\chi}\partial\!\!\!/\chi = \frac{1}{2}\int dx^0 dx^1 \left(-i\chi_{\dot{+}}(-\partial_0 + \partial_1)\chi_+ - i\chi_{\dot{-}}(\partial_0 + \partial_1)\chi_-\right). \tag{A.28}$$

In curved (torsion free) space-time, the Dirac action takes the modified form,

$$S_\chi = \frac{1}{2}\int dx^0 dx^1 [e^{-1}]\, i\overline{\chi} D\!\!\!\!/\chi\,, \tag{A.29}$$

where $[e^{-1}]$ stands for the determinant of the inverse zweibein, $\det[e^a_m]$, related to the metric by $g_{mn} = e^a_m e^b_n \eta_{ab}$. The spinor covariant derivative is $D_m = (\partial_m + \frac{1}{2}\omega_m\gamma_5)$, where the connection is defined as follows:

$$\omega_m = \epsilon^{ab} e^n_a \partial_n e_{b,m} \quad . \tag{A.30}$$

In the case of space-time with torsion, for instance in SUGRA theories, additional gravitino terms appear in the connection (A.30).

The conformal gauge zweibein may be parametrized as follows:

$$e^a_m = e^\phi \delta^a_m \quad . \tag{A.31}$$

The explicit form of the connection may be then derived as,

$$\omega_m = \tilde{\epsilon}_{mn}\partial^n\phi\,, \tag{A.32}$$

where $\tilde{\epsilon}_{mn}$ is a genuine tensor, related to Levi-Civita tensor density by $\epsilon_{mn}\sqrt{-g}$.

Using the explicit form for the connection and the zweibein (metric), it is straightforward to derive the form of the action useful for anomaly calculations,

$$\begin{aligned} S_\chi &= \frac{1}{2}\int d^2x\sqrt{-g}(i\chi^\dagger\gamma^1)e_a^m\gamma^a D_m\chi \\ &= \frac{1}{2}\int d^2x\left\{-i\chi_{\ddagger}e^{\phi/2}(-\partial_0+\partial_1)e^{\phi/2}\chi_+ - i\chi_{=}e^{\phi/2}(\partial_0+\partial_1)e^{\phi/2}\chi_-\right\} \end{aligned} \tag{A.33}$$

A.6 Lightcone coordinates

Lightcone coordinates are defined in agreement with reference [21],

$$x^{\ddagger} = \frac{1}{2}(x^0 + x^1) \qquad x^{=} = \frac{1}{2}(-x^0 + x^1) \quad , \tag{A.34}$$

and the corresponding derivatives are defined by $[\partial_a, x^b] = \delta_a^b$:

$$\partial_{\ddagger} = \partial_0 + \partial_1 \qquad \partial_{=} = -\partial_0 + \partial_1 \quad . \tag{A.35}$$

The integration measure may be expressed as,

$$\frac{1}{2}\int dx^0 dx^1 = \int dx^{\ddagger}dx^{=} \quad . \tag{A.36}$$

The definition of lightcone coordinates (A.34) has the advantage of keeping the same orientation in the transformation from $\{x^0, x^1\}$ to $\{x^{\ddagger}, x^{=}\}$ basis, therefore signs of action are identical in both.

The Minkowski space metric takes the form:

$$\eta_{\ddagger =} = \eta_{= \ddagger} = 2 \quad , \tag{A.37}$$

with the inverse metric

$$\eta^{\ddagger =} = \eta^{= \ddagger} = \frac{1}{2} \quad , \tag{A.38}$$

and the Levi-Civita symbol becomes,

$$\epsilon^{\ddagger =} = -\epsilon^{= \ddagger} = \frac{1}{2} \quad . \tag{A.39}$$

The Dirac action (A.28) becomes

$$S_\chi = \int dx^{\ddagger} dx^{=} \left(-i\chi_{\dot{+}}\partial_{=}\chi_{+} - i\chi_{\dot{-}}\partial_{=}\chi_{-}\right) \quad , \tag{A.40}$$

which may be rewritten in lightcone coordinates as

$$S_\chi = \int dx^{\ddagger} dx^{=}\, i\overline{\chi}\partial\!\!\!/\chi\,, \tag{A.41}$$

provided that

$$\gamma^{\ddagger} = \begin{pmatrix} 0 & i \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad \gamma^{=} = \begin{pmatrix} 0 & 0 \\ -i & 0 \end{pmatrix} \quad . \tag{A.42}$$

It is useful to note some properties of lightcone "Gamma" algebra:

$$\begin{aligned} \left(\gamma^{\ddagger}\right)^2 &= 0 \\ (\gamma^{=})^2 &= 0 \\ \gamma^{\ddagger}\gamma^{=} &= \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \frac{1}{2}(1+\gamma_5) = P_+ \\ \gamma^{=}\gamma^{\ddagger} &= \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \frac{1}{2}(1-\gamma_5) = P_- \quad . \end{aligned} \tag{A.43}$$

A.7 (1,1) Superspace

In this section a derivation for supersymmetric action is presented from superspace. The first subsection is focused on global superspace, and the second on local superspace.

A.7.1 Global Superspace

$(1,1)$ superspace is a four dimensional space with coordinates $\{x^{\ddagger}, x^{=}, \theta^{+}, \theta^{-}\}$, where $\theta^{\pm}$ are anticommuting (spinor) coordinates with the Hermitian property $(\theta^{\pm})^{\dagger} = \theta^{\pm}$.

Derivatives with respect to the spinor coordinates are defined by the anticommutation relation $\{\partial_\alpha, \theta^\beta\} = \delta_\alpha^\beta$. Where as bosonic derivatives are anti-Hermitian, spinor derivatives are Hermitian.

In superspace, both Poincaré and supersymmetry algebra (4.2) may be represented as operators acting on the space,

$$P_{\ddagger} = i\partial_{\ddagger} \qquad\qquad P_{=} = i\partial_{=}$$

$$Q_{+} = i\left(\partial_{+} - \frac{i}{2}\theta^{+}\partial_{\ddagger}\right) \qquad Q_{-} = i\left(\partial_{-} - \frac{i}{2}\theta^{-}\partial_{=}\right) \quad . \tag{A.44}$$

Note that the overall factor (i) in the supersymmetry charges $Q_{\pm}$ makes them anti-Hermitian.

Supersymmetry covariant spinor derivatives are defined to commute with supercharges (A.44),

$$D_{+} = \partial_{+} + \frac{i}{2}\theta^{+}\partial_{\ddagger} \qquad D_{-} = \partial_{-} + \frac{i}{2}\theta^{-}\partial_{=} \quad , \tag{A.45}$$

and they satisfy the following anticommutation relation,

$$\{D_{+}, D_{+}\} = i\partial_{\ddagger} \qquad \{D_{-}, D_{-}\} = i\partial_{=} \quad . \tag{A.46}$$

In principle, the notion of tangent space, coordinate and non-coordinate bases (and vielbein) can already be introduced for flat superspace as follows,

$$\partial_{A} = \hat{E}_{A}^{M} D_{M} \tag{A.47}$$

where,

$$\hat{E}_{A}^{M} = \delta_{A}^{M} \tag{A.48}$$

relates coordinate $(D_{M}) = (\partial_{m}, D_{\mu})$ to non-coordinate (∂_{A}) basis of tangent space. If instead of supersymmetry covariant derivatives, (D_{M}), one chooses to parameterize the coordinate basis of the tangent space with the naive derivatives $(\partial_{M}) = (\partial_{m}, \partial_{\mu})$, the vielbein takes the following form,

$$\hat{E}_{A}^{M} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \frac{i}{2}\theta^{+} & 0 & 1 & 0 \\ 0 & \frac{i}{2}\theta^{-} & 0 & 1 \end{pmatrix} \quad . \tag{A.49}$$

Non-trivial geometry of rigid superspace is defined in terms of field strengths (torsion T and curvature R),

$$[\partial_{A}, \partial_{B}] = T_{AB}^{C}\partial_{C} + R_{AB}^{CD}J_{CD} \tag{A.50}$$

Above definition is consistent for either choice of the coordinate basis, (D_{M}) or (∂_{M}). With the corresponding vielbein in (A.49) or (A.48), the same non-zero torsion of superspace is seen in the non-coordinate basis, $T_{++}^{\ddagger} = T_{--}^{=} = i$.

One may define fields on superspace, superfields, depending on all four superspace coordinates. A scalar superfield Φ may be Taylor expanded in the anticommuting coordinates as follows,

$$\Phi(x^{\ddagger}, x^{=}, \theta^{+}, \theta^{-}) \;=\; \phi(x) + \frac{1}{\sqrt{2}}\theta^{+}\chi(x)_{+} + \frac{1}{\sqrt{2}}\theta^{-}\chi(x)_{-} + \frac{i}{2}\theta^{+}\theta^{-}N(x) \quad . \qquad \text{(A.51)}$$

The expansion terminates due to the anticommuting property of the spinor coordinates, and space-time fields are recovered as the coefficients of expansion. In the case of scalar superfield, one recovers the multiplet of fields representing (1,1) supersymmetry in eq. (4.4).

It is useful to introduce an equivalent definition of space-time fields, by the method of projections,

$$\begin{aligned} \Phi\,| &= \phi \\ \sqrt{2}D_{+}\Phi\,| &= \chi_{+} \\ \sqrt{2}D_{-}\Phi\,| &= \chi_{-} \\ i[D_{+}, D_{-}]\Phi\,| &= N \quad , \end{aligned} \qquad \text{(A.52)}$$

where the vertical bar denotes taking the limit $\theta^{\pm} \to 0$.

