# APPLICATIONS OF BRAUER INDUCTION TO ARTIN $L$-FUNCTIONS

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# CONTENTS

## ABSTRACT

In this thesis we introduce Brauer's induction theorem and its application to Artin's $L$-functions. We also see several zeta functions, including Artin's $L$-functions, and their relations.

## RÉSUMÉ

Dans cette thèse nous introduisons le théorème induction de Brauer et son application aux fonctions $L$ d'Artin. Nous faisons aussi une étude de plusieur fonctions zêta, inclus les fonctions $L$ d'Artin, et leurs relations.

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# 1 Introduction

Many special functions named zeta functions have been defined since the nineteenth century. We will state four key points about them. We will see some of them more precisely in chapter 3.

I) The method of defining zeta-functions
II) The properties of zeta-functions.
Normally, a zeta-function has the following properties.
(1) It is a single-valued meromorphic function.
(2) It is expressed as Dirichlet series.
(3) It has an Euler's infinite product representation.
(4) It satisfies a functional equation. Moreover, it is important to decide the poles, residues and zeros of a zeta-function.
III) The applications of zeta-functions to Number Theory.
IV) The relations between zeta-functions.

Here as an introduction, we will classify the main zeta-functions which have been defined at present.

1) The zeta-functions and the $L$-functions of algebraic number fields:
Riemann zeta-function, Dirichlet zeta-function, Hecke $L$-function, Hecke $L$-function with Grössencharacter $\chi$, Artin $L$-function, Weil $L$-function
2) P-adic $L$-functions
3) The zeta functions of quadratic forms:
Epstein zeta function, the zeta functions of indefinite quadratic form by C. L. Siegel.
4) The zeta functions and $L$-functions of algebras:
Hey zeta function, The zeta functions and $L$-functions of R. Godement and Tsuneo Tamagawa.
5) The zeta functions defined by Hecke operators:
6) The zeta functions and $L$-functions of algebraic varieties defined over finite fields and the zeta functions and $L$ functions of a scheme.
7) Hasse zeta function
8) The zeta functions attached to discontinuous groups:
Selberg zeta function, Eisenstein series by A. Selberg, Godement, and I. M. Gel'fand.
9) Ihara zeta function.

10) The zeta functions of prehomogeneous vector spaces.

These zeta functions or $L$-functions are related to one another.
In this paper, we will prove Brauer's induction theorem and see its application to the Artin $L$-functions and lastly mention the Dedekind conjecture.

## 2 Brauer's Induction Theorem

We will introduce a theorem proved by Brauer in 1947. This theorem has a lot of important applications. We will see one of the applications later. Before starting to state and prove the theorem, we need some preparations.

### 2.1. Preliminaries

Let $Irr(G)$ denote the set of irreducible characters of a finite group $G$.
Suppose that

$$Irr(G) = \{\chi_1, \chi_2, \cdots, \chi_h\},$$

where $h$ is the number of the conjugacy classes of $G$.
We call a linear combination with integer coefficients of $\chi_1, \ldots, \chi_h$
$\sum_{i=1}^{h} a_i\chi_i$ ($a_i \in \mathbf{Z}$) a general character of $G$.
Let $Cl_{\mathbf{Z}}(G)$ denote the set of general characters of $G$.
If $a_i \geq 0$ for $i = 1, \ldots, h$, then $\sum_{i=1}^{h} a_i\chi_i$ is a character of a representation of $G$.
Since $\chi_i\chi_j$ is also a character of $G$, we have

$$\chi_i\chi_j \in Cl_{\mathbf{Z}}(G).$$

Therefore, we see $Cl_{\mathbf{Z}}(G)$ is a ring with identity $\mathbf{1}_G$, where $\mathbf{1}_G$ is a identity character.
We call $Cl_{\mathbf{Z}}(G)$ the character ring of $G$.
Let $Cl(G)$ denote the set of class functions of $G$, and let $\mathbf{C}(G)$ denote the set of complex valued functions on $G$.

**Definition 1** *If a subgroup $E$ of $G$ can be expressed as $E = P \times C$, where $P$ is a p-group with a prime number p, and $C$ is a cyclic group s.t. $(|C|, p) = 1$, then $E$ is called an elementary subgroup of $G$.*

An elementary subgroup is a nilpotent group.
Let $\varepsilon_G$ denote the set of all elementary subgroups of $G$.

Let $H$ be a subgroup of $G$. For $x \in G$ and $f \in \mathbf{C}(H)$, put

$$f^0(x) = \begin{cases} f(x) & if\ x \in H \\ 0 & otherwise \end{cases}$$

Let $\chi$ be a character of a representation of $H$ over $\mathbf{C}$.
Then, we call

$$\chi^G = \frac{1}{|H|} \sum_{y \in G} \chi^0(y^{-1}xy)$$

the induced character of $\chi$. Put

$$R(G, \varepsilon_G) = \{f \in Cl(G) |\ f|_E \in Cl_{\mathbf{Z}}(E)\ for\ \forall E \in \varepsilon_G\}$$

and

$$I(G, \varepsilon_G) = \{f \in Cl(G) |\ f = \sum_i a_i \varphi^G,\ \ \varphi_i \in Irr(E_i), E_i \in \varepsilon_G\}$$

i.e. the set of a linear combination with integer coefficients of the induced character ${\varphi_i}^G$ of an irreducible character $\varphi_i$ of an elementary subgroup $E_i$. Clearly, we have

$$I(G, \varepsilon_G) \subseteq Cl_{\mathbf{Z}}(G) \subseteq R(G, \varepsilon_G).$$

If we define, for $f, g \in R(G, \varepsilon_G)$ *and* $x \in G$,

$$(f+g)(x) = f(x) + g(x)$$

$$(fg)(x) = f(x)g(x)\ ,$$

then $R(G, \varepsilon_G)$ is a commutative ring with identity $\mathbf{1}_G$.

**Lemma 1** *If $H$ and $K$ are subgroups of a finite group $G$, then*

$$(i)\ \ \varphi^G \psi = (\varphi \psi_H)^G\ for\ \varphi \in Cl(H)\ and\ \psi \in Cl(G).$$

$$(ii)\ \varphi^{KG} = \varphi^G\ if\ \ H \subseteq K\ and\ \varphi \in Cl(H).$$

**Proof)** (i)

$$(\varphi\psi_H)^G(x) = \frac{1}{|H|}\sum_{y\in G}\varphi^0(y^{-1}xy)\psi(y^{-1}xy)$$

$$= \psi(x)\frac{1}{|H|}\sum_{y\in G}\varphi^0(y^{-1}xy) = \psi(x)\cdot\varphi^G(x) = \varphi^G\psi(x).$$

(ii) Put $\tau = \varphi^K$. Then,

$$\varphi^{KG}(x) = \tau^G(x) = \frac{1}{|K|}\sum_{y\in G}\tau^0(y^{-1}xy) = \frac{1}{|K||H|}\sum_{y\in G}\sum_{a\in K}\varphi^0(a^{-1}y^{-1}xya)$$

$$= \frac{1}{|K||H|}\sum_{z\in G}|K|\varphi^0(z^{-1}xz) = \frac{1}{|H|}\sum_{z\in G}\varphi^0(z^{-1}xz) = \varphi^G(x).$$

