SOME PROPERTIES OF THREE-WAY LAYOUTS

by

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Abstract

We consider various properties of the general three-way layout in experimental design, and begin with the information or C-matrices that play a key role, particularly with row-column and two-way elimination of heterogeneity designs. We introduce a new class of three-way layouts that satisfy a certain "generalized" decomposability property and obtain several new results for such layouts. Four different types of canonical correlations and associated canonical efficiency factors are considered and their connection with connectedness and orthogonality examined. We obtain a new inequality involving the average efficiency factors of a two-way elimination of heterogeneity design and of its two subdesigns. The concepts of variance and efficiency balance are characterized, while that of general balance is studied in the context of row-column designs; we point out the correspondence between general balance and commutativity of the efficiency matrices in two-way elimination of heterogeneity designs.

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Résumé

Nous considérons un nombre de propriétés des plans à classification triple. En premier lieu, nous décrivons les matrices soi-disant d'information ou C qui jouent un rôle clé, particulièrement chez les plans "row-column" et les plans à classification double avec élimination de l'hétérogenéité. Nous présentons une nouvelle classe de plans à classification triple qui satisfont une certaine propriété de décomposabilité "généralisée" et nous obtenons plusieurs nouveaux résultats pour de tels plans à classification triple. Nous considérons quatre genres différents de corrélations canoniques et facteurs d'efficacité canoniques associés et examinons leur rapport avec les propriétés de connectivité et d'orthogonalité. Nous obtenons une inégalité nouvelle, impliquant les facteurs d'efficacité moyens d'un plan à classification double avec élimination de l'hétérogenéité et de ses deux sous-plans. Nous caractérisons les concepts d'équilibre-variance et d'équilibre-efficacité tandis que le concept d'équilibre general est étudié dans le contexte des plans "row-column"; nous indiquons la correspondance entre équilibre general et la commutativité des matrices d'efficacité pour les plans à classification double avec élimination de l'hétérogenéité.

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TABLE OF CONTENTS

CHAPTER 1. INTRODUCTION

SECTION 1.1: SUMMARY

The three-way layout, in one form or another, has been used and studied for many years. In experimental design comparing several treatments the three-way layout has often been employed to remove two non-interacting sources of variation, e.g., the treatments may be arranged in the form of a grid or two sets of non-interacting treatments may be applied to a block design. The simplest form of such designs, the "Latin Squares", has been discussed for over two hundred years while more elaborate designs have been studied for at least the last sixty years.

In Chapter 2, we introduce the general three-way layout as a fixed-effects model with no interaction; a number of more specific models are also mentioned, e.g., row-column designs, two-way elimination of heterogeneity. In solving the normal equations for the factor estimates of interest, a very important role is played by the so-called information or C-matrices; we give a description of the different information matrices in Section 2.2. We stress the significance of the relationships between the information matrix for the factor of interest (usually treatments) in the whole layout and the information matrices for the subdesigns obtained by ignoring factors which are considered nuisance parameters (usually rows and columns). More specifically, we look at a class of designs, apparently new, for which the information matrix for the whole design decomposes into a linear function of the information matrices for the subdesigns. We compare our decomposition with the less general one introduced by Baksalary and Shah (1990) and the alternate decomposition introduced by Baksalary and Siatkowski (1990). These decompositions are of interest because they simplify the analysis of the design and make the relationships between properties of the full design and of its subdesigns easier to identify. We provide several examples of designs illustrating the different decomposability properties.

In Chapter 3, we first present four different types of canonical correlations in the context of the general three-way layout. These are obtained according as we ignore, include, adjust for or adjust only partially for some or all of the other factors. In Section 3.2 we give special attention, for all four types of canonical correlations, to the number that are equal to one, and the number that are nonzero and strictly less than one; these numbers are seen to be very useful in assessing different properties of the three-way layout. The concept of connectedness is then introduced in Section 3.3 and we show its importance in terms of estimability of the factors. In the three-way layout, several different types of connectedness may be defined which are not independent. We point out a number of relationships which exist between them. Similarly, there are different concepts of orthogonality in the three-way layout. In Section 3.4, we introduce two versions of orthogonality: weak orthogonality [cf. Chakrabarti (1962)] and strict orthogonality [cf. Eccleston and Russell (1975, 1977)]. We look at these two concepts of orthogonality in the different situations where the third factor is either ignored, included, adjusted for or only partially adjusted for. Some results involving these different concepts of orthogonality are then presented. We also look at a relationship between connectedness and orthogonality. In Section 3.5, we introduce canonical efficiency factors associated with the three-way design and we point out their relationship to canonical correlations. An average efficiency factor is then defined as the harmonic mean of the canonical efficiency factors; we present a sharp upper bound for the average efficiency factor of a two-way elimination of heterogeneity design in terms of the average efficiency factors for the subdesigns.

In Chapter 4, we first characterize the two concepts of variance balance and efficiency balance in the two-way elimination of heterogeneity design. We then present some results relating these two concepts of balance to equireplication and the corresponding balance properties in the subdesigns. We then introduce general balance in the mixed-model setting [cf. Nelder (1965a, b)] and develop this concept further in the context of row-column designs. We point out the correspondence between general balance and commutativity of the efficiency matrices in two-way elimination of heterogeneity designs. The importance of this commutativity property in terms of a common spectral decomposition for the efficiency matrices is stressed and a number of theorems making use of this are presented. In particular, stricter bounds for the average efficiency factor of a two-way elimination of heterogeneity are obtained.

Appendix 1 contains a series of what we call *mutzE*-tables, one associated with every three-way layout example presented in this thesis; these tables give the numbers m, u, and t, respectively, of nonzero, unit and nonunit nonzero canonical correlations, as well as the associated numbers z and E related to the canonical efficiency factors. In Appendix 2, we give the computer programs (written in Mathematica 1.2) that we prepared in order to compute these *mutzE*-tables and to assess the associated decomposability and commutativity properties. Finally, in Appendix 3 we present several matrix and linear algebra results, mostly with proofs, that we used in this thesis.

SECTION 1.2: NOTATION AND TERMINOLOGY

All scalars, vectors and matrices considered in this thesis are real, unless stated otherwise. Vectors, denoted by bold-face Roman lower case letters, are always taken to be column vectors—for example the $n \times 1$ vector $\mathbf{y}$. Matrices are denoted by bold-face Roman upper case letters—for example $\mathbf{A}$, $\mathbf{B}$, $\mathbf{X}$, $\mathbf{H}$. Scalars are denoted by light-face letters in italics.

For a given matrix $\mathbf{A}$, the corresponding lower case letter with the subscript ij refers to the $(i, j)^{th}$ element, and we write $\mathbf{A} = \{a_{ij}\}$. The symbols $\mathcal{C}(\mathbf{A})$ and $\mathcal{N}(\mathbf{A})$ represent, respectively, the column space or range of $\mathbf{A}$ (the space spanned by the columns of $\mathbf{A}$), and the null space of $\mathbf{A}$ (the set of all vectors $\mathbf{x}$ which transform $\mathbf{A}$ into the zero vector, i.e., which satisfy $\mathbf{Ax} = \mathbf{0}$). The rank and the nullity (number of columns minus the rank) of the matrix $\mathbf{A}$ are denoted by $\mathrm{r}(\mathbf{A})$ and $\nu(\mathbf{A})$, respectively. Transposition is denoted by a prime, e.g., $\mathbf{A}'$ is the transpose of $\mathbf{A}$ and so $\mathbf{A}' = \{a_{ji}\}$. We denote by dim the dimension of a subspace, i.e., the unique number of vectors in any basis of the subspace—for example $\dim\mathcal{C}(\mathbf{A})$ is the dimension of the column space of $\mathbf{A}$. The vector space $\mathcal{V}$ is said to be the direct sum of $\mathcal{V}_1$ and $\mathcal{V}_2$, i.e., $\mathcal{V} = \mathcal{V}_1 \oplus \mathcal{V}_2$, if $\mathcal{V}_1$ and $\mathcal{V}_2$ are subspaces of $\mathcal{V}$ satisfying $\mathcal{V}_1 \cap \mathcal{V}_2 = \{\mathbf{0}\}$ and $\mathcal{V}_1 \cup \mathcal{V}_2 = \mathcal{V}$. We use $\oplus^{\perp}$ to denote the direct sum of two orthogonal vector spaces.

Given an $m \times n$ matrix $\mathbf{A}$ and an $m \times p$ matrix $\mathbf{B}$, we write $(\mathbf{A} : \mathbf{B})$ for the $m \times (n+p)$ partitioned matrix with $\mathbf{A}$ placed next to $\mathbf{B}$. For a square matrix $\mathbf{B}$, we use $|\mathbf{B}|$, $\text{tr}(\mathbf{B})$, $\{\text{ch}(\mathbf{B})\}$ and $\text{ch}_h(\mathbf{B})$ to denote, respectively, the determinant, trace, set of characteristic roots or eigenvalues of $\mathbf{B}$ and the h^{th} largest (real) characteristic root or eigenvalue of $\mathbf{B}$. For an $m \times n$ matrix $\mathbf{A}$, $\text{sg}_h(\mathbf{A})$ denotes the h^{th} largest singular value of $\mathbf{A}$, i.e., the (positive) square root of the h^{th} largest nonzero (positive) characteristic root of $\mathbf{A}'\mathbf{A}$ or $\mathbf{A}\mathbf{A}'$.

The $n \times 1$ vector of ones is denoted by $\mathbf{e}^{(n)}$ or just by $\mathbf{e}$ when the dimension is clear from the context. Similarly, the $n \times n$ identity matrix is denoted by $\mathbf{I}_n$ or just by $\mathbf{I}$. The $n \times n$ matrix with all elements equal to $1/n$ is denoted by $\mathbf{J}_n$, i.e., $\mathbf{J}_n = \mathbf{e}^{(n)}\mathbf{e}^{(n)\prime}/n$, and the $n \times n$ centering matrix by $\mathbf{C}_n$, i.e., $\mathbf{C}_n = \mathbf{I}_n - \mathbf{J}_n$. Again the subscript n may be dropped.

If $\mathbf{A}$ is square and nonsingular ($|\mathbf{A}| \neq 0$), its inverse is denoted by $\mathbf{A}^{-1}$. We define a generalized inverse [see, e.g., Rao and Mitra (1971)] of an $m \times n$ matrix $\mathbf{A}$ as an $n \times m$ matrix $\mathbf{A}^-$, where $\mathbf{A}^-$ is any solution to $\mathbf{A}\mathbf{A}^-\mathbf{A} = \mathbf{A}$. The unique generalized inverse, denoted by $\mathbf{A}^+$, which satisfies the four equations $\mathbf{A}\mathbf{A}^+\mathbf{A} = \mathbf{A}$, $\mathbf{A}^+\mathbf{A}\mathbf{A}^+ = \mathbf{A}^+$, $(\mathbf{A}\mathbf{A}^+)' = \mathbf{A}\mathbf{A}^+$ and $(\mathbf{A}^+\mathbf{A})' = \mathbf{A}^+\mathbf{A}$ is called the Moore-Penrose inverse; then the "hat matrix" $\mathbf{H}_\mathbf{A} = \mathbf{A}\mathbf{A}^+$ and the "residual matrix" $\mathbf{M}_\mathbf{A} = \mathbf{I}_n - \mathbf{A}\mathbf{A}^+$ are the orthogonal projectors, respectively, on $\mathcal{C}(\mathbf{A})$ and on its orthocomplement $\mathcal{C}^\perp(\mathbf{A}) = \mathcal{N}(\mathbf{A}')$.

We represent a Kronecker (or Zehfuss) product by the symbol $\otimes$, i.e., given an $m \times n$ matrix $\mathbf{A}$ and a $p \times q$ matrix $\mathbf{B}$, the Kronecker product of $\mathbf{A}$ and $\mathbf{B}$ is the $mp \times nq$ matrix

$$\mathbf{A} \otimes \mathbf{B} = \begin{pmatrix} a_{11}\mathbf{B} & \ldots & a_{1n}\mathbf{B} \\ \ldots & \ldots & \ldots \\ a_{m1}\mathbf{B} & \ldots & a_{mn}\mathbf{B} \end{pmatrix}$$

and the direct sum of $\mathbf{A}$ and $\mathbf{B}$ is the $(m+p) \times (n+q)$ matrix

$$\mathbf{A} \oplus \mathbf{B} = \begin{pmatrix} \mathbf{A} & \mathbf{0} \\ \mathbf{0} & \mathbf{B} \end{pmatrix}.$$

For real square matrices **A** and **B** we denote the Löwner or nonnegative definite partial ordering by $\leq_L$. Thus $\mathbf{A} \leq_L \mathbf{B} \Leftrightarrow \mathbf{B} - \mathbf{A}$ is symmetric nonnegative definite, i.e., there exists a real matrix **C** such that $\mathbf{B} - \mathbf{A} = \mathbf{CC}'$ [cf. e.g., Hartwig and Styan (1987)].

We denote the expectation of a random vector **x** by $\mathcal{E}(\mathbf{x})$ and its dispersion or covariance matrix by $\mathcal{D}(\mathbf{x})$. The (not necessarily unique) least squares estimate of α is denoted by $\hat{\alpha}$. When summing over an index we replace the index of summation by a dot (or period), e.g., $n_{ij.} = \sum_{k=1}^{v} n_{ijk}$. The end of a proof is indicated by the halmos symbol □.

CHAPTER 2. INFORMATION MATRICES AND DECOMPOSABILITY

SECTION 2.1: THE THREE-WAY LAYOUT

It is usually necessary to use a three-way layout when three factors are needed to explain the variations in a set of the observations. Often, two of the factors are used to block the experimental units in two directions while the levels of the third factor represent the treatment (or variety) effects. Frequently, one of the two blocking factors represents rows while the other represents columns (sometimes called blocks). Each intersection of a row and a column constitutes a plot or experimental unit to which we apply one or more treatments.

The first type of three-way layout ever to be studied is probably the Latin square, in which the numbers of treatments, rows and columns are all the same and each treatment is applied only once in each row and in each column. According to Dénes and Keedwell (1974, p. 138), the enumeration of Latin squares was first discussed in 1779 by Leonhard Euler, and according to Freeman (1988), the first experiment using a Latin-square design was performed in 1788 by Cretté de Palluel. The requirements for a Latin square are, however, very restrictive and so a more general form of design is often needed. Yates (1936) introduced the first design with unequal numbers of rows, columns and treatments, but still with equally replicated treatments; he called it an incomplete Latin square—the design being a Latin square from which a row is missing. This concept was extended by Youden (1937), who developed the designs now called Youden squares, in which the number of treatments is greater than the column size. The Youden squares are obtained by rearranging the plots of certain balanced incomplete block (BIB) designs [for more on balanced incomplete block designs see, e.g., Boothroyd (1988)]. Since then, a large number of different types of three-way layouts have been introduced and studied; we now describe the most general form that they may take.

Let y_{ijkl} represent the yield corresponding to treatment k being applied to the l^{th} experimental unit in the j^{th} column and i^{th} row. The model can then be written as

$$y_{ijkl} = \alpha_i + \beta_j + \tau_k + \varepsilon_{ijkl}\,,\ i = 1, \ldots, r, j = 1, \ldots, c, k = 1, \ldots, v, l = 1, \ldots, n_{ijk}\,, \quad (2.1.1)$$

where the α_i, β_j and τ_k are fixed parameters representing the effects due to the i^{th} row, j^{th} column and k^{th} treatment, respectively, and n_{ijk} is the number of times that the k^{th} treatment is applied to the $(i, j)^{th}$ plot. The numbers n_{ijk} need not all be equal, and some may be 0. We will assume, however, that for all $i = 1, \ldots, r$, for all $j = 1, \ldots, c$, and for all $k = 1, \ldots, v$,

$$n_{i..} = \sum_{j=1}^{c} \sum_{k=1}^{v} n_{ijk} > 0, \quad n_{.j.} = \sum_{i=1}^{r} \sum_{k=1}^{v} n_{ijk} > 0 \quad \text{and} \quad n_{..k} = \sum_{i=1}^{r} \sum_{j=1}^{c} n_{ijk} > 0, \qquad (2.1.2)$$

i.e., every row and every column contains at least one treatment and every treatment is applied at least once.

It is often assumed that at most one treatment is applied to each experimental unit. An important special case occurs when the $n_{ij.}$'s are equal to one for all i and j; such a design is called a "row-(and-)column design" [cf. e.g., Pearce (1975), Freeman (1988)]. Another instance where the model in (2.1.1) may be used with the $n_{ij.}$ equal to one for all i and j is when a set of treatments is applied to a block design, and then a further set of treatments is applied, the assumption being made that there is no interaction between the current effects of the second set of treatments and the residual effects of the first [cf. Freeman (1959)].

We will assume that the model contains fixed effects and that the error terms ε_{ijkl} are uncorrelated random variables, each with mean 0, and are homoscedastic with unknown common variance σ^2 (white noise). The expectation of y_{ijkl} is given, therefore, by

$$\mathcal{E}(y_{ijkl}) = \alpha_i + \beta_j + \tau_k\,. \qquad (2.1.3)$$

Let $\mathbf{y}_{ijk} = (y_{ijk1}, \ldots, y_{ijkn_{ijk}})'$ be the $n_{ijk} \times 1$ vector of observations in row i and column j for treatment k, provided $n_{ijk} > 0$; otherwise $\mathbf{y}_{ijk} = \varnothing$ and hence absent. We then write

$$\mathbf{y} = (\mathbf{y}'_{111}, \mathbf{y}'_{112}, \ldots, \mathbf{y}'_{211}, \ldots, \mathbf{y}'_{rcv})'$$

as the $n \times 1$ vector of all the observations, where $n = \sum_{i=1}^{r} \sum_{j=1}^{c} \sum_{k=1}^{v} n_{ijk}$. Thus, in matrix notation, equation (2.1.3) may be written as

$$\mathcal{E}(\mathbf{y}) = \mathbf{X}_1\boldsymbol{\alpha} + \mathbf{X}_2\boldsymbol{\beta} + \mathbf{X}_3\boldsymbol{\tau} = (\mathbf{X}_1 : \mathbf{X}_2 : \mathbf{X}_3)\begin{pmatrix}\boldsymbol{\alpha}\\ \boldsymbol{\beta}\\ \boldsymbol{\tau}\end{pmatrix} = \mathbf{X}\boldsymbol{\gamma}, \qquad (2.1.4)$$

where the vectors $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_r)'$, $\boldsymbol{\beta} = (\beta_1, \ldots, \beta_c)'$, and $\boldsymbol{\tau} = (\tau_1, \ldots, \tau_v)'$ consist of the row, column and treatment effects, respectively. The matrices $\mathbf{X}_1, \mathbf{X}_2, \mathbf{X}_3$ are $n \times r$, $n \times c$ and $n \times v$ "design matrices" identifying the correspondence between the elements of y and, respectively, the rows, columns and treatments of the three-way layout. Thus the partitioned matrix $\mathbf{X} = (\mathbf{X}_1 : \mathbf{X}_2 : \mathbf{X}_3)$ is the $n \times (r + c + v)$ design matrix for the whole layout. Since exactly one treatment is applied to each observation which appears in precisely one row and one column, we have $\mathbf{X}_1\mathbf{e}^{(r)} = \mathbf{X}_2\mathbf{e}^{(c)} = \mathbf{X}_3\mathbf{e}^{(v)} = \mathbf{e}^{(n)}$. In the special case of row-column designs, the design matrices $\mathbf{X}_1$ and $\mathbf{X}_2$ can be expressed as $\mathbf{X}_1 = \mathbf{e}^{(c)} \otimes \mathbf{I}_r$ and $\mathbf{X}_2 = \mathbf{I}_c \otimes \mathbf{e}^{(r)}$, where $\otimes$ denotes the Kronecker product.

We will write $\mathbf{N}_{12} = \mathbf{X}_1'\mathbf{X}_2$ for the incidence matrix whose $(i, j)^{th}$ element, $n_{ij.}$, is the number of units treated in the i^{th} row and j^{th} column. We then denote its transpose by interchanging the two subscripts, i.e., $\mathbf{N}_{12}' = \mathbf{N}_{21} = \mathbf{X}_2'\mathbf{X}_1$. Similarly we let $\mathbf{N}_{13} = \mathbf{X}_1'\mathbf{X}_3$ be the incidence matrix whose $(i, k)^{th}$ element, $n_{i.k}$, is the number of units in the i^{th} row to which the k^{th} treatment has been applied, and we let $\mathbf{N}_{23} = \mathbf{X}_2'\mathbf{X}_3$ be the incidence matrix whose $(j, k)^{th}$ element, $n_{.jk}$, is the number of units in the j^{th} column to which the k^{th} treatment has been applied. Their transposes are respectively, $\mathbf{N}_{31} = \mathbf{X}_3'\mathbf{X}_1$ and $\mathbf{N}_{32} = \mathbf{X}_3'\mathbf{X}_2$.

We will let $\mathbf{k}_1 = \mathbf{X}_1'\mathbf{e}^{(n)} = (n_{1..}, \ldots, n_{r..})'$ denote the vector of row sizes where, as we assumed above, $n_{i..} > 0$ for all $i = 1, \ldots, r$. The vector of column sizes is denoted by $\mathbf{k}_2 = \mathbf{X}_2'\mathbf{e}^{(n)} = (n_{.1}, \ldots, n_{.c.})'$ where $n_{.j.} > 0$ for all $j = 1, \ldots, c$, and the vector of treatment sizes or replications is $\mathbf{k}_3 = \mathbf{X}_3'\mathbf{e}^{(n)} = (n_{..1}, \ldots, n_{..v})'$, where $n_{..k} > 0$ for all $k = 1, \ldots, v$. The three matrices $\mathbf{D}_1 = \mathbf{X}_1'\mathbf{X}_1$, $\mathbf{D}_2 = \mathbf{X}_2'\mathbf{X}_2$ and $\mathbf{D}_3 = \mathbf{X}_3'\mathbf{X}_3$ are all diagonal and positive definite, with the successive elements of $\mathbf{k}_1$, $\mathbf{k}_2$ and $\mathbf{k}_3$, respectively, as their diagonal elements. When $\mathbf{D}_1$ and $\mathbf{D}_2$ are both scalar matrices, i.e., multiples of the identity matrix, and so all row sizes and column sizes are equal, Raghavarao and Federer (1975) call the design "ordinary". Since for a row-column design, the row sizes are all equal to the number of columns, i.e., $n_{1..} = \ldots = n_{r..} = c$, and the column sizes are all equal to the number of rows, i.e., $n_{.1.} = \ldots = n_{.c.} = r$, a row-column design is ordinary and also satisfies $\mathbf{N}_{12} = \mathbf{e}^{(r)}\mathbf{e}^{(c)'}$. If each treatment is applied in k plots so that $n_{..1} = \ldots = n_{..v} = k$, then we call the design "equireplicate"; otherwise it may be called "unequireplicate".

With this notation, the normal equations $\mathbf{X}'\mathbf{X}\hat{\boldsymbol{\gamma}} = \mathbf{X}'\mathbf{y}$ for the least squares estimates of the row, column and treatment effects can be written as

$$\begin{pmatrix} \mathbf{D}_1 & \mathbf{N}_{12} & \mathbf{N}_{13} \\ \mathbf{N}_{21} & \mathbf{D}_2 & \mathbf{N}_{23} \\ \mathbf{N}_{31} & \mathbf{N}_{32} & \mathbf{D}_3 \end{pmatrix} \begin{pmatrix} \hat{\boldsymbol{\alpha}} \\ \hat{\boldsymbol{\beta}} \\ \hat{\boldsymbol{\tau}} \end{pmatrix} = \begin{pmatrix} \mathbf{y}_{\mathrm{rt}} \\ \mathbf{y}_{\mathrm{ct}} \\ \mathbf{y}_{\mathrm{vt}} \end{pmatrix}, \tag{2.1.5}$$

where $\mathbf{y}_{\mathrm{rt}} = \mathbf{X}_1'\mathbf{y} = (y_{1..}, \ldots, y_{r..})'$ is the vector of row totals, $\mathbf{y}_{\mathrm{ct}} = \mathbf{X}_2'\mathbf{y} = (y_{.1.}, \ldots, y_{.c.})'$ is the vector of column totals, and $\mathbf{y}_{\mathrm{vt}} = \mathbf{X}_3'\mathbf{y} = (y_{..1}, \ldots, y_{..v})'$ is the vector of treatment totals. Since $\mathbf{e}^{(n)} \in \mathcal{C}(\mathbf{X}_1) \cap \mathcal{C}(\mathbf{X}_2) \cap \mathcal{C}(\mathbf{X}_3)$, it follows that the rank $\mathrm{r}(\mathbf{X}'\mathbf{X}) = \mathrm{r}(\mathbf{X}) \le r + c + v - 2$ and so the normal equations do not have a unique solution; the normal equations are, however, consistent since $\mathrm{r}(\mathbf{X}'\mathbf{X} : \mathbf{X}'\mathbf{y}) = \mathrm{r}(\mathbf{X}'\mathbf{X})$. [To see this, write $\mathrm{r}(\mathbf{X}'\mathbf{X} : \mathbf{X}'\mathbf{y}) = \mathrm{r}[\mathbf{X}'(\mathbf{X} : \mathbf{y})] \le \mathrm{r}(\mathbf{X}') = \mathrm{r}(\mathbf{X}'\mathbf{X}) \le \mathrm{r}(\mathbf{X}'\mathbf{X} : \mathbf{X}'\mathbf{y})$].

SECTION 2.2: INFORMATION OR C-MATRICES

Let us now consider rows and columns as sets of nuisance parameters in the model (2.1.4) (we could similarly consider rows and treatments, or columns and treatments, as the sets of nuisance parameters, but in most experiments the interest is in comparing effects of different treatments after having removed the row and column effects). In this situation, we call the three-way layout a "two-way elimination of heterogeneity design" [cf. e.g., Agrawal (1966b)]. A very important role is then played by the matrix $\mathbf{S}_{3.12}$, obtained by eliminating the matrix of row and column effects, i.e.,

$$(\mathbf{X}_1 : \mathbf{X}_2)'(\mathbf{X}_1 : \mathbf{X}_2) = \begin{pmatrix} \mathbf{D}_1 & \mathbf{N}_{12} \\ \mathbf{N}_{21} & \mathbf{D}_2 \end{pmatrix} \tag{2.2.1}$$

from the full normal equations (2.1.5). The matrix $\mathbf{S}_{3.12}$ is then given by

$$\mathbf{S}_{3.12} = \mathbf{X}_3'\mathbf{X}_3 - \mathbf{X}_3'\mathbf{H}_{12}\mathbf{X}_3 = \mathbf{X}_3'\mathbf{M}_{12}\mathbf{X}_3, \tag{2.2.2}$$

where $\mathbf{H}_{12}$ is the hat matrix for the augmented matrix $(\mathbf{X}_1 : \mathbf{X}_2)$, and $\mathbf{M}_{12} = \mathbf{I} - \mathbf{H}_{12}$. Because of the relation $\mathcal{C}(\mathbf{X}_1 : \mathbf{X}_2) = \mathcal{C}(\mathbf{X}_1) \oplus^{\perp} \mathcal{C}(\mathbf{M}_1\mathbf{X}_2) = \mathcal{C}(\mathbf{X}_2) \oplus^{\perp} \mathcal{C}(\mathbf{M}_2\mathbf{X}_1)$, where $\oplus^{\perp}$ denotes the direct sum of the two orthogonal subspaces, the hat matrix for or orthogonal projector $\mathbf{H}_{12}$ onto $\mathcal{C}(\mathbf{X}_1 : \mathbf{X}_2)$ can be expressed as

$$\mathbf{H}_{12} = \mathbf{H}_1 + \mathbf{H}_{2|1} = \mathbf{H}_1 + \mathbf{M}_1\mathbf{X}_2(\mathbf{X}_2'\mathbf{M}_1\mathbf{X}_2)^{-}\mathbf{X}_2'\mathbf{M}_1 \tag{2.2.3}$$

$$= \mathbf{H}_2 + \mathbf{H}_{1|2} = \mathbf{H}_2 + \mathbf{M}_2\mathbf{X}_1(\mathbf{X}_1'\mathbf{M}_2\mathbf{X}_1)^{-}\mathbf{X}_1'\mathbf{M}_2, \tag{2.2.4}$$

where $\mathbf{H}_{2|1}$ and $\mathbf{H}_{1|2}$ are the orthogonal projectors onto $\mathcal{C}(\mathbf{M}_1\mathbf{X}_2)$ and $\mathcal{C}(\mathbf{M}_2\mathbf{X}_1)$, respectively. From (2.2.3) and (2.2.4) it follows directly that

$$\mathbf{M}_{12} = \mathbf{M}_1 - \mathbf{H}_{2|1} = \mathbf{M}_2 - \mathbf{H}_{1|2}. \tag{2.2.5}$$

Substituting $\mathbf{M}_{12} = \mathbf{M}_1 - \mathbf{H}_{2|1}$ in (2.2.2) we get a further representation for the matrix $\mathbf{S}_{3.12}$,

$$\mathbf{S}_{3.12} = \mathbf{D}_3 - \mathbf{N}_{31}\mathbf{D}_1^{-1}\mathbf{N}_{13} - (\mathbf{N}_{32} - \mathbf{N}_{31}\mathbf{D}_1^{-1}\mathbf{N}_{12})(\mathbf{D}_2 - \mathbf{N}_{21}\mathbf{D}_1^{-1}\mathbf{N}_{12})^{-}(\mathbf{N}_{23} - \mathbf{N}_{21}\mathbf{D}_1^{-1}\mathbf{N}_{13}). \tag{2.2.6}$$

The dual of the representation (2.2.6) is obtained by substituting $\mathbf{M}_{12} = \mathbf{M}_2 - \mathbf{H}_{1|2}$ into (2.2.2),

$$\mathbf{S}_{3.12} = \mathbf{D}_3 - \mathbf{N}_{32}\mathbf{D}_2^{-1}\mathbf{N}_{23} - (\mathbf{N}_{31} - \mathbf{N}_{32}\mathbf{D}_2^{-1}\mathbf{N}_{21})(\mathbf{D}_1 - \mathbf{N}_{12}\mathbf{D}_2^{-1}\mathbf{N}_{21})^{-}(\mathbf{N}_{13} - \mathbf{N}_{12}\mathbf{D}_2^{-1}\mathbf{N}_{23}) \tag{2.2.7}$$

The matrix $\mathbf{S}_{3.12}$ is often called the "information matrix" [John (1987), pp. 8, 95], the "C-matrix" [Raghavarao and Federer (1975)] or the "coefficient matrix" [Pearce (1983), p.59] from which both row and column effects have been eliminated. We also note that the matrix $\mathbf{S}_{3.12}$ is the (unique) Schur complement of (2.2.1) in $\mathbf{X}'\mathbf{X}$; for more on Schur complements, see Ouellette (1981) and Styan (1985).

The reduced normal equations for estimating treatment effects are given by

$$\mathbf{S}_{3.12}\hat{\boldsymbol{\tau}} = \mathbf{z}_{3.12}; \tag{2.2.8}$$

we will call $\mathbf{z}_{3.12} = \mathbf{X}_3'\mathbf{y} - \mathbf{X}_3'\mathbf{H}_{12}\mathbf{y} = \mathbf{X}_3'\mathbf{M}_{12}\mathbf{y}$ the vector of "adjusted treatment totals". We note that the adjectival position of the word "adjusted" here seems to imply that the vector $\mathbf{X}_3'\mathbf{y}$ of treatment totals in the original data vector $\mathbf{y}$ is being adjusted for rows and columns; the "adjustment", however, actually occurs first yielding the vector $\mathbf{M}_{12}\mathbf{y}$ of the original data adjusted for rows and columns and then its components corresponding to treatments are summed to form the vector $\mathbf{X}_3'\mathbf{M}_{12}\mathbf{y}$, which really contains "treatment totals of the data which have first been adjusted for rows and columns". In this thesis, as it seems in all of the literature, the vector $\mathbf{X}_3'\mathbf{M}_{12}\mathbf{y}$ is referred to as the "vector of adjusted treatment totals"; we will adopt this convention also for all other vectors of adjusted totals.

Other information matrices of importance are those obtained by ignoring one of the two sets of nuisance parameters. When we ignore the column effects, we call the resulting design the "treatment-row subdesign". Here, the treatment effects are estimated after eliminating the row effects, and the information matrix is given by

$$\mathbf{S}_{3.1} = \mathbf{X}_3'\mathbf{M}_1\mathbf{X}_3. \tag{2.2.9}$$

For the treatment-column subdesign (where the row effects are ignored), the treatment effects are estimated after eliminating the column effects, and the information matrix is given by

$$\mathbf{S}_{3.2} = \mathbf{X}_3'\mathbf{M}_2\mathbf{X}_3. \tag{2.2.10}$$

The information matrix for the model in which both rows and columns are ignored will be denoted by $\mathbf{S}_{3.0}$, where

$$\mathbf{S}_{3.0} = \mathbf{X}_3'\mathbf{X}_3 - \frac{\mathbf{k}_3\mathbf{k}_3'}{n} = \mathbf{X}_3'\mathbf{C}_n\mathbf{X}_3, \tag{2.2.11}$$

and $\mathbf{C}_n = \mathbf{I}_n - \mathbf{J}_n = \mathbf{I}_n - (1/n)\mathbf{e}^{(n)}\mathbf{e}^{(n)'}$ is the $n \times n$ centering matrix.

In the context of row-column designs, we also have information matrices for the model where the treatment effects are estimated from an orthonormal set of row contrasts and from an orthonormal set of column contrasts, i.e.,

$$\mathbf{S}_{3.r} = \mathbf{X}_3'(\mathbf{H}_1 - \mathbf{J}_n)\mathbf{X}_3 \quad \text{and} \quad \mathbf{S}_{3.c} = \mathbf{X}_3'(\mathbf{H}_2 - \mathbf{J}_n)\mathbf{X}_3, \tag{2.2.12}$$

respectively [cf. Shah and Eccleston (1986)].

None of the information matrices above has full rank since the rows and columns of $\mathbf{S}_g$, g = 3.12, 3.1, 3.2, 3.0, 3.r or 3.c, all sum to 0, implying that $r(\mathbf{S}_g) \leq v - 1$. Solutions to the reduced normal equations in (2.2.8) can be written as $\hat{\boldsymbol{\tau}} = \mathbf{S}_{3.12}^{-}\mathbf{z}_{3.12} = \mathbf{S}_{3.12}^{-}\mathbf{X}_3'\mathbf{M}_{12}\mathbf{y}$ for some choice of generalized inverse $\mathbf{S}_{3.12}^{-}$; there is no unique solution, however, as the generalized inverse $\mathbf{S}_{3.12}^{-}$, and hence $\mathbf{S}_{3.12}^{-}\mathbf{X}_3'\mathbf{M}_{12}$, may be chosen in many different ways.

SECTION 2.3: DECOMPOSABILITY

An apparently new, and we believe important, subclass of two-way elimination of heterogeneity designs is specified by the information matrix $\mathbf{S}_{3.12}$ being decomposable in the following way,

$$\mathbf{S}_{3.12} = \xi_1 \mathbf{S}_{3.1} + \xi_2 \mathbf{S}_{3.2} - \xi_0 \mathbf{S}_{3.0}, \quad \xi_1, \xi_2 \text{ and } \xi_0 > 0, \tag{2.3.1}$$

cf. Bérubé and Styan (1990b). This subclass comprises designs for which the study of relationships between properties of the three-way design itself, and corresponding properties of its treatment-row and treatment-column subdesigns, is simplified. As the study of block designs is more straightforward than that of three-way designs, we can see that when (2.3.1) is satisfied, the level of difficulty in analyzing the design would be reduced from one three-way level to two two-way level designs. As we will see later in this thesis, our decomposability property (2.3.1) seems to be, up to now, probably the most general form of designs for which certain results on connectedness, orthogonality and balance hold.

The special case of condition (2.3.1) when $\xi_1 = \xi_2 = \xi_0 = 1$ was introduced very recently in Baksalary and Shah (1989), where the two-way elimination of heterogeneity design is then said to satisfy the "decomposability property," i.e.,

$$\mathbf{S}_{3.12} = \mathbf{S}_{3.1} + \mathbf{S}_{3.2} - \mathbf{S}_{3.0} \,. \tag{2.3.2}$$

We will say that the set of designs for which (2.3.1) holds, but for which (2.3.2) does not hold, satisfy the "generalized decomposability property," while those for which (2.3.2) holds, and hence also (2.3.1), we will say satisfy the "reduced decomposability property."

Agrawal (1966c) constructed designs for which each of $\mathbf{S}_{3.12}$, $\mathbf{S}_{3.1}$, $\mathbf{S}_{3.2}$ and $\mathbf{S}_{3.0}$ has the form $a\mathbf{I} + b\mathbf{J}$, i.e., all diagonal elements equal and all off-diagonal elements equal. Although this kind of design does not necessarily satisfy the reduced decomposability property (2.3.2), it very often satisfies our generalized decomposability property (2.3.1).