Integration on the spinor coordinates is defined by the corresponding derivatives,

$$\int d\theta^{+}d\theta^{-}\mathcal{L}(x,\theta) \;=\; \partial_{+}\partial_{-}\mathcal{L}(x,\theta) \;=\; D_{+}D_{-}\mathcal{L}(x,\theta)\,| \quad . \qquad \text{(A.53)}$$

Therefore, the action of scalar superfield in (1,1) superspace takes the following form:

$$S_0 \;=\; 4\int dx^{\ddagger}dx^{=}d\theta^{+}d\theta^{-}\;\Phi(x,\theta)D_{+}D_{-}\Phi(x,\theta) \quad . \qquad \text{(A.54)}$$

The spinor integration may be performed according to the definition (A.53); using the definition of space-time field components (A.52) the following space-time action is recovered,

$$S_0 \;=\; \int dx^{\ddagger}dx^{=}\left(\phi\,\partial_{\ddagger}\partial_{=}\phi - i\chi_{-}\partial_{\ddagger}\chi_{-} - i\chi_{+}\partial_{=}\chi_{+} - N^{2}\right) \quad , \qquad \text{(A.55)}$$

which after converting from lightcone coordinates agrees with the supersymmetric action in eq. (4.4).

$$S_0 = \frac{1}{2}\int d^2x \left(-(\partial_a\phi)^2 + i\overline{\chi}\partial\!\!\!/\chi - N^2\right) \quad . \tag{A.56}$$

Global supersymmetry transformation on the superfield Φ is defied as follows,

$$\delta_Q\Phi = -i\sqrt{2}\overline{\varepsilon}Q\Phi \quad . \tag{A.57}$$

To recover component field transformation rules, it is convenient to work with their definitions as projections. One finds,

$$\begin{aligned} \delta_Q\phi &= -i\sqrt{2}\overline{\varepsilon}Q\Phi\,| \\ &= \sqrt{2}\left(\varepsilon^+(D_+ + ...) + \varepsilon^-(D_- + ...)\right)\Phi\,| \\ &= \varepsilon^+\chi_+ + \varepsilon^-\chi_- \\ &= \overline{\varepsilon}\chi \quad . \end{aligned} \tag{A.58}$$

Proceeding similarly, one finds,

$$\delta_Q\begin{pmatrix}\chi_+\\ \chi_-\end{pmatrix} = \begin{pmatrix}-\partial_{\ddagger}\phi\varepsilon_- + N\varepsilon_+\\ \partial_{=}\phi\varepsilon_+ + N\varepsilon_-\end{pmatrix} = i\partial\!\!\!/\phi\varepsilon + N\varepsilon \quad , \tag{A.59}$$

and

$$\delta_Q N = -\varepsilon^+\partial_{\ddagger}\chi_- + \varepsilon^-\partial_{=}\chi_+ = i\overline{\varepsilon}\partial\!\!\!/\chi \quad . \tag{A.60}$$

A.7.2 Local Superspace

In order to describe SUGRA in a superspace context, it is necessary to introduce the notion of tangent space. Superspace is now a curved manifold, and is parametrized by the coordinates $\{x^M\} = \{x^m, \theta^\mu\} = \{x^{\ddagger}, x^{\approx}, \theta^{+}, \theta^{\sim}\}$. The corresponding derivatives define a local, coordinate basis of the tangent space $D_M = \{\partial_{\ddagger}, \partial_{\approx}, D_{+}, D_{\sim}\}$. Note, that one can also choose to parameterize the spinor part of the basis in terms of the naive spinor derivatives $\{\partial_M\} = \{\partial_{\ddagger}, \partial_{\approx}, \partial_{+}, \partial_{\sim}\}$. Vector (tensor) fields are defined as scalars on tangent space, and their transformation rules are defined implicitly by the transformations of the coordinate basis.

It is convenient to introduce a non-coordinate, flat basis of the tangent space, $\{\partial_A\}$ $= \{\partial_a, \partial_\alpha\} = \{\partial_{\ddagger}, \partial_{=}, \partial_+, \partial_-\}$. The coordinate and non-coordinate bases are related with the vielbein superfield, E_A^M, which contains the SUGRA degrees of freedom.

$$\partial_A = E_A^M D_M = E_A^M \hat{E}_M^N \partial_N \quad , \tag{A.61}$$

where $\hat{E}_M^N$ is the flat superspace vielbein (A.49).

SUGRA is a theory of local symmetries: (super)coordinate transformations act on the (curved) coordinate indices, and local Lorentz symmetry acts on (flat) tangent space. The Lorentz generator $\mathcal{M}$ has the following commutation relations:

$$\begin{aligned} [\mathcal{M}, \chi_\pm] &= \pm\frac{1}{2}\chi_\pm \\ \left[\mathcal{M}, V_{\ddagger}\right] &= V_{\ddagger} \\ [\mathcal{M}, V_{=}] &= -V_{=} \quad . \end{aligned} \tag{A.62}$$

(1, 1) Covariant derivatives are defined as follows,

$$\nabla_A = \partial_A + \omega_A \mathcal{M} = E_A^M D_M + \omega_A \mathcal{M} \quad , \tag{A.63}$$

where ω_A is the spin connection. The superspace torsion (T_{AB}^C) and curvature (R_{AB}) superfields are defined super-covariantly by,

$$[\nabla_A, \nabla_B\} = T_{AB}^C \nabla_C + R_{AB}\mathcal{M} \quad , \tag{A.64}$$

where the square/curly bracket notation refers to choosing anticommutation for two spinor objects, and commutation otherwise.

Consider next the vielbein superfield E_A^M. It has 16 superfield degrees of freedom. Using local super-coordinate and local Lorentz symmetries one may gauge away 5 superfield degrees of freedom. Since the conformal gauge field content of supergravity, as in eq. (4.17), may be grouped into a single superfield, there still remain 10 spurious superfield degrees of freedom in the vielbein. Additionally, one has 4 superfield degrees of freedom in the connection ω_A. It can be seen, therefore, that superspace formulation of supergravity is highly redundant. The name of the game is to remove the unwanted degrees of freedom, while maintaining supersymmetry and gauge covariance. Therefore, constraints should be placed on field strengths, i.e., on torsion and/or on curvature superfields. The goal of the constraint analysis is to express all components of superspace torsion and curvature in terms of a single superfield.

In the definition of torsion and curvature in eq. (A.64), one may consider taking either spinor or bosonic covariant derivatives on the left hand side. In order that local superspace reduces to global superspace in absence of curvature, the following constraint may be placed on the spinor-spinor part of eq. (A.64),

$$\begin{aligned} \{\nabla_+, \nabla_+\} &= i\nabla_{\mp\!\!\!\!\!\!+} \\ \{\nabla_-, \nabla_-\} &= i\nabla_{=} \\ \{\nabla_+, \nabla_-\} &= R\mathcal{M} \quad , \end{aligned} \tag{A.65}$$

which imply, respectively,

$$\begin{aligned} T_{++}^{\mp\!\!\!\!\!\!+} &= i\,, \qquad T_{++}^{=} = T_{++}^{\alpha} = R_{++} = 0 \quad , \\ T_{--}^{=} &= i\,, \qquad T_{--}^{\mp\!\!\!\!\!\!+} = T_{--}^{\alpha} = R_{--} = 0 \quad , \\ R_{+-} &= R\,, \qquad T_{+-}^{A} = 0 \quad . \end{aligned} \tag{A.66}$$

Using graded Bianchi identities for the general bosonic (B) and fermionic (F) operators [24],

$$\begin{aligned} 0 &= [\{F_1, F_2\}, F_3] + [\{F_1, F_3\}, F_2] + [\{F_2, F_3\}, F_1] \quad , \\ 0 &= \{[B_1, F_2], F_3\} + \{[B_1, F_3], F_2\} + [\{F_2, F_3\}, B_1] \quad , \end{aligned} \tag{A.67}$$

together with the constraint relations (A.65), one finds further consequences of chosen constrains:

$$\begin{aligned} \left[\nabla_{\mp\!\!\!\!\!\!+}, \nabla_+\right] &= 0 \quad \Rightarrow \quad T_{+\mp\!\!\!\!\!\!+}^{A} = R_{+\mp\!\!\!\!\!\!+} = 0 \\ [\nabla_{=}, \nabla_-] &= 0 \quad \Rightarrow \quad T_{-=}^{A} = R_{-=} = 0 \\ \left[\nabla_-, \nabla_{\mp\!\!\!\!\!\!+}\right] &= 2i(\nabla_+ R)\mathcal{M} - iR\nabla_+ \quad \Rightarrow \\ & T_{-\mp\!\!\!\!\!\!+}^{+} = -iR, \; R_{-\mp\!\!\!\!\!\!+} = 2i(\nabla_+ R), \; T_{-\mp\!\!\!\!\!\!+}^{-} = T_{-\mp\!\!\!\!\!\!+}^{\mp\!\!\!\!\!\!+} = T_{-\mp\!\!\!\!\!\!+}^{=} = 0 \end{aligned}$$

$$
\begin{aligned}
[\nabla_+, \nabla_=] &= 2i(\nabla_- R)\mathcal{M} - iR\nabla_- \quad \Rightarrow \\
& \qquad T_{+=}^{+} = iR,\ R_{+=} = 2i(\nabla_- R),\ T_{+=}^{+} = T_{+=}^{\ddagger} = T_{+=}^{=} = 0 \\
\left[\nabla_{\ddagger}, \nabla_=\right] &= 4(\nabla_+\nabla_- R)\mathcal{M} + 2(\nabla_- R)\nabla_+ + 2(\nabla_+ R)\nabla_- + 2R^2\mathcal{M} \quad \Rightarrow \\
& \qquad R_{\ddagger=} = (\nabla_+\nabla_- R) + 2R^2,\ T_{\ddagger=}^{+} = 2(\nabla_- R),\ T_{\ddagger=}^{-} = 2(\nabla_+ R),\ T_{\ddagger=}^{a} = 0
\end{aligned}
\tag{A.68}
$$

The choice of constraints (A.65) is sufficient and does not overdetermine SUGRA, since all curvature and torsion components are now expressed in terms of a single superfield R.

The Wess-Zumino gauge for superspace SUGRA is used to identify the space-time field content, and it is defined by projections of covariant derivatives,

$$
\begin{aligned}
\nabla_\pm \mid &= \partial_\pm \\
\nabla_{\ddagger} \mid &= \mathbf{D}_{\ddagger} + \frac{1}{\sqrt{2}}\psi_{\ddagger}^{\alpha}\nabla_\alpha \mid \\
\nabla_= \mid &= \mathbf{D}_= + \frac{1}{\sqrt{2}}\psi_=^{\alpha}\nabla_\alpha \mid \quad ,
\end{aligned}
\tag{A.69}
$$

where ψ_a^α is the gravitino field. Space-time covariant derivatives,

$$
\begin{aligned}
\mathbf{D}_{\ddagger} &= \partial_{\ddagger} + \omega_{\ddagger}\mathcal{M} \\
\mathbf{D}_= &= \partial_= + \omega_=\mathcal{M}
\end{aligned}
\tag{A.70}
$$

contain spin connection (ω) which depend on torsion (gravitino) terms, to be computed shortly.

Component supergravity action is recovered from superspace by taking the generalization of global superspace action (A.54),

$$
S_{(1,1)} = 4\int d^2x d^2\theta \, [E^{-1}] \, \Phi\nabla_+\nabla_-\Phi \quad , \tag{A.71}
$$

together with the SUGRA covariant generalization of the definition of component fields (A.52),

$$
\Phi \mid = \phi
$$

$$\begin{aligned} \sqrt{2}\nabla_+\Phi\,| &= \chi_+ \\ \sqrt{2}\nabla_-\Phi\,| &= \chi_- \\ i[\nabla_+, \nabla_-]\Phi\,| &= N \quad . \end{aligned} \tag{A.72}$$

In flat superspace fermionic integration may be replaced by the projections with SUSY covariant derivatives (A.53): $\int d\theta^+ d\theta^- \mathcal{L} \longrightarrow D_+ D_- \mathcal{L}\,|$. A similar projection may be attempted in the SUGRA case. The complication in SUGRA is that in addition to spinor derivatives, one also has dimensionful SUGRA component fields, which may appear in the projection formula.