**Lemma 2**

*$I(G,\varepsilon_G)$ is an ideal of $R(G,\varepsilon_G)$.*

**Proof)** Put

$$\varphi = \sum_i a_i{\varphi_i}^G \in I(G,\varepsilon_G),$$

*where $a_i \in \mathbf{Z}$, $\varphi_i \in Irr(E_i)$ and $E_i \in \varepsilon_G$.*

For $\theta \in R(G,\varepsilon_G)$, by Lemma 1 (i) we have

$$\theta\varphi = \sum a_i\theta{\varphi_i}^G = \sum a_i(\varphi_i\theta_{E_i})^G.$$

Since

$$\varphi_i\theta_{E_i} \in Cl_{\mathbf{Z}}(E_i),$$

we have

$$\theta{\varphi_i}^G \in I(G,\varepsilon_G).$$

Then,

$$\theta\varphi \in I(G,\varepsilon_G).$$

Therefore, $I(G,\varepsilon_G)$ is an ideal of $R(G,\varepsilon_G)$.

Let $N$ be a normal subgroup of $G$.

**Definition 2** *We say that $\theta^x \in \mathbf{C}(N)$ is a conjugate of $\theta \in \mathbf{C}(N)$ with respect to $G$ if $\theta^x(y) = \theta(xyx^{-1})$ for $\forall y \in N$.*

We define the inner product on $\mathbf{C}(G)$ by

$$(f,g)_G = \frac{1}{|G|}\sum_{x\in G} f(x)\overline{g(x)} \quad for \quad f,g \in \mathbf{C}(G).$$

There is an important result about the decomposition of $\chi_N$, where $\chi$ is an irreducible character of $G$, namely Clifford's Theorem.

**Theorem 1 (Clifford's Theorem)** *Let $N$ be a normal subgroup of $G$. Let $\chi \in Irr(G)$ and let $\theta \in Irr(N)$ s.t. $\theta$ is a component of $\chi_N$. Let $\theta_1, \ldots, \theta_t$ be conjugates of $\theta$ with respect to $G$. Then, we have*

$$\chi_N = e\sum_{i=1}^{t}\theta_i, \quad e = (\chi_N, \theta)_N$$

We need the following Lemma to show Theorem 1.

**Lemma 3** *Let $G \triangleright N$, $\varphi, \psi \in Cl(N)$, $x \in G$. Then, the following (i)-(iv) hold.*

*(i)* $\varphi^x \in Cl(N)$
*(ii)* $(\varphi^x, \psi^x)_N = (\varphi, \psi)_N$
*(iii) If* $\chi \in Cl(N)$, *then* $(\chi_N, \varphi^x) = (\chi_N, \varphi)_N$
*(iv) If* $\varphi$ *is a character of* $N$, *then* $\varphi^x$ *is a character of* $N$

**Proof)**
$(i)$ For $y$ and $z \in N$, putting $z' = xzx^{-1} \in N$, we have

$$\varphi^x(zyz^{-1}) = \varphi(xzy(xz)^{-1}) = \varphi(z'xyx^{-1}z'^{-1}) = \varphi(xyx^{-1}) = \varphi^x(y)\ .$$

Therefore, $\varphi^x \in Cl(N)$.
$(ii)$ We have

$$(\varphi^x, \psi^x)_N = \frac{1}{|N|}\sum_{y\in N}\varphi^x(y)\overline{\psi^x(y)} = \frac{1}{|N|}\sum_{y\in N}\varphi(xyx^{-1})\overline{\psi(xyx^{-1})}\ldots(*)$$

If y runs over all elements of $N$, then $xyx^{-1}$ runs over all elements of $N$. Therefore,

$$(*) = \frac{1}{|N|}\sum_{y\in N}\varphi(y)\overline{\psi(y)} = (\varphi,\psi)_N.$$

*(iii)* If $\chi \in Cl(G)$, then ${\chi_N}^x = \chi_N$. By *(ii)*, we have

$$(\chi_N, \varphi^x)_N = ({\chi_N}^x, \varphi^x)_N = (\chi_N, \varphi)_N \ .$$

*(iv)* If $\varphi$ is the character of the matrix representation $y \longmapsto R(y)$ for $y \in N$, then $y \longmapsto R(xyx^{-1})$ for $y \in N$ is also a representation of $N$, and the character is $\varphi^x$.

**The proof of Theorem 1)**
Put $e = (\chi_N, \theta)$. By *(iii)* of Lemma 3,

$$e = (\chi_N, \theta_i)_N \quad for \quad i = 1, 2, \ldots, t \ .$$

To show this theorem, it's sufficient to show that the irreducible characters of $N$ whichi are included in $\chi_N$ are $\theta_1, \theta_2, \ldots, \theta_N$ only.
By Frobenius reciprocity law,

$$e = (\chi_N, \theta)_N = (\chi, \theta^G)_G \ .$$

Then,

$$\theta^G = e\chi + \ldots \quad and \quad \theta^G|_N = e\chi_N + \ldots \quad (*) \ .$$

On the other hand, for $g \in \mathbf{C}(H)$, by

$$g^G(x) = \frac{1}{|H|}\sum_{y\in G} g^0(y^{-1}xy) \ ,$$

if $n \in N$, then we have

$$|N|\theta^G(n) = \sum_{x\in G}\theta^0(xnx^{-1}) = \sum_{x\in G}\theta(xnx^{-1}) = \sum_{x\in G}\theta^x(n) \ .$$

Therefore, the irreducible characters of $N$ included in $\theta^G|_N$ are only the conjugates of $\theta$. By $(*)$, the irreducible characters of $N$ included in $\chi_N$ are only the conjugates of $\theta$.

Put

$$I_G(\theta) = \{x \in G \mid \theta^x = \theta\ \}.$$

$I_G(\theta)$ is a subgroup of $G$ s.t. $N \subset I_G(\theta)$.
We call it the inertial group of $\theta$ with respect to $G$. We know that the number of the conjugates of $G$ is $[G : I_G(\theta)]$. We have the following theorem.

