The role of intuition in mathematics

Emily Carson
Department of Philosophy
McGill University, Montréal
December, 1988

A Thesis submitted to the Faculty of Graduate Studies and Research in partial fulfillment of the requirements for the degree of M.A.

Abstract

In Part one, I examine the naive platonist view, attributed to Gödel on the basis of the analogy he draws between mathematics and physics, according to which mathematical objects are abstract objects existing independently of us, and we have a faculty of mathematical intuition which gives us access to and knowledge of these objects. While this view is easily shown to be incoherent, I then look two other possibilities for developing an account of mathematical knowledge on the basis of Gödel's analogy. First, there is the recent claim that we perceive mathematical objects in the same way as we perceive physical objects. The second more plausible view is that mathematical objects are abstractions from physical objects. The reasons for the failure of these views point to a fundamental misunderstanding of the nature of **empirical** knowledge which was first recognized by Kant. In Part two, I examine Kant's attempt to resolve this misunderstanding and the implications of this resolution for his account of mathematical knowledge, in the hope that this will suggest an alternative explanation of Gödel's analogy and of the notion of mathematical intuition.

Abrégé

Dans la première partie, j'examine l'opinion de la platoniste, attribuée à Gödel à cause de son analogie entre les mathématiques et la physique, selon les objets des mathématiques sont des objects abstraits qui ont une existence indépendante, et nous avons une faculté d'intuition mathématique qui nous habilite à avoir connaissance de ces objets. Quoique cette opinion est incohérente, j'examine deux autres possibilités pour interpreter l'analogie de Gödel. Premièrement, il y a une suggestion récente que nous percevions les objets mathématiques de la même façon que nous percevions les objets physiques. Une deuxième opinion est que les objets de la mathématique sont des abstractions des objects physiques. Les raisons pour la faillite de ces interpretations révélent une mésintelligence fondamentale du caractère de la connaissance **empirique**, une mésintelligence qui a été identifée par Kant. Dans la deuxième partie, j'examine la tentative de Kant à résoudre la mèsintelligence, et les implications de cette tentative pour son idée de la connaissance mathématique, en espérant de suggérer une explication alternative de l'analogie de Gödel et de la notion de l'intuition mathématique.

THE ROLE OF INTUITION IN MATHEMATICS

Emily Carson

Introduction

Most current treatments of mathematical intuition are based on a famous passage in Gödel's 'What is Cantor's continuum problem?' where Gödel somewhat vaguely draws an analogy between mathematics and physics. This analogy is often taken to be the cornerstone of a platonist philosophy of mathematics, in one form or another, depending on exactly how the analogy is interpreted. The role set out for intuition as an epistemological principle also varies with the different explanations of the relationship of mathematics to the physical world. In an attempt to locate and articulate what there is in common between empirical and mathematical knowledge, and therefore what role the notion of intuition plays in both of these, I want to look at four different ways in which this analogy may be understood. The question is interesting for two reasons. First, mathematics has always been taken to be a paradigm of certain knowledge, with its certainty dependent upon its abstract nature. In other words, it is usually **contrasted** with empirical knowledge in that its certainty is based upon what distinguishes it from our knowledge of the physical world. But one of the most important, and mysterious, facts about mathematics is its applicability to the physical world. As Whitehead puts it:

> Nothing is more impressive than the fact that as mathematics withdrew increasingly into the upper regions of ever greater extremes of abstract thought, it returned back to earth with a corresponding growth of importance for the analysis of concrete fact.... The paradox is now fully established that the utmost abstractions are the true weapons with which to control our thought of concrete fact.[1]

This question of applicability can be seen as the heart of the issue of how best to draw out the analogy between mathematics and the empirical sciences. Perhaps this paradox can be resolved.

The second motivation for investigating this question is a foundational one. We would like to discover whether there is some

---

[1] Whitehead [1925], pp.36-7.

common foundation for mathematics and physics which would suggest a uniform account of human knowledge. This is precisely what Kant claimed to have accomplished in his **Critique of Pure Reason**. Interestingly, in the same papers in which Gödel suggests this analogy, he makes explicit references to Kant's doctrine of the Categories. This suggests that the correct interpretation of the analogy will have a somewhat Kantian tone. Thus I think that by taking Gödel's allusions to Kant more seriously, we can absolve him of the metaphysical realist position thrust upon him by his critics, and perhaps uncover a more plausible and philosophically more interesting account of the relationship between mathematical knowledge and our knowledge of the physical world, and the role of intuition as an epistemological principle.

PART 1: Three interpretations of Gödel's analogy

1. The standard platonist problem

Gödel draws the analogy with which we are concerned in the following two well-known passages. In the first, in 'What is Cantor's continuum problem?', he draws an analogy between the epistemology of mathematics and that of physics, supposedly based on sense perception.
Gödel says:

> But despite their remoteness from sense experience, we do have something like a perception also of the objects of set theory, as is seen from the fact that the axioms force themselves upon us as being true. I don't see any reason why we should have less confidence in this kind of perception, i.e. in mathematical intuition, than in sense perception, which induces us to build up physical theories and to expect that future sense perceptions will agree with them and, moreover, to believe that a question not decidable now has meaning and may be decided in the future.[2]

In an earlier paper on Russell's mathematical logic, we find a similar analogy between the ontology of mathematics and that of physics:

---

[2] Gödel [1964], pp. 483-4.

> Classes and concepts may, however, also be conceived as real *objects*, namely classes as "pluralities of things" or as structures consisting of a plurality of things and concepts as the properties and relations of things, existing independently of our definitions and constructions.
>
> It seems to me that the assumption of such objects is quite as legitimate as the assumption of physical bodies and there is quite as much reason to believe in their existence. They are in the same sense necessary to obtain a satisfactory system of mathematics as physical bodies are necessary for a satisfactory theory of our sense perceptions and in both cases it is impossible to interpret the propositions one wants to assert about these entities as propositions about the "data", i.e., in the latter case the actually occurring sense perceptions.[3]

Thus it seems that, for Gödel, mathematics is the attempt to describe a special realm of reality, just as natural science is the attempt to describe physical reality. On the basis of this comparison of the apprehension of mathematical truth to the perception of physical objects, and of mathematical reality to the physical universe, Gödel is commonly construed as an arch-platonist who must posit the mysterious faculty of mathematical intuition just to make epistemological sense out of his ontological views.[4]

The argument for this view usually attributed to the platonist begins with the common claim that the statements of mathematics are truths, and that we can know at least some of these truths. As we have seen in the previous passage, Gödel himself takes as a premise that the axioms of mathematics "force themselves upon us as being true". The comparison with natural science first crops up in the platonist conception of truth. The logical form of mathematical statements is assimilated to that of empirical ones: both contain predicates, singular terms, quantifiers, etc. Thus, just as the empirical statement

The cat is on the mat

is true if a certain relation, *is-on*, holds between the two things named by the singular terms, the mathematical statement

3 Gödel [1944], pp.456-7.

4 See for example, Benacerraf [1973], pp.415-6; Chihara [1973], p.62; Dummett [1967], p.202.

3 is greater than 1

is true if and only if the relation *is-greater-than* holds between the two things named by the expressions `3' and `1'. Thus the platonist is taken as applying a Tarskian semantics to mathematical statements, which requires that there be corresponding mathematical objects which those statements describe.[5] But not only does the platonist want to account for the truth of mathematical statements, he also wants mathematical knowledge to be a priori, possessing universal and necessary validity, as opposed to the contingent status of empirical knowledge. Clearly then, the objects described by the propositions of mathematics cannot be objects in the physical realm. Thus in order to account for the objectivity and apodeictic nature of mathematical knowledge, the platonist is forced to postulate a realm of abstract mathematical objects "existing independently of our definitions and constructions"[6] which it is the task of the mathematician to describe.

Just as this semantical and ontological picture is developed analogously with that of the empirical sciences, the platonist provides a corresponding epistemological view to account for how we can have knowledge of these abstract objects. Where the scientist has access to the physical objects he studies through sense perception, the mathematician has access to abstract objects through "something like a perception", which the platonist calls `mathematical intuition'. Now the picture and the analogy are complete. Mathematics is viewed as reporting the facts about mind-independent objects. Unfortunately, the picture as presented here is subject to numerous well-known and fatal criticisms.

In "Mathematical truth", Benacerraf praises Gödel for at least having recognized the fundamental problem which motivates his

[5] I take this to be the standard presentation of the argument as presented in Benacerraf [1973], p.410ff. Benacerraf's unargued-for acceptance of a Tarskian semantics as the best theory we have will be important later.

[6] Gödel [1944], p.456.

attempt to draw a parallel between mathematics and empirical science.

> He sees, I think, that something must be said to bridge the chasm, created by his realistic and platonistic interpretation of mathematical propositions, between the entities that form the subject matter of mathematics and the human knower ... [thus] he postulates a special faculty through which we "interact" with these objects.[7]

The problem is that no platonist has given an account of what mathematical intuition could consist in, or of **how** we interact with abstract objects. The postulation of this special faculty seems to be a purely ad hoc move to fill the gap between the knower and what he knows. It is interesting to note that Plato's doctrine of recollection can be seen as an attempt to provide a positive account of how this gap is bridged. We have certain knowledge of the timeless truths of, for example, geometry; therefore the objects of this knowledge must be immutable, timeless objects, just as for the mathematical platonist. If we were mere physical entities, we could not interact with the pure immaterial "Ideas". Yet we do have knowledge of them. This is taken to prove the existence and immortality of the soul in that only this could explain our scientific and mathematical knowledge. The immaterial, immortal soul "having seen all things that exist, whether in this world or in the world below, has knowledge of them all".[8] While this clearly just pushes the problem back a step, leaving us with platonism of the soul instead of platonism of the mind, at least Plato recognized that some positive account must be given.

But it appears that the kind of account demanded of the platonist is impossible for him to provide, given the notion of abstract object which is forced upon him by his views on semantics and the nature of mathematical knowledge. We have seen that belief in the existence of mathematical objects is a consequence of the platonist

---

[7] Benacerraf [1973], p. 416. We shall see later that taking Gödel to be postulating a special mathematical faculty is not the best way to interpret his remarks, and that, in fact, he can be taken to be making the opposite claim.

[8] **Meno**, 81.

view of mathematical truth. But the corresponding truth conditions for mathematical propositions are such that it impossible for us to know if they are satisfied. Recall that the platonist wants mathematical knowledge to be a priori and apodeictic; this seems to require that the objects of that knowledge be immutable, non-spatial, non-temporal, and, consequently, **acausal**. This is clearly illustrated in the following passage from Frege:

> The theorems of arithmetic are never about symbols, but about the things they represent. True, these objects are neither palpable, visible, nor even real, if what is called real is what can exert or suffer an influence. Numbers do not undergo change, for the theorems of arithmetic embody eternal truths. We can say, therefore, that these objects are outside time...[9]

This requirement, combined with a causal theory of knowledge according to which a pre-condition for knowledge is that we have some causal interaction with the object of that knowledge, presents the platonist with a dilemma: either mathematical objects exist, and consequently statements about them are unknowable, or mathematical objects do not exist, with the result that statements about them are not true.[10] Thus,says Benacerraf,

> If, for example, numbers are the kinds of entities they are normally taken to be, then the connection between the truth conditions for the statements of number theory and any relevant events connected with the people who are supposed to have mathematical knowledge cannot be made out.[11]

And if numbers are not the kinds of entities they are normally taken to be, then this form of platonism collapses.[12]

---

9 Frege [1895], reprinted in Frege [1984], p.230..

10 Steiner [1975], p.110. Steiner here attempts to defend the Platonist by examining different interpretations of the causal theory of knowledge, concluding that "the most plausible version of the causal theory of knowledge admits Platonism, and the version most antagonistic to Platonism is implausible" [ibid., p.116]. Because I am not attempting a defense of this form of Platonism, I won't examine his arguments here.

11 Benacerraf [1973], p.414.

12 In the next two sections, we shall see what happens in two cases where mathematical objects are taken to be **another** kind of 'entity'.

While Benacerraf has shown that, given this view of platonist ontology, no coherent epistemology can be made out, Lear argues in his "Sets and Semantics" that no account of reference can be given either. Not only are we unable to know facts about mathematical objects, but we are unable even to **refer** to them. Lear shows that the belief that sets are abstract objects is incompatible with the belief that set theoretic discourse is **about** these abstract objects. His argument, like Benacerraf's, is based on the kind of causal theory of reference put forth by Kripke and Putnam. On this view, a necessary condition for reference to, for example, gold, is that the speaker stand in some causal relation to a sample of gold. This relation is established by a causal chain of communication linking the speaker with an initial baptism, where the baptist presumably isolates samples and dubs them `gold'. There are two ways in which this initial dubbing may be achieved: either by ostension, where the dubber causally interacts with a sample of gold, or by description. As we have seen, mathematical objects are taken to be acausal, so the first way is out of the question. The second route, the dubbing-by-description, also seems to be closed to the platonist: it is difficult to see how a sample of the natural kind *set* could be picked out by a description without presupposing that the extension is already picked out. It seems that the description would have to be something like: `a set is the kind of which the set of my hands is a sample.'[13] As Lear says:

> There is no standard set with which one stands in the necessary causal relation to make it vulnerable to the appropriate dubbing. Any theory of how the extension of `set' is fixed, compatible with (1) [the claim that sets are abstract objects] will be denied the Kripke-Putnam explanation of how the extension of a term can be fixed before anyone has a fully adequate understanding of what one is talking about.[14]

Furthermore, if one accepts the previous argument that we can have no epistemological access to these abstract objects, and thus no understanding of them, clearly the extension of the term cannot

---

13 Maddy [1980], p.167.
14 Lear [1977], p.88.

be fixed by description. How can we describe something to which we have no access? It seems that the intuition to which Lear is attempting to do justice - that statements about sets are just that: statements about **sets** - must be abandoned by the platonist.[15] As Lear puts it:

> For the supposition that one can succeed in talking about abstract objects about which no one has or has ever had any understanding leads to absurd consequences. In such a situation, it might be the case that in making set-theoretic assertions, one succeeded in talking about mysterious objects having nothing to do with the membership relation these objects would witness the falsity of even the most trivially true set-theoretic assertions.[16]

This objection thus seems to come down to the same fundamental point as the previous one; that is, the inaccessibility of the states of affairs said to exist by the truth conditions of mathematical statements.

Another argument showing the platonist's inability to account for reference to mathematical objects has been put forth by Benacerraf in "What numbers could not be". Granting the platonist claim that mathematics describes a realm of objects, we might ask what kind of objects these are. One answer to this question, given by set-theoretic investigations, is that all mathematical objects can be identified as sets. Benacerraf takes the example of the natural numbers: if we know that the natural numbers are sets, then it seems natural to ask precisely which sets the numbers are. Unfortunately, the aforementioned set-theoretic investigations have provided us with too many answers. We have infinitely many correct set-theoretic accounts of the natural numbers "to which no exception could be taken and on the basis of which all that we know about or do with numbers could be explained".[17] Each account tells us that number terms `really refer' to a particular set of objects, and

15 This gives rise to the obvious question `Just what are these statements about', to which it appears the only remaining response is some form of nominalism, either reductionist or, like Hartry Field's, eliminationist, where he simply denies that mathematical statements are about anything.

16 ibid. p.89.

17 Benacerraf [1965], p.279.

allows us to prove the independent existence of the object which is, for example, the number 3. Unfortunately, on each account, the set of objects and the particular object which is the number 3 are different. On one account, the progression which comprises the set of objects to which all number terms `really refer' is 0, {0}, {0,{0}}, {0,{0},{0,{0}}},... where 3 is {0,{0},{0,{0}}}, whereas on the other account, the relevant progression is {0}, {{0}}, {{{0}}},... and 3 is {{{0}}}. Our arithmetic purposes are served equally well by either of these accounts, since both admit of the structure necessary and sufficient for any correct account of the natural numbers. The platonist is again faced with a dilemma. Either he accepts that 3={{{0}}}, and 3={0,{0},{0,{0}}} or he must provide some argument to show that one or the other is the correct account.

> ... if the number 3 is really one set rather than another, it must be possible to give some cogent reason for thinking so; for the position that this is an unknowable truth is hardly tenable. But there seems little to choose among the accounts.[18]

As Putnam would put it, both theories satisfy all our operational and theoretical constraints: what more could we ask for?

In "Models and Reality", Putnam argues along the same lines that the Löwenheim-Skolem theorem shows that the intended interpretation of `set' is not captured by the formal system. "But", Putnam asks, "if axioms cannot capture the intuitive notion of a set, what possibly could?"[19] According to Putnam, the platonist takes this as evidence for his position:

> The platonist will reply that what this really shows is that we have some mysterious faculty of `grasping concepts' (or `intuiting mathematical objects') and it is this that enables us to fix a model as *the* model, and not just operational and theoretical constraints.[20]

But it is precisely this notion of mathematical intuition which has been shown to be incompatible with our best theory of knowledge.

---

18 ibid.,p.284.
19 Putnam [1980], p.423.
20 ibid., p.430.

Furthermore, for the platonist, this faculty is one which "the naturalistically minded philosopher will never succeed in giving an account of".[21] Thus it seems that the platonist is stuck with either indeterminacy of reference - at best - or mysticism.

Putnam is correct that platonists sometimes do take the limitative results of logic as support for their view. As we have seen, a primary ground for platonism was the conviction that mathematics is the investigation of **existing** abstract structures. As Dummett points out, this view seems to be

> ...reinforced by our inability to give a formally determinate characterization of them [the structures]. Our grasp of these structures appears to outrun our ability to describe them. Hence it appears inescapable that we possess an intuitive faculty by which we can observe that these structures are there and can apprehend them as a whole: a faculty which guides, but is not exhausted by, our recognition of the truth of axioms and the validity of methods of proof.[22]

When we do number theory, we apparently have in mind one determinate structure which we intend to investigate. The fact that the theory of the structure cannot be completely formalized shows only the limitations of formalization. Unfortunately, as Dummett goes on to point out, this view engenders as many absurdities as it replaces. The claim that our intuitive grasp of the natural numbers cannot be communicated by formal systems can be tolerated only if we can suggest an alternative way of communicating it since it seems obvious that when different people talk of the standard model, they are all talking about the same model. Certainly, the platonist at least wants to say that our agreement on results arises from the fact that we all have in mind the **same** structure. So there must be some way of communicating our intuitions of that structure. The only alternative is the second-order induction principle, where the predicate variables range over all properties. But this only

21 ibid., p.424. As we shall see, Penelope Maddy attempts to give a version of Platonism according to which this faculty of intuition is naturalized. But, as we shall also see, she fails to refute Putnam's claim.

