Foliations on Shimura varieties in positive characteristic

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ABSTRACT

In this thesis, we investigate some foliations on the positive characteristic fibres of certain Shimura varieties. First, we review the general theory of foliations in positive characteristic, especially looking at torus-equivariant foliations on toric varieties. In particular, we will provide an explicit description for the singular locus of a torus-equivariant foliation on a toric variety. Secondly, we apply these observations to the tautological foliations on Hilbert Modular Varieties in both characteristic zero and positive characteristic, thus giving a description of the singular locus of these foliations on a toroidal compactification of a Hilbert Modular Variety. We will also investigate the behaviour of the V-foliation on unitary Shimura varieties of signature (n, m) and show that certain high-dimensional Ekedahl–Oort strata are integral varieties with respect to this foliation.

Résumé

Dans cette thèse, on étudie certains feuilletages sur les fibres de caractéristique positive de certaines variétés de Shimura. On commence par une introduction à la théorie générale de les feuilletages en caractéristique positive, en particulier on considère les feuilletages sur des variétés toriques. On donne une description explicite du lieu singulier d'un feuilletage torique. Àpres, on applique ces observations aux feuilletages tautologiques sur les variétés modulaires de Hilbert à la fois en caractéristique nulle et en caractéristique positive, donnant ainsi une description du lieu singulier de ces feuilletages sur une compactification toroïdale d'une variété modulaire de Hilbert. On étudie également la V-feuilletage sur les variétés unitaires de Shimura de signature (n, m), et on montre que certaines strates Ekedahl–Oort de haute dimension sont des variétés intégrales par rapport à ce feuilletage.

CONTENTS

1. INTRODUCTION

Let X be a smooth variety over a field κ with tangent bundle $\mathcal{T}X$. A foliation on X is defined as an involutive saturated subsheaf of $\mathcal{T}X$, that is, a saturated subsheaf that is closed under the Lie bracket. If κ is a field such that $\mathrm{char}(\kappa) = p > 0$, we have a map $\zeta \mapsto \zeta^p$ on $\mathcal{T}X$. If $\mathscr{F}$ is a foliation on X that is closed under this map, it is called a p-foliation. In [Eke87], it was shown that p-foliations have a deep connection with inseparable morphisms. In this thesis we will be examining two examples of p-foliations on Shimura varieties, namely the tautological foliations on Hilbert modular varieties, and the V-foliation on unitary Shimura varieties.

A Hilbert modular variety $\mathcal{M}$ can be viewed as a moduli space parameterizing polarized abelian varieties with real multiplication by $\mathcal{O}_L$, where L is a totally real field. The complex points of $\mathcal{M}$ can be described by the uniformization $\mathcal{M}(\mathbb{C}) \cong \Gamma\backslash\mathfrak{h}^g$, where Γ is an arithmetic subgroup of $\mathrm{SL}_2(L)$ acting on g copies of the upper half-plane. Let $(z_1, \ldots, z_g)$ be the coordinates of $\mathfrak{h}^g$, then for any subset $J \subseteq \{1, \ldots g\}$, we can define a foliation $\mathscr{F}_J$ on $\mathcal{M}(\mathbb{C})$ by considering the subbundle of $\mathcal{T}\mathcal{M}$ spanned by $\{\frac{\partial}{\partial z_j}\}_{j\in J}$. We call these the tautological foliations on $\mathcal{M}$. In [GdS23], de Shalit and Goren gave an algebraic description of the tautological foliations and studied them on the positive characteristic fibres of $\mathcal{M}$. We will build on their work by extending the tautological foliations to toroidal compactifications of Hilbert modular varieties. These extensions are generally not smooth foliations, and our main result about them is to explicitly describe the singular locus such a foliation on a given toroidal compactification.

One of the main components in the construction of toroidal compactifications of Hilbert modular varieties are toric varieties. In [Kly90], a correspondence between torus-equivariant vector bundles on a toric variety and certain multi-filtrations of a vector space is given. In the case of toric foliations, this classification reduces to a correspondence between toric foliations on a toric variety X with subspaces of certain vector space. In characteristic zero, this theory has been developed and used in, for example, [Per04], [Pan15] and [Wan23]. In this thesis we further extend this idea to positive characteristic and prove the correspondence between toric p-foliations on a toric variety X defined over a field κ of positive characteristic and subspaces of a vector space defined over κ.

Using this correspondence, we can compute the singular locus of toric foliations, and in particular, the singular locus of a tautological foliation on a toroidal compactification of a Hilbert modular variety. This leads to some interesting results in the positive characteristic case. For example, it is well-known that the singular locus of a tautological foliation on any toroidal compactification of a Hilbert modular surface defined over $\mathbb{C}$ is precisely the zero-dimensional toric strata. However, when working over a field κ with $\mathrm{char}(\kappa) = p > 0$, we demonstrate that it is possible for the tautological foliations to be smooth at some of the zero-dimensional strata. Indeed, in characteristics 2 and 3, we prove it is always possible to choose a toroidal compactification of a given Hilbert modular surface such that one of the tautological foliations is smooth everywhere.

We will also be considering a p-foliation defined on unitary Shimura varieties. Given a quadratic imaginary field E, the unitary Shimura variety $\mathcal{M}$ of signature (n, m) can be viewed as a moduli space parameterizing polarized abelian varieties A with an endomorphism structure $\mathcal{O}_E \hookrightarrow \mathrm{End}(A)$ of signature (n, m). Let κ be a field of characteristic p. The Ekedahl–Oort stratification, first defined over $\mathcal{A}_g$ in [Oor01], classifies the points in $\mathcal{M}(\kappa)$ by the isomorphism class of the p-torsion of the abelian variety parameterized.

For $m < n$, de Shalit and Goren constructed a natural height 1 foliation of rank m^2 over unitary Shimura varieties of signature (n, m) in [dSG18]. This foliation, known as the V-foliation, was first defined over the open Ekedahl–Oort stratum, but was shown to extend deeper to a particular stratum, denoted S_{fol}. Furthermore, they showed that S_{fol} is an integral variety for the V-foliation. In this thesis, we look specifically at the Ekedahl–Oort strata that lie between the open stratum and S_{fol}. Using Dieudonné theory, and Zink's theory of displays, we give an explicit description of the V-foliation and the tangent spaces of the individual Ekedahl–Oort strata, and use this to demonstrate that each of these strata are invariant with respect to the V-foliation.

1.1 Structure of the Thesis

We first examine the behavior of the tautological foliations on a Hilbert modular variety $\mathcal{M}$ as they are extended to toroidal compactification of $\mathcal{M}$. In chapter 2, we will review some of the background material relevant to this topic, including results on toric varieties, formal schemes and Hilbert modular varieties. These results are not new, but they provide the foundation of the work in chapters 3 and 4.

Chapter 3 begins by reviewing the theory of foliations and p-foliations, with a particular focus on toric foliations. The work done by Klyachko in [Kly90] regarding the classification of toric vector bundles is extended to the case of positive characteristic. A new criterion for explicitly calculating the singular locus of a toric foliation is proven. Also, a description of the quotient of a toric variety by a toric p-foliation is given.

In chapter 4, we apply the results of chapter 3 to the case of the tautological foliations of a Hilbert modular variety, both when working over $\mathbb{C}$ and when working over a field κ of positive characteristic. We explicitly compute some examples in dimension 2, and describe when a tautological foliation on a Hilbert modular surface is smooth at certain zero-dimensional toric strata.

Secondly, we will examine the interaction of the V-foliation on a unitary Shimura variety with the Ekedahl–Oort stratification. In chapter 5, we will set up our notations, and review some important results of Dieudonné theorey and the theory of displays that will be used later on. Then, in chapter 6, we will look at the Ekedahl–Oort stratification, using the elementary sequences defined by Oort, as in [Oor01] and [Moo01] to parameterized the EO-strata. While more general techniques exist for working with the EO-stratification, (e.g. [VW13], [PWZ15], [Zha13]), we are choosing to use the elementary sequences here as their explicit descriptions of the actions of the Frobenius and Verscheibung operators make them well suited for computations involving the V-foliation, which is defined explicitly in terms of the action of Verschiebung.

In chapter 7, the theory of displays is used to explicitly compute the universal deformations over the individual EO-strata. Finally in chapter 8, we conclude by discussing the V-foliation on a unitary Shimura variety. Using the results of chapter 7, we demonstrate that every strata larger than S_{fol} is invariant with respect to this foliation.

Note that chapters 5–8 are completely independent from chapters 2–4 and can be read separately. As such, there may be some small overlap in the introductory material introducted in these two sections of the thesis, particularly with regards to the basic theory of foliations.

2. PRELIMINARIES

2.1 Algebraic Lemmas

Let R be a (commutative) ring. An R-module M is said to be **free** if there exists some subset $\mathcal{B}$ of M, such that every element of M can be uniquely written in the form $m = r_1b_1 + \cdots + r_nb_n$ for some $b_1, \ldots, b_n \in \mathcal{B}$, and $r_1, \ldots, r_n \in R$. If $\mathcal{B}$ is finite, with cardinality r, then M is called a free R-module of rank r. Such a subset $\mathcal{B}$ is called a basis for M. If $\mathcal{B}' \subset M$ is such that $\mathcal{B}'$ is a basis for some submodule of M, then $\mathcal{B}'$ is said to be **R-linearly independent**.

Lemma 2.1. *Let R be an integral domain, with field of fractions F. If M is an R-module containing an R-linearly-independent subset $\{b_1, \ldots, b_r\}$, then $M \otimes F$ is an F-vector space containing the F-linearly independent set $\{b_i \otimes 1 : 1 \leq i \leq r\}$.*

Proof. Let $x \in M \otimes F$ be such that x is in the span of $\{b_i \otimes 1 : 1 \leq i \leq r\}$. Then, if $x = \sum_{j=1}^{r} r_j(b_j \otimes 1) = \sum_{j=1}^{r} s_j(b_j \otimes 1)$, for some $r_j, s_j \in F$, let $C \in R$ be chosen such that $Cr_j, Cs_j \in R$ for all $1 \leq j \leq r$ and $C \neq 0$. Then $Cx = \sum_{j=1}^{r} Cr_jb_j = \sum_{j=1}^{r} Cs_jb_j$. Then, since $\{b_1, \ldots, b_r\}$ is R-linearly independent, it must be that $Cr_j = Cs_j$ for all $1 \leq j \leq r$. Thus $r_j = s_j$ for all $1 \leq j \leq r$.

Thus, $\{b_i \otimes 1 : 1 \leq i \leq r\}$ is F-linearly independent in $M \otimes F$. $\square$

Let N be a sub-R-module of an R-module M. Then N is said to be **saturated in** M if for every $m \in M$ such that there exists some nonzero $r \in R$ with $rm \in N$, also $m \in N$.

Lemma 2.2. *Let R be an integral domain, and let M be a free R-module of rank r. Let N be a saturated, free sub-R-module of M of rank r. Then $N = M$.*

Proof. Let $\{a_1, \ldots, a_r\}$ be a basis for M and let $\{b_1, \ldots, b_r\}$ be a basis for N. Then there exists $t_{ij} \in R$ such that $b_i = \sum_{j=1}^{r} t_{ij} a_i$. Let F be the field of fractions of R. Then, since $\{b_1, \ldots, b_r\}$ is an R-linearly independent set in N, then the set $\{b_1 \otimes 1, \ldots, b_r \otimes 1\}$ must be F-linearly independent in $N \otimes F$ by Lemma 2.1. Thus, the columns of the matrix $T = [t_{ij}]$ must be F-linearly independent, and is thus and invertible matrix over F. By the adjugate formula for the inverse, we know that:

$$\mathrm{adj}(T)T = \det(T) I_r.$$

However, since each $t_{ij} \in R$, it must be that each component of $\mathrm{adj}(T)$ is also in R. Hence if $\mathrm{adj}(T) = [s_{ij}]$, then $\det(T)a_i = \sum_{j=1}^{r} s_{ij} b_j$. Thus $\det(T)a_i \in N$, for all $1 \leq i \leq r$. But since N is saturated in M, it must be that $a_i \in N$ for all $1 \leq i \leq n$. Thus $M \subseteq N$. Hence $N = M$. $\square$

Lemma 2.3. *Let A be a $g \times g$ invertible matrix over a field k. Let $I, J \subseteq \{1, 2, \ldots, g\}$. Define A_{IJ} as the submatrix of A given by the rows in I and columns in J. Then nullity(A_{IJ}) = nullity$((A^{-1})_{\bar{J}\bar{I}})$, where $\bar{I}$ (resp. $\bar{J}$) refers to the complement of I (resp. J) in $\{1, 2, \ldots, g\}$.*

Proof. Let E_I be the matrix of size $g \times |I|$ with columns e_i, the elementary basis vectors for $i \in I$. Define E_J similarly. Then $A_{IJ} = {}^tE_I A E_J$. Now, since A is invertible, and E_J has full column rank, we see that $\ker(AE_J) = \{0\}$. Thus

$$AE_J(\ker(A_{IJ})) = \ker({}^tE_I) \cap \mathrm{im}(AE_J).$$

Let U_I be the subspace of k^g generated by the elementary basis vectors e_i for $i \in I$, and define U_J similarly. So $\operatorname{im}(E_J) = U_J$, and $\ker({}^tE_I) = U_{\bar{I}}$, where $\bar{I}$ is the complement of I in $\{1, 2, \ldots, g\}$. Thus,

$$\operatorname{nullity}(A_{IJ}) = \dim(\ker(A_{IJ})) = \dim(AE_J(\ker(A_{IJ})) = \dim(\ker({}^tE_I) \cap \operatorname{im}(AE_J))$$

$$= \dim(U_{\bar{I}} \cap A(U_J)).$$

By replacing I with $\bar{J}$, J with $\bar{I}$, and A with A^{-1} the same argument shows that:

$$\operatorname{nullity}((A^{-1})_{\bar{J}\bar{I}}) = \dim(U_J \cap A^{-1}(U_{\bar{I}})).$$

However, since A is invertible, we see that:

$$\dim(U_J \cap A^{-1}(U_{\bar{I}})) = \dim(A(U_J \cap A^{-1}(U_{\bar{I}}))) = \dim(U_{\bar{I}} \cap A(U_J)).$$

Thus $\operatorname{nullity}(A_{IJ}) = \operatorname{nullity}((A^{-1})_{\bar{J}\bar{I}})$. □

We remark that this Lemma includes the cases where I, J or their complements are empty. In such a case, we define the nullity of an $m \times 0$ matrix to be 0, and the nullity of a $0 \times n$ matrix to be n. This convention is in line with viewing and $m \times n$ matrix as a linear map $k^n \to k^m$ with the nullity defined as the dimension of its kernel. Along the same lines, the product of an $m \times 0$ matrix by a $0 \times n$ matrix to defined to be the zero matrix of size $m \times n$.

After a previous version of this thesis was written, we discovered that this result has previously appeared in [Gus84] and independently in [FM86].

Let A be a k-algebra, $\mathfrak{p}$ be a prime ideal of A, and $A_{\mathfrak{p}}$ the localization of A at $\mathfrak{p}$. A k-**derivation** on A is a k-linear map $\delta : A \to A$ such that for any $f, g \in A$, the map δ satisfies the Leibniz property $\delta(fg) = f\delta(g) + g\delta(f)$. Note that this implies $\delta(1^2) = \delta(1) + \delta(1)$, we must have $\delta(1) = 0$, and thus for any $c \in k$, we have $\delta(c) = 0$.

Lemma 2.4. *Let A be a k-algebra, S a multiplicative subset of A, and δ a k-derivation on A. Then δ extends to a unique k-derivation δ_S on $S^{-1}A$ by the formula*

$$\delta_S\left(\frac{f}{g}\right) = \frac{g\delta(f) - f\delta(g)}{g^2}.$$

Proof. We first show that δ_S is well-defined. Suppose $\frac{f_1}{g_1} = \frac{f_2}{g_2}$ in $S^{-1}A$. Then by the definition of localization, there exists some $s \in S$ such that $s(g_2 f_1 - f_2 g_1) = 0$. If we apply δ to this equation, we obtain

$$s(g_2\delta(f_1) + f_1\delta(g_2) - f_2\delta(g_1) - g_1\delta(f_2)) = \delta(s)(f_2 g_1 - g_2 f_1).$$

Now, we will show that $\delta_S(\frac{f_1}{g_1}) = \delta_S(\frac{f_2}{g_2})$. Using the equations above, we get:

$$\begin{aligned}
&s^2(g_2^2 g_1\delta(f_1) - g_2^2 f_1\delta(g_1) - g_1^2 g_2\delta(f_2) + g_1^2 f_2\delta(g_2))\\
&= s^2(g_2^2 g_1\delta(f_1) - g_2 f_2 g_1\delta(g_1) - g_1^2 g_2\delta(f_2) + g_1 g_2 f_1\delta(g_2))\\
&= (sg_1g_2)\cdot s(g_2\delta(f_1) - f_2\delta(g_1) - g_1\delta(f_2) + f_1\delta(g_2))\\
&= sg_1g_2\delta(s)(f_2g_1 - g_2f_1)\\
&= 0.
\end{aligned}$$

Thus

$$\delta_S\left(\frac{f_1}{g_1}\right) = \frac{g_1\delta(f_1) - f_1\delta(g_1)}{g_1^2} = \frac{g_2\delta(f_2) - f_2\delta(g_2)}{g_2^2} = \delta_S\left(\frac{f_2}{g_2}\right).$$

So δ_S is well-defined. From this definition it is clear that δ_S is k-linear, and satisfies the Leibniz property. Thus δ_S is a well-defined derivation on $S^{-1}A$ that extends the derivation δ on A.

Finally, we can show that this is the unique such extension, as if δ_S is any extension of δ to $S^{-1}A$, we must have $\delta_S(f) = \delta_S(g \cdot \frac{f}{g})$, so

$$\delta_S(f) = \frac{f}{g}\delta_S(g) + g\delta_S\left(\frac{f}{g}\right).$$

By rearranging we have

$$\delta_S\left(\frac{f}{g}\right) = \frac{1}{g}\left(\delta_S(f) - \frac{f}{g}\delta_S(g)\right) = \frac{g\delta_S(f) - f\delta_S(g)}{g^2}.$$

□

The n^{th} **symbolic power** of $\mathfrak{p}$, denoted $\mathfrak{p}^{(n)}$ is defined as $\mathfrak{p}^n A_{\mathfrak{p}} \cap A$, as in [Mat80]. It is clear that $\mathfrak{p}^n \subseteq \mathfrak{p}^{(n)}$, however equality may not always hold. For example, let A be the ring $k[X,Y,Z]/\langle YZ - X^2\rangle$, and coosider the ideal $\mathfrak{p} = \langle X, Y\rangle$. We see that $\mathfrak{p}^{(2)} \neq \mathfrak{p}^2$ as follows. Note that $Z \notin \mathfrak{p}$, so $Z^{-1} \in A_{\mathfrak{p}}$. So $Y = Z^{-1}X^2 \in \mathfrak{p}^{(2)}$, since $Z^{-1}X \in \mathfrak{p}A_{\mathfrak{p}}$, but $Y \notin \mathfrak{p}^2$.

Proposition 2.5. *Let A be a k-algebra, and let $\mathfrak{p}$ be a prime ideal of A. Let δ be a k-derivation on A such that $\delta(\mathfrak{p}) \subseteq \mathfrak{p}$. Then for any $n \in \mathbb{N}$,we have $\delta(\mathfrak{p}^{(n)}) \subseteq \mathfrak{p}^{(n)}$.*

Proof. First, we can show that $\delta(\mathfrak{p}^n) \subseteq \mathfrak{p}^n$. We proceed by induction. It is given that $\delta(\mathfrak{p}) \subseteq \mathfrak{p}$. So, suppose that $\delta(\mathfrak{p}^{(n-1)}) \subseteq \mathfrak{p}^{(n-1)}$. Then, if $f \in \mathfrak{p}^n$, there exists such $r \in \mathbb{N}$ and elements $f_i \in \mathfrak{p}^{n-1}$ and $g_i \in \mathfrak{p}$ for $1 \leq i \leq r$ such that $f = \sum_{i=1}^r f_i g_i$. Thus:

$$\delta(f) = \sum_{i=1}^r \delta(f_i g_i) = \sum_{i=1}^r \left(f_i\delta(g_i) + g_i\delta(f_i)\right).$$

Since each $g_i \in \mathfrak{p}, \delta(g_i) \in \mathfrak{p}$, Thus $f_i\delta(g_i) \in \mathfrak{p}^n$. Similarly, since each $f_i \in \mathfrak{p}^{n-1}$ we have $\delta(f_i) \in \mathfrak{p}^{n-1}$ so $g_i\delta(f_i) \in \mathfrak{p}^n$. Thus $\delta(f) \in \mathfrak{p}^n$ as required.

It remains to show that $\delta(\mathfrak{p}^n A_{\mathfrak{p}}) \subseteq \mathfrak{p}^n A_{\mathfrak{p}}$. So suppose that $f \in \mathfrak{p}^n$, and $g \notin \mathfrak{p}$. Then, by Lemma 2.4:

$$\delta\left(\frac{f}{g}\right) = \frac{g\delta(f) - f\delta(g)}{g^2}.$$

Since $\delta(f) \in \mathfrak{p}^n$ and $f \in \mathfrak{p}^n$ it must be that $\frac{g\delta(f) - f\delta(g)}{g^2} \in \mathfrak{p}^n A_{\mathfrak{p}}$.

Thus, $\delta(\mathfrak{p}^{(n)}) \subseteq \mathfrak{p}^{(n)}$. □

2.2 Toric Varieties

Let k be an algebraically closed field. A **toric variety** over k is defined to be a normal variety X/k, along with a dense open subset T, such that T is an algebraic torus, and there is an action of T on X that extends the natural action of T on itself. We will review the basic construction and notations that will be used in the sequel. See [Ful93, Oda88] for more details. Many of these results can be extended to define toric schemes over any commutative ring. For more details, see [KKMSD73].

2.2.1 Strongly convex rational polyhedral cones

Let N be a lattice, isomorphic to $\mathbb{Z}^r$, and let M be the dual lattice, $M := \mathrm{Hom}(N, \mathbb{Z})$. Denote the natural pairing $\langle \cdot, \cdot \rangle : M \times N \to \mathbb{Z}$. A subset $\sigma \subset N_{\mathbb{R}} := N \otimes \mathbb{R}$ is called a **strongly convex rational polyhedral cone** if there exists a subset $\{n_1, \ldots, n_s\} \subset N$ such that

$$\sigma = \mathbb{R}_{\geq 0} n_1 + \cdots + \mathbb{R}_{\geq 0} n_s$$

and $\sigma \cap (-\sigma) = \{0\}$. We will abbreviate this by calling σ a **scrp** cone. If σ is generated by a single element of N, then it is called a **ray**.

Let σ be an scrp cone in $N_{\mathbb{R}}$. Then the **dual cone**, denoted $\sigma^\vee \subseteq M_{\mathbb{R}}$ is the set of vectors:

$$\sigma^\vee = \{u \in M_{\mathbb{R}} : \langle u, v \rangle \geq 0, v \in \sigma\}.$$

A **face** of an scrp cone σ is a subset $\tau \subseteq \sigma$ such that $\tau = \sigma \cap \{m_0\}^\perp$ for some $m_0 \in \sigma^\vee$. Note that this implies τ is itself an scrp cone [Oda88, Prop 1.3]. We define the set $\sigma(1)$ to be the set of faces τ of σ such that τ is a ray.

Lemma 2.6. *Let σ be an scrp cone in $\mathbb{R}^r$. Then $\{0\}$ is a face of σ.*

Proof. Choose some $m_1 \in \sigma^\vee$. Consider the hyperplane $\rho_1 = \{m_1\}^\perp$. If $\rho_1 \cap \sigma = \{0\}$, we are done, since $\{0\} = \sigma \cap \{m_1\}^\vee$. Otherwise, choose some nonzero $n_2 \in \sigma \cap \rho_1$. If

$\langle m, n_2 \rangle = 0$ for all $m \in \sigma^\vee$, then $\langle m, -n_2 \rangle = 0$ for all $m \in \sigma^\vee$, so $-n_2 \in \sigma$. But this cannot be, since σ is strongly convex. That is, $\sigma \cap (-\sigma) = \{0\}$. So, choose $m_2 \in \sigma^\vee$ such that $\langle m_2, n_2 \rangle > 0$. Then consider the hyperplane $\rho_2 = \{m_2\}^\perp$. If $\sigma \cap \rho_1 \cap \rho_2 = \{0\}$, set $m = m_1 + m_2$. Otherwise, continue this process until $\sigma \cap \rho_1 \cap \rho_2 \cap \cdots \cap \rho_s = \{0\}$. Note that this process will take at most r steps, since each ρ_j is chosen such that $\bigcap_{i=1}^{j-1} \rho_i \not\subset \rho_j$. So $\bigcap_{i=1}^{j} \rho_i$ has codimension j in $\mathbb{R}^r$. Then, we set $m = m_1 + \cdots + m_s$.

Since $\sigma \cap \rho_1 \cap \cdots \cap \rho_s = \{0\}$, we see that for any nonzero $n \in \sigma$, we must have some i such that $\langle n, m_i \rangle > 0$. Also, since each m_i was chosen in $\sigma^\vee$, we have $\langle n, m_i \rangle \geq 0$ for all $1 \leq i \leq s$. Thus $\langle n, m \rangle > 0$ for all nonzero $n \in \sigma$, giving $\sigma \cap \{m\}^\perp = \{0\}$. So $\{0\}$ is a face of σ. $\square$

Lemma 2.7. *Let σ be an scrp cone generated by $\{n_1, \ldots, n_s\}$, where this set of generators is minimal. Then $\sigma(1) = \{\tau_i\}_{i=1}^s$ where $\tau_i = \mathbb{R}_{\geq 0} n_i$.*

Proof. First we will show that τ_i is a face of σ. Without loss of generality, suppose $i = 1$. Since $\{n_1, \ldots, n_s\}$ is a minimal set of generators for σ, the cone $\sigma' = \mathbb{R}_{\geq 0} n_2 + \cdots + \mathbb{R}_{\geq 0} n_s$ is a proper subset of σ. Thus $\sigma^\vee \subsetneq (\sigma')^\vee$. So let $m_0 \in (\sigma')^\vee$ but $m_0 \notin \sigma^\vee$. Thus $-\alpha := \langle m_0, n_1 \rangle < 0$, but $\langle m_0, n_i \rangle \geq 0$ for $2 \leq i \leq s$.

Now, by the previous lemma, we know that there exists some $m \in M$ such that $\sigma \cap \{m\}^\perp = \{0\}$. So $\langle m, n_i \rangle > 0$ for $1 \leq i \leq s$. Let $\beta = \langle m, n_1 \rangle$. Note that:

$$\langle \alpha m + \beta m_0, n_1 \rangle = \alpha\beta - \beta\alpha = 0.$$

$$\langle \alpha m + \beta m_0, n_i \rangle = \alpha \langle m, n_i \rangle + \beta \langle m_0, n_i \rangle > 0, \qquad 2 \leq i \leq s.$$

Thus for any $n \in \sigma$, we have $\langle \alpha m + \beta m_0, n \rangle = 0$ if and only if $n = \mathbb{R}_{\geq 0} n_1 = \tau_1$. Hence $\tau_1 = \sigma \cap \{\alpha m + \beta m_0\}^\perp$. Thus the ray τ_1 is a face of σ, so $\tau_1 \in \sigma(1)$. Since this argument works for all τ_i, we have $\{\tau_i\}_{i=1}^s \subseteq \sigma(1)$.

Conversely, if $\mathbb{R}_{\geq 0} n \in \sigma(1)$, there exists some $m_0 \in \sigma^\vee$ such that $\langle m_0, n' \rangle = 0$ if and only if n' is a multiple of n. Since σ is generated by $\{n_1, \ldots, n_s\}$, write $n = \sum_{i=1}^s c_i n_i$

for $c_i \geq 0$. Then

$$\langle m_0, n\rangle = \sum_{i=1}^{s} c_i \langle m_0, n_i\rangle.$$

Since $m_0 \in \sigma^\vee$, we know that $\langle m_0, n_i\rangle \geq 0$. So for each $1 \leq i \leq s$, we must have either $c_i = 0$ or $\langle m_0, n_i\rangle = 0$, that is n_i is a multiple of n. Since $n \neq 0$, it must be that some n_i is a multiple of n, so $\mathbb{R}_{\geq 0}n = \mathbb{R}_{\geq 0}n_i = \tau_i$.

Thus $\sigma(1) = \{\tau_i\}_{i=1}^{s}$. □

2.2.2 Toric varieties and fans

Given a cone $\sigma \in N_\mathbb{R}$, let S_σ denote the semigroup $\sigma^\vee \cap M$.

Proposition 2.8. *S_σ is a finitely generated additive subsemigroup of M that generates M as a group.*

Proof. See [Oda88, Prop 1.1]. □

Define $U_\sigma := \mathrm{Spec}(k[S_\sigma])$. Then U_σ is an affine toric variety. The inclusion of the open dense torus in U_σ is induced by the natural map $k[S_\sigma] \to k[M]$. Indeed, every affine toric variety X is of the form U_σ for some scrp cone σ in N, where N is the co-character lattice of the torus T in X. Given an affine toric variety X, we can define S as the semigroup of characters on T that extend to regular functions on X. Then $X = \mathrm{Spec}(k[S])$, and $S^\vee$ is an scrp cone in N.

A **fan** Σ in $N_\mathbb{R}$ is defined as a finite collection of scrp cones, such that if $\sigma \in \Sigma$, then every face of σ is in Σ, and if $\sigma, \tau \in \Sigma$, then $\sigma \cap \tau$ is a face of σ and τ. Note that if τ is a face of σ, then $U_\tau \subseteq U_\sigma$, since $M \cap \sigma^\vee \subseteq M \cap \tau^\vee$. Then we can construct the toric variety $X(\Sigma)$ by taking U_σ for each $\sigma \in \Sigma$, and gluing U_σ and U_τ along $U_{\sigma\cap\tau}$. Indeed, given a toric variety X with torus T, there exists a unique choice of fan Σ in N such that X is equivariantly isomorphic to $X(\Sigma)$. [Oda88, Theorem 1.5].

Given a toric variety $X(\Sigma)$, then we call the lattices M and N the character and co-character lattices of X respectively. Indeed, the elements of M are precisely the characters of T, and for any $\sigma \in \Sigma$, the elements of S_σ are precisely the characters of T

that extend as regular functions to U_σ. In order to clarify notations, for an element $m \in M$, we will denote the regular function on U_σ by χ^m. This reflects the fact that the character corresponding to the sum $m_1 + m_2$ in the lattice M is the product of the characters $\chi^{m_1}\chi^{m_2}$ as functions.

Similarly, the elements of N correspond precisely to the one-parameter subgroups of T. For $n \in N$, we can define the one-parameter subgroup $\gamma_n : \mathbb{G}_m \to T$ such that $m(\gamma_n(\lambda)) = \lambda^{\langle m,n\rangle}$ for any $m \in M$ and $\lambda \in k$.

Example: Consider the projective plane $\mathbb{P}^2_k$, along with the embedding $T \to \mathbb{P}^2_k$ of a rank 2 split torus given by $(t_1, t_2) \mapsto [t_1 : t_2 : 1]$. Then the action of T on $\mathbb{P}^2_k$ that extends the translation action of T on itself is the action

$$(t_1, t_2) \cdot [x : y : z] = [t_1 x : t_2 y : z].$$

The characters of T are precisely the maps of the form $(t_1, t_2) \mapsto t_1^a t_2^b$ for $a, b \in \mathbb{Z}^2$. We will denote such a character by the point $(a, b) \in \mathbb{Z}^2$. The character (a, b) thus extends to the rational function $[x : y : z] \mapsto \frac{x^a y^b}{z^{a+b}}$ on $\mathbb{P}^2_k$. To determine the fan associated with this toric variety, we will consider which of these characters extend as regular functions to the different T-invariant affine charts of $\mathbb{P}^2_k$.

First, consider the affine open U_1 given by $z \neq 0$. Then, the character (a, b) acts on this open by $[x : y : 1] \mapsto x^a y^b$. This is a regular function if and only if $a \geq 0$ and $b \geq 0$, that is if (a, b) is in the semigroup S_1 generated by $(1, 0)$ and $(0, 1)$. Let N, the dual of the character lattice M, also be described as $\mathbb{Z}^2$ with the standard pairing. Then the dual cone σ_1 to S_1 is generated by $(1, 0)$ and $(0, 1)$ in N.

Next, consider the affine open U_2 given by $y \neq 0$. So the character given as the pair (a, b) acts by $[x : 1 : z] \mapsto \frac{x^a}{z^{a+b}}$. This is a regular function if and only if $a \geq 0$ and $b \leq -a$. That is, if (a, b) is in the semigroup S_2 generated by $(1, -1)$ and $(0, -1)$. The dual cone σ_2 to S_2 is generated by $(1, 0)$ and $(-1, -1)$ in N.

Finally, consider the affine open U_3 given by $x \neq 0$. Then the character given as the pair (a, b) acts by $[1 : y : z] \mapsto \frac{y^b}{z^{a+b}}$. This is a regular function if and only if $b \geq 0$

and $a \leq -b$. That is, if (a, b) is in the semigroup S_3 generated by $(-1, 1)$ and $(-1, 0)$. The dual cone σ_3 to S_3 is generated by $(0, 1)$ and $(-1, -1)$ in N.

So $\mathbb{P}^2_k$ is equivariantly isomorphic to the toric variety $X[\Sigma]$ where Σ is the fan containing $\sigma_1, \sigma_2, \sigma_3$ and their faces.

Proposition 2.9. *Let T be an algebraic torus with character lattice M and co-character lattice N. Let Σ and Σ' be fans in N such that there exists a (not-necessarily equivariant) isomorphism $\varphi : X(\Sigma) \to X(\Sigma')$ such that $\varphi(T) = T$. Then there exists some $g \in \mathrm{GL}(N)$ such that $g\Sigma = \Sigma'$. Conversely, given any $g \in \mathrm{GL}(N)$, there exists an isomorphism $\varphi : X(\Sigma) \to X(g\Sigma)$ such that $\varphi(T) = T$ and $\varphi|_T = g$ as an element of $Aut(T)$.*

Proof. Recall that $\mathrm{Aut}(T) = \mathrm{GL}(N)$. Since $\varphi(T) = T$, let $g = \varphi|_T \in \mathrm{GL}(N)$. Since $M = N^\vee$, for any $g \in \mathrm{GL}(N)$, it has an unique adjoint $g^T \in \mathrm{GL}(M)$ such that $(g^T\chi)(t) = \chi(g \cdot t)$.

Let $\chi \in M$ be a character of T. Then $\chi(\varphi(t)) = \chi(g \cdot t) = (g^T\chi)(t)$. Thus, since $\chi(\varphi(t)) = (g^T\chi)(t)$ for any $t \in T$, we must have $\chi(\varphi(x)) = (g^T\chi)(x)$ for any $x \in X(\Sigma)$.

Let σ be a cone in Σ, and let $U_\sigma \in X$ be the corresponding affine open chart. Since $\varphi(U_\sigma)$ is also a T-invariant affine open of $X(\Sigma)$, let σ' be the corresponding cone in Σ'. Then, $\chi \in M$ extends as a regular function to U_σ if and only if $\chi \in M \cap \sigma^\vee$. Further, since $\chi(\varphi(x)) = (g^T\chi)(x)$, we see that χ extends as a regular function to U'_σ if and only if $g^T\chi$ extends to U_σ. So $\chi \in S_{\sigma'}$ if and only if $g^T\chi \in S_\sigma$. Thus:

$$\begin{aligned}
\sigma &= \{n \in N : \langle \chi, n \rangle \geq 0, \forall \chi \in S_\sigma\} \\
&= \{n \in N : \langle g^T\chi, n \rangle \geq 0, \forall \chi \in S_{\sigma'}\} \\
&= \{n \in N : \langle \chi, gn \rangle \geq 0, \forall \chi \in S_{\sigma'}\} \\
&= \{n \in N : gn \in \sigma'.\}
\end{aligned}$$

Thus $g\sigma = \sigma'$. As this holds for every $\sigma \in \Sigma$, we must have $g\Sigma = \Sigma'$. Conversely, given any cone σ, and $g \in \mathrm{GL}(N)$, we can define the cone

$$\sigma' = \{gn : n \in \sigma\}.$$

Then, by the same calculation as above, we see that $\chi \in S_{\sigma'}$ if and only if $g^T\chi \in S_\sigma$. The map $\chi \mapsto g^T\chi$ induces an isomorphism $k[S_{\sigma'}] \to k[S_\sigma]$, which after taking Spec of both sides, induces an isomorphism $g : U_\sigma \to U_{\sigma'}$ extending the automorphism $g : T \to T$.

Therefore by the functoriality of this construction, if $g\Sigma = \Sigma'$, the automorphism $g \in \mathrm{GL}(N)$ induces an isomorphism $X(\Sigma) \to X(\Sigma')$. $\square$

Example: Let $X_1 = \mathbb{P}^2_k$ be the toric variety with the torus action given as in the previous example. Now define $X_2 = \mathbb{P}^2_k$, but with the torus $T \hookrightarrow \mathbb{P}^2_k$ embedded as $(t_1, t_2) \mapsto (t_1 : t_1t_2 : 1)$. Then let $\varphi : X_1 \to X_2$ be the identity map on $\mathbb{P}^2_k$. Note that for both X_1 and X_2 the image of T is the open subset of $\mathbb{P}^2_k$ given by $x, y, z \neq 0$. However, since the torus action is different on X_1 and X_2 this is not an equivariant isomorphism. Indeed, if we restrict φ to the image of T in X_1, we map the image of (t_1, t_2) to the image of $(t_1, t_1^{-1}t_2)$ in X_2. Thus, g is the automorphism that maps (t_1, t_2) to $(t_1, t_1^{-1}t_2)$, and is given by the matrix $\begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$.

We can compute the fan corresponding to X_2 just as we did for X_1. Note that if (a, b) is the character on T that maps $(t_1, t_2) \to t_1^a t_2^b$, then (a, b) acts as a rational function on X_2 via $(a, b) \cdot [x : y : z] \mapsto \frac{x^{a-b}y^b}{z^a}$.

Consider first the open affine chart U_1 given by $z \neq 0$. Then (a, b) acts on U_1 via $(a, b) \cdot [x : y : 1] \mapsto x^{a-b}y^b$. So this is a regular function if and only if $a \geq b$ and $b \geq 0$. This describes the semigroup $S_1' \subseteq M$ generated by $(1, 0), (1, 1)$. This is dual to the cone σ_1' generated by $(0, 1)$ and $(1, -1)$ in N.

Next consider the open affine chart U_2 given by $y \neq 0$. Then (a, b) acts on U_2 via $(a, b) \cdot [x : 1 : z] \mapsto \frac{x^{a-b}}{z^a}$. So this is a regular function if and only if $a \leq 0$ and $b \leq a$. This describes the semigroup $S_2' \subseteq M$ generated by $(0, -1), (-1, -1)$. This is dual to the cone σ_2' generated by $(-1,0)$ and $(1,-1)$ in N.

Finally consider the open affine chart U_3 given by $x \neq 0$. Then (a,b) acts on U_3 via $(a,b)\cdot[1:y:z] \mapsto \frac{y^b}{z^a}$. So this is a regular function if and only if $a \leq 0$ and $b \geq 0$. This describes the semigroup $S_3' \subseteq M$ generated by $(-1,0)$ and $(0,1)$. This is dual to the cone σ_3' generated by $(-1,0)$ and $(0,1)$ in N.

Thus X_2 is the toric variety given by the fan Σ' consisting of the cones σ_1, σ_2 and σ_3. Note that

$$\begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} -1 \\ -1 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \end{pmatrix}.$$

So if we consider the fan Σ with cones $\sigma_1', \sigma_2', \sigma_3'$ as in the previous example, we see that

$$g(\sigma_1) = \sigma_1', \qquad g(\sigma_2) = \sigma_2', \qquad g(\sigma_3) = \sigma_3'.$$

Thus $g\Sigma = \Sigma'$, as described in the proposition above.

Corollary 2.10. *Let U be a subgroup of Aut(T), and let Σ be a fan in the cocharacter lattice N such that $u\Sigma = \Sigma$ for all $u \in U$. Then U is a subgroup of Aut$(X(\Sigma))$.*

Proof. Let $u \in U$. Then by Proposition 2.9, the automorphism u induces an isomorphism $X(\Sigma) \to X(u\Sigma)$. But $\Sigma = u\Sigma$. Thus u induces an automorphism $X(\Sigma) \to X(\Sigma)$. So $U \subseteq \mathrm{Aut}(X(\Sigma))$. □

Proposition 2.11. *The toric variety $X(\Sigma)$ is nonsingular if and only if each cone σ is non-singular, in the sense that for each σ, there exists a $\mathbb{Z}$-basis $\{n_1, \ldots, n_r\}$ of N such that for some $s \leq r$, the cone $\sigma = \mathbb{R}_{\geq 0}n_1 + \cdots + \mathbb{R}_{\geq 0}n_s$. Such a fan Σ is also called non-singular.*

Proof. See [Oda88, Theorem 1.10]. □

The **support** of a fan Σ, denoted Supp(Σ) is defined as the subset:

$$\mathrm{Supp}(\Sigma) = \bigcup_{\sigma \in \Sigma} \sigma \subseteq N_{\mathbb{R}}$$

Proposition 2.12. *The toric variety $X(\Sigma)$ is compact if and only if Σ is a complete fan, i.e., Supp$(\Sigma) = N$.*

Proof. See [Oda88, Theorem 1.11]. □

Lemma 2.13. *Let Σ be a non-singular fan in $\mathbb{Z}^2$ such that $(1,0) \in$ Supp(Σ), but $(0,\pm 1) \notin$ Supp(Σ). Then the ray generated by $(1,0)$ is in Σ.*

Proof. Suppose $(1,0)$ does not generate a ray in Σ. Then, since $(1,0) \in \text{Supp}(\Sigma)$, the point $(1,0)$ must be in some $\sigma \in \Sigma$. Let (a,c) and (b,d) be the generators of σ. Since $(1,0) \in \sigma$, we must have either $c < 0, d > 0$ or $c > 0, d < 0$. Without loss of generality, suppose the generators are ordered such that $c < 0 < d$. Further, since $(0,\pm 1) \notin \text{Supp}(\Sigma)$, it must be that $(0,\pm 1) \notin \sigma$. Thus $a, b > 0$.

But then, $ad - bc \geq 1 + 1 = 2$. However, since Σ is a non-singular fan, σ must be generated by a basis of $\mathbb{Z}^2$. So $ad - bc = 1$. This cannot be, therefore for any non-singular fan Σ containing $(1,0)$ in its support, but not $(0,\pm 1)$, the ray generated by $(1,0)$ must be in Σ. □

Let $X(\Sigma)$ be a toric variety. As part of its defining data, $X(\Sigma)$ is equipped with an action by the torus T. We can classify the T-orbits on $X(\Sigma)$ as follows:

Proposition 2.14. *The orbits of the T-action on $X(\Sigma)$ are in one-to-one correspondence with the cones $\sigma \in \Sigma$ as follows. For $\sigma \in \Sigma$, define*

$$\sigma^{\perp} = \{u \in M_{\mathbb{R}} : \langle u, v\rangle = 0, v \in \sigma\}.$$

Then the closed subvariety $Z_\sigma :=$ Spec$(k[M \cap \sigma^{\perp}])$ of U_σ is a T-orbit in X. All T-orbits are on this form. Further:

(i) If $\sigma = 0$ is the zero cone, then $Z_0 =$ Spec$(k[M]) = T$ is the embedding of T into $X(\Sigma)$.

(ii) If σ has codimension r in N, then Z_σ has dimension r.

(iii) Z_σ *is the unique closed* T*-orbit in* U_σ*, and we have* $U_\sigma = \bigsqcup_{\tau < \sigma} Z_\tau$.

Proof. See [Oda88, Proposition 1.6]. □

Note that while Z_σ is closed in U_σ, it is not necessarily closed in $X(\Sigma)$. Indeed, the closure of Z_σ in $X(\Sigma)$ is the union of all orbits Z'_σ for $\sigma' \in \Sigma$ such that $\sigma \leq \sigma'$.

Note that the requirement that the fan Σ is finite is only required to ensure that $X(\Sigma)$ is an algebraic variety. We will need to also consider the case when Σ is an infinite collection of cones. In this case, $X(\Sigma)$ can still be constructed as a normal scheme, separated and locally of finite type [Oda88]. $X(\Sigma)$ is still covered by affine toric varieties, however this cover may be infinite. All of the above results, with the exception of Proposition 2.12, still hold true as they were proven locally at the level of affine toric varieties.

2.2.3 *Ideals of toric varieties*

Let T be a split torus over a field k of arbitrary characteristic with character lattice M and co-character lattice N. Recall that elements $m \in M$ are notated as functions $\chi^m \in k[M]$.

Let τ be a ray in N generated by some $n_\tau \in N$. Then we can define a $\mathbb{Z}$-gradation on $k[M]$:

$$k[M] = \bigoplus_{\alpha \in \mathbb{Z}} k[S_{\tau,\alpha}].$$

where $S_{\tau,\alpha}$ is the k-module generated by $\{m \in M : \langle m, n_\tau \rangle = \alpha\}$. Note that for any monomial $m_1 \in S_{\tau,\alpha_1}$ and $m_2 \in S_{\tau,\alpha_2}$, we have $\chi^{m_1}\chi^{m_2} = \chi^{m_1+m_2}$, and

$$\langle m_1 + m_2, n_\tau \rangle = \langle m_1, n_\tau \rangle + \langle m_2, n_\tau \rangle = \alpha_1 + \alpha_2.$$

Thus $k[S_{\tau,\alpha_1}] \cdot k[S_{\tau,\alpha_2}] = k[S_{\tau,\alpha_1+\alpha_2}]$, so this is indeed a $\mathbb{Z}$-gradation.

Now, define a valuation ν_τ on $k[M]$ by

$$\nu_\tau(f) = \max\{r : f \in \bigoplus_{\alpha \geq r} k[S_{\tau,\alpha}]\}.$$

That is, $\nu_\tau(f)$ is the minimum value of $\langle m, n_\tau \rangle$ for monomials χ^m in f.

Let $f, g \in k[M]$ so $f = f_{\alpha_1} + \cdots + f_{\alpha_r}$ and $g = g_{\beta_1} + \cdots + g_{\beta_s}$, where f_{α_i} (resp. g_{β_j}) is the α_i (resp. β_j) part of f (resp. g) with respect to the gradation given above. In particular $f_{\alpha_i}, g_{\beta_j} \neq 0$ Further suppose these are sorted such that $\alpha_1 < \alpha_2 < \cdots < \alpha_r$ and $\beta_1 < \beta_2 < \cdots < \beta_s$. Then $\nu_\tau(f) = \alpha_1$ and $\nu_\tau(g) = \beta_1$. Let $\gamma = \min\{\alpha_1, \beta_1\}$ then,

$$f + g \in \bigoplus_{\alpha \geq \gamma} k[S_{\tau,\alpha}].$$

Thus $\nu_\tau(f + g) \geq \min\{\nu_\tau(f), \nu_\tau(g)\}$. Also, for any $1 \leq i \leq r$ and $1 \leq j \leq s$, we have $\alpha_i + \beta_j \geq \alpha_1 + \beta_1$ with equality only if $i = j = 1$. Since $f_{\alpha_1} g_{\beta_1} \neq 0$, as $k[M]$ is an integral domain, we see that

$$fg \in \bigoplus_{\alpha \geq \alpha_1 + \beta_1} k[S_{\tau,\alpha}].$$

Thus, this is indeed a valuation.

Recall the semigroup $S_\tau \subset M$ defined above as $S_\tau = \{m \in M : \langle m, n_\tau \rangle \geq 0\}$. From this definition it is clear that $k[S_\tau] = \bigoplus_{\alpha \geq 0} k[S_{\tau,\alpha}]$. Let $I_\tau := \bigoplus_{\alpha \geq 1} k[S_{\tau,\alpha}]$. Then I_τ is a prime ideal of $k[S_\tau]$.

Now, let σ be an scrp cone in N generated by $\{n_1, \ldots, n_r\}$. Then $S_\sigma = \{m \in M : \langle m, n \rangle \geq 0, n \in \sigma\}$. Note that $\langle m, n \rangle \geq 0$ for all $n \in \sigma$ if and only if $\langle m, n_i \rangle \geq 0$ for all $1 \leq i \leq r$. Thus, for any $1 \leq i \leq r$, if τ_i is the ray generated by n_i, we have:

$$k[S_\sigma] \subseteq k[S_{\tau_i}].$$

Let I_σ be the ideal of $k[S_\sigma]$ generated by $\{m \in S_\sigma : \langle m, n \rangle \geq 1, n \in \sigma\}$. Then I_σ is the ideal corresponding to the complement of the torus T in U_σ. Let $\tau \in \sigma(1)$ be a ray of σ, and let $n_\tau \in N$ be the generator of τ. Then define $I_{\sigma,\tau}$ to be the ideal of $k[S_\sigma]$ generated by $\{m \in S_\sigma : \langle m, n_\tau \rangle \geq 1\}$. So $I_{\sigma,\tau} = I_\tau \cap k[S_\sigma]$ and is thus a prime ideal of $k[S_\sigma]$.

Lemma 2.15. *The minimal prime decomposition of I_σ is:*

$$I_\sigma = \bigcap_{\tau \in \sigma(1)} I_{\sigma,\tau}.$$

Proof. Suppose $f \in I_\sigma$. Then $f = c_1\chi^{m_1} + \ldots c_r\chi^{m_r}$ such that$\langle m_i, n \rangle \geq 1$, for each $1 \leq i \leq r$. In particular, for each generator n_τ for $\tau \in \sigma(1)$, we have $\langle m_i, n_\tau \rangle \geq 1$. Thus $f \in \bigcap_{\tau \in \sigma(1)} I_{\sigma,\tau}$.

On the other hand, if $f \in \bigcap_{\tau\in\sigma(1)} I_{\sigma,\tau}$, then for each monomial χ^{m_i} in f, it must be that $\langle m_i, n_\tau \rangle \geq 1$ for each $\tau \in \sigma(1)$. But if $n \in \sigma$, then n is a positive linear combination of the generators n_τ for the rays $\tau \in \sigma(1)$. Thus $\langle m_i, n \rangle > 0$.

Therefore, since $\langle m_i, n \rangle \in \mathbb{Z}$, we must have $\langle m_i, n \rangle \geq 1$, so $\chi^{m_i} \in I_\sigma$. Thus $f \in I_\sigma$. So

$$I_\sigma = \bigcap_{\tau \in \sigma(1)} I_{\sigma,\tau}.$$

Since each $I_{\sigma,\tau}$ is prime, this is a prime decomposition.

By Lemma 2.7, each τ is a face of σ, for there exists some $m_\tau \in \sigma^\vee$ such that $\langle m_\tau, n_\tau \rangle = 0$ but $\langle m_\tau, n_{\tau'} \rangle > 0$ for any $\tau' \neq \tau$. So $m_\tau \in \bigcap_{\tau' \in \sigma(1), \tau' \neq \tau} I_{\sigma,\tau'}$ but not in I_σ. This thus is a minimal prime decomposition. □

Lemma 2.16. *The n^{th} symbolic power of $I_{\sigma,\tau}$ can be decomposed as:*

$$I^{(n)}_{\sigma,\tau} = \bigoplus_{\alpha \geq n} k[S_{\tau,\alpha}] \cap k[S_\sigma].$$

That is, $I^{(n)}_{\sigma,\tau}$ is generated by $\{m \in S_\sigma : \langle m, n_\tau \rangle \geq n\}$.

Proof. Suppose $f \in I_{\sigma,\tau}^{(n)}$. So, by the definition of symbolic powers, we have $f \in I_{\sigma,\tau}^n k[S_\sigma]_{I_{\sigma,\tau}} \cap k[S_\sigma]$. Thus, there exist $g_1, \ldots, g_n, h_1, h_2 \in k[S_\sigma]$ such that $g_i \in I_{\sigma,\tau}$, and $h_2 \notin I_{\sigma,\tau}$ such that:

$$h_2 f = g_1 g_2 \ldots g_n h_1.$$

Since $h_2 \in S_\sigma$ but not in $I_{\sigma,\tau}$, we have $\nu_\tau(h_2) = 0$. Thus $\nu_\tau(h_2 f) = \nu_\tau(f)$. Also, as $h_1 \in S_\sigma$, we have $\nu_\tau(h_1) \geq 0$. Further, since each $g_i \in I_{\sigma,\tau}, \nu_\tau(g_i) \geq 1$. Thus:

$$\nu_\tau(f) = \sum_{i=1}^{n} \nu_\tau(g_i) + \nu_\tau(h_1) \geq n.$$

Thus $f \in \bigoplus_{\alpha \geq n} k[S_{\tau,\alpha}]$. So

$$I_{\sigma,\tau}^{(n)} \subseteq \bigoplus_{\alpha \geq n} k[S_{\tau,\alpha}] \cap k[S_\sigma].$$

On the other hand, suppose $f \in \bigoplus_{\alpha \geq n} k[S_\tau, \alpha] \cap k[S_\sigma]$. Let $\{n_\tau, n_2, \ldots n_r\}$ be the generators of σ. Note that n_τ is one of the generators as $\tau \in \sigma(1)$. Let $\tau_2, \ldots, \tau_r$ be the rays corresponding to the generators $n_2, \ldots n_r$ respectively. Let $m, m' \in S_\sigma$ be chosen such that $\langle m, n_\tau \rangle = 1$ and $\langle m', n_\tau \rangle = 0$ but $\langle m', n_i \rangle > 0$ for $2 \leq i \leq r$. So $m \in I_{\sigma,\tau}$, but $m' \notin I_{\sigma,\tau}$. Note that m' exists since τ is a face, by Lemma 2.7, thus $\tau = \sigma \cap \{m'\}^\perp$ for such an element $m' \in S_\sigma$.