The following steps lead one to derive the correct projection. One computes $[\nabla_{\ddagger}, \nabla_{=}]\,|$ in two different ways, first by taking the projection of the consequence of constraints (A.68), (with $R\,| := S$), and second by considering the Wess-Zumino gauge expansion (A.69). Matching $\nabla_+\,|, \nabla_-\,|$, and $\mathcal{M}$, components one obtains the following results:

$$\begin{aligned} \nabla_- R\,| &= \frac{1}{2\sqrt{2}}\Big(\mathbf{D}_{\ddagger}(\psi_{=}^{+}) - \mathbf{D}_{=}(\psi_{\ddagger}^{+}) + \frac{i}{2}\psi_{\ddagger}^{-}\psi_{=}^{-}\psi_{=}^{+} + i\psi_{=}^{-}S\Big) \\ \nabla_+ R\,| &= \frac{1}{2\sqrt{2}}\Big(\mathbf{D}_{\ddagger}(\psi_{=}^{-}) - \mathbf{D}_{=}(\psi_{\ddagger}^{-}) - \frac{i}{2}\psi_{\ddagger}^{+}\psi_{\ddagger}^{-}\psi_{=}^{+} + i\psi_{\ddagger}^{+}S\Big) \\ \nabla_+\nabla_- R\,| &= -\frac{1}{2}S^2 + \frac{1}{8}\Big\{2[\mathbf{D}_{\ddagger}, \mathbf{D}_{=}]\,|_{\mathcal{M}} - \psi_{\ddagger}^{+}\psi_{=}^{-}S + \psi_{\ddagger}^{-}\psi_{=}^{+}S + i\psi_{\ddagger}^{+}(\mathbf{D}_{\ddagger}\psi_{=}^{+}) + \\ &\qquad + i\psi_{=}^{-}(\mathbf{D}_{=}\psi_{\ddagger}^{-}) - i\psi_{\ddagger}^{+}(\mathbf{D}_{=}\psi_{\ddagger}^{+}) - i\psi_{=}^{-}(\mathbf{D}_{\ddagger}\psi_{=}^{-}) - \psi_{\ddagger}^{+}\psi_{\ddagger}^{-}\psi_{=}^{-}\psi_{=}^{+}\Big\} \end{aligned} \tag{A.73}$$

where $[\mathbf{D}_{\ddagger}, \mathbf{D}_{=}]\,|_{\mathcal{M}}$ refers to the $\mathcal{M}$ component of the commutator of covariant derivatives. Next the following anzats is taken for the projection formula,

$$\int d^2x d^2\theta [E^{-1}]\mathcal{L} = \int d^2x [e^{-1}]\left(\nabla_+\nabla_- + a^-\nabla_- + a^+\nabla_+ + b + cS\right)\mathcal{L}\,| \quad , \tag{A.74}$$

where on dimensional grounds, a^+ and a^- are linear in gravitini, b is a bilinear in gravitini, and c is dimensionless. In order to fix the parameters in the anzats (A.74), one requires that the superspace action,

$$S = 4\int d^2x d^2\theta\,[E^{-1}]\,R \quad , \tag{A.75}$$

reduces to the pure supergravity component action, namely the sum of the Einstein-Hilbert term, the gravitino action and the auxiliary field action. As remarked in Chapter 4, the gravitino action (4.6) vanishes identically in $(1+1)$ dimensions. The Einstein-Hilbert action is a total derivative, and it may be expressed as the commutator of covariant derivatives $\int d^2x[e^{-1}][\mathbf{D}_{\ddagger}, \mathbf{D}_{=}]\,|_{\mathcal{M}}$, where the subscript $\mathcal{M}$ implies the projection of the coefficient of the Lorentz generator. Finally the auxiliary field action must vanish in order to maintain SUGRA invariance. Using relations (A.73), it is now possible to uniquely fix the coefficients in (A.74), leading to the following projection formula,

$$\begin{aligned}\int d^2x d^2\theta [E^{-1}]\mathcal{L} &= \int d^2x[e^{-1}]\Big(\nabla_+\nabla_- - \frac{i}{2\sqrt{2}}\psi_{\ddagger}^{+}\nabla_- + \frac{i}{2\sqrt{2}}\psi_{=}^{-}\nabla_+ \\ &\quad -\frac{1}{8}\left(\psi_{\ddagger}^{+}\psi_m m^- + \psi_{\ddagger}^{-}\psi_{=}^{+}\right) + \frac{1}{2}S\Big)\mathcal{L}\,| \quad . \qquad \text{(A.76)}\end{aligned}$$

Applying the projection formula (A.76) and the matter component field definitions (A.72) to the scalar superfield action, the following component action is recovered:

$$\begin{aligned}S_{(1,1)} &= -4\int d^2x d^2\theta[E^{-1}](\nabla_+\Phi)(\nabla_-\Phi) \\ &= \int dx^{\ddagger}dx^{=}[e^{-1}]\Big(-(\mathbf{D}_{\ddagger}\phi)(\mathbf{D}_{=}\phi) - i\chi_+\mathbf{D}_{=}\chi_+ - i\chi_-\mathbf{D}_{\ddagger}\chi_- - N^2 \\ &\quad -(\mathbf{D}_{\ddagger}A)\psi_{=}^{+}\chi_+ - (\mathbf{D}_{=}A)\psi_{\ddagger}^{-}\chi_- - \frac{1}{2}\psi_{\ddagger}^{-}\psi_{=}^{+}\chi_+\chi_-\Big) \quad , \qquad \text{(A.77)}\end{aligned}$$

which may be rewritten in an explicitly covariant form (4.7),

$$S_{(1,1)} = \frac{1}{2}\int d^2x[e^{-1}]\left\{\phi\Box_e\phi + i\overline{\chi}\not{\mathbf{D}}\chi - N^2 - \kappa\overline{\psi_a}\not{\partial}A\gamma^a\chi + \frac{\kappa^2}{8}\overline{\psi}_n^B\gamma^m\gamma^n\psi_m^B\overline{\chi}\chi\right\} \quad . \qquad \text{(A.78)}$$

To obtain torsion t and curvature r, it is useful to consider the defining relation,

$$[\mathbf{D}_{\ddagger}, \mathbf{D}_{=}] = t_{\ddagger=}^{\ddagger}\mathbf{D}_{\ddagger} + t_{\ddagger=}^{=}\mathbf{D}_{=} + r_{\ddagger=}\mathcal{M} \quad . \qquad \text{(A.79)}$$

In the above definition one substitutes the explicit form of the gravitational covariant derivative, $\mathbf{D}_a = \partial_a + \omega_a\mathcal{M}$, where $\partial_a = e_a^m\partial_m$, to find,

$$\begin{aligned}t_{\ddagger=}^{\ddagger} &= c_{\ddagger=}^{\ddagger} - \omega_{=} \\ t_{\ddagger=}^{=} &= c_{\ddagger=}^{=} - \omega_{\ddagger} \\ r_{\ddagger=} &= e_{\ddagger}\omega_{=} - e_{=}\omega_{\ddagger} - c_{\ddagger=}^{\ddagger}\omega_{\ddagger} - c_{\ddagger=}^{=}\omega_{=} \quad , \qquad \text{(A.80)}\end{aligned}$$

and where so-called anholomy coefficients are defined by,

$$[\partial_a, \partial_b] = c_{ab}^d \partial_d \quad . \tag{A.81}$$

Finally, matching $[\nabla_{\ddagger}, \nabla_{=}] \mid$ evaluated with the constraints (A.68) and with the Wess-Zumino gauge condition (A.69), one finds,

$$\omega_{\ddagger} = c_{\ddagger=}^{=} + i\psi_{\ddagger}^{-}\psi_{=}^{-} \,, \qquad \omega_{=} = c_{\ddagger=}^{\ddagger} + i\psi_{\ddagger}^{+}\psi_{=}^{+} \quad , \tag{A.82}$$

which further implies,

$$t_{\ddagger=}^{=} = -i\psi_{\ddagger}^{-}\psi_{=}^{-} \,, \qquad t_{\ddagger=}^{\ddagger} = -i\psi_{\ddagger}^{+}\psi_{=}^{+} \quad . \tag{A.83}$$

The local supersymmetry transformations for matter fields may be derived in a similar way to the global supersymmetry case (A.58). Action of the transformation on the scalar superfield is $\delta\Phi = \sqrt{2}\bar{\varepsilon}\nabla\Phi$ [22], and the component field transformation are derived from their definition as projections (A.72). The following results are obtained,

$$\begin{aligned}
\delta\phi &= \bar{\varepsilon}\chi \\
\delta\chi &= \begin{pmatrix} i\varepsilon^{+}(\mathbf{D}_{\ddagger}\phi + \frac{1}{2}\psi_{\ddagger} + \alpha\chi_{\alpha}) + i\varepsilon^{-}N \\ i\varepsilon^{-}(\mathbf{D}_{=}\phi + \frac{1}{2}\psi_{=} + \alpha\chi_{\alpha}) - i\varepsilon^{+}N \end{pmatrix} \\
\delta N &= \varepsilon^{+}\left(\mathbf{D}_{\ddagger}\chi_{-} + \frac{i}{2}\psi_{\ddagger}^{+}N - \frac{i}{2}\psi_{\ddagger}^{-}(\mathbf{D}_{=}\phi + \frac{1}{2}\psi_{=}^{\alpha}\chi_{\alpha})\right) + \\
&\quad \varepsilon^{-}\left(\mathbf{D}_{=}\chi_{+} + \frac{i}{2}\psi_{=}^{-}N - \frac{i}{2}\psi_{=}^{+}(\mathbf{D}_{\ddagger}\phi + \frac{1}{2}\psi_{\ddagger}^{\alpha}\chi_{\alpha})\right)
\end{aligned} \tag{A.84}$$

In order to derive the local supersymmetry transformation rules for the component supergravity fields, it is useful to recall the projection of the covariant derivative in Wess-Zumino gauge,

$$\nabla_a \mid = e_a^m \partial_m + \frac{1}{\sqrt{2}}\psi_a^{\alpha}\nabla_{\alpha} \mid + \omega_a \mathcal{M} \tag{A.85}$$