22 Dummett [1967], p.208.

postpones the problem because we now have to explain how *this* notion is communicated.

> Since, for any given formalization of second-order logic, there will be a non-standard interpretation, we cannot know that other people understand the notion of all properties (of some set of individuals) as we do, and hence have the same model of the natural numbers as we do.[23]

Thus we are left with the notion of an intuitive grasp of these abstract structures which is private and incommunicable. While we know nothing that would count as uncertainty over whether some mathematical object is or is not a natural number, we are completely unable to verify that what someone else refers to as the natural number system is the same as, or isomorphic to, the system we have in mind. The platonist is left only with "the myth of an ineffable intuition of the standard model."[24]

A very important motivation behind the platonist position is the need to explain the usefulness of mathematics. In particular, mathematical formulations are successfully used to describe the physical world. We can see that it is this consideration which prompts the platonist to treat mathematical statements as bearing truth values,[25] particularly if we compare the fictionalist position that mathematical statements are meaningless marks produced by playing formal games.[26] Such an account cannot explain why the games mathematicians play are so useful. How is it that we can combine mathematical and physical premises to yield empirically testable conclusions? This can be explained if we take mathematical statements as statements with truth values, as the platonist does. As we saw earlier, it is to account for the truth of mathematical statements that the platonist finds it necessary to posit a realm of abstract objects which render those statements true or false.

23 ibid. p.210.

24 ibid., p.211.

25 In **Science without numbers**, Hartry Field, and unsympathetic commentator on the platonist position, calls this the best argument for the claim that mathematics is a body of truths.

26 Kitcher [1984], p.110.

Ironically, this second step actually undermines the claim it is supposed to account for. If the world is bifurcated, as the platonist claims, into a physical realm, described by science, and an abstract realm, described by mathematics, how is it that the truths of one realm have any application in the other supposedly independent realm? Once again, the platonist has to explain how such interaction is possible and successful. Until he does, he must appeal either to some form of pre-established harmony or mysticism.

As we have seen, the primary objection to the platonist view is that no account is given of the mathematical intuition which is supposed to fill the epistemological gap between the knower and the abstract mathematical objects. However, two of Gödel's many critics, Kitcher and Chihara, do suggest ways - albeit weakly - in which such an account might be developed from Gödel's comments in the passages quoted earlier. As one might expect, they come up empty-handed.

Kitcher concedes the possibility of a non-inferential knowledge-generating process which he calls mathematical intuition with the intention of showing that such a process is incoherent and that it could not yield a priori knowledge.[27] He claims that there are only two strategies for responding to the skeptic who questions whether anyone has `performed' such intuitions: one way is to appeal to authority by asking mathematicians if they recognize in themselves processes of intuition which give them knowledge of mathematical objects. The second way is to proceed by something akin to transcendental argument like that outlined earlier. That is, to account for mathematical truth, we must postulate mathematical objects; given this, there must be some process of this kind. If the platonist follows this route, then the notion of mathematical intuition ends up as a theoretical entity, inasmuch as we believe in its existence because we hold a particular theory about mathematical truth. So all we know about the notion of intuition is what the theory tells us. This, according to Kitcher, amounts to a denial that the process is at all identifiable: no one knows what it is

27 Kitcher [1984], pp.58-64.

like to have an intuition; we just know that we must have them. It follows then, even if we accept that mathematicians do have platonist intuitions, that we cannot discriminate genuine mathematical intuitions, yielding knowledge of some fact about mathematical objects, from processes known to yield false beliefs. Here, Kitcher cites as evidence the universal comprehension principles hailed by Frege, Dedekind, and Cantor as 'intuitively evident', but which turned out to be false. Thus intuition characterized according to the second strategy fails to meet Kitcher's criteria for a priori warrants.

This line of argument defeats the first strategy for characterizing intuition, the appeal to mathematicians, since it has shown that they can be mistaken, and it significantly weakens the second strategy in that, even if intuition is granted as a theoretical entity, characterized only by its role within a more general theory, it does not fulfill all that the platonist requires of it: it does not issue in a priori mathematical knowledge. Thus, Kitcher concludes, intuition "whether clearly specified or ill-defined, will not do the job which the a priorist demands of it".[28]

Chihara identifies mathematical intuition with what he calls the mathematical experience of an axiom's 'forcing itself upon us as being true'. He turns Gödel's argument around by claiming that Gödel postulates the abstract entities in order to explain this experience. Then again, it is up to the platonist to characterize this experience if he wants to avoid the charge of mysticism. Who, if anyone, has had these experiences and under what conditions? One possible case is that of a student given a description of the intuitive notion of set.[29] Subsequently, upon being shown the regularity axioms, the student immediately recognizes the truth of those axioms. But all that has happened in this case is that the student has recognized that the universe of sets described by the 'intuitive notion of set' just is a universe in which, say, the regularity axiom holds, but this does nothing to establish the **objective** truth of the regularity axiom,

---

28 ibid., p.64.
29 Chihara [1982], p.216.

that is, that the regularity axiom states a fact about the world. The regularity axiom is objectively true only if the intuitive notion of set is an accurate description of an independently existing mathematical object. But this is what the objective truth of statements including the regularity axiom is supposed to establish. Now if this student's experience is what constitutes an axiom's 'forcing itself upon us as true', then

> I don't see why one should be tempted to postulate a perception of sets to explain such experiences, any more than one would be tempted to postulate a perception of angels in order to explain someone's experience of coming to realize that, according to Dante's conception or notion of Heaven, there are angels in Heaven.[30]

In other words, we are not led to believe in the existence of certain objects just because some theory demands it unless we are given good evidence for the theory itself.[31] But, according to Chihara, Gödel's evidence for the objective truth of mathematics is the mathematical experience which the existence of mathematical objects is supposed to explain! Chihara seems to be saying that Gödel's argument here is circular: the postulation of mathematical objects is necessary to obtain a satisfactory theory of mathematical experience, but that mathematical experience provides evidence for the claim that mathematical statements are objective, i.e. about objects.[32]

It seems that the main problem with the platonist view of mathematics is best illustrated in terms of Benacerraf's dichotomy between views of mathematics motivated by epistemological concerns and those motivated by ontological concerns. Platonism is an example of the latter, so the platonist is left with the task of providing a satisfactory epistemology to go along with his ontology. But it appears that this ontology precludes the possibility of a reasonable (i.e. non-mystical) epistemology. The objects prescribed by the platonist ontology (as dictated by his view of mathematical

30 ibid.

31 Chihara [1972], p.76.

32 Chihara [1982], p.213.

truth) are abstract objects, which are incapable of any causal interaction. Here, Gödel's analogy between mathematics and science breaks down. It appears that no account of reference to or knowledge of these objects is possible. We looked at two unsympathetic attempts to provide some further account of the nature of mathematical intuition, which is supposed to do the trick here, but these attempts failed to meet other requirements of the platonist programme.

In response to these problems with so-called Gödelian platonism, Maddy develops a naturalistic platonism with the intention, she claims, of getting the analogy straight. In the next section, I want to examine how successful her explanation is.

2. Maddy and Naturalized Platonism

The particular form of the analogy with which Maddy is concerned is the one suggested by Gödel in the passage quoted earlier: that mathematical intuition plays a role in set theory similar to that played by sense-perception in physics. Furthermore, Gödel claims that there is, in our experience of physical objects, something besides the sensations and qualitatively different from them, which is immediately given, as, for example, the idea of object itself. Finally, we are given a clue as to what mathematical intuition may be intuition of: "Evidently, the given underlying mathematics is closely related to the abstract elements contained in our empirical ideas."[33] Maddy is on the right track in taking this as a suggestion that

> the nature of mathematical intuition can be better understood in light of an investigation of the origin and role of the abstract elements of perceptual experience.[34]

Maddy believes that Gödel's analogy is best served by examining our best theory of the perception of physical objects, and then showing that this can be extended so as to include the

33 Gödel [1964], p.484.
34 Maddy [1980], p.169.

perception of mathematical objects. Then we can deal with the central question posed by the anti-platonist, that is: "What legitimates the gap between what is causally interacted with and what is known about, or what's kind is dubbed, in the case the causal theory accepts?"[35] If the answer to this question in the case of physical objects also applies to the case of mathematical objects (in this case, sets), then the platonist has a response to Benacerraf and Lear.

Thus Maddy first explains how, according to the theory, it is that we **perceive** physical objects, as opposed to simply undergoing sensory stimulation. Whatever it is that constitutes the difference between sensations caused by a physical object and the perception of that physical object is the `abstract element' we are looking for. This will be what gives us the `idea of an object itself'. According to Maddy, our best theory of perception tells us that human beings develop neural `object-detectors' which are responsible for our acquiring perceptual beliefs about physical objects. Thus two distinct events occur when we perceive a hand, for example: first, an aspect of the hand causally interacts with the retinas; secondly, this stimulates the appropriate detector. Put crudely, this object-detector is what transforms our sensations, or the pattern of sensory stimulation, into the perception of a hand. Maddy's bold claim is now that we similarly develop `set-detectors' which enable us to perceive **sets** of physical objects given sensory stimulation from ordinary physical objects. The idea must be, then, that the same causal interaction with the retina which stimulates the object-detector also stimulates the set-detector,[36] thus allowing us to **perceive a set** of five fingers. When someone looks into an egg carton, he perceives both the eggs and the set of eggs. Thus he **knows** that there is a set of eggs before him. That the presence of the eggs sets off the set-detector is supposed to be enough to allow us to say that the fact that there is a set of eggs before him is

[35] ibid., pp.169-70.

[36] Although Maddy does speak of neural gating mechanisms which correspond to the perceiver's degree of attention or inattention, so that he may, on a given occasion, perceive only the five fingers, and not the set of them.

"appropriately causally responsible" for his belief in the statement 'There is a set of eggs before me'. As Maddy says:

> As in the case of knowledge of physical objects, it is the presence of the appropriate detector which legitimizes the gap between what is causally interacted with, and what is known about.[37]

Presumably, we also have sufficient causal interaction to admit that, when our perceiver makes a statement about the set of eggs, he is in fact **referring** to the set of eggs. Thus Maddy claims to have rescued our intuition that we can talk about and have knowledge of sets, just as we talk about and have knowledge of eggs.

Obviously, Maddy's claim to have solved the problems facing the platonist depends upon her more basic claim that we can perceive sets. Thus we are faced with two issues. First, is this basic claim defensible? And secondly, is it sufficient to rescue the platonist? More succinctly, "can we perceive sets and does it matter?"[38] I want to claim that the answers to these are 'no' and, more importantly, 'no'. While clearly, a negative answer to the second question will render any argument against the first claim superfluous, it might nevertheless be instructive to examine briefly why I think the thesis that we can and do perceive sets is wrong-headed.

While there is no completely precise characterization of sethood, there are two properties which are fundamental. First, sets are extensional, i.e. sets with the same members are identical, and secondly, they are objects. Thus any account which purports to show that we can perceive sets must show that we perceive collections as extensional objects.[39] Again, we may appeal to the analogy with physical objects. Certainly, when I perceive an apple on the table, I see it as one object. The issue then, is whether, when I perceive three apples on the table, I also perceive one extensional object - the **set** of three apples. Of course, we may **treat** a collection as a

---

37 ibid., p.183.

38 The following argument is based on an unpublished paper by Michael Hallett (Hallett [1983]).

39 Hallett [1983], p.13.

single object; for example, we easily talk about two shoes as **one** pair, or of wolves as forming **a** pack. But the difference here is that we consciously **impose** this grouping on the multiplicity that we perceive. Conceiving of a multiplicity of wolves as a pack is "a conceptual innovation that we introduce as an intellectual simplification and **not** at all something we perceive".[40] In the case of the apple on the table, we take it naturally, automatically, to be (i.e. we perceive it as) one object. Nevertheless, it can be conceptually analyzed by physicists into a multiplicity of primitive objects. On the other hand, in the case of a collection, we naturally take it as a multiplicity of objects, we perceive the multiplicity, but it can be conceptually treated as one object. But no matter how successful this conceptual imposition is, the collection remains a multiplicity to our senses. So where Maddy sees an analogy, I suggest the converse relationship holds. We naturally perceive a physical object as one object, but we can analyze it into a multiplicity of parts. However, we naturally perceive a collection as a multiplicity, but we can consider it as one object. It is this fact which makes the notion of an object-detector less unintuitive than that of a set-detector.

The preceding claims about sets are consistent with Cantor's remarks about the unity of sets; for example: "by a `manifold' or `aggregate', I generally understand every multiplicity which **can be thought of** as a one."[41] This suggests that the unity is intended as purely intellectual, and imposed by us: we "create" the set out of the elements we perceive. But this conceptual exercise will not turn the multiplicity into one **perceivable** thing.[42] An essential feature of sets is their objecthood, but since we do not perceive collections as objects, therefore we do not perceive sets. The claim that we do "perceive" sets is no more plausible than the claim that we "perceive" electrons. Of course, as I mentioned earlier, the acceptance of this basic claim turns out to be irrelevant because,

---

40 Hallett [1983], p.16.

41 Cantor [1883], p.93.

42 cf. Hallett [1984], pp.34-5.

even if we do perceive sets, as far as mathematical set theory (or even the desperate platonist) is concerned, it doesn't matter.

The argument against knowledge of and reference to abstract objects is based on the more general claim that we cannot causally interact with abstract objects because they don't exist in space and time. Maddy claims that she has shown how abstract objects can exist in space and time, as sets of physical objects do. She is not claiming that we causally interact with these sets, but rather that the sets play a role in generating our perceptual beliefs about them analogous to the role that physical objects play in our generation of perceptual beliefs about them.

Thus far, Maddy claims to have shown how we come to know, by observation, simple facts about particular physical sets. But, as she points out, she also has to account for our knowledge of general truths about sets, truths like the basic axioms of set theory. This is where she introduces the notion of intuition. She claims that the structure of the set-detector determines some general beliefs about sets; these she calls `intuitive beliefs'. So the claim must be that, by interacting with collections of physical objects, we develop set-detectors which allow us to perceive these collections as sets, and which embody intuitive beliefs about sets, such as that the number property of a set is not changed by moving its elements, sets other than singletons have many proper subsets, any property determines a set of things which have that property, etc. Because these beliefs are built into the structure of the detector which allows us to perceive sets, the linguistic formulations of these prelinguistic beliefs `force themselves upon us as being true'.

Unfortunately, Maddy's solution seems to be no less mysterious than the platonist's claim that intuition just is what gives us knowledge of the axioms. Unless she gives some explanation of how these particular intuitive beliefs come to be built into the set-detector, this neural mechanism seems to be no more than an ad hoc gap-filler. It is this type of explanation the anti-platonist demands, and the fact that we have some contact with the objects doesn't help Maddy here. The objection seems related to standard objections to an abstractionist account of mathematical knowledge. As we shall see in the next section, Maddy is simply loading the problems with abstraction onto the set-detector, thereby merely pushing the problem back a step. This should become clearer when we look at the last of Maddy's claims, that she has given an account of how we can refer to sets.

We come now to Maddy's solution to Lear's problem about reference. As we have seen, Maddy believes that the problem of reference to sets is solved by her claim that we perceive physical sets and thus that a causal theory of reference can be maintained. Reference to sets is possible because they form a natural kind which we pick out by interacting with physical samples, i.e. by ostension. A standard example of this type of dubbing is the case of water. The natural kind water was originally picked out by an initial baptism with the baptist pointing at a sample and stating "This is water and from now on we shall call anything like this stuff `water'". Thus the reference of `water' is fixed independently of any specific knowledge of water. Maddy claims that we can have an analogous baptism for sets:

> We imagine our baptist in his study saying things like: "All the books on this shelf, taken together, regardless of order, form a set," and "The globe, the inkwell, and the pages in this notebook, taken together in no particular order, form a set." By this process, the baptist picks out samples of a kind. The word `set' refers to the kind of which these samples are members. One important feature of this treatment is that we can refer to sets without knowing much about them, just as we first referred to gold without knowing that it has atomic number 79, or how it differs from iron pyrites.[43]

The main problems with this stage in Maddy's argument are more general in that, even if we accept this account of how an initial baptism might be possible, it falls far short of what is needed for an epistemology of set theory. I will deal with this later, but for now I want to show that granting Maddy this account is already too charitable.

One central problem here is that the standard objections to ostensive definition apply; that is, if the initial baptism is to

---

43 Maddy, p.167.

ensure that the word 'set' refers to the correct objects, then it must be clear that the intended referent, what is being pointed at, is just that object of which the books on the shelf are the elements.[44] Just pointing to the individual books will not do, because nothing precludes the possibility that 'set' is intended to refer to a physical complex, such as a library, comprised of the individual books 'taken together, in no particular order'. In order to interpret the ostensive act, then, it appears as though one must already understand the set-membership relation and the distinction between elements and what they are elements of. In other words, to recognize the intended referent of the ostensive act, one must already possess the concept of set!

I suspect that Maddy would count on her set-detector to defuse this objection. We can compare the case of ostensive dubbing of, for example, lemons, where it must be understood that the intended referent is not the front side of a time slice of a lemon, but rather a persisting, three-dimensional object. In response to a different objection,[45] Maddy claims that "the realist could argue that the relation of element to set is no more objectionable than the relation of fleeting aspect to temporally extended object".[46] And in fact it is the object-detector which is responsible for our perceptual beliefs about physical **objects** as objects, that is, for our seeing a series of sensory patterns as aspects of one thing. To have the concept of an object in general just is to have a fully developed object-detector, and the object-detector ensures that when the dubber points to the lemon and says "That and anything like it is to be called 'lemon'", we correctly pick out as the referent the whole lemon, and not just a time slice of it. Similarly, it must be the set-detector which is responsible for our ability to pick out as the correct referent of 'set' the **set** of books on the shelf, and not a physical complex, or any other plausible construal. Thus, the possession of a fully-developed set-detector simply amounts to the possession of the concept of set.