**Theorem 2** *Let $G \rhd N$, $Irr(N) \ni \theta$ and $T = I_G(\theta)$. Put*

$$\mathbf{\Theta}_T = \{\psi \in Irr(T)| \quad (\psi, \theta^T) \neq 0\}$$

*and*

$$\mathbf{\Theta}_G = \{\chi \in Irr(G)| \quad (\chi, \theta^G) \neq 0\}$$

*Then, $\psi \longmapsto \psi^G$ is a bijection map from $\mathbf{\Theta}_T$ to $\mathbf{\Theta}_G$. If $N = I_G(\theta)$, then $\theta^G$ is an irreducible character or $G$.*

**Proof)** Let $\chi$ be an irreducible component of $\psi^G$, where $\psi \in \mathbf{\Theta}_T$. By Frobenious reciprocity theorem, $\psi$ is an irreducible component of $\chi_T$.
Let $\theta_1, \theta_2, \ldots, \theta_t$ $(t = [G : T])$ be the set of all conjugates of $\theta$ with respect to $G$.
By Theorem 1, we have

$$\chi_N = e\sum_{i=1}^{t}\theta_i\ ,\ e = (\chi_N, \theta)\ .$$

Since the conjugate of $\theta$ with respect to $T$ is only $\theta$, by Theorem 1, we have

$$\psi_N = f\theta, \quad f = (\psi_N, \theta).$$

Since $\psi$ is an irreducible component, $f \leq e$.
Then, we have

$$et\theta(1) = \chi(1) \leq \psi^G(1) = \psi(1)t = ft\theta(1) \leq et\theta(1)$$

Therefore, these inequalities become equalities, namely,

$$\chi(1) = \psi^G(1) \ldots (*)$$

and

$$(\chi_N, \theta) = e = f = (\psi_N, \theta) \ldots (**)\ .$$

By $(*)$, $\chi = \psi^G$ i.e. $\psi^G$ is an irreducible character of $G$. Since

$$(\psi^G, \theta^G) = (\chi, \theta^G) = (\chi_N, \theta) \neq 0,$$

we have $\psi^G \in \Theta_G$. Then, $\psi \longmapsto \psi^G$ is well defined.
To show that this map is injective, let

$$\varphi^G = \psi^G = \chi, \ where\ \varphi, \psi \in \Theta_T .$$

By $(**)$, we have

$$(\varphi_N, \theta) = (\psi_N, \theta) = (\chi_N, \theta) .$$

If $\varphi \neq \psi$, then

$$(\chi_N, \theta) \geq ((\varphi + \psi)_N, \theta) = 2e .$$

This is a contradiction. Then, $\psi \longmapsto \psi^G$ ($\psi \in \Theta_T$) is an injection.
To show that this map is surjective, let $\chi \in \Theta_{\mathbf{G}}$.
Since $(\chi, \theta^G) = (\chi_N, \theta) \neq 0$, $\exists$ an irreducible component $\psi$ of $\chi_T$ s.t.

$$(\psi, \theta^T) = (\psi_N, \theta) \neq 0 .$$

We have $\psi \in \Theta_T$. By Frobenius reciprocity theorem,

$$(\psi^G, \chi) = (\psi, \chi_T) \neq 0 ,$$

and since $\psi^G \in \Theta_G$, we have $\psi^G = \chi$. Therefore, $\psi \longmapsto \psi^G$ ($\psi \in \Theta_T$) is surjective.

If $\rho$ is an induced representation of 1-dimensional representation of a subgroup of $G$, then we call $\rho$ a monomial representation. If an irreducible character $\rho$ is not an induced character of any proper subgroup of $G$, we call $\rho$ primitive.

**Lemma 4** *Let $G \rhd N$ and let $\chi$ be a primitive character of $G$. Then, the following hold:*

$$(i)\ \exists a \in \mathbf{Z}\ and\ \exists \psi \in Irr(N)\ s.t.\ a > 0\ and\ \chi_N = a\psi$$

$$(ii)\ If\ N\ is\ abelian\ and\ \chi\ is\ injective,\ then\ N \subseteq Z(G).$$

**Proof)**
(i) By Theorem 1 (Clifford's Theorem), we can assume that

$$\chi_N = e\sum_{i=1}^{t}\theta_i \ , \ where\ e = (\chi_N, \theta)_N \ and\ \theta_i \in Irr(N) \ .$$

It is sufficient to show that $t = 1$. By Theorem 2,

$$\exists\psi \in Irr(I_G(\theta_1))\ s.t.\ \chi = \psi^G \ .$$

Since $\chi$ is primitive, we have $G = I_G(\theta_1)$, namely, $t = 1$.
(ii) By (i), we have $\chi_N = e\theta$ where $\theta \in Irr(N)$. Since $N$ is an abelian group, $\theta$ is a character of degree 1. Moreover, since $\chi$ is injective, we have $N \subseteq Z(G)$.

Now we will explain Mackey Decomposition Theorem.
Let $k$ be a field and $k[G]$ a group ring of $G$.
For a subgroup $H$ of $G$ and a left $k[G]$-module $U$, let $U_H$ denote $U$ as a left $k[H]$-module.
Let

$$V^G = k[G] \otimes_{k[H]} V$$

be an induced module of a left $k[H]$-module $V$, and let $(V^G)_K$ be the restriction to $K$ of $V^G$, where $K$ is a subgroup of $G$.
Then, Mackey Decomposition is the direct sum decomposition of $(V^G)_K$ into $k[K]$-modules.
What we need here is a special case of Mackey Decomposition and we will describe it.
If $G$ acts on an finite set $\Omega$, then we have the permutation representation $\rho$.
Let $\chi$ be the character of $\rho$. Then, $\chi(x)$ $(x \in G)$ is the number of the fixed point of $\Omega$ by $x$, namely

$$\chi(x) = \ \sharp\ \{v \in \Omega | x \circ v = v\}$$

Let $\Omega_1, \Omega_2, \ldots, \Omega_t$ be all the orbits of $\Omega$.
Then, $\rho$ is the sum of $\rho_i$ $(i = 1, \ldots, t)$, where $\rho_i$ are the permutation representation from $(G, \Omega_i)$ $(i = 1, \ldots, t)$, and we have $\chi_\rho = \chi_{\rho_1} + \ldots + \chi_{\rho_t}$.
Let $H$ be a subgroup of $G$ and let

$$G = x_1 H + \ldots + x_n H \ ,$$

where $n = [G : H]$, be the left coset decomposition of $G$ by $H$.
If we let $1_H{}^G$ also denote the character of the permutation representation $1_H{}^G$, then we have

$$1_H{}^G(x) = (\textit{ the number of } i \textit{ such that } xx_iH = x_iH\ ).$$

Specially, if $H = \{1\}$, then the character of the regular representation $1^G$ is

$$1^G(x) = \begin{cases} |G| & if\ x = 1 \\ 0 & otherwise \end{cases} \quad \ldots (A)$$

Now, let $K$ be a subgroup of $G$ and we consider the permutation representaion of $K$ by its acting on $G/H$.
Let

$$G = Kx_1H + Kx_2H + \ldots + Kx_tH$$

be the double coset decomposition of $G$ by $K$ and $H$.
For each $i$, the set of left cosets of $H$ which are included in $Kx_iH$ is an orbit of $(K, G/H)$, and the stabilizer of $x_iH$ is $x_iHx_i{}^{-1} \cap K$.
If we put

$$\hat{K_i} = x_iHx_i{}^{-1} \cap K\ ,$$

then K's action on this orbit is isomorphic to $(K, K/\hat{K_i})$. We have

$$1_H{}^G(x) = \sum_{i=1}^{t} 1_{\hat{K_i}}{}^K(x) \quad for \quad x \in K$$

If we let $1_H{}^G|_K$ denote the of $1_H{}^G$ to K, we have

$$1_H{}^G|_K = \sum_{i=1}^{t} 1_{\hat{K_i}}{}^K \ \ldots\ (\#)$$

This is a special case of Mackey Decomposition.