Since in our generalized decomposability property, the matrices $\mathbf{S}_{3.12}$, $\mathbf{S}_{3.1}$, $\mathbf{S}_{3.2}$ and $\mathbf{S}_{3.0}$ need have no particular form, the class of designs satisfying our generalized decomposability property is more general than this special class of designs considered by Agrawal (1966c).

If the two-way elimination of heterogeneity design is ordinary (equal row sizes $\mathbf{k}_1 = k_1\mathbf{e}^{(r)}$ and equal column sizes $\mathbf{k}_2 = k_2\mathbf{e}^{(c)}$ for some positive integers k_1 and k_2 such that $k_1 r = k_2 c = n$), then the reduced decomposability property (2.3.2) is equivalent to

$$\mathbf{S}_{3.12} = \mathbf{D}_3 - \frac{\mathbf{N}_{31}\mathbf{N}_{13}}{k_1} - \frac{\mathbf{N}_{32}\mathbf{N}_{23}}{k_2} + \frac{\mathbf{k}_3\mathbf{k}_3'}{n} . \qquad (2.3.3)$$

Any row-column design, i.e., any three-way layout with incidence matrix $\mathbf{N}_{12} = \mathbf{e}^{(r)}\mathbf{e}^{(c)\prime}$, provides a simple example of a design satisfying the reduced decomposability property. Since now the row sizes $k_1 = c$, the column sizes $k_2 = r$, and the total number of observations $n = rc$, the equation (2.3.3) becomes

$$\mathbf{S}_{3.12} = \mathbf{D}_3 - \frac{\mathbf{N}_{31}\mathbf{N}_{13}}{c} - \frac{\mathbf{N}_{32}\mathbf{N}_{23}}{r} + \frac{\mathbf{k}_3\mathbf{k}_3'}{rc} .$$

A somewhat different decomposition of the information matrix $\mathbf{S}_{3.12}$ was introduced in Baksalary and Siatkowski (1990) with designs for which the information matrix takes the form

$$\mathbf{S}_{3.12} = \mathbf{D}_3 - \upsilon_1\mathbf{N}_{31}\mathbf{N}_{13} - \upsilon_2\mathbf{N}_{32}\mathbf{N}_{23} + \rho\,\mathbf{k}_3\mathbf{k}_3' , \quad \upsilon_1,\ \upsilon_2,\ \rho > 0, \qquad (2.3.4)$$

of which clearly (2.3.3) is the special case with $\upsilon_1 = 1/k_1$, $\upsilon_2 = 1/k_2$, and $\rho = 1/n$. We will say that designs for which (2.3.4) holds satisfy the "extended decomposability property."

Our generalized decomposability property (2.3.1), the extended decomposability property (2.3.4) and the reduced decomposability property (2.3.2) are not equivalent, as we will show in the following two examples.

EXAMPLE 2.3.1: As an example of a three-way layout that satisfies both (2.3.4) and (2.3.1) but not (2.3.2), we consider the following design with seven rows, seven columns and seven treatments, taken from Agrawal (1966c), see also Table A.1.1 in Appendix A,

*	3	5	*	2	*	*
*	*	4	6	*	3	*
*	*	*	5	7	*	4
5	*	*	*	6	1	*
*	6	*	*	*	7	2
3	*	7	*	*	*	1
2	4	*	1	*	*	*

(2.3.5)

where * denotes an empty cell. For this design (2.3.5), $S_{3.1} = S_{3.2} = (7/3)C_7$, $S_{3.0} = 3C_7$ and $S_{3.12} = C_7$, and so (2.3.4) holds with $\rho = 2/21$ and any υ_1 and υ_2 such that $\upsilon_1 + \upsilon_2 = 1$, $\upsilon_1, \upsilon_2 > 0$. Baksalary and Siatkowski (1990) use (2.3.5) as an example of a design satisfying (2.3.4) but not (2.3.2), since obviously here $S_{3.12} \neq S_{3.1} + S_{3.2} - S_{3.0}$. We can, however, express $S_{3.12}$ as in (2.3.1), i.e., this design satisfies our generalized decomposability property (2.3.1) with, for example, $\xi_1 + \xi_2 = 1$, $\xi_1, \xi_2 > 0$ and $\xi_0 = 4/9$.

The following example exhibits a design which is not ordinary, and which satisfies our generalized decomposabilitv property (2.3.1) but not the extended decomposability property (2.3.4). We have, however, not yet found a design which satisfies the extended decomposability property (2.3.4) but not our generalized decomposability property (2.3.1), nor have we been able to show whether or not there exists such a design.

EXAMPLE 2.3.2: Consider the following design with three rows, three columns, and three treatments, see also Table A.1.2 in Appendix A,

1	2	3
*	1	2
3	*	1

(2.3.6)

where again * denotes an empty cell. It is straightforward to show that the associated information matrix for the full design

$$\mathbf{S}_{3.12} = (1/15)\begin{pmatrix} 24 & -12 & -12 \\ -12 & 16 & -4 \\ -12 & -4 & 16 \end{pmatrix},$$

while the information matrices for the treatment-row and treatment-column subdesigns are equal and are given by

$$\mathbf{S}_{3.1} = \mathbf{S}_{3.2} = (1/6)\begin{pmatrix} 10 & -5 & -5 \\ -5 & 7 & -2 \\ -5 & -2 & 7 \end{pmatrix} \qquad (2.3.7)$$

and the information matrix ignoring both rows and columns is given by

$$\mathbf{S}_{3.0} = (1/7)\begin{pmatrix} 12 & -6 & -6 \\ -6 & 10 & -4 \\ -6 & -4 & 10 \end{pmatrix}. \qquad (2.3.8)$$

Hence,

$$\mathbf{S}_{3.12} = \xi_1\mathbf{S}_{3.1} + \xi_2\mathbf{S}_{3.2} - (7/30)\mathbf{S}_{3.0},$$

for any positive ξ_1, ξ_2 such that $\xi_1 + \xi_2 = 6/5$.

However, there exist no υ_1, υ_2, $\rho > 0$ such that $\mathbf{S}_{3.12}$ could satisfy the extended decomposability property, for

$$\mathbf{N}_{31}\mathbf{N}_{13} = \mathbf{N}_{32}\mathbf{N}_{23} = \begin{pmatrix} 3 & 2 & 2 \\ 2 & 2 & 1 \\ 2 & 1 & 2 \end{pmatrix} \quad \text{and} \quad \mathbf{k}_3\mathbf{k}_3' = \begin{pmatrix} 9 & 6 & 6 \\ 6 & 4 & 4 \\ 6 & 4 & 4 \end{pmatrix};$$

trying to solve for the unknowns in (2.3.4) gives rise to the following inconsistent system of equations,

$$-\upsilon + 3\rho = -7/15,$$

$$-\upsilon + 3\rho = -2/5.$$

There are, however, special cases when our generalized decomposability property (2.3.1) and the extended decomposability property (2.3.4) are equivalent. For example, the case of ordinary two-way elimination of heterogeneity designs, i.e., designs which have row sizes all equal to k_1 and column sizes all equal to k_2. For such designs, as Baksalary and Siatkowski (1990) point out, if we postmultiply (2.3.4) by $\mathbf{e}^{(\nu)}$, we obtain the equality

$$\mathbf{0} = (1 - \upsilon_1 k_1 - \upsilon_2 k_2 + \rho n)\mathbf{k}_3, \tag{2.3.9}$$

implying that

$$1 = \upsilon_1 k_1 + \upsilon_2 k_2 - \rho n. \tag{2.3.10}$$

The extended decomposability property can then be rewritten as

$$\mathbf{S}_{3.12} = \upsilon_1 k_1(\mathbf{D}_3 - \frac{\mathbf{N}_{31}\mathbf{N}_{13}}{k_1}) + \upsilon_2 k_2(\mathbf{D}_3 - \frac{\mathbf{N}_{32}\mathbf{N}_{23}}{k_2}) - \rho n(\mathbf{D}_3 - \frac{\mathbf{k}_3\mathbf{k}_3'}{n}), \tag{2.3.11}$$

which is equivalent to (2.3.1) with $\xi_1 = \upsilon_1 k_1$, $\xi_2 = \upsilon_2 k_2$ and $\xi_0 = \rho n$. Substituting into (2.3.10), yields

$$\xi_0 = \xi_1 + \xi_2 - 1. \tag{2.3.12}$$

Our generalized decomposability property (2.3.1) can then be rewritten as

$$\mathbf{S}_{3\,12} = \xi_1\mathbf{S}_{3.1} + \xi_2\mathbf{S}_{3.2} - (\xi_1 + \xi_2 - 1)\mathbf{S}_{3.0}. \tag{2.3.13}$$

For example, if we look again at the design (2.3.5) in Example 2.3.1, where $k_1 = k_2 = 3$ and $n = 21$, we see that the extended decomposability property is satisfied with $\upsilon_1 + \upsilon_2 = 1$, $\upsilon_1, \upsilon_2 > 0$ and $\rho = 2/21$. This implies that we can have $\xi_1 + \xi_2 = 3$, $\xi_1, \xi_2 > 0$ and $\xi_0 = 2$; in this case, it is obvious that our generalized decomposability property is equivalent to (2.3.13), i.e., $-2\mathbf{C}_7 = -(2/3)\xi_1\mathbf{C}_7 - (2/3)\xi_2\mathbf{C}_7$, with $\xi_1 + \xi_2 = 3$, $\xi_1, \xi_2 > 0$.

For designs where our generalized decomposability property (2.3.1) holds irrespective of the application of treatments, i.e., designs for which

$$\mathbf{H}_{12} = \xi_1\mathbf{H}_1 + \xi_2\mathbf{H}_2 - \xi_0\mathbf{J}_n, \tag{2.3.14}$$

then (2.3.13) also holds since, again, if we postmultiply (2.3.14) by $\mathbf{e}^{(n)}$, we obtain $1 = \xi_1 + \xi_2 - \xi_0$ and hence $\xi_0 = \xi_1 + \xi_2 - 1$ as in (2.3.12).

Later on in this thesis we will use our generalized decomposability property (2.3.1) to generalize theorems that previously assumed either the reduced or the extended decomposability property.

CHAPTER 3. CANONICAL CORRELATIONS AND EFFICIENCY FACTORS

SECTION 3.1: INTRODUCTION

Let us consider the following four different types of canonical correlations:

(i) $\rho_h^{(i,j)}$ between $\mathbf{X}_i'\mathbf{y}$ and $\mathbf{X}_j'\mathbf{y}$,

(ii) $\rho_h^{(i,jk)}$ between $\mathbf{X}_i'\mathbf{y}$ and $(\mathbf{X}_j : \mathbf{X}_k)'\mathbf{y}$,

(iii) $\rho_h^{(i;j|k)}$ between $\mathbf{X}_i'\mathbf{y}$ and $\mathbf{X}_j'\mathbf{M}_k\mathbf{y}$,

(iv) $\rho_h^{(i,j|k)}$ between $\mathbf{X}_i'\mathbf{M}_k\mathbf{y}$ and $\mathbf{X}_j'\mathbf{M}_k\mathbf{y}$,

where $i \neq j$, $i \neq k$, $j \neq k$; $i, j, k = 1, 2, 3$; $h = 1, \ldots, m$. The upper limits m of the index h denote the numbers of positive canonical correlations; we write $m = 0$ when the associated vectors are uncorrelated. There are fifteen kinds of canonical correlations ρ_h in all, three each of types (i), (ii) and (iv), plus six of type (iii).

Canonical correlations of type

(i) $\rho_h^{(i,j)}$ where we *ignore* factor k,
(ii) $\rho_h^{(i,jk)}$ where we *include* factor k, and
(iv) $\rho_h^{(i,j|k)}$ where we *adjust* for factor k,

were studied in some detail in Styan (1986); those of type

(iii) $\rho_h^{(i;j|k)}$ where we *adjust* only *partially* for factor k,

were introduced in Worsley, Styan and Bérubé (1990).

Canonical correlations of type (i), since the third factor is ignored, apply directly to two-way layouts or subdesigns; cf. e.g., Latour and Styan (1985). Then, because of the white noise assumption, the h^{th} largest nonzero canonical correlation between $\mathbf{X}_i'\mathbf{y}$ and $\mathbf{X}_j'\mathbf{y}$ is given by

$$\rho_h^{(i,j)} = \mathrm{ch}_h^{1/2}(\mathbf{D}_i^{-1}\mathbf{N}_{ij}\mathbf{D}_j^{-1}\mathbf{N}_{ji}) = \mathrm{ch}_h^{1/2}(\mathbf{H}_i\mathbf{H}_j) = \mathrm{sg}_h(\mathbf{D}_i^{-1/2}\mathbf{N}_{ij}\mathbf{D}_j^{-1/2}), \tag{3.1.1}$$

where $\mathrm{ch}_h(.)$ and $\mathrm{sg}_h(.)$ denote the h^{th} largest (real) characteristic root (eigenvalue) and singular value, respectively; cf. Khatri (1976), Seshadri and Styan (1980), Rao (1981) and Styan (1985).

For canonical correlations of type (ii), augmenting the j^{th} factor with the k^{th} factor, we write (2.1.4) as

$$\mathcal{E}(\mathbf{y}) = \mathbf{X}_i\boldsymbol{\gamma}_i + (\mathbf{X}_j : \mathbf{X}_k)\begin{pmatrix}\boldsymbol{\gamma}_j\\ \boldsymbol{\gamma}_k\end{pmatrix}, \tag{3.1.2}$$

where $\boldsymbol{\gamma}_i$, $\boldsymbol{\gamma}_j$ and $\boldsymbol{\gamma}_k$ are any permutation of the vectors $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$ and $\boldsymbol{\tau}$ of row, column and treatment effects. We can consider (3.1.2) as a two-way layout and apply (3.1.1) to obtain the h^{th} largest nonzero canonical correlation between $\mathbf{X}_i'\mathbf{y}$ and $(\mathbf{X}_j : \mathbf{X}_k)'\mathbf{y}$,

$$\begin{aligned}\rho_h^{(i,jk)} &= \mathrm{ch}_h^{1/2}\{\mathbf{D}_i^{-1}\mathbf{X}_i'(\mathbf{X}_j : \mathbf{X}_k)[(\mathbf{X}_j : \mathbf{X}_k)'(\mathbf{X}_j : \mathbf{X}_k)]^{-}(\mathbf{X}_j : \mathbf{X}_k)'\mathbf{X}_i\}\\ &= \mathrm{ch}_h^{1/2}(\mathbf{H}_i\mathbf{H}_{jk}).\end{aligned} \tag{3.1.3}$$

For canonical correlations of type (iii), we consider the joint dispersion matrix of $\mathbf{X}_i'\mathbf{y}$ and $\mathbf{X}_j'\mathbf{M}_k\mathbf{y}$, given by:

$$\mathcal{D}\begin{pmatrix}\mathbf{X}_i'\mathbf{y}\\ \mathbf{X}_j'\mathbf{M}_k\mathbf{y}\end{pmatrix} = \sigma^2\begin{pmatrix}\mathbf{D}_i & \mathbf{X}_i'\mathbf{M}_k\mathbf{X}_j\\ \mathbf{X}_j'\mathbf{M}_k\mathbf{X}_i & \mathbf{X}_j'\mathbf{M}_k\mathbf{X}_j\end{pmatrix}. \tag{3.1.4}$$

The h^{th} largest nonzero canonical correlation is then given by:

$$\rho_h^{(i;j|k)} = \mathrm{ch}_h^{1/2}\{\mathbf{D}_i^{-1}\mathbf{X}_i'\mathbf{M}_k\mathbf{X}_j(\mathbf{X}_j'\mathbf{M}_k\mathbf{X}_j)^{-}\mathbf{X}_j'\mathbf{M}_k\mathbf{X}_i\} = \mathrm{ch}_h^{1/2}(\mathbf{H}_i\mathbf{H}_{j|k}), \tag{3.1.5}$$

where $\mathbf{H}_{j|k}$ is the orthogonal projector onto $\mathcal{C}(\mathbf{M}_k\mathbf{X}_j)$.

Finally, for canonical correlations of type (iv), if we premultiply (3.1.2) by $\mathbf{M}_k$ for $k \neq i, j$, we obtain,

$$\mathcal{E}(\mathbf{M}_k\mathbf{y}) = \mathbf{M}_k\mathbf{X}_i\boldsymbol{\gamma}_i + \mathbf{M}_k\mathbf{X}_j\boldsymbol{\gamma}_j, \tag{3.1.6}$$

which we may consider as the "two-way layout" corresponding to the design for factors i and j after adjusting (both) for, or eliminating, factor k. We again apply (3.1.1) to obtain the h^{th} largest nonzero canonical correlation between $\mathbf{X}_i'\mathbf{M}_k\mathbf{y}$ and $\mathbf{X}_j'\mathbf{M}_k\mathbf{y}$,

$$\rho_h^{(i,j|k)} = \mathrm{ch}_h^{1/2}\{(\mathbf{X}_i'\mathbf{M}_k\mathbf{X}_i)^{-}(\mathbf{X}_i'\mathbf{M}_k\mathbf{X}_j)(\mathbf{X}_j'\mathbf{M}_k\mathbf{X}_j)^{-}(\mathbf{X}_j'\mathbf{M}_k\mathbf{X}_i)\}$$

$$= \mathrm{ch}_h^{1/2}(\mathbf{H}_{i|k}\mathbf{H}_{j|k}). \tag{3.1.7}$$

Several efficient numerical methods for computing canonical correlations are given in Björck and Golub (1973); cf. also Golub and Van Loan (1989, pp. 584-585).

As in Latour and Styan (1985) and Styan (1986) we define, for each of the fifteen kinds of canonical correlations,

t = the number of *nonzero* canonical correlations *strictly less* than 1

u = the number of canonical correlations *equal* to 1.

The number of *nonzero* canonical correlations is then obviously

$$m = u + t. \tag{3.1.8}$$

We will say that the corresponding layout is "orthogonal" whenever $t = 0$, and "connected" whenever $u = 1$ for canonical correlations of types (i) and (ii) and $u = 0$ for canonical correlations of types (iii) and (iv). We may thus interpret t as the "degree of nonorthogonality" and u as the "degree of disconnectedness". Whenever we need to be more specific, we add subscripts to m, u and t, corresponding to the superscripts in (i), (ii), (iii) and (iv), e.g., for type (i), with $i = 1, j = 2$, we would write $m_{1\,2} = u_{1\,2} + t_{1\,2}$ The fifteen m, u, and t values for the canonical correlations of types (i) to (iv) may then be collected in what we refer to from now on as a "*mut*-" or "*mutzE*-table", cf. Styan (1986), Bérubé and Styan (1990a), Worsley, Styan and Bérubé (1990); the numbers z and E will be defined in Section 3.5. In Appendix 1, we present several such tables, corresponding to designs given as examples throughout this thesis.

SECTION 3.2: SOME RESULTS FOR THE NUMBERS m, u AND t.

The numbers m of nonzero canonical correlations, u of unit canonical correlations, and t of nonzero canonical correlations strictly less than one, are closely related to the following rank formula:

$$r(\mathbf{X}_i'\mathbf{X}_j) = r(\mathbf{X}_i) + r(\mathbf{X}_j) - r(\mathbf{X}_i : \mathbf{X}_j) + r(\mathbf{X}_i'\mathbf{M}_j\mathbf{M}_i\mathbf{X}_j), \tag{3.2.1}$$

where the matrices are all real. A short proof of this formula for arbitrary complex matrices $\mathbf{X}_i$ and $\mathbf{X}_j$ (and with prime denoting conjugate transpose) is given in Baksalary and Styan (1990). Since we are only interested in matrices that are real, we consider only these in the following lemma.

Lemma 3.2.1 (Baksalary and Styan, 1990). *For any real matrices* $\mathbf{A}_{p\times q}$ *and* $\mathbf{B}_{p\times s}$ *the rank of the product* $\mathbf{A}'\mathbf{B}$ *admits the representation*

$$r(\mathbf{A}'\mathbf{B}) = r(\mathbf{A}) + r(\mathbf{B}) - r(\mathbf{A} : \mathbf{B}) + r(\mathbf{A}'\mathbf{M_B}\mathbf{M_A}\mathbf{B}), \tag{3.2.2}$$

where the matrices $\mathbf{M_A}$ *and* $\mathbf{M_B}$ *are the orthogonal projectors on the orthocomplements of the column spaces, respectively, of* $\mathbf{A}$ *and* $\mathbf{B}$.

Proof. Using (A.3.2) and (A.3.8), we have

$$r(\mathbf{A}'\mathbf{M_B}\mathbf{M_A}\mathbf{B}) = r(\mathbf{A}'\mathbf{M_B}\mathbf{H_A}\mathbf{B}) = r(\mathbf{M_B}\mathbf{H_A}\mathbf{B}) = r(\mathbf{B} : \mathbf{H_A}\mathbf{B}) - r(\mathbf{B}). \tag{3.2.3}$$

Since elementary column operations do not change the rank, it follows that

$$r(\mathbf{B} : \mathbf{H_A}\mathbf{B}) = r(\mathbf{M_A}\mathbf{B} : \mathbf{H_A}\mathbf{B}) = r(\mathbf{M_A}\mathbf{B}) + r(\mathbf{H_A}\mathbf{B}) = r(\mathbf{A} : \mathbf{B}) - r(\mathbf{A}) + r(\mathbf{A}'\mathbf{B}). \tag{3.2.4}$$

Combining (3.2.4) with (3.2.3) yields (3.2.2). ❒

As was observed in Seshadri and Styan (1980), the number of positive canonical correlations, $m_{i,j}$, equals the rank $r(\mathbf{X}_i'\mathbf{X}_j)$ of the cross-covariance matrix between the vectors $\mathbf{X}_i'\mathbf{y}$ and $\mathbf{X}_j'\mathbf{y}$ assuming that the dispersion matrix $\mathfrak{D}(\mathbf{y}) = \sigma^2\mathbf{I}$. Moreover, $m_{i,j}$ also satisfies the following equation (3.2.5), stated but not proven in Styan (1986), but which follows at once from a more general matrix result in Baksalary and Styan (1990, Corollary 2).

Lemma 3.2.2: *The rank of the cross-covariance matrix between the vectors of centered row and column totals in a two-way layout equals*

$$r(\mathbf{X}_i'\mathbf{X}_j) = r(\mathbf{X}_i'\mathbf{C}_n\mathbf{X}_j) + 1. \tag{3.2.5}$$

Proof: Since $\mathbf{e}^{(n)} \in \mathcal{C}(\mathbf{X}_i) \cap \mathcal{C}(\mathbf{X}_j)$, we obtain from (A.3.8), $r(\mathbf{C}_n\mathbf{X}_i) = r(\mathbf{X}_i) - 1$, $r(\mathbf{C}_n\mathbf{X}_j) = r(\mathbf{X}_j) - 1$, and $r(\mathbf{C}_n\mathbf{X}_i : \mathbf{C}_n\mathbf{X}_j) = r(\mathbf{X}_i : \mathbf{X}_j) - 1$. Moreover, $\mathbf{M}_{(\mathbf{X}_i : \mathbf{e})} = \mathbf{M}_i$ and $\mathbf{M}_{(\mathbf{X}_j : \mathbf{e})} = \mathbf{M}_j$. Replacing $\mathbf{A}$ and $\mathbf{B}$ in (3.2.1) with $\mathbf{C}_n\mathbf{X}_i$ and $\mathbf{C}_n\mathbf{X}_j$, respectively, yields

$$r(\mathbf{X}_i'\mathbf{C}_n\mathbf{X}_j) = r(\mathbf{X}_i) - 1 + r(\mathbf{X}_j) - r(\mathbf{X}_i : \mathbf{X}_j) + r(\mathbf{X}_i'\mathbf{M}_j\mathbf{M}_i\mathbf{X}_j). \tag{3.2.6}$$

Equation (3.2.5) is then obtained by combining (3.2.1) and (3.2.6). ❒

Similarly, we get the number of nonzero canonical correlations of type (ii) from the rank of the cross-covariance matrix between $\mathbf{X}_i'\mathbf{y}$ and $(\mathbf{X}_j : \mathbf{X}_k)'\mathbf{y}$, i.e.,

$$m_{i.jk} = r[\mathbf{X}_i'(\mathbf{X}_j : \mathbf{X}_k)]. \tag{3.2.7}$$

Since $\mathbf{M}_k$ is symmetric and idempotent, the cross-covariance matrix $\sigma^2\mathbf{X}_i'\mathbf{M}_k\mathbf{X}_j$ between $\mathbf{X}_i'\mathbf{y}$ and $\mathbf{X}_j'\mathbf{M}_k\mathbf{y}$ is the same as the cross-covariance matrix between $\mathbf{X}_i'\mathbf{M}_k\mathbf{y}$ and $\mathbf{X}_j'\mathbf{M}_k\mathbf{y}$, and so

$$m_{i.j\,|k} = r(\mathbf{X}_i'\mathbf{M}_k\mathbf{X}_j) = m_{i;j\,|k}. \tag{3.2.8}$$

We see, therefore, that the numbers of nonzero canonical correlations of types (iii) and (iv) are the same; as we will see below, the numbers of unit and positive nonunit canonical correlations of type (iii) and (iv), however, are not necessarily equal.

The number $u_{i.j}$ of unit canonical correlations of type (i), i.e., when factor k is ignored and we are in a two-way layout or block design setting, is given by the nullity of the information matrix $\mathbf{S}_{ij}$

$$\psi(\mathbf{S}_{i.j}) = u_{i.j} = r(\mathbf{X}_i) + r(\mathbf{X}_j) - r(\mathbf{X}_i : \mathbf{X}_j), \tag{3.2.9}$$

cf. Seshadri and Styan (1980), Latour and Styan (1985). We notice that $u_{i.j}$ is equal to the first three terms on the right-hand side of equation (3.2.1). Moreover

$$u_{i.j} = \dim[\mathcal{C}(\mathbf{X}_i) \cap \mathcal{C}(\mathbf{X}_j)] \geq 1, \tag{3.2.10}$$

since $\mathbf{e}^{(n)} \in \mathcal{C}(\mathbf{X}_i) \cap \mathcal{C}(\mathbf{X}_j)$.

We may also express $u_{i.j}$ in terms of the hat matrices $\mathbf{H}_i$ and $\mathbf{H}_j$ as follows:

$$u_{i.j} = r(\mathbf{H}_i\mathbf{H}_j) - r(\mathbf{H}_i\mathbf{M}_j\mathbf{M}_i\mathbf{H}_j) = r(\mathbf{H}_i) + r(\mathbf{H}_j) - r(\mathbf{H}_i + \mathbf{H}_j).$$

To prove this it suffices to show that $r(\mathbf{H}_i + \mathbf{H}_j) = r(\mathbf{X}_i : \mathbf{X}_j) = r(\mathbf{H}_i : \mathbf{H}_j)$, since $r(\mathbf{H}_i) = r(\mathbf{X}_i)$, $r(\mathbf{H}_j) = r(\mathbf{X}_j)$, $r(\mathbf{H}_i\mathbf{H}_j) = r(\mathbf{X}_i'\mathbf{X}_j)$, and $r(\mathbf{H}_i\mathbf{M}_j\mathbf{M}_i\mathbf{H}_j) = r(\mathbf{X}_i'\mathbf{M}_j\mathbf{M}_i\mathbf{X}_j)$. However,

$$r(\mathbf{H}_i : \mathbf{H}_j) = r\left[(\mathbf{H}_i : \mathbf{H}_j)\begin{pmatrix}\mathbf{H}_i' \\ \mathbf{H}_j'\end{pmatrix}\right] = r(\mathbf{H}_i + \mathbf{H}_j).$$

Similarly, we obtain the number of unit canonical correlations of type (ii), i.e., where the third factor is included,

$$\begin{aligned} u_{i.jk} &= \psi(\mathbf{S}_{i.jk}) = r(\mathbf{X}_i) + r(\mathbf{X}_j : \mathbf{X}_k) - r(\mathbf{X}_i : \mathbf{X}_j : \mathbf{X}_k) \\ &= \dim[\mathcal{C}(\mathbf{X}_i) \cap \mathcal{C}(\mathbf{X}_j : \mathbf{X}_k)] = r(\mathbf{H}_i) + r(\mathbf{H}_{jk}) - r(\mathbf{H}_i + \mathbf{H}_{jk}) \geq 1, \end{aligned} \tag{3.2.11}$$

since $\mathbf{e}^{(n)} \in \mathcal{C}(\mathbf{X}_i) \cap \mathcal{C}(\mathbf{X}_j : \mathbf{X}_k)$.

For canonical correlations of type (iii), where there is partial adjustment only of factor j for factor k, we have

$$\begin{aligned} u_{i;j|k} &= \psi(\mathbf{X}_i'\mathbf{M}_{j|k}\mathbf{X}_i) = r(\mathbf{X}_i) + r(\mathbf{M}_k\mathbf{X}_j) - r(\mathbf{X}_i : \mathbf{M}_k\mathbf{X}_j) \\ &= \dim[\mathcal{C}(\mathbf{X}_i) \cap \mathcal{C}(\mathbf{M}_k\mathbf{X}_j)] = r(\mathbf{H}_i) + r(\mathbf{H}_{j|k}) - r(\mathbf{H}_i + \mathbf{H}_{j|k}) \geq 0\,; \end{aligned} \tag{3.2.12}$$

we note that $u_{i;j|k}$ may equal 0 since the vector $\mathbf{e}^{(n)}$ does not belong to $\mathcal{C}(\mathbf{X}_i) \cap \mathcal{C}(\mathbf{M}_k\mathbf{X}_j)$, and that $u_{i;j|k} = 0$ if and only if $\mathrm{r}(\mathbf{H}_i + \mathbf{H}_{j|k}) = \mathrm{r}(\mathbf{H}_i) + \mathrm{r}(\mathbf{H}_{j|k})$.

For the adjusted canonical correlations of type (iv), where there is partial adjustment of both factors i and j for factor k,

$$\begin{aligned} u_{i.j|k} &= \psi[\mathbf{X}_i'\mathbf{M}_k\mathbf{X}_i - \mathbf{X}_i'\mathbf{M}_k\mathbf{X}_j(\mathbf{X}_j'\mathbf{M}_k\mathbf{X}_j)^{-}\mathbf{X}_j'\mathbf{M}_k\mathbf{X}_i] \\ &= \mathrm{r}(\mathbf{M}_k\mathbf{X}_i) + \mathrm{r}(\mathbf{M}_k\mathbf{X}_j) - \mathrm{r}(\mathbf{M}_k\mathbf{X}_i : \mathbf{M}_k\mathbf{X}_j) \\ &= \dim[\mathcal{C}(\mathbf{M}_k\mathbf{X}_i)\cap\mathcal{C}(\mathbf{M}_k\mathbf{X}_j)] = \mathrm{r}(\mathbf{H}_{i|k}) + \mathrm{r}(\mathbf{H}_{j|k}) - \mathrm{r}(\mathbf{H}_{i|k} + \mathbf{H}_{j|k}) \geq 0\,; \end{aligned} \tag{3.2.13}$$

again $u_{i.j|k}$ may equal 0 since the vector $\mathbf{e}^{(n)}$ does not belong to $\mathcal{C}(\mathbf{M}_k\mathbf{X}_i) \cap \mathcal{C}(\mathbf{M}_k\mathbf{X}_j)$, and $u_{i.j|k} = 0$ if and only if $\mathrm{r}(\mathbf{H}_{i|k} + \mathbf{H}_{j|k}) = \mathrm{r}(\mathbf{H}_{i|k}) + \mathrm{r}(\mathbf{H}_{j|k})$. As $\mathrm{r}(\mathbf{M}_k\mathbf{X}_i) \leq \mathrm{r}(\mathbf{X}_i)$ and $\mathrm{r}[\mathbf{M}_k(\mathbf{X}_i : \mathbf{M}_k\mathbf{X}_j)] \leq \mathrm{r}(\mathbf{X}_i : \mathbf{M}_k\mathbf{X}_j)$ with the differences not necessarily equal, we see from (3.2.12) and (3.2.13) that $u_{i;j|k}$ and $u_{i.j|k}$ need not be equal.

Furthermore, we can determine the numbers t of canonical correlations which are strictly less than one. In the block design setting, Latour and Styan (1985) prove that $t_{i.j}$ is the rank of the cross-covariance matrix between the vectors of adjusted totals:

$$t_{i.j} = \mathrm{r}(\mathbf{X}_i'\mathbf{M}_j\mathbf{M}_i\mathbf{X}_j). \tag{3.2.14}$$

Comparing (3.2.14) with (3.2.1) we see that (3.2.14) is the last term on the right-hand side of (3.2.1).

Similarly we get numbers t for canonical correlations of types (ii), (iii) and (iv):

$$t_{i.jk} = \mathrm{r}[\mathbf{X}_i'\mathbf{M}_{jk}\mathbf{M}_i(\mathbf{X}_j : \mathbf{X}_k)], \tag{3.2.15}$$

$$t_{i;j|k} = \mathrm{r}\{\mathbf{X}_i'[\mathbf{I} - \mathbf{M}_k\mathbf{X}_j(\mathbf{X}_j'\mathbf{M}_k\mathbf{X}_j)^{-}\mathbf{X}_j'\mathbf{M}_k]\mathbf{M}_i\mathbf{M}_k\mathbf{X}_j\} = \mathrm{r}(\mathbf{X}_i'\mathbf{M}_{j|k}\mathbf{M}_i\mathbf{M}_k\mathbf{X}_j) \tag{3.2.16}$$

and

$$t_{i,j|k} = \mathrm{r}[\mathbf{X}_i'\mathbf{M}_k\mathbf{M}_{j|k}\mathbf{M}_{i|k}\mathbf{M}_k\mathbf{X}_j] = \mathrm{r}(\mathbf{X}_i'\mathbf{M}_{jk}\mathbf{M}_{ik}\mathbf{X}_j). \tag{3.2.17}$$

This last equality holds since the column space $\mathcal{C}(\mathbf{X}_i : \mathbf{X}_k) = \mathcal{C}(\mathbf{X}_i) \oplus^{\perp} \mathcal{C}(\mathbf{M}_i\mathbf{X}_k)$, as noted earlier, and hence

$$\mathbf{M}_{ik}\mathbf{X}_j = [(\mathbf{I} - \mathbf{H}_k - \mathbf{H}_{i|k})\mathbf{M}_k\mathbf{X}_j] = \mathbf{M}_{i|k}\mathbf{M}_k\mathbf{X}_j. \tag{3.2.18}$$

We can see from equations (3.2.16) and (3.2.17) that the numbers of canonical correlations of types (iii) and (iv) which are strictly less than one need not be equal since the ranks $\mathrm{r}(\mathbf{X}_i'\mathbf{M}_{j|k}\mathbf{M}_i\mathbf{M}_k\mathbf{X}_j)$ and $\mathrm{r}(\mathbf{X}_i'\mathbf{M}_{jk}\mathbf{M}_{ik}\mathbf{X}_j)$ are not necessarily equal.

Theorem 3.2.3: *The number $t_{i,j}$ of nonzero canonical correlations between factor i and factor j that are strictly less than* 1 *satisfies*

$$t_{i,j} = \mathrm{rank}[(\mathbf{H}_i\mathbf{H}_j)^2 - \mathbf{H}_i\mathbf{H}_j] = \mathrm{rank}(\mathbf{H}_i\mathbf{H}_j - \mathbf{H}_i\mathbf{H}_j\mathbf{H}_i) = (1/2)\mathrm{rank}(\mathbf{H}_i\mathbf{H}_j - \mathbf{H}_j\mathbf{H}_i). \tag{3.2.19}$$

Moreover, $t_{i,j} = 0$ if and only if $\mathbf{H}_i\mathbf{H}_j = \mathbf{H}_j\mathbf{H}_i$.