---

44 Hale, pp.81-2.

45 Or, as Hale would have it, a misunderstanding of this argument.

46 Maddy, p.168.

So Maddy would agree that the success of the ostensive baptism of sets does depend on the prior possession of the concept of set. But in response to, for example, Hale's objection that one who already possesses the concept of set "has no need of ostensive exemplification of it"[47] and therefore the relevant notion of ostensive baptism is either incoherent or superfluous, Maddy would just refer us to the set-detector. The set-detector develops pre-linguistically, so "it is possible to acquire these concepts and intuitive beliefs without acquiring linguistic forms with which to express them".[48] So the function of the ostensive dubbing is to allow the subject to associate the word with the appropriate detector or, equivalently, with the concept. Note once again, that the set-detector is made to do all the work in an apparently ad hoc fashion. This sense of `ad hoc-ness' is not suggested in the case of ostensive baptism of physical objects because of the general applicability of the conceptual equipment required for the success of the baptism. The subject does not require the concept of lemon to ensure the successful identification of the intended referent of `lemon', but just something like the concept of an object in general which, perhaps, can be argued for on independent grounds. We shall come back to this in later sections. But to say that the correct identification of the intended referent of `set' is ensured by the prior possession of the concept of set, is not to explain very much, unless some fuller explanation is given of how we acquire this concept. And as we have seen, the set-detector doesn't do the job.

Thus far, we have looked at the more specific claims of Maddy's programme:

(i) that we can and do perceive physical sets in the same way as we perceive physical objects;

(ii) consequently, we can maintain a causal theory of knowledge according to which we acquire knowledge of some simple facts about particular physical sets through perception of them;

---

47 Hale, p.82.

48 Maddy, p.185.

(iii) we also acquire knowledge of more general truths, such as some of the basic axioms of set theory, through intuition;

(iv) we can hold on to a causal theory of reference by maintaining that sets form a natural kind which we pick out by causally interacting with, i.e. perceiving, samples.

I have attempted to show that each of these individual claims falls because they each rely on the naturalistic construal of 'Gödelian' intuition, which can't fulfill its role. However, I now want to show that even if we grant Maddy all of these claims, they cannot be transformed into an adequate account of mathematical epistemology.

The key point is just this: what does this rather scant knowledge of **physical** sets have to do with our knowledge of mathematical sets? Set theory is about sets in general, but all Maddy has accounted for is limited knowledge of a particular kind of set, i.e. those which exist in space and time. What is needed here is some connection between this limited realm and sets in general, including pure, abstract mathematical sets. Towards this end, Maddy argues that our "mathematical experiences" - our direct interaction with physically given mathematical objects, i.e. sets - "lend support to the theoretical parts of set theory".[49] This suggests an analogy with the notion of empirical support in physics in that certain axioms are observationally confirmed with respect to the physical sets, and this gives us confidence in the correctness of the axioms for pure mathematical sets. But we still need some explanation of how confirmation of certain axioms for physical sets tells us anything about pure sets. For example, how can observation of physical, and therefore finite, sets 'force on us as true' the axiom of infinity, which is crucial to the mathematical strength of pure set theory, and which claims existence of a set which even Maddy could not argue is perceived? Without the axiom of infinity, set theory just collapses to arithmetic. Yet it is not at all clear how Maddy could justify acceptance of it.

---

49 Maddy, p.193.

Furthermore, while observation may confirm some axioms, e.g. union or power set, for finite sets, why should we then accept them for infinite sets? The only way this can be made to work is by claiming that the knowledge we acquire of these physical sets is applicable to mathematical sets because the former are representative of the latter, that is, they are of the same kind:

> And, just as gold on other planets, and gold that doesn't look like gold, are included within the scope of the kind *gold*, pure sets, and infinite sets are included within the kind *set*; all that matters is that they are the same kind of thing.[50]

Unfortunately, this cannot work unless we are given some explanation of what the appropriate similarity relation between physical sets and pure sets might be. What basis could there be for claiming that physical sets and pure or infinite sets are of the same kind? It certainly cannot be the perceptual similarity relation Maddy speaks of, since what we want is a relation between perceived and unperceivable sets. The only answer seems to be that they satisfy certain conditions: they have number properties, they have many proper subsets, etc. In other words, they satisfy the conditions embodied in the set-detector. But, returning to the original problem, we see how little help this can offer. What we want is an explanation of how we can justify accepting for infinite sets the axioms which are observationally confirmed for finite sets. What we get is the claim that infinite sets are of the same kind as finite sets. But how do we know this? Well, they satisfy the same axioms. Not only would this reply be circular, but it also represents a return to the "traditional theory according to which a set is anything which satisfies certain conditions"[51] which Maddy wants to replace with her natural kind claim.

Unless the notion of set-detector can be clarified and explained, Maddy has simply replaced the platonism of the mind with a platonism of neurophysiology: that is, she has simply posited the

---

50 Maddy, p.167.

51 Maddy [1980], p.183.

existence of a neurological faculty which fills the gap left by the failure of the faculty of mathematical intuition of section 1.

3. Abstraction

I want to look now at a third way in which an analogy between mathematical epistemology and that of natural science has been supported. According to the two views we have looked at so far, mathematical knowledge was taken to be knowledge of objects existing independently of the mind and apart from the objects studied by physicists. The theories begin with objects, that is, by setting out an ontology. The explanatory problem they then face is to provide an epistemology to fit this. Given this, the main problem is to find in mathematical epistemology an analogue for the role played by perception, or contact, in physical science; that is, to explain how we have access to these objects. In one case, this role was played by an as-yet-unexplained, and perhaps unexplainable, faculty of mathematical intuition. In the second case, the analogy was dispensed with by Maddy's assimilation of mathematical to physical epistemology with the claim that we perceive sets in the same way as we perceive physical objects. But neither case gives us a satisfactory epistemology. This third view differs in that it attempts to account for mathematical knowledge while doing away with an independent realm of mathematical objects, and it does so by beginning this time with epistemology rather than ontology.

For the abstractionist, mathematics is (loosely) determined by the physical world in that mathematical objects or properties are derivative on physical objects. The analogue to perception, and thus the process by which we derive the subject-matter of mathematics, is abstraction. The new twist introduced by the abstractionist is the active participation of the mind which in a sense unites the ontology and the epistemology in a way in which the two previous views could not. For Maddy and the naive platonist, the knower is supposed to perceive **passively** things which are really there. But for the abstractionist, mathematical 'objects' or properties are constructions by the mind out of what is really there. Thus he appears to have a ready-made cognitive link between the knower and the subject matter of his knowledge.

There is no doubt that abstractionism has some attractions, particularly for those who hold that our knowledge is ultimately derived from the natural world around us. But even if abstractionism can be made clear, it is open to severe difficulties. These are perhaps best seen by looking at how abstractionism might work with respect to number, in particular, the concept of number as based on the traditional account derived from the Greeks. The traditional definition of number goes back at least as far as Euclid, who writes that "a number is a multitude composed of units"[52]. Now for the platonist who doesn't shrink from ontological commitment, and for Plato himself, the units which compose the numbers are pure, independently existing mathematical objects, different from the objects comprising the collections they number. Thus, the numbers, the things we count with, are given the same ontological status as the objects we count, if not higher.[53] The abstractionist account is developed in opposition to this view of number. For the abstractionist, the units **depend** for their existence on the perceptible objects they number; they are abstractions from collections of **real** objects. The explanation of this is based on a quasi-psychological account of what is involved in the process of counting.[54]

Consider the primitive act of counting a small finite collection of objects. We may think of it as a process of replacement where we mark each object with a finger or a stroke, and then count the strokes. Thus we replace the set of physical objects, say an apple, a pencil, and a book, with a set of strokes, ///, isomorphic to the original collection. The strokes are then `idealised' as units, at least in the Greek account, in recognition of the fact that, if mathematical truths are eternal truths, then the objects of mathematics must be

---

52 Elements, Book VII, definition 2.

53 In fact, since for Plato explanatory priority implies ontological priority, and since these pure units are required to explain the possibility of counting, the numbers are considered to be more real than the physical objects they number.

54 See Hallett [1984], pp.124-5.

unchanging and therefore not physical. In fact, Kline credits the Greeks with first having recognized this:

> One of the great Greek contributions to the very concept of mathematics was the conscious recognition and emphasis of the fact that mathematical entities, numbers, and geometrical figures are abstractions, ideas entertained by the mind and sharply distinguished from physical objects or pictures.[55]

The problem for the abstractionist, though, is to explain how the stroke is transformed into the unit **without** presupposing platonism of another kind. Now it is important that each object makes the same contribution to the total number, so we must consider this replacement as the repetition of the same stroke or unit; this leads naturally to the conception of number as consisting of a multiplicity of identical units. So far so good. The role of abstraction now is to explain how we transform each physical element of a set into a unit. In the context of the Greek account of number, one way to explain this is to treat abstraction as based on a faculty of the mind which allows us to strip away the particular properties of any object until we are left with something common to all objects, the unit.

It should be clear from this that, unlike the platonist of sections 1 and 2, the abstractionist is not committed to the mind-independent existence of the natural numbers. Instead, the numbers are more like conceptual objects arrived at by the exercise of an active faculty of the mind on physical reality. Moreover, abstractionism ought to foster one other benefit, namely to provide the beginnings of an answer to the question of the applicability of mathematics to the physical world. Like the physicist, the mathematician studies **directly** some properties of physical objects, in this case, their most general properties. His faculty of abstraction allows him to isolate the properties which are the particular concern of mathematics.

---

55 Kline [1972], p.29. For a much better and more detailed account of this, see Hallett [1984], pp.132-33.

Unfortunately, in the case of the account of number we are using for illustration, this faculty falls victim to Frege's fatal attack in both the *Grundlagen der arithmetik* and his review of Husserl's *Philosophie der arithmetik*. Frege tells us that according to Husserl we may arrive at the number of an arbitrary concrete multiplicity if we

> ...abstract from the peculiar constitution of the individual contents that make up the multiplicity and retain each one only in so far as it is a something or a one, thus obtaining with respect to their collective connection the general form of multiplicity that corresponds to the present multiplicity, i.e., the corresponding number. This process of abstracting the number goes hand in hand with a process of emptying of all content.[56]

Thus collections of physical objects are transformed into collections of units by a cleansing of their peculiarities in "the wash-tub of the mind" (see below). As is clear from Frege's ironic tone, he has doubts about the plausibility of this alleged faculty; furthermore, even if we grant the ability to abstract away all peculiarities from an object, this will not issue in a satisfactory account of number as a multiplicity of units. The question confronting the abstractionist here, like the one confronting Maddy, is two-fold: Can we perform this process of abstraction, and does it matter? Again, the answers are, respectively, no and no.

As we have seen, this process of abstraction is supposed to account for the intellectual transformation of a physical object into a unit devoid of particular properties. According to Husserl, "to abstract from something simply means: not to attend to it especially". So for example, suppose there are two cats, one black and one white, sitting side by side. Husserl's claim must be that, if we do not attend especially to their colour, we render them colourless. If we do not attend especially to their position, they no longer sit. We do not attend especially to their place, and they cease to occupy one. The repeated application of this procedure transforms

---

56 Frege [1984], p.196.

these objects into "bloodless phantoms", emptied of all content. But, according to Frege, this does not in fact leave us with the concept of the number of things considered, but rather with the general concept under which those things fall, in this case, the general concept of cat:

> Even if I proceed to bring them both under this concept and call them, I suppose, units, the white one still remains white just the same, and the black black. I may not think about their colours, or I may propose to make no inference from their difference in this respect, but for all that the cats do not become colourless and they remain different precisely as before. The concept *cat*, no doubt, which we have arrived at by abstraction no longer includes the special characteristics of either, but of it, for just that reason, there is only one.[57]

The intellectual transformation of a collection of objects into a collection of identical units is impossible to perform: "we cannot succeed in making different things identical simply by dint of operations with concepts".

But Frege's attack does not end there: even if we could perform this process of abstraction, it cannot be the basis for a satisfactory account of number. The main problem arises from the impossibility of reconciling the identity of the units with their distinguishability. Abstraction is supposed to strip away all distinguishing properties of the separate objects to render them identical units, so that each object makes the same contribution to the total number. But the plurality required of number arises out of diversity. In set-theoretic terms, say for example we want to number a collection A={a,b,c}. By abstraction we supposedly arrive at the set A'={1,1,1} where a, b, and c have been reduced to the same unit 1. But according to the principle of extensionality, {1,1,1}={1}, so that the number associated with **any** collection is {1}. If, on the other hand, we try to bring in the necessary diversity by providing the 1 with differentiating marks, e.g. indices, then we are left with the set A'={1,1',1"}. But then the identity of the units is lost completely, and we are left with a new plurality of distinct objects in all their

57 Frege [1884], pp.45-6.

particularity; we may as well stick with the original set {a,b,c}. The abstractionist is faced with a dilemma:

> If we try to produce the number by putting together different distinct objects, the result is an agglomeration in which the objects contained remain still in possession of precisely those properties which serve to distinguish them from one another; and that is not the number. But if we try to do it in the other way, by putting together identicals, the result runs perpetually together into one and we never reach a plurality.[58]

Forced to somehow ascribe to units two contradictory qualities, the abstractionist must take "the road of magic rather than of science".[59] He simply posits the existence of an ad hoc faculty which

> just has the marvellous and very useful property of making things absolutely identical without changing them. This is possible only in the wash-tub of the mind.[60]

Just as Maddy's set-detector and the naive platonist's notion of intuition are ad hoc faculties postulated to `explain' how we come to know the axioms of set theory, abstraction in this numerical case looks like an equally ad hoc faculty introduced to explain what numbers are. The question of epistemological access is supposed to be what motivates the abstractionist account. The truths of mathematics are taken to be arrived at by the operation of the mind on the physical world; hence there is supposedly no need to postulate a separate realm of abstract objects. But where Maddy and the naive platonist must fill the gap between the knower and some abstract realm, the abstractionist must fill the gap between the physical world and the knower. He must explain exactly what this operation of the mind consists in. Hence the epistemological benefits are by no means as great as appears at first sight.

---

58 Frege [1884], p.50.
59 Frege [1895], p.205.
60 Ibid.

Whatever its impact on traditional theories of number, the core of Frege's attack on abstractionism should not have come as a surprise, since it had been foreshadowed by Berkeley's criticism of Locke's notion of a general triangle. As is well known, Locke developed a theory of abstract ideas to explain, given his doctrine that all existent things are only particulars, how we come to acquire and use general terms. On Locke's empiricist conception of language, words act as labels for our mental images or ideas, which are derived from particular things in the physical world. Now if every particular thing - i.e. idea - were assigned a distinct name, says Locke, names would be "endless". But we do use general terms,and there must therefore be some idea to which they correspond: this is an abstract idea. Thus, according to Locke, it must be the case that

> ...the mind makes the particular ideas received from particular objects to become general; which is done by considering them as they are in the mind such appearances, - separate from all other existences, and the circumstances of real existence, as time, place, or any other concomitant ideas. This is called ABSTRACTION, whereby ideas taken from particular beings become general representatives of all of the same kind; and their names general names, applicable to whatever exists conformable to such abstract ideas.... and thus universals, whether ideas or terms, are made.[61]

The empiricist tendency here is quite clear. To use Locke's example, the mind today receiving sensations from chalk or snow which it received yesterday from milk considers "that appearance alone" which is common to them (i.e. the same colour) and makes it a representative of all of that kind, things of that colour. So by abstracting from, or not especially attending to, the particular properties of the samples, i.e. the shape of the chalk, the location of the snow, the fluidity of the milk, the mind provides itself with an image of whiteness alone, to which it affixes the label "whiteness".

Not only does this doctrine of abstraction permit an empirically based explanation of our use of general terms, but it also serves an epistemological purpose, for it addresses the question "how is it

61 Locke [1689], (II,xi,9). Cf. also (III,iii,6).

that we come to have universal knowledge when we observe only particulars?" If abstraction does any serious work, then it is here that it is applicable to mathematics. For example, the statements of geometry are taken to be universally true: Pythagoras' theorem is taken to be true of all particular triangles. But we do not, and could not, verify this proposition for every particular triangle, so what justification do we have for holding it to be universally true? Locke would claim that we demonstrate it of the general idea of the triangle, which represents all particular triangles.[62]

So the empiricist picture must be something like this. Through interaction with physical samples of particular oblique, equilateral, and scalene triangles, drawn, for example, with chalk, or formed from pieces of wire,or drawn in the sand, the mind abstracts away from the incidental properties and extracts only the properties which are common to all triangles. By joining these common properties together, the mind constructs an "image" or representation of a general triangle, "neither equilateral, equicrural, nor scalenon, but all and none of these at once".[63] Clearly then, any properties possessed by this general triangle are possessed by any particular triangle since the general triangle is just the `embodiment' of all properties common to all triangles. In other words, all universal truths about triangles are built into the structure of the general triangle, and are thus to be discovered by investigation of this mental image. It should be clear from this that the ability to construct a general triangle is the mental equivalent for geometry of Maddy's set-detector in that both amount simply to the possession of the concept of triangle and set, respectively. Thus, as we shall see, abstractionism with respect to geometry is subject to the same basic criticism as Maddy's account of set theoretic knowledge.

62 See ibid., p.336 (IV,vi,16), where Locke concludes his discussion of universal propositions with the claim that "general certainty is never to be found but in our ideas. Whenever we go to seek it elsewhere, in experiment or observations without us, our knowledge goes not beyond particulars. It is the contemplation of our own abstract ideas that alone is able to afford us general knowledge."