Let $\beta \in \mathbb{Z}$ be chosen such that $\beta \geq \max\limits_{2 \leq i \leq r} \{n \cdot \nu_{\tau_i}(m) - \nu_{\tau_i}(f)\}$. Since χ^m is a monomial, it is invertible in $k[M]$. Thus the quotient $\frac{f}{m^n} \in k[M]$. Further:

$$\nu_\tau(\chi^{\beta m'} \frac{f}{\chi^{nm}}) = \beta \cdot \nu_\tau(m') + \nu_\tau(f) - n \cdot \nu_\tau(m) \geq 0$$

as f was chosen such that $\nu_\tau(f) \geq n$, and $\nu_\tau(m) = 1, \nu_\tau(m') = 0$. Also, for $2 \leq i \leq r$:

$$\nu_{\tau_i}(\chi^{\beta m'} \frac{f}{\chi^{nm}}) = \beta \cdot \nu_{\tau_i}(m') - (\nu_{\tau_i}(f) - n \cdot \nu_{\tau_i}(m)) \geq 0$$

by choice of β. Thus $(\chi^{\beta m'}\frac{f}{\chi^{nm}}) \in k[S_\sigma]$. Therefore as

$$(\chi^{\beta m'} f = (\chi^{nm})(\chi^{\beta m'}\frac{f}{m^n}),$$

we must have $f \in I^n_{\sigma,\tau} k[S_\sigma]_{I_{\sigma,\tau}}$. As we also have $f \in k[S_\sigma]$, we must have $f \in I^{(n)}_{\sigma,\tau}$. So

$$I^{(n)}_{\sigma,\tau} = \bigoplus_{\alpha \geq n} k[S_{\tau,\alpha}] \cap k[S_\sigma].$$

□

Lemma 2.17. *Let σ be an scrp cone in N, and let $m \in S_\sigma$. Then the ideal $\langle \chi^m \rangle \subseteq k[S_\sigma]$ is*

$$\langle \chi^m \rangle = \bigcap_{\tau \in \sigma(1)} I^{(\langle m, n_\tau \rangle)}_{\sigma,\tau}.$$

where $n_\tau \in N$ is the generator of ray τ.

Proof. Suppose $f \in \langle \chi^m \rangle$. Then $f = \chi^m g$ for some $g \in k[S_\sigma]$. Thus for each $\tau \in \sigma(1)$, we have

$$\nu_\tau(f) = \nu_\tau(\chi^m) + \nu_\tau(g) \geq \nu_\tau(m) = \langle m, n_\tau \rangle.$$

Thus $f \in \bigcap_{\tau \in \sigma(1)} k[S_{\tau,\langle m, n_\tau \rangle}]$. Since $f \in k[S_\sigma]$ As well, by Lemma 2.16, we have $f \in \bigcap_{\tau \in \sigma(1)} I^{(\langle m, n_\tau \rangle)}_{\sigma,\tau}$.

On the other hand, if $f \in \bigcap_{\tau \in \sigma(1)} I^{(\langle m, n_\tau \rangle)}_{\sigma,\tau}$, then $\nu_\tau(f) - \nu_\tau(\chi^m) \geq 0$ for each $\tau \in \sigma(1)$. Since χ^m is a monomial, $\frac{f}{\chi^m} \in k[M]$. Therefore, $\frac{f}{\chi^m} \in k[S_\sigma]$. Hence, $f \in \langle \chi^m \rangle$.

Therefore $\langle \chi^m \rangle = \bigcap_{\tau \in \sigma(1)} I^{(\langle m, n_\tau \rangle)}_{\sigma,\tau}$. □

2.3 Farey diagrams

There is a good exposition of Farey diagrams and their relationship with classical problems in number theory involving binary quadratic forms in [Hat22]. Here we will be using a correspondence between the Farey diagram and the set of smooth rational cones in $\mathbb{Z}^2$ in order to reason about classes of smooth toric surfaces. We review some basic results about the Farey diagram, and its relationship to cones in $\mathbb{Z}^2$.

The Farey diagram is a graph for which each vertex is labeled by a pair of integers, written as fractions $\frac{a}{b}$. The edges are produced using the following inductive procedure. Note that all edges are undirected in this construction.

In stage 0, we start with two edges $(\frac{1}{0}, \frac{0}{1})$ and $(\frac{0}{1}, \frac{-1}{0})$. At stage 1, we then add four edges $(\frac{1}{0}, \frac{1}{1}), (\frac{1}{1}, \frac{0}{1}), (\frac{0}{1}, \frac{-1}{1})$ and $(\frac{-1}{1}, \frac{-1}{0})$. Continuing inductively for $i \geq 1$, we look at each edge $(\frac{a}{c}, \frac{b}{d})$ added in stage i, and in stage $i+1$ we add edges $(\frac{a}{c}, \frac{a+b}{c+d})$ and $(\frac{a+b}{c+d}, \frac{b}{d})$.

We then have the following facts about this diagram, proved in [Hat22].

Proposition 2.18. *For each pair of fractions $\frac{a}{c}$ and $\frac{b}{c}$, including $\frac{\pm 1}{0}$, there exists an edge in the Farey diagram between $\frac{a}{c}$ and $\frac{b}{d}$ if and only if $ad - cb = \pm 1$.*

Proof. [Hat22, Theorem 1.1]. □

Corollary 2.19. *The fraction $\frac{a}{b}$ appears as a label in the Farey diagram if and only if $\frac{a}{b} = \frac{\pm 1}{0}$ or $\frac{a}{b}$ is a rational number in lowest terms.*

Proof. See Corollary 1.2 and Proposition 1.3 in [Hat22]. □

We can identify the vertices $\frac{1}{0}$ and $\frac{-1}{0}$ together, and thus get a graph where the set of vertices is identified with $\mathbb{P}^1(\mathbb{Q})$. The first few iterations of this graph is shown in the figure above.

As the set of vertices of the Farey diagram are identified with $\mathbb{P}^1(\mathbb{Q})$, we can extend of the action of $\mathrm{SL}_2(\mathbb{Z})$ on $\mathbb{P}^1(\mathbb{Q})$ via Mobius transformations to the Farey diagram. In order for this action to be well-defined, we need to show that for any $\gamma \in \mathrm{SL}_2(\mathbb{Z})$, and edge (p, q) in the Farey diagram, then $(\gamma p, \gamma q)$ is also an edge in the diagram. Indeed, as $(\frac{a}{c}, \frac{b}{d})$ is an edge if and only if $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{SL}_2(\mathbb{Z})$, and γ acts on the set of edges by matrix multiplication, this group action is well-defined.

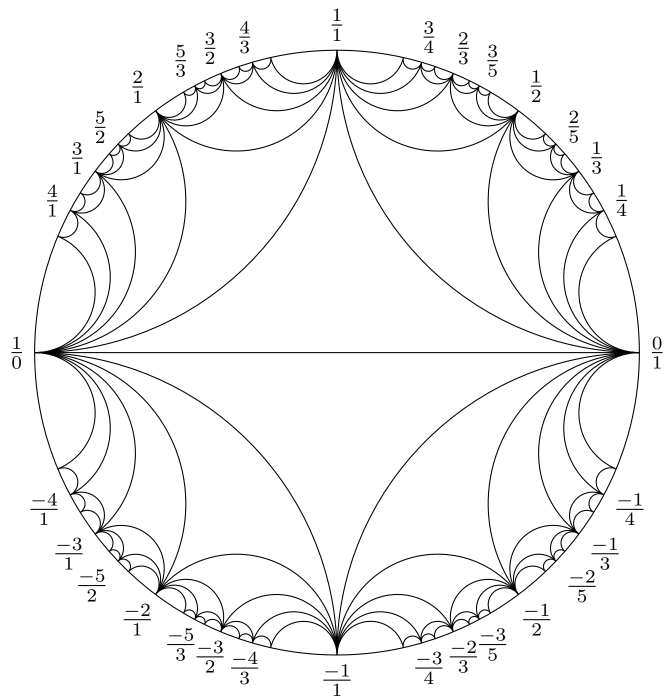


Fig. 2.1: The Farey Diagram

Proposition 2.20. *The action of* $\mathrm{PSL}_2(\mathbb{Z})$ *on the set of edges of the Farey diagram is free and transitive.*

Proof. From the previous proposition, we know that every edge, $(\frac{a}{c}, \frac{b}{d})$ satisfies $ad - cb = \pm 1$. If, $ad - bc = -1$, we can rewrite this edge as $(\frac{a}{c}, \frac{-b}{-d})$. Then, we have $a(-d) - c(-b) = -(ad - cb) = 1$. So every edge can be written such that $ad - cb = 1$. But then:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \left(\frac{1}{0}, \frac{0}{1}\right) = \left(\frac{a}{c}, \frac{b}{d}\right).$$

Thus, the action of $\mathrm{PSL}_2(\mathbb{Z})$ is transitive on the edges of the Farey diagram.

Also, if $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot (\frac{1}{0}, \frac{0}{1}) = (\frac{1}{0}, \frac{0}{1})$, then we must have $b = c = 0$, and $ad = 1$. So we must have $\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \pm I$, which is the identity element in $\mathrm{PSL}_2(\mathbb{Z})$. Thus $\mathrm{PSL}_2(\mathbb{Z})$ acts freely on the Farey diagram. $\square$

The **Farey height** of a rational number is defined to be the stage in which the number first appears as the endpoint of the edge.

Proposition 2.21. *Let $[a_0; a_1, a_2, \ldots, a_n]$ be the simple continued fraction expansion of a rational number q, then the Farey height of q is the sum $\sum_{i=0}^{n} a_i$.*

Proof. Follows directly from [Hat22, Theorem 2.1]. □

As the vertices of the Farey diagram are elements of $\mathbb{P}^1(\mathbb{Q})$, we can extend the natural order on $\mathbb{Q}$ to $\mathbb{P}^1(\mathbb{Q})$ by saying that $\frac{a}{b} \leq \frac{1}{0}$ for all $\frac{a}{b} \in \mathbb{P}^1(\mathbb{Q})$. Given this ordering, we say that a path $(p_1, p_2, \ldots, p_r)$ is **decreasing** if $p_1 > p_2 > \cdots > p_r$ with respect to this ordering.

Proposition 2.22. *Suppose $P = \{p_1, p_2, \ldots, p_r\}$ is an decreasing path on the Farey diagram. Then there exists some $i \in \{1, 2, \ldots, r\}$ such that the Farey height of p_i is less than the Farey height of any other rational number in the interval (p_r, p_1).*

Proof. Note that for any $\frac{a}{c} > \frac{b}{d}$, we have $\frac{a}{c} > \frac{a+b}{c+d} > \frac{b}{d}$. Thus, for each edge (α, β) in the Farey diagram added at stage i, there are no points strictly between α and β that are already in the Farey diagram, that is, having height less than i. Therefore, for each $q \in (p_{i+1}, p_i)$ the height of q must be greater than the heights of both p_i and p_{i+1}. Hence the point of lowest height in (p_r, p_1) must be one of the p_i.

Now, suppose that p_i and p_j have the same height, and that this is the lowest height in the interval (p_r, p_1). Then, as in the construction of the Farey diagram only 1 point of height i is added between each consecutive pair of points of height less than i , there must be a point of lesser height between p_i and p_j. But this contradictions the minimality of the height of p_i and p_j. Thus, there is a unique point p_i of lowest height on the path P, such that the height of p_i is less than the height of any $q \in (p_r, p_1)$ such that $q \neq p_i$. □

Proposition 2.23. *There is a 2-1 correspondence between smooth rational cones in $\mathbb{Z}^2$ and the directed edges of the Farey diagram that is equivariant with respect to the action of $\mathrm{SL}_2(\mathbb{Z})$.*

Proof. Let C be a smooth cone on $\mathbb{Z}^2$ with generators (a, c) and (b, d) ordered such that $ad - bc = 1$. We associate this cone to the edge of the Farey diagram from $\frac{a}{c}$ to $\frac{b}{d}$. Note these are the inverse slopes of the faces of this cone. This edge exists by Proposition 2.18, since $ad - bc = 1$. Let φ be this correspondence. So

$$\varphi\{(a, c), (b, d)\} = (\frac{a}{c}, \frac{b}{d}).$$

Further, again by Proposition 2.18, if there is an edge in the Farey diagram going from $\frac{a}{c}$ to $\frac{b}{d}$, we must have $ad - bc = \pm 1$. If $ad - bc = 1$, then it corresponds to the cone generated by (a, c) and (b, d), and if $ad - bc = -1$, it corresponds to the cone generated by (a, c) and $(-b, -d)$. So φ is surjective.

Now, suppose that two cones C and C' correspond to the same edge. Then if C has generators (a, c) and (b, d), the cone C' must have generators $(\lambda a, \lambda c)$ and $(\mu b, \mu d)$ such that $\lambda\mu ad - \lambda\mu bc = 1$. Thus, $\lambda\mu = 1$. The only integer solutions to this equation are $\lambda = \mu = \pm 1$. Thus, precisely two distinct cones correspond to each edge. So this is a $2 - 1$ correspondence.

Now, we can check the equivariance with respect to the natural actions of $\mathrm{SL}_2(\mathbb{Z})$ on $\mathbb{Z}^2$ and $\mathbb{P}^1(\mathbb{Q})$ as follows. Suppose that $\gamma = \begin{bmatrix} w & y \\ x & z \end{bmatrix} \in \mathrm{SL}_2(\mathbb{Z})$.

Then:

$$\begin{aligned}
\varphi\left(\gamma \cdot \{(a, c), (b, d)\}\right) &= \varphi\left(\{\gamma \cdot (a, c), \gamma \cdot (b, d)\}\right) \\
&= \varphi\left(\{(wa + xc, ya + zc), (wb + xd, yb + zd)\}\right) \\
&= \left\{\frac{wa + xc}{ya + zc}, \frac{wb + xd}{yb + zd}\right\} \\
&= \left\{\frac{w\frac{a}{c} + x}{y\frac{a}{c} + z}, \frac{w\frac{b}{d} + x}{y\frac{b}{d} + z}\right\} \\
&= \left\{\gamma \cdot \frac{a}{c}, \gamma \cdot \frac{b}{d}\right\} \\
&= \gamma \cdot \varphi\left(\{(a, c), (b, d)\}\right).
\end{aligned}$$

Thus φ is $\mathrm{SL}_2(\mathbb{Z})$-equivariant. ☐

Let $(p_1, p_2, \ldots, p_r)$ be a path in the Farey diagram, so each $p_i \in \mathbb{P}^1(\mathbb{Q})$. We will define an action of $\mathrm{PGL}_2(\mathbb{Z})$ on these paths as follows.

$$\gamma(p_1, p_2, \ldots, p_r) = \begin{cases} (\gamma p_1, \ldots, \gamma p_r) & \det(\gamma) = 1 \\ (\gamma p_r, \ldots, \gamma p_1) & \det(\gamma) = -1 \end{cases}.$$

Proposition 2.24. *There is a 1-1 correspondence between non-equivariant isomorphism classes of smooth compact toric surfaces containing a given 2-dimensional torus T and $PGL_2(\mathbb{Z})$ orbits of cycles $(p_1, p_2, \ldots, p_r, p_1)$ in the Farey diagram such that there are precisely two indices i, j such that $p_i < p_{i+1}$ and $p_j < p_{j+1}$.*

Proof. By Proposition 2.12, a compact toric surface X is given by a finite and complete fan Σ in $N \cong \mathbb{Z}^2$. Such a fan can be described by choosing a ray τ_1, and listing the remaining rays in Σ in clockwise order $\{\tau_1, \tau_2, \ldots, \tau_r, \tau_1\}$. Since Σ is finite, such a list can be constructed, and since Σ is complete, for each $1 \leq i \leq r$, the cone generated by $\{\tau_i, \tau_{i+1}\} \in \Sigma$. Since X is smooth, we can use Proposition 2.23 to associate each cone to an edge in the Farey diagram. Thus, this fan corresponds to a path $\{p_1, p_2, \ldots, p_r, p_1\}$, where $(p_i, p_{i+1}) = \varphi(\{\tau_i, \tau_{i+1}\})$.

Since the τ_i were taken in clockwise order, and since p_i corresponds to the reciprocal of the slope of τ_i, we see $p_i < p_{i+1}$ if and only if $(\pm 1, 0)$ is in the interior of the cone generated by $\{\tau_i, \tau_{i+1}\}$ or $(\pm 1, 0) = \tau_{i+1}$. Since $(1, 0)$ and $(-1, 0)$ cannot be in the same scrp cone, there are exactly two distinct cones $\{\tau_i, \tau_{i+1}\}$ and $\{\tau_j, \tau_{j+1}\}$ such that $p_i < p_{i+1}$ and $p_j < p_{j+1}$.

Similarly, given a path $(p_1, p_2, \ldots, p_r, p_1)$ in the Farey diagram as in the statement, we can produce a fan Σ in $\mathbb{Z}^2$ by adding a ray τ_1 with inverse slope p_1 such that either the second coordinate of τ_1 is positive, or $\tau_1 = (1, 0)$. Then, for each p_i, add a ray τ_i of inverse slope p_i in the clockwise direction from p_{i-1}. Note that $p_i < p_{i+1}$ will thus correspond precisely to including $(1, 0)$ or $(-1, 0)$ in the cone $\{\tau_i, \tau_{i+1}\}$ and $(\pm 1, 0) \neq \tau_i$. Thus, since there are precisely two edges such that $p_i < p_{i+1}$, this process will wrap around $\mathbb{Z}^2$ precisely one time. So Σ is a complete fan.

By Proposition 2.23, the correspondence φ is $\mathrm{SL}_2(\mathbb{Z})$-equivariant, so for any $\gamma \in \mathrm{SL}_2(\mathbb{Z})$, the γ-translate of the fan Σ will correspond to the γ-translate of the corresponding path as constructed above. Further, since $\det(\gamma) = 1$, the clockwise cyclic ordering of the rays is unchanged, so the γ-translate of this path corresponding to a given fan Σ, will be the path given by the above construction using the fan $\gamma\Sigma$ and beginning at the ray $\gamma(p_1)$.

If $\gamma \in \mathrm{GL}_2(\mathbb{Z})$ has determinant -1, then the clockwise ordering of the rays in the fan $\gamma\Sigma$ will be reversed from the ordering in Σ, and by the definition of the $\mathrm{PGL}_2(\mathbb{Z})$ action on cycles, the ordering of the cycle will also be reversed. This new cycle will again correspond to the clockwise ordering of rays for a complete fan in $\mathbb{Z}^2$, and thus satisfy the condition that precisely two indices satisfy $p_i < p_{i+1}$ and $p_j < p_{j+1}$. Therefore, $\mathrm{GL}_2(\mathbb{Z})$-orbits of finite, complete fans in $\mathbb{Z}^2$ are in 1-1 correspondence with $\mathrm{PGL}_2(\mathbb{Z})$-orbits of cycles in the Farey diagram satisfying this condition.

Thus, by Proposition 2.9, non-equivariant isomorphism classes of compact toric surfaces containing T are in 1-1 correspondence with $\mathrm{PGL}_2(\mathbb{Z})$ orbits of cycles in the Farey diagram such that precisely two edges (p_i, p_{i+1}) and (p_j, p_{j+1}) satisfy $p_i < p_{i+1}$ and $p_j < p_{j+1}$. $\square$

2.4 Formal Schemes

Let X be a noetherian scheme with a closed reduced subscheme X'. Let $\mathscr{I}$ be the ideal sheaf defining X'. The **completion $\widehat{X}$ of X along X'** is defined as the locally ringed space with underlying topological space X' and sheaf of rings $\mathcal{O}_{\widehat{X}} := \varprojlim \mathcal{O}_X/\mathscr{I}^n$.

A **locally noetherian formal scheme** is a locally ringed space $(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}})$ such that there exists of cover $(\mathfrak{U}_i)$, for which each pair $(\mathfrak{U}_i, \mathcal{O}_{\mathfrak{X}}|_{\mathfrak{U}_i})$ is the completion of some noetherian scheme along a closed subscheme.

An **ideal of definition** for $\mathfrak{X}$ is a sheaf of ideals $\mathscr{I}$ such that the support of $\mathcal{O}_{\mathfrak{X}}/\mathscr{I}$

is all of $\mathfrak{X}$, and the locally ringed space $(\mathcal{O}_{\mathfrak{X}}/\mathscr{I})$ is a locally noetherian scheme.

Lemma 2.25. *Let $\mathfrak{X}$ be a locally noetherian formal scheme, then a unique maximal ideal of definition $\mathscr{I}$ exists, characterized by the fact that $(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}}/\mathscr{I})$ is a reduced scheme.*

Proof. See [DG67, I.10.5.4]. □

Lemma 2.26. *Let $\widehat{X}$ be the completion of X along a closed reduced subscheme X', and let $\mathscr{I}$ be the ideal sheaf defining X'. Then then image of $\mathscr{I}$ in $\mathcal{O}_{\widehat{X}}$ is the unique maximal ideal of definition, and $\mathcal{O}_{\widehat{X}}/\mathscr{I} = \mathcal{O}_{X'}$.*

Proof. See [DG67, I.10.8.5]. □

Given a coherent $\mathcal{O}_X$-module $\mathcal{H}$, the completion $\widehat{\mathcal{H}}$ of $\mathcal{H}$ along a subscheme X' is defined to be the restriction to X' of the sheaf $\varprojlim(\mathcal{H} \otimes_{\mathcal{O}_X} \mathcal{O}_X/\mathscr{I}^n)$, where $\mathscr{I}$ is the defining ideal sheaf of X' [DG67, I.10.8.4]. We recall some basic results about the completions of coherent $\mathcal{O}_X$-modules.

Lemma 2.27. *The function $\mathcal{H} \to \widehat{\mathcal{H}}$ is exact.*

Proof. See [DG67, I.10.8.8]. □

Lemma 2.28. *Let $\iota : \widehat{X} \to X$ be the natural inclusion of locally ringed spaces. Then $\iota : \widehat{X} \to X$ is flat, and for any coherent $\mathcal{O}_X$ module, $\iota^*(\mathcal{H}) \to \widehat{H}$ is an isomorphism of $\mathcal{O}_{\widehat{X}}$-modules. That is, $\mathcal{H} \otimes_{\mathcal{O}_X} \mathcal{O}_{\widehat{X}} \cong \widehat{H}$.*

Proof. See [DG67, I.10.8.8-9]. □

Lemma 2.29. *Let $\mathcal{H}$ be a coherent $\mathcal{O}_X$-module, then $\widehat{\mathcal{H}}$ is a coherent $\mathcal{O}_{\widehat{X}}$-module.*

Proof. See [DG67, I.10.10.5]. □

Lemma 2.30. *Let $\widehat{X}$ and X be as above, then $\widehat{(\mathcal{O}_X)^n} \cong (\mathcal{O}_{\widehat{X}})^n$.*

Proof.

$$\widehat{(\mathcal{O}_X)^n} \cong (\mathcal{O}_X)^n \otimes_{\mathcal{O}_X} \mathcal{O}_{\widehat{X}} \cong (\mathcal{O}_X \otimes_{\mathcal{O}_X} \mathcal{O}_{\widehat{X}})^n \cong (\mathcal{O}_{\widehat{X}})^n.$$

□

Lemma 2.31. *Let X' be a closed subscheme of X, and $\widehat{X}$ the completion of X along X'. Let $\mathcal{H}$ be a coherent $\mathcal{O}_X$-module, and $x \in X'$. Then $\mathcal{H}_x$ is a free $\mathcal{O}_{X,x}$-module if and only if $\widehat{\mathcal{H}_x}$ is a free $\widehat{\mathcal{O}_{X,x}}$-module.*

Proof. Since $\mathcal{H}$ is coherent, we know that $\mathcal{H}_x$ is finitely generated. Thus $\mathcal{H}_x$ is a free $\mathcal{O}_{X,x}$-module if and only if $\mathcal{H}_x \cong (\mathcal{O}_{X,x})^n$.

By Lemma 2.28, the morphism $\iota : \widehat{X} \to X$ is flat. Thus, for each $x \in X'$, the map on stalks $\mathcal{O}_{X,x} \to \mathcal{O}_{\widehat{X},x}$ is flat. Further, as we are considering only the local rings at $x \in X'$, the corresponding map on spectra, $\iota|_x : \operatorname{Spec}(\mathcal{O}_{\widehat{X},x}) \to \operatorname{Spec}(\mathcal{O}_{X,x})$, is clearly surjective. Therefore, $\mathcal{O}_{\widehat{X},x}$ is a faithfully flat $\mathcal{O}_{X,x}$ module. So

$$\mathcal{H}_x \cong (\mathcal{O}_{X,x})^n \text{ if and only if } \mathcal{H}_x \otimes_{\mathcal{O}_{X,x}} \mathcal{O}_{\widehat{X},x} \cong (\mathcal{O}_{X,x})^n \otimes_{\mathcal{O}_{X,x}} \mathcal{O}_{\widehat{X},x}.$$

By Lemma 2.28, we can reduce this statement to:

$$\mathcal{H}_x \cong (\mathcal{O}_{X,x})^n \text{ if and only if } \widehat{\mathcal{H}_x} \cong \widehat{(\mathcal{O}_{X,x})^n}.$$

Using Lemma 2.30, we get $\widehat{(\mathcal{O}_{X,x})^n} = (\mathcal{O}_{\widehat{X},x})^n$. Thus, we have:

$$\mathcal{H}_x \cong (\mathcal{O}_{X,x})^n \text{ if and only if } \widehat{\mathcal{H}_x} \cong (\mathcal{O}_{\widehat{X},x})^n.$$

Since $\mathcal{H}$ is coherent, then by Lemma 2.29, we know that $\widehat{\mathcal{H}}$ is a coherent $\widehat{\mathcal{O}_X}$-module. Thus if $\widehat{\mathcal{H}}_x$ is free, it must be that $\widehat{\mathcal{H}}_x \cong (\mathcal{O}_{\widehat{X},x})^n$ for some $n \in \mathbb{N}$. Hence, via the equivalences above, $\mathcal{H}_x$ is a free $\mathcal{O}_{X,x}$-module if and only if $\widehat{\mathcal{H}_x}$ is a free $\mathcal{O}_{\widehat{X},x}$. □

Proposition 2.32. *Let X, Y be locally noetherian schemes with closed reduced subschemes X', Y' respectively. Let $\widehat{X}, \widehat{Y}$ be the completions of X and Y with respect* *to X' and Y' respectively. Then, if $\varphi : \widehat{X} \to \widehat{Y}$ is an isomorphism, φ induces an isomorphism $X' \to Y'$.*

Proof. By the definition above, an isomorphism $\widehat{X} \to \widehat{Y}$ is a topological homeomorphism of the underlying spaces of $X' \to Y'$, along with a sheaf isomorphism $\varphi^* : \mathcal{O}_{\widehat{Y}} \to \mathcal{O}_{\widehat{X}}$. Let $\mathscr{J}$ be the maximal ideal of definition for $\widehat{Y}$. Then, by Lemma 2.26, $\mathcal{O}_{Y'} = \mathcal{O}_{\widehat{Y}}/\mathscr{J}$. Further, as φ^* is an isomorphism, $\varphi^*(\mathscr{J})$ is the maximal ideal of definition for $\widehat{X}$. So $\mathcal{O}_{\widehat{X}}/\varphi^*(\mathscr{J}) = \mathcal{O}_{X'}$. Thus φ^* induces a sheaf isomorphism $\mathcal{O}'_Y \to \mathcal{O}'_X$.

Thus φ induces a scheme isomorphism $X' \to Y'$. □

Proposition 2.33. *Consider the setup in Proposition 2.32. Then $\widehat{\mathcal{T}X} \cong \varphi^*\widehat{\mathcal{T}Y}$.*

Proof. Since a map between sheaves that is locally an isomorphism is an isomorphism, we can suppose that X is affine without loss of generality. So, suppose that $X = \operatorname{Spec}(A)$, then for any ideal $I \subseteq A$, we have the exact sequence:

$$I/I^2 \to \Omega^1_{X/k} \otimes A/I \to \Omega^1_{\operatorname{Spec}(A/I)/k} \to 0.$$

In particular, if we let I be the reduced ideal defining the subscheme X', and consider the system of ideals $\{I^n\}_{n\in\mathbb{N}}$, we get a system of exact sequences:

$$I^n/I^{2n} \to \Omega^1_{X/k} \otimes A/I^n \to \Omega^1_{\operatorname{Spec}(A/I^n)/k} \to 0.$$

Further, note that the system $\{I^n/I^{2n}\}$ is Mittag-Leffler, since for any $k \in \mathbb{N}$, and any $n \geq 2k$, the map $I^n/I^{2n} \to I^k/I^{2k}$ is the zero map. Recall the proposition in [DG67, 0.13.2.2], which states that given a sequence

$$0 \to A_n \to B_n \to C_n \to 0$$

of projective systems such that A_n is Mittag-Leffler, the sequence

$$0 \to \varprojlim A_n \to \varprojlim B_n \to \varprojlim C_n \to 0$$

is also exact. Therefore, as the system $\{I^n/I^{2n}\}$ is Mittag-Leffler, we get:

$$\varprojlim I^n/I^{2n} \to \varprojlim(\Omega^1_{X/k} \otimes A/I^n) \to \varprojlim \Omega^1_{\mathrm{Spec}(A/I^n)/k} \to 0.$$

But $\varprojlim I^n/I^{2n}$ is zero, since A is noetherian. Also,

$$\varprojlim(\Omega^1_{X/k} \otimes A/I^n) = \widehat{\Omega^1_{X/k}}.$$

Finally, since I is reduced, its image in $\hat{A}$ is precisely the unique reduced ideal of definition $\mathcal{I}$. Thus $A/I^n = \hat{A}/\mathcal{I}^n$. So we get:

$$\widehat{\Omega^1_{X/k}} \cong \varprojlim \Omega^1_{\hat{A}/\mathcal{I}^n}.$$

Recall that $\widehat{\mathcal{H}om}_{\mathcal{O}_X}(\mathscr{F}, \mathcal{G}) \cong \mathcal{H}om_{\mathcal{O}_{\mathfrak{X}}}(\widehat{\mathscr{F}}, \widehat{\mathcal{G}})$, by [DG67, III.4.5.1].

Thus, we have:

$$\begin{aligned}
\widehat{\mathcal{T}X} &:= \widehat{\mathcal{H}om}_{\mathcal{O}_X}(\Omega^1_{X/k}, \mathcal{O}_X) \\
&\cong \mathcal{H}om_{\mathcal{O}_{\widehat{X}}}(\widehat{\Omega^1_{X/k}}, \widehat{\mathcal{O}_X}) \\
&\cong \mathcal{H}om_{\mathcal{O}_{\widehat{X}}}(\varprojlim \Omega^1_{\widehat{X}/\mathscr{I}^n}, \mathcal{O}_{\widehat{X}}) \\
&\cong \mathcal{H}om_{\mathcal{O}_{\widehat{X}}}(\varprojlim \varphi^*(\Omega^1_{\widehat{Y}/\mathscr{J}^n}), \mathcal{O}_{\widehat{X}}) \\
&\cong \mathcal{H}om_{\mathcal{O}_{\widehat{X}}}(\varphi^*(\varprojlim \Omega^1_{\widehat{Y}/\mathscr{J}^n}), \varphi^*(\mathcal{O}_{\widehat{Y}})) \\
&\cong \varphi^*\mathcal{H}om_{\mathcal{O}_{\widehat{Y}}}(\varprojlim \Omega^1_{\widehat{Y}/\mathscr{J}^n}, \mathcal{O}_{\widehat{Y}}) \\
&\cong \varphi^*\mathcal{H}om_{\mathcal{O}_{\widehat{Y}}}(\widehat{\Omega^1_{Y/k}}, \widehat{\mathcal{O}_Y}) \\
&\cong \varphi^*\widehat{\mathcal{H}om}_{\mathcal{O}_Y}(\Omega^1_{Y/k}, \mathcal{O}_Y) =: \varphi^*\widehat{\mathcal{T}Y}.
\end{aligned}$$

□

2.5 Hilbert Modular Varieties

We review the construction of Hilbert modular varieties along with the notations that we will be using. This treatment is based on the constructions in [VdG88, Cha90] and [Gor02]. See also the first chapter of [Kat78].

Let S be a scheme defined over a field k. Let L be a totally real number field of degree g, with ring of integers $\mathcal{O}_L$. An **abelian scheme with real multiplication by** $\mathcal{O}_L$ over S is a couple (A, ι), where A is an abelian scheme of dimension g over S, along with a ring embedding $\iota : \mathcal{O}_L \to \mathrm{End}_S(A)$. Further, we require that (A, ι) satisfies the Deligne–Pappas condition:

$$A \otimes_{\mathcal{O}_L} P_A \cong A^\vee$$

where

$$P_A := \mathrm{Hom}_{\mathcal{O}_L}(A, A^\vee)^{sym} = \{\lambda : A \to A^\vee : \lambda = \lambda^\vee, \lambda \circ \iota(\alpha) = \iota(\alpha)^\vee \circ \lambda, \forall \alpha \in \mathcal{O}_L\}.$$

The module P_A is equipped with a notion of positivity, by declaring the positive elements to be the $\mathcal{O}_L$-equivariant polarizations of A. Given a fractional ideal $\mathfrak{c}$ of L, a **$\mathfrak{c}$-polarization** of (A, ι) is an $\mathcal{O}_L$-isomorphism $\lambda : P_A \to \mathfrak{c}$ such that the positive elements of P_A correspond to the totally positive elements of $\mathfrak{c}$. Such a triple (A, ι, λ) is called a **$\mathfrak{c}$-polarized abelian scheme with real multiplication**. Note that multiplying $\mathfrak{c}$ by some principal ideal with a totally positive generator will not change this construction, so we can consider $\mathfrak{c}$ as an element of $Cl(L)^+$.

Lemma 2.34. *Suppose (A, ι) satisfies the Deligne–Pappas condition as above, and suppose that k is either a field of characteristic 0, or a field of characteristic p where p is unramified in L. Then, $Lie(A)$ is a locally free $\mathcal{O}_L \otimes \mathcal{O}_S$ module of rank 1, and P_A is a projective $\mathcal{O}_L$-module of rank 1.*

Proof. See [Gor02, Ch 3, Lemma 5.5]. □

There exists a coarse moduli space, which we will denote $\mathcal{M}(\mathfrak{c})$, parameterizing triples (A, ι, λ) as above. $\mathcal{M}(\mathfrak{c})$ is called a **Hilbert modular variety**.

We now consider Hilbert modular varieties over $\mathbb{C}$. Define the group:

$$\Gamma_{\mathfrak{c}} := \mathrm{SL}(\mathcal{O}_L \oplus \mathfrak{c})^+ = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, d \in \mathcal{O}_L, b \in \mathfrak{c}^{-1}, c \in \mathfrak{c}, ad - bc = 1 \right\}.$$

Let $\{\sigma_1, \ldots, \sigma_g\}$ be the set of embeddings of L into $\mathbb{R}$. If $\mathfrak{h}$ is the upper half plane, the group $\Gamma_{\mathfrak{c}}$ acts on $\mathfrak{h}^g$ by:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot (z_1, \ldots, z_g) = \left(\frac{\sigma_1(a)z_1 + \sigma_1(b)}{\sigma_1(c)z_1 + \sigma_1(d)}, \ldots, \frac{\sigma_g(a)z_g + \sigma_g(b)}{\sigma_g(c)z_g + \sigma_g(d)} \right).$$

Proposition 2.35. $\mathcal{M}(\mathfrak{c})(\mathbb{C}) \cong \Gamma_{\mathfrak{c}} \backslash \mathfrak{h}^g$.

Proof. See [Gor02, Theorem 2.2.17]. □

The cusps of $\mathcal{M}(\mathfrak{c})(\mathbb{C})$ are parameterized by the $\Gamma_{\mathfrak{c}}$-orbits of $\mathbb{P}^1(L)$. Note that two points $(\alpha_1 : \beta_1)$ and $(\alpha_2 : \beta_2)$ are in the same $\Gamma_{\mathfrak{c}}$-orbit if and only if the ideals $\alpha_1\mathcal{O}_L + \beta_1\mathfrak{c}^{-1}$ and $\alpha_2\mathcal{O}_L + \beta_2\mathfrak{c}^{-1}$ are equivalent in the class group of L, as shown in the proof of [Gor02, Prop 2.2.22]. So, we can identify the set of cusps of $\mathcal{M}(\mathfrak{c})(\mathbb{C})$ with the class group of L.

Consider the cusp at ∞. The isotropy group of this cusp is

$$\Gamma' := \left\{ \begin{pmatrix} \epsilon & \mu \\ 0 & \epsilon^{-1} \end{pmatrix} : \epsilon \in \mathcal{O}_L^\times, \mu \in \mathfrak{c}^{-1} \right\}.$$

As a transformation group, this is the same as the group:

$$\left\{ \begin{pmatrix} \epsilon^2 & \mu \\ 0 & 1 \end{pmatrix} : \epsilon \in \mathcal{O}_L^\times, \mu \in \mathfrak{c}^{-1} \right\} \cong \mathfrak{c}^{-1} \rtimes (\mathcal{O}_L^\times)^2.$$

So, locally around the cusp at ∞, the variety $\mathcal{M}(\mathfrak{c})(\mathbb{C})$ has a neighbourhood homeomorphic to a neighbourhood of $(i\infty, \ldots, i\infty)$ in $\Gamma'\backslash\mathfrak{h}^g$.

In general, let M be a $\mathbb{Z}$-module in L of maximal rank, and let V be a finite index subgroup of $\mathcal{O}_L^{\times,+}$. Around a cusp of type (M, V), that is, a cusp with isotropy group $M \rtimes V$, the variety $\mathcal{M}(\mathbb{C})$ thus looks like $M \rtimes V\backslash\mathfrak{h}^g$, where the action is defined such that M acts on $\mathfrak{h}^g$ as

$$\mu \cdot (z_1, z_2, \ldots, z_g) = (z_1 + \sigma_1(\mu), z_2 + \sigma_2(\mu), \ldots, z_g + \sigma_g(\mu)),$$

and V will act on $\mathfrak{h}^g$ as

$$\epsilon \cdot (z_1, z_2, \ldots, z_g) = (\sigma_1(\epsilon)z_1, \sigma_2(\epsilon)z_2, \ldots, \sigma_g(\epsilon)z_g).$$

As a moduli space, $\mathcal{M}(\mathfrak{c})$ can be defined over any field k. Later on we will consider the case where κ is a field of characteristic p, for p unramified in L. As such, we will need a different description of the cusps of $\mathcal{M}(\mathfrak{c})$ that works over arbitrary fields. As described, $\mathcal{M}(\mathfrak{c})$ is only a course moduli space, and not a scheme. In order to get a moduli space tThat is representable by a scheme, we must consider a level structre as well. Let n be an integer such that $(n, p) = 1$. An $\mathcal{O}_L$-equivariant embedding $\alpha : \mu_n \otimes_{\mathbb{Z}} \mathfrak{d}_L^{-1} \to A[n]$ is then called a **rigid** $\Gamma_{00}(n)$**-level structure**. We define $\mathcal{M}_n(\mathfrak{c})$ to be the moduli space parameterizing tuples $(A, \iota, \lambda, \alpha)$ as above. For $n \geq 4$, $\mathcal{M}_n(\mathfrak{c})$ is indeed a scheme.

There exists a universal $\mathfrak{c}$-polarized abelian scheme with real multiplication and $\Gamma_{00}(n)$-level structure over $\mathcal{M}_n(\mathfrak{c})$, which we denote as $(A_{\text{univ}}, \iota_{\text{univ}}, \lambda_{\text{univ}}, \alpha_{\text{univ}})$.

The cusps of $\mathcal{M}_n(\mathfrak{c})$ are defined as follows. (See for example [Cha90]). Let $(\mathfrak{a}, \mathfrak{b})$ be two fractional ideals of L, such that $\mathfrak{b}^{-1}\mathfrak{a} = \mathfrak{c}$, and let H be a projective $\mathcal{O}_L$-module forming a short exact sequence of $\mathcal{O}_L$-modules:

$$0 \to (\mathfrak{a}\mathfrak{d})^{-1} \to H \to \mathfrak{b} \to 0$$

where $\mathfrak{d}$ is the different ideal of L. Also, let consider an isomorphism

$$\gamma : \mathfrak{a}^{-1}/n\mathfrak{a}^{-1} \to (\mathcal{O}_L/n\mathcal{O}_L).$$

Note that in the case that $\mathfrak{a}$ is coprime to n, such an isomorphism can be given canonically. A cusp of $\mathcal{M}_n(\mathfrak{c})$ is then given by an isomorphism class of triples $(\mathfrak{a}, \mathfrak{b}, \gamma)$. Note that the isomorphism class of $(\mathfrak{a}, \mathfrak{b})$ is determined precisely by the ideal class of $\mathfrak{a}$, so we can always choose a representative $(\mathfrak{a}, \mathfrak{b}, \gamma)$ such that $\mathfrak{a}$ is coprime to n. In the following, we will generally supress the level structure in the notation, and describe cusps by the pair $(\mathfrak{a}, \mathfrak{b})$.

The previous description of cusps as orbits of points in $\mathbb{P}^1(L)$ can be connected to this description as given in [VdG88]. To the cusp at ∞ we associate the exact sequence $0 \to \mathcal{O}_L \to \mathcal{O}_L \oplus (\mathfrak{c}\mathfrak{d})^{-1} \to (\mathfrak{c}\mathfrak{d})^{-1} \to 0$. This corresponds to the pair $(\mathfrak{d}, (\mathfrak{c}\mathfrak{d})^{-1})$ of ideals. Any other cusp can be written in the form $A \cdot \infty$ for some $A \in \Gamma_{\mathfrak{c}} \subset \mathrm{SL}_2(L)$. We can also use A to transform our exact sequence and this will give us the exact sequence corresponding to the cusp $A \cdot \infty$.

2.6 Toroidal Compactification of Hilbert Modular Varieties

The main result of toroidal compactification of Hilbert modular varieties is given below. It was originally proven by Rapoport in [Rap78]. See also [FC90, Cha90]. For a statement for more general Shimura varieties, see [Lan13].

The principal tool we will use to construct the toroidal compactification is the Mumford construction, as originally formulated in [Mum72]. Let A be an excellent integrally closed noetherian ring, with an ideal I such that A is I-adically complete. For our purposes, it is enough to let A be a complete discrete valuation ring, and is thus integrally closed and noetherian. Let K be the fraction field of A. Let $S = \mathrm{Spec}(A)$, with closed subscheme $S_0 = \mathrm{Spec}(A/I)$, and generic point η. If G is a group scheme over S, then we will denote the generic fibre of G by G_η, and the fibre over S_0 by G_0.

Let G be a split torus of rank r over S. We say that an (abstract) subgroup $Y \subset G(K)$ is a **period subgroup**, if $Y \cong \mathbb{Z}^r$, and there exists a homomorphism $\iota : Y \to X(G)$ into the character group of G, such that for all $x, y \in Y$, we have $\iota(x)(y) = \iota(y)(x)$ and $\iota(y)(y) \in I$, for all $y \neq 0$. This homomorphism is called a **polarization** of the period subgroup.

Theorem 2.36 (Mumford Construction)**.** *Let G be a split torus over S with period subgroup Y. Then there exists a semi-abelian group scheme G/Y over S such that $(G/Y)_0 \cong G_0$ and $(G/Y)_\eta$ is an abelian variety.*

Proof. See [Mum72]. □

The Mumford construction first embeds G as an open subset of what is called a relatively complete model P of G with respect to Y, then defines G/Y as an open subset of the quotient P/Y.

For example, consider the case where $A = \mathbb{Z}[[q]]$ and $I = qA$. Thus A is I-adically complete with fraction field $K = \mathbb{Z}((q))$. Let G be the one-dimensional torus $\mathbb{G}_m$, with period subgroup $Y = q^{\mathbb{Z}}$. Then the map which takes q^m to the character $(x \mapsto x^m)$ on $\mathbb{G}_m(K) = K^*$ is a polarization, since

$$\iota(q^m)(q^n) = (q^n)^m = q^{mn} = \iota(q^n)(q^m).$$

and for $n \neq 0$, we have

$$\iota(q^n)q^n = q^{n^2} \in I.$$

In this example, the relatively complete model P is a toric scheme containing the torus G, such that the closed fibre of P is an infinite union of non-singular rational curves, connected in a chain and crossing each other transversely.

The quotient P/Y in this example is known as the Tate curve. Note that is defined over the ring of formal power series $\mathbb{Z}[[q]]$. By changing the base to any complete field k, the Tate curve describes a degenerating family of elliptic curves, where for any $c \in k^*$

such that $|c| < 1$, the fibre over $q = c$ is an elliptic curve over k, while the special fibre, that is the fibre over $q = 0$, is a rational curve with an ordinary double point. Then G/Y is the open subset formed by removing only the double point in the special fibre. The generic fibre of G/Y is the same as the generic fibre of the Tate curve, while the special fibre is a torus.

Let C be a cusp of $\mathcal{M}_n(\mathfrak{c})$, given by ideals $(\mathfrak{a}, \mathfrak{b})$. Let $M_C = \mathfrak{a}\mathfrak{b}$, and $N_C = \mathrm{Hom}(M_C, \mathbb{Z}) = (\mathfrak{a}\mathfrak{b}\mathfrak{d})^{-1}$. Let M_C^+ and N_C^+ denote the totally positive elements of M_C and N_C respectively.

A $\Gamma(n)$**-admissible decomposition** $\{\Sigma_C\}$ for $\mathcal{M}_n(\mathfrak{c})$ is a collection of fans indexed by cusps C of $\mathcal{M}_n(\mathfrak{c})$. For each cusp C, it is required that Σ_C be a fan on N_C with support $(N_C)^+_{\mathbb{R}}$. Furthermore, the fan Σ_C should be invariant under the natural action of U_n^2, where $U_n = \{x \in \mathcal{O}_L^\times : x \equiv 1 \bmod n\mathcal{O}_L\}$. For simplicity, we will also assume that for any $\sigma \in \Sigma_C$ and $u \in U_n^2$, the intersection $\sigma \cap u \cdot \sigma = \{0\}$. Moreover, the collection of cones Σ_C/U_n^2 should be finite.

We will then use this data to construct the quotient $X(\Sigma_C)/U_n^2$. Note that by Corollary 2.10, U_n^2 does have a well-defined action on $X(\Sigma_C)$, since Σ_C is invariant under U_n^2. If we are working over $\mathbb{C}$, this will be the quotient as a complex analytic manifold, well-defined since the definition of $\Gamma(n)$-admissible decomposition ensures that the action of U_n^2 is free and discontinuous. On the other hand, if we are working in mixed or positive characteristic, we will need to use the rigid analytic quotient, as described in [Rap78]. This quotient can still be covered by the affine charts U_σ as σ ranges over the orbits of cones in Σ_C/U_n^2. While the interior of U_σ in the quotient $X(\Sigma_C)/U_n^2$ is now the quotient of a torus by U_n^2, the boundary remains the same, since $u \cdot \sigma \cap \sigma = \{0\}$ for any $u \in U_n^2$.

We would like to use the Mumford construction to produce a semi-abelian scheme over the quotient $X(\Sigma_C)/U_n^2$ of the toric scheme built from this fan, however the affine charts of this fan are not complete with respect to the ideal defining the boundary. As such, we make the following modifications.

Let $R_C^0 = \mathbb{Z}[q^m : m \in M_C]$, so we have the torus $S_C^0 = \mathrm{Spec}(R_C^0)$. Let $\sigma \in \Sigma_C$ be a rational cone. So we can define $R_C(\sigma) = \mathbb{Z}[q^m : m \in \sigma^\vee \cap M_C]$, with the toric scheme $S_C(\sigma) = \mathrm{Spec}(R_C)(\sigma)$. Let $S_C(\sigma)^\infty := S_C(\sigma)\backslash S_C^0$ be the boundary of this toric scheme. Then we can define the formal completion $\widehat{S_C(\sigma)}$ as the completion of $S_C(\sigma)$ along $S_C(\sigma)^\infty$.

This is a formal scheme with coordinate ring

$$\widehat{R_C(\sigma)} := R_C^0[\![q^m : \langle m, n\rangle > 0, n \in \sigma, m \in M_C]\!].$$

Now we can define $\widehat{S_C(\sigma)} := \mathrm{Spec}(\widehat{R_C(\sigma)})$. The ring $\widehat{R_C(\sigma)}$ is complete over the ideal that defines the subscheme $S_C(\sigma)^\infty$, so we can perform the Mumford construction over $\widehat{R_C(\sigma)}$.

We have a tautological homomorphism $q : M_C = \mathfrak{ab} \to \mathbb{G}_m(\widehat{R_C(\sigma)})$, given by $m \mapsto q^m$. Note the isomorphisms:

$$\begin{aligned}
\mathrm{Hom}(\mathfrak{ab}, \mathbb{G}_m(\widehat{R_C(\sigma)})) &\cong \mathrm{Hom}_{\mathcal{O}_L}(\mathfrak{b}, \mathrm{Hom}(\mathfrak{a}, \mathbb{G}_m(\widehat{R_C(\sigma)}))) \\
&\cong \mathrm{Hom}_{\mathcal{O}_L}(\mathfrak{b}, (\mathbb{G}_m \otimes \mathrm{Hom}(\mathfrak{a}, \mathbb{Z}))(\widehat{R_C(\sigma)})) \\
&\cong \mathrm{Hom}_{\mathcal{O}_L}(\mathfrak{b}, (\mathbb{G}_m \otimes \mathfrak{a}^\vee)(\widehat{R_C(\sigma)}))
\end{aligned}$$

where $\mathfrak{a}^\vee = (\mathfrak{a}\mathfrak{d})^{-1}$ is the dual of $\mathfrak{a}$ with respect to the trace pairing. Thus the map q induces an $\mathcal{O}_L$-module homomorphism:

$$q : \mathfrak{b} \to (\mathbb{G}_m \otimes \mathfrak{a}^\vee)(\widehat{R_C(\sigma)}).$$

If $\{u_1, \ldots, u_r\}$ is any $\mathbb{Z}$-basis of $\mathfrak{a}$, and $\{v_1, \ldots, v_r\}$ is the dual basis of $\mathfrak{a}^\vee$ with respect to the trace, then we can explicitly describe q as the map:

$$q(b) = \sum_{i=1}^{r} q^{u_i b} \otimes v_i.$$

Note that the character group of $\mathbb{G}_m \otimes \mathfrak{a}^\vee$ is $\mathfrak{a}$, so even without a basis, we can realize $q(b)$ as the element of $\mathbb{G}_m \otimes \mathfrak{a}^\vee$ such that for any character $a \in \mathfrak{a}$, we have $\chi^a(q(b)) = q^{ab}$. Furthermore, the homomorphism q realizes $\mathfrak{b}$ as a period subgroup of this split torus.

To see this, let $d \in \mathbb{Z}$ be such that $d\mathfrak{b} \subseteq \mathfrak{a}$, and let $\iota : \mathfrak{b} \to \mathfrak{a}$ be the multiplication by d map. Then for $b_1, b_2 \in \mathfrak{b}$:

$$\iota(b_1)q(b_2) = q^{db_1b_2} = \iota(b_2)q(b_1).$$

Also, since $\iota(b_1)q(b_1) = q^{db^2}$ and σ is a subset of the totally positive cone, for any $n \in \sigma$, we must have

$$\langle db^2, n\rangle = \mathrm{Tr}(db^2 n) > 0.$$

Thus $\iota(b_1)q(b_1)$ is in the ideal defining $S_C(\sigma)^\infty$, so q is indeed a period map.

Let $\tilde{G}_\sigma := (\mathbb{G}_m \otimes \mathfrak{a}^\vee)/q^{\mathfrak{b}}$ be the quotient built via the Mumford construction. Then, by the functoriality of Mumford's construction we can glue together each of the $\tilde{G}_\sigma$ for $\sigma \in \Sigma$ to produce a scheme G_{mum} over the completion $\widehat{X(\Sigma)}$ along all the boundary components, which is then considered over the quotient $\widehat{X(\Sigma)}/U_n^2$.

The polarization, endomorphism structure and level structure all pass through this construction, providing the **Mumford family** $(G_{\mathrm{mum}}, \lambda_{\mathrm{mum}}, \iota_{\mathrm{mum}}, \alpha_{\mathrm{mum}})$ over the quotient $(\widehat{X(\Sigma_C)}/U_n^2)$. Indeed, we have the following theorem.

Theorem 2.37. *Let $n \geq 4$, and let $\{\Sigma_C\}$ be a $\Gamma(n)$-admissible decomposition for $\mathcal{M}_n(\mathfrak{c})$. There exists a scheme $\mathcal{M}_n^{TC}(\mathfrak{c})$, called the toroidal compactification given by $\{\Sigma_C\}$, such that there exists an open immersion $j : \mathcal{M}_n(\mathfrak{c}) \to \mathcal{M}_n^{TC}(\mathfrak{c})$, and an isomorphism:*

$$\varphi : \bigsqcup_{\textit{cusps } C} (\widehat{X(\Sigma_C)}/U_n^2) \times \mathrm{Spec}(\mathbb{Z}[1/n]) \to \widehat{\mathcal{M}_n^{TC}}(\mathfrak{c})$$

where $\widehat{\mathcal{M}_n^{TC}}(\mathfrak{c})$ is the completion of $\mathcal{M}_n^{TC}(\mathfrak{c})$ along the complement of $j(\mathcal{M}_n(\mathfrak{c}))$. Furthermore, there exists a semi-abelian scheme $(G, \lambda, \iota, \alpha)$ with real multiplication over $\mathcal{M}_n^{TC}(\mathfrak{c})$ extending the universal abelian scheme with real multiplication over $\mathcal{M}_n(\mathfrak{c})$, *such that the pullback of* $(G, \lambda, \iota, \alpha)$ *over* φ *is the Mumford family descibed above over* $\widehat{X(\Sigma_C)}$.

Proof. See [Rap78, Theorem 5.1]. See also [Lan13, Theorem 6.4.1.1] for a more general statement. □

We will also need the following corollary regarding the Lie algebra of G_{mum}.

Corollary 2.38. *Notation as above, for any cone* σ_α *in* Σ_C*, there is a canonical isomorphism*

$$Lie(\tilde{G}_\sigma) \cong \mathfrak{a}^\vee \otimes \mathcal{O}_{\widehat{S_C(\sigma)}}.$$

such that the action induced by ι_{mum} *is given by the natural action of* $\mathcal{O}$ *on* $\mathfrak{a}^\vee$.

Proof. [Rap78, Corollary 4.4] □

2.7 Hilbert Modular Varieties with Iwahori Level Structure

We may also want to consider level structures that are not prime to p. Let $\underline{A} := (A, \iota, \lambda, \alpha)$ be a polarized abelian variety with real multiplication, and thus parameterized by a point in $\mathcal{M}_N(\mathfrak{c})$. Then we can use this to produce a perfect pairing $A[p] \otimes A[p] \to \mu_p$. We begin with the standard Weil pairing $\langle \_, \_ \rangle_w : A[p] \otimes A^\vee[p] \to \mu_p$. By the Deligne–Pappas condition, we have an isomorphism $A^\vee \cong A \otimes_{\mathcal{O}_L} \mathcal{P}_A$. So passing the Weil pairing through this isomorphism produces a map $A[p] \otimes A[p] \otimes_{\mathcal{O}_L} \mathcal{P}_A \to \mu_p$.