Proof: From (3.2.14),

$$\begin{aligned} t_{i,j} &= \mathrm{r}(\mathbf{X}_i'\mathbf{M}_j\mathbf{M}_i\mathbf{X}_j) = \mathrm{r}[\mathbf{X}_i'(-\mathbf{H}_i + \mathbf{H}_j\mathbf{H}_i)\mathbf{X}_j] \\ &= \mathrm{r}[\mathbf{X}_i'(\mathbf{H}_j\mathbf{H}_i - \mathbf{I})\mathbf{X}_j] = \mathrm{r}[\mathbf{H}_i(\mathbf{H}_j\mathbf{H}_i - \mathbf{I})\mathbf{H}_j] = \mathrm{r}[(\mathbf{H}_i\mathbf{H}_j)^2 - \mathbf{H}_i\mathbf{H}_j], \end{aligned} \tag{3.2.20}$$

since $\mathbf{H}_i = \mathbf{X}_i(\mathbf{X}_i'\mathbf{X}_i)^-\mathbf{X}_i'$ and $\mathrm{r}(\mathbf{H}_i) = \mathrm{r}(\mathbf{X}_i')$; this proves the first equality in (3.2.19). We may, however, rewrite (3.2.20) as

$$t_{i.j} = \mathrm{r}(\mathbf{H}_i\mathbf{H}_j\mathbf{M}_i\mathbf{H}_j) = \mathrm{r}(\mathbf{H}_i\mathbf{H}_j\mathbf{M}_i) = \mathrm{r}(\mathbf{H}_i\mathbf{H}_j - \mathbf{H}_i\mathbf{H}_j\mathbf{H}_i), \qquad (3.2.21)$$

since $\mathrm{r}(\mathbf{H}_j\mathbf{M}_i\mathbf{H}_j) = \mathrm{r}(\mathbf{H}_j\mathbf{M}_i)$. This establishes the second equality in (3.2.19).

To prove* the third equality in (3.2.19), we write

$$\mathbf{T}_{ij} = \mathbf{H}_i\mathbf{H}_j - \mathbf{H}_i\mathbf{H}_j\mathbf{H}_i = \mathbf{H}_i\mathbf{H}_j\mathbf{M}_i$$

and observe that $\mathbf{T}_{ij}^2 = \mathbf{0}$. From (3.2.2) it follows that

$$\mathrm{r}(\mathbf{T}_{ij} : \mathbf{T}'_{ij}) = \mathrm{r}(\mathbf{T}_{ij}) + \mathrm{r}(\mathbf{T}'_{ij}) = 2\mathrm{r}(\mathbf{T}_{ij}) = 2t_{i.j}$$

and so

$$2t_{i.j} = \mathrm{r}[(\mathbf{T}_{ij} : \mathbf{T}'_{ij})(\mathbf{T}_{ij} : \mathbf{T}'_{ij})'] = \mathrm{r}(\mathbf{T}_{ij}\mathbf{T}'_{ij} + \mathbf{T}'_{ij}\mathbf{T}_{ij}) = \mathrm{r}[(\mathbf{T}_{ij} - \mathbf{T}'_{ij})(\mathbf{T}_{ij} - \mathbf{T}'_{ij})']$$

$$= \mathrm{r}(\mathbf{T}_{ij} - \mathbf{T}'_{ij}) = \mathrm{r}(\mathbf{H}_i\mathbf{H}_j - \mathbf{H}_j\mathbf{H}_i).$$

which completes the proof. ❒

EXAMPLE 3.2.1: For the layout

1	2	3
1	*	*
*	2	*
*	*	3

(3.2.22)

considered by Baksalary (1990) we have $t_{1.2} = 2$ (cf. Table A.1.3), while $\mathrm{r}(\mathbf{H}_1\mathbf{H}_2 - \mathbf{H}_2\mathbf{H}_1) = 4$. More generally, in the layout

* My thanks go to Dr. Robert E. Hartwig for help with this proof.

1	2	...	*b*
1	*	...	*
*	2	...	*
...	...	...	...
*	*	...	*b*

it is straightforward to show directly that $t_{1.2} = b - 1$, while $r(\mathbf{H}_1\mathbf{H}_2 - \mathbf{H}_2\mathbf{H}_1) = 2(b-1) = 2t_{1.2}$.

We note that equation (3.2.1) is very useful to summarize, for all four types of canonical correlations listed at the beginning of Section 3.1, the relationship between the associated numbers *m*, *u* and *t*. We have the following four equations, obtained according as we ignore, include, adjust partially or adjust totally for factor *k*.

$$\text{(i)}\quad r(\mathbf{X}_i'\mathbf{X}_j) = r(\mathbf{X}_i) + r(\mathbf{X}_j) - r(\mathbf{X}_i : \mathbf{X}_j) + r(\mathbf{X}_i'\mathbf{M}_j\mathbf{M}_i\mathbf{X}_j)$$
$$m_{i.j} = u_{i.j} + t_{i.j}$$

$$\text{(ii)}\quad r[\mathbf{X}_i'(\mathbf{X}_j : \mathbf{X}_k)] = r(\mathbf{X}_i) + r(\mathbf{X}_j : \mathbf{X}_k) - r(\mathbf{X}_i : \mathbf{X}_j : \mathbf{X}_j) + r[\mathbf{X}_i'\mathbf{M}_{jk}\mathbf{M}_i(\mathbf{X}_j : \mathbf{X}_k)]$$
$$m_{i.jk} = u_{i.jk} + t_{i.jk}$$

$$\text{(iii)}\quad r(\mathbf{X}_i'\mathbf{M}_k\mathbf{X}_j) = r(\mathbf{X}_i) + r(\mathbf{M}_k\mathbf{X}_j) - r(\mathbf{X}_i : \mathbf{M}_k\mathbf{X}_j) + r(\mathbf{X}_i'\mathbf{M}_{j|k}\mathbf{M}_i\mathbf{M}_k\mathbf{X}_j)$$
$$m_{i;j|k} = u_{i;j|k} + t_{i;j|k}$$

$$\text{(iv)}\quad r(\mathbf{X}_i'\mathbf{M}_k\mathbf{X}_j) = r(\mathbf{M}_k\mathbf{X}_i) + r(\mathbf{M}_k\mathbf{X}_j) - r(\mathbf{M}_k\mathbf{X}_i : \mathbf{M}_k\mathbf{X}_j) + r(\mathbf{X}_i'\mathbf{M}_{jk}\mathbf{M}_{ik}\mathbf{X}_j)$$
$$m_{i.j|k} = u_{i.j|k} + t_{i.j|k}$$

We note again that while $m_{i;j|k} = m_{i.j|k} = r(\mathbf{X}_i'\mathbf{M}_k\mathbf{X}_j)$ it is not necessary that $u_{i;j|k} = u_{i.j|k}$ or that $t_{i;j|k} = t_{i.j|k}$.

SECTION 3.3: CONNECTEDNESS

In the context of two-way layouts or block designs, Bose (1947) introduced the concept of connectedness, which has subsequently been widely studied.

Definition 3.3.1: A two-way layout or block design is connected for treatments whenever all elementary treatment contrasts $\mathbf{c}'\boldsymbol{\tau}$, for any $v \times 1$ vector satisfying $\mathbf{c}'\mathbf{e}^{(v)} = 0$, are unbiasedly estimable.

This means that unless the two-way layout is connected for treatments, certain contrasts in the treatments are not unbiasedly estimable. There is also a combinatorial definition of connectedness, cf. e.g., Raghavarao (1971, p.49), which says that the two-way layout is connected whenever, given any two treatments τ and τ', we can construct a chain of treatments numbered $\tau = \tau_0, \tau_1, \ldots, \tau_p = \tau'$, say, such that every consecutive pair of treatments in the chain occurs together in at least one block. Bose (1947) gave both the statistical and combinatorial formulations for connectedness.

For the three-way layout, however, additional definitions are required since the properties and characteristics of connectedness in a (single) block design are not necessarily equivalent to those in a three-way layout, though we can easily extend the above Definition 3.3.1 to a similar one for connectedness of the treatments in a two-way elimination of heterogeneity design.

Definition 3.3.2: A two-way elimination of heterogeneity design is connected for treatments whenever all elementary treatment contrasts $\mathbf{c}'\boldsymbol{\tau}$, for any $v \times 1$ vector satisfying $\mathbf{c}'\mathbf{e}^{(v)} = 0$, are unbiasedly estimable in the design.

Raghavarao and Federer (1975) call such a two-way elimination of heterogeneity design "doubly connected", while Styan (1986) says that then "treatments are connected with both rows and columns".

Even when the condition in Definition 3.3.2 is satisfied, however, not all the elementary row and column contrasts are necessarily estimable. To guarantee that they are estimable, we also need to have connectedness for rows and for columns, the definition of which is similar to that of connectedness for treatments. Therefore, when we are interested in all three factors—rows, columns and treatments—the overall connectedness of the design is required. Eccleston and Russell (1975) give the following definition.

Definition 3.3.3: A three-way layout is completely connected if it is connected for rows, for columns and for treatments.

This means that if the design is completely connected, every row, column and treatment contrast is estimable.

It is well known [cf. Chakrabarti (1962)] that a two-way layout or block design is connected for factor 1 (of size r) if and only if $r(\mathbf{S}_{1.2}) = r - 1$ or only one canonical correlation $\rho_h^{(1.2)} = 1$, which we denote by $u_{1.2} = 1$. In the case of a block design, this corresponds to the design itself being connected, i.e., connected for both factors 1 and 2, as $u_{1.2} = u_{2.1}$. Since the treatment-row and treatment-column subdesigns are essentially block designs, they are each connected if and only if $u_{1.3} = 1$ and $u_{2.3} = 1$, respectively. Similarly, a two-way elimination of heterogeneity design is connected for treatments if and only if $r(\mathbf{S}_{3.12}) = v - 1$ or only one $\rho_h^{(3.12)} = 1$, which we denote by $u_{3.12} = 1$.

If the design is connected for treatments then $\mathbf{S}_{3.12} + \mathbf{J}_v$ is nonsingular (cf. Lemma A.3.6). In this situation, the reduced normal equations for studying treatment effects have a simple solution since the matrix $(\mathbf{S}_{3.12} + \mathbf{J}_v)^{-1}$ is then a generalized inverse of $\mathbf{S}_{3.12}$. This can be shown by first noticing that $\mathbf{J}_v = (\mathbf{S}_{3.12} + \mathbf{J}_v)\mathbf{J}_v$, which implies that $\mathbf{J}_v = (\mathbf{S}_{3.12} + \mathbf{J}_v)^{-1}\mathbf{J}_v$. From this we then get the following equality because of the nonsingularity of $\mathbf{S}_{3.12} + \mathbf{J}_v$,

$$(\mathbf{S}_{3.12} + \mathbf{J}_v)^{-1}\mathbf{S}_{3.12} = (\mathbf{S}_{3.12} + \mathbf{J}_v)^{-1}[(\mathbf{S}_{3.12} + \mathbf{J}_v) - \mathbf{J}_v] = \mathbf{I}_v - \mathbf{J}_v = \mathbf{C}_v. \qquad (3.3.1)$$

Premultiplying (3.3.1) by $\mathbf{S}_{3.12}$ shows that indeed $\mathbf{S}_{3.12}(\mathbf{S}_{3.12} + \mathbf{J}_v)^{-1}\mathbf{S}_{3.12} = \mathbf{S}_{3.12}$. We can, therefore, write a solution to the reduced normal equations in (2.2.8) as

$$\hat{\boldsymbol{\tau}} = (\mathbf{S}_{3.12} + \mathbf{J}_v)^{-1}\mathbf{z}_{3.12}. \tag{3.3.2}$$

More generally, John (1965) has shown that provided $r(\mathbf{S}_{3.12}) = v - 1$, then the matrix $\mathbf{S}_{3.12} + a\mathbf{hh}'$ is positive definite, where $a > 0$ and $\mathbf{h}$ is any $v \times 1$ vector such that $\mathbf{h}$ does not belong to $\mathcal{C}(\mathbf{X}_3'\mathbf{M}_{12})$ or equivalently, the columns of $\mathbf{S}_{3.12}$ together with $\mathbf{h}$, span a space of dimension v. For example, one such positive definite matrix is the $\boldsymbol{\Omega}$–matrix introduced by Tocher (1952),

$$\boldsymbol{\Omega} = [\mathbf{X}_3'(\mathbf{M}_{12} + \mathbf{J}_n)\mathbf{X}_3]^{-1} = (\mathbf{S}_{3.12} + \frac{\mathbf{k}_3\mathbf{k}_3'}{n})^{-1}. \tag{3.3.3}$$

This matrix is now often called Tocher's $\boldsymbol{\Omega}$–matrix and is frequently used both in block designs {in the form $\boldsymbol{\Omega} = [\mathbf{X}_1'(\mathbf{M}_2 + \mathbf{J}_n)\mathbf{X}_1]^{-1}$ if interest resides in factor 1} and in two-way elimination of heterogeneity designs [cf. for example, Caliński (1971), Pearce (1975) and Singh and Dey (1978)].

Another type of connectedness is that found between rows and columns adjusted for treatments. This is characterized by $u_{1.2|3} = 0$, i.e., all the nonzero canonical correlations $\rho_h^{(1.2|3)}$ being less than 1 (the common vector $\mathbf{e}^{(n)}$ having been eliminated). Khatri and Shah (1986) call $u_{i.j|k}$ the degree of disconnectedness of classification (factor) i with respect to j after adjusting for k.

An important consideration in three-way layouts is to know how connectedness for one factor (or all three) is related to connectedness in the subdesigns and connectedness between two factors adjusted for the third. These three different properties are not independent of each other. Our next theorem, due to Styan (1986), gives a relationship between the numbers u of unit canonical correlations in these three situations. From this, we can see how different types of connectedness might follow from one and another [cf. also Khatri and Shah (1986)].

Theorem 3.3.1 (Styan, 1986): *For $i \neq j$, $i \neq k$ and $j \neq k$, the following equalities hold*

$$u_{i.j|k} = u_{i.jk} - u_{i.k} = u_{j.ik} - u_{j.k}. \tag{3.3.4}$$

Proof: From equation (3.2.11) we have

$$\begin{aligned} u_{i.jk} &= r(\mathbf{X}_i) + r(\mathbf{X}_j : \mathbf{X}_k) - r(\mathbf{X}_i : \mathbf{X}_j : \mathbf{X}_k) \\ &= u_{i.k} - r(\mathbf{X}_k) + r(\mathbf{X}_i : \mathbf{X}_k) + r(\mathbf{X}_j : \mathbf{X}_k) - r(\mathbf{X}_i : \mathbf{X}_j : \mathbf{X}_k). \end{aligned}$$

Similarly, we get

$$u_{i.j|k} = r(\mathbf{M}_k\mathbf{X}_i) + r(\mathbf{M}_k\mathbf{X}_j) - r(\mathbf{M}_k\mathbf{X}_i : \mathbf{M}_k\mathbf{X}_j) \tag{3.3.5}$$

from equation (3.2.13). Since $\mathbf{M}_k = \mathbf{I} - \mathbf{X}_k\mathbf{X}_k^+$, we can use result (A.3.8) three times in (3.3.5) to get

$$\begin{aligned} u_{i.j|k} &= [r(\mathbf{X}_i : \mathbf{X}_k) - r(\mathbf{X}_k)] + [r(\mathbf{X}_j : \mathbf{X}_k) - r(\mathbf{X}_k)] - [r(\mathbf{X}_i : \mathbf{X}_j : \mathbf{X}_k) - r(\mathbf{X}_k)] \\ &= u_{i.jk} - u_{i.k}, \end{aligned}$$

which proves the first equality in (3.3.4). The proof for the second equality in (3.3.4) is similar. ❒

From equation (3.3.4) we see that the number $u_{i.j|k}$ gives the difference between the number of linearly independent linear contrasts in factor i which are estimable in the whole two-way elimination of heterogeneity design, and the number which are estimable in the subdesign obtained by ignoring the j^{th} factor. If $u_{i.j|k} = 0$, then the first equality in (3.3.4) means that the design is connected for factor i if it is connected for factor i in the subdesign obtained by ignoring factor j, while the second equality in (3.3.4) means that the design is connected for factor j if it is connected for factor j when factor i is ignored.

To assess the overall connectedness of the design, we define

$$u_{1.2.3} = \mathrm{r}(\mathbf{X}_1) + \mathrm{r}(\mathbf{X}_2) + \mathrm{r}(\mathbf{X}_3) - \mathrm{r}(\mathbf{X}_1 : \mathbf{X}_2 : \mathbf{X}_3). \tag{3.3.6}$$

Since $\mathbf{e}^{(n)} \in \mathcal{C}(\mathbf{X}_1) \cap \mathcal{C}(\mathbf{X}_2) \cap \mathcal{C}(\mathbf{X}_3)$ it follows that $u_{1.2.3} \geq 2$. Using Corollary 3.3.2, it is easy to see that a design is completely connected if and only if $u_{1.2.3} = 2$. In our *mutzE*-tables in Appendix 1, we have included a bottom line for the value of $u_{1.2.3}$ so that we can immediately identify the inherent connectedness or degree of disconnectedness.

The number $u_{1.2.3}$ is related to the other numbers u of unit canonical correlations in the following ways.

Corollary 3.3.2 (Styan, 1986). *For all $i \neq j$, $i \neq k$ and $j \neq k$; $i, j, k = 1, 2, 3$*

$$u_{1.2.3} = u_{i.jk} + u_{j.k} = u_{i.k} + u_{j.k} + u_{i.j|k} \tag{3.3.7}$$

Proof: The first equality follows at once from

$$u_{i.jk} + u_{j.k} = [\mathrm{r}(\mathbf{X}_i) + \mathrm{r}(\mathbf{X}_j : \mathbf{X}_k) - \mathrm{r}(\mathbf{X}_i : \mathbf{X}_j : \mathbf{X}_k)] + [\mathrm{r}(\mathbf{X}_j) + \mathrm{r}(\mathbf{X}_k) - \mathrm{r}(\mathbf{X}_j : \mathbf{X}_k)]$$

$$= \mathrm{r}(\mathbf{X}_i) + \mathrm{r}(\mathbf{X}_j) + \mathrm{r}(\mathbf{X}_k) - \mathrm{r}(\mathbf{X}_i : \mathbf{X}_j : \mathbf{X}_k) = u_{1.2.3}.$$

The second equality is obtained directly by combining (3.3.4) and the first equality in (3.3.7). ❐

Eccleston and Russell (1975) show in their Theorem 2 that a two-way elimination of heterogeneity design is completely connected, i.e., $u_{1.2.3} = 2$, if and only if it is connected for factor i and has its subdesign ignoring factor i connected, i.e.,

$$u_{ij.k} = 1 \text{ and } u_{j.k} = 1,\ i \neq j,\ i \neq k,\ j \neq k, \text{ for any } i, j, k = 1, 2, 3 \iff u_{1.2.3} = 2 \tag{3.3.8}$$

(cf. also Raghavarao and Federer, 1975). Result (3.3.8) follows directly from the first equality in Corollary 3.3.2. In particular, we see from both (3.3.7) and (3.3.8) that a row-column design is completely connected if and only if it is connected for treatments. It also follows from (3.3.7) and (3.3.8) that if a two-way elimination of heterogeneity design is completely connected, then $u_{1.2} = u_{1.3} = u_{2.3} = u_{1.23} = u_{2.13} = u_{3.12} = 1$ and $u_{1.2\,|\,3} = u_{1.3\,|\,2} = u_{2.3\,|\,1} = 0$.

If our interest lies only in the treatment effects (two-way elimination of heterogeneity designs), then we are only concerned with connectedness for treatments. A problem which seems not yet to have been completely solved, however, concerns the relationship between connectedness for treatments ($u_{3.12} = 1$) in a two-way elimination of heterogeneity design and connectedness in its treatment-row ($u_{1.3} = 1$) and treatment-column subdesigns ($u_{2.3} = 1$). Raghavarao and Federer (1975) showed that if a two-way elimination of heterogeneity design is connected for treatments, then the treatment-row and treatment-column subdesigns are also connected (the row-column subdesign need not, however, be connected, i.e., $u_{1.2}$ need not be equal to 1). However, the converse of this statement is not generally true as was shown by Shah and Khatri (1973).

EXAMPLE 3.3.1 (Shah and Khatri, 1973): For the following design

1	2	5	6
3	4	7	8
8	6	1	3
7	5	2	4

both the treatment-row and treatment-column subdesigns are connected, i.e., both $u_{1.3} = 1$ and $u_{2.3} = 1$, but the overall design itself is not connected for treatments, i.e., $u_{3.12} \neq 1$. Associated with this design is Table A.1.4, given in part in Styan (1986, Table 2.1). We can see that although $u_{1.3} = u_{2.3} = 1$, the design is not connected for treatments since $u_{3.12} = 2$. We can also verify that the vector $\mathbf{a} = (0\ 1\ -1\ 0\ 1\ 0\ 0\ -1)'$ is orthogonal to each row of

$\mathbf{S}_{3.12}$, i.e., $\mathbf{S}_{3.12}\mathbf{a} = \mathbf{0}$, implying that $r(\mathbf{S}_{3.12}) \leq v-2$, since $\mathbf{a}$ is not a scalar multiple of $\mathbf{e}$. Thus, the treatment contrast

$$(2)-(3)+(5)-(8) \tag{3.3.9}$$

is not estimable. We can show this by noting that $r(\mathbf{X}' : \mathbf{a}) = 14$ while $r(\mathbf{X}) = 13$, i.e., $r(\mathbf{X}' : \mathbf{a}) \neq r(\mathbf{X})$, where $\mathbf{X}$ is the full design matrix and $\mathbf{a}'\boldsymbol{\gamma}$ gives the linear function of treatments in (3.3.9). This inequality between the ranks implies that $\mathbf{a}'\boldsymbol{\gamma}$ is not estimable [cf. e.g., Alalouf and Styan (1979)].

Furthermore, from (3.3.7) and (3.3.8), we have the equality $u_{3.12} = u_{1.3} + u_{2.3} - u_{1.2} + u_{1.2|3}$, which implies that indeed $u_{1.3} = 1$ and $u_{2.3} = 1$ together are not sufficient for $u_{3.12} = 1$ to hold unless $u_{1.2} = u_{1.2|3} + 1$ also holds. In particular, Raghavarao and Federer (1975) show that for equireplicate row-column designs satisfying the condition $\mathbf{N}_{13}\mathbf{N}_{32} = k\mathbf{e}^{(r)}\mathbf{e}^{(n)\prime}$ (which implies $u_{1.2|3} = 0$ as we will see in the next Section 3.4 on orthogonality), connectedness of the treatment-row and treatment-column subdesigns does lead to treatment-connectedness. This result was first strengthened by Sia (1977) who showed that when $\mathbf{S}_{3.1}$ and $\mathbf{S}_{3.2}$ commute in an equireplicate row-column design (or equivalently when $\mathbf{N}_{31}\mathbf{N}_{13}$ and $\mathbf{N}_{32}\mathbf{N}_{23}$ commute, and $\mathbf{N}_{12} = \mathbf{e}^{(r)}\mathbf{e}^{(c)\prime}$), then $u_{1.3} = u_{2.3} = 1$ implies $u_{3.12} = 1$ if and only if the sums of the eigenvalues of $\mathbf{S}_{3.1}$ and $\mathbf{S}_{3.2}$ corresponding to the same eigenvectors are different from k, the number of replications of each treatment. The commutativity of $\mathbf{S}_{3.1}$ and $\mathbf{S}_{3.2}$ by itself is not sufficient for this result to still hold, however, as was again shown by the design in Shah and Khatri (1973), cf. our Example 3.3.1 and Table A.1.4, where $\mathbf{S}_{3.1}$ and $\mathbf{S}_{3.2}$ do commute.

The equireplicate condition was relaxed in Baksalary and Kala (1980), where the more general commutativity condition

$$\mathbf{S}_{3.1}\mathbf{D}_3^{-1}\mathbf{S}_{3.2} = \mathbf{S}_{3.2}\mathbf{D}_3^{-1}\mathbf{S}_{3.1} \tag{3.3.10}$$

was considered. Introducing the efficiency matrices

$$\mathbf{A}_{3.1} = \mathbf{D}_3^{-1/2}\mathbf{S}_{3.1}\mathbf{D}_3^{-1/2} \quad \text{and} \quad \mathbf{A}_{3.2} = \mathbf{D}_3^{-1/2}\mathbf{S}_{3.2}\mathbf{D}_3^{-1/2} \tag{3.3.11}$$

(cf. Section 3.5), we may express (3.3.10) as

$$\mathbf{A}_{3.1}\mathbf{A}_{3.2} = \mathbf{A}_{3.2}\ \mathbf{A}_{3.1}$$

which from now on we will refer to as the "commutativity property" (for more about this commutativity property see Section 4.4).

If the commutativity property holds then the efficiency matrices $\mathbf{A}_{3.1}$, $\mathbf{A}_{3.2}$ and $\mathbf{A}_{3.0}$ are all spanned by the same set of eigenvectors, i.e., there exists an orthogonal matrix $\mathbf{U}$ such that $\mathbf{U}'\mathbf{A}_g\mathbf{U}$, $g = 3.1, 3.2, 3.0$, are all diagonal matrices; the efficiency matrix

$$\mathbf{A}_{3.0} = \mathbf{D}_3^{-1/2}\mathbf{S}_{3.0}\mathbf{D}_3^{-1/2}.$$

Furthermore, if our generalized decomposability property (2.3.1) is also satisfied, then the matrix $\mathbf{U}'\mathbf{A}_{3.12}\mathbf{U}$ will be diagonal. In the following theorem, we give an extension for two-way elimination of heterogeneity designs with equal row sizes and equal column sizes, satisfying our generalized decomposability property (2.3.1). Our proof follows that of Baksalary and Kala (1980).

Theorem 3.3.3: *Consider a two-way elimination of heterogeneity design which is ordinary, i.e., with equal row and column sizes:* $\mathbf{k}_1 = k_1\mathbf{e}^{(r)}$ *and* $\mathbf{k}_2 = k_2\mathbf{e}^{(c)}$, *which satisfies both the generalized decomposability property*

$$\mathbf{S}_{3.12} = \xi_1\mathbf{S}_{3.1} + \xi_2\mathbf{S}_{3.2} - \xi_0\mathbf{S}_{3.0},\ \ \xi_1, \xi_2 \text{ and } \xi_0 > 0,$$

and the commutativity property

$$\mathbf{A}_{3.1}\mathbf{A}_{3.2} = \mathbf{A}_{3.2}\mathbf{A}_{3.1}.$$

If the treatment-row and treatment-column subdesigns are connected, then the design itself is connected for treatments if and only if

$$\xi_1\phi_s^{(3.1)} + \xi_2\phi_s^{(3.2)} \neq \xi_0 \qquad s = 1, \ldots, v-1, \tag{3.3.12}$$

where $\phi_s^{(3.1)}$ *and* $\phi_s^{(3.2)}$ *are eigenvalues of, respectively,* $\mathbf{A}_{3.1}$ *and* $\mathbf{A}_{3.2}$ *corresponding to the same eigenvector.*

Furthermore, (3.3.12) *is also equivalent to*

$$\xi_1 k_2 \mu_s + \xi_2 k_1 \omega_s \neq k_1 k_2(\xi_1 + \xi_2 - \xi_0), \quad s = 1, \ldots, v-1, \tag{3.3.13}$$

where μ_s *is an eigenvalue of* $\mathbf{N}_{31}\mathbf{N}_{13}\mathbf{D}_3^{-1}$ *not equal to* k_1, *and* ω_s *is that eigenvalue of* $\mathbf{N}_{32}\mathbf{N}_{23}\mathbf{D}_3^{-1}$ *not equal to* k_2 *and which corresponds to the same eigenvector as does the eigenvalue* μ_s.

Proof: We have a two-way elimination of heterogeneity design with efficiency matrices satisfying the following relation:

$$\mathbf{A}_{3.12} = \xi_1\mathbf{A}_{3.1} + \xi_2\mathbf{A}_{3.2} - \xi_0\mathbf{A}_{3.0}, \qquad \xi_1, \xi_2 \text{ and } \xi_0 > 0 \tag{3.3.14}$$

in view of our generalized decomposability property (2.3.1). Since we assume that the design satisfies the commutativity property (3.3.10), the three matrices $\mathbf{A}_{3.1}$, $\mathbf{A}_{3.2}$ and $\mathbf{A}_{3.0}$ have a common set of eigenvectors. The zero eigenvalue for each matrix corresponds to the same eigenvector $\mathbf{D}_3^{1/2}\mathbf{e}^{(v)}$. The other $v-1$ eigenvalues of $\mathbf{A}_{3.0} = \mathbf{I} - (1/n)\mathbf{D}_3^{-1/2}\mathbf{e}^{(v)}\mathbf{e}^{(v)\prime}\mathbf{D}_3^{-1/2}$ are all equal to 1. If the treatment-row and treatment-column subdesigns are connected, then the remaining eigenvalues of $\mathbf{A}_{3.1}$ and $\mathbf{A}_{3.2}$ are all nonzero, and equal, respectively, to

$$\phi_s^{(3.1)} = 1 - \frac{\mu_s}{k_1} \quad \text{and} \quad \phi_s^{(3.2)} = 1 - \frac{\omega_s}{k_2}; \qquad s = 1, \ldots, v-1. \tag{3.3.15}$$

From (3.3.14) we find that the design itself is connected for treatments if and only if the $v - 1$ eigenvalues of $\mathbf{A}_{3.12}$

$$\xi_1\phi_s^{(3.1)} + \xi_2\phi_s^{(3.2)} - \xi_0 \neq 0, \qquad s = 1, \ldots, v-1, \tag{3.3.16}$$

or equivalently (3.3.12) holds. Furthermore, substituting (3.3.15) in (3.3.16) yields the inequality

$$\xi_1\left(1 - \frac{\mu_s}{k_1}\right) + \xi_2\left(1 - \frac{\omega_s}{k_2}\right) \neq \xi_0, \qquad s = 1, \ldots, v-1,$$

which implies (3.3.13). ❐

We illustrate the result in Theorem 3.3.3 with the following example.

EXAMPLE 3.3.2: The design

2	*	*	*	5	3	*	*	4	*
1	3	*	*	*	*	4	*	*	5
*	2	4	*	*	1	*	5	*	*
*	*	3	5	*	*	2	*	1	*
*	*	*	4	1	*	*	3	*	2

given by Worsley (1990) is ordinary with equal row sizes $k_1 = 4$ and equal column sizes $k_2 = 2$, and satisfies the conditions in Theorem 3.3.3, i.e., $\mathbf{S}_{3.1}\mathbf{D}_3^{-1}\mathbf{S}_{3.2} = \mathbf{S}_{3.2}\mathbf{D}_3^{-1}\mathbf{S}_{3.1}$, $u_{1.3} = u_{2.3} = 1$ and $\mathbf{S}_{3.12} = \mathbf{S}_{3.1} + \mathbf{S}_{3.2} - (25/16)\mathbf{S}_{3.0}$. From Table A.1.5, we see that $\phi_s^{(3.1)} = 15/16$ and $\phi_s^{(3.2)} = 5/8$ and so, since $\phi_s^{(3.1)} + \phi_s^{(3.2)} = 25/16$, we conclude that this design is not connected for treatments—indeed $u_{3.12} = 5$.

The design considered in our Example 2.3.1 also satisfies the conditions in Theorem 3.3.4, where $\mathbf{S}_{3.12}$ can be written as $\mathbf{S}_{3.12} = (1/2)\mathbf{S}_{3.1} + (1/2)\mathbf{S}_{3.2} - (4/9)\mathbf{S}_{3.0}$. From Table A.1.1 we have that $\phi_s^{(3.1)} = \phi_s^{(3.2)} = 7/9 \neq 4/9$, and so the design is connected for treatments.

Baksalary and Kala (1980) obtained the special case of our Theorem 3.3.4 for row-column designs, i.e., with $\xi_1 = \xi_2 = \xi_0 = 1$, $k_1 = c$ and $k_2 = r$.

Russell (1976) also proved a similar result but only for the class of equireplicated row-column designs characterized by a treatment-column subdesign being pairwise balanced, i.e.,

$$\mathbf{N}_{32}\mathbf{N}_{23} = p\mathbf{I} + q\mathbf{e}^{(v)}\mathbf{e}^{(v)\prime}, \qquad p, q > 0 \tag{3.3.17}$$

(cf. e.g., Hedayat and Federer, 1974). Any design of this type satisfies $u_{2.3} = 1$ since every treatment is applied together in the columns with every other treatment.

Theorem 3.3.4 (Russell, 1976): *An equireplicated row-column design such that* $\mathbf{N}_{32}\mathbf{N}_{23} = p\mathbf{I} + q\mathbf{e}^{(v)}\mathbf{e}^{(v)\prime}$, $p, q > 0$, *is completely connected* ($u_{1.2.3} = 2$) *if and only if* $c(kr - p)/r$ *is not an eigenvalue of* $\mathbf{N}_{31}\mathbf{N}_{13}$.

Proof: Because we are dealing with an equireplicated row-column design, the information matrix has the form

$$\mathbf{S}_{3.12} = k\mathbf{I} - (1/c)\mathbf{N}_{31}\mathbf{N}_{13} - (1/r)\mathbf{N}_{32}\mathbf{N}_{23} + (k^2/cr)\mathbf{e}^{(v)}\mathbf{e}^{(v)\prime}$$

$$= [(kr - p)/r]\mathbf{I} - (1/c)\mathbf{N}_{31}\mathbf{N}_{13} - [(cq - k^2)/cr]\mathbf{e}^{(v)}\mathbf{e}^{(v)\prime}, \tag{3.3.18}$$

since $\mathbf{N}_{32}\mathbf{N}_{23} = p\mathbf{I} + q\mathbf{e}^{(v)}\mathbf{e}^{(v)\prime}$. It is straightforward to show that the matrices $\mathbf{N}_{31}$, $\mathbf{N}_{13}$ and $\mathbf{e}^{(v)}\mathbf{e}^{(v)\prime}$ commute. The eigenvalues of $\mathbf{e}^{(v)}\mathbf{e}^{(v)\prime}$ are v with multiplicity 1 and zero with multiplicity $v - 1$; the distinct eigenvalue v is associated with the eigenvector $\mathbf{e}^{(v)}$ which is also the eigenvector corresponding to a zero eigenvalue of $\mathbf{S}_{3.12}$. The design is connected for treatments, therefore, if and only if the other $v-1$ eigenvalues of $\mathbf{S}_{3.12}$ satisfy

$$\mathrm{ch}(\mathbf{S}_{3.12}) = [(kr-p)/r] - (1/c)\mathrm{ch}(\mathbf{N}_{31}\mathbf{N}_{13}) \neq 0$$

$$\Longleftrightarrow \mathrm{ch}(\mathbf{N}_{31}\mathbf{N}_{13}) \neq c(kr-p)/r. \qquad \square$$

EXAMPLE 3.3.3: Theorem 3.3.4 can be applied to the row-column design

3	3	1
3	2	2
1	2	1

which is equireplicated. Here $\mathbf{N}_{31}\mathbf{N}_{13} = \mathbf{N}_{32}\mathbf{N}_{23} = 3\mathbf{I} + 2\mathbf{e}^{(v)}\mathbf{e}^{(v)'}$ and so $c(kr-p)/r = 6$ is not an eigenvalue of $\mathbf{N}_{31}\mathbf{N}_{13}$. This design is completely connected, i.e., $u_{1.2.3} = 2$, as Theorem 3.3.4 implies.

If the treatment-row subdesign is connected, then Theorem 3.3.4 is comparable to the special case of Theorem 3.3.3 for row-column designs. From the condition that $\mathbf{N}_{32}\mathbf{N}_{23} = p\mathbf{I} + q\mathbf{e}^{(v)}\mathbf{e}^{(v)'}, p, q > 0$, it follows that $\mathbf{S}_{3.2} = [k-(p/c)]\mathbf{I} - (qv/c)\mathbf{J}$ and so $\phi_s^{(3.2)} = (1/k)[k - (p/r)]$, i.e., $\omega_s = p/k$, $s = 1, \ldots, v-1$, where ω_s is as in Theorem 3.3.3. It then follows that since the eigenvalues of $\mathbf{N}_{31}\mathbf{N}_{13}$ are $k\mu_s$,

$$r\mu_s + c\,\omega_s \neq rc \Longleftrightarrow k\mu_s \neq c(kr-p)/r, \qquad s = 1, \ldots, v-1.$$

A more specific subclass of equireplicated row-column designs considered by Russell (1976, 1980) is characterized by a treatment-column subdesign being a balanced incomplete block design (BIBD) with parameters $\{c, v, k, r, \lambda = k(r-1)/(v-1)\}$. This means that the v treatments are replicated k times in c columns of size r ($< v$) in such a fashion that no treatment appears more than once in a column but appears in λ columns with each other treatment. Roy and Shah (1961) call such a design a two-way design with column balance. Here, the incidence matrix $\mathbf{N}_{32}$ satisfies (3.3.17) with $p = k - \lambda$ and $q = \lambda$ and Russell (1976) shows that this design's treatment-row subdesign is connected. We then have the following corollary.

Corollary 3.3.5 (Russell, 1976): *An equireplicated row-column design with a* BIBD$\{c, v, k, r, \lambda = k(r-1)/(v-1)\}$ *as a treatment-column subdesign is completely connected if and only if $cv\lambda/r$ is not an eigenvalue of* $\mathbf{N}_{31}\mathbf{N}_{13}$.