Taking the commutator $[\sqrt{2}\bar{\varepsilon}\nabla, \nabla_a]$, and projecting out ∂_m and $\nabla_{\alpha} \mid$ components gives the transformations for e_a^m and ψ_a^{α} respectively. Projecting $\sqrt{2}\bar{\varepsilon}\nabla R \mid$ gives the transformation of the SUGRA auxiliary field S. After some algebra, one obtains,

$$
\begin{aligned}
\delta e_a^m &= i\left(\varepsilon^+\psi_a^+ e_{\ddagger}^m + \varepsilon^-\psi_a^- e_{=}^m\right)\\
\delta\psi_{\ddagger} &= \begin{pmatrix} -2\mathbf{D}_{\ddagger}\varepsilon_+ - i\psi_{\ddagger}^+\psi_{\ddagger}^-\varepsilon_- + i\psi_{\ddagger}^-\psi_{=}^-\varepsilon_+ \\ -2\mathbf{D}_{\ddagger}\varepsilon_- + 2iS\varepsilon_+ - i\psi_{\ddagger}^-\psi_{=}^+\varepsilon_+ \end{pmatrix}\\
\delta\psi_{=} &= \begin{pmatrix} -2\mathbf{D}_{=}\varepsilon_+ - 2iS\varepsilon_- - i\psi_{=}^+\psi_{\ddagger}^-\varepsilon_- \\ -2\mathbf{D}_{=}\varepsilon_- + i\psi_{=}^+\psi_{\ddagger}^+\varepsilon_- - i\psi_{=}^-\psi_{=}^+\varepsilon_+ \end{pmatrix}\\
\delta S &= \frac{1}{2}\varepsilon^+\left(\mathbf{D}_{\ddagger}\psi_{=}^- - \mathbf{D}_{=}\psi_{\ddagger}^- - \frac{i}{2}\psi_{\ddagger}^+\psi_{\ddagger}^-\psi_{=}^+ + i\psi_{\ddagger}^+ S\right) + \\
&\quad \frac{1}{2}\varepsilon^-\left(\mathbf{D}_{\ddagger}\psi_{=}^+ - \mathbf{D}_{=}\psi_{\ddagger}^+ + \frac{i}{2}\psi_{\ddagger}^-\psi_{=}^-\psi_{=}^+ + i\psi_{=}^- S\right)
\end{aligned}
\tag{A.86}
$$

After some more algebra, both (A.84) and (A.86) may be rewritten in the explicitly covariant form (4.8), (where ψ_m and S are rescaled by the SUGRA coupling constant κ),

$$
\begin{aligned}
\delta\phi &= \bar{\varepsilon}\chi \qquad \delta\chi = (i\hat{\not{\mathbf{D}}}\phi + N)\varepsilon \qquad \delta N = i\bar{\varepsilon}\hat{\not{\mathbf{D}}}\chi\\
\delta e_m^a &= i\kappa\bar{\varepsilon}\gamma^a\psi_m \qquad \delta\psi_m = \frac{-2}{\kappa}\mathbf{D}_m\varepsilon - S\gamma_m\varepsilon + \frac{i\kappa}{2}\overline{\psi}_a\eta^{ab}\psi_b\gamma_m\varepsilon + i\kappa\left(\overline{\psi}_b\gamma_5\gamma^b\varepsilon\right)\eta_{ad}\epsilon^{de}\psi_e\\
\delta S &= \frac{i}{2}\bar{\varepsilon}\left(-2\epsilon^{ab}\gamma_5\mathbf{D}_a\psi_b + (iA + \kappa^2\overline{\psi}_a\gamma^b\gamma^a\psi_b)\gamma^c\psi_c\right).
\end{aligned}
\tag{A.87}
$$

Appendix B

Measure Matters

In this Appendix a detailed construction of the (two dimensional) path integral measure is presented. The construction is carried out in Riemannian space, with the idea that results are to be Wick-rotated to Lorentzian space-times. The reason for working in Riamannian space is that the anomaly calculations presented in this chapter require a positive definite field operator.

The construction of a consistent path integral measure is carried in several steps. First, one chooses the (preferred) symmetry which is to be maintained by the QFT. Next, an explicit form of the measure is constructed in terms of normal mode functions. Finally, the complement to the preferred symmetry is shown to be anomalous, and the anomaly is evaluated. The discussion is then specified to the choice of GCT invariant measure, and the corresponding conformal scaling anomaly. Next, an outline of the $(1, 1)$ superconformal anomaly is presented.

Appendix B ends with an explanation of the difference in $(2, 2)$ superconformal anomaly between this thesis and the original reference [19].

B.1 Preferred Symmetry of a QFT

In this section the first steps are taken in constructing the path integral measure for a free scalar QFT in curved background. The program of Riemannian QFT may be stated with the path integral,

$$Z[g_{mn}] = \int [\mathcal{D}\phi]_g e^{-S[\phi, g_{mn}]} = e^{-\Sigma} \quad , \tag{B.1}$$

where one is interested in the contribution to the effective action Σ. In order to make sense of eq. (B.1) one needs to specify the path integral measure, $[\mathcal{D}\phi]_g$, or equivalently, the prescription of how to sum up contributions from different field configurations. This "summation" process should respect certain preferred symmetries of the QFT, which is shown now to be a subset of the symmetries of the action.

Consider the free, scalar action for concreteness,

$$S_\phi = -\frac{1}{2}\int d^2x\,[g]^{1/2}\phi(x)\Box_g\phi(x) = -\frac{1}{2}\int d^2x[g]^{k/2}\phi(x)\Box_g^k\phi(x)\ , \tag{B.2}$$

where $\Box_g^k = g^{(1-k)/2}\Box_g$ and $[g]$ stands for the determinant of the metric. In the limit $k = 1$ one recovers the usual Laplacian. The reason for writing the second equality in eq. (B.2) is to demonstrate the freedom in the choice of the path integral measure, (or the choice of preferred symmetry in the QFT).

Following Fujikawa [54, 31, 32], the path integral measure is defined based on a mode expansion of the field operator of the quadratic part of the action. For the action (B.2), the field operator is $\Box_g^k$, and it defines the following scalar eigenvalue problem,

$$\Box_g^k \xi_N^k = \lambda_N^k \xi_N^k \quad . \tag{B.3}$$

The corresponding inner product is

$$(\phi_1(x), \phi_2(x))^{k/2} = \int d^2x[g]^{k/2}\phi_1(x)\phi_2(x) \quad , \tag{B.4}$$

and the completeness relation is

$$\sum_N \xi_N^k(x)\xi_N^k(y) = \frac{\delta^2(x-y)}{[g]^{k/2}} \quad . \tag{B.5}$$

In eq. (B.3), eigenfunction (ξ_N^k) are scalar quantities, therefore the scalar field $\phi(x)$ may be expanded as

$$\phi(x) = \sum_N a_N^k \xi_N^k(x), \tag{B.6}$$

and the action (B.2) takes the following form,

$$S_\phi = -\frac{1}{2}\sum_N \lambda_N^k (a_N^k)^2 \quad . \tag{B.7}$$

To derive the preferred symmetry of the system one varies the definition of the inner product (B.4) under diffeomorphisms with the vector parameter (v^l) and under Weyl scaling with the corresponding scalar parameter β, and sets the result to zero.

$$\begin{aligned}\delta_{v,\beta}\left(\phi(x),\phi(x)\right)^{k/2} &= \delta_{v,\beta}\int d^2x\,[g]^{k/2}\,\phi(x)^2 \\ &= \int d^2x\,[g]^{k/2}\left[(k-1)\partial_l v^l + 2k\beta\right]\phi(x)^2 \;=\; 0\end{aligned} \tag{B.8}$$

So the condition $(k-1)\partial_l v^l + 2k\beta = 0$, for a particular choice of $k(x)$, defines the preferred symmetry. In the limit $k=1$ the condition reads $\beta = 0$, which implies that GCT invariance is preserved but that Weyl scaling is not.

An equivalent way to see the preferred symmetry condition is to vary the related eigenvalue problem (B.3),

$$\delta_{v,\beta}([g]^{\frac{1-k}{2}}\Box_g \xi_N^k - \lambda_N^k \xi_N^k) = 0\,, \tag{B.9}$$

which reduces to

$$[(1-k)\partial_p v^p - 2k\beta][g]^{\frac{1-k}{2}}\Box_g \xi_N^k = 0 \quad , \tag{B.10}$$

implying the same condition as in eq. (B.8).

B.2 Consistent Definition of the Measure and its Anomaly

In this section the construction of the path integral measure is presented. A consistency requirement for the measure is stated and verified.

The first step in the construction of the path integral measure is making a choice of the preferred symmetry for the QFT, i.e. some particular choice of $k(x)$ from the previous section. DeWitt [29] (and Polyakov [8]) define the measure implicitly via the inner product (B.4) and Gaussian integration,

$$\int [\mathcal{D}\phi(x)]^k_g \exp\{-\frac{1}{2}\int d^2x[g]^{k/2}\phi(x)^2\} \equiv 1 \quad . \tag{B.11}$$

In the limit $k(x) \to 1$, the measure is GCT invariant, which is the case of interest for DeWitt and Polyakov. For $k(x) \to 0$, the path integral measure satisfies volume-preserving diffeomorphisms and Weyl invariance [55].

The above construction of the path integral measure is implicit in nature; where as a more satisfactory result ought to be its explicit construction. The construction of the measure should pass the test of consistency, namely, that a commutation of anomalies for two non-preferred transformations is the anomaly for the commutation of the same two transformations [56]. An anomalous transformation is a symmetry of the action which is in the complement to the preferred symmetries of the path integral measure. Performing an anomalous transformation yields a non-trivial Jacobian.

Fujikawa [54] constructs the path integral based on the normal mode expansion. The main conclusion of his analysis (which is generalized here to a space dependent $k(x)$) is that one should integrate over the normal modes of the scalar density $g^{k/4}\phi(x)$, rather than over the normal modes of $\phi(x)$, in order to preserve the chosen symmetry parametrized by $k(x)$. In general terms, Fujikawa proves so by demonstrating that for GCT invariant ($k = 1$) measure, the (regularized) Jacobian for infinitesimal GCT's is a total derivative, hence it drops out under the assumptions of reasonable boundary conditions. To transform the measure, one needs to know how the scalar density field $g^{k/4}\phi(x)$ varies under a GCT, and the corresponding variation for the normal mode basis.

Here is how all this works in practice. Choosing the mode basis for the preferred symmetry $k(x)$, one writes the measure in terms of the field modes in eq. (B.6).

$$[\mathcal{D}\phi(x)]^k_g = \prod_N da^k_N \tag{B.12}$$

To determine the transformation properties of the measure under diffeomorphisms and Weyl scaling, consider the effects of the transformations on the coefficients $\{a^k_N\}$

from eq. (B.6), which may be defined by projecting $\phi(x)$ onto the basis of ξ_N^k, as follows,

$$a_N^k = \int d^2x\, [g]^{k/2}\, \phi(x)\xi_N^k(x) \quad . \tag{B.13}$$

The transformed expansion coefficient takes the following form,

$$(a_N^k)' = \int d^2x(1+\delta_{v,\beta})[g]^{k/2}\phi(x)\xi_N^k(x) = \sum_P C_{NP}a_P^k \quad , \tag{B.14}$$

where

$$C_{NP} = \int d^2x\, [g]^{k/2}\, [1 + (k-1)\partial_p v^p + 2k\beta]\xi_P^k \xi_N^k \quad . \tag{B.15}$$

Therefore, the path integral measure (B.12) transforms as

$$(1+\delta_{v,\beta})\, [\mathcal{D}\phi(x)]_g^k = \prod_n da_n^k \det[C_{NP}] \quad , \tag{B.16}$$

where the determinant is regularized with the field operator in (B.3).