The polarization λ is defined as an $\mathcal{O}_L$-isomorphism $\mathcal{P}_A \to \mathfrak{c}$, so we can use this isomorphism to produce a map $A[p] \otimes A[p] \otimes_{\mathcal{O}_L} \mathfrak{c} \to \mu_p$. But since $\mathfrak{c}$ is prime to p, there exists a canonical isomorphism $A[p] \otimes_{\mathcal{O}_L} \mathfrak{c} \to A[p]$ given by $(u \otimes \alpha) \mapsto \iota(\alpha)u$. So after passing the second coordinate of the Weil pairing through each of these isomorphisms, we have a pairing $\langle \_, \_ \rangle_\lambda : A[p] \otimes A[p] \to \mu_p$.

Let $\mathfrak{P}$ be a product of prime ideals $\mathfrak{p}_1 \dots \mathfrak{p}_r$ over p. The Hilbert modular variety with Iwahori level structure by $\mathfrak{P}$, denoted $\mathcal{M}_{n\mathfrak{P}}(\mathfrak{c})$ parameterizes pairs $(\underline{A}, H)$ where $\underline{A}$

is a point of $M_n(\mathfrak{c})$, and H is a finite flat $\mathcal{O}_L$-invariant subgroup scheme of $A[\mathfrak{P}]$ of order p^f, where f is $\sum_{i=1}^r f(\mathfrak{p}_i/p)$, and H is isotropic with respect to the pairing $\langle -, - \rangle_\lambda$.

Alternatively, $\mathcal{M}_{n\mathfrak{P}}(\mathfrak{c})$ can be thought of as parameterizing pairs $(\underline{A_1}, \underline{A_2})$, of points in $\mathcal{M}_n(\mathfrak{c})$, and $\mathcal{M}_n(\mathfrak{c}\mathfrak{P})$ respectively, equipped with a $\mathfrak{P}$-isogeny $A_1 \to A_2$, respecting the polarization and endomorphism structures.

The cusps of $\mathcal{M}_{n\mathfrak{P}}(\mathfrak{c})$ can be parameterized by isomorphism classes of maps α between exact sequences:

$$\begin{tikzcd}
0 \arrow[r] & \mathfrak{a}_1^\vee \arrow[r] \arrow[d] & H_1 \arrow[r] \arrow[d] & \mathfrak{b}_1 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathfrak{a}_2^\vee \arrow[r] & H_2 \arrow[r] & \mathfrak{b}_2 \arrow[r] & 0
\end{tikzcd}$$

such that $H_2/\alpha(H_1) \cong \mathcal{O}_L/\mathfrak{P}\mathcal{O}_L$ (cf. [Dia22, §3.2]). Under the forgetful morphism $(\underline{A}, H) \to \underline{A} : \mathcal{M}_{n\mathfrak{P}}(\mathfrak{c}) \to \mathcal{M}_n(\mathfrak{c})$, such a cusp will map to the cusp given by the pair $(\mathfrak{a}_1, \mathfrak{b}_1)$. The isomorphism classes of cusps lying over $(\mathfrak{a}_1, \mathfrak{b}_1)$ are precisely determined by the ideal $\mathfrak{q}$ such that $\mathfrak{b}_2/\alpha(\mathfrak{b}_1) \cong \mathcal{O}_L/\mathfrak{q}\mathcal{O}_L$, which can be any ideal of $\mathcal{O}_L$ containing $\mathfrak{P}$. As such, we will denote the cusps of $\mathcal{M}_{n\mathfrak{p}}(\mathfrak{c})$ by triples $(\mathfrak{a}, \mathfrak{b}, \mathfrak{q})$.

We can construct the toroidal compactification of $\mathcal{M}_{n\mathfrak{P}}(\mathfrak{c})$ just as we did for $\mathcal{M}_n(\mathfrak{c})$. Over the cusp $C = (\mathfrak{a}, \mathfrak{b}, \mathfrak{q})$, we will need to build our admissible polyhedral decomposition using the lattices $M_C = \mathfrak{a}\mathfrak{b}\mathfrak{q}^{-1}$, and $N_C = (\mathfrak{a}\mathfrak{b}\mathfrak{d})^{-1}\mathfrak{q}$. Here the semi-abelian scheme built over the cusp C will still be $(\mathbb{G}_m \otimes \mathfrak{a}^\vee)/q^{\mathfrak{b}}$, but now equipped with a subgroup scheme H of the $\mathfrak{P}$-torsion, for which the choice of $\mathfrak{q}$ will describe the multiplicative and étale parts of H.

Now consider the case where $\mathfrak{P} = \mathfrak{p}$ is prime. Then the ordinary locus of $\mathcal{M}_{n\mathfrak{p}}(\mathfrak{c})$ can be decomposed into two disjoint smooth varieties. The first is denoted $\mathcal{M}_{n\mathfrak{p}}(\mathfrak{c})^{\mathrm{ord,m}}$, paramterizing pairs $(\underline{A}, H)$ where H is multiplicative. The second is denoted $\mathcal{M}_{n\mathfrak{p}}(\mathfrak{c})^{\mathrm{ord,ét}}$, parameterizing pairs $(\underline{A}, H)$ where H is étale.

Similarly, when classifying the cusps of $\mathcal{M}_{n\mathfrak{p}}(\mathfrak{c})$ the only options for $\mathfrak{q}$ are $\mathfrak{p}$ or $\mathcal{O}_L$.

Over the cusp $C = (\mathfrak{a}, \mathfrak{b}, \mathfrak{p})$, we have the semi-abelian scheme $(\mathfrak{a}^\vee \otimes \mathbb{G}_m)/q^{\mathfrak{b}}$ where H will be the étale part of the $\mathfrak{p}$-torsion, whereas, over the cusp $(\mathfrak{a}, \mathfrak{b}, \mathcal{O}_L)$, we will have the semi-abelian scheme $(\mathfrak{a}^\vee \otimes \mathbb{G}_m)/q^{\mathfrak{b}}$ where H is the multiplicative part of the $\mathfrak{p}$-torsion. Thus the cusps of $\mathcal{M}_{n\mathfrak{p}}(\mathfrak{c})^{\mathrm{ord,m}}$ are of parameterized by tuples $(\mathfrak{a}, \mathfrak{b}, \mathcal{O}_L)$, and the cusps of $\mathcal{M}_{n\mathfrak{p}}(\mathfrak{c})^{\mathrm{ord,ét}}$ are parameterized by tuples $(\mathfrak{a}, \mathfrak{b}, \mathfrak{p})$.

3. FOLIATIONS

3.1 Vector Bundles

Let Y be a scheme. A **pre-vector bundle on** Y is a Y-scheme X, such that for all Y-schemes T, the set $X(T) := \mathrm{Hom}_Y(T, X)$ has an $\mathcal{O}_T(T)$-module structure, such that for all Y-morphisms $T \to T'$, the induced map $X(T') \to X(T)$ is $\mathcal{O}_T(T)$-linear.

Let X, X' be pre-vector bundles. A **morphism** of pre-vector bundles is a Y-scheme morphism $X \to X'$ such that for all Y-schemes T, the induced map on the T-points $X(T) \to X'(T)$ is $\mathcal{O}_T(T)$-linear.

For example, $\mathbb{A}^n_Y$ is a pre-vector bundle. This is called the **trivial** bundle. A **vector bundle on** Y is pre-vector bundle on Y that is locally trivial of constant finite rank n. That is, a pre-vector bundle X, such that there exists an open cover $\{U_i\}$ of Y, and isomorphisms $\psi_i : X_{U_i} \to \mathbb{A}^n_{U_i}$, such that the transition maps $\psi_i \circ \psi_j^{-1}$ are linear on $\mathbb{A}^n_{U_{ij}}$.

A **subbundle** X' of a vector bundle X is a vector bundle such that there exists a closed immersion $X' \hookrightarrow X$ of Y-schemes that is also a morphism of (pre-)vector bundles.

Proposition 3.1. *Let X and X' be vector bundles over Y, of ranks n and m respectively, such that there exists a closed immersion of Y-schemes $\phi : X' \to X$. Then X' is a subbundle of X if and only if there exists a cover $\{U_i\}$ of Y and isomorphisms ψ_i, ξ_i satisfying the following relation:*

$$\begin{tikzcd}
X_{U_i} \arrow[r, "\psi_i"] & \mathbb{A}^n_{U_i} \\
X'_{U_i} \arrow[u, hook, "\phi_{U_i}"] \arrow[r, "\xi_i"] & \mathbb{A}^m_{U_i} \arrow[u, hook]
\end{tikzcd}$$

where the inclusion $\mathbb{A}^m_{U_i}$ into $\mathbb{A}^n_{U_i}$ is the standard inclusion of the first m coordinates.

Proof. Let X' be a subbundle of X. Choose an affine cover $\{U_i\}$ of Y that trivializes both X and X', and let $U_i = \mathrm{Spec}(A_i)$. Then we have isomorphisms $\tilde{\psi}_i : X_{U_i} \to \mathbb{A}^n_{U_i}$ and $\xi_i : X'_{U_i} \to \mathbb{A}^m_{U_i}$. Let $\alpha_i := \tilde{\psi}_i \circ \phi_{U_i} \circ \xi_i^{-1}$. Then, if ι_i is the standard inclusion of the first m coordinates into $\mathbb{A}^n_{U_i}$, we are looking for a linear isomorphism γ that makes the following diagram commute:

$$\begin{tikzcd}
X_{U_i} \arrow[r, "\tilde{\psi}_i"] & \mathbb{A}^n_{U_i} \arrow[r, dashed, "\gamma"] & \mathbb{A}^n_{U_i} \\
X'_{U_i} \arrow[u, hook, "\phi_{U_i}"] \arrow[r, "\xi_i"] & \mathbb{A}^m_{U_i} \arrow[u, hook, "\alpha_i"] \arrow[ur, hook, "\iota_i"'] &
\end{tikzcd}$$

Let $T = U_i$, then by the definition of morphism of pre-vector bundles, we know that the map $X'(T) \to X(T)$ is linear, but this is just the map ϕ_{U_i}. Since $\tilde{\psi}_i$ and ξ_i are isomorphisms of modules, they are linear. Thus α_i is a linear map. Furthermore, since ϕ is a closed immersion, we see that α_i is a closed immersion. Thus, when passing from affine schemes to commutative rings, we see that α_i is induced by a surjective, linear map

$$\alpha_i^* : A_i[x_1, \ldots, x_n] \to A_i[x_1, \ldots, x_m].$$

Note that $\iota_i^* : A_i[x_1, \ldots, x_n] \to A_i[x_1, \ldots, x_m]$ is the map such that $\iota_i^*(x_i) = x_i$ for $1 \leq i \leq m$, and $\iota_i^*(x_i) = 0$ for $m < i \leq n$. Thus $\ker(\iota_i^*) = \langle x_{m+1}, \ldots, x_n \rangle$.

Recall that $A_i[x_1, \ldots, x_n]$ is just the symmetric algebra of the free A_i-module A_i^n, and that the linear maps on $A_i[x_1, \ldots, x_n]$ are precisely those induced by module maps on A_i^n. So, let $\tilde{\alpha_i^*}$ be the surjective module map $A_i^n \to A_i^m$ that induces α_i^*. Note that the kernel of $\tilde{\alpha_i^*}$ need not be free. However, since $\tilde{\alpha_i^*}$ is a surjective morphism of free modules, the kernel is projective, and hence locally free. So, let $\{V_i\}$ be a refinement of $\{U_i\}$ such that the kernel of $\tilde{\alpha_i^*}$ is free over each V_i. Let $V_i = \mathrm{Spec}(B_i)$. So over V_i, we have $\ker(\tilde{\alpha_i^*}) \cong B_i^{n-m}$. Then, we have a comparison of exact sequences:

$$\begin{tikzcd} 0 \arrow[r] & B_i^{n-m} \arrow[r] \arrow[d] & B_i^n \arrow[r, "\tilde{\iota_i^*}"] \arrow[d, dashed, "\tilde{\gamma^*}"] & B_i^m \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & \ker(\tilde{\alpha_i^*}) \arrow[r] & B_i^n \arrow[r, "\tilde{\alpha_i^*}"] & B_i^m \arrow[r] & 0 \end{tikzcd}$$

Since $\operatorname{Ext}(B_i^{n-m}, B_i^m) = 0$, we know that there must exist such an isomorphism $\tilde{\gamma^*}$. Then, we can define $\gamma : \mathbb{A}^n_{U_i} \to \mathbb{A}^n_{U_i}$ as the map induced by $\operatorname{Sym}(\tilde{\gamma^*})$, so $\gamma \circ \alpha_i = \iota_i$, as required.

Now, suppose that $X' \hookrightarrow X$ is a closed immersion of Y-schemes, such that there exists a cover $\{U_i\}$ and isomorphisms ψ_i, ξ_i such that the diagram in the statement commutes. In order to show that X' is a subbundle of X is suffices to show that for each Y-scheme T, the map $\operatorname{Hom}_Y(T, X') \to \operatorname{Hom}_Y(T, X)$ is linear. Note that for each U_i, $X_{U_i} \cong \mathbb{A}^n_{U_i}$, and $X'_{U_i} \cong \mathbb{A}^m_{U_i}$, and the induced map $\operatorname{Hom}_Y(T_{U_i}, \mathbb{A}^m_{U_i}) \to \operatorname{Hom}_Y(T_{U_i}, \mathbb{A}^n_{U_i})$ is defined by post-composing with ι_i. This is clearly linear. Since the support of the image of any Y-morphism $T_{U_i} \to X$ must lie over U_i, we then see that this map is isomorphic to $\operatorname{Hom}_Y(T_{U_i}, X') \to \operatorname{Hom}_Y(T_{U_i}, X)$, which is thus linear.

By the definition of pre-vector bundle, the map $\operatorname{Hom}_Y(T, \mathbb{A}^m_{U_i}) \to \operatorname{Hom}_Y(T_{U_i}, \mathbb{A}^m_{U_i})$ given by the precomposition by the inclusion $T_{U_i} \hookrightarrow T$ is also linear. Therefore, for each pair U_i, U_j we have the following commuting square of linear maps:

$$\begin{tikzcd} \operatorname{Hom}_Y(T, X') \arrow[r] \arrow[d] & \operatorname{Hom}_Y(T_{U_i}, X') \arrow[r] & \operatorname{Hom}_Y(T_{U_i}, X) \arrow[d] \\ \operatorname{Hom}_Y(T_{U_j}, X') \arrow[r] & \operatorname{Hom}_Y(T_{U_j}, X) \arrow[r] & \operatorname{Hom}_Y(T_{U_i \cap U_j}, X) \end{tikzcd}$$

Thus, by the universal property of sheaves of modules, we must have a unique linear map $\operatorname{Hom}_Y(T, X') \to \operatorname{Hom}_Y(T, X)$, such that the maps $\operatorname{Hom}_Y(T, X') \to \operatorname{Hom}_Y(T_{U_i}, X)$ factor through it. But this is precisely the condition that $X' \to X$ is a morphism of pre-vector bundles. Therefore X' is a subbundle of X. $\square$

Recall that there is a correspondence between vector bundles and locally free sheaves over a scheme Y (cf. [Har77, Exercise II.5.18]). In particular, if $\mathscr{E}$ is a locally free sheaf on Y, then $\mathrm{Spec}(\mathrm{Sym}(\mathscr{E}^\vee))$ is the vector bundle with sheaf of sections $\mathscr{E}$. This construction is functorial, giving a categorical equivalence between locally free sheaves on Y and vector bundles over Y. However, this equivalence need not preserve subobjects.

Let $V(\mathscr{E}) := \mathrm{Spec}(\mathrm{Sym}(\mathscr{E}^\vee))$.

Proposition 3.2. *Let $\mathscr{F}$ be a locally free subsheaf of some locally free sheaf $\mathscr{E}$ on Y. Then $V(\mathscr{F})$ is a subbundle of $V(\mathscr{E})$ if and only if $\mathscr{E}/\mathscr{F}$ is locally free.*

Proof. Define $\mathscr{G} := \mathscr{E}/\mathscr{F}$. We then have an exact sequence:

$$0 \to \mathscr{F} \to \mathscr{E} \to \mathscr{G} \to 0.$$

Recall that the dual of a coherent sheaf $\mathscr{E}$ is defined as $\mathscr{E}^\vee := \mathscr{H}om(\mathscr{E}, \mathcal{O}_Y)$. Thus, by applying the dual, we get a long exact sequence:

$$0 \to \mathscr{G}^\vee \to \mathscr{E}^\vee \to \mathscr{F}^\vee \to \mathscr{E}xt^1(\mathscr{G}, \mathcal{O}_Y) \to \mathscr{E}xt^1(\mathscr{E}, \mathcal{O}_Y).$$

Note that if $\mathscr{G}$ is locally free, then $\mathscr{E}xt^1(\mathscr{G}, \mathcal{O}_Y) = 0$. Thus $\mathscr{E}^\vee \to \mathscr{F}^\vee$ is surjective. Since forming the symmetric algebra is an exact functor, the map $\mathrm{Sym}(\mathscr{E}^\vee) \to \mathrm{Sym}(\mathscr{F}^\vee)$ is surjective. Thus $V(\mathscr{F}) \to V(\mathscr{E})$ is a closed immersion. Since it is induced by a map of locally free sheaves, it must be a morphism of vector-bundles. Therefore $V(\mathscr{F})$ is a subbundle of $V(\mathscr{E})$,

On the other hand, if $V(\mathscr{F})$ is a subbundle of $V(\mathscr{E})$, then $\mathrm{Sym}(\mathscr{E}^\vee) \to \mathrm{Sym}(\mathscr{F}^\vee)$ is surjective. Thus $\mathscr{E}^\vee \to \mathscr{F}^\vee$ is surjective, since is this is just the degree 1-summand of the graded symmetric algebra. Therefore $\mathscr{E}xt^1(\mathscr{G}, \mathcal{O}_Y)$ injects into $\mathscr{E}xt^1(\mathscr{E}, \mathcal{O}_Y)$. But since $\mathscr{E}$ is locally free, we know that $\mathscr{E}xt^1(\mathscr{E}, \mathcal{O}_Y) = 0$. So $\mathscr{E}xt^1(\mathscr{G}, \mathcal{O}_Y) = 0$.

Therefore, since $\mathscr{F}$ is locally free, we see that $\mathscr{E}xt^1(\mathscr{G}, \mathscr{F})$ is trivial. Hence, the exact sequence

$$0 \to \mathscr{F} \to \mathscr{E} \to \mathscr{G} \to 0$$

splits. Hence, $\mathscr{G}$ is a local direct summand of $\mathscr{E}$, and thus locally free. $\square$

3.2 Foliations

Let $\mathscr{E}$ be a coherent sheaf on a noetherian scheme X, and let η be the generic point of X. Then $\mathscr{E}$ is said to be a **torsion sheaf** if $\mathscr{E}_\eta = \{0\}$. Recall that the **support** of a sheaf $\mathscr{E}$, denote $\mathrm{Supp}(\mathscr{E})$ is defined as the set of points $x \in X$ on which the stalk $\mathscr{E}_x$ is non-zero While the $\mathrm{Supp}(\mathscr{E})$ may not be closed, the codimension of $\mathrm{Supp}(\mathscr{E})$ can be defined as the codimension of the smallest subscheme of X containing $\mathrm{Supp}(\mathscr{E})$.

Lemma 3.3. *Let $\mathscr{E}$ be a coherent sheaf on a noetherian scheme X such that the support of $\mathscr{E}$ is nonzero. Then $\mathscr{E}$ is a torsion sheaf.*

Proof. This follows directly from the definition. If $\zeta \in \mathscr{E}_\eta$ is non-zero, then η is in the support of $\mathscr{E}$, so the codimension of $\mathscr{E}$ would have to be 0. Thus, $\mathscr{E}_\eta = \{0\}$, so $\mathscr{E}$ is torsion. $\square$

Let $\mathscr{E}$ be a coherent sheaf. The **torsion subsheaf** $\mathscr{T}$ of $\mathscr{E}$ is the maximal subsheaf of $\mathscr{E}$ that is torsion. $\mathscr{E}$ is said to be **torsion-free** if the torsion subsheaf of $\mathscr{E}$ is zero. Note that $\mathscr{E}/\mathscr{T}$ is torsion-free.

Let $\mathscr{F}$ be a subsheaf of some coherent sheaf $\mathscr{E}$ on a scheme X. Then $\mathscr{F}$ is said to be **saturated** if the quotient $\mathscr{E}/\mathscr{F}$ is torsion-free. Equivalently, $\mathscr{F}$ is saturated if for any section δ of $\mathscr{E}$ over an open subset U of X and $f \in \mathcal{O}_X(U)$ such that $f\delta$ is a section of $\mathscr{F}$, then δ is also a section of $\mathscr{F}$ over U.

Let $\mathscr{T} \subseteq \mathscr{E}/\mathscr{F}$ be the torsion subsheaf of $\mathscr{E}/\mathscr{F}$. Then the **saturation of** $\mathscr{F}$, denoted $\mathscr{F}^{\mathrm{sat}}$ is defined to be kernel of the map $\mathscr{E} \to (\mathscr{E}/\mathscr{F})/\mathscr{T}$. Note that $\mathscr{F} \subseteq \mathscr{F}^{\mathrm{sat}}$, and that $\mathscr{F}^{\mathrm{sat}}$ is the minimal saturated subsheaf of $\mathscr{E}$ containing $\mathscr{F}$.

Let X be a variety over k. The **tangent bundle** of X, denoted $\mathcal{T}X$ is defined as $\mathcal{H}om(\Omega^1_{X/k}, \mathcal{O}_X)$. For any affine open $U = \mathrm{Spec}(A)$ of X, the $\Gamma(U, \mathcal{T}X) = \mathrm{Der}_k(A)$ [DG67, IV.16.5.7], that is the k-derivations on A. This gives $\Gamma(U, \mathcal{T}X)$ the structure of a Lie algebra, with Lie bracket given by $[\delta,\zeta]=\delta\circ\zeta-\zeta\circ\delta$ for k-derivations δ,ζ on A [DG67, IV.16.5.9].

If X is smooth, then $\mathcal{T}X$ is locally free, and is thus a vector bundle. A subbundle $\mathscr{F}\subseteq\mathcal{T}X$ is said to be **involutive** if $\mathscr{F}$ is closed under the Lie bracket. An involutive subbundle of $\mathcal{T}X$ is called a **smooth foliation**. That is, $\mathscr{F}$ is a smooth foliation on X if it is a locally free subsheaf of $\mathcal{T}X$ such that $\mathscr{F}$ is closed under the Lie bracket, and $\mathcal{T}X/\mathscr{F}$ is locally free.

Lemma 3.4. *Let $\mathscr{F}$ be an involutive subsheaf of $\mathcal{T}X$, then $\mathscr{F}^{\mathrm{sat}}$ is also involutive.*

Proof. Suppose that δ,ζ are sections of $\mathscr{F}^{\mathrm{sat}}$. Then there exists $f,g\in\mathcal{O}_X$ such that $f\delta, g\zeta$ are sections of $\mathscr{F}$. So $[f\delta,g\zeta]$ is a section of $\mathscr{F}$, and in particular a section of $\mathscr{F}^{\mathrm{sat}}$. But:

$$[f\delta,g\zeta]=(f\delta)\circ(g\zeta)-(g\zeta)\circ(f\delta)=fg(\delta\circ\zeta)+f(\delta(g))\zeta-fg(\zeta\circ\delta)-g(\zeta(f))\delta$$

$$=fg([\delta,\zeta])+f(\delta(g))\zeta-g(\zeta(f))\delta.$$

Since δ,ζ are sections of $\mathscr{F}^{\mathrm{sat}}$, it must be that $fg([\delta,\zeta])$ is a section of $\mathscr{F}^{\mathrm{sat}}$. Thus $[\delta,\zeta]$ is a section of $\mathscr{F}^{\mathrm{sat}}$, as $\mathscr{F}^{\mathrm{sat}}$ is a saturated subsheaf of $\mathcal{T}X$. Thus $\mathscr{F}^{\mathrm{sat}}$ is involutive. $\square$

A **foliation** $\mathscr{F}$ is defined as an involutive saturated subsheaf of $\mathcal{T}X$ such that there exists some open dense subset U of X such that $\mathscr{F}$ is a smooth foliation on U. If $\mathscr{F}$ is a foliation, then we denote by $S(\mathscr{F})$ the singular set of $\mathscr{F}$, which is defined as

$$S(\mathscr{F}):=\{x\in X:(\mathcal{T}X/\mathscr{F})_x \text{ is not free}\}.$$

In particular, consider a smooth variety $\overline{X}$ with dense subvariety X. If $\mathscr{F}$ is a foliation on X, then $(\iota_*\mathscr{F})^{\mathrm{sat}}$ is a saturated subsheaf of $\overline{X}$. By Lemma 3.4, it is also an involutive subsheaf. Further, since any dense open subvariety $U\subseteq X$ is also a dense open subvariety of $\overline{X}$, and $\mathscr{F}|_U=(\iota_*\mathscr{F})^*|_U$, we see that $(\iota_*\mathscr{F})^{\mathrm{sat}}$ is a foliation on $\overline{X}$.

When working over a field k of positive characteristic p, we have also the notion of a p-foliation. [Eke87, Miy87]. In this context, the p-fold composition of a derivation with itself is again a derivation. A foliation that is closed under p-fold composition is said to be **p-closed**, and is called a **p-foliation**.

In [Eke87], Ekedahl describes both the involutivity and p-closed conditions on a subsheaf $\mathscr{E}$ of the $\mathcal{T}X$ in terms of $\mathcal{O}_X$-linear morphisms. This provides a useful criterion for determining when a subsheaf of $\mathcal{T}X$ is a p-foliation.

Lemma 3.5. *Let $\mathscr{E}$ be a subsheaf of $\mathcal{T}X$.*

(i) The Lie bracket on $\mathcal{T}X$ induces an $\mathcal{O}_X$-linear morphism

$$\Lambda^2\mathscr{E} \to \mathcal{T}X/\mathscr{E}.$$

(ii) Suppose $\mathscr{E}$ is involutive. Then the p-th power morphism induces an $\mathcal{O}_X$-linear morphism

$$F^*\mathscr{E} \to \mathcal{T}X/\mathscr{F},$$

where $F : X \to X$ is the absolute Frobenius morphism.

In particular, if $\mathcal{T}X/\mathscr{E}$ is torsion-free and

$$\mathcal{H}om_{\mathcal{O}_X}(\Lambda^2\mathscr{E}, \mathcal{T}X/\mathscr{E}) = \mathcal{H}om_{\mathcal{O}_X}(F^*\mathscr{E}, \mathcal{T}X/\mathscr{E}) = 0,$$

then $\mathscr{E}$ is a p-foliation on X.

Proof. See [Eke87, Lemma 4.2]. □

Unlike in characteristic zero, we have the notion of the quotient by a p-foliation. Let $\mathscr{F}$ be a p-foliation on X, and define the annihilator of $\mathscr{F}$ to be

$$\mathrm{Ann}(\mathscr{F}) := \{f \in \mathcal{O}_X : \delta(f) = 0, \forall \delta \in \mathscr{F}\}.$$

It is shown in [Miy87] that $\mathrm{Ann}(\mathscr{F})$ is an integrally closed $\mathcal{O}_X^p$-subalgebra of $\mathcal{O}_X$

containing $\mathcal{O}_X$, and as such $\mathrm{Spec}(\mathrm{Ann}(\mathscr{F}))$ gives rise to a normal variety, which we will denote $X/\mathscr{F}$, such that there exists a factorization $X \to X/\mathscr{F} \to X^{(p)}$ of the Frobenius morphism on X.

Proposition 3.6. *Let $\mathscr{F}$ be a p-foliation on a smooth variety X.*

(i) If $\mathscr{F}$ has rank r, then $[k(X) : k(X/\mathscr{F})] = p^r$.

(ii) There is a one-to-one correspondence between p-foliations on X and normal varieties between X and $X^{(p)}$, by the association $\mathscr{F}$ with $X/\mathscr{F}$.

(iii) The variety $X/\mathscr{F}$ is smooth if and only if $\mathscr{F}$ is a smooth foliation.

Proof. See [Miy87, Proposition 1.9]. □

3.3 Examples

Let k be a field and consider the affine plane over k, defined as $\mathbb{A}^2_k = \mathrm{Spec}(k[x, y])$. Then the tangent bundle $\mathcal{T}\mathbb{A}^2_k$ is generated by the derivations $\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y} \rangle$.

Example 1: Let $\mathscr{F}_1 = \langle \frac{\partial}{\partial x} \rangle$. Since $\mathscr{F}_1$ is rank 1, it is trivally involutive. Further, $\mathscr{F}_1$ is a saturated subbundle of $\mathcal{T}\mathbb{A}^2_k$ since $\mathcal{T}\mathbb{A}^2_k/\mathscr{F}_1 = \langle \frac{\partial}{\partial y} \rangle$ is torsion-free. Indeed, the stalks of $\mathcal{T}\mathbb{A}^2_k/\mathscr{F}_1$ are free at every point of $\mathbb{A}^2_k$, thus $\mathscr{F}_1$ is a smooth foliation.

Example 2: Consider the subsheaf $\mathscr{F}_2 = \langle x\frac{\partial}{\partial x} \rangle \subset \mathcal{T}\mathbb{A}^2_k$. Let p be a point of $\mathbb{A}^2_k$ such that $x = 0$. Then, we note that $\frac{\partial}{\partial x} \notin \mathscr{F}_{2,p}$, since if $ax\frac{\partial}{\partial x} = \frac{\partial}{\partial x}$, we must have $a = x^{-1} \notin \mathcal{O}_{\mathbb{A}^2_k,p}$. However, $x\frac{\partial}{\partial x} \in \mathscr{F}_{2,p}$, by definition. Thus $\mathscr{F}_2$ is not saturated everywhere, so $\mathscr{F}_2$ is not a foliation on $\mathbb{A}^2_k$. However, the saturation of $\mathscr{F}_2$ in $\mathcal{T}\mathbb{A}^2_k$ is just $\mathscr{F}_1$, which is a smooth foliation.

Example 3: Consider $\mathscr{F}_3 = \langle x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} \rangle \subset \mathcal{T}\mathbb{A}^2_k$. We can show that $\mathscr{F}_3$ is saturated. Let $f\frac{\partial}{\partial x} + g\frac{\partial}{\partial y} \in \mathcal{T}\mathbb{A}^2_k$ and suppose there exists $a \in k[x, y]$ such that $af\frac{\partial}{\partial x} + ag\frac{\partial}{\partial y} \in \mathscr{F}_3$. Then there must be some $h \in k[x, y]$ such that

$$af\frac{\partial}{\partial x} + ag\frac{\partial}{\partial y} = xh\frac{\partial}{\partial x} + yh\frac{\partial}{\partial y}.$$

For $p \in k[x,y]$, let $|p|_x$ be the highest power of x dividing p. Since $af = xh$, we have $|a|_x + |f|_x = 1 + |h|_x$. Similarly, $ag = yh$. So $|a|_x + |g|_x = |h|_x$. Thus $|f|_x = 1 + |g|_x$. In particular, $|f|_x \geq 1$. Thus $x|f$.

Similarly, $y|g$. But $\frac{af}{x} = h = \frac{ag}{y}$. In particular $\frac{h}{a} \in k[x,y]$. Thus

$$f\frac{\partial}{\partial x} + g\frac{\partial}{\partial y} = \frac{h}{a}\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right).$$

Thus $\mathscr{F}_3$ is saturated. So $\mathscr{F}_3$ is a foliation. However, it is not a smooth foliation, as the quotient $\mathcal{T}\mathbb{A}^2_k/\mathscr{F}_3$ is not locally free. In particular the stalk of this quotient at the origin is not free. This will be proven for a somewhat more general version of this foliation in Theorem 3.21.

So $\mathscr{F}_3$ is a singular foliation, with singular locus equal to the origin.

Example 4: Consider now the case that k is a field of characteristic p. Then we can calculate:

$$\frac{\partial}{\partial x}^{(p)}(x^a y^b) = a(a-1)(a-2)\dots(a-p+1)x^{a-p}y^b.$$

Note that since $a \in \mathbb{Z}$, at least one of $a, (a-1), (a-2), \dots, (a-p+1)$ must vanish modulo p, thus $\frac{\partial}{\partial x}^{(p)} = 0$. Also, we can calculate:

$$\left(x\frac{\partial}{\partial x}\right)^{(p)}(x^a y^b) = a^p x^a y^b$$

But, as $a \in \mathbb{Z}$, and $a^p \equiv a$ modulo p, we have $(x\frac{\partial}{\partial x})^{(p)} = x\frac{\partial}{\partial x}$.

These calculations show us that each of the foliations in the previous examples are p-closed, and are thus p-foliations, when considered over a field k of characteristic p. As such, we can compute the quotients $\mathbb{A}^2_k/\mathscr{F}_1$ and $\mathbb{A}^2_k/\mathscr{F}_3$. In the case of the smooth foliation $\mathscr{F}_1 = \langle\frac{\partial}{\partial x}\rangle$, it is easy to see that

$$\mathrm{Ann}(\mathscr{F}_1) = k[x^p, y].$$

Thus, $\mathbb{A}^2_k/\mathscr{F}_1 = \operatorname{Spec}(k[x^p, y])$, which is just another affine plane. On the other hand, when we consider a foliation with a singularity, such as $\mathscr{F}_3 = \langle x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\rangle$, we compute that

$$\operatorname{Ann}(\mathscr{F}_3) = k[x^p, x^{p-1}y, \dots, xy^{p-1}, y^p].$$

Thus $\mathbb{A}^2_k/\mathscr{F}_3 = \operatorname{Spec}(k[x^p, x^{p-1}y, \dots, xy^{p-1}, y^p])$, a surface with a cyclic quotient singularity at the origin.

However, not all foliations are p-closed. For example, let $\alpha \in k$ such that $\alpha \notin \mathbb{F}_p$, and consider the foliation $\mathscr{F}_4 = \langle x\frac{\partial}{\partial x} + \alpha y\frac{\partial}{\partial y}\rangle$. We can see that $\mathscr{F}_4$ is not p-closed by computing $(x\frac{\partial}{\partial x} + \alpha y\frac{\partial}{\partial y})^{(p)} = x\frac{\partial}{\partial x} + \alpha^p y\frac{\partial}{\partial y} \notin \mathscr{F}_4$. So $\mathscr{F}_4$ is not a p-foliation.

3.4 Foliations on Toric Varieties

Let X be a toric variety over a field k of arbitrary characteristic, containing an open dense torus T. A **torus-equivariant vector bundle** $\mathscr{E}$ on X is a vector bundle along with a map $T \times \mathscr{E} \to \mathscr{E}$, such that the following diagram commutes:

$$\begin{tikzcd} T \times \mathscr{E} \arrow[r] \arrow[d] & \mathscr{E} \arrow[d] \\ T \times X \arrow[r] & X \end{tikzcd}.$$

We further require that for any $x \in X$ and $t \in T(k)$, the map $\mathscr{E}_x \to \mathscr{E}_{t\cdot x}$ is a linear isomorphism.

Proposition 3.7. *A bundle $\mathscr{E}$ on a toric variety X can be endowed with a toric structure if and only if $\mathscr{E} \cong t^*\mathscr{E}$ for every $t \in T(k)$.*

Proof. [Kly90, Proposition 1.2.1]. □

Note that this does not imply that the given isomorphisms $\mathscr{E} \cong t^*\mathscr{E}$ are the toric structure, merely that if such a family of isomorphisms exists, then a toric structure also exists.

3.4.1 Klyachko Filtrations

In [Kly90], Klyachko classified vector bundles on a toric variety $X(\Sigma)$ in terms of a collection of filtrations indexed by the rays $\tau \in \Sigma(1)$. We review the relevant results here.

Let T be a split torus over k, and let σ be a cone in the cocharacter lattice N of T. Then $X = X(\sigma) := \operatorname{Spec}(k[S_\sigma])$ is an affine toric variety. X has a unique closed T-orbit. Let $T|_\sigma \subseteq T$ be the stabilizer of a point in the closed orbit.

Proposition 3.8. *Let $X = X(\sigma)$ be an affine toric variety. Then all toric bundles $\mathscr{E}$ on X take the form $\mathscr{E} = E \times X$, where E is a representation of T, with toric action on $\mathscr{E}$ given by $t \cdot (e, x) = (t \cdot e, tx)$.*

Proof. [Kly90, Proposition 2.1.1(i)] The Quillen–Suslin Theorem states that every projective module over a polynomial ring is free. Equivalently, every vector bundle over affine space is trivial. This was extended by Gubeladze in [Gub87] to affine toric varieties. Thus $\mathscr{E}$ is a trivial vector bundle over X, that is $\mathscr{E} = E' \times X$ for some vector space E'. We would like to show that there is a T-action on E' that extends to the toric action on $\mathscr{E}$. We will do this by defining a vector space E with torus action, and showing that it is isomorphic to E', in such a way that the T-action on E induces the toric action on $\mathscr{E}$.

Let x_σ be a point in the closed orbit of X. Then, consider the map $p : \Gamma(X, \mathscr{E}) \to \mathscr{E}(x_\sigma) \cong E'$, such that $p(s) = s(x_\sigma)$. This is a surjective map, since $\mathscr{E}$ is a trivial bundle.

First, we claim that there is a T-invariant subspace $E \subseteq \Gamma(X, \mathscr{E})$ on which p is an isomorphism. Let E be a maximal T-invariant subspace on which p is injective. By the diagonalizability of torus representations [Mil17, Theorem 12.12], the representation $\Gamma(X, \mathscr{E})$ of the torus T decomposes into character spaces $\Gamma(X, \mathscr{E}) = \bigoplus_\chi E_\chi$. So $\bigoplus_\chi p(E_\chi) = \mathscr{E}(x_\sigma)$, since p is surjective. If $p(E) \neq \mathscr{E}(x_\sigma)$, there must be some χ and $\gamma \in E_\chi$ such that $p(\gamma) \notin p(E)$. Then $\gamma \notin E$. Thus p is injective on $E + \langle \gamma \rangle$. Also, for any $t \in T$ we have $t \cdot \gamma = \chi(t)\gamma$. thus $E + \langle \gamma \rangle$ is T-invariant. But this contradicts the maximality of E. Thus $p(E) = \mathscr{E}(x_\sigma)$. In other words, $p : E \to \mathscr{E}(x_\sigma)$ is an isomorphism.

Now, let $s_i \in E$ be such a basis of T-eigenvectors. Clearly, they are linearly independent over x_σ. Suppose there exists some point x_0 over which the s_i are not linearly independent. Then, since the s_i are T-eigenvectors, we see that they are not linearly independent over any point tx_0. Thus, they are not linearly independent over the orbit containing x_0. However, since such a linear dependency is a closed condition, we know that the s_i must not be linearly independent over the closure of this orbit. But the unique closed orbit (containing x_σ) is in the closure of every T-orbit on X, by Proposition 2.14. So this contradicts the fact that the s_i are linearly independent over x_σ. Therefore, they are linearly independent over all $x \in X$. Thus $\mathscr{E} = E \times X$ as a toric bundle. That is, $\mathscr{E}$ extends the T-action on E. $\square$

Proposition 3.9. *Let $\mathscr{E} = E \times X$ and $\mathscr{F} = F \times X$ be toric bundles on an affine toric variety $X = X(\sigma)$. For each ray $\tau \in \sigma(1)$, let α_τ be the generator of τ and define a decreasing $\mathbb{Z}$-filtration on E, (similarly on F), as follows:*

$$E^\tau(i) = \bigoplus_{\langle \chi, \alpha_\tau \rangle \geq i} E_\chi$$

where, for $\chi \in M$, the vector space E_χ is the χ-isotypical component of E. Then, $Hom_T(\mathscr{E}, \mathscr{F})$ is canonically isomorphic to:

$$\{\varphi \in Hom(E, F) : \varphi(E^\tau(i)) \subset F^\tau(i), \tau \in \sigma(1), i \in \mathbb{Z}\}.$$

Proof. [Kly90, Proposition 2.1.1(iii)] Consider the case $\dim(E) = \dim(F) = 1$. Let χ^{m_E} and χ^{m_F} be the characters by which T acts on E and F respectively. Then, we see that a bundle morphism $f : \mathscr{E} \to \mathscr{F}$, is a family of maps $\varphi_x : E \to F$, parameterized by $x \in X$. For such a morphism to be T-equivariant, it must satisfy:

$$\varphi_{tx}(te) = t \cdot \varphi_x(e).$$

That is:

$$\chi^{m_E}(t)\varphi_{tx}(e) = \chi^{m_F}(t)\varphi_x(e).$$

Thus, if we fix x_0 in the open T-orbit of X, and denote $\varphi := \varphi_{x_0}$. Then for any point $x_1 = tx_0$ in the open orbit, we must have:

$$f(e, tx_0) = (\varphi_{tx_0}(e), tx_0) = \left(\frac{\chi^{m_F}}{\chi^{m_E}}(t)\varphi(e), tx_0\right).$$

Note that this formula determines f uniquely, as it gives the value of f on an open dense subset of X. Furthermore, since f is regular, we know that either $\varphi = 0$ or the character $\frac{\chi^{m_F}}{\chi^{m_E}} = \chi^{m_F - m_E}$ is a regular function on X. Since $X = \mathrm{Spec}(k[S_\sigma])$, we see that $\chi^{m_F - m_E}$ is a regular function if and only if it is in S_σ, i.e. $\langle m_F - m_E, \alpha_\tau\rangle \geq 0$ for all $\tau \in \sigma(1)$. Equivalently, that $\langle m_F, \alpha_\tau\rangle \geq \langle m_E, \alpha_\tau\rangle$ for all $\tau \in \sigma(1)$. However, this is equivalent to saying that $\varphi(E^\tau(i)) \subseteq F^\tau(i)$ for all $\tau \in \sigma(1)$ and $i \in \mathbb{Z}$, as required.

On the other hand, suppose there exists some non-zero $\varphi \in \mathrm{Hom}(E, F)$, such that $\varphi(E^\tau(i)) \subseteq F^\tau(i)$ for all $\tau \in \sigma(1)$, and $i \in \mathbb{Z}$, we must have $F^\tau(i) = 0$ only if $E^\tau(i) = 0$. Thus, $\langle m_E, \alpha_\tau\rangle \leq \langle m_F, \alpha_\tau\rangle$ for all $\tau \in \sigma(1)$. So the character $\chi^{m_F - m_E} \in S_\sigma$ and thus extends as a regular function to X. Thus, the map

$$f(e, tx_0) = \left(\frac{\chi^{m_F}}{\chi^{m_E}}(t)\varphi(e), tx_0\right)$$

extends to a regular map on $E \times X$.

Thus we get a canonical isomorphism $\mathrm{Hom}_T(\mathscr{E}, \mathscr{F})$ with the subset of $\mathrm{Hom}(E, F)$ that respects the filtrations.

Now, if $\dim(E)$ or $\dim(F)$ is greater than 1, we see from the proof of the previous proposition, that E and F are generated by T-eigenvectors. Thus $\mathscr{E}$ and $\mathscr{F}$ are sums of T-invariant line bundles, and these sums respect the filtrations given. Thus, this result generalizes to E and F of arbitrary dimension. $\square$

3.4.2 Classifying Toric Foliations

We will now specialize to looking specifically at the tangent bundle, in order to classify the toric foliations. Let X be a toric variety of dimension n with open dense torus T. Let Σ be the fan defining X, and let $\sigma \in \Sigma$. Let $\tau \in \sigma(1)$ be a ray of σ. Then τ determines a closed codimension-1 subvariety of U_σ, corresponding to the prime ideal:

$$I_{\sigma,\tau} := \langle \chi^m; m \in S_\sigma, \langle m, n\rangle > 0, n \in \tau\rangle \subseteq k[S_\sigma].$$

This subvariety is precisely the closure of the T-orbit of the point induced by the map:

$$S_\sigma \to k : m \mapsto \begin{cases} 1 & m \in \tau^\perp \\ 0 & m \notin \tau^\perp \end{cases}$$

This is the codimension-1 T-orbit Z_τ. Call this subvariety $V(\tau)$.

Let D be the T-invariant Weil divisor $\sum_{\tau\in\Sigma(1)} V(\tau)$, where $\Sigma(1)$ is the set of rays in the fan Σ. Let $\mathscr{I}$ be the sheaf of ideals of $\mathcal{O}_X$ determining D. In particular, for $\sigma \in \Sigma$, we see that

$$\mathscr{I}(U_\sigma) = I_\sigma = \bigcap_{\tau\in\sigma(1)} I_{\sigma,\tau},$$

as described in Lemma 2.15.

Now, we define $\mathcal{T}X(-\log(D))$ to be the subsheaf of $\mathcal{T}X$ consisting of derivations δ such that $\delta(\mathscr{I}) \subseteq \mathscr{I}$.

Proposition 3.10. *Let X be a toric variety, corresponding to the fan Σ. Then there is a T-equivariant isomorphism $\mathcal{T}X(-\log(D)) \cong N \otimes_{\mathbb{Z}} \mathcal{O}_X$, where N is the cocharacter lattice of T.*

Proof. [Oda88, Proposition 3.1] Suppose that for each cone $\sigma \in \Sigma$ we have such an isomorphism $\mathcal{T}U_\sigma(-\log(D)) \cong N \otimes_{\mathbb{Z}} \mathcal{O}_{U_\sigma}$, that respects the inclusions $\tau \subset \sigma$ for any face τ of σ. Then we can glue these isomorphisms together to get an isomorphism of sheaves $\mathcal{T}X(-\log(D)) \cong N \otimes_{\mathbb{Z}} \mathcal{O}_X$. We will show that such an isomorphism exists for each cone $\sigma \in \Sigma$. Note that after restriction to the affine open U_σ, the ideal sheaf $\mathscr{I}$ is just the coherent sheaf associated to the ideal $I_\sigma \subseteq k[S_\sigma]$.

For each $\sigma \in \Sigma$, define a homomorphism $\Delta_\sigma : N \otimes \mathcal{O}_{U_\sigma} \to \mathcal{T}X(-\log(D))$ by:

$$n \otimes f \mapsto f\delta_n.$$

where δ_n is defined to be the k-derivation such that

$$\delta_n(\chi^m) = \langle m, n\rangle \chi^m.$$

Since $\chi^{m_1}\chi^{m_2} = \chi^{m_1+m_2}$ we need to check that δ_n is a well-defined derivation. We see that:

$$\delta_n(\chi^{m_1}\chi^{m_2}) = \chi^{m_1}\langle m_2, n\rangle\chi^{m_2} + \chi^{m_2}\langle m_1, n\rangle\chi^{m_1}$$

$$= \langle m_1 + m_2, n\rangle\chi^{m_1}\chi^{m_2} = \delta_n(\chi^{m_1+m_2}).$$

Thus δ_n is a well-defined k-derivation.

Further, if $\varphi = \sum_{i=1}^n c_i\chi^{m_i} \in I_\sigma$, then each $\chi^{m_i} \in I_\sigma$, as each monomial in φ must be a product of a monomial in $\chi^{m_{i1}} \in k[S_\sigma]$, and a generator $\chi^{m_{i2}}$ of I_σ. So $\langle m_{i1}, n\rangle \geq 0$ for all $n \in \sigma$ and $\langle m_{i2}, n\rangle > 0$ for all $n \in \sigma$. Therefore, we see that $\langle m_i, n\rangle = \langle m_{i1} + m_{i2}, n\rangle > 0$, which implies $\chi^{m_i} \in I_\sigma$.

Then, since $\delta_n(\chi^{m_i})$ is a multiple of χ^{m_i}, we must have $\delta_n(\chi^{m_i}) \in I_\sigma$. Therefore $\delta_n(I_\sigma) \subseteq I_\sigma$. So the image of Δ_σ is contained in $\mathcal{T}X(-\log(D))$. Thus Δ_σ is a well-defined map $N \otimes \mathcal{O}_{U_\sigma} \to \mathcal{T}X(-\log(D))$.

Next, we will verify that Δ_σ is indeed a homomorphism. Given $n_1, n_2 \in N$, we have:

$$\delta_{n_1+n_2}(\chi^m) = \langle n_1 + n_2, m\rangle\chi^m = \langle n_1, m\rangle\chi^m + \langle n_2, m\rangle\chi^m$$

$$= \delta_{n_1}(\chi^m) + \delta_{n_2}(\chi^m).$$

Thus the map $N \to \mathcal{T}U_\sigma(-\log(D))$ given by $n \mapsto \delta_n$ is a homomorphism of $\mathbb{Z}$-modules.

Thus Δ_σ, as defined above, is a homomorphism of $\mathcal{O}_{U_\sigma}$-modules. We would like to show that it is an isomorphism.

The injectivity is clear, since if $n_1, n_2 \in N$ such that $\langle m, n_1\rangle = \langle m, n_2\rangle$ for all $m \in S_\sigma$, then $\langle m, n_1\rangle = \langle m, n_2\rangle$ for all $m \in M$, as S_σ generates M as a group, by Proposition 2.8. Thus $n_1 = n_2$, as M is the dual lattice to N. It remains to show surjectivity.

Suppose $\delta \in \mathcal{T}U_\sigma(-\log(D))$. Then $\delta(I_\sigma) \subseteq I_\sigma$. Recall $I_\sigma = \bigcap_{\tau\in\sigma(1)} I_{\sigma,\tau}$ is a minimal prime decomposition, by Lemma 2.15. We will then show that $\delta(I_{\sigma,\tau}) \subseteq I_{\sigma,\tau}$ for each $\tau \in \sigma(1)$.

Let $\tau_0 \in \sigma(1)$. Then, since $I_\sigma = \bigcap_{\tau\in\sigma(1)} I_{\sigma,\tau}$ is a minimal prime decomposition, for any $x \in I_{\sigma,\tau_0}$ we can find some $y \notin I_{\sigma,\tau_0}$ such that $xy \in I_\sigma$. Since $\delta(I_\sigma) \subseteq I_\sigma$, we know that $\delta(xy) \in I_\sigma \subseteq I_{\sigma,\tau_0}$. But:

$$\delta(xy) = x\delta(y) + y\delta(x).$$

Thus, since $\delta(xy)$ and $x\delta(y)$ are in I_{σ,τ_0}, it must be that $y\delta(x)$ is in I_{σ,τ_0}. Further, as $y \notin I_{\sigma,\tau_0}$, and I_{σ,τ_0} is prime, it must be that $\delta(x) \in I_{\sigma,\tau_0}$. Thus $\delta(I_{\sigma,\tau_0}) \subseteq I_{\sigma,\tau_0}$.

By Lemma 2.17, we also know that for any $m \in S_\sigma$, the ideal $\langle \chi^m\rangle \subseteq k[S_\sigma]$ has primary ideal decomposition:

$$\langle \chi^m\rangle = \bigcap_{\tau\in\sigma(1)} I_{\sigma,\tau}^{(\langle m,n_\tau\rangle)}.$$

where n_τ is a generator of the ray $\tau \subseteq N$. Since δ preserves $I_{\sigma,\tau}$, it also preserves its symbolic powers, by Lemma 2.5. Hence $\delta(\langle\chi^m\rangle) \subseteq \langle\chi^m\rangle$. In particular, for any $m \in S_\sigma$, there exists $\xi(m) \in k[S_\sigma]$ such that $\delta(\chi^m) = \xi(m)\chi^m$.

So $\xi : S_\sigma \to k[S_\sigma]$. Note that this is an additive map, since for any $m_1, m_2 \in S_\sigma$:

$$\delta(\chi^{m_1}\chi^{m_2}) = \delta(\chi^{m_1+m_2}) = \xi(m_1+m_2)\chi^{m_1+m_2} = \xi(m_1+m_2)\chi^{m_1}\chi^{m_2}.$$

Also

$$\delta(\chi^{m_1}\chi^{m_2}) = \chi^{m_1}\delta(\chi^{m_2}) + \chi^{m_2}\delta(\chi^{m_1}) = \chi^{m_1}\xi(m_1)\chi^{m_2} + \chi^{m_2}\xi(m_2)\chi^{m_1}$$

$$= (\xi(m_1) + \xi(m_2))\chi^{m_1}\chi^{m_2}.$$

Therefore, since S_σ generates M as a group, ξ extends to a homomorphism $M \to k[S_\sigma]$. Further, as N is the dual of M. Any homomorphism $M \to k[S_\sigma]$ can be viewed as an element of $k[S_\sigma] \otimes_{\mathbb{Z}} N$.

Thus, there exists $\sum_{i=1}^n a_i n_i$ such that $\xi(m) = \sum_{i=1}^n a_i \langle m, n_i \rangle$. Thus

$$\sum_{i=1}^n a_i \delta_{n_i}(\chi^m) = \sum_{i=1}^n a_i \langle m, n_i \rangle \chi^m = \xi(m)\chi^m = \delta(\chi^m).$$

So $\delta = \sum_{i=1}^n a_i \delta_{n_i}$. Thus, Δ_σ is surjective.

Further, from the definition of Δ_σ, it is clear that for any face $\tau \subset \sigma$ the following square commutes:

$$\begin{array}{ccc} k[S_\sigma] \otimes N & \xrightarrow{\Delta_\sigma} & \mathcal{T}U_\sigma(-\log(D)) \\ \downarrow & & \downarrow \\ k[S_\tau] \otimes N & \xrightarrow{\Delta_\tau} & \mathcal{T}U_\tau(-\log(D)) \end{array}$$

where the vertical arrows are restrictions from U_σ to U_τ. Therefore, this we can glue the isomorphisms Δ_σ to get an isomorphism $\Delta : \mathcal{O}_X \otimes N \to \mathcal{T}X(-\log(D))$.