Proof: Follows at once by replacing p with $k - \lambda$ and q with λ in Theorem 3.3.4. ❒

Russell (1980) then extended this result by looking at BIBDs with specific parameter sets for which connectedness can be determined without having to find the eigenvalues of $\mathbf{N}_{31}\mathbf{N}_{13}$.

Theorem 3.3.6 (Russell, 1980): *If the parameter set of an equireplicated row-column design with a* BIBD$\{c, v, k, r, \lambda = k(r-1)/(v-1)\}$ *as a treatment-column subdesign is such that $cv\lambda/r$ is not an integer or is unreduced, i.e., the parameters are given by* $c = \binom{v}{r}$, $k = \binom{v-1}{r-1}$ *and* $\lambda = \binom{v-2}{r-2}$, *then the design is connected for treatments.*

Proof: Since the elements of $\mathbf{N}_{31}\mathbf{N}_{13}$ are integers, the characteristic polynomial $|\mathbf{N}_{31}\mathbf{N}_{13} - \lambda\mathbf{I}|$ is a monic polynomial in the set of integers $\mathbb{Z}[x]$, i.e., a polynomial in x with integer coefficients and leading coefficient equal to 1. The only possible roots, therefore, are integers and irrational numbers. This means that if the parameters of the equireplicated row-column design are such that $cv\lambda/r$ is not an integer, then $cv\lambda/r$ cannot be an eigenvalue of $\mathbf{N}_{31}\mathbf{N}_{13}$ and the design is connected. Now, if the parameter set of the design is unreduced, then $c = \binom{v}{r}$, i.e., the columns consist of all the different ways of choosing r treatments out of v. From every such design we can obtain, by removing a certain number of columns, a sub row-column design with also a BIBD as its treatment-column subdesign and with parameters $\{v = c = (r+1), k = r, \lambda = (r-1)\}$. The columns of this design consist of the $r + 1$ different ways of choosing r out of $r + 1$ treatments. For this sub row-column design, $cv\lambda/r = (r - 1)(r + 1)^2/r$ is not an integer and so the sub row-column design is completely connected. In particular, it is connected for rows and this implies that the whole design is connected for rows. Now since both treatment-row and treatment-column subdesigns are connected (cf. Russell, 1976), the connectedness for rows implies from equation (3.3.8) that the design is completely connected. ❐

Fisher and Yates (1963, Table XVIII) list 63 parameter sets for BIBDs with treatment replication $k \leq 10$. Applying Theorem 3.3.6, we find that the equireplicated row-column designs with treatment-column subdesign a BIBD corresponding to 44 out of these 63 sets are connected for treatments. Russell (1980) lists the 19 parameter sets to which Theorem 3.3.6 does not apply and for which no general results appear to be known.

SECTION 3.4: ORTHOGONALITY

The following two types of orthogonality for a multi-way design with f factors have been considered in the literature [cf. Chakrabarti (1962), Eccleston and Russell (1975, 1977), Khatri and Shah (1986)].

Definition 3.4.1: Factors i and j are weakly orthogonal if all the covariances (or the cross-covariance matrix) between the two factor totals, each adjusted for the other $f - 1$ factors, are zero.

Definition 3.4.2: Factors i and j are strictly orthogonal if the covariance between the two factor totals, each adjusted for the remaining $f - 2$ factors (when $f \geq 3$) or for the mean (when $f = 2$) is zero.

In a three-way layout, Definition 3.4.1 is equivalent to

$$\mathrm{cov}(\mathbf{X}_i'\mathbf{M}_{jk}\mathbf{y} : \mathbf{X}_j'\mathbf{M}_{ik}\mathbf{y}) = \mathbf{0} \tag{3.4.1}$$

or

$$\mathbf{X}_i'\mathbf{M}_{jk}\mathbf{M}_{ik}\mathbf{X}_j = \mathbf{0}, \tag{3.4.2}$$

while Definition 3.4.2 is equivalent to

$$\mathrm{cov}(\mathbf{X}_i'\mathbf{M}_k\mathbf{y} : \mathbf{X}_j'\mathbf{M}_k\mathbf{y}) = \mathbf{0} \tag{3.4.3}$$

or

$$\mathbf{X}_i'\mathbf{M}_k\mathbf{X}_j = \mathbf{0}. \tag{3.4.4}$$

However, equations (3.4.1) to (3.4.4) do not cover the only kind of orthogonality that might be of interest in a three-way layout. For example, there might also be some questions concerning orthogonality in the subdesigns. For each pair of factors, it is possible to define four different kinds of strict and weak orthogonality [cf. Styan (1986), Worsley, Styan and Bérubé (1990)] according as we ignore, include, adjust or adjust only partially for the third factor.

When the third factor, say k, is ignored or included (i.e., we are now dealing with two-way designs), characterizations of weak orthogonality, according to Definition 3.4.1, are given, respectively, by

$$\mathbf{X}_i'\mathbf{M}_j\mathbf{M}_i\mathbf{X}_j = \mathbf{0}, \tag{3.4.5}$$

and

$$\mathbf{X}_i'\mathbf{M}_{jk}\mathbf{M}_i(\mathbf{X}_j : \mathbf{X}_k) = \mathbf{0}. \tag{3.4.6}$$

It is easy to see from (3.2.14) and (3.2.15) that (3.4.5) and (3.4.6) are equivalent to $t_{i.j} = 0$ and $t_{i.jk} = 0$ respectively. Again, if factor k, say, is ignored or included, Definition 3.4.2 for strict orthogonality requires that the cross-covariance matrix between the two sets of factor totals, each adjusted for the mean, be zero, i.e.,

$$\mathbf{X}_i' \mathbf{C}_n \mathbf{X}_j = \mathbf{0} \quad \text{and} \quad \mathbf{X}_i' \mathbf{C}_n (\mathbf{X}_j : \mathbf{X}_k) = \mathbf{0},$$

respectively. Comparing these two equations with the result in Lemma 3.2.2, we see that equivalent characterizations for those two types of strict orthogonality are given by $m_{i.j} = 1$ and $m_{i.jk} = 1$, respectively.

When there is adjustment for the third factor, then the characterizations for weak and strict orthogonality are (3.4.1) and (3.4.3), respectively, or equivalently $t_{i.j|k} = 0$ and $m_{i.j|k} = 0$, cf. (3.2.17) and (3.2.8).

We can summarize these different kinds of orthogonality as follows. Respectively

(i) $t_{i.j} = 0$ and $m_{i.j} = 1$, weak and strict "pairwise orthogonality,"

(ii) $t_{i.jk} = 0$ and $m_{i.jk} = 1$, weak and strict "augmented orthogonality"

(iii) $t_{i;j|k} = 0$ and $m_{i;j|k} = 0$, weak and strict "partially adjusted orthogonality."

(iv) $t_{i.j|k} = 0$ and $m_{i.j|k} = 0$, weak and strict "completely adjusted orthogonality."

Since $m_{i;j|k} = m_{i.j|k}$, cf. (3.2.8), it follows that strict completely adjusted orthogonality is equivalent to strict partially adjusted orthogonality, and we will then refer to this situation as just strict adjusted orthogonality; this was originally introduced as just "adjusted orthogonality" in Eccleston and Russell (1975, 1977), who also pointed out that our strict pairwise orthogonality, characterized by $m_{i.j} = 1$, is a particular case of their adjusted orthogonality. To see this, we note that if we ignore the third factor k, then strict pairwise orthogonality requires that the covariance between the two other factor totals, each adjusted for the mean, be zero, i.e., $\text{cov}(\mathbf{X}_i'\mathbf{C}_n\mathbf{y} : \mathbf{X}_j'\mathbf{C}_n\mathbf{y}) = \mathbf{0}$. Eccleston and Russell (1975, 1977) refer to this as traditional or pairwise orthogonality between two factors, and this has been studied by many authors including Yates (1933), Pearce (1970) and John (1971). The necessary and sufficient condition for strict pairwise orthogonality, i.e., $\mathbf{X}_i'\mathbf{C}_n\mathbf{X}_j = \mathbf{0}$ is often written as $\mathbf{N}_{ij} = \mathbf{k}_i\mathbf{k}_j'/n$ or equivalently, $n_{pq} = (n_{p.}n_{.q})/n$ for all p and q, where n_{pq} is the $(p, q)^{th}$ element of $\mathbf{N}_{ij}$. In the case where the factor k is included, strict augmented orthogonality, i.e., $m_{i.jk} = 1$, is also a special case of adjusted orthogonality which requires both $m_{i.j} = 1$ and $m_{i.k} = 1$. For more on adjusted orthogonality see Baksalary and Pukelsheim (1990).

Because of the identity $m = u + t$, it is clear that strict orthogonality implies weak orthogonality, and that both weak orthogonality and connectedness together are equivalent to strict orthogonality.

The condition for weak pairwise orthogonality, i.e., $t_{i.j} = 0$, is given among a list of 45 algebraic characterizations established in Baksalary (1987) for the commutativity of two orthogonal projectors, i.e., $\mathbf{H}_i\mathbf{H}_j = \mathbf{H}_j\mathbf{H}_i$ [cf. Theorem 3.2.3]. Any one of the other 44 equivalent conditions can, therefore, be used to express weak pairwise orthogonality. We will need the following condition, due originally to Rao and Yanai (1979).

Lemma 3.4.1: *Factor i and factor j are weakly orthogonal, i.e., $t_{i.j} = 0$, if and only if*

$$\mathbf{H}_{ij} = \mathbf{H}_i + \mathbf{H}_j - \mathbf{H}_i\mathbf{H}_j .$$

Proof: First, we assume that $t_{i.j} = 0$ and write out the following equation,

$$\mathbf{H}_{i|j}(\mathbf{I} - \mathbf{H}_i) = \mathbf{M}_j\mathbf{X}_i'(\mathbf{X}_i'\mathbf{M}_j\mathbf{X}_i)^{-}\mathbf{X}_i'\mathbf{M}_j(\mathbf{I} - \mathbf{H}_i). \qquad (3.4.7)$$

We then look only at $\mathbf{X}_i'\mathbf{M}_j(\mathbf{I} - \mathbf{H}_i)$ and use the commutativity of the two orthogonal projectors $\mathbf{H}_i$ and $\mathbf{H}_j$ to yield

$$\mathbf{X}_i'\mathbf{M}_j\mathbf{M}_i = \mathbf{X}_i'\mathbf{H}_j - \mathbf{X}_i'\mathbf{H}_j\mathbf{H}_i = \mathbf{X}_i'\mathbf{H}_j - \mathbf{X}_i'\mathbf{H}_i\mathbf{H}_j = \mathbf{0}. \tag{3.4.8}$$

The left hand side of (3.4.7) can therefore be written as

$$(\mathbf{H}_{ij} - \mathbf{H}_j)(\mathbf{I} - \mathbf{H}_i) = \mathbf{H}_{ij} - \mathbf{H}_j - \mathbf{H}_i - \mathbf{H}_i\mathbf{H}_j = \mathbf{0}.$$

The converse follows at once since $\mathbf{H}_{ij} = \mathbf{H}_{ji}$, cf. (2.2.3) and (2.2.4). ❐

Whenever weak pairwise orthogonality holds, certain equalities among the canonical correlations follow. This next theorem lists some of these equalities. Parts (i) and (ii) are stated, but not proven, in Styan (1986), while part (iii) is new.

Theorem 3.4.2: *Let factor i and factor j be weakly pairwise orthogonal, i.e.,* $t_{i.j} = 0$, *and let k be a third factor. Then*

(i) $\{\rho^{(i.jk)}\} = \{\rho^{(i.k|j)}\} + \{u_{i.j} \text{ ones}\}$,

(ii) $\{\rho^{(j.ik)}\} = \{\rho^{(j.k|i)}\} + \{u_{i.j} \text{ ones}\}$,

(iii) $\{\rho^{(i;k|j)}\} = \{\rho^{(i.k|j)}\}$ *and* $\{\rho^{(j;k|i)}\} = \{\rho^{(j.k|i)}\}$.

Proof: (i) Let the matrix $\mathbf{X}_{jk}$ denote the augmented matrix $(\mathbf{X}_j : \mathbf{X}_k)$. The set of canonical correlations between factor i and the other two factors j and k is denoted by $\{\rho^{(i.jk)}\}$

$$\{\rho^{(i.jk)}\} = \{\mathrm{ch}^{1/2}[\mathbf{X}_i'\mathbf{X}_{jk}(\mathbf{X}_{jk}'\mathbf{X}_{jk})^{-}\mathbf{X}_{jk}'\mathbf{X}_i(\mathbf{X}_i'\mathbf{X}_i)^{-}]\}$$

$$= \{\mathrm{ch}^{1/2}(\mathbf{H}_i\mathbf{H}_{jk})\}. \tag{3.4.9}$$

Using the decomposition of the projectors as given in (2.2.3), we can rewrite (3.4.9) as

$$\{\rho^{(i.jk)}\} = \{\mathrm{ch}^{1/2}[\mathbf{H}_i(\mathbf{H}_j + \mathbf{H}_{k|j})]\}.$$

We now consider the characteristic polynomial of $\mathbf{H}_i(\mathbf{H}_j + \mathbf{H}_{k|j})$

$$|\lambda\mathbf{I} - \mathbf{H}_i(\mathbf{H}_j + \mathbf{H}_{k|j})| = |\lambda\mathbf{I} - \mathbf{H}_i\mathbf{H}_j|\cdot|\mathbf{I} - (\lambda\mathbf{I} - \mathbf{H}_i\mathbf{H}_j)^{-1}\mathbf{H}_i\mathbf{H}_{k|j}| \qquad (3.4.10)$$

for all $\lambda \neq \mathrm{ch}(\mathbf{H}_i\mathbf{H}_j) = \rho_h^{(i.j)}$; $h = 1, \ldots, m_{i.j}$. We can simplify the last term in (3.4.10) using the commutativity of the projectors $\mathbf{H}_i$ and $\mathbf{H}_j$ (since $t_{i.j} = 0$) and so

$$(\lambda\mathbf{I} - \mathbf{H}_i\mathbf{H}_j)\mathbf{H}_i\mathbf{H}_{k|j} = \lambda\mathbf{H}_i\mathbf{H}_{k|j} - \mathbf{H}_i\mathbf{H}_j\mathbf{H}_i\mathbf{H}_{k|j} = \lambda\mathbf{H}_i\mathbf{H}_{k|j} - \mathbf{H}_i\mathbf{H}_j\mathbf{H}_{k|j}$$

$$= \lambda\mathbf{H}_i\mathbf{H}_{k|j}.$$

The characteristic polynomial (3.4.10) is, therefore, equivalent to

$$|\lambda\mathbf{I} - \mathbf{H}_i\mathbf{H}_j|\cdot|\mathbf{I} - (1/\lambda)\mathbf{H}_i\mathbf{H}_{k|j}| = (\lambda - 1)^{u_{i.j}}\lambda^{n-u_{i.j}}\lambda^{-n}|\lambda\mathbf{I} - \mathbf{H}_i\mathbf{H}_{k|j}| \qquad (3.4.11)$$

for $\lambda \neq 0$ and $\lambda \neq \rho_h^{(i.j)}$; $h = 1, \ldots, m_{i.j}$. We recall that the nonzero eigenvalues of $\mathbf{H}_i\mathbf{H}_j$ are the squares of the nonzero canonical correlations between $\mathbf{X}_i'\mathbf{y}$ and $\mathbf{X}_j'\mathbf{y}$ and are all equal to 1 since $\mathbf{H}_i\mathbf{H}_j$ is idempotent whenever $t_{i.j} = 0$.

We may use Lemma 3.4.1 in order to rewrite the right hand side of (3.4.11), i.e.,

$$\mathbf{H}_i\mathbf{H}_{k|j} = (\mathbf{H}_i - \mathbf{H}_i\mathbf{H}_j)\mathbf{H}_{k|j} = (\mathbf{H}_{ij} - \mathbf{H}_j)\mathbf{H}_{k|j} = \mathbf{H}_{i|j}\mathbf{H}_{k|j}. \qquad (3.4.12)$$

The right hand side of (3.4.11) equals

$$\left(\frac{\lambda - 1}{\lambda}\right)^{u_{i.j}}|\lambda\mathbf{I} - \mathbf{H}_{i|j}\mathbf{H}_{k|j}|,$$

and so we obtain the required result.

The proof is similar for (ii), and result (iii) follows at once from the equality in (3.4.12) and the fact that $t_{i.j} = t_{j.i}$. ❐

Now, if one factor, say j, is pairwise orthogonal to both of the other two factors, a further set of equalities between canonical correlations can be obtained.

Corollary 3.4.3: *Let* $t_{i.j} = t_{j.k} = 0$. *Then*

(a) $\{\rho^{(i.jk)}\} = \{\rho^{(k.ij)}\} + \{u_{i.j} - u_{j.k} \text{ ones}\}$ *if* $u_{i.j} \geq u_{j.k}$

$= \{\rho^{(i.k|j)}\} + \{u_{i.j} \text{ ones}\}$

(b) $\{\rho^{(j.ik)}\} = \{\rho^{(i.j|k)}\} + \{u_{j.k} \text{ ones}\} = \{\rho^{(j.k|i)}\} + \{u_{i.j} \text{ ones}\}$.

If in addition $u_{i.j} = u_{j.k}$, *then*

(c) $\{\rho^{(i.jk)}\} = \{\rho^{(k.ij)}\}$

(d) $\{\rho^{(i.j|k)}\} = \{\rho^{(j.k|i)}\} = \{\rho^{(i;j|k)}\} = \{\rho^{(j;k|i)}\}$

Proof: To prove the second equality in (a), we start with (3.4.11), i.e.,

$$\left(\frac{\lambda - 1}{\lambda}\right)^{u_{i.j}} |\lambda \mathbf{I} - \mathbf{H}_i \mathbf{H}_{k|j}|.$$

This characteristic polynomial is equivalent to

$$\left(\frac{\lambda - 1}{\lambda}\right)^{u_{i.j}} |\lambda \mathbf{I} - \mathbf{H}_i(\mathbf{H}_k - \mathbf{H}_j \mathbf{H}_k)| = \left(\frac{\lambda - 1}{\lambda}\right)^{u_{i.j}} |\lambda \mathbf{I} + \mathbf{H}_j \mathbf{H}_k - \mathbf{H}_{ij} \mathbf{H}_k|, \quad (3.4.13)$$

since $t_{i.j} = 0$ is equivalent to $\mathbf{H}_i\mathbf{H}_j = \mathbf{H}_i + \mathbf{H}_j - \mathbf{H}_{ij}$ (cf. Lemma 3.4.1). We may rewrite (3.4.13) as

$$\left(\frac{\lambda - 1}{\lambda}\right)^{u_{i.j}} |\lambda\mathbf{I} - \mathbf{H}_{ij}\mathbf{H}_k||\mathbf{I} + (\lambda\mathbf{I} - \mathbf{H}_{ij}\mathbf{H}_k)^{-1}\,\mathbf{H}_j\mathbf{H}_k|. \tag{3.4.14}$$

The last term in (3.4.14) can be simplified since $(\lambda\mathbf{I} - \mathbf{H}_{ij}\mathbf{H}_k)\mathbf{H}_j\mathbf{H}_k = (\lambda - 1)\mathbf{H}_j\mathbf{H}_k$, using the commutativity of $\mathbf{H}_j$ and $\mathbf{H}_k$. Therefore, we rewrite (3.4.14) as

$$\left(\frac{\lambda - 1}{\lambda}\right)^{u_{i.j}} |(\lambda - 1)\mathbf{I} + \mathbf{H}_j\mathbf{H}_k||\lambda\mathbf{I} - \mathbf{H}_{ij}\mathbf{H}_k| = \left(\frac{\lambda - 1}{\lambda}\right)^{u_{i.j} - u_{j.k}} |\lambda\mathbf{I} - \mathbf{H}_{ij}\mathbf{H}_k|.$$

The equalities in (b) can be obtained directly from Theorem 3.4.2, while the equalities in (c) and (d) follow at once from (a) and (b). □

The 45 equivalent conditions given by Baksalary (1987) can also be used to characterize weak adjusted orthogonality, i.e., $t_{i.j\,|\,k} = 0$. Here, the commuting orthogonal projectors are $\mathbf{H}_{i|k}$ and $\mathbf{H}_{j|k}$. It follows that weak orthogonality implies the following equality

$$\mathbf{X}_i'\mathbf{M}_k\mathbf{X}_j = \mathbf{X}_i'\mathbf{M}_k\mathbf{X}_j(\mathbf{X}_j'\mathbf{M}_k\mathbf{X}_j)^{-}\mathbf{X}_j'\mathbf{M}_k\mathbf{X}_i(\mathbf{X}_i'\mathbf{M}_k\mathbf{X}_i)^{-}\mathbf{X}_i'\mathbf{M}_k\mathbf{X}_j. \tag{3.4.15}$$

In the following theorem we show the equivalence of four characterizations to the property of strict adjusted orthogonality, i.e., $m_{i.j\,|\,k} = 0$. The three characterizations (ii), (iii) and (iv) are due to Eccleston and Russell (1977); characterization (v) is due to Siatkowski (1990).

Theorem 3.4.4: *The following nine characterizations are equivalent.*

(i) $\mathbf{X}_i'\mathbf{M}_k\mathbf{X}_j = \mathbf{0}$

(ii) $\mathbf{S}_{i.k} = \mathbf{S}_{i.jk}$

(iii) $(\mathbf{X}_i'\mathbf{M}_k\mathbf{X}_i)^-\mathbf{X}_i'\mathbf{M}_k\mathbf{y} = (\mathbf{X}_i'\mathbf{M}_{jk}\mathbf{X}_i)^-\mathbf{X}_i'\mathbf{M}_{jk}\mathbf{y}$

(iv) $(\mathbf{X}_i'\mathbf{M}_k\mathbf{y})'(\mathbf{X}_i'\mathbf{M}_k\mathbf{X}_i)^-(\mathbf{X}_i'\mathbf{M}_k\mathbf{y}) = (\mathbf{X}_i'\mathbf{M}_{jk}\mathbf{y})'(\mathbf{X}_i'\mathbf{M}_{jk}\mathbf{X}_i)^-(\mathbf{X}_i'\mathbf{M}_{jk}\mathbf{y})$

(v) $\mathbf{X}_i'\mathbf{M}_k = \mathbf{X}_i'\mathbf{M}_{jk}$

(vi) $\mathbf{S}_{j.k} = \mathbf{S}_{j.ik}$

(vii) $(\mathbf{X}_j'\mathbf{M}_k\mathbf{X}_j)^-\mathbf{X}_j'\mathbf{M}_k\mathbf{y} = (\mathbf{X}_j'\mathbf{M}_{ik}\mathbf{X}_j)^-\mathbf{X}_j'\mathbf{M}_{ik}\mathbf{y}$

(viii) $(\mathbf{X}_j'\mathbf{M}_k\mathbf{y})'(\mathbf{X}_j'\mathbf{M}_k\mathbf{X}_j)^-(\mathbf{X}_j'\mathbf{M}_k\mathbf{y}) = (\mathbf{X}_j'\mathbf{M}_{ik}\mathbf{y})'(\mathbf{X}_j'\mathbf{M}_{ik}\mathbf{X}_j)^-(\mathbf{X}_j'\mathbf{M}_{ik}\mathbf{y})$

(ix) $\mathbf{X}_j'\mathbf{M}_k = \mathbf{X}_j'\mathbf{M}_{ik}$

Proof: Condition (ii) is equivalent to $\mathbf{X}_i'\mathbf{H}_{j|k}\mathbf{X}_i = \mathbf{0}$, since $\mathbf{M}_{jk} = \mathbf{M}_k - \mathbf{H}_{j|k}$. Now, using rank cancellation rule (A.3.5), $\mathbf{X}_i'\mathbf{H}_{j|k}\mathbf{X}_i = \mathbf{0}$ is equivalent to $\mathbf{X}_i'\mathbf{H}_{j|k} = \mathbf{0}$. We can expand $\mathbf{X}_i'\mathbf{H}_{j|k} = \mathbf{0}$ as $\mathbf{X}_i'\mathbf{M}_k\mathbf{X}_j(\mathbf{X}_j'\mathbf{M}_k\mathbf{X}_j)^-\mathbf{X}_j'\mathbf{M}_k = \mathbf{0}$, and postmultiplying by $\mathbf{X}_j$ we obtain the equivalent condition (i).

Postmultiplying both sides of (v) by $\mathbf{X}_j$ yields condition (ii). To go the other way we premultiply $\mathbf{M}_{jk} = \mathbf{M}_k - \mathbf{H}_{j|k}$ by $\mathbf{X}_i'$, noting that $\mathbf{X}_i'\mathbf{H}_{j|k} = \mathbf{0}$ whenever $\mathbf{S}_{i.k} = \mathbf{S}_{i.jk}$. Conditions (ii) and (v) are therefore equivalent.

It is easy to see that condition (v) is sufficient for both conditions (iii) and (iv) to hold. Finally, we want to show that (iii) and (iv) are both sufficient for (i). If we postmultiply

$$(\mathbf{X}_i'\mathbf{M}_k\mathbf{X}_i)^-\mathbf{X}_i'\mathbf{M}_k = (\mathbf{X}_i'\mathbf{M}_{jk}\mathbf{X}_i)^-\mathbf{X}_i'\mathbf{M}_{jk} \tag{3.4.16}$$

and

$$\mathbf{M}_k\mathbf{X}_i(\mathbf{X}_i'\mathbf{M}_k\mathbf{X}_i)^-\mathbf{X}_i'\mathbf{M}_k = \mathbf{M}_{jk}\mathbf{X}_i(\mathbf{X}_i'\mathbf{M}_{jk}\mathbf{X}_i)^-\mathbf{X}_i'\mathbf{M}_{jk} \tag{3.4.17}$$

by $\mathbf{X}_j$, the right-hand sides of both equations (3.4.16) and (3.4.17) become $\mathbf{0}$. We can then use

$$(\mathbf{X}_i'\mathbf{M}_k\mathbf{X}_i)^{-}\mathbf{X}_i'\mathbf{M}_k\mathbf{X}_j = \mathbf{0} \Longleftrightarrow \mathbf{M}_k\mathbf{X}_i(\mathbf{X}_i'\mathbf{M}_k\mathbf{X}_i)^{-}\mathbf{X}_i'\mathbf{M}_k\mathbf{X}_j = \mathbf{0}$$

where the last equality is equivalent to $\mathbf{X}_i'\mathbf{M}_k\mathbf{X}_j = \mathbf{0}$ and so conditions (iii) and (iv) are sufficient for (i).

We note that (vi) through (ix) are just (ii) through (v) with i and j interchanged. We can interchange i and j since from (i) by transposition

$$\mathbf{X}_i'\mathbf{M}_k\mathbf{X}_j = \mathbf{0} \Longleftrightarrow \mathbf{X}_j'\mathbf{M}_k\mathbf{X}_i = \mathbf{0},$$

i.e., i is orthogonal to j after adjusting for k whenever j is orthogonal to i after adjusting for k. □

We may interpret the characterizations (ii), (iii) and (iv) in the following way:

(ii) The information matrices of the reduced normal equations for $\hat{\boldsymbol{\alpha}}$ [$\hat{\boldsymbol{\beta}}$] are the same whether or not $\boldsymbol{\beta}$ [$\boldsymbol{\alpha}$] is included in the model, i.e.,

$$\mathbf{S}_{i.k} = \mathbf{S}_{i.jk} \quad [\mathbf{S}_{j.k} = \mathbf{S}_{j.ik}]$$

(iii) A least squares solution $\hat{\boldsymbol{\alpha}}$ [$\hat{\boldsymbol{\beta}}$] is the same whether or not $\boldsymbol{\beta}$ [$\boldsymbol{\alpha}$] is included in the model, i.e.,

$$(\mathbf{X}_i'\mathbf{M}_k\mathbf{X}_i)^{-}\mathbf{X}_i'\mathbf{M}_k\mathbf{y} = (\mathbf{X}_i'\mathbf{M}_{jk}\mathbf{X}_i)^{-}\mathbf{X}_i'\mathbf{M}_{jk}\mathbf{y}$$

$$[(\mathbf{X}_j'\mathbf{M}_k\mathbf{X}_j)^{-}\mathbf{X}_j'\mathbf{M}_k\mathbf{y} = (\mathbf{X}_j'\mathbf{M}_{ik}\mathbf{X}_j)^{-}\mathbf{X}_j'\mathbf{M}_{ik}\mathbf{y}]$$

(iv) The regression sum of squares for factor i is the same whether or not $\boldsymbol{\beta}$ $[\boldsymbol{\alpha}]$ is included in the model, i.e.,

$$(\mathbf{X}_i'\mathbf{M}_k\mathbf{y})'(\mathbf{X}_i'\mathbf{M}_k\mathbf{X}_i)^-(\mathbf{X}_i'\mathbf{M}_k\mathbf{y}) = (\mathbf{X}_i'\mathbf{M}_{jk}\mathbf{y})'(\mathbf{X}_i'\mathbf{M}_{jk}\mathbf{X}_i)^-(\mathbf{X}_i'\mathbf{M}_{jk}\mathbf{y})$$

$$[(\mathbf{X}_j'\mathbf{M}_k\mathbf{y})'(\mathbf{X}_j'\mathbf{M}_k\mathbf{X}_j)^-(\mathbf{X}_j'\mathbf{M}_k\mathbf{y}) = (\mathbf{X}_j'\mathbf{M}_{ik}\mathbf{y})'(\mathbf{X}_j'\mathbf{M}_{ik}\mathbf{X}_j)^-(\mathbf{X}_j'\mathbf{M}_{ik}\mathbf{y})]$$

For example, if the treatment-row subdesign of a row-column design is a complete block design and the treatment-column subdesign is a binary incomplete block design, then $\mathbf{S}_{3.12} = \mathbf{S}_{3.2}$ and $\mathbf{z}_{3\ 12} = \mathbf{z}_{3.2}$, i.e., treatments and rows adjusted for columns are strictly orthogonal ($m_{2.3|1} = 1$). In this situation, the least squares estimates of the treatment parameters are the same as those obtained from a model where the rows are removed. John (1987) calls such row-column designs, row-orthogonal designs.

In a row-column design, where factors i, j and k are rows, columns and treatments respectively, condition (i) in Theorem 3.4.4 is equivalent to

$$\mathbf{N}_{13}\mathbf{D}_3^{-1}\mathbf{N}_{32} = \mathbf{e}^{(r)}\mathbf{e}^{(c)\prime}. \tag{3.4.18}$$

If the treatments are equireplicated, strict adjusted orthogonality in a row-column design reduces to

$$\mathbf{N}_{13}\mathbf{N}_{32} = k\mathbf{e}^{(r)}\mathbf{e}^{(c)\prime}, \tag{3.4.19}$$

i.e., each row has k treatments in common with each column. Equireplicate row-column designs satisfying (3.4.19) have been studied by many authors including Agrawal (1966a), Raghavarao and Federer (1975), Shah (1977), John and Eccleston (1986) and Lewis and Dean (1989).

If all three factors are strictly pairwise orthogonal, i.e., $m_{1.2} = m_{1.3} = m_{2.3} = 1$, then Eccleston and Russell (1977) showed that this implies $m_{1.2|3} = m_{1.3|2} = m_{2.3|1} = 0$ and the design is said to be orthogonal. Here the incidence matrices reduce to

$$\mathbf{N}_{31} = \frac{\mathbf{k}_3\mathbf{k}_1'}{rc},\ \mathbf{N}_{32} = \frac{\mathbf{k}_3\mathbf{k}_2'}{rc} \text{ and } \mathbf{N}_{12} = \mathbf{e}^{(r)}\mathbf{e}^{(c)\prime} \tag{3.4.20}$$

and we can write the information matrix for the design eliminating rows and columns as

$$\mathbf{S}_{3.12} = \mathbf{D}_3 - \frac{\mathbf{k}_3\mathbf{k}_3'}{n} = \mathbf{S}_{3.0}. \tag{3.4.21}$$

We can easily see that $\mathbf{D}_3^{-1}$ is then a generalized inverse of $\mathbf{S}_{3.12}$ and so a solution to the normal equations given in (2.2.5) can be written as

$$\hat{\boldsymbol{\tau}} = \mathbf{D}_3^{-1}\mathbf{z}_{3.12}, \tag{3.4.22}$$

where $\mathbf{z}_{3.12} = \mathbf{y}_{vt} - \bar{y}\mathbf{k}_3$ and $\bar{y} = (1/n)\sum_{k=1}^{n} y_{ijk}$. This means that in an orthogonal design, estimates of the treatment parameters are the same as those obtained if both the rows and columns are ignored. Examples of orthogonal row-column designs are Latin squares and F-squares, i.e., designs for which each treatment is applied the same number of times in each row and each column [see Hedayat and Seiden (1970)].

The next theorem given in Siatkowski (1990) and Baksalary and Styan (1990) is concerned with relationships between the properties of orthogonality and connectedness.

Theorem 3.4.5: *A three-way layout has factors i and j adjusted for factor k strictly orthogonal and its subdesign ignoring factor j (factor i) connected if and only if the factors i and j adjusted for factor k are weakly orthogonal and the layout is connected for factor i (factor j), i.e.,*

$$r(\mathbf{X}_i'\mathbf{M}_k\mathbf{X}_j) = 0 \text{ and } u_{i.k} = 1 \ (u_{j.k} = 1) \Longleftrightarrow r(\mathbf{X}_i'\mathbf{M}_{jk}\mathbf{M}_{ik}\mathbf{X}_j) = 0 \text{ and } u_{i.jk} = 1 \ (u_{j.ik} = 1). \tag{3.4.23}$$

Proof: When the left hand side of (3.4.23) holds, i.e., $m_{i.j|k} = 0$, then the equality

$$m_{i.j|k} = u_{i.j|k} + t_{i.j|k} \tag{3.4.24}$$

forces $u_{i.j|k} = t_{i.j|k} = 0$, and so the right hand side of (3.4.23) holds using the first equality in (3.3.4). Now suppose that the right-hand side of (3.4.23) holds, then again using the first equality in (3.3.4) forces $u_{i.j|k} = 0$ and $u_{i.k} = 1$, therefore $m_{i.j|k} = 0$ from (3.4.24). ❐

Our Theorem 3.4.5 is an extension of Theorem 1 in Eccleston and Russell (1975), where the condition of strict adjusted orthogonality is used on both sides of (3.4.23), while here we use only the condition of weak adjusted orthogonality on the right hand side of (3.4.23). It is easy to see that if we replace connectedness for factor i, i.e., $u_{i.jk} = 1$, on the right-hand side of (3.4.23) by complete connectedness, i.e., $u_{1.2.3} = 1$, then the left-hand side holds with both $u_{i.k} = 1$ and $u_{j.k} = 1$.

SECTION 3.5: CANONICAL EFFICIENCY FACTORS AND AVERAGE EFFICIENCY FACTORS

In a two-way layout, we define the "efficiency matrix" as

$$\mathbf{A}_{i.j} = \mathbf{D}_i^{-1/2}\mathbf{S}_{i.j}\mathbf{D}_i^{-1/2} = \mathbf{I} - \mathbf{D}_i^{-1/2}\mathbf{N}_{ij}\mathbf{D}_j^{-1}\mathbf{N}_{ji}\mathbf{D}_i^{-1/2}, \tag{3.5.1}$$

cf. (3.3.11). The nonzero eigenvalues $\phi_s^{(i.j)}$, $s = 1, \ldots, m_{i.j}$, of the matrix $\mathbf{A}_{i\,j}$ are called "canonical efficiency factors" [cf. James and Wilkinson (1971), John (1987, p. 35)]. Clearly $m_{i.j} = \mathrm{r}(\mathbf{A}_{i.j}) = \mathrm{r}(\mathbf{S}_{i.j})$.

For canonical correlations of type (i), equation (3.1.1) implies that the following relationship holds between the canonical correlations and the canonical efficiency factors

$$\begin{aligned} \phi_s^{(i.j)} &= 1; & s &= 1, \ldots, z_{i.j} \\ &= 1 - [\rho_{d_i+1-s}^{(i.j)}]^2; & s &= z_{i.j}+1, \ldots, z_{i.j}+t_{i.j}, \end{aligned} \tag{3.5.2}$$

where $z_{i.j} = d_i - m_{i.j}$, the number d_i being the dimensionality of factor i. The $z_{i.j}$ unit canonical efficiency factors correspond to zero canonical correlations.