For the present purpose of determining consistency of the definition of the measure, it is most convenient to work with the anomaly defined as the effect of varying the contribution to the effective action Σ in eq. (B.1) [56]. Considering a GCT with the parameter (v^l), one finds

$$\delta_v \Sigma^k \;=\; H_v = -\frac{1}{48\pi}\int d^2x\, [g]^{k/2}(k(x)-1)\partial_l v^l \mathcal{R}(\hat{g}_{mn}^k) \tag{B.17}$$

where $\mathcal{R}(\hat{g}_{mn}^k)$ is the Ricci curvature scalar calculated from the conformally rescaled metric $\hat{g}_{mn}^k = g^{\frac{k-1}{2}} g_{mn}$. Notice, that only for $k(x) = 1$ there is no GCT anomaly. For a general $k(x)$, The GCT anomaly in (B.17) should satisfy the following consistency condition [56]:

$$\delta_v H_w - \delta_w H_v = H_{[v,w]} \quad , \quad [v,w]_p = v^l \partial_l w_p - w^l \partial_l v_p \quad , \tag{B.18}$$

which indeed holds for the measure definition in eq. (B.12).

Similarly, it may be shown for general k, the Weyl anomaly is consistent for the definition (B.12).

In passing, it is interesting to make the following observation. The path integral measure may be thought of as taking a lattice and allowing the field ϕ to take arbitrary values at each lattice point. The construction of the path integral measure in this section suggests that a consistent definition of the position-space representation of the measure ought to be *defined* as the change of basis from the mode representation. Therefore, the position space path integral measure should contain a Jacobian factor, as follows,

$$[\mathcal{D}\phi(x)]_g^k = \prod_x d\phi_x \det[\xi_N^k(x)]^{-1} \quad , \tag{B.19}$$

where ξ_N^k are the normal mode functions.

B.3 Conformal Anomaly

In this section the conformal anomaly is calculated for arbitrary choice of the parameter $k(x)$. The result is then specified to the GCT invariant case, namely, $k = 1$. The plan is to evaluate the anomaly infinitesimally, and then integrate the result to obtain the finite anomaly.

The anomaly is defined as the Jacobian factor for Weyl scaling transformation, namely, the β dependent part of the determinant of matrix C_{NP} in eq. (B.16). Writing $\det[C_{NP}]$, with $v^l \to 0$, as the exponentiated trace of the logarithm, and considering infinitesimal β, one obtains the infinitesimal Jacobian,

$$J_\beta = \exp\left\{\int d^2x [g]^{\frac{1-k}{2}} \beta(x) B(x)\right\} \quad , \tag{B.20}$$

where $B(x)$ is the trace which needs to be regularized with some regulating operator (Reg) and a cut-off M, as follows,

$$B(x) = \sum_m \left(\xi_m^k(x)\right)^2 = \lim_{M\to\infty} \sum_m \xi_m^k(x) e^{-(Reg)/M^2} \xi_m^k(x) \quad . \tag{B.21}$$

Recall that $\xi_m^k(x)$ are eigenfunctions of the operator $g^{\frac{1}{2}(1-k)}\Box_g$, It is this operator which is identified with the regulator (Reg) in eq. (B.21), so that the preferred symmetry is preserved by the regularization procedure. The fact that "taking the trace" is an operation independent of the basis, allows a substitution of the plane wave expansion in place of ξ_P^k.

Specifying (Reg) to the conformal gauge implies $Reg = e^{-2k\sigma}\Box$. Therefore, one finds

$$B(x) = \lim_{M\to\infty} \int \frac{d^2p}{(2\pi)^2} e^{-2k\sigma} \exp\left\{\frac{e^{-2k\sigma}}{M^2}(-\vec{p}^2 + 2ip^a\partial_a + \Box)\right\} \quad , \tag{B.22}$$

where indices $(a, b, c...)$ refer to flat space.

Following the standard procedure due to Fujikawa the operator in the exponent is split up as follows,

$$H_0 = e^{-2k\sigma}\vec{p}^2 \qquad H_1^{\mathcal{A}} = e^{-2k\sigma}\frac{2ip_a\partial^a}{M} \qquad H_1^{\mathcal{B}} = e^{-2k\sigma}\frac{\Box}{M^2} \quad . \tag{B.23}$$

Note that the dangling operators in the exponent of eq. (B.22) may not be dropped because of the position dependence of $e^{-2k\sigma}$. In addition, the expression may be thought of as acting on identity, so that after expanding derivatives in the exponent dangling derivatives do not remain in the final expression for $B(x)$.

Next, after rescaling $\vec{p}$ by M in (B.22), it is useful to define the function $f(x)$ to satisfy

$$\exp\{-H_0 + H_1^{\mathcal{A}} + H_1^{\mathcal{B}}\} = \exp\{-H_0\}f(1) \quad . \tag{B.24}$$

As can be checked by direct substitution in (B.24), $f(1)$ can be expressed as a recursive expansion in powers of operators $H_1^{\mathcal{A}}$ and $H_1^{\mathcal{B}}$, which in turn depend inversely on the cut-off M,

$$f(t) = t + \int_0^t dt' \left(H_1^{\mathcal{A}}(t') + H_1^{\mathcal{B}}(t')\right) f(t')dt' \quad \text{where} \quad H_1(t) = e^{H_0 t}H_1 e^{-H_0 t} \, . \tag{B.25}$$

One should evaluate $f(1)$ up to $O(\frac{1}{M^3})$ to account for the divergent and finite part of $B(x)$ in the limit of large M. Therefore,

$$f(1) = 1 + \int_0^1 \left\{H_1^{\mathcal{A}}(t)\left(1 + \int_0^t H_1^{\mathcal{A}}(t')dt'\right) + H_1^{\mathcal{B}}(t)\right\} dt + O\left(\frac{1}{M^3}\right) = 1 + f(1)^{\mathcal{A}} + f(1)^{\mathcal{B}}, \tag{B.26}$$

where $f(1)^{\mathcal{A}}$ contains operators $H_1^{\mathcal{A}}$ only, and $f(1)^{\mathcal{B}}$ depends on $H_1^{\mathcal{B}}$.

$$\begin{aligned} f(1)^{\mathcal{A}} \;\; &= \;\; e^{-4k\sigma}\frac{2i}{M}p^a\partial_a(k\sigma)\vec{p}^2 - e^{8k\sigma}\frac{8}{M^2}\vec{p}^4(p^a\partial_a(k\sigma))^2 + e^{-6k\sigma}\frac{16}{3M^2}\vec{p}^2(p^a\partial_a(k\sigma))^2 \\ &\quad - e^{-6k\sigma}\frac{4}{3M^2}\vec{p}^2(p^a\partial_a)^2(k\sigma) \end{aligned} \tag{B.27}$$

$$f(1)^{\mathcal{B}} = e^{-6k\sigma}\frac{4}{3M^2}\vec{p}^4(\partial_a(k\sigma))^2 - e^{-4k\sigma}\frac{4}{2M^2}\vec{p}^2(\partial_a(k\sigma))^2 + e^{-4k\sigma}\frac{2}{2M^2}\vec{p}^2\Box(k\sigma) \quad \text{(B.28)}$$

Expressions for $f(1)^{\mathcal{A}}$ and $f(1)^{\mathcal{B}}$ are substituted into $B(x)$, the integration variable is shifted $p \to e^{k\sigma}p$, which is allowed since no derivatives act on $\vec{p}$ in (B.27) and (B.28). Next, terms odd in powers of $\vec{p}$ are dropped from p integration, so that the following form of $B(x)$ is determined,

$$\begin{aligned}
B(x) &= \lim_{M\to\infty} M^2 \int \frac{d^2p}{(2\pi)^2}e^{-\vec{p}^2}\frac{1}{M^2}\Big\{1 + e^{-2k\sigma}\Big[-2\vec{p}^4(p^a\partial_a(k\sigma))^2 + \frac{16}{3}\vec{p}^2(p^a\partial_a(k\sigma))^2 \\
&\quad -\frac{4}{3}\vec{p}^2(p^a\partial_a)^2 k\sigma + \frac{4}{3}\vec{p}^4\partial_a(k\sigma)\partial^a(k\sigma) - 2\vec{p}^2\partial_a(k\sigma)\partial^a(k\sigma) + \vec{p}^2\Box(k\sigma)\Big]\Big\} \\
&= \frac{M^2}{4\pi} + \frac{e^{-2k\sigma}}{4\pi}\Big\{-2\times 3(\partial_a(k\sigma))^2 + \frac{16}{3}(\partial_a(k\sigma))^2 + \frac{4}{3}\times 2(\partial_a(k\sigma))^2 \\
&\quad -2(\partial_a(k\sigma))^2 - \frac{4}{3}\Box(k\sigma) + \Box(k\sigma)\Big\} \\
&= \frac{M^2}{4\pi} - \frac{e^{-2k\sigma}}{12\pi}\Box(k\sigma)
\end{aligned} \quad \text{(B.29)}$$

Therefore, the infinitesimal Jacobian takes the following form,

$$\begin{aligned}
J_\beta &= \exp\left\{\int d^2x\left(e^{2k\sigma}\frac{M^2}{4\pi} - \frac{1}{12\pi}\Box(k\sigma)\right)\beta(x)\right\} \\
&= \exp\left\{\int d^2x e^{2k\sigma}\left(\frac{M^2}{4\pi} - \frac{1}{24\pi}\mathcal{R}(g^k_{mn})\right)\beta(x)\right\} \quad .
\end{aligned} \quad \text{(B.30)}$$

In order to obtain the effective action for the finite anomalous transformation, the infinitesimal transformation leading to the anomaly (B.30) is iterated. The infinitesimal parameter β from eq. (B.30) is replaced with $\frac{2k\sigma}{N}$ with the implied limit of $N \to \infty$. With each infinitesimal Weyl transformation on the metric, the conformal background seen by the scalar field is reduced. Therefore the finite anomaly is,

$$\begin{aligned}
J &= \lim_{N\to\infty}\exp\left\{\int d^2x\frac{M^2}{4\pi}\left(\frac{2k\sigma}{N}\right)\left(e^{2k\sigma} + e^{2k\sigma\frac{N-1}{N}} + \dots + e^{2k\sigma\frac{1}{N}}\right)\right. \\
&\quad \left. -\int d^2x\left(\frac{2k\sigma}{N}\right)\left(\frac{\Box(k\sigma)}{12\pi}\right)\left(1 + \frac{N-1}{N} + \dots + \frac{1}{N}\right)\right\} \\
&= \exp\left\{\frac{M^2}{4\pi}\int d^2x(k\sigma)\int_0^1 d\rho e^{2k\sigma\rho} - \frac{1}{24\pi}\int d^2x(k\sigma)\Box(k\sigma)\right\} \\
&= \exp\left\{\frac{M^2}{8\pi}\int d^2x\left(e^{2k\sigma} - 1\right) - \frac{1}{24\pi}\int d^2x(k\sigma)\Box(k\sigma)\right\}
\end{aligned} \quad \text{(B.31)}$$

Specifying the result in eq. (B.31) to the GCT invariant measure (i.e., $k = 1$), one recovers the usual conformal anomaly relation, which may be expressed covariantly as the Liouville action (3.3).

