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LANDAUER'S PRINCIPLE IN REPEATED INTERACTION SYSTEMS

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Contents

Résumé.

Le principe de Landauer affirme qu'il existe une borne inférieure à l'énergie requise pour changer l'état d'un petit système d'un état initial à un certain état final en intéragissant avec un réservoir thermondynamique. La situation dans laquelle cette borne est saturée, et donc le coût énergétique minimisé, pour une transformation donnée est d'un intérêt particulier. Nous étudions le principe de Landauer dans le contexte des systèmes à intéractions répétées (RIS), un classe de systèmes physiques dans laquelle un petit système d'intérêt intéragit avec une suite de sondes thermales. En particulier, nous démontrons que, pour les RIS, la borne de Landauer n'est généralement pas saturée dans la limite adiabatique, dans laquelle l'évolution se fait, en un certain sens, infiniement lentement. Ce résultat présente un contraste au cas d'un système intéragissant avec un seul réservoir thermondynamique. Toutefois, pour un RIS spécifique modelant le petit système et les sondes par des systèmes à 2 niveaux intéragissants dans l'approximation *rotating wave*, la borne de Landauer est saturée adiabatiquement. Dans ce travail, nous formulons et démontrons aussi un théorème adiabatique discret et non-unitaire pour usage dans les RIS.

Abstract.

Landauer's Principle states that there is a lower bound on the energy required to change the state of a small system from an initial state to a final state by interacting with a thermodynamic reservoir; of particular interest is when the bound is saturated and the minimal energy cost obtained for a given state transformation. We investigate Landauer's Principle in the context of repeated interaction systems (RIS), a class of physical systems in which a small system of interest interacts with a sequence of thermal probes. In particular, we show that for RIS, Landauer's bound is not saturated generically in the adiabatic limit, in which time evolution can be thought of as proceeding infinitely slowly, in contrast to the case of the interaction of a system and a single thermodynamic reservoir. However, for a specific RIS which models the small system and the probes as 2-level systems interacting via a dipole interaction in the rotating wave approximation, Landauer's bound is saturated adiabatically. In this work, we also formulate and prove a discrete non-unitary adiabatic theorem to use for RIS.

Acknowledgments.

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Introduction

We aim to investigate Landauer's Principle in the context of finite-dimensional *repeated interaction systems*, a class of quantum systems with several amenable properties. Briefly, repeated interaction systems consist of a system of interest interacting with a chain of probes, one probe at a time. This work is based on the preprint [HJPR15]; my coauthors Alain Joye, Yan Pautrat, and Renaud Raquépas are responsible for most of the ideas, their mathematical formulation, and their proof. In fact, this document does not prove many of the results of the important small coupling limit section of that preprint. However, this report presents more details in the proofs and hopefully provides more intuition for the ideas. We'll use finite-dimensional perturbation theory extensively, as formulated in chapters 1 and 2 of Tosio Kato's *Perturbation Theory for Linear Operators* [Kat76].

We'll first discuss the context of this work, by introducing Landauer's Principle in section 1.1, adiabatic theorems in section 1.2, and repeated interaction systems in section 1.3. With this background, we're ready to formulate Landauer's Principle for repeated interaction systems in the adiabatic limit in section 2.1. With definitions in hand, we will proceed to our first attempts to look for the saturation of Landauer's bound in section 2.2. Our partial success there will lead us to develop two tools: a discrete non-unitary adiabatic theorem (chapter 3), and a perturbation result for the relative entropy function (chapter 4). Both seem to be novel, although the methods and tools used to prove them are standard. Empowered by these results, we will develop a criterion to check for the adiabatic saturation of Landauer's bound at each step of a repeated interaction system in section 5.1, and return to our initial explorations in section 5.3. We will also discuss the small coupling limit in section 5.4, and apply those results to an example system in section 6.1.

1
The setting

In order to describe Landauer's principle for repeated interaction systems, we need to understand both. We will have reason to suspect that Landauer's principle in an adiabatically evolving system is of particular interest, so we will introduce the adiabatic limit here as well. These well-trod grounds are included here to establish notation, highlight relevant features, and provide a unified exposition.

1.1 Landauer's Principle

Landauer's principle states that there is a minimal energetic cost for a state transformation $\rho^{\mathrm{i}} \to \rho^{\mathrm{f}}$ on a system $\mathcal{S}$ via the action of a thermal reservoir $\mathcal{E}$ at temperature $(k_B\beta)^{-1}$. In particular, if $\Delta S_{\mathcal{S}}$ is the change of entropy of the system $\mathcal{S}$, and $\Delta Q_{\mathcal{E}}$ is the change in energy of the reservoir $\mathcal{E}$, then

See [Lan61] for Rolf Landauer's original description.

$k_B \approx 1.38 \times 10^{-23}$ Joules per Kelvin is Boltzmann's constant.

$$\Delta Q_{\mathcal{E}} \geq \beta^{-1} \Delta S_{\mathcal{S}}. \tag{1.1}$$

This principle has generated interest since its inception in 1961; see [RW14, Section I] for a recent summary. First, the bound has allusions to practicality: perhaps the energy efficiency of our computers will be limited. For changing the state of a classical or quantum bit however, the bound is at most

$$\Delta Q_{\mathcal{E}} \geq k_B \cdot T \log 2 \approx (9.6 \times 10^{-24} J/K) \cdot T$$

which is extremely small for reasonable temperatures T; yet, modern processors are within several orders of magnitude of this limit, as shown to the right in fig. 1.1. Moreover, in 1973 Bennett showed that any Turing machine program may be implemented in a reversible manner [Ben73], so that $\Delta S_{\mathcal{S}} = 0$. Reversible computing is an area of considerable practical interest and continuing theoretical work [Vos10; Jea15].

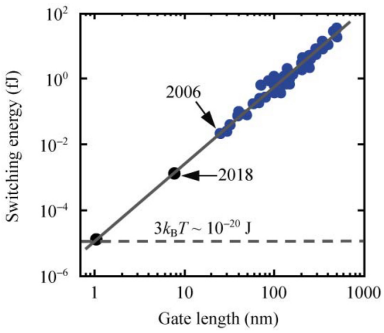

Figure 1.1: Energy cost of changing state for modern silicon transistors, as compared to a theoretical minimum for classical bits encoded in electron charge at room temperature ([CZH+06]). Figure reproduced from [Pop10, Figure 1(a)].

More fundamentally, Landauer's bound is a direct relationship between energy and information (entropy). In fact, Landauer's principle follows from the *entropy balance equation*

$$\Delta S_{\mathcal{S}} + \sigma = \beta \Delta Q_{\mathcal{E}} \tag{1.2}$$

where σ is the *entropy production*.

From now on, we will use natural units so that $k_B = \hbar = 1$, and describe temperature in terms of β, where $\beta^{-1} = T$.

We will define σ and prove eq. (1.2), in a finite dimensional quantum unitary setup, following [RW14] and [JP14, Section 2]. We assume the system $\mathcal{S}$ is described by a finite dimensional Hilbert space $\mathcal{H}_{\mathcal{S}}$, with self-adjoint Hamiltonian $h_{\mathcal{S}}$. The initial state on the system is given by a density matrix[1] ρ^{i}. Likewise, we assume the environment is described by a finite dimensional Hilbert space $\mathcal{H}_{\mathcal{E}}$ with self-adjoint Hamiltonian $h_{\mathcal{E}}$, and initial state

[1] non-negative trace-one operator on $\mathcal{H}_{\mathcal{S}}$

$$\xi^{\mathrm{i}} = \frac{\exp(-\beta h_{\mathcal{E}})}{\mathrm{Tr}(\exp(-\beta h_{\mathcal{E}}))} \tag{1.3}$$

the Gibbs state[2] at temperature β^{-1}. The system and environment start uncoupled, so the joint initial state is $\rho^{\mathrm{i}} \otimes \xi^{\mathrm{i}}$. The evolution of the joint system is given by a unitary operator $U \in \mathcal{B}(\mathcal{H}_{\mathcal{S}} \otimes \mathcal{H}_{\mathcal{E}})$, leading to the final joint state $U\rho^{\mathrm{i}} \otimes \xi^{\mathrm{i}} U^*$. We decouple the systems, yielding

[2] Gibbs states on $\mathcal{E}$ are invariant under the free dynamics $h_{\mathcal{E}}$; in this finite dimensional context, they are uniquely so. They thus have the interpretation of thermal equilibrium states.

$$\rho^{\mathrm{f}} = \mathrm{Tr}_{\mathcal{E}}(U\rho^{\mathrm{i}} \otimes \xi^{\mathrm{i}} U^*), \qquad \xi^{\mathrm{f}} = \mathrm{Tr}_{\mathcal{S}}(U\rho^{\mathrm{i}} \otimes \xi^{\mathrm{i}} U^*)$$

as the final state on the system, environment, respectively.

We identify two quantities of interest during this process: $\Delta S_{\mathcal{S}}$, the change of entropy of the system of interest, and $\Delta Q_{\mathcal{E}}$, the change of energy of the environment, defined as

Note the sign convention.

$$\Delta S_{\mathcal{S}} := S(\rho^{\mathrm{i}}) - S(\rho^{\mathrm{f}}), \qquad \Delta Q_{\mathcal{E}} := \mathrm{Tr}(h_{\mathcal{E}}\xi^{\mathrm{f}}) - \mathrm{Tr}(h_{\mathcal{E}}\xi^{\mathrm{i}}),$$

where $S(\rho) := -\mathrm{Tr}\,\rho\log\rho$ is the von Neumann entropy. Recall the relative entropy $S(\eta|\nu) = \mathrm{Tr}(\eta\log\eta - \log\nu))$ of two faithful states η and ν has $S(\eta|\nu) \geq 0$ with equality if and only if $\eta = \nu$. With this function, we define the entropy production

See [JOPP12, Sections 2.5–2.6] for a review of entropy functions in finite dimensional quantum mechanics.

$$\sigma := S(U\rho^{\mathrm{i}} \otimes \xi^{\mathrm{i}} U^* | \rho^{\mathrm{f}} \otimes \xi^{\mathrm{i}}). \tag{1.4}$$

We may proceed to derive eq. (1.2), simply by expanding σ:

$$\begin{aligned}
\sigma &= -S(U\rho^{\mathrm{i}} \otimes \xi^{\mathrm{i}} U^*) - \mathrm{Tr}\left(U\rho^{\mathrm{i}} \otimes \xi^{\mathrm{i}} U^* \left(\log \rho^{\mathrm{f}} \otimes \mathrm{Id}\right)\right) - \mathrm{Tr}\left(U\rho^{\mathrm{i}} \otimes \xi^{\mathrm{i}} U^* \left(\mathrm{Id} \otimes \log \xi^{\mathrm{i}}\right)\right) \\
&= -S(\rho^{\mathrm{i}} \otimes \xi^{\mathrm{i}}) + S(\rho^{\mathrm{f}}) - \mathrm{Tr}(\xi^{\mathrm{f}} \log \xi^{\mathrm{i}}) \\
&= -S(\rho^{\mathrm{i}}) - S(\xi^{\mathrm{i}}) + S(\rho^{\mathrm{f}}) - \mathrm{Tr}(\xi^{\mathrm{f}} \log \xi^{\mathrm{i}}) \\
&= -\Delta S_{\mathcal{S}} + \beta \Delta Q_{\mathcal{E}}.
\end{aligned}$$

For a more detailed derivation, consult appendix B.

We may interpret eq. (1.2) as a microscopic Clausius formulation of the Second Law of Thermodynamics [BHN+15]. More specifically, we may interpret $\beta\Delta Q_{\mathcal{E}} = \int_{\mathrm{i}}^{\mathrm{f}} \frac{\mathrm{d}Q_{\mathcal{E}}}{T} = \Delta S_{\mathcal{E}}^{\text{Clausius}}$ as the Clausius entropy change of the environment. Then, with a minus sign to account for our sign convention, $\Delta S_{\mathcal{S}}^{\text{Clausius}} = -\Delta S_{\mathcal{S}}$, and the Second Law is

$$\Delta S_{\mathcal{E}}^{\text{Clausius}} + \Delta S_{\mathcal{S}}^{\text{Clausius}} = \text{entropy production} \geq 0.$$

In this language then, σ serves as the entropy production. The classical Second Law, however, is a statement about macroscopic quantities obtained from the behavior of $\gtrsim 10^{23}$ particles. Within the theory of quantum mechanics and our assumptions, however, the balance equation eq. (1.2) is exact on a microscopic level.

We are interested in the case of equality: when is $\sigma = 0$? In fact, in this finite dimensional framework, only in the case $\Delta S_{\mathcal{S}} = \Delta Q_{\mathcal{E}} = 0$. In nontrivial cases, tighter bounds exist [RW14].

1.2 The adiabatic limit

Let us consider a finite dimensional quantum system described by a Hilbert space $\mathcal{H}$ with $\dim \mathcal{H} < \infty$, and for each time $s \in [0,1]$, a self-adjoint Hamiltonian $h(s) \in \mathcal{B}(\mathcal{H})$. Time evolution is governed by the solution to the Schrödinger equation

$$i\frac{\mathrm{d}}{\mathrm{d}s}U(s) = h(s)U(s),\ s \in [0,1],\ \text{with } U(0) = \mathrm{Id}.$$

The *adiabatic limit* concerns the evolution of the rescaled Schrödinger equation

$$i\frac{\mathrm{d}}{\mathrm{d}t}U_T(t) = h(t/T)U_T(t),\ t \in [0,T],\ \text{with } U_T(0) = \mathrm{Id} \tag{1.5}$$

in the limit $T \to \infty$. The *adiabatic parameter* T corresponds to a physical time scale over which the process takes place; the limit $T \to \infty$ corresponds to the process being "infinitely slow", or quasi static.

Adiabatic theorems generally correspond to assumptions which can be made on $h(s)$ and $\mathcal{H}$ to guarantee properties of the solution $U_T(t)$. In particular, often assumptions are chosen so that $U_T(t)$ approximately transports states starting in an eigenspace of $h(0)$ to the states in the corresponding eigenspace of $h(t/T)$, as demonstrated in the following result due to Tosio Kato [Kat50].

Theorem 1.1 (Kato, 1950)**.** *Let $e(s)$ be an eigenvalue of $h(s)$ separated from the rest of spectrum of $h(s)$ by a gap. Let $P(s)$ be its projection onto the associated eigenspace. Assume that $e(s)$ and $P(s)$ are continuous functions of $s \in [0,1]$, and that $\frac{\mathrm{d}P}{\mathrm{d}s}$ and $\frac{\mathrm{d}^2P}{\mathrm{d}^2s}$ are piecewise continuous. Then there exists a unitary operators $W(t)$ for $t \in [0,T]$ such that*

$$W(t)P(0) = P(t/T)W(t) \tag{1.6}$$

$$\Big(U_T(t) - \exp\big(-iT\int_0^{t/T} e(s)\,\mathrm{d}s\big)W(t)\Big)P(0) = O(T^{-1}),\ \textit{uniformly in } t \in [0,T]. \tag{1.7}$$

The condition (1.6) is an *intertwining* relation; if a state η begins in the $P(0)$ subspace (that is, $P(0)\eta = \eta$), then

$$W(t)\eta = W(t)P(0)\eta = P(t/T)W(t)\eta$$

lies in the $P(t/T)$ subspace at time t. Equation (1.7) then has the interpretation that $U_T(t)$ acting on the $P(0)$ subspace is approximated up to an error of $O(1/T)$ by the operator $W(t)$ when augmented by the *dynamical phase* factor $\exp\big(-iT\int_0^{t/T} e(s)\,\mathrm{d}s\big)$. This makes precise

the notion that $U_T(t)$ approximately transports states from an eigenspace of $h(0)$ to the corresponding eigenspace of $h(t/T)$.

In the context of this work, a natural application of an adiabatic theorem is to the setup of Landauer's Principle in section 1.1. Given a self-adjoint time-dependent Hamiltonian

$$[0,1] \ni s \mapsto h(s) \in \mathcal{B}(\mathcal{H}_\mathcal{S} \otimes \mathcal{H}_\mathcal{E})$$

for the system and environment together, we may define $U_T(t)$ the adiabatic time evolution operator as the solution to eq. (1.5). By choosing the time evolution unitary U of section 1.1 to be $U_T = U_T(1)$, we obtain Landauer's Principle in an adiabatic setup: Just as before, we obtain the change in entropy of the system, $\Delta S_{\mathcal{S},T}$, change in energy of the environment, $\Delta Q_{\mathcal{E},T}$, and entropy production σ_T, related by the balance equation

$$\Delta S_{\mathcal{S},T} + \sigma_T = \beta \Delta Q_{\mathcal{E},T},$$

where we have explicitly written the T-dependence inherited from U_T.

In [JP14], the authors formulate Landauer's Principle in a setup where the environment is described by an infinite-dimensional reservoir. They derive a balance equation analogous to eq. (1.2). With an ergodicity assumption to ensure the system and environments interaction mixes thoroughly enough and the Avron-Elgart adiabatic theorem, they show that $\sigma_T \to 0$ as $T \to \infty$. In this case then, Landauer's bound is saturated in the adiabatic limit.

1.3 Repeated interaction systems (RIS)

A repeated interaction system (RIS) consists of a system of interest $\mathcal{S}$ which is coupled to a sequence (or *chain*) of probes $\{\mathcal{E}_k\}_{k=1}^\infty$. The system $\mathcal{S}$ interacts with each probe, one at a time, for some duration τ. Each probe is discarded after it interacts with the system; mathematically, this is modeled by tracing out the probe. This is a type of open quantum system which has the advantage of a simple mathematical model while being relevant to experiments in quantum optics. For a recent review, see [BJM14].

In an RIS, the system $\mathcal{S}$ has an associated Hilbert space $\mathcal{H}_\mathcal{S}$, and self-adjoint Hamiltonian $h_\mathcal{S} \in \mathcal{B}(\mathcal{H}_\mathcal{S})$. Each probe $\mathcal{E}_k$ is described by a Hilbert space $\mathcal{H}_{\mathcal{E}_k}$ with self-adjoint Hamiltonian $h_{\mathcal{E}_k} \in \mathcal{B}(\mathcal{H}_{\mathcal{E}_k})$. We will assume each probe's Hilbert space is identical: $\mathcal{H}_{\mathcal{E}_k} \equiv \mathcal{H}_\mathcal{E}$. Additionally, for this work we assume $\dim \mathcal{H}_\mathcal{S} < \infty$ and $\dim \mathcal{H}_\mathcal{E} < \infty$. We specify the state of the kth probe

ξ_k^{i} to be the Gibbs state at inverse temperature β_k:

$$\xi_k^{\mathrm{i}} := \frac{\exp(-\beta_k h_{\mathcal{E}_k})}{\mathrm{Tr}(\exp(-\beta_k h_{\mathcal{E}_k}))}. \tag{1.8}$$

The system $\mathcal{S}$ begins in an initial state ρ^{i}. Then the system couples to the first probe $\mathcal{E}_0$ yielding an initial joint state $\rho^{\mathrm{i}} \otimes \mathcal{E}_0^{\mathrm{i}}$. The coupling is described by a self-adjoint potential $v_0 \in \mathcal{B}(\mathcal{H}_{\mathcal{S}} \otimes \mathcal{H}_{\mathcal{E}})$ and coupling constant $\lambda_0 > 0$. The joint state is evolved to time τ by

$$U_0 = \exp(-i\tau_0(h_{\mathcal{S}} \otimes \mathrm{Id} + \mathrm{Id} \otimes h_{\mathcal{E}_0} + \lambda_0 v_0)).$$

Then the joint state of $\mathcal{S} + \mathcal{E}_0$ at the end of their interaction is $U_0(\rho^{\mathrm{i}} \otimes \xi_0^{\mathrm{i}})U_0^*$. We trace out $\mathcal{E}_0$ to obtain the final state on the system

$$\rho_1 = \mathrm{Tr}_{\mathcal{H}_{\mathcal{E}}}(U_0(\rho^{\mathrm{i}} \otimes \xi_0^{\mathrm{i}})U_0^*).$$

This state is then the *initial* state of the system for its interaction with $\mathcal{E}_1$. Now, assume the system is in state ρ_{k-1} after interacting with the first $k-1$ probes. Then the kth probe is in the state ξ_k^{i}, and the joint step at the start of the kth interaction is $\rho_{k-1} \otimes \xi_k^{\mathrm{i}}$. Time evolution to the end of the step is governed by

$$U_k = \exp\left(-i\tau_k(h_{\mathcal{S}} \otimes \mathrm{Id} + \mathrm{Id} \otimes h_{\mathcal{E}_k} + \lambda_k v_k)\right) \tag{1.9}$$

for $\lambda_k > 0$ and $v_k \in \mathcal{B}(\mathcal{H}_{\mathcal{S}} \otimes \mathcal{H}_{\mathcal{E}})$. This yields the joint state $U_k \rho_{k-1} \otimes \xi_k^{\mathrm{i}} U_k^*$ and hence the state of the system after interacting with the kth probe is $\rho_k = \mathrm{Tr}_{\mathcal{H}_{\mathcal{E}}}(U_k \rho_{k-1} \otimes \xi_k^{\mathrm{i}} U_k^*)$. This process is depicted in fig. 1.2.

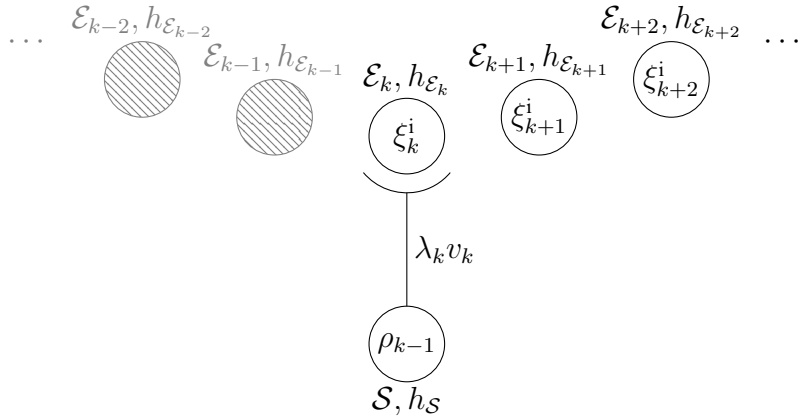

Figure 1.2: A schematic indicating a repeated interaction system at the beginning of the kth step, at time $\sum_{n=1}^{k-1} \tau_n$.

This process defines a family of maps $\{\mathcal{L}_k\}_k$ on $\mathcal{I}_1(\mathcal{H}_\mathcal{S})$, the trace-class operators on $\mathcal{H}_\mathcal{S}$. Since $\dim \mathcal{H}_\mathcal{S} < \infty$, $\mathcal{I}_1(\mathcal{H}_\mathcal{S}) \cong \mathcal{B}(\mathcal{H}_\mathcal{S})$, i.e. every operator is trace class. The notation is used here to convey that spiritually, $\mathcal{L}_k$ acts on states, not observables; additionally, we will equip $\mathcal{I}_1(\mathcal{H}_\mathcal{S})$ with the trace norm, and $\mathcal{B}(\mathcal{H}_\mathcal{S})$ with the uniform norm. We define

$$\begin{array}{rccc} \mathcal{L}_k : & \mathcal{I}_1(\mathcal{H}_\mathcal{S}) & \to & \mathcal{I}_1(\mathcal{H}_\mathcal{S}) \\ & \eta & \mapsto & \mathrm{Tr}_\mathcal{E}\left(U_k(\eta \otimes \xi_k^{\mathrm{i}})U_k^*\right). \end{array} \tag{1.10}$$

These maps are called the *reduced dynamics*. Then the state of the system after step k is

$$\rho_k = \mathcal{L}_k\mathcal{L}_{k-1}\cdots\mathcal{L}_1\rho^{\mathrm{i}}.$$

This is a Markovian form for the sequence of states of the system $(\rho_k)_k$. Thus, when considering the state of the system at times $(\sum_{n=1}^k \tau_n)_k$ the exact state of the system is described by a Markovian process, which is determined by $h_\mathcal{S}, h_{\mathcal{E}_k}, v_k, \lambda_k, \tau_k$, and the initial state ξ_k^{i} of the probe.

Often, open quantum systems, which consist of a system of interest interacting with an environment (in this case, the chain of probes), can only be approximated by Markovian dynamics, and are only precisely described by Hamiltonian dynamics. Here, the two approaches coincide at the times of interest, namely at the end of each step.

Remark. We will assume without loss of generality that the coupling $\lambda_k \equiv \lambda > 0$ is constant. We will consider the small coupling limit (which involves $\lambda \to 0$) in section 5.4. Additionally, we will assume the interaction durations $\tau_k \equiv \tau > 0$ are constant.

The maps $\mathcal{L}_k$ are completely positive and trace preserving (CPTP). Let us equip $\mathcal{I}_1(\mathcal{H}_\mathcal{S})$ with the trace norm $\|\eta\|_1 = \mathrm{Tr}\,|\eta| = \mathrm{Tr}\,\sqrt{\eta^*\eta}$. Denote $\|\mathcal{L}_k\| = \sup_{\|\eta\|_1=1}\|\mathcal{L}_k(\eta)\|_1$ its uniform norm as an operator on $\mathcal{I}_1(\mathcal{H}_\mathcal{S})$. Then $\mathcal{L}_k$ is a contraction. In fact, we may prove this only using that the $\mathcal{L}_k$ are CPTP.

Proposition 1.2. *Let $\mathcal{H}$ be a Hilbert space and $L : (\mathcal{I}_1(\mathcal{H}), \|\cdot\|_1) \to (\mathcal{I}_1(\mathcal{H}), \|\cdot\|_1)$ be a CPTP map. Then $\|L\| = \sup_{\|\eta\|_1\le 1}\|L(\eta)\|_1 = 1$, i.e. L is a contraction.*

Remark. For the initiated: this proof is simply an application of the Russo-Dye theorem.

Proof. Recall that the topological dual of trace class operators is the set of bounded operators: $(\mathcal{I}_1(\mathcal{H}), \|\cdot\|_1)^* = (\mathcal{B}(\mathcal{H}), \|\cdot\|)$ [RS81, Theorem VI.26], where $\|A\| = \sup_{\psi\in\mathcal{H},\,\|\psi\|\le 1}\|A\psi\|$. In this duality, we identify $A \in \mathcal{B}(\mathcal{H})$ with the map $\eta \mapsto \mathrm{Tr}(\eta A)$. Then we have the Banach space adjoint of L,

$$L^* : \quad (\mathcal{I}_1(\mathcal{H}), \|\cdot\|_1)^* \to (\mathcal{I}_1(\mathcal{H}), \|\cdot\|_1)^*$$

so that for all $A \in (\mathcal{I}_1(\mathcal{H}), \|\cdot\|_1)^*$ and $\eta \in (\mathcal{I}_1(\mathcal{H}), \|\cdot\|_1)$, we have

$$\mathrm{Tr}(AL(\eta)) = A(L(\eta)) = L^*(A)(\eta) = \mathrm{Tr}(L^*(A)\eta)$$

where on the far left and far right we have recalled our identification $(\mathcal{I}_1(\mathcal{H}), \|\cdot\|_1)^* = (\mathcal{B}(\mathcal{H}), \|\cdot\|)$. If we take $A = \mathrm{Id}$, then using that L is trace preserving, we have

$$\mathrm{Tr}(\eta) = \mathrm{Tr}(L(\eta)) = \mathrm{Tr}(L^*(\mathrm{Id})\eta).$$

Since the map $A \mapsto \mathrm{Tr}(A\cdot)$ is an isometric isomorphism of $\mathcal{B}(\mathcal{H}) \to (\mathcal{I}_1(\mathcal{H}), \|\cdot\|_1)^*$ and Id satisfies $\mathrm{Tr}(\mathrm{Id}\cdot\eta) = \mathrm{Tr}(\eta)$ for all $\eta \in \mathcal{I}_1(\mathcal{H})$, we must have that $L^*(\mathrm{Id}) = \mathrm{Id}$, and thus L^* is unital. This map is also completely positive, as we can see from the following argument.

Let $n \in \mathbb{N}$. If $A \in \mathcal{B}(\mathcal{H} \otimes \mathbb{C}^n)$ has $A \geq 0$, then for all $\eta \in \mathcal{I}_1(\mathcal{H} \otimes \mathbb{C}^n)$,

$$(L^* \otimes \mathrm{Id}_n)(A)(\eta) = \mathrm{Tr}((L^* \otimes \mathrm{Id}_n)(A)\eta) = \mathrm{Tr}(A(L \otimes \mathrm{Id}_n)(\eta)).$$

Let us choose $\eta = |\psi\rangle\langle\psi| = \langle\psi, \cdot\rangle\,\psi$ for $\psi \in \mathcal{H} \otimes \mathbb{C}^n$. Then η is a rank one projection and positive semi-definite. Then

$$\begin{aligned}\langle\psi, (L^* \otimes \mathrm{Id})(A)\psi\rangle &= \mathrm{Tr}((L^* \otimes \mathrm{Id}_n)(A)\eta) = \mathrm{Tr}(A(L \otimes \mathrm{Id}_n)(\eta)) \\ &= \mathrm{Tr}(A^{1/2}[(L \otimes \mathrm{Id}_n)(\eta)]^{1/2}[(L \otimes \mathrm{Id}_n)(\eta)]^{1/2}A^{1/2}) \\ &= \mathrm{Tr}(([(L \otimes \mathrm{Id}_n)(\eta)]^{1/2}A^{1/2})^*[(L \otimes \mathrm{Id}_n)(\eta)]^{1/2}A^{1/2}) \geq 0\end{aligned}$$

where we have used that L is completely positive map, so that $L \otimes \mathrm{Id}_n(\eta)$ is positive semi-definite, and the fact that positive semi-definite operators admit positive semi-definite square roots. Thus, $L^* : (\mathcal{B}(\mathcal{H}), \|\cdot\|) \to (\mathcal{B}(\mathcal{H}), \|\cdot\|)$ is completely positive.

Since $(\mathcal{B}(\mathcal{H}), \|\cdot\|)$ is a C^*-algebra, we may apply the Russo-Dye theorem [RD66, Corollary 1] to conclude that because L^* is a positive unital map, we must have $\|L^*\| = 1$. But since the mapping $T \to T^*$ of operators to their Banach space adjoints is an isometric isomorphism [RS81, Theorem VI.2], we have $\|L\| = \|L^*\| = 1$.

□

Remark. This proof also shows us that $1 \in \mathrm{sp}(L^*) = \overline{\mathrm{sp}(L)}$. Thus, 1 is an eigenvalue of $\mathcal{L}_k$.

On the other hand, if we equip $\mathcal{I}_1(\mathcal{H}_\mathcal{S})$ with the Hilbert Schmidt norm $\|\cdot\|_2$ induced by the inner product $(A, B) \mapsto \mathrm{Tr}(A^*B)$, we do not in general have that $\mathcal{L}_k$ is a contraction. In fact, $\mathcal{L}_k$ being a contraction in the uniform norm induced by $\|\cdot\|_2$ is equivalent to $\mathcal{L}_k(\mathrm{Id}) = \mathrm{Id}$ [PWPR06]. Additionally, since $\mathcal{L}_k$ is positive, $\mathcal{L}_k(\eta)^* = \mathcal{L}_k(\eta^*)$ ([JOPP12, Ch. 2]) and

consequently $\mathrm{sp}(\mathcal{L}_k) = \overline{\mathrm{sp}(\mathcal{L}_k)}$, i.e., the spectrum of $\mathcal{L}_k$ is symmetric about the real axis.