**Proposition 1** *Any irreducible representation of a finite nilpotent group $G$ is a monomial representation.*

**Proof)** First of all, we know that if $G$ is a non-commutative nilpotent group, then $\exists$ a commutative normal subgroup $N$ of $G$ s.t. $N$ is not included in $Z(G)$.
In fact, let $1 \subsetneq Z(G) \subsetneq Z_2(G) \subsetneq \ldots$ be a series and take $a \in Z_2(G) \setminus Z(G)$. Then $< a, Z(G) >$ is a commutative normal subgroup of $G$ and is not included in $Z(G)$.
Let $\chi$ be an irreducible character of $G$ and let $K$ be a subgroup of $G$ s.t. $\exists \psi \in Irr(K) \;\; with \;\; \chi = \psi^G$.
(For example, we can take $G = K$ and $\psi = \chi$.) We let $H$ denote a subgroup with the minimum order among the above $K$'s. Then, $H$ is primitive by Lemma 1 (ii).
Put $\bar{H} = H/Ker\psi$. Since $\psi$ is an injective primitive character of $\bar{H}$, by Lemma 4 (ii), a commutative normal subgroup of $\bar{H}$ is included in $Z(\bar{H})$.
Therefore, $\bar{H}$ is an abelian group and $\psi$ is a 1-dimensional character of $H$.

**Lemma 5** *Suppose that a p-group acts on a finite set $\Omega$ with $(|\Omega|, p) = 1$. Then, there exists a fixed point $a \in \Omega$ s.t. $g \circ a = a \;\; for \;\; \forall g \in G$.*

**Proof)**
Let the orbits of the action be $\Omega_1, \ldots, \Omega_t$. Then,

$$|\Omega| = |\Omega_1| + \ldots + |\Omega_t|.$$

Since the order of each orbit is a divisor of $|G|$ and $G$ is a $p$-group, $|\Omega_i|$ is a multiple of $p$ or 1.
If $|\Omega_i| \neq 1$ for $i = 1, \ldots, t$, then $(|\Omega|, p) \geq p$. This is a contradiction.
Therefore, $\exists i_0 \in \{1, \ldots, t\}$ s.t. $|\Omega_{i_0}| = 1$. Namely,

$$\exists a \in \Omega \;\; s.\; t. \;\; g \circ a = a \;\; for \;\; \forall g \in G.$$

## 2.2. Brauer's Induction Theorem

Now we will introduce the Theorem.

**Theorem 3 (Brauer)** *(i) $Cl_{\mathbf{Z}}(G) = R(G, \varepsilon_G)$*
*(ii) Any character of $G$ can be expressed by a linear combination with integer coefficients of 1-dimensional induced characters of some elementary subgroups.*

To prove Theorem 3, we need to prove Theorem 4 first. Theorem 4 is as follows.

**Theorem 4**

$$I(G,\varepsilon_G) \ni 1_G$$

We claim that Theorem 4 implies Theorem 3. As for (i) in Theorem 3, it's clear by Lemma 2. We will show that (ii) is also true under Theorem 4.
Since $I(G,\varepsilon_G) \subseteq Cl_{\mathbf{Z}}(G) \subseteq R(G,\varepsilon_G)$ *and* $I(G,\varepsilon_G) = R(G,\varepsilon_G)$ ,
we have

$$Cl_{\mathbf{Z}}(G) = R(G,\varepsilon_G).$$

Therefore we can express any character $\chi$ of $G$ by $\Sigma a_i {\varphi_i}^G$ ($a_i \in \mathbf{Z}$) for $\varphi_i \in Irr(E_i)$, where $E_i \in \varepsilon_G$.
Since $E_i$ is nilpotent, by Proposition 1, $\varphi_i$ is an induced character of a 1-dimensional character $\lambda_i$ of a subgroup $F_i$ of $E_i$ i.e. $\varphi_i = {\lambda_i}^{E_i}$.
Then,

$$\chi = \Sigma a_i {\varphi_i}^G = \Sigma a_i ({\lambda_i}^{E_i})^{G_i} = \Sigma a_i {\lambda_i}^G$$

by Lemma 1.
Therefore, Theorem 4 implies Theorem 3.

Now we will prove Theorem 4. At first, Theorem 4 was proved by Brauer in 1947, but it was complicated. Later, it was differently proved by Brauer and Tate in [BT], and usually this proof has been introduced in many books. Here we will introduce other proof by Issacs in [I].
First, we need some preparations and a theorem.

Let $F$ be a subgroup of $G$.

**Definition 3** *We say that $F$ is a p-quasi-elementary subgroup for the prime $p$ if*

$$F = PC \rhd C \text{ and } P \cap C = 1,$$

*where $P$ is a p-group and $C$ is an cyclic group such that $(|C|, p) = 1$.*

A subgroup of a p-quasi-elementary subgroup is also a p-quasi-elementary subgroup. Clearly, an elementary subgroup is also quasi-elementary.

Let $\varepsilon_{G'}$ denote the set of quasi-elementary subgroups of $G$. We put

$$P(G) = \{\sum_i a_i 1_{H_i}{}^G | a_i \in \mathbf{Z},\ H_i \ is\ a\ subgroup\ of\ G\}$$

$$P(G, \varepsilon_{G'}) = \{\sum_i a_i 1_{F_i}{}^G | a_i \in \mathbf{Z}, F_i \in \varepsilon_{G'}\}$$

**Theorem 5 (L.Solomon)** $P(G) = P(G, \varepsilon_{G'})$

**Proof)**
We will prove this theorem with three steps $(\alpha)$,$(\beta)$ and $(\gamma)$.

$(\alpha)$ *$P(G)$ is a ring and $P(G, \varepsilon_{G'})$ is an ideal of $P(G)$.*
**(Proof)** Let $H$ and $K$ be subgroups of $G$ and put $1_K{}^G = \theta$.
By Lemma 1, we have

$$1_H{}^G \cdot 1_K{}^G = \theta \cdot 1_H{}^G = \theta_H{}^G.$$

On the other hand, by (#),

$$\theta_H = 1_K{}^G|_H = \sum_i 1_{\hat{K}_i}{}^H\ ,\ where\ \hat{K}_i = x_i K x_i{}^{-1} \cap H \subseteq H.$$

Then, by Lemma 3, we have

$$\theta_H{}^G = \sum_i (1_{\hat{K}_i}{}^H)^G = \sum_i 1_{\hat{K}_i}{}^G \in P(G).$$

Therefore $P(G)$ is a ring.
If we take an element of $\varepsilon_{G'}$ as $H$, since $\hat{K}_i \in \varepsilon_{G'}$, then $P(G, \varepsilon_{G'})$ is an ideal of $P(G)$.