Finally, recall that the action of T on $\mathcal{T}X$ is such that for any derivation δ, we have:

$$(t \cdot \delta)(f)(x) = \delta(f \circ t^{-1})(t \cdot x).$$

So, for any $n \in N$, we can calculate for any $m \in M$ and $x_0 \in X$:

$$\begin{aligned}
(t \cdot \delta_n)(\chi^m)(x_0) &= \delta_n(\chi^m \circ t^{-1})(t \cdot x_0) \\
&= \delta_n(\chi^m(t^{-1}) \cdot \chi^m)(t \cdot x_0) \\
&= \chi^m(t^{-1})\delta_n(\chi^m)(t \cdot x_0) \\
&= \chi^m(t^{-1})\langle m, n\rangle \chi^m(t \cdot x_0) \\
&= \chi^m(t^{-1})\chi^m(t)\langle m, n\rangle \chi^m(x_0) \\
&= \langle m, n\rangle \chi^m(x_0) \\
&= \delta_n(\chi^m)(x_0).
\end{aligned}$$

Thus $t \cdot \delta_n = \delta_n$ for any $n \in N$. Further, for any $f \in \mathcal{O}_X$ and $\delta \in \mathcal{T}X$, we have:

$$t \cdot (f\delta)(g)(x_0) = (f\delta)(g \circ t^{-1})(t \cdot x_0) = f(t \cdot x_0)\delta(g \circ t^{-1})(t \cdot x_0)$$

$$= (t \cdot f)(t \cdot \delta)(g)(x_0).$$

Thus the isomorphism Δ is equivariant with respect to the action of T on $\mathcal{T}X(-\log(D))$ derived from the action of T on X, and the action of T on $N \otimes \mathcal{O}_X$ given by the trivial action on N, and the action on $\mathcal{O}_X$ derived from the action of T on X. □

Lemma 3.11. *For any $n_1, n_2 \in N$, we have $[\delta_{n_1}, \delta_{n_2}] = 0$. Thus, any subsheaf of $\mathcal{T}X(-\log(D))$ generated by $\{\delta_{n_i}\}$ for some subset $\{n_i\} \subseteq N$ is involutive.*

Proof. For any $m \in M$, we have:

$$\begin{aligned}
[\delta_{n_1}, \delta_{n_2}](\chi^m) &= \delta_{n_1}(\delta_{n_2}(\chi^m)) - \delta_{n_2}(\delta_{n_1}(\chi^m)) \\
&= \delta_{n_1}(\langle n_2, m\rangle \chi^m) - \delta_{n_2}(\langle n_1, m\rangle \chi^m) \\
&= \langle n_1, m\rangle\langle n_2, m\rangle \chi^m - \langle n_2, m\rangle\langle n_1, m\rangle \chi^m \\
&= 0.
\end{aligned}$$

Thus, since $k[S_\sigma] \subseteq k[M]$ is generated by the χ^m as a k-vector space,

$$[\delta_{n_1}, \delta_{n_2}] = 0.$$

$\square$

Example: Consider the affine toric variety U_σ over k given by the cone σ generated by $n_1 = (1,0)$ in $N = \mathbb{Z}^2$. So $S_\sigma = \{(a,b) \in \mathbb{Z}^2 : a \geq 0\}$. Thus, naming the first coordinate X and the second Y, we have $U_\sigma = \mathrm{Spec}(k[S_\sigma]) = \mathrm{Spec}(k[X, Y^{\pm 1}])$. This is just the product of the rank 1 torus with the affine line.

Now, δ_{n_1} is the derivation on $k[X, Y^{\pm 1}]$ defined such that $\delta_{n_1}(X^a Y^b) = aX^a Y^b$. So $\delta_{n_1} = X\frac{\partial}{\partial X}$. Similarly, if $n_2 = (0,1) \in N$, then δ_{n_2} is the derivation on $k[X, Y^{\pm 1}]$ defined such that $\delta_{n_2}(X^a Y^b) = bX^a Y^b$. So $\delta_{n_2} = Y\frac{\partial}{\partial Y}$.

We know that the tangent space $\mathcal{T}U_\sigma$ of $U_\sigma = \mathrm{Spec}(k[X, Y^{\pm 1}])$ is generated by $\frac{\partial}{\partial X}$ and $Y\frac{\partial}{\partial Y}$, that is $\frac{1}{X}\delta_{n_1}$ and δ_{n_2}. Then $\mathcal{T}U_\sigma(-\log(D))$, where D is given by the divisor of $X = 0$ is generated by $X\frac{\partial}{\partial X} = \delta_{n_1}$ and $Y\frac{\partial}{\partial Y} = \delta_{n_2}$. So the map $n_1 \mapsto \delta_{n_1}, n_2 \mapsto \delta_{n_2}$ is indeed an isomorphism $N \otimes \mathcal{O}_X \to \mathcal{T}U_\sigma(-\log(D))$ in this example.

Example: This construction works in general to find the tangent space $\mathcal{T}U_\sigma$ for some smooth σ in N, the co-character lattice of a rank g split torus. Let $\{u_{q+1}, \ldots, u_g\}$ be a basis for σ as a semigroup that extends to a basis $\{u_1, \ldots, u_g\}$ for N as a group. Let $\{m_1, \ldots, m_g\} \subset M$ be a dual basis to $\{u_1, \ldots, u_g\}$. Then $S_\sigma = k[m_1^{\pm 1}, \ldots, m_q^{\pm 1}, m_{q+1}, \ldots, m_g]$. So $U_\sigma = k[S_\sigma]$.

So Z_σ is the closed subvariety given by $\chi^{m_j} = 0$ for $q+1 \leq j \leq g$. As in Proposition 3.8, we want to construct the tangent space $\mathcal{T}U_\sigma$ as a product $E \times U_\sigma$ where E is a representation of T. Consider the point $x_\sigma \in Z_\sigma$ where $\chi^{m_j}(x_\sigma) = 1$ or $1 \leq j \leq q$. Also, since $x_\sigma \in Z_\sigma$, we must have $\chi^{m_j} = 0$ for $q+1 \leq j \leq g$.

Then the fibre $\mathcal{T}U_\sigma(x_\sigma) = \mathrm{span}_k\{\frac{\partial}{\partial \chi^{m_i}}\}_{i=1}^{g}$. Following a similar computation as in the previous example, we can deduce that for each u_i, we have $\delta_{u_i} = \chi^{m_i}\frac{\partial}{\partial \chi^{m_i}}$. Note that since each δ_{u_i} is T-invariant, each of the δ_{u_i} are T-eigenvectors, with respect to the trivial character. So, we can begin constructing our E using the δ_{u_i}. Since $\chi^{m_i}(x_\sigma) = 1$

for $1 \le i \le q$, the image of δ_{u_i} in the fibre $\mathcal{T}U_\sigma(x_\sigma)$ is just $\frac{\partial}{\partial \chi^{m_i}}$. However, $\chi^{m_j}(x_\sigma)$ for $q+1 \le j \le g$ is 0. So the image of δ_{u_j} in this fibre is zero.

Consider instead the derivations $\frac{1}{\chi^{m_j}}\delta_{u_j}$ for $q+1 \le j \le g$. Then for $t \in T$ we have

$$t \cdot (\frac{1}{\chi^{m_j}}\delta_{u_j}) = \frac{\chi^{m_j}(t)}{\chi^{m_j}}(t \cdot \delta_{u_j}) = \chi^{m_j}(t)\frac{1}{\chi^{m_j}}\delta_{u_j}.$$

Thus $\frac{1}{\chi^{m_j}}\delta_{u_j}$ is a T-eigenvector with character χ^{m_j}. Further, the image of $\frac{1}{\chi^{m_j}}\delta_{u_j}$ on the fibre over x_σ is just $\frac{\partial}{\partial \chi^{m_j}}$.

So, by Proposition 3.8, we have $\mathcal{T}U_\sigma = E \times U_\sigma$ where E is

$$E = \mathrm{span}_k\{\delta_{u_1}, \dots, \delta_{u_q}, \frac{1}{\chi^{m_{q+1}}}\delta_{u_{q+1}}, \dots, \frac{1}{\chi^{m_g}}\delta_{u_g}\}.$$

So we get the following corollary:

Corollary 3.12. *Let U_σ be an affine toric variety, with notation as above. Then*

$$\mathcal{T}U_\sigma = \mathrm{span}_{\mathcal{O}_{U_\sigma}}\left\{\delta_{u_1}, \dots, \delta_{u_q}, \frac{1}{\chi^{m_{q+1}}}\delta_{u_{q+1}}, \dots, \frac{1}{\chi^{m_g}}\delta_{u_g}\right\}.$$

Further, since we have clearly defined the T-action on E, we can explicitly describe the filtrations from Proposition 3.9 for $\mathcal{T}U_\sigma$ here. Note that the eigenspace decomposition of E is

$$E_1 = \langle \delta_{u_1}, \dots, \delta_{u_q} \rangle, \qquad E_{\chi^{m_j}} = \left\langle \frac{1}{\chi^{m_j}}\delta_{u_j}, q+1 \le j \le g \right\rangle.$$

For each $\tau_j = \langle u_j \rangle \in \sigma(1)$, that is, for $q+1 \le j \le g$, we have:

$$E^{\tau_j}(i) = \begin{cases} E & i \le 0 \\ \left\langle \frac{1}{\chi^{m_j}}\delta_{u_j} \right\rangle & i = 1 \\ 0 & i \ge 2 \end{cases}.$$

Proposition 3.13. *There is a 1-1 correspondence between the following sets:*

$$\{\textit{Foliations of } X\} \Leftrightarrow \{\textit{Involutive saturated subsheaves of } \mathcal{T}X(-\log(D))\ \}.$$

Proof. First, note that $\mathcal{T}X(-\log(D))$ is involutive. If $\mathscr{I}$ is the ideal sheaf defining D, and f is a section of $\mathscr{I}$. Then for any $\delta, \xi \in \mathcal{T}X(-\log(D)$, we have

$$[\delta, \xi](f) = \delta(\xi(f)) - \xi(\delta(f)).$$

But since ξ and δ are in $\mathcal{T}X(-\log(D))$, they map sections of $\mathscr{I}$ to sections of $\mathscr{I}$. Thus $[\delta, \xi](f)$ is a section of $\mathscr{I}$. Thus $[\delta, \xi] \in \mathcal{T}X(-\log(D))$. So $\mathcal{T}X(-\log(D))$ is involutive.

Let $\mathscr{F}$ be a foliation on X. So $\mathscr{F}$ is an involutive saturated subsheaf of $\mathcal{T}X$. As such, we can associate it with its restriction to $\mathcal{T}X(-\log(D))$, which remains involutive and saturated in $\mathcal{T}X(-\log(D))$.

On the other hand, let $\mathscr{G}$ be an involutive saturated subsheaf of $\mathcal{T}X(-\log(D))$. Consider the saturation $\mathscr{G}^{\text{sat}}$ of $\mathscr{G}$ in $\mathcal{T}X$. Then $\mathscr{G}^{\text{sat}}$ remains involutive by Lemma 3.4, and is a saturated subsheaf of $\mathcal{T}X$. That is, $\mathscr{G}^{\text{sat}}$ is a foliation.

Finally, note that these are inverse operations. Let $\mathscr{I}$ be the ideal sheaf defining D, and let $\mathscr{F}$ be a foliation on X. Then if δ is a section of $\mathscr{F}$, and $f \in \mathscr{I}$, we see that $f\delta \in \mathcal{T}X(-\log(D))$. Thus, $f\delta$ is a section of the restriction of $\mathscr{F}$ to $\mathcal{T}X(-\log(D))$. Thus, δ is in the saturation in $\mathcal{T}X$ of this restriction.

On the other hand, suppose δ is in the saturation in $\mathcal{T}X$ of the restriction of $\mathscr{F}$ to $\mathcal{T}X(-\log(D))$. Then there is some $f \in \mathcal{O}_X$ such that $f\delta \in \mathscr{F}(-\log(D))$. So $f\delta \in \mathscr{F}$, which implies $\delta \in \mathscr{F}$, since $\mathscr{F}$ is saturated in $\mathcal{T}X$. $\square$

Proposition 3.14. *Let X be a toric variety with torus T defined over a field k. Let D be the divisor as described previously. There is a 1-1 correspondence between the fol-*

lowing sets:

$$\left\{\begin{array}{c} T\text{-equivariant saturated} \\ \text{subsheaves of } \mathcal{T}X(-\log(D)) \end{array}\right\} \Leftrightarrow \{\text{subspaces of } N \otimes_{\mathbb{Z}} k\}$$

where N is the cocharacter lattice of T.

Proof. By Proposition 3.10, we know that

$$\mathcal{T}X(-\log(D)) \cong N \otimes_{\mathbb{Z}} \mathcal{O}_X.$$

Let V be a subspace of $N \otimes_{\mathbb{Z}} k$. So $V \otimes_k \mathcal{O}_X$ is a subsheaf of $\mathcal{T}X(-\log(D))$. Let $\mathscr{F}$ be the saturation of $V \otimes_k \mathcal{O}_X$ in $\mathcal{T}X(-\log(D))$. Note that $\mathscr{F}$ is T-invariant, as the action of T on $N \otimes_{\mathbb{Z}} \mathcal{O}_X$ is defined by acting on the $\mathcal{O}_X$ part over k. Thus restricting to a subspace V of $N \otimes_{\mathbb{Z}} k$ remains T-invariant.

On the other hand, let $\mathscr{F}$ be a T-equivariant saturated subsheaf of $\mathcal{T}X(-\log(D))$. Let K be the function field of X. Then, if η is the generic point of X, the stalk $\mathscr{F}_\eta$ is a K-subspace of $\mathcal{T}X(-\log(D))_\eta$. So let $\{v_1, \ldots, v_r\}$ be a basis of $\mathscr{F}_\eta$ that extends to a K-basis $\{v_1, \ldots, v_r, \ldots, v_g\}$ of $\mathcal{T}X(-\log(D))_\eta$. Then, since $\mathcal{T}X(-\log(D)) \cong N \otimes_{\mathbb{Z}} \mathcal{O}_X$, we must have $\mathcal{T}X(-\log(D))_\eta \cong N \otimes_{\mathbb{Z}} K$. Let $\{n_1, \ldots, n_g\}$ be a basis of N. Then we can find $a_{ij}, b_{ij} \in \mathcal{O}_X$ such that:

$$v_i = \sum_{j=1}^{g} \frac{a_{ij}}{b_{ij}} \delta_{n_j}.$$

Thus $\mathcal{T}X(-\log(D))/\mathscr{F}$ is free away from the codimension 1 subscheme defined by $\{b_{ij} = 0\}_{1 \le i,j \le g}$.

But, since $\mathscr{F}$ is T-equivariant, the stalks $(\mathcal{T}X(-\log(D))/\mathscr{F})_x$ must be isomorphic along T-orbits. Since $(\mathcal{T}X(-\log(D))/\mathscr{F})_x$ is free for all x away from a codimension 1 subscheme, it is free for all x on the open T-orbit $U_0 \cong T$ of X. So, after restricting to U_0, the subsheaf $\mathscr{F}$ is a subbundle of $\mathcal{T}X(-\log(D))$.

So by Proposition 3.9, the inclusion $\mathscr{F} \hookrightarrow \mathcal{T}X(-\log(D))$ corresponds to a vector space morphism $V \hookrightarrow N \otimes k$. Over the open orbit, this is the inverse of the construction above. It remains to show that if two T-equivariant, involutive, saturated subsheaves of $\mathcal{T}X(-\log(D))$ coincide on U_0, then they are equal.

Suppose $\mathscr{F}_1, \mathscr{F}_2$ are T-equivariant, saturated subsheaves of $\mathcal{T}X(-\log(D))$ such that $\mathscr{F}_1$ and $\mathscr{F}_2$ coincide on U_0. Then let $\mathscr{F}_3 = \mathscr{F}_1 + \mathscr{F}_2$. Since $\mathscr{F}_1, \mathscr{F}_2$ coincide on U_0, the quotients $\mathscr{F}_3/\mathscr{F}_1$ and $\mathscr{F}_3/\mathscr{F}_2$ must be supported away from U_0, and thus by Lemma 3.3, they are torsion subsheaves of $\mathcal{T}X/(-\log(D))/\mathscr{F}_1$ and $\mathcal{T}X(-\log(D))/\mathscr{F}_2$ respectively. But $\mathscr{F}_1$ and $\mathscr{F}_2$ are saturated in $\mathcal{T}X(-\log(D))$. So $\mathscr{F}_3/\mathscr{F}_1 = 0$ and $\mathscr{F}_2/\mathscr{F}_1 = 0$. Thus $\mathscr{F}_1 = \mathscr{F}_3 = \mathscr{F}_2$ on X.

Thus we have a 1-1 correspondence between T-equivariant involutive saturated subsheaves of $\mathcal{T}X(-\log(D))$ and subspaces of $N \otimes_{\mathbb{Z}} k$. $\square$

Note that the subsheaf corresponding to $V \otimes_{\mathbb{Z}} \mathcal{O}_X$ is involutive, since $[\delta_{v_1}, \delta_{v_2}] = 0$ for any $v_1, v_2 \in N$ by Lemma 3.11. Thus:

$$\left[\sum f_i \delta_{v_i}, \sum g_j \delta_{v_j}\right] = \sum_{i,j} [f_i \delta_{v_i}, g_i \delta_{v_j}]$$

$$= \sum_{i,j} \left(f_i g_j [\delta_{v_i}, \delta_{v_j}] + f_i \delta_{v_i}(g_j)\delta_{v_j} - g_j \delta_{v_j}(f_i)\delta_{v_i}\right) \in V \otimes_{\mathbb{Z}} \mathcal{O}_X.$$

Thus combining the above two propositions gives us a 1-1 correspondence between subspaces of $N \otimes_{\mathbb{Z}} k$ and T-equivariant foliations of X, which we will call **toric foliations**. Given a subspace V of $N \otimes_{\mathbb{Z}} k$, we will denote the associated toric foliation by $\mathscr{F}_V$, that is

$$\mathscr{F}_V := \langle \delta_n : n \in V \rangle^{\text{sat}} \subseteq \mathcal{T}X.$$

Since $\mathscr{F}_V$ is generated by the derivations δ_n for $n \in V$, we see that the rank of $\mathcal{F}_V$ as an $\mathcal{O}_X$-module is equal to the k-dimension of V.

Lemma 3.15. *Let $u \in Aut(T)$, and let $\delta_n \in \mathcal{T}X$ be defined as above. Then, the differential $du : \mathcal{T}X_p \to \mathcal{T}X_{u(p)}$ maps $\delta_n|_p$ to $\delta_{un}|_{u(p)}$.*

Proof. Recall that a map $u : T \to T$ induces a map u^* on the character lattice M of T, such that for any $p \in T$, the character $(u^*\chi^m)(p) := \chi^m(u(p))$. The map u^* then has a dual map $(u^*)^\vee : N \to N$ on the cocharacter lattice N of T, such that for any $m \in M$ and $n \in N$, we have $\langle m, (u^*)^\vee n\rangle = \langle u^*m, n\rangle$. For ease of notation, we will denote $(u^*)^\vee$ as u.

Then we can calculate:

$$\delta_{un}(\chi^m)(u(p)) = \langle m, un\rangle \chi^m(u(p)) = \langle u^*m, n\rangle \chi^m(u(p)).$$

Also

$$du(\delta_n)(\chi^m)(p) = \delta_n(u^*\chi^m)(p) = \langle u^*m, n\rangle (u^*\chi^m)(p) = \langle u^*m, n\rangle \chi^m(u(p)).$$

Thus, for any $p \in T$, the differential du maps $\delta_n|_p$ to $\delta_{un}|_{u(p)}$, as required. □

Corollary 3.16. *Let U be a subgroup of $Aut(T)$, and let Σ be a U-invariant fan in the cocharacter lattice N of T. Then there is a 1-1 correspondence between the following sets*

$$\{\textit{Toric foliations of } X(\Sigma)/U\} \Leftrightarrow \{U\textit{-invariant subspaces of } N \otimes k\}.$$

Proof. Let V be a U-invariant subspace of $N \otimes k$, then the projection of $\mathscr{F}_V$ on $X(\Sigma)$ to $X(\Sigma)/U$ is well defined, since $\delta_n \in \mathscr{F}_V$ if and only if $\delta_{un} \in \mathscr{F}$ for $u \in U$, by the U-invariance of V. So V corresponds to a toric foliation of $X(\Sigma)/U$.

Similarly, let $\mathscr{F}$ is a toric foliation on $X(\Sigma)/U$. Then $\mathscr{F}$ pulls back to a toric foliation on $X(\Sigma)$. Thus, $\mathscr{F} = \mathscr{F}_V$ for some subspace V of $N \otimes k$. By the construction in Proposition 3.14, we know that $\mathscr{F} = \mathscr{F}_V$ where $V = \{n : \delta_n \in \mathscr{F}\}$. Further, since $\mathscr{F}$ is defined on $X(\Sigma)/U$, it must be that $\delta_n \in \mathscr{F}$ if and only if $\delta_{un} \in \mathscr{F}$ for any $u \in U$, by Lemma 3.15. Thus V is U-invariant.

Therefore, there is a 1-1 correspondence between toric foliations of $X(\Sigma)/U$ and U-invariant subspaces of $N \otimes k$. □

3.4.3 Classifying Toric p-Foliations

Let k be a perfect field of characteristic p. We will now provide an extension of the above results to p-foliations. Let N be the cocharacter lattice of a split torus T over k. Consider the action of Frobenius on $N \otimes k$, given by the trivial action on N, and Frobenius on k. Then a subspace V of $N \otimes k$ is called **p-closed** if it is closed under this action of Frobenius.

Lemma 3.17. *Let k be an finite extension of $\mathbb{F}_p$, and let N be a $\mathbb{F}_p$-vector space. Let $\sigma \in Gal(k, \mathbb{F}_p)$ denote the Frobenius automorphism. Suppose V is a k-subspace of $N \otimes k$ such that $\sigma(V) \subseteq V$. Then there exists an $\mathbb{F}_p$-subspace $V_0 \subseteq N$ such that $V = V_0 \otimes k$.*

Proof. This is exactly Galois descent for vector spaces (cf. [Bou81, V.10.4]). □

Proposition 3.18. *There is a 1-1 correspondence between the following sets:*

$$\{\textit{Toric } p\textit{-foliations on } X\} \Leftrightarrow \{p\textit{-closed subspaces of } N \otimes_{\mathbb{Z}} k\}.$$

Proof. To prove this proposition, we will start with the following Lemma:

Lemma 3.19. *Let X be a variety defined over a field of characteristic p. Let $\mathscr{F}$ be a foliation on $\mathcal{O}_X$ that is generated by $\{D_1, \ldots, D_s\}$ as an $\mathcal{O}_X$-module. Then $\mathscr{F}$ is a p-foliation if and only if $D_i^p \in \mathscr{F}$ for $1 \leq i \leq s$.*

Proof. If $\mathscr{F}$ is a p-foliation, then $D_i^p \in \mathscr{F}$ for any $D_i \in \mathscr{F}$ by the definition of p-closed.

On the other hand, let $\{D_1, \ldots, D_s\}$ be a generating set for $\mathscr{F}$, and suppose that $D_i^p \in \mathscr{F}$ for each generator D_i. By Deligne's identity [Kat70, Prop 5.3], we know that for any $g \in \mathcal{O}_X$, we have $(gD_i)^p = g^p D_i^p - g D_i^{p-1}(g^{p-1}) D_i$. Therefore, since D_i and D_i^p are both in $\mathscr{F}$. we see that $(gD_i)^p \in \mathscr{F}$.

By Jacobson's identity [Jac62, p. 187], we have

$$(D + D')^p = D^p + D'^p + \sum_{i-1}^{p-1} s_i(D, D'),$$

where each $s_i(D, D')$ is in the Lie subalgebra of $\mathcal{T}X$, generated by D and D'. Therefore, if $D, D' \in \mathscr{F}$, then $s_i(D, D') \in \mathscr{F}$, since $\mathscr{F}$ is involutive.

Thus, $(D + D')^p \in \mathscr{F}$, as $D^p, D'^p, s_i(D, D') \in \mathscr{F}$.

Thus, if $\{D_1, \ldots, D_k\}$ is a generating set of $\mathscr{F}$ and $D_i^p \in \mathscr{F}$ for each $1 \leq i \leq s$, then $\mathscr{F}$ is p-closed and thus is a p-foliation. □

We now continue with the proof of Proposition 3.18. Let $\mathscr{F}$ be some toric foliation on X, and let V be the subspace of $N \otimes_{\mathbb{Z}} k$ corresponding to $\mathscr{F}$, via the correspondence in Proposition 3.14. Let $\{v_1, v_2, \ldots, v_s\}$ be a basis of V. Then, given any $\mathbb{Z}$-basis $\{n_1, \ldots, n_d\}$ of N, we can write $v_j = \sum_{i=1}^d n_i \otimes a_{ij}$ for some $a_{ij} \in k$, since $V \subseteq N \otimes k$.

Since $\mathscr{F}$ is the foliation associated to V, it must be generated by derivations δ_{v_j} where $\delta_{v_j} := \sum_{i=1}^d a_{ij}\delta_{n_i}$. Note that for any $m \in M$, we have

$$\delta_{n_i}^p(\chi^m) = \langle m, n_i\rangle^p \chi^m.$$

Note that $\langle m, n_i\rangle \in \mathbb{Z}$, since the $n_i \in N$. Since we are working over a field with characteristic p, and as $\langle m, n_i\rangle \equiv \langle m, n_i\rangle^p \bmod p$, we must have $\delta_{n_i}^p = \delta_{n_i}$.

Since $\delta_{n_i}^p = \delta_{n_i}$ and $[\delta_m, \delta_n] = 0$ by Lemma 3.11, we see that $\delta_{v_j}^p = \sum_{i=1}^n a_{ij}^p \delta_{n_i}$. Note that the p-action on $N \otimes k$ is such that $v_j^{(p)} := \sum_{i=1}^d n_i \otimes a_{ij}^p$. Thus $\delta_{v_j}^p = \delta_{v_j^{(p)}}$.

Therefore $\mathscr{F}$ is p-closed if and only if V is p-closed, as required. □

Corollary 3.20. *Let U be a closed subgroup of Aut(T), and let Σ be a U-invariant fan in the cocharacter lattice N of T. Then there is a 1-1 correspondence between the following sets:*

$$\{\textit{Toric } p\textit{-foliations on } X(\Sigma)/U\} \Leftrightarrow \{p\textit{-closed } U\textit{-invariant subspaces of } N \otimes_{\mathbb{Z}} k\}.$$

Proof. Follows from Proposition 3.18 and Corollary 3.16. □

3.5 Singular Locus of Toric Foliations

Let k be a field of arbitrary characteristic, and let T be a split torus of rank g over k, with character lattice M and co-character lattice N.

Let $\sigma \in \Sigma$ be a smooth scrp cone in $N \otimes k$ with codimension q. Since σ is smooth, we know by Lemma 2.11 that σ is generated by part of a $\mathbb{Z}$-basis of N. Let $\{u_{q+1}, \ldots, u_g\}$ be a basis of σ that extends to $\{u_1, \ldots, u_g\}$ a basis of $N \otimes k$. Note that we are placing the basis of σ at the end of this basis, and not the beginning.

Let $\{m_1, \ldots, m_g\}$ be the basis of M that is dual to $\{u_1, \ldots, u_g\}$. That is,

$$\langle u_i, m_j \rangle = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}.$$

The affine open chart $U_\sigma \subseteq X$ is given by:

$$U_\sigma = \operatorname{Spec}(k[(\chi^{m_1})^{\pm 1}, \ldots, (\chi^{m_q})^{\pm 1}, \chi^{m_{q+1}}, \ldots, \chi^{m_g}]) = \operatorname{Spec}(k[S_\sigma]).$$

Thus, as computed in Corollary 3.12, the tangent space of U_σ is generated over $k[S_\sigma]$ by:

$$\mathcal{T}U_\sigma = \operatorname{span}_{k[S_\sigma]} \left\{ \delta_{u_1}, \ldots, \delta_{u_q}, \frac{1}{\chi^{m_{q+1}}}\delta_{u_{q+1}}, \ldots, \frac{1}{\chi^{m_g}}\delta_{u_g} \right\}.$$

Let Z_σ be the stratum in X corresponding to the cone σ. That is, Z_σ is the unique closed stratum in U_σ. Note that $\dim(Z_\sigma) = q$, and is isomorphic to a torus of dimension q. By construction, given any point $x \in Z_\sigma$, the function χ^{m_i} for $q < i \leq g$, is not invertible in $\mathcal{O}_{X,x}$, as Z_σ is defined by the equations $\chi^{m_i} = 0$ for $q < i \leq g$. On the other hand, for $1 \leq i \leq q$, the function χ^{m_i} is invertible over all of U_σ, so we could also have written the generating set of $\mathcal{T}U_\sigma$ as

$$\mathcal{T}U_\sigma = \operatorname{span}_{k[S_\sigma]} \left\{ \frac{1}{\chi^{m_1}}\delta_{u_1}, \ldots, \frac{1}{\chi^{m_g}}\delta_{u_g} \right\}.$$

Let V be the subspace of $N \otimes k$ corresponding to some toric foliation $\mathscr{F}$ of rank p

on a toric variety $X = X(\Sigma)$. Let $\{v_1, \ldots, v_p\}$ be a k-basis of V. Note that the basis elements $v_i \in N \otimes k$ need not be of the form $n_i \otimes 1$ for elements n_i of N. $\mathscr{F}$ is thus generated over $k[S_\sigma]$ by $\{\delta_{v_1}, \ldots, \delta_{v_p}\}$. Now define $c_{ij} := \langle v_i, m_j \rangle$. Then we have

$$v_i = \sum_{j=1}^{g} c_{ij} u_j$$

, for constants $c_{ij} \in k$. Note that this implies that $\delta_{v_i} = \sum_{j=1}^{g} c_{ij} \delta_{u_j}$.

So $C = [c_{ij}]$ is a $p \times g$ matrix with coefficients in k. Let us break C into two blocks

$$C = [C'_t \;\; C'_a]$$

where C'_t is size $p \times q$ and C'_a is size $p \times g - q$.

After possibly re-ordering the basis $\{u_{q+1}, \ldots, u_g\}$, let $0 \le s \le g - q$ be such that the i^{th} column of C'_a is in the column space of C'_t if and only if $i \le s$. Let $q' = q + s$ and consider the block decomposition

$$C = [C_t \;\; C_a]$$

where C_t is $p \times q'$ and C_a is $p \times g - q'$. Note that C_t has the same rank as C'_t, and no column of C_a is in the column space of C_t.

Theorem 3.21. *With notation as above, the foliation $\mathscr{F}_V$ extends smoothly to Z_σ if and only if rank$(C_t) = p + q' - g$.*

Before proceeding with the proof, we will look at a couple small examples. Let σ be a codimension q cone in Σ, so Z_σ is a q-dimensional toric stratum in X. Consider the case that $\mathscr{F}$ is the full tangent space. $V = N \otimes k$, so C is a $g \times g$ invertible matrix. Thus all of the columns of C are independent, and hence $q' = q$. Also, the rank of C_t will be q, as it is a $g \times q$ matrix, with q independent columns. Since $p = g$ and $q' = q$, the condition $\text{rank}(C_t) = p + q' - g$ is indeed satisfied. As $\mathscr{F}$ is the full tangent space, and X is smooth, $\mathscr{F}$ extends smoothly to all of X, and in particular, it extends smoothly to Z_σ.

As another example, let X be the affine plane $\mathbb{A}^2_k$. This is the affine toric variety corresponding to a smooth cone with generators $\{u_1, u_2\}$ in $N = \mathbb{Z}^2$. If we let $\{x, y\}$ be the corresponding dual basis of M, we have $\delta_{u_1} = x\frac{\partial}{\partial x}$ and $\delta_{u_2} = y\frac{\partial}{\partial y}$.

Let $\mathscr{G}$ be the rank 1-foliation generated by $x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}$ on $\mathbb{A}^2_k$, and let Z_σ be the 0-dimensional boundary stratum, that is the origin of $\mathbb{A}^2_k$. So $C = \begin{bmatrix} 1 & 1 \end{bmatrix}$. Since we are considering the 0-dimensional boundary stratum, C_t is the first 0 columns of C, and as such has rank 0. Since neither column of C is the zero column, $q' = q = 0$. So we can calculate $p + q' - g = 1 + 0 - 2 = -1$, which is not equal to $\operatorname{rank}(C_t) = 0$. So the condition in the theorem is not satisfied, and indeed $\mathscr{G}$ does not extend smoothly to Z_σ.

Proof. Let $X = X(\Sigma)$ be a toric variety with cocharacter lattice N, and let $\sigma \in \Sigma$ be a codimension q cone with basis $\{u_{q+1}, \ldots, u_g\}$. So we have a basis $\{u_1, \ldots, u_g\}$ of N. We will be looking at a point $x \in Z_\sigma$, which is a q-dimensional toric stratum in X.

Let $V \subseteq N \otimes k$ be generated by k-basis $\{v_1, \ldots, v_p\}$, and let $\mathscr{F}_V$ be the foliation associated to V. Recall that by construction we have

$$\begin{bmatrix} \delta_{v_1} \\ \vdots \\ \delta_{v_p} \end{bmatrix} = C \begin{bmatrix} \delta_{u_1} \\ \vdots \\ \delta_{u_g} \end{bmatrix}.$$

Thus, the stalk $\mathscr{F}_{V,x}$ is defined as the saturation in $\mathcal{T}X_x$ of the $\mathcal{O}_{X,x}$ row span of:

$$C \begin{bmatrix} \delta_{u_1} \\ \vdots \\ \delta_{u_g} \end{bmatrix}.$$

As above, we have the block decomposition $C = \begin{bmatrix} C_t & C_a \end{bmatrix}$ into a $p \times q'$ block C_t and a $p \times (g - q')$ block C_a. Since elementary row operations will not change the rowspace of the matrix, we can reduce the k-matrix C to the row reduced echelon form R, and $\mathscr{F}_{V,x}$ will still be the saturation of the rowspace of:

$$R \begin{bmatrix} \delta_{u_1} \\ \vdots \\ \delta_{u_g} \end{bmatrix}.$$

Let r :=rank(C_t). Since R is in row reduced echelon form, we can write it as

$$R = \begin{bmatrix} S & * \\ 0 & T \end{bmatrix},$$

where S and T are in row reduced echelon form, where S is size $r \times q'$, and T is size $p - r \times g - q'$. Furthermore, since no column in C_a is in the column space of C_t, every column of T must be nonzero. Also, as C consists of p linearly-independent rows, it must have full row rank. Thus T has full row rank as well.

Now, rename the elements of $\{u_i\}_{i=1}^g$ to $\{a_i\}_{i=1}^p$ and $\{b_i\}_{i=1}^{g-p}$, where the a_i correspond to columns in R with leading ones, and the b_i correspond to columns without leading ones. Similarly rename the dual basis elements $\{m_i\}_{i=1}^g$ to $\{\hat{a}_i\}_{i=1}^p$ and $\{\hat{b}_i\}_{i=1}^{g-p}$ respectively.

Since S has rank r, this implies that $a_i \in \{u_j\}_{j=1}^q$ for $1 \leq i \leq r$, and $a_i \in \{u_j\}_{j=q'+1}^g$ for $r < i \leq p$. In particular, there cannot be a leading one in the columns corresponding to u_j for $q < j \leq q'$, as these columns are in the columns space of the first q columns, by the construction of C_t. Thus, if $d_{ij} \in k$ are the entries of R in the columns corresponding to the b_j, we get:

$$\text{rowsp}_{\mathcal{O}_{X,x}} R \begin{bmatrix} \delta_{u_1} \\ \vdots \\ \delta_{u_g} \end{bmatrix} = \text{span}_{\mathcal{O}_{X,x}} \left\{ \delta_{a_i} + \sum_{j=1}^{g-p} d_{ij} \delta_{b_j} \right\}.$$

Suppose $x \in Z_\sigma$, we wish to determine if $\mathscr{F}_{V,x}$ is a direct summand of $\mathcal{T}X_x$. We have two cases. Either T is square, or it is not square.

Suppose that T is square. Then, since T is in row reduced echelon form, and has full row rank, it must be the identity matrix of size $g - q' = p - r$. Thus we have:

$$\mathscr{F}_{V,x} = \left(\operatorname{span}_{\mathcal{O}_{X,x}}\left(\left\{\delta_{a_i} + \sum_{j=1}^{g-p} d_{ij}\delta_{b_j}\right\}_{1\leq i\leq r} \cup \{\delta_{a_i}\}_{r<i\leq p}\right)\right)^{\text{sat}}.$$

Even though χ^{m_i} is not invertible in $\mathcal{O}_{X,x}$ for $q < i \leq g$, we have the derivations $\frac{1}{\chi^{m_i}}\delta_{u_i} \in \mathcal{T}X_x$. Thus for $r < i \leq p$, we have $\frac{1}{\chi^{\hat{a}_i}}\delta_{a_i} \in \mathscr{F}_{V,x}$ by saturation. So:

$$\mathscr{F}_{V,x} = \left(\operatorname{span}_{\mathcal{O}_{\mathcal{X},x}}\left(\left\{\delta_{a_i} + \sum_{j=1}^{g-p} d_{ij}\delta_{b_j}\right\}_{1\leq i\leq r} \cup \left\{\frac{1}{\chi^{\hat{a}_i}}\delta_{a_i}\right\}_{r<i\leq p}\right)\right)^{\text{sat}}.$$

Recall that over $x \in Z_\sigma$, we have

$$\mathcal{T}X_x = \operatorname{span}_{\mathcal{O}_{X,x}}\left\{\delta_{u_1}, \ldots, \delta_{u_q}, \frac{1}{\chi^{m_{q+1}}}\delta_{u_{q+1}}, \ldots, \frac{1}{\chi^{m_g}}\delta_{u_g}\right\}.$$

Let us define the map $\phi : \mathcal{T}X_x \to \operatorname{span}_{\mathcal{O}_{X,x}}\left\{\frac{1}{\chi^{\hat{b}_j}}\delta_{b_j}\right\}$, by:

$$\begin{aligned}
\delta_{a_i} &\mapsto \sum_{j=1}^{g-p} -d_{ij}\delta_{b_j} && 1 \leq i \leq r \\
\frac{1}{\chi^{\hat{a}_i}}\delta_{a_i} &\mapsto 0 && r < i \leq p \\
\delta_{b_j} &\mapsto \delta_{b_j} && 1 \leq j \leq q - r \\
\frac{1}{\chi^{\hat{b}_j}}\delta_{b_j} &\mapsto \frac{1}{\chi^{\hat{b}_j}}\delta_{b_j} && q - r < j \leq g - p.
\end{aligned}$$

Let $\zeta = \sum_{i=1}^{r} f_i\delta_{a_i} + \sum_{i=r+1}^{p} f_i\frac{1}{\chi^{\hat{a}_i}}\delta_{a_i} + \sum_{j=1}^{g-p} h_j\frac{1}{\chi^{\hat{b}_j}}\delta_{b_j}$. Since $\chi^{\hat{b}_j}$ is invertible in $\mathcal{O}_{X,x}$ for $1 \leq j \leq q - r$, this is indeed a generic element of $\mathcal{T}X_x$. Then, we can compute:

$$\phi(\zeta) = \left(\sum_{i=1}^{r}\sum_{j=1}^{g-p} -d_{ij}f_i\delta_{b_j}\right) + \left(\sum_{j=1}^{g-p} h_j\frac{1}{\chi^{\hat{b}_j}}\delta_{b_j}\right)$$

$$= \sum_{j=1}^{g-p} \frac{h_j}{\chi^{\hat{b}_j}} - \sum_{i=1}^{r} (d_{ij} f_i) \delta_{b_j}.$$

So $\zeta \in \ker(\phi)$ if and only if $\frac{h_j}{\chi^{\hat{b}_j}} = \sum_{i=1}^{r} d_{ij} f_i$. That is, if

$$\zeta = \sum_{i=1}^{r} f_i \delta_{a_i} + \sum_{i=r+1}^{p} \frac{f_i}{\chi^{\hat{a}_i}} \delta_{a_i} + \sum_{j=1}^{g-p} \sum_{i=1}^{r} d_{ij} f_i \delta_{b_j}$$

$$= \sum_{i=1}^{r} f_i \left(\delta_{a_i} + \sum_{j=1}^{g-p} d_{ij} \delta_{b_j} \right) + \sum_{i=r+1}^{p} \frac{f_i}{\chi^{\hat{a}_i}} \delta_{a_i}.$$

So $\ker(\phi) = \mathscr{F}_{V,x}$. Thus we get a split exact sequence:

$$0 \to \mathscr{F}_{V,x} \to \mathcal{T} X_x \to \operatorname{span}_{\mathcal{O}_{X,x}} \left\{ \frac{1}{\chi^{\hat{b}_j}} \delta_{b_j} \right\} \to 0.$$

As $\operatorname{span}_{\mathcal{O}_{X,x}} \left\{ \frac{1}{\chi^{\hat{b}_j}} \delta_{b_j} \right\}$ is clearly free, we see that $\mathscr{F}_{V,x}$ is a direct summand of $\mathcal{T} X_x$ for all $x \in Z_\sigma$, and thus $\mathscr{F}_V$ extends smoothly to Z_σ.

On the other hand, suppose that T is not square. Since it has full row rank, this implies that it must have more columns than rows. That is, it implies that $g-q' > p-r$. In particular, since $g - q' > 0$, it implies that not all the columns of C are in C_t. Thus $\operatorname{rank}(C_t) < \operatorname{rank}(C)$. So $r < p$, and T has at least one row. Also, since T has no zero columns, and more columns than rows, that implies there is a row in T with at least 2 non-zero components.

We then can describe $\mathscr{F}_{V,x}$ as:

$$\mathscr{F}_{V,x} = \left(\operatorname{span}_{\mathcal{O}_{X,x}} \left\{ \delta_{a_i} + \sum_{j=1}^{g-p} d_{ij} \delta_{b_j} \right\} \right)^{\text{sat}}.$$

Since some row in T has at least 2 non-zero components, there exists some $i > r$ and $1 \leq j \leq g-p$ such that $d_{ij} \neq 0$. We now prove that $\mathscr{F}_{V,x}$ is not a direct summand of $\mathcal{T} X_x$ as follows. First, note that if $\mathscr{F}_{V,x}$ were a direct summand of $\mathcal{T} X_x$, then by the additivity of rank we would know that $\mathcal{T} X_x / \mathscr{F}_{V,x}$ would be free of rank $g-p$. Secondly, note that the classes $\langle \frac{1}{\chi^{\hat{b}_j}}\delta_{b_j}\rangle$ modulo $\mathscr{F}_{V,x}$ provide a saturated, free submodule of $\mathcal{T}X_x/\mathscr{F}_{V,x}$ of rank $g-p$, as it is generated by a sub-basis of $\mathcal{T}X_x$ and does not intersect $\mathscr{F}_{V,x}$ in $\mathcal{T}X_x$. Therefore, by Lemma 2.2, we must have $\mathcal{T}X_x/\mathscr{F}_{V,x} = \langle \frac{1}{\chi^{\hat{b}_j}}\delta_{b_j}\rangle$. It remains to show that $\mathcal{T}X_x/\mathscr{F}_{V,x} \neq \langle \frac{1}{\chi^{\hat{b}_j}}\delta_{b_j}\rangle$. We will do this by demonstrating that the class of $\frac{1}{\chi^{\hat{a}_i}}\delta_{a_i}$ modulo $\mathscr{F}_{V,x}$ is not in $\langle \frac{1}{\chi^{\hat{b}_j}}\delta_{b_j}\rangle$. This will prove that $\mathscr{F}_{V,x}$ is not a direct summand of $\mathcal{T}X_x$.

Suppose that $\frac{1}{\chi^{\hat{a}_i}}\delta_{a_i} \equiv \sum_{j=1}^{g-p} f_j \frac{1}{\chi^{\hat{b}_j}}\delta_{b_j}$ modulo $\mathscr{F}_{V,x}$. for some $f_j \in \mathcal{O}_{X,x}$. Then, there exists some $g \in \mathcal{O}_{X,x}$, $\xi \in \mathscr{F}_{V,x}$, and $t_k \in \mathcal{O}_{X,x}$ such that:

$$\frac{1}{\chi^{\hat{a}_i}}\delta_{a_i} + \xi = \sum_{j=1}^{g-p} f_j \frac{1}{\chi^{\hat{b}_j}}\delta_{b_j}$$

$$g\xi = \sum_{\ell=1}^{p} t_\ell \left(\delta_{a_\ell} + \sum_{j=1}^{g-p} d_{\ell j}\delta_{b_j}\right).$$

Thus:

$$g\frac{1}{\chi^{\hat{a}_i}}\delta_{a_i} + \sum_{j=1}^{g-p} -gf_j \frac{1}{\chi^{\hat{b}_j}}\delta_{b_j} = \sum_{\ell=1}^{p} t_\ell \left(\delta_{a_\ell} + \sum_{j=1}^{g-p} d_{\ell j}\delta_{b_j}\right).$$

Now, since the δ_{a_i} and δ_{b_j} are all $\mathcal{O}_{X,x}$-independent, we see that for all $\ell \neq i$ we must have $t_\ell = 0$, since there is no δ_{a_ℓ} term on the left hand side. So this equation reduces to:

$$g\frac{1}{\chi^{\hat{a}_i}}\delta_{a_i} + \sum_{j=1}^{g-p} -gf_j \frac{1}{\chi^{\hat{b}_j}}\delta_{b_j} = t_i\delta_{a_i} + \sum_{j=1}^{g-p} d_{ij}t_i\delta_{b_j}.$$

This gives us a system of equations in $\mathcal{O}_{X,x}$:

$$\begin{aligned} g &= t_i\chi^{\hat{a}_i} & \qquad & 1 \leq i \leq p \\ -gf_j &= d_{ij}t_i\chi^{\hat{b}_j} & & 1 \leq j \leq g-p. \end{aligned}$$

Now, since $i > r$, we know that $\chi^{\hat{a}_i}\mathcal{O}_{X,x}$ is a prime ideal in $\mathcal{O}_{X,x}$, thus we have a valuation on $\mathcal{O}_{X,x}$ given by $\chi^{\hat{a}_i}$ Let us call this valuation ν. So, taking valuations, we see that

$$\nu(g) = \nu(t_i) + \nu(\chi^{\hat{a}_i}) = \nu(t_i) + 1.$$

Choose j such that $d_{ij} \neq 0$. Recall that by choice of i there is such a j. Then:

$$\nu(gf_j) = \nu(d_{ij} t_i \chi^{\hat{b}_j})$$

$$\nu(g) = \nu(t_i) + \nu(\chi^{\hat{b}_j}) - \nu(f_j) = \nu(t_i) - \nu(f_j).$$

Thus $\nu(t_i) \geq \nu(g) = \nu(t_i) + 1$. Since $g \neq 0$ this is a contradiction. Hence,

$$\frac{1}{\chi^{\hat{a}_i}} \delta_{a_i} \notin \left\langle \frac{1}{\chi^{\hat{b}_j}} \delta_{b_j} \right\rangle \text{ modulo } \mathscr{F}_{V,x}.$$

Putting this altogether gives us the statement that for $x \in Z_\sigma$, the stalk $\mathscr{F}_{V,x}$ is a direct summand of $\mathcal{T}X_x$ if and only if T is square, that is to say, if $g - q' = p - r$. Thus $\mathscr{F}_V$ extends smoothly to Z_σ if and only if $r = p + g - q'$. □

Corollary 3.22. *Suppose $V \neq N \otimes k$ and suppose further that there exists a basis $\{v_1, v_2, \ldots, v_g\}$ of N, extending the given basis of V, such that there exists $a_{ij} \in k$ all non-zero, such that $u_i = \sum_{j=1}^{g} a_{ij} v_j$. Then $\mathscr{F}_V$ extends smoothly to Z_σ if and only if rank $(C_t) = p$.*

Proof. Let A be the matrix of a_{ij} as in the corollary statement. Thus

$$\begin{bmatrix} u_1 \\ \vdots \\ u_g \end{bmatrix} = A \begin{bmatrix} v_1 \\ \vdots \\ v_g \end{bmatrix}.$$

As such, by the construction of C above, we have the following block matrix for A^{-1}:

$$A^{-1} = \left[\begin{array}{c|c} C_t & C_a \\ \hline * & * \end{array}\right],$$

where C_t and C_a are as above, so C_t is a block of size $p \times q'$. Then, we can let A' be the bottom right $g - q' \times g - p$ block of A. So Lemma 2.3 tells us that $\text{nullity}(C_t) = \text{nullity}(A')$.

Now, note that:

$$\begin{aligned}\text{rank}(C_t) &= q' - \text{nullity}(C_t)\\ &= q' - \text{nullity}(A')\\ &= q' - (g - p - \text{rank}(A'))\\ &= p + q' - g + \text{rank}(A').\end{aligned}$$

Thus, $\text{rank}(C_t) = p + q' - g$ if and only if $\text{rank}(A') = 0$. But, since all of the a_{ij} are non-zero, the only way for $\text{rank}(A') = 0$ to occur is if it is an empty matrix. Since A' is $g - q' \times g - p$ block, we must have either $g = p$ or $g = q'$. Since $V \neq N \otimes k$, we know that $p < g$. Also, from the construction of q' we know that $g = q'$ if and only if $\text{rank}(C_t) = p$. Hence, $\mathscr{F}_V$ extends smoothly to Z_σ if and only if $\text{rank}(C_t) = p$. $\square$

3.6 Quotients by Toric p-foliations

Let T be a split torus over k, where k is a perfect field of characteristic p. Recall that the Frobenius map is defined on k as $x \mapsto x^p$. Let $T^{(p)}$ be the base change of T by Frobenius. That is, define $T^{(p)} := T \otimes_{\text{Spec}(k),F} \text{Spec}(k)$. Then $T^{(p)}$ is also a split torus over k, since $T \cong T^{(p)}$.

Let g be the rank of T, so $T = \text{Spec}(k[X_1^{\pm 1}, \dots X_g^{\pm 1}])$. Similarly, let $Y_1, \dots, Y_g$ be such that $T^{(p)} = \text{Spec}(k[Y_1^{\pm 1}, \dots, Y_g^{\pm 1}])$. Then the canonical map $T \to T^{(p)}$ is induced by the ring map $Y_i \mapsto X_i^p$.

Let $M := X(T)$ be the character group of T, and let $M' := X(T^{(p)})$ be the character group of $T^{(p)}$. Recall that $X(T) := \text{Hom}(T, \mathbb{G}_m)$. So $X(T) \cong \mathbb{Z}^g$, where the vector $(m_1, \dots, m_g)$ corresponds to the character inducted by $X \mapsto \prod_{i=1}^g X_i^{m_i}$. The map

$T \to T^{(p)}$ induces a map of character groups $X(T^{(p)}) \to X(T)$. After passing over the isomorphism with $\mathbb{Z}^g$, we see that the image of this map is $p\mathbb{Z}^g$.

Thus, if $M = X(T)$, we can naturally identify $X(T^{(p)})$ with the lattice pM. Considering the dual lattices, we see that if N is the co-character group of T, we can naturally identify the co-character group of $T^{(p)}$ with the lattice $p^{-1}N$.

Now, let us consider an affine toric variety X_σ, where σ is a scrp cone in N. Then $X_\sigma := \mathrm{Spec}(k[S_\sigma])$. Again, as k is a perfect field, we know that $X_\sigma \cong X_\sigma^{(p)}$, and we have a natural map $X_\sigma \to X_\sigma^{(p)}$, such that when we restrict to T, we get the map $T \to T^{(p)}$.

By definition, the characters of X_σ are the semigroup $S_\sigma = \sigma^\vee \cap M$. Then, by the same argument as for T, we see that the characters of $X_\sigma^{(p)}$ are naturally identified with $\sigma^\vee \cap pM$. Note that $\sigma^\vee$ is a cone, thus for any $m \in M, pm \in \sigma^\vee$ if and only if $m \in \sigma^\vee$. Hence, $X_\sigma^{(p)}$ is the toric variety given by the cone σ in the lattice $p^{-1}N$. Note that since $\sigma \subseteq N_\mathbb{R}$, and $N_\mathbb{R} = p^{-1}N_\mathbb{R}$, we could equally view σ as a cone in the space generated by the cocharacter lattice of $T^{(p)}$. Thus $X_\sigma^{(p)}$ is the toric variety containing torus $T^{(p)}$ given by the cone σ. Furthermore, if $\{n_1, \dots n_r\}$ are generators of σ in N, then $\{p^{-1}n_1, \dots, p^{-1}n_r\}$ are generators of σ in $p^{-1}N$. Thus $\sigma^\vee \cap M \cong \sigma^\vee \cap pM$, so $X_\sigma \cong X_\sigma^{(p)}$ as required.

Finally, since the functor $X \to X^{(p)}$ commutes with colimits, if X_Σ is the toric variety with torus T corresponding to a fan Σ in $N_\mathbb{R}$, we must have $X_\Sigma^{(p)}$ as the toric variety with torus $T^{(p)}$ corresponding to the same fan Σ in $N_\mathbb{R}$.

Now, let us consider a toric p-foliation $\mathscr{F}$ over a torus T. By Proposition 3.18 there exists a vector space $V \subseteq N \otimes \mathbb{F}_p$ such that $\mathscr{F} = \mathscr{F}_V$. Note that since N is a lattice, we have $N/pN \cong N \otimes \mathbb{F}_p$. Thus, we have a surjection: $\pi : N \to N \otimes \mathbb{F}_p$.

Proposition 3.23. *Define* $V^{\perp_p} := \{m \in M : \langle m, n\rangle \equiv_p 0, \forall n \in \pi^{-1}(V)\}$. *Then* $pM \subseteq V^{\perp_p} \subseteq M$ *and* $T/\mathscr{F}$ *is a torus such that the projection map* $T \to T/\mathscr{F}$ *induces an injective map* $X(T/\mathscr{F}) \to X(T)$ *with image* $V^{\perp_p}$.

Proof. Recall that $\mathscr{F}$ is the foliation defined by $\mathrm{span}_{\mathcal{O}_T}\{\delta_n\}_{n\in V}$. Then, by the definition of quotient by a p-foliation, $T/\mathscr{F}$ will be the torus such that the character group of

$T/\mathscr{F}$ is the subgroup of $X(T)$ annihilated by every derivation in $\mathscr{F}$. Since $\delta_n(\chi^m) := \langle m, n\rangle\chi^m$, we see that $\delta_n(\chi^m) = 0$ for all $n \in V$ if and only if $\langle m, n\rangle \equiv 0 \bmod p$ for all $n \in V$. Thus the inclusion $X(T/F) \to X(T)$ is just the inclusion of $V^{\perp_p}$ in M. Note that since $\langle pm, n\rangle = p\langle m, n\rangle$, we have $pM \subseteq V^{\perp_p}$. □

Proposition 3.24. *Let Σ be a fan in $N_{\mathbb{R}}$, and let $N' = (V^{\perp_p})^\vee$ be the viewed as a lattice in $N_{\mathbb{R}}$ containing N. So let Σ' be the fan Σ, now considered over the lattice N'. Then $X/\mathscr{F}$ is the toric variety corresponding to the fan Σ'.*

Proof. Let $\sigma \in \Sigma$. The characters of $X_\sigma/\mathscr{F}$ are precisely the characters of $T/\mathscr{F}$ that are also in $\sigma^\vee$. Since the characters of $T/\mathscr{F}$ are precisely $V^{\perp_p} = N'^\vee$ by Proposition 3.23, the space $(\sigma \cap N')^\vee$ describes the characters of $X_\sigma/\mathscr{F}$. So $X_\sigma/\mathscr{F}$ is the affine toric variety constructed from lattice N' and cone σ.