1.3.1 An example

Before moving on, let us consider an example, the simplest non-trivial RIS. We will return to this example throughout this report. In this case, both the system and probes are 2-level systems, so $\mathcal{H}_\mathcal{S} = \mathcal{H}_{\mathcal{E}_k} \equiv \mathcal{H}_\mathcal{E} = \mathbb{C}^2$. We choose Hamiltonians $h_\mathcal{S} = Ea^*a$ and $h_{\mathcal{E}_k} \equiv h_\mathcal{E} = E_0 b^* b$ where a/a^*, respectively b/b^* are the annihilation/creation operators for $\mathcal{S}$, resp. $\mathcal{E}$. That is, choosing the basis (ground state, excited state), we have

$$a = b = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad a^* = b^* = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \qquad a^*a = b^*b = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

We will consider two choices of potential; in either case, we will take the potential to be the same for each interaction. The full dipole interaction is given by

$$v_{\mathrm{FD}} = \frac{1}{2}(a^* + a) \otimes (b^* + b) = \frac{1}{2}(a \otimes b + a^* \otimes b + a \otimes b^* + a^* \otimes b^*).$$

If we drop the two "counter-rotating" terms $a^* \otimes b^*$ and $a \otimes b$, we obtain the *rotating wave approximation*

$$v_{\mathrm{RW}} = \frac{1}{2}(a^* \otimes b + a \otimes b^*).$$

This is a common approximation in the regime $|E - E_0| \ll \min\{E, E_0\}$ and $\lambda \ll |E_0|$. This potential has the property that it commutes with the total number operator

$$N_{\mathrm{tot}} = a^*a \otimes \mathrm{Id} + \mathrm{Id} \otimes b^*b.$$

This can be checked by hand, although by inspection we have the physical interpretation that either a quanta of particle is created on the system and annihilated on the probe or vice-versa, so the total number of quanta is preserved.

In these examples then, the only parameter which will change from probe to probe is the inverse temperature β_k. In the rotating wave case, we may compute $\mathcal{L}_k$ (as defined in eq. (1.10)) by diagonalizing h using $[h, N_{\mathrm{tot}}] = 0$. The results of this are shown in Example 2.4 of [BJM14]. Computationally the full dipole case is much more complicated. However, using Mathematica to perform the symbolic manipulation, we may compute matrix representations of $\mathcal{L}_k$ in either

case (see [Han16] for the code). We identify $\mathcal{I}_1(\mathcal{H}_\mathcal{S}) \cong \mathrm{Mat}_{2\times 2}(\mathbb{C}) \cong \mathbb{C}^4$ via

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}$$

where the basis of the matrices is (ground state, excited state) basis from earlier. Then we obtain $\mathcal{L}_k^{\mathrm{RW}}$ ($\mathcal{L}_k$ when $v_k \equiv v_{\mathrm{RW}}$) to be

$$\begin{pmatrix} \frac{(\cos(\nu\tau)-1)\lambda^2}{2\left(1+e^{E_0\beta_k}\right)\nu^2}+1 & 0 & 0 & -\frac{e^{E_0\beta_k}\lambda^2(\cos(\nu\tau)-1)}{2\left(1+e^{E_0\beta_k}\right)\nu^2} \\ 0 & \alpha & 0 & 0 \\ 0 & 0 & \beta & 0 \\ \frac{\lambda^2\sin^2\left(\frac{1}{2}\nu\tau\right)}{\left(1+e^{E_0\beta_k}\right)\nu^2} & 0 & 0 & \frac{2\nu^2+e^{E_0\beta_k}\left(2\nu^2+\lambda^2\cos(\nu\tau)\right)}{2\left(1+e^{E_0\beta_k}\right)\nu^2} \end{pmatrix},$$

where

$$\alpha := \frac{e^{-\frac{1}{2}i(E_0+E+\nu)\tau}\left(\left(-1+e^{i\nu\tau}\right)E_0+E+e^{i\nu\tau}(\nu-E)+\nu\right)}{2\nu},$$
$$\beta := \frac{e^{-\frac{1}{2}i(-E_0-E+\nu)\tau}\left(-e^{i\nu\tau}E_0+E_0-E+e^{i\nu\tau}(E+\nu)+\nu\right)}{2\nu},$$

and $\nu = \sqrt{(E_0-E)^2+\lambda^2} > 0$. We also find in the case $v_k \equiv v_{\mathrm{FD}}$,

$$\mathcal{L}_k^{\mathrm{FD}} = \begin{pmatrix} \frac{e^{E_0\beta_k}\left(2(E_0+E)^2+\lambda^2+\lambda^2\cos(\eta\tau)\right)}{\eta^2}+\frac{2(E_0-E)^2+\lambda^2+\lambda^2\cos(\nu\tau)}{\nu^2} & 0 & 0 & \frac{\lambda\left(-\frac{2e^{-E_0\beta_k}\lambda(\cos(\eta\tau)-1)}{\eta^2}-\frac{2\lambda(\cos(\nu\tau)-1)}{\nu^2}\right)}{4\left(1+e^{-E_0\beta_k}\right)} \\ 0 & & & 0 \\ 0 & & C & 0 \\ \frac{\lambda\left(-\frac{2\lambda(\cos(\eta\tau)-1)}{\eta^2}-\frac{2e^{-E_0\beta_k}\lambda(\cos(\nu\tau)-1)}{\nu^2}\right)}{4\left(1+e^{-E_0\beta_k}\right)} & 0 & 0 & \frac{e^{-E_0\beta_k}\left(2(E_0+E)^2+\lambda^2+\lambda^2\cos(\eta\tau)\right)}{\eta^2}+\frac{2(E_0-E)^2+\lambda^2+\lambda^2\cos(\nu\tau)}{\nu^2} \\ & & & \frac{}{2\left(1+e^{-E_0\beta_k}\right)} \end{pmatrix},$$

where

$$C = \begin{pmatrix} \frac{\left(i\eta\cos\left(\frac{\eta\tau}{2}\right)+(E_0+E)\sin\left(\frac{\eta\tau}{2}\right)\right)\left((E_0-E)\sin\left(\frac{\nu\tau}{2}\right)-i\nu\cos\left(\frac{\nu\tau}{2}\right)\right)}{\sqrt{E_0^4+2\left(\lambda^2-E^2\right)E_0^2+\left(E^2+\lambda^2\right)^2}} & \frac{\lambda^2\sin\left(\frac{\eta\tau}{2}\right)\sin\left(\frac{\nu\tau}{2}\right)}{\sqrt{E_0^4+2\left(\lambda^2-E^2\right)E_0^2+\left(E^2+\lambda^2\right)^2}} \\ \frac{\lambda^2\sin\left(\frac{\eta\tau}{2}\right)\sin\left(\frac{\nu\tau}{2}\right)}{\sqrt{E_0^4+2\left(\lambda^2-E^2\right)E_0^2+\left(E^2+\lambda^2\right)^2}} & \frac{e^{-\frac{1}{2}i\nu\tau}\left(-e^{i\nu\tau}E_0+E_0-E+\nu+e^{i\nu\tau}(E+\nu)\right)\left(\eta\cos\left(\frac{\eta\tau}{2}\right)+i(E_0+E)\sin\left(\frac{\eta\tau}{2}\right)\right)}{2\sqrt{\eta^2\nu^2}} \end{pmatrix},$$

and $\eta = \sqrt{(E+E_0)^2+\lambda^2}$. We include these matrix representations here for completeness and concreteness. This explicit form for the reduced dynamics of this repeated interaction system with the full dipole interaction is probably novel; perhaps it might be of use to others.

2
Rising action

In section 2.1, we will tie together Landauer's Principle, adiabatic limits, and repeated interaction systems. From there, we will attempt to consider two simple systems of interest in section 2.2. This investigation will motivate the tools and results of the rest of this report.

2.1 Formulation of Landauer's Principle for repeated interaction systems

We have three subjects to reconcile. First, we'll discuss Landauer's bound in an RIS context. Then, motivated by saturation of Landauer's bound, we'll interpret the adiabatic limit of repeated interaction systems.

During each step, an RIS obeys unitary time evolution in exactly the same setup as that of Landauer's Principle (at least, as formulated in section 1.1). Thus, we may define at each step the change in entropy of the system

$$\Delta S_{\mathcal{S},k} = S(\rho_{k-1}) - S(\rho_k),$$

and the change of energy of the probe

$$\Delta Q_{\mathcal{E},k} = \mathrm{Tr}(h_{\mathcal{E}_k}\xi_k^{\mathrm{f}}) - \mathrm{Tr}(h_{\mathcal{E}_k}\xi_k^{\mathrm{i}}),$$

where $\xi_k^{\mathrm{f}} = \mathrm{Tr}_{\mathcal{H}_\mathcal{S}}(U_k\rho_{k-1}\otimes\xi_k^{\mathrm{i}}U_k^*)$ is the final state of the kth probe. As before, these quantities are related by the balance equation

$$\Delta S_{\mathcal{S},k} + \sigma_k = \beta_k \Delta Q_{\mathcal{E},k}, \tag{2.1}$$

where σ_k is the entropy production defined as

$$\sigma_k := S\big(U_k(\rho_{k-1}\otimes\xi_k^{\mathsf{i}})U_k^*|\mathcal{L}_k(\rho_{k-1})\otimes\xi_k^{\mathsf{i}}\big).$$

Note $\sigma_k \geq 0$. If consider T steps, we may sum over k to obtain

$$\sum_{k=1}^{T}\Delta S_{\mathcal{S},k} + \sum_{k=1}^{T}\sigma_k = \sum_{k=1}^{T}\beta_k\Delta Q_k.$$

Since $\sigma_k \geq 0$ for each k, we have

$$\sum_{k=1}^{T}\Delta S_k \leq \sum_{k=1}^{T}\beta_k\Delta Q_k$$

which is Landauer's bound in an RIS setup. In particular, we are interested in the case of equality, which occurs when $\sum_{k=1}^{T}\sigma_k = 0$. Since each $\sigma_k \geq 0$, we must then have $\sigma_k \equiv 0$.

LET US NOTE here that alternatively, we could describe this process in the large Hilbert space of the entire chain up to step T along with the small system. On this space, time evolution at step k is given by

$$\tilde{U}_k = e^{-i\tau_1 h_{\mathcal{E}_1}}\otimes\cdots\otimes e^{-i\tau_{k-1}h_{\mathcal{E}_{k-1}}}\otimes U_k\otimes e^{-i\tau_{k+1}h_{\mathcal{E}_{k+1}}}\otimes\cdots\otimes e^{-i\tau_T h_{\mathcal{E}_T}},$$

that is, the free evolution on each $\mathcal{E}_j$ for $j \neq k$, and unitary evolution according to U_k as defined in eq. (1.9), omitting tensor products with the identity operator. Then

$$\tilde{U}_{\text{tot}} = \tilde{U}_T\tilde{U}_{T-1}\cdots\tilde{U}_1$$

is the joint time evolution operator from the first step to the end of step T. Define

$$\mathcal{L}_{\text{tot}}(\rho) := \mathrm{Tr}_{\mathcal{E}_1,\mathcal{E}_2,\ldots,\mathcal{E}_T}\left(\tilde{U}_{\text{tot}}(\rho\otimes\xi_{\mathcal{E}_1,\ldots,\mathcal{E}_T})\tilde{U}_{\text{tot}}^*\right)$$

where $\xi_{\mathcal{E}_1,\ldots,\mathcal{E}_T} = \xi_1\otimes\xi_2\otimes\cdots\otimes\xi_T$. Then we have

$$\sum_{k=1}^{T}\sigma_k = S(\tilde{U}_{\text{tot}}(\rho^{\mathsf{i}}\otimes\xi_{\mathcal{E}_1,\ldots,\mathcal{E}_T})\tilde{U}_{\text{tot}}^*|\mathcal{L}_{\text{tot}}(\rho^{\mathsf{i}})\otimes\xi_{\mathcal{E}_1,\ldots,\mathcal{E}_T})$$

by direct computation. In this way, we may delay tracing out the probes until the end of the

process. If we consider our system and chain of probes interacting through the unitary operator $\tilde{U}_{\text{tot}}$, the final state of the system is

$$\rho_T^{\text{1-step}} := \mathcal{L}_{\text{tot}}(\rho^{\text{i}}) = \rho_T$$

where we use the 1-step superscript to indicate we are considering time evolution by $\tilde{U}_{\text{tot}}$ and are tracing out the probes at the end. Since the change entropies

$$\Delta S_{\mathcal{S}}^{\text{1-step}} := S(\rho^{\text{i}}) - S(\rho_T^{\text{1-step}}) = \sum_{k=1}^{T} \Delta S_k$$

telescope, we may sum the balance equation eq. (2.1) to obtain

$$\Delta S_{\mathcal{S}}^{\text{1-step}} + \sigma^{\text{1-step}} = \sum_{k=1}^{T} \beta_k \Delta Q_{\mathcal{E},k},$$

defining

$$\sigma^{\text{1-step}} := S(\tilde{U}_{\text{tot}}(\rho^{\text{i}} \otimes \xi_{\mathcal{E}_1,\ldots,\mathcal{E}_T})\tilde{U}_{\text{tot}}^* | \mathcal{L}_{\text{tot}}(\rho^{\text{i}}) \otimes \xi_{\mathcal{E}_1,\ldots,\mathcal{E}_T}) = \sum_{k=1}^{T} \sigma_k.$$

Thus, we may model the process as occurring via unitary time evolution. This is a concrete manifestation of an idea described well by Michael Wolf [Wol12, p. 7]:

> [T]he division of a physical process into preparation and measurement is ambiguous…but, fortunately, in the case of quantum mechanics predictions do not depend on this choice.

One may wonder then why the considerations of RIS here differ from those of Landauer's Principle in the case of a small system interacting with an infinitely extended thermal reservoir via unitary time evolution, as remarked on at the end of section 1.2 ([JP14]). The key differences here are the state of the chain

$$\xi_1 \otimes \xi_2 \otimes \cdots \otimes \xi_T$$

which in general is *not* a KMS state for the probes' free Hamiltonian at a single temperature β, and the specific unitary dynamics

$$\tilde{U}_{\text{tot}} = \tilde{U}_k \tilde{U}_{k-1} \cdots \tilde{U}_1$$

which reflect that the chain interacts with the system one probe at a time. Since this operator depends on the properties of the first T environments, it is not clear how to control its spectral

properties as $T \to \infty$ in order to apply an adiabatic theorem. Instead, it is more natural to use the special structure of the RIS system and model the system's evolution by the operators $\mathcal{L}_k$.

NEXT, WE WISH to formulate an adiabatic limit of an RIS. At each step, could replace U_k with the solution to the rescaled Schrödinger's equation corresponding to the Hamiltonian $h_{\mathcal{S}} + h_{\mathcal{E}_k} + \lambda v_k$. This would consider the RIS to be a sequence of essentially unrelated adiabatic processes, each step of which proceeds infinitely slowly and takes infinite time. There is little to say here; each step simply consists of the well-studied adiabatic time evolution of a small system interacting with a thermal reservoir. In particular, Landauer's principle in this context is thoroughly understood [JP14].

Instead, we'll try to formulate an adiabatic process natural to RIS. Given

$$[0,1] \ni s \mapsto h_{\mathcal{E}}(s), \qquad [0,1] \ni s \mapsto \beta(s), \qquad [0,1] \ni s \mapsto v(s)$$

C^2 functions, we'll define an RIS process with respect to a (fixed, large) adiabatic parameter T by sampling:

$$h_{\mathcal{E},k,T} = h_{\mathcal{E}}\Big(\frac{k}{T}\Big), \quad \beta_{k,T} = \beta\Big(\frac{k}{T}\Big), \quad v_{k,T} = v\Big(\frac{k}{T}\Big), \qquad k = 1, 2, \ldots, T. \tag{ADRIS}$$

In this setup, the chain consists of T probes. Thus, for each T, we have an RIS consisting of a system $\mathcal{S}$ (with Hilbert space $\mathcal{H}_{\mathcal{S}}$), a chain of probes $\{\mathcal{E}_k\}_{k=1}^T$ (with identical associated Hilbert spaces $\mathcal{H}_{\mathcal{E},k} \equiv \mathcal{H}_{\mathcal{E}}$), and parameters $\{h_{\mathcal{E},k,T}, \beta_{k,T}, v_{k,T}\}$ chosen according to (ADRIS), as well as fixed parameters $\{h_{\mathcal{S}}, \tau, \lambda, \rho^{\mathrm{i}}\}$. That an RIS is of this form will be known as the assumption (ADRIS), short for adiabatic RIS. The limit $T \to \infty$ is a *double* limit: we consider interactions with T probes, whose parameters from step to step change by $O(1/T)$ by the mean value theorem. We require the functions $h_{\mathcal{E}}(s), \beta(s), v(s)$ to be C^2 instead of C^1 so that the second differences, e.g.

$$(\beta_{k+1,T} - \beta_{k,T}) - (\beta_{k,T} - \beta_{k-1,T}),$$

are $O(T^{-2})$. We will use this assumption in section 3.2 to apply a discrete time non-unitary adiabatic theorem to RIS. Since the Hamiltonians of the probes, the temperatures of the probes, and the interaction Hamiltonians each depend on T, the change in entropy $\Delta S_{\mathcal{S},k,T}$ of the system at each step, the change in energy $\Delta Q_{\mathcal{E},k,T}$ of each probe, and the entropy production $\sigma_{k,T}$ each depend on T. We again have the balance equation eq. (2.1) at each step, but now each parameter depends on T:

$$\Delta S_{\mathcal{S},k,T} + \sigma_{k,T} = \beta_{k,T} \Delta Q_{\mathcal{E},k,T}. \tag{2.2}$$

We are interested in

$$\sigma_{k,T} := S\big(U_{k,T}(\rho_{k-1,T} \otimes \xi^{\mathrm{i}}_{k,T})U^*_{k,T} | \mathcal{L}_{k,T}(\rho_{k-1,T}) \otimes \xi^{\mathrm{i}}_{k,T}\big) \tag{2.3}$$

and in particular, the limit $T \to \infty$ of $\sigma^{\mathrm{tot}}_T := \sum_{k=1}^T \sigma_{k,T}$.

NOTE THAT COMPUTATIONALLY the step-wise structure of the RIS only comes to play in the sum $\sigma_T := \sum_{k=1}^T \sigma_{k,T}$ and in computing the state of the system at a given step. At the level of an individual step of the RIS process however, the parameters are constant or sampled at a single point from a C^2 function; thus, regarding $\sigma_{k,T}$ as a function of the parameters $\{h_{\mathcal{S}}, \tau, \lambda, \rho^{\mathrm{i}}, h_{\mathcal{E}_{k,T}}, \beta_{k,T}, \nu_{k,T}\}$ we can substitute the k, T dependence for s dependence, and consider $\sigma(s)$. In this language, $\sigma^{\mathrm{tot}}_T = \sum_{k=1}^T \sigma(k/T)$. Likewise, we may consider $\mathcal{L}(s)$ instead of $\mathcal{L}_{k,T}$, or other quantities which depend on k and T via k/T only. We should be clear that the functional notation $\mathcal{L}(s)$ does not reflect a continuity of the underlying physical process; simply, for each s we have a choice of parameters from which we can generate a step of an RIS process with corresponding reduced dynamics $\mathcal{L}(s)$.

2.2 *A first attempt*

RECALL OUR EXAMPLE from section 1.3.1, with $v = v_{\mathrm{RW}}$. Since $\mathcal{H}_{\mathcal{S}} = \mathcal{H}_{\mathcal{E}} = \mathbb{C}^2$, we may interpret the system and each probe as being a qubit (2-level quantum system).

Can we determine $\sigma_{k,T}$? Given a step k, we know all the parameters to generate $U_{k,T}$ and compute $\mathcal{L}_{k,T}$. But for some large k, we have the task of determining

$$\rho_{k-1,T} = \mathcal{L}_{k-1,T}\mathcal{L}_{k-2,T} \cdots \mathcal{L}_{1,T}\rho^{\mathrm{i}}.$$

Given the matrix representation from section 1.3.1, theoretically we could compute this state for any fixed step k, but we have little hope of a closed form. This motivates a consideration of the spectral properties of $\mathcal{L}_{k,T}$. From our computation of a matrix form of $\mathcal{L}_{k,T}$ in section 1.3.1, we may compute the eigenvalues and eigenstates (see [Han16] for the code). We obtain for eigenvalues $\theta^1, \theta^2, \theta^3$,

$$\mathcal{L}_{k,T}(a) = \theta^1 a, \qquad \mathcal{L}_{k,T}(a^*) = \theta^2 a^*, \qquad \mathcal{L}_{k,T}(\mathrm{Id} - 2a^*a) = \theta^3(\mathrm{Id} - 2a^*a)$$

as well as the eigenvalue one: $\mathcal{L}_{k,T}(\rho_{\beta^*_{k,T}}) = \rho_{\beta^*_{k,T}}$ where $\rho_{\beta^*_{k,T}}$ is the Gibbs state at inverse tem-

perature $\beta^*_{k,T} := \frac{E_0}{E}\beta_{k,T}$, namely

$$\rho_{\beta^*} = \frac{\exp(-\beta^*_{k,T} h_{\mathcal{S}})}{\mathrm{Tr}(\exp(-\beta^*_{k,T} h_{\mathcal{S}}))}. \tag{2.4}$$

We have

$$\begin{aligned}
\theta^1 &= \frac{e^{-\frac{1}{2}i\tau(E_0+E+\nu)}\left(E_0\left(-1+e^{i\nu\tau}\right)+(\nu-E)e^{i\nu\tau}+E+\nu\right)}{2\nu},\\
\theta^2 &= \frac{e^{-\frac{1}{2}i\tau(-E_0-E+\nu)}\left(-E_0\left(-1+e^{i\nu\tau}\right)+(\nu+E)e^{i\nu\tau}-E+\nu\right)}{2\nu},\\
\theta^3 &= \frac{2(E_0-E)^2+\lambda^2\cos(\nu\tau)+\lambda^2}{2\nu^2},
\end{aligned}$$

where, as before, $\nu = \sqrt{(E-E_0)^2+\lambda^2}$. Note that these eigenvalues are independent of $\beta_{k,T}$ and thus of k and T. In fact, θ^2 is θ^1 after the substitutions $E_0 \to -E_0$, $E \to -E$. Thus, $\theta^1 \neq \theta^2$ unless $\theta^1 = f(E, E_0)$ for some even function of E and E_0; since we may write θ^1 as a power series in, say, E with non-zero odd terms, this must enot be the case. Similarly, for all but a countable set of parameters, θ^1 and θ^2 have imaginary components, whereas for every choice of parameters, θ^3 is real, and for $\nu\tau \notin 2\pi\mathbb{Z}$, is strictly less than one. We will assume these restrictions. Then $\mathcal{L}_{k,T}$ has only simple eigenvalues, and we have that

$$\max_{j=1,2,3}|\theta^j| \leq \left(1-\frac{\lambda^2}{\nu^2}\sin^2\frac{\nu\tau}{2}\right) =: \ell < 1.$$

Then we may write

$$\mathcal{L}_{k,T} = P + \sum_{i=1}^{3}\theta^i Q^i \tag{2.5}$$

the Jordan form of $\mathcal{L}_{k,T}$, where Q^i is the eigenprojection associated to θ^i. Then, using $Q^jP = PQ^j = Q^jQ^{j'} = 0$ for $j \neq j'$,

$$\mathcal{L}^n_{k,T} = P + \sum_{j=1}^{3}(\theta^j)^n Q^j.$$

But since $|\theta^j| < \ell < 1$ the terms in the sum vanish exponentially fast. Thus, we see repeated applications of $\mathcal{L}_{k,T}$ to a state will drive it exponentially quickly towards the invariant state $\rho_{\beta^*_{k,T}}$. We will return to this particular property in section 5.3.1.

This gives a hint to a possible approach to computing $\sigma_{k,T}$: since we assume successive probes are close together (in some parameter space, at least), perhaps $\mathcal{L}_{k,T}\mathcal{L}_{k-1,T}\cdots\mathcal{L}_{1,T}$ acts similar to $\mathcal{L}^n_{k,T}$. Then we might expect $\rho_{k-1,T} \sim \rho_{k,T} \sim \rho_{\beta^*_{k,T}}$. Since $\sigma_{k,T}$ only depends on parameters at

step k and on $\rho_{k-1,T}$, we would then have that $\sigma_{k,T}$ approximately only depends on the parameters at step k. In this case, we could compute $\sigma_{k,T}$ as a function of T, λ, τ, E, E_0 and $\beta_{k,T}$, and see if $\sigma_{k,T} \to 0$, and if so, how quickly.

Remark. For those disappointed with the use of the rotating wave approximation, rest assured that its use here was not at all essential; with a different choice of $\ell < 1$ and a different (countable) set of excluded parameters, we would recover the same exponential driving towards the (different) invariant state when considering, say, the full dipole interaction. In fact if we assume only that $\mathcal{L}$ is some reduced dynamics with a unique invariant state which is the only eigenvalue on the unit circle, then we may write

$$\mathcal{L} = P + \sum_j \theta^j Q^j + D^j$$

where P is the projection onto the invariant state, the θ^j are the other eigenvalues, Q^j their eigenprojections, and D^j their eigennilpotents, in the language of [Kat76, Section I.5.4]. Set $\dim \mathcal{H}_S = d$; since there are at most d^2 eigenvalues of $\mathcal{L}$, the sum is finite. Then, using $Q^j P = PQ^j = Q^j Q^{j'} = D^j Q^{j'} = Q^j D^{j'} = 0$ for $j \neq j'$,

$$\mathcal{L}_{k,T}^n = P + \sum_j (\theta^j Q^j + D^j)^n = P + \sum_{j=1}^{j} \sum_{m=0}^{n} \binom{n}{m} (\theta^j)^{n-m} Q^j (D^j)^m$$

by the binomial theorem. Since the D^j are nilpotent operators on $\mathbb{C}^{d^2}$, they have degree at most d^2, so for $n \geq d^2$,

$$\mathcal{L}_{k,T}^n = P + \sum_j \sum_{m=0}^{d^2} \binom{n}{m} (\theta^j)^{n-m} Q^j (D^j)^m.$$

Since $\binom{n}{m} \leq c_1 n^{d^2}$ and $(\theta^j)^{n-m} \leq c_2 (\max_j |\theta^j|)^{n-d^2}$ for constants $c_1, c_2 > 0$ and large n, the at most d^4 terms in the sums tend to zero exponentially fast (since $\max_j |\theta^j| < 1$). Thus, even in a fairly general case, iterating a single choice of $\mathcal{L}_{k,T}$ provides a strong control on the resulting state, just as with the example system with the RW approximation. For further discussion, see [BJM14, Section 3].

We will return to both the full dipole interaction example and the rotating wave example, and in fact compute the entropy production $\sigma_{k,T}$ in each case (to some kind of leading order). See section 5.3 for the rotating wave case, and section 6.1 for the full dipole interaction.

3
Turning point: An adiabatic theorem

To proceed past the explorations of the section 2.2, we would like a way to approximate the entropy production at each step by a quantity which only depends on the parameters at that step, and not all of the steps before it. The entropy production

$$\sigma_{k,T} = S(U_{k,T}\rho_{k-1,T} \otimes \xi^{\mathrm{i}}_{k,T} U^*_{k,T} | \rho_{k,T} \otimes \xi^{\mathrm{i}})$$

depends both on the parameters at step k and on $\rho_{k-1,T} = \mathcal{L}_{k-1,T} \cdots \mathcal{L}_{0,T}\rho^{\mathrm{i}}$. We will thus first try to approximate $\rho_{k-1,T}$ by a state depending on only the parameters at step k, and then learn how to propagate this uncertainty through the relative entropy function. In this chapter, we'll formulate a discrete time non-unitary adiabatic theorem in section 3.1, and then apply the theorem to repeated interaction systems in section 3.2. In chapter 4, we will apply perturbation theory to the relative entropy function to find a leading order term, and then in chapter 6 put these results together to compute the entropy production of repeated interaction systems.

3.1 Discrete non-unitary adiabatic theorem (DNUAT)

Our setting will be a finite-dimensional Banach space X, with a norm $\|\cdot\|$. We consider an operator-valued function

$$[0,1] \ni s \mapsto \mathcal{L}(s) \in \mathcal{B}(X).$$

Remark. We do not assume here that $\mathcal{L}(s)$ is the reduced dynamics of an RIS, nor an underlying Hilbert space structure.

Here and in what follows, we say a function f on $[0,1]$ is C^2 on $[0,1]$ if f is continuous on $[0,1]$, twice continuously differentiable on $(0,1)$, and $\lim_{s\downarrow 0} f'(s)$, $\lim_{s\uparrow 1} f'(s)$, $\lim_{s\downarrow 0} f''(s)$ and $\lim_{s\uparrow 1} f''(s)$ exist and are finite.

We consider the following hypotheses:

H1. For each $s \in [0,1]$, $\mathcal{L}(s)$ is a contraction, i.e. $\|\mathcal{L}(s)\| \leq 1$.

H2. There is a uniform gap $\epsilon > 0$ such that, for $s \in [0,1]$ $|e^j(s) - e^i(s)| > 2\epsilon$ for any *peripheral eigenvalues* $e^j(s) \neq e^i(s)$ in $\operatorname{sp}\mathcal{L}(s) \cap S^1$.

H3. Let $P^j(s)$ be the spectral projector associated with $e^j(s) \in \operatorname{sp}\mathcal{L}(s) \cap S^1$, and $P(s) = \sum_j P^j(s)$ the *peripheral spectral projector*. The map $s \mapsto \mathcal{L}^P(s) := \mathcal{L}(s)P(s)$ is C^2 on $[0,1]$.

H4. There is a uniform bound on the strictly contracting part of $\mathcal{L}(s)$, i.e. if $Q(s) := \mathrm{Id} - P(s)$,

$$\ell := \sup_{s\in[0,1]} \|\mathcal{L}(s)Q(s)\| < 1.$$

3.1.1 Implications of H1–H4.

Let us see what we can get with these assumptions. First, we have a useful lemma.

Lemma 3.1. *The peripheral eigenvalues of a contraction L on a finite-dimensional Banach space X are semi-simple.*

Proof. Write L it its Jordan canonical form, $L = \sum_i e_i P_i + D_i$, where each e_i is an eigenvalue of L, P_i is the associated eigenprojection, and D_i the associated nilpotent, summed from $i = 1$ to $\dim X$. Let $m_i = \operatorname{order} D_i$, and $m = \max\{m_i : 1 \leq i \leq \dim X\}$. Assume for some eigenvalue e_i with $|e_i| = 1$ that $m_i \geq 2$, i.e., e_i has an eigennilpotent. We wish to derive a contradiction, implying $D_i \equiv 0$.

Note that for $n > m$, using $D_iP_j = P_jD_i = \delta_{ij}D_i$, and $P_iP_j = P_jP_i = \delta_{ij}P_i$, a binomial expansion yields

$$L^n = \sum_i \sum_{k=0}^{m} \binom{n}{k} e_i^{n-k} P_i (D_i)^k.$$

The plan of attack is to use that the binomial coefficient $\binom{n}{k}$ becomes large, while e_i^n stays on the unit circle, and $\|L^n\| \leq \|L\|^n = 1$.