63 Ibid., p.339. (IV,vii,9).

Berkeley's attack on this abstract general triangle amounts simply to the claim that its construction is impossible. Just as Frege rejected the possibility of ascribing to units - mental constructions in the same sense as the alleged general triangle - the two contradictory properties of identity and distinguishability, Berkeley denies that we can conceive of a triangle with both **all** and **none** of a group of mutually incompatible properties:

> I find indeed I have a faculty of imagining, or representing to myself, the ideas of those particular things I have perceived, and of variously compounding and dividing them... I can consider the hand, the eye, the nose, each by itself abstracted or separated from the rest of the body. But then whatever hand or eye I imagine, it must have some particular shape and colour.
>
> ...I own myself able to abstract in one sense, as when I consider some particular parts or qualities separated from others, with which, though they are united in some object, yet it is possible they may **really exist** without them. But I deny that I can abstract from one another, or conceive separately, those qualities which it is impossible should exist so separated.[64]

We shall come back to this emphasis on real existence later, but for now I want to compare Berkeley's objections to this account of abstraction to the objections to the various accounts we have seen thus far. Because abstractionism was developed to tackle the same epistemological problem which intuition is supposed to solve, we should determine whether or not it is a more plausible construal of the analogy between mathematical and empirical knowledge. First, Berkeley's criticism is reminiscent of Frege's objection to numerical abstraction, where he simply shows that it is apparently impossible to perform the necessary "acts" of abstraction. Berkeley too challenges the Lockean empiricist to find someone who claims to have a faculty which would allow him to frame the requisite ideas. Secondly, an objection raised against Maddy's set-detector is also applicable here. We saw that the postulation of the presence of a

64 Berkeley [1710], p.407, (Introduction,10).

set-detector simply amounts to the claim that, by interacting with physical samples of sets, we develop the concept of set, with no non-ad hoc explanation of how this is supposed to happen. Furthermore, any explanation would seem to require the prior possession of the concept of set, in that the mind would somehow have to sift out the essential from the incidental properties of physical sets. The notion of similarity, used by Locke and implicitly by Maddy, is always a problem for any abstractionist account of concept formation. Consider, for example, Locke's attempt to explain how we acquire the concept of whiteness. He claims that we do this by comparing different objects - milk, snow, chalk - to see in what way they are similar. But we cannot simply compare two items to see if they are similar; we have to see if they are similar **with respect to something**, some concept, in this case, whiteness. Thus it seems as if we could never discover or form the concept *whiteness* in this way, because we need to fix *whiteness* before we can do any stripping away of other properties. Thus we already need to possess the concept of whiteness.

What we are left with then is the claim that this Lockean faculty of abstraction, this "wash-tub of the mind", is just described ad hoc as that faculty which provides us with the general idea of a triangle from our exposure to particular instantiations. But again, the acquisition of the general idea of a triangle seems to presuppose the possession of the concept of triangle, since the mind must be able to recognize which properties of each particular triangle are merely incidental and which are essential, i.e., belong to the concept of triangle! Thus it seems that Locke cannot retain both his doctrine of the general triangle, and his `tabula rasa' conception of the mind. We also saw this in Maddy's case: the claim that we develop set-detectors cannot be made coherent without presupposing our possession of the concept of set. But in possessing the concept, we already know all the general truths about triangles, so the intermediate step, the idea of a general triangle, is superfluous, and the acquisition of concepts has not been explained. Again, we have been led along "the road of magic rather than of science". The upshot of this is that we find ourselves back at the starting point. The challenge was to fill the epistemological gap between knower and known in a way which avoids the mystery of naive platonism. But the attempts we have looked at hitherto have involved appeals to another kind of mysterious platonism, platonism of neurophysiology or of the mind. The naive platonist's faculty of mathematical intuition is replaced by equally unexplanatory - and therefore equally mystical - faculties of abstraction or set-detection. We have yet to make any explanatory progress.

Berkeley's claim that Locke's abstract general ideas are impossible results from his more fundamental claim that all concepts must have their basis in perception, i.e. in sensible intuition. All we have are particular instantiations of triangles, each of which must be **either** scalene **or** equilateral **or** oblique. For him, if it can't be represented spatially, then it can't be conceived either: in other words, imagination cannot outstrip perception, hence the emphasis on real existence. Fortunately for the geometer, in this case it makes no difference. We do not need the notion of an abstract general triangle to account for the universality of, for example, the theorem which states that the three angles of a triangle are equal to two right angles. For Berkeley, all we have are particular physically instantiated triangles (or ideas of these); but in demonstrating of a particular isosceles triangle whose sides are of determinate length that its three angles are equal to two right angles, the geometer makes no appeal to the right angle, the equality of the two sides, or the determinate length of the sides of the particular triangle. Thus the right angle might have been oblique, and the sides unequal, and the demonstration would still hold good. By considering the figure merely as triangular, without attending to the particular qualities of the angles or relations of the sides, one may show a proposition to be true of all particular triangles, without the intervention of a general triangle.

However, there are other areas of mathematics where this sort of appeal to some faculty of abstraction leads the mathematician to make what Berkeley considers to be erroneous claims. For example, in _The analyst_, Berkeley suggests that all arguments for the infinite divisibility of finite extension require taking general abstract ideas as the object of geometry.[65] Thus by showing that we do not have such ideas (in this case the abstract idea of extension "which is neither line, surface, nor solid, nor has any figure or magnitude"[66]), he has also disproved the infinite divisibility of finite extension. The fallacy arises from taking the particular lines used in diagrams as standing for innumerable others of different sizes.[67] Suppose we take a particular line, one inch long, drawn on paper. Clearly this line itself is not divisible into a thousand parts, because we can neither perform, nor conceive of, so many divisions. But if, as the geometer does in demonstration, we take this particular line as a sign for all finite lines in general - i.e. if we abstract from the magnitude of this line -then it may stand for a line of any arbitrary finite length. And we may certainly conceive of a line greater than this one, the diameter of the earth for example, as being divided into ten thousand parts or more. But then, the properties of the lines **signified** by this sign are transferred to the sign itself, and therein lies the fallacy:

> Because there is no number of parts so great but it is possible there may be a line containing more, the inch line is said to contain parts more than any assignable number; which is true, not of the inch taken absolutely, but only for the things signified by it. But men, not retaining that distinction in their thoughts, slide into a belief that the small particular line described on paper contains in itself parts innumerable.[68]

In the case of the general triangle, the complaint was that the geometer took as his object an abstract triangle whose existence in the physical world is impossible. But according to Berkeley, part of what makes geometry "an excellent Logic" is "the distinct contemplation and comparison of figures" which allows us to derive their properties from a well-connected chain of consequences "the objects being still kept **in view**, and the attention ever fixed upon

65 Berkeley [1734], p.97, (Qu.20).
66 ibid., p.406, (Intro.,7).
67 This and the following example are from Berkeley [1710], pp.438-9, (sections 125-128).
68 ibid.

them".[69] But if it is impossible that the objects exist, then they certainly cannot be kept "in view". Berkeley is not speaking metaphorically here. He does insist that the mathematician operate only with notions which it is in our power to perceive or, what almost amounts to the same thing for him, conceive.

The belief in the infinite divisibility of finite extension which results from the illegitimate extension of geometry beyond what is grounded in our ability to perceive is therefore false. It is also on precisely these grounds that Berkeley bases his attack on the infinitesimal calculus, i.e. on Newton's fluxions and Leibniz's differentials. While the underlying concepts of geometry are grounded in our ability to perceive - we can, for example, **draw** geometrical objects - there is no such foundation for the underlying concepts of early analysis. It requires the use of infinitely small quantities, "that is, infinitely less than any sensible or imaginable quantity, or than any the least finite magnitude", or of "a thing which hath no magnitude".[70] For Berkeley, such "shadowy entities" are

> so difficult to imagine or conceive distinctly, that (to say the least) they cannot be admitted as principles or objects of clear and accurate science ... this obscurity and incomprehensibility of your metaphysics had been alone sufficient to allay your pretensions to evidence ....[71]

Neither through our senses nor our imagination (which is derived from our senses) can we frame a clear idea of a fluxion or a differential, therefore these cannot be objects of study. We can only study what we can perceive.

Put in this way, it is clear that Berkeley's position is consistent with, and probably the logical consequence of, the motivation behind the abstractionist position. It could be looked at as what would result from the unqualified acceptance of the perception-based interpretation of Gödel's analogy. The abstractionist wants to do away with both a separate realm of mathematical entities and any

69 Berkeley [1734], p.65, (section 2).
70 Berkeley [1734], pp.68-9.
71 ibid., p.95.

special mathematical faculty which is supposed to provide access to it. So he assimilates mathematical and empirical knowledge, by claiming that the subject of both is the physical world, and that ordinary perception gives us access to it. But what Berkeley's work seems to show is that there is then no warrant for making any claims which reach beyond perceptual verification, for "believing further than one can see".[72] In fact we have seen that any extrapolation beyond the limits of perception requires an appeal to some faculty like abstraction to ground it; supposedly we can believe further than we can see because the faculty of abstraction provides us with the objects of study. Unfortunately, this faculty is no less mysterious or ad hoc than the previous ones we have looked at. To believe in it, just as to believe in the set-detector, mathematical intuition, or the existence of God, is equivalent to a leap of faith across the epistemological gap. This is the essential problem faced by any mathematical epistemologist with a physicalist bent who also wants to adhere to the claim that mathematics is apodeictic. If mathematics is assimilated to empirical science in this perceptual way, then what can account for the difference in the nature of mathematical knowledge? Hence the appeal to mysterious faculties. It is this need to account for the apodeictic nature of mathematics which gave rise to the platonist picture in the first place.

I should mention here that this criticism of abstraction is based on a very restricted view of empirical science. Using Berkeley's criteria, much of modern physics, for example, would suffer the same fate from Berkeley as analysis did. But this point shows that, if we want to support some kind of analogy between mathematics and empirical science, then Berkeley's view of science will not do. We want an account that will admit, in both science and

[72] Berkeley's intention is to show that our scientific and mathematical knowledge is no more firmly grounded than the mysteries in faith - and that in fact the objects of science are just such obscure mysteries. He is attacking those scientists and mathematicians, the "rigourous exactors of evidence in religion" who "pretend to believe no further than they can see", and on that basis reject religious mysteries and points of faith:

> ...he who can digest a second or third fluxion, a second or third difference, need not methinks, be squeamish about any point in divinity. [The analyst, p.68]

mathematics, what Berkeley rejects as obscure mysteries. I hope to suggest the lines along which such an account may be developed in part 2, where Berkeley will come up once again. Nevertheless, the argument from similarity which we looked at earlier is much more general and is therefore sufficient for our purposes, i.e. to expose the inadequacy of the abstractionist position.

4. Platonism and the physical world

We have now looked at three attempts to interpret the analogy suggested by Gödel in the passage quoted at the beginning of this paper. In a paper examining the relationship between truth and proof in mathematics,[73] Tait suggests that the search for the analogy has been misguided. It has heretofore been based on a misconception of discourse about sensible objects as being privileged,[74] where the challenge to mathematical epistemology has been to match the standards of this supposedly privileged discourse. Tait characterizes the general complaint against naive platonism in terms of the "truth/proof problem", and suggests that it is not a real problem but rather a consequence of a confusion which has an analogue in the case of the sensible world. Tait suggests an alternative way of interpreting Gödel's analogy, an alternative which I hope to develop, in one form or another, in the rest of this paper.

The truth/proof problem results from two apparently incompatible beliefs the platonist might want to hold. The first is the characteristic platonist belief about the truth of mathematical statements which we examined in section 1. That is, for example, the belief that an arithmetical statement is **about** or refers to a certain structure - the natural numbers - and if true, is true in virtue of a certain fact about this structure which may obtain whether we can know it or not. On the other hand, we learn that the ultimate warrant for a mathematical statement is a proof of it: thus we are justified in asserting the truth of a particular mathematical statement only when we have a proof of it.

---

73 Tait [1986]

74 Berkeley's criticism is the epitome of this.

This then leaves us with two criteria for the truth of a mathematical statement: it is true if it holds in the system of numbers, and it is true if we can prove it. "But", Tait asks, "what has what we have learned or agreed to count as a proof got to do with what obtains in the system of numbers?"[75] It looks as if we have another mystical platonist appeal to pre-established harmony, unless the platonist can explain the relationship between truth and proof; the challenge is to fill this gap by providing some evidence that the canons of proof apply to the structure. The platonist fails to meet this challenge because he fails to explain the analogy between mathematics and empirical science. To illustrate this, let us reformulate the example on p.2 in terms of the truth/proof problem. This problem is taken to be that the arithmetical statement

3 is greater than 1

cannot really be about the system of numbers because our warrant for it is a proof, and that has nothing to do with the system of numbers. On the other hand, the statement

The cat is on the mat

really is about the sensible world because we verify it by looking and **seeing** a cat and a mat. Thus the whole platonist project rests on finding an analogue to observation in the case of mathematics: i.e. some kind of mathematical intuition. But there was no analogue to be found, therefore, the argument goes, platonism is incoherent.

This argument is, of course, based on the implicit assumption that the experience of seeing a cat on the mat warrants the assertion of the second sentence more than a proof that 3 is greater than 1 warrants the first. In other words, the (supposedly) direct relationship between experience and assertions about the physical world gives to discourse about this world a privileged status over discourse about the world of abstract objects to which we have no

---

75 ibid., p.341.

immediate access. The truth/proof problem does not apply in the case of the physical world because there is no gap between what holds in the physical world, the world of cats and mats, and what we experience. The two are related by what Benacerraf considers one of "the better understood means of human cognition", sense perception. We have direct experience of cats and mats, where, as we saw in section 1, we do not bear any such immediate relation to threes and ones.

It should be clear that so far the truth/proof problem is just a reworking of the basic problem of the gap between the knower and the known. But Tait raises a traditional skeptical question which suggests that the gap-filler in the case of empirical knowledge, sense perception, does not fulfill its role: what have our experiences got to do with physical objects and their relationships at all? For the sentence "There is a cat on the mat" is about physical objects, but I have **direct** access only to my sensations. Now the challenge is thrown back to the platonist's critic: to explain the link between the knower and the known, in this case, physical objects, he must first explain the relationship between our sensations and physical objects. Why should these two things have anything to do with each other? Why should my cat- and mat-like experiences count as verification for a statement about the real world of cats and mats?

Thus there is nothing special to mathematics about the truth/proof problem for we can describe a truth/verification problem which is its analogue in the case of the physical world. Moreover, the problems with reference to mathematical objects can also be shown to apply to physical objects as well. How can we be referring to cats and mats if we have no causal interaction with actual cats and actual mats? But of course the correct response here would not be to become a skeptic about the physical world as well, but rather to attempt to dissolve the twin problems by exposing the confusion which led to them. Benacerraf et al. start with a picture of empirical knowledge and, showing that it cannot account for our mathematical knowledge as construed by the platonist, take this as a proof of the incoherence of platonism. Tait's point is that this picture of empirical knowledge cannot even account for our

empirical knowledge, and that it is therefore this picture which must be replaced:

> Every reasonably schooled child understands the language of arithmetic. It is the schizophrenic parent of the child who, motivated by an inappropriate picture of meaning and knowledge, develops 'ontological qualms'. The picture is read into platonism and then, because it is inappropriate, Platonism, i.e., our ordinary conception of mathematics, is rejected. *The fact that the picture is generally inappropriate is simply ignored.*[76]

In what follows, I want to develop Tait's suggestion that this might be the analogy Gödel had in mind when he wrote that "the question of the existence of the objects of mathematical intuition...is the exact replica of the question of the objective existence of the outer world".[77] This is also consistent with Gödel's claim, quoted earlier, that the assumption of the objects of mathematics is "quite as legitimate as the *assumption* of physical bodies... in both cases it is impossible to interpret the propositions one wants to assert about these entities as propositions about the "data", i.e., in the latter case the actually occurring sense perceptions." In other words, sense perceptions alone are insufficient to account for our knowledge of the physical world. There must be something over and above our sensory faculties and the objects themselves which gives us such knowledge. This was recognized at least as far back as Plato's "Theaetetus" where Socrates seeks to show his opponent that perception of the objects of sense cannot amount to our knowledge of the physical world. The idea is essentially that through the various sense organs, we are directly acquainted with the objects fitted to each of those organs. For example, through touch, we perceive the objects of touch and through sight we perceive the objects of sight. But then, in reaching out and touching the apple I see on my desk, I am not actually touching the same object I am seeing, for there is no faculty through which I perceive both visual and tactile properties. The question then is how do we come by a

---

76 Tait [1986], p.353.

77 Gödel [1964], p.484.

"common perception" of the objects of sight and touch? In Kantian terms, and for future reference, we want to know how these disparate perceptions are combined to give us the experience of **one object**, an apple. Plato's answer, already presaging Kant, is that "knowledge does not consist in impressions of sense, but in reasoning about them" [186]. Thus we have an early nod towards the mind-dependence of experience, and it is interesting in this context that the nod comes from Plato. One might want to say that it is in this sense that Gödel was a platonist.

It is in her recognition of this aspect of Gödel's thought where I think Maddy is absolutely correct. She wants to uncover the "abstract element" which constitutes the difference between our simply undergoing sensory stimulation, for example, hearing purring sounds and seeing grey patches, and our perceiving a cat. Her solution is to postulate an object-detector which combines, and thereby transforms, these disparate sensations into the perception of one object. The important insight, hinted at by Maddy, is the active participation of something other than the sensory faculties - for Maddy, the brain; for Kant and Gödel, the mind - even in ordinary experience. In this sense, the key to Gödel's analogy is to be found in the following passage:

> It should be noted that mathematical intuition need not be conceived of as a faculty giving an immediate knowledge of the objects concerned. Rather it seems that, as in the case of physical experience, we form our ideas also of those objects on the basis of something else which is immediately given. Only this something else here is not, or not primarily, the sensations. That something besides the sensations actually is immediately given follows (independently of mathematics) from the fact that even our ideas referring to physical objects contain constituents qualitatively different from sensations or mere combinations of sensations, e.g., the idea of object itself, whereas, on the other hand, by our thinking we cannot create any qualitatively new elements, but only reproduce and combine those that are given. Evidently the "given" underlying mathematics is closely related to the abstract elements contained in our empirical ideas.[78]

78 ibid.