In a three-way layout, we can define efficiency matrices as in (3.5.1) for subdesigns and similarly obtain further efficiency matrices for designs eliminating two factors. For example, the two efficiency matrices for the treatment-row and treatment-column subdesigns in the two-way elimination of heterogeneity are given by, cf. (3.3.11),

$$\mathbf{A}_{3.1} = \mathbf{D}_3^{-1/2}\mathbf{S}_{3.1}\mathbf{D}_3^{-1/2} \quad \text{and} \quad \mathbf{A}_{3\,2} = \mathbf{D}_3^{-1/2}\mathbf{S}_{3.2}\mathbf{D}_3^{-1/2} \tag{3.5.3}$$

respectively, while the efficiency matrix for the full design, after eliminating rows and columns, is given by

$$\mathbf{A}_{3.12} = \mathbf{D}_3^{-1/2}\mathbf{S}_{3.12}\mathbf{D}_3^{-1/2}. \tag{3.5.4}$$

From the efficiency matrix for the design eliminating two factors, say j and k, we get the following canonical efficiency factors associated with canonical correlations of type (ii),

$$\begin{aligned} \phi_s^{(i.jk)} &= 1; && s = 1, \ldots, z_{i.jk} \\ &= 1 - [\rho_{d_i+1-s}^{(i.jk)}]^2; && s = z_{i.jk}+1, \ldots, z_{i.jk}+t_{i.jk}, \end{aligned} \tag{3.5.5}$$

where $z_{i.jk} = d_i - m_{i.jk}$. Canonical efficiency factors associated with canonical correlations of types (iii) and (iv) can similarly be defined as

$$\begin{aligned} \phi_s^{(i.j|k)} &= 1; && s = 1, \ldots, z_{i.j|k} \\ &= 1 - [\rho_{d_i+1-s}^{(i.j|k)}]^2; && s = z_{i.j|k}+1, \ldots, z_{i.j|k}+t_{i.j|k}, \end{aligned} \tag{3.5.6}$$

and

$$\begin{aligned} \phi_s^{(i;j|k)} &= 1; && s = 1, \ldots, z_{i;j|k} \\ &= 1 - [\rho_{d_i+1-s}^{(i;j|k)}]^2; && s = z_{i;j|k}+1, \ldots, z_{i;j|k}+t_{i;j|k}, \end{aligned} \tag{3.5.7}$$

where $z_{i.j|k} = d_i + m_{i.j|k}$ and $z_{i;j|k} = d_i + m_{i;j|k}$.

The nonzero canonical correlations of types (i), (ii) and (iv) are symmetric in their arguments, e.g.,

$$\rho_h^{(i.j)} = \rho_h^{(j.i)} \tag{3.5.8}$$

so that $t_{i.j} = t_{j.i}$ and $u_{i.j} = u_{j.i}$ and hence the nonunit canonical efficiency factors are also symmetric in this way, e.g.,

$$\phi_s^{(i.j)} = \phi_s^{(j.i)}; \quad s = z_{i.j} + 1, \ldots, z_{i.j} + t_{i.j}. \tag{3.5.9}$$

However, the numbers $z_{i.j} = d_i - m_{i.j}$ and $z_{j.i} = d_j - m_{j.i} = d_j - m_{i.j}$ of unit canonical efficiency factors are not necessarily equal as the dimensions of the factors i and j need not be the same. Obviously, $d_i = d_j \Leftrightarrow z_{i.j} = z_{j.i}$. In view of this, we write $z'_{i.j} = z_{j.i}$.

We define the "average efficiency factor" of the design eliminating two factors, say j and k, as the harmonic mean of the corresponding canonical efficiency factors [cf. e.g., John (1987, p. 27)]:

$$E_{i.jk} = \frac{v - u_{i.jk}}{(v - m_{i.jk}) + \sum_{s=v-m_{i.jk}+1}^{v-u_{i.jk}} \frac{1}{\phi_s^{(i.jk)}}} = \frac{\mathrm{r}(\mathbf{S}_{i.jk})}{\mathrm{tr}(\mathbf{A}_{i.jk}^{+})} \tag{3.5.10}$$

where the superscript $^+$ denotes the Moore-Penrose inverse and the efficiency matrix $\mathbf{A}_{i.jk}$ is defined analogously to (3.5.4). We will assume that $\mathbf{S}_{i.jk} \neq \mathbf{0}$ and so $v > u_{i.jk}$ and hence

$$0 < E_{i.jk} \leq 1,$$

with

$$E_{i.jk} = 1 \Leftrightarrow t_{i.jk} = 0 \Leftrightarrow m_{i.jk} = u_{i.jk}.$$

When $u_{i.jk} = 1$ we say that the design is "connected for treatments" and then $E_{i.jk}$ is the ratio of the average variance of the elementary contrasts for factor i, compared to those of an equivalent Latin square design [cf. e.g., Anderson and Eccleston (1985)].

Similarly, we define the average efficiency factors

$$E_{i.j} = \frac{v - u_{i.j}}{(v - m_{i.j}) + \sum_{s=u_{i.j}+1}^{m_{i.j}} \frac{1}{\phi_s^{(i.j)}}} = \frac{r(\mathrm{S}_{i.j})}{\mathrm{tr}(\mathrm{A}_{i.j}^{+})} \qquad (3.5.11)$$

and

$$E_{i.k} = \frac{v - u_{i.k}}{(v - m_{i.k}) + \sum_{s=u_{i.k}+1}^{m_{i.k}} \frac{1}{\phi_s^{(i.k)}}} = \frac{r(\mathrm{S}_{i.k})}{\mathrm{tr}(\mathrm{A}_{i.k}^{+})}, \qquad (3.5.12)$$

respectively, for the two subdesigns corresponding to either factor j or k being ignored. We can see that if $v - u = t$ or $v = m$, and if the non-unit canonical correlations ρ_h are all equal to ρ, then the average efficiency factor simplifies to $E = 1 - \rho^2$. We also note that $E_{i.j}$ need not be equal to $E_{j.i}$; clearly $E_{i.j} = E_{j.i} \Leftrightarrow d_i = d_j$. And so we will write $E'_{i.j} = E_{j.i}$ (following $z'_{i.j} = z_{j.i}$).

When $u_{i.j} = 1$ and the row-column subdesign is connected then $E_{i.j}$ is the ratio of the average variance of the elementary contrasts for factor i with those of an equivalent complete block design.

Our next result provides an upper bound for this average efficiency factor $E_{i.jk}$. A first version of the result was obtained by Roy and Shah (1961) for equireplicated row-column designs; Shah and Eccleston (1986) showed that equal replication was not needed. Our version is an extension to two-way elimination of heterogeneity designs which need not be connected; we require only that the degree of nonorthogonality $t_{i.jk}$ in the full design be no less than the degrees $t_{i.j}$ and $t_{i.k}$ of nonorthogonality in the two corresponding subdesigns.

Theorem 3.5.1 (Bérubé and Styan, 1990a): *For any three-way design satisfying* $t_{i.jk} \geq t_{i.g}$, $g = j$ *or* k, $i \neq j$, $i \neq k$, $j \neq k$, $i, j, k = 1, 2, 3$, *the average efficiency factors* $E_{i.jk}$, $E_{i.j}$ *and* $E_{i.k}$ *satisfy*

$$E_{i.jk} \leq \min(E_{i.j}, E_{i.k}). \tag{3.5.13}$$

Moreover, equality holds in (3.5.13) *if and only if* $u_{i.jk} = u_{i.g}$, $t_{i.jk} = t_{i.g}$, $m_{i.jk} = m_{i.g}$, *and* $\phi_s^{(i.jk)} = \phi_s^{(i.g)}$, $s = 1, \ldots, t_{i.g}$, *where* $g = j$ *or* k.

Proof: By the definition of canonical correlation

$$\rho_h^{(i.jk)} \geq \rho_h^{(i.g)}, \qquad h = 1, \ldots, m_{i.g},$$

where the subscript h indicates the h^{th} largest canonical correlation and $g = j$ or k. If $t_{i.jk} \geq t_{i.g}$, $g = j$ or k, then the following two sets of inequalities hold,

$$\frac{1}{\phi_s^{(i.jk)}} \geq \frac{1}{\phi_s^{(i.g)}}, \qquad s = 1, \ldots, t_{i.g}, \tag{3.5.14}$$

and

$$\frac{1}{\phi_s^{(i.jk)}} \geq 1, \qquad s = t_{i.g} + 1, \ldots, t_{i.jk}. \tag{3.5.15}$$

Now, the inequality in (3.5.13) holds if and only if the following inequality is true for $g = j$ or k,

$$E_{i.g} = \frac{d_i - u_{i.g}}{(d_i - m_{i.g}) + \sum_{s=u_{i.g}+1}^{m_{i.g}} \frac{1}{\phi_s^{(i.g)}}} \geq \frac{d_i - u_{i.jk}}{(d_i - m_{i.jk}) + \sum_{s=u_{i.jk}+1}^{m_{i.jk}} \frac{1}{\phi_s^{(i.jk)}}} = E_{i.jk}.$$

The above inequality holds if and only if

$$(d_i - u_{i.g})\left(d_i - m_{i.jk} + \sum_{s=u_{i.jk}+1}^{m_{i.jk}} \frac{1}{\phi_s^{(i.jk)}}\right) \geq (d_i - u_{i.jk})\left(d_i - m_{i.g} + \sum_{s=u_{i.g}+1}^{m_{i.g}} \frac{1}{\phi_s^{(i.g)}}\right)$$

$\Leftrightarrow$

$$(d_i - u_{i.g})(d_i - u_{i.jk}) + (d_i - u_{i.g})\left(\sum_{s=u_{i.jk}+1}^{m_{i.jk}} \frac{1}{\phi_s^{(i.jk)}} - t_{i.jk}\right)$$

$$\geq (d_i - u_{i.jk})(d_i - u_{i.g}) + (d_i - u_{i.jk})\left(\sum_{s=u_{i.g}+1}^{m_{i.g}} \frac{1}{\phi_s^{(i.g)}} - t_{i.g}\right)$$

$$\Leftrightarrow \quad (d_i - u_{i.g})\left[\sum_{s=u_{i.jk}+1}^{m_{i.jk}}\left(\frac{1}{\phi_s^{(i.jk)}} - 1\right)\right] \geq (d_i - u_{i.jk})\left[\sum_{s=u_{i.g}+1}^{m_{i.g}}\left(\frac{1}{\phi_s^{(i.g)}} - 1\right)\right].$$

This inequality holds since by the definition of canonical correlation $u_{i.g} \leq u_{i.jk}$ for $g = j$ or k, while from (3.5.14) and (3.5.15) we have

$$\sum_{s=u_{i.jk}+1}^{m_{i.jk}}\left(\frac{1}{\phi_s^{(i.jk)}} - 1\right) \geq \sum_{s=u_{i.g}+1}^{m_{i.g}}\left(\frac{1}{\phi_s^{(i.g)}} - 1\right).$$

Equality holds if and only if $u_{i.jk} = u_{i.g}$, $t_{i.jk} = t_{i.g}$, $m_{i.jk} = m_{i.g}$, and $\phi_s^{(i.jk)} = \phi_s^{(i.g)}$, $s = 1, \ldots, t_{i.g}$, and $g = j$ or k. ❐

When strict adjusted orthogonality holds in an equireplicated row-column design, the information matrices satisfy $\mathbf{S}_{3.12} = (1/k)\mathbf{S}_{3.1}\mathbf{S}_{3.2}$ and admit a common spectral decomposition. We then see that the canonical efficiency factors of such a design satisfy the two relationships

$$\phi_s^{(3.12)} = \phi_s^{(3.1)} + \phi_s^{(3.2)} - 1 \quad \text{and} \quad \phi_s^{(3.12)} = \phi_s^{(3.1)}\phi_s^{(3.2)},$$

i.e., at least one of $\phi_s^{(3.1)}$ or $\phi_s^{(3.2)}$ is always equal to one, for all $s = 1, \ldots, v-1$. We can use this to obtain the relationship

$$E_{3.12} = \frac{1}{E_{3.1}^{-1} + E_{3.2}^{-1} - 1}$$

between the average efficiency factors $E_{3.12}$, $E_{3.1}$ and $E_{3.1}$, cf. Eccleston and McGilchrist (1986).

The following theorem establishes relationships between canonical correlations of types (iii) and (iv).

Theorem 3.5.2: *The canonical correlations* $\rho_h^{(i;j|k)}$, $\rho_h^{(j;i|k)}$ *and* $\rho_h^{(i.j|k)}$ *satisfy the following inequality strings:*

$$\rho_h^{(i;j|k)} \le \rho_h^{(i.j|k)} \le \frac{\rho_h^{(i;j|k)}}{\sqrt{1 - [\rho_1^{(i.k)}]^2}}, \qquad h = 1, \ldots, m_{i.j|k}, \qquad (3.5.16)$$

$$\rho_h^{(j;i|k)} \le \rho_h^{(i.j|k)} \le \frac{\rho_h^{(j;i|k)}}{\sqrt{1 - [\rho_1^{(j.k)}]^2}}, \qquad h = 1, \ldots, m_{i.j|k}, \qquad (3.5.17)$$

where $\rho_1^{(i.k)}$ *and* $\rho_1^{(j.k)}$ *are, respectively, the largest non-unit canonical correlations between* $\mathbf{X}_i'\mathbf{y}$ *and* $\mathbf{X}_k'\mathbf{y}$, *and between* $\mathbf{X}_j'\mathbf{y}$ *and* $\mathbf{X}_k'\mathbf{y}$.

Proof: To prove the inequality on the left of (3.5.16), we first observe the Löwner partial ordering

$$\mathbf{M}_k\mathbf{H}_i\mathbf{M}_k \leq_{\mathrm{L}} \mathbf{M}_k, \tag{3.5.18}$$

since $\mathbf{H}_i$ and $\mathbf{M}_k$ are symmetric idempotent matrices. Hence

$$\mathbf{X}_i'\mathbf{M}_k\mathbf{H}_i\mathbf{M}_k\mathbf{X}_i \leq_{\mathrm{L}} \mathbf{X}_i'\mathbf{M}_k\mathbf{X}_i = \mathbf{X}_i'\mathbf{M}_k\mathbf{X}_i(\mathbf{X}_i'\mathbf{M}_k\mathbf{X}_i)^-\mathbf{X}_i'\mathbf{M}_k\mathbf{X}_i = \mathbf{X}_i'\mathbf{H}_{i\,|k}\mathbf{X}_i. \tag{3.5.19}$$

Using Lemma A.3.5 with $r(\mathbf{X}_i'\mathbf{M}_k\mathbf{X}_i) = r(\mathbf{M}_k\mathbf{X}_i) = r(\mathbf{X}_i'\mathbf{M}_k)$ we obtain

$$\mathbf{M}_k\mathbf{H}_i\mathbf{M}_k \leq_{\mathrm{L}} \mathbf{H}_{i\,|k}$$

and hence

$$\mathbf{H}_{j\,|k}\mathbf{M}_k\mathbf{H}_i\mathbf{M}_k\mathbf{H}_{j\,|k} \leq_{\mathrm{L}} \mathbf{H}_{j\,|k}\mathbf{M}_k\mathbf{X}_i(\mathbf{X}_i'\mathbf{M}_k\mathbf{X}_i)^-\mathbf{X}_i'\mathbf{M}_k\mathbf{H}_{j\,|k} = \mathbf{H}_{j\,|k}\mathbf{H}_{i\,|k}\mathbf{H}_{j\,|k}; \tag{3.5.20}$$

since $\mathbf{M}_k\mathbf{H}_{j\,|k} = \mathbf{H}_{j\,|k}$ it follows that

$$\mathbf{H}_{j\,|k}\mathbf{H}_i\mathbf{H}_{j\,|k} \leq_{\mathrm{L}} \mathbf{H}_{j\,|k}\mathbf{H}_{i\,|k}\mathbf{H}_{j\,|k},$$

proving the inequality on the left of (3.5.16). To establish the inequality on the right of (3.5.16), we use a result in Styan (1985), who shows that

$$\mathbf{H}_{i\,|k} \leq_{\mathrm{L}} \frac{\mathbf{M}_k\mathbf{H}_i\mathbf{M}_k}{1 - (\rho_1^{(i.k)})^2}. \tag{3.5.21}$$

Hence

$$\mathbf{H}_{j|k}\mathbf{H}_{i|k}\mathbf{H}_{j|k} \leq_{\mathrm{L}} \frac{\mathbf{H}_{j|k}\mathbf{H}_{i}\mathbf{H}_{j|k}}{1-(\rho_1^{(i.k)})^2}, \tag{3.5.22}$$

which establishes the inequality on the right of (3.5.16). The inequality string (3.5.17) follows at once from (3.5.16) by interchanging i and j and noting that $\rho_h^{(i.j|k)} = \rho_h^{(j.i|k)}$. ❐

A similar result to our Theorem 3.5.2 is given in Latour and Styan (1985) and Styan (1985) for sums of squares $S_h = \mathbf{y}'\mathbf{H}_{1|2}\mathbf{y}$ and $S_h^* = \mathbf{y}'\mathbf{M}_2\mathbf{H}_1\mathbf{M}_2\mathbf{y}$ in a two-way layout; they show that, with probability one, S_h and S_h^* satisfy the inequality string

$$S_h^* \leq S_h \leq \frac{S_h^*}{1-(\rho_1^{(i.k)})^2}, \tag{3.5.23}$$

cf. (3.5.16) and (3.5.17).

As shown in Baksalary (1987), we have equality in (3.5.19) if and only if the matrices $\mathbf{H}_i$ and $\mathbf{H}_k$ commute or equivalently $t_{i.k} = 0$ and we then have equality on the left of (3.5.16). Equality holds in (3.5.19) if and only if either $\rho_1^{(i.k)} = 0$ (since then the left-hand side and right-hand side of (3.5.16) are exactly the same), or the incidence matrix $\mathbf{N}_{ik}$ has full row rank equal to $m_{i.k} = t_{i.k} + u_{i.k}$ and $\rho_1^{(i.k)} = \ldots = \rho_{t_{i.k}}^{(i.k)}$. We then get equality on the right of (3.5.16).

EXAMPLE 3.5.1: We illustrate these conditions for equality with the following design given in Eccleston and Russell (1977),

$$\begin{matrix} 1 & 1 & 2 \\ 2 & 3 & 3 \end{matrix}\;.$$

From Table A.1.7 we see that $t_{1.2} = 0$ and indeed $\rho_h^{(1.3|2)} = \rho_h^{(1;3|2)} = (1/3)\sqrt{8}$. Also, this design satisfies $r(N_{23}) = 3 = u_{2.3} + t_{2.3}$ and $\rho_1^{(2.3)} = \rho_2^{(2.3)} = 1/2$, and as expected, we have that $\rho^{(1.2|3)} = [1 - (1/4)]^{-1/2}\rho^{(2;1|3)}$.

Since canonical correlations are nonnegative and cannot exceed 1, we have

$$0 \le \rho_h^{(i;j|k)} \le \rho_h^{(i.j|k)} \le 1$$

and

$$0 \le \rho_h^{(j;i|k)} \le \rho_h^{(i.j|k)} \le 1,$$

which forces $\rho_h^{(i.j|k)} = 1$ whenever either $\rho_h^{(i;j|k)}$ or $\rho_h^{(j;i|k)}$ is equal to 1, and furthermore both $\rho_h^{(i;j|k)} = 0$ and $\rho_h^{(j;i|k)} = 0$ whenever $\rho_h^{(i.j|k)} = 0$. Then, we must have that

$$0 \le u_{i;j|k} \le u_{i.j|k} \quad \text{and} \quad 0 \le u_{j;i|k} \le u_{i.j|k}. \tag{3.5.24}$$

We also know that the number of nonzero canonical correlations are the same, i.e., $m_{i;j|k} = m_{j;i|k} = m_{i.j|k}$ and so it follows that

$$0 \le t_{i.j|k} \le t_{i;j|k} \quad \text{and} \quad 0 \le t_{i.j|k} \le t_{j;i|k}. \tag{3.5.25}$$

Combining equations (3.5.24) and (3.5.25), we obtain the results

$$u_{i;j|k} = u_{i.j|k} \Longleftrightarrow t_{i;j|k} = t_{i.j|k},$$

and

$$u_{j;i|k} = u_{i.j|k} \Longleftrightarrow t_{j;i|k} = t_{i.j|k}.$$

CHAPTER 4. BALANCE AND COMMUTATIVITY

SECTION 4.1: VARIANCE AND EFFICIENCY BALANCE

The term "balanced design" can have a number of different meanings [cf. Preece (1982)]. In this first section, we will restrict ourselves to only two types of balance. In the next section, we will introduce another type of balance.

Definition 4.1.1: A two-way elimination of heterogeneity design is said to be variance balanced or to have variance balance whenever the ordinary least squares estimators of all normalized contrasts in the treatments have the same variance.

Following Jones (1959) we have a second type of balance where the concept of efficiency replaces that of variance in Definition 4.1.1.

Definition 4.1.2: A two-way elimination of heterogeneity design is said to be efficiency balanced or to have efficiency balance whenever the ordinary least squares estimators of all normalized contrasts in the treatments have the same efficiency.

A well-known necessary and sufficient condition [cf. Kshirsagar (1957), Singh, Dey and Nigam (1979)] for a two-way elimination of heterogeneity design, connected for treatments, to be variance balanced is that the information matrix $\mathbf{S}_{3.12}$ be a scalar multiple of the centering matrix:

$$\mathbf{S}_{3.12} = \lambda \mathbf{C}_{\nu} = \lambda[\mathbf{I} - (1/\nu)\mathbf{e}^{(\nu)}\mathbf{e}^{(\nu)\prime}], \qquad \lambda > 0, \tag{4.1.1}$$

i.e., the matrix $\mathbf{S}_{3.12}$ has all of its off-diagonal elements equal and all of its diagonal elements equal. This form guarantees that all the $\nu - 1$ nonzero eigenvalues of $\mathbf{S}_{3.12}$ are equal.

Similarly, a well-known characterization [cf. Jones (1959)] for an efficiency-balanced, treatment-connected two-way elimination of heterogeneity design is

$$\mathbf{S}_{3.12} = \vartheta\mathbf{S}_{3.0} = \vartheta[\mathbf{D}_3 - (1/n)\mathbf{k}_3\mathbf{k}_3'], \qquad \vartheta \in (0, 1], \qquad (4.1.2)$$

where ϑ represents the efficiency with which each treatment contrast $\mathbf{c}'\boldsymbol{\tau}$ is estimated. This representation was apparently first given by Williams (1975) in the context of block designs. Singh and Dey (1978) give an alternative representation of efficiency balance in terms of a matrix we will denote as $\mathbf{Q}_0$. This matrix $\mathbf{Q}_0$ is also used outside of the context of balance [cf. Caliński (1971) and Pearce (1975)], and is defined as

$$\mathbf{Q}_0 = \mathbf{I} - \mathbf{D}_3^{-1}\mathbf{S}_{3.12} - (1/n)\mathbf{e}\mathbf{k}_3'$$

$$= \mathbf{D}_3^{-1}\{\mathbf{N}_{31}[\mathbf{X}_1'\mathbf{M}_2\mathbf{X}_1 + (1/n)\mathbf{k}_1\mathbf{k}_1']^{-1}\mathbf{N}_{13} + \mathbf{N}_{32}[\mathbf{X}_2'\mathbf{M}_1\mathbf{X}_2 + (1/n)\mathbf{k}_2\mathbf{k}_2']^{-1}\mathbf{N}_{23}$$

$$- \mathbf{N}_{32}\mathbf{D}_2^{-1}\mathbf{N}_{21}[\mathbf{X}_1'\mathbf{M}_2\mathbf{X}_1 + (1/n)\mathbf{k}_1\mathbf{k}_1']^{-1}\mathbf{N}_{13} - \mathbf{N}_{31}\mathbf{D}_1^{-1}\mathbf{N}_{12}[\mathbf{X}_2'\mathbf{M}_1\mathbf{X}_2 + (1/n)\mathbf{k}_2\mathbf{k}_2']^{-1}\mathbf{N}_{23}\}.$$

A further representation of efficiency balance is given in term of Tocher's $\boldsymbol{\Omega}$–matrix [cf. (3.3.3)]. With respect to the matrices $\mathbf{Q}_0$ and $\boldsymbol{\Omega}$, efficiency balance is characterized by

$$\mathbf{Q}_0 = (1 - \vartheta)[\mathbf{I} - (1/n)\mathbf{e}\mathbf{k}_3'], \qquad \vartheta \in (0, 1], \qquad (4.1.3)$$

and

$$\boldsymbol{\Omega} = [\mathbf{D}_3^{-1} - (1 - \vartheta)\mathbf{J}_n]/\vartheta, \qquad \vartheta \in (0, 1]. \qquad (4.1.4)$$

We define variance balance and efficiency balance for the treatment-row and treatment-column subdesigns in a similar fashion, i.e., the subdesigns are variance balanced whenever all the ordinary least squares estimators of all the normalized contrasts in the treatments have the same variance, and efficiency balanced whenever the ordinary least squares estimators of all the normalized contrasts in the treatments have the same efficiency. Whenever the subdesigns are connected we can say that they have

$$\text{variance balance} \iff \mathbf{S}_{3.h} = \vartheta_h \mathbf{C}_v, \qquad h = 1, 2, \tag{4.1.5}$$

$$\text{efficiency balance} \iff \mathbf{S}_{3.h} = \vartheta_h \mathbf{S}_{3.0}, \qquad h = 1, 2. \tag{4.1.6}$$

With the following theorem, Singh, Dey and Nigam (1979) obtained a result for two-way elimination of heterogeneity designs that is equivalent to results established by Puri and Nigam (1975) and Williams (1975) in a block-design setting. However, as pointed out by Baksalary, Shah and Siatkowski (1990), the assumption that there be at least 3 treatments is essential—even though not stated in Singh, Dey and Nigam (1979).

Theorem 4.1.1 (Baksalary, Shah and Siatkowski, 1990): *For a treatment-connected two-way elimination of heterogeneity design with the number of treatments $v \geq 3$, any two of the following properties imply the third.*

(i) *The design is efficiency balanced,*

(ii) *The design is variance balanced,*

(iii) *The design is equireplicated.*

Proof: From (4.1.1) and (4.1.2) we see that if $v \geq 3$, then $\mathbf{S}_{3.0}$ is proportional to $\mathbf{C}_v$ if and only if the design is equireplicated. ❐

For an equireplicated row-column design, we have both variance balance and efficiency balance whenever

$$\frac{\mathbf{N}_{31}\mathbf{N}_{13}}{c} - \frac{\mathbf{N}_{32}\mathbf{N}_{23}}{r} = \eta_1\mathbf{I} + \eta_2\mathbf{e}^{(v)}\mathbf{e}^{(v)\prime} \text{ for some } \eta_1 \neq 0 \text{ and some } \eta_2.$$

The simplest design of this type is the "Youden square", cf. Youden (1937), in which the treatment-column subdesign is arranged as a symmetrical ($c = v$) balanced incomplete block design (BIBD) and the treatment-row subdesign is a complete block design, i.e., $\mathbf{N}_{31}\mathbf{N}_{13} = r\mathbf{e}^{(v)}\mathbf{e}^{(v)\prime}$.

With our next theorem, we present a relationship between efficiency balance in a two-way elimination of heterogeneity design and efficiency balance in its subdesigns.

Theorem 4.1.2: *For a treatment-connected two-way elimination of heterogeneity design satisfying the generalized decomposability property* $\mathbf{S}_{3.12} = \xi_1\mathbf{S}_{3.1} + \xi_2\mathbf{S}_{3.2} - \xi_0\mathbf{S}_{3.0}$, ξ_1, ξ_2 *and* $\xi_0 > 0$, *cf.*(2.3.1), *any two of the following properties imply the third.*

(i) *The design is efficiency balanced,*

(ii) *The treatment-row subdesign is efficiency balanced,*

(iii) *The treatment-column subdesign is efficiency balanced.*

Proof: Follows at once from the characterizations in (4.1.2) and (4.1.6). ❐

A form of this theorem was first given in the first part of Theorem 2 in Singh, Dey and Nigam (1979). Our version is a slight extension of the version given in Baksalary, Shah and Siatkowski (1990), since we have replaced the more restrictive reduced decomposability property, cf. (2.3.2),

$$\mathbf{S}_{3.12} = \mathbf{S}_{3.1} + \mathbf{S}_{3.2} - \mathbf{S}_{3.0}$$

by our less restrictive generalized decomposability property (2.3.1), i.e.,

$$\mathbf{S}_{3.12} = \xi_1\mathbf{S}_{3.1} + \xi_2\mathbf{S}_{3.2} - \xi_0\mathbf{S}_{3.0}, \quad \xi_1, \xi_2 \text{ and } \xi_0 > 0.$$

Corollary 4.1.3 (Ceranka and Kozlowska, 1985): *A treatment connected two-way elimination of heterogeneity design with treatments and rows strictly orthogonal after adjusting for columns ($m_{1.3|2} = 0$) is efficiency (or variance) balanced if and only if its treatment-row subdesign is efficiency (or variance) balanced.*

Proof: Follows directly from condition (ii) in Theorem 3.4.4. ❒

For an orthogonal row-column design (i.e., all three factors strictly pairwise orthogonal), $\mathbf{S}_{3.12} = \mathbf{S}_{3.0}$ and the design is efficiency balanced (with full efficiency for all contrasts). Nigam (1976) shows that if any one column is removed from an orthogonal design, then the design is still efficiency balanced (but now with reduced efficiency). Since removing this column does not affect the orthogonality in the treatment-column subdesign, the new information matrix for the reduced design, $\mathbf{S}^*_{3.12}$, say, is equal to the new treatment-row subdesign information matrix $\mathbf{S}^*_{3.1}$. Let $\mathbf{N}^*_{3.1}$ denote the new treatment-row incidence matrix and let $\mathbf{D}^*_3$ denote the new diagonal matrix with $\mathbf{k}^*_3$ the new vector of treatment replications on its diagonal. Then

$$\mathbf{S}^*_{3.1} = \mathbf{D}^*_3 - [1/(c-1)]\mathbf{N}^*_{3.1}\mathbf{N}^*_{1.3},$$

where the "new" incidence matrix $\mathbf{N}^*_{3.1} = [c\mathbf{k}^*_3\mathbf{e}^{(r)\prime}/r(c-1)] - \mathbf{W}$ is obtained from the incidence matrix of the original design by subtracting a matrix $\mathbf{W}$ which is equal to the submatrix in $\mathbf{X}_3'$ corresponding to the deleted column, i.e., a $v \times r$ matrix $\mathbf{W}$ such that $\mathbf{W}\mathbf{e}^{(r)} = \mathbf{k}^*_3/(c-1)$, $\mathbf{W}'\mathbf{e}^{(v)} = \mathbf{e}^{(r)}$ and $\mathbf{W}\mathbf{W}' = \mathbf{D}^*_3/(c-1)$. Therefore,

$$\mathbf{N}^*_{3.1}\mathbf{N}^*_{1.3} = \frac{c^2\mathbf{k}^*_3\mathbf{k}'^*_3}{r(c-1)^2} - \frac{2c\mathbf{k}^*_3\mathbf{k}'^*_3}{r(c-1)} + \frac{\mathbf{D}^*_3}{(c-1)} = \frac{\mathbf{D}^*_3}{(c-1)} - \frac{c(c-2)\mathbf{k}^*_3\mathbf{k}'^*_3}{r(c-1)^2}$$

and so

$$\mathbf{S}^*_{3.12} = [c(c-2)/(c-1)^2]\mathbf{S}^*_{3.0}. \tag{4.1.7}$$

EXAMPLE 4.1.1: To illustrate (4.1.7), we consider the following F-square design, cf. Table A.1.8,

1	2	2	3	4
4	1	2	2	3
3	4	1	2	2
2	3	4	1	2
2	2	3	4	1

Here $\mathbf{S}_{3.12} = \mathbf{S}_{3.0}$, i.e., the design has full efficiency. If the last column is deleted, then the treatment-row incidence matrix becomes

$$\mathbf{N}^*_{3.1} = \begin{pmatrix} 1 & 2 & 1 & 0 \\ 1 & 2 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 2 & 1 & 1 \end{pmatrix}$$

and the information matrix becomes $\mathbf{S}^*_{3.12} = (15/16)\mathbf{S}^*_{3.0}$, i.e., the design is still efficiency balanced but with an efficiency, now, of 15/16.

Although orthogonal row-column designs from which an arbitrary column is removed are still efficiency balanced, this does not hold true when at least two columns are deleted, cf. Nigam (1976).

A theorem similar to Theorem 4.1.2, but with one further condition, holds for designs that are variance balanced. We extend Theorem 3 in Baksalary, Shah and Siatkowski (1990) [which is a corrected version of the second part of Theorem 2 in Singh, Dey and Nigam (1979)] with our:

Theorem 4.1.4: *For a treatment-connected two-way elimination of heterogeneity design with the number of treatments* $v \geq 3$ *and satisfying the generalized decomposability property* $S_{3.12} = \xi_1 S_{3.1} + \xi_2 S_{3.2} - \xi_0 S_{3.0}$, ξ_1, ξ_2 *and* $\xi_0 > 0$, *cf.* (2.3.1), *any three of the following properties imply the fourth.*

(i) *The design is variance balanced,*

(ii) *The treatment-row subdesign is variance balanced,*

(iii) *The treatment-column subdesign is variance balanced,*

(iv) *The design is equireplicated.*

Proof: Follows from the characterizations in (4.1.1) and (4.1.5). ❒

EXAMPLE 4.1.2: The interesting equireplicated variance-balanced row-column design (cf. Table A.1.9)

2	4	8	7	6	3
3	5	7	2	4	9
5	6	3	1	7	8
4	9	1	8	3	5
6	7	9	5	1	2
1	8	2	4	9	6

(4.1.8)

given by Kshirsagar (1957), has both its subdesigns unbalanced. According to Baksalary, Shah and Siatkowski (1990) it is the only such design that has so far appeared in the literature. The eigenvalues of $S_{3.2}$ are 0, 7/2 repeated 4 times, and 4 repeated 4 times, those of $S_{3.1}$ are 0, 4 repeated 4 times, and 7/2 repeated 4 times. However, the 8 nonzero eigenvalues of $S_{3.12}$ are all 7/2.

For an equireplicated row-column design to be variance-balanced it is necessary that the off-diagonal elements of $r\mathbf{N}_{31}\mathbf{N}_{13} + c\mathbf{N}_{32}\mathbf{N}_{23}$ all be equal and that the diagonal elements all be equal (cf. Nigam, 1987).

Baksalary, Shah and Siatkowski (1990) point out that certain special cases of the results given in our Theorems 4.1.2 and 4.1.4 hold for designs with equal row sizes and equal column sizes satisfying the extended decomposability property (2.3.4), i.e.,

$$\mathbf{S}_{3.12} = \mathbf{D}_3 - \upsilon_1\mathbf{N}_{31}\mathbf{N}_{13} - \upsilon_2\mathbf{N}_{32}\mathbf{N}_{23} + \rho\mathbf{k}_3\mathbf{k}_3', \quad \upsilon_1, \upsilon_2, \rho > 0.$$

But as was pointed out in Section 2.3, since the row and column sizes are all equal, this is only a special case of designs satisfying our generalized decomposability property (2.3.1), i.e.,

$$\mathbf{S}_{3.12} = \xi_1\mathbf{S}_{3.1} + \xi_2\mathbf{S}_{3.2} - \xi_0\mathbf{S}_{3.0}, \quad \xi_1, \xi_2 \text{ and } \xi_0 > 0.$$

SECTION 4.2: GENERAL BALANCE

Commutativity of the efficiency matrices, cf. (3.3.10), is an important property for a design to possess since it is a necessary and sufficient condition for $\mathbf{A}_{3.1}$, $\mathbf{A}_{3.2}$ and $\mathbf{A}_{3.0}$ to admit a common spectral decomposition. The commutativity property was first introduced under the name of "general balance" by Nelder (1965a, b) in the context of equireplicated mixed linear models. It was further developed by Nelder (1968), Houtman and Speed (1983) and Payne and Tobias (1990) and is nicely summarized by Speed (1983).

In a mixed linear model setting, we have a vector of observations $\mathbf{y}$ for which the expected value is $\mathcal{E}(\mathbf{y}) = \mathbf{X}\boldsymbol{\tau}$ and the dispersion matrix is $\mathcal{D}(\mathbf{y}) = \mathbf{V}(\boldsymbol{\theta})$, where $\boldsymbol{\theta} = (\theta_1, \ldots, \theta_s)'$, with $s < n$, i.e., the dispersion matrix depends on s unknown parameters. Before considering general balance, we first consider the blocking structure of the design, i.e., the part which does not depend on the application of treatments.

Definition 4.2.1: The $n \times n$ dispersion matrix $\mathbf{V}$ is said to have *orthogonal block structure* if it can be written as

$$\mathbf{V} = \sum_{i=1}^{s} \theta_i \mathbf{V}_i,$$

a linear combination of s distinct, known, idempotent, symmetric, pairwise orthogonal matrices $\mathbf{V}_i$ of rank r_i, which sum to the identity matrix $\mathbf{I}_n$.