$(\beta)$ *Let $G \ni x$ and let $p$ be a prime. Then, $\exists F \in \varepsilon_{G'}$ s.t. $p$ does not divide $1_F{}^G(x)$.*
**(Proof)**
Put $< x >= P \times C$, where $P$ is a p-group and $p$ does not divide $|C|$.
Putting $N = N_G(C)$, we have $x \in N$.
We take a Sylow p-subgroup $F/C$ of $N/C$ such that $x \in F$. $F$ is a p-quasi-elementary subgroup. We will show that $p$ doesn't divide $1_F{}^G(x)$. We know that

$$1_F{}^G(x) = (the\ number\ of\ the\ left\ cosets\ yF\ s.t.\ xyF = yF).$$

Then, if $xyF = yF$, since $y^{-1}xy \in F$, we have $y^{-1}Cy \subseteq F$.
Since $F$ is a p-quasi-elementary subgroup, $y^{-1}Cy = C$, namely, $y \in N$. This implies that

$$1_F{}^G(x) = \textit{ the number of the fixed points of the action of } x \textit{ on } N/F \ .$$

Since $N \rhd C$ and $C$ fixes each element of $N/F$,
$C \subseteq$ *the kernel of the action of* $N$ *on* $N/F$ .
Since $< x > /C$ is a p-group, we have

$$[N : F] \equiv 1_F{}^G(x)(\ mod\ p)$$

by considering the decomposition to the orbits of the action of $x$ on $N/F$.
Since $p$ does not divide $[N : F]$, $p$ does not divide $1_F{}^G(x)$.

$(\gamma)$

$$1_G \in P(G, \varepsilon_{G'})$$

**(Proof)** We put

$$I_x = \{f(x) | f \in P(G, \varepsilon_{G'})\} \ for\ x \in G.$$

$I_x$ is a subgroup of $\mathbf{Z}$.
We will prove that

$$I_x = \mathbf{Z} \ \ for \ \ \forall x \in G \quad (*).$$

If, for some $x \in G$, $I_x =< m >$, where $1 \neq m \in \mathbf{Z}$, then we take a prime $p$ such that $p|m$ and we have

$$p|f(x) \ \ for \ \ \forall f \in P(G, \varepsilon_{G'}) \ .$$

But if we take $F \in \varepsilon_{G'}$ as in $(\beta)$, then

$$p \textit{ does not divide } 1_F{}^G(x) \ \ and \ \ 1_F{}^G \in P(G, \varepsilon_{G'}).$$

This is a contradiction.
Therefore, $(*)$ holds. Then, there exists

$$f_x \in P(G, \varepsilon_{G'}) \ \ s.t. \ \ f_x(x) = 1 \ \ for\ \forall x \in G.$$

By expanding

$$\prod_{x \in G} (f_x - 1_G) = 0 \ ,$$

we have $1_G \in P(G, \varepsilon_{G'})$.
Then, by $(\alpha)$ and $(\gamma)$, we have $P(G, \varepsilon_{G'}) = P(G)$.

Now we will prove Theorem 4.
**Proof of Theorem 4)**

It's obvious that if $G$ is an elementary subgroup then Theorem 4 holds.
**(I)** Suppose that $G$ is quasi-elementary subgroup.
Put $G = PC \rhd C$, where $P$ is a p-group and $C$ is a cyclic group with $(|C|, p) = 1$.
We also put $Z = C_c(P)$ and $E = ZP$. We have

$$G \rhd Z \ \ and \ \ E \in \varepsilon_G.$$

Since we assume that $G$ is not an elementary subgroup, we have $G \neq E$.
Let $\chi$ be an irreducible component of $\varphi$, where ${1_E}^G = 1_G + \varphi$. If we prove that

$$\chi \in I(G, \varepsilon_G),$$

then

$$1_G = {1_E}^G - \varphi \in I(G, \varepsilon_G),$$

which is the conclusion.
Now we need to show that $\chi \in I(G, \varepsilon_G)$.
Let $\lambda$ be an irreducible components of $\varphi|_C$.
First we will show that

$$\lambda \neq 1_c \ \ and \ \ Z \subseteq Ker\lambda. \quad (*)$$

Since $G = CE$ and $C \cap E = Z$, by the special case (#) of Mackey decomposition, we have

$$1_C + \varphi_C = ({1_E}^G)_C = {1_Z}^C \ .$$

${1_Z}^C$ is a regular representation of $C/Z$ and, by (A) in p.12,

$$({1_Z}^C(x), 1_C) = 1_C(1) = 1,$$

and $1_Z{}^C$ includes $1_c$ with multiplicity 1.
Therefore, $\varphi_C$ doesn't include $1_C$ and $\lambda \neq 1_C$.
Since $E \supseteq Z \lhd G$, we have

$$Z \subseteq Ker((1_E{}^G)_C) \ \ and \ \ Z \subseteq Ker\lambda.$$

Next, we will show that

$$G \neq I_G(\lambda) \quad (**).$$

Put $T = I_G(\lambda)$ and $K = Ker\lambda$. Suppose that $G = T$.
Then, we claim that $xK (x \in C)$ is $P$-invariant i.e.

$$y^{-1}(xK)y = xK \quad for \ \ \forall y \in P.$$

The reason is as follows.
$\lambda$ is a character of degree 1 and

$$\lambda(x_1K) \neq \lambda(x_2K) \ \ if \ \ x_1K \neq x_2K \ .$$

Since $G = T$,

$$\lambda^y = \lambda \ for \ \forall y \in P, \text{ i.e., } \lambda(y^{-1}xKy) = \lambda(xK).$$

Therefore,

$$y^{-1}xKy = xK \ \ ,$$

i.e., $xK$ is $P$-invariant.
Consider the action of $P$ by conjugation on $xK$. Then, $P$ has a fixed point on $xK$ by Lemma 5, and

$$Z \cap xK = C_c(P) \cap xK \neq \phi.$$

Since $Z \subseteq K$, we have $x \in K$.
Since $x$ is an arbitrary element of $C$, we have $K = C$ and $\lambda = 1_c$. This is a contradiction to $(*)$. Then, $(**)$ holds.
By Theorem 2,

$$\exists \psi \in Irr(T) \ s.t. \ \chi = \psi^G.$$

By $(**)$, $|T| < |G|$.
Then, we may assume that

$$\psi \in I(T, \varepsilon_T)$$

by the mathematical induction with respect to the order of $G$.
By Lemma 1 (ii), we have

$$\chi = \psi^G \in I(G, \varepsilon_G).$$

**(II)** In the case that G is not p-quasi elementary subgroup, we will prove the desired result by the mathematical induction with respect to the order of $G$.
If $H$ is a proper subgroup of G, we may asssume that

$$1_H \in I(H, \varepsilon_H).$$

In particular,

$$1_F \in I(F, \varepsilon_F) \ for \ \forall F \in \varepsilon_{G'} \quad (***) \ .$$

By Theorem 5, we have

$$1_G = \sum a_i 1_{F_i}{}^{G_i} \ \ for \ \ F_i \in \varepsilon_{G'} \ \ and \ \ a_i \in \mathbf{Z}.$$

By $(***)$ and Lemma 1, we have

$$1_{F_i}{}^G \in I(G, \varepsilon_G).$$

Therefore,

$$1_G \in I(G, \varepsilon_G).$$

Theorem 3 has very important applications. The following are two of the most important applications.