Now, since $D_i^{m_i-1} \neq 0$, for some vector v we have $D_i^{m_i-1}v \neq 0$. Let $n \in \mathbb{N}$ large enough so that $n\|D_i^{m_i-1}v\| > \|D_i^{m_i-2}v\|$ and $n > m$. Then, since L is a contraction,

$$\|D_i^{m_i-2}v\| \geq \|L^n D_i^{m_i-2}v\| = \left\| \sum_j \sum_{k=0}^{m} \binom{n}{k} e_j^{n-k} P_j D_j^k D_i^{m_i-2} v \right\|.$$

Note $P_jP_i = 0$ implies $P_jD_i^{m_i-2}v = P_jP_iD_i^{m_i-2}v = 0$ for $i \neq j$, so we have

$$\|D_i^{m_i-2}v\| \geq \left\| \sum_{k=0}^{m} \binom{n}{k} e_i^{n-k} P_i (D_i)^k D_i^{m_i-2} v \right\|.$$

But, by choice of m_i, only two terms in the sum survive: $k = 0$ and $k = 1$:

$$\|D_i^{m_i-2}v\| \geq \left\| e_i^n D_i^{m_i-2}v + n e_i^{n-1} D_i^{m_i-1} v \right\|.$$

Then, by reverse triangle inequality and using $|e_i| = 1$,

$$\|D_i^{m_i-2}v\| \geq \left| \left\| D_i^{m_i-2}v \right\| - n \left\| D_i^{m_i-1}v \right\| \right| \geq n \left\| D_i^{m_i-1}v \right\|.$$

By our choice of v, we have $\left\| D_i^{m_i-1}v \right\| \neq 0$, hence

$$\frac{\|D_i^{m_i-2}v\|}{\left\| D_i^{m_i-1}v \right\|} \geq n$$

for all n large enough. This is a contradiction to our choice of n; moreover, we could take $n \to \infty$, while the LHS remains bounded. □

Remark. We saw that the reduced dynamics for qubits with the RW approximation was simple in eq. (2.5); this lemma shows that in general, the *peripheral* reduced dynamics $\mathcal{L}^P$ is always semi-simple. With this in hand, we may proceed to extend differentiability to the individual eigenvalues and eigenprojections.

Lemma 3.2. *Assume* **H1** *to* **H4**. *Then the peripheral eigenvalues $e^j(s)$ and eigenprojectors $P^j(s)$ of $\mathcal{L}^P(s)$ are C^2 as functions of s on $[0, 1]$.*

Remark. The assumption **H4** here may be weakened; see lemma 3.8.

Proof. By theorem A.9, the set of eigenvalues $\{e_j(s)\}$ is continuous in s. Since the peripheral eigenvalues are isolated by assumption **H2**, we can parametrize them by continuous functions $s \mapsto e^j(s)$.

Next, as shown in [Kat76], if an operator-valued function $T(s)$ is differentiable and invertible in a neighborhood of s, then $T^{-1}(s)$ is differentiable in that same neighborhood, and

$$\frac{\mathrm{d}}{\mathrm{d}s} T(s)^{-1} = -T(s)^{-1} T'(s) T(s)^{-1}.$$

Applying this to $R^P(s,z) := (z - \mathcal{L}^P(s))^{-1}$, we obtain that $s \mapsto R^P(s,z)$ is twice differentiable on any interval of $[0,1]$ on which z is not an eigenvalue of $\mathcal{L}^P(s)$, using **H3**. Choose some peripheral eigenvalue $e^j(s)$ and fix s_0. From our gap and bound assumptions **H2** and **H4**, there exists a circle Γ and $\delta > 0$ such that Γ encircles $e^j(s)$ for $|s - s_0| < \delta$, but stays a uniform distance away from $e_i(s)$ for any $i \neq j$. Then for any s in the above neighborhood of s_0, the spectral projector onto $e^j(s)$ is equal to

$$P^j(s) = \frac{1}{2\mathrm{i}\pi} \int_\Gamma R^P(s,z)\,\mathrm{d}z,$$

and the preceding discussion shows that $P^j(s)$ is C^2 on $[0,1]$.

Next, because $e^j(s)$ is semi-simple (using **H1** and lemma 3.1), we may write

$$\mathcal{L}^P(s)P^j(s) = e^j(s)P^j(s).$$

Again, fix s_0. Since $P^j(s_0) \neq 0$, in a matrix representation of $P^j(s_0)$ in some fixed basis, some matrix element $a(s)$ must be nonzero at s_0: $a(s_0) \neq 0$. But since $P^j(s)$ is C^2, $s \mapsto a(s)$ is C^2 and in particular continuous, so on a neighborhood of s_0, $a(s) \neq 0$. Choosing the corresponding matrix element $b(s)$ of $\mathcal{L}^P(s)P^j(s)$, we have

$$b(s) = e^j(s)a(s)$$

with $a(s) \neq 0$ on a neighborhood of s_0. Then on that neighborhood,

$$e^j(s) = \frac{b(s)}{a(s)}$$

is C^2. □

Remark. In particular, there exists $c_p > 0$ such that for all $s \in [0,1]$,

$$\max\left(\left\|\frac{\mathrm{d}^2P(s)}{\mathrm{d}s^2}\right\|, \left|\frac{\mathrm{d}^2e(s)}{\mathrm{d}s^2}\right|, \left\|\frac{\mathrm{d}P(s)}{\mathrm{d}s}\right\|, \left|\frac{\mathrm{d}e(s)}{\mathrm{d}s}\right|\right) \leq c_P, \tag{3.1}$$

and for some $N \in \mathbb{N}$, we have

$$\forall\, s \in [0,1], \quad \dim P(s) \equiv N \tag{3.2}$$

by corollary A.4. Note by **H2**, $N \leq \min\left(\frac{2\pi}{\epsilon}, \dim X\right)$.

Now, for $T \in \mathbb{N}$ and $0 \leq k \leq T$, we set

$$\mathcal{L}_{k,T} := \mathcal{L}(k/T), \qquad P_{k,T} := P(k/T), \qquad e^j_{k,T} := e^j(k/T).$$

Let us suppress the subscript T for the rest of the section.

Lemma 3.3. *Assume* **H1**, **H2** *and* **H4** *and let P_k^j be the eigenprojector corresponding to a peripheral eigenvalue e_k^j of $\mathcal{L}_k$ and $P_k = \sum_j P_k^j$. Then*

1. *for each j, $\|P_k^j\| = 1$, and $\|P_k\| = 1$, so that $\|P_k \mathcal{L}_k\| \leq 1$,*
2. *if in addition $\mathcal{L}_k$ is CPTP, then both P_k and $P_k\mathcal{L}_k$ are CPTP.*

Proof. Omit the subscript k. Using that $\mathcal{L}^P$ is simple, we may write

$$\bar{e}^j \mathcal{L} = \sum_i \bar{e}^j e^i P^i + \bar{e}^j \mathcal{L}^Q.$$

Then using $P^i P^j = P^j P^i = \delta_{ij} P^i$, we compute the ergodic sum

$$\frac{1}{M} \sum_{n=0}^{M-1} (\bar{e}^j \mathcal{L})^n = P^j + \frac{1}{M} \sum_{i \neq j} \frac{1 - (\bar{e}^j . e^i)^M}{1 - (\bar{e}^j . e^i)} P^i + \frac{1}{M} \sum_{n=0}^{M-1} (\bar{e}^j)^n \, (\mathcal{L}^Q)^n. \tag{3.3}$$

where we have used the geometric series $\sum_{k=0}^{n-1} r^k = \frac{1-r^n}{1-r}$ for $|r| < 1$. The LHS is a contraction: $\left\| \frac{1}{M} \sum_{n=0}^{M-1} (\bar{e}^j \mathcal{L})^n \right\| \leq \frac{1}{M} \sum_{n=0}^{M-1} \|L\|^n \leq 1$. The second term is of the form $\frac{1}{M}$ times a bounded function of M. For the third term, the spectral radius satisfies

$$\lim_{n \to \infty} \|(\mathcal{L}^Q)^n\|^{1/n} = \ell < 1$$

so for $\epsilon > 0$ such that $\ell + 2\epsilon < 1$, for some n_0 and $n \geq n_0$, $\|(\mathcal{L}^Q)^n\| \leq (\ell + \epsilon)^n < 1$. Since $|\bar{e}^j| = 1$, we have the bound

$$\begin{aligned} \left\| \frac{1}{M} \sum_{n=0}^{M-1} (\bar{e}^j)^n (\mathcal{L}^Q)^n \right\| &\leq \left\| \frac{1}{M} \sum_{n=0}^{n_0} (\bar{e}^j)^n (\mathcal{L}^Q)^n \right\| + \frac{1}{M} \sum_{n=n_0+1}^{M-1} (\ell + \epsilon)^n \\ &\leq \frac{1}{M} \left\| \sum_{n=0}^{n_0} (\bar{e}^j)^n (\mathcal{L}^Q)^n \right\| + \frac{1}{M} \frac{1}{1 - (\ell + \epsilon)} = O(M^{-1}). \end{aligned}$$

Hence, the third term vanishes with M as well. Thus, taking the norm of eq. (3.3), we obtain $\|P^j\| \leq 1$. Since P^j is a projection, $\|P^j\| = \|(P^j)^2\|$ and $\|P^j\| \geq 1$.

Next, we wish to show $\|P\| = 1$. let us write $e^j = e^{2\pi i x_j}$ for $x_j \in \mathbb{R}$, $j = 1, \dots, r$. By Dirichlet's approximation theorem for each $q \in \mathbb{R} \setminus \{0\}$ there exists a set of integers $\{p_1, p_2, \dots, p_r\}$ and $n_q \geq q$ in $\mathbb{N}$ such that $|n_q x_j - p_j| \leq \frac{1}{q}$ for all $j = 1, \dots, r$. Then,

$$(e^j)^{n_q} = e^{2\pi p_j} e^{2\pi(n_q x_j - p_j)} \leq 1(1 + (n_q x_j - p_j) + \frac{1}{2}\exp(\zeta)(n_q x_j - p_j)^2)$$

by Taylor's theorem, where $\zeta \in (0, n_q x_j - p_j)$; then $\exp(\zeta) \leq \exp(1)$. Thus, $(e^j)^{n_q} = 1 + O(1/q)$.

Then

$$\mathcal{L}^n = \sum_j (e^j)^n P^j + Q\mathcal{L}^n \implies \mathcal{L}^{n_q} = P + O(1/q) + Q\mathcal{L}^n.$$

Now, consider an increasing subsequence $(\tilde{n}_q)_q$ of $(n_q)_q$. Then as before for $\epsilon > 0$ such that $\ell + 2\epsilon < 1$, we have for sufficiently large q, $\|Q\mathcal{L}^{\tilde{n}_q}\| \leq (\ell + \epsilon_0)^{\tilde{n}_q}$. Thus,

$$\|\mathcal{L}^{\tilde{n}_q} - P\| = O(1/q) + (\ell + \epsilon_0)^{\tilde{n}_q} \to 0.$$

So $\lim_{q\to\infty} \mathcal{L}^{\tilde{n}_q} = P$. Thus, similarly to before, since $\mathcal{L}$ is a contraction, $\|P\| = 1$.

Lastly, if $\mathcal{L}$ is CPTP, then $\mathcal{L}^{\tilde{n}_q}$ is CPTP for each $\tilde{n}_q$. Note that the composition of CPTP is CPTP, because if at each step (complete) positivity is preserved and trace is preserved, then the composite map preserves those properties. Then since $\mathcal{L}^{\tilde{n}_q} \to P$, by proposition A.10, P is CPTP as well. Then the composition $\mathcal{L}^P$ is CPTP. □

The arguments showing $\mathcal{L}^{\tilde{n}_q} \to P$ are from [Wol12]. We should also note that although $\|P^j\| = 1$, these projectors may not be self-adjoint, since $\|\cdot\|$ is induced by a norm on a Banach space, not a Hilbert space.

We will proceed to state the main result of this chapter.

Theorem 3.4. *If $\mathcal{L}(s)$ satisfies* **H1** *to* **H4**, *then there exists constants $T_0 > 0$ and $C^P > 0$ depending only on c_P defined by eq. (3.1) and on N defined by eq. (3.2) such that for all $T \geq T_0$ there exists two family of maps $(A_{k,T})_{k=1,\dots,T}$ and $(A^\dagger_{k,T})_{k=1,\dots,T}$ with uniform bounds*

$$\sup_{k=0,\dots,T} \max(\|A_{k,T}\|, \|A^\dagger_{k,T}\|) \leq N\big(1 - \frac{c_P^2}{T_0^2}\big)^{-T_0/2} \tag{3.4}$$

satisfying

$$A_{k,T}^\dagger A_{k,T} = P_0, \qquad A_{k,T} A_{k,T}^\dagger = P_{k,T},$$
$$A_{k,T} P_0^j = P_{k,T}^j A_{k,T}, \qquad A_{k,T}^\dagger P_{k,T}^j = P_0^j A_{k,T}^\dagger, \qquad A_{k,T} Q_0 = Q_0 A_{k,T}^\dagger = 0,$$

and such that for all $k \leq T$,

$$\|\mathcal{L}_{k,T}\mathcal{L}_{k-1,T}\cdots\mathcal{L}_{1,T} - A_{k,T}\| \leq \frac{C^P}{T(1-\ell)} + 2\ell^k. \tag{3.5}$$

Additionally,

$$\|\mathcal{L}_{k,T}\mathcal{L}_{k-1,T}\cdots\mathcal{L}_{1,T} - A_{k,T} - \mathcal{L}_{k,T}^Q\mathcal{L}_{k-1,T}^Q\cdots\mathcal{L}_{1,T}^Q\| \leq \frac{C^P}{T(1-\ell)}. \tag{3.6}$$

Remark. From the point of view of adiabatic approximations, eq. (3.5) is the central result of this chapter: an approximation of adiabatically-stretched non-unitary time evolution from time $t = 0$ to some discrete time $t = k\tau$ via an operator which intertwines with the spectral projectors of the dynamics. While less satisfying, eq. (3.6) provides a useful description of the $2\ell^k$ error for later use.

We will prove this result in two steps: first by approximating the reduced dynamics $\mathcal{L}_{k,T}\cdots\mathcal{L}_{1,T}$ by the peripheral reduced dynamics $\mathcal{L}_{k,T}^P\cdots\mathcal{L}_{1,T}^P$ in proposition 3.5. This is essentially a combinatorial result. Then we will approximate the peripheral dynamics by operators which intertwine with the peripheral eigenprojectors in proposition 3.6; this is modeled on a unitary discrete time adiabatic theorem by Tanaka [Tan11]. The construction of the families $(A_{k,T})_{k=1,\dots,T}$ and $(A_{k,T}^\dagger)_{k=1,\dots,T}$ is detailed in section 3.1.3.

Remark. All statements of this chapter hold for X an *infinite* dimensional Banach space, assuming the peripheral eigenvalues $e^j(s)$ are isolated and the differentiability conditions are understood in the norm sense. These assumptions imply that $\sup_s \dim P(s)X < +\infty$.

3.1.2 Approximation by peripheral dynamics

Proposition 3.5. *If $\mathcal{L}(s)$ satisfies* **H1** *to* **H4**, *then there exists a constant C^P depending only on c_P defined by eq.* (3.1), *such that for any $T \geq 1$ and $k \leq T$,*

$$\|\mathcal{L}_{k,T}\mathcal{L}_{k-1,T}\cdots\mathcal{L}_{1,T} - \mathcal{L}_{k,T}^P\mathcal{L}_{k-1,T}^P\cdots\mathcal{L}_{1,T}^P P_0 - \mathcal{L}_{k,T}^Q\mathcal{L}_{k-1,T}^Q\cdots\mathcal{L}_{1,T}^Q Q_0\| \leq \frac{C^P}{T(1-\ell)}, \tag{3.7}$$

where we have defined $\mathcal{L}^P_{n,T} = \mathcal{L}_{n,T}P_{n,T}$ *and* $\mathcal{L}^Q_{n,T} = \mathcal{L}_{n,T}Q_{n,T}$. *Moreover, we have the bound*

$$\|\mathcal{L}^Q_{k,T}\mathcal{L}^Q_{k-1,T}\cdots\mathcal{L}^Q_{1,T}Q_0\| \leq 2\ell^k. \tag{3.8}$$

Remark. Note first that eq. (3.7) implies that for some operator A with $\|A\| \leq \frac{C^P}{T(1-\ell)}$, and the definitions

$$L := \mathcal{L}_{k,T}\mathcal{L}_{k-1,T}\cdots\mathcal{L}_{1,T}, \quad L^P := \mathcal{L}^P_{k,T}\mathcal{L}^P_{k-1,T}\cdots\mathcal{L}^P_{1,T}P_0, \quad L^Q := \mathcal{L}^Q_{k,T}\mathcal{L}^Q_{k-1,T}\cdots\mathcal{L}^Q_{1,T}Q_0,$$

we have

$$L = L^P + L^Q + A.$$

Multiplying from the left and right by projectors P_n or Q_n, for $n = 0, k$ yields

$$\max(\|P_n\mathcal{L}_n\cdots\mathcal{L}_1Q_0\|, \|Q_m\mathcal{L}_m\cdots\mathcal{L}_1P_0\|) \leq \frac{2C^P}{T(1-\ell)}, \tag{3.9}$$

$$\|\mathcal{L}_{k,T}\mathcal{L}_{k-1,T}\cdots\mathcal{L}_{1,T}P_0 - \mathcal{L}^P_{k,T}\mathcal{L}^P_{k-1,T}\cdots\mathcal{L}^P_{1,T}P_0\| \leq \frac{C^P}{T(1-\ell)}. \tag{3.10}$$

Equation (3.9) shows that there is little transition from the Q_0 subspace to the P_n subspace under the action of the reduced dynamics, nor from P_0 to Q_m, for any $n, m \in \mathbb{N}$. Moreover, eq. (3.10) has the interpretation that the action of the reduced dynamics on an initial state in the peripheral subspace is well approximated by the peripheral reduced dynamics. If the initial state is not in the peripheral subspace, then by eq. (3.8) we see the non-peripheral part is exponentially suppressed, although *not* vanishing in the limit $T \to \infty$.

Proof. We may rewrite eq. (3.7) as

$$\left\|\left(\mathcal{L}_k\mathcal{L}_{k-1}\cdots\mathcal{L}_1 - \mathcal{L}^P_k\mathcal{L}^P_{k-1}\cdots\mathcal{L}^P_1\right)P_0 + \left(\mathcal{L}_k\mathcal{L}_{k-1}\cdots\mathcal{L}_1 - \mathcal{L}^Q_k\mathcal{L}^Q_{k-1}\cdots\mathcal{L}^Q_1\right)Q_0\right\| \leq \frac{C^P}{T(1-\ell)}.$$

Let us bound

$$\left(\mathcal{L}_k\mathcal{L}_{k-1}\cdots\mathcal{L}_1 - \mathcal{L}^P_k\mathcal{L}^P_{k-1}\cdots\mathcal{L}^P_1\right)P_0. \tag{3.11}$$

For each $0 \leq n \leq k$, we write $\mathcal{L}_n = \mathcal{L}^P_n + \mathcal{L}^Q_n$. This yields 2^k terms, corresponding to a choice of $\mathcal{L}^P_n$ or $\mathcal{L}^Q_n$ for each $1 \leq n \leq k$; since $P_0Q_0 = Q_0P_0 = 0$, the choice for $n = 0$ must be P_0. In addition to the term $\mathcal{L}^Q_k\mathcal{L}^Q_{k-1}\cdots\mathcal{L}^Q_1P_0$, we identify four forms, corresponding to starting with

P/Q and to ending with P/Q:

$$(\prod_{a\in A_d}\mathcal{L}_a^Q)(\prod_{b\in B_d}\mathcal{L}_b^P)\dots(\prod_{a\in A_1}\mathcal{L}_a^Q)(\prod_{b\in B_1}\mathcal{L}_b^P)\,P_0, \tag{3.12}$$

$$(\prod_{b\in B_{d+1}}\mathcal{L}_b^P)(\prod_{a\in A_d}\mathcal{L}_a^Q)(\prod_{b\in B_d}\mathcal{L}_b^P)\dots(\prod_{a\in A_1}\mathcal{L}_a^Q)(\prod_{b\in B_1}\mathcal{L}_b^P)\,P_0, \tag{3.13}$$

$$(\prod_{a\in A_{d+1}}\mathcal{L}_a^Q)(\prod_{b\in B_{d+1}}\mathcal{L}_b^P)\dots(\prod_{a\in A_2}\mathcal{L}_a^Q)(\prod_{b\in B_2}\mathcal{L}_b^P)(\prod_{a\in A_1}\mathcal{L}_a^Q)\,P_0, \tag{3.14}$$

$$(\prod_{b\in B_{d+1}}\mathcal{L}_b^P)(\prod_{a\in A_d}\mathcal{L}_a^Q)(\prod_{b\in B_{d-1}}\mathcal{L}_b^P)\dots(\prod_{a\in A_2}\mathcal{L}_a^Q)(\prod_{b\in B_2}\mathcal{L}_b^P)(\prod_{a\in A_1}\mathcal{L}_a^Q)\,P_0, \tag{3.15}$$

where $d \geq 1$, and each A_n and B_n is a nonempty set of consecutive elements of $\{1,\dots,k\}$ such that the $\{A_n, B_m\}_{n,m}$ are a partition of $\{1,\dots,k\}$. The products are ordered as

$$\prod_{a\in\{a_0+1,\dots,a_0+t\}}\mathcal{L}_a^Q := \mathcal{L}_{a_0+t}^Q\dots\mathcal{L}_{a_0+1}^Q, \qquad \prod_{b\in\{b_0+1,\dots,b_0+t\}}\mathcal{L}_b^P := \mathcal{L}_{b_0+t}^P\dots\mathcal{L}_{b_0+1}^P.$$

The partition $\{A_n, B_m\}_{n,m}$ is also ordered in a particular sense: for eq. (3.12), for example, $\max B_n = \min A_n - 1$ for each $1 \leq n \leq d$. The key fact is that for $c = 2c_P$,

$$\|P_nQ_{n-1}\| \leq c/T, \qquad \|Q_nP_{n-1}\| \leq c/T. \tag{3.16}$$

This follows from $\|P_n - P_{n-1}\| \leq c_P/T$ by the mean value theorem and definition of c_P (eq. (3.1)). Then, e.g., $P_nQ_{n-1} = (P_n - P_{n-1})Q_{n-1}$, and $\|Q_{n-1}\| \leq \|\mathrm{Id} - P_{n-1}\| \leq 2$.

We also have that

$$\|\prod_{a\in A_n}\mathcal{L}_a^Q\| \leq \ell^{|A_n|}, \qquad \|\prod_{b\in B_n}\mathcal{L}_b^P\| \leq 1,$$

using **H4** and lemma 3.3. This norm bound along with the estimate $\|Q_0\| \leq 2$ yields eq. (3.8). Additionally, $\|\mathcal{L}_k^Q\mathcal{L}_{k-1}^Q\cdots\mathcal{L}_1^QP_0\| \leq c\ell^kT^{-1}$, and

$$\|(3.12)\| \leq (c/T)^{2d-1}\,\ell^{\sum_n|A_n|}, \qquad \|(3.13)\| \leq (c/T)^{2d}\,\ell^{\sum_n|A_n|},$$
$$\|(3.14)\| \leq (c/T)^{2d+1}\,\ell^{\sum_n|A_n|}, \qquad \|(3.15)\| \leq (c/T)^{2d}\,\ell^{\sum_n|A_n|}.$$

Thus, it remains to count the number of terms of each form; we'll only consider the case (3.12) as the others are very similar. The set of valid choices of d is $\{1,\dots,\lfloor\frac{k}{2}\rfloor\}$. If we constrain $\sum_n|A_n| = \alpha$, then we need $\alpha \geq d$ and $k - \alpha \geq d$ as $\sum_n|B_n| = k - \alpha$. Given such an α

and d, the only remaining freedom is in choice of $|A_1|,\dots, |A_d|$ and $|B_1|, \dots, |B_d|$, since once the cardinalities are chosen, the particular elements are determined by the ordering. There are respectively $\binom{\alpha-1}{d-1}$ and $\binom{k-\alpha-1}{d-1}$ such choices. We may see this with the "stars and bars" argument: if we write the elements of $\bigcup_n A_n$ as stars

$$\star\star\star\star\star\cdots\star$$

then our task is to put $d-1$ bars between the stars and thus partition the elements into $A_1, \dots, A_d$ (since the ordering is fixed).

$$\underset{A_1}{\star}\,|\,\underset{A_2}{\star\star\star}\,|\,\star\cdots\star$$

There are $\alpha - 1$ gaps between the stars of which we must choose $d - 1$ to place bars. Partitioning $\bigcup_n B_n$ follows the same logic. Thus, the number of terms of form (3.12) at fixed d with $\sum_n |A_n| = \alpha$ is

$$\binom{\alpha-1}{d-1}\binom{k-\alpha-1}{d-1}$$

and each such term has norm $\left(\frac{c}{T}\right)^{2d-1}\ell^\alpha$ as estimated above. Thus, the sum of all terms of the form (3.12) has norm at most

$$\begin{aligned}
&\sum_{d=1}^{\lfloor \frac{k}{2}\rfloor}\sum_{\alpha=d}^{k-d}\left(\frac{c}{T}\right)^{2d-1}\ell^\alpha\binom{\alpha-1}{d-1}\binom{k-\alpha-1}{d-1}\\
&\qquad= \frac{T}{c}\sum_{\alpha=1}^{k-1}\ell^\alpha\sum_{d=1}^{\inf(\alpha,k-\alpha)}\left(\frac{c^2}{T^2}\right)^d\binom{\alpha-1}{d-1}\binom{k-\alpha-1}{d-1} \qquad (3.17)\\
&\qquad\le \frac{T}{c}\sum_{\alpha=1}^{k-1}\ell^\alpha\left(\sum_{d=1}^{\alpha}\left(\frac{c^2}{T^2}\right)^{d/2}\binom{\alpha-1}{d-1}\right)\left(\sum_{d=1}^{k-\alpha}\left(\frac{c^2}{T^2}\right)^{d/2}\binom{k-\alpha-1}{d-1}\right)\\
&\qquad\le \frac{c}{T}\left(1+\frac{c}{T}\right)^{k-2}\sum_{\alpha=1}^{k-1}\ell^\alpha \le \frac{c\,\exp c}{T(1-\ell)}, \qquad (3.18)
\end{aligned}$$

We can expand the other forms (3.13) to (3.15) similarly, and obtain the same type of upper bound. To bound

$$(\mathcal{L}_k\mathcal{L}_{k-1}\cdots\mathcal{L}_1 - \mathcal{L}_k^Q\mathcal{L}_{k-1}^Q\cdots\mathcal{L}_1^Q)Q_0$$

we may again substitute $\mathcal{L}_n = \mathcal{L}_n^P + \mathcal{L}_n^Q$ and obtain the term $\mathcal{L}_k^P\mathcal{L}_{k-1}^P\cdots\mathcal{L}_1^PQ_0$ plus terms of the forms (3.12) to (3.15) with P and Q interchanged. We note $\|\mathcal{L}_k^P\mathcal{L}_{k-1}^P\cdots\mathcal{L}_1^PQ_0\| \le c/T$ and we

may bound the terms of the other forms in the same way as above.

□

Remark. We see that following this method of proof, we cannot obtain a better dependence on T than $1/T$. The expansion of eq. (3.11) via $\mathcal{L}_n = \mathcal{L}_n^P + \mathcal{L}_n^Q$ yields terms

$$\mathcal{L}_k^Q \cdots \mathcal{L}_{n+1}^Q \mathcal{L}_n^P \cdots \mathcal{L}_1^P P_0$$

which have exactly one $Q_n P_{n-1}$ part yielding a $1/T$ factor. Moreover, there are $k-1$ such terms; since we may have $k = T$, the assumption $\|\mathcal{L}^Q\| \leq \ell < 1$ is essential.

Additionally, the dependence on ℓ in eq. (3.18) will be important in section 5.4, but may not be improved substantially. In fact, eq. (3.17) is bounded below by the term with $d = 1$, namely $\frac{c}{T}\sum_{\alpha=1}^{k-1} \ell^\alpha = \frac{c\ell(1-\ell^{k-1})}{T(1-\ell)}$, which has the essential features of the bound.

3.1.3 *Approximation of peripheral dynamics*

THE RESULTS of the previous section, namely proposition 3.5, motivates us to restrict our attention to $\mathcal{L}_{k,T}^P \mathcal{L}_{k-1,T}^P \cdots \mathcal{L}_{1,T}^P P_0$. This operator is a product of contractions, each with all eigenvalues on the unit circle.

In order to construct the families of operators $(A_{k,T})_{k=0,\ldots,T}$ and $(A_{k,T}^\dagger)_{k=0,\ldots,T}$, we will construct several intermediate families. First, define $(W_{k,T})_{k=0,\ldots,T}$ and $(W_{k,T}^\dagger)_{k=0,\ldots,T}$ by $W_{0,T} = W_{0,T}^\dagger = P_{0,T}$, and

$$\begin{aligned} W_{k+1,T} &:= \sum_j P_{k+1,T}^j P_{k,T}^j \big(\mathrm{Id} - (P_{k+1,T}^j - P_{k,T}^j)^2\big)^{-1/2}, \\ W_{k+1,T}^\dagger &:= \sum_j P_{k,T}^j P_{k+1,T}^j \big(\mathrm{Id} - (P_{k+1,T}^j - P_{k,T}^j)^2\big)^{-1/2}. \end{aligned} \tag{3.19}$$

Note for an operator R with $\operatorname{spr} R < 1$, the operator $(\mathrm{Id} - R)$ is invertible, and its inverse is positive definite, thus admitting a unique positive definite square root.