We note that the θ_i's are eigenvalues of $\mathbf{V}$ with multiplicities r_i and associated eigenvector sets that span the columns of $\mathbf{V}_i$. The range or column space $\mathfrak{C}(\mathbf{V}_i)$ is said to be the i^{th} stratum with $\mathbf{V}_i$ being the orthogonal projection onto this stratum. The projections of the observations into these different strata are uncorrelated and each has a single unknown variance parameter. In each stratum, an estimator $\hat{\boldsymbol{\tau}}$ of $\boldsymbol{\tau}$ can be found which is the generalized least squares estimator.

We may now define general balance as follows.

Definition 4.2.2: A design with orthogonal block structure is said to have *general balance* whenever the matrices $\mathbf{HV}_1\mathbf{H}, \ldots, \mathbf{HV}_s\mathbf{H}$ commute.

Designs with orthogonal block structure are, therefore, those for which the matrices $\mathbf{HV}_1\mathbf{H}, \ldots, \mathbf{HV}_s\mathbf{H}$ are spanned by a common set of eigenvectors. In more practical terms, if the $n \times v$ design matrix $\mathbf{X}$ has full column rank $v < n$ then the design has general balance if the columns of $\mathbf{X}$ are linear transformations of a subset of v eigenvectors of $\mathbf{V}$. In other words, $\mathbf{X}$ can be transformed by a nonsingular matrix $\mathbf{T}$ to an orthonormal matrix $\mathbf{X}^* = \mathbf{XT}$ with the property that $\mathbf{X}^{*\prime}\mathbf{V}_i\mathbf{X}^* = \mathbf{\Lambda}_i$, a diagonal matrix, for all $i = 1, \ldots, s$. We can see that since $(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{V}_i\mathbf{XT} = (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{X}^*\mathbf{\Lambda}_i = \mathbf{T}\mathbf{\Lambda}_i$, the columns of $\mathbf{T}$ are the eigenvectors of $(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{V}_i\mathbf{X}$ and the components of the diagonal matrix $\mathbf{\Lambda}_i$ are the eigenvalues of $(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{V}_i\mathbf{X}$ or equivalently of $\mathbf{HV}_i\mathbf{H}$. When we have general balance, estimation of the treatment parameters $\boldsymbol{\tau}$ and the variance components θ_i is simple and direct.

SECTION 4.3: GENERAL BALANCE IN ROW-COLUMN DESIGNS

In the context of row-column designs, the appropriate mixed model may be expressed as

$$\mathbf{y} = \mathbf{X}_1\mathbf{a} + \mathbf{X}_2\mathbf{b} + \mathbf{X}_3\boldsymbol{\tau} + \mathbf{u},$$

where the components a_i, b_j, u_{ijk} of $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{u}$ are uncorrelated random variables with zero means and, respectively, variances $\mathrm{Var}(a_i) = \sigma_a^2$, $\mathrm{Var}(b_j) = \sigma_b^2$ and $\mathrm{Var}(u_{ijk}) = \sigma^2$. The dispersion matrices are $\mathfrak{D}(\mathbf{a}) = \sigma_a^2\mathbf{I}$, $\mathfrak{D}(\mathbf{b}) = \sigma_b^2\mathbf{I}$ and $\mathfrak{D}(\mathbf{u}) = \sigma^2\mathbf{I}$. The treatment parameters τ_k in $\boldsymbol{\tau}$ are (unknown) constant parameters and so

$$\mathrm{Var}(y_{ij}) = \sigma^2 + \sigma_a^2 + \sigma_b^2,$$

$$\mathrm{Cov}(y_{ij}, y_{i'j'}) = \begin{cases} \sigma_a^2 & i = i' \quad j \neq j' \\ \sigma_b^2 & i \neq i' \quad j = j' \\ 0 & i \neq i' \quad j \neq j' \end{cases}$$

This gives the following expected value and dispersion matrix for the observations,

$$\mathcal{E}(\mathbf{y}) = \mathbf{X}_3\boldsymbol{\tau}$$

and

$$\mathbf{V} = \mathfrak{D}(\mathbf{y}) = \mathbf{X}_1\mathbf{X}_1'\sigma_a^2 + \mathbf{X}_2\mathbf{X}_2'\sigma_b^2 + \sigma^2\mathbf{I}_{rc}$$

$$= c\sigma_a^2(\mathbf{J}_c \otimes \mathbf{I}_r) + r\sigma_b^2(\mathbf{I}_c \otimes \mathbf{J}_r) + \sigma^2\mathbf{I}_{rc}.$$

The spectral form of this dispersion matrix can then be written as

$$\mathbf{V} = \mathbf{J}_n\sigma_0^2 + \left(\frac{\mathbf{X}_1\mathbf{X}_1'}{c} - \mathbf{J}_n\right)\sigma_1^2 + \left(\frac{\mathbf{X}_2\mathbf{X}_2'}{r} - \mathbf{J}_n\right)\sigma_2^2 + \left(\mathbf{I}_n - \frac{\mathbf{X}_1\mathbf{X}_1'}{c} - \frac{\mathbf{X}_2\mathbf{X}_2'}{r} + \mathbf{J}_n\right)\sigma_3^2$$

$$= \mathbf{J}_n\sigma_0^2 + (\mathbf{H}_1 - \mathbf{J}_n)\sigma_1^2 + (\mathbf{H}_2 - \mathbf{J}_n)\sigma_2^2 + (\mathbf{I}_n - \mathbf{H}_1 - \mathbf{H}_2 + \mathbf{J}_n)\sigma_3^2$$

$$= \mathbf{J}_n\sigma_0^2 + (\mathbf{C}_r \otimes \mathbf{J}_c)\sigma_1^2 + (\mathbf{J}_r \otimes \mathbf{C}_c)\sigma_2^2 + (\mathbf{C}_c \otimes \mathbf{C}_r)\sigma_3^2$$

$$= \sigma_0^2\mathbf{V}_0 + \sigma_1^2\mathbf{V}_1 + \sigma_2^2\mathbf{V}_2 + \sigma_3^2\mathbf{V}_3,$$

say, where $\sigma_0^2 = \sigma_1^2 + \sigma_2^2 - \sigma_3^2 = c\sigma_a^2 + r\sigma_b^2 + \sigma^2$, $\sigma_3^2 = \sigma^2$, $\sigma_1^2 = c\sigma_a^2 + \sigma^2$ and $\sigma_2^2 = r\sigma_b^2 + \sigma^2$. The dispersion matrix $\mathbf{V}$ has orthogonal block structure since $\mathbf{V}_0$, $\mathbf{V}_1$, $\mathbf{V}_2$ and $\mathbf{V}_3$ are idempotent, symmetric, pairwise orthogonal matrices which sum to $\mathbf{I}$. In a row-column design, we call the four strata: the mean stratum, the row-stratum, the column-stratum and the row-by-column stratum. We note that $\mathbf{X}_3'\mathbf{V}_i\mathbf{X}_3$, $i = 0, 1, 2, 3$, is the information matrix in the i^{th} stratum and we therefore have $\mathbf{S}_{3.12} = \mathbf{X}_3'\mathbf{V}_3\mathbf{X}_3 = \mathbf{S}_{3.1} + \mathbf{S}_{3.2} - \mathbf{S}_{3.0}$, $\mathbf{S}_{3.r} = \mathbf{X}_3'\mathbf{V}_1\mathbf{X}_3$ and $\mathbf{S}_{3.c} = \mathbf{X}_3'\mathbf{V}_2\mathbf{X}_3$ (cf. Section 2.2).

Since for row-column designs $\mathbf{H}_3\mathbf{V}_0\mathbf{H}_3\mathbf{V}_i\mathbf{H}_3 = \mathbf{0}$ holds for all $i = 1, 2, 3$, the condition of general balance is satisfied if and only if

$$\mathbf{H}_3\mathbf{H}_1\mathbf{H}_3\mathbf{H}_2\mathbf{H}_3 = \mathbf{H}_3\mathbf{H}_2\mathbf{H}_3\mathbf{H}_1\mathbf{H}_3,$$

or equivalently, using the result in (A.3.4),

$$\mathbf{N}_{31}\mathbf{N}_{13}\mathbf{D}_3^{-1}\mathbf{N}_{32}\mathbf{N}_{23} = \mathbf{N}_{32}\mathbf{N}_{23}\mathbf{D}_3^{-1}\mathbf{N}_{31}\mathbf{N}_{13}.$$

If the design is equireplicated and has general balance, then the matrices $\mathbf{X}_3'\mathbf{V}_0\mathbf{X}_3$, $\mathbf{X}_3'\mathbf{V}_1\mathbf{X}_3$, $\mathbf{X}_3'\mathbf{V}_2\mathbf{X}_3$ and $\mathbf{X}_3'\mathbf{V}_3\mathbf{X}_3$ can be simultaneously diagonalized and so the strata information matrices can also be simultaneously diagonalized. This is the form introduced by Nelder (1965a, b). Speed (1983) claims that "all row-column designs ever used in practice (involving random effects) satisfy the property of general balance"; designs that do not satisfy this condition do, however, exist.

EXAMPLE 4.3.1: The equireplicated row-column design (cf. Table A.1.10)

2	1	1	1
1	3	3	2
2	2	4	3
4	4	4	3

is a row-column design which does not satisfy the property of general balance, cf. Speed (1983). We can easily show that general balance does not hold by noting that the commutator

$$\mathbf{N}_{31}\mathbf{N}_{13}\mathbf{N}_{32}\mathbf{N}_{23} - \mathbf{N}_{32}\mathbf{N}_{23}\mathbf{N}_{31}\mathbf{N}_{13} = \begin{pmatrix} 0 & 6 & 0 & -6 \\ -6 & 0 & 2 & 4 \\ 0 & -2 & 0 & 2 \\ 6 & -4 & -2 & 0 \end{pmatrix} \neq \mathbf{0}.$$

If the condition of general balance is satisfied, the efficiency matrices $\mathbf{A}_{3\,r}$, $\mathbf{A}_{3\,c}$, $\mathbf{A}_{3\,1}$, $\mathbf{A}_{3.2}$, $\mathbf{A}_{3\,0}$ and $\mathbf{A}_{3\,12}$ are all spanned by the same set of eigenvectors, i.e., there exists an orthogonal matrix $\mathbf{U}$ such that $\mathbf{U}'\mathbf{A}_g\mathbf{U}$, $g = 3.r, 3.c, 3.1, 3.2, 3.0, 3.12$, are diagonal matrices. We then find that the s^{th} canonical efficiency factor $\phi_s^{(3\,12)}$ of the row-column design which corresponds to the eigenvector $\mathbf{u}_s$ is given by

$$\phi_s^{(3.12)} = \phi_s^{(3.1)} + \phi_s^{(3.2)} - 1$$

$$= 1 - \phi_s^{(3.r)} - \phi_s^{(3.c)}, \qquad (4.3.1)$$

where $\phi_s^{(3.r)} = \text{ch}_s(\mathbf{A}_{3.r})$ and $\phi_s^{(3.c)} = \text{ch}_s(\mathbf{A}_{3.c})$. From the first equation in (4.3.1) we observe that a row-column design with row-treatment and column-treatment subdesigns having high canonical efficiency factors must itself be highly efficient. Also, since $\phi_s^{(3.12)} + \phi_s^{(3.r)} + \phi_s^{(3.c)} = 1$, any canonical contrast $\mathbf{D}_3^{-1/2}\mathbf{u}_s'\boldsymbol{\tau}$ is estimable in at least one stratum, cf. Shah and Eccleston (1986).

SECTION 4.4: COMMUTATIVITY

When the specific case of row-column designs is replaced by the more general one of three-way designs, the term general balance might not be applicable since three-way designs do not necessarily have an orthogonal block structure. However, commutativity of the efficiency matrices is still an important property for a design to possess. In the context of fixed effect two-way elimination of heterogeneity designs, Baksalary and Shah (1990) simply call this the "commutativity property". The terms general balance and commutativity property have both been used when row-column designs are considered as fixed-effect models [cf. Shah and Eccleston (1986), Eccleston and McGilchrist (1985) and Lewis and Dean (1990)].

Our Theorem 4.4.1 gives nine conditions, each of which is sufficient for a design to satisfy the commutativity property.

Theorem 4.4.1: *Any one of the following nine conditions is sufficient for a two-way elimination of heterogeneity design to fulfill the commutativity property:*

(a) $t_{1.2} = 0$ *and* $m_{1.2|3} = 0$,

(b) $t_{1.2} = 0$ *and* $t_{1.3} = 0$,

(c) $t_{1.2} = 0$ *and* $t_{2.3} = 0$,

(d) $m_{1.2|3} = 0$ *and* $m_{1.3|2} = 0$,

(e) $m_{1.2|3} = 0$ *and* $m_{2.3|1} = 0$,

(f) $t_{2.3} = 0$ *and* $m_{2.3|1} = 0$,

(g) $t_{2.3} = 0$ *and* $m_{1.3|2} = 0$,

(h) $t_{1.3} = 0$ *and* $m_{1.3|2} = 0$,

(i) $t_{1.3} = 0$ *and* $m_{2.3|1} = 0$.

Proof: The sufficiency of each of the above conditions may be established using the result that weak pairwise orthogonality, i.e., $t_{ij} = 0$ for any $i, j = 1, 2, 3$; $i \neq j$, is equivalent to the commutativity of the orthogonal projectors $\mathbf{H}_i$ and $\mathbf{H}_j$, cf. Theorem 3.2.3, and/or the result that strict adjusted orthogonality, i.e., $m_{i.j|k} = 0$ for $i, j, k = 1, 2, 3$; $i \neq j, i \neq k, j \neq k$, is equivalent to the equality $\mathbf{H}_i\mathbf{H}_k\mathbf{H}_j = \mathbf{H}_i\mathbf{H}_j$, cf. (3.4.4).

We prove only the sufficiency of condition (a): The commutativity property can be expressed as

$$\mathbf{H}_3\mathbf{H}_1\mathbf{H}_3\mathbf{H}_2\mathbf{H}_3 = \mathbf{H}_3\mathbf{H}_2\mathbf{H}_3\mathbf{H}_1\mathbf{H}_3\,; \tag{4.4.1}$$

since $m_{1.2|3} = 0$ is equivalent to $\mathbf{H}_1\mathbf{H}_3\mathbf{H}_2 = \mathbf{H}_1\mathbf{H}_2$ the left hand side of (4.4.1) simplifies to $\mathbf{H}_3\mathbf{H}_1\mathbf{H}_2\mathbf{H}_3$, which equals $\mathbf{H}_3\mathbf{H}_2\mathbf{H}_1\mathbf{H}_3$ when $t_{1.2} = 0$, since then $\mathbf{H}_1\mathbf{H}_2 = \mathbf{H}_2\mathbf{H}_1$. Replacing $\mathbf{H}_2\mathbf{H}_1 = \mathbf{H}_2\mathbf{H}_3\mathbf{H}_1$ yields the right hand side of (4.4.1). ❐

Baksalary and Shah (1990) point out that strict adjusted orthogonality implies commutativity for all treatment-connected two-way elimination of heterogeneity designs such that $r(\mathbf{N}_{12}) = 1$. Siatkowski (1990) strengthens this result by allowing the design to be disconnected for treatments. Our result with condition (a) in Theorem 4.4.1 is stronger than both these results since the assumption that $r(\mathbf{N}_{12}) = 1$ is replaced by the weaker condition $t_{12} = 0$.

EXAMPLE 4.4.1: The design

1	2	*	*
2	1	*	*
*	*	3	4
*	*	4	3

was considered by Eccleston and Russell (1975). Here, cf. Table A.1.11, $t_{12} = 0$ and $m_{1.2|3} = 0$ but $r(\mathbf{N}_{12}) = 2$. This design does satisfy the commutativity property, i.e., $\mathbf{A}_{3.1}\mathbf{A}_{3.2} = \mathbf{A}_{3.2}\mathbf{A}_{3.1}$.

It was pointed out by Shah (1977) and Sia (1977) that equireplicate row-column designs satisfying strict adjusted orthogonality form a subclass of designs satisfying the commutativity property. The complementary subclass of designs satisfying the commutativity property and not the strict adjusted orthogonality property is, however, not empty.

EXAMPLE 4.4.2: The design

1	1	2
2	3	3

considered by Eccleston and Russell (1977) belongs to this complementary subclass of designs, i.e., it satisfies the commutativity property but $m_{1.2|3} = 1$ (cf. Table A.1.7).

Shah and Eccleston (1986) observe that unequireplicate row-column designs satisfying strict adjusted orthogonality ($m_{12|3} = 0$) also form a subclass of designs satisfying the commutativity property. This can be shown by recalling that if $m_{12|3} = 0$, then $\mathbf{N}_{13}\mathbf{D}_3^{-1}\mathbf{N}_{32} = \mathbf{e}^{(r)}\mathbf{e}^{(c)\prime}$, cf. (3.3.17). Pre- and post-multiplying by $\mathbf{N}_{31}$ and $\mathbf{N}_{23}$ respectively, we obtain $\mathbf{N}_{31}\mathbf{N}_{13}\mathbf{D}_3^{-1}\mathbf{N}_{32}\mathbf{N}_{23} = \mathbf{k}_3\mathbf{k}_3'$ which is symmetric. For row-column designs, Shah and Eccleston (1986) present a result similar to that given by Shah (1977) for equireplicate row-column designs only. It gives a characterization for designs which satisfy strict adjusted orthogonality in the set of designs satisfying the commutativity property. Although Shah and Eccleston (1990) present their result without the commutativity property being satisfied, we believe the result only holds if this property is present. This result uses the matrices defined in (2.2.11).

Theorem 4.4.2 (Shah and Eccleston, 1986): *A row-column design satisfying the commutativity property also satisfies strict adjusted orthogonality if and only if for $s = 1, \dots, v-1$,*

$$\phi_s^{(3.r)}\phi_s^{(3.c)} = 0.$$

Proof: If the design satisfies strict adjusted orthogonality, then

$$\mathbf{N}_{31}\mathbf{N}_{13}\mathbf{D}_3^{-1}\mathbf{N}_{32}\mathbf{N}_{23} = \mathbf{k}_3\mathbf{k}_3' \tag{4.4.2}$$

and so $\mathbf{A}_{3.r}\mathbf{A}_{3.c} = \mathbf{0}$. If we let $\mathbf{u}_s$ be the eigenvector corresponding to both the eigenvalues $\phi_s^{(3.r)}$ and $\phi_s^{(3.c)}$, then we can write

$$\mathbf{A}_{3.r}\mathbf{A}_{3.c} = \sum_{s=1}^{v-1}\phi_s^{(3.r)}\phi_s^{(3.c)}\mathbf{u}_s\mathbf{u}_s',$$

and it follows that $\phi_s^{(3.r)}\phi_s^{(3.c)} = 0$ for $s = 1, \dots, v-1$.

Conversely, if $\phi_s^{(3.r)}\phi_s^{(3.c)} = 0$ for $s = 1, \ldots, v-1$, then $\mathbf{N}_{31}\mathbf{N}_{13}\mathbf{D}_3^{-1}\mathbf{N}_{32}\mathbf{N}_{23} = \mathbf{k}_3\mathbf{k}_3'/n$ and so $r(\mathbf{N}_{31}\mathbf{N}_{13}\mathbf{D}_3^{-1}\mathbf{N}_{32}\mathbf{N}_{23}) = 1$. We then have that

$$r(\mathbf{N}_{23}\mathbf{D}_3^{-1}\mathbf{N}_{31}\mathbf{N}_{13}\mathbf{D}_3^{-1}\mathbf{N}_{32}\mathbf{N}_{23}\mathbf{D}_3^{-1}\mathbf{N}_{31}\mathbf{N}_{13}\mathbf{D}_3^{-1}\mathbf{N}_{32})$$

$$= r(\mathbf{N}_{23}\mathbf{D}_3^{-1}\mathbf{N}_{31}\mathbf{N}_{13}\mathbf{D}_3^{-1}\mathbf{N}_{32}) = r(\mathbf{N}_{13}\mathbf{D}_3^{-1}\mathbf{N}_{32}) = 1,$$

and so the expression $\mathbf{N}_{13}\mathbf{D}_3^{-1}\mathbf{N}_{32}$ must take the form $\mathbf{pq}' = \{p_i q_j\}$. We can postmultiply $\mathbf{N}_{13}\mathbf{D}_3^{-1}\mathbf{N}_{32}$ by $\mathbf{e}^{(r)}$ and obtain $c\mathbf{e}^{(r)} = \mathbf{p}\sum_{j=1}^{c} q_j$. We thus find that $\mathbf{p}$ is proportional to the vector $\mathbf{e}^{(r)}$. In a similar way, we can show that $\mathbf{q}$ is proportional to $\mathbf{e}^{(c)}$ and so $\mathbf{N}_{13}\mathbf{D}_3^{-1}\mathbf{N}_{32}$ is proportional to $\mathbf{e}^{(r)}\mathbf{e}^{(c)\prime}$. However, since $\mathbf{N}_{13}\mathbf{D}_3^{-1}\mathbf{N}_{32}\mathbf{e}^{(c)} = c\mathbf{e}^{(r)}$ it follows that

$$\mathbf{N}_{13}\mathbf{D}_3^{-1}\mathbf{N}_{32} = \mathbf{e}^{(r)}\mathbf{e}^{(c)\prime}$$

and so the design satisfies strict adjusted orthogonality. ❐

In our next theorem, we extend a result given by Baksalary and Shah (1990) for designs satisfying the reduced decomposability property, i.e., $\mathbf{S}_{3\,12} = \mathbf{S}_{3.1} + \mathbf{S}_{3.2} - \mathbf{S}_{3.0}$, to designs satisfying our less restrictive generalized decomposability property, i.e., $\mathbf{S}_{3\,12} = \xi_1\mathbf{S}_{3.1} + \xi_2\mathbf{S}_{3.2} - \xi_0\mathbf{S}_{3.0}$, ξ_1, ξ_2 and $\xi_0 > 0$.

Theorem 4.4.3: *If a treatment-connected ($u_{3\,12} = 1$) two-way elimination of heterogeneity design satisfying the generalized decomposability property, i.e., $\mathbf{S}_{3\,12} = \xi_1\mathbf{S}_{3.1} + \xi_2\mathbf{S}_{3.2} - \xi_0\mathbf{S}_{3.0}$, ξ_1, ξ_2 and $\xi_0 > 0$, is efficiency balanced or if its treatment-row or treatment-column subdesign is efficiency-balanced, then the commutativity property holds.*

Proof: We first suppose that the treatment-row subdesign is efficiency-balanced. Then we can write

$$\mathbf{A}_{3.1} = \vartheta\mathbf{A}_{3\,0},$$

and, postmultiplying by $A_{3.2}$, yields

$$A_{3.1}A_{3.2} = \vartheta A_{3.0}A_{3.2} = \vartheta A_{3.2},$$

which is symmetric and so the commutativity property holds. The proof for the column-treatment subdesign is similar.

When the row-column design itself is efficiency-balanced, then we have

$$A_{3.12} = \vartheta A_{3.0},$$

which is equivalent to

$$A_{3.2} = \frac{(\xi_0 + \vartheta)A_{3.0} - \xi_1 A_{3.1}}{\xi_2}. \qquad (4.4.3)$$

Therefore, if we now premultiply (4.4.3) by $A_{3.1}$, we obtain

$$A_{3.1}A_{3.2} = \frac{(\xi_0 + \vartheta)A_{3.1}A_{3.0} - \xi_1 A_{3.1}^2}{\xi_2} = \frac{(\xi_0 + \vartheta)A_{3.1} - \xi_1 A_{3.1}^2}{\xi_2},$$

which is symmetric and so the commutativity property holds. ❐

Another relationship between efficiency balance and the commutativity property is given in the following lemma.

Lemma 4.4.4. *A treatment-connected two-way elimination of heterogeneity design, which satisfies* $S_{3.12} = \xi_1 S_{3.1} + \xi_2 S_{3.2} - \xi_0 S_{3.0}$ *for some* ξ_1, ξ_2 *and* $\xi_0 > 0$, *is efficiency balanced if and only if it satisfies the commutativity property and* $\xi_1\phi_s^{(3\,1)} + \xi_2\phi_s^{(3\,2)}$ *is the same for all* $s = 1, \ldots, v-1$, *where the nonzero eigenvalues* $\phi_1^{(3\,h)}, \ldots, \phi_{v-1}^{(3\,h)}$, $h = 1, 2$, *are ordered correspondingly to a fixed set of common eigenvectors of* $A_{3.1}$ *and* $A_{3.2}$.

Proof: If $\mathbf{A}_{3.12} = \vartheta\mathbf{A}_{3.0}$ for some ϑ, then $\mathbf{S}_{3.12} = \xi_1\mathbf{S}_{3.1} + \xi_2\mathbf{S}_{3.2} - \xi_0\mathbf{S}_{3.0}$ implies that

$$\xi_1\mathbf{A}_{3.1} + \xi_2\mathbf{A}_{3.2} = (\xi_0 + \vartheta)\mathbf{A}_{3.0}$$

and so $\mathbf{A}_{3.1}\mathbf{A}_{3.2} = \mathbf{A}_{3.2}\mathbf{A}_{3.1}$. This in turn implies that

$$\xi_1\phi_s^{(3.1)} + \xi_2\phi_s^{(3.2)} = \xi_0 + \vartheta \qquad \text{for } s = 1, \ldots, \nu - 1.$$

Conversely, if $\mathbf{A}_{3.1}\mathbf{A}_{3.2} = \mathbf{A}_{3.2}\mathbf{A}_{3.1}$ and $\xi_1\phi_s^{(3.1)} + \xi_2\phi_s^{(3.2)}$ is equal to a constant c, say, then $\mathbf{S}_{3.12} = \xi_1\mathbf{S}_{3.1} + \xi_2\mathbf{S}_{3.2} - \xi_0\mathbf{S}_{3.0}$ implies that

$$\phi_s^{(3.12)} = \xi_1\phi_s^{(3.1)} + \xi_2\phi_s^{(3.2)} - \xi_0 = c - \xi_0$$

and so $\mathbf{S}_{3.12}$ is a scalar multiple of $\mathbf{S}_{3.0}$. ❐

EXAMPLE 4.4.3: We apply Lemma 4.4.4 to the design

1	2	*	*	*	*	3	4
2	1	*	*	*	*	4	3
*	*	1	*	*	*	*	*
*	*	*	2	*	*	*	*
*	*	*	*	3	*	*	*
*	*	*	*	*	4	*	*
3	4	*	*	*	*	1	2
4	3	*	*	*	*	2	1

The lemma's conditions are satisfied, i.e., $u_{3.12} = 1$, and we can write $\mathbf{S}_{3.12} = \mathbf{S}_{3.1} + \mathbf{S}_{3.2} - (4/5)\mathbf{S}_{3.0}$. Also, the commutativity property holds and from Table A.1.12 we see that $\phi_s^{(3.1)} + \phi_s^{(3.2)} = 4/5$. This implies that the design is efficiency balanced and so $\mathbf{S}_{3.12} = (5/4)\mathbf{S}_{3.0}$.

Our Lemma 4.4.4 above is a slight modification of Theorem 4.2 in Baksalary and Shah (1990) and the Lemma on page 7 in Baksalary and Siatkowski (1990). In their Theorem 4.2, Baksalary and Shah (1990) assume the decomposability property only, and in the Lemma on page 7, Baksalary and Siatkowski (1990) assume equal row and column sizes with the information matrix satisfying $\mathbf{S}_{3.12} = \mathbf{D}_3 - \upsilon_1\mathbf{N}_{31}\mathbf{N}_{13} - \upsilon_2\mathbf{N}_{32}\mathbf{N}_{23} + \rho\, \mathbf{k}_3\mathbf{k}_3'$, υ_1, υ_2, $\rho > 0$.

A number of results for bounds on the average efficiency factor can be obtained for treatment-connected designs satisfying the commutativity property. In particular, for two-way elimination of heterogeneity designs of that type, stronger bounds than those in (3.5.13) can be obtained for the average efficiency factor. A first bound was given by Eccleston and McGilchrist (1985) for equireplicate row-column designs satisfying the commutativity property,

$$E_{3.12}^{-1} \geq E_{3.1}^{-1} + E_{3.2}^{-1} - 1. \tag{4.4.4}$$

Shah and Eccleston (1986) showed that (4.4.4) also holds for unequireplicated row-column designs; Baksalary and Shah (1990) extended this result to designs satisfying the reduced decomposability property, i.e., $\mathbf{S}_{3.12} = \mathbf{S}_{3.1} + \mathbf{S}_{3.2} - \mathbf{S}_{3.0}$, while Baksalary and Siatkowski (1990) showed that (4.4.4) holds for designs with equal row sizes and equal column sizes and such that $\mathbf{S}_{3.12} = \mathbf{D}_3 - \upsilon_1\mathbf{N}_{31}\mathbf{N}_{13} - \upsilon_2\mathbf{N}_{32}\mathbf{N}_{23} + \rho\, \mathbf{k}_3\mathbf{k}_3'$, υ_1, υ_2, $\rho > 0$. Here we prove yet another extension following the proof of Theorem 4.2 in Baksalary and Shah (1990).

Theorem 4.4.5. *If a treatment-connected two-way elimination of heterogeneity design satisfies the commutativity property and* $\mathbf{S}_{3.12} = \xi_1\mathbf{S}_{3.1} + \xi_2\mathbf{S}_{3.2} - \xi_0\mathbf{S}_{3.0}$ *for some* ξ_1, ξ_2 *and* $\xi_0 > 0$, $\xi_0 \geq \xi_1$ *and* $\xi_0 \geq \xi_2$ *or* $\xi_0 \leq \xi_1$ *and* $\xi_0 \leq \xi_2$, *then the average efficiency factor of the design and of the two subdesigns satisfy the inequality*

$$E_{3.12}^{-1} \geq (\xi_1 E_{3.1})^{-1} + (\xi_2 E_{3.2})^{-1} - 1/\xi_0, \tag{4.4.5}$$

with equality if and only if

$$(\xi_0 \mathbf{A}_{3.0} - \xi_1 \mathbf{A}_{3.1})(\xi_0 \mathbf{A}_{3.0} - \xi_2 \mathbf{A}_{3.2}) = \mathbf{0} \tag{4.4.6}$$

or equivalently

$$\mathbf{N}_{31}\mathbf{D}_1^{-1}\mathbf{N}_{13}\mathbf{D}_3^{-1}\mathbf{N}_{32}\mathbf{D}_2^{-1}\mathbf{N}_{23} = -[(\xi_1 - \xi_0)(\xi_2 - \xi_0)/\xi_1\xi_2]\mathbf{D}_3 + [1 - (\xi_0/\xi_2)]\mathbf{N}_{31}\mathbf{D}_1^{-1}\mathbf{N}_{13}$$

$$+ [1 - (\xi_0/\xi_1)]\mathbf{N}_{32}\mathbf{D}_2^{-1}\mathbf{N}_{23} + (\xi_0^2/n\xi_1\xi_2)\mathbf{k}_3\mathbf{k}_3'. \tag{4.4.7}$$

Proof: Because of the commutativity property and connectedness, there exists a $v \times (v-1)$ matrix $\mathbf{U}$ such that $\mathbf{U}'\mathbf{U} = \mathbf{I}$, $\mathbf{A}_{3.0} = \mathbf{U}\mathbf{U}'$ and $\mathbf{A}_{3.h} = \mathbf{U}\Lambda_h\mathbf{U}'$, $h = 1, 2$, where $\Lambda_h = \text{diag}(\phi_1^{(3.h)}, \ldots, \phi_{v-1}^{(3.h)})$ with $\phi_s^{(3.h)} \in (0, 1]$, $h = 1, 2$ and $s = 1, \ldots, v-1$. Therefore

$$\frac{1}{\xi_1\phi_s^{(3.1)} + \xi_2\phi_s^{(3.2)} - \xi_0} - \left[\frac{1}{\xi_1\phi_s^{(3.1)}} + \frac{1}{\xi_2\phi_s^{(3.2)}} - \frac{1}{\xi_0}\right]$$

$$= \frac{(\xi_1\phi_s^{(3.1)} + \xi_2\phi_s^{(3.2)})(\xi_0 - \xi_1\phi_s^{(3.1)})(\xi_0 - \xi_1\phi_s^{(3.2)})}{(\xi_1\phi_s^{(3.1)} + \xi_2\phi_s^{(3.2)} - \xi_0)\xi_1\phi_s^{(3.1)}\xi_2\phi_s^{(3.2)}\xi_0} \geq 0$$

for all $s = 1, \ldots, v-1$, because of the restrictions on the ξ's. This implies that

$$\text{tr}(\mathbf{A}_{3.12}^{+}) = \sum_{s=1}^{v-1} \frac{1}{\xi_1\phi_s^{(3.1)} + \xi_2\phi_s^{(3.2)} - \xi_0} \geq \sum_{s=1}^{v-1} \left[\frac{1}{\xi_1\phi_s^{(3.1)}} + \frac{1}{\xi_2\phi_s^{(3.2)}} - \frac{1}{\xi_0}\right]$$

$$= [\text{tr}(\mathbf{A}_{3.1}^{+})/\xi_1] + [\text{tr}(\mathbf{A}_{3.2}^{+})/\xi_2] - [(v-1)/\xi_0],$$

which is equivalent to (4.4.5). Equality holds if and only if $(\xi_0 - \xi_1\phi_s^{(3.1)})(\xi_0 - \xi_1\phi_s^{(3.2)}) = 0$ for all $s = 1, \ldots, v-1$, which is equivalent to (4.4.6). ❐

If $\xi_0 = \xi_1 = \xi_2 = 1$, i.e., the design satisfies the reduced decomposability property, then the result in our Theorem 4.4.5 reduces to that of Baksalary and Shah (1990), with (4.4.5) becoming

$$E_{3.12}^{-1} \geq E_{3.1}^{-1} + E_{3.2}^{-1} - 1; \tag{4.4.8}$$

from (4.4.7) we see that equality holds in (4.4.8) if and only if

$$\mathbf{N}_{31}\mathbf{D}_1^{-1}\mathbf{N}_{13}\mathbf{D}_3^{-1}\mathbf{N}_{32}\mathbf{D}_2^{-1}\mathbf{N}_{23} = (1/n)\mathbf{k}_3\mathbf{k}_3'$$

or equivalently if and only if $r(\mathbf{N}_{13}\mathbf{D}_3^{-1}\mathbf{N}_{32}) = 1$ or $\mathbf{N}_{13}\mathbf{D}_3^{-1}\mathbf{N}_{32} = (1/n)\mathbf{k}_1\mathbf{k}_2'$.

Now if the two-way elimination of heterogeneity design is ordinary (equal row sizes k_1 and equal column sizes k_2), and satisfies the conditions of Theorem 4.4.5 (ignoring the restrictions on the ξ's), then as pointed out in Section 2.3, we can write

$$\mathbf{S}_{3.12} = \mathbf{D}_3 - (\xi_1/k_1)\mathbf{N}_{31}\mathbf{N}_{13} - (\xi_2/k_2)\mathbf{N}_{32}\mathbf{N}_{23} + (\xi_0/n)\mathbf{k}_3\mathbf{k}_3'$$

or equivalently

$$\mathbf{S}_{3.12} - \mathbf{S}_{3.0} = \xi_1(\mathbf{S}_{3.1} - \mathbf{S}_{3.0}) + \xi_2(\mathbf{S}_{3.2} - \mathbf{S}_{3.0}).$$

With this we obtain, cf. Baksalary and Siatkowski (1990),

$$E_{3.12} \leq [\xi_1(E_{3.1}^{-1} - 1) + \xi_2(E_{3.2}^{-1} - 1) + 1]^{-1}$$

with equality if and only if

$$[(\xi_1 - 1)k_1^2]\mathbf{N}_{31}\mathbf{N}_{13}\mathbf{D}_3^{-1}\mathbf{N}_{31}\mathbf{N}_{13} + [\xi_2/k_1k_2]\mathbf{N}_{31}\mathbf{N}_{13}\mathbf{D}_3^{-1}\mathbf{N}_{32}\mathbf{N}_{23} = (\xi_0/n)\mathbf{k}_3\mathbf{k}_3'$$

and

$$[\xi_1/k_1k_2]\mathbf{N}_{31}\mathbf{N}_{13}\mathbf{D}_3^{-1}\mathbf{N}_{32}\mathbf{N}_{23} + [(\xi_2 - 1)k_2^2]\mathbf{N}_{32}\mathbf{N}_{23}\mathbf{D}_3^{-1}\mathbf{N}_{32}\mathbf{N}_{23} = (\xi_0/n)\mathbf{k}_3\mathbf{k}_3'.$$

If, furthermore, the two-way elimination of heterogeneity design has efficiency balance, then a lower bound can also be found for the average efficiency factor.