(i) Every Artin's $L$ function extends to a meromorphic function on the whole complex plane.
(ii)For every representation of a finite group $G$ on $\mathbf{C}$, there exist a matrix representation such that the components of the matrix can be included in $\mathbf{Q}(\zeta_n)$, where $n = |G|$ and $\zeta_n$ is the n-th root of unity.

In the next chapter, we will see (i).

# 3 Artin's $L$-function and Artin's conjecture

## 3.1 The Riemann zeta function and Hecke $L$-functions

As we see in chapter 1, we have a lot of zeta functions, which are related to one another. In this chapter, we will introduce three of them, namely Riemann zeta function, Hecke $L$-function and Artin's $L$-function. Moreover, in the next chapter, we will introduce one more zeta function, which is Dedekind's zeta function.

In the eighteenth century, Euler had already known that the convergent series with a real variable $s > 1$

$$\zeta(s) = 1 + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \ldots + \frac{1}{n^s} + \ldots$$

can be written as an infinite product

$$\prod_p (1 - p^{-s})^{-1}\ , \textit{ where } p \textit{ runs over all primes.}$$

This is called Euler's infinite product representation or just Euler product. About one hundred years after Euler, the foundation of function theory had been almost created by Cauchy and other mathematicians, and Riemman took this advantage and studied $\zeta(s)$ with $s$ as a complex variable.
We will state some properties of the Riemann zeta function $\zeta(s)$.
$(i)$

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

has the abscissa of convergence 1. It converges uniformly and absolutely in $Re(s) > 1 + \delta$ for every $\delta > 0$.
$(ii)$ For $Re(s) > 1$, we have

$$\zeta(s) = \prod_p \frac{1}{1 - p^{-s}} \quad (\textit{Euler product})$$

where $p$ runs over all prime numbers.
$(iii)$ $\zeta(s)$ has a meromorphic continuation to the whole complex plane and

satisfies a functional equation, namely if $\Gamma(s)$ is the gamma function, then the function

$$\xi(s) = \pi^{-\frac{s}{2}}\Gamma(\frac{s}{2})\zeta(s)$$

is meromorphic in the entire $s$-plane, holomorphic except for simple poles at $s = 0$ and $s = 1$, and we have the functional equation

$$\xi(s) = \xi(1-s) \ .$$

(*iv*) $\zeta(s)$ has no zero for $Re(s) \geq 1$, and $\zeta(s)$ has zeros with order 1 at $s = -2, -4, \ldots, -2n, \ldots$ for $Re(s) \leq 0$.
For $0 < Re(s) < 1$, $\zeta(s)$ has infinitely many zeros, which are called non-trivial zeros.
We have the famous hypothesis:

**(Riemann Hypothesis)** If $s$ is a non-trivial zero of $\zeta(s)$, then

$$Re(s) = \frac{1}{2}.$$

In the case of an algebraic number field, we obtain the generalization of the Riemann zeta function, namely Dedekind zeta function. We will see it in Chapter 4.

Now we will go on to Hecke $L$-functions.

Let $m_0$ be an ideal of the ring $\theta$ of integers of an algebraic number field $k$ of finite degree. Let $m_\infty$ be a product of some real infinite primes. Then, for $m = m_0 m_\infty$, we put $a \equiv 1(\text{mod } m)$ if $a \in k$ satisfies the two following conditions.
(1) If $m_0 = \wp^{e_1} \ldots \wp^{e_t}$, then we have

$$(a, \wp_i) = 1 \ and \ a - 1 \in {\wp_i}^{e_i} \ for \ \forall i = 1, \ldots, t \ .$$

(2) If $a$ is embedded into $\mathbf{R}$ by using $\wp_\infty$, which is a component of $m_\infty$, then $\wp_\infty(a) > 0$.

Let $I(m)$ be the set of the fractional ideals of $k$ which is prime to $m_0$.
Put

$$S(m) = \{a \in k | a \equiv 1 \ (mod \ m)\} \ .$$

Then, $S(m) \subseteq I(m)$ and $|I(m)/S(m)| < \infty$ by a theorem of Minkowski. We call $I(m)/S(m)$ an ideal group mod m. Let $\chi$ be a group character of $I(m)/S(m)$ and we extend $\chi$ for any integral ideal $\alpha \subset I_k$ by defining

$$\chi(\alpha) = \begin{cases} \chi(C) & \alpha \in C \quad (C \in I(m)/S(m)) \\ 0 & (\alpha, m) \neq I_k \end{cases}$$

We call it a Hecke class character.

**Definition 4** *The Hecke L-series is defined as*

$$L_k(s, \chi) = \sum_{\alpha} \frac{\chi(\alpha)}{(N\alpha)^s} \quad (\alpha \textit{ runs all ideals of } \theta)$$

The Hecke $L$-series converges on $Re(s) > 1$ and has the Euler product

$$L_k(s, \chi) = \prod_{\wp} \frac{1}{1 - \chi(\wp)/N(\wp)^s} \quad (\wp \textit{ runs all prime ideals})$$

Hecke showed $L_k(s, \chi)$ extends to an entire function and satisfies a functional equation like $\zeta(s)$.

### 3.2 Artin's $L$-function and Artin's conjecture

We will define the Artin $L$-functions, which are associated to Galois extensions.

Let $L/K$ be a Galois extension with group $G$ and let $(\rho, V)$ be a representation of $G$. For every ideal $\alpha$ of the ring $\theta_K$ of the intergers of $K$. We denote by $N(\alpha) = \sharp\theta_K/\alpha$ the absolute norm of $\alpha$. We denote the action of $\sigma \in G$ on $v \in V$ by $\sigma v$. Let $\wp$ be a prime ideal of $K$ and let $\beta$ a prime ideal of $L$ lying above $\wp$. Let $G_\beta$ be the decomposition group and $T_\beta$ the inertia group of $\beta/\wp$. Then the factor group $G_\beta/T_\beta$ is generated by the Frobenius automorphism $\varphi_\beta$. $\varphi_\beta$ is an endomorphism of the fixed module $V^{T_\beta}$. The characteristic polynomial

$$det(1 - \varphi_\beta t; V^{T_\beta})$$

depends only on the prime $\wp$. The determinant depends only on the character $\chi$ of $\rho$, since two representations with the same character are equivalent. Then, we have the following definition.

**Definition 5** *Let $L/K$ be a Galois extension with group $G$ and let $(\rho, V)$ be a representation of $G$ with character $\chi$. Then the Artin $L$-function of $\rho$ or $\chi$ is defined by*

$$L(s, \chi, L/K) = \prod_{\wp} \frac{1}{det(1 - \varphi_\beta N(\wp)^{-s}; V^{T_\beta})}$$

For every $\delta > 0$, the Artin $L$-function converges absolutely and uniformly on the half plane $Re \geq 1 + \delta$ by comparison with the ordinary zeta function.