By lemma 3.2 and the mean value theorem, $\|P_{k+1,T}^j - P_{k,T}^j\| \leq \frac{c_P}{T}$, so the operators $W_{k+1,T}$ and $W_{k+1,T}^\dagger$ are well-defined for $T \geq T_0(c_P) := 2c_p$ (that is, the spectral radius condition is met). Each $(P_{k+1,T}^j - P_{k,T}^j)^2$ commutes with both $P_{k,T}^j$ and $P_{k+1,T}^j$ (as shown in the proof of lemma A.3), yielding

$$W_{k+1,T} P_{k,T}^j = P_{k+1,T}^j P_{k,T}^j (\mathrm{Id} - (P_{k+1,T}^j - P_{k,T}^j)^2)^{-1/2} P_{k,T}^j = P_{k+1,T}^j W_{k+1,T}.$$

We also have the identity

$$\begin{aligned} P^j_{k,T}(\mathrm{Id} - (P^j_{k+1,T} - P^j_{k,T})^2) &= P^j_{k,T}(\mathrm{Id} - P^j_{k+1,T} - P^j_{k,T} + P^j_{k+1,T}P^j_{k,T} + P^j_{k,T}P^j_{k+1,T}) \\ &= P^j_{k,T}P^j_{k+1,T}P^j_{k,T}. \end{aligned}$$

which we may use to compute

$$\begin{aligned} W^\dagger_{k+1,T}W_{k+1,T} &= \sum_j P^j_{k,T}P^j_{k+1,T}P^j_{k,T}(\mathrm{Id} - (P^j_{k+1,T} - P^j_{k,T})^2)^{-1} \\ &= \sum_j P^j_{k,T}(\mathrm{Id} - (P^j_{k+1,T} - P^j_{k,T})^2)(\mathrm{Id} - (P^j_{k+1,T} - P^j_{k,T})^2)^{-1} = P_{k,T}. \end{aligned}$$

By interchanging $W^\dagger_{k,T}$ and $W_{k,T}$, we find similar results; all together, we have

$$\begin{aligned} W_{k+1,T}\, P^j_{k,T} &= P^j_{k+1,T}\, W_{k+1,T}, \qquad & W^\dagger_{k+1,T}\, P^j_{k+1,T} &= P^j_{k,T}\, W^\dagger_{k+1,T}, \\ W^\dagger_{k+1,T}\, W_{k+1,T} &= P_{k,T}, \qquad & W_{k+1,T}\, W^\dagger_{k+1,T} &= P_{k+1,T}. \end{aligned} \tag{3.20}$$

Remark. The operator $W^\dagger_{k+1,T}$ is a pseudo-adjoint of $W_{k+1,T}$, in the sense that we would have $W^*_{k+1,T} = W^\dagger_{k+1,T}$ if the spectral projectors $P^j_{k,T}$ were self-adjoint. We continue with this notation throughout this section, and every operator $Y^\dagger$ will be a pseudo-adjoint of Y, depending on $\{P^j_{k,T}\}_{k,j}$.

From these families, we have the natural constructions $K_{0,T} := K^\dagger_{0,T} := \mathrm{Id}$, and

$$K_{k,T} := W_{k,T} \dots W_{1,T}, \qquad K^\dagger_{k,T} := W^\dagger_{1,T} \dots W^\dagger_{k,T}. \tag{3.21}$$

From eq. (3.20), we have

$$\begin{aligned} K_{k,T}\, P^j_0 &= P^j_{k,T}\, K_{k,T}, \qquad & K^\dagger_{k,T}\, P^j_{k,T} &= P^j_0\, K^\dagger_{k,T}, \\ K^\dagger_{k,T}\, K_{k,T} &= P_0, \qquad & K_{k,T}\, K^\dagger_{k,T} &= P_{k,T}. \end{aligned} \tag{3.22}$$

These intertwining relations have the interpretation that $K_{k,T}$ accounts for the motion of the spectral projections of the dynamics from step 1 to step k.

We may bound their norm using that $P^j_{k,T}P^{j'}_{\ell,T} = 0$ for $j \neq j'$ and any k, ℓ, so by eq. (3.21)

we have for $T \geq T_0$ and any $k = 0, \dots, T$,

$$\begin{aligned}\|K_{k,T}\| &\leq \left\| \sum_j \prod_{\ell=0}^{k-1} P^j_{\ell+1,T} P^j_{\ell,T} \big(\mathrm{Id} - (P^j_{\ell+1,T} - P^j_{\ell,T})^2\big)^{-1/2} \right\| \\ &\leq N \left\| \prod_{\ell=0}^{k-1} \big(\mathrm{Id} - (P^j_{\ell+1,T} - P^j_{\ell,T})^2\big)^{-1/2} \right\| \\ &\leq N \prod_{\ell=0}^{k-1} \big(1 - (c_p/T)^2\big)^{-1/2} = N\big(1 - (c_p/T)^2\big)^{-(k-1)/2} \\ &\leq N\big(1 - (c_p/T)^2\big)^{-T/2}.\end{aligned}$$

Since $K^\dagger_{k,T}$ is bounded in the same way, we have

$$\sup_{k=0,\dots,T} \max(\|K_{k,T}\|, \|K^\dagger_{k,T}\|) \leq N_{\max}(1 - \frac{c_P^2}{T^2})^{-T/2}. \tag{3.23}$$

Next, define two families $(\Phi_{k,T})_{k=1,\dots,T}$ and $(\Phi^\dagger_{k,T})_{k=1,\dots,T}$ by $\Phi_{0,T} = \Phi^\dagger_{0,T} = P_{0,T}$, and

$$\Phi_{k,T} = \sum_j \big(\prod_{n=1}^{k} e^j_{n,T}\big)\, P^j_0, \qquad \Phi^\dagger_{k,T} = \sum_j \big(\prod_{n=1}^{k} \overline{e}^j_{n,T}\big)\, P^j_0. \tag{3.24}$$

These operators have $\Phi_{k,T}\Phi^\dagger_{k,T} = P_0 = \Phi^\dagger_{k,T}\Phi_{k,T}$, and have the interpretation of accounting for the motion of the phase within each spectral subspace of the dynamics from step 1 to step k. The construction is finished by setting $A_{k,T} = K_{k,T}\Phi_{k,T}$ and $A^\dagger_{k,T} = \Phi^\dagger_{k,T}K^\dagger_{k,T}$. Note that by eq. (3.22),

$$A_{k,T}A^\dagger_{k,T} = P_k, \qquad A^\dagger_{k,T}A_{k,T} = P_0. \tag{3.25}$$

Remark. The bound eq. (3.4) in theorem 3.4 follows from the bound eq. (3.23) using that the RHS is monotone decreasing with T.

With this construction in hand, we are ready to state the result of this section.

Proposition 3.6. *If $\mathcal{L}(s)$ satisfies* **H1** *to* **H4**, *then there are positive constants T_0 and C^P depending only on c_P defined by eq. (3.1) and N defined by eq. (3.2) such that the adiabatic approximation $(A_{k,T})_k$ defined above satisfies for all $T \geq T_0$,*

$$\|\mathcal{L}^P_{k,T}\mathcal{L}^P_{k-1,T} \cdots \mathcal{L}^P_{1,T}P_0 - A_{k,T}\| \leq C^P/T, \tag{3.26}$$

where $A_{k,T}P^j_0 = P^j_{k,T}$, $A_{k,T}Q_0 = 0$, and $\|A_{k,T}\|$ is uniformly bounded in k, T for $k \leq T$.

The proof follows a similar strategy to that of [Tan11], a unitary discrete-time adiabatic theorem. To simplify notation, we will omit T subscripts, and say an expression in $O^P(T^{-\alpha})$ if there exists a numeric constant C depending on c_P and N only such that the expression is bounded by $C \cdot T^{-\alpha}$. In this language, we wish to show

$$\mathcal{L}_k^P \mathcal{L}_{k-1}^P \cdots \mathcal{L}_1^P P_0 - K_k \Phi_k P_0 = O^P(T^{-1}). \tag{$\star$}$$

If we define

$$\Omega_k := \Phi_k^\dagger K_k^\dagger \mathcal{L}_k^P \mathcal{L}_{k-1}^P \cdots \mathcal{L}_1^P P_0$$

then ($\star$) is equivalent to

$$\Omega_k = P_0 + O^P(T^{-1}) \tag{$\star\star$}$$

by eq. (3.25) and eq. (3.23). Note that $\Omega_0 = P_0$, $P_0 \Omega_k = \Omega_k P_0 = \Omega_k$, and that Ω_k is bounded uniformly in k by eq. (3.4).

Define $\Theta_k := \Phi_k^\dagger K_k^\dagger \mathcal{L}_k^P K_{k-1} \Phi_{k-1}$ so that $\Omega_k = \Theta_k \Omega_{k-1}$ and $\Theta_0 = P_0$. Then we may write the telescoping sum

$$\Omega_k = P_0 + \sum_{n=1}^{k} \Theta_n \Omega_{n-1} - \Omega_{n-1} = P_0 + \sum_{n=1}^{k} (\Theta_n - P_0)\Omega_{n-1}.$$

If we define $V_n := \sum_{m=1}^{n} (\Theta_m - P_0)$ and $V_0 := 0$, we have

$$\Omega_k = P_0 + \sum_{n=1}^{k} (V_n - V_{n-1})\Omega_{n-1}.$$

Now, we are in a position to use summation by parts:

$$\Omega_k = P_0 + V_k \Omega_{k-1} - \sum_{n=1}^{k-1} V_n(\Omega_n - \Omega_{n-1}) = P_0 + V_k \Omega_{k-1} - \sum_{n=1}^{k-1} V_n(\Theta_n - \Theta_0)\Omega_{n-1}.$$

Now, we will show that each $\Theta_n - \Theta_0 = O^P(T^{-1})$ in claim 1, and $V_n = O^P(T^{-1})$ in claim 2, which proves ($\star\star$), as the sum is over $k - 1 \leq T$ terms.

Claim 1. *We have*

1. $Q_0(\Theta_k - \Theta_0) = (\Theta_k - \Theta_0)Q_0 = 0$,
2. $P_0^j(\Theta_k - \Theta_0)P_0^\ell = O^P(T^{-1})$ *for any* $j \neq \ell$,
3. $P_0^j(\Theta_k - \Theta_0)P_0^j = O^P(T^{-2})$ *for any* j.

Remark. This claim shows $\Theta_k - \Theta_0 = O^P(T^{-1})$, since we may write $\mathrm{Id} = Q_0 + \sum_j P_0^j$ to obtain

$$\Theta_k - \Theta_0 = \big(\sum_j P_0^j + Q_0\big)(\Theta_k - \Theta_0)\big(\sum_\ell P_0^\ell + Q_0\big) = \sum_{j,\ell} P_0^j(\Theta_k - \Theta_0)P_0^\ell.$$

Here, we only need $P_0^j(\Theta_k - \Theta_0)P_0^j = O^P(T^{-1})$, but the sharper estimate is used in claim 2.

Proof of claim 1. The first relation follows from $Q_0\Phi_k\dagger = \Phi_{k-1}Q_0 = 0$ and the definition of Θ_k and Θ_0. For the second and third, fix j and ℓ. Then

$$P_0^j\Theta_k P_0^\ell = P_0^j A_k^\dagger \mathcal{L}_k^P A_{k-1} P_0^\ell = A_k^\dagger P_k^j \mathcal{L}_k^P P_{k-1}^\ell A_{k-1}$$

Now, $\mathcal{L}_k^P = \sum_{j'} P_k^{j'} e_k^{j'}$, so $P_k^j \mathcal{L}_k^P = P_k^j e_k^j$. Then, $P_0^j(\Theta_k - \Theta_0)P_0^\ell = e_k^j A_k^\dagger P_k^j P_{k-1}^\ell A_{k-1}$.

Now, if $j \neq \ell$ we have

$$\|P_k^j P_{k-1}^\ell\| = \|(P_k^j - P_{k-1}^j)P_{k-1}^\ell\| \leq \|P_{k-1}^\ell\| \cdot \|P_k^j - P_{k-1}^j\| = O^P(T^{-1}),$$

hence $P_0^j(\Theta_k - \Theta_0)P_0^\ell = O^P(T^{-1})$, using that $P_0^j\Theta_0 P_0^\ell = P_0^j P_0^\ell = 0$. On the other hand, if $j = \ell$, we have

$$P_0^j\Theta_k P_0^\ell = e_k^j A_k^\dagger P_k^j P_{k-1}^j A_{k-1} = A_k^\dagger P_k^j W_k^\dagger W_k K_{k-1} e_k^j \Phi_{k-1} P_0^j = A_k^\dagger P_k^j W_k^\dagger K_k \Phi_k = A_k^\dagger P_k^j W_k^\dagger A_k$$

Then,

$$\begin{aligned} P_0^j(\Theta_k - \Theta_0)P_0^j &= A_k^\dagger P_k^j W_k^\dagger A_k - P_0^j = A_k^\dagger P_k^j W_k^\dagger A_k - A_k^\dagger P_k^j A_k \\ &= A_k^\dagger (P_k^j W_k^\dagger - P_k^j) A_k = A_k^\dagger P_k^j \big[(\mathrm{Id} - (P_k^j - P_{k-1}^j)^2)^{-1/2} - \mathrm{Id}\big] A_k. \end{aligned}$$

Since $\|P_k^j - P_{k-1}^j\| \leq c_p/T$, and

$$(\mathrm{Id} - (P_k^j - P_{k-1}^j)^2)^{-1/2} = \mathrm{Id} - \frac{1}{2}(P_k^j - P_{k-1}^j)^2 + O^P((P_k^j - P_{k-1}^j)^4) \tag{3.27}$$

we have the result. □

Lastly, the property $V_n = O^P(T^{-1})$ follows from the next claim.

Claim 2. *For any $k \in \{1, \ldots, T\}$, we have*

1. $Q_0 V_k = V_k Q_0 = 0$.
2. $P_0^j V_k P_0^\ell = O^P(T^{-1})$ *for any* j, ℓ.

Proof. The first is immediate. For the second, in the case $j = \ell$, summing over k in the third part of claim 1 yields the result. Next, fix j and ℓ with $j \neq \ell$. We have

$$P_0^j V_k P_0^\ell = \sum_{m=1}^{k} e_m^j A_m^\dagger P_m^j P_{m-1}^\ell A_{m-1}.$$

as shown in the proof of claim 1. We may write

$$\begin{aligned} e_m^j A_m^\dagger P_m^j P_{m-1}^\ell A_{m-1} &= e_m^j P_m^j \Phi_m^\dagger K_m^\dagger P_m^j P_{m-1}^\ell \Phi_{m-1} K_{m-1} \\ &= \Big(\prod_{z=1}^{m-1} \bar{e}_z^j\Big) K_m^\dagger P_m^j P_{m-1}^\ell \Big(\prod_{z=1}^{m-1} e_z^\ell\Big) K_{m-1} \\ &= \Big(\prod_{z=1}^{m-1} \bar{e}_z^j \cdot e_z^\ell\Big) K_m^\dagger P_m^j P_{m-1}^\ell K_{m-1}. \end{aligned}$$

Define $Z_{m-1} = \big(\prod_{z=1}^{m-1} \bar{e}_z^j \cdot e_z^\ell\big)$ and $R_m = K_m^\dagger P_m^j P_{m-1}^\ell K_{m-1}$. Then

$$P_0^j V_k P_0^\ell = \sum_{m=1}^{k} Z_{m-1} R_m.$$

Note that $Z_{m-1} = \frac{Z_m - Z_{m-1}}{\bar{e}_m^j e_m^\ell - 1}$, so that we may sum by parts to obtain

$$\begin{aligned} P_0^j V_k P_0^\ell &= \sum_{m=1}^{k} (Z_m - Z_{m-1}) \frac{R_m}{\bar{e}_m^j e_m^\ell - 1} \\ &= \frac{Z_k R_k}{\bar{e}_k^j e_k^\ell - 1} - \frac{Z_0 R_1}{\bar{e}_1^j e_1^\ell - 1} - \sum_{m=1}^{k-1} Z_m \left(\frac{R_{m+1}}{\bar{e}_{m+1}^j e_{m+1}^\ell - 1} - \frac{R_m}{\bar{e}_m^j e_m^\ell - 1} \right). \end{aligned} \tag{3.28}$$

The first two terms (the boundary terms) are $O^P(T^{-1})$ by the gap assumption **H2**, and that $R_m = O^P(T^{-1})$ for each m, which was shown in claim 1 in different language. Next,

$$Z_m \left(\frac{R_{m+1}}{\bar{e}_{m+1}^j e_{m+1}^\ell - 1} - \frac{R_m}{\bar{e}_m^j e_m^\ell - 1} \right) = \frac{Z_m}{\bar{e}_m^j e_m^\ell - 1} (R_{m+1} - R_m) + O^P(T^{-2})$$

since we may expand

$$\begin{aligned}\frac{1}{\bar{e}^j_{m+1}e^\ell_{m+1}-1} &= \frac{1}{\bar{e}^j_m e^\ell_m + (\bar{e}^j_{m+1}e^\ell_{m+1} - \bar{e}^j_m e^\ell_m) - 1} \\ &= \frac{1}{\bar{e}^j_m e^\ell_m} - \frac{(\bar{e}^j_{m+1}e^\ell_{m+1} - \bar{e}^j_m e^\ell_m)}{\bar{e}^j_m e^\ell_m} + O^P((\bar{e}^j_{m+1}e^\ell_{m+1} - \bar{e}^j_m e^\ell_m)^2)\end{aligned}$$

and use that $(\bar{e}^j_{m+1}e^\ell_{m+1} - \bar{e}^j_m e^\ell_m) = O^P(T^{-1})$ by lemma 3.2, and $R_{m+1} = O^P(T^{-1})$ as stated earlier. Thus, if we show $R_{m+1} - R_m = O^P(T^{-2})$, we sum may over $k-1 \leq T$ terms in eq. (3.28) to recover $P^j_0 V_k P^\ell_0 = O^P(T^{-1})$. By definition,

$$\begin{aligned}R_{m+1} - R_m &= K^\dagger_{m+1}P^j_{m+1}P^\ell_m K_m - K^\dagger_m P^j_m P^\ell_{m-1}K_{m-1} \\ &= K^\dagger_m(W^\dagger_{m+1}P^j_{m+1}P^\ell_m W_m - P^j_m P^\ell_{m-1})K_{m-1}.\end{aligned}$$

Recalling the expansion eq. (3.27), we have both $W^\dagger_{m+1}P^j_{m+1} = P^j_m P^j_{m+1} + O^P(T^{-2})$ and $P^\ell_m W_m = P^\ell_m P^\ell_{m-1} + O^P(T^{-2})$, hence

$$R_{m+1} - R_m = K^\dagger_m P^j_m [P^j_{m+1}P^\ell_m - P^j_m P^\ell_{m-1}]P^\ell_{m-1}K_{m-1} + O^P(T^{-2}).$$

By strategically adding zero,

$$\begin{aligned}P^j_m P^j_{m+1}P^\ell_m P^\ell_{m-1} - P^j_m P^\ell_{m-1} &= P^j_m P^j_{m+1}(P^\ell_m - P^\ell_{m+1})P^\ell_{m-1} - P^j_m(P^\ell_{m-1} - P^\ell_m) \\ &= P^j_m[P^j_{m+1}(P^\ell_m - P^\ell_{m+1}) - (P^\ell_{m-1} - P^\ell_m)]P^\ell_{m-1}.\end{aligned}$$

Now, write $P^j_{m+1} = P^j_m + A$, i.e. $A = P^j_{m+1} - P^j_m = O^P(T^{-1})$. Then,

$$P^j_m P^j_{m+1}P^\ell_m P^\ell_{m-1} - P^j_m P^\ell_{m-1} = P^j_m[(P^\ell_m - P^\ell_{m+1}) - (P^\ell_{m-1} - P^\ell_m)]P^\ell_{m-1} + P^j_m A(P^\ell_m - P^\ell_{m+1})P^\ell_{m-1}.$$

But

$$(P^\ell_m - P^\ell_{m+1}) - (P^\ell_{m-1} - P^\ell_m) = (P^\ell(\tfrac{m}{T}) - P^\ell(\tfrac{m+1}{T})) - (P^\ell(\tfrac{m-1}{T}) - P^\ell(\tfrac{m}{T})) = O^P(T^{-2}))$$

by the Taylor-Lagrange theorem, and $\|A(P^\ell_m - P^\ell_{m+1})\| \leq \|A\| \cdot \|P^\ell_m - P^\ell_{m+1}\| = O^P(T^{-2})$. This concludes the proof. □

3.1.4 *Relaxing H4*

The assumption **H4** requires a uniform bound $\sup_{s\in[0,1]} \|\mathcal{L}(s)Q(s)\| = \ell < 1$. However, in our examples in section 2.2, we don't have an easy way to check this condition. The spectral radius, on the other hand, is simple to compute. Motivated by this, we define the following weaker version of **H4**:

wH4. We have the uniform spectral bound $\ell := \sup_{s\in[0,1]} \operatorname{spr} \mathcal{L}(s)Q(s) < 1$.

Then we have the following result.

Lemma 3.7. *Assume that $[0,1] \ni s \mapsto \mathcal{L}(s)$ is continuous and satisfies **H1** to **H3** and **wH4**. Then there exists $m_0 \in \mathbb{N}$ such that for any $m \geq m_0$, the map $[0,1] \ni s \mapsto \mathcal{L}^m(s)$ satisfies **H1** to **H4**.*

Remark. Note we have the additional condition that the whole map $s \mapsto \mathcal{L}(s)$ is continuous.

Proof. For any $m \in \mathbb{N}$, we have $\|\mathcal{L}^m\| \leq \|\mathcal{L}\|^m \leq 1$ and hence **H1**. Since $(\mathcal{L}^m)^P = (\mathcal{L}^P)^m$, we also have **H3**. Let $\epsilon_0 > 0$ so that $1 - \ell > 2\epsilon_0 > 0$, and fix $s \in [0,1]$. Since $\operatorname{spr}(\mathcal{L}(s)Q(s)) = \lim_{m\to\infty} \|\mathcal{L}^m(s)Q(s)\|^{1/m} \leq \ell$ for some $m(s)$,

$$\|\mathcal{L}^m(s)Q(s)\| \leq (\ell + \epsilon_0)^m \leq \ell + \epsilon_0$$

for any $m \geq m(s)$, using that $\ell + \epsilon_0 < 1$. Since $P(s) = \sum_j P^j(s)$ is C^2 by lemma 3.2, $Q(s) = \mathrm{Id} = P(s)$ is C^2, and hence $s \mapsto \mathcal{L}^m(s)Q(s)$ is continuous, using the assumption that $s \mapsto \mathcal{L}(s)$ is continuous. Thus, there exists an open interval $I_s \ni s$ such that if $s' \in I_s$, then $\|\mathcal{L}^{m(s)}(s')Q(s')\| \leq \ell + 2\epsilon_0$. Then for $m \geq m(s)$, we have for $s' \in I_s$,

$$\|\mathcal{L}^m(s')Q(s')\| \leq \|\mathcal{L}(s')^{m-m(s)}\| \|\mathcal{L}^{m(s)}(s')Q(s')\| \leq \ell + 2\epsilon_0.$$

Since $[0,1]$ is compact and $\{I_s\}_{s\in[0,1]}$ is an open cover of $[0,1]$, we may consider a finite subcover $I_{s_1}, \ldots, I_{s_p}$ and take $m_0 = \max\{m(s_1, \ldots, m(s_p)\}$. □

Let us also note that the assumptions for lemma 3.2 may be weakened. The lemma states that peripheral eigenvalues and eigenprojections of $\mathcal{L}(s)$ are themselves C^2, under **H1** to **H4**. However, in the proof assumption **H4** is only used to ensure the eigenvalues stay away from the unit circle, which is achieved simply by **wH4**. Thus, we have the following.

Lemma 3.8. *Assume **H1** to **H3** and **wH4**. Then the peripheral eigenvalues $e^j(s)$ and eigenprojectors $P^j(s)$ of $\mathcal{L}^P(s)$ are C^2 as functions of s on $[0,1]$.*

3.2 Applying DNAUT to RIS

We've formulated theorem 3.4 in terms of **H1** to **H4**, but we would like to obtain assumptions more naturally formulated for the reduced dynamics $\mathcal{L}(s)$ of an RIS (defined in eq. (1.10), written as a function of s as discussed in the remark at the end of section 2.1). As discussed in section 1.3, the reduced dynamics $\mathcal{L}(s)$ are CPTP contractions. Let us take $\mathcal{H}$ to be a finite dimensional Hilbert space, $X = \mathcal{I}_1(\mathcal{H})$ the trace-class operators on $\mathcal{H}$, equipped with the trace norm $\|\eta\|_1 = \mathrm{Tr}((\eta^*\eta)^{1/2})$.

Def 3.9. A CPTP map $\mathcal{L}$ on $\mathcal{I}_1(\mathcal{H})$ is called *irreducible* if the only self-adjoint projections $P \in \mathcal{B}(\mathcal{H})$ satisfying $\mathcal{L}(P\mathcal{I}_1(\mathcal{H})P) \subset P\mathcal{I}_1(\mathcal{H})P$ are $P = 0$ and $P = \mathrm{Id}$.

Recalling our identification of $(\mathcal{I}_1(\mathcal{H}), \|\cdot\|_1)^*$ with $(\mathcal{B}(\mathcal{H}), \|\cdot\|)$ from chapter 6, we have the following lemma.

Lemma 3.10. *If $\mathcal{L}$ on $\mathcal{I}_1(\mathcal{H})$ is CPTP and irreducible, then $\mathcal{L}^*$ is CP, unital, and irreducible on $\mathcal{B}(\mathcal{H})$.*

Remark. That $\mathcal{L}^*$ is irreducible on $\mathcal{B}(\mathcal{H})$ means that the only self-adjoint projections $P \in \mathcal{B}(\mathcal{H})$ satisfying $\mathcal{L}^*(P\mathcal{B}(\mathcal{H})P) \subset P\mathcal{B}(\mathcal{H})P$ are $P = 0$ and $P = \mathrm{Id}$.

Proof. We've shown directly by hand that if $\mathcal{L}$ is CPTP then $\mathcal{L}^*$ is CP and unital in proposition 1.2. Here, we'll use the Kraus representation (see, e.g. [Sch01]): $\mathcal{L}$ is CPTP iff it has a Kraus representation $\mathcal{L}(\eta) = \sum_{i\in I} V_i \eta V_i^*$, where $V_i \in \mathcal{B}(\mathcal{H})$ for all $i \in I$. In the finite dimensional case we have here, we may take $|I| \leq (\dim \mathcal{H})^2$, and we will restrict to this case. As $\mathcal{L}$ is trace-preserving, $\sum_{i\in I} V_i^* V_i = \mathrm{Id}$.

For $A \in \mathcal{B}(\mathcal{H})$ and $\eta \in \mathcal{I}_1(\mathcal{H})$,

$$\mathcal{L}^*(A)(\eta) = \mathrm{Tr}(\mathcal{L}^*(A)\eta) = \mathrm{Tr}(A\mathcal{L}(\eta)) = \sum_{i\in I} \mathrm{Tr}(AV_i\eta V_i^*)$$
$$= \sum_{i\in I} \mathrm{Tr}(V_i^* A V_i \eta) = \left(\sum_{i\in I} V_i^* A V_i\right)(\eta).$$

So, $\mathcal{L}^*(A) = \sum_{i\in I} V_i^* A V_i$ is a Kraus representation for $\mathcal{L}^*$ on $\mathcal{B}(\mathcal{H})$, where we have recalled our identification $(\mathcal{I}_1(\mathcal{H}))^* = \mathcal{B}(\mathcal{H})$. Thus, $\mathcal{L}^*$ is CP. Since $\sum_{i\in I} V_i^* V_i = \mathrm{Id}$, we have $\mathcal{L}^*(\mathrm{Id}) = \mathrm{Id}$, so $\mathcal{L}^*$ is unital.

Now, to prove the equivalence of irreducibility, we will use the following claim, proven in [Far96] with different language.

Claim. *$\mathcal{L}$ is irreducible iff there is no non-trivial subspace of $\mathcal{H}$ left invariant by all V_i.*

Proof of claim. Let $P \in \mathcal{B}(\mathcal{H})$ be a self-adjoint projector. Assume $V_i P\mathcal{H} = P\mathcal{H}$ for all $i \in I$. Fix $i \in I$. Then for all $\psi \in \mathcal{H}$, $V_i P\psi = P\phi$ for some $\phi \in \mathcal{H}$. But, left multiplying by P, we have $PV_iP\psi = P\phi = V_iP\psi$. Thus, we may take $\phi = V_iP\psi$. We have $V_iP\psi = PV_iP\psi$ for all $\psi \in \mathcal{H}$. That is, $V_iP = PV_iP$. On the other hand, if $V_iP = PV_iP$, then $V_iP\mathcal{H} = PV_iP\mathcal{H} \subset P\mathcal{H}$, so $V_iP = PV_iP$ is equivalent to V_i leaves $P\mathcal{H}$ invariant.

Now, assuming each V_i leaves $P\mathcal{H}$ invariant, for any $\eta \in \mathcal{I}_1(\mathcal{H})$, we have

$$\mathcal{L}(P\eta P) = \sum_{i\in I} V_iP\eta PV_i^* = \sum_{i\in I} V_iP\eta(V_iP)^* = \sum_{i\in I} PV_iP\eta(PV_iP)^* = P\mathcal{L}(P\eta P)P.$$

That is, $\mathcal{L}(P\mathcal{I}_1(\mathcal{H})P) \subset P\mathcal{I}_1(\mathcal{H})P$. So if $\mathcal{L}$ is irreducible, then $P \in \{0, \mathrm{Id}\}$, and there must be no non-trivial subspace of $\mathcal{H}$ left invariant by all V_i.

On the other hand, assume there is no non-trivial subspace left invariant by all V_i. Assume $P \in \mathcal{B}(\mathcal{H})$ is a self-adjoint projector such that $\mathcal{L}(P\mathcal{I}_1(\mathcal{H})P) \subset P\mathcal{I}_1(\mathcal{H})P$. Let $\eta \in \mathcal{I}_1^+(\mathcal{H})$ be positive semi-definite and trace-class. Let $\xi \in \ker P$.

$$\langle \xi, \mathcal{L}(P\eta P)\xi\rangle = \sum_{i\in I} \langle \xi, V_iP\eta PV_i^*\xi\rangle = \sum_{i\in I} \langle \xi, PV_iP\eta PV_i^*P\xi\rangle = 0$$

since $P\xi = 0$. But, $\langle \xi, V_iP\eta PV_i^*\xi\rangle = \langle PV_i^*\xi, \eta PV_i^*\xi\rangle \geq 0$ since $\eta \geq 0$. Thus, $\langle \xi, V_iP\eta PV_i^*\xi\rangle = 0$ for all i. Since $\eta \geq 0$, there exists unique $\eta^{1/2} \geq 0$. We have

$$\langle \eta^{1/2}PV_i^*\xi, \eta^{1/2}PV_i^*\xi\rangle = 0$$

Thus, the vector $\eta^{1/2}PV_i^*\xi = 0$ for each $i \in I$. Choose $\eta = \mathrm{Id}$; since we are in finite dimensions, the identity operator is trace-class. Then $PV_i^*\xi = 0$ for all $\xi \in \ker P$. That is, $PV_i^*Q = 0$, and hence, $PV_i^* = PV_i^*P$ for each $i \in I$. By taking the adjoint, we have $V_iP = PV_iP$. Then $V_iP\mathcal{H} = PV_iP\mathcal{H} \subset P\mathcal{H}$. Hence, V_i leaves $P\mathcal{H}$ invariant; by assumption then, $P \in \{0, \mathrm{Id}\}$. Thus, $\mathcal{L}$ is irreducible.

□

The same proof (up to change of symbols $\mathcal{I}_1(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$) shows that $\mathcal{L}^*$ is irreducible iff there is no non-trivial subspace left invariant by all V_i^*.