Using a similar procedure the Weyl anomaly for Dirac fermions may be evaluated, with the same result for the finite part of the anomaly as in eq. (B.31), but with the regulator-scale dependent term proportional to $-M^2/(4\pi)\int d^2x....$ Analogous result is available for Majorana spinor, yielding the same anomaly as for Dirac spinor, but for the overall multiplicative factor of $\frac{1}{2}$.

B.4 (1,1) Superconformal Anomaly

In this section, the outline of the anomaly calculation is presented. The gauge-fixed, $(1,1)$ supergravity sector (4.8) has a residual $(1,1)$ global supersymmetry. This global supersymmetry must be present in the effective action, meaning that the result of performing the path integral is globally $(1,1)$ invariant. The simplest such action is the Wess-Zumino action in eq. (4.4), but it is important to check if additional terms appear in the effective action. This is accomplished by calculating the Weyl anomaly for the supersymmetric matter multiplet (by setting to zero the gravitino and the auxiliary scalar in eq. (4.7)). Anomaly fields appear in supersymmetric multiplets, so it is possible to deduce the form of the effective action from the leading (Weyl) component of the anomaly action.

As evaluated in Section B.3, scalar field and Majorana spinor contribute the following coefficients to the Weyl anomaly $-\frac{1}{24\pi}$ and $-\frac{1}{48\pi}$, which add up to $-\frac{1}{16\pi}$. The auxiliary field does not contribute to Weyl anomaly. Further, it is interesting to notice that terms proportional to the regulating scale M cancel in the Weyl anomaly for the supersymmetric multiplet. Therefore the Weyl anomaly for the (1,1) matter multiplet is precisely in the form of the finite part of Liouville action (3.3), and no additional terms appear that depend on the regularization parameter. Taking (1,1) supersymmetric extension of the Weyl anomaly, namely the Wess-Zumino type multiplet, one recovers the super-Liouville action in eq. (4.31).

B.5 (2,2) Anomalies

In this section some comments are offered for the $(2,2)$ supercovariant anomaly action, which is used in the section on $(2,2)$ superduality and which differs from the previous evaluations in the literature.

The discrepancy with literature concerns the relative sign of the Weyl $(\partial_m\sigma\partial^m\sigma)$ and chiral$(\partial_m\rho\partial^m\rho$) anomaly terms in the action (4.46). In particular, the in ref. [19, 13], the relative signs are opposite, which means that the anomaly action does not have the same structure as the matter multiplet.

This discrepancy is based on the parameterization of the anomaly action, which is used in the Euclidean-signature calculations of refs. [19, 13]. To see what is going on notice that the ρ-dependent term really starts its life as the two-dimensional anomaly action for a gauge potential, A_m,

$$\mathcal{L}_{\rm anom} = \frac{1}{4\pi}\, F^{mn}\, \left(\frac{1}{\Box}\right)\, F_{mn}. \tag{B.32}$$

Eq. (4.46) is obtained from this when A_m is restricted (in Minkowski signature) to be transverse:

$$A_m = \frac{1}{2}\, \epsilon_{mn}\, \partial^n\, \rho. \tag{B.33}$$

Now comes the key point. If the same replacement, eq.(B.33), were to be made in *Euclidean* signature, as is done in ref. [19, 13], then one instead obtains a ρ kinetic term having the *opposite* sign as in eq. (4.46). One obtains opposite-sign actions depending on whether or not eq. (B.33) is applied in Minkowski or Euclidean signature.

Since the duality calculation is performed in Lorentzian signature, the substitution (B.33) is applied there to the effect of eq. (4.46).

If the Euclidean convention is to be *defined* to reproduce Minkowski space-time result, then the correct substitution which restricts A_m to be transverse in Euclidean signature is

$$A_m = i\, \epsilon_{mn}\, \partial^n \rho, \tag{B.34}$$

where the key difference from eq. (B.33) is the factor of 'i'. Besides ensuring the equivalence of the Minkowski- and Euclidean-signature anomaly actions, this factor of 'i' is required for the unitarity of the Euclidean action. That is, terms linear in A_m, such as $i\bar{\chi}\gamma^m\chi\ A_m$, do not satisfy the Osterwalder-Schraeder (OS) positivity condition [57] unless eq.(B.34) is used instead of eq. (B.33). The OS condition is the Euclidean equivalent of the Minkowski-signature condition of the reality of the action, and if it is violated, it implies the failure of unitarity.

Appendix C

Multiple Gamma Functions

In this Appendix principal formulae pertaining to the multiple gamma functions, G_n, are collected. A new integral representation for these functions is derived and used to obtain closed forms for the integral moments of the $\Psi = \partial_x \ln \Gamma(x)$ function.

C.1 defining properties

Historically, Barnes introduces the "G-function" in ref. [58]. It is a generalization of the standard Γ function and satisfies the following relations:

$$\begin{aligned} G(z+1) = & \quad (2\pi)^{1/2\,z}\, e^{-1/2\,z(z+1)-1/2\,\gamma\, z^2} \prod_{n=1}^{\infty} \left(1+\frac{z}{n}\right)^n e^{-z-1/2\,\frac{z^2}{n}} \\ G(z+1) = & \quad \Gamma(z)G(z) \\ G(1) = & \quad 1 \end{aligned} \tag{C.1}$$

Similar functions are further generalized by Vignéras in ref. [41], who proved the following theorem on the hierarchy of Multiple Gamma functions, $\{G_n\}$.

Theorem (Vignéras)

There exists a unique hierarchy of functions G_n which satisfy the following properties:

$$\begin{aligned} &(1) \quad G_n(z+1) = G_{n-1}(z)G_n(z), \\ &(2) \quad G_n(1) = 1, \\ &(3) \quad \frac{d^{n+1}}{dz^{n+1}} \log G_n(z+1) \geq 0 \quad for \quad z \geq 0, \end{aligned}$$

$$(4) \quad G_0(z) = z \tag{C.2}$$

$G_1(z)$ is the Γ function and $G_2(z)$ is Barnes' G-function.

C.2 Product and Asymptotic Representations

Vignéras [41] derived a Weistrass product representation for the multiple Gamma functions. Another infinite product representation is derived by Ueno and Nishizawa in ref. [59], which is quoted here for the first few functions.

$$G_1(z+1) = \Gamma(z+1) = e^{-\gamma z} \prod_{k=1}^{\infty} \left\{ \left(1 + \frac{z}{k}\right)^{-1} e^{\frac{z}{k}} \right\}. \tag{C.3}$$

This is precisely the Weierstrass product representation for the Γ function.

$$G_2(z+1) = G(z+1) = e^{-z\zeta'(0) - \frac{z^2}{2}\gamma - \frac{z^2+z}{2}} \prod_{k=1}^{\infty} \left\{ \left(1 + \frac{z}{k}\right)^{k} \exp\left(-z + \frac{z^2}{2k}\right) \right\}. \tag{C.4}$$

Since $\zeta'(0) = -\frac{1}{2}\log(2\pi)$, this result is in agreement with the Weierstrass product representation for the Barnes' G-function in ref. [58].

For $n = 3, 4, 5$ one finds the following results.

$$\begin{aligned} G_3(z+1) &= \exp\left\{ -\frac{z^3}{4} + \frac{z^2}{8} + \frac{7}{24}z + \zeta'(-1) - \frac{z(z-1)}{2}\zeta'(0) - \left(\frac{z^3}{6} - \frac{z^2}{4}\gamma\right) \right\} \\ &\quad \times \prod_{k=1}^{\infty} \left[\left(1 + \frac{z}{k}\right)^{-\frac{k(k+1)}{2}} \exp\left\{ \left(\frac{z^3}{6} - \frac{z^2}{4}\right)\frac{1}{k} - \left(\frac{z^2}{4} - \frac{z}{2}\right) + \frac{z}{2}k \right\} \right] \end{aligned} \tag{C.5}$$

$$\begin{aligned} G_4(z+1) &= \exp\left\{ \frac{61}{144}z^4 + \frac{13}{18}z^3 + \frac{19}{144}z^2 - \frac{5}{24}z \right. \\ &\quad \left. -\frac{z}{2}\zeta'(-2) + \frac{z^2 - 2z}{3}\zeta'(-1) - \frac{z^3 - 3z^2 + 2z}{6}\zeta'(0) - \frac{z^4 - 4z^3 + 4z^2}{24}\gamma \right\} \\ &\quad \times \prod_{k=1}^{\infty} \left[\left(1 + \frac{z}{k}\right)^{\frac{k(k+1)(k+2)}{6}} \exp\left\{ \left(\frac{z^4}{24} - \frac{z^3}{6} + \frac{z^2}{6}\right)\frac{1}{k} \right.\right. \\ &\quad \left.\left. - \left(\frac{z^3}{18} - \frac{z^2}{4} - \frac{z}{3}\right) + \left(\frac{z^2}{12} - \frac{z}{2}\right)k - \frac{z}{6}k^2 \right\} \right] \end{aligned} \tag{C.6}$$

$$
\begin{aligned}
G_5(z+1) &= \exp\left\{-\frac{5}{288}z^5+\frac{7}{64}z^4-\frac{173}{864}z^3-\frac{z^2}{36}+\frac{2827}{17280}z\right.\\
&\quad +\frac{z}{6}\zeta'(-3)-\frac{z^2-3z}{4}\zeta'(-2)+\frac{2z^3-9z^2+11z}{12}\zeta'(-1)\\
&\quad \left.-\frac{z^4-6z^3+11z^2-6z}{24}\zeta'(0)-\frac{6z^5-45z^4+110z^3-90z^2}{720}\gamma\right\}\\
&\quad \times\prod_{k=1}^{\infty}\left[\left(1+\frac{z}{k}\right)^{-\frac{k(k+1)(k+2)(k+3)}{24}}\exp\left\{\left(\frac{z^5}{120}-\frac{z^4}{16}+\frac{11}{72}z^3-\frac{z^2}{8}\right)\frac{1}{k}\right.\right.\\
&\qquad -\left(\frac{z^4}{96}-\frac{z^3}{12}+\frac{11}{48}z^2-\frac{z}{4}\right)+\left(\frac{z^3}{72}-\frac{z^2}{8}+\frac{11}{24}\right)k\\
&\qquad \left.\left.-\left(\frac{z^2}{24}-\frac{z}{4}\right)k^2+\frac{z}{24}k^3\right\}\right] \qquad \text{(C.7)}
\end{aligned}
$$