The Artin $L$-functions have the following formalism.

(1) If $K/k$ is an abelian extension and $\chi$ is of degree 1, then by class field theory $\chi(\wp)$ coincides with a class character of $k$ modulo a conductor of $K/k$ and the Artin's $L$-functions coincides with the Hecke's $L$-functions.

(2) If $L' \supseteq L \supseteq K$ is a bigger Galois extension, and $\chi$ is a character of $G(L/K)$, also viewed as character of $G(L'/K)$, then

$$L(s, \chi, L/K) = L(s, \chi, L'/K)$$

(3) If $F$ is an intermediate field, and let $\chi$ be a character of $G(L/F)$. Let $\chi_\psi$ be the induced character of $G(L/K)$. Then

$$L(s, \psi, L/F) = L(s, \chi_\psi, L/K)$$

(4) We have $L(s, 1, L/K) = \zeta_K(S)$, which is the Dedekind zeta function.

(5) If $\chi_1$, $\chi_2$ are characters of G, then

$$L(s, \chi_1 + \chi_2, L/K) = L(s, \chi_1, L/K) L(s, \chi_2, L/K)$$

It is easy to prove (4) and (5).

Let $\rho$ be the trivial representation $\rho : G \longmapsto GL(C)$ defined by $\rho(\sigma) \equiv 1$. Then, we have

$$det(1 - \varphi_\beta N(\wp)^{-s}; C) = 1 - N(\wp)^{-s} .$$

Therefore

$$L(s, 1, L/K) = \zeta_K(s) .$$

Next, let $(\rho_1, V_1)$ and $(\rho_2, V_2)$ are representations of $G(L/K)$ with character $\chi_1$ and $\chi_2$.

Then, we have $(\rho_1 \oplus \rho_2, V_1 \oplus V_2)$ is a representation with character $\chi_1 + \chi_2$ and

$$det(1 - \varphi_\beta t; V_1 \oplus V_2) = det(1 - \varphi_\beta t; V_1) \cdot det(1 - \varphi_\beta t; V_2) \ .$$

Therefore,

$$L(s, \chi_1 + \chi_2, L/K) = L(s, \chi_1, L/K) \cdot L(s, \chi_2, L/K) \ .$$

The brilliant thing about Artin's discovery is that it is not a generalization purely for the sake of generalization.
Just as the Riemann zeta function is intimately connected to the distribution of prime numbers, the Artin $L$-functions are connected to subtler questions about prime numbers. For example, how often is 2 a primitive root $(mod\, p)$? Are there infinitely many such primes $p$?
These questions can only be answered by looking at the Artin $L$-function of the Galois extension $\mathbf{Q}(\zeta_m, \sqrt[m]{2})$.
In striking contrast to the work of Dirichlet and Hecke, Artin was not able to show his new $L$-functions extend to entire functions. It is still a major problem of number theory that the Artin $L$-function $L(s, \chi, L/K)$ attached to an irreducible character $\chi \neq 1$ of the Galois group extends to an entire function. This is the famous Artin's conjecture.

**Artin's conjecture**
$L(s, \chi, K/k)$ extends to an entire function if $\chi$ is irreducible and $\chi \neq 1$.

Therefore, it was definitely a major breakthrough when Brauer showed using group theory that every Artin $L$-function extends to a meromorphic function. That is the application of Brauer's induction theorem that we mentioned in Chapter 2.

**Theorem 6** *The Artin's L-function extends to a meromorphic function defined on the whole complex plane.*

**Proof)** By Brauer's induction theorem, we have $\chi = \Sigma m_i \chi_{\psi_i}$ $(m_i \in \mathbf{Z})$, where $\psi_i$ $(i = 1, 2, \ldots)$ is 1-dimensional character of subgroups $C \times P$ of G. Then, by (2),(4) of the formalism of the Artin's $L$-functions, $L(s, \chi, K/k)$ is expressed by a product of Hecke $L$-functions $L_{\Omega_i}(s, \psi_i)$, namely

$$L(s, \chi, K/k) = \prod_i L_{\Omega_i}(s, \psi_i)^{m_i} \ (m_i \in \mathbf{Z}).$$

The theorem now follows since each Hecke $L$-function is entire.

When $\chi$ is 1-dimensional, Artin showed that his $L$-function extends to an entire function. He did this by showing that in fact in this case his $L$-function coincides with a Hecke $L$-function $L(s, \psi)$ for some character $\psi$ of the ideal class group. This is called the Artin reciprocity law and we mentioned this as (1) in the formalism of the Artin's $L$-functions.

# 4 Dedekind's conjecture

The starting point of Artin's analytical investigations was the question whether the Dedekind's conjecture is true or not. To state the Dedekind's conjecture, we will start this chapter by defining the Dedekind zeta function.
As we mentioned in chapter 3, Dedekind zeta function is a generalization of Riemann zeta function.

**Definition 6** *Let $K$ be an algebraic number field. For every ideal $\alpha$ of the ring $\theta_k$ of integers of $K$, we denote by $N(\alpha) = \sharp\theta_K/\alpha$ the absolute norm of $\alpha$. Then, the Dedekind zeta function of $K$ is defined by*

$$\zeta_K(s) = \sum_{\alpha} \frac{1}{N(\alpha)^s}$$

*where $\alpha$ runs overall integral ideals of $K$.*

The Dedekind's zeta function has the following properties similar to the Riemann zeta function.
(1) $\zeta_K(s)$ converges uniformly and absolutely in $Re(s) \geq 1 + \delta$ for every $\delta > 0$.
(2) $\zeta_K(s)$ has an analytic continuation to $Re(s) \geq 1 - \frac{1}{N}$, $N = [K : \mathbf{Q}]$ except for a simple pole at $s = 1$ with residue

$$\lim_{s \to 1+0} (s-1)\zeta_K(s) = \kappa \cdot h \ \ \textit{with} \ \ \kappa = \frac{2^{r_1}(2\pi)^{r_2} R}{m \cdot \sqrt{D}},$$

where $h$ is the class number of $K$, $r_1$ is the number of the real and $r_2$ the number of complex places of $K$, $R$ is the regulator, $m$ the number of roots of unity in $K$ and $D$ the discriminant.
(3) For $Re(s) > 1$, we have the Euler product

$$\zeta_K(s) = \prod_{\wp} \frac{1}{1 - N(\wp)^{-s}},$$

where the product is taken over all prime ideals of $K$.
(4) $\zeta_K(s)$ has a meromorphic continuation to the whole complex plane and satisfies a functional equation, namely putting

$$G_1(s) = \pi^{-\frac{s}{2}}\Gamma(\frac{s}{2}) \ \ \textit{and} \ \ G_2(s) = (2\pi)^{-s}\Gamma(s),$$

the function

$$\zeta_K(s) = G_1(s)^{r_1} G_2(s)^{r_2} \zeta_K(s)$$

is meromorphic in the entire $s$-plane, holomorphic except for simple poles at $s = 0$ *and* $s = 1$, and satisfies the functional equation

$$\xi_K(s) = |D|^{\frac{1}{2}-s} \xi_K(1-s).$$

Now we will introduce the Dedekind conjecture.