It is from this standpoint that the platonist characterized in Section 1 and attacked by Benacerraf et al. deserves to be called the **naive** platonist. For we have arguments going back as far as Plato which show that this picture of the object existing, in all its 'object-hood', independently of the passive perceiver is much too simplistic. As Hallett points out in his forthcoming book, these critics of the so-called Gödelian platonist have won their case only by loading onto our notion of **both** physical and mathematical object "more than they can bear, more than we should want them to bear, and much more than Gödel intended them to bear".[79] The criticisms of the various interpretations of Gödel's analogy all assume that he wants to assimilate the epistemology of mathematics to that of the physical world, where the latter is based on some notion of direct contact - given by perception - with the objects of knowledge. But we have just seen that perception does not give us direct contact with the objects as we know them. Thus even on a realist-empiricist view, there must be some conceptual apparatus mediating between these sense perceptions and what we experience. And this is precisely what Gödel seems to be saying when he claims that even our ideas of physical objects contain constituents qualitatively different from sensations. It should be clear from this that Gödel was not a metaphysical realist about the physical world, as his critics suppose, insofar as he does not believe that we encounter physical objects directly as they are in themselves. In other words, he would not have claimed that we have a faculty giving us immediate knowledge of or direct access to the objects of physics. Therefore he should not be taken to be claiming that we have an analogous faculty which gives us immediate knowledge of or direct access to the objects of mathematics. Rather, the analogy should be developed in terms of the mind-dependence of experience even of the physical world.[80] Taken this way, our experience of the world is determined by the conceptual apparatus according to which our sensations are organized; our minds contribute or impose the notion

---

79 Hallett [forthcoming], p.118.

80 see ibid., pp.118-122.

of an enduring physical object. Here we appeal to Gödel's claim that the assumption of mathematical objects is just as legitimate as the assumption of physical objects: the general notion of mathematical object plays the same role in the way we construe mathematics as the notion of physical object does in the way we construe sensation. Thus, "our conceptual machinery is such that we inevitably interpret our mathematics as being about a realm of persisting objects".[81] Furthermore, unlike Maddy, who posited the object-detector as playing a role in our experience of the physical world, and the set-detector which was to explain our mathematical experience, Gödel suggests that similar concepts underlie both our mathematics and our experience of the physical world. As I mentioned earlier, such a uniform account dispels the ad hoc nature of a special mathematical faculty. As Parsons puts it,

> If a positive account of mathematical intuition is to get anywhere, it has to make clear, as its advocates [i.e. Kant and Gödel] intended, that mathematical intuition is not an isolated epistemological concept, to be applied only to pure mathematics, but must be so closely related to the concepts by which we describe perception and our knowledge of the physical world that the "faculty" involved will be seen to be at work when one is not consciously doing mathematics.[82]

Moreover, this provides a solution to the question of the applicability of mathematics to the physical world: the concepts underlying that mathematics are also concepts according to which we interpret the physical world.

Thus the basic idea common to Plato, Gödel and, with some qualifications, Maddy, is this claim that the mind must act on what is received by the senses in order to transform the manifold of sensations into the perception of an object. This idea, the synthesis of the manifold of sensations, was given its most extensive formulation by Kant, who bases his theory of knowledge upon it, as is clear from the following passage:

---

81 ibid., p.121.

82 Parsons [1980], p.155.

> **It is to synthesis, therefore, that we must first direct our attention if we would determine the first origin of our knowledge.[83]**

Let us thus direct our attention now.

83 CPR, B130-1.

PART TWO: Kant, sensible intuition, and the philosophy of mathematics

1. The problem of knowledge

Kant's notion of synthesis arose in response to the fundamental epistemological problem of the Enlightenment. Descartes had made the distinction between the world as it is and the world as it is represented to the mind. He saw as his "principal task" to consider,

> in respect to those ideas which appear to me to proceed from certain objects that are outside me, what are the reasons which cause me to think them similar to these objects.[84]

The problem, then, as expressed by Kant in a letter to Marcus Herz, was to discover "what is the ground of the relation of that in us which we call `representation' to the object?"[85] To put it another way, how can our scientific knowledge, for example of mathematical physics, derived by reason, give us an accurate account of the way the physical world outside us is? This is of course the same question that Tait posed in linguistic terms: what is the connection between the statements we make about the physical world and the objects in that world? Before Kant, there were two different attempts to solve this problem: one by the rationalists and one by the empiricists.

In the rationalist tradition, the bond between thought and its object was established by attributing them both to the same origin. Knowledge can be explained only in terms of the ideas, or primitive notions, that the mind finds within itself. The structures of thought and reality coincide because both our concepts and their objects arise out of a single divine source; the applicability of mathematical concepts, like extension, to objects in the world is guaranteed by a veracious God. Thus for Descartes, "the certainty and truth of all knowledge depends alone on the knowledge of the true God."[86]

---

[84] Descartes [1641], p.83.
[85] Zweig [1967], p.71.
[86] Descartes [1641], p.96.

However, this introduces an element of inference into the move from representation to reality, thereby undermining the absolute certainty sought for natural science. Furthermore, this appeal to a metaphysical guarantor could not survive the secularizing tendencies of the Enlightenment, and the quest for a direct, unmediated connection between knowledge and reality on the basis of experience alone. According to Kant (in the same letter to Herz), "the deus ex machina is the greatest absurdity one could hit upon in the determination of the origin and validity of our knowledge."[87]

With the rejection of any transcendent guarantee of the objective validity of the troublesome connection, it appeared to the empiricists that the only route left was to establish a direct influence between the mind and the material world. Against the Descartes-Leibniz claim that our ideas are placed in our minds by God, the empiricists held, like Aristotle, that those ideas or concepts are abstracted from sensation: nothing is in the intellect which was not first in the senses. Thus we have the claim that

> Nothing is proved in metaphysics, and we know nothing either concerning our intellectual faculties or concerning the origin and progress of our knowledge, if the old principle: nihil est in intellectu, etc., is not evidence of a first axiom.[88]

The correctness of our ideas is assured by their direct connection, through perception, with the physical world. Unfortunately, Hume showed that there is nothing we could abstract from sensation, nothing in the object itself, which would guarantee the necessity of scientific knowledge. There can be no rational justification for the inference from our perception of contiguity and succession to the claim that things **actually** occur in some determinate order, the principles of which can be known: it is an assumption we are

---

[87] Zweig [1967], p.73.

[88] Diderot, ***Apologie de l'Abbé de Prades***, sect. XII; cited in Cassirer [1951], p.99. Even Hume, the skeptic, accepts this as a `first principle':

> ...no discovery could have been made more happily for deciding all controversies concerning ideas than this, that impressions always take the precedence of them, and that every idea with which the imagination is furnished first makes its appearance in a correspondent impression. (Hume [1740], p.185]

inclined to believe, but having no objective necessity. Neither reason nor experience alone can demonstrate that one event **must** be followed by another. As Hume says,

> It is not, therefore, reason which is the guide of life, but custom. That alone **determines the mind** in all instances to suppose the future conformable to the past.[89]

It is the common ground between the rationalist and empiricist attempts to establish the connection between thought and reality that is the failing of both. The rationalist and empiricist epistemologies both have their basis in subjectivist assumptions. Both reduce all human knowledge to a single source within the subject: for Descartes and Leibniz, that source is the faculty of reason; for Locke, it is sensation. The obvious subjectivist implications of these positions were avoided only by an appeal to unexplainable belief, thus opening the door for Hume's skepticism.

The parallels between this debate about knowledge in general and the debate about mathematical knowledge as presented in part 1 should, I hope, be clear. We start from the naive 'metaphysical realist' position that there is a world of independent physical objects of which we somehow have knowledge. The problem is to explain the possibility of such knowledge. How is it that our mental images are accurate representations of the external world? What bridges the gap between thought and its object? This is of course the physical counterpart to the truth/proof problem where the challenge is to bridge the gap between abstract mathematical objects and the knower. Presumably then, by examining the proposed solutions to this earlier problem, we may shed some light on its mathematical counterpart. Interestingly, the pattern of responses bears a striking similarity to the attempted solutions to the truth/proof problem. The rationalist-Cartesian solution, the appeal to a transcendent guarantor, has its modern reformulation in the naive platonist's postulation of a mysterious faculty of intuition, and is equally unexplanatory. The empiricist account requires the

---

89 Hume [1740], p.189.

now-familiar notion of abstraction, and moreover, is shown by Hume to be incapable of accounting for the necessity of the laws of nature. Once again, the only available explanations of the relationship between the knower and what he knows fail to **explain** anything: they are merely ad hoc postulations of mysterious faculties. However, Hume's allusion in the preceding passage to the role of the mind in experience points us in the direction in which we want to go. In fact, it is Hume who first woke Kant from his "dogmatic slumber" and gave his "investigations in the field of speculative philosophy a quite new direction"[90]. Let us turn now to these investigations.

2. The redefinition of knowledge

For Kant, then, the empiricist (and Humean) claim that knowledge contains nothing other than that which is given in sense experience effectively denies that we can have knowledge, because judgments based only on sense experience lack the necessary character required of true knowledge. On the other hand, the rationalist claim that the only source of knowledge is reason led to the irresolvable contradictions, or antinomies, of the speculative metaphysicians. Furthermore, if speculative metaphysics were indeed possible, then the laws which reason has discovered to hold in the world of experience would be applicable also to the world beyond experience. A fundamental law of natural science in Kant's day was that of universal causal determinism. If all knowledge derived from reason alone is valid, then this law is universally applicable, and we must give up our claims to freedom required for moral responsibility. If, on the other hand, we can have no knowledge beyond what is given in sense experience, then we must give up our claims to knowledge. Kant rejects both these conclusions; even if we grant the divine guarantor or the faculty of abstraction, neither one will give us an adequate account of knowledge. If knowledge is not possible on these assumptions, then we must redefine what knowledge is for us. Kant diagnoses two fundamental mistakes made by his predecessors. The first was in assuming that knowledge could be achieved only by

90 Kant [1783], p.8.

getting to the essence of things as they are in themselves. The redefinition of knowledge rests on the denial of the common presupposition of the rationalists and the empiricists that there is a ready-made reality of things to which the contents of consciousness must be directly related: given what this reality is, we must understand how we can come to know it. But as Kant points out:

> ...if intuition must conform to the constitution of the objects, I do not see how we could know anything of the latter a priori; but if the object (as object of the senses) must conform to the constitution of our faculty of intuition, I have no difficulty in conceiving such a possibility.[91]

Thus Kant instead asks what the nature of the object must be in order for us to know it.

> Hitherto it has been assumed that all our knowledge must conform to objects. But all attempts to extend our knowledge of objects by establishing something in regard to them a priori, by means of concepts, have, on this assumption, ended in failure. We must therefore make trial whether we may have more success in the tasks of metaphysics, if we suppose that objects must conform to our knowledge.[92]

The claim that objects must conform to our cognition is made possible by Kant's redefinition of knowledge: because objects can be present to us only as we represent them, we can only have universal, necessary knowledge of our **representations** of objects. Kant starts from the premise that we do in fact have certain knowledge, and he takes Hume's argument to show that such knowledge cannot be of things as they are in themselves, but rather it must be of those things as they appear to us.[93] On this account then, Kant's predecessors were simply asking the wrong question: there is no **ascertainable** necessary connection between our thought or knowledge and things as they are in themselves, between the 'real world' and the phenomenal world. The only world which we

---

91 CPR, Bxvii.
92 CPR, Bxvi.
93 CPR, [Bxx].

experience, of which we can have knowledge, and of which we can meaningfully speak is the phenomenal world.

In other words, Kant recognizes that Hume's argument should not be taken to show that certain knowledge is not possible, but rather that this picture of knowledge must be wrong. This is precisely the point Tait makes against the platonist's critics: the inability of their picture of knowledge to account for our knowledge of mathematical 'objects' should be taken, not to show that we cannot have such knowledge, but rather that the critics' picture of knowledge is simply inadequate. Compare Tait's strategy in the following passage:

> The fact is that we do know how to apply mathematics and we do not causally interact with mathematical objects. *Why doesn't this fact simply refute a theory of knowing how that implies otherwise?* [94]

The final step in Kant's modification of the rationalist and the empiricist traditions, is to do away with their respective attempts to reduce knowledge to a single source, their second mistake. Because knowledge is to be necessary, it cannot be derived from sense-experience alone; because it is to be universally applicable to experience, its source cannot be found in thought alone. Kant makes this clear in a letter to J. S. Beck:

> For knowledge, two sorts of representations are required: (1) intuition, by means of which an object is given, and (2) conception, by means of which an object is thought.[95]

And in the **Critique**:

> Without sensibility no object would be given to us, without understanding no object would be thought. Thoughts without content are empty, intuitions without concepts are blind.[96]

---

[94] Tait [1986], p.351.
[95] Zweig [1967], p.184.
[96] CPR, A51,B75.

Because the only intuitions human beings can have are sensible, knowledge is inextricably bound up with sensation. It is in this way that Kant limits the use of reason to the realm of experience:

> If knowledge is to have objective reality, that is, to relate to an object, and is to acquire meaning and significance in respect to it, the object must be capable of being in some manner given. Otherwise the concepts are empty; through them we have indeed thought, but in this thinking we have really *known nothing*; we have merely played with representations. That an object be given...means simply that the representation through which the object is thought relates to actual or possible experience.[97]

Or, more succinctly,

> we therefore demand that a bare concept be made *sensible*, that is, that an object corresponding to it be presented in intuition. Otherwise the concept would, as we say, be without sense, that is, without meaning.[98]

These passages express what Strawson calls the "principle of significance", by means of which Kant has fulfilled his two goals: he has guaranteed the validity of our knowledge against the skeptical attack by claiming that we have necessary, universal knowledge of appearances; he has also protected his moral and religious beliefs, and diffused the antinomies, by limiting our ability to know to the realm of the sensible.

Thus Kant's epistemology was developed to counter the inadequate **general** picture of knowledge which led either to skepticism or dogmatism. The task confronting us now is to see if this new picture of knowledge can also account for our mathematical knowledge. There are two aspects to Kant's treatment of mathematics within this framework. It has a heuristic function as a paradigm of the kind of knowledge which Kant is attempting to protect against skeptical attack. However, these mathematical examples can also be combined to construct a philosophy of mathematics consistent with the strict requirements for knowledge set forth in the **Critique**. The following is an attempt to show how

---

97 CPR, A155-6,B194-5.

98 CPR, A240, B299.

Kant conceived mathematical knowledge, specifically, as grounded on sensible intuition. Two questions must be answered in connection with this: (1) **why** does Kant claim that "pure mathematics, and especially pure geometry, can have objective reality only on condition that they refer merely to objects of sense" [**Prol**.286-7] and (2) **how** can such a conception be made to account for our mathematical knowledge?

The answer to the first question - that is, **why** does Kant insist that mathematical knowledge depends on sensible intuition - should be clear by now. It is that, for Kant, **all** non-analytic knowledge is so dependent insofar as a concept is only meaningful if it relates to the empirical or experiential conditions of its application.[99] The next problem, then, is to explain **how** Kant applies the `principle of significance' to mathematics. Given the view that geometry is a theory about real space and given the role that `ecthesis' plays in Euclidean proofs, it is not so difficult to see how Kant might have tried to account for the role of sensibility in geometry. The question becomes more difficult, and more interesting, with respect to arithmetic; I shall also attempt to locate the place of algebra within this framework. It will be seen that Kant, in his zeal to ground human knowledge while at the same time limiting the pretensions of speculative metaphysicians, also severely restricts the science of mathematics.

3. Mathematical knowledge

The validity of mathematical knowledge had never been in doubt. According to Kant, however, this was due to "an erroneous view". Mathematics had been protected from the skeptical attack by the claim that it was a priori, and therefore analytic; mathematical truths were truths of reason. Leibniz, for example, held that all mathematical truths were demonstrable from definitions and the law of contradiction, which is a self-evident innate truth. In the **New Essays on Human Understanding**, Leibniz gives a proof of the proposition 'two and two are four':

[99] Strawson [1966], p.16.

Definitions. (1) Two is one and one.
(2) Three is two and one.
(3) Four is three and one.

Axiom. If equals be substituted for equals, the equality remains.

Demonstration.	2 and 2 is 2 and 1 and 1	(def.1)	2 + 2
	2 and 1 and 1 is 3 and 1	(def.2)	2 + 1 + 1
	3 and 1 is 4	(def.3)	3 + 1
			4

Therefore (by the Axiom)
2 and 2 is 4. Which is what was to be demonstrated.[100]

Kant deals with this general claim in the following passage:

> *All mathematical judgments, without exception, are synthetic.* This fact, though incontestably certain and in its consequences very important, has hitherto escaped the notice of those who are engaged in the analysis of human reason, and is indeed, directly opposed to all their conjectures. For as it was found that all mathematical inferences proceed in accordance with the principle of contradiction (which the nature of all apodeictic certainty requires), it was supposed that the fundamental propositions of the science can themselves be known to be true through that principle. This is an erroneous view. [B15]

Kant goes on to say that the concept of 12 cannot be already thought in thinking the union of 7 and 5. Similarly, with respect to the proposition that the straight line between two points is the shortest, the concept of straight contains nothing of quantity, only of quality, and the concept of the shortest cannot be derived, through any process of analysis, from the concept of the straight line.[101] Thus mathematical judgments, as universal and necessary, are a priori; but because they are not reducible to the principle of non-contradiction (the predicate concept is not contained in the subject-

---

[100]Leibniz [1685], Book IV,vii,10; pp.413-4.

[101] On the analyticity of geometry, compare Leibniz [1685], Book IV,i,9; pp.360-1. It is universal propositions, i.e. definitions and axioms and theorems which have already been demonstrated, that make up the reasoning, and they would sustain it even if there were no diagram.

concept), they must also be synthetic. Mathematics, therefore, is a paradigm of synthetic, a priori knowledge, the possibility of which Kant is trying to defend against Hume. The task now is to explain the syntheticity of mathematics. The subsequent development of consistent (i.e. non-contradictory) non-Euclidean geometries seems to support Kant's claim for the non-analyticity of geometry;[102] furthermore, for the reasons mentioned earlier, geometry does seem at first glance to be more amenable to such treatment.

(i) Geometry

In the "Discipline of Pure Reason in its Dogmatic Employment", Kant distinguishes between philosophical knowledge, which is gained by reason from concepts, and mathematical knowledge, which is gained by reason from the **construction** of concepts. To construct a concept is to "exhibit a priori the intuition which corresponds to the concept" [A714,B742]. An intuition is "such a representation as would immediately depend upon the presence of the object" [**Prol.**8]; it is particular, as opposed to concepts, which are general [A320,B376]. Thus the syntheticity of mathematics results from the use of constructions in intuition, that is, of particulars immediately present in sensible intuition.