Taking the colimit of this construction gives us the same result for any fan Σ. Thus, $X_\Sigma/\mathscr{F}$ is the toric variety with lattice N' and fan Σ, with the canonical map $X_\Sigma \to X_\Sigma/\mathscr{F}$ given by the inclusion of N into N'. □

4. TAUTOLOGICAL FOLIATIONS OF HILBERT MODULAR VARIETIES

We will now give a definition of the tautological foliations on Hilbert modular varieties following [GdS23]. Then we will then compute the singular locus of the tautological foliations on toroidal compactifications of Hilbert modular varieties with some examples both over $\mathbb{C}$ and in characteristic p. In the case of Hilbert modular surfaces in characteristic $2, 3$, we will show that for any tautological foliations, there is a toroidal compactification on which the foliation is smooth. We will also briefly look at behavior of the toroidal compactification when taking the quotient by a tautological p-foliation.

4.1 Tautological Foliations and Toric Foliations

First, consider the case $k = \mathbb{C}$. Then $\mathcal{M}(\mathfrak{c})(\mathbb{C})$ is isomorphic to a quotient $\Gamma_{\mathfrak{c}} \backslash \mathfrak{h}^g$, by Proposition 2.35. Recall that $\Gamma_{\mathfrak{c}}$ acts on each $\mathfrak{h}$ by way of the natural embeddings $\mathrm{SL}_2(L)$ into $\mathrm{SL}_2(\mathbb{R})$ induced by the g embeddings $L \hookrightarrow \mathbb{R}$.

Let $\{z_i\}$ be the coordinates of $\mathcal{M}(\mathfrak{c})(\mathbb{C})$ given by the natural coordinates on $\mathfrak{h}^g$. Then the tangent space $\mathcal{T}\mathcal{M}(\mathfrak{c})(\mathbb{C})$ is canonically generated by the derivations $\frac{\partial}{\partial z_i}$. For any subset $J \subseteq \{1, \ldots, g\}$ we define the foliation $\mathscr{F}_J$ of $\mathcal{M}(\mathfrak{c})(\mathbb{C})$ as the subbundle of $\mathcal{T}\mathcal{M}(\mathfrak{c})(\mathbb{C})$ generated by $\{\frac{\partial}{\partial z_i}\}_{i \in J}$. The foliations of the form $\mathscr{F}_J$ are called **tautological foliations**.

In the case $g = 2$, these foliations play a role in McQuillen's classification of foliated surfaces. In particular, the rank 1 tautological foliations on a Hilbert modular surface, extended to the minimal compactification, are canonical models of foliations with numerical Kodaira dimension 1, and foliated Kodaira dimension $-\infty$, see [McQ08, Theorem IV.5.11]. Allowing $g \geq 2$, the tautological foliations on Hilbert modular varieties have been used to prove geometric results, such as a Green-Griffiths-Lang principle for Hilbert modular varieties, [RT18].

We can construct a generalization of these tautological foliations to Hilbert modular varieties defined in positive characteristic. Let κ be a field of positive characteristic p, and let L be a totally real field such that p is unramified in L. Suppose κ is sufficiently large as to contain the residue fields $\mathcal{O}_L/\mathfrak{p}\mathcal{O}_L$ for each prime $\mathfrak{p}$ containing p. Let $\mathcal{M}_n(\mathfrak{c})(W(\kappa))$ be the Hilbert modular scheme over $W(\kappa)$ with $\Gamma_{00}(n)$-level structure. We will care especially about the special fibre $\mathcal{M}_n(\mathfrak{c})(\kappa)$. Recall that we have a universal $\mathfrak{c}$-polarized abelian scheme A_{univ} over $\mathcal{M}_n(\mathfrak{c})(W(\kappa))$. Also recall that we defined our moduli problem such that by Lemma 2.34, the action of $\mathcal{O}_L$ makes $\text{Lie}(A_{\text{univ}})$ a locally free $\mathcal{O}_L \otimes \mathcal{O}_{\mathcal{M}_n(\mathfrak{c})}$ module of rank 1.

Let $\mathbb{B}$ be the set of embeddings of L into $W(\kappa)[1/p]$. Since p does not divide the discriminant of $\mathcal{O}_L$, the action of $\mathcal{O}_L$ decomposes $\text{Lie}(A_{\text{univ}})$ into a direct sum

$$\text{Lie}(A_{\text{univ}}) = \bigoplus_{\sigma \in \mathbb{B}} \mathcal{L}_\sigma^{-1},$$

where the line bundle $\mathcal{L}_\sigma^{-1}$ is defined by the piece of $\text{Lie}(A_{\text{univ}})$ on which the action of $\mathcal{O}_L$ corresponds with the natural $W(\kappa)$-action under the embedding σ.

Now, as shown in [Kat78], there is a Kodaira–Spencer isomorphism

$$\text{Lie}(A_{\text{univ}})^{\otimes 2} \otimes_{\mathcal{O}_L} \mathfrak{d}\mathfrak{c} \to \mathcal{T}\mathcal{M}_n(\mathfrak{c}).$$

Note that $\text{Lie}(A_{\text{univ}}) \otimes_{\mathcal{O}_L} \mathfrak{d}\mathfrak{c} \cong \text{Lie}(A_{\text{univ}})$, thus we have a natural decomposition:

$$\mathcal{T}\mathcal{M}_n(\mathfrak{c}) = \bigoplus_{\sigma \in \mathbb{B}} \mathcal{L}_\sigma^{-2}.$$

For any subset $J \subseteq \mathbb{B}$, we define a foliation $\mathscr{F}_J$ on $\mathcal{M}_N(\mathfrak{c})(W(\kappa))$ by:

$$\mathscr{F}_J := \bigoplus_{\sigma \in J} \mathcal{L}_\sigma^{-2}.$$

This is clearly a direct summand of $\mathcal{TM}_n(\mathfrak{c})(W(\kappa))$ and by [GdS23, Lemma 3.1] is indeed a smooth foliation. The foliations $\mathscr{F}_J$ as described here are also called **tautological foliations**. When we restrict to the special fiber, each of the tautological foliations $\mathscr{F}_J$ can now be viewed as foliations on $\mathcal{M}_n(\mathfrak{c})(\kappa)$. Since we are now working over a field of positive characteristic, we would like to know when $\mathscr{F}_J$ is a p-foliation. This is given as part of Theorem 3.2 in [GdS23], reproduced here.

Theorem 4.1. *[GdS23, Theorem 3.2 (i)] The smooth foliation $\mathscr{F}_J$ is p-closed if and only if J is invariant under the action of Frobenius on $\mathbb{B}$.*

By Theorem 2.37, we have $\mathcal{M}_n^{TC}$, a toroidal compactification of $\mathcal{M}_n(\mathfrak{c})$, as well as a toric scheme $X(\Sigma_C)/U_n^2$, and an isomorphism

$$\varphi : \widehat{X(\Sigma_C)}/U_n^2 \times \operatorname{Spec}(\mathbb{Z}[1/n]) \to \widehat{\mathcal{M}_n^{TC}},$$

where the completion of $\mathcal{M}_n^{TC}$ is performed over the cusp C. For ease of notation, let $X = X(\Sigma_C)/U_n^2 \times Spec(\mathbb{Z}[1/n])$, and $\mathcal{M} = \mathcal{M}_n^{TC}$, with $\widehat{X}$, and $\widehat{M}$ the completions over the boundary, and the cusp C respectively.

Proposition 4.2. *Let $\mathscr{F}$ be a tautological foliation on $\mathcal{M}$, and let $\mathscr{G}$ be some foliation on X, smooth away from the boundary, such that $\varphi^*\widehat{\mathscr{F}} = \widehat{\mathscr{G}}$. Then $S(\mathscr{F}) = \varphi(S(\mathscr{G}))$.*

Proof. Recall that completion is an exact functor. Therefore, we have the exact sequence:

$$0 \to \widehat{\mathscr{F}} \to \widehat{\mathcal{TM}} \to \widehat{\mathcal{TM}/\mathscr{F}} \to 0.$$

Since φ is an isomorphism, it is flat. Thus, we can pullback the above sequence by φ to get:

$$0 \to \varphi^*\widehat{\mathscr{F}} \to \varphi^*\widehat{\mathcal{TM}} \to \varphi^*\widehat{\mathcal{TM}/\mathscr{F}} \to 0.$$

Recall that by hypothesis, we have $\varphi^*\widehat{\mathscr{F}} = \widehat{\mathscr{G}}$. Also, by Proposition 2.33, we have $\varphi^*\widehat{\mathcal{T}M} = \widehat{\mathcal{T}X}$, we must have $\varphi^*(\widehat{\mathcal{T}\mathcal{M}/\mathscr{F}}) = \widehat{\mathcal{T}X/\mathscr{G}}$.

Note that $\mathcal{T}\mathcal{M}/\mathscr{F}$ is a coherent $\mathcal{O}_{\mathcal{M}}$-module, so we will apply Lemma 2.31. We then see that for any $x \in X$ lying over the boundary, $x \notin S(\mathscr{G})$ if and only if $\mathscr{G}_x$ and $(\mathcal{T}X/\mathscr{G})_x$ are free, which holds if and only if $\widehat{\mathscr{G}_x}$ and $(\widehat{\mathcal{T}X/\mathscr{G}})_x$ are free. But, since $\varphi^*(\widehat{\mathscr{F}}) = \widehat{\mathscr{G}}$ and $\varphi^*(\widehat{\mathcal{T}\mathcal{M}/\mathscr{F}}) = \widehat{\mathcal{T}X/\mathscr{G}}$, this holds if and only if $\widehat{\mathscr{F}}_{\varphi(x)}$ and $(\widehat{\mathcal{T}\mathcal{M}/\mathscr{F}})_{\varphi(x)}$ are free, which, by Lemma 2.31 again, is true if and only if $\mathscr{F}_{\varphi(x)}$ and $(\mathcal{T}\mathcal{M}/\mathscr{F})_{\varphi(x)}$ are free. That is, if and only if $\varphi(x) \notin S(\mathscr{F})$. Thus $S(\mathscr{F}) = \varphi(S(\mathscr{G}))$. $\square$

So, we would like to find some foliation on $X(\Sigma_C)$, invariant under U_n^2, such that after completion over the boundary, it is isomorphic to the pullback of a given tautological foliation on $\mathcal{M}_n^{TC}$, completed at the cusp C.

If we are working over $\mathbb{C}$, then we know by Proposition 3.14, that foliations over $X(\Sigma_C)$ correspond to subspaces of $N_C \otimes \mathbb{C} \cong \mathbb{C}^g$. So let $J \subseteq \{1, \dots, g\}$, and define the foliation $\mathscr{G}_J$ as the foliation corresponding to the subspace V_J defined as the span of elementary basis vectors $\langle e_j \rangle_{j \in J} \subseteq \mathbb{C}^g$. Note that for any $u \in U_n^2$, and indeed for any $u \in O_L$, the action of u on N_C is given by $u \cdot e_j = \sigma_j(u) e_j$. Thus V_J is U_n^2 invariant, and thus by Corollary 3.16, induces a foliation on $X(\Sigma_C)/U_n^2$.

On the other hand, if we are working over a field κ with positive characteristic p, we can choose $\mathfrak{a}$ and $\mathfrak{b}$ such that the fractional ideal $N_C = (\mathfrak{a}\mathfrak{b}\mathfrak{d})^{-1}$ is prime to p. Thus, for each $\sigma \in \mathbb{B}$, we have $\sigma(N_C) \subseteq W(\kappa)$. Thus we have an embedding $N_C \hookrightarrow W(\kappa)^g$, given by the maps $\sigma_i \in \mathbb{B}$. These maps can be reduced modulo p to get $\overline{\sigma}_i : (\mathfrak{a}\mathfrak{b}\mathfrak{d})^{-1} \to \kappa$ So we have can reduce the above embedding to a map $N_C \to \kappa^g$. Note that this is no longer an embedding, however after tensoring with κ we do get an isomorphism $N_C \otimes \kappa = \kappa^g$.

Again using Proposition 3.14, the foliations (not necessarily p-closed) are given by subspaces of $N_C \otimes \kappa \cong \kappa^g$. We would like to know when $\mathscr{G}_J$ is p-closed in this context.

Note that we have an action of Frobenius on $\mathbb{B}$, given by $\sigma \mapsto \mathrm{Fr} \circ \sigma$, where Fr is the Frobenius on $W(\kappa)$. This gives an action on $N \otimes \kappa$ where $\mathrm{Fr}(n \otimes a) = n \otimes a^p$.

We would like to explicitly understand the action thus induced on κ^g via the isomorphism $N \otimes \kappa \cong \kappa^g$. Let $\mathbb{B} = \{\sigma_1, \dots, \sigma_g\}$. Recall that by this isomorphism, any element of κ^g can be written in the form:

$$\begin{bmatrix} x_1 \\ \vdots \\ x_g \end{bmatrix} = \sum_{i=1}^{r} \left(\begin{bmatrix} \overline{\sigma}_1(\alpha_i) \\ \vdots \\ \overline{\sigma}_g(\alpha_i) \end{bmatrix} \otimes a_i \right) = \sum_{i=1}^{r} \begin{bmatrix} a_i\overline{\sigma}_1(\alpha_i) \\ \vdots \\ a_i\overline{\sigma}_g(\alpha_i) \end{bmatrix}$$

for some $a_i \in \kappa$ and $\alpha_i \in N_C = (\mathfrak{a}\mathfrak{b}\mathfrak{d})^{-1}$. Let τ denote the permutation on $\{1, \dots, g\}$ such that $\mathrm{Fr}(\sigma_i) = \sigma_{\tau(i)}$. Then, we see that:

$$\begin{aligned} \mathrm{Fr}\left(\begin{bmatrix} x_1 \\ \vdots \\ x_g \end{bmatrix} \right) &= \mathrm{Fr}\left(\sum_{i=1}^{r} \begin{bmatrix} a_i\overline{\sigma}_1(\alpha_i) \\ \vdots \\ a_i\overline{\sigma}_g(\alpha_i) \end{bmatrix} \right) \\ &= \sum_{i=1}^{r} \begin{bmatrix} a_i^p\overline{\sigma}_1(\alpha_i) \\ \vdots \\ a_i^p\overline{\sigma}_g(\alpha_i) \end{bmatrix} \\ &= \sum_{i=1}^{r} \begin{bmatrix} (a_i\overline{\sigma}_{\tau^{-1}(1)}(\alpha_i))^p \\ \vdots \\ (a_i\overline{\sigma}_{\tau^{-1}(g)}(\alpha_i))^p \end{bmatrix} \\ &= \begin{bmatrix} x_{\tau^{-1}(1)}^p \\ \vdots \\ x_{\tau^{-1}(g)}^p \end{bmatrix}. \end{aligned}$$

Using this action, we can now use Proposition 3.18 to determine which subsets $J \subseteq \mathbb{B}$ induce a p-foliation $\mathscr{G}_J$. Since $\mathscr{G}_J$ is induced by the subspace $\langle e_j \rangle_{j \in J}$, and since the action above maps $e_j \mapsto e_{\tau^{-1}(j)}$, we see that this subspace is preserved under Frobenius exactly when J is stable under τ. So $\mathscr{G}_J$ is a p-foliation if and only if J is stable under the action of Frobenius on $\mathbb{B}$.

Lemma 4.3. *Let $\mathfrak{p}$ be an ideal of $\mathcal{O}_L$ containing p, and let $\mathfrak{p}'$ be the conjugate ideal such that $\mathfrak{p}\mathfrak{p}' = (p)$. Also let $J_\mathfrak{p} \subseteq \mathbb{B}$ consist of the mappings $\sigma_j \in \mathbb{B}$ such that $\sigma_j^{-1}(pW(\kappa)) \cap \mathcal{O}_L = \mathfrak{p}$. Let $V_{J_\mathfrak{p}} = \langle e_j \rangle_{j \in J_\mathfrak{p}}$ as a subspace of $N_C \otimes \kappa$. Then $\alpha \otimes 1 \in V_{J_\mathfrak{p}}$ if and only if $\alpha \in \mathfrak{p}' N_C$.*

Proof. Let $\alpha \in N_C$. Then, by the isomorphism $N_C \otimes \kappa \to \kappa^g$ described above, $\alpha \otimes 1 = \sum_{i=1}^g \overline{\sigma}_i(\alpha) e_i$. So $\alpha \otimes 1 \in V_{J_\mathfrak{p}}$ if and only if $\overline{\sigma}_i(\alpha) = 0$ for $i \notin J_\mathfrak{p}$. That is $\sigma_i(\alpha) \in pW(\kappa)$ for each $i \notin J_\mathfrak{p}$. Since $(\sigma_i)^{-1}(pW(\kappa)) \cap \mathcal{O}_L \subseteq \mathfrak{p}'$ for $i \notin J_\mathfrak{p}$, it must be that $\alpha \in \mathfrak{p}' \cap N_C$. Since N_C is relatively prime with p, this is $\alpha \in \mathfrak{p}' N_C$. □

Proposition 4.4. *For any $J \subseteq \{1, 2, \ldots, g\}$, we have $\widehat{\mathscr{G}}_J = \varphi^* \widehat{\mathscr{F}}_J$.*

Proof. Recall from Theorem 2.37 that the pullback of the universal semi-abelian variety over $\mathcal{M}_n^{TC}$ is precisely the semi-abelian variety from the Mumford construction.

$$\begin{tikzcd}
(G_{\mathrm{mum}}, \lambda_{\mathrm{mum}}, \iota_{\mathrm{mum}}) \arrow[r] \arrow[d] & (A_{\mathrm{univ}}, \lambda_{\mathrm{univ}}, \iota_{\mathrm{univ}}) \arrow[d] \\
(\widehat{X(\Sigma_C)}/U_n^2 \arrow[r, "\varphi"] & \mathcal{M}_n^{TC}(\mathfrak{c})
\end{tikzcd}$$

and this pullback commutes with the Kodaira–Spencer isomorphisms:

$$\begin{tikzcd}
\mathrm{Lie}(G_{\mathrm{mum}})^{\otimes 2} \otimes_{\mathcal{O}_L} \mathfrak{d}\mathfrak{c} \arrow[r] \arrow[d, "KS"] & \mathrm{Lie}(A_{\mathrm{univ}})^{\otimes 2} \otimes_{\mathcal{O}_L} \mathfrak{d}\mathfrak{c} \arrow[d, "KS"] \\
\mathcal{T}(\widehat{X(\Sigma_C)}/U_n^2) \arrow[r] & \mathcal{T}\mathcal{M}_N^{TC}(\mathfrak{c}).
\end{tikzcd}$$

Recall that we had a decomposition $\mathrm{Lie}(A_{\mathrm{univ}}) = \bigoplus_{\sigma \in \mathbb{B}} \mathcal{L}_\sigma^{-1}$ which was defined by the $\mathcal{O}_L$-action as induced by ι_{univ}. Since ι_{mum} is the pullback of ι_{univ} in the above diagram, we see that if $\mathrm{Lie}(G_{\mathrm{mum}}) = \bigoplus_{\sigma \in \mathbb{B}} {\mathcal{L}'_\sigma}^{-1}$ is the decomposition defined by the action of $\mathcal{O}_L$ on $\mathrm{Lie}(G_{\mathrm{mum}})$ induced by ι_{mum}, the pullback of each $\mathcal{L}_\sigma^{-2} \otimes \mathfrak{d}\mathfrak{c}$ is precisely ${\mathcal{L}'_\sigma}^{-2} \otimes \mathfrak{d}\mathfrak{c}$.

Now, note that $\mathrm{Lie}(G_{\mathrm{mum}}) \cong \mathfrak{a}^\vee \otimes \mathcal{O}_{\widehat{X(\Sigma_C)}}$, where the action induced by ι_{mum} is the natural action of $\mathcal{O}_L$ on $\mathfrak{a}^\vee$ by Corollary 2.38. But then

$$\mathrm{Lie}(G_{\mathrm{mum}})^{\otimes 2} \otimes_{\mathcal{O}_L} \mathfrak{d}\mathfrak{c} = \mathfrak{a}^\vee \mathfrak{a}^\vee \mathfrak{d}\mathfrak{c} \otimes \mathcal{O}_{\widehat{X(\Sigma_C)}} = (\mathfrak{a}\mathfrak{b}\mathfrak{d})^{-1} \otimes \mathcal{O}_S \cong N \otimes \mathcal{O}_{\widehat{X(\Sigma_c)}}.$$

So let $\{e_1, \dots, e_g\}$ be the natural basis for $N \otimes \kappa$ such that the $\mathcal{O}_L$ action on N is given by $a \cdot e_j = \sigma_j(a) e_j$ for any $j \in \{1, \dots, g\}$. Then, for any $J \subseteq \{1, \dots, g\}$ the foliation $\mathscr{G}_J$, given by $\langle e_j \rangle_{j \in J}$ is the pullback $\varphi^* \mathscr{F}_J$, as it was defined using the pullback of the $\mathcal{O}_L$ action that defined $\mathscr{F}_J$ in the same construction. $\square$

4.2 Computing the Singular Locus of Tautological Foliations

To determine the singular locus of $\mathscr{G}_J$, we will consider each boundary piece individually. We will work in the positive characteristic context here, although the construction over $\mathbb{C}$ is similar. Let C be a cusp of $\mathcal{M}_n(\mathfrak{c})(\kappa)$ associated with the ideal pair $(\mathfrak{a}, \mathfrak{b})$. So a toroidal compactification at the cusp C is given by an admissible $\Gamma(n)$-admissible polyhedral decomposition of $N_C := (\mathfrak{a}\mathfrak{b}\mathfrak{d})^{-1}$. Let σ be a cone in Σ generated by $\{\mu_{q+1}, \dots, \mu_g\}$ in N_C. Since σ is smooth, we can extend this to a basis $\{\mu_1, \dots, \mu_g\}$ of N_C.

Recall that we $\mathbb{B}$ is the set of embeddings $\sigma_i : L \hookrightarrow W(\kappa)[1/p]$. We denote by $\overline{\sigma}_i : N_C \to \kappa$ the reduction of σ_i modulo p, after restriction to N_C. Thus, the element $\mu_j \in N_C$ corresponds to the vector $[\overline{\sigma}_i(\mu_j)] \in \kappa^n$, under the map $N_C \to N_C \otimes \kappa \cong \kappa^n$.

Without loss of generality, we suppose that $J = \{\sigma_1, \dots, \sigma_r\} \subseteq \mathbb{B}$. That is, let J be such that $\{e_1, \dots, e_r\}$ is the basis for the vector space V that induces $\mathscr{G}_J$. Note that in section 3.5, we used p to denote the rank of the foliation, but here we are using r to denote the rank of the foliation $\mathscr{G}_J$ to avoid confusion, as p already denotes the characteristic of κ.

Since $N_C \to N_C \otimes \kappa \cong \kappa^g$ is given by the map $\mu \to [\overline{\sigma_i}(\mu)]$, we have the equations $\mu = \sum_{i=1}^{g} \overline{\sigma_i}(\mu) e_i$ for any $\mu \in N_C$. Thus, if $\{\mu_1, \dots, \mu_g\}$ is the basis of N_C including the generating set of σ, we have the equations:

$$\begin{bmatrix} \mu_1 \\ \vdots \\ \mu_g \end{bmatrix} = [\overline{\sigma}_i(\mu_j)] \begin{bmatrix} e_1 \\ \vdots \\ e_g \end{bmatrix}.$$

Note that this is the reverse of the equation that defined C in section 3.5, so we can invert this matrix to get:

$$\begin{bmatrix} e_1 \\ \vdots \\ e_g \end{bmatrix} = [\overline{\sigma}_i(\mu_j)]^{-1} \begin{bmatrix} \mu_1 \\ \vdots \\ \mu_g \end{bmatrix}.$$

So let $C := [\overline{\sigma}_i(\mu_j)]^{-1}$. Then, as before, we can divide C into blocks $C = \left[\begin{array}{c|c} C_t & C_a \\ \hline * & * \end{array}\right]$, where C_t is size $r \times q$. After possibly reordering $\{\mu_{q+1}, \ldots, \mu_g\}$, choose $0 \leq s \leq g - q$ such that the i^{th} column of C_a is in the column space of C_t if and only if $i \leq s$. Then let $q' = q + s$, so the upper $r \times q'$ block of C has the same rank as C_t. Thus, by Theorem 3.21, we see that the toric stratum corresponding to the cone σ is in $S(\mathscr{G}_J)$ if and only if $\operatorname{rank}(C_t) = r + q' - g$.

Note that if we do this construction over $\mathbb{C}$, rather than working in characteristic p, we can get a cleaner result. The only difference in the construction is that we now consider $\sigma_i(\mu_j) \in \mathbb{C}$ to be the image of $\mu_j \in (\mathfrak{abd})^{-1}$ under the i^{th} embedding of L into $\mathbb{C}$. Note that this implies that $\sigma_i(\mu_j) \neq 0$, since $\mu_j \neq 0$. Note that this may fail in the characteristic p case, since we had reduced modulo p rather than working directly in $W(\kappa)[1/p]$. Thus it is possible for $\overline{\sigma}_i(\mu_j) = 0 \in \kappa$, even though $\mu_j \neq 0 \in L$.

This gives us precisely the extra condition required to apply Corollary 3.22. So over $\mathbb{C}$ we can say that the toric stratum corresponding to the cone σ is in $S(\mathscr{G}_J)$ if and only if $\operatorname{rank}(C_t) = r$.

Proposition 4.5. *Consider a cusp C of the Hilbert modular variety $\mathcal{M}_n(\mathfrak{c})(\mathbb{C})$ associated with the ideal pair $(\mathfrak{a}, \mathfrak{b})$. So $N_C = (\mathfrak{a}\mathfrak{b}\mathfrak{d})^{-1}$. Let V be the subspace of $N \otimes \mathbb{C}$ defining a tautogical foliation $\mathscr{F}_J$ of rank r, and let Z_σ be a boundary piece of the toroidal compactification of M corresponding to the cone σ, where the dimension of $Z_\sigma = q < r$. Then $\mathscr{F}_J$ is singular on Z_σ.*

Proof. Note that if μ is a generator of σ, then $\sigma_i(\mu) \neq 0$ for all embeddings $\sigma_i \in \mathbb{B}$. Thus, the hypothesis of Corollary 3.22 holds. So, by this Corollary, we see that $\mathscr{F}_J$ is singular on Z_σ if and only if the rank of $C'_t = r$. But C'_t is a $r \times q$ matrix, thus if $q < r$, it is impossible for it to be rank r. $\square$

Recall that a square matrix A is said to be **totally invertible** if every square submatrix of A is invertible.

Proposition 4.6. *Using the same notation as above, if there exists a maximal cone having a face σ with a generating set that extends to a basis $\{\mu_j\}$ of N_C, such that the matrix $[\sigma_i(\mu_j)]$ is totally invertible, then $\mathscr{F}_J$ extends smoothly to Z_σ if and only if the rank of $\mathscr{F}_J \leq dim(Z_\sigma)$.*

Proof. Note that by Lemma 2.3, the inverse of a totally invertible matrix is also totally invertible. Hence, since $[\sigma_i(\mu_j)]$ is totally invertible, so is the matrix C. As above, since the hypotheses of Corollary 3.22 always hold in characteristic zero, we know that $\mathscr{F}_J$ is smooth if and only if the rank of C'_t is p. So, suppose that $p \leq q$. Then since C'_t is a submatrix of a totally invertible matrix, we know that any $p \times p$ submatrix of C'_t has rank p. Thus the rank of C'_t is also p. Thus $\mathscr{F}_J$ is extends smoothly to Z_σ. $\square$

Conversely, in the case that $[\sigma_i(\mu_j)]$ is not totally invertible, it is possible for the singular locus of $\mathscr{F}_J$ to have components with dimension greater than or equal to the rank of $\mathscr{F}_J$.

Example: Consider the totally real field $L = \mathbb{Q}[\sqrt{2}, \sqrt{3}]$, and consider the cusp corresponding to the fractional ideal $\mathcal{O}_L = \langle 1, \sqrt{2}, \sqrt{3}, \frac{\sqrt{2}+\sqrt{6}}{2} \rangle$. Take a $\Gamma(n)$-admissible decomposition of the totally positive cone of $\mathcal{O}_L$ that contains the cone generated by

$$\{1, 2+\sqrt{2}, 3+\sqrt{3}, 4+\sqrt{6}\}.$$

Consider the rank 2 foliation $\mathscr{F}_J$, where $J \subseteq \mathbb{B}$ is the embeddings $\sqrt{2} \mapsto -\sqrt{2}$, and $\sqrt{3} \mapsto -\sqrt{3}$. We will show that the dimension 2 boundary stratum corresponding to the cone σ generated by $\{1, 4+\sqrt{6}\}$ is in the singular locus of $\mathscr{F}_J$.

First, we extend the generators of σ to a basis of N_C,

$$\{2+\sqrt{2}, 3+\sqrt{3}, 1, 4+\sqrt{6}\}$$

as given by the maximal cone. Let $\{u_1, u_2, u_3, u_4\}$ be the dual basis in the character lattice M_C to this basis. These are thus the coordinates of the affine chart given by $U_\sigma \cong (\mathbb{C}^\times)^2 \times \mathbb{C}^2$. So the generators of $\mathcal{T}U_\sigma$ are $\langle u_1\frac{\partial}{\partial u_1}, u_2\frac{\partial}{\partial u_2}, \frac{\partial}{\partial u_3}, \frac{\partial}{\partial u_4}\rangle$. Also, the tautological foliations are given by $\langle \frac{\partial}{\partial z_i}\rangle_{i\in J}$, where J is this set of embeddings $L \hookrightarrow \mathbb{C}$, and $u_i\frac{\partial}{\partial u_i} = \sum_{j=1}^{r} \sigma_i(\mu_j)\frac{\partial}{\partial z_j}$.

Now, let us look at $[\sigma_i(\mu_j)]^{-1}$. We can compute that:

$$\begin{bmatrix} 2-\sqrt{2} & 3+\sqrt{3} & 1 & 4-\sqrt{6} \\ 2+\sqrt{2} & 3-\sqrt{3} & 1 & 4-\sqrt{6} \\ 2+\sqrt{2} & 3+\sqrt{3} & 1 & 4+\sqrt{6} \\ 2-\sqrt{2} & 3-\sqrt{3} & 1 & 4+\sqrt{6} \end{bmatrix}^{-1} = \frac{1}{24}\begin{bmatrix} -3\sqrt{2} & 3\sqrt{2} & 3\sqrt{2} & -3\sqrt{2} \\ 2\sqrt{3} & -2\sqrt{3} & 2\sqrt{3} & -2\sqrt{3} \\ -\sqrt{6} & -\sqrt{6} & \sqrt{6} & \sqrt{6} \\ \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 \end{bmatrix}.$$

For some $\alpha_i \in \mathbb{Q}[\sqrt{2}, \sqrt{3}]$. Note that the α_i are all conjugates in $\mathbb{Q}[\sqrt{2}, \sqrt{3}]$, but the exact value is not needed for this compuation.

Since

$$\begin{bmatrix} \frac{\partial}{\partial z_1} \\ \frac{\partial}{\partial z_2} \\ \frac{\partial}{\partial z_3} \\ \frac{\partial}{\partial z_4} \end{bmatrix} = [\sigma_i(\mu_j)]^{-1} \begin{bmatrix} u_1\frac{\partial}{\partial u_1} \\ u_2\frac{\partial}{\partial u_2} \\ u_3\frac{\partial}{\partial u_3} \\ u_4\frac{\partial}{\partial u_4} \end{bmatrix},$$

we see that the matrix C used in Theorem 3.21 is just the first two rows of $[\sigma_i(\mu_j)]^{-1}$, broken into 2×2 blocks. Thus we get $C'_t = \frac{1}{24}\begin{bmatrix} -3\sqrt{2} & 3\sqrt{2} \\ 2\sqrt{3} & -2\sqrt{3} \end{bmatrix}$, which has rank 1. Further, note that neither of the columns of C'_a are in the column space of C'_t. Thus this is also C_t, and $q' = 2$. Note also that $\operatorname{rank}(C_t) = 1$, but $r + q' - g = 2 + 2 - 4 = 0$. So by Theorem 3.21, the boundary piece given by $u_1, u_2 \neq 0$ and $u_3, u_4 = 0$ is part of the singular locus of $\mathscr{F}_J$. Indeed, we can explicitly compute that:

$$\begin{aligned} \mathscr{F}_J = \Big\langle &-3\sqrt{2}u_1\frac{\partial}{\partial u_1} + 3\sqrt{2}u_2\frac{\partial}{\partial u_2} + 3\sqrt{2}u_3\frac{\partial}{\partial u_3} - 3\sqrt{2}u_4\frac{\partial}{\partial u_4}, \\ &2\sqrt{3}u_1\frac{\partial}{\partial u_1} - 2\sqrt{3}u_2\frac{\partial}{\partial u_2} + 2\sqrt{3}u_3\frac{\partial}{\partial u_3} - 2\sqrt{3}u_4\frac{\partial}{\partial u_4}\Big\rangle \\ &= \Big\langle u_1\frac{\partial}{\partial u_1} - u_2\frac{\partial}{\partial u_2}, u_3\frac{\partial}{\partial u_3} - u_4\frac{\partial}{\partial u_4}\Big\rangle. \end{aligned}$$

From this, we see by the singularity of $u_3\frac{\partial}{\partial u_3} - u_4\frac{\partial}{\partial u_4}$ at $u_3 = u_4 = 0$, that $\mathscr{F}_J$ is singular on the piece of the boundary given by $u_3 = u_4 = 0$, which is exactly the toric stratum Z_σ.

4.2.1 Examples in Characteristic p

In this section, to keep the computations simple, we will be working at the level $N = 1$, thus the our cone decompositions of the totally positive cone will be $\Gamma(1)$-admissible. Thus, they will need to be stable under translations by the group

$$(\mathcal{O}_L^\times)^2 := \{u^2 | u \in \mathcal{O}_L^\times\}.$$

Note that any cone decomposition that is $(\mathcal{O}_L^\times)^2$-invariant, will also be U_n^2 invariant, but the resulting toroidal compactification will be a finite cover of order $[U_n^2 : (\mathcal{O}_L^\times)^2]$ over the examples we are giving here. This will not affect the calcuations regarding singularity and smoothness as the calculations are done locally, and the tautological foliations are $(\mathcal{O}_L^\times)^2$-invariant regardless of the chosen level.

Let $L = \mathbb{Q}(\sqrt{13})$, and let $p = 3$. Since p is split in L, we can set $\kappa = \mathbb{F}_3$, as $\mathcal{O}_L/\mathfrak{p}\mathcal{O}_L \cong \mathbb{F}_3$ for each prime ideal $\mathfrak{p}$ containing 3. Indeed, if we let $\omega = \frac{1+\sqrt{13}}{2}$, these ideals are $\mathfrak{p}_1 = \langle\omega\rangle$ and $\mathfrak{p}_2 = \langle 1-\omega\rangle$.

Consider $\mathcal{M}(\mathfrak{c})(\kappa)$, the Hilbert modular surface parameterizing $\mathfrak{c}$-polarized abelian varieties with real multiplication by L. Since the class number of L is 1, there is only one cusp C on M, so let us choose $\mathfrak{a}$ and $\mathfrak{b}$ such that $N_C = \mathcal{O}_L = \mathbb{Z}[\omega]$. Further, note that the fundamental unit of $\mathbb{Z}[\omega]$ is $1+\omega$. Also, note that $(1+\omega)^2 = (4+3\omega)$. So the squared unit group $(\mathcal{O}_L^\times)^2$ is generated by $4+3\omega$. In order to construct the toroidal compactification of M around C, we need to find an admissible cone decomposition of the totally positive cone that is invariant under the action of $(\mathcal{O}_L^\times)^2$.

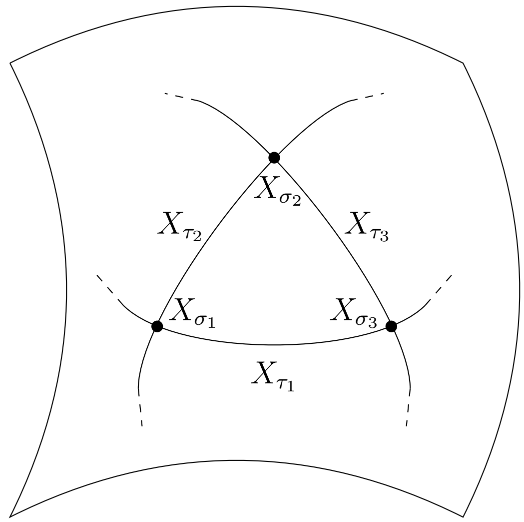
Let us consider the cone decomposition Σ given by translating the 3 cones

$$\sigma_1 = \langle 1, 2+\omega\rangle, \sigma_2 = \langle 2+\omega, 3+2\omega\rangle, \sigma_3 = \langle 3+2\omega, 4+3\omega\rangle$$

by $(\mathcal{O}_L^\times)^2$. Let us call the faces

$$\tau_1 = \langle 1\rangle, \tau_2 = \langle 2+\omega\rangle, \tau_3 = \langle 3+2\omega\rangle.$$

Since Σ is $(\mathcal{O}_L)^2$-invariant, we can take the quotient $X(\Sigma)/(\mathcal{O}_L)^2$. This quotient is then covered by the affine toric varieties corresponding to cones in $\Sigma/(\mathcal{O}_L)^2$. As such, we need only look at the cones σ_1, σ_2 and σ_3, while identifying the ray generated by 1 with the ray generated by $4+3\omega$. Thus the cone σ_3 will have faces τ_3 and τ_1. So the fibre of the cusp C in the toroidal compactification will be 3 rational nonsingular curves X_{τ_i} that intersect each other once in the intersection points X_{σ_i}, as in the diagram below:



Recall that since $p = 3$ is split in $\mathcal{O}_L$, it decomposes as $(3) = \mathfrak{p}_1\mathfrak{p}_2$, where $\mathfrak{p}_1 = \langle \omega \rangle$ and $\mathfrak{p}_2 = \langle 1 - \omega \rangle$. Thus $\mathbb{B}$ consists of the two embeddings $L \hookrightarrow W(\mathbb{F}_3)[1/3]$. Since we are working over $\kappa = \mathbb{F}_3$, we will restrict these maps to $\mathcal{O}_L$ and project them to $\mathbb{F}_3$, become the quotient maps onto $\mathcal{O}_L/\mathfrak{p}_1$ and $\mathcal{O}_L/\mathfrak{p}_2$. Since Fr is the identity on $\mathbb{F}_3$, the rank 1 tautological foliations on $\mathcal{M}(\mathfrak{c})(\mathbb{F}_3)$ corresponding to each of these maps are indeed p-foliations. Under the map $N_C \to N_C \otimes \mathbb{F}_3 \cong \mathbb{F}_3^2$, we see that

$$1 \mapsto \begin{pmatrix} 1 \\ 1 \end{pmatrix} \qquad 2 + \omega \mapsto \begin{pmatrix} 2 \\ 0 \end{pmatrix}$$
$$3 + 2\omega \mapsto \begin{pmatrix} 0 \\ 2 \end{pmatrix} \qquad 4 + 3\omega \mapsto \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Further, by the definition of the tautological foliations, we know that $\mathscr{F}_1$ corresponds to the subspace $V = \left\langle \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\rangle$ in $N_C \otimes \mathbb{F}_3$. Using this, we can now explicitly compute the singular locus of $\mathscr{F}_1$. We will consider the 6 boundary pieces individually. Let Z_{τ_i} be toric stratum corresponding to the ray τ_i. In particular Z_{τ_i} is the curve X_{τ_i} with the intersection points removed. Also, let $Z_{\sigma_i} = X_{\sigma_i}$, as these are the zero-dimensional toric strata.

For Z_{τ_1}, we get $\left\{\begin{pmatrix}1\\1\end{pmatrix}\right\}$ as a basis for τ_1. We can extend this to a basis of N_C as $\left\{\begin{pmatrix}2\\0\end{pmatrix},\begin{pmatrix}1\\1\end{pmatrix}\right\}$. Note that this is the affine chart for the toroidal compactification pictured above, with the origin at Z_{σ_1}, and the u_1 axis being X_{τ_2} and the u_2 axis being X_{τ_1}. In these coordinates, Z_{τ_1} is the u_1 axis excluding the origin.

Since V is generated by $v_1 = \begin{pmatrix}1\\0\end{pmatrix}$, we see that $v_1 = 2u_1$. So the matrix C in Theorem 3.21 is $\left[\begin{array}{c|c}2 & 0\end{array}\right]$. Thus $\operatorname{rank}(C_t) = 1$. Since the second column is in the column space of C_t, we must have $q' = 2$. But $r + q' - g = 1 + 2 - 2 = 1$. Thus $\mathscr{F}_1$ extends smoothly to Z_{τ_1}. Indeed, in coordinates u_1 and u_2, we can explicitly write $\mathscr{F}_1 = \langle 2u_1\frac{\partial}{\partial u_1}\rangle^{\mathrm{sat}} = \langle \frac{\partial}{\partial u_1}\rangle$, which is indeed non-singular along Z_{τ_1}.

Similarly, for Z_{τ_2}, note that τ_2 generated by $\left\{\begin{pmatrix}2\\0\end{pmatrix}\right\}$, so if we extend the basis to $\left\{\begin{pmatrix}0\\2\end{pmatrix},\begin{pmatrix}2\\0\end{pmatrix}\right\}$, we get the affine chart with origin at Z_{σ_2}, where the u_1 axis is X_{τ_3} and the u_2 axis is X_{τ_2}. In these coordinates we have $v_1 = 2u_2$. So $C = \left[\begin{array}{c|c}0 & 2\end{array}\right]$. Thus $\operatorname{rank}(C_t) = 0$. Since the second column is not in the column space of C_t, we have $q' = 1$, so we compute: $r + q' - g = 1 + 1 - 2 = 0$. which equals the rank of C_t. So $\mathscr{F}_1$ extends smoothly to Z_{τ_2}. Indeed, $\mathscr{F}_1 = \langle 2u_2\frac{\partial}{\partial u_2}\rangle^{\mathrm{sat}} = \langle \frac{\partial}{\partial u_2}\rangle$ which is non-singular everywhere, in particular along Z_{τ_2}.

Now, for X_{τ_3}, note that τ_3 is generated by $\left\{\begin{pmatrix}0\\2\end{pmatrix}\right\}$, so we can extend the basis to $\left\{\begin{pmatrix}1\\1\end{pmatrix},\begin{pmatrix}0\\2\end{pmatrix}\right\}$. This now corresponds to the affine chart with origin at X_{τ_1}, with u_1 axis given by X_{τ_1} and u_2 axis given by X_{τ_3}. So $v_1 = u_1 + u_2$. Thus $C = \left[\begin{array}{c|c}1 & 1\end{array}\right]$. Once again, we have $\operatorname{rank}(C_t) = 1$, so the second column is in the column space of C_t. Thus $q' = 2$. So in this case we have $r + q' - g = 1 + 2 - 2 = 1$ $=\operatorname{rank}(C_t)$. So $\mathscr{F}_1$ does extend smoothly to Z_{τ_3}. Indeed, $\mathscr{F}_1 = \langle u_1\frac{\partial}{\partial u_1} + u_2\frac{\partial}{\partial u_2}\rangle$. Note that this foliation is indeed non-singular along Z_{τ_3}, that is, the strata defined by equations $u_1 = 0, u_2 \neq 0$.

These calculations also describe at which intersection points $\mathscr{F}_1$ is smooth. From the above descriptions, it is clear that $\mathscr{F}$ is smooth at Z_{σ_1} and Z_{σ_2}, but not at Z_{σ_3}. We can verify this using Theorem 3.21. Consider the affine chart corresponding to the cone σ_1. In this case, we care about the zero-dimensional stratum. So, the matrix C is still $\begin{bmatrix} 2 & 0 \end{bmatrix}$, but now C_t consists of the first zero columns. Thus the column space of C_t is just $\{0\}$. Since the second column is in this space, we get $q' = 1$. So $r + q' - g = 1 + 1 - 2 = 0$, which equals the rank of C_t. Thus Theorem 3.21 confirms that $\mathscr{F}_1$ is smooth at Z_{σ_1}.

The calculation for Z_{σ_2} is very similar. However, for Z_{σ_3}, we see that the matrix C formed using the cone σ_3 is $\begin{bmatrix} 1 & 1 \end{bmatrix}$. Since C_t is empty, its column space is still $\{0\}$, so none of the columns are in that space. Thus $q' = 0$. So $r+q'-g = 1+0-2 = -1$ which is not equal to the rank of C_t. Therefore, the theorem confirms that $\mathscr{F}_1$ is singular at Z_{σ_3}. This matches are earlier calculuation, giving $\mathscr{F}_1 = \langle u_1 \frac{\partial}{\partial u_1} + u_2 \frac{\partial}{\partial u_2} \rangle$ on the affine chart corresponding to σ_3.

Thus the singular locus for $\mathscr{F}_1$ is just the point X_{σ_3}. Similar calculations show that the singular locus for $\mathscr{F}_2$ is just the point X_{σ_1}.

Example: Let $L = \mathbb{Q}(\sqrt{17})$. The narrow class number of L is 1, so there is only one possible choice for the fractional ideal $\mathfrak{c}$ that determines the polarization module. Also, there is only one cusp C on $\mathcal{M}(\mathfrak{c})$. We will represent that cusp with ideals $(\mathfrak{a}, \mathfrak{b})$ such that $N_C = (\mathfrak{a}\mathfrak{b}\mathfrak{d})^{-1} = \mathcal{O}_L = \mathbb{Z}[\omega]$ where $\omega = \frac{1+\sqrt{17}}{2}$.

The fundamental unit of $\mathbb{Z}[\omega]$ is $3 + 2\omega$. Since $(3 + 2\omega)^2 = 25 + 16\omega$, this is the generator of $(\mathcal{O}_L^\times)^2$. Thus, we would like to find an admissible polyhedral decomposition of the totally positive cone in $N_C = \mathbb{Z}[\omega]$ that is preserved under the action of $(\mathcal{O}_L^\times)^2 = (25 + 16\omega)^{\mathbb{Z}}$. A fundamental domain for this action is the cone $\langle 1, 25 + 16\omega \rangle$, so any decomposition of this cone can be extended to the totally positive cone in a way that preserves the action of the units.

The minimal smooth polyhedral decomposition of $\langle 1, 25+16\omega\rangle$ is:

$$\langle 1, 2+\omega\rangle, \langle 2+\omega, 5+3\omega\rangle, \langle 5+3\omega, 8+5\omega\rangle,$$

$$\langle 8+5\omega, 11+7\omega\rangle, \langle 11+7\omega, 25+16\omega\rangle.$$

Then the fibre of the point at the cusp C of the toroidal compactification given by this decomposition over the minimal compactification consists of 5 rational nonsingular curves intersecting each other in a cycle, thus 5 intersection points, each given by one of the cones listed above.

We will compute this example in characteristic 2. Note that the ideal $\langle 2\rangle$ decomposes in $\mathcal{O}_L$ as $\langle 2\rangle = \langle 2+\omega\rangle\langle 3-\omega\rangle$. Since 2 is split, we see that $\kappa = \mathbb{F}_2$, and thus both of the rank 1 tautological foliations are p-foliations.

Consider the tautological foliation $\mathscr{F}_1$ on $\mathcal{M}(\mathfrak{c})(\mathbb{F}_2)$ defined by the ideal $\langle 2+\omega\rangle$. Then $\mathscr{F}_1$ is smooth away from the intersection points. Also, via the same type of calculuations as in the previous example, $\mathscr{F}_1$ is smooth at the intersection point given by the cone $\langle\alpha, \beta\rangle$ if and only if either α or β are elements of the conjugate ideal $\langle 3-\omega\rangle$.

Since $5+3\omega$ and $11+7\omega$, are elements of $\langle 3-\omega\rangle$, we see that $\mathscr{F}_1$ is smooth at 4 of the intersection points, but singular at the point corresponding to the cone $\langle 1, 2+\omega\rangle$. However, if we blowup the surface at this point, we now have 6 rational curves over the cusp intersecting in a cycle, with intersection points corresponding to the cones:

$$\langle 1, 3+\omega\rangle, \langle 3+\omega, 2+\omega\rangle, \langle 2+\omega, 5+3\omega\rangle, \langle 5+3\omega, 8+5\omega\rangle,$$

$$\langle 8+5\omega, 11+7\omega\rangle, \langle 11+7\omega, 25+16\omega\rangle.$$

Since $3+\omega \in \langle 3-\omega\rangle$, we see that $\mathscr{F}_1$ is smooth at all 6 of these intersection points.

On the other hand, if we want $\mathscr{F}_2$, defined by the ideal $\langle 3-\omega\rangle$ to be smooth everywhere, we must instead blow up the point corresponding to the cone generated by $\langle 11+7\omega, 25+16\omega\rangle$. We will then have the toroidal compactification defined by:

$$\langle 1, 2+\omega\rangle, \langle 2+\omega, 5+3\omega\rangle, \langle 5+3\omega, 8+5\omega\rangle, \langle 8+5\omega, 11+7\omega\rangle,$$

$$\langle 11+7\omega, 36+23\omega\rangle, \langle 36+23\omega, 25+16\omega\rangle.$$

Since $\mathscr{F}_2$ is smooth at any intersection point given by a cone with a generator in $\langle 2+\omega\rangle$, and since $2+\omega, 8+5\omega$, and $36+23\omega$ are all elements of this ideal, we see that with this blowup, $\mathscr{F}_2$ is smooth.

In the next section, we will see that in characteristic 2 or 3, blowups can always be done to make either $\mathscr{F}_1$ or $\mathscr{F}_2$ smooth, but they cannot be made smooth simultaneously.

4.3 Smoothness of Tautological Foliations on Hilbert Modular Surfaces

In this section we will show that for characteristic $p=2$ or $p=3$, it is possible to refine any admissible polyhedral decomposition such that the tautological foliations on a Hilbert modular surface are smooth everywhere. However, for $p \geq 5$, blowing-up the singular points will not generally be able to remove the singularities of the tautological foliations.

Let ρ be the reduction map $\mathbb{P}^1(\mathbb{Q}) \to \mathbb{F}_p^2$, for which given some $q \in \mathbb{P}^1(\mathbb{Q})$ that can be written in lowest terms $q = \frac{m}{n}$ with $n \geq 0$, then $\rho(q) = (\overline{m}, \overline{n})$.

Let ℓ be a linear functional $\ell : \mathbb{F}_p^2 \to \mathbb{F}_p$, then we can define

$$\tilde{\ell} = \ell \circ \rho : \mathbb{P}^1(\mathbb{Q}) \to \mathbb{F}_p^2.$$

We define an **ℓ-path** on the Farey diagram to be a path on the Farey diagram such that at least one endpoint q of each edge on the path satisfies $\tilde{\ell}(q) = 0$.

Lemma 4.7. *Suppose ℓ is non-trivial. Then for each edge on the Farey diagram, at most one end satisfies $\tilde{\ell}(q) = 0$.*

Proof. Suppose $\left\{\frac{a}{c}, \frac{b}{d}\right\}$ is an edge on the Farey diagram. Then $ad - bc = \pm 1$. Let $\ell(x,y) = mx + ny$, and suppose that both endpoints of the edge satisfy $\tilde{\ell}(q) = 0$. Then, we have $p|(ma + nc)$ and $p|(mb + nd)$, where m and n are considered here as integers. Since ℓ is non-trivial, we must have either $p \nmid m$ or $p \nmid n$. If $p \nmid m$, then:

$$p|(ma + nc)d - (mb + nd)c = m(ad - bc).$$

But since $p \nmid m$ and $p \nmid (ad - bc)$, this cannot happen. Similarly, if $p \nmid n$, then:

$$p|(mb + nd)a - (ma + nc)b = n(ad - bc).$$

But since $p \nmid n$ and $p \nmid (ad - bc)$, this cannot occur. So we have a contradiction.

This argument is essentially saying that the product over $\mathbb{F}_p$ of the rank 1 matrix representing ℓ, namely $\begin{bmatrix} m & n \end{bmatrix}$ and the rank 2 matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ representing the edge on the Farey diagram cannot be zero.

Therefore, given any edge on the Farey diagram, at most one endpoint q satisfies $\tilde{\ell}(q) = 0$. □

Corollary 4.8. *If ℓ is non-trivial, then an ℓ-path is one such that every other vertex q satisfies $\tilde{\ell}(q) = 0$.*

Proof. Since every edge in the path must have exactly one endpoint that satisfies $\tilde{\ell}(q) = 0$, this guarantees that every other vertex on the path satisfies this condition. □

Proposition 4.9. *Suppose that $p = 2$ or $p = 3$. Then every path in the Farey diagram has a refinement that is an ℓ-path.*

Proof. First, we will consider the case $p = 2$. Let (p, q) be an edge on the Farey diagram. By the lemma, we know that we cannot have both $\tilde{\ell}(p) = \tilde{\ell}(q) = 0$. So, we have 3 possibilities. If $\tilde{\ell}(p) = 0$ and $\tilde{\ell}(q) = 1$, then this edge can be part of an ℓ-path. Similarly, if $\tilde{\ell}(p) = 1$ and $\tilde{\ell}(q) = 0$.

Suppose that $\tilde{\ell}(p) = 1$ and $\tilde{\ell}(q) = 1$. Let $p = \frac{a}{c}$ and $q = \frac{b}{d}$. If (p, q) was produced in stage i of building the Farey diagram, then in stage $i + 1$, then we construct the mediant $m_{pq} = \frac{a+b}{c+d}$ between p and q and build the edges (p, m_{pq}) and (m_{pq}, q). Note however, that

$$\tilde{\ell}(m_{pq}) = \ell((a, c) + (b, d)) = \ell(a, c) + \ell(b, d) = 1 + 1 = 0.$$

Thus, if we replace every edge (p, q) of a path in the Farey diagram such that $\tilde{\ell}(p) = \tilde{\ell}(q) = 1$, with the two edges (p, m_{pq}) and (m_{pq}, q), the refinement is an ℓ-path.

In the case $p = 3$, the construction is similar.

Consider a path in the Farey diagram. Then, there are four ways that an edge (p, q) can fail to have an endpoint that vanishes under $\tilde{\ell}$. Suppose first that $\tilde{\ell}(p) = 1$ and $\tilde{\ell}(q) = 2$. Then, as above, $\tilde{\ell}(m_{pq}) = 1 + 2 = 0$. Similarly if $\tilde{\ell}(p) = 2$ and $\tilde{\ell}(q) = 1$. So we can refine the edge (p, q) to the two edges (p, m_{pq}) and (m_{pq}, q).

Now, if $\tilde{\ell}(p) = \tilde{\ell}(q) = 1$. Then we have $\tilde{\ell}(m_{pq}) = 2$, so when we refine (p, q) to the two edges (p, m_{pq}) and (m_{pq}, q), both of these edges are in one of the first two cases. Similarly, if $\tilde{\ell}(p) = \tilde{\ell}(q) = 2$, we compute $\tilde{\ell}(m_{pq}) = 1$, which also reduces the problem to the earlier cases.