Theorem 4.4.6. *If a two-way elimination of heterogeneity design is connected for treatments, satisfies* $\mathbf{S}_{3.12} = \xi_1\mathbf{S}_{3.1} + \xi_2\mathbf{S}_{3.2} - \xi_0\mathbf{S}_{3.0}$ *for some* ξ_1, ξ_2 *and* $\xi_0 > 0$, *and is efficiency balanced, i.e.,* $\mathbf{S}_{3.12} = \vartheta\mathbf{S}_{3.0}$ *for some* ϑ, *then*

$$\xi_1E_{3.1} + \xi_2E_{3.2} - \xi_0 \le E_{3.12} \le [(\xi_1E_{3.1})^{-1} + (\xi_2E_{3.2})^{-1} - (1/\xi_0)]^{-1}, \qquad (4.4.9)$$

where for the inequality on the right $\xi_0 \ge \xi_1$ *and* $\xi_0 \ge \xi_2$ *or* $\xi_0 \le \xi_1$ *and* $\xi_0 \le \xi_2$. *Equality holds on the left if and only if both the treatment-row and treatment-column subdesigns are efficiency balanced and equality on the right if and only if either* $\xi_1\mathbf{S}_{3.1} = \xi_2\mathbf{S}_{3.2} = \xi_0\mathbf{S}_{3.0}$ *or the nonzero eigenvalues of* $\mathbf{A}_{3.1}$ *and* $\mathbf{A}_{3.2}$ *corresponding to the same eigenvector satisfy*

$$\phi_s^{(3.1)} = \vartheta/\xi_1,\ \phi_s^{(3.2)} = \xi_0/\xi_2 \ \text{ or } \ \phi_s^{(3.1)} = \xi_0/\xi_1,\ \phi_s^{(3.2)} = \vartheta/\xi_2,\ s = 1, \ldots, v-1. \qquad (4.4.10)$$

Proof: If $\mathbf{A}_{3.12} = \vartheta\mathbf{A}_{3.0}$, then from Lemma 4.4.4,

$$\xi_1\phi_s^{(3.1)} + \xi_2\phi_s^{(3.2)} = \xi_0 + \vartheta \quad \text{for } s = 1, \ldots, v-1, \qquad (4.4.11)$$

and so, applying the Minkowski inequality (cf. Theorem A.3.6 with $p = 1$), this implies that

$$\left[\sum_{s=1}^{v-1} \frac{1}{\xi_1\phi_s^{(3.1)}}\right]^{-1} + \left[\sum_{s=1}^{v-1} \frac{1}{\xi_2\phi_s^{(3.2)}}\right]^{-1} \le \frac{\xi_0 + \vartheta}{v-1}.$$

Then from the definition of the average efficiency factor and $\mathbf{A}_{3.12} = \vartheta\mathbf{A}_{3.0}$, we have

$$\xi_1 E_{3.1} + \xi_2 E_{3.2} \leq \xi_0 + E_{3.12}, \tag{4.4.12}$$

which is the inequality on the left of (4.4.9). Equality holds in (4.4.12) if and only if there exist c_1 and c_2 not both zero such that

$$\xi_1 c_1 \phi_s^{(3.1)} + c_2 \xi_2 \phi_s^{(3.2)} = 0, \quad s = 1, \ldots, \nu - 1.$$

Combining this with (4.4.11) shows that equality holds if and only if both designs have efficiency balance.

The inequality on the right of (4.4.9) follows from Lemma 4.4.4 and Theorem 4.4.5. As was shown in Theorem 4.4.5, a necessary and sufficient condition for equality to hold is that

$$(\xi_0\mathbf{A}_{3.0} - \xi_1\mathbf{A}_{3.1})(\xi_0\mathbf{A}_{3.0} - \xi_2\mathbf{A}_{3.2}) = \mathbf{0}.$$

We can transform this using $\xi_1\mathbf{A}_{3.1} + \xi_2\mathbf{A}_{3.2} = (\xi_0 + \vartheta)\mathbf{A}_{3.0}$ to yield

$$\xi_h^2\mathbf{A}_{3.h}^2 - \xi_h(\xi_0 + \vartheta)\mathbf{A}_{3.h} + \xi_0\vartheta\mathbf{A}_{3.0} = \mathbf{0}, \quad h = 1, 2,$$

which is equivalent to (4.4.10). ❐

Our Theorem 4.4.6 is an extension of Theorem 5.2 in Baksalary and Shah (1990) where the reduced decomposability property is assumed; the result then simplifies to

$$E_{3.1} + E_{3.2} - 1 \leq E_{3.12} \leq [(E_{3.1})^{-1} + (E_{3.2})^{-1} - 1]^{-1},$$

with equality on the left, again, if and only if both treatment-row and treatment-column subdesigns have efficiency balance and equality on the right if and only if either $\mathbf{S}_{3.1} = \mathbf{S}_{3.2} = \mathbf{S}_{3.0}$ or the non-zero eigenvalues of $\mathbf{A}_{3.1}$ and $\mathbf{A}_{3.2}$ corresponding to the same eigenvector satisfy

$$\phi_s^{(3.1)} = \vartheta \text{ and } \phi_s^{(3.2)} = 1 \quad \text{or} \quad \phi_s^{(3.1)} = 1 \text{ and } \phi_s^{(3.2)} = \vartheta, \qquad s = 1, \ldots, v-1.$$

Baksalary and Siatkowski (1990) also give a result similar to that in our Theorem 4.4.6 for designs with equal row sizes and equal column sizes and which satisfy the extended decomposability property, i.e., $\mathbf{S}_{3.12} = \mathbf{D}_3 - \upsilon_1\mathbf{N}_{31}\mathbf{N}_{13} - \upsilon_2\mathbf{N}_{32}\mathbf{N}_{23} + \rho\,\mathbf{k}_3\mathbf{k}_3'$, υ_1, υ_2, $\rho > 0$.

APPENDIX 1. *mutzE*-TABLES

This appendix contains 11 tables, which we call *mutzE*-tables, one associated with every three-way layout example presented in this thesis. In the heading for each table, we give the three-way layout to be analyzed, its source in the literature and to which example(s) in the thesis it corresponds. We then give a summary of certain properties: commutativity, generalized decomposability, reduced decomposability and extended decomposability. In each table we present for each of the fifteen different kinds of canonical correlations the numbers m, u and t, respectively, of nonzero, unit and nonunit nonzero canonical correlations, as well as the associated numbers z and E related to the corresponding canonical efficiency factors; also included are the fifteen different kinds of canonical efficiency factors ϕ, and the degree of disconnectedness u_{123}, cf. (3.3.6).

Table	Example(s)	Source
A.1.1	2.3.1, 3.3.2	Agrawal (1966c)
A.1.2	2.3.2	—
A.1.3	3.2.1	Baksalary (1990)
A.1.4	3.3.1	Shah and Khatri (1973)
A.1.5	3.3.2	Worsley (1990)
A.1.6	3.3.3	—
A.1.7	3.5.1, 4.4.2	Eccleston and Russell (1977)
A.1.8	4.1.1	—
A.1.9	4.1.2	Kshirsagar (1957)
A.1.10	4.3.1	Speed (1983)
A.1.11	4.4.1	Eccleston and Russell (1975)
A.1.12	4.4.3	—

Table A.1.1 (Examples 2.3.1 and 3.3.2).

Source: Agrawal (1966c).

*	3	5	*	2	*	*
*	*	4	6	*	3	*
*	*	*	5	7	*	4
5	*	*	*	6	1	*
*	6	*	*	*	7	2
3	*	7	*	*	*	1
2	4	*	1	*	*	*

Commutativity: yes. **Decomposability—General** yes; **Extended** yes; **Reduced** no.

$S_{3.12} = \xi_1 S_{3.1} + \xi_2 S_{3.2} - \xi_0 S_{3.0}$, ξ_1, ξ_2 and $\xi_0 > 0$, such that $\xi_1 = 3/7 - \xi_2 + (9/7)\xi_0$.

$= D_3 - \upsilon_1 N_{31} N_{13} - \upsilon_2 N_{32} N_{23} + (2/21)\, k_3 k_3'$, υ_1 and $\upsilon_2 > 0$, such that $\upsilon_1 + \upsilon_2 = 1$.

	m	u	t	ϕ	z	z'	E	E'
1.2	7	1	6	$\{\frac{7}{9}, \frac{7}{9}, \frac{7}{9}, \frac{7}{9}, \frac{7}{9}, \frac{7}{9}\}$	0	0	0.777778	0.777778
1.3	7	1	6	$\{\frac{7}{9}, \frac{7}{9}, \frac{7}{9}, \frac{7}{9}, \frac{7}{9}, \frac{7}{9}\}$	0	0	0.777778	0.777778
2.3	7	1	6	$\{\frac{7}{9}, \frac{7}{9}, \frac{7}{9}, \frac{7}{9}, \frac{7}{9}, \frac{7}{9}\}$	0	0	0.777778	0.777778
1.23	7	1	6	$\{\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}\}$	0	7	0.333333	0.52
2.13	7	1	6	$\{\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}\}$	0	7	0.333333	0.52
3.12	7	1	6	$\{\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}\}$	0	7	0.333333	0.52
1;2|3	6	0	6	$\{\frac{5}{9}, \frac{5}{9}, \frac{5}{9}, \frac{5}{9}, \frac{5}{9}, \frac{5}{9}\}$	1	1	0.59322	0.59322
2;1|3	6	0	6	$\{\frac{5}{9}, \frac{5}{9}, \frac{5}{9}, \frac{5}{9}, \frac{5}{9}, \frac{5}{9}\}$	1	1	0.59322	0.59322
1;3|2	6	0	6	$\{\frac{5}{9}, \frac{5}{9}, \frac{5}{9}, \frac{5}{9}, \frac{5}{9}, \frac{5}{9}\}$	1	1	0.59322	0.59322
3;1|2	6	0	6	$\{\frac{5}{9}, \frac{5}{9}, \frac{5}{9}, \frac{5}{9}, \frac{5}{9}, \frac{5}{9}\}$	1	1	0.59322	0.59322
2;3|1	6	0	6	$\{\frac{5}{9}, \frac{5}{9}, \frac{5}{9}, \frac{5}{9}, \frac{5}{9}, \frac{5}{9}\}$	1	1	0.59322	0.59322
3;2|1	6	0	6	$\{\frac{5}{9}, \frac{5}{9}, \frac{5}{9}, \frac{5}{9}, \frac{5}{9}, \frac{5}{9}\}$	1	1	0.59322	0.59322
1.2|3	6	0	6	$\{\frac{3}{7}, \frac{3}{7}, \frac{3}{7}, \frac{3}{7}, \frac{3}{7}, \frac{3}{7}\}$	1	1	0.466667	0.466667
1.3|2	6	0	6	$\{\frac{3}{7}, \frac{3}{7}, \frac{3}{7}, \frac{3}{7}, \frac{3}{7}, \frac{3}{7}\}$	1	1	0.466667	0.466667
2.3|1	6	0	6	$\{\frac{3}{7}, \frac{3}{7}, \frac{3}{7}, \frac{3}{7}, \frac{3}{7}, \frac{3}{7}\}$	1	1	0.466667	0.466667
1.2.3		2						

Table A.1.2 (Example 2.3.2).

1	2	3
*	1	2
3	*	1

Commutativity: yes. **Decomposability—General** yes; **Extended** no; **Reduced** no.

$S_{3.12} = \xi_1 S_{3.1} + \xi_2 S_{3.2} - (7/30) S_{3.0}$, for any positive ξ_1, ξ_2 such that $\xi_1 + \xi_2 = 6/5$.

	m	u	t	ϕ	z	z'	E	E'
1.2	3	1	2	$\{\frac{3}{4}, \frac{35}{36}\}$	0	0	0.846774	0.846774
1.3	3	1	2	$\{\frac{3}{4}, \frac{35}{36}\}$	0	0	0.846774	0.846774
2.3	3	1	2	$\{\frac{3}{4}, \frac{35}{36}\}$	0	0	0.846774	0.846774
1.23	3	1	2	$\{\frac{2}{3}, \frac{14}{15}\}$	0	3	0.777778	0.897436
2.13	3	1	2	$\{\frac{2}{3}, \frac{14}{15}\}$	0	3	0.777778	0.897436
3.12	3	1	2	$\{\frac{2}{3}, \frac{14}{15}\}$	0	3	0.777778	0.897436
1;2|3	2	0	2	$\{\frac{173}{180}, \frac{11}{12}\}$	1	1	0.958047	0.958047
2;1|3	2	0	2	$\{\frac{173}{180}, \frac{11}{12}\}$	1	1	0.958047	0.958047
1;3|2	2	0	2	$\{\frac{173}{180}, \frac{11}{12}\}$	1	1	0.958047	0.958047
3;1|2	2	0	2	$\{\frac{173}{180}, \frac{11}{12}\}$	1	1	0.958047	0.958047
2;3|1	2	0	2	$\{\frac{173}{180}, \frac{11}{12}\}$	1	1	0.958047	0.958047
3;2|1	2	0	2	$\{\frac{173}{180}, \frac{11}{12}\}$	1	1	0.958047	0.958047
1.2|3	2	0	2	$\{\frac{24}{25}, \frac{8}{9}\}$	1	1	0.947368	0.947368
1.3|2	2	0	2	$\{\frac{24}{25}, \frac{8}{9}\}$	1	1	0.947368	0.947368
2.3|1	2	0	2	$\{\frac{24}{25}, \frac{8}{9}\}$	1	1	0.947368	0.947368
1.2.3		2						

Table A.1.3 (Example 3.2.1).

Source: Baksalary (1990).

1	2	3
1	*	*
*	2	*
*	*	3

Commutativity: yes. **Decomposability—General** yes; **Extended** yes; **Reduced** no.

$$\mathbf{S}_{3.12} = \xi_1\mathbf{S}_{3.1} + \xi_2\mathbf{S}_{3.2} - \xi_0\mathbf{S}_{3.0},\ \xi_1,\ \xi_2 \text{ and } \xi_0 > 0,\ \xi_1 = 2\xi_0.$$

$$= \mathbf{D}_3 - \upsilon_1\mathbf{N}_{31}\mathbf{N}_{13} - \upsilon_2\mathbf{N}_{32}\mathbf{N}_{23} + \rho\mathbf{k}_3\mathbf{k}_3',\upsilon_1,\ \upsilon_2 \text{ and } \rho > 0,\ \upsilon_1 = 4\rho,\ \upsilon_2 = 1/2 - \rho.$$

	m	u	t	ϕ	z	z'	E	E'
1.2	3	1	2	$\{\frac{1}{2}, \frac{1}{2}\}$	1	0	0.6	0.5
1.3	3	1	2	$\{\frac{1}{2}, \frac{1}{2}\}$	1	0	0.6	0.5
2.3	3	3	0	{ }	0	0	-	-
1.23	3	1	2	$\{\frac{1}{2}, \frac{1}{2}\}$	1	3	0.6	0.714286
2.13	3	3	0	{ }	0	4	-	1.
3.12	3	3	0	{ }	0	4	-	1.
1;2|3	0	0	0	{ }	4	3	1.	1.
2;1|3	0	0	0	{ }	3	4	1.	1.
1;3|2	0	0	0	{ }	4	3	1.	1.
3;1|2	0	0	0	{ }	3	4	1.	1.
2;3|1	2	0	2	$\{\frac{1}{2}, \frac{1}{2}\}$	1	1	0.6	0.6
3;2|1	2	0	2	$\{\frac{1}{2}, \frac{1}{2}\}$	1	1	0.6	0.6
1.2|3	0	0	0	{ }	4	3	1.	1.
1.3|2	0	0	0	{ }	4	3	1.	1.
2.3|1	2	2	0	{ }	1	1	1.	1.
1.2.3		4						

Table A.1.4 (Example 3.3.1).

Source: Khatri and Shah (1986).

1	2	5	6
3	4	7	8
8	6	1	3
7	5	2	4

Commutativity: yes. **Decomposability—General** yes; **Extended** yes; **Reduced** yes.

$$\mathbf{S}_{3.12} = \mathbf{S}_{3.1} + \mathbf{S}_{3.2} - \mathbf{S}_{3.0}.$$

$$= \mathbf{D}_3 - (1/4)\mathbf{N}_{31}\mathbf{N}_{13} - (1/4)\mathbf{N}_{32}\mathbf{N}_{23} + (1/16)\,\mathbf{k}_3\mathbf{k}_3'.$$

	m	u	t	ϕ	z	z'	E	E'
1.2	1	1	0	{ }	3	3	1.	1.
1.3	3	1	2	$\{\frac{1}{2}, \frac{1}{2}\}$	1	5	0.6	0.777778
2.3	3	1	2	$\{\frac{1}{2}, \frac{1}{2}\}$	1	5	0.6	0.777778
1.23	3	2	1	$\{\frac{1}{2}\}$	1	9	0.666667	0.909091
2.13	3	2	1	$\{\frac{1}{2}\}$	1	9	0.666667	0.909091
3.12	4	2	2	$\{\frac{1}{2}, \frac{1}{2}\}$	4	4	0.75	0.75
1;2|3	1	0	1	$\{\frac{1}{2}\}$	3	3	0.8	0.8
2;1|3	1	0	1	$\{\frac{1}{2}\}$	3	3	0.8	0.8
1;3|2	2	1	1	$\{\frac{1}{2}\}$	2	6	0.75	0.875
3;1|2	2	0	2	$\{\frac{1}{2}, \frac{1}{2}\}$	6	2	0.8	0.666667
2;3|1	2	1	1	$\{\frac{1}{2}\}$	2	6	0.75	0.875
3;2|1	2	0	2	$\{\frac{1}{2}, \frac{1}{2}\}$	6	2	0.8	0.666667
1.2|3	1	1	0	{ }	3	3	1.	1.
1.3|2	2	1	1	$\{\frac{1}{2}\}$	2	6	0.75	0.875
2.3|1	2	1	1	$\{\frac{1}{2}\}$	2	6	0.75	0.875
1.2.3		3						

Table A.1.5 (Example 3.3.2).

Source: Worsley (1990).

2	*	*	*	5	3	*	*	4	*
1	3	*	*	*	*	4	*	*	5
*	2	4	*	*	1	*	5	*	*
*	*	3	5	*	*	2	*	1	*
*	*	*	4	1	*	*	3	*	2

Commutativity: yes. **Decomposability—General** yes; **Extended** yes; **Reduced** no.

$$\mathbf{S}_{3\ 12} = \xi_1\mathbf{S}_{3.1} + \xi_2\mathbf{S}_{3.2} - \xi_0\mathbf{S}_{3.0},\ \xi_1,\ \xi_2 \text{ and } \xi_0 > 0,\ \xi_1 = -(2/3)\xi_2 + (16/15)\xi_0.$$

$$= \mathbf{D}_3 - \upsilon_1\mathbf{N}_{31}\mathbf{N}_{13} - \upsilon_2\mathbf{N}_{32}\mathbf{N}_{23} + \rho\,\mathbf{k}_3\mathbf{k}_3',\ \upsilon_1,\ \upsilon_2 \text{ and } \rho > 0,$$

$$4\upsilon_1 = -1/2 + 6\rho, \upsilon_2 = 3/2 - 2\rho.$$

	m	*u*	*t*	ϕ	*z*	*z'*	*E*	*E'*
1.2	5	1	4	$\{\frac{5}{8}, \frac{5}{8}, \frac{5}{8}, \frac{5}{8}\}$	0	5	0.625	0.789474
1.3	5	1	4	$\{\frac{15}{16}, \frac{15}{16}, \frac{15}{16}, \frac{15}{16}\}$	0	0	0.9375	0.9375
2.3	5	1	4	$\{\frac{5}{8}, \frac{5}{8}, \frac{5}{8}, \frac{5}{8}\}$	5	0	0.789474	0.625
1.23	5	5	0	{ }	0	10	-	1.
2.13	5	5	0	{ }	5	5	1	1.
3.12	5	5	0	{ }	0	10	-	1.
1;2|3	4	0	4	$\{\frac{1}{16}, \frac{1}{16}, \frac{1}{16}, \frac{1}{16}\}$	1	6	0.0769231	0.142857
2;1|3	4	0	4	$\{\frac{3}{8}, \frac{3}{8}, \frac{3}{8}, \frac{3}{8}\}$	6	1	0.6	0.428571
1;3|2	4	0	4	$\{\frac{3}{8}, \frac{3}{8}, \frac{3}{8}, \frac{3}{8}\}$	1	1	0.428571	0.428571
3;1|2	4	0	4	$\{\frac{3}{8}, \frac{3}{8}, \frac{3}{8}, \frac{3}{8}\}$	1	1	0.428571	0.428571
2;3|1	4	0	4	$\{\frac{3}{8}, \frac{3}{8}, \frac{3}{8}, \frac{3}{8}\}$	6	1	0.6	0.428571
3;2|1	4	0	4	$\{\frac{1}{16}, \frac{1}{16}, \frac{1}{16}, \frac{1}{16}\}$	1	6	0.0769231	0.142857
1.2|3	4	4	0	{ }	1	6	1.	1.
1.3|2	4	4	0	{ }	1	1	1.	1.
2.3|1	4	4	0	{ }	6	1	1.	1.
1.2.3		6						

Table A.1.6 (Example 3.3.3).

3	3	1
3	2	2
1	2	1

Commutativity: yes. **Decomposability—General** yes; **Extended** yes; **Reduced** yes.

$S_{3.12} = \xi_1 S_{3.1} + \xi_2 S_{3.2} - \xi_0 S_{3.0}$, ξ_1, ξ_2 and $\xi_0 > 0$, $\xi_1 = (1/2) - \xi_2 + (3/2)\xi_0$.

$= \mathbf{D}_3 - \upsilon_1 \mathbf{N}_{31}\mathbf{N}_{13} - \upsilon_2 \mathbf{N}_{32}\mathbf{N}_{23} + (1/9)\,\mathbf{k}_3\mathbf{k}_3^{;}$, υ_1, υ_2 and $\rho > 0$, $\upsilon_1 = 2/3 - \upsilon_2$,

	m	*u*	*t*	ϕ	*z*	*z'*	*E*	*E'*
1.2	1	1	0	{ }	2	2	1.	1.
1.3	3	1	2	$\{\frac{2}{3}, \frac{2}{3}\}$	0	0	0.666667	0.666667
2.3	3	1	2	$\{\frac{2}{3}, \frac{2}{3}\}$	0	0	0.666667	0.666667
1.23	3	1	2	$\{\frac{1}{2}, \frac{1}{2}\}$	0	3	0.5	0.714286
2.13	3	1	2	$\{\frac{1}{2}, \frac{1}{2}\}$	0	3	0.5	0.714286
3.12	3	1	2	$\{\frac{1}{3}, \frac{1}{3}\}$	0	3	0.333333	0.555556
1;2|3	2	0	2	$\{\frac{5}{6}, \frac{5}{6}\}$	1	1	0.882353	0.882353
2;1|3	2	0	2	$\{\frac{5}{6}, \frac{5}{6}\}$	1	1	0.882353	0.882353
1;3|2	2	0	2	$\{\frac{1}{2}, \frac{1}{2}\}$	1	1	0.6	0.6
3;1|2	2	0	2	$\{\frac{2}{3}, \frac{2}{3}\}$	1	1	0.75	0.75
2;3|1	2	0	2	$\{\frac{1}{2}, \frac{1}{2}\}$	1	1	0.6	0.6
3;2|1	2	0	2	$\{\frac{2}{3}, \frac{2}{3}\}$	1	1	0.75	0.75
1.2|3	2	0	2	$\{\frac{3}{4}, \frac{3}{4}\}$	1	1	0.818182	0.818182
1.3|2	2	0	2	$\{\frac{1}{2}, \frac{1}{2}\}$	1	1	0.6	0.6
2.3|1	2	0	2	$\{\frac{1}{2}, \frac{1}{2}\}$	1	1	0.6	0.6
1.2.3		2						

Table A.1.7 (Examples 3.5.1 and 4.4.2).

Source: Eccleston and Russell (1977).

1	1	2
2	3	3

Commutativity: yes. **Decomposability—General** yes; **Extended** yes; **Reduced** yes.

$$S_{3.12} = S_{3.1} + \xi_2 S_{3.2} - \xi_0 S_{3.0},\ \xi_2 \text{ and } \xi_0 > 0,\ \xi_2 = -(1/3) + (4/3)\xi_0.$$
$$= D_3 - (1/3)N_{31}N_{13} - (1/2)N_{32}N_{23} + (1/6)\,\mathbf{k}_3\mathbf{k}_3'.$$

	m	u	t	ϕ	z	z'	E	E'
1.2	1	1	0	{ }	1	2	1.	1.
1.3	2	1	1	$\{\frac{1}{3}\}$	0	1	0.333333	0.5
2.3	3	1	2	$\{\frac{3}{4}, \frac{3}{4}\}$	0	0	0.75	0.75
1.23	2	1	1	$\{\frac{1}{9}\}$	0	4	0.111111	0.384615
2.13	3	1	2	$\{\frac{1}{4}, \frac{3}{4}\}$	0	2	0.375	0.545455
3.12	3	1	2	$\{\frac{1}{12}, \frac{3}{4}\}$	0	2	0.15	0.26087
1;2|3	1	0	1	$\{\frac{7}{9}\}$	1	2	0.875	0.913043
2;1|3	1	0	1	$\{\frac{1}{2}\}$	2	1	0.75	0.666667
1;3|2	1	0	1	$\{\frac{1}{9}\}$	1	2	0.2	0.272727
3;1|2	1	0	1	$\{\frac{1}{3}\}$	2	1	0.6	0.5
2;3|1	2	0	2	$\{\frac{1}{4}, \frac{3}{4}\}$	1	1	0.473684	0.473684
3;2|1	2	0	2	$\{\frac{3}{4}, \frac{3}{4}\}$	1	1	0.818182	0.818182
1.2|3	1	0	1	$\{\frac{1}{3}\}$	1	2	0.5	0.6
1.3|2	1	0	1	$\{\frac{1}{9}\}$	1	2	0.2	0.272727
2 3|1	2	0	2	$\{\frac{1}{4}, \frac{3}{4}\}$	1	1	0.473684	0.473684
1.2.3		2						

Table A.1.8 (Example 4.1.1).

1	2	2	3	4
4	1	2	2	3
3	4	1	2	2
2	3	4	1	2
2	2	3	4	1

Commutativity: yes. **Decomposability—General** yes; **Extended** yes; **Reduced** yes.

$$\mathbf{S}_{3.12} = \xi_1\mathbf{S}_{3.1} + \xi_2\mathbf{S}_{3.2} - \xi_0\mathbf{S}_{3.0},\ \xi_1,\ \xi_2 \text{ and } \xi_0 > 0,\ \xi_1 + \xi_2 - \xi_0 = 1.$$
$$= \mathbf{D}_3 - \upsilon_1\mathbf{N}_{31}\mathbf{N}_{13} - \upsilon_2\mathbf{N}_{32}\mathbf{N}_{23} + (1/9)\,\mathbf{k}_3\mathbf{k}_3',\ \upsilon_1,\ \upsilon_2 \text{ and } \rho > 0,\ \upsilon_1 = 1/5 - \upsilon_2 + 5\rho.$$

	m	u	t	ϕ	z	z'	E	E'
1.2	1	1	0	{}	4	4	1.	1.
1.3	1	1	0	{}	4	3	1.	1.
2.3	1	1	0	{}	4	3	1.	1.
1.23	1	1	0	{}	4	8	1.	1.
2.13	1	1	0	{}	4	8	1.	1.
3.12	1	1	0	{}	3	9	1.	1.
1;2|3	0	0	0	{}	5	5	1.	1.
2;1|3	0	0	0	{}	5	5	1.	1.
1;3|2	0	0	0	{}	5	4	1.	1.
3;1|2	0	0	0	{}	4	5	1.	1.
2;3|1	0	0	0	{}	5	4	1.	1.
3;2|1	0	0	0	{}	4	5	1.	1.
1.2|3	0	0	0	{}	5	5	1.	1.
1.3|2	0	0	0	{}	5	4	1.	1.
2.3|1	0	0	0	{}	5	4	1.	1.
1.2.3		2						

Table A.1.9 (Example 4.1.2).

Source: Kshirsagar (1957).

2	4	8	7	6	3
3	5	7	2	4	9
5	6	3	1	7	8
4	9	1	8	3	5
6	7	9	5	1	2
1	8	2	4	9	6

Commutativity: yes. **Decomposability—General** yes; **Extended** yes; **Reduced** yes.

$$\mathbf{S}_{3.12} = \xi_1\mathbf{S}_{3.1} + \xi_2\mathbf{S}_{3.2} - \xi_0\mathbf{S}_{3.0},\ \xi_1,\ \xi_2 \text{ and } \xi_0 > 0,\ \xi_1 = 7/15 + (8/15)\xi_0,$$
$$\xi_2 = 7/15 + (8/15)\xi_0$$
$$= \mathbf{D}_3 - (1/6)\mathbf{N}_{31}\mathbf{N}_{13} - (1/6)\mathbf{N}_{32}\mathbf{N}_{23} + (1/36)\,\mathbf{k}_3\mathbf{k}_3'.$$

	m	u	t	ϕ	z	z'	E	E'
1.2	1	1	0	{ }	5	5	1.	1.
1.3	5	1	4	$\{\frac{7}{8}, \frac{7}{8}, \frac{7}{8}, \frac{7}{8}\}$	1	4	0.897436	0.933333
2.3	5	1	4	$\{\frac{7}{8}, \frac{7}{8}, \frac{7}{8}, \frac{7}{8}\}$	1	4	0.897436	0.933333
1.23	5	1	4	$\{\frac{7}{8}, \frac{7}{8}, \frac{7}{8}, \frac{7}{8}\}$	1	10	0.897436	0.960784
2.13	5	1	4	$\{\frac{7}{8}, \frac{7}{8}, \frac{7}{8}, \frac{7}{8}\}$	1	10	0.897436	0.960784
3.12	9	1	8	$\{\frac{7}{8}, \frac{7}{8}, \frac{7}{8}, \frac{7}{8}, \frac{7}{8}, \frac{7}{8}, \frac{7}{8}, \frac{7}{8}\}$	0	3	0.875	0.905882
1;2|3	0	0	0	{ }	6	6	1.	1.
2;1|3	0	0	0	{ }	6	6	1.	1.
1;3|2	4	0	4	$\{\frac{7}{8}, \frac{7}{8}, \frac{7}{8}, \frac{7}{8}\}$	2	5	0.913043	0.940299
3;1|2	4	0	4	$\{\frac{7}{8}, \frac{7}{8}, \frac{7}{8}, \frac{7}{8}\}$	5	2	0.940299	0.913043
2;3|1	4	0	4	$\{\frac{7}{8}, \frac{7}{8}, \frac{7}{8}, \frac{7}{8}\}$	2	5	0.913043	0.940299
3;2|1	4	0	4	$\{\frac{7}{8}, \frac{7}{8}, \frac{7}{8}, \frac{7}{8}\}$	5	2	0.940299	0.913043
1.2|3	0	0	0	{ }	6	6	1.	1.
1.3|2	4	0	4	$\{\frac{7}{8}, \frac{7}{8}, \frac{7}{8}, \frac{7}{8}\}$	2	5	0.913043	0.940299
2.3|1	4	0	4	$\{\frac{7}{8}, \frac{7}{8}, \frac{7}{8}, \frac{7}{8}\}$	2	5	0.913043	0.940299
1.2.3		2						

Table A.1.10 (Example 4.3.1).

Source: Speed (1983).

2	1	1	1
1	3	3	2
2	2	4	3
4	4	4	3

Commutativity: no. **Decomposability—General** yes; **Extended** yes; **Reduced** yes.

$$\mathbf{S}_{3\ 12} = \mathbf{S}_{3.1} + \mathbf{S}_{3.2} - \mathbf{S}_{3.0}.$$
$$= \mathbf{D}_3 - (1/4)\mathbf{N}_{31}\mathbf{N}_{13} - (1/4)\mathbf{N}_{32}\mathbf{N}_{23} + (1/16)\,\mathbf{k}_3\mathbf{k}_3'.$$

	m	u	t	ϕ	z	z'	E	E'
1.2	1	1	0	{ }	3	3	1.	1.
1.3	4	1	3	{0.345, $\frac{3}{4}$, 0.904}	0	0	0.563	0.563
2.3	3	1	2	{$\frac{13}{16}$, $\frac{13}{16}$}	1	1	0.867	0.867
1.23	4	1	3	{0.285, 0.716, 0.883}	0	4	0.497	0.698
2.13	3	1	2	{0.644, 0.789}	1	5	0.785	0.895
3.12	4	1	3	{0.255 , 0.637 , 0.732 }	0	4	0.438	0.645
1;2|3	2	0	2	{0.285, 0.716, 0.383}	2	2	0.97	0.97
2;1|3	2	0	2	{0.644, 0.789}	2	2	0.946	0.946
1;3|2	3	0	3	{0.285, 0.716, 0.883}	1	1	0.569	0.569
3;1|2	3	0	3	{0.345, $\frac{3}{4}$, 0.904}	1	1	0.632	0.632
2;3|1	2	0	2	{0.644, 0.789}	2	2	0.83	0.83
3;2|1	2	0	2	{$\frac{13}{16}$, $\frac{13}{16}$}	2	2	0.897	0.897
1.2|3	2	0	2	{0.793, 0.971}	2	2	0.932	0.932
1.3|2	3	0	3	{0.285, 0.716, 0.883}	1	1	0.569	0.569
2.3|1	2	0	2	{0.644, 0.789}	2	2	0.83	0.83
1.2.3		2						

Table A.1.11 (Example 4.4.1).

Source: Eccleston and Russell (1975).

1	2	*	*
2	1	*	*
*	*	3	4
*	*	4	3

Commutativity: yes. **Decomposability—General** no; **Extended** no; **Reduced** no.

	m	u	t	ϕ	z	z'	E	E'
1.2	2	2	0	{ }	2	2	1	1
1.3	2	2	0	{ }	2	2	1	1
2.3	2	2	0	{ }	2	2	1	1
1.23	2	2	0	{ }	2	6	1	1
2.13	2	2	0	{ }	2	6	1	1
3.12	2	2	0	{ }	2	6	1	1
1;2|3	0	0	0	{ }	4	4	1	1
2;1|3	0	0	0	{ }	4	4	1	1
1;3|2	0	0	0	{ }	4	4	1	1
3;1|2	0	0	0	{ }	4	4	1	1
2;3|1	0	0	0	{ }	4	4	1	1
3;2|1	0	0	0	{ }	4	4	1	1
1.2|3	0	0	0	{ }	4	4	1	1
1.3|2	0	0	0	{ }	4	4	1	1
2.3|1	0	0	0	{ }	4	4	1	1
1.2.3		4						

Table A.1.12 (Example 4.4.3).