As an example of the construction of a geometrical figure, Kant provides us with Euclid's proof of the angle-sum theorem [A716,B744]. The geometer begins by instantiating the concept of a triangle, and then performing constructions on that instantiation (prolonging one side, dividing the external angle by drawing a line, etc.). From this, he can deduce that the sum of the three internal angles of this particular triangle is 180 degrees. Furthermore, this construction, as intuited, must

> be a single object, and yet none the less, as the construction of a concept (a universal representation), it must in its representation express universal validity for all possible intuitions which fall under the same concept. [A713,B742]

[102]That is, if we assume that geometry is 'about the world'.

This gives rise to the rather obvious objection that the Aristotelian logic which Kant regarded as "a closed and completed body of doctrine" [Bviii] did not allow for the derivation of universal propositions from premises involving particulars. However, the subsequent expansion of traditional logic to predicate logic permits such a move, leading such commentators as Russell, Beth and Hintikka to claim that, if Kant had had predicate logic at his disposal, he would not have needed to introduce constructions into his account of mathematical reasoning. According to Russell,

> Kant, having observed that the geometers of his day could not prove their theorems by unaided argument, invented a theory of mathematical reasoning according to which the inference is never strictly logical, but always requires the support of what is called 'intuition'.[103]

For Russell, then, what Kant believed were synthetic truths are, when properly viewed from the standpoint of modern logic, actually analytic. Beth and Hintikka however, retain the syntheticity of Kant's geometry by having it depend on intuition rather than on construction, or more accurately, by reducing the notion of construction to the introduction of intuitions. A geometrical proof is synthetic if it can be converted into a quantificational argument in which new free individual constants are introduced.

For Hintikka, such an argument would take the form of existential instantiation. For example, by instantiating the existentially quantified proposition Ex(Ax), we get A(a/x) where a is a free individual constant and A(a/x) is the proposition which results from replacing all occurrences of x in A with a. The condition on the use of this rule is that a must be a new symbol in the proof. This introduction of new individuals is, for Hintikka, identical with Kant's introduction of intuitions into the proof.

Hintikka followed a similar approach which had been taken by Beth, where the emphasis was placed on the rule of universal generalization. The procedure follows that of the Euclidean proof outlined by Kant at B744: say, for example, we want to prove (x)(Ax-

[103]Russell [1919], p.145.

->Bx).[104] We assume a particular a such that Aa (e.g. a is a triangle). From Aa we deduce Ba (the sum of the interior angles of a is 180 degrees); thus we have Aa-->Ba independently of the hypothesis. Because, in both these types of proofs, a refers to an **arbitrary** individual, and none of its specific characteristics have been used in the proof, we may conclude that (x)(Ax-->Bx) or 'if x is **any** triangle, then the sum of the interior angles of x is 180 degrees'. Like Hintikka, Beth identifies the arbitrary individual with a Kantian intuition, and similarly claims that any proof in which a new individual is introduced is synthetic. The Beth-Hintikka view captures something of the point of the use of intuition for Kant, but clearly not all of it. This latter claim would appear to be consistent with Kant's claim at B15 that we must "call in the aid of intuition" in the proof of the proposition that the straight line between two points is the shortest, and with the following passage from the "Enquiry concerning the clarity of the principles of natural theology and ethics":

> Mathematics reaches all its definitions synthetically, philosophy analytically... The concept which I am explaining is not given before the definition. A cone may signify elsewhere whatever it will; in mathematics it originates from the arbitrary representation of a right-angled triangle rotated on one of its sides. [p.6]

However, the kind of intuition being invoked here conflicts with Kant's argument in the Transcendental Aesthetic that human intuitions are necessarily sensible [A43,B60]. (I will develop this objection later.) For Beth and Hintikka, as for Russell, the physical nature of the constructed figures is inessential; their use simply reflects the inadequacy of traditional logic.

Another attempt to explain Kant's insistence on the use of constructions in geometry attributes it to the fact that Kant did not have the existence axioms introduced by Hilbert in the late nineteenth century. This approach focuses on the necessarily spatio-

---

[104]See Parsons [1969], p.124.

temporal (i.e. sensible) nature of construction as emphasized by Kant in the following passage:

> I cannot represent to myself a line, however small, without drawing it in thought, that is, generating from a point all its parts one after another. Only in this way can the intuition be obtained... The mathematics of space (geometry) is based upon this successive synthesis of the productive in the generation of figures.

Since Hilbert's axiomatization of geometry, no such spatio-temporal construction has been necessary in the proof of geometric propositions. They serve merely as intuitive and dispensable aids to our understanding of the purely formal proof. For Kant, as we have seen, the claim that mathematical truths can be proved by the conjunction of definitions and axioms according to the principle of contradiction is an "erroneous view". Friedman points out that, for Kant, Euclid's axioms do not imply Euclid's theorems by logic alone, because Euclid's axioms, unlike Hilbert's axioms for Euclidean geometry, do not contain an explicit theory of order.

A modern (anachronistic) objection to Euclid's proof that an equilateral triangle can be constructed with any given line segment as base illustrates the difference between the two systems. The proof involves the drawing of two lines from the point of intersection of two circles drawn from the same radius. However, from a modern point of view, Euclid did not prove the existence of that point of intersection. Modern axiomatic formulations preclude such a non-intersection by a continuity axiom. For Euclid, on the other hand, the existence of points is not logically deduced from existential axioms. Rather, the points are generated by defined construction operations with a ruler and compass:[105]

1. To draw a straight line from any point to any point.
2. To produce a finite straight line continuously in a straight line.
3. To describe a circle with any centre and distance.

The indefinite iterability of these construction operations guarantees that the infinity of the set of points may be generated.

---

[105] **Elements**, Book 1, Postulates.

But with the logic of relations, the necessary infiniteness conditions can be explicitly expressed by a theory of dense linear order without endpoints. Rather than Euclid's postulates, we have axioms such as:

(a)(Eb)(a < b) and (b)(Ea)(a < b) - no endpoints;

(a)(b)(Ec)(a < b --> (a < c < b)) - denseness.

Because Kant was limited by monadic logic, he could not represent denseness conceptually since it is intrinsically a relational concept; instead, Friedman says, for Kant,

> denseness is represented by a definite fact about my intuitive capacities: namely, whenever I can represent (construct) two distinct points a and b on a line, I can represent (construct) a third point c between them... this procedure of generating new points by the iterative application of constructive functions takes the place, as it were, of our use of intricate rules of quantification theory such as existential instantiation. Since the methods involved go far beyond the essentially monadic logic available to Kant, he views the inferences in question as synthetic rather than analytic.[106]

It seems then that Friedman has gone even further than Hintikka; for Hintikka, quantificational arguments which introduce new individuals are synthetic, so mathematical arguments of that form will still be called synthetic. Friedman, on the other hand, seems to be claiming in this passage that Kant's conception of the syntheticity of mathematical inferences derives solely from the inadequacy of monadic logic: when converted into quantificational arguments, the propositions in question can be rendered analytic. But Friedman, in his reconstruction of Kant's theory of geometry, does find a role for sensible intuition which is retained in the very nature of modern axiomatic theory as well, and therefore accounts for the synthetic nature of mathematical knowledge.

Friedman rejects the argument that the syntheticity of geometry is based on the synthetic character of its axioms, because this could not account for the syntheticity of arithmetic, which Kant explicitly asserts does not have axioms [A165,B206]. Arithmetic propositions are synthetic because they are established by the successive

---

[106]Friedman [1985], p.468.

addition of unit to unit [B16], that is, by calculation. This method is essentially synthetic because it is temporal, it involves a "successive advance from one moment to another" [A163,B203]. For Friedman then, "intuition underlies the step-by-step process of calculation".[107] Moreover, in the "Enquiry concerning the clarity of the principles of natural theology and ethics", Kant describes the arithmetical method as the examination of "the universal under symbols *in concreto* ", wherein one "proceeds with these signs according to easy and secure rules... so that the things symbolized are here completely ignored" [p.8].

Thus arithmetic proceeds by symbolic construction where the signs, potentially standing for objects, are manipulated according to certain specified operations. The role of intuition here is simply to check perceptually that these operations have been applied correctly. Similarly, the postulates of Euclidean geometry can be viewed as construction operations potentially standing for objects. For example, we start with two points, x and y, to which we apply the line-forming operation $f_L(x,y)$. This operation, along with the other basic construction operations, may be iterated indefinitely, a procedure analogous to arithmetical computation. The geometer, like the arithmetician, handles the individual symbols themselves, instead of the "universal concept of things". the certainty of mathematics as opposed to philosophy is due to this symbolic construction, that is, to the "clearer and easier representation of signs"[108]. The role of intuition, again, is simply to secure "all inferences from error by setting each one before our eyes" [A734,B762] in the form of a symbolic construction, a purely syntactic object to which visually verifiable rules are applied.

Once again, Kant's notion of intuition seems to have been stripped of its essential features. The problem with the Beth-Hintikka reduction of construction in intuition to the introduction of individuals into a logical argument was that it made the physical construction only necessary relative to the prevailing logical theory:

---

[107] Ibid., p.492.
[108] "Enquiry...", pp.9-10.

But with the logic of relations, the necessary infiniteness conditions can be explicitly expressed by a theory of dense linear order without endpoints. Rather than Euclid's postulates, we have axioms such as:

(a)(Eb)(a < b) and (b)(Ea)(a < b) - no endpoints;

(a)(b)(Ec)(a < b --> (a < c < b)) - denseness.

Because Kant was limited by monadic logic, he could not represent denseness conceptually since it is intrinsically a relational concept; instead, Friedman says, for Kant,

> denseness is represented by a definite fact about my intuitive capacities: namely, whenever I can represent (construct) two distinct points a and b on a line, I can represent (construct) a third point c between them... this procedure of generating new points by the iterative application of constructive functions takes the place, as it were, of our use of intricate rules of quantification theory such as existential instantiation. Since the methods involved go far beyond the essentially monadic logic available to Kant, he views the inferences in question as synthetic rather than analytic.[106]

It seems then that Friedman has gone even further than Hintikka; for Hintikka, quantificational arguments which introduce new individuals are synthetic, so mathematical arguments of that form will still be called synthetic. Friedman, on the other hand, seems to be claiming in this passage that Kant's conception of the syntheticity of mathematical inferences derives solely from the inadequacy of monadic logic: when converted into quantificational arguments, the propositions in question can be rendered analytic. But Friedman, in his reconstruction of Kant's theory of geometry, does find a role for sensible intuition which is retained in the very nature of modern axiomatic theory as well, and therefore accounts for the synthetic nature of mathematical knowledge.

Friedman rejects the argument that the syntheticity of geometry is based on the synthetic character of its axioms, because this could not account for the syntheticity of arithmetic, which Kant explicitly asserts does not have axioms [A165,B206]. Arithmetic propositions are synthetic because they are established by the successive

---

[106]Friedman [1985], p.468.

addition of unit to unit [B16], that is, by calculation. This method is essentially synthetic because it is temporal, it involves a "successive advance from one moment to another" [A163,B203]. For Friedman then, "intuition underlies the step-by-step process of calculation".[107] Moreover, in the "Enquiry concerning the clarity of the principles of natural theology and ethics", Kant describes the arithmetical method as the examination of "the universal under symbols *in concreto* ", wherein one "proceeds with these signs according to easy and secure rules... so that the things symbolized are here completely ignored" [p.8].

Thus arithmetic proceeds by symbolic construction where the signs, potentially standing for objects, are manipulated according to certain specified operations. The role of intuition here is simply to check perceptually that these operations have been applied correctly. Similarly, the postulates of Euclidean geometry can be viewed as construction operations potentially standing for objects. For example, we start with two points, x and y, to which we apply the line-forming operation $f_L(x,y)$. This operation, along with the other basic construction operations, may be iterated indefinitely, a procedure analogous to arithmetical computation. The geometer, like the arithmetician, handles the individual symbols themselves, instead of the "universal concept of things". the certainty of mathematics as opposed to philosophy is due to this symbolic construction, that is, to the "clearer and easier representation of signs"[108]. The role of intuition, again, is simply to secure "all inferences from error by setting each one before our eyes" [A734,B762] in the form of a symbolic construction, a purely syntactic object to which visually verifiable rules are applied.

Once again, Kant's notion of intuition seems to have been stripped of its essential features. The problem with the Beth-Hintikka reduction of construction in intuition to the introduction of individuals into a logical argument was that it made the physical construction only necessary relative to the prevailing logical theory:

---

[107] Ibid., p.492.
[108] "Enquiry...", pp.9-10.

had quantificational logic been available to Kant, he would not have insisted on physical construction of concepts. Rather, 'empty' symbolic representation would do. As a result, this approach takes no account of what I have argued is a necessary connection between synthetic (including mathematical) knowledge and sensibility. It can be shown that the notion of a physical construction is essential not only to Kant's account of geometry, but also to his claims about the nature of space. While Friedman does seem to maintain that the proof of a mathematical proposition is necessarily spatio-temporal, and that the intuition involved in the proof is necessarily sensible (i.e. visual), his account of geometry does not fulfill the requirements of the Transcendental Exposition of the Concept of Space. He neglects what I earlier referred to as the heuristic function of mathematics in Kant's overall epistemological solution.

The original problem which Beth, Hintikka, and Friedman all tried to resolve was to explain how we can come to have necessary universal knowledge from the intuition of a particular construction. Clearly, such apodeictic knowledge cannot be obtained by examining the empirical properties of a particular figure, since empirical knowledge lacks necessity. In the Transcendental Exposition of the Concept of Space, Kant provides a principle from which the possibility of a priori synthetic geometrical knowledge can be understood [B40]. Briefly, the 'exposition' runs as follows.

Given that geometry is a science which determines the properties of space synthetically, and a priori, what must be our representation of space in order that such knowledge be possible? Because such knowledge cannot proceed from concepts alone, this representation must in its origin be an intuition. Because such knowledge is apodeictic and not empirical, the intuition must be a priori, i.e. it must be prior to any perception of an object. This is only possible insofar as the intuition has its seat in the subject only, as the form of sensibility in general.

For Kant, then, the only explanation of the possibility of geometry is that we possess a special faculty of pure intuition by which the Euclidean constructions are guided. It is therefore the properties of this faculty which comprise the subject matter of geometry. This alone, however, is not sufficient to account for the possibility of geometry as a science. Benacerraf suggests two conditions a putative account must meet:[109]

> (1) The account should imply truth conditions for mathematical propositions that are evidently conditions of their truth.
> (2) the conditions of the truth of mathematical propositions cannot make it impossible for us to know that they are satisfied.

Kant meets these criteria with his thesis of the transcendental ideality of space. In order to clarify this claim, it may be useful to distinguish two "subtheses"[110] which conveniently correspond to Benacerraf's requirements. The argument of the Transcendental Exposition presented above establishes what we might call (loosely) the metaphysical thesis that the only explanation of the possibility of geometry is a special faculty of pure intuition to which geometrical truths refer, and in virtue of which these propositions are true. While this establishes that geometrical truths must be propositions which are true of this faculty, it is left to Kant's epistemological "subthesis" (again, loosely called) to solve the problem of access to these facts, that is, how we can have knowledge of them.

The problem facing Kant here is essentially to show, given that the geometer investigates this pure faculty of intuition, (1) how he can come to know the properties of this faculty, and (2) how this can give him knowledge of the properties of **real** space. This is achieved by Kant's identification of real space with represented space. He begins by refuting Berkeley's naive realist view of our concept of space as derived empirically from experience [A24,B38]. This cannot be the case because we can't have outer experience at all without entailing a distinction between the subject and the object; the distance between the subject and the object is spatial; thus the concept of space is a **necessary presupposition of experience**, and cannot therefore be epistemologically prior to it. (Here Kant derives the nature of space without appealing to the certainty of

[109] Benacerraf [1973], pp.408-9.
[110] Following Kitcher [1975], p.29.

geometry.) In fact, Kant goes on to claim, we can have no knowledge of real space, if it is thought to exist over and above represented space; because we have access only to our representation of space, knowledge of an independently existing real space could only be inferential, and therefore contingent. Thus, by what Kant called a transcendental argument, based on the premise that geometry gives us universal knowledge of the properties of real space, we may conclude that real space is identical with represented space:

> Were this representation of space a concept acquired a posteriori, and derived from outer experience in general, the first principles of mathematical determination would be nothing but perceptions. They would therefore all share in the contingent character of perception... [A24]

The problem of access to geometrical facts about real space is on its way to being solved: we can have knowledge of our representations of space, and these representations are identical with real space. Furthermore, our representations, as intuited, are necessarily subject to the form of our external intuition, space. Combining these two theses results in the claim that the knowledge we have of this faculty of pure intuition is knowledge of the properties of real space. Finally we arrive at the role of construction in geometry:

> This pure intuition is in fact easily perceived in geometrical axioms, and any mental construction of postulates or even problems That in space there are no more than three dimensions, that between two points there is but one straight line, that in a plane surface from a given point with a given right line a circle is describable, are not conclusions from some universal notion of space, but only *discernible* in space as in the concrete. [ID 142]

We do not learn about this faculty of pure intuition by introspection, but by reflecting on objects which are constructed in space, and which therefore exhibit the properties of space, the form of intuition. We have epistemological access to this faculty only by means of the construction of an object revealing its properties. This explains Kant's tribute to the first man who demonstrated the properties of the isosceles triangle:

> The true method, so he found, was not to inspect what he discerned either in the figure, or in the bare concept of it, and from this, as it were, to read off its properties; but to bring out what was necessarily implied in the concepts that he had himself formed a priori, and had put into the figure in the construction by which he presented it to himself. If he is to know anything with a priori certainty he must not ascribe to the figure anything save what necessarily follows from what he has himself set into it in accordance with his concept. [Bxii]