Thus for $p = 2, 3$, every path in a Farey diagram can be refined to an ℓ-path. $\square$

Let L be a real quadratic field, and let N_C be a fractional ideal in L with generators α and β. Further, suppose that α is totally positive, but neither of $\pm\beta$ are totally positive. We can view N_C as the lattice $\mathbb{Z}^2 \subseteq \mathbb{R}^2$ through the correspondence $a\alpha + b\beta \mapsto (a, b)$. Consider the totally postive cone in N_C, denoted N_C^+. When viewed in $\mathbb{R}^2$, the cone N_C^+ contains $(1, 0)$, but not $(0, \pm 1)$. Thus, the totally positive cone is bounded below by a ray of negative slope λ, and above by a ray of positive slope μ.

Since every non-zero element γ of N_C satisfies $\sigma_1(\gamma) \neq 0$ and $\sigma_2(\gamma) \neq 0$, the bounding rays of the totally positive cone, defined by the equations $\sigma_1(a\alpha + b\beta) = 0$ and $\sigma_2(a\alpha + b\beta) = 0$, must not intersect N_C away from the origin. Thus λ and μ are irrational.

Note that U_n^2 acts on N_C by multiplication, so we can view $U_n^2 \subset \mathrm{Aut}(N_C) = \mathrm{SL}_2(\mathbb{Z})$. Further, since every element of U_n^2 is totally positive, this action preserves the totally positive cone.

Let $a < 0 < b$ be irrational numbers. We will say that an infinite path

$$(\ldots, p_{-2}, p_{-1}, p_0, p_1, p_2, \ldots)$$

in the Farey diagram is **decreasing from** a **to** b or $D_{a,b}$ if the following conditions hold:

(i) $p_0 = \frac{1}{0}$,

(ii) $\lim\limits_{n\to-\infty} p_n = a$,

(iii) $\lim\limits_{n\to\infty} p_n = b$,

(iv) $p_i > p_{i+1}$ for all $i \neq -1$.

With this notation, we now have the following correspondence.

Lemma 4.10. *There is a 1-1 correspondence between:*

$$\left\{ \begin{array}{c} \textit{Smooth } \Gamma(n) - \textit{admissible cone} \\ \textit{decompositions of the totally positive cone} \end{array} \right\}$$

$$\leftrightarrow \left\{ \ U_n^2 - \textit{invariant } D_{\lambda^{-1},\mu^{-1}}\textit{-paths.} \ \right\}$$

$$\leftrightarrow \left\{ \begin{array}{c} \textit{Finite decreasing paths in the Farey diagram from} \\ \frac{1}{0} \textit{ to } \frac{a}{b} \textit{ where } \gamma\alpha = a\alpha + b\beta \\ \textit{and } \gamma \textit{ is the generator of } U_n^2 \textit{ such that } n > 0. \end{array} \right\}.$$

Proof. Suppose Σ is a smooth $\Gamma(n)-$admissible decomposition of the totally positive cone. Since we are working over a rank 2 lattice, this decomposition is a collection of rays with slopes between λ and μ. Note that $(1, 0)$ is a ray in Σ, by Lemma 2.13, so let $\rho_0 = (1, 0)$. By the definition of $\Gamma(n)-$admissible, we know that the quotient Σ/U_n^2 is finite. Since U_n^2 is an infinite cyclic group, it has only two generators inverse to each other. Let γ be the generator of U_n^2, such that $\gamma\rho_0$ has positive slope. Then there is a finite collection, ordered by increasing slope $\{\rho_1, \ldots, \rho_s\}$ of rays in Σ with slopes between ρ_0 and $\gamma\rho_0$. Thus, the rays in Σ can be listed in order of increasing slope:

$$\{\ldots, \gamma^{-2}\rho_s, \gamma^{-1}\rho_0, \gamma^{-1}\rho_1, \ldots, \gamma^{-1}\rho_s, \rho_0, \rho_1, \ldots, \rho_s, \gamma\rho_0, \gamma\rho_1, \ldots, \gamma\rho_s, \gamma^2\rho_0, \ldots\}.$$

Relabel these such that $\rho_n = \gamma^q\rho_r$ for $n = q(s+1) + r$, $0 \leq r < s+1$. So, the rays in Σ form an infinite sequence of increasing slope:

$$\{\ldots, \rho_{-2}, \rho_{-1}, \rho_0, \rho_1, \rho_2, \ldots\}$$

such that $\lim_{n\to-\infty}(\text{slope of } \rho_n) = \lambda$ and $\lim_{n\to\infty}(\text{slope of } \rho_n) = \mu$.

By the correspondence of Proposition 2.23, each maximal cone in Σ corresponds to an edge in the Farey diagram with endpoints equal to the inverse slopes of the faces of the cone. Thus we can associate to Σ the path in the Farey diagram $\{\ldots, p_{-2}, p_{-1}, p_0, p_1, p_2, \ldots\}$ where p_i is the inverse slope of ρ_i. This is then a path in the Farey diagram with limiting values λ^{-1} and μ^{-1}. Since ρ_0 has a slope of zero, we have $p_0 = \frac{1}{0}$. Finally, since the slopes are increasing, the inverse slopes are decreasing, except for the edge (p_{-1}, p_0). So this is a $D_{\lambda^{-1},\mu^{-1}}$ path. Also, since Σ is invariant under the action of U_n^2, and as the action of $\mathrm{SL}_2(\mathbb{Z})$ on N carries through the correspondence of Proposition 2.23, this path is also U_n^2-invariant.

On the other hand, given any decreasing U_n^2-invariant path on the Farey diagram from λ^{-1} to μ^{-1}, we can build a smooth $\Gamma(n)$-admissible cone decomposition of the totally positive cone in N_C, by taking the cones that correspond to each edge in the path. Note that while the correspondence from Proposition 2.23 is a 2-1 correspondence, each edge on this path corresponds to a cone with slopes between λ and μ, and thus correspond to one totally positive cone and one totally negative cone. We choose the totally positive one for each edge. By the U_n^2-invariance of the path, the cone decomposition is also U_n^2-invariant, as the correspondence between edges on the Farey diagram and smooth rational cones is equivariant with respect to U_n^2. Furthermore, the quotient by U_n^2 is finite, since there are only finitely many edges in a path between any vertex a_0 and its γ-translate γa_0.

Thus, we have a $1-1$ correspondence between smooth $\Gamma(n)$-admissible decompositions of the totally positive cone in N, and U_n^2-invariant $D_{\lambda^{-1},\mu^{-1}}$ paths.

By definition, $\frac{1}{0}$ is on any $D_{\lambda^{-1},\mu^{-1}}$ path. So let $\gamma = a\alpha + b\beta$ be the generator of U_n^2 such that $\gamma \cdot \frac{1}{0}$ is positive, where this is the induced action on $\mathbb{P}^1(\mathbb{Q})$, by the correspondence $\langle \alpha, \beta \rangle \cong \mathbb{Z}^2$. In particular, $\gamma \cdot \frac{1}{0} = \frac{a}{b}$.

Then, since $\frac{1}{0}$ and $\gamma \cdot \frac{1}{0} = \frac{a}{b}$ are on the decreasing path, we can look at the sub-path from $\frac{1}{0}$ to $\frac{a}{b}$. On the other hand, given any decreasing path on the Farey diagram from $\frac{1}{0}$ to $\frac{a}{b}$, labeled $(\frac{1}{0}, a_1, a_2, \ldots, a_s, \frac{a}{b})$, we can extend it via the U_n^2-action to an infinite path from λ^{-1} to μ^{-1} on the Farey diagram:

$$(\ldots, \gamma^{-2}a_s, \gamma^{-1}0, \gamma^{-1}a_1, \ldots, \gamma^{-1}a_s, 0, a_1, \ldots, a_s, \gamma 0, \gamma a_1, \ldots, \gamma a_s, \gamma^2 0, \ldots).$$

Indeed, if we define the sequences (a_r, b_r) such that $\gamma^r = a_r\alpha + b_r\beta$, we know that $\lim_{r\to-\infty} \frac{a_r}{b_r} = \lambda^{-1}$ and $\lim_{r\to\infty} \frac{a_r}{b_r} = \mu^{-1}$. So this is a $D_{\lambda^{-1},\mu^{-1}}$-path.

Thus, we have a 1-1 correspondence between U_n^2-invariant decreasing paths from λ^{-1} to μ^{-1}, and finite decreasing paths from $\frac{1}{0}$ to $\frac{a}{b}$. $\square$

Let L be a real quadratic field in which the rational prime p is split into $p = \mathfrak{p}_1\mathfrak{p}_2$. Define the linear relation $\ell_i(a, b) = a + b\omega \mod \mathfrak{p}_i$.

Consider the corresponding Hilbert modular surface $\mathcal{M}_n(\mathfrak{c})(\kappa)$, where $\kappa = \mathbb{F}_p$. Then there is a toroidal compactification of $\mathcal{M}_n(\mathfrak{c})(\kappa)$ given by a $\Gamma(n)$-admissible decomposition Σ_C for each cusp C. We want to determine when the tautological foliation $\mathscr{F}_1$, corresponding to the ideal $\mathfrak{p}_1$ is smooth on the toroidal compactifcation. Since the singular locus of a foliation must always have codimension 2, we only need to examine the 0-dimensional boundary strata.

Proposition 4.11. *The tautological foliation $\mathscr{F}_1$ is smooth at a zero-dimensional boundary stratum if and only if one of the faces of the cone σ corresponding to this stratum is in $\mathfrak{p}_2$. That is, if $\ell_2(\mu_1, \mu_2) = 0$ for one of the generators (μ_1, μ_2) of the cone σ.*

Proof. Let σ be a cone in the $\Gamma(n)$-admissible decomposition Σ, and let μ_1, μ_2 be the generators of σ. Suppose that $\mu_1 \in \mathfrak{p}_2$. So, if we reduce μ_1 mod p to a vector in $\mathbb{F}_p^2$, we will get $\mu_1 = \begin{bmatrix} a \\ 0 \end{bmatrix}$ for some $a \neq 0$. Since $\mathscr{F}_{\mathfrak{p}_1}$ corresponds to the vector space generated by $v_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, we have $v_1 = a^{-1}\mu_1 + 0\mu_2$, regardless of what μ_2 is. Thus, over the $\mathbb{F}_p$-fibre, $\mathscr{F}_{\mathfrak{p}_1} = \langle a^{-1} u_1 \frac{\partial}{\partial u_1} \rangle^{\text{sat}} = \langle a^{-1} \frac{\partial}{\partial u_1} \rangle$, which is smooth at $u_1 = u_2 = 0$.

Alternatively, we can use Theorem 3.21. Using this approach, we have matrix $C = \begin{bmatrix} a^{-1} & 0 \end{bmatrix}$. Since we are looking at a zero-dimensional stratum, C'_t is the first zero columns of C so rank $C_t = 0$. However, as there is a zero column, we have $q' = 1$. So $r + q' - g = 1 + 1 - 2 = 0 =\text{rank}(C_t)$, thus $\mathscr{F}_1$ is smooth at this zero-dimensional boundary piece.

Similarly, $\mathscr{F}_1$ is smooth when $\mu_2 \in \mathfrak{p}_2$.

Now, suppose that neither μ_1 nor μ_2 are in $\mathfrak{p}_2$. Then, for $v_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, we must have $v_1 = a\mu_1 + b\mu_2$ with $a, b \neq 0$. So, using Theorem 3.21, we have the matrix $C = \begin{bmatrix} a & b \end{bmatrix}$, so rank $C_t = 0$, but as there are no zero columns, we also have $q' = 0$. So $r + q' - g = 1 + 0 - 2 = -1 \neq\text{rank}(C_t)$. Thus $\mathscr{F}_1$ extends smoothly to the zero dimensional boundary piece corresponding to cone $\langle \mu_1, \mu_2 \rangle$ if and only if either μ_1 or μ_2 are elements of $\mathfrak{p}_2$. □

Corollary 4.12. *$\mathscr{F}_1$ is smooth everywhere if and only if the admissible polyhedral decomposition defining the toroidal compactification corresponds to an ℓ_2-path.*

Proof. By Proposition 4.11, $\mathscr{F}_1$ is smooth everywhere if and only if every cone in the defining $\Gamma(n)$-admissible decomposition has one of the faces satisfying $\ell_2(a, b) = 0$. But, by the correspondence in Lemma 4.10, these cone decompositions correspond precisely to that paths such that every other vertex q satisfies $\ell_2(q) = 0$. In other words, it corresponds to an ℓ_2-path. □

Corollary 4.13. *If $p = 2$ or 3, then for every Hilbert modular surface and tautological foliation thereon, there exists a smooth toroidal compactification such that the tautological foliation is smooth.*

Proof. This is a direct result of Propostion 4.9 and Corollary 4.12. □

Corollary 4.14. *For $p \geq 5$, given a Hilbert modular surface $\mathcal{M}_n(\mathfrak{c})(\kappa)$ and tautological foliation $\mathscr{F}$, then either $\mathscr{F}$ can be made smooth on the $\mathbb{F}_p$- fibre of the minimal smooth toroidal compactification after blowing up each singularity once, or there is no toroidal compactification for which $\mathscr{F}$ is smooth.*

Proof. Consider the minimal smooth toroidal compactification of $\mathcal{M}_n(\mathfrak{c})(\kappa)$. This is the toroidal compactification formed by the Hirzebruch resolution of the cusp singularities of the minimal compactification, as in [VdG88]. In the case of surfaces, every smooth toroidal compactification is a blowup of the minimal one. Let Σ be the cone decomposition corresponding to the minimal smooth compactification. Note that the rays in Σ are precisely the rays generated by points on the convex hull of the totally positive cone in N_C. Let P be the path on the Farey diagram corresponding to Σ.

Let $\mathfrak{p}_1$ be the ideal defining $\mathscr{F}$, and let $\mathfrak{p}_2 = p/\mathfrak{p}_1$. Let α, β be the generators of N_C, giving the isomorphism $N_C \cong \mathbb{Z}^2$, and define $\ell(a, b) := a\alpha + b\beta \bmod \mathfrak{p}_2$. For each edge (p, q) in P there are 3 possibilities.

Case 1: Either $\tilde{\ell}(p) = 0$ or $\tilde{\ell}(q) = 0$. In this case, $\mathscr{F}$ is already smooth at the zero-dimensional boundary point corresponding to the cone corresponding to the edge (p, q).

Case 2: $\tilde{\ell}(p) + \tilde{\ell}(q) = 0$. Recall that if m_{pq} is the mediant of p, q then $\tilde{\ell}(m_{pq}) = \tilde{\ell}(p) + \tilde{\ell}(q) = 0$. Also, if we perform a single blowup at the point corresponding to this edge, we get the path with edges (p, m_{pq}) and (m_{pq}, q). Since $\tilde{\ell}(m_{pq}) = 0$, we must have $\mathscr{F}$ smooth at both of the zero-dimensional strata over the original point after the blowup.

Case 3: $\tilde{\ell}(p), \tilde{\ell}(q), \tilde{\ell}(p)+\tilde{\ell}(q) \neq 0$. Since $\tilde{\ell}(m_{pq}) = \tilde{\ell}(p)+\tilde{\ell}(q) \neq 0$, it must be that the edges (p, m_{pq}) and (m_{pq}, q) are in either cases 2 or 3. Suppose that both (p, m_{pq}) and (m_{pq}, q) are in Case 2. That is, suppose that $\tilde{\ell}(p)+\tilde{\ell}(m_{pq}) = 0 = \tilde{\ell}(m_{pq})+\tilde{\ell}(q)$. Then, we must have $\tilde{\ell}(p) = \tilde{\ell}(q)$. Thus $\tilde{\ell}(m_{pq}) = 2\tilde{\ell}(p)$. So $3\tilde{\ell}(p) = 0$. But $p \geq 5$, so 3 is invertible in $\mathbb{F}_p$. Thus $\tilde{\ell}(p) = 0$. But this contradicts our initial hypothesis for Case 3. Thus, at least one of the edges $(p, m_{pq}), (m_{pq}, q)$ is in Case 3. Therefore, no matter how many blow-ups are performed, there will always be an edge in P that is in Case 3.

Thus, if one of the edges in the path corresponding to the minimal smooth toroidal compactification is in Case 3, $\mathscr{F}$ will not be smooth for any toroidal compactification.

□

4.4 Quotients by the Tautological Foliations

Let L be a totally real field containing a fractional ideal $\mathfrak{c}$, and let $n \geq 4$. Then we have the Hilbert modular variety $\mathcal{M}_n(\mathfrak{c})$ as constructed in section 2.5. Let p be a rational prime unramified in L, and let κ be a field of characteristic p large enough to contain the residue fields $\mathcal{O}_L/\mathfrak{p}_i$ for prime ideals $\mathfrak{p}_i$ of $\mathcal{O}_L$ containing p.

As noted previously, we have a collection $\mathbb{B}$ of embeddings $\sigma_i : L \hookrightarrow W(\kappa)[1/p]$. Let $\mathfrak{p}$ be a prime ideal of $\mathcal{O}_L$ containing p, and as before let $\mathbb{B}_{\mathfrak{p}}$ be the collection of elements σ_j in $\mathbb{B}$ such that $\sigma_j^{-1}(pW(\kappa)) \cap \mathcal{O}_L = \mathfrak{p}$.

Let $\mathfrak{p}'$ be the complement ideal to $\mathfrak{p}$, that is the ideal $\mathfrak{p}'$ such that $\mathfrak{p}\mathfrak{p}' = (p)$. Define the subset $J = \bigsqcup_{\mathfrak{q} \neq \mathfrak{p}} \mathbb{B}_{\mathfrak{q}} \subseteq \mathbb{B}$. We will consider the foliation $\mathscr{F}_J$ as constructed in section 4.1

Consider the cusp C of $\mathcal{M}_n(\mathfrak{c})$ given by the pair $(\mathfrak{a}, \mathfrak{b})$ of fractional ideals. Then, we have defined the foliation $\mathscr{G}_J$ on $X(\Sigma)$ where Σ is an admissible polyhedral cone decomposition of $N_C^+ = (\mathfrak{a}\mathfrak{b}\mathfrak{d})^{-1,+}$.

By Lemma 4.3, the foliation $\mathcal{G}_J$ is induced by the subspace $V = \mathfrak{p}N_C \otimes \kappa$ of $N_C \otimes \kappa$. So $V^{\perp_p} = \mathfrak{p}' M_C \subseteq M_C = \mathfrak{ab}$. By Proposition 3.24, the quotient $X(\Sigma)/\mathcal{G}_J$ is $X(\Sigma')$ where Σ' is the fan Σ, now considered on the lattice $N'_C = (\mathfrak{abdp}')^{-1}$. Note that even if Σ was a smooth fan on N_C, the fan Σ' will generally not be smooth on N'_C.

The quotient of $\mathcal{M}_n(\mathfrak{c})$ by $\mathscr{F}_J$ is described in [GdS23]. There it is shown that

$$\mathcal{M}_n(\mathfrak{c})/\mathscr{F}_J \cong \mathcal{M}_{n\mathfrak{p}}(\mathfrak{c})^{\text{ét}}.$$

Over the ordinary locus, the quotient map $\theta : \mathcal{M}_n(\mathfrak{c}) \to \mathcal{M}_{n\mathfrak{p}}(\mathfrak{c})^{\text{ét}}$ is defined as the composition $\theta = \theta' \circ \omega \circ \sigma$ where θ', ω and σ have the following moduli definitions:

$$\sigma : \mathcal{M}_n(\mathfrak{c}) \to \mathcal{M}_{n\mathfrak{p}}(\mathfrak{c})^m, \quad \underline{A} \mapsto (\underline{A}, A[\mathrm{Fr}] \cap A[\mathfrak{p}])$$

$$\omega : \mathcal{M}_{n\mathfrak{p}}(\mathfrak{c})^m \to \mathcal{M}_{n\mathfrak{p}}(\mathfrak{cp})^{\text{ét}}, \quad (\underline{A}, H) \mapsto (\underline{A}/H, A[\mathfrak{p}]/H)$$

$$\theta' \colon \mathcal{M}_{n\mathfrak{p}}(\mathfrak{cp})^{\text{ét}} \to \mathcal{M}_{n\mathfrak{p}}(\mathfrak{c})^{\text{ét}}, \quad (\underline{A}, H) \mapsto (\underline{A}/A[\mathrm{Fr}] \cap A[\mathfrak{p}'], H \bmod A[\mathrm{Fr}] \cap A[\mathfrak{p}']).$$

We can examine the behaviour of this map on the toroidal compactification by applying these maps to the semi-abelian schemes over the cusps.

Theorem 4.15. *Let C be the cusp of $\mathcal{M}_n(\mathfrak{c})$ given by $(\mathfrak{a}, \mathfrak{b})$, where the representative ideals $\mathfrak{a}$ and $\mathfrak{b}$ are chosen to be prime to p. Let $\mathcal{M}_n^{TC}(\mathfrak{c})$ be the toroidal compactification of $\mathcal{M}_n(\mathfrak{c})$ using the $\Gamma(n)$-admissible decomposition Σ of $N = (\mathfrak{abd})^{-1}$. Then the quotient of $\mathcal{M}_n^{TC}(\mathfrak{c})$ by $\mathcal{F}_J$ is $\mathcal{M}_{n\mathfrak{p}}^{TC}(\mathfrak{c})^{ét}$, where the toroidal compactification at the cusp C', corresponding to $(\mathfrak{a}, \mathfrak{b}, \mathfrak{p})$, is given by the $\Gamma(n)$-admissible decomposition Σ' of $N' = (\mathfrak{abdp}')^{-1}$ induced from Σ by the natural inclusion $N \hookrightarrow N'$.*

Proof. Let C be the cusp given by $(\mathfrak{a}, \mathfrak{b})$, we can look at the semi-abelian scheme G over C, given by the Mumford construction $G = (\mathbb{G}_m \otimes \mathfrak{a}^\vee)/\iota(\mathfrak{b})$.

Note that the $\mathfrak{p}$-torsion of G has a multiplicative and an étale part. The multiplicative part corresponds to the $\mathfrak{p}$-torsion of $\mathbb{G}_m \otimes \mathfrak{a}^\vee$, and the étale part corresponds to the lattice $\iota(\mathfrak{b}\mathfrak{p}^{-1})$. So, the map σ, as described above, takes G to the tuple consisting of G and the multiplicative part of the $\mathfrak{p}$-torsion. This is the semi-abelian scheme over the cusp $C_1 = (\mathfrak{a}, \mathfrak{b}, \mathcal{O}_L)$ of $\mathcal{M}_{n\mathfrak{p}}(\mathfrak{c})$.

In order to apply the map ω, we need to take the quotient of $(\mathbb{G}_m \otimes \mathfrak{a}^\vee)/\iota(\mathfrak{b})$ by the multiplicative part of its $\mathfrak{p}$-torsion. This will just be the $\mathfrak{p}$-torsion of $(\mathbb{G}_m \otimes \mathfrak{a}^\vee)$.

First, note that the p-torsion of $(\mathbb{G}_m \otimes \mathfrak{a}^\vee)$ is $\mu_p \otimes \mathfrak{a}^\vee = \mu_p \otimes \mathfrak{a}^\vee/p\mathfrak{a}^\vee$. Further, for a simple tensor $q \otimes \alpha \in \mu_p \otimes \mathfrak{a}^\vee$ to be in the $\mathfrak{p}$-torsion, we require $c\alpha \in p\mathfrak{a}^\vee$ for all $c \in \mathfrak{p}$. Thus, we will need $\alpha \in \mathfrak{p}'\mathfrak{a}^\vee$. So the $\mathfrak{p}$-torsion of $(\mathbb{G}_m \otimes \mathfrak{a}^\vee)$ is $\mu_p \otimes \mathfrak{p}'\mathfrak{a}^\vee$.

Consider the map $\tilde{\omega} : (\mathbb{G}_m \otimes \mathfrak{a}^\vee) \to (\mathbb{G}_m \otimes \mathfrak{p}^{-1}\mathfrak{a}^\vee)$ induced by $\mathfrak{a}^\vee \hookrightarrow \mathfrak{p}^{-1}\mathfrak{a}^\vee$. The kernel of this map will be generated by the simple tensors of the form $q \otimes \alpha$ where $q \in \mu_p$ and $\alpha/p \in \mathfrak{p}^{-1}\mathfrak{a}^\vee$. But $\alpha/p \in \mathfrak{p}^{-1}\mathfrak{a}^\vee$ if and only if $\alpha \in \mathfrak{p}'\mathfrak{a}^\vee$. So $\mu_p \otimes \mathfrak{p}'\mathfrak{a}^\vee$ is indeed the kernel of this map. Thus we have an isomorphism from the quotient of $(\mathbb{G}_m \otimes \mathfrak{a}^\vee)$ by its $\mathfrak{p}$-torsion to $(\mathbb{G}_m \otimes \mathfrak{p}^{-1}\mathfrak{a}^\vee)$.

Recall from section 2.6 that $q : \mathfrak{b} \to (\mathbb{G}_m \otimes \mathfrak{a}^\vee)$ sends any element $b \in \mathfrak{b}$ to the unique element q^b of the torus $\mathbb{G}_m \otimes \mathfrak{a}^\vee$ such that for any character $\chi^a \in \mathfrak{a}$, we have $\chi^a(q^b)) = q^{ab}$. After applying $\tilde{\omega}$, the lattice $q^\mathfrak{b}$ still satisfies $\chi^a(q^b) = q^{ab}$ for any χ^a, with the χ^a now ranging over $\mathfrak{p}\mathfrak{a}$. Thus we can extend this map to the quotient

$$\omega : (\mathbb{G}_m \otimes \mathfrak{a}^\vee)/q^\mathfrak{b} \to (\mathbb{G}_m \otimes \mathfrak{p}^{-1}\mathfrak{a}^\vee)/q^\mathfrak{b},$$

the image is the semi-abelian scheme over a cusp of $\mathcal{M}_n(\mathfrak{c}\mathfrak{p})$ given by the ideals $(\mathfrak{p}\mathfrak{a}, \mathfrak{b})$.

The quotient of the $\mathfrak{p}$-torsion of G by its multiplicative part will be precisely the étale part of the $\mathfrak{p}$-torsion of our new semi-abelian scheme. So $\omega(\sigma(G)) = (\mathbb{G}_m \otimes \mathfrak{p}^{-1}\mathfrak{a}^\vee)/q^\mathfrak{b}$ along with the étale part of its $\mathfrak{p}$-torsion. This is the tuple over the cusp C_2 of $\mathcal{M}_{n\mathfrak{p}}(\mathfrak{c}\mathfrak{p})$ given by $(\mathfrak{p}\mathfrak{a}, \mathfrak{b}, \mathfrak{p})$.

Finally, we will apply the map θ', in which we do much the same quotient as we did for ω, but this time with the multiplicative part of $\mathfrak{p}'$-torsion. Thus, we will get the semi-abelian scheme over the cusp $C' = (p\mathfrak{a}, \mathfrak{b}, \mathfrak{p})$ of $\mathcal{M}_{n\mathfrak{p}}(\mathfrak{c}p) = \mathcal{M}_{n\mathfrak{p}}(\mathfrak{c})$.

Note that over the cusps C_1 and C_2, the toroidal compactification is defined using an admissible polyhedral decomposition of $N_1 = (\mathfrak{a}\mathfrak{b}\mathfrak{d})^{-1}$, and $N_2 = (\mathfrak{a}\mathfrak{b}\mathfrak{d}\mathfrak{p})^{-1}\mathfrak{p}$, both of which are equal to N_C. So we can set these decompositions to be $\Sigma_1 = \Sigma_2 = \Sigma$. For the cusp, C' we must have an admissible polyhedral decomposition of $(p\mathfrak{a}\mathfrak{b}\mathfrak{d})^{-1}\mathfrak{p} = (\mathfrak{a}\mathfrak{b}\mathfrak{d}\mathfrak{p}')^{-1} = N'_C$. This will just be the decomposition Σ' given in the statement. Thus, we have the following diagram:

$$\begin{tikzcd}
X(\Sigma)/U_n^2 \arrow[r] & X(\Sigma_1)/U_n^2 \arrow[r] & X(\Sigma_2)/U_n^2 \arrow[r] & X(\Sigma')/U_n^2 \\
\widehat{X(\Sigma)/U_n^2} \arrow[r] \arrow[u, hook] & \widehat{X(\Sigma_1)/U_n^2} \arrow[r] \arrow[u, hook] & \widehat{X(\Sigma_2)/U_n^2} \arrow[r] \arrow[u, hook] & \widehat{X(\Sigma')/U_n^2} \arrow[u, hook] \\
{}^{C}\widehat{\mathcal{M}_n^{TC}(\mathfrak{c})} \arrow[r] \arrow[u, equal] \arrow[d, hook] & {}^{C_1}\widehat{\mathcal{M}_{n\mathfrak{p}}^{TC}(\mathfrak{c})}^{\mathrm{m}} \arrow[r] \arrow[u, equal] \arrow[d, hook] & {}^{C_2}\widehat{\mathcal{M}_{n\mathfrak{p}}^{TC}(\mathfrak{c}\mathfrak{p})}^{\text{ét}} \arrow[r] \arrow[u, equal] \arrow[d, hook] & {}^{C'}\widehat{\mathcal{M}_{n\mathfrak{p}}^{TC}(\mathfrak{c})}^{\text{ét}} \arrow[u, equal] \arrow[d, hook] \\
\mathcal{M}_n^{TC}(\mathfrak{c}) \arrow[r] & \mathcal{M}_{n\mathfrak{p}}^{TC}(\mathfrak{c})^{\mathrm{m}} \arrow[r] & \mathcal{M}_{n\mathfrak{p}}^{TC}(\mathfrak{c}\mathfrak{p})^{\text{ét}} \arrow[r] & \mathcal{M}_{n\mathfrak{p}}^{TC}(\mathfrak{c})^{\text{ét}} \\
\mathcal{M}_n(\mathfrak{c}) \arrow[r] \arrow[u, hook] & \mathcal{M}_{n\mathfrak{p}}(\mathfrak{c})^{\mathrm{m}} \arrow[r] \arrow[u, hook] & \mathcal{M}_{n\mathfrak{p}}(\mathfrak{c}\mathfrak{p})^{\text{ét}} \arrow[r] \arrow[u, hook] & \mathcal{M}_{n\mathfrak{p}}(\mathfrak{c})^{\text{ét}} \arrow[u, hook]
\end{tikzcd}$$

Here ${}^{C}\widehat{\mathcal{M}_n^{TC}(\mathfrak{c})}$ denotes the completion of $\mathcal{M}_n^{TC}(\mathfrak{c})$ along the fibre over the cusp C of $\mathcal{M}_n(\mathfrak{c})$, and $\widehat{X(\Sigma)/U_n^2}$ denotes the completion of $X(\Sigma)/U_n^2$ along its boundary, as in Theorem 2.37. The bottom arrows are the quotient by $\mathscr{F}_J$ and the top arrows are the quotient by $\mathcal{G}_J$. The vertical arrows correspond the toroidal compactification, and the third row is induced by the maps we just described on the semi-abelian schemes defined over the cusps.

This gives us a full description of the quotient of the toroidal compactification of $\mathcal{M}_n(\mathfrak{c})$ by the tautological foliations $\mathscr{F}_J$. In particular, we see that this quotient gives us a toroidal compactification, not necessarily smooth, of $\mathcal{M}_{n\mathfrak{p}}(\mathfrak{c})^{et}$. $\square$

5. DIEUDONNÉ MODULES AND DISPLAYS

5.1 Unitary Shimura varieties

Fix a prime $p > 2$, an imaginary quadratic field E in which p is inert, and let $m, n \in \mathbb{Z}$ with $0 < m \leq n$. Let $*$ be the non-trivial automorphism of $E/\mathbb{Q}$. Let $\Lambda = \mathcal{O}_E^{m+n}$, and $V = \Lambda \otimes E$, along with the Hermitian pairing:

$$\langle u, v \rangle = \overline{u}^T \begin{pmatrix} & & 1_m \\ & 1_{n-m} & \\ 1_m & & \end{pmatrix} v.$$

Let $\mathbf{G}$ denote the group of E-linear symplectic similitudes of $(E, \langle ., . \rangle)$. Note that $\mathbf{G}_{\mathbb{R}}$ is isomorphic to $GU(n, m)$, since $\left(\begin{smallmatrix} & & 1_m \\ & 1_{n-m} & \\ 1_m & & \end{smallmatrix}\right)$ is similar to $\left(\begin{smallmatrix} 1_n & \\ & -1_m \end{smallmatrix}\right)$. That is, $\mathbf{G}$ is the subgroup of GL_{m+n} such that for each $g \in \mathbf{G}$ there exist some $\mu(g)$, satisfying $\langle gu, gv \rangle = \mu(g) \langle u, v \rangle$.

Let $\mathbb{S}$ be the algebraic group given by the restriction of scalars from $\mathbb{C}$ to $\mathbb{R}$. Thus $\mathbb{S}_{\mathbb{R}} = \mathbb{C}^{\times}$. Now, let $h \colon \mathbb{S} \to \mathbf{G}$ be the homomorphism defined on $\mathbb{R}$-points, by taking $z \mapsto \left(\begin{smallmatrix} z \cdot 1_n & 0 \\ 0 & \overline{z} \cdot 1_m \end{smallmatrix}\right)$. So h is a Hodge structure of type $(-1, 0), (0, -1)$ on $V \otimes_{\mathbb{Q}} \mathbb{R}$.

Let C^p be an open compact subgroup of $\mathbf{G}(\mathbb{A}_f^p)$, where $\mathbb{A}_f^p$ denotes the finite adeles, trivial at p.

The data $(\mathbf{G}, h)$ is a Shimura datum, from which a unitary Shimura variety can be constructed. This variety can be viewed as a moduli space, as described by Kottwitz [Kot92].

Fix an embedding $\Sigma : E \hookrightarrow \mathbb{C} \hookrightarrow \mathbb{C}_p$, and let $\overline{\Sigma}$ be the conjugate embedding. Let S be a locally noetherian scheme over $\mathcal{O}_E \otimes_{\mathbb{Z}} \mathbb{Z}_p$, and let A be an abelian scheme of dimension g over S.

Given an injection $\iota\colon \mathcal{O}_E \hookrightarrow \mathrm{End}(A)$, we say that ι has signature (n,m) if, for all $b \in \mathcal{O}_E$, the characteristic polynomial of $\iota(b)$, when viewed as acting on $\mathrm{Lie}(A)$ is:

$$(X - \Sigma(b))^n (X - \overline{\Sigma}(b))^m \in \mathcal{O}_S[X]$$

Note that as A has dimension g over S, we must have $n + m = g$. From here on we will also assume that $0 < m \leq n$.

We can now formulate the moduli problem. Consider the set-valued contravariant functor from the category of locally noetherian schemes S over $\mathcal{O}_E \otimes_{\mathbb{Z}} \mathbb{Z}_p$ that associates to S, the set of isomorphism classes of quadruples (A, ι, ϕ, η), where:

— A is an abelian scheme of dimension $g = m + n$ over S;

— $\iota\colon \mathcal{O}_E \hookrightarrow \mathrm{End}(A)$ has signature (n,m);

— $\zeta\colon A \to A^*$ is a principal polarization whose Rosati involution induces $\iota(a) \mapsto \iota(\overline{a})$ on the image of ι;

— η is a rigid C^p level structure as in [Kot92].

Then, we know that this functor is representable by a quasi-projective scheme $\mathcal{M}$ over $\mathcal{O}_E \otimes_{\mathbb{Z}} \mathbb{Z}_p$.

Our object of study is the special fibre $\mathcal{M}_p$ of $\mathcal{M}$ at p, which is a smooth variety over κ, the residue field of p. As we will not need $\mathcal{M}$ itself in the following, we will denote $\mathcal{M}_p$ by $\mathcal{M}$ from here on.

5.2 Dieudonné modules

Let k be a perfect field of characteristic p, and let $W(k)$ be the ring of Witt vectors over k, with Frobenius $\sigma\colon W(k) \to W(k)$. Then a **Dieudonné module** is defined as a finitely-generated $W(k)$-module along with a σ-linear operator F, and σ^{-1}-linear operator V such that $FV = VF = p$.

Proposition 5.1 (Dieudonné)**.** *There is an equivalence $\mathcal{G} \mapsto \mathbb{D}(\mathcal{G})$ of categories be-*

tween the category of p-divisible groups over k and the category of Dieudonné modules that are free and finite rank as $W(k)$-modules.

There is also a contravariant version of this equivalence, which is precisely dual to the covariant theory. The construction of the contravariant equivalence can be found in the literature, for example [Dem72]. We can use this equivalence to study p-divisible groups by considering their associated Dieudonné modules

Note that if $\mathcal{G}$ is a p-divisible group over k, $\mathcal{F}\colon \mathcal{G} \to \mathcal{G}^{(p)}$ is the Frobenius morphism and $\mathcal{V}\colon \mathcal{G}^{(p)} \to \mathcal{G}$ is the Verschiebung morphism, then if we denote $\mathbb{D}(\mathcal{G}) = N$, then

$$\mathbb{D}(\mathcal{F}\colon \mathcal{G} \to \mathcal{G}^{(p)}) = V : N \to N^{(p)}$$

and

$$\mathbb{D}(\mathcal{V}\colon \mathcal{G}^{(p)} \to \mathcal{G}) = F : N^{(p)} \to N.$$

This equivalence continues modulo p.

Proposition 5.2. *There is an equivalence of categories between the category of finite commutative k-group schemes that are killed by p with the category of Dieudonné modules N killed by p such that N is finite-dimensional as a k-vector space.*

If a finite commutative k-group scheme $\mathcal{G}$ that is annihilated by p also satisfies the condition that the sequence $\mathcal{G} \xrightarrow{\mathcal{F}} \mathcal{G}^{(p)} \xrightarrow{\mathcal{V}} \mathcal{G}$ is exact, then $\mathcal{G}$ is called a **truncated Barsotti–Tate group of level 1**, which we abbreviate as BT_1. When carried across the Dieudonné equivalence, this condition becomes $\mathrm{im}(V) = \ker(F)$ and $\ker(V) = \mathrm{im}(F)$. A Dieudonné module that is finite-dimensional as a k-vector space, annihilated by p, and satisfies the condition that $\mathrm{im}(V) = \ker(F)$ and $\ker(F) = \mathrm{im}(V)$ is called a **regular Dieudonné space** [Wed01]. Thus, we have an equivalence between the category of BT_1, and the category of regular Dieudonné spaces.

Note also that if A is an abelian variety, then $A[p]$ is a BT_1. As we care about groups of the form $A[p]$, we will be looking primarily at the theory of BT_1's and regular Dieudonné spaces from here on.

We now wish to consider what happens when we consider the polarization on A. First, a **polarized** BT_1 is defined to be a BT_1 group $\mathcal{G}$, along with a non-degenerate alternating pairing $\langle , \rangle : \mathbb{D}(\mathcal{G}) \times \mathbb{D}(\mathcal{G}) \to k$ such that $\langle Fx, y\rangle = \langle x, Vy\rangle^p$, for $x, y \in \mathbb{D}(\mathcal{G})$. A Dieudonné module with such a pairing is said to be **symmetric**. Note that if A is a principally polarized abelian scheme, then $A[p]$ is a polarized BT_1, where $\langle , \rangle$ is induced by the polarization on A [Oor01, 9.2,12.2].

Now let k be algebraically closed. We wish to consider what happens when we incorporate an endomorphism structure on A along with a principal polarization. Let E be a quadratic imaginary field, such that p is inert in E, and let κ be the residue field of p. As above, let S be a locally noetherian $\mathcal{O}_E \otimes_{\mathbb{Z}} \mathbb{Z}_p$ scheme, and let A be an abelian scheme over S. Suppose there exists an action $\iota : \mathcal{O}_E \hookrightarrow A$ with signature (n, m). Furthermore, suppose that this action is compatible with the principal polarization, that is, the Rosati involution induced by the polarization must act as $\iota(a) \mapsto \iota(\overline{a})$ on the image of ι in $\mathrm{End}(A)$.

Now let N be the regular Dieudonné space associated with $A[p]$. Due to the endomorphism structure ι on A, we see that N can be seen not only as k-vector space, but as an $\kappa \otimes k$-vector space. Recall that we have two embeddings $\Sigma, \overline{\Sigma} : E \hookrightarrow \mathbb{C}^p$. By taking quotients by (p), these embeddings induce

$$\Sigma, \overline{\Sigma} : \kappa \hookrightarrow \overline{\mathbb{F}}_p \hookrightarrow k.$$

Thus, we have $\kappa \otimes k \cong k \oplus k$, and a decomposition of N as $N(\Sigma) \oplus N(\overline{\Sigma})$, where Σ and $\overline{\Sigma}$ are the two embeddings of κ into k [Moo01, 4.3].

5.3 Displays

In order to study the geometry of $\mathcal{M}$, we will need to understand deformations of abelian varieties with polarizations and endomorphism structures. By a theorem of Serre–Tate, it is known that deformations of abelian varieties over a field of characteristic p are equivalent to the deformations of their p-divisible groups. Furthermore, by the Dieudonné equivalence, we can study the deformations of $A[p^\infty]$ by looking at the deformations of the related Dieudonné modules. These deformations have been explicitly described in the work of Norman and Oort [Nor75, NO80]. These methods have since been generalized and extended via Zink's theory of displays in [Zin02].

Let (A, ι, ζ, η) be as in the previous section, and let N be the Dieudonné module of $A[p^\infty]$. Then a **displayed basis** of N is defined as a set of generators $\{e_1, \dots, e_{2g}\}$ for N as a $W(k)$-modules such that there exists $\{a_{ij}\} \in W(k)$ that satisfy:

$$\begin{aligned} F(e_i) &= \sum_{j=1}^{2g} a_{ij} e_j && 1 \le i \le g \\ e_i &= V\left(\sum_{j=1}^{2g} a_{ij} e_j\right) && g+1 \le i \le 2g. \end{aligned}$$

The matrix $(a_{ij}) = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ is called the **display matrix** for N. In particular, note that since $FV = p$, we have

$$F(e_i) = \sum_{j=1}^{2g} p a_{ij} e_j \qquad g+1 \le i \le 2g$$

Thus, the action of F on N in the displayed basis is given by the matrix $\left(\begin{smallmatrix} A & pB \\ C & pD \end{smallmatrix}\right)$. When working with this matrix representation it is important to remember that this is not a linear operator on N, but a σ-linear operator.

Such objects can be defined over more general rings than just perfect fields. We follow the construction of displays as introduced by Zink in [Zin02]. Let R be a commutative unitary ring, such that p is nilpotent in R. Then $W(R)$ is the ring of Witt vectors over R. Let $\sigma : W(R) \to W(R)$ be the Frobenius map, and let $I_R \subset W(R)$ be the Witt vectors $(x_0, x_1, \dots) \in W(R)$ such that $x_0 = 0$. Let ${}^V : W(R) \to W(R)$ be the Verschiebung map, that is ${}^V(x_0, x_1, \dots) = (0, x_0, x_1, \dots)$.

A display over R is defined to be a quadruple (P, Q, F, V^{-1}), where P is a finitely generated $W(R)$-module, $Q \subset P$ is a submodule, and F and V^{-1} are σ-linear maps $F\colon P \to P$ and $V^{-1} :: Q \to P$ that satisfy the following properties:

— $I_R P \subset Q \subset P$, and P/Q is a direct summand of the $W(R)$ module $P/I_R P$.

— $V^{-1} : Q \to P$ is a σ-linear epimorphism.

— For $x \in P$ and $w \in W(R)$ we have $V^{-1}({}^V wx) = wFx$.

Proposition 5.3. [Zin02] *Over a perfect field k, there is an equivalence between the category of displays over k and the category of Dieudonné modules over k.*

We wish to study the deformations of N through these displays. In this context, we define a deformation of N to be a display over a local Artinian $W(k)$-algebra R with residue field k, such that it specializes to N after base change to k.

Proposition 5.4. [Nor75] *Let R be a local Artinian $W(k)$-algebra with residue field k. Every deformation of N over R is isomorphic to a deformation with display matrix:*

$$\begin{pmatrix} A+TC & B+TD \\ C & D \end{pmatrix}$$

where $t_{ij} \in R$ are uniquely determined, and $T = (\hat{t}_{ij})$, for $\hat{t}_{ij} := (t_{ij}, 0, \dots) \in W(R)$.

From this result, we see that the universal deformation of N is over the local Artinian $W(k)$-algebra $R = k[[t_{ij}]]$, with the displayed basis $\{e_1, \dots, e_{2g}\}$ and matrix

$$\begin{pmatrix} A+TC & B+TD \\ C & D \end{pmatrix}$$

where $T = (\hat{t}_{ij})$ and $\hat{t}_{ij} = (t_{ij}, 0, \dots) \in W(k[[t_{ij}]])$.

If we wish to take the polarization structure into account, we must insist that the displayed basis be a symplectic basis with respect to the polarization. That is, we must have $\langle e_i, e_{g+j} \rangle = \delta_{ij}$ for $1 \le i, j \le g$. In this case, the deformations of N over R that preserve the polarization are given precisely by the condition that $t_{ij} = t_{ji}$ [NO80].

Furthermore, we will need to preserve the action of $\mathcal{O}_E$. As such, we cite the following Lemma, which gives the deformation of N that is universal for deformations that respect the polarization and endomorphism structure on N.

Lemma 5.5. [Woo16, Lemma 2.2.8] *There exists a displayed basis for the Dieudonné module N of $A[p^\infty]$ of the form $\{e_1, \ldots, e_g; f_1, \ldots, f_g\}$ such that*

— $\mathcal{B}_1 = \{e_1, \ldots, e_m, f_{m+1}, \ldots f_g\}$ *is a basis for* $N[\Sigma]$

— $\mathcal{B}_2 = \{e_{m+1}, \ldots, e_g, f_1, \ldots, f_m\}$ *is a basis for* $N[\overline{\Sigma}]$

— $V(N) = span\{f_1, \ldots, f_g\}$

— $\langle e_i, f_j\rangle = \delta_{ij} = -\langle f_j, e_i\rangle$

and the displayed matrix for N has the form

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} 0 & A_1 & B_1 & 0 \\ A_2 & 0 & 0 & B_2 \\ C_2 & 0 & 0 & D_2 \\ 0 & C_1 & D_1 & 0 \end{pmatrix}$$

with respect to the basis $\{e_1, \ldots, e_g; f_1, \ldots, f_g\}$. Also,it satisfies the relation

$$\begin{pmatrix} A & pB \\ C & pD \end{pmatrix} \begin{pmatrix} -pD^t & pB^T \\ C^t & -A^t \end{pmatrix} = pI_{2g}.$$

Furthermore, the universal display of N preserving $\mathcal{O}_E$-action and prime-to-p polarization has the form

$$\begin{pmatrix} A+TC & B+TD \\ C & D \end{pmatrix}$$

where

$$T = \begin{pmatrix} 0_m & T' \\ T'^t & 0_n \end{pmatrix}.$$

Here T' refers to the matrix

$$\begin{pmatrix} \hat{t}_{1(n+1)} & \dots & \hat{t}_{1g} \\ \vdots & \ddots & \vdots \\ \hat{t}_{n(n+1)} & \dots & \hat{t}_{ng} \end{pmatrix}.$$

5.4 Permutations and Shuffles

In the sequel, we will be using the permutation group S_g, along with a quotient of that group. The purpose of this section is to set down notation for discussing these groups.

Elements of S_g will be notated using one-line notation. For example, the identity element of S_g is $[1\ 2\ 3 \dots g]$. When giving particular examples, the brackets may be suppressed, for example, the identity element of S_6 is 123456.

Every element of S_g can be written as a product of transpositions of neighbouring elements. Define the length of $\omega \in S_g$ to be the smallest k such that ω can be written as a product of k transpositions of neighbouring elements. We denote the length of ω as $\ell(\omega)$. Note that $\ell(\omega)$ can be computed as

$$\ell(\omega) = \#\{(i,j) : 1 \leq i < j \leq g; \omega(j) > \omega(i), \} \tag{5.1}$$

Let $n + m = g$ as above. View S_g as acting on the set $\{1, 2, 3, \dots, g\}$. Consider the subgroup $S_n \times S_m \subset S_g$, where S_n acts on the subset $\{1, 2, 3, \dots, n\}$, and S_m acts on the subset $\{n+1, \dots, g-1, g\}$. Define $\mathrm{Shf}_{n,m} = S_n \times S_m \backslash S_g$. Each coset of $\mathrm{Shf}_{n,m}$ has a unique minimal length representative. These minimal length representatives are characterized by the property that $\omega \in S_g$ is a minimal length representative of its coset in $\mathrm{Shf}_{n,m}$ if and only if:

$$\omega(1) < \omega(2) < \cdots < \omega(n) \quad \text{and} \quad \omega(n+1) < \cdots < \omega(g-1) < \omega(g).$$

Such a permutation is called an (n, m)**-shuffle**. Elements of $\mathrm{Shf}_{n,m}$ will be denoted by their minimal length representative.

6. THE EKEDAHL–OORT STRATIFICATION

In this chapter we will describe the Ekedahl–Oort stratification of $\mathcal{M}$. This stratification was first defined for $\mathcal{A}_g$ [Oor01]. It has since been extended to (good reductions of) Shimura varieties of PEL-type [Moo01, Wed01] and more generally to (good reductions of) Shimura varieties of Hodge-type [Zha18]. Further, the abelian case has been treated in [She20].

In section 6.1, we will look at Oort's construction of this stratification for $\mathcal{A}_g$. In section 6.2, we will consider some results pertaining to Moonen's extension of this stratification to Shimura varieties of PEL-type. Then in section 6.3, we will look a little more explicitly at these results for unitary Shimura varieties.

6.1 The Ekedahl–Oort Stratification of $\mathcal{A}_g$

In this section we review the construction of the Ekedahl–Oort stratification of $\mathcal{A}_g$, the moduli space of principally polarized abelian varieties of dimension g. The construction provided here is based on the work of Oort [Oor01], and much of the extension to unitary Shimura varieties will be based on it.

Let k be a perfect field of characteristic p, and let A be a principally polarized abelian variety over k of dimension g. Then $A[p]$ is a BT_1, and as such there is a regular symmetric Dieudonné space N corresponding to $A[p]$. Note that since we are using the covariant Dieudonné theory, F corresponds to the Verschiebung map on $A[p]$, and V corresponds to the Frobenius map on $A[p]$.

There is a minimal filtration

$$0 = N_0 \subset N_1 \subset \dots N[V] = N_r = F(N) \subset \cdots \subset N_{d-1} \subset N_d = N$$

that is stable under F and V^{-1} [Oor01]. This is called the **canonical filtration** of N.

Given such a filtration, we define a **canonical type** $\tau = \{\rho, f, v\}$ as a triple of functions

$$\rho\colon \{0, \dots, d\} \to \mathbb{Z}_{\geq 0}$$
$$f\colon \{0, \dots, d\} \to \{0, \dots, r\}$$
$$v\colon \{0, \dots, d\} \to \{r, \dots, d\}$$

are defined such that:

$$\mathrm{rk}(N_i) = p^{\rho(i)}$$
$$F(N_i) = N_{f(i)}$$
$$V^{-1}(N_i) = N_{v(i)}.$$

Furthermore, we define a permutation $\pi\colon \{1, \dots, d\} \to \{1, \dots, d\}$ by

$$\pi(i) = \begin{cases} f(i) & f(i) > f(i-1) \\ v(i) & v(i) > v(i-1). \end{cases}$$

Conversely, consider any such functions $\{\rho, f, v\}$, with induced permutation π, that satisfy the properties:

— $\rho\colon \{0, \dots, d\} \to \mathbb{Z}_{\geq 0}$ is strict monotone with $\rho(0) = 0$.

— v and f are monotone and surjective, with

$$v(i+1) > v(i) \Leftrightarrow f(i+1) = f(i).$$

— $\rho(i+1) - \rho(i) = \rho(\pi(i+1)) - \rho(\pi(i))$ for every $i \in \{0, \ldots, d\}$.

Then, there exists a BT_1 group $\mathcal{G}$ with covariant Dieudonné module having canonical type $\{\rho, f, v\}$ [Oor01, 2.8].

In our case, as N is the Dieudonné module of a polarized BT_1 we have some additional structure on this filtration. In particular, there exists a non-degenerate alternating pairing $\langle,\rangle\colon N \times N \to k$ such that $\langle Fx, y\rangle = \langle x, Vy\rangle^p$, for $x, y \in N$. Suppose that T is a subspace of N with inclusion map $\iota\colon T \to N$. We write $\perp(T) := \{y \in N : \langle x, y\rangle = 0, x \in T\}$.

Proposition 6.1. [Oor01, 5.2] *Let $T \subset N$ be as above. Then*

(1) $\perp(\perp(T)) = T$;

(2) $\perp(F(N)) = F(N)$;

(3) The set $\{w(N)\}$, where w ranges over finite words in the symbols F and $\perp$, is a finite filtration on N;

(4) $\perp(F(T)) = V^{-1}(\perp(T))$ for every submodule $T \subset N$;

(5) The filtration in part (3) of the above is the canonical filtration.

In particular, this proposition tells us that for any regular symmetric Dieudonné module, the minimal filtration that is stable under F and V^{-1} is the same as the minimal filtration that is stable under F and $\perp$.

We will call any filtration that is stable under F and V^{-1} an **admissible filtration**. So, as the canonical filtration is the unique minimal filtration stabilizing F and V^{-1}, we note that any admissible filtration of N is a refinement of the canonical filtration. If $0 = T_0 \subsetneq T_1 \subsetneq T_2 \subsetneq \cdots \subsetneq T_g = N$ is an admissible filtration such that $\dim(T_i) = i$ for each step in the filtration, we say that $T_\bullet$ is a maximal admissible filtration for N.

Let (ρ, f, v) be the canonical type for some symmetric regular Dieudonné space N.

Then $d = 2r$ and for all $0 \leq j < r$ we must have:

$$v(j) = 2r - f(2r - j)$$
$$\rho(j+1) - \rho(j) = \rho(2r - j) - \rho(2r - j - 1).$$

We call such a canonical type **symmetric**.

An **elementary sequence** is a map $\varphi\colon \{0, \ldots, g\} \to \mathbb{Z}_{\geq 0}$, such that $\varphi(0) = 0$, and

$$\varphi(i) \leq \varphi(i+1) \leq \varphi(i) + 1, \quad 0 \leq i < g.$$

Given N as above, we can produce an elementary sequence as follows. Consider a maximal admissible filtration of N

$$N_\bullet \colon 0 = N_0 \subset N_1 \subset \cdots \subset N_{2g-1} \subset N_{2g} = N.$$

Note that each N_i is a subspace of dimension i. Then, define $\varphi\colon \{1, \ldots g\} \to \mathbb{Z}_{\geq 0}$ such that $V(N_i) = N_{\varphi(i)}$.