In ref. [59] one also finds asymptotic expansions for general $\{G_n\}$, which are the analogue of Sterling formula for the Γ function, quoted here for low values of n,

In the case $n=1$, the usual Stirling formula is recovered,

$$
\begin{aligned}
\log G_1(z+1) &= \log\Gamma(z+1)\\
&\sim \left(z+\frac{1}{2}\right)\log(z+1)-(z+1)-\zeta'(0)+\sum_{r=1}^{\infty}\frac{B_{2r}}{[2r]_2}\frac{1}{(z+1)^{2r-1}}\,, \qquad \text{(C.8)}
\end{aligned}
$$

where $[2r]_n$ stands for $\Gamma(2r+1)/\Gamma(2r-n+1)$. For $n=2$, one finds the formula first derived by Barnes [58],

$$
\begin{aligned}
\log G_2(z+1) &\sim \left(\frac{z^2}{2}-\frac{1}{12}\right)\log(z+1)-\frac{3}{4}z^2-\frac{z}{2}+\frac{1}{4}-z\zeta'(0)+\zeta'(-1)\\
&\quad -\frac{1}{12}\frac{1}{z+1}+\sum_{r=2}^{\infty}\frac{B_{2r}}{[2r]_3}\frac{1}{(z+1)^{2r-1}}(z-2r+1)\,. \qquad \text{(C.9)}
\end{aligned}
$$

For $n=3$, 4 and 5 the asymptotic expansions are as follows:

$$
\begin{aligned}
\log G_3(z+1) &\sim \left(\frac{z^3}{6}-\frac{z^2}{4}+\frac{1}{24}\right)\log(z+1)-\frac{11}{36}z^3+\frac{5}{24}z^2+\frac{z}{3}-\frac{13}{72}\\
&\quad -\frac{z^2-z}{2}\zeta'(0)+\frac{2z-1}{2}\zeta'(-1)-\frac{1}{2}\zeta'(-2)+\frac{1}{12}\frac{1}{z+1}\\
&\quad +\sum_{r=2}^{\infty}\frac{B_{2r}}{[2r]_4}\frac{1}{(z+1)^{2r-1}}\left\{z^2-(6r-11)z+(4r^2-16r+16)\right\}\,,
\end{aligned}
$$

$$
\begin{aligned}
\log G_4(z+1) \quad\sim\quad & \left(\frac{z^4}{24}-\frac{z^3}{6}+\frac{z^2}{6}-\frac{19}{720}\right)\log(z+1)-\frac{4}{72}z^4+\frac{2}{9}z^3+\frac{z^2}{8}-\frac{11}{36}z \\
+\quad & \frac{31}{144}-\frac{z^3-3z^2+2z}{6}\zeta'(0)+\frac{3z^2-6z+2}{6}\zeta'(-1)-\frac{z-1}{2}\zeta'(-2) \\
+\quad & \frac{1}{6}\zeta'(-3)-\frac{1}{12}\frac{1}{z+1}+\frac{1}{720}\frac{1}{(z+1)^3}\left(6z^2+\frac{13}{2}z+\frac{5}{2}\right) \\
+\quad & \sum_{r=3}^{\infty}\frac{B_{2r}}{[2r]_5}\frac{1}{(z+1)^{2r-1}}\left\{z^3-(12r-27)z^2+(20r^2-94r+111)z\right. \\
& \qquad \left.-(8r^3-56r^2+134r-109)\right\}, \\
\log G_5(z+1) \quad\sim\quad & \left(\frac{z^5}{120}-\frac{z^4}{16}+\frac{11}{72}z^3-\frac{z^3}{8}+\frac{3}{160}\right)\log(z+1) \\
-\quad & \frac{137}{7200}z^5+\frac{39}{320}z^4-\frac{461}{2160}z^3+\frac{z^2}{1440}-\frac{323}{1440}z+\frac{5639}{43200} \\
-\quad & \frac{z^4-6z^3+11z^2-6z}{24}\zeta'(0)+\frac{4z^3-18z^2+22z-6}{24}\zeta'(-1) \\
-\quad & \frac{6z^2-18z+11}{24}\zeta'(-2)+\frac{2z-3}{12}\zeta'(-3)-\frac{1}{24}\zeta'(-4) \\
+\quad & \frac{1}{12}\frac{1}{z+1}-\frac{1}{720}\frac{1}{(z+1)^3}\left(\frac{35}{4}z^2+\frac{45}{4}z+\frac{9}{2}\right) \\
+\quad & \sum_{r=3}^{\infty}\frac{B_{2r}}{[2r]_6}\frac{1}{(z+1)^{2r-1}}\left\{z^4-(20r-54)\,z^3\right. \\
& + \left(70r^2-375r+506\right)z^2-\left(\frac{200}{3}r^3-540r^2+\frac{4420}{3}r-1354\right)z \\
& \left.+16r^4-\frac{536}{3}r^3+754r^2-\frac{4279}{3}r+1021\right\}. \qquad \text{(C.10)}
\end{aligned}
$$

C.3 Integral Relations

Theorem *The following line integral represents the logarithm of the multiple Gamma functions G_n:*

$$
\ln G_n(z+1)=\int_0^{\infty}dt\,\frac{e^{-t}}{t}(-1)^n\left(\frac{1-e^{-zt}}{(1-e^{-t})^n}+\sum_{m=1}^{n}\frac{(-1)^m}{(1-e^{-t})^{n-m}}\binom{z}{m}\right) \qquad \text{(C.11)}
$$

Proof: It is shown explicitly that the defining conditions in C.2 are satisfied. The proof follows by induction on n and from the uniqueness of the hirarchy of $\{G_n\}$ (C.2).

(1) $\ln G_n(z+2) = \ln G_{n-1}(z+1) + \ln G_n(z+1)$ follows from the binomial relation:

$$\begin{pmatrix} z+1 \\ m \end{pmatrix} = \begin{pmatrix} z \\ m-1 \end{pmatrix} + \begin{pmatrix} z \\ m \end{pmatrix} \tag{C.12}$$

The integrand splits up as follows:

$$\begin{aligned} (-1)^n \quad & \left(\frac{1-e^{-zt}e^{-t}}{(1-e^{-t})^n} + \sum_{m=1}^{n} \frac{(-1)^m}{(1-e^{-t})^{n-m}} \begin{pmatrix} z+1 \\ m \end{pmatrix} \right) = \\ & = (-1)^n \left(\frac{1-e^{-zt}}{(1-e^{-t})^n} + \sum_{m=1}^{n} \frac{(-1)^m}{(1-e^{-t})^{n-m}} \begin{pmatrix} z \\ m \end{pmatrix} \right) \\ & +(-1)^{n-1} \left(\frac{1-e^{-zt}}{(1-e^{-t})^{n-1}} + \sum_{m=1}^{n-1} \frac{(-1)^m}{(1-e^{-t})^{n-m-1}} \begin{pmatrix} z \\ m \end{pmatrix} \right) . \end{aligned} \tag{C.13}$$

where the index on the second sum has been shifted to bring it to the standard form.

(2) $\ln G_n(1) = 0$ follows from the vanishing integrand in the limit $z \to 0$.

(3)) $\partial_z^{n+1} \ln G_n(z+1) \geq 0$ follows from the absolute positivity of the integrand:

$$\int_0^\infty dt \, \frac{e^{-t}}{t} \frac{-(-t)^n + 1}{(1-e^{-t})^n} \geq 0 \quad . \tag{C.14}$$

(4) Setting $n \to 0$ reduces to an integral representation of $\ln(z+1)$ and $n \to 1$ to a standard representation of the logarithm of the Γ function, thereby completing the proof by induction on n.

Corollary

Using the integral representation of G_n it is possible to derive the following tower of relations among the logarithmic derivatives $\psi_n(z+1) := \partial_z \ln G_n(z+1)$:

$$\begin{aligned} & \psi_2(z+1) - z\psi_1(z+1) = C_2 - \frac{z}{2} \\ & \psi_3(z+1) - z\psi_2(z+1) + \frac{z(z+1)}{2!}\psi_1(z+1) = C_3 + \frac{3}{4}z + \frac{1}{4}z^2 \\ & \psi_4(z+1) - z\psi_3(z+1) + \frac{z(z+1)}{2!}\psi_2(z+1) - \frac{z(z+1)(z+2)}{3!}\psi_1(z+1) = \\ & \qquad\qquad C_4 - \frac{11}{18}z - \frac{1}{3}z^2 - \frac{1}{18}z^3 \end{aligned} \tag{C.15}$$

where $C_2 = -\zeta'(0) - \frac{1}{2} = \frac{1}{2}ln(2\pi) - \frac{1}{2}$, $C_3 = -.3332237448...$, $C_4 = .2786248832...$, etc.

Substituting lower order relations in the higher order ones, and integrating with respect to z, further relations are found:

$$\begin{aligned}
&\int^a z\psi_1(z+1)dz = \ln G_2(z+1) - aC_2 + \frac{1}{4}a^2 \\
&\int^a \frac{1}{2!}\,z\,(z-1)\,\psi_1(z+1)dz = \ln G_3(z+1) + \frac{1}{12}a^3 - \left(\frac{1}{2}C_2 + \frac{3}{8}\right)a^2 - aC_3 \\
&\int^a \frac{1}{3!}\,z\,(z-1)\,(z-2)\,\psi_1 dz = \ln G_4(z+1) + \frac{1}{72}a^4 - \left(\frac{1}{6}C_2 + \frac{2}{9}\right)a^3 \\
&\qquad - \left(\frac{1}{2}C_3 - \frac{11}{36} - \frac{1}{4}C_2\right)a^2 - C_4 a
\end{aligned} \tag{C.16}$$

Finally, the integrals in eqs. (C.16) may be rewritten as integral moments of the ψ function, as follows:

$$\int^a z^n\psi(z+1)dz = \begin{cases} n=0: & \ln G_1(a+1) \\ n=1: & \ln G_2(a+1) - aC_2 + \frac{1}{4}a^2 \\ n=2: & \frac{1}{6}a^3 + \left(-\frac{1}{2} - C_2\right)a^2 + (-C_2 - 2\,C_3)\,a + 2\,\ln G_3(a+1) + \\ & \quad \ln G_2(a+1) \\ n=3: & \frac{1}{12}a^4 + \left(-C_2 - \frac{5}{6}\right)a^3 + \left(-\frac{1}{6} - \frac{3}{2}C_2 - 3\,C_3\right)a^2 + \\ & \quad (-6\,C_3 - C_2 - 6\,C_4)\,a + 6\,\ln G_4(a+1) + \\ & \quad 6\,\ln G_3(a+1) + \ln G_2(a+1) \end{cases} \tag{C.17}$$

Bibliography

[1] C.P. Burgess, M.T. Grisaru, M. Kamela, M.E. Knutt-Wehlau, P. Page, F. Quevedo and M. Zebarjad, "Space-time duality and superduality," Nucl. Phys. **B542**, 195 (1999) [hep-th/9809085].