But,

$$V_iP = PV_iP \iff QV_iP = 0 \iff PV_i^*Q = 0 \iff V_i^*Q = QV_i^*Q$$

That is, all the V_i leave $P\mathcal{H}$ invariant iff all the V_i^* leave $Q\mathcal{H}$ invariant. Thus, $\mathcal{L}$ is irreducible iff

$\mathcal{L}^*$ is irreducible. □

Then if $\mathcal{L}$ is CPTP and irreducible, $\mathcal{L}^*$ is CP, unital, and irreducible. Then by [Wol12, Theorem 6.6], for some $z \in \mathbb{N}$, the peripheral spectrum of $\mathcal{L}^*$ is $S_z := \{\exp(2\pi ik/z) : k = 1, 2, \ldots, z-1\}$, and each peripheral eigenvalue is simple. Note $S^1 \subset \mathbb{C}$ is the unit circle, i.e. $S^1 = \exp(i\mathbb{R})$. Since $\mathrm{sp}(\mathcal{L}^*) = \overline{\mathrm{sp}\,\mathcal{L}} = \mathrm{sp}\,\mathcal{L}$, we have that

$$\mathrm{sp}(\mathcal{L}) \cap S^1 = S_z := \{\exp(2\pi ik/z) : k = 1, 2, \ldots, z-1\}, \tag{3.29}$$

and each eigenvalue $\exp(2\pi ik/z)$ is simple. Additionally, by, e.g. [Wol12, Theorem 6.3], the eigenvector for the eigenvalue 1 may be chosen to be positive definite. With simplicity we then have that there exists a unique invariant state ρ^{inv}.

We may now formulate (stronger) assumptions for theorem 3.4 relevant to this context.

Proposition 3.11. *Assume X is a finite dimensional Banach space, $[0,1] \ni s \mapsto \mathcal{L}(s) \in \mathcal{B}(X)$ is an operator-valued function such that for each $s \in [0,1]$, $\mathcal{L}(s)$ is an irreducible CPTP map. If $s \mapsto \mathcal{L}(s)$ satisfies* **H3** *and* **H4** *then it satisfies* **H1** *to* **H4**. *If $s \mapsto \mathcal{L}(s)$ is continuous and satisfies* **H3** *and* **wH4**, *then there exists $m \in \mathbb{N}$ such that $s \mapsto \mathcal{L}^m(s)$ satisfies* **H1** *to* **H4**.

In either case, there exists $z \in \mathbb{N}$ such that $\mathrm{sp}(\mathcal{L}(s)) \cap S^1 = S_z$ for all $s \in [0,1]$.

Proof. By proposition 1.2, the CPTP map $\mathcal{L}(s)$ is a contraction for each s, and hence $s \mapsto \mathcal{L}(s)$ satisfies **H1**.

As discussed above, since $\mathcal{L}(s)$ is irreducible, for each s there exists $z_s \in \mathbb{N}$ such that $\mathrm{sp}(\mathcal{L}(s)) \cap S^1 = S_{z_s}$, and each peripheral eigenvalue is simple. We have that $1 \leq z_s \leq \dim X$ for each s. Thus, $\mathrm{sp}(\mathcal{L}(s)) \cap S^1 \subset \bigcup_{z=1}^{\dim X} S_z$ for each $s \in [0,1]$. Since $|\bigcup_{z=1}^{\dim X} S_z| \leq \sum_{z=1}^{\dim X} z = \frac{1}{2}\dim X(\dim X + 1) < \infty$, the minimal distance

$$\epsilon := \frac{1}{2}\min\left\{|e_1 - e_2| : e_1, e_2 \in \bigcup_{z=1}^{\dim X} S_z,\ e_1 \neq e_2\right\}$$

has $\epsilon > 0$. For any s then, any two distinct peripheral eigenvalues $e^j(s)$, $e^i(s)$ have $|e^j(s) - e^i(s)| > 2\epsilon$, which is **H2**. This proves the first statement, and aided by lemma 3.7, the second.

Now, we have that $s \mapsto \mathcal{L}(s)$ satisfies at least **H1** to **H3** and **wH4**, so by lemma 3.8, $s \mapsto P(s)$ is C^2. Since $\dim P(s) = z_s$, by corollary A.4, continuity implies that z_s is constant: $z_s \equiv z$ for some $z \in \mathbb{N}$. This proves the last statement. □

Remark. In the case where $s \mapsto \mathcal{L}(s)$ satisfies **H1** to **wH4** we may (and will) choose m so that the peripheral eigenvalues of $\mathcal{L}^m(s)$ are simple. If $\mathrm{sp}(\mathcal{L}(s)) \cap S^1 = S_z$, then simply choose $m \geq m_0$ such that $\gcd(m, z) = 1$, where m_0 is the exponent guaranteed by lemma 3.7.

4
Perturbation of relative entropy

Let η be a faithful state on a finite-dimensional Hilbert space $\mathcal{H}$, and let D_1 and D_2 be two operators on $\mathcal{H}$. We call these operators D_1 and D_2 corrections to our state, although for the sake of generality we won't make any assumptions on them yet; eventually, we will be interested in the case when both $\eta + D_1$ and $\eta + D_2$ are themselves states, i.e., non-negative and trace 1, implying in particular that each D_j is traceless and self-adjoint. We wish to expand the relative entropy $S(\eta + D_1 | \eta + D_2)$ in terms of $\|D_1\|$ and $\|D_2\|$, for sufficiently small $\|D_1\|, \|D_2\|$.

Let's note right away that if $\eta + D_1$ and $\eta + D_2$ are states, there is a known lower bound which does not require perturbation theory, the Quantum Pinsker Inequality:

$$S(\eta + D_1 | \eta + D_2) \geq \frac{1}{2} \|D_1 - D_2\|_1^2 = \frac{1}{2} \operatorname{Tr}(|D_1 - D_2|)^2. \tag{4.1}$$

See Theorem I.1.15 of [OP93] for a proof of this lower bound in algebraic terms, e.g. commutative subalgebras, and Theorem 11.9.2 of [Wil11] for essentially the same proof in language probably more familiar to quantum information theorists, e.g. states diagonal in the same basis.

For our purposes, we will use perturbation theory to expand the relative entropy, and obtain a leading term, for a lower bound alone will not always suffice. To do so, we wish to use the Dunford-Taylor holomorphic functional calculus to write the relative entropy as an integral around the spectrum of $\eta + D_1$ and $\eta + D_2$ (see e.g. [Kat76, Section I.5.6] for reference). The choice of path for the integral will be important, but we won't know what we require from the path until further into the computation.

Our strategy will be to expand the resolvent $R_\ell(\zeta) := R_{\eta + D_\ell}(\zeta) = (\eta + D_\ell - \zeta)^{-1}$ of $\eta + D_\ell$

for $\ell = 1, 2$, in terms of the resolvent $R_\eta(\zeta)$ of η, using the Neumman expansion

$$R_\ell(\zeta) = R_\eta(\zeta)(\mathrm{Id} + D_\ell R_\eta(\zeta))^{-1} = R_\eta(\zeta) \sum_{n\geq 0} (-D_\ell R_\eta(\zeta))^n$$

which converges absolutely if $\|D_\ell R_\eta(\zeta)\| < 1$ [Kat76, Section II.1.3]. Let us assume we may choose a curve Γ encircling $\mathrm{sp}(\eta + D_\ell)$ and $\mathrm{sp}(\eta)$ so that for each $\zeta \in \Gamma$ we have $\|D_\ell R_\eta(\zeta)\| < 1$, for $\ell = 1, 2$. We also require that $\Gamma \subset \{z \in \mathbb{C} : \mathrm{Re}\, z > 0\}$ so that the logarithm is well-defined. These are the minimal requirements we have of a path for our integration to use the holomorphic calculus and this expansion; we will find another requirement later. Then, in lemma 4.4 we will prove that a path satisfying our requirements exists.

Now, we may write

$$\begin{aligned} S(\eta + D_1|\eta + D_2) &= \mathrm{Tr}\left(-\frac{1}{2\pi i} \int_\Gamma \zeta \log \zeta R_1(\zeta) + (\eta + D_1) \frac{1}{2\pi i} \int_\Gamma \log \zeta R_2(\zeta)\, \mathrm{d}\zeta \right) \\ &= \mathrm{Tr}\Big(-\frac{1}{2\pi i} \int_\Gamma \zeta \log \zeta R_\eta(\zeta) \sum_{n\geq 0} (-D_1 R_\eta(\zeta))^n \, \mathrm{d}\zeta \\ &\qquad + (\eta + D_1) \frac{1}{2\pi i} \int_\Gamma \log \zeta R_\eta(\zeta) \sum_{n\geq 0} (-D_2 R_\eta(\zeta))^n \, \mathrm{d}\zeta \Big). \end{aligned}$$

Separating the higher order terms, we have

$$\begin{aligned} &= \mathrm{Tr}\left(\frac{1}{2\pi i} \int_\Gamma \zeta \log \zeta R_\eta(\zeta) D_1 R_\eta(\zeta) - \zeta \log \zeta R_\eta(\zeta) (D_1 R_\eta(\zeta))^2 \, \mathrm{d}\zeta \right) \\ &\quad - \mathrm{Tr}(D_1 \log \eta) \\ &\quad - \mathrm{Tr}\Big((\eta + D_1) \frac{1}{2\pi i} \int_\Gamma (\log \zeta R_\eta(\zeta D_2 R_\eta(\zeta) \\ &\qquad\qquad - \log \zeta R_\eta(\zeta) D_2 R_\eta(\zeta)) D_2 R_\eta(\zeta)) \Big) \\ &\quad + r(\Gamma, \eta, D_1, D_2), \end{aligned}$$

where the remainder term is

$$\begin{aligned} r(\Gamma, \eta, D_1, D_2) :&= \mathrm{Tr}\left(-\frac{1}{2\pi i} \int_\Gamma \zeta \log \zeta R_\eta(\zeta) \sum_{n\geq 3} (-D_1 R_\eta(\zeta))^n \, \mathrm{d}\zeta \right) \\ &\quad + \mathrm{Tr}\left(-\frac{\eta + D_1}{2\pi i} \int_\Gamma \zeta \log \zeta R_\eta(\zeta) \sum_{n\geq 3} (-D_2 R_\eta(\zeta))^n \right). \end{aligned}$$

We may group the earlier terms by the number of factors of D_ℓ they contain.

Def 4.1. Let f be a function of ζ holomorphic in some open domain $\Omega \subset \mathbb{C}$ containing $\operatorname{sp}\eta$, and M and D two matrices on $\mathcal{H}$. Then for any curve $\Gamma \subset \Omega$ encircling $\operatorname{sp}\eta$, define

$$T_1(M, D, f) := \operatorname{Tr}\left(-\frac{1}{2\pi i} M \int_\Gamma R_\eta(\zeta) D R_\eta(\zeta) f(\zeta)\, \mathrm{d}\zeta\right)$$
$$T_2(M, D, f) := \operatorname{Tr}\left(-\frac{1}{2\pi i} M \int_\Gamma R_\eta(\zeta) D R_\eta(\zeta) D R_\eta(\zeta) f(\zeta)\, \mathrm{d}\zeta\right).$$

Now, let us return to our curve Γ defined above. We may define $f(\zeta) = \zeta \log \zeta$, and $g(\zeta) = \log \zeta$; both functions are holomorphic in $\Omega = \{z \in \mathbb{C} : z > 0\}$, which contains Γ, which, in turn, encircles $\operatorname{sp}\eta$. Returning to our expansion of $S(\eta + D_1|\eta + D_2)$, we have

$$\begin{aligned} S(\eta + D_1|\eta + D_2) = &-T_1(\mathrm{Id}, D_1, f) + T_2(\mathrm{Id}, D_1, f) - \operatorname{Tr}(D_1 \log \eta) \\ &+ T_1(\eta, D_2, g) + T_1(D_1, D_2, g) - T_2(\eta, D_2, g) \\ &- T_2(D_1, D_2, g) + r(\Gamma, \eta, D_1, D_2). \end{aligned} \tag{4.2}$$

It thus remains to compute or estimate T_1, T_2, and r. To do so, we will write the spectral decomposition of η as $\eta = \sum_i \mu_i p_i$.

Lemma 4.2. *As defined in definition 4.1, T_1 and T_2 satisfy*

$$T_1(M, D, f) = -\sum_i \operatorname{Tr}(M p_i D p_i) f'(\mu_i) - \sum_{i<j} \operatorname{Tr}(M(p_i D p_j + p_j D p_i)) \frac{f(\mu_i) - f(\mu_j)}{\mu_i - \mu_j},$$

and, if $[M, \eta] = 0$,

$$\begin{aligned} T_2(M, D, f) = &\sum_i \operatorname{Tr}(M D p_i D p_i) \frac{f''(\mu_i)}{2} \\ &+ \sum_{i \neq j} \operatorname{Tr}(M D p_j D p_i) \frac{f'(\mu_i)}{\mu_i - \mu_j} + \operatorname{Tr}(M D p_j D p_i) \frac{f(\mu_i) - f(\mu_j)}{(\mu_i - \mu_j)^2}. \end{aligned}$$

Proof. Since $\eta = \sum_i p_i \mu_i$, we can write

$$R_\eta(\zeta) = \sum_i (\mu_i - \zeta)^{-1} p_i$$

by the spectral theorem. Then we have

$$\begin{aligned} T_1(M,D,f) &= \operatorname{Tr}\left(-\frac{1}{2\pi i}M\sum_{i,j}\int_\Gamma (\mu_i-\zeta)^{-1}(\mu_j-\zeta)^{-1}p_iDp_jf(\zeta)\,\mathrm{d}\zeta\right) \\ &= \operatorname{Tr}\left(-\frac{1}{2\pi i}M\sum_{i,j}p_iDp_j\int_\Gamma (\mu_i-\zeta)^{-1}(\mu_j-\zeta)^{-1}f(\zeta)\,\mathrm{d}\zeta\right). \end{aligned}$$

The integral is only over $\mathbb{C}$-valued functions, and we may apply Cauchy's theorem:

$$\frac{1}{2\pi i}(\zeta-\mu_i)^{-1}(\zeta-\mu_j)^{-1}f(\zeta)\,\mathrm{d}\zeta = \begin{cases} f'(\mu_i) & \text{if } i=j \\ \frac{f(\mu_i)-f(\mu_j)}{\mu_i-\mu_j} & \text{if } i\neq j. \end{cases}$$

Thus, we have

$$T_1(M,D,f) = -\operatorname{Tr}\left(M\sum_{i\neq j}p_iDp_j\frac{f(\mu_i)-f(\mu_j)}{\mu_i-\mu_j}\right) - \operatorname{Tr}\left(M\sum_i p_iDp_if'(\mu_i)\right).$$

Lastly, the symmetry

$$\sum_{i\neq j}\operatorname{Tr}(Mp_iDp_j)\frac{f(\mu_i)-f(\mu_j)}{\mu_i-\mu_j} = \sum_{i<j}\operatorname{Tr}(M(p_iDp_j+p_jDp_i))\frac{f(\mu_i)-f(\mu_j)}{\mu_i-\mu_j}$$

yields the result. Next,

$$\begin{aligned} T_2(M,D,f) :&= \operatorname{Tr}\left(-\frac{1}{2\pi i}M\int_\Gamma R_\eta(\zeta)DR_\eta(\zeta)DR_\eta(\zeta)f(\zeta)\,\mathrm{d}\zeta\right) \\ &= \operatorname{Tr}\left(-\frac{1}{2\pi i}M\sum_{i,j,k}\int_\Gamma (\mu_i-\zeta)^{-1}p_iD(\mu_j-\zeta)^{-1}p_jD(\mu_k-\zeta)^{-1}p_kf(\zeta)\,\mathrm{d}\zeta\right) \\ &= \operatorname{Tr}\left(-\frac{1}{2\pi i}Mp_iDp_jDp_k\sum_{i,j,k}\int_\Gamma (\mu_i-\zeta)^{-1}(\mu_j-\zeta)^{-1}(\mu_k-\zeta)^{-1}f(\zeta)\,\mathrm{d}\zeta\right). \end{aligned}$$

Now, we use the assumption $[M,\eta]=0$ and the cyclicity of the trace to write

$$T_2(M,D,f) = \operatorname{Tr}\left(-\frac{1}{2\pi i}\sum_{i,j,k}Mp_kp_iDp_jD\int_\Gamma (\mu_i-\zeta)^{-1}(\mu_j-\zeta)^{-1}(\mu_k-\zeta)^{-1}f(\zeta)\,\mathrm{d}\zeta\right).$$

But $p_k p_i = \delta_{ik} p_i$, so we may sum over k to obtain

$$T_2(M, D, f) = \operatorname{Tr}\left(-\frac{1}{2\pi i} \sum_{i,j} M p_i D p_j D \int_\Gamma (\mu_i - \zeta)^{-2} (\mu_j - \zeta)^{-1} f(\zeta)\, \mathrm{d}\zeta \right).$$

Another application of Cauchy's theorem yields

$$\frac{1}{2\mathrm{i}\pi} \int_\Gamma (\zeta - \mu_i)^{-2} (\zeta - \mu_j)^{-1} f(\zeta)\, \mathrm{d}\zeta = \begin{cases} \frac{f''(\mu_i)}{2} & \text{if } i = j \\ \frac{f(\mu_j) - f(\mu_i)}{(\mu_j - \mu_i)^2} + \frac{f'(\mu_i)}{(\mu_i - \mu_j)} & \text{if } i \neq j, \end{cases}$$

which yields our result. □

Recalling the specific terms we need to compute from eq. (4.2), from lemma 4.2 we can obtain simpler forms of two types of terms which appear often.

Corollary 4.3. *In the setup of lemma 4.2, if* $[M, \eta] = 0$*, we have*

$$T_1(M, D, f) = -\operatorname{Tr}(M D f'(\eta)).$$

Additionally, when $M = \mathrm{Id}$ *we have*

$$T_2(\mathrm{Id}, D, f) = \sum_i \operatorname{Tr}((D p_i)^2) \frac{f''(\mu_i)}{2} + \sum_{i<j} \operatorname{Tr}(D p_j D p_i) \frac{f'(\mu_i) - f'(\mu_j)}{\mu_i - \mu_j}.$$

Proof. For the T_1 result, we note if $[M, \eta] = 0$, then $[M, p_i] = 0$ for each i. Then $\operatorname{Tr}(M p_i D p_j) = \operatorname{Tr}(M D p_j p_i) = 0$, using the cyclicity of the trace and that $p_j p_i = 0$. So the second term vanishes, and the first term is

$$T_1(M, D, f) = -\sum_i \operatorname{Tr}(M p_i D p_i) f'(\mu_i) = -\sum_i \operatorname{Tr}(M D p_i f'(\mu_i)) = -\operatorname{Tr}(M D f'(\eta))$$

as we wanted. For the T_2 result, $\operatorname{Tr}(D p_j D p_i) = \operatorname{Tr}(p_j D p_i D) = \operatorname{Tr}(D p_i D p_j)$ using cyclicity of the trace. Then the symmetry

$$\operatorname{Tr}(D p_j D p_i) \frac{f(\mu_i) - f(\mu_j)}{(\mu_i - \mu_j)^2} = -(i \leftrightarrow j)$$

yields that the sum over $i \neq j$ of this term is zero. Lastly, given (i, j) with $i \neq j$, then

$$\operatorname{Tr}(D p_j D p_i) \frac{f(\mu_i)}{\mu_i - \mu_j} + (i \leftrightarrow j) = \operatorname{Tr}(D p_j D p_i) \frac{f(\mu_i) - f(\mu_j)}{\mu_i - \mu_j}.$$ □

Using lemma 4.2 and corollary 4.3, we find

$$\begin{aligned}
T_1(\mathrm{Id}, D_1, f) &= -\operatorname{Tr}\big(D_1(\log\eta + \mathrm{Id})\big) = -\operatorname{Tr}(D_1\log\eta) - \operatorname{Tr}(D_1),\\
T_1(\eta, D_2, g) &= -\operatorname{Tr}(\eta D_2\eta^{-1}) = -\operatorname{Tr}(D_2),\\
T_1(D_1, D_2, g) &= -\sum_i \operatorname{Tr}(D_1 p_i D_2 p_i)\frac{1}{\mu_i} - \sum_{i<j}\operatorname{Tr}\big(D_1(p_i D_2 p_j + p_j D_2 p_i)\big)\frac{\log(\mu_i)-\log(\mu_j)}{\mu_i-\mu_j},\\
T_2(\mathrm{Id}, D_1, f) &= \sum_i \operatorname{Tr}\big((D_1 p_i)^2\big)\frac{1}{2\mu_i} + \sum_{i<j}\operatorname{Tr}(D_1 p_j D_1 p_i)\frac{\log(\mu_i)-\log(\mu_j)}{\mu_i-\mu_j},\\
T_2(\eta, D_2, g) &= \operatorname{Tr}\Big(-\sum_i (D_2 p_i)^2\frac{1}{2\mu_i} + \sum_{i<j} D_2 p_j D_2 p_i\frac{\log(\mu_j)-\log(\mu_i)}{\mu_i-\mu_j}\Big).
\end{aligned}$$

We only have left to bound the remainder term $r(\Gamma, \eta, D_1, D_2)$ and the term $T_2(D_1, D_2, g)$. To do so, we will make use of our remaining freedom to choose the shape of Γ; so far, we've only required that Γ encircle η, $\eta + D_1$, and $\eta + D_2$ with $\|D_\ell R_\eta(\zeta)\|^{-1} < 1$ for each $\zeta \in \Gamma$ and both $\ell = 1, 2$.

We'll call a stadium shaped path two semi-circles joined by straight lines, as shown in fig. 4.1.

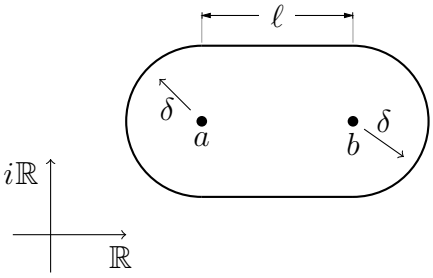

Figure 4.1: Illustration of a stadium shaped path. As depicted, a stadium shaped path consists of two semicircles at a and at b with radii δ, connected by segments of length ℓ. The length of the path is $2(\ell + \pi\delta)$.

Lemma 4.4. *There exists a stadium shaped path $\Gamma \subset \{z \in \mathbb{C} : \operatorname{Re} z > 0\}$ with the following properties:*

1. *For some constant c_η only depending on η, then for $\ell = 1, 2$ if $\|D_\ell\| < c_\eta$, then Γ encircles $\eta + D_\ell$.*
2. *Additionally, for $\|D_\ell\| < c_\eta$, we have $\|D_\ell R_\ell(\zeta)\| < 1$.*
3. *Lastly, defining $B := 2^9 \dim \mathcal{H}\frac{\log(2/\mu_1)}{\mu_1^4}(\|D_1\| + \|D_2\|)^3$, where $\mu_1 := \inf\operatorname{sp}\eta$, we have the bounds*

$$|r(\Gamma, \eta, D_1, D_2)| \le B, \qquad |T_2(D_1, D_2, g)| \le B.$$

Proof. Let the set of eigenvalues of η be $\mathfrak{S}(\eta) = \{\mu_1, \ldots, \mu_N\} \subset \mathbb{R}$, with $\mu_1 = \inf\operatorname{sp}\eta > 0$, using that η is a faithful state. Let Γ be the stadium shaped path depicted in fig. 4.2.

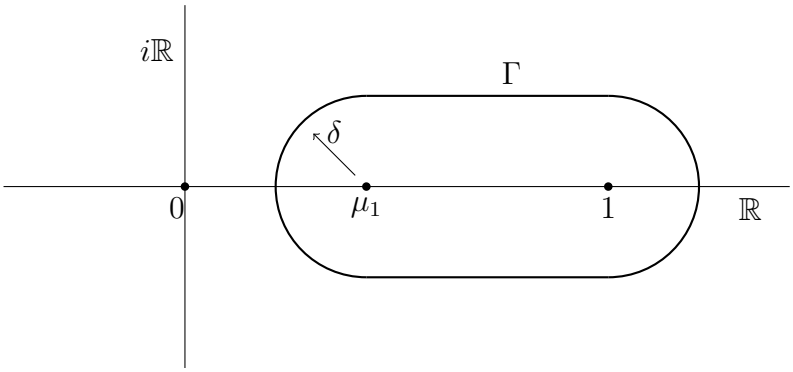

Figure 4.2: We define Γ to be the stadium shaped path with radii $\delta < \mu_1$, with semicircles centered on μ_1 and 1, as shown. Note $\operatorname{sp}\eta \subset [\mu_1, 1] \subset \mathbb{R}$. We may choose any δ with $0 < \delta < \mu_1$; we choose to keep it arbitrary for clarity, to see the role this radius plays.

Then $\sup_{\zeta\in\Gamma} \|R_\eta(\zeta)\| = \delta^{-1}$, by proposition A.7.

Since $\mathfrak{S}(\eta + D_\ell)$ the set of eigenvalues of $\eta + D_\ell$, is continuous by theorem A.9, for small enough D_ℓ, we have $\operatorname{dist}(\mathfrak{S}(\eta + D_\ell), \mathfrak{S}(\eta)) < \delta$. Then in some ordering of the eigenvalues of $\eta + D_\ell$, say $(\mu_1^{(\ell)}, \dots, \mu_N^{(\ell)})$, we have $|\mu_k^{(\ell)} - \mu_k| < \delta$ for each $k = 1, \dots, N$. Thus, Γ encircles the spectrum of $\eta + D_\ell$, for small enough $\|D_\ell\|$, for $\ell = 1, 2$.

Moreover, if we choose $\|D_\ell\| < \frac{1}{2} \inf_{\zeta\in\eta} \|R_\eta(\zeta)\|^{-1} = \frac{1}{2}\delta$ for $\ell = 1, 2$, then

$$\|D_\ell R_\eta(\zeta)\| \leq \|D_\ell\| \cdot \|R_\eta(\zeta)\| < \frac{1}{2}\|R_\eta(\zeta)\|^{-1}\|R_\eta(\zeta)\| = \frac{1}{2}$$

for all $\zeta \in \Gamma$. Now, for any function f and $\ell = 1, 2$, we may bound

$$\begin{aligned}
\left\| -\frac{1}{2\pi i} \int_\Gamma f(\zeta) R_\eta(\zeta) \sum_{n\geq 3} (-D_\ell R_\eta(\zeta))^n \, \mathrm{d}\zeta \right\| &\leq \frac{1}{2\pi} \operatorname{length}(\Gamma) \sup_{\zeta\in\Gamma} |f(\zeta)| \cdot \delta^{-1} \cdot \sum_{n\geq 3} \|D_\ell\|^n \cdot \delta^{-n} \\
&= \frac{1}{2\pi} \operatorname{length}(\Gamma) \sup_{\zeta\in\Gamma} |f(\zeta)| \cdot \delta^{-1} \cdot \frac{\|D_\ell\|^3 \delta^{-3}}{1 - \|D_\ell\|\delta^{-1}} \\
&\leq \frac{1}{\pi}(2 + 2\pi\delta) \sup_{\zeta\in\Gamma} |f(\zeta)| \, \|D_\ell\|^3 \delta^{-4}.
\end{aligned}$$

Now, to bound r, we take $f(\zeta) = \zeta \log \zeta$. Then

$$|f(\zeta)| = |\zeta| \cdot |\log \zeta| = |\zeta| \cdot |\log|\zeta| + i \arg \zeta| \leq |\zeta|(|\log|\zeta|| + \pi).$$

But by monotonicity of the logarithm, for $\zeta \in \Gamma$, we have

$$\log(\mu_1 - \delta) \leq \log|\zeta| \leq \log(1 + \delta). \tag{4.3}$$

For simplicity, let us take $\delta = \frac{\mu_1}{2}$. This fixes $c_\eta = \mu_1/4$. Note that if $\dim \mathcal{H} \geq 2$, we must have

$\mu_1 \leq \frac{1}{2}$, otherwise $\mathrm{Tr}(\eta) > 1$. But for $x \in (0, \frac{1}{2}]$,

$$-\log(x) \geq \log(1+x) \iff \frac{1}{x} \geq 1 + x \iff 1 \geq x(1+x)$$

and $x(1+x) \leq \frac{3}{4}$. Thus, our bound eq. (4.3) becomes

$$\log \frac{\mu_1}{2} \leq \log|\zeta| \leq \log(1 + \frac{\mu_1}{2}) \leq -\log \frac{\mu_1}{2}.$$

and we have $|\log|\zeta|| \leq \log \frac{2}{\mu_1}$. Hence, for all $\zeta \in \Gamma$,

$$|f(\zeta)| = |\zeta \log \zeta| \leq \big(1 + \frac{\mu_1}{2}\big)\big(\log \frac{2}{\mu_1} + \pi\big)$$

using that $|\zeta| \leq 1 + \delta = 1 + \frac{\mu_1}{2}$. We may simplify the expression by noting $\pi < 3 \log 4 \leq 3 \log \frac{2}{\mu_1}$, since $\mu_1 \leq \frac{1}{2}$. Also, $1 + \frac{\mu_1}{2} < 2$, so $|f(\zeta)| < 2^3 \log \frac{2}{\mu_1}$. Then,

$$\begin{aligned}\left\| -\frac{1}{2\pi i} \int_\Gamma f(\zeta) R_\eta(\zeta) \sum_{n \geq 3} (-D_\ell R_\eta(\zeta))^n \, \mathrm{d}\zeta \right\| &\leq 2^8 \frac{1}{\pi}(1 + \frac{\pi\mu_1}{2}) \log \frac{2}{\mu_1} \cdot \|D_\ell\|^3 \frac{1}{\mu_1^4}. \\ &\leq 2^9 \|D_\ell\|^3 \frac{1}{\mu_1^4} \log \frac{2}{\mu_1}.\end{aligned}$$

Next, we use that for an operator A on $\mathcal{H}$, $|\mathrm{Tr}(A)| \leq \mathrm{Tr}\,|A| \leq \dim \mathcal{H} \|A\|$to obtain

$$|r(\Gamma, \eta, D_1, D_2)| \leq \dim \mathcal{H} \cdot 2^9 \frac{1}{\mu_1^4} \log \frac{2}{\mu_1} \left(\|D_1\|^3 + \|\eta + D_1\| \cdot \|D_2\|^3 \right).$$

Using $\|\eta\| \leq 1$, as η is self-adjoint with all eigenvalues at most 1,

$$\begin{aligned}&\leq \dim \mathcal{H} \cdot 2^9 \frac{1}{\mu_1^4} \log \frac{2}{\mu_1} \left(\|D_1\|^3 + (1 + \|D_1\|) \cdot \|D_2\|^3 \right) \\ &\leq \dim \mathcal{H} \cdot 2^9 \frac{1}{\mu_1^4} \log \frac{2}{\mu_1} \left(\|D_1\| + \|D_2\| \right)^3.\end{aligned}$$

Finally,

$$T_2(D_1, D_2, \log \zeta) := \mathrm{Tr}\left(-\frac{1}{2\pi i} D_1 \int_\Gamma R_\eta(\zeta) D_2 R_\eta(\zeta) D_2 R_\eta(\zeta) \log \zeta \, \mathrm{d}\zeta \right).$$

Clearly, we may use our estimate of $|\log \zeta|$ on Γ and our other bounds to obtain the (even

looser) estimate of

$$|T_2(D_1, D_2, \log\zeta)| \leq \dim\mathcal{H} \cdot 2^9 \frac{1}{\mu_1^4} \log\frac{2}{\mu_1} \left(\|D_1\| + \|D_2\|\right)^3. \qquad \square$$

With this choice of path Γ, we have that

$$\begin{aligned} S(\eta + D_1|\eta + D_2) = \operatorname{Tr}(D_1 - D_2) + \sum_i \operatorname{Tr}\Big(\big((D_1 - D_2)p_i\big)^2\Big)(2\mu_i)^{-1} \\ + \sum_{i<j} \operatorname{Tr}((D_1 - D_2)p_j(D_1 - D_2)p_i)\frac{\log(\mu_i) - \log(\mu_j)}{\mu_i - \mu_j} + O_\eta(\|D\|^3). \end{aligned} \tag{4.4}$$

where we write $O_\eta(\|D\|^k)$ to mean a term which is bounded by $C_\eta(\|D_1\| + \|D_2\|)^k$ for $\|D_1\|$ and $\|D_2\|$ small enough, where C_η is a constant only depending on η. We may compile the main results of the chapter thus far in the following proposition.