The faculty of pure intuition provides the "universal conditions of construction" [A714,B742], which consequently are also what Thales himself had put into the figure. He can have a priori knowledge of these properties because he contributes them to experience. While the geometer has represented to himself a particular, he ascribes to that particular only the properties which belong to all instances in virtue of these universal conditions. Thus the construction in intuition solves the problem of epistemological access to the faculty of pure intuition. While the properties of space are necessarily exhibited in all our intuitions (because they constitute the form of intuition), and therefore in all our experience [ID 142], the geometer requires a **determinate** representation which makes these properties evident. This representation must be a priori, i.e. constructed "without having borrowed the pattern from any experience" [A713,B741]. We must consider "only the act whereby we construct the concept, and abstract from the many determinations". The key notion here is that this act must be performed in space; the properties must be instantiated. One more rather lengthy passage should clarify this point:

> It does, indeed, seem as if the possibility of a triangle could be known from its concept in and by itself (the concept is certainly independent of experience), for we can, as a matter of fact, give it an object completely a priori, that is can construct it. But since this is only the form of an object, it would remain a mere product of imagination, and the possibility of its object would still be doubtful. To determine its possibility, something more is required, namely, that such a figure be thought under no conditions save those upon which all objects of experience rest. [A224,B271]

As we saw in the passage from the **Inaugural Dissertation**, Kant identified the universal conditions of construction issuing from the faculty of pure intuition as the axioms of geometry. A comparison of

Kant's conception of geometrical axioms with those of modern axiomatics should shed light on his call for constructions. In a completely axiomatized theory in which all the concepts of that theory are explicitly given as primitives, and all assumptions are stated in or provable from the axioms, the validity of the deductions is independent of the actual meaning of the primitive terms and axioms. That is, we can usually think of more than one possible model for this theory for which the axioms come out true. The development by Hilbert of the conception of a formal system carries this abstraction from meaning even farther: the axioms of a formal system are a class of strings of symbols, specified with no reference to interpretation. The modern conception of axioms then seems to be as free postulates of thought, out of which an unlimited number of systems (and therefore an unlimited number of geometries) can be constructed. For Kant, on the other hand, there is only one geometry - Euclidean - the principles of which alone can be counted as axioms. The axioms of geometry provide insights into the essential nature of space.[111] While the concept of a figure enclosed by two straight lines is not contradictory, it is an impossible object because it defies the conditions of space and its determinations, which constitute the form of experience in general [A220,B268]. We can think such an object, but it will never be found in experience, and is therefore meaningless. While Hilbert's formalization of geometry is supposedly completely independent of space, Kant views geometry precisely as a theory about space. Consequently, the properties of its figures must be capable of being exhibited in space. This points to an important distinction made by Kant between geometry and algebra [A717,B745]. Whereas in algebra, construction is symbolic, and abstracts completely from the properties of the object, the geometer proceeds by constructing its object itself by means of an ostensive construction. It is just this elimination of the spatial elements in Kant's account of geometry that makes the previous reconstructions by Beth, Hintikka, and Friedman inadequate. Kant's general epistemological problem is to show how we can have

[111] Against Martin's claim in his [1955], p.20.

apodeictic knowledge of the objects of experience, or, how synthetic a priori knowledge is possible. Mathematics is given as the paradigmatic example of such knowledge, because it is a "great and established branch of knowledge... carrying with it thoroughly apodeictic certainty" [Prol.281]. If Kant's account of mathematical knowledge required no (essential) reference to our experience, then it would not fulfill its heuristic function as a paradigm of synthetic a priori knowledge. This connection with experience is established, in geometry, through Kant's doctrine of pure intuition as the form of our experience. Clearly, if the role of intuition is reduced to the introduction of new logical individuals in a formal proof, or to the verification of mechanical procedures applied on syntactic objects, then the connection with experience cannot be made.

It is also in this way that physical constructions are essential. As we saw earlier, Kant recognizes the possibility of a figure contained by two straight lines, but

> since the mere form of knowledge, however completely it may be in agreement with logical laws, is far from being sufficient to determine the material (objective) truth of knowledge, no one can venture with the help of logic alone to judge regarding objects, or to make any assertion. [A60,B85]

Mathematics differs from logic in that it has existential import;[112] it has content, and is therefore synthetic [A598,B626]. Whether an object of a geometrical proposition exists depends on whether or not it is constructible; one cannot construct a figure enclosed by two straight lines, so it is an impossible figure about which we can make no meaningful assertions. Thus the construction in intuition supplies the evidence for geometrical assertions. We see that the axioms are true and applicable to real space, because in performing the act of construction, we necessarily apply them; conversely, the exhibition

---

[112]Cf. Kant's "Open letter on Fichte's **Wissenschaftslehre**", August 7, 1799, in Zweig [1967], p.253.:

> ...the principles of logic cannot lead to any material knowledge. Since logic... abstracts from the *content* of knowledge, the attempt to cull a real object out of logic is a vain effort and therefore a thing that no one has ever done.

in concreto of the concepts of mathematics immediately exposes "everything unfounded and arbitrary in them" [A711,B739]. Through construction in intuition, the axioms of geometry are demonstrated [A734,B762]. Finally, because these axioms are truths about the way we must intuit objects in space, about the form of intuition, their application is limited to what we can intuit:

> The concept itself is always a priori in origin, and so likewise are the synthetic principles or formulas derived from such concepts; but their employment and their relation to their professed objects can in the end be sought nowhere but in experience, of whose possibility they contain the formal conditions. [A240,B299]

In other words, mathematical judgments are subject to the principle of significance; through construction of his object in intuition, the geometer meets this demand.

Friedman claims that the syntheticity of geometry could not be explained by the synthetic nature of the axioms because this could not explain the syntheticity of arithmetic. Instead, he holds that the synthetic nature of both arithmetic and geometry is based on the necessity for visually surveyable proofs. I have tried to show that, with respect to geometry, this view does not suffice for Kant's epistemological project. The proffered alternative to Friedman's account, which does mesh with Kant's project, attributes the syntheticity of of geometry to the fact that its operations have spatial content, and that the axioms are therefore demonstrable: their truth is evident.[113] It may well be that the theorems of geometry are derived by the calculational procedure described by Friedman, but it is essential to Kant's theory that the spatial content remains, at least at the level of the axioms. It is in virtue of its dependence on spatial content that geometry is based on sensible intuition. The task now is to show how Kant conceived of arithmetic as based on sensible intuition.

[113]In fact, one might say that they "force themselves upon us as true".

(ii) Arithmetic

The initial appearance of dissimilarity between arithmetic and geometry arises from this notion of content. While geometry is considered as a theory about space, it is not so clear what, for Kant, arithmetical judgments are about. Unlike the intuitionists who claim descendance from Kant, he does not hold that we can have intuition of the natural numbers themselves. Arithmetical judgments cannot be judgments about numbers construed as individual objects, because objects can only be given in space and time; Kant would not admit Frege's conception of number as an abstract object because the whole purpose of his project is to invalidate claims to knowledge beyond the limits of sense experience. While Kant did not consider numbers as denoting objects, he did not have available the alternative of taking `is ten', for example, as a predicate applicable to sets."[114] For this reason, propositions such as `7 + 5 = 12' could not be construed as propositions about a general class of objects (i.e. sets); consequently, they cannot be called axioms [A164,B205]. Kant explains this fact himself in a letter to Johann Schultz:

> certainly arithmetic has no axioms, since its object is actually not any *quantum*, that is, any quantitative object of intuition, but rather *quantity as such*, that is, it considers the concept of a thing in general by means of quantitative determination. [115]

In other words, arithmetical propositions are not about numbers or any other object of intuition; rather, they are about the pure concept of quantity. On the other hand, the axioms and propositions of geometry have as their objects forms such as points, lines and figures, through which the connection with the form of intuition - and thus with sensibility - is established. How can arithmetical propositions about the "concept of a thing in general" be dependent in this way upon sensibility?

---

[114]Tait [1986], pp.597-9. Furthermore, Kant would not have recognized judgments about abstract sets as objectively valid or meaningful.
[115]Zweig [1967], p.130.

The attempt to answer this question requires a brief detour into Kant's Transcendental Deduction, in which he is concerned with answering his "chief question", that is, "what and how much can the understanding and reason know apart from all experience" [Axvii]. He attempts to establish the objective validity of the pure concepts of the understanding (the Categories) by showing that they are antecedent - and therefore a priori - conditions through which alone experience is possible [A93,B125], just as we saw earlier that the forms of intuition are the conditions under which alone objects can be intuited. Thus the categories, as the form of thought, specify the concept of an object in general. Their objective validity rests upon their being the a priori conditions of a possible experience, and of an empirical object [A96].

The Transcendental Deduction of the first edition begins with the assumption that all of our representations are subject to time: "in it they must all be ordered, connected, and brought into relation" [A99]. In all our experience, we are conscious of this temporal ordering; sensibility alone is not sufficient for experience, then, because it is merely passive. For the manifold given in sensible intuition to be temporally ordered, we require an active synthesis, that is, Kant's three-fold synthesis.

While the passive faculty of sensibility presents the manifold, it cannot produce it as a manifold, i.e. as both unity and multiplicity. It must first be "run through and held together" in a synthesis of apprehension by the mind. Furthermore, since our awareness of an object is a temporal process, we must be able to retain before our minds what has been given previously; we must be able to represent an object even when it is no longer present in intuition through the synthesis of reproduction in imagination. Kant provides the significant (for my purposes) example of representing to oneself a particular number [A102]. To be aware of a number of units, each particular unit must be apprehended in thought one after the other; but they must also be reproduced - or kept in mind - while advancing to other units. Otherwise, one could never be aware of the total that had been counted. If, for example, one is counting up to 5, we must remember that the units which we ultimately apprehend as 5 are the same units we apprehended in the act of counting up to 5. However, we must also recognize that this total was produced by one act of counting, or the "successive addition of unit to unit". To recognize that this is the number 5, we must recognize the unity of the synthesis through which the total was produced: this is the synthesis of recognition under concepts. Thus the concept of the number consists precisely in the consciousness of the unity of the synthesis [A103].

The process here described as the synthesis of the manifold of intuition which is necessary for the application to intuition of the concept of an object in general (the Categories) corresponds to Kant's account in the Introduction of how we come to know that 7 + 5 = 12. We saw earlier that Kant holds, against Leibniz, that the concept of 12 cannot be thought in the union of 7 and 5; instead,

> We have to go outside these concepts, and call in the aid of the intuition which corresponds to one of them, our five fingers, for instance...For starting with the number 7, and for the concept of 5 calling in the aid of the fingers of my hand as intuition, I now add one by one to the number 7 the units which I previously took together to form the number 5, and with the aid of that figure see the number 12 come into being. [B15-16]

The passage describes a process of enumeration enabling us to combine (or synthesize) an intuited plurality (the fingers) and to apprehend it as a totality (a collection of 12 things), just as we might combine a series of appearances of windows, bricks, and doors under the concept of a house. As the concept of the house consists in the consciousness of the unity of the synthesis of the windows, bricks, and doors, the concept of the number 12 consists in consciousness of the unity of the synthesis of the fingers in counting.

In the Transcendental Aesthetic, Kant sought to establish the necessary conditions for intuition. By reflecting on experience, he determined that the pure form of outer intuition (that which makes outer intuition possible) must be space. While this form necessarily underlies all our experience, in order to know its properties, the geometer must present himself with a determinate representation of those properties by constructing a figure which exhibits them.

In the Transcendental Deduction, Kant seeks to establish the necessary conditions of experience of objects. This requires that the manifold given in sensible intuition be held together in a necessary synthetic unity. This unity is achieved by bringing the manifold under the pure concepts of the understanding, or the 'concept of an object in general'. Thus the act of synthesis according to the categories necessarily underlies all our experience.

As the propositions of geometry are about space, the propositions of arithmetic are about the pure concept of quantity. As the geometer has no direct epistemic access to the pure form of sensibility which is necessary for intuition, the arithmetician has no direct epistemic access to the pure concept of quantity which implicitly underlies all our experience. For both the geometer and the arithmetician, this access is gained by exhibiting a priori the intuition corresponding to the concept - that is, by constructing the concept. We have seen that the necessary process of the understanding by which the intuited manifold is brought under concepts is the same procedure employed in counting or adding, as presented by Kant in the preceding passage. In his letter to Schultz, Kant identifies this procedure with the **construction of the concept** of quantity:

> In the problem, conjoin 3 and 4 in one number, the number 7 must arise not out of an analysis of the constituent concepts but rather by means of a construction, that is, synthetically. This construction, a single counting up in an a priori intuition, presents the concept of the conjunction of two numbers. Here we have the construction of the concept of quantity rather than that of a quantum. For the idea that the conjoining of 3 and 4, as distinct quantitative concepts, could yield the concept of "one" magnitude was only a thought. The number 7 is thus the presentation of this thought in an act of counting together.[116]

As was the case with geometry, the synthetic nature of arithmetic results from the use of constructions in the proof of propositions.

---

[116] Zweig [1967], p.131.

But where the essential feature of the geometrical construction was that it be exhibited spatially, the essential feature of the arithmetic construction is that it is a temporal process. It is thus through time as a form of our intuition that arithmetic is dependent on sensibility.

It is clear from Kant's example of the addition of 7 and 5 that the construction of the sum necessarily takes place in time (as, in fact, does all experience): it requires a succession of mental acts of "running through" the collection of things. For example, to represent the number 5, I would proceed by perceiving, one by one, different aspects of a manifold (given in pure or empirical intuition), until I would "see" the number 5 come into being. This operation would require five successive acts of apprehension, so that whether I construct the sequence empirically or in my head, the structure of the number is already represented in this sequence of operations. In this way, time provides a "universal source of models for the numbers".[117] In "considering only the act whereby we construct the concept" [A714,B742], we can provide ourselves with determinate representations of the concept of quantity (i.e. the concept of a thing in general by means of a quantitative determination).

Kant explicitly recognizes the connection of arithmetic with time in (once again) his letter to Schultz:

> The science of numbers, notwithstanding the succession that every construction of quantity requires, is a pure intellectual synthesis, which we represent to ourselves in thought. But insofar as specific quantities (quanta) are to be determined in accordance with this science, they must be given to us in such a way that we can grasp their intuition successively; and thus this grasping is subjected to the time condition. So that when all is said and done, we cannot subject any object other than an object of a possible *sensible* intuition to quantitative, numerical assessment, and it thus remains a principle without exception that mathematics can be applied only to *sensibilia*.[118]

[117]Parsons [1969], p.140. Similarly, space provides the model for geometrical concepts.
[118]Zweig [1967], p.131.

The construction of mathematical concepts is governed by the same conditions that govern our perception of physical objects, that is, by the forms of sensible intuition,and the concept of an object in general, which are the conditions of possible experience. Consequently, all mathematical knowledge is limited to the objects of possible experience. While the concepts always originate a priori,

> their employment and their relation to their professed objects can in the end be sought nowhere but in experience, of whose possibility they contain the formal conditions. [A240,B299]

As we saw earlier, it was the primary aim of Kant's **Critique** to prove that our synthetic knowledge is limited to objects of possible experience. In the case of arithmetic, he gives a plausible account of how we might proceed in an intuitive verification by construction of elementary propositions involving small numbers. Unfortunately, his conception of mathematics in general fails to account for higher levels of mathematical knowledge. For example, the generation of the natural number series obviously extends beyond the numbers that we could actually construct. Similarly, for Kant, space is necessarily Euclidean; consequently, it must also be infinitely divisible. In fact, he asserts that the infinite divisibility of lines and angles is a rule of construction in space; any objection to this is "only the chicanery of a falsely instructed reason" [A165,B206]. But surely the verification of such a rule is beyond the realm of possible experience.

These difficulties prompt Parsons to attempt to salvage Kant's account by attributing to him a more sophisticated conception of `possible experience':

> if this analysis is to yield its result, the limitation of our knowledge to objects of possible experience must mean more than that the objects should be such as might present themselves in some way or other in a possible experience.[119]

---

[119]Parsons [1964].

If it does not mean more than this, then Kant's claim about infinite divisibility, and in fact any infinitary notions, comes into conflict with the limited acuteness of our senses. If infinite divisibility is indeed "a rule of construction in space", then it must be a property of the faculty of pure intuition and therefore a condition of the possibility of experience. The question then arises as to what kind of possibility Kant is talking about. It cannot be practical possibility based on human capacities because there are natural circumstances which would prevent us from actually carrying out the infinite division of a line segment: we would eventually reach the limit of the acuteness of our senses, or run out of ink, or eventually die.

Using a method somewhat akin to a transcendental argument, Parsons argues that Kant must mean by `the possibility of experience' an abstract kind of possibility. He suggests that this possibility might be defined by an idealized form of intuition, such as might be possessed by a creature who could perform infinitely many acts in a finite time, and who could increase the acuteness of his senses beyond any limit. However, this certainly involves an unjustifiable extrapolation beyond any experience that a human being might have; we could have no way of verifying that the process of division can continue into infinity. As Parsons points out, if **this** is Kant's conception of the possibility of experience, then the notion of a form of intuition of which we have synthetic a priori knowledge is deprived of any explanatory force, and becomes a merely ad hoc explanation for our knowledge of space.

Correspondingly, the natural number series would have to be conceived as an abstract extension of the mathematical forms embodied in our experience analogous to the extension of the objective world beyond what we actually experience.[120] In the **Inaugural Dissertation** [123n], Kant alludes to the possibility of a creature, not bound by the smallness of the human intellect, who might perceive a multiplicity by a single insight, and who therefore could conceive of an infinite totality. But to the human intellect

[120] Parsons [1967].

it is given to attain to a definite concept of multiplicity only by the successive addition of unit to unit, and to the sum total called a number only by going through with this progress *within a finite time*...[ibid.]