Proposition 6.2. [Oor01, 5.7] *There is a natural bijection of sets between symmetric canonical types with $\rho(d) = 2g$ and elementary sequences of length g.*

Given N, a regular symmetric Dieudonné space, we denote the elementary sequence corresponding to the symmetric canonical type of its canonical filtration as $ES(N)$.

Theorem 6.3. [Oor01, 9.4] *Suppose k is an algebraically closed field of characteristic p, and let φ be an elementary sequence. Then there exists a polarized BT_1 $(\mathcal{G}, \zeta)$ of rank p^{2g} defined over k, such that $\varphi = ES(\mathbb{D}(\mathcal{G}))$. Furthermore, $(\mathcal{G}, \zeta)$ is unique up to non-unique isomorphism.*

This theorem tells us that elementary sequences classify isomorphism classes of BT_1 over k.

Consider $\mathcal{A}_g$, the moduli space of principally polarized abelian varieties over k. The Ekedahl–Oort stratification is given by

$$\{S_\varphi : \varphi \text{ is an elementary sequence on } \{0,\dots,g\}\},$$

where S_φ is the locally closed subset of $\mathcal{A}_g$ such that for all geometric points x, the abelian variety $\mathcal{A}_{g,x}$ belongs to S_φ if and only if $A_x[p]$ has elementary sequence φ. It is known that each S_φ is regular, quasi-affine and equidimensional. The dimension of each stratum is given by the following formula.

Proposition 6.4. [Oor01, 11.2] *$dim(S_\varphi) = \sum_{i=0}^{g} \varphi(i)$.*

6.2 *The Ekedahl–Oort Stratification of Shimura varieties of PEL-type*

We now consider an extension of this stratification to Shimura varieties of PEL-type. This construction was done by Moonen in [Moo01, Moo04]. Here, we do not need the results in their original generality, so we will consider the following moduli problem. Let $\mathcal{B}$ be a number field of degree n such that p is inert in $\mathcal{O}_\mathcal{B}$, along with a positive involution $x \mapsto \overline{x}$. Thus $\kappa = \mathcal{O}_\mathcal{B}/(p)$ is a field of characteristic p. Let S be a locally noetherian $\mathcal{O}_\mathcal{B} \otimes_\mathbb{Z} \mathbb{Z}_p$ scheme, let $N \geq 3$, and consider the following data.

— A, an abelian scheme up to prime-to-p isogeny over S;

— ζ, a prime-to-p polarization, considered modulo $\mathbb{Z}_p^\times$;

— $\iota : \mathcal{O}_\mathcal{B} \to \mathrm{End}(A)$, such that if $\dagger$ is the Rosati involution on $\mathrm{End}(A)$ corresponding to ζ, then $\iota(\overline{b}) = \iota(b)^\dagger$.

— η, a level-N structure on A

Let $\mathcal{M}_{\mathcal{B},p}$ be the moduli space parameterizing this data. Note that the $\mathcal{M}$ we defined earlier further specializes this problem to the case that $\mathcal{B}$ is an imaginary quadratic field. To study the Ekedahl–Oort stratification of $\mathcal{M}_{\mathcal{B},p}$, we will need to classify the p-torsion of such abelian schemes. So let $\mathcal{G}$ be $A[p]$ for some (A,ζ,ι,η) as above. So $\mathcal{G}$ is a polarized BT_1, and $\iota : \kappa \to \mathrm{End}(\mathcal{G})$. In this section, we will look at classifying such tuples $(\mathcal{G}, \zeta, \iota)$.

By the Dieudonné equivalence, we know that there exists a regular Dieudonné space N corresponding to $\mathcal{G}$. We also see that the action $\iota \colon \kappa \to \mathrm{End}(\mathcal{G})$ induces a $K := \kappa \otimes_{\mathbb{F}_p} k$-module structure on N that commutes with F and V. Let $\mathcal{I} := \mathrm{Hom}(\kappa, k)$. This gives a canonical decomposition $N = \oplus_{i \in \mathcal{I}} N_i$, where

$$N_i = \{n \in N : i(b)n = b(n), \forall b \in \kappa\}$$

where we write $b(n)$ for $\iota(b)(n)$.

Note that we have a canonical involution on $\mathcal{I}$, given by $i \mapsto \bar{i}$, where

$$\bar{i}(b) := i(\bar{b}) = i(b)^{\dagger}.$$

Let $d \in \mathbb{Z}_{>0}$ be given by $d = \dim(N_i)$. Note that since N is a regular Dieudonné space, $\dim(N_i) = \dim(N_j)$ for any $i, j \in \mathcal{I}$. Consider $L := \ker(F) \subset N$. Since F commutes with the K-module structure, we see that L decomposes as $\oplus_{i \in \mathcal{I}} L_i$, with $L_i \subset N_i$. Define $\mathfrak{f} \colon \mathcal{I} \to \mathbb{Z}_{\geq 0}$ by $\mathfrak{f}(i) = \dim(L_i)$. We say that $(d, \mathfrak{f})$ is the **multiplication type** of $(\mathcal{G}, \zeta, \iota)$; it determines (N, L) up to isomorphism [Moo01, Ch. 4].

Let $G := U_{\kappa \otimes k}(N)$, the group of automorphisms of N that preserves the pairing $\langle ., . \rangle$. [Moo04, 1.3]. Then $P := \mathrm{Stab}(L)$ is a parabolic subgroup of the reductive group G. Define $\mathbb{X}$ to be the conjugacy class of parabolic subgroups containing P. Thus, if W_G denotes the Weyl group of G, there is a subgroup $W_{\mathbb{X}} \subset W_G$ that corresponds to $\mathbb{X}$. Note that $\mathbb{X}$ depends only on the multiplication type $(d, \mathfrak{f})$, and not on the particular subspace $L \subseteq N$. Thus, any two regular Dieudonné spaces with the same multiplication type will produce the same subgroup $W_X \subset W_G$ under this construction. So once a multiplication type $(d, \mathfrak{f})$ is fixed, $W_{\mathbb{X}}$ is well-defined, without regard for which $(\mathcal{G}, \zeta, \iota)$ of that type is used to construct it.

To any tuple $(\mathcal{G}, \zeta, \iota)$ of type $(d, \mathfrak{f})$, we can associate an element of $W_{\mathbb{X}} \backslash W_G$, as follows. Let N be the regular Dieudonné space corresponding to $\mathcal{G}$, and consider the canonical filtration $N_\bullet$ of N, as in the previous section. Furthermore, note that since F and V commute with the K-module structure, this is not only a filtration of k-vector spaces, but of K-modules. In other words, the endomorphism structure is preserved in the canonical filtration.

Let Q be the parabolic subgroup that fixes this filtration, and let the Weyl group of Q be $W_Q \subset W_G$. Then the relative position of L and $N_\bullet$ is given by an element $\underline{\omega}(\mathcal{G}, \zeta, \iota) \in W_{\mathbb{X}} \backslash W_G / W_Q$. Moonen proves that the canonical filtration is *in optimal position* with respect to L [Moo01, Ch. 4]. That is, for any Borel subgroup B of Q, the relative position of L and the complete flag fixed by B does not depend on the choice of B. In particular, this means that we can consider the relative position to be an element of $W_{\mathbb{X}} \backslash W_G$.

Theorem 6.5. [Moo01, 5.5,6.7] *The map $(\mathcal{G}, \zeta, \iota) \to \underline{\omega}(\mathcal{G}, \zeta, \iota)$ is a bijection*

$$\left\{ \begin{array}{c} \textit{isomorphism classes of tuples} \\ (\mathcal{G}, \zeta, \iota), \ \textit{of multiplication type } (d, \mathfrak{f}) \end{array} \right\} \to W_X \backslash W_G.$$

Also, note that this construction decomposes across $\mathcal{I}$. Given a tuple $(\mathcal{G}, \iota)$ of type $(d, \mathfrak{f})$, where N is the Dieudonné space associated with $\mathcal{G}$, we can decompose $N = \oplus_{i \in \mathcal{I}} N[i]$. Furthermore, since F and V commute with the K-module structure on N, each piece of the canonical filtration is a K-module. Therefore, each $N[i]$ has a filtration $0 = N[i]_0 \subsetneq N[i]_1 \subsetneq \cdots \subsetneq N[i]_{r_i} = N[i]$ [Moo01, 4.3]. For $1 \leq j \leq r_i$ let $B_{i,j} := N[i]_j / N[i]_{j-1}$. Then, since the canonical filtration is stable under F and V^{-1}, we see that for each $B_{i,j}$ we either have an isomorphism $F \colon B_{i,j} \to B_{i+1,j'}$ for some $1 \leq j' \leq r_{i+1}$, or $F(B_{i,j}) = 0$. Similarly, we either have an isomorphism $V \colon B_{i,j} \to B_{i-1,j'}$ for some $1 \leq j' \leq r_{i-1}$ or $V(B_{i,j}) = 0$.

Note that this produces an equivalence relation on the blocks. Following [Moo01], we say that B_{i_1,j_1} and B_{i_2,j_2} are in the same orbit, if there is a sequence of isomorphisms F and V^{-1} taking $B_{i_1,j_1} \xrightarrow{\sim} B_{i_2,j_2}$. These orbits respect duality. That is, if B_{i_1,j_1} and B_{i_2,j_2} are in the same orbit, then $B_{\overline{i_1}, r_{i_1} - j_1}$ is in the same orbit as $B_{\overline{i_2}, r_{i_2} - j_2}$. In the case that $B_{i,j}$ and $B_{\bar{i},r_i-j}$ are in the same orbit, we say that this orbit is self-dual. On the other hand if $B_{i,j}$ and $B_{\bar{i},r_i-j}$ are in different orbits, we say that these two orbits are dual to each other.

Proposition 6.6. [Moo01, see 4.11] *For every pair $i \in \mathcal{I}, j \in \{1, \dots, r_i\}$ we can simultaneously choose ordered bases $\beta_{i,j}$ for each $B_{i,j}$ such that every isomorphism $F\colon B_{i,j} \to B_{i+1,j'}$ maps $\beta_{i,j} \xrightarrow{\sim} \beta_{i+1,j'}$, and every isomorphism $V\colon B_{i,j} \to B_{i-1,j'}$ maps $\beta_{i,j} \xrightarrow{\sim} \beta_{i-1,j'}$.*

This proposition implies that we can refine the filtrations $N[i]$ to maximal filtrations:

$$0 = N[i]_0 \subsetneq N[i]_1 \subsetneq \cdots \subsetneq N[i]_d = N[i]$$

with the property that for each $i \in \mathcal{I}, 0 \leq j \leq d$, there exists j' and j'' such that $F(N[i]_j) = N[i+1]_{j'}$ and $V^{-1}(N[i]_j) = N[i+1]_{j''}$. We will call a collection of filtrations of the $N[i]$ that satisfies this property, **$\mathcal{I}$-admissible**. The actions of F and V can be made explicit as follows.

Proposition 6.7. [Moo01, see 4.9,4.17] *Let ω be the minimal length representative of $\underline{\omega}(\mathcal{G}, \iota) \in W_X \backslash W_G$. Then the $\beta_{i,j}$ can be lifted to ordered bases $\{e_{i,1}, \dots, e_{i,d}\}$ of N_i, such that*

$$F(e_{i,j}) = \begin{cases} 0 & \omega(j) \leq \mathfrak{f}(i) \\ e_{i+1,m} & \omega(j) = \mathfrak{f}(i) + m \end{cases}$$

and

$$V(e_{i+1,j}) = \begin{cases} 0 & j \leq d - \mathfrak{f}(i) \\ e_{i,n} & j = d - \mathfrak{f}(i) + \omega(n). \end{cases}$$

Note that if we start with some $\omega \in W_{\mathbb{X}} \backslash W_G$, define a Dieudonné space N by endowing the k-span of $\{e_{i,j} : i \in \mathcal{I}, 1 \leq j \leq d\}$ with a κ-action by $b(e_{i,j}) = i(b)e_{i,j}$, for all $b \in \kappa$, and maps F and V as in the above proposition, then ω is indeed the relative position of $L = F^{-1}(0)$ and the canonical filtration of this space. Furthermore, we can define a polarization on this space following [Moo01]:

Recall that the $B_{i,j}$ were divided into orbits. Denote the set of orbits by $\mathcal{A}$. Partition $\mathcal{A}$ into 3 (possibly empty) pieces, $\mathcal{A} = \mathcal{A}_1 \cup \mathcal{A}_2 \cup \mathcal{A}_{sd}$, such that $\mathcal{A}_{sd}$ consists of all self dual orbits, and for every orbit $O \in \mathcal{A}_1$, the dual orbit $\overline{O} \in \mathcal{A}_2$.

For each (i,j) we define $c_{i,j}$ to be a root of unity as follows. Recall that the basis $\{e_{i,j}\}$ of N was lifted from the collection $\beta_{i,j}$ of bases for the $B_{i,j}$. So suppose for some (i,j), $e_{i,j}$ was lifted from the basis $\beta_{i,j'}$ and is in an orbit O. If $O \in \mathcal{A}_1$, let $c_{i,j} = 1$. If $O \in \mathcal{A}_2$, let $c_{i,j} = -1$. If $O \in \mathcal{A}_{sd}$, and O has length $2s$, choose $c_{i,j}$ to be a root of unity in k such that $c_{i,j}^{p^s} = -c$. Furthermore, consider the unique j_0 such that either $F(e_{i,j}) = e_{i+1,j_0}$ or $V(e_{i+1,j_0}) = e_{i,j}$. We require that $c_{i+1,j_0} = c_{i,j}^p$. Now we can define the polarization on N by:

$$\langle e_{i_1,j_1}, e_{\overline{i_2},d+1-j_2} \rangle = c_{i_1,j_1}\delta_{i_1,i_2}\delta_{j_1,j_2}$$

One can check that this definition satisfies the relations $\langle x, y \rangle = -\langle y, x \rangle$ and $\langle Fx, y \rangle = \langle x, Vy \rangle^p$.

Proposition 6.8. [Moo01, see 5.8,6.9] *The polarized Dieudonné space constructed above corresponds to ω under the bijection in Theorem 6.5.*

Consider again the moduli space $\mathcal{M}_{\mathcal{B},p}$. For each $\omega \in W_{\mathbb{X}} \backslash W_G$, we can define S_ω to be the locally closed subset of $\mathcal{M}_{\mathcal{B},p}$ such that for all geometric points x, $\mathcal{M}_{\mathcal{B},p,x}$ belongs to S_ω if and only if $A_x[p]$ maps to ω under the bijection in Theorem 6.5. This produces a stratification of $\mathcal{M}_{\mathcal{B},p}$, that generalizes the Ekedahl–Oort stratification defined on $\mathcal{A}_g$ [Wed01, 6.7]. This stratification is also called the Ekedahl–Oort stratification.

Recall that if we fix a generating set of reflections $S \subset W_G$, every coset $\omega \in W_X \backslash W_G$ has a unique minimal length representative $\dot{\omega} \in W_G$. While the choice of $\dot{\omega}$ depends on S, its length, denoted $\ell(\dot{\omega})$ does not.

Proposition 6.9. [Moo04] *Let $\omega \in W_X \backslash W_G$, and let $\dot{\omega}$ be the minimal length representative of $\omega \in W_G$, with respect to some generating set of reflections. If $A_\omega \neq \emptyset$, then the dimension of A_ω is $\ell(\dot{\omega})$.*

6.3 The Ekedahl–Oort Stratification of Unitary Shimura varieties

Now we will specialize these results to the case of unitary Shimura varieties. That is, we will specialize the results from the previous section by requiring that $\mathcal{B} = E$ is an imaginary quadratic field, in which p is inert. Recall that we are considering the moduli space of quadruples (A, ι, ζ, η), where A is an abelian scheme of dimension $m+n$, $\iota\colon \mathcal{O}_E \hookrightarrow \mathrm{End}(A)$ with signature (n, m), ζ is a principal polarization compatible with complex conjugation on $\mathcal{O}_E$, and η is an appropriate level structure.

We will consider the p-torsion of A. Since the action of $p\mathcal{O}_E$ on $A[p]$ is zero, we can let $\kappa = \mathcal{O}_E/p\mathcal{O}_E \cong \mathbb{F}_{p^2}$. Thus $\mathcal{I} = \mathrm{Hom}(\kappa, \overline{\mathbb{F}}_p) = \{\Sigma, \overline{\Sigma}\}$, as described earlier.

Let N be the regular Dieudonné space corresponding to some such $A[p]$. There is a decomposition $N = N[\Sigma] \oplus N[\overline{\Sigma}]$. Note that $\dim(N) = 2g$, and $\dim(N[\Sigma]) = \dim(N[\overline{\Sigma}]) = g$. If we consider $L = \ker(F) \subset N$, then the fact that the action of $\mathcal{O}_E$ on $\mathrm{End}(A)$ has signature (n, m) implies that $\dim(L[\Sigma]) = n$ and $\dim(L[\overline{\Sigma}]) = m$. Thus, the multiplication type of $(A[p], \iota)$ is $(d, \mathfrak{f})$ where $d = g$, $\mathfrak{f}(\Sigma) = n$ and $\mathfrak{f}(\overline{\Sigma}) = m$.

We continue with the notation as above. Consider the canonical filtration of N.

$$0 = N_0 \subset N_1 \subset \dots N[V] = N_r = F(N) \subset \cdots \subset N_{2r} = N$$

Furthermore, we note that the action of F and V directly maps $N[\Sigma] \to N[\overline{\Sigma}]$ and vice versa. Thus, each piece of this filtration has a decomposition $N_i = N_i[\Sigma] \oplus N_i[\overline{\Sigma}]$. So we have two filtrations:

$$0 = N[\Sigma]_0 \subsetneq N[\Sigma]_1 \subsetneq \cdots \subsetneq N[\Sigma]_r = N[\Sigma]$$

$$0 = N[\overline{\Sigma}]_0 \subsetneq N[\overline{\Sigma}]_1 \subsetneq \cdots \subsetneq N[\overline{\Sigma}]_r = N[\overline{\Sigma}]$$

such that for all $0 \leq i \leq 2r$, there exist $0 \leq j, k \leq r$ such that $N_i[\Sigma] = N[\Sigma]_j$ and $N_i[\overline{\Sigma}] = N[\overline{\Sigma}]_k$.

Refine these two filtrations to a maximal $\mathcal{I}$-admissible pair of filtrations for N (c.f.

Propositions 6.6 and 6.7). For ease of notation, we will denote these two filtrations as follows:

$$\begin{aligned} A_\bullet &: \; 0 = A_0 \subset A_1 \subset \cdots \subset A_g = N[\Sigma] \\ B_\bullet &: \; 0 = B_0 \subset B_1 \subset \cdots \subset B_g = N[\overline{\Sigma}] \end{aligned} \tag{6.1}$$

Furthermore, Proposition 6.7 provides more than just filtrations, but bases for the maximal $\mathcal{I}$-admissible pair of filtrations $A_\bullet$ and $B_\bullet$. Let $\{a_1, \ldots, a_g\}$ be this basis of $N[\Sigma]$, and $\{b_1, \ldots, b_g\}$ be this basis for $N[\overline{\Sigma}]$. So:

$$\begin{aligned} A_i &= \mathrm{span}_k\{a_1, \ldots, a_i\} \\ B_i &= \mathrm{span}_k\{b_1, \ldots, b_i\} \end{aligned} \tag{6.2}$$

Furthermore, we have a polarization on N given by:

$$\langle a_i, b_{g+1-j} \rangle = c_i \delta_{ij} \tag{6.3}$$

where c_i is defined as $c_{\Sigma,i}$ from the discussion following Proposition 6.7.

We can now directly compute $W_{\mathbb{X}} \backslash W_G$.

Proposition 6.10. [Woo16, Corollary 3.4.2] *The set $W_{\mathbb{X}} \backslash W_G$ can be presented as the set*

$$\{(\omega_1, \omega_2) : \omega_2 = \omega_0 \omega_1 \omega_0\} \subset \mathrm{Shf}_{n,m} \times \mathrm{Shf}_{m,n}$$

where $\omega_0 = [g \;\; g-1 \;\; \ldots \;\; 1]$.

Note that since ω_2 can be directly computed from ω_1, there is an isomorphism given by projection to the first coordinate $W_{\mathbb{X}} \backslash W_G \to \mathrm{Shf}_{n,m}$.

As before, we set $L := F^{-1}(0)$. The bijection between EO-strata and elements of $W_{\mathbb{X}} \backslash W_G$ is given by setting ω_1 to be the relative position of the filtration $A_\bullet$ with $L[\Sigma]$, and ω_2 to be the relative position of the filtration $B_\bullet$ with $L[\overline{\Sigma}]$.

Thus as projection to the first coordinate is an isomorphism, we need only look at the Σ part of the filtration. Then ω_1 can be seen as an element of $\mathrm{Shf}_{n,m}$ directly by considering the relative positions of the flag $A_\bullet$ with $L[\Sigma]$ as follows:

Set

$$\eta(j) := \dim(A_j \cap L[\Sigma]).$$

So η describes the relative positions of the flags $A_\bullet$ and $0 \subset L[\Sigma] \subset N[\Sigma]$. Note that for all $1 \leq j \leq g$, we have either $\eta(j) = \eta(j-1)$ or $\eta(j) = \eta(j-1)+1$. Let $1 \leq j_1 < j_2 < \cdots < j_n$ be the indices j such that $\eta(j) = \eta(j-1)+1$, and let $1 \leq i_1 < i_2 < \cdots < i_m$ be the indices such that $\eta(i) = \eta(i-1)$. Then ω_1 is given by $\omega_1(j_\alpha) = \alpha$ and $\omega_1(i_\alpha) = n + \alpha$.

This can also be done in reverse. Let ω be an (n,m)-shuffle. Now, define a function $\eta_\omega \colon \{1,\ldots,g\} \to \{1,\ldots,n\}$ by

$$\eta_\omega(j) := \#\{i \in \{1,\ldots,j\} | \omega(i) \leq n\}. \tag{6.4}$$

Note that if ω is constructed by the function η given above, then $\eta_\omega = \eta$. For ease of notation we will suppress the ω and simply write η, when the shuffle being used is clear.

Thus, using Theorem 6.5, we see that the EO-strata of the unitary Shimura variety of signature (n,m) are classified by $\mathrm{Shf}_{n,m}$.

This provides a nice specialization of the classification of EO-strata for PEL-type Shimura varieties, to the case of unitary Shimura varieties. Another useful approach is to generalize the elementary sequences used to classify the EO-strata of $\mathcal{A}_g$.

Define a function $\varphi \colon \{0,\ldots,g\} \to \{0,\ldots,m\}$ such that

$$F(A_i) = B_{\varphi(i)}.$$

Note that φ is an elementary sequence, as defined earlier. However, when this was defined for the $\mathcal{A}_g$ case, the elementary sequence was defined by the low-dimension half of the canonical filtration, whereas here it is defined by the Σ part of the whole canonical filtration. Furthermore, requiring that the action of $\mathcal{O}_E$ has signature (n,m)

enforces the condition that $\varphi(g) = \dim(F(N[\Sigma])) = m$.

Proposition 6.11. *The EO-stratification of M is classified by elementary sequences $\varphi\colon \{0, \dots, g\} \to \{0, \dots, m\}$ such that $\varphi(g) = m$.*

Proof. Let $\omega \in W_{\mathbb{X}} \backslash W_G$ be an (n, m)-shuffle corresponding to some BT_1 group N, with polarization and endomorphism structures as above. Continuing the notation from above, we know that $\eta(i) = \dim(L[\Sigma] \cap A_i)$. But $L[\Sigma] \cap A_i = \ker(F|_{A_i})$. Since A_i is the i-dimensional piece of $A_\bullet$, we can compute:

$$\dim(F(A_i)) = \dim(A_i) - \eta(i) = i - \eta(i)$$

As such, the elementary sequence corresponding to any point in S_ω is:

$$\varphi(i) = i - \eta(i) \tag{6.5}$$

Therefore, the elementary sequence of some (A, ι, ζ, η) is determined completely by the EO-strata it is in.

Now, let φ be some elementary sequence such that $\varphi(g) = m$. Define a shuffle ω by

$$\omega(i) = \begin{cases} i - \varphi(i) & \varphi(i) = \varphi(i-1) \\ \varphi(i) + n & \varphi(i) = \varphi(i-1) + 1. \end{cases} \tag{6.6}$$

This is an inverse to the map from shuffles to elementary sequences given above. To see this, we take ω to be the shuffle defined by some φ. and compute $\eta_\omega(i)$. We see that

$$\begin{aligned} \eta(i) &= \#\{j \in \{1, \dots, i\} | \omega(j) \leq n\} \\ &= \#\{j \in \{1, \dots, i\} | \varphi(j) = \varphi(j-1)\} \\ &= i - \#\{j \in \{1, \dots, i\} | \varphi(j) = \varphi(j-1) + 1\}. \end{aligned}$$

But since φ increases by either 1 or 0 at each step, and $\varphi(0) = 0$, we know that $\#\{j \in \{1, \ldots, i\} | \varphi(j) = \varphi(j-1)+1\} = \varphi(i)$. So $\eta(i) = i - \varphi(i)$. That is $\varphi(i) = i - \eta(i)$. Thus, setting $\varphi(i) = i - \eta(i)$ from this shuffle will return the elementary sequence we started with. $\square$

Note that this besides giving us another way of classifying the EO-strata of a unitary Shimura variety, it provides an explicit formula for converting between the two classification schemes. Given an (n, m)-shuffle, (6.4) and (6.5) describe an elementary sequence, and (6.6) provides the inverse operation. As such, we will also denote S_ω by S_φ, where φ is the elementary sequence parameterizing S_ω.

We now wish to have a formula for the dimension of S_φ is terms of φ directly. We already know the dimension of S_φ in terms of ω, by Proposition 6.9, thus it is a straightforward computation to write this in terms of φ.

Lemma 6.12. *For an (n, m)-shuffle ω, we have*

$$\sum_{i=1}^{g} \eta(i) = \frac{n(n+1)}{2} + mn - \ell(\omega).$$

Proof. We will compute $\sum_{i=1}^{g} \eta(i)$ by splitting into two cases. First, we note that if $\omega(i) \leq n$, then $\eta(i) = \omega(i)$. Thus

$$\sum_{i, \omega(i) \leq n} \eta(i) = \sum_{i, \omega(i) \leq n} \omega(i) = \sum_{k=1}^{n} k = \frac{n(n+1)}{2}.$$

On the other hand, if $\omega(i) > n$, we see that:

$$\begin{aligned}\sum_{i,\omega(i)>n} \eta(i) &= \sum_{i,\omega(i)>n} \#\{j \leq i | \omega(j) \leq n\} \\ &= \sum_{i,\omega(i)>n} \#\{j < i | \omega(j) \leq n\} \\ &= \sum_{i,\omega(i)>n} n - \#\{i < j | \omega(j) \leq n\} \\ &= mn - \sum_{i=1}^{g} \#\{j | i < j, \omega(i) > \omega(j)\}.\end{aligned}$$

Note that for an (n, m)-shuffle, the only way for $i < j$ and $\omega(i) > \omega(j)$ to occur is if $\omega(j) \leq n < \omega(i)$. Thus the equality above is justified. Now using (5.1), we have:

$$\begin{aligned}\sum_{i,\omega(i)>n} \eta(i) &= mn - \sum_{i=1}^{g} \#\{j | i < j, \omega(i) > \omega(j)\} \\ &= mn - \#\{(i,j) | i < j, \omega(i) > \omega(j)\} \\ &= mn - \ell(\omega).\end{aligned}$$

Thus,

$$\sum_{i=1}^{g} \eta(i) = \sum_{i,\omega(i)\leq n} \eta(i) + \sum_{i,\omega(i)>n} \eta(i) = \frac{n(n+1)}{2} + mn - \ell(\omega).$$

$\square$

Proposition 6.13. *Let S_φ be the EO-stratum corresponding to the elementary sequence φ. Then*

$$dim(S_\varphi) = \left(\sum_{i=1}^{g} \varphi(i)\right) - \frac{m(m+1)}{2}.$$

Proof. Given Lemma 6.12, we see that if ω is the shuffle corresponding to S_φ, we have

$$\begin{aligned}
\dim(S_\varphi) &= \ell(\omega) \\
&= \frac{n(n+1)}{2} + mn - \sum_{i=1}^{g} \eta(i) \\
&= \frac{n(n+1)}{2} + mn - \sum_{i=1}^{g} (i - \varphi(i)) \\
&= \sum_{i=1}^{g} \varphi(i) - \frac{m(m+1)}{2}.
\end{aligned}$$

□

7. UNIVERSAL DEFORMATIONS OF EKEDAHL–OORT STRATA

7.1 Combinatorics of Elementary Sequences

Let E be an imaginary quadratic field, where p is an inert prime in E. Choose any $0 \leq m \leq n$, and let $g = m + n$. Consider the moduli space $\mathcal{M}$ parameterizing quadruples (A, ι, ζ, η) as described above. That is, where A is a g-dimensional abelian variety, with principal polarization ζ, and ι is an action of $\mathcal{O}_E$ on A with signature (n, m) compatible with ζ.

Now, let φ be any elementary sequence. So φ can be seen as a function from $\{0, \ldots, g\} \to \{0, \ldots, m\}$ such that $\varphi(0) = 0$, $\varphi(g) = m$, and $\varphi(i) \leq \varphi(i+1) \leq \varphi(i) + 1$.

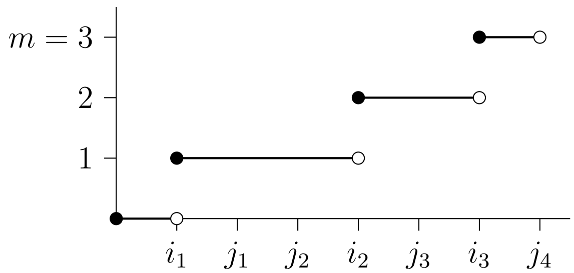
Define:

$$I_\varphi := \{i : \varphi(i) = \varphi(i-1) + 1\}, \quad J_\varphi := \{i : \varphi(i) = \varphi(i-1)\} \tag{7.1}$$

Thus $|I_\varphi| = m$ and $|J_\varphi| = n$. Denote the elements of I_φ in ascending order as $\{i_1, i_2, \ldots, i_m\}$. Similarly, $J_\varphi = \{j_1, j_2, \ldots, j_n\}$, Also set $i_0 = j_0 = 0$.

For example, suppose we have signature $(n, m) = (4, 3)$. We can consider the $(4, 3)$-shuffle 5126374 (written in one-line notation). By (6.4) and (6.5), this corresponds to the elementary sequence $\varphi = (0, 1, 1, 1, 2, 2, 3, 3)$. So here we would have $J_\varphi = \{2, 3, 5, 7\}$ and $I_\varphi = \{1, 4, 6\}$.

Note that I_φ and J_φ can be easily read off of the graph of φ. In this example, the graph of φ becomes:



Note that I_φ enumerates the "jumps" of φ, while J_φ gives the locations where φ remains constant.

Lemma 7.1. *Given φ, I and J as above, the following hold:*

(a) $\varphi(i_\alpha) = \alpha$

(b) $\varphi(j_\beta) = \max(0, \max\{\alpha | i_\alpha < j_\beta\})$

(c) $\varphi(j_\beta) = j_\beta - \beta$

Proof. (a) and (b) are trivial, and can be seen from the graph of φ.

For (c), first consider the case that $j_\beta < i_1$. So $\varphi(j_\beta) = 0$. But then the whole set $\{1, 2, \ldots, j_\beta\} \subseteq J$. Thus $j_\beta = \beta$. So $\varphi(j_\beta) = 0 = j_\beta - \beta$. Now, if $j_\beta > i_1$, we see that $\#\{i_\alpha | i_\alpha < j_\beta\} = \max\{\alpha | i_\alpha < j_\beta\}$, since $\{i_1, \ldots, i_n\}$ is increasing. Thus $\{1, 2, \ldots, j_\beta\} \cap I$ contains precisely $\varphi(j_\beta)$ elements. Thus, as I and J partition the set $\{1, 2, \ldots, g\}$, we see that $\{1, 2, \ldots, j_\beta\}$ contains precisely $j_\beta - \varphi(j_\beta)$ elements from J. Since $\{j_1, \ldots, j_\beta\}$ are increasing, this implies there are exactly β elements from J. Hence $j_\beta - \varphi(j_\beta) = \beta$. That is, $\varphi(j_\beta) = j_\beta - \beta$ thus proving (c). □

We can use this lemma to produce an explicit description of the action of F and V on the Dieudonné module N associated to $A[p]$ for some $A \in \mathcal{M}$.

Lemma 7.2. *Let N be as above, and let $\{a_1, \ldots, a_g, b_1, \ldots, b_g\}$ be a basis for N as in (6.2). Let φ be the elementary sequence for N, and I_φ and J_φ as defined in (7.1).*

Then:

$$\begin{array}{llll} F(a_{i_\alpha}) = b_\alpha & i_\alpha \in I_\varphi & F(a_{j_\beta}) = 0 & j_\beta \in J_\varphi \\ F(b_{g+1-i_\alpha}) = 0 & i_\alpha \in I_\varphi & F(b_{g+1-j_\beta}) = a_{n+1-\beta} & j_\beta \in J_\varphi \\ V(a_i) = 0 & 1 \le i \le n & V(a_{g+1-\alpha}) = b_{g+1-i_\alpha} & 1 \le \alpha \le m \\ V(b_i) = 0 & 1 \le i \le m & V(b_{m+\beta}) = a_{j_\beta} & 1 \le \beta \le n \end{array}$$

Proof. This is just the specialization of Proposition 6.7. We know that

$$F(a_j) = \begin{cases} 0 & \omega(j) \le n \\ b_\alpha & \omega(j) = n + \alpha. \end{cases}$$

Now, recall that for any index $j_\beta \in J$, we know that $\varphi(j) = \varphi(j-1)$. Thus, by the construction in Proposition 6.11, we know that

$$\omega(j_\beta) = j_\beta - \varphi(j_\beta) = j_\beta - j_\beta + \beta = \beta \le n.$$

So $F(a_{j_\beta}) = 0$.

On the other hand, for any index $i_\alpha \in I$, we know that $\varphi(i_\alpha) = \varphi(i_\alpha - 1) + 1$. Thus:

$$\omega(i_\alpha) = \varphi(i_\alpha) + n = \alpha + n.$$

Thus, $F(a_{i_\alpha}) = b_\alpha$.

In order to compute $F(b_j)$, we must look at ω_2, which is the conjugate of the shuffle produced in Proposition 6.11 by ω_0, where $\omega_0(i) = (g+1) - i$. Thus:

$$\omega_2(j) = \begin{cases} (g+1) - ((g+1-j) - \varphi(g+1-j)) & \varphi(g+1-j) = \varphi(g-j) \\ g+1-(\varphi(g+1-j)+n) & \varphi(g+1-j) = \varphi(g-j)+1 \end{cases}$$

$$= \begin{cases} j + \varphi(g-j+1) & \varphi(g-j+1) = \varphi(g-j) \\ m+1-\varphi(g-j+1) & \varphi(g-j+1) = \varphi(g-j)+1. \end{cases}$$

By Proposition 6.7, we have

$$F(b_j) = \begin{cases} 0 & \omega_2(j) \leq m \\ a_\beta & \omega_2(j) = m+\beta. \end{cases}$$

Now, note that for any $i_\alpha \in I$, so $\varphi(i_\alpha) = \varphi(i_\alpha - 1) + 1$, thus $\omega_2(g+1-i_\alpha) = m+1-\varphi(i_\alpha) \leq m$. So $F(b_{g+1-i_\alpha}) = 0$.

On the other hand, for $j_\beta \in J$, we have $\varphi(j_\beta) = \varphi(j_\beta - 1)$. Thus

$$\omega_2(g+1-j_\beta) = g+1-j_\beta+\varphi(j_\beta) = g+1-j_\beta+j_\beta-\beta = m+(n+1-\beta).$$

So $F(b_{g+1-j_\beta}) = a_{n+1-\beta}$.

Now, to compute V, we know that:

$$V(a_j) = \begin{cases} 0 & j \leq n \\ b_\alpha & j = n+\omega_2(\alpha) \end{cases}$$

and

$$V(b_j) = \begin{cases} 0 & j \leq m \\ a_\beta & j = m+\omega(\beta). \end{cases}$$

Thus $V(a_i) = 0$ for $1 \leq i \leq n$ and $V(b_i) = 0$ for $1 \leq i \leq m$.

Now, let $1 \leq \alpha \leq n$. Then

$$\begin{aligned} & n + \omega_2(g+1-i_\alpha) \\ & = n + (g+1-\omega(i_\alpha)) \\ & = n + (g+1-(\varphi(i_\alpha)+n)) \\ & = g+1-\alpha. \end{aligned}$$

So $V(a_{g+1-\alpha}) = b_{g+1-i_\alpha}$.

Also, if $1 \leq \beta \leq m$, then

$$m + \omega(j_\beta) = m + j_\beta - \varphi(j_\beta) = m + j_\beta - j_\beta + \beta = m + \beta.$$

Thus $V(b_{m+\beta}) = a_{j_\beta}$. □

Given an elementary sequence φ, consider the Dieudonné space N' formed by starting with the basis $\{a_1, \ldots, a_g, b_1, \ldots, b_g\}$ where $\mathcal{O}_E$ acts by Σ on the a_i and by $\overline{\Sigma}$ on the b_i, with F and V defined as in the statement of the Lemma. Let $A_i = \operatorname{span}\{a_1, \ldots, a_i\}$, and $B_i = \operatorname{span}\{b_1, \ldots, b_i\}$. Then, one can compute that

$$\begin{aligned} & 0 \subset A_1 \subset A_2 \subset \cdots \subset A_g = N'[\Sigma] \\ & 0 \subset B_1 \subset B_2 \subset \cdots \subset B_g = N'[\overline{\Sigma}] \end{aligned}$$

will be a maximal $\mathcal{I}$-admissible pair of filtrations for N. Furthermore, the elementary sequence corresponding to N' can be seen to be φ.

7.2 Computing the Universal Display

Next, we would like to directly compute the universal display $\mathcal{N}$ for N, as discussed in section 5.3. In particular, we need to know the action of F on $\mathcal{N}$, as we will need that to determine the deformations that preserve the elementary sequence of N.

Lemma 7.3. *Let $\mathcal{N}$ be the universal display of N as in Lemma 5.5. Then the action of F on $\mathcal{N}$ is:*

$$F(a_{i_\alpha}) = \begin{cases} b_\alpha & g+1-\alpha \in J \\ b_\alpha + \sum_{\beta=1}^{n} c_{g+1-\alpha} t_{\beta,n+x} b_{g+1-j_{n+1-\beta}} & g+1-\alpha = i_x \in I \end{cases}$$

$$F(b_{g+1-j_\beta}) = \begin{cases} a_{n+1-\beta} & n+1-\beta \in I \\ a_{n+1-\beta} - \sum_{\alpha=1}^{m} c_{n+1-\beta} t_{n+1-y,n+\alpha} a_{i_\alpha} & n+1-\beta = j_y \in J. \end{cases}$$

Proof. We first wish to construct a displayed basis for D. Recall that this is a symplectic basis such that the second half is a basis for the kernel of F. Using the information from the previous lemma, and the c_i as in polarization formula in Equation 6.3, we compute that

$$\begin{aligned} &\{a_{i_1}, a_{i_2}, \ldots, a_{i_m}, b_{g+1-j_n}, \ldots, b_{g+1-j_1}; \\ &c_{i_1}^{-1} b_{g+1-i_1}, c_{i_2}^{-1} b_{g+1-i_2}, \ldots, c_{i_m}^{-1} b_{g+1-i_m}, -c_{j_n}^{-1} a_{j_n}, \ldots, -c_{j_1}^{-1} a_{j_1}\} \end{aligned}$$

is a displayed basis such that the display matrix is of the form:

$$\left[\begin{array}{c|c||c|c} & A_1 & B_1 & \\ \hline A_2 & & & B_2 \\ \hline\hline C_2 & & & D_2 \\ \hline & C_1 & D_1 & \end{array}\right].$$

Now, as in Lemma 5.5, we consider the quotient of the universal deformation ring that preserves the polarization and endomorphism structure on N. This quotient is the ring $R = k[[t_{ij} : 1 \le i \le n < j \le g]]$. Set

$$T = \left[\begin{array}{ccc|ccc} & & & t_{1,n+1} & \dots & t_{1,g} \\ & & & \vdots & \ddots & \vdots \\ & & & t_{n,n+1} & \dots & t_{n,g} \\ \hline t_{1,n+1} & \dots & t_{n,n+1} & & & \\ \vdots & \ddots & \vdots & & & \\ t_{1,g} & \dots & t_{n,g} & & & \end{array}\right] = \left[\begin{array}{c|c} & T' \\ \hline T'^t & \end{array}\right].$$

We then see by Lemma 5.5 that the display matrix for $\mathcal{N}$ is:

$$\left[\begin{array}{c|c||c|c} & A_1 + T'C_1 & B_1 + T'D_1 & \\ \hline A_2 + T'^tC_2 & & & B_2 + T'^tD_2 \\ \hline\hline C_2 & & & D_2 \\ \hline & C_1 & D_1 & \end{array}\right].$$

The Lemma follows by explicitly computing the above matrix. $\square$

For the results we have regarding canonical filtrations and elementary sequences of Dieudonné modules and regular Dieudonné spaces, we need to be working over a perfect field. Thus let:

$$R_\varphi = R/\langle t_{\alpha\beta} : j_{n+1-\alpha} < i_{\beta-n} \rangle. \quad 1 \le \alpha \le n < \beta \le g$$

Furthermore, define $\mathbb{F}_\varphi := \mathrm{Frac}(R_\varphi)$ and let $\mathbb{F}_\varphi^{\mathrm{perf}}$ denote the perfect closure of $\mathrm{Frac}(R_\varphi)$. Note that a display over R_φ, is also a display over $\mathbb{F}_\varphi^{\mathrm{perf}}$.

Lemma 7.4. *Let $\mathcal{N}'$ be the Dieudonné module over R_φ given by restricting the universal deformation $\mathcal{N}$ to R_φ. Then the elementary sequence for $\mathcal{N}'$, viewed as a regular Dieudonné space over $\mathbb{F}_\varphi^{perf}$ is φ.*

Proof. Recall that by Proposition 6.7 and (6.1), the canonical filtration for D can be refined to a maximal admissible filtration that decomposes into its Σ and $\overline{\Sigma}$ parts. Thus we have the filtrations:

$$0 = A_0 \subset A_1 \subset \cdots \subset A_g = D[\Sigma]$$
$$0 = B_0 \subset B_1 \subset \cdots \subset B_g = D[\overline{\Sigma}].$$

We have a basis $\{a_1, \ldots a_g, b_1, \ldots, b_g\}$ for N such that $A_k = \langle a_1, \ldots, a_k\rangle$ and $B_k = \langle b_1, \ldots b_k\rangle$. Let $A'_k := A_k \otimes \mathbb{F}_\varphi^{\mathrm{perf}}$, and $B'_k := B_k \otimes \mathbb{F}_\varphi^{\mathrm{perf}}$. We will demonstrate that after base change, these filtrations are stable under F and $\perp$. That is, we have $A_i^\perp = D[\Sigma] \cup B_{g-i}$, and $B_i^\perp = D[\overline{\Sigma}] \cup A_{g-i}$. Thus by Proposition 6.1, they form an $\mathcal{I}$-admissible pair of filtrations for $\mathcal{N}'$. Furthermore, this computation will demonstrate that the elementary sequence of $\mathcal{N}'$ is φ.

To avoid confusion in this proof, we will denote the map F in the Dieudonné space N as F_N, and the map F in the Dieudonné space $\mathcal{N}'$ as $F_{\mathcal{N}'}$.

Note that the displayed basis used in the proof of Lemma 7.3 is a symplectic basis over k for N, and remains a symplectic basis over $\mathbb{F}_\varphi^{\mathrm{perf}}$ for $\mathcal{N}'$. Thus, the pair of filtrations above is stable under duality, even after base change. It remains to show that it is stable under F. This is non-trivial, as $F_{\mathcal{N}'} \neq F_N \otimes \mathrm{id}_{\mathbb{F}_\varphi^{\mathrm{perf}}}$ on $\mathcal{N}'$, despite $\mathcal{N}' \cong N \otimes \mathbb{F}_\varphi^{\mathrm{perf}}$ as vector spaces.

First we will show that $F_{\mathcal{N}'}(A'_k) = B'_{\varphi(k)}$. Clearly this is true for $k = 0$, as $A'_0 = B'_0 = \{0\}$. Now, we proceed by induction. Assume $F_{\mathcal{N}'}(A_{k-1})' = B'_{\varphi(k-1)}$. We now split into cases.

Case 1: Suppose $k \in J = J_\varphi$. Then, by definition of J, we know that $\varphi(k) = \varphi(k-1)$. Also, as $\ker(F_{\mathcal{N}'}) = \ker(F_N) \otimes \mathbb{F}_\varphi^{\mathrm{perf}}$, we know that $F_N(a_k) = 0$, so $F_{\mathcal{N}'}(a_k) = 0$

as well. Thus $F_{\mathcal{N}'}(A'_k) = F_{\mathcal{N}'}(A'_{k-1}) = B'_{\varphi(k-1)} = B'_{\varphi(k)}$.

Case 2a: Suppose $k = i_\alpha \in I = I_\varphi$, and suppose that $g+1-\alpha \in J$. Since $k \in I$, we know that $\varphi(k) = \varphi(k-1)+1$. Also, as $g+1-\alpha \in J$, we see by Lemma 7.3 that $F_{\mathcal{N}'}(a_k) = b_\alpha$. By Proposition 7.1, we know that $\varphi(k) = \varphi(i_\alpha) = \alpha$. Thus $F_{\mathcal{N}'}(A'_k) = F_{\mathcal{N}'}(\langle a_k \rangle) \oplus F_{\mathcal{N}'}(A'_{k-1}) = \langle b_\alpha \rangle \oplus B'_{\varphi(k-1)} = B'_{\varphi(k)}$.

Case 2b: Suppose $k = i_\alpha \in I$, and suppose that $g+1-\alpha = i_x \in I$. Then, by Lemma 7.3 we know that $F_{\mathcal{N}'}(a_k) = b_\alpha + \sum_{\beta=1}^n t_{\beta,n+x} b_{g+1-j_{n+1-\beta}}$. However, if $t_{\beta,n+x} \neq 0$ in R_φ, then by definition of R_φ, we must have $i_x < j_{n+1-\beta}$. But $i_x = g+1-\alpha$. Thus $g+1-j_{n+1-\beta} < \alpha = \varphi(k)$. Therefore the term $\sum_{\beta=1}^n t_{\beta,n+x} b_{g+1-j_{n+1-\beta}} \in B_{\varphi(k-1)'}$. So we still get the formula $F_{\mathcal{N}'}(A'_k) = B'_{\varphi(k)}$.

Thus, we see that $F_{\mathcal{N}'}(A'_k) = B'_{\varphi(k)}$ for all k.

Next, we will demonstrate that $F_{\mathcal{N}'}(B'_k) = A'_{\psi(k)}$ where ψ is the unique function such that $F_N(B_k) = A_{\psi(k)}$ Again, this is clearly true for $k = 0$, as $A'_0 = B'_0 = \{0\}$. Here we have $\psi(0) = 0$. Again, we proceed by induction. Assume that $F_{\mathcal{N}'}(B'_{k-1}) = A'_{\psi(k-1)}$. Again, we split into cases:

Case 1: Suppose $g+1-k \in I$. Then by Lemma 7.2, $F_N(b_k) = 0$, so $F_{\mathcal{N}'}(b_k) = 0$ also. Thus $F_{\mathcal{N}'}(B'_k) = F_{\mathcal{N}'}(B'_{k-1}) = A_{\psi(k-1)'}$. Since $F_N(b_k) = 0$ we have $\psi(k) = \psi(k-1)$, and thus $F_{\mathcal{N}'}(B'_k) = A'_{\psi(k)}$.

Case 2a: Suppose that $g+1-k = j_\beta \in J$, and suppose that $n+1-\beta \in I$. Then, by Lemma 7.3, we know that $F_{\mathcal{N}'}(b_k) = a_{n+1-\beta}$. By Lemma 7.2, we see that $F_N(b_k) = a_{n+1-\beta}$ as well, so $\psi(k) = n+1-\beta$, as the filtrations in (6.1) are stable under F_N. Thus$F_{\mathcal{N}'}(B_k) = F_{\mathcal{N}'}(\langle b_k \rangle) \oplus F_{\mathcal{N}'}(B'_{k-1}) = \langle a_{n+1-\beta} \rangle \oplus A'_{\psi(k-1)} = A'_{\psi(k)}$.

Case 2b: Suppose $g+1-k = j_\beta \in J$, and suppose that $n+1-\beta = j_y \in J$. Then by Lemma 7.3, we have $F_{\mathcal{N}'}(b_k) = a_{n+1-\beta} + \sum_{\alpha=1}^m t_{n+1-y,n+\alpha} a_{i_\alpha}$. By Lemma 7.2, we know that $F_N(b_k) = a_{n+1-\beta}$, thus $\psi(k) = n+1-\beta$. So it remains to show that if $t_{n+1-y,n+\alpha} \neq 0$ in R_φ, then $i_\alpha < n+1-\beta$. But, by the definition of R_φ,we see that $t_{n+1-y,n+\alpha} \neq 0$ implies that $i_\alpha \leq j_y = n+1-\beta$. Thus $i_\alpha < n+1-\beta$, as required. Thus $F_{\mathcal{N}'}(B'_k) = A'_{\psi(k)}$ in this case as well.

Therefore $F_{\mathcal{N}'}(B'_k) = A'_{\psi(k)}$ for all k.

Hence the pair of filtrations in (6.1) is stable under $F_{\mathcal{N}'}$ and duality. As such it is a maximal admissible pair of filtrations of $\mathcal{N}'$. Also, since $F_{\mathcal{N}'}(A'_k) = B'_{\varphi(k)}$ for all k, the elementary sequence corresponding to $\mathcal{N}'$ must be φ. □

7.3 The Universal Deformation of S_φ

We will now show that for any $x \in S_\varphi$, the module R_φ is naturally isomorphic to $\widehat{\mathcal{O}}_{S_\varphi,x}$. This will tell us that R_φ parameterizes all deformations of x along S_φ.

Proposition 7.5. *$dim(R_\varphi) = (\sum_{i=1}^{g} \varphi(i)) - \frac{n(n+1)}{2}$.*

Proof. To compute the dimension of R_φ, we just need to count the pairs (α, β) that satisfy the conditions $1 \leq \alpha \leq n < \beta \leq g$, and $j_{n+1-\alpha} > i_{\beta-n}$. Or, equivalently, the pairs $j_b > i_a$ for $1 \leq a \leq n$, $1 \leq b \leq m$. Recall from Lemma 7.1 that $\varphi(j_\beta) = \max(0, \max\{\alpha | i_\alpha < j_\beta\})$. So $\varphi(j_\beta)$ is equal to the number of i_α such that $i_\alpha < j_\beta$. Thus, the number of pairs $j_b > i_a$ is precisely, $\sum_{b=1}^{m} \varphi(j_b)$.

Also from Lemma 7.1, we know that $\varphi(i_\alpha) = \alpha$. Thus

$$\sum_{a=1}^{n} \varphi(i_a) = \sum_{a=1}^{n} a = \frac{n(n+1)}{2}.$$

Therefore, since each integer between 1 and g is either of the form i_a or j_b, we see that:

$$\sum_{b=1}^{m} \varphi(j_b) = \sum_{i=1}^{g} \varphi(i) - \sum_{a=1}^{n} \varphi(i_a) = \left(\sum_{i=1}^{g} \varphi(i) \right) - \frac{n(n+1)}{2}.$$

□

Theorem 7.6. *Let k be an algebraically closed field, and let $x \in S_\varphi(k)$. Then there exists a natural isomorphism $R_\varphi \cong \widehat{\mathcal{O}}_{S_\varphi,x}$.*

Proof. Let $\underline{A} = (A, \iota, \zeta, \eta)$ be the object corresponding to $x \in S_\varphi$. Now let $\mathcal{A}$ be the universal deformation of $\underline{A}$, which is a formal principally polarized abelian variety over $\mathrm{Spf}(R)$.

Since R_φ is a quotient of R, there is a closed immersion $\mathrm{Spf}(R_\varphi) \hookrightarrow \mathrm{Spf}(R)$. By pulling back the universal deformation $\mathcal{A} \to \mathrm{Spf}(R)$ over this map, we get a deformation of $\underline{A}$ over $\mathrm{Spf}(R_\varphi)$. By Grothendieck's existence theorem ([EGA] III.5), this deformation is uniquely algebrizable to a principally polarized abelian variety, which by a slight abuse of notation, we will also denote $\mathcal{A} \to \mathrm{Spec}(R_\varphi)$.

This gives us a map $\mathrm{Spec}(R_\varphi) \to \mathcal{M}$. We would like to show that this map factors through S_φ. By Lemma 7.4, we know that if we base change up to $\mathbb{F}_\varphi^{\mathrm{perf}}$, $\mathcal{N}$ is associated to a point in $S_\varphi(\mathbb{F}_\varphi^{\mathrm{perf}})$, thus, we get a commutative diagram as below.

$$\begin{tikzcd}
\mathrm{Spec}(\mathbb{F}_\varphi^{\mathrm{perf}}) \arrow[r] \arrow[d] & S_\varphi \arrow[dd] \\
\mathrm{Spec}(\mathbb{F}_\varphi) \arrow[d] & \\
\mathrm{Spec}(R_\varphi) \arrow[r] & \mathcal{M}.
\end{tikzcd}$$

Since $\mathbb{F}_\varphi^{\mathrm{perf}}$ is a field, the image of $\mathrm{Spec}(\mathbb{F}_\varphi^{\mathrm{perf}})$ in $\mathcal{M}$ will be a single point, which we have just shown to be in S_φ. However, $\mathbb{F}_\varphi$ is also a field, thus the image of $\mathrm{Spec}(\mathbb{F}_\varphi)$ is also a single point in $\mathcal{M}$. By the above diagram, we know that the image of $\mathrm{Spec}(\mathbb{F}_\varphi)$ must contain the image of $\mathrm{Spec}(\mathbb{F}_\varphi^{\mathrm{perf}})$. So it must be the same point, which is in S_φ. Thus, we have can factor the map $\mathrm{Spec}(\mathbb{F}_\varphi) \to \mathcal{M}$ through S_φ.