**The Dedekind conjecture:**
Let $K$ be an algebraic number field over $k$. Then, $\zeta_K(s)/\zeta_k(s)$ is entire.

The Dedekind conjecture follows from the Artin's conjecture in Chapter 3. We will see it. By the formalism of the Artin's $L$-functions, which we have mentioned Chapter 3, we will have the following theorem.

**Theorem 7** *Let $\hat{K}/k$ be its normal closure with $G = Gal(\hat{K}/k)$ and $H = Gal(\hat{K}/K)$. Let $\chi_1$ be the character of $G(\hat{K}/K)$ induced by the principal character $1_H$ of $H$, and let $\chi_1 = \sum_\alpha \gamma_\alpha \chi_\alpha (0 \leq \gamma_\alpha \in \mathbf{Z})$ be its decomposition into the irreducible characters $\chi_\alpha$. Then,*

$$\zeta_K(s) = \zeta_k(s) \cdot \prod_{\chi_\alpha \neq 1} L(s, \chi_\alpha, L/k)^{r_\alpha}$$

**Proof)** By (3) in the formalism of Artin $L$-function in Chapter 3, we have

$$L(s, \psi, \hat{K}/K) = L(s, \chi_1, \hat{K}/k).$$

On the other hand, by (4) and (5) in the formalism of Artin $L$-function,

$$L(s, \psi, \hat{K}/K) = \zeta_K(s)$$

and

$$L(s, \sum_\alpha \gamma_\alpha \chi_\alpha, \hat{K}/k) = \zeta_k(s) \prod_{\chi_\alpha \neq 1} L(s, \chi_\alpha, \hat{K}/k)^{\gamma_\alpha} .$$

Therefore, we have

$$\zeta_K(s) = \zeta_k(s) \prod_{\chi_\alpha \neq 1} L(s, \chi_\alpha, \hat{K}/k)^{\gamma_\alpha} .$$

By this theorem, if the Artin's conjecture is true, then the Dedekind's conjecture is true. Moreover, we have a theorem which has been proved by Aramata in 1933 and by Brauer in 1947, that is

**Theorem 8** *Let $K/k$ be a Galois extention. Then, the Dedekind's conjecture is true.*

**Proof)**
By the formalism of Artin's L-functions described in Chapter 3, the theorem follows from the following proposition.

**Proposition 2** *Let $1^G$ be the character of the regular representation of $G$. Then, there are subgroups $\{H_i\}$ of $G$, 1-dimentional character $\psi_i$ of $H_i$ and $0 \leq m_i \in \mathbf{Z}$ so that*

$$1^G - 1_G = \sum m_i {\psi_i}^G$$

**Proof)**
Since $1^G = {1_{\{e\}}}^G$, by Frobenius reciprocity,

$$(1^G, 1_G) = ({1_{\{e\}}}^G, 1_G) = (1_{\{e\}}, 1_G|_{\{e\}}) = 1.$$

For any cyclic subgroup $A$, define $\theta_A : A \longmapsto \mathbf{C}$ by

$$\theta_A(\sigma) = \begin{cases} |A| & if \ <\sigma>= A \\ 0 & otherwise \end{cases}$$

and

$$\lambda_A = \phi(|A|)1^A - \theta_A.$$

Thus,

$$\lambda_A(\sigma) = \begin{cases} \phi(|A|)|A| & if \ \sigma = 1 \\ -\theta_A(\sigma) & if \ \sigma \neq 1. \end{cases}$$

This proposition will be proved in two steps.

**Step 1**
$\lambda_A = \sum m_\chi \chi$ with $m_\chi \geq 0$, $m_\chi \in \mathbf{Z}$ and $\chi$ runs over the characters of $A$.
**(proof)** It is enough to show that $(\lambda_A, \chi) \geq 0$ for any irreducible character $\chi$ of $A$. But,

$$(\lambda_A, \chi) = \phi(|A|) - (\theta_A, \chi) = \phi(|A|) - \sum_{\sigma \in A, <\sigma>=A} \chi(\sigma)$$

$$= \sum_{\sigma \in A, <\sigma>=A} (1 - \chi(\sigma)) = Tr(1 - \chi(\sigma)) \in \mathbf{Z} \ \ \textit{for any generator } \sigma \textit{ of } A.$$

Now for $\chi \neq 1$,

$$Re(1-\chi(\sigma)) > 0 \; if \; \sigma \neq e \; and \; Re(1-\chi(\sigma)) = 0 \; if \; \sigma = e.$$

Then, if $A \neq \{1\}$, $(\lambda_A, \chi)$ is positive for all $\chi \neq 1$ and $(\lambda_A, \chi) = 0$ if $\chi = 1$. If $A = \{1\}$, then $\lambda_A = 0$.

**Step 2**

$$1^G - 1_G = \frac{1}{|G|}\sum_A {\lambda_A}^G,$$

where the sum is over all cyclic subgroups $A$ of $G$.
**(Proof)** It is enough to show that for any irreducible character $\psi$ of $G$, both sides have the same inner product with $\psi$. Now,

$$(|G|(1^G - 1_G), \psi) = \sum (1^G - 1_G)(g)\overline{\psi(g)} = |G|\psi(1) - \sum_{g \in G} \psi(g).$$

Also, by Frobenius reciprocity,

$$\sum_A ({\lambda_A}^G, \psi) = \sum_A (\lambda_A, \psi|_A)$$

$$= \sum_A \{\phi(|A|)\psi(1) - \sum_{\sigma \in A, <\sigma>=A} \psi(\sigma)\} = \psi(1)\sum_A \phi(|A|) - \sum_{\sigma \in G} \psi(\sigma).$$

Now,

$$\sum_A \phi(|A|) = \sum_A \sum_{\sigma \in A, <\sigma>=A} 1 = \sum_{\sigma \in G} 1 = |G|.$$

This completes step 2 and the proof of Proposition 2.

Lastly, we will introduce the Langlands-Tunnell theorem because that is now the limit of our knowledge on Artin's conjecture at present.
For certain classes of groups, like monomial groups, we can again by group theory establish the Artin's conjecture. But the decisive next step was taken by Langlands. He showed that if the Galois group is solvable and $\chi$ is 2-dimensional, then $L(s, \chi, L/K)$ extends to an entire function and in fact is the $L$-function attached to a modular form. This should be seen as a 2-dimensional version of Artin's reciprocity law.

**Theorem 9** *A two-dimensional representation*

$$\rho : Gal(\overline{\mathbf{Q}}/\mathbf{Q}) \longmapsto GL_2(\mathbf{C})$$

*whose image in* $PGL_2(\mathbf{C})$ *is* $A_4$ *arises from a modular form of weight* 1.

We will stress that this theorem is the starting point of Wiles' proof of Fermat's Last Theorem. Some experts believe that the work of Wiles contains some new ideas that may be perhaps used to show that every 2-dimensional Artin $L$-function extends to an entire function.

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