Proposition 4.5. *Let η be a faithful state with spectral decomposition $\eta = \sum_i \mu_i p_i$, where μ_j are the eigenvalues and p_j the associated spectral projections. Let D_1, D_2 be two perturbations of η. There exist constants $C_\eta > 0$ and $D_\eta > 0$ depending only on η, such that if D_1, D_2 satisfy $\|D_j\| \leq D_\eta$, $j = 1, 2$, then the relative entropy $S(\eta + D_1|\eta + D_2)$ satisfies*

$$\left|S(\eta + D_1|\eta + D_2) - F_\eta(D_1 - D_2) - \operatorname{Tr}(D_1 - D_2)\right| \leq C_\eta(\|D_1\| + \|D_2\|)^3 \tag{4.5}$$

where $F_\eta(A) := F_\eta(A, A)$ for

$$F_\eta(A, B) := \sum_i \operatorname{Tr}(Ap_iBp_i)\frac{1}{2\mu_i} + \sum_{i<j} \operatorname{Tr}(Ap_jBp_i)\frac{\log(\mu_i) - \log(\mu_j)}{\mu_i - \mu_j}. \tag{4.6}$$

Moreover, we may take $D_\eta = \inf \operatorname{sp}(\eta)/4$.

Remark. We see that if $\eta = \eta_0 + \Delta$ with $\operatorname{Tr}\Delta = \operatorname{Tr}(D_1 + D_2) = 0$, then

$$S(\eta + D_1|\eta + D_2) = F_{\eta_0}(D_1 - D_2) + O_{\eta_0}\big((\|D_1\| + \|D_2\| + \|\Delta\|)^3\big).$$

Note that this proposition holds true if any of the norms $\|\cdot\|$ are replaced by the trace norm $\|\cdot\|_1$ due to the inequalities

$$\|D\| = \sup_{\substack{\psi\in\mathcal{H} \\ \|\psi\|=1}} \|D\psi\| \leq \sqrt{\operatorname{Tr}(D^*D)} \leq \operatorname{Tr}|D| = \|D\|_1.$$

E.g. if $\|D\|_1 < D_\eta$ then $\|D\| < D_\eta$, and if $F(D) \leq C(\|D\|)^3$ then $F(D) \leq C(\|D\|_1)^3$.

We'll conclude this chapter with some properties of the leading order term $F_\eta(A, B)$.

Corollary 4.6. *The map $F_\eta(\cdot, \cdot)$ defined in eq. (4.6) has the following properties:*

1. *F_η is a bilinear form.*
2. *We have the bound*

$$|F_\eta(A, B)| \leq 2\|A\|\|B\|\frac{(\dim \mathcal{H})^2}{\mu_1}$$

   where $\mu_1 > 0$ is the smallest eigenvalue of η.
3. *If A is self-adjoint, then $F_\eta(A) \geq 0$ with equality iff $A = 0$.*

Proof. The first point is immediate from the definition. For the second, we will first employ the result

$$|\operatorname{Tr}(AB)| \leq \|A\| \operatorname{Tr}|B|. \qquad \text{([JOPP12, Inequality 2.4])}$$

Using this and $\|p_i\| = 1$,

$$\begin{aligned}\sum_i |\operatorname{Tr}(Ap_iBp_i)| &\leq \sum_i \|Ap_iB\| \operatorname{Tr}|p_i| \\ &\leq \sum_i \|A\|\|p_i\|\|B\| \operatorname{Tr}|p_i| \leq \|A\|\|B\| \sum_i \operatorname{Tr}|p_i|.\end{aligned}$$

By proposition A.2, $\operatorname{Tr}|p_i| = \dim \operatorname{Ran} p_i$. Furthermore, $\sum_i \dim \operatorname{Ran} p_i = \dim \mathcal{H}$, since $\mathcal{H} = \oplus_i \operatorname{Ran} p_i$. We have thus bounded the first term.

To bound the second, we'll use the mean value theorem: for each $1 \leq i, j \leq \dim \mathcal{H}$, there exists some $c_{ij} \in [\mu_1, 1]$ such that

$$\frac{\log(\mu_i) - \log(\mu_j)}{\mu_i - \mu_j} = \frac{1}{c_{ij}} \leq \frac{1}{\mu_1}.$$

Then,

$$\begin{aligned}\left|\sum_{i<j} \operatorname{Tr}(Ap_jBp_i)\frac{\log(\mu_i) - \log(\mu_j)}{\mu_i - \mu_j}\right| &= \left|\sum_{i<j} \operatorname{Tr}(Ap_jBp_i)\frac{1}{c_{ij}}\right| \leq \frac{1}{\mu_1}\sum_{i<j} |\operatorname{Tr}(p_iAp_jBp_i)| \\ &\leq \frac{1}{\mu_1}\|A\|\|\mathcal{B}\| \sum_{i<j} \operatorname{Tr}|p_i| \leq \frac{1}{\mu_1}\|A\|\|B\|(\dim \mathcal{H})^2,\end{aligned}$$

which completes the bound. For the third point,

$$\begin{aligned}
F_\eta(A) &= \sum_i \mathrm{Tr}(Ap_iAp_i)\frac{1}{2\mu_i} + \sum_{i<j} \mathrm{Tr}(Ap_jAp_i)\frac{\log(\mu_i)-\log(\mu_j)}{\mu_i-\mu_j} \\
&= \sum_i \mathrm{Tr}(p_iAp_ip_iAp_i)\frac{1}{2\mu_i} + \sum_{i<j} \mathrm{Tr}(p_iAp_jp_jAp_i)\frac{\log(\mu_i)-\log(\mu_j)}{\mu_i-\mu_j} \\
&= \sum_i \mathrm{Tr}((p_iAp_i)^*p_iAp_i)\frac{1}{2\mu_i} + \sum_{i<j} \mathrm{Tr}((p_jAp_i)^*p_jAp_i)\frac{\log(\mu_i)-\log(\mu_j)}{\mu_i-\mu_j} \\
&= \sum_i \mathrm{Tr}(|p_iAp_i|^2)\frac{1}{2\mu_i} + \sum_{i<j} \mathrm{Tr}(|p_jAp_i|^2)\frac{\log(\mu_i)-\log(\mu_j)}{\mu_i-\mu_j}.
\end{aligned}$$

Each term is thus non-negative, using the monotonicity of the logarithm. If $F_\eta(A) = 0$ then (since $(X,Y) \mapsto \mathrm{Tr}(X^*Y)$ is an inner product), for all i,j we must have $p_jAp_i = 0$. Summing over i and j yields $A = 0$. □

5
Assemblage

5.1 General results

Combining our adiabatic theorem, theorem 3.4, with the perturbative expansion of the relative entropy, proposition 4.5, we may compute the entropy production

$$\sigma_{k,T} := S\big(U_{k,T}(\rho_{k-1,T}\otimes\xi^{\mathrm{i}}_{k,T})U^*_{k,T}|\mathcal{L}_{k,T}(\rho_{k-1,T})\otimes\xi^{\mathrm{i}}_{k,T}\big)$$

to leading order.

Proposition 5.1. *Consider an RIS process obeying the assumption* (ADRIS)*, such that the reduced dynamics $\mathcal{L}(s)$ is irreducible for all $s\in[0,1]$ and satisfies* **H4**. *Let $\rho^{\mathrm{inv}}_{k,T}$ denote the (unique) invariant state of $\mathcal{L}_{k,T}$, and let $P^1_{k,T}$ denote the associated spectral projector. Let ρ^{i} be the initial state of the system, and assume $(P^1_{0,T}+Q_{0,T})\rho^{\mathrm{i}}=\rho^{\mathrm{i}}$. Define:*

$$X_{k,T} := U_{k,T}\rho^{\mathrm{inv}}_{k,T}\otimes\xi_{k,T}U^*_{k,T}-\rho^{\mathrm{inv}}_{k,T}\otimes\xi_{k,T}, \tag{5.1}$$

$$D_{k,T} := \mathcal{L}_{k,T}(\rho_{k-1,T}-\rho^{\mathrm{inv}}_{k,T})\otimes\xi_{k,T}-U_{k,T}((\rho_{k-1,T}-\rho^{\mathrm{inv}}_{k,T})\otimes\xi_{k,T})U^*_{k,T}. \tag{5.2}$$

Then there exists $T_0>0$ and $D_0>0$ such that for $T\geq\max(T_0,(D_0(1-\ell))^{-1})$ and $k\leq T$ large enough so that $\|Q_{0,T}\rho^{\mathrm{i}}\|\ell^k<D_0$, and assuming $\|X_{k,T}\|_1<D_0$, one has

$$|\sigma_{k,T}-F_{k,T}(X_{k,T}-D_{k,T})|\leq C^P_{\eta_{k,T}}\big((\|X_{k,T}\|_1+\|Q_{0,T}\rho^{\mathrm{i}}\|_1\ell^k+(T(1-\ell)^{-1})^3)\big), \tag{5.3}$$

where $F_{k,T}=F_{\eta_{k,T}}$ is associated to $\eta_{k,T}:=\rho^{\mathrm{inv}}_{k,T}\otimes\xi_{k,T}$, D_0 depends on $\eta_{k,T}$ only, T_0 on c_P defined by eq. (3.1) *and $N=\dim P(s)$ only, and $C^P_{\eta_{k,T}}$ on c_P, N, and on $\eta_{k,T}$ only. In particular, we may take $D_0=\inf\mathrm{sp}(\eta_{k,T})/4$.*

Remark. This detailed proposition calls for some remarks. First, the assumption $(P^1_{0,T}+Q_{0,T})\rho^{\mathrm{i}}=$

ρ^{i} ensures that

$$(\mathrm{Id} - P^1_{0,T})\rho^{\mathrm{i}} = (\mathrm{Id} - P^1_{0,T})(P^1_{0,T} + Q_{0,T})\rho^{\mathrm{i}} = Q_{0,T}\rho^{\mathrm{i}}$$

which we may interpret as the non-invariant part of the state is strictly contracting. We must be careful with such language as the eigenprojections need not be self-adjoint. This way, the adiabatic theorem theorem 3.4 will transport ρ^{i} to $\rho^{\mathrm{inv}}_{k,T}$, up to loss from the strictly contracting part. Of particular interest is the case $P^1_{0,T}\rho^{\mathrm{i}} = \rho^{\mathrm{i}}$, when $Q_{0,T}\rho^{\mathrm{i}} = 0$. Next, $X_{k,T}$ characterizes the invariance of $\rho^{\mathrm{inv}}_{k,T} \otimes \xi_{k,T}$ under the full dynamics, i.e. how invariant the invariant state of the *reduced dynamics* is under the full dynamics when coupled with the probe. This term is inherent to the non-equilibrium nature of the RIS dynamics. The quantity $D_{k,T}$ quantifies the error in both arguments of $\sigma_{k,T} = S(\cdot|\cdot)$ when approximating $\rho_{k-1,T} \sim \rho_{k,T} \sim \rho^{\mathrm{inv}}_{k,T}$. Note that $\|D_{k,T}\|_1 \leq 2\|\rho_{k-1,T} - \rho^{\mathrm{inv}}_{k,T}\|$. The result of the proposition, eq. (5.3), yields a leading order term $F_{k,T}(X_{k,T} - D_{k,T})$ to the entropy production $\sigma_{k,T}$ in a sense the proposition makes precise.

In section 5.4, we will consider the small coupling version of this proposition; in particular, we will show the condition on $X_{k,T}$ holds for λ small.

Proof. We will drop the subscript T, and use the notation $O^P_\eta(Z)$ for any term bounded by $C^P_\eta Z$ for non-negative Z small enough and a constant C^P_η depending only on a given state η and c_P, $z =: \dim P(s)$, and $\dim X$. If such a quantity does not depend on a state η, we'll write $O^P(Z)$ instead.

Claim. $\|\rho^{\mathrm{inv}}_k - \rho_{k-1}\| \leq 2\|Q_0\rho^{\mathrm{i}}\|_1 + O^P((T(1-\ell))^{-1})$.

First, we'll show $\|\rho^{\mathrm{inv}}_{k-1} - \rho_{k-1}\|$ has a similar bound, and then approximate the difference between the consecutive invariant states. By theorem 3.4,

$$\mathcal{L}_{k-1}\mathcal{L}_{k-2}\cdots\mathcal{L}_1 = A_{k-1} + \mathcal{L}^Q_{k-1}\mathcal{L}^Q_{k-2}\cdots\mathcal{L}^Q_1 + O^P((T(1-\ell))^{-1}).$$

Then applying this to ρ^{i},

$$\|\rho^{\mathrm{inv}}_{k-1} - \rho_{k-1}\|_1 \leq \|\rho^{\mathrm{inv}}_{k-1} - A_{k-1}\rho^{\mathrm{i}}\|_1 + \|Q_0\rho^{\mathrm{i}}\|_1\ell^k + O^P((T(1-\ell))^{-1})$$

using **H4**. Note we can obtain $\|Q_0\rho^{\mathrm{i}}\|_1$ instead of $\|Q_1\rho^{\mathrm{i}}\|_1$ by using $Q_0 + P_0 = \mathrm{Id}$ and $Q_1P_0 = O^P(T^{-1})$. Now, we wish to show the first term is small. We have that $A_{k-1}\rho^{\mathrm{i}} = A_{k-1}P^1_0\rho^{\mathrm{i}} = P^1_{k-1}A_{k-1}\rho^{\mathrm{i}} \in \mathrm{Ran}\, P^1_{k-1} = \mathbb{C}\rho^{\mathrm{inv}}_{k-1}$. So for some $\alpha \in \mathbb{C}$, $A_{k-1}\rho^{\mathrm{i}} = \alpha\rho^{\mathrm{inv}}_{k-1}$. Then,

$$\|\rho^{\mathrm{inv}}_{k-1} - A_{k-1}\rho^{\mathrm{i}}\|_1 = \rho^{\mathrm{inv}}_{k-1} - \alpha\rho^{\mathrm{inv}}_{k-1}\|_1 = |1-\alpha|\|\rho^{\mathrm{inv}}_{k-1}\|_1 = |1-\alpha|$$

using that $\|\rho^{\mathrm{inv}}_{k-1}\|_1 = 1$ as $\rho^{\mathrm{inv}}_{k-1}$ is positive semi-definite and trace one. Next, because each $\mathcal{L}_n$ is

trace preserving,

$$1 = \mathrm{Tr}(\rho^{\mathrm{i}}) = \mathrm{Tr}(\mathcal{L}_{k-1}\cdots\mathcal{L}_1\rho^{\mathrm{i}}) = \alpha + \mathrm{Tr}(\mathcal{L}_{k-1}^Q\cdots\mathcal{L}_1^Q\rho^{\mathrm{i}}) + O^P((T(1-\ell))^{-1}).$$

Then, using $\mathrm{Tr}(\cdot) \le \mathrm{Tr}\,|\cdot| = \|\cdot\|_1$, we have $|1-\alpha| \le \|Q_0\rho^{\mathrm{i}}\|_1\ell^k + O^P((T(1-\ell))^{-1})$ and hence,

$$\|\rho_{k-1}^{\mathrm{inv}} - \rho_{k-1}\|_1 \le \|Q_0\rho^{\mathrm{i}}\|_1 + O^P((T(1-\ell))^{-1}).$$

To prove the second step, we again use that the invariant state is unique: $\mathrm{Ran}\,P^1(s) = \mathbb{C}\rho_s^{\mathrm{inv}}$, so for any $\eta \in \mathcal{I}_1(\mathcal{H}_\mathcal{S})$, we have $P^1(s)\eta = \alpha_s(\eta)\rho_s^{\mathrm{inv}}$ for some $\alpha_s(\eta) \in \mathbb{C}$. Taking the trace, we have $\mathrm{Tr}(P^1(s)\eta) = \alpha_s(\eta)$. In particular, $P^1(s)\rho_{s_0}^{\mathrm{inv}} = \mathrm{Tr}(P^1(s)\rho_{s_0}^{\mathrm{inv}})\rho_s^{\mathrm{inv}}$. Define γ by $s \mapsto \gamma_s = \mathrm{Tr}(P^1(s)\rho_{s_0}^{\mathrm{inv}})$. Then $[0,1] \ni s \mapsto \gamma_s$ is a C^2 function with $\gamma_{s_0} = 1$. In particular, $\gamma_s \neq 0$ for s near s_0. Then

$$\rho_s^{\mathrm{inv}} - \rho_{s_0}^{\mathrm{inv}} = (\gamma(s)^{-1}P^1(s) - P^1(s_0))\rho_{s_0}^{\mathrm{inv}} = O^P(|s-s_0|).$$

Then, for sufficiently large T, taking $s = k/T$ and $s_0 = (k-1)/T$ yields the claim:

$$\|\rho_k^{\mathrm{inv}} - \rho_{k-1}\|_1 \le \|\rho_k^{\mathrm{inv}} - \rho_{k-1}^{\mathrm{inv}}\|_1 + \|\rho_{k-1}^{\mathrm{inv}} - \rho_{k-1}\|_1 \le 2\|Q_0\rho^{\mathrm{i}}\|_1 + O^P((T(1-\ell))^{-1}).$$

Now, define

$$\begin{aligned} D_k' =& \mathcal{L}_k(\rho_{k-1})\otimes\xi_k - \rho_k^{\mathrm{inv}}\otimes\xi_k = \mathcal{L}_k(\rho_{k-1} - \rho_k^{\mathrm{inv}})\otimes\xi_k, \\ D_k'' =& U_k(\rho_{k-1}\otimes\xi_k)U_k^* - \rho_k^{\mathrm{inv}}\otimes\xi_k = U_k\big((\rho_{k-1} - \rho_k^{\mathrm{inv}})\otimes\xi_k\big)U_k^* + X_k, \end{aligned}$$

so that $\sigma_{k,T} = S(\eta_k + D_k''|\eta_k + D_k')$ for $\eta_k = \rho_k^{\mathrm{inv}}\otimes\xi_k$. We also have $\mathrm{Tr}(D_k') = \mathrm{Tr}(D_k'') = 0$, and, importantly,

$$D_k - X_k = D_k' - D_k''. \tag{5.4}$$

Note that

$$\|D_k\|_1 \le 2\|\rho_k^{\mathrm{inv}} - \rho_{k-1}\|_1 \le 4\|Q_0\rho^{\mathrm{i}}\|_1 + O^P((T(1-\ell))^{-1}) \tag{5.5}$$

by the claim.

To apply proposition 4.5, we must assume T and $k \le T$ are sufficiently large so that $\|D_k\|_1$ and $\|X_k\|_1$ are smaller than a constant D_0 which we may take to be $\inf \mathrm{sp}(\eta_{k,T})/4$. Then proposition 4.5 yields

$$\sigma_k = F_k(D_k - X_k) + O_{\eta_k}^P\Big((\|Q_0\rho^{\mathrm{i}}\|_1\ell^k + \|X_k\|_1 + (T(1-\ell))^{-1})^3\Big). \tag{5.6}$$

□

Remark.
- If $Q_{0,T}\rho^{\mathrm{i}} = 0$ then eq. (5.6) holds even for $k = 1$.
- Moreover, eq. (5.4) yields an interesting lower bound via the Quantum Pinsker Inequality, eq. (4.1):

$$\begin{aligned}\sigma_{k,T} &\geq \frac{1}{2}\|X_{k,T} - D_{k,T}\|_1^2 \geq \frac{1}{2}(|\|X_{k,T}\|_1 - \|D_{k,T}\|_1|)^2 \\ &= \frac{1}{2}\left(\|X_{k,T}\|_1^2 + \|D_{k,T}\|_1^2 - 2\|X_{k,T}\|_1\|D_{k,T}\|_1\right).\end{aligned}$$

If we consider the case $Q_{0,T}\rho^{\mathrm{i}} = 0$, then we have

$$\sigma_{k,T} \geq \frac{1}{2}\|X_{k,T}\|_1^2 + O^P((T(1-\ell))^{-1}) \tag{5.7}$$

using eq. (5.5). In particular, if $\inf_{s\in[0,1]} \|X(s)\|_1 = C > 0$ for some constant C, then $\sigma_{k,T} \geq \frac{1}{2}C^2 + O^P((T(1-\ell))^{-1})$. We recall our notation O^P indicates that there is no dependence on k or T. Then we may sum over steps to obtain

$$\sigma_T^{\mathrm{tot}} := \sum_{k=1}^{T} \sigma_{k,T} \geq \frac{1}{2}TC^2 + O(1).$$

Thus, σ_T^{tot} diverges as T in the adiabatic limit $T \to \infty$. Proposition 5.1 treats the case when $X_{k,T}$ is sufficiently small at each step k; the Quantum Pinsker Inequality along with an analysis of the above proof yields the case when $\inf_s \|X(s)\|_1 > 0$. We'll return to this in corollary 5.3.

5.2 *m-RIS*

Motivated by our considerations in section 3.1.4 and the results of proposition 3.11, we wish to formulate a repeated interaction system to take advantage of **wH4**.

Consider an RIS with constant elements $\mathcal{H}_{\mathcal{S}}, h_{\mathcal{S}}, \lambda, \tau, \mathcal{H}_{\mathcal{E}}$, and variables $(h_{\mathcal{E}_k})_{k=1,\dots,T}$, $(v_k)_{k=1,\dots,T}$, and $(\beta_k)_{k=1,\dots,T}$. We define the *m-repeated version* of this RIS to be the RIS associated to $\mathcal{H}_{\mathcal{S}}, h_{\mathcal{S}}, \lambda, \tau, \mathcal{H}_{\mathcal{E}}$, and variables $(h_{\mathcal{E}_{\lfloor (k'-1)/m \rfloor + 1}})_{k'=1,\dots,mT}$, $(v_{\lfloor (k'-1)/m \rfloor + 1})_{k'=1,\dots,mT}$, and $(\beta_{\lfloor (k'-1)/m \rfloor + 1})_{k'=1,\dots,mT}$. That is, we obtain the m-RIS from the original by repeating each probe m times.

Now, let us consider the adiabatic setup (ADRIS) with an RIS corresponding to parameters $h_{\mathcal{S}}, \lambda, \tau$, $s \mapsto h_{\mathcal{E}}(s)$, $s \mapsto \beta(s)$, $s \mapsto v(s)$ such that the induced map $\mathcal{L}(s)$ obeys **H1** to **H3**

and **wH4**. Note that since $\mathcal{L}$ is obtained via (ADRIS), $s \mapsto \mathcal{L}(s)$ is continuous, so all the assumptions of lemma 3.7 are satisfied. Then $\mathcal{L}_{k,T} = \mathcal{L}(k/T)$ describes the time evolution of $\mathcal{S}$ from time $(k-1)\tau$ to $k\tau$ at fixed adiabatic parameter T.

Lemma 3.7 yields an $m \in \mathbb{N}$ such that $\mathcal{L}^m(s)$ obeys **H1** to **H4**. This is then a natural candidate for an m-RIS. Choosing the m of lemma 3.7, we have that the time evolution of the m-RIS from time $(k-1)m\tau$ to $km\tau$ at fixed adiabatic parameter T is given by $\mathcal{L}^m(k/T)$. We may consider in an ordinary 1-RIS that the way the parameters are sampled from the curves of (ADRIS) is by stretching the curves by a factor of T; in this language, an m-RIS approximately stretches the curves by a factor mT, as illustrated in fig. 5.1. In this sense, we may consider an m-RIS as a further slowing of the process, in the spirit of the adiabatic limit.

Note as well that an m-RIS is still an RIS in the sense of the original definition; it is simply a constraint on the choice of parameters. This constraint manifests itself in the way parameters are sampled from functions in the adiabatic setup of (ADRIS).

We should emphasize that although the origins of the idea of an m-RIS are mathematical (**wH4** instead of **H4**), the "stretching" of an RIS to the corresponding m-RIS is a change of physical setup. One experimentally-oriented interpretation is that if the assumptions of theorem 3.4 are hard to check or false for a given RIS, the result may still be able to be applied to a modified setup.

We may formalize the above discussion by defining the following assumption.

***m*ADRIS** The repeated interaction system is the m-repeated version of a system satisfying (ADRIS), i.e. we have for $k = 1, \dots, T$,

$$h_{\mathcal{E},k',T} = h_{\mathcal{E}}\Big(\frac{[(k'-1)/m]+1}{T}\Big), \quad \beta_{k',T} = \beta\Big(\frac{[(k'-1)/m]+1}{T}\Big), \quad v_{k',T} = v\Big(\frac{[(k'-1)/m]+1}{T}\Big), \tag{mADRIS}$$

where $s \mapsto h_{\mathcal{E}}(s)$, $\beta(s)$, $v(s)$ are C^2 functions on $[0,1]$ and m is given by lemma 3.7.

We may now formulate an analog of proposition 5.1 for m-RIS.

Corollary 5.2 (m-RIS). *Consider a repeated interaction system satisfying* (mADRIS) *where the CPTP map $\mathcal{L}(s)$ is irreducible for all $s \in [0,1]$ and satisfies* **wH4**. *Assume that the initial state ρ^{i} satisfies $(P^1_{0,T} + Q_{0,T})\rho^{\mathrm{i}} = \rho^{\mathrm{i}}$. Define*

$$D^{(j)}_{k,T} := \mathcal{L}_{k,T}(\rho^{(j)}_{k,T} - \rho^{\mathrm{inv}}_{k,T}) \otimes \xi_{k,T} - U_{k,T}\big((\rho^{(j)}_{k,T} - \rho^{\mathrm{inv}}_{k,T}) \otimes \xi_{k,T}\big)U^*_{k,T}.$$

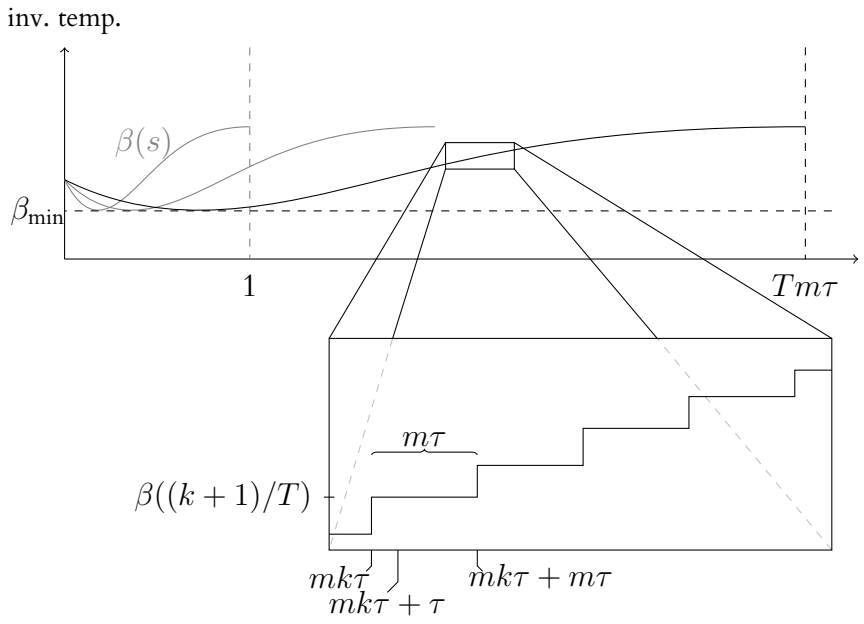

Figure 5.1: For a C^2 curve $\beta(s)$ on $[0,1]$ with positive minimum, the m-repeated RIS is such that, between time $mk\tau$ and time $mk\tau+m\tau$, the system $\mathcal{S}$ interacts one-by-one with m atoms at the same inverse temperature $\beta((k+1)/T)$.

With the same conditions on T, $k\le T$, and $\|X_{k,T}\|_1$ as in proposition 5.1, we have

$$\left|\sigma_{k,T}^{(j)} - F_{k,T}(X_{k,T} - D_{k,T}^{(j)})\right| \le C_{\eta_k}^P\big((\|X_{k,T}\|_1 + \|Q_{0,T}\rho^{\mathrm{i}}\|_1\ell^k + (T(1-\ell)^{-1})^3\big), \tag{5.8}$$

where $F_{k,T} = F_{\eta_{k,T}}$ and $C_{\eta_{k,T}}^P$ are the same as in proposition 5.1.

Remark. Then an m-RIS has approximately an m-fold increase in entropy as compared to the associated 1-RIS.

Proof. For T and $k\le T$ large enough, as described in the proof of proposition 5.1, after $k-1$ steps each consisting of m interactions with external probes, the system is in the state $\rho_{k-1}^{(m)} = \mathcal{L}_{k-1}^m\cdots L_1^m\rho^{\mathrm{i}}$, and from the proof of proposition 5.1 applied to $\mathcal{L}^m$, we have

$$\|\rho_k^{\mathrm{inv}} - \rho_{k-1}\|_1 \le 2\|Q_0\rho^{\mathrm{i}}\|_1\ell^k + O^P((T(1-\ell))^{-1})$$

where ρ_k^{inv} is the unique invariant state for $\mathcal{L}_k^m$ and thus for $\mathcal{L}_k$. This thus approximates the state of the m-RIS after switching steps (i.e., at times $(km\tau)_k$). For $1\le j\le m$,

$$\|\rho_k^{\mathrm{inv}} - \rho_k^{(j)}\|_1 = \|\mathcal{L}_k^j(\rho_k^{\mathrm{inv}} - \rho_{k-1})\|_1 \le \|\rho_k^{\mathrm{inv}} - \rho_{k-1}\|_1 \le 2\|Q_0\rho^{\mathrm{i}}\|_1\ell^k + O^P\big((T(1-\ell))^{-1}\big). \tag{5.9}$$

Thus, within a single step, we have the same bound between the state of the system and the

invariant state of the start of the step. Thus, we could proceed as in the proof of eq. (5.3) to obtain eq. (5.8), noting that $X_{k,T}$ is independent of j. □

Remark. In either proposition 5.1 or corollary 5.2, if the assumptions are met and we additionally have $X_{k,T} \equiv 0$ and $Q_{0,T}\rho^{\mathrm{i}} = 0$, then

$$\left|\sigma_{k,T}^{(j)} - F_{k,T}(D_{k,T}^{(j)})\right| \leq C_{\eta_k,T}^{P}\big((T(1-\ell)^{-3}\big).$$

If we make the additional assumption $\inf_{s\in[0,1]} \inf \mathrm{sp}\, \rho^{\mathrm{inv}}(s) > 0$, then we may remove the $\eta_{k,T}$ dependence from the constant on the RHS as shown by the explicit form computed in lemma 4.4. Lastly, we make use of the bound on F_η given in corollary 4.6 to obtain

$$\sigma_{k,T}^{(j)} = F_{k,T}(D_{k,T}^{(j)}) + O^P\big((T(1-\ell)^{-3}\big) = O^P(\|D_{k,T}^{(j)}\|^2) + O^P\big((T(1-\ell)^{-3}\big).$$

Lastly, the bound eq. (5.9) yields $\|D_{k,T}^{(j)}\| = O^P((T(1-\ell))^{-1})$, and we have

$$\sigma_{k,T}^{(j)} = O^P\big((T(1-\ell))^{-2}\big).$$

Summing over j and k yields

$$\sigma_T^{\mathrm{tot}} = \sum_{k=1}^{T}\sum_{j=1}^{m} \sigma_{k,T}^{(j)} = O^P\big((T(1-\ell)^{-1}\big) \tag{5.10}$$

and in particular $\sigma_T^{\mathrm{tot}} \to 0$ as $T \to \infty$.