1	2	*	*	*	*	3	4
2	1	*	*	*	*	4	3
*	*	1	*	*	*	*	*
*	*	*	2	*	*	*	*
*	*	*	*	3	*	*	*
*	*	*	*	*	4	*	*
3	4	*	*	*	*	1	2
4	3	*	*	*	*	2	1

Commutativity: yes. **Decomposability—General** yes; **Extended** yes; **Reduced** no.

$$\mathbf{S}_{3\,12} = \xi_1\mathbf{S}_{3.1} + \xi_2\mathbf{S}_{3.2} - \xi_0\mathbf{S}_{3.0}\,,\ \xi_1,\ \xi_2 \text{ and } \xi_0 > 0,\ \xi_1 = 1 - \xi_2 + (5/4)\xi_0$$
$$= \mathbf{D}_3 - \upsilon_1\mathbf{N}_{31}\mathbf{N}_{13} - \upsilon_2\mathbf{N}_{32}\mathbf{N}_{23} + (3/25)\,\mathbf{k}_3\mathbf{k}_3',\ \upsilon_1 \text{ and } \upsilon_2 > 0,\ \upsilon_1 = 1 - \upsilon_2.$$

	m	*u*	*t*	ϕ	*z*	*z'*	*E*	*E'*
1.2	5	5	0	{ }	3	3	1.	1.
1.3	4	1	3	$\{\frac{4}{5}, \frac{4}{5}, \frac{4}{5}\}$	4	0	0.903226	0.8
2.3	4	1	3	$\{\frac{4}{5}, \frac{4}{5}, \frac{4}{5}\}$	4	0	0.903226	0.8
1.23	5	5	0	{ }	3	7	1.	1.
2.13	5	5	0	{ }	3	7	1.	1.
3.12	4	1	3	$\{\frac{4}{5}, \frac{4}{5}, \frac{4}{5}\}$	0	12	0.8	0.952381
1;2|3	4	1	3	$\{\frac{1}{5}, \frac{1}{5}, \frac{1}{5}\}$	4	4	0.368421	0.368421
2;1|3	4	1	3	$\{\frac{1}{5}, \frac{1}{5}, \frac{1}{5}\}$	4	4	0.368421	0.368421
1;3|2	0	0	0	{ }	8	4	1.	1.
3;1|2	0	0	0	{ }	4	8	1.	1.
2;3|1	0	0	0	{ }	8	4	1.	1.
3;2|1	0	0	0	{ }	4	8	1.	1.
1.2|3	4	4	0	{ }	4	4	1.	1.
1.3|2	0	0	0	{ }	8	4	1.	1.
2.3|1	0	0	0	{ }	8	4	1.	1.
1.2.3		6						

APPENDIX 2. COMPUTER PROGRAMS

This appendix contains two computer programs written in Mathematica 1.2 [cf. Wolfram (1988)]. With the first program we calculate the *mutzE*-tables proper (cf. Appendix 1), while with the second we verify if a particular design satisfies the commutativity property and the various decomposability properties. In each program, the input consists of an array symbolizing the design under investigation. The output from the first program consists of a table displaying the numbers m, u, t, z, z', E, E' and the canonical efficiency factors ϕ_S. The output from the second program contains the commutator matrix $\mathbf{H}_3\mathbf{H}_1\mathbf{H}_3\mathbf{H}_2\mathbf{H}_3 - \mathbf{H}_3\mathbf{H}_2\mathbf{H}_3\mathbf{H}_1\mathbf{H}_3$ and solutions (if they exist) to the generalized and extended decomposability equations. The two programs are available on diskette and may be obtained from the author upon request.

```
(* PROGRAM 1 *)
(* this program calculates a design's mutzE-table *)
(**********************************************************)

<<matrixmut.m
TableForm[m]
OutputForm[TableForm[m]]>>mutzetable.m

{r,c}=Dimensions[m];
n=0;t=0;

(* get number n of plots and number t of treatments *)

Block[{i,j},Do[If[m[[i,j]]!=0,n=n+1];
        If[m[[i,j]]>t,t=m[[i,j]]],{i,r},{j,c}]]

count=0;
x1=Table[0,{n},{r}];x2=Table[0,{n},{c}];
        x3=Table[0,{n},{t}];
jn=Table[1,{n},{n}];
```

```
(* design matrices for rows and columns *)

Block[{i,j},Do[If[m[[i,j]]!=0,
            count=count+1;
            x1[[count,i]]=1;
            x2[[count,j]]=1],
        {j,c},{i,r}]]
(* design matrix for treatments *)

count=0;
Block[{i,j,k},Do[If[m[[i,j]]==k,
              count=count+1;
              x3[[count,k]]=1],
         {j,c},{i,r},{k,t}]]
d[m_]:=Length[m[[1]]]
rank[m_]:=d[m]-Length[NullSpace[m]]

(* hat matrices for x1, x2 and x3 *)

h1=x1.Inverse[Transpose[x1].x1].Transpose[x1]
h2=x2.Inverse[Transpose[x2].x2].Transpose[x2]
h3=x3.Inverse[Transpose[x3].x3].Transpose[x3];

(* singular value decomposition of the design matrices *)

{u1,s1,v1}=SingularValues[N[x1]]
{u2,s2,v2}=SingularValues[N[x2]]
{u3,s3,v3}=SingularValues[N[x3]];
u1h=Transpose[u1]
u2h=Transpose[u2]
u3h=Transpose[u3];

(* canonical efficiency factors for matrices S3.r and S3.c *)

d3=Transpose[x3].x3;
s3r=Sqrt[Inverse[d3]].Transpose[x3].(h1-jn).x3.
        Sqrt[Inverse[d3]];
s3c=Sqrt[Inverse[d3]].Transpose[x3].(h2-jn).x3.
        Sqrt[Inverse[d3]];
phi3r=Eigenvalues[s3r]//N
phi3c=Eigenvalues[s3c]//N

(* positive canonical correlations of type 1 *)

s12=Sort[SingularValues[N[Transpose[u1h].u2h]][[2]]]
s13=Sort[SingularValues[N[Transpose[u1h].u3h]][[2]]]
s23=Sort[SingularValues[N[Transpose[u2h].u3h]][[2]]]

(* function which augments a matrix x by a matrix y *)

augment[x_,y_]:=Transpose[Join[Transpose[x],Transpose[y]]]

(* m, u, t and z values of type 1 *)

m12=rank[Transpose[x1].x2]
m13=rank[Transpose[x1].x3]
m23=rank[Transpose[x2].x3];
u12=rank[x1]+rank[x2]-rank[augment[x1,x2]]
u13=rank[x1]+rank[x3]-rank[augment[x1,x3]]
```

```
u23=rank[x2]+rank[x3]-rank[augment[x2,x3]];
t12=m12-u12
t13=m13-u13
t23=m23-u23;
z12=d[x1]-m12
z21=d[x2]-m12
z13=d[x1]-m13
z31=d[x3]-m13
z23=d[x2]-m23
z32=d[x3]-m23;

(* canonical efficiencies of type 1 *)

phi12=Chop[Rationalize[Sort[1-s12^2]]]
phi13=Chop[Rationalize[Sort[1-s13^2]]]
phi23=Chop[Rationalize[Sort[1-s23^2]]]
phi12=Take[phi12,{u12+1,m12}]
phi13=Take[phi13,{u13+1,m13}]
phi23=Take[phi23,{u23+1,m23}]

(* average canonical efficiencies of type 1 *)

Block[{i},If[u12!=d[x1],
        e12=(d[x1]-u12)/(z12+Sum[1/phi12[[i]],{i,1,t12}]),
        e12="-"];
        If[u12!=d[x2],
        e21=(d[x2]-u12)/(z21+Sum[1/phi12[[i]],{i,1,t12}]),
        e21="-"]]//N
Block[{i},If[u13!=d[x1],
        e13=(d[x1]-u13)/(z13+Sum[1/phi13[[i]],{i,1,t13}]),
        e13="-"];
        If[u13!=d[x3],
        e31=(d[x3]-u13)/(z31+Sum[1/phi13[[i]],{i,1,t13}]),
        e31="-"]]//N
Block[{i},If[u23!=d[x2],
        e23=(d[x2]-u23)/(z23+Sum[1/phi23[[i]],{i,1,t23}]),
        e23="-"];
        If[u23!=d[x3],
        e32=(d[x3]-u23)/(z32+Sum[1/phi23[[i]],{i,1,t23}]),
        e32="-"]]//N

(* augmented matrices and their singular value decompositions *)

x4=augment[x1,x2]
x5=augment[x1,x3]
x6=augment[x2,x3];
{u4,s4,v4}=SingularValues[N[x4]]
{u5,s5,v5}=SingularValues[N[x5]]
{u6,s6,v6}=SingularValues[N[x6]];
u4h=Transpose[u4]
u5h=Transpose[u5]
u6h=Transpose[u6];

(* positive canonical correlations of type 2 *)

s1a23=Sort[SingularValues[N[Transpose[u1h].u6h]][[2]]]
s2a13=Sort[SingularValues[N[Transpose[u2h].u5h]][[2]]]
s3a12=Sort[SingularValues[N[Transpose[u3h].u4h]][[2]]]
```

```
(* m, u, t and z values of type 2 *)

m1a23=rank[Transpose[x1].x6]
m2a13=rank[Transpose[x2].x5]
m3a12=rank[Transpose[x3].x4];
u1a23=rank[x1]+rank[x6]-rank[augment[x1,x6]]
u2a13=rank[x2]+rank[x5]-rank[augment[x2,x5]]
u3a12=rank[x3]+rank[x4]-rank[augment[x3,x4]];
u123=rank[x1]+rank[x2]+rank[x3]-rank[augment[x1,x6]]     (* overall connectedness *)
t1a23=m1a23-u1a23
t2a13=m2a13-u2a13
t3a12=m3a12-u3a12;
z1a23=d[x1]-m1a23
z23a1=d[x6]-m1a23
z2a13=d[x2]-m2a13
z13a2=d[x5]-m2a13
z3a12=d[x3]-m3a12
z12a3=d[x4]-m3a12;

(* canonical efficiencies of type 2 *)

phi1a23=Chop[Rationalize[Sort[1-s1a23^2]]]
phi2a13=Chop[Rationalize[Sort[1-s2a13^2]]]
phi3a12=Chop[Rationalize[Sort[1-s3a12^2]]]

(* average canonical efficiencies of type 1 *)

Block[{i},If[u1a23!=d[x1],
        e1a23=(d[x1]-u1a23)/
        (z1a23+Sum[1/phi1a23[[i]],{i,u1a23+1,m1a23}]),
        e1a23="-"];
        If[u1a23!=d[x6],
        e23a1=(d[x6]-u1a23)/
        (z23a1+Sum[1/phi1a23[[i]],{i,u1a23+1,m1a23}]),
        e23a1="-"]]//N
Block[{i},If[u2a13!=d[x2],
        e2a13=(d[x2]-u2a13)/
        (z2a13+Sum[1/phi2a13[[i]],{i,u2a13+1,m2a13}]),
        e2a13="-"];
        If[u2a13!=d[x5],
        e13a2=(d[x5]-u2a13)/
        (z13a2+Sum[1/phi2a13[[i]],{i,u2a13+1,m2a13}]),
        e13a2="-"]]//N
Block[{i},If[u3a12!=d[x3],
        e3a12=(d[x3]-u3a12)/
        (z3a12+Sum[1/phi3a12[[i]],{i,u3a12+1,m3a12}]),
        e3a12="-"];
        If[u2a13!=d[x4],
        e12a3=(d[x4]-u3a12)/
        (z12a3+Sum[1/phi3a12[[i]],{i,u3a12+1,m3a12}]),
        e12a3="-"]]//N

(* non unit canonical efficiencies of type 2 *)

phi1a23=Take[phi1a23,{u1a23+1,m1a23}]
phi2a13=Take[phi2a13,{u2a13+1,m2a13}]
phi3a12=Take[phi3a12,{u3a12+1,m3a12}]
```

```
(* M matrices *)

m1=IdentityMatrix[n]-h1
m2=IdentityMatrix[n]-h2
m3=IdentityMatrix[n]-h3;

(* singular value decomposition of the adjusted design matrices *)

If[m2.x1!=Table[0,{n},{r}],
                        {u1m2,s1m2,v1m2}=SingularValues[N[m2.x1]];
                               u1m2h=Transpose[u1m2]];
If[m3.x1!=Table[0,{n},{r}],
        {u1m3,s1m3,v1m3}=SingularValues[N[m3.x1]];
        u1m3h=Transpose[u1m3]];
 If[m1.x2!=Table[0,{n},{c}],
        {u2m1,s2m1,v2m1}=SingularValues[N[m1.x2]];
        u2m1h=Transpose[u2m1]];
If[m3.x2!=Table[0,{n},{c}],
        {u2m3,s2m3,v2m3}=SingularValues[N[m3.x2]];
        u2m3h=Transpose[u2m3]];
If[m1.x3!=Table[0,{n},{t}],
        {u3m1,s3m1,v3m1}=SingularValues[N[m1.x3]];
        u3m1h=Transpose[u3m1]];
If[m2.x3!=Table[0,{n},{t}],
        {u3m2,s3m2,v3m2}=SingularValues[N[m2.x3]];
        ; u3m2h=Transpose[u3m2]]

(* check for adjusted orthogonality *)

adj3=Transpose[x1].m3.x2
adj2=Transpose[x1].m2.x3
adj1=Transpose[x2].m1.x3;

(* positive canonical correlations of type 3 & 4 *)

If[adj3==Table[0,{r},{c}],
        sm3={0};s12m3={0};s21m3={0},
        sm3=Sort[SingularValues[N[Transpose[u1m3h].u2m3h]][[2]]],
        s12m3=Sort[SingularValues[N[Transpose[u2m3h].u1h]][[2]]];
        s21m3=Sort[SingularValues[N[Transpose[u1m3h].u2h]][[2]]]
        ]
If[adj2==Table[0,{r},{t}],
        sm2={0};s13m2={0};s31m2={0},
        sm2=Sort[SingularValues[N[Transpose[u1m2h].u3m2h]][[2]]];
        s13m2=Sort[SingularValues[N[Transpose[u3m2h].u1h]][[2]]];
        s31m2=Sort[SingularValues[N[Transpose[u1m2h].u3h]][[2]]]
        ]
If[adj1==Table[0,{r},{t}],
        sm1={0};s23m1={0};s32m1={0},
        sm1=Sort[SingularValues[N[Transpose[u2m1h].u3m1h]][[2]]];
        s23m1=Sort[SingularValues[N[Transpose[u3m1h].u2h]][[2]]];
        s32m1=Sort[SingularValues[N[Transpose[u2m1h].u3h]][[2]]]
        ]

(* m, u, t and z values of type 3 and 4 *)

mm1=rank[adj1]
mm2=rank[adj2]
mm3=rank[adj3];
```

```
um1=rank[m1.x2]+rank[m1.x3]-rank[augment[m1.x2,m1.x3]]
um2=rank[m2.x1]+rank[m2.x3]-rank[augment[m2.x1,m2.x3]]
um3=rank[m3.x1]+rank[m3.x2]-rank[augment[m3.x1,m3.x2]]
u12m3=rank[x1]+rank[m3.x2]-rank[augment[x1,m3.x2]]
u21m3=rank[x2]+rank[m3.x1]-rank[augment[x2,m3.x1]]
u13m2=rank[x1]+rank[m2.x3]-rank[augment[x1,m2.x3]]
u31m2=rank[x3]+rank[m2.x1]-rank[augment[x3,m2.x1]]
u23m1=rank[x2]+rank[m1.x3]-rank[augment[x2,m1.x3]]
u32m1=rank[x3]+rank[m1.x2]-rank[augment[x3,m1.x2]];
tm3=mm3-um3
tm2=mm2-um2
tm1=mm1-um1;
t12m3=mm3-u12m3
t21m3=mm3-u21m3
t13m2=mm2-u13m2
t31m2=mm2-u31m2
t23m1=mm1-u23m1
t32m1=mm1-u32m1;
z12m3=d[x1]-mm3
z2m31=d[m3.x2]-mm3   (* z2m31m3 *)
z21m3=d[x2]-mm3
z1m32=d[m3.x1]-mm3   (* z1m32m3 *)
z13m2=d[x1]-mm2
z3m21=d[m2.x3]-mm2   (* z3m21m2 *)
z31m2=d[x3]-mm2
z1m23=d[m2.x1]-mm2   (* z1m23m2 *)
z23m1=d[x2]-mm1
z3m12=d[m1.x3]-mm1   (* z3m12m1 *)
z32m1=d[x3]-mm1
z2m13=d[m1.x2]-mm1;  (* z2m13m1 *)

(* canonical efficiencies of type 3 and 4 *)

phim3=Take[Chop[Rationalize[Sort[1-sm3^2]]],{um3+1,mm3}]
phi12m3=Take[Chop[Rationalize[Sort[1-s12m3^2]]],{u12m3+1,mm3}]
phi21m3=Take[Chop[Rationalize[Sort[1-s21m3^2]]],{u21m3+1,mm3}]
phim2=Take[Chop[Rationalize[Sort[1-sm2^2]]],{um2+1,mm2}]
phi13m2=Take[Chop[Rationalize[Sort[1-s13m2^2]]],{u13m2+1,mm2}]
phi31m2=Take[Chop[Rationalize[Sort[1-s31m2^2]]],{u31m2+1,mm2}]
phim1=Take[Chop[Rationalize[Sort[1-sm1^2]]],{um1+1,mm1}]
phi23m1=Take[Chop[Rationalize[Sort[1-s23m1^2]]],{u23m1+1,mm1}]
phi32m1=Take[Chop[Rationalize[Sort[1-s32m1^2]]],{u32m1+1,mm1}]

(* average canonical efficiencies of type 3 and 4 *)

Block[{i},e2m31m3=(d[m3.x2]-um3)/
        (z2m31+Sum[1/phim3[[i]],{i,1,tm3}])]//N
Block[{i},e1m32m3=(d[m3.x1]-um3)/
        (z1m32+Sum[1/phim3[[i]],{i,1,tm3}])]//N
Block[{i},e12m3=(d[x1]-u12m3)/
        (z12m3+Sum[1/phi12m3[[i]],{i,1,t12m3}])]//N
Block[{i},e2m31=(d[m3.x2]-u12m3)/
        (z2m31+Sum[1/phi12m3[[i]],{i,1,t12m3}])]//N
Block[{i},e21m3=(d[x2]-u21m3)/
        (z21m3+Sum[1/phi21m3[[i]],{i,1,t21m3}])]//N
Block[{i},e1m32=(d[m3.x1]-u21m3)/
        (z1m32+Sum[1/phi21m3[[i]],{i,1,t21m3}])]//N
Block[{i},e3m21m2=(d[m2.x3]-um2)/
        (z3m21+Sum[1/phim2[[i]],{i,1,tm2}])]//N
```

```
Block[{i},c1m23m2=(d[m2.x1]-um2)/
        (z1m23+Sum[1/phim2[[i]],{i,1,tm2}])]//N
Block[{i},c13m2=(d[x1]-u13m2)/
        (z13m2+Sum[1/phi13m2[[i]],{i,1,t13m2}])]//N
Block[{i},c3m21=(d[m2.x3]-u13m2)/
        (z3m21+Sum[1/phi13m2[[i]],{i,1,t13m2}])]//N
Block[{i},c31m2=(d[x3]-u31m2)/
        (z31m2+Sum[1/phi31m2[[i]],{i,1,t31m2}])]//N
Block[{i},c1m23=(d[m2.x1]-u31m2)/
        (z1m23+Sum[1/phi31m2[[i]],{i,1,t31m2}])]//N
Block[{i},e3m12m1=(d[m1.x3]-um1)/
        (z3m12+Sum[1/phim1[[i]],{i,1,tm1}])]//N
Block[{i},e2m13m1=(d[m1.x2]-um1)/
        (z2m13+Sum[1/phim1[[i]],{i,1,tm1}])]//N
Block[{i},c23m1=(d[x2]-u23m1)/
        (z23m1+Sum[1/phi23m1[[i]],{i,1,t23m1}])]//N
Block[{i},e3m12=(d[m1.x3]-u23m1)/
        (z3m12+Sum[1/phi23m1[[i]],{i,1,t23m1}])]//N
Block[{i},c32m1=(d[x3]-u32m1)/
        (z32m1+Sum[1/phi32m1[[i]],{i,1,t32m1}])]//N
Block[{i},e2m13=(d[m1.x2]-u32m1)/
        (z2m13+Sum[1/phi32m1[[i]],{i,1,t32m1}])]//N

(* write out the results in the form of a mutzE-table *)

OutputForm[SequenceForm[
ColumnForm[{"  ","-----"," ",1.2," "," ",1.3," "," ",2.3," ","-----"," ",1.23," "," ",
        2.13," "," ",3.12," ","-----"," ","1;2|3"," "," ","2;1|3"," "," ","1;3|2"," "," ",
        "3;1|2"," "," ","2;3|1"," "," ","3;2|1"," ","-----"," ","1.2|3"," "," ","1.3|2"," ",
        " ","2.3|1"," ","-----"," ","1.2.3"},
        Center],
ColumnForm[{"m","-----"," ",m12," "," ",m13," "," ",m23," ","-----"," ",m1a23,
        " "," ",m2a13," "," ",m3a12," ","-----"," ",mm3," "," ",mm3," "," ",mm2,
        " "," ",mm2," "," ",mm1," "," ",mm1," ","-----"," ", mm3," "," ",mm2," ",
        " ",mm1," ","-----"},
        Center],
ColumnForm[{"u","-----"," ",u12," "," ",u13," "," ",u23," ","-----"," ",u1a23," ",
        " ",u2a13," "," ",u3a12," ","-----"," ",u12m3," "," ",u21m3," "," ",u13m2,
        " "," ",u31m2," "," ",u23m1," "," ",u32m1," ","-----"," ",um3," "," ",um2,
        " "," ",um1," ","-----"," ",u123},
        Center],
ColumnForm[{"t","-----"," ",t12," "," ",t13," "," ",t23," ","-----"," ",t1a23," "," ",
        t2a13," "," ",t3a12," ","-----"," ",t12m3," "," ",t21m3,  "," ",t13m2," "," ",
        t31m2," "," ",t23m1," "," ",t32m1," ","-----"," ",tm3," "," ",tm2," "," ",tm1,
        " ","-----"},
        Center],
ColumnForm[{"phi","----------------"," ",phi12," ",phi13,phi23,"----------------",
        phi1a23,phi2a13,phi3a12,"----------------",phi12m3,phi21m3,phi13m2,
        phi31m2,phi23m1,phi32m1,"----------------"," ",phim3," ",phim2,phim1,
        "----------------"},
        Center],
ColumnForm[{"z","-----"," ",z12," "," ",z13," "," ",z23," ","-----"," ",z1a23," ",
        " ",z2a13," "," ",z3a12," ","-----"," ",z12m3," "," ",z21m3," "," ",z13m2,
        " "," ",z31m2," "," ",z23m1," "," ",z32m1," ","-----"," ",z1m32," "," ",
        z1m23," "," ",z2m13," ","-----"},
        Center],
```

```
ColumnForm[{"z'","-----"," ",z21," "," ",z31," "," ",z32," ","-----"," ",z23a1," ",
        " ",z13a2," "," ",z12a3," ","-----"," ",z2m31," "," ",z1m32," "," ",z3m21,
        " "," ",z1m23," "," ",z3m12," "," ",z2m13," ","-----"," ",z2m31," "," ",
        z3m21," "," ",z3m12," ","-----"},
        Center],
ColumnForm[{"E","--------"," ",e12//N," "," ",e13//N," "," ",e23//N," ","--------",
        " ",e1a23//N," "," ",e2a13//N," "," ",e3a12//N," ","--------"," ",e12m3//N," ",
        " ",e21m3//N," "," ",e13m2//N," "," ",e31m2//N," "," ",e23m1//N," "," ",
        e32m1//N," ","--------"," ",e1m32m3//N," "," ",e1m23m2//N," "," ",
        e2m13m1//N," ","--------"},
        Center],
ColumnForm[{" ","---"," "," "," "," "," "," "," "," "," "," "," ","---"," "," "," "," "," "," "," ",
        " "," "," ","---"," "," "," "," "," "," "," "," "," "," "," "," "," "," "," "," "," "," "," "," ",
        " "," "," "," ","---"," "," "," "," "," "," "," "," "," "," "," "," "," ","---"},
        Center],
ColumnForm[{"E'","-------"," ",e21//N," "," ",e31//N," "," ",e32//N," ","-------",
        " ",e23a1//N," "," ",e13a2//N," "," ",e12a3//N," ","-------"," ",e2m31//N,
        " "," ",e1m32//N," "," ",e3m21//N," "," ",e1m23//N," "," ",e3m12//N," ",
        " ",e2m13//N," ","------ " " e2m31m3//N," "," ",e3m21m2//N," "," ",
        e3m12m1//N," ","------- },
        Center]
]]>>>mutzetable.m
```

!!*mutzE*-table.m

(*PROGRAM 2*)

(* this program assesses a design's decomposability and commutativity *)

(***)

```
<<matrixmut.m
TableForm[m]

{r,c}=Dimensions[m];
n=0;t=0;

(* number n of plots, number t of treatments *)

Block[{i,j},Do[If[m[[i,j]]!=0,n=n+1];
          If[m[[i,j]]>t,t=m[[i,j]]],{i,r},{j,c}]]
count=0;
x1=Table[0,{n},{r}];x2=Table[0,{n},{c}];
              x3=Table[0,{n},{t}];


(* design matrices for rows and columns *)

Block[{i,j},Do[If[m[[i,j]]!=0,
             count=count+1;
             x1[[count,i]]=1;
             x2[[count,j]]=1],
         {j,c},{i,r}]]

(* design matrix for treatments *)

count=0;
Block[{i,j,k},Do[If[m[[i,j]]==k,
              count=count+1;
              x3[[count,k]]=1],
          {j,c},{i,r},{k,t}]]
en=Table[1,{n}]
jn=Table[1,{n},{n}]
in=IdentityMatrix[n];
cn=in-jn/n;
d[m_]:=Length[m[[1]]]
rank[m_]:=d[m]-Length[NullSpace[m]]

(* residual matrices *)

m1=in-x1.Inverse[Transpose[x1].x1].Transpose[x1];
m2=in-x2.Inverse[Transpose[x2].x2].Transpose[x2];
m3=in-x3.Inverse[Transpose[x3].x3].Transpose[x3];

(* incidence matrices and treatment replication matrix *)

n12=Transpose[x1].x2
n13=Transpose[x1].x3
n23=Transpose[x2].x3
d3=Transpose[x3].x3;
k3=Transpose[x3].en
k3k3p=d3.Table[k3,{t}];
```

```
(* information matrices for the subdesigns *)

s31=Transpose[x3].m1.x3
s32=Transpose[x3].m2.x3
s30=Transpose[x3].cn.x3;

(* check for commutativity *)

comm=N[Inverse[Sqrt[d3]].s31.Inverse[Sqrt[d3]].
    Inverse[Sqrt[d3]].s32.Inverse[Sqrt[d3]]-
    Inverse[Sqrt[d3]].s32.Inverse[Sqrt[d3]].
    Inverse[Sqrt[d3]].s31.Inverse[Sqrt[d3]]]

(* function which augments a matrix x by a matrix y *)

augment[x_,y_]:=Transpose[Join[Transpose[x],Transpose[y]]]
x4=augment[x1,x2];
{u4,s4,v4}=SingularValues[N[x4]];
u4h=Transpose[u4]
m4=in-u4h.Transpose[u4h];

(* information matrix for the two-way elimination of heterogeneity design *)

s312=Chop[Rationalize[Transpose[x3].m4.x3]];

(* solve for generalized decomposability *)

Solve[x s31+y s32-z s30 == s312,{x,y,z}]

(* solve for extended decomposability *)

Solve[d3-aa Transpose[n13].n13-bb Transpose[n23].n23
    +cc k3k3p == s?12,{aa,bb,cc}]
```

APPENDIX 3. SOME MATRIX AND LINEAR ALGEBRA RESULTS

We present here some matrix and linear algebra results that are used in this thesis (see also Lemma 3.2.1).

Theorem A.3.1 (Marsaglia and Styan, 1974): *Let the matrix* **C** *have a left inverse (full column rank) and let the matrix* **R** *have a right inverse (full row rank). Then for any conformable matrix* **A**,

$$r(\mathbf{A}) = r(\mathbf{CA}) = r(\mathbf{AR}). \tag{A.3.1}$$

Moreover, for conformable matrices **A**, **X** *and* **Y**

$$r(\mathbf{XA}) = r(\mathbf{A}) \Rightarrow r(\mathbf{XAE}) = r(\mathbf{AE}) \qquad \textit{for every possible } \mathbf{E} \tag{A.3.2}$$

and

$$r(\mathbf{AY}) = r(\mathbf{A}) \Rightarrow r(\mathbf{KAY}) = r(\mathbf{KA}) \qquad \textit{for every possible } \mathbf{K}. \tag{A.3.3}$$

In addition,

$$r(\mathbf{XA}) = r(\mathbf{A}) \textit{ and } \mathbf{XAF} = \mathbf{XAG} \Rightarrow \mathbf{AF} = \mathbf{AG}, \tag{A.3.4}$$

and

$$r(\mathbf{AY}) = r(\mathbf{A}) \textit{ and } \mathbf{KAY} = \mathbf{LAY} \Rightarrow \mathbf{KA} = \mathbf{LA}. \tag{A.3.5}$$

Proof: To prove (A.3.1), let **B** be a left inverse of **C** and so **BC** = **I**. Then r(**A**) = r(**BCA**) ≤ r(**CA**) ≤ r(**A**). We prove r(**A**) = r(**AR**) similarly. Now let r(**XA**) = r(**A**) and let **A** = **PQ** be a full rank decomposition. Then $r(\mathbf{P}) = r(\mathbf{A}) = r(\mathbf{XA}) = r(\mathbf{XPQ}) = r(\mathbf{XP})$ and $\mathbf{XP}$ has full column rank. Thus $r(\mathbf{XAE}) = r(\mathbf{XPQE}) = r(\mathbf{QE}) = r(\mathbf{PQE}) = r(\mathbf{AE})$, establishing (A.3.2). If $\mathbf{E} = \mathbf{F} - \mathbf{G}$ then (A.3.4) follows from $r(\mathbf{XAE}) = r(\mathbf{XAF} - \mathbf{XAG}) = 0 = r(\mathbf{AE}) = r(\mathbf{AF} - \mathbf{AG})$. Results (A.3.3) and (A.3.5) are obtained similarly. ❐

The column space of the augmented matrix $(\mathbf{A} : \mathbf{B})$ is $\mathcal{C}(\mathbf{A} : \mathbf{B}) = \mathcal{C}(\mathbf{A}) + \mathcal{C}(\mathbf{B})$. Assuming that, for two vector spaces $\mathcal{A}$ and $\mathcal{B}$, the dimension

$$\dim(\mathcal{A} + \mathcal{B}) = \dim(\mathcal{A}) + \dim(\mathcal{B}) - \dim(\mathcal{A} \cap \mathcal{B}),$$

then it follows that

$$\dim(\mathcal{C}(\mathbf{A}) \cap \mathcal{C}(\mathbf{B})) = r(\mathbf{A}) + r(\mathbf{B}) - r(\mathbf{A} : \mathbf{B}). \qquad (A.3.6)$$

Lemma A.3.2: *Let* $\mathbf{A}$ *denote a matrix, not necessarily symmetric, with* $r(\mathbf{GA}) = r(\mathbf{A})$. *Then, the Löwner partial ordering*

$$\mathbf{GAXA'G'} \leq_L \mathbf{GAYA'G'}, \qquad (A.3.7)$$

where $\mathbf{X}$ *and* $\mathbf{Y}$ *are symmetric matrices, implies that*

$$\mathbf{AXA'} \leq_L \mathbf{AYA'}. \qquad (A.3.8)$$

Proof: We need to show that $\mathbf{GAYA'G'} - \mathbf{GAXA'G'} = \mathbf{GA(Y - X)A'G'} \geq_L \mathbf{0}$ implies that $\mathbf{A(Y - X)A'} \geq_L \mathbf{0}$. Let $\mathbf{A} = \mathbf{PQ}$ be a full rank decomposition. Then $r(\mathbf{GA}) = r(\mathbf{GPQ}) = r(\mathbf{GP}) = r(\mathbf{A})$ and $\mathbf{GP}$ has full column rank and hence a left inverse. Therefore $\mathbf{GA(Y - X)A'G'} = \mathbf{GPQ(Y - X)Q'P'G'} \geq_L \mathbf{0}$ implies that $\mathbf{Q(Y - X)Q'} \geq_L \mathbf{0}$ and so pre- and post-multiplying respectively by $\mathbf{P}$ and $\mathbf{P'}$ proves the result. ❐

Theorem A.3.3 (Marsaglia and Styan, 1974): *For any choice of generalized inverse* $\mathbf{A}^-$ *and every possible* **B** *and* **C**,

$$r(\mathbf{AB} : \mathbf{M_A C}) = r(\mathbf{AB}) + r(\mathbf{M_A C}), \tag{A.3.9}$$

where $\mathbf{M_A} = \mathbf{I} - \mathbf{AA}^-$.

Proof: Suppose there exist vectors **a** and **b** such that $\mathbf{ABa} = \mathbf{M_A Cb} \neq \mathbf{0}$; then premultiplying by $\mathbf{M_A}$ yields $\mathbf{M_A Cb} = \mathbf{0}$. Hence $\mathcal{C}(\mathbf{AB}) \cap \mathcal{C}(\mathbf{M_A C}) = \{\mathbf{0}\}$, and so the equality (A.3.9) holds. ❐

Theorem A.3.4 (Marsaglia and Styan, 1974): *For conformable matrices* **A** *and* **B**, *and for any choices of their generalized inverses* $\mathbf{A}^-$ *and* $\mathbf{B}^-$,

$$r(\mathbf{A} : \mathbf{B}) = r(\mathbf{A}) + r(\mathbf{M_A B}) = r(\mathbf{M_B A}) + r(\mathbf{B}), \tag{A.3.10}$$

where $\mathbf{M_A} = \mathbf{I} - \mathbf{AA}^-$ and $\mathbf{M_B} = \mathbf{I} - \mathbf{BB}^-$.

Proof: Using (A.3.1), we write

$$r(\mathbf{A} : \mathbf{B}) = r\left[(\mathbf{A} : \mathbf{B})\begin{pmatrix} \mathbf{I} & -\mathbf{A}^-\mathbf{B} \\ \mathbf{0} & \mathbf{I} \end{pmatrix}\right] = r(\mathbf{A} : \mathbf{M_A B}) = r(\mathbf{A}) + r(\mathbf{M_A B})$$

where the last equality follows from Theorem A.3.3. The second equality in (A.3.10) is proved similarly. ❐

Lemma A.3.5 (Brauer 1952; Paige, Styan and Wachter, 1975): *Let the $n \times n$ matrix* $\mathbf{A}$ *have characteristic roots $\alpha_1 = 0, \alpha_2, \ldots, \alpha_n$ and suppose that* $\mathbf{Ae}^{(n)} = \mathbf{0}$. *Let* $\mathbf{u}$ *be an $n \times 1$ column vector. Then* $\mathbf{A} + \mathbf{e}^{(n)}\mathbf{u}'$ *has characteristic roots* $\mathbf{u}'\mathbf{e}^{(n)}, \alpha_2, \ldots, \alpha_n$.

Proof: The characteristic polynomial of $\mathbf{A} + \mathbf{e}^{(n)}\mathbf{u}'$ is

$$|\lambda \mathbf{I} - (\mathbf{A} + \mathbf{e}^{(n)}\mathbf{u}')| = |(\lambda \mathbf{I} - \mathbf{A})[\mathbf{I} - (\lambda \mathbf{I} - \mathbf{A})^{-1}\mathbf{e}^{(n)}\mathbf{u}']| = |\lambda \mathbf{I} - \mathbf{A}||\mathbf{I} - (\lambda \mathbf{I} - \mathbf{A})^{-1}\mathbf{e}^{(n)}\mathbf{u}'| \tag{A.3.11}$$

for all $\lambda \neq \mathrm{ch}(\mathbf{A})$. Since $\mathbf{I} - (\lambda \mathbf{I} - \mathbf{A})^{-1}\mathbf{e}^{(n)}\mathbf{u}'$ has only one nonunit characteristic root equal to $1 - \mathbf{u}'(\lambda \mathbf{I} - \mathbf{A})^{-1}\mathbf{e}^{(n)}$, (A.3.11) becomes

$$|\lambda \mathbf{I} - \mathbf{A}|(1 - \mathbf{u}'(\lambda \mathbf{I} - \mathbf{A})^{-1}\mathbf{e}^{(n)}). \tag{A.3.12}$$

Since $\mathbf{Ae}^{(n)} = \mathbf{0}$, it follows that $(\lambda \mathbf{I} - \mathbf{A})\mathbf{e}^{(n)} = \lambda \mathbf{e}^{(n)}$, and so $(\lambda \mathbf{I} - \mathbf{A})^{-1}\mathbf{e}^{(n)} = \mathbf{e}^{(n)}/\lambda$, provided $\lambda \neq 0$. Now $|\lambda \mathbf{I} - \mathbf{A}| = \prod_{i=1}^{n}(\lambda - \alpha_i) = \lambda \prod_{i=2}^{n}(\lambda - \alpha_i)$, and so (A.3.12) becomes

$$|\lambda \mathbf{I} - (\mathbf{A} + \mathbf{e}^{(n)}\mathbf{u}')| = \lambda \prod_{i=2}^{n}(\lambda - \alpha_i)(1 - \mathbf{u}'\mathbf{e}^{(n)}/\lambda) = (\lambda - \mathbf{u}'\mathbf{e}^{(n)}) \prod_{i=2}^{n}(\lambda - \alpha_i) \tag{A.3.13}$$

for all but the finite number of values $\lambda = 0, \mathrm{ch}_1(\mathbf{A}), \ldots, \mathrm{ch}_n(\mathbf{A})$. Hence, (A.3.12) holds for all real λ and thus the characteristic roots of $\mathbf{A} + \mathbf{e}^{(n)}\mathbf{u}'$ are $\mathbf{u}'\mathbf{e}^{(n)}, \alpha_2, \ldots, \alpha_n$. ❐

Theorem A.3.6 (Minkowski's Inequality): *If $x_i, y_i \geq 0$, $p > 1$, then*

$$\left[\sum_{i=1}^{n}(x_i + y_i)^p\right]^{1/p} \leq \left[\sum_{i=1}^{n} x_i^p\right]^{1/p} + \left[\sum_{i=1}^{n} y_i^p\right]^{1/p}. \tag{A.3.14}$$

If $0 \neq p < 1$, then the inequality is reversed, (for $p < 0$ the $x_i, y_i > 0$). Equality holds if and only if the sets of x_i and y_i are proportional.

Proof: See, e.g., Beckenbach and Bellman (1965, Theorem 3, pp. 19-20).

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