[2] M. Kamela and C.P. Burgess, "Massive scalar effective actions on Anti-de Sitter space-time," Can. J. Phys. **77**, 85 (1999) [hep-th/9808107].

[3] N.D. Birrell and P.C. Davies, "Quantum Fields In Curved Space," *Cambridge, Uk: Univ. Pr. (1982) 340p*; S.A. Fulling, "Aspects Of Quantum Field Theory In Curved Space-Time," *Cambrigde, UK: Univ. Pr. (1989) 315 P. (London Mathematical Society Student Texts, 17)*; R.M. Wald, "Quantum field theory in curved space-time and black hole thermodynamics," *Chicago, USA: Univ. Pr. (1994) 205 p*; I.L. Buchbinder, S.D. Odintsov and I.L. Shapiro, "Effective action in quantum gravity," *Bristol, UK: IOP (1992) 413 p.*

[4] S. Weinberg, "The Quantum theory of fields. Vol. 1: Foundations," *Cambridge, UK: Univ. Pr. (1995) 609 p.*

[5] T.H. Buscher, "Path Integral Derivation Of Quantum Duality In Nonlinear Sigma Models," Phys. Lett. **201B**, 466 (1988).

[6] C.P. Burgess and F. Quevedo, "Bosonization as duality," Nucl. Phys. **B421**, 373 (1994) [hep-th/9401105]; C.P. Burgess and F. Quevedo, "NonAbelian bosonization as duality," Phys. Lett. **B329**, 457 (1994) [hep-th/9403173].

[7] R.P. Feynman, “Quantum Theory Of Gravitation,” Acta Phys. Polon. **24**, 697 (1963); L.D. Faddeev and V.N. Popov, “Feynman Diagrams For The Yang-Mills Field,” Phys. Lett. **25B**, 29 (1967); B.S. Dewitt, “Quantum Theory Of Gravity. II. The Manifestly Covariant Theory,” Phys. Rev. **162**, 1195 (1967).

[8] A.M. Polyakov, “Quantum Geometry Of Bosonic Strings,” Phys. Lett. **103B**, 207 (1981); “Quantum Geometry Of Fermionic Strings,” Phys. Lett. **103B**, 211 (1981).

[9] A.M. Polyakov, “Gauge Fields And Strings,” *Chur, Switzerland: Harwood (1987) 301 p. (Contemporary Concepts in Physics, 3).*

[10] J. Distler and H. Kawai, “Conformal Field Theory And 2-D Quantum Gravity Or Who's Afraid Of Joseph Liouville?,” Nucl. Phys. **B321**, 509 (1989).

[11] F. David, “Conformal Field Theories Coupled To 2-D Gravity In The Conformal Gauge,” Mod. Phys. Lett. **A3**, 1651 (1988).

[12] E. D'Hoker, “Equivalence of Liouville theory and 2-D quantum gravity,” Mod. Phys. Lett. **A6**, 745 (1991).

[13] H. Leutwyler, “Gravitational Anomalies: A Soluble Two-Dimensional Model,” Phys. Lett. **153B**, 65 (1985). Erratum-ibid.155B:469,1985.

[14] see for instance P.G. Freund, “Introduction To Supersymmetry,” *Cambridge, Uk: Univ. Pr. (1986) 152 P. (Cambridge Monographs On Mathematical Physics).*

[15] M. Nakahara, “Geometry, Topology And Physics,” *Bristol, UK: Hilger (1990) 505 p. (Graduate student series in physics).*

[16] U. Lindstrom, N.K. Nielsen, M. Rocek and P. van Nieuwenhuizen, “The Supersymmetric Regularized Path Integral Measure In X Space,” Phys. Rev. **D37**, 3588 (1988).

[17] L. Brink and J.H. Schwarz, "Local Complex Supersymmetry In Two-Dimensions," Nucl. Phys. **B121**, 285 (1977).

[18] K. Higashijima, T. Uematsu, and Y-Z Yu, Phys. Lett. **139B**,161 (1984).

[19] E.S. Fradkin and A.A. Tseytlin, "Quantization Of Two-Dimensional Supergravity And Critical Dimensions For String Models," Phys. Lett. **106B**, 63 (1981).

[20] P.S. Howe and G. Papadopoulos, "N=2, D = 2 Supergeometry," Class. Quant. Grav. **4**, 11 (1987).

[21] S.J. Gates, M.T. Grisaru and M.E. Wehlau, "A Study of general 2-D, N=2 matter coupled to supergravity in superspace," Nucl. Phys. **B460**, 579 (1996) [hep-th/9509021].

[22] S.J. Gates, M.T. Grisaru, M. Rocek and W. Siegel, "Superspace Or One Thousand And One Lessons In Supersymmetry," *Reading, Usa: Benjamin/cummings (1983) 548 P. (Frontiers In Physics, 58).*

[23] B. DeWitt, "Supermanifolds," *Cambridge, UK: Univ. Pr. (1992) 407 p. (Cambridge mongraphs on mathematical physics). (2nd ed.),.*

[24] P. West, "Introduction To Supersymmetry And Supergravity," *Singapore, Singapore: World Scientific (1990) 425 p.*

[25] M.T. Grisaru and M.E. Wehlau, "Prepotentials for (2,2) supergravity," Int. J. Mod. Phys. **A10**, 753 (1995) [hep-th/9409043]; "Superspace measures, invariant actions, and component projection formulae for (2,2) supergravity," Nucl. Phys. **B457**, 219 (1995) [hep-th/9508139].

[26] M. Grisaru and M. Knutt, work in progress (private communication).

[27] J. Maldacena, "The Large N limit of superconformal field theories and supergravity," Adv. Theor. Math. Phys. **2**, 231 (1998) [hep-th/9711200].

[28] see for instance S. Weinberg, "Gravitation and Cosmology", *New York ; Toronto : Wiley, (1972).*

[29] B.S. de Witt, "Dynamical Theory of Groups and Fields", *New York : Gordon and Breach, (1965).*

[30] P. Candelas and D.J. Raine, "Feynman Propagator In Curved Space-Time," Phys. Rev. **D15**, 1494 (1977).

[31] K. Fujikawa, "Path Integral Measure For Gravitational Interactions," Nucl. Phys. **B226**, 437 (1983).

[32] K. Fujikawa, U. Lindstrom, N.K. Nielsen, M. Rocek and P. van Nieuwenhuizen, "The Regularized Brst Coordinate Invariant Measure," Phys. Rev. **D37**, 391 (1988).

[33] P. Candelas and D.J. Raine, "General Relativistic Quantum Field Theory - An Exactly Soluble Model," Phys. Rev. **D12**, 965 (1975).

[34] B. Allen and T. Jacobson, "Vector Two Point Functions In Maximally Symmetric Spaces," Commun. Math. Phys. **103**, 669 (1986).

[35] S.J. Avis, C.J. Isham and D. Storey, "Quantum Field Theory In Anti-De Sitter Space-Time," Phys. Rev. **D18**, 3565 (1978).

[36] P. Breitenlohner and D.Z. Freedman, "Stability In Gauged Extended Supergravity," Ann. Phys. **144**, 249 (1982).

[37] C.P. Burgess and C.A. Lutken, Phys. Lett. **153B** (1985) 137.

[38] S.W. Hawking, "Zeta Function Regularization Of Path Integrals In Curved Space-Time," Commun. Math. Phys. **55**, 133 (1977).

[39] S. Hawking and G. Ellis, "The large scale structure of space-time", *Cambrigde Unioversity Press, (1972).*

[40] E.T. Whittaker and G.N. Watson, "Modern Analysis, Part II" 4th ed., *Cambridge University Press, (1927), part II (1934).*

[41] M. Vignéras, Asterisque **61**, 235 (1979).

[42] R. Camporesi, "Zeta function regularization of one loop effective potentials in anti-de Sitter space-time," Phys. Rev. **D43**, 3958 (1991).

[43] Symbollic Math Package: Waterloo Maple 5, Version 4.

[44] B. Allen, "Phase Transitions In De Sitter Space," Nucl. Phys. **B226**, 228 (1983).

[45] R. Camporesi and A. Higuchi, "Arbitrary spin effective potentials in anti-de Sitter space-time," Phys. Rev. **D47**, 3339 (1993).

[46] J. Cruz, "Hidden conformal symmetry of a massive scalar field in two-dimensional AdS," Class. Quant. Grav. **16**, L23 (1999), [hep-th/9806145].

[47] E. D'Hoker and R. Jackiw, "Liouville Field Theory," Phys. Rev. **D26**, 3517 (1982).

[48] M. Henneaux and C. Teitelboim, "Quantization of gauge systems," *Princeton, USA: Univ. Pr. (1992) 520 p.*

[49] A. Smailagic and E. Spallucci, "Space-time duality and two-dimensional gauge field theory," [hep-th/9812108].

[50] M.J. Duff, "Ultraviolet Divergences In Extended Supergravity," *Lectures given at Spring School on Supergravity, Trieste, Italy, Apr 22 - May 6, 1981.*

[51] C. Misner, K. Thorne, and J. Wheeler, "Gravitation", *San Francisco: Freeman and Co., (1973).*

[52] S. Weinberg, "Covariant Path Integral Approach To String Theory," *Lectures given at 3rd Jerusalem Winter School of Theoretical Physics, Jerusalem, Israel, Dec 30, 1986 - Jan 8, 1987.*

[53] Private communication with my thesis advisor.

[54] K. Fujikawa, "On The Path Integral Of Relativistic Strings," Phys. Rev. **D25**, 2584 (1982).

[55] D.R. Karakhanian, R.P. Manvelian and R.L. Mkrtchian, "Area preserving structure of 2-d gravity," Phys. Lett. **B329**, 185 (1994) [hep-th/9401031]; R. Jackiw, "Another view on massless matter - gravity fields in two-dimensions," [hep-th/9501016]; G. Amelino-Camelia, D. Bak and D. Seminara, "Area preserving structure and anomalies in (1+1)-dimensional quantum gravity," Phys. Lett. **B354**, 213 (1995) [hep-th/9505136]; H. La, "Area preserving diffeomorphisms and 2-D gravity," [hep-th/9510147].

[56] W.A. Bardeen and B. Zumino, "Consistent And Covariant Anomalies In Gauge And Gravitational Theories," Nucl. Phys. **B244**, 421 (1984).

[57] K. Osterwalder and R. Schrader, "Axioms For Euclidean Green's Functions," Commun. Math. Phys. **31**, 83 (1973); Helv. Phys. Acta, **46**, 277 (1973); "Axioms For Euclidean Green's Functions. 2," Commun. Math. Phys. **42**, 281 (1975).

[58] E. Barnes, Quaterly Journal of Mathematics, **31**, 264 (1899).

[59] Ueno, K., Nishizawa, M., [q-alg/9605002].