On the other hand, if $\inf_{s\in[0,1]} \|X(s)\|_1 > 0$ and $Q_{0,T}\rho^{\mathrm{i}} = 0$, we may generalize our analysis after proposition 5.1 to conclude $\sigma_T^{\mathrm{tot}} \to \infty$ as $T \to \infty$, by replacing eq. (5.5) with eq. (5.9). We may summarize this discussion in a corollary.

Corollary 5.3. *Consider a repeated interaction system satisfying either*

- *assumption* (ADRIS)*, such that the reduced dynamics $\mathcal{L}(s)$ is irreducible for all $s \in [0,1]$ and satisfies* **H4***, or*
- *assumption* (mADRIS)*, such that the reduced dynamics $\mathcal{L}(s)$ is irreducible for all $s \in [0,1]$ and satisfies* **wH4***.*

Denote ρ_s^{inv} by the unique invariant state for $\mathcal{L}(s)$, and define

$$X(s) = U(s)\rho_s^{\mathrm{inv}} \otimes \xi_s U(s)^* - \rho_s^{\mathrm{inv}} \otimes \xi_s.$$

Assume the initial state ρ^{i} satisfies $P^1(0)\rho^{\mathrm{i}} = \rho^{\mathrm{i}}$. Then,

- *If* $\sup_{s\in[0,1]} \|X(s)\|_1 = 0$*, then* $\sigma_T^{tot} \to 0$ *in the limit* $T \to \infty$*, assuming* $\inf_{s\in[0,1]} \operatorname{sp} \rho_s^{\mathrm{inv}} > 0$.
- *If* $\inf_{s\in[0,1]} \|X(s)\|_1 > 0$*, then* $\sigma_T^{tot} \to \infty$ *in the limit* $T \to \infty$.

5.3 Qubits with RW interaction

Let us return to our example of section 2.2. We will use that, as established in section 2.2,

$$\operatorname{spr}(\mathcal{L}_{k,T}Q_{k,T}) \leq \left(1 - \frac{\lambda^2}{\nu^2}\sin^2\frac{\nu\tau}{2}\right) := \ell,$$

and $\ell < 1$ for $\nu\tau \notin 2\pi\mathbb{N}$, which we will assume for now on. Since $\mathcal{L}_{k,T}$ has a trivial peripheral spectrum, in the sense that $\operatorname{sp}(\mathcal{L}_{k,T}) \cap S^1 = \{1\}$ and 1 is a simple eigenvalue, and the corresponding eigenvector $\rho_{k,T}^{\mathrm{inv}} = \rho_{\beta_{k,T}^*}$ the Gibbs state at temperature $\beta_{k,T}^* = \frac{E_0}{E}\beta_{k,T}$ is positive definite, we have that $\mathcal{L}_{k,T}$ is an *ergodic* (or primitive) CPTP map, by e.g. [Wol12, Theorem 6.7]. In particular, it is irreducible. Thus, we may apply proposition 3.11 to conclude there exists $m \in \mathbb{N}$ such that $\mathcal{L}(s)$ satisfies **H1** to **H4**. We will consider the resulting m-RIS for the rest of this section.

Motivated by corollary 5.2, we compute $X_{k,T}$.

$$\begin{aligned} U_{k,T}\rho_{k,T}^{\mathrm{inv}} \otimes \xi_{k,T}U_{k,T}^* &= \frac{1}{\operatorname{Tr}(\cdots)}U_{k,T}\exp(-\beta\frac{E_0}{E}Ea^*a)\exp(-\beta E_0 b^*b)U_{k,T}^* \\ &= \frac{1}{\operatorname{Tr}(\cdots)}U_{k,T}\exp(-\beta E_0(a^*a + b^*b))U_{k,T}^*. \end{aligned}$$

But, as discussed in section 1.3.1, v_{RW} commutes with $N_{\mathrm{tot}} = a^*a + b^*b$. Thus, $U_{k,t} = \exp(-\tau(Ea^*a + E_0b^*b + \frac{\lambda}{2}(a^* \otimes b + a \otimes b^*)$ commutes with $\exp(-\beta E_0 N_{\mathrm{tot}})$, and we have

$$U_{k,T}\rho_{k,T}^{\mathrm{inv}} \otimes \xi_{k,T}U_{k,T}^* = \rho_{k,T}^{\mathrm{inv}} \otimes \xi_{k,T}U_{k,T}U_{k,T}^* = \rho_{k,T}^{\mathrm{inv}} \otimes \xi_{k,T}.$$

Thus, $X_{k,T} = 0$. If we choose the initial state $\rho^{\mathrm{i}} = \rho_{0,T}^{\mathrm{inv}}$, then $Q_{0,T}\rho^{\mathrm{i}} = 0$. Since $\rho_s^{\mathrm{inv}} > 0$ for all $s \in [0,1]$ as a Gibbs state at positive temperature, by corollary 5.3, we conclude $\sigma_T^{\mathrm{tot}} \to 0$.

On the other hand, if we start in a different initial state, the first steps alone may produce entropy which will not vanish in the adiabatic limit, although this contribution will exponentially decrease with the step number k.

Remark. The qubit setup with the full dipole interaction (introduced in section 1.3.1) does not have $X_{k,T} \equiv 0$; we will postpone its analysis to section 6.1.

5.3.1 An aside on adiabatic cooling

THE FORM OF THE INVARIANT STATE as a Gibbs state

$$\rho_{\beta^*} = \frac{\exp(-\beta^*_{k,T} h_S)}{\mathrm{Tr}(\exp(-\beta^*_{k,T} h_S))}.$$

prompts a question. Since $\beta^* = \frac{E}{E_0}\beta$ where β is the temperature of the probes, if $E_0 < E$, we may cool the system *below* the temperature of the probes (i.e., reach higher inverse temperature β). In particular, we can *adiabatically* drive the initial state of the system to the thermal state at temperature $\beta^* < \beta$, that is, without entropy production. Given Landauer's bound, this enables the preparation of such a state with a minimal energetic cost, making the cooling effect more surprising. To truly produce no entropy production, we must start with ρ^{i} as the invariant state for $\mathcal{L}_0$; thus, we must start with the system in a thermal state which is already cooler than the initial probe. If are willing to allow some entropy production, however, we may start in a different initial state, and produce $\sim \ell^k$ entropy production at each step k, i.e. exponentially decaying entropy production at each step. If we need no control over the entropy production, then we could simply fix the target temperature β^* and apply $\mathcal{L}(\beta)$ many times to drive the system to ρ_{β^*} exponentially fast, as in section 2.2. We could also investigate a system with $E_0 = E_0(s)$ varying.

Before being carried away, however, we should be suspicious: it does not seem plausible to cool, say, a fermion by interacting with a chain of hot fermions. In fact, the larger E_0 is compared to E, the larger the cooling. However, the rotating wave approximation is known to hold best when $E_0 \approx E$. Thus, the cooling phenomenon improves just as the RWA is known to break down. We will return to this when we consider the full dipole interaction in section 6.1.

5.4 Discussion of the small coupling limit

COROLLARY 5.3 ESSENTIALLY CHARACTERIZES the entropy production of RIS in two extremal cases when T is independent of the other parameters. We are now interested in looking for an intermediate case: $X_{k,T}$ decreasing with T. We will see shortly in lemma 5.4 that $X_{k,T} = O(\lambda)$. This suggests considering the small coupling regime of $|\lambda| \ll 1$, which is the focus of this section. However, small λ is technically challenging.

For an RIS with $\lambda = 0$, the reduced dynamics $\mathcal{L}(s)$ is simply the free evolution of the small system, and any density matrix which commutes with h_S will be invariant. In the 2×2 exam-

ple with the RW approximation considered in section 5.3, then both $\rho = a^*a$ and $\rho = \mathrm{Id} - a^*a$ will be invariant states of $\mathcal{L}(s)$ at every $s \in [0,1]$, and we no longer have a unique invariant state. In general, as $\lambda \to 0$, we expect the eigenvalues of $\mathcal{L}(s)$ to move towards the unit circle, which makes the ℓ-dependence in the discrete non-unitary adiabatic theorem theorem 3.4 relevant: we approximate the state of $\mathcal{S}$ up to an error of order $((1-\ell)T)^{-1}$, which may or may not be small.

A brief aside on the conceptual structure of analyzing repeated interaction systems. We wish to consider the mathematical behavior of a fixed RIS where we change the parameter λ and understand the behavior of the entropy production. What do we mean by this? Implicitly, we've already done something similar by considering the large T limit: for each T, we obtain a "different" RIS, from which we compute the entropy production. We are taking T large then in this "RIS-parameter-space" and evaluating the resultant total entropy production. This may be clarified by the following diagram.

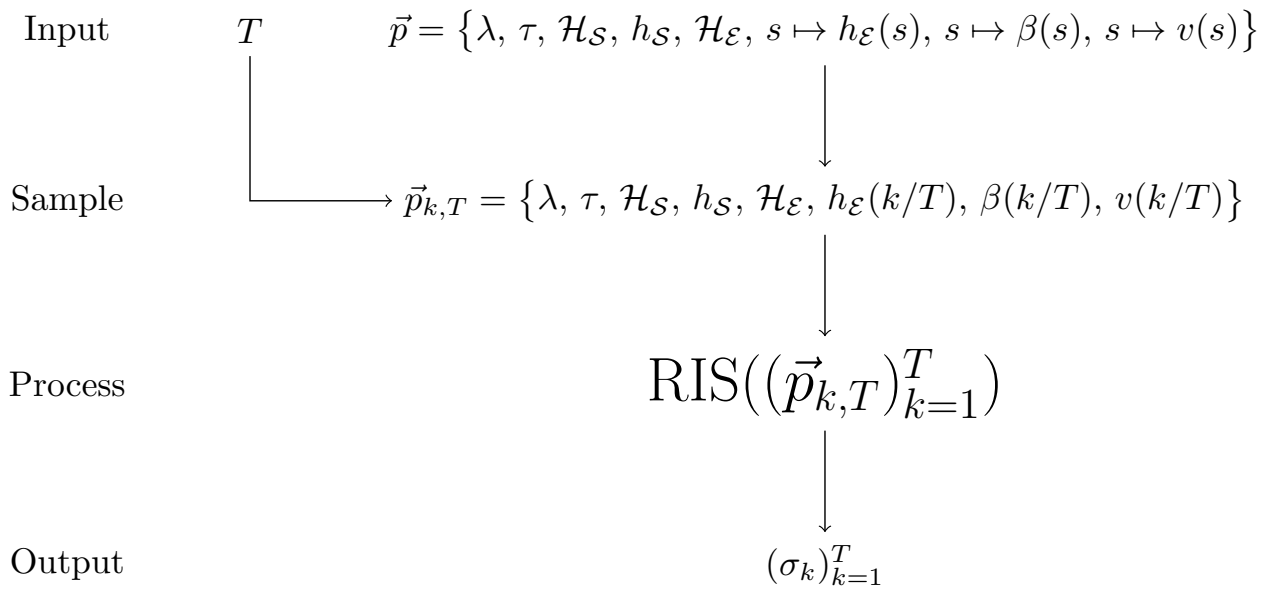

Figure 5.2: Diagram of an adiabatic RIS process. Inputs T, $\vec{p}$ are selected which meet some assumptions A. These are sampled to determine the parameters for T interactions. The system $\mathcal{S}$ interacts with the T probes as outlined in section 1.3, yielding an entropy production σ_k for each step. .

We make assumptions on the inputs: on the constant parameters, on the C^2 functions, and on T. Then we sample the functions to obtain the list of parameters for the T steps of the RIS process. We run the RIS process for all T steps, and finally output the entropy productions from each step: $(\sigma_k)_{k=1}^T$.

Mathematically, we compute (σ_k) by making some assumptions A on the set of inputs (for the sake of discussion, we won't be specific). Then, via e.g. eq. (5.3), we obtain information $I(\vec{p}, T)$ about σ_k, e.g. an expansion like eq. (5.3), which is a function of the inputs. If we change the inputs to $\vec{p}', T'$ in a way such that they still obey our assumptions A (e.g., taking $T+1$ instead of T), then running this whole process again must yield the same information about σ_k evaluated at the new inputs: $I(\vec{p}', T')$.

This is why we may take the limit $T \to \infty$ in eq. (5.10) to conclude the entropy production of an RIS vanishes in the adiabatic limit $T \to \infty$: at each increasingly large T, we meet our assumptions A and obtain the same type of expansion such that $\sigma_T^{\text{tot}} = O(1/T)$. Since the constants hidden in the 'O' do not depend on T, the parameter which is changing, we find that changing T as an *input* yields an *output* σ_T^{tot} which yields a vanishing sequence of real numbers.

However, if we change our inputs $\vec{p}, T$ such that our assumptions are violated (e.g. taking $\lambda = 0$), we cannot simply evaluate I which was derived for $\lambda > 0$ and obtain a valid result for the entropy production of an RIS without coupling. On the other hand, we may simply find that our information I is unhelpful for some choices of parameters, as in the case when $\ell \to 1$ and we obtain estimates up to order $((1-\ell)T)^{-1}$. In fact, we are additionally troubled by the fact that the constants here depend on the regularity of the spectral projectors of the reduced dynamics, which then depends on λ. Thus, they are not "constant" as we move λ in RIS-parameter space.

RETURNING TO THE SMALL COUPLING LIMIT, we will outline the plan of approach to the small coupling limit taken in [HJPR15] and quote the main result.

1. We wish to ensure taking λ small yields small $X_{k,T}$ to ameliorate the assumption $\|X_{k,T}\|_1 < D_0$ from proposition 5.1. We show the first step of this in lemma 5.4.
2. Then, we need to control the λ-dependence of c_P as defined in eq. (3.1) so that our adiabatic theorem, theorem 3.4, yields helpful estimates of the state of the RIS at each step. Note that since $N :\equiv \dim P(s) \leq \min(\frac{2\pi}{\epsilon}, \dim X)$, this parameter is already bounded uniformly in λ. This would allow us to formulate an analog of theorem 3.4 to estimate the the state of $\mathcal{S}$ at each step of the RIS process.
3. As discussed in section 3.1.2, the $(1-\ell)^{-1}$ dependence of the adiabatic theorem cannot be improved in our approach. Since $\ell \geq \operatorname{spr}(\mathcal{L}(s)Q(s))$, and the spectral radius approaches 1 as $\lambda \to 0$, even the newly hardened adiabatic theorem of step 2 will not suffice to estimate σ_k in the small coupling limit. We will have to resort to **wH4** and switch to an m-RIS with $m = m(\lambda)$ increasing as $\lambda \to 0$. We can interpret this as a further adiabatic slowing of the process.

4. Finally, we will be able to prove an analog of corollary 5.2 to estimate $\sigma_{k,T}^{(j)}$ with a clear understanding of the λ-dependence of the error. Thus we may estimate

$$\sigma_{T,\lambda} = \sum_{k=1}^{T} \sum_{j=1}^{m(\lambda)} \sigma_{k,T}^{(j)}$$

with control over both the T and λ dependence. This allows, e.g., computation of the limit $\lambda \to 0$, $T \to \infty$, $\lambda T \equiv 1$.

First, we will show that $X_{k,T}$ should be small as $\lambda \to 0$.

Lemma 5.4. *Assume we may expand*

$$\rho_{k,T}^{\text{inv}} = (\rho_{k,T}^{\text{inv}})^{(0)} + \lambda(\rho_{k,T}^{\text{inv}})^{(1)} + O(\lambda^2) \tag{5.11}$$

uniformly in k. Then $X_{k,T}$ as defined in eq. (5.1) satisfies

$$X_{k,T} = \lambda M_{k,T} + O(\lambda^2)$$

where

$$M_{k,T} := U^{(0)}\big((\rho_{k,T}^{\text{inv}})^{(1)} \otimes \xi_{k,T}\big)(U^{(0)})^* - (\rho_{k,T}^{\text{inv}})^{(1)} \otimes \xi_{k,T} \\ - \Big[(\rho_{k,T}^{\text{inv}})^{(0)} \otimes \xi_{k,T}, \sum_i \pi_{i,k,T} v \pi_{i,k,T}(-\mathrm{i}\tau) + \sum_{i \neq j} \pi_{i,k,T} v \pi_{j,k,T}\Big(\frac{\exp(-\mathrm{i}\tau(E_{i,k,T} - E_{j,k,T})) - 1}{E_{i,k,T} - E_{j,k,T}}\Big)\Big],$$

$\pi_{j,k,T}$, resp. $E_{j,k,T}$, are the spectral projectors, resp. eigenvalues, of $h_0 = h_{\mathcal{S}} + h_{\mathcal{E},k,T}$, and $U^{(0)} := \exp(-i\tau h_0)$. Note that $M_{k,T}$ is traceless, self-adjoint, and depends on $\rho_{k,T}^{\text{inv}}$, $\xi_{k,T}$, $v(k/T)$, and τ, but is independent of λ, and is bounded uniformly in k.

Remark. We will not address assumption eq. (5.11) here, although it is shown in [HJPR15] to follow from an investigation of the operator

$$\mathcal{L}^{\lambda}(s)(\cdot) = \mathrm{Tr}_{\mathcal{E}}\left(e^{-i\tau(h_0(s)+\lambda v(s))}(\cdot) \otimes \xi(s) e^{i\tau(h_0(s)+\lambda v(s))}\right) \tag{5.12}$$

where $\xi(s) = \exp(-\beta(s) h_{\mathcal{E}}(s))$ and $h_0(s) := h_{\mathcal{S}} + h_{\mathcal{E}}(s)$.

Proof. Let us drop the subscripts k and T for notational simplicity. Since $U = \exp(-\mathrm{i}\tau(h_0 + \lambda v))$, the operator U is analytic in λ. Since by assumption $\omega := \rho^{\text{inv}} \otimes \xi$ admits a second order

expansion in λ, we have

$$\omega = \omega^\lambda = \omega^{(0)} + \lambda\omega^{(1)} + O(\lambda^2), \qquad U = U^\lambda = U^{(0)} + \lambda U^{(1)} + O(\lambda^2).$$

Let $R^\lambda(z) = (h_0 + \lambda v - z)^{-1}$ to be the resolvent of the coupled Hamiltonian and $R_0(z) = (h_0 - z)^{-1}$. The holomorphic functional calculus yields

$$U^\lambda \omega^\lambda (U^\lambda)^* = \frac{1}{(2\mathrm{i}\pi)^2} \int_\Gamma \int_{\Gamma'} \exp(-\mathrm{i}\tau(\zeta - \zeta')) R^\lambda(\zeta) \omega^\lambda R^\lambda(\zeta') \,\mathrm{d}\zeta' \,\mathrm{d}\zeta,$$

where Γ' is a contour contained in the interior of Γ, and both contain the spectrum of the coupled Hamiltonian h. Note that since U is entire such contours trivially exist, unlike with the relative entropy in chapter 4 where we need to use the faithfulness of the state. Next, we substitute the Neumann expansion $R^\lambda(z) = R_0(z)[\mathrm{Id} + \lambda v R_0(z)]^{-1} = R_0(z)[\mathrm{Id} - \lambda v R_0(z) + O(\lambda^2)]$ for $z \in \Gamma$, to obtain

$$U^\lambda \omega^\lambda (U^\lambda)^* = \frac{1}{(2\mathrm{i}\pi)^2} \int_\Gamma \int_{\Gamma'} \exp(-\mathrm{i}\tau(\zeta - \zeta')) R_0(\zeta)[\mathrm{Id} - \lambda v R_0(\zeta)] \omega^\lambda R_0(\zeta')[\mathrm{Id} - \lambda v R_0(\zeta')] \,\mathrm{d}\zeta' \,\mathrm{d}\zeta + O(\lambda^2).$$

We insert our expansion for ω^λ and use $U^{(0)}\omega^{(0)}(U^{(0)})^* = \omega^{(0)}$ to rearrange $U^\lambda \omega^\lambda (U^\lambda)^*$ as

$$\omega^{(0)} + \lambda U^{(0)} \omega^{(1)} (U^{(0)})^* - \frac{\lambda}{(2\mathrm{i}\pi)^2} \int_\Gamma \int_{\Gamma'} \exp(-\mathrm{i}\tau(\zeta - \zeta')) R_0(\zeta)[v R_0(\zeta)\omega^{(0)} + \omega^{(0)} R_0(\zeta') v] R_0(\zeta') \,\mathrm{d}\zeta' \,\mathrm{d}\zeta + O(\lambda^2).$$

We compute these integrals using standard techniques. For example, the first term is

$$I := \frac{1}{(2\mathrm{i}\pi)^2} \int_\Gamma \int_{\Gamma'} \exp(-\mathrm{i}\tau(\zeta - \zeta')) R_0(\zeta) v R_0(\zeta) R_0(\zeta') \,\mathrm{d}\zeta' \,\mathrm{d}\zeta.$$

We apply the first resolvent identity (see proposition A.6) on the last factor $R_0(\zeta)R_0(\zeta')$, then perform the ζ' integral. Next, we write remaining resolvents using the spectral representation $h_0 = \sum_i \pi_i E_i$, and use Cauchy's integral formula to obtain

$$I = -\sum_i \pi_i v \pi_i (-\mathrm{i}\tau) - \sum_{i \neq j} \pi_i v \pi_j \left(\frac{\exp(-\mathrm{i}\tau(E_i - E_j))}{E_i - E_j} - \frac{1}{E_i - E_j} \right).$$

We deal with the other term in the same way to obtain, using that $[\omega^{(0)}, R_0(z)] = 0$,

$$U^\lambda \omega^\lambda (U^\lambda)^* = \omega^{(0)} + \lambda U^{(0)} \omega^{(1)} (U^{(0)})^* \\ - \lambda \Big[\omega^{(0)}, \sum_i \pi_i v \pi_i (-\mathrm{i}\tau) + \sum_{i \neq j} \pi_i v \pi_j \Big(\frac{\exp(-\mathrm{i}\tau(E_i - E_j)) - 1}{E_i - E_j} \Big) \Big] + O(\lambda^2).$$

□

Now, let us consider eq. (5.12). When $\lambda = 0$, we have

$$\mathcal{L}^0(s)(\cdot) = e^{-i\tau h_{\mathcal{S}}} \cdot e^{i\tau h_{\mathcal{S}}}.$$

This map is independent of s. Since $h_{\mathcal{S}}$ is self-adjoint, it has $\dim \mathcal{H}_{\mathcal{S}}$ linearly independent eigenvectors $\{\phi_j\}$ satisfying $h_{\mathcal{S}}\phi_j = E_j\phi_j$ for some $E_j \in \mathbb{R}$. Then

$$\mathcal{L}^0(s)(|\phi_j\rangle \langle\phi_k|) = e^{i\tau(E_k - E_j)} |\phi_j\rangle \langle\phi_k| .$$

Then $\mathcal{L}^0(s)$ has eigenvalue 1 with multiplicity at least $\dim \mathcal{H}_{\mathcal{S}}$ (corresponding to $E_k = E_j$), and all $(\dim \mathcal{H}_{\mathcal{S}})^2$ eigenvalues in $\operatorname{sp} \mathcal{L}^0(s)$ are on the unit circle. In order to perform steps 2 and 3, we need to make a genericity assumption to forbid accidental degeneracy.

wGEN The spectrum of $\mathcal{L}^0(s)$ consists of $\dim \mathcal{H}_{\mathcal{S}}(\dim \mathcal{H}_{\mathcal{S}} - 1)$ simple eigenvalues different from 1, and 1 which is $\dim \mathcal{H}_{\mathcal{S}}$-fold degenerate. Furthermore, hypotheses **H1** to **H3** and **wH4** hold for all $\lambda \in \mathbb{R}^*$ small enough, uniformly in $s \in [0, 1]$.

Remark. In particular, we assume $h_{\mathcal{S}}$ is not degenerate. This assumption also ensures that $v(s)$ effectively couples the system and probes in the sense that for small but strictly positive coupling λ, **H4** still holds, and the strictly contracting part of $\mathcal{L}^\lambda(s)$ is separated from the peripheral part.

This assumption and an analysis of the operator eq. (5.12) along with wGEN allows the tasks in step 2 and 3 to be performed, yielding the following results.

Lemma 5.5 (m-RIS control of $\ell(\lambda)$)**.** *Assume wGEN. For all $0 < G < 1$, there exists $m(\lambda) \in \mathbb{N}$ such that for any $\lambda \in \mathbb{R}^*$ small enough,*

$$\ell(\lambda) := \sup_{s \in [0,1]} \| \mathcal{L}^\lambda(s)^{m(\lambda)} Q(\lambda, s) \| \leq 1 - G.$$

We can take $m(\lambda) \geq M_0' \frac{\ln(1/|\lambda|)}{|\lambda|^r}$ *where* $M_0' > 0$ *is a constant, and* $r > 0$ *satisfies the estimate*

$$\sup_{s\in[0,1]} \operatorname{spr}(\mathcal{L}^\lambda(s)Q(\lambda,s)) \leq 1 - S_0|\lambda|^r, \text{ for some } S_0 > 0.$$

If $\dim \mathcal{H}_S = 2$, $\ell(\lambda) \leq 1 - G$ *for* $m(\lambda) \geq \frac{M_0}{\lambda^2}$ *for some* $M_0 > 0$, *for* $\lambda \in \mathbb{R}^*$ *small enough.*

Then by choosing an $m(\lambda)$-RIS, we may eliminate the λ-dependence from ℓ. However, these estimates yield $m(\lambda)$ with $m(\lambda) \to \infty$ as $\lambda \to 0$. We may think of this as a further adiabatic slowing of the process, just as with a constant m-RIS. Now, we may formulate the final result of the section (i.e. step 4).

Proposition 5.6 (SCL m-RIS)**.** *Consider a repeated interaction system satisfying assumption* (mADRIS) *such that the induced CPTP map* $\mathcal{L}^\lambda(s)$ *is irreducible for all* $s \in [0,1]$, *and satisfies wGEN. Denote by* $\rho^{\text{inv}}_{k,T}(\lambda)$ *the unique invariant state of* $\mathcal{L}^\lambda_{k,T}$ *and* $P^1_{k,T}(\lambda)$ *the associated spectral projector of* $\mathcal{L}^\lambda_{k,T}$. *Assume this state is faithful up to* $\lambda = 0$. *Let* ρ^{i} *be the initial state of the system, and assume* $(P^1_{0,T}(\lambda) + Q_{0,T}(\lambda))\rho^{\text{i}} = \rho^{\text{i}}$. *Then, using the notation of corollary 5.2, for* T *large enough,* λ *small enough, and* $k \leq T$ *large enough, we have* $\rho^{\text{inv}}_{k,T}(\lambda) = (\rho^{\text{inv}}_{k,T})^{(0)} + \lambda(\rho^{\text{inv}}_{k,T})^{(1)} + O(\lambda^2)$, *and for all* $1 \leq j \leq m(\lambda)$,

$$\begin{aligned}\sigma^{(j)}_{k,T} = \lambda^2 F^{(0)}_{k,T}(M_{k,T}, M_{k,T}) + F^{(0)}_{k,T}(D^{(j)}_{k,T}, D^{(j)}_{k,T}) - \lambda F^{(0)}_{k,T}(D^{(j)}_{k,T}, M_{k,T}) \\ - \lambda F^{(0)}_{k,T}(M_{k,T}, D^{(j)}_{k,T}) + O(\{\lambda + \|Q_{0,T}\rho^{\text{i}}\|\ell(\lambda)^k + T^{-1}\}^3),\end{aligned} \tag{5.13}$$

where $F^{(0)}_{k,T}(\cdot,\cdot) = F_{(\rho^{\text{inv}}_{k,T})^{(0)}}(\cdot,\cdot)$ *and* $M_{k,T}$ *is defined in lemma 5.4.*

This allows the computation of $\sigma^{(j)}_{k,T}$ at every step of the RIS, as a function of all the parameters, with control over both λ and T. Let us apply this to the special case with $Q_0\rho^{\text{i}} = 0$ so that $D_{k,T} = O(1/T)$, using eq. (5.5).

We assume that $\inf \operatorname{sp}(\rho^{\text{inv}}_s)^{(0)} > 0$, which yields a lower bound for the spectrum of $(\rho^{\text{inv}}_{k,T})^{(0)}$ uniform in k and T. Then our bound on the function F_η from corollary 4.6 yields e.g., ${}^{(0)}_{k,T}(D^{(j)}_{k,T}) = O(T^{-2})$ uniformly in k and T. Thus we have

$$\sigma^{(j)}_{k,T} = \lambda^2 F^{(0)}_{k,T}(M_{k,T}) + O(1/T^2) + O(\lambda/T) + O(\lambda^3).$$

Then,

$$\begin{aligned}\sigma^{\text{tot}}_{\lambda,T} &:= \sum_{k=1}^{T}\sum_{j=1}^{m(\lambda)} \sigma^{(j)}_{k,T} = m(\lambda)\sum_{k=1}^{T}\left[\lambda^2 F^{(0)}_{k,T}(M_{k,T}) + O(1/T^2) + O(\lambda/T) + O(\lambda^3)\right] \\ &= m(\lambda)\left[\lambda^2\sum_{k=1}^{T} F^{(0)}_{k,T}(M_{k,T}) + O(1/T) + O(\lambda) + O(T\lambda^3)\right]. \\ &= m(\lambda)\left[\lambda^2 T F_0 + O(1/T) + O(\lambda) + O(T\lambda^3)\right]. \end{aligned} \tag{5.14}$$

where

$$F_0 := \lim_{T\to\infty}\frac{1}{T}\sum_{k=1}^{T} F^{(0)}_{k,T}(M_{k,T}).$$

In fact, if we consider the C^2 function $M(s)$,

$$\begin{aligned}M(s) = \Big[(\rho^{\text{inv}}(s))^{(0)}\otimes\xi(s), \sum_i \pi_i(s)v\pi_i(s)(-\mathrm{i}\tau) + \sum_{i\neq j}\pi_i(s)v\pi_j(s)\left(\frac{\exp(-\mathrm{i}\tau(E_i(s)-E_j(s)))-1}{E_i(s)-E_j(s)}\right)\Big] \\ + U^{(0)}(s)(\rho^{\text{inv}}(s))^{(1)}\otimes\xi(s)(U^{(0)}(s))^*.\end{aligned}$$

then we may write

$$F_0 = \lim_{T\to\infty}\sum_{k=1}^{T} F_{\rho^{\text{inv}}(k/T)^{(0)}}(M(K/T))(k/T-(k-1)/T) = \int_0^1 F_{\rho^{\text{inv}}(s)^{(0)}}(M(s))\,\mathrm{d}s. \tag{5.15}$$

Since $M(s)$ is C^2 and $F_{([s]^{\text{inv}})^{(0)}}$ bounded independently of s, the integral is finite. Additionally, since $M(s)$ is self-adjoint, by corollary 4.6, the integrand is non-negative, so $0 \leq F_0 < \infty$.