On the other hand,

the infinity of a series consists in the fact that it can never be completed through successive synthesis.[A426,B454]

The infinity of the number series and the notion of an infinitely divisible line segment fail to satisfy Kant's principle of significance because they cannot be given intuitive content. In considering this, it is important to remember Kant's goals. The reason he claims that our experience consists only of appearances, thereby shunting the `thing-in-itself' into an unknowable realm of its own, is that he recognizes that we can have only inferential knowledge of objects existing completely independently of ourselves. Hume showed that there is no rational guarantee of the necessity or universality of this inferential knowledge. Kant's entire epistemology consists of the attempt to show that we can have non-inferential and therefore apodeictic knowledge of the objects of experience. With this purpose in view, we certainly could not accept an interpretation of the possibility of experience and the forms of intuition as based on inferential extrapolations beyond our experience. Moreover, such an interpretation would seem to conflict with the many passages in the **Critique** and in Kant's correspondence which assert that concepts must be given empirical meaning. At A240,B299 for example, Kant demands that the bare concepts of geometry "would mean nothing, were we not able to present their meaning in appearances, that is, in *empirical objects*." Later in the same passage, this point is made even more strongly:

In the same science [mathematics] the concept of magnitude seeks its support and sensible meaning in number, and this in turn in the fingers, in the beads of the abacus, or in strokes and points which can be placed before the eyes.

Surely, Cantor's aleph numbers could not find support in the beads of any empirical abacus. We have already seen how, in **The Analyst**, Berkeley used this same phenomenalist criterion to discredit Newton's doctrine of fluxions and Leibniz's doctrine of the differential calculus. Recall that Berkeley's complaints against early analysis, as represented by these theories, were based on their use of infinitely small quantities, "that is, infinitely less than any sensible or imaginable quantity, or than any the least finite magnitude", the conception of which is beyond human capacity. Certainly Berkeley's intention was radically different from Kant's; his purpose was to show that our scientific and mathematical 'knowledge' is no more firmly grounded than the mysteries in faith. Kant on the other hand, wants to show that we **can** and **do** have firmly grounded mathematical and scientific reason. Because such notions as the velocities of nascent and evanescent quantities, **abstracted from time and space**, may not be comprehended or demonstrated, we must - if we are to avoid running up against the 'eternal crevices of unreason' - recognize that to talk of investigating, obtaining, and considering the proportions of such velocities, exclusively of time and space, is to talk unintelligibly.[121] Thus for Berkeley, the extension of mathematical reasoning beyond our capacity to perceive is illegitimate.

Kant, however, recognizes the usefulness of such extensions, and therefore finds a legitimate role for them in his doctrine of the Transcendental Ideas of pure reason. The principle of these Ideas is to "find for the conditioned knowledge obtained through the understanding the unconditioned whereby its unity is brought to completion" [A307,B364];

> they are concerned with something to which all experience is subordinate, but which is never itself an object of experience - something to which reason leads in its inferences from experience, and in accordance with which it estimates and gauges the degree of its empirical employment, but which is never itself a member of the empirical synthesis.[A311,B367]

---

[121] Berkeley [1634], p.85.

There is then no necessity for postulating a superhuman creature (Parsons' `U') to describe our form of intuition in order to account for the infinite divisibility of the line segment, or the infinity of the natural number series. They are given a definite function as Ideas which supplement our intuitions by providing an overall framework to satisfy what Weyl calls our "desire urging towards totality". This also seems to be the role Kant has in mind for algebra. In a letter to Rehberg,[122] Kant distinguishes between the algebraic symbol `a', which requires no synthesis in time (and therefore has no content), and the number for which `a' stands, which is dealt with in arithmetic, and requires a pure intuition (i.e. it must have content). Although the symbols of algebra do not themselves have content, they are manipulated according to fixed rules which may be given content by geometry and arithmetic. Nevertheless, with respect to these ideal notions, we must be satisfied with the symbol; we mustn't "expect the transcendent to fall within the lighted circle of intuition".[123] Kant issues a reminder of this point in a letter to Reinhold, in which he condemns Eberhard for claiming that the mathematicians have succeeded in designing entire sciences without even a single word of the reality of the objects of their concepts:

> When intuitions are lacking, we must be resigned to forego the claim that our concepts have the status of cognitions (of objects). We must admit that they are only ideas, mere regulative principles for the use of reason directed toward objects given in intuition, objects that, however, can never be known completely, since they are conditioned.[124]

Thus, even though nothing in experience could justify it, the mind simply assumes that the process of bisecting a line segment or of successively adding unit to unit can be indefinitely continued. In this respect, Kant's approach to mathematics resembles that of Hilbert. Hilbert accepts certain elementary mathematical assertions as "immediately intuitable and understandable without recourse to

---

[122] Zweig [1967], pp.166-9.
[123] Weyl [1949].
[124] Zweig [1967], p.146.

anything else"; but he also recognized that assertions involving references to the infinite were necessarily not subject to this perceptual verification. Like Kant, Hilbert considered the assertions belonging to the former class to be meaningful, and those belonging to the latter as meaningless. Furthermore, as Brouwer showed, certain fundamental notions of logic, particularly the law of the excluded middle and negation, lose their validity in the transition to the infinite. However, because the infinite occupies a justified place in our thinking, and it plays the role of an indispensable concept, Hilbert allowed the introduction of infinitary concepts into mathematics as **ideal** elements: "to preserve the simple formal rules of ordinary Aristotelian logic, we must *supplement the finitary statements with ideal statements* ":

> The role that remains for the infinite to play is solely that of an idea - if one means by an idea, in Kant's terminology, a concept of reason which transcends all experience and which completes the concrete as a totality - that of an idea which we may unhesitatingly trust within the framework erected by our theory.[125]

Since Kant warned us that we must forego the claim that such concepts have the status of cognitions, Hilbert is satisfied to guarantee, not the truth, but merely the consistency of classical mathematics. Given Kant's determination to show that the synthetic a priori propositions of mathematics amount to apodeictic knowledge, it seems that he could not have accepted this development of his account. The introduction of the Transcendental Ideas allows him to complete the picture of mathematics as based on sensible intuition, but at the expense of severely restricting our claims to mathematical and, by extension, synthetic a priori **knowledge**. To return to an earlier example, if we cannot know that a line is infinitely divisible, then how can we know that two circles drawn from the same radius will intersect? How can we know that Euclid's proof goes through if we can't know (i.e. empirically verify) that the assumption of infinite divisibility is valid?

---

[125]Hilbert [1925], p.195; p.201.

I have tried to emphasize that Kant's account of mathematics is secondary to his more general epistemological theory. His primary purpose is to grant knowledge its independence from the empirical, thereby securing its universal necessity, while at the same time guaranteeing its application to the empirical, and limiting claims to speculative-metaphysical knowledge. He sees geometry and arithmetic as confirmations of his new method of thought, and therefore as useful tools to prove his main point. It is with this purpose in mind that he constructs his theory of mathematical knowledge. Not even the mathematician can

> free himself from the demand, so troublesome yet so unavoidable for all dogmatism, that no concept be admitted to the class of cognitions if its objective reality is not made evident by the possibility of the object's being presented in a corresponding intuition.[126]

While Kant has succeeded in formulating a conception of mathematics as based on sensible intuition, and therefore satisfying his criterion of significance, by denying that we can have **knowledge** beyond what is intuitively evident, he has restricted our claims to mathematical knowledge. It is thus not clear that he has fulfilled his own broader aim, insofar as he has shown the range of synthetic a priori knowledge, at least in mathematics, to be severely limited. This "great and established branch of knowledge" which carries with it "thoroughly apodeictic certainty" falls victim to Kant's finding [Bxx] that it is "necessary to deny knowledge, in order to make room for faith".

[126]Letter to Reinhold, May 19, 1789; Zweig [1967], p.145.

Conclusion

The point of this lengthy detour into Kantian metaphysics was to try to shed some light on Gödel's allusions to Kant in the presentation of his platonist views, and to see if a more plausible interpretation of the platonist analogy between mathematical and scientific knowledge could be made out. First of all, it should be clear that Kant and Gödel are addressing the same question in two different realms. Kant wants to discover how it is that we have experience of independently existing, enduring physical objects, and how we can have a priori, universal knowledge of these objects. Gödel's platonism consists in his belief that mathematical knowledge is knowledge of objects, and is a priori. Although we have just seen that Kant's attempted solution to his question is inadequate, at least as an account of mathematical knowledge, it does suggest a route worth investigating. The inadequacy of Kant's response is a consequence of his desire to dispel speculative metaphysics, which leads him to place overly restrictive conditions - i.e. the principle of significance - on what we may call knowledge. Thus his construal of intuition underdetermines mathematics. Nevertheless, Kant's fundamental insight about the contribution of the mind to experience could be generalized to form the core of a much broader and more satisfactory account of mathematical knowledge.

In the paper on Cantor's continuum problem quoted earlier, Gödel explicitly notes the similarity between his conception of set and Kant's categories of the understanding:

> ... there is a close relationship between the [iterative] concept of set and the categories of pure understanding in Kant's sense. Namely, the function of both is "synthesis", i.e., the generating of unities out of manifolds (e.g., in Kant, of the idea of *one* object out of its various aspects).[127]

It is in this context that Gödel remarks that "evidently the 'given' underlying mathematics is closely related to the abstract elements

[127] Gödel [1964], p.484n.

contained in our empirical ideas". For Kant, these abstract elements are of course the categories of the understanding, which comprise the conceptual framework we impose on the manifold of sensation, and which determine the form of what we experience. The categories, i.e., the concept of an object in general, are what bring together the multiplicity of sensations - for example, yellowness, sourness, oval-shape, solid - to give us the experience of one object, in this case, a lemon. We also determined that for Kant, mathematics is the study of these abstract elements: geometry is **about** our faculty of spatial intuition, and arithmetic is **about** the pure concepts of the understanding. Mathematics, then, is the study of the structure of the human mind which underlies and determines all of experience. Thus, Kant maintains that

> in every special doctrine of nature only so much science proper can be found as there is mathematics in it.[128]

This, I would like to suggest, is the sense in which Gödel's analogy should be understood. The fundamental concepts underlying mathematics, such as the concept of set, are closely related to the concepts underlying the empirical sciences, and all of these are similar to the concepts which shape our experience. In this connection, Gödel quotes with approval from Russell's **Introduction to Mathematical Philosophy**, where Russell claims that

> logic is concerned with the real world just as truly as zoology, though with its more abstract and general features.[129]

One essential feature of our concept of set is that a set is an **object**; as Cantor put it, a set is any multiplicity "which can be thought of as a one". Gödel's point, then, must be that something like this concept of set, which requires the synthesis of manifold elements into one object, is also what underlies our ability to organize and combine our manifold perceptions of the physical world

---

[128]Kant [1786], 470.

[129]Russell [1919], p.169. Cited in Gödel [1944], p.449.

into the experience of one physical object. This ability is fundamental to all experience.

I would like to suggest that if platonism is characterized as a thesis about the **objectivity** and **a priori nature** of mathematical knowledge, then, once the importance of the contribution of the mind to experience is recognized, platonism no longer seems so absurd. The objectivity of mathematical discourse is no more mysterious than the objectivity of discourse about the physical world. We have seen that sense perception alone is insufficient to account for the latter; there must be something else which makes it objective - about objects. It is this same 'something else' which makes our knowledge of mathematics knowledge of objects. Objectivity is conferred on both kinds of discourse by the abstract elements basic to all experience. And because, like Thales in Kant's example at Bxii, we necessarily contribute these elements to our experience of objects, we can have a priori knowledge of them. Finally, mathematical intuition is not an "isolated epistemological faculty" used only in doing mathematics, but rather it is comprised of just these abstract elements underlying experience. Mathematical intuition then, is the modern counterpart to Kant's categories of the understanding.

BIBLIOGRAPHY

Benacerraf, Paul [1965] 'What numbers could not be', Philosophical Review, 74, pp.47-73. Reprinted in Benacerraf and Putnam [1983], pp.272-94.

---------------- [1973] 'Mathematical truth', Journal of Philosophy 70, pp.661-80. Reprinted in Benacerraf and Putnam [1983], pp.403-420.

---------------- and Putnam, Hilary [1983] **Philosophy of Mathematics: Selected Readings**, second edition, Cambridge: Cambridge University Press.

Berkeley, George [1710] **A Treatise Concerning the Principles of Human Knowledge** in **The works of George Berkeley, Bishop of Cloyne**, vol. II, eds. A.A. Luce and T.E. Jessop, London: Nelson (1949).

---------------- [1734] **The analyst** in **The works of George Berkeley, Bishop of Cloyne**, vol. IV, eds. A.A. Luce and T.E. Jessop, London: Nelson (1951).

Cassirer, Ernst [1951] **The philosophy of the Enlightenment** Princeton: Princeton University Press.

Chihara, Charles [1972] **Ontology and the vicious circle principle** New York: Cornell University Press.

---------------- [1982] "A Gödelian thesis regarding mathematical objects: do they exist? and can we perceive them?", Philosophical Review, XCI, No.2, pp.211-227.

Descartes, Réné [1641] **Meditations on First Philosophy** in **The Philosophical Works of Descartes**, vol. 1, tr. E.S. Haldane and G.R.T. Ross, Cambridge: Cambridge University Press (1911).

Dummett, Michael [1967] 'Platonism', in Dummett [1978], pp.202-214.

---------------- [1978] **Truth and other enigmas**, Cambridge: Harvard University Press.

Frege, Gottlob [1884] ***Grundlagen der Arithmetik***, translated by J.L. Austin as **The Foundations of Arithmetic**, Oxford: Basil Blackwell, second edition, 1953.

-------------- [1894] `Review of E.G. Husserl, ***Philosophie der Arithmetik***', *Zeitschrift für Philosophie und Philosophische Kritik* 103, pp.313-32; English translation in Frege [1894], pp.195-209.

-------------- [1895] `Whole numbers', *Revue de metaphysique et de morale 3,* pp.73-8; English translation in Frege [1984], pp.229-233.

-------------- [1984] **Gottlob Frege: Collected Papers on Mathematics, Logic and Philosophy**, Oxford: Basil Blackwell.

Friedman, Michael [1985] 'Kant's theory of geometry', Philosophical Review, 94, pp.455-506.

Gödel, Kurt [1964] `What is Cantor's continuum problem?', in Benacerraf and Putnam [1983], pp.470-485.

----------- [1944] `Russell's mathematical logic', **The Philosophy of Bertrand Russell**, Evanston and Chicago: Northwestern University, pp.125-53. Reprinted in Benacerraf and Putnam [1983], pp.447-469.

Hale, Bob [1987] **Abstract Objects**, Oxford: Basil Blackwell.

Hallett, Michael [1983] `Can we perceive sets and does it matter?', unpublished.

---------------- [1984] **Cantorian set theory and limitation of size**, Oxford: Clarendon Press.

---------------- [forthcoming] **Mathematics and Mind: an essay on Kant and Hilbert**, Oxford: Oxford University Press.

Hilbert, David [1926] *'Über das unendliche' Mathematische Annalen*, vol. 95. Tr. as 'On the infinite' by E. Putnam and G. J. Massey in Benacerraf and Putnam [1983], pp. 183-201.

Hume, David [1740] **An Enquiry Concerning Human Understanding**, ed. L.A. Selby-Bigge, Oxford: Clarendon Press (1967).

Kant, Immanuel [1763] 'Enquiry concerning the clarity of the principles of natural theology and ethics' in **Kant: Selected pre-critical writings**, pp.3-35.

------------- [1770] *'De Mundi Sensibilis atque intelligibilis forma et Principiis Dissertatio*, translated as 'The Inaugural Dissertation' in **Kant: Selected pre-critical writings**. pp.45-92.

------------- [1783] ***Prolegomena zu einer jeden künftigen Metaphysik die als Wissenschaft auftreten können***, translated by Carus as **Prolegomena to any future metaphysics**, New York: Bobbs-Merrill Co. (1950).

------------- [1786] ***Metaphysische Anfangsgründe der Naturwissenschaft***, translated by James Ellington as **Metaphysical Foundations of Natural Science**, New York: Bobbs-Merrill Co. (1970).

------------- [1787] ***Kritik der reinen Vernunft***, translated by Norman Kemp Smith as **The Critique of Pure Reason** , London: Macmillan (1929).

Kitcher, Philip [1975] 'Kant and the foundations of mathematics', Philosophical Review, 84, pp.23-50.

------------- [1984] **The nature of mathematical knowledge**, Oxford: Oxford University Press.

Kline, Morris [1972] **Mathematical thought from ancient to modern times**, New York: Oxford University Press.

Lear, Jonathan [1977] 'Sets and Semantics', Journal of Philosophy, 74, no.2, pp.86-102.

Leibniz, G. W. [1685] **Nouveaux Essais sur l'entendement humaine,** translated by P. Remnant and J. Bennett as **New Essays Concerning Human Understanding**, Cambridge: Cambridge University Press (1981).

Locke, John [1689] **An Essay Concerning Human Understanding** ed. A.C. Frazer, New York: Dover (1959).

Maddy, Penelope [1980] `Perception and mathematical intuition', Philosophical Review, LXXXIX, no.2, pp.163-196.

Martin, Gottfried [1955] **Arithmetic and Combinatorics: Kant and his contempories** Chicago: Southern Illinois University Press.

Parsons, Charles [1964] 'Infinity and Kant's conception of the 'possibility of experience''; Philosophical Review, 73, pp.182-97. Reprinted in Parsons [1983].

--------------- [1969] 'Kant's Philosophy of Arithmetic', **Philosophy, Science and Method: Essays in honour of Ernest Nagel**, ed. Morgenbesser et al., New York, St. Martin's Press. Reprinted in Parsons [1983].

--------------- [1980] 'Mathematical Intuition', Proceedings of the Aristotelian Society, 1980, pp.145-168.

--------------- [1983] **Mathematics in Philosophy: Selected Essays**, New York: Cornell University Press.

Putnam, Hilary [1980] `Models and Reality', Journal of Symbolic Logic 45.3, pp.464-82. Reprinted in Benacerraf and Putnam [1983], pp.421-444.

Russell, Bertrand [1919] **Introduction to Mathematical Philosophy** London: George Allen and Unwin.

Steiner, Mark [1975] **Mathematical Knowledge** Ithaca and London: Cornell University Press.

Strawson, P. F. [1966] **The Bounds of Sense** London: Methuen and Co.

Tait, W. W. [1986] 'Truth and proof: the platonism of mathematics', Synthese 69, pp.341-70.

Weyl, Herman [1949] **Philosophy of Mathematics and Natural Science** Princeton: Princeton University Press.

Whitehead, Alfred North [1925] **Science and the Modern World**, New York: The Macmillan Company.

Zweig, Arnulf [1967] **Kant: Philosophical Correspondence 1759-99** Chicago: University of Chicago Press.