Consider the map $\mathrm{Spec}(R_\varphi) \to \mathcal{M}$. We wish to prove that this map also factors through S_φ. We know that the generic point of $\mathrm{Spec}(R_\phi)$ corresponds to the map $\mathrm{Spec}(\mathbb{F}_\varphi) \to \mathrm{Spec}(R_\phi)$, as $\mathbb{F}_\varphi$ is the fraction field of R_φ. Therefore, as the map $\mathbb{F}_\varphi \to \mathcal{M}$ factors through $S_\varphi \subset \mathcal{M}$, we know that the generic point of $\mathrm{Spec}(R_\varphi)$ maps into S_φ. Thus, as $\mathrm{Spec}(R_\varphi)$ is the closure of its generic point, $\mathrm{Spec}(R_\varphi) \to \mathcal{M}$ factors through $\overline{S_\varphi}$, the closure of S_φ.

Furthermore, the special point corresponding to the ideal $t_{ij} = 0$ maps to $x \in S_\varphi$. Since the special point is contained in the closure of any point in $\mathrm{Spec}(R_\varphi)$, the closure of any point in the image of $\mathrm{Spec}(R_\varphi)$ must intersect S_φ non-trivially. As S_φ is a stratum in the EO-stratification, it is relatively open in $\overline{S_\varphi}$, thus the image of $\mathrm{Spec}(R_\varphi)$ must be in S_φ itself. Therefore, the map $\mathrm{Spec}(R_\varphi) \to \mathcal{M}$ factors through S_φ. As such, we have a diagram

$$\begin{tikzcd}
\mathrm{Spec}(\mathbb{F}_\varphi^{\mathrm{perf}}) \arrow[r] \arrow[d] & S_\varphi \arrow[dd] \\
\mathrm{Spec}(\mathbb{F}_\varphi) \arrow[d] \arrow[ur, dashed] & \\
\mathrm{Spec}(R_\varphi) \arrow[r] \arrow[uur, dashed] & \mathcal{M}.
\end{tikzcd}$$

Now, let us consider the induced map on the completed local ring at x. As $\mathrm{Spf}(R_\varphi) \hookrightarrow \mathrm{Spf}(R)$ is a closed immersion, and $R \cong \widehat{O}_{\mathcal{M},x}$, we have a surjective map $\widehat{O}_{\mathcal{M},x} \to R_\varphi$. Furthermore, as this immersion factors through $\widehat{S}_{\varphi,x}$, we see that this map factors as

$$\widehat{O}_{\mathcal{M},x} \to \widehat{O}_{S_\varphi,x} \to R_\varphi.$$

Thus, there is a surjective morphism $\widehat{O}_{S_\varphi,x} \to R_\varphi$. But $\widehat{O}_{S_\varphi,x}$ is a power ring of the form $k[[t_1, \dots, t_{\dim(S_\varphi)}]]$, and R_φ is a power ring of the form $k[[t_1, \dots, t_{\dim(R_\varphi)}]]$

Furthermore, by Lemma 7.5 and Proposition 6.12, we know that

$$\dim(R_\varphi) = \left(\sum_{i=1}^{g} \varphi(i)\right) - \frac{n(n+1)}{2} = \dim(S_\varphi).$$

Therefore, this map is a surjective map of noetherian rings of the same dimension, and is hence an isomorphism. $\square$

7.4 Examples

Example 1: $U(n,1)$: The Ekedahl–Oort stratification in the case of $U(n,1)$ is very straightforward. For each $0 \leq k \leq n$, there exists a unique k-dimensional EO-stratum. It is given by the elementary sequence where $\varphi(i) = 0$ for $i \leq n-k$, and $\varphi(i) = 1$ for

$i > n-k$. By the computation in Proposition 6.11, we have:

$$\omega(i) = \begin{pmatrix} 1 & 2 & \dots & n-k & n-k+1 & n-k+2 & \dots & n+1 \\ 1 & 2 & \dots & n-k & n+1 & n-k+1 & \dots & n \end{pmatrix}$$

Note that Proposition 6.13 does indeed recover the dimension, as:

$$\sum_{i=1}^{n+1} \varphi(i) - \frac{1(1+1)}{2} = \sum_{i=1}^{n-k} 0 + \sum_{i=n-k+1}^{n+1} 1 - 1 = (n+1) - (n-k) - 1 = k.$$

Now, let (A, ι, ζ, η) be parameterized by a point in the k-dimensional stratum, and let N be the Dieudonné module of $A[p]$. From these representations, we compute:

$$I = \{n-k+1\}, J = \{1, 2, \dots, \widehat{n-k+1}, \dots, n+1\}.$$

By Proposition 6.7, there exists a basis $\{a_1, \dots, a_{n+1}, b_1, \dots, b_{n+1}\}$ of N such that $\{a_1, \dots, a_{n+1}\}$ generates $N[\Sigma]$ and $\{b_1, \dots, b_{n+1}\}$ generates $N[\overline{\Sigma}]$. Furthermore, we know that F and V act as:

$$F(a_i) = \begin{cases} b_1 & i = n-k+1 \\ 0 & i \neq n-k+1, \end{cases} \qquad V(a_i) = \begin{cases} 0 & i < n+1 \\ b_{k+1} & i = n+1, \end{cases}$$

$$F(b_i) = \begin{cases} a_i & i < k+1 \\ 0 & i = k+1 \\ a_{i-1} & i > k+1, \end{cases} \qquad V(b_i) = \begin{cases} 0 & i = 1 \\ a_{i-1} & 2 \leq i \leq n-k+1 \\ a_i & i > n-k+1. \end{cases}$$

Using I and J we can find a displayed basis for N. This gives us:

$$\{a_{n-k+1}, b_1, \dots, b_k, b_{k+2}, \dots, b_{n+1};$$
$$c_{n-k+1}^{-1} b_{k+1}, -c_{n+1}^{-1} a_{n+1}, \dots, -c_{n-k+2}^{-1} a_{n-k+2}, -c_{n-k}^{-1} a_{n-k}, \dots, -c_1^{-1} a_1\}.$$

The above data can be used to give us the display matrix. We will look at the case where $n = 5$ and $k = 2$. We compute that there are two orbits given by $\{a_1, b_2, a_2, b_3, a_6, b_6, a_5, b_5, a_4, b_1\}$ and $\{a_3, b_4\}$ Note that these are both self-dual. So let $\zeta \in k$ be a such that $\zeta^{p^5} = -\zeta$, and let $\eta \in k$ such that $\eta^p = -\eta$. Then we get a display basis of:

$$\{a_4, b_1, b_2, b_4, b_5, b_6; \zeta^{-p^3} b_3, -\zeta^{-p^4} a_6, -\zeta^{-p} a_5, -\eta^{-1} a_3, -\zeta^{-p^2} a_2, -\zeta^{-1} a_1\}$$

and the display matrix is:

$$\left[\begin{array}{c|ccccc||c|ccccc}
 & 0 & 0 & 0 & 1 & 0 & 0 & & & & & \\
\hline
1 & & & & & & & 0 & 0 & 0 & 0 & 0 \\
0 & & & & & & & 0 & 0 & 0 & 0 & -\zeta^{-p} \\
0 & & & & & & & 0 & 0 & \eta^{-1} & 0 & 0 \\
0 & & & & & & & 0 & -\zeta^{-p^2} & 0 & 0 & 0 \\
0 & & & & & & & \zeta^{-1} & 0 & 0 & 0 & 0 \\
\hline\hline
0 & & & & & & & 0 & 0 & 0 & -1 & 0 \\
\hline
 & 0 & 0 & 0 & 0 & 0 & -1 & & & & & \\
 & 0 & 0 & 0 & 0 & -\zeta^p & 0 & & & & & \\
 & 0 & 0 & -\eta & 0 & 0 & 0 & & & & & \\
 & 0 & -\zeta^{p^2} & 0 & 0 & 0 & 0 & & & & & \\
 & -\zeta & 0 & 0 & 0 & 0 & 0 & & & & &
\end{array}\right].$$

The universal deformation $\mathcal{N}$ of N over $\mathcal{M}$, as described in Lemma 7.4, is defined over $R = k[[t_{i,(n+1)} : 1 \leq i \leq n]]$. In particular, for the 2-dimensional stratum in the case $n = 5$, the display of $\mathcal{N}$ is:

$$\left[\begin{array}{c|ccccc||c|ccccc}
 & -\zeta t_{56} & -\zeta^{p^2} t_{46} & -\eta t_{36} & 1 & -\zeta^p t_{26} & -t_{16} & & & & & \\
\hline
1 & & & & & & & 0 & 0 & 0 & -t_{16} & 0 \\
0 & & & & & & & 0 & 0 & 0 & -t_{26} & -\zeta^{-p} \\
0 & & & & & & & 0 & 0 & \eta^{-1} & -t_{36} & 0 \\
0 & & & & & & & 0 & -\zeta^{-p^2} & 0 & -t_{46} & 0 \\
0 & & & & & & & \zeta^{-1} & 0 & 0 & -t_{56} & 0 \\
\hline\hline
0 & & & & & & & 0 & 0 & 0 & -1 & 0 \\
\hline
 & 0 & 0 & 0 & 0 & 0 & -1 & & & & & \\
 & 0 & 0 & 0 & 0 & -\zeta^p & 0 & & & & & \\
 & 0 & 0 & -\eta & 0 & 0 & 0 & & & & & \\
 & 0 & -\zeta^{p^2} & 0 & 0 & 0 & 0 & & & & & \\
 & -\zeta & 0 & 0 & 0 & 0 & 0 & & & & &
\end{array}\right].$$

We can compute the action of F on $\mathcal{N}$ directly from such a display. When doing this for a general stratum in the case of $U(n, 1)$, this computation gives us 3 cases:
Case 1: $k = 0$

$$F(a_i) = \begin{cases} b_1 + \sum_{\beta=2}^{n} c_{n+1} t_{\beta-1,n+1} b_\beta & i = n+1 \\ 0 & i \neq n-k+1 \end{cases}$$

$$F(b_i) = \begin{cases} 0 & i = 1 \\ a_{i-1} - c_{i-1} t_{i-1,n+1} a_{n+1} & i > 1. \end{cases}$$

Case 2: $0 < k < n-k+1$

$$F(a_i) = \begin{cases} b_1 & i = n-k+1 \\ 0 & i \neq n-k+1 \end{cases}$$

$$F(b_i) = \begin{cases} a_i - c_i t_{n-i+1,n+1} a_{n-k+1} & i < k+1 \\ 0 & i = k+1 \\ a_{i-1} - c_{i-1} t_{n-i+2,n+1} a_{n+k-1} & k+1 < i < n-k+2 \\ a_{n-k+1} & i = n-k+2 \\ a_{i-1} - c_{i-1} t_{n-i+3,n+1} a_{n-k+1} & i > n-k+2. \end{cases}$$

Case 3: $n-k+1 \leq k \leq n$

$$F(a_i) = \begin{cases} b_1 & i = n-k+1 \\ 0 & i \neq n-k+1 \end{cases}$$

$$F(b_i) = \begin{cases} a_i - c_i t_{n-i+1,n+1} a_{n-k+1} & i < n-k+1 \\ a_{n+k-1} & i = n-k+1 \\ a_i - c_i t_{n-i+2,n+1} a_{n-k+1} & n-k+1 < i \leq k \\ 0 & i = k+1 \\ a_{i-1} - c_{i-1} t_{n-i+3,n+1} a_{n-k+1} & i > k+1. \end{cases}$$

Now, consider working over R_φ. The deformations given above are universal over the whole of $\mathcal{M}$. If we restrict to working over R_φ, we will get the universal deformation over each individual stratum. We know by the definition of R_φ that if $\beta > k+1$ then $t_{1,\beta} = 0$ in R_φ. Note that this gives us $R_\varphi = k[[t_{1,n+1}, t_{2,n+1}, \ldots, t_{k,n+1}]]$ which is precisely k-dimensional, as expected. We can reduce the above formulae to get the universal deformations over each EO-stratum in $U(n,1)$. Again, we consider the 3 cases:

Case 1: $k = 0$

$$F(a_i) = \begin{cases} b_1 & i = n+1 \\ 0 & i \neq n-k+1 \end{cases}$$

$$F(b_i) = \begin{cases} 0 & i = 1 \\ a_{i-1} & i > 1. \end{cases}$$

Case 2: $0 < k < n-k+1$

$$F(a_i) = \begin{cases} b_1 & i = n-k+1 \\ 0 & i \neq n-k+1 \end{cases}$$

$$F(b_i) = \begin{cases} a_i & i < k+1 \\ 0 & i = k+1 \\ a_{i-1} & k+1 < i < n-k+2 \\ a_{n-k+1} & i = n-k+2 \\ a_{i-1} - c_{i-1} t_{n-i+3,n+1} a_{n-k+1} & i > n-k+2. \end{cases}$$

Case 3: $n-k+1 \leq k \leq n$

$$F(a_i) = \begin{cases} b_1 & i = n-k+1 \\ 0 & i \neq n-k+1 \end{cases}$$

$$F(b_i) = \begin{cases} a_i & i < n-k+1 \\ a_{n+k-1} & i = n-k+1 \\ a_i - c_i t_{n-i+2,n+1} a_{n+k-1} & n-k+1 < i \leq k \\ 0 & i = k+1 \\ a_{i-1} - c_{i-1} t_{n-i+3,n+1} a_{n-k+1} & i > k+1. \end{cases}$$

Example 2: A higher-dimensional example: Let $\mathcal{M}$ be as above, with signature $(n, m) = (4, 3)$. Let φ be the elementary sequence $(0, 1, 1, 1, 2, 2, 3, 3)$, and consider the Ekedahl–Oort stratum S_φ. By the computation in Proposition 6.11, we have:

$$\omega(i) = \begin{cases} i - \varphi(i) & \varphi(i) = \varphi(i-1) \\ \varphi(i) + 4 & \varphi(i) = \varphi(i-1) + 1. \end{cases}$$

This gives us $\omega = 5126374$.

By Proposition 6.13, we have:

$$\dim(S_\varphi) = \sum_{i=1}^{7} \varphi(i) - \frac{3(3+1)}{2} = (1+1+1+2+2+3+3) - 6 = 7.$$

So S_φ is a 7-dimensional stratum.

Now, let $(A, \iota, \zeta, \eta) \in S_\varphi$, and let N be the Dieudonné module of $A[p]$. Then by Proposition 6.7, there exists a basis $\{a_1, \ldots, a_7, b_1, \ldots, b_7\}$ of N, such that $\{a_1, \ldots, a_7\}$ generates $N[\Sigma]$, and $\{b_1, \ldots, b_7\}$ generates $N[\overline{\Sigma}]$. Also, F and V act as:

$$\begin{array}{llll}
F(a_1) = b_1 & V(a_1) = 0 & F(b_1) = a_1 & V(b_1) = 0 \\
F(a_2) = 0 & V(a_2) = 0 & F(b_2) = 0 & V(b_2) = 0 \\
F(a_3) = 0 & V(a_3) = 0 & F(b_3) = a_2 & V(b_3) = 0 \\
F(a_4) = b_2 & V(a_4) = 0 & F(b_4) = 0 & V(b_4) = a_2 \\
F(a_5) = 0 & V(a_5) = b_2 & F(b_5) = a_3 & V(b_5) = a_3 \\
F(a_6) = b_3 & V(a_6) = b_4 & F(b_6) = a_4 & V(b_6) = a_5 \\
F(a_7) = 0 & V(a_7) = b_7 & F(b_7) = 0 & V(b_7) = b_7.
\end{array}$$

By looking at φ, we see that $I = \{1, 4, 6\}$ and $J = \{2, 3, 5, 7\}$. The only self-dual orbit here is computed to contain only $\{a_3, b_5\}$. So let $\eta \in k$ be such that $\eta^p = -\eta$. The other orbits can be divided into $\mathcal{A}_1 = \{\{a_1, b_1\}, \{a_2, b_4, a_6, b_3\}\}$ and their duals

$\mathcal{A}_2 = \{\{a_7, b_7\}, \{a_4, b_2, a_5, b_6\}\}$. Thus, the displayed basis for N is

$$\{a_1, a_4, a_6, b_1, b_3, b_5, b_6; b_7, -b_4, b_2, a_7, a_5, -\eta^{-1}a_3, -a_2\}.$$

The display matrix of N is:

$$\left[\begin{array}{ccc|cccc||ccc|cccc}
 & & & 1 & 0 & 0 & 0 & 0 & 0 & 0 & & & & \\
 & & & 0 & 0 & 0 & 1 & 0 & 0 & 0 & & & & \\
 & & & 0 & 0 & 0 & 0 & 0 & -1 & 0 & & & & \\
\hline
1 & 0 & 0 & & & & & & & & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & & & & & & & & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & & & & & & & & 0 & 0 & \eta^{-1} & 0 \\
0 & 0 & 0 & & & & & & & & 0 & 1 & 0 & 0 \\
\hline\hline
0 & 0 & 0 & & & & & & & & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & & & & & & & & 0 & 0 & 0 & 1 \\
0 & 1 & 0 & & & & & & & & 0 & 0 & 0 & 0 \\
\hline
 & & & 0 & 0 & 0 & 0 & 1 & 0 & 0 & & & & \\
 & & & 0 & 0 & 0 & 0 & 0 & 0 & 1 & & & & \\
 & & & 0 & 0 & -\eta & 0 & 0 & 0 & 0 & & & & \\
 & & & 0 & -1 & 0 & 0 & 0 & 0 & 0 & & & &
\end{array}\right].$$

The universal deformation $\mathcal{N}$ of N over $\mathcal{M}$, as described in Lemma 7.4, is defined over $R = k[[t_{ij} : 1 \leq i \leq 4 < j \leq 7]]$. The display of $\mathcal{N}$ is:

$$\left[\begin{array}{ccc|cccc||ccc|cccc}
 & & & 1 & -t_{45} & -\eta t_{35} & 0 & t_{15} & 0 & t_{25} & & & & \\
 & & & 0 & -t_{46} & -\eta t_{36} & 1 & t_{16} & 0 & t_{26} & & & & \\
 & & & 0 & -t_{47} & -\eta t_{37} & 0 & t_{17} & 1 & t_{27} & & & & \\
\hline
1 & t_{17} & 0 & & & & & & & & t_{15} & 0 & 0 & t_{16} \\
0 & t_{27} & 1 & & & & & & & & t_{25} & 0 & 0 & t_{26} \\
0 & t_{37} & 0 & & & & & & & & t_{35} & 0 & \eta^{-1} & t_{36} \\
0 & t_{47} & 0 & & & & & & & & t_{45} & -1 & 0 & t_{46} \\
\hline\hline
0 & 0 & 0 & & & & & & & & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & & & & & & & & 0 & 0 & 0 & 1 \\
0 & 1 & 0 & & & & & & & & 0 & 0 & 0 & 0 \\
\hline
 & & & 0 & 0 & 0 & 0 & 1 & 0 & 0 & & & & \\
 & & & 0 & 0 & 0 & 0 & 0 & 0 & 1 & & & & \\
 & & & 0 & 0 & -\eta & 0 & 0 & 0 & 0 & & & & \\
 & & & 0 & -1 & 0 & 0 & 0 & 0 & 0 & & & &
\end{array}\right].$$

We would now like restrict to working over R_φ. We produce this by setting all $t_{\alpha\beta}$ such that $j_{5-\alpha} < i_{\beta-4}$ to zero. This occurs precisely for $t_{\alpha\beta} \in \{t_{27}, t_{36}, t_{37}, t_{46}, t_{47}\}$. So consider $\mathcal{N}$ to now be the Dieudonné space over $\mathbb{F}_\varphi^{\mathrm{perf}}$ given by base change from the universal deformation described above. Then, this display tells us that the action of F

on $\mathcal{N}$ is given by:

$$
\begin{aligned}
F(a_1) &= b_1 & F(b_1) &= a_1 \\
F(a_2) &= 0 & F(b_2) &= 0 \\
F(a_3) &= 0 & F(b_3) &= a_2 - t_{45}a_1 \\
F(a_4) &= b_2 + t_{17}b_1 & F(b_4) &= 0 \\
F(a_5) &= 0 & F(b_5) &= a_3 - \eta t_{35}a_1 \\
F(a_6) &= b_3 & F(b_6) &= a_4 \\
F(a_7) &= 0 & F(b_7) &= 0.
\end{aligned}
$$

Note that this shows that the pair of flags

$$0 \subseteq A_1 \subseteq A_2 \subseteq \cdots \subseteq A_7$$

and

$$0 \subseteq B_1 \subseteq B_2 \subseteq \cdots \subseteq B_7$$

is stable under F. Also, as per the reasoning in Lemma 3.4, the elementary sequence of this Dieudonné space is $(0, 1, 1, 1, 2, 2, 3, 3) = \varphi$. So $\mathcal{N}$ is a deformation of N that lies within S_φ. Furthermore, as $R_\varphi = k[[t_{17}, t_{16}, t_{26}, t_{15}, t_{25}, t_{35}, t_{45}]]$, we see that R_φ is 7-dimensional, as is S_φ. Thus $\mathcal{N}$ is the universal deformation for N that preserves the elementary sequence φ.

8. THE V-FOLIATION

8.1 Foliations

We now wish to study a particular foliation on M. First, we will state some general facts about foliations in characteristic p. See [Eke87] for more details.

Let k be an algebraically closed field of characteristic p, and let X be a non-singular variety over k of dimension n. Let $\mathcal{T}X$ be the tangent sheaf of X. Note that $\mathcal{T}X$ can be seen as a p-Lie algbera over k, with operations given by

$$[\xi, \eta] = \xi \circ \eta - \eta \circ \xi, \xi^{(p)} = \xi \circ \xi \circ \cdots \circ \xi$$

where $\xi, \eta \in \mathcal{T}X$ are viewed as vector fields defined in some open $U \subset X$, and regarded as operators on $\mathcal{O}_X(U)$.

A **foliation of height 1** on X is a sub-bundle $\mathcal{E} \subset \mathcal{T}X$, which is a p-Lie subalgebra. That is, $\mathcal{E}$ is closed under the Lie bracket and $\xi \mapsto \xi^{(p)}$. As we only work with foliations of height 1 here, we will simply refer to them as **foliations** from here on.

Given a subvariety $Y \subset X$, we say that Y is a **integral subvariety** for the foliation $\mathcal{E}$ if $\mathcal{E}|_Y = \mathcal{T}Y$.

One of the main reasons we care about height 1 foliations is their connection with height 1 morphisms, as given in the following proposition.

Proposition 8.1. [Eke87, 2.4] *Let X be a non-singular k-variety. There is a natural one-to-one correspondence, between finite flat height 1 morphisms $f : X \to Y$ and height 1 foliations $\mathcal{E} \subset \mathcal{T}\mathcal{X}$. One has deg$(f) = p^{rk(\mathcal{E})}$.*

8.2 The V-foliation Over $S_\sharp$

Let $\mathcal{A}$ be the universal abelian scheme of $\mathcal{M}$, with structure map $\pi\colon \mathcal{A} \to \mathcal{M}$, equipped with ι, ζ, η as above, and let $\mathcal{A}^t$ be its dual scheme. Then we can construct the Hodge filtration:

$$0 \to \omega_{\mathcal{A}/\mathcal{M}} \to H^1_{dR}(\mathcal{A}/\mathcal{M}) \to \omega^\vee_{\mathcal{A}^t/\mathcal{M}} \to 0$$

where $\omega_{\mathcal{A}/\mathcal{M}} = R^0\pi_*\Omega^1_{\mathcal{A}/\mathcal{M}}$, and $\omega^\vee_{\mathcal{A}^t/\mathcal{M}} = R^1\pi_*\mathcal{O}_{\mathcal{A}}$.

Recall that at a geometric point $x \in \mathcal{M}$, there is a canonical identification of $H^1_{dR}(\mathcal{A}_x/k)$ with the contravariant Dieudonné module D of $\mathcal{A}_x[p]$. In particular, under this correspondence, $\omega_{\mathcal{A}_x/k} \cong D[F]$.

Note that since p is unramified in E, the Hodge bundle $\omega_{\mathcal{A}/\mathcal{M}}$ decomposes into its Σ and $\overline{\Sigma}$ parts. Define the vector bundles

$$P := \omega_{\mathcal{A}/\mathcal{M}}[\Sigma], \qquad Q := \omega_{\mathcal{A}/\mathcal{M}}[\overline{\Sigma}].$$

Note that P is a bundle of rank n,and Q is a bundle of rank m.

Consider the Gauss–Manin connection

$$\nabla\colon H^1_{dR}(\mathcal{A}/\mathcal{M}) \to \Omega^1_{\mathcal{M}} \otimes_{\mathcal{O}_{\mathcal{M}}} H^1_{dR}(\mathcal{A}/\mathcal{M}).$$

If we restrict this map to $\omega_{\mathcal{A}/\mathcal{M}} \subset H^1_{dR}(\mathcal{A}/\mathcal{M})$, and project the second component of the result to $\omega^\vee_{\mathcal{A}^t/\mathcal{M}}$, we obtain the Kodaira–Spencer map:

$$\omega_{\mathcal{A}/\mathcal{M}} \to \Omega^1_{\mathcal{M}} \otimes \omega^\vee_{\mathcal{A}^t/M}.$$

Since the polarization ζ induces an isomorphism $Q^\vee := \omega^\vee_{\mathcal{A}/\mathcal{M}}[\overline{\Sigma}] \cong \omega^\vee_{\mathcal{A}^t/M}[\Sigma]$, we see that restricting the Kodaira–Spencer map to the Σ part, we get a morphism

$$P \to \Omega^1_{\mathcal{M}} \otimes Q^\vee.$$

This induces an isomorphism

$$KS\colon P \otimes Q \to \Omega^1_{\mathcal{M}}.$$

Recall that we have isogenies $\mathcal{F}\colon \mathcal{A} \to \mathcal{A}^{(p)}$, and $\mathcal{V}\colon \mathcal{A}^{(p)} \to \mathcal{A}$. These induce maps on cohomology, which by a slight abuse of notation, we will also denote as

$$\mathcal{F}\colon H^1_{dR}(\mathcal{A}^{(p)}/\mathcal{M}) \to H^1_{dR}(\mathcal{A}/\mathcal{M})$$

and

$$\mathcal{V}\colon H^1_{dR}(\mathcal{A}/\mathcal{M}) \to H^1_{dR}(\mathcal{A}^{(p)}/\mathcal{M}).$$

Consider $P_0 := P[\mathcal{V}] = \ker(\mathcal{V}\colon P \to Q^{(p)})$. We would like to say that P_0 is a sub-bundle of P, however its rank may increase when specializing from one EO-stratum to a smaller one. As such, we will first consider P_0 over only the open stratum, denoted S_{ord}. Over S_{ord}, the sheaf P_0 is a sub-bundle of P, for which the fibers have constant rank $n-m$. Thus, after restriction to S_{ord}, $KS(P_0 \otimes Q)$ is a sub-bundle of $\Omega^1_{\mathcal{M}}$ with rank $(n-m)m$. Accordingly the V- foliation $\mathcal{T}^+ \subset \mathcal{T}S_{\mathrm{ord}}$ is defined

$$\mathcal{T}^+ = KS(P_0 \otimes Q)^{\perp}.$$

Proposition 8.2. *$\mathcal{T}^+$ is a foliation of height 1.*

Proof. [dSG18, Proposition 3]. □

The following will be a useful lemma in explicitly computing $\mathcal{T}^+$.

Lemma 8.3. *$\xi \in \mathcal{T}^+$ if and only of $\nabla_\xi(\mathcal{P}_0) \subset \mathcal{P}_0$.*

Proof. [dSG18, Corollary 5]. □

We would like to extend $\mathcal{T}^+$ to all of $\mathcal{M}$, but it becomes singular over some of the deeper EO-stratum. As such, it can only be properly extended to a certain union of

EO-stratum. The details of this extension can be found in [dSG18]. We will state some useful notation and results here.

Consider the EO-stratum corresponding to the shuffle

$$\omega_{\text{fol}} = [1\ 2\ \ldots\ n-m\ \ n+1\ \ldots\ g\ \ n-m+1\ \ \ldots\ n]$$

or equivalently, the elementary sequence

$$\varphi_{fol}(i) = \begin{cases} 0 & 0 \le i < n-m \\ i-(n-m) & n-m \le i \le n \\ m & n < i \le g. \end{cases}$$

This stratum will be denoted as S_{fol}, and has dimension m^2. Let $S_\sharp := \bigcup_{S_{\text{fol}} \subseteq \overline{S_\varphi}} S_\varphi$. This is a locally closed subset of $\mathcal{M}$. Also, $\mathcal{T}^+$ extends to $S_\sharp$ with the same definitions for S_{ord}.

Note that Lemma 8.3 characterizes $\mathcal{T}^+$ over all of $S_\sharp$,

Theorem 8.4. *The EO stratum S_{fol} is an integral subvariety of the foliation $\mathcal{T}^+$ i.e. $\mathcal{T}^+|_{S_{fol}} = \mathcal{T}S_{fol}$.*

Proof: [dSG18, Theorem 25].

8.3 The V-foliation Over S_φ

We would like to directly compute $\mathcal{T}^+|_{S_\varphi}$ for any stratum $S_\varphi \subset S_\sharp$. In order to use Lemma 8.3, we need to evaluate $\nabla_\xi(P_0)$ for ξ in the tangent space of $\mathcal{M}$ at x. As such, it would be beneficial to have a basis for $\mathcal{T}\mathcal{M}$ over some neighbourhood of x that is horizontal with respect to the Gauss–Manin connection. In particular, as $\mathrm{Spf}(R) \cong \mathrm{Spf}(\widehat{\mathcal{O}}_{\mathcal{M},x})$ (c.f. Theorem 7.6), we would like to find a horizontal basis for P_0 over $\mathrm{Spec}(R)$ from here on out.

Unfortunately, computing such a basis for $P_0(\mathrm{Spec(R)})$ can be difficult. However, if we only consider first-order deformations, and define $R' := R/\mathfrak{m}_R^2$, we will be able to find a basis for $P_0(\mathrm{Spec}(R'))$ that is horizontal with respect to ∇; this will be sufficient for our purposes.

Note that since the Gauss–Manin connection respects isogenies, such as F and V, and as the basis given in Proposition 6.7 was chosen to respect F and V, we know that this is a horizontal basis for $H^1_{dR}(\mathcal{A}_x/k)$.

Now, we need a horizontal basis for the universal first-order deformation, that is the universal deformation over R'. When we computed the universal deformation, we seemed to be using the same basis for both N and $\mathcal{N}$. However, this may be a little misleading. While the basis remained the same, the action of F was deformed. This will not produce a horizontal basis for $\mathcal{N}$. In order for the basis to be horizontal, we must instead deform the basis elements in such a way the the action of F is preserved. For more details on this process, see [AG04, §5].

Recall that the action of F was given in block matrix form as:

$$\begin{pmatrix} A & pB \\ C & pD \end{pmatrix}$$

such that the universal display was

$$\begin{pmatrix} A+TC & p(B+TD) \\ C & pD \end{pmatrix}.$$

We can consider the automorphism of the universal display given by:

$$\begin{pmatrix} I & -T \\ 0 & I \end{pmatrix} \begin{pmatrix} A+TC & p(B+TD) \\ C & pD \end{pmatrix} \begin{pmatrix} I & T^\sigma \\ 0 & I \end{pmatrix} = \begin{pmatrix} A & AT^\sigma + pB \\ C & CT^\sigma + pD \end{pmatrix}.$$

Note that if we consider this equation over $R' = R/\mathfrak{m}^2$, then T^σ vanishes, so this produces the original equations for F. Note that this automorphism corresponds to the change of basis corresponding to $\begin{pmatrix} I & -T \\ 0 & I \end{pmatrix}$. So, if we reduce modulo p, we see that the the basis elements not in the kernel of F remain the same, where as the basis elements in the kernel of F have become

$$\{c_{i_\alpha}^{-1} b_{g+1-i_\alpha} - \sum_{k=1}^{n} t_{i,n+\alpha} b_{g+1-j_{n+1-k}}\}_{1\le\alpha\le m} \cup \{-c_{j_\beta}^{-1} a_{j_\beta} - \sum_{k=1}^{m} t_{n+1-\beta,n+k} a_{i_k}\}_{1\le\beta\le n}.$$

Thus, by the work in [AG04, §5], this provides us with a horizontal basis with respect to ∇ for $\omega_{\mathcal{A}/\mathcal{M}}(\mathrm{Spec}(R'))$.

Lemma 8.5. *An EO-stratum $S_\varphi \subset S_\sharp$, that is $S_{fol} \subset \overline{S_\varphi}$, if and only if $\varphi(n) = m$.*

Proof. Recall that $S_\sharp$ is a union of the EO-strata such that $P[\mathcal{V}]$ has rank $n-m$ on S_φ. Let $x \in S_\varphi$, and let N be the covariant Dieudonné module corresponding to $\mathcal{A}_x[p]$, and let D be the contravariant Dieudonné module of the same. Given that $\omega_{\mathcal{A}_x/k} \cong D[\mathcal{F}]$, we can view $P \cong D[\mathcal{F}][\Sigma] \cong N[V][\Sigma]$. So let

$$0 = A_0 \subset A_1 \subset \cdots \subset A_g = N[\Sigma]$$

be the Σ part of the canonical filtration of N, as in (6.1). We know that $N[V][\Sigma]$ has dimension n, and the canonical fitration is stable under V^{-1}. Thus $N[V][\Sigma] = A_n$.

Also note that when we pass through the isomorphism $\omega_{\mathcal{A}_x/k} \cong N[V]$, the map $\mathcal{V}$ on P becomes the map F on $N[V][\Sigma]$. Thus $S_\varphi \subset S_\sharp$ if and only if the kernel of $F|_{A_n}$ has dimension $n-m$. As $\dim(A_n) = n$, we see that this is equivalent to stating that $\dim(F(A_n)) = m$. But by the definition of the canonical filtration, this means that $\varphi(n) = m$. □

Lemma 8.6. *Let $1 \le \alpha \le m \le n < \beta \le g$. If $\varphi(n) = m$, then $t_{\alpha\beta} \neq 0$ in R_φ.*

Proof. Recall that $t_{\alpha\beta} \neq 0$ in R_φ if and only if $i_{\beta-n} < j_{n+1-\alpha}$. Also, by the definition of elementary sequences, we know that $\varphi(g) = m$. Thus, as φ is monotonic, we know that $\varphi(x) = m$ for $n \leq x \leq g$. Thus $x \in J$ for $n < x \leq g$.

Therefore $i_m \leq n$, and for $1 \leq \alpha \leq m$, we have $j_{n+1-\alpha} = g + 1 - \alpha$. Thus, if $1 \leq \alpha \leq m \leq n < \beta \leq g$, we have:

$$i_{\beta-n} \leq i_{g-n} = i_m \leq n = g - m < g + 1 - \alpha = j_{n+1-\alpha}.$$

□

Note that by Theorem 7.6, we have an explicit description of $\widehat{O}_{S_\varphi,x}$. Thus, using the coordinates of R_φ, we see that at a point $x \in S_\varphi$, we have

$$\mathcal{T}S_\varphi|_x = \left\langle \frac{\partial}{\partial t_{\alpha\beta}} : t_{\alpha\beta} \neq 0 \in R_\varphi \right\rangle.$$

We can now also view $\mathcal{T}^+|_x$ in these same coordinates.

Theorem 8.7. *Let $S_\varphi \subset S_\sharp$ be an EO-stratum, and let $x \in S_\varphi$. Then*

$$\mathcal{T}^+|_x = \left\langle \frac{\partial}{\partial t_{\alpha\beta}} : 1 \leq \alpha \leq m \leq n < \beta \leq g \right\rangle.$$

Therefore $\mathcal{T}^+|_{S_\varphi} \subset \mathcal{T}S_\varphi$.

Proof. Let $x \in S_\varphi$, and let N be the covariant Dieudonné module of $\mathcal{A}_x[p]$, and let D be the contravariant Dieudonné module. Note that these are dual as vector spaces. Furthermore, the dual of F on N is $\mathcal{V}$ on D, and the dual of V on N is $\mathcal{F}$ on D.

In order to compute $\mathcal{T}^+|_x$, we want to use Lemma 8.3. As such, we will need to compute P_0 at x.

Note that P is the Σ part of $\omega_{\mathcal{A}/\mathcal{M}} = D[\mathcal{F}]$. Since the covariant Dieudonné module is dual to the contravariant Dieudonné module, we see that this is naturally isomorphic to $N[V]$. So, if we let $\{a_1, \ldots, a_g, b_1, \ldots, b_g\}$ be the basis for N as in Proposition 6.6, we see by Lemma 7.2 that $N[V] = \langle a_1, \dots, a_n, b_1, \dots, b_m \rangle$. Thus, if we let $a'_i \in D$ be the dual vector to $a_i \in N$, we see that $P = \langle a'_1, \dots, a'_n \rangle$.

Also, if we have I and J as in chapter 7, we see that $P_0 = P[\mathcal{V}]$. Thus, we can compute it by finding the kernel of F on $\operatorname{span}\{a_1, \dots, a_n\}$, which is $\{a_{j_1}, \dots, a_{j_r}\}$, where j_r is the maximal element of J such that $j_r \leq n$. However, by Lemma 8.5, we know that $\varphi(n) = m$. Thus $\varphi(k) = \varphi(k-1)$ for all $k > n$. Thus $j_{n-m+1}, \dots, j_n \geq n$. So $r = n - m$. Thus

$$P_0 = \operatorname{span}\{a'_{j_1}, \dots, a'_{j_{n-m}}\}.$$

Now, if we consider the general deformations of P and P_0 around x, we get:

$$P(\operatorname{Spec}(R)) = \operatorname{Span}_R\{-c_{j_\beta}^{-1} a'_{j_\beta} - \sum_{k=1}^{m} t_{n+1-\beta,n+k} a'_{i_k}\}_{\beta=1}^{n}$$

and

$$P_0(\operatorname{Spec}(R)) = \operatorname{Span}_R\{-c_{j_\beta}^{-1} a'_{j_\beta} - \sum_{k=1}^{m} t_{n+1-\beta,n+k} a'_{i_k}\}_{\beta=1}^{n-m}.$$

Thus, we see that $\nabla_{\frac{\partial}{\partial t_{\alpha\beta}}}(P_0) \subset P_0$ if and only if $\alpha \leq m$, in which case $\nabla_{\frac{\partial}{\partial t_{\alpha\beta}}}(P_0) = 0$. Thus

$$\mathcal{T}^+|_x = \left\langle \frac{\partial}{\partial t_{\alpha\beta}} : 1 \leq \alpha \leq m \leq n < \beta \leq g \right\rangle.$$

Therefore, by Lemma 8.6 we can conclude that $\mathcal{T}^+|_{S_\varphi} \subset \mathcal{T} S_\varphi$. □

Corollary 8.8. *$\mathcal{T}^+$ induces a height 1 surjective morphism $f : \mathcal{M} \to \mathcal{M}'$ of degree p^{m^2}, where $\mathcal{M}'$ is a non-singular scheme. The restriction of f to any $S_\varphi \subseteq S_\sharp$ is likewise a height 1 morphism of degree p^{m^2} onto its image in $\mathcal{M}'$.*

Proof. This follows directly from Proposition 8.1 and Theorem 8.7. □

8.4 Example

Consider the example of $U(n, 1)$ described in the previous section. Since $m = 1$ this tells us that S_{fol} is the unique 1-dimensional stratum, and $\mathcal{T}^+$ has rank 1.

So, if x is in the k-dimensional stratum for $k > 0$, we have:

$$P_x = \mathrm{Span}_k\{a'_1, a'_2, \ldots, \widehat{a'_{n-k+1}}, \ldots, a'_{n+1}\},$$

$$Q_x = \mathrm{Span}_k\{b'_{k+1}\},$$

and

$$(P_0)_x = \mathrm{Span}_k\{a'_1, a'_2, \ldots, \widehat{a'_{n-k+1}}, \ldots, a'_n\}.$$

Using the formula from Theorem 8.7, we see that a basis for $P_0(Spec(R))$ is:

$$\{-c_i^{-1}a'_i - t_{n+1-i,n+1}a'_{n-1+k}\}_{i=1}^{n-k} \cup \{-c_i^{-1}a'_i - t_{n+2-i,n+1}a'_{n-1+k}\}_{i=n-k+2}^{n}$$

Note that for $i \leq n-k$, we know that $n+1-i \geq k+1 > 1$, and for $n-k+2 \leq i \leq n$, we have $n+2-i \geq 2 > 1$. Thus $t_{1,n+1}$ does not appear in any of these generating elements for $P_0(\mathrm{Spec}(R))$. Thus

$$\nabla_{\frac{\partial}{\partial t_{1,n+1}}}(P_0(\mathrm{Spec(R)})) = \{0\} \subset P_0(\mathrm{Spec(R)}).$$

Therefore $\frac{\partial}{\partial t_{1,n+1}} \in \mathcal{T}^+$.

On the other hand, for $1 < j \leq k$, we see that $-c_{n+2-j}^{-1}a'_{n+2-j} - t_{j,n+1}a'_{n-1+k} \in P_0(\mathrm{Spec}(R))$. But:

$$\nabla_{\frac{\partial}{\partial t_{j,n+1}}}(-c_{n+2-j}^{-1}a'_{n+2-j} - t_{j,n+1}a'_{n-1+k})) = a'_{n-1+k} \notin P_0(\mathrm{Spec(R)}).$$

Also, for $k < j \leq n$, we see that $-c_{n+1-j}^{-1}a'_{n+1-j} - t_{j,n+1}a'_{n-1+k} \in P_0(\mathrm{Spec(R)}))$. But

$$\nabla_{\frac{\partial}{\partial t_{j,n+1}}}(-c_{n+1-j}^{-1}a'_{n+1-j} - t_{j,n+1}a'_{n-1+k})) = a'_{n-1+k} \notin P_0(\mathrm{Spec(R)}).$$

Thus $\frac{\partial}{\partial t_{j,n+1}} \notin \mathcal{T}^+$. This makes sense, as $\mathcal{T}^+$ is known to be one-dimensional in this case.

Now, recall that if φ is the elementary sequence parameterizing the unique k-dimensional stratum, we found that $R_\varphi = k[[t_{i,n+1}]]_{i=1}^k$. Thus $t_{1,n+1} \neq 0 \in R_\varphi$ for all strata with dimension at least 1. But this is precisely the strata that have the unique 1-dimensional stratum in their boundary, i.e. the strata in $S_\sharp$. Thus for any $x \in S_\varphi \subset S_\sharp$, we have $\mathcal{T}^+|_x \subset \mathcal{T}\mathcal{S}_\varphi|_x$.

9. CONCLUSION

In this thesis, we examined two classes of p-foliations on Shimura varieties. We first examined the tautological foliations on toroidal compactifications a Hilbert modular variety $\mathcal{M}_n(\mathfrak{c})$. Using Klyachko's classification of toric vector bundles by multifiltrations, and in particular the correspondence between toric foliations $\mathscr{F}_V$ on a toric variety X and subspaces V of a certain vector space, we described the singular locus of $\mathscr{F}_V$ in terms of the relationship between V and the cone decomposition defining X. By relating the tautological foliations on $\mathcal{M}_n(\mathfrak{c})$ to certain toric foliations on the toric varieties defining a toroidal compactification of $\mathcal{M}_n(\mathfrak{c})$, we used these results to describe the singular loci of the tautological foliations.

Generically, we saw that the tautological foliations tend to extend smoothly to the toric strata of dimension at least the rank of the foliation, but become singular at lower dimension strata. However, there are exceptions to this characterization. In Proposition 4.6, we saw that this characterization holds when the matrices describing the relation between V and cones $\sigma \in \Sigma$ are totally invertible. While total invertibility is an open condition, and thus holds generically, it is not hard to find examples where this hypothesis fails.

This leads to some questions for further investigation. Let L be a totally real field with embeddings $\{\sigma_1, \ldots, \sigma_g\}$ into $\mathbb{R}$ (or $W(\kappa)[1/p]$) and let N be a fractional ideal of L with basis $\{\mu_1, \ldots, \mu_g\}$. We could ask when the matrix $[\sigma_i(\mu_j)]$ is totally invertible. One way in which total invertibility could fail is if some subset of $\{\mu_1, \ldots, \mu_g\}$ lie within a proper subfield of L, as happens in the example given in section 4.2. A solution to this question could lead to a characterization of the toric strata with dimension at least that of the rank of a tautological foliations $\mathscr{F}$, on which $\mathscr{F}$ does not extend smoothly.

In positive characteristic, we have even more interesting behavior. When working over $\mathbb{C}$, we saw that the tautological foliations never extend smoothly to toric strata of dimension less than the rank of the foliation. On the other hand, it can be possible to have these smooth extensions in positive characteristic. In the case of a Hilbert modular surface, we found that in characteristic 2 and 3, there always exists a toroidal compactification on which a given tautological foliation extends smoothly. In Corollary 4.14, we gave a further characterization of when a Hilbert modular surface over a field of positive characteristic at least 5 has a toroidal compactification on which a tautological foliation extends smoothly. Moving forward, I would like to examine whether the case in which $\mathscr{F}$ can be made smooth can be removed from Corollary 4.14. Preliminary calculations have failed to find a case in which such an $\mathscr{F}$ can be made smooth everywhere in characteristic at least 5, but proving it never happens may require methods beyond those found in this thesis.

We also considered the V-foliation $\mathcal{T}^+$ on a unitary Shimura variety $\mathcal{M}$ of signature (n, m). In order to study the geometry of the Ekedahl–Oort strata on $\mathcal{M}$, we used Zink's theory of displays to compute the universal display for the Dieudonné module associated with a point in a given Ekedahl–Oort stratum. This provided a useful description of the tangent space of the stratum and how it lies within the full tangent space of the $\mathcal{M}$. By combining this description with an explicit description of $\mathcal{T}^+$ in these same coordinates, we were able to show that the $\mathcal{T}^+$ lies within the tangent space for each stratum lying between S_{fol} and the open stratum. That is, each of these strata are invariant with respect to $\mathcal{T}^+$.

This still leaves room for further investigations. The Ekedahl–Oort stratum S_{fol} is but one of the integral subvarieties of $\mathcal{T}^+$. In the recent work of Goren and de Shalit in [GdS23], it is shown that there are a multitude of integral subvarieties for any p-foliation. As such, we may ask if one can define a "stronger" version of integrality, for which only certain integral subvarieties, such as S_{fol}, that have arithmetic significance to the underlying moduli problem remain integral under this stronger definition.

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BIBLIOGRAPHY

[AG04] Fabrizio Andreatta and Eyal Z. Goren. Hilbert modular varieties of low dimension. *Geometric aspects of Dwork theory, Vol. I, II*, pages 113–175, 2004.

[Bou81] Nicolas Bourbaki. *Algebra II: Chapters 4 - 7.* Masson, Paris, 1981.

[Cha90] C. L. Chai. Appendix to "The Iwasawa conjecture for totally real fields": Arithmetic minimal compactification of the Hilbert-Blumenthal moduli spaces. *Annals of Mathematics*, 131(3):541–554, 1990.

[Dem72] Michel Demazure. *Lectures on p-divisible groups.* Springer-Verlag Berlin Heidelberg, 1972.

[DG67] Jean Dieudonné and Alexander Grothendieck. Éléments de géométrie algébrique. *Inst. Hautes Études Sci. Publ. Math.*, 4, 8, 11, 17, 20, 24, 28, 32, 1961–1967.

[Dia22] Fred Diamond. Compactifications of Iwahori-level Hilbert modular varieties. 2022.

[dSG18] Ehud de Shalit and Eyal Z. Goren. Foliations on unitary Shimura varieties in positive characteristic. *Compos. Math*, 154:2267 – 2304, 2018.

[Eke87] Torsten Ekedahl. Foliations and inseparable morphisms. In *Algebraic Geometry - Bowdoin 1985*, volume 46, Part 2 of *Proc. Sympos. Pure Math*, pages 139–149. Amer. Math. Soc., 1987.

[FC90] Gerd Faltings and Ching-Li Chai. *Degeneration of Abelian Varieties.* Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge / A Series of Modern Surveys in Mathematics. Springer Berlin, Heidelberg, 1990.

[FM86] Miroslav Fiedler and Thomas L. Markham. Completing a matrix when certain entries of its inverse are specified. *Linear Algebra and its Applications*, 74:225–237, 1986.

[Ful93] William Fulton. *Introduction to Toric Varieties*. Annals of Mathematics Studies. Princeton University Press, 1993.

[GdS23] Eyal Z. Goren and Ehud de Shalit. Foliations on Shimura varieties in positive characteristic. Published electronically in *J. Algebraic Geometry*, September 2023.

[Gor02] Eyal Z. Goren. *Lectures on Hilbert Modular Varieties and Modular Forms*, volume 14 of *CRM Monograph Series*. American Mathematical Society, 2002.

[Gub87] I. Dzh. Gubeladze. The Anderson conjecture and projective modules over monoid algebras. *Soobshch. Akad. Nauk Gruzin. SSR*, 125(2):289–291, 1987.

[Gus84] William H. Gustafson. A note of matrix inversion. *Linear Algbera and its Applications*, 57:71–73, 1984.

[Har77] Robin Hartshorne. *Algebraic Geometry*. Graduate Texts in Mathematics. Springer New York NY, 1977.

[Hat22] Allen Hatcher. *Topology of Numbers*. American Mathematical Society, 2022.

[Jac62] Nathan Jacobson. *Lie Algebras*, volume 10 of *Interscience Tracts in Pure and Applied Mathematics*. John Wiley and Sons, New York, 1962.

[Kat70] Nicholas M. Katz. Nilpotent connections and the monodromy theorem: Applications of a result of Turrittin. *Publications Mathematiques de L'Institut des Hautes Scientifiques*, 39:175–232, 1970.

[Kat78] Nicholas M. Katz. p-Adic L-Functions for CM Fields. *Inventiones Mathematicae*, 49:199–297, 1978.

[KKMSD73] G. Kempf, F. Knudson, D. Mumford, and B. Saint-Donat. *Toroidal Embeddings 1*, volume 339 of *Lecture Notes in Mathematics*. Springer Berlin, Heidelberg, 1973.

[Kly90] Alexander A. Klyachko. Equivariant bundles on toral varieties. *Mathematics of The Ussr-izvestiya*, 35:337–375, 1990.

[Kot92] Robert Kottwitz. Points on some Shimura varieties over finite fields. *Amer. Math. Soc.*, 5(2):373–444, 1992.

[Lan13] Kai-Wen Lan. *Arithmetic Compactifications of PEL-Type Shimura Varieties*. London Mathematical Society Monographs. Princeton University Press, 2013.

[Mat80] Hideyuki Matsumura. *Commutative Algebra*. W. A. Benjamin, Inc., second edition, 1980.

[McQ08] Michael McQuillan. Canonical models of foliations. *Pure and Applied Mathematics Quarterly*, 4:877–1012, 2008.

[Mil17] James Milne. *Algebraic Groups: The Theory of Group Schemes of Finite Type over a Field*. Cambridge Studies in Advanced Mathematics. Cambridge University Press, 2017.

[Miy87] Yoichi Miyaoka. Deformations of a morphism along a foliation and applications. In *Algebraic Geometry - Bowdoin 1985*, volume 46, Part 1 of *Proceedings of Symposia in Pure Mathematics*, pages 245–268. American Mathematical Society, 1987.

[Moo01] Ben Moonen. Group schemes with additional structures and Weyl group cosets. In *Moduli of Abelian Varieties*, volume 195 of *Progress in Mathematics*, pages 255–298. Birkhäuser, Basel, 2001.

[Moo04] Ben Moonen. A dimension formula for Ekedahl-Oort strata. *Annales de l'Institut Fourier*, 54(3):666–698, 2004.

[Mum72] David Mumford. An analytic construction of degenerating abelian varieties over complete rings. *Compositio Mathematica*, 24(3):239–272, 1972.

[NO80] Peter Norman and Frans Oort. Moduli of abelian varieties. *Annals of Mathematics*, 112:413–439, Sep 1980.

[Nor75] Peter Norman. An algorithm for computing local moduli of abelian varieties. *Annals of Mathematics*, 101:499–509, May 1975.

[Oda88] Tadao Oda. *Convex Bodies and Algebraic Geometry*. Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge / A Series of Modern Surveys in Mathematics. Springer Berlin, Heidelberg, 1988.

[Oor01] Frans Oort. A stratification of a moduli space of abelian varieties. In *Moduli of Abelian Varieties*, volume 195 of *Progress in Mathematics*, pages 346–416. Birkhäuser, Basel, 2001.

[Pan15] Thiam-Sun Pang. *The Harder-Narasimhan Filtrations and Rational Contractions*. PhD thesis, Albert-Ludwigs-Universit at Freiburg, 2015.

[Per04] Markus Perling. Graded rings and equivariant sheaves on toric varieties. *Mathematische Nachrichten*, 263-264(1):181–197, 2004.

[PWZ15] Richard Pink, Torsten Wedhorn, and Paul Ziegler. F-zips with additional structure. *Pacific Journal of Mathematics*, 274(1):183–236, 2015.

[Rap78] Michael Rapoport. Compactifications de l'espace de modules de Hilbert-Blumenthal. *Compositio Mathematica*, 36(3):255–335, 1978.

[RT18] Erwan Rousseau and Frédéric Touzet. Curves in Hilbert modular varieties. *Asian J. Math*, 22(4):673–690, 2018.

[She20] Xu Shen. On some generalized Rapoport–Zink spaces. *Canadian Journal of Mathematics*, 72(5):1111–1187, 2020.

[VdG88] Gerard Van der Geer. *Hilbert Modular Surfaces*. Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge / A Series of Modern Surveys in Mathematics. Springer Berlin, Heidelberg, 1988.

[VW13] Eva Viehmann and Torsten Wedhorn. Ekedahl-Oort and Newton strata for Shimura varieties of PEL type. *Mathematische Annalen*, 356:1493–1550, 2013.

[Wan23] Weikun Wang. Toric foliated minimal model program. *Journal of Algebra*, 632:70–86, 2023.

[Wed01] Torsten Wedhorn. The dimension of Oort strata of Shimura varieties of PEL-type. In *Moduli of Abelian Varieties*, volume 195 of *Progress in Mathematics*, pages 441 – 471. Birkhäuser, Basel, 2001.

[Woo16] Amy Wooding. *The Ekedahl-Oort stratification of unitary Shimura varieties*. PhD thesis, McGill University, 2016.

[Zha13] Chao Zhang. *G-zips and Ekedahl-Oort strata for Hodge-type Shimura varieties*. PhD thesis, Universiteit Leiden, 2013.

[Zha18] Chao Zhang. Ekedahl-Oort strata for good reductions of Shimura varieties of Hodge type. *Canadian Journal of Mathematics*, 70:451–480, 2018.

[Zin02] Thomas Zink. The display of a formal p-divisible group. In Berthelot P., Fontaine J., Illusie L., Kato K., and Rapoport M., editors, *Cohomologies p-adiques et applications arithmétiques (I)*, number 278 in Astérisque. Société mathématique de France, 2002.