Equation (5.14) shows that for any small but finite $\lambda_0 > 0$, the adiabatic limit $T \to \infty$ yields divergent entropy production. This agrees with our analysis in terms of the Quantum Pinsker Inequality.

Moreover, if $\dim \mathcal{H}_{\mathcal{S}} = 2$, then we may choose $m(\lambda) = \lceil M_0/\lambda^2 \rceil$, and obtain for $F_0 > 0$

$$\sigma^{\text{tot}}_{\lambda,T} = TM_0F_0(1 + O(1/(T^2\lambda^2)) + O(1/T\lambda) + O(\lambda)).$$

Then $\sigma^{\text{tot}}_{\lambda,T}$ diverges as T even in the limit $T \to \infty$, $\lambda \to 0$, $\lambda T =$ constant. In higher dimensional spaces, lemma 5.5 yields larger estimates for $m(\lambda)$, so the corresponding $m(\lambda)$-RIS also has divergent entropy production in this limit.

6
Dénouement

6.1 Qubits with full dipole interaction

We now have the tools to return to the example of qubits with the full dipole interaction considered in section 1.3.1. In that section, we computed the reduced dynamics, $\mathcal{L}$ as a function of the parameters. We may compute its eigenvalues and eigenvectors (see [Han16]). Since $h_{\mathcal{S}} = \begin{pmatrix} 0 & 0 \\ 0 & E \end{pmatrix}$, for $\tau E \notin 2\pi\mathbb{Z}$ we have $\mathrm{sp}(\mathcal{L}^0) = \{e^{\pm i\tau E}, 1\}$ where 1 has multiplicity two. We may check via perturbation theory or computationally that for $\lambda > 0$, 1 is a simple eigenvalue and the only peripheral eigenvalue of $\mathcal{L}^\lambda(s)$. We also see that all eigenvalues of $\mathcal{L}^\lambda(s)$ are in fact independent of β and thus of $s \in [0,1]$. We may compute the eigenvector associated to 1 for $\lambda > 0$:

$$\rho_s^{\mathrm{inv}} = \begin{pmatrix} \frac{e^{\beta(s)E_0}(1-\cos(\nu\tau))\eta^2+\nu^2(1-\cos(\eta\tau))}{\left(1+e^{\beta(s)E_0}\right)((1-\cos(\nu\tau))\eta^2+\nu^2(1-\cos(\eta\tau)))} & 0 \\ 0 & \frac{(1-\cos(\nu\tau))\eta^2+e^{\beta(s)E_0}\nu^2(1-\cos(\eta\tau))}{\left(1+e^{\beta(s)E_0}\right)((1-\cos(\nu\tau))\eta^2+\nu^2(1-\cos(\eta\tau)))} \end{pmatrix}$$

where $\nu = \sqrt{(E_0-E)^2+\lambda^2}$ and $\eta = \sqrt{(E+E_0)^2+\lambda^2}$, using the basis described in section 1.3.1, the eigenbasis of the unperturbed Hamiltonian $h_0 = h_{\mathcal{S}} + h_{\mathcal{E}}$. Since this matrix is strictly positive-definite for whenever either $\nu\tau \notin 2\pi\mathbb{Z}$ or $\eta\tau \notin 2\pi\mathbb{Z}$. Thus, if we exclude a countable set from parameter space, this invariant state is faithful for small enough λ, uniformly in s. Thus, similarly to the RW case discussed in section 5.3, since $\mathcal{L}^\lambda(s)$ has trivial peripheral spectrum with positive-definite eigenvector to eigenvalue 1, we have that $\mathcal{L}^\lambda(s)$ is an ergodic CPTP map, and in particular irreducible. Additionally, these considerations show that wGEN is satisfied.

By lemma 5.5, there exists $m(\lambda) \in \mathbb{N}$ for each $\lambda \neq 0$ small enough to bound $\ell(\lambda)$ uniformly away from one. The lemma shows we may take $m(\lambda) = \lceil M_0 \frac{1}{\lambda^2} \rceil$ for some constant $M_0 > 0$.

We will consider the corresponding $m(\lambda)$-RIS, where each probe is repeated $m(\lambda)$ times.

First, let us consider corollary 5.3: do we have $\inf_s \|X(s)\|_1 > 0$? Even with Mathematica, a simplified form of $\|X_{k,T}\|_1$ was not found. However, we may recognize that if $M_{k,T} \neq 0$, as will be shown below, then $X_{k,T}$ can only vanish with a conspiracy of the higher order terms, which can only happen for a countable set of λ. For any other choice of parameters then,

$$\sigma^{\text{tot}}_{T,\lambda} \to \infty$$

as $T \to \infty$ when $Q_{0,T}\rho^{\text{i}} = 0$.

Now, let us consider the small coupling limit. Let $\rho^{\text{i}} = (\rho_0^{\text{inv}})^{(0)}$, the invariant state at $s = 0$. Then $Q_{0,T}\rho^{\text{i}} = 0$, and we are in the case computed after proposition 5.6. We may explicitly compute $M(s)$, using that ρ_s^{inv} has no term which is first order in λ. We obtain

$$M_{k,T} = A_{k,T} \begin{pmatrix} 0 & 0 & 0 & -\left(e^{i\tau\eta_0}-1\right)\sin^2(\nu_0\tau/2)\eta_0 \\ 0 & 0 & -\left(e^{i\tau\nu_0}-1\right)\sin^2(\eta_0\tau/2)\nu_0 & 0 \\ 0 & e^{-i\nu_0\tau}\left(e^{i\tau\nu_0}-1\right)\sin^2(\eta_0\tau/2)\nu_0 & 0 & 0 \\ e^{-i\eta_0\tau}\left(e^{i\tau\eta_0}-1\right)\sin^2(\nu_0\tau/2)\eta_0 & 0 & 0 & 0 \end{pmatrix},$$

where

$$A_{k,T} = \frac{\frac{1}{2}\tanh\left(\frac{\beta_{k,T}E_0}{2}\right)}{2E_0E\sin(E_0\tau)\sin(E\tau) - (E_0^2+E^2)\left(1-\cos(E_0\tau)\cos(E\tau)\right)}$$

and $\nu_0 = |E - E_0|$ and $\eta_0 = |E + E_0|$.

Since $[\rho_s^{\text{inv}}, \xi(s)^{\text{i}}, h_0] = 0$, using that ρ_s^{inv} and ξ_s are diagonal in the eigenbasis of h_0, the spectral projectors in the definition of $F_{\rho_s^{\text{inv}}\otimes\xi_s^{\text{i}}}$ are the same as those for h_0. We use this to compute $F_{\rho_s^{\text{inv}}\otimes\xi_s^{\text{i}}}(M(s))$. We sample at $s = k/T$ to conclude for small λ that

$$\sigma^{(j)}_{k,T} = F_{\rho^{\text{inv}}_{k,T}\otimes\xi^{\text{i}}_{k,T}}(M_{k,T}) + O(\lambda T^{-1}) + O(T^{-2}) + O(\lambda^3),$$

$$F_{\rho^{\text{inv}}_{k,T}\otimes\xi^{\text{i}}_{k,T}}(M_{k,T}) = \begin{cases} \lambda^2\frac{\beta_{k,T}E_0}{2}\tanh\left(\frac{\beta_{k,T}E_0}{2}\right)\frac{(\cos(E_0\tau)-\cos(E\tau))^2}{2E_0E\sin(E_0\tau)\sin(E\tau)-\left(E_0^2+E^2\right)(1-\cos(E_0\tau)\cos(E\tau))} & E \neq E_0 \\ \lambda^2\beta_{k,T}E_0\tanh\left(\frac{\beta_{k,T}E_0}{2}\right)\frac{\tau^2\sin^2(E_0\tau)}{1+2E_0^2\tau^2-\cos(2E_0\tau)} & E = E_0. \end{cases}$$

Given a curve $s \mapsto \beta(s)$, we may then compute F_0 via eq. (5.15). However, we notice that F_0 vanishes or not depending only on E, E_0, τ, and not depending on β or λ. As pointed out at the end of section 5.4, if $F_0 \neq 0$ then $\sigma^{\text{tot}}_{\lambda,T}$ diverges as T in the limit $T \to \infty$, $\lambda \to 0$, $\lambda T =$ constant. In particular, for any small $\lambda > 0$, the entropy production $\sigma^{\text{tot}}_{\lambda,T}$ diverges as $T \to \infty$.

Naively, when would we have $F_0 = 0$ by the expression above? This occurs exactly when

$(E-E_0)\tau \in 2\pi\mathbb{Z}$. But this implies $\nu_0 = \nu|_{\lambda=0} \in 2\pi\mathbb{Z}$ violating our assumption that the invariant state is faithful for all λ small enough, including $\lambda = 0$. Thus, for any choice of parameters meeting our assumptions we obtain divergent entropy production for the $m(\lambda)$-RIS.

6.1.1 A return to cooling

LET US RETURN to our considerations of cooling in the RW system discussed in section 5.3.1. First, we have seen we may not drive the system to the invariant state without entropy production in this way. Next, we may rewrite the invariant state in the full dipole system as a Gibbs state as some effective temperature β_{eff}, which may not be positive. We can then check to see if $\beta_{\text{eff}} > \beta$.

Writing $\rho^{\text{inv}} = \begin{pmatrix} \gamma & \\ & 1-\gamma \end{pmatrix}$, we equate

$$\begin{pmatrix} \gamma & \\ & 1-\gamma \end{pmatrix} = \exp(-\beta^{\text{eff}} h_{\mathcal{E}})/\operatorname{Tr}(\cdots) = \begin{pmatrix} 1 & \\ & e^{-\beta^{\text{eff}}E_0} \end{pmatrix} \frac{1}{1+e^{-\beta^{\text{eff}}E_0}}.$$

This generates two equivalent conditions, one of which is

$$\gamma = \frac{1}{1+e^{-\beta^{\text{eff}}E_0}} \iff e^{-\beta^{\text{eff}}E_0} = \frac{1}{\gamma} - 1 \iff -\beta^{\text{eff}}E_0 = \log\left(\frac{1}{\gamma}-1\right)$$

That is, $\beta^{\text{eff}} = -\frac{1}{E_0}\log\left(\frac{1}{\gamma}-1\right)$. First, when is $\beta^{\text{eff}} > 0$? We have

$$\beta^{\text{eff}} > 0 \iff \log\left(\frac{1}{\gamma}-1\right) < 0 \iff \frac{1}{\gamma} - 1 < 1 \iff \gamma > \frac{1}{2}.$$

Substituting the definition of γ into this inequality, we see this occurs when

$$\begin{aligned}
e^{\beta E_0}(1-\cos(\nu\tau))\eta^2 + \nu^2(1-\cos(\eta\tau)) &> \frac{1}{2}\left(1+e^{\beta E_0}\right)\left((1-\cos(\nu\tau))\eta^2 + \nu^2(1-\cos(\eta\tau))\right) \\
\frac{1}{2}e^{\beta E_0}(1-\cos(\nu\tau))\eta^2 + \frac{1}{2}\nu^2(1-\cos(\eta\tau)) &> \frac{1}{2}(1-\cos(\nu\tau))\eta^2 + \frac{1}{2}e^{\beta E_0}\nu^2(1-\cos(\eta\tau)) \\
\left(e^{\beta E_0}-1\right)(1-\cos(\nu\tau))\eta^2 &> \left(e^{\beta E_0}-1\right)\nu^2(1-\cos(\eta\tau)) \\
(1-\cos(\nu\tau))\eta^2 &> \nu^2(1-\cos(\eta\tau))
\end{aligned}$$

which we can rewrite and simplify as

$$\left((E_0-E)^2+\lambda^2\right)\cos\left(\tau\sqrt{(E_0+E)^2+\lambda^2}\right) + 4E_0E > \left((E_0+E)^2+\lambda^2\right)\cos\left(\tau\sqrt{(E_0-E)^2+\lambda^2}\right).$$

By inspection, in the RWA regime, λ small and $E_0 \approx E$, for example, this condition holds. It does not always hold; take, say, $\lambda = 2$, $E_0 = 2$, $E = 1$, and $\tau = 3$, and we have the LHS as $13\cos(3\sqrt{5}) \approx 11.84$ and the RHS as $8 + 5\cos(3\sqrt{5}) \approx 7.11$. Moving on, can we have $\beta^{\text{eff}} > \beta$?

$$\begin{aligned}
\beta^{\text{eff}} > \beta &\iff -\beta^{\text{eff}} E_0 < -\beta E_0 \iff e^{-\beta^{\text{eff}} E_0} < e^{-\beta E_0} \\
&\iff e^{-\beta^{\text{eff}} E_0} + 1 < e^{-\beta E_0} + 1 \\
&\iff \frac{1}{e^{-\beta E_0} + 1} < \frac{1}{e^{-\beta^{\text{eff}} E_0} + 1} = \gamma \\
&\iff \frac{1}{e^{-\beta E_0} + 1} < \frac{e^{\beta E_0}(1 - \cos(\nu\tau))\eta^2 + \nu^2(1 - \cos(\eta\tau))}{(1 + e^{\beta E_0})\left((1 - \cos(\nu\tau))\eta^2 + \nu^2(1 - \cos(\eta\tau))\right)} \\
&\iff e^{\beta E_0} = \frac{e^{\beta E_0} + 1}{e^{-\beta E_0} + 1} < \frac{e^{\beta E_0}(1 - \cos(\nu\tau))\eta^2 + \nu^2(1 - \cos(\eta\tau))}{(1 - \cos(\nu\tau))\eta^2 + \nu^2(1 - \cos(\eta\tau))} \\
&\iff e^{\beta E_0}\left((1 - \cos(\nu\tau))\eta^2 + \nu^2(1 - \cos(\eta\tau))\right) < e^{\beta E_0}(1 - \cos(\nu\tau))\eta^2 + \nu^2(1 - \cos(\eta\tau)) \\
&\iff e^{\beta E_0}\nu^2(1 - \cos(\eta\tau)) < \nu^2(1 - \cos(\eta\tau)) \\
&\iff e^{\beta E_0} < 1 \iff \beta E_0 < 0
\end{aligned}$$

which we've assumed to not be the case. Therefore, the cooling seen in section 5.3.1 cannot occur with the full dipole interaction, at least in the sense shown here. That cooling may thus indicate a failure of the rotating wave approximation.

6.2 Retrospective

Repeated interaction systems have two sources of entropy production: the difference between the actual dynamics and the adiabatic dynamics, and the failure of the invariant state of the reduced dynamics to be invariant under the full dynamics. We would expect the first for any adiabatic limit. The second is indicative of the non-equilibrium nature of repeated interaction systems, which is due to the probes being swapped out every step. Even if one starts in the invariant state for the first step, we expect order λ^2 entropy to be produced from each subsequent step. As the number of steps is given by the adiabatic parameter T, in the adiabatic limit we obtain infinite entropy production, growing as $m(\lambda)\lambda^2 T$ for small but finite coupling λ. This more or less resolves our initial question: Landauer's Principle is not saturated in the adiabatic limit. However, we saw when the system and probes are given by two level systems interacting through their dipoles, in the rotating wave approximation, the entropy production vanishes even for large coupling.

A
Linear algebra

A.1 Projections

Def A.1. If X is a vector space, a bounded operator $P \in \mathcal{B}(X)$ is a *projection* if $P^2 = P$.

Projections have the following properties.

Proposition A.2. *If X is a finite-dimensional vector space and $P \in \mathcal{B}(X)$ a projection, then*

1. $\operatorname{sp} P \subset \{0, 1\}$
2. $\operatorname{Tr} P = \operatorname{Tr} |P| = \operatorname{rank} P = \dim \operatorname{Ran} P$.

Lemma A.3. *Let X be a finite-dimensional vector space. Let P and Q be projections on X with $\operatorname{spr}(P - Q)^2 < 1$. Then P is similar to Q.*

Proof. We wish to find an invertible operator U such that $P = U^{-1}QU$. Consider

$$U' = QP + (1 - Q)(1 - P).$$

On the P subspace, U' maps to Q, and on the $1 - P$ subspace, U' maps to $1 - Q$; in particular, $U'P = Q$. But the map isn't bijective: $U'P(1 - Q) = Q(1 - Q) = 0$. But $P(1 - Q) \neq 0$ in general. We could similarly consider

$$V' = PQ + (1 - P)(1 - Q),$$

which then has $V' : QX \to PX$. We have

$$\begin{aligned} U'V' &= QPQ + (1-Q)(1-P)(1-Q) \\ &= QPQ + (1-Q)(1-P-Q+PQ+QP) \\ &= \cancel{QPQ} + 1 - P - Q + PQ + \cancel{QP} - \cancel{Q} + QP + \cancel{Q} - \cancel{QPQ} - \cancel{QP} \\ &= 1 - P - Q + PQ + QP \\ &= 1 - (P-Q)^2. \end{aligned}$$

Since the RHS is invariant under $P \leftrightarrow Q$, the LHS is too, and we have $V'U' = 1 - (P-Q)^2$. Let $R = (P-Q)^2$. Then $U'V' = V'U' = 1 - R$. If the far right side was 1 instead of $1 - R$, then U' and V' would be inverses to each other, and we would have found a bijective map. We will try to modify U' and V' to let this happen. Note that

$$PR = P + PQ - PQ - PQP = P - PQP = P + QP - QP - PQP = RP$$

and similarly, R commutes with Q. Thus, R commutes with U' and V'. With our assumption $\operatorname{spr} R < 1$, we have the absolutely convergent series

$$(1-R)^{-1/2} = \sum_{n=0}^{\infty} \binom{-1/2}{n} (-R)^n$$

where we define $\binom{\alpha}{n} := \frac{\alpha(\alpha-1)(\alpha-2)\cdots(\alpha-n+1)}{n!}$ for $\alpha \in \mathbb{C}$. This sum satisfies $((1-R)^{-1/2})^2 = (1-R)^{-1}$ as we would expect, according to [Kat76]. Alternatively, we may see that since $\operatorname{spr} R < 1$, at least in this finite dimensional case, $1 - R$ is positive definite. Then its inverse $(1-R)^{-1}$ is positive definite and so admits a unique positive square root, which we will call $(1-R)^{-1/2}$. Now, define $U = U'(1-R)^{-1/2}$, and $V = V'(1-R)^{-1/2}$. Then $UV = VU = (1-R)(1-R)^{-1} = 1$, so $V = U^{-1}$.

Finally,

$$UP = U'P(1-R)^{-1/2} = QP(1-R)^{-1/2} = QU'(1-R)^{-1/2} = QU$$

So $P = U^{-1}QU$, and the two operators are similiar. □

Corollary A.4. *Let Y and X be finite dimensional vector spaces, and $P : \mathcal{B}(Y) \to \mathcal{B}(X)$, $A \mapsto P_A$ be a projector-valued map, which is continuous at $A_0 \in \mathcal{B}(Y)$. Then for sufficiently small $\|A - A_0\|$, the range $P_A X$ is isomorphic to $P_{A_0} X$. In particular, $\dim P_A X$ is constant.*

Proof. Since $A \mapsto P_A$ is continuous at A_0, there exists $\delta > 0$ such that for $\|A - A_0\| < \delta$, the norm $\|P_A - P_{A_0}\| < 1$. Fix A with $\|A - A_0\| < \delta$. Then

$$\mathrm{spr}(P_A - P_{A_0})^2 \leq [\mathrm{spr}(P_A - P_{A_0})]^2 \leq \|P_A - P_{A_0}\|^2 < 1.$$

Then by lemma A.3, $P_A = U^{-1} P_{A_0} U$ for some invertible $U \in \mathcal{B}(X)$. Then $UP_A = P_{A_0}U$, so $U : P_A X \to P_{A_0} X$ is a vector space isomorphism. □

A.2 Resolvents

Def A.5. For X a finite dimensional vector space, and $T \in \mathcal{B}(X)$, define the *resolvent* as the operator-valued map

$$R_T : P(T) \to \mathcal{B}(X)$$
$$\zeta \mapsto R_T(\zeta) = (T - \zeta)^{-1},$$

where $P(T)$ is the *resolvent set* of T, namely $\mathbb{C} \setminus \mathrm{sp}(T)$ where $\mathrm{sp}(T)$ is (in finite dimensions) the set of eigenvalues of T.

We will follow convention and also call the operator $R_T(\zeta)$ the resolvent of T at ζ.

Remark. The resolvent encodes all of the information of T in a way that allows us to access the powerful results of complex analysis.

Proposition A.6 (Properties of the resolvent)**.** *For X a finite dimensional vector space and $T, A, B \in \mathcal{B}(X)$, we have the following properties.*

1. *For $\zeta \in P(T)$, the resolvent $R_T(\zeta)$ commutes with T and has exactly the eigenvalues $\{(\lambda - \zeta)^{-1} : \lambda$ is an eigenvalue of $T\}$.*
2. *For $\zeta_1, \zeta_2 \in P(T)$, we have the first resolvent equation*

$$R_T(\zeta_1) - R_T(\zeta_2) = R_T(\zeta_1)(\zeta_1 - \zeta_2)R_T(\zeta_2) = (\zeta_1 - \zeta_2)R_T(\zeta_1)R_T(\zeta_2).$$

   In particular, $R_T(\zeta_1)$ and $R_T(\zeta_2)$ commute.

3. *The second resolvent equation: for $\zeta \in P(A) \cap P(B)$,*

$$R_A(\zeta) - R_B(\zeta) = R_A(\zeta)(B - A)R_B(\zeta).$$

Proposition A.7. *Let X be a Hilbert space with $\dim X = N < \infty$ and $T \in \mathcal{B}(X)$ be a normal operator on X with eigenvalues $\{\lambda_1, \ldots, \lambda_N\}$ counted with multiplicity. Then*

$$\|R_T(\zeta)\| = \min_{k=1,\ldots,N} \frac{1}{|\zeta - \lambda_k|}.$$

Remark. This is problem 6.43 of [Kat76], page 60.

A.3 Continuity of eigenvalues

Def A.8. Let $\mathbb{C}^{(n)} = \{\{z_1, z_2, \ldots, z_n\} : z_j \in \mathbb{C},\ z_j \neq z_i \text{ for } i, j = 1, 2, \ldots, n\}$ be the set of *unordered* n-tuples of complex numbers. Given two tuples $A = \{a_1, \ldots, a_n\} \in \mathbb{C}^{(n)}$ and $B = \{b_1, \ldots, b_n\} \in \mathbb{C}^{(n)}$, define

$$\operatorname{dist}(A, B) = \min_{\sigma \in S_n} \max_{1 \leq k \leq n} |a_{\sigma(k)} - b_k|$$

where S_n is the symmetric group on n letters.

Remark. It is straightforward to check that the function $\operatorname{dist} : \mathbb{C}^{(n)} \times \mathbb{C}^{(n)} \to \mathbb{R}_+$ is a metric. This distance is a natural one for sets; we rearrange A and B so that their largest elementwise distance is minimal, and use that distance for the distance between the tuples.

Theorem A.9 (Continuity of eigenvalues)**.** *Let X be an n-dimensional vector space. Let $\mathfrak{S} : (\mathcal{B}(X), \|\cdot\|) \to (\mathbb{C}^{(n)}, \operatorname{dist})$ map $T \in \mathcal{B}(X)$ to its n-tuple of eigenvalues, repeatedly according to their algebraic multiplicies, in any order. Then $\mathfrak{S}$ is continuous.*

Proof. This follows from the continuity of zeros of polynomials, e.g. [Zed65, Theorem 1]. □

A.3.1 CPTP maps

Proposition A.10. *Let $\mathcal{H}$ be a finite dimensional Hilbert space. Then set $\mathcal{C}$ of CPTP maps on $(\mathcal{B}(\mathcal{H}), \|\cdot\|_2) \to (\mathcal{B}(\mathcal{H}), \|\cdot\|_2)$ is a compact subset of $\mathcal{B}(\mathcal{B}(\mathcal{H}))$ with the norm $\|\Phi\| = \sup_{\|\rho\|_2 = 1} \|\Phi(\rho)\|_2$.*

Remark. Since all norms are equivalent on these finite dimensional spaces, we recover compactness of CPTP maps on $(\mathcal{I}_1(\mathcal{H}), \|\cdot\|_1)$.

Proof. See [Cro13, Lemma 1.9]. □

B
Detailed derivation of eq. (1.2)

By definition of relative entropy, the entropy production (eq. (1.4)) may be written

$$\sigma = \mathrm{Tr}\left(U\rho^{\mathrm{i}} \otimes \xi^{\mathrm{i}} U^* \log(U\rho^{\mathrm{i}} \otimes \xi^{\mathrm{i}} U^*)\right) - \mathrm{Tr}\left(U\rho^{\mathrm{i}} \otimes \xi^{\mathrm{i}} U^* \log(\rho^{\mathrm{f}} \otimes \xi^{\mathrm{i}})\right).$$

We then recognize the first term as an entropy, and expand the second term using the following lemma.

Lemma B.1. *If A, B are strictly positive operators, then*

$$\log(A \otimes B) = \log(A) \otimes \mathrm{Id} + \mathrm{Id} \otimes \log(B).$$

Proof. If A, B have spectral decompositions $A = \sum_i \mu_i P_i$ and $B = \sum_j \lambda_j Q_j$, then $A \otimes B = \sum_{ij} \mu_i \lambda_j P_i \otimes Q_j$. With this,

$$\begin{aligned}
\log(A \otimes B) &= \sum_{ij} \log(\mu_i \lambda_j) P_i \otimes Q_j \\
&= \sum_{ij} (\log \mu_i + \log \lambda_j) P_i \otimes Q_j \\
&= \sum_{ij} \log \mu_i P_i \otimes Q_j + \sum_{ij} \log \lambda_j P_i \otimes Q_j \\
&= \sum_{i} \log \mu_i P_i \otimes \mathrm{Id} + \sum_{j} \log \lambda_j \mathrm{Id} \otimes Q_j \\
&= \log(A) \otimes \mathrm{Id} + \mathrm{Id} \otimes \log(B).
\end{aligned}$$

□

This yields

$$\sigma = -S(U\rho^{\mathrm{i}} \otimes \xi^{\mathrm{i}}U^*) - \operatorname{Tr}\left(U\rho^{\mathrm{i}} \otimes \xi^{\mathrm{i}}U^* \left(\log \rho^{\mathrm{f}} \otimes \mathrm{Id}\right)\right) - \operatorname{Tr}\left(U\rho^{\mathrm{i}} \otimes \xi^{\mathrm{i}}U^* \left(\mathrm{Id} \otimes \log \xi^{\mathrm{i}}\right)\right).$$

Since entropy is invariant under a unitary transformation , we have $S(U\rho^{\mathrm{i}} \otimes \xi^{\mathrm{i}}U^*) = S(\rho^{\mathrm{i}} \otimes \xi^{\mathrm{i}})$. Furthermore, by definition of the partial trace,

$$\operatorname{Tr}\left(U\rho^{\mathrm{i}} \otimes \xi^{\mathrm{i}}U^* \left(\log \rho^{\mathrm{f}} \otimes \mathrm{Id}\right)\right) = \operatorname{Tr}\left(\operatorname{Tr}_{\mathcal{S}}(U\rho^{\mathrm{i}} \otimes \xi^{\mathrm{i}}U^*) \log \rho^{\mathrm{f}}\right),$$

which is simply $\operatorname{Tr}(\rho^{\mathrm{f}} \log \rho^{\mathrm{f}}) = -S(\rho^{\mathrm{f}})$. Using this argument for the third term as well, we are left with

$$\sigma = -S(\rho^{\mathrm{i}} \otimes \xi^{\mathrm{i}}) + S(\rho^{\mathrm{f}}) - \operatorname{Tr}(\xi^{\mathrm{f}} \log \xi^{\mathrm{i}}).$$

But

$$\begin{aligned} S(\rho^{\mathrm{i}} \otimes \xi^{\mathrm{i}}) &= -\operatorname{Tr}\left(\rho^{\mathrm{i}} \otimes \xi^{\mathrm{i}} \log(\rho^{\mathrm{i}} \otimes \xi^{\mathrm{i}})\right) \\ &= -\operatorname{Tr}\left(\rho^{\mathrm{i}} \otimes \xi^{\mathrm{i}}(\log \rho^{\mathrm{i}} \otimes \mathrm{Id})\right) - \operatorname{Tr}\left(\rho^{\mathrm{i}} \otimes \xi^{\mathrm{i}}(\mathrm{Id} \otimes \log \xi^{\mathrm{i}})\right) \\ &= -\operatorname{Tr}\left(\rho^{\mathrm{i}} \log \rho^{\mathrm{i}} \otimes \xi^{\mathrm{i}}\right) - \operatorname{Tr}\left(\rho^{\mathrm{i}} \otimes \xi^{\mathrm{i}} \log \xi^{\mathrm{i}}\right) \\ &= -\operatorname{Tr}(\rho^{\mathrm{i}} \log \rho^{\mathrm{i}}) \operatorname{Tr}(\xi^{\mathrm{i}}) - \operatorname{Tr}(\xi^{\mathrm{i}} \log \xi^{\mathrm{i}}) \operatorname{Tr}(\rho^{\mathrm{i}}) \\ &= S(\rho^{\mathrm{i}}) + S(\xi^{\mathrm{i}}), \end{aligned}$$

using that $\operatorname{Tr}(\rho^{\mathrm{i}}) = \operatorname{Tr}(\xi^{\mathrm{i}}) = 1$. Then,

$$\begin{aligned} \sigma &= -S(\rho^{\mathrm{i}}) + S(\rho^{\mathrm{f}}) - S(\xi^{\mathrm{i}}) - \operatorname{Tr}(\xi^{\mathrm{f}} \log \xi^{\mathrm{i}}) \\ &= -\Delta S_{\mathcal{S}} - \operatorname{Tr}\left((\xi^{\mathrm{f}} - \xi^{\mathrm{i}}) \log \xi^{\mathrm{i}}\right). \end{aligned}$$

Additionally, using eq. (1.3),

$$\log \xi^{\mathrm{i}} = -\beta h_{\mathcal{E}} - \log(\operatorname{Tr}(\exp(-\beta h_{\mathcal{E}})),$$

so, using that $\operatorname{Tr}(\xi^{\mathrm{f}} - \xi^{\mathrm{i}}) = 1 - 1 = 0$,

$$\begin{aligned} \sigma &= -\Delta S_{\mathcal{S}} + \beta \operatorname{Tr}\left((\xi^{\mathrm{f}} - \xi^{\mathrm{i}}) h_{\mathcal{E}}\right) + \log(\operatorname{Tr}(\exp(-\beta h_{\mathcal{E}}))) \cancel{\operatorname{Tr}(\xi^{\mathrm{f}} - \xi^{\mathrm{i}})} \\ &= -\Delta S_{\mathcal{S}} + \beta \Delta Q_{\mathcal{E}}. \end{aligned}$$

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