Two-Dimensional Gravity and the Sachdev-Ye-Kitaev Model

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Statement of Contribution

This thesis is a literature review, except for section Section 3.1 "An Attempt at a Canonical Analysis", which is original. The original research question of finding a Hilbert space interpretation for the Schwarzian path integral was initiated by my thesis supervisor, Alexander Maloney.

Abstract

We briefly introduce and review the Sachdev-Ye-Kitaev (SYK) model. This model exhibits quenched disorder, and many quantities, like the Euclidean path integral, are computed as an average over disorder or an average over many different unitary theories corresponding to different realizations of the disorder couplings. This places the interpretation of the Euclidean path integral as a thermal partition function under scrutiny, which we promptly investigate. We reach he seeming conclusion that the Euclidean path integral indeed does not support a Hilbert space interpretation, which is corroborated by other investigations in the literature. We then move one dimension higher, to two-dimensional Jackiw-Teitelboim (JT) gravity, investigating its salient features and most importantly, the problem of Hilbert space factorization. This puzzle implies that, due to constraints coming from gauge symmetry, a bulk gravitational theory cannot be dual to a theory with a tensor-product structure on disconnected boundary components. We then briefly introduce and review spin glasses, and review the investigations into the presence of a spin glass phase in the SYK model. Finally, we discuss some natural outgrowths of ideas presented in this thesis.

Abrégè

Nous présentons brièvement et examinons le modèle Sachdev-Ye-Kitaev (SYK). Ce modèle présente le désordre, et de nombreuses quantités, comme l'intégrale du chemin Euclidien, sont calculées comme une moyenne sur le désordre ou une moyenne sur de nombreuses théories unitaires différentes. Nous étudions l'interprétation de l'intégrale du chemin Euclidien comme fonction de partition thermique. Nous concluons que l'intégrale du chemin Euclidien n'a pas d'interprétation de l'espace de Hilbert, ce qui est corroboré par d'autres recherches dans la littérature. Nous passons ensuite d'une dimension supérieure à la gravité bidimensionnelle de Jackiw-Teitelboim (JT), en examinant ses caractéristiques principales et, plus important encore, le problème de la factorisation de l'espace de Hilbert. Nous avons ensuite brièvement présenté et passé en revue les verres de spin et passé en revue les enquêtes sur la présence d'une phase de verre de spin dans le modèle SYK. Enfin, nous discutons de certaines extensions naturelles des idées présentées dans cette thèse.

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First and foremost, I would like to thank my advisor Alexander Maloney. Whenever I read the acknowledgments sections of his students, they would describe his crystal clear way of thinking about physics, his passion and enthusiasm, and his ability to distill complex ideas to their essence. Over the past three years, I have come to understand these sentiments, and experience them firsthand. He has suggested many great ideas and avenues of research over the past two years, and although they were very exciting and intriguing, I found most to be hard to capitalize on. I hope in that respect I was not a disappointment.

I would also genuinely like to thank the physics department at McGill, primarily Simon Caron-Huot for sponsoring the weekly graduate seminars and for many interesting discussions on the infrared structure of gauge theories and spin chains. I would like to thank Yan Gobeil for being an all-around awesome office-mate and for always being there for help with anything and everything. I would like to thank Matt Hodel, for always being infectiously enthusiastic, and Yiannis for organizing the group meetings and giving really clear talks. I still remember Yiannis' visible excitement when giving talks. Although he is not around anymore, Henry Maxfield was always extremely clear and willing to share his very deep knowledge. I would also like to thank Kale Colville for interesting discussions on $T\bar{T}$, and Anh-Khoi Trinh for always dropping by to ask us to get lunch, because eating at your desk is boring. I would like to thank Waleed (hep-ex guy) for always inviting me to pray with him; there's more to life than just materialism. So, thank you hep-th group for being awesome and fostering an amazing environment to learn and really immerse yourself in theoretical physics goodness.

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Contents

1 Introduction

The real world is complex. It is so complex that if you just consider two black holes colliding or proteins interacting in the living cell, you immediately lose all hope of understanding the processes involved. It can be a bit disheartening. It turns out however, quite fortuitously, that the physical world can be *modeled* and understood by the use of a language invented by humans, *mathematics*. We emphasize the notion of modeling, which attempts to recognize the most important aspects of a physical system, while ignoring the complicated and messy details. There is no particular process or methodology that can be systematically applied to all systems, and one has to abstract away the details of systems of interest to reduce them to their essence on a case-by-case scenario. In that regard, it is more of an art than it is a science. In abstracting away the details and keeping what you believe are *universal* characteristics, you could make the system too simple. The idea is to balance solvability and relevance. This thesis is a study of models. The Sachdev-Ye-Kitaev model is a model in condensed matter physics; Jackiw-Teitelboim gravity is a model of Einstein gravity that arises in certain fiber-structure space-times. Are these systems *too* simple? Or do they capture interesting features of, and thus can teach us lessons about, more realistic systems of interest? That is what we hope to find out by studying these toy models.

2 The SYK Model: Definition and Overview

The Sachdev-Ye-Kitaev (SYK) model is a sufficiently simple model that lives at the intersection of many areas of physics. In spite of its simplicity, it is sufficiently rich to capture many interesting, universal features. This theme will permeate what follows in this thesis: connections between seemingly disparate areas through *universality*. Our initial study of the

SYK model will lead us to Random Matrix Theory, Co-adjoint orbits of the Virasoro Group, black holes and gravity in two dimensions, and spin glasses. This section is largely based on [1].

The SYK model is a model of N Majorana fermions interacting through all-to-all couplings in clusters of q [2]. The N Majorana fermions obey the Clifford algebra

$$\{\psi_i, \psi_j\} = \delta_{ij}. \tag{1}$$

For N Majorana fermions, the matrices ψ_i need to be $2^{N/2}$-dimensional, which is the dimension of the Hilbert space. The Hamiltonian describing the system is [1]:

$$H_{SYK} = (i)^{\frac{q}{2}} \sum_{i_N < i_N < \cdots < i_q}^{N} j_{i_1 \ldots i_q} \psi_{i_1} \ldots \psi_{i_q}. \tag{2}$$

The factor of $i^{\frac{q}{2}}$ upfront ensures the Hamiltonian is time-reversal symmetric for $q \equiv 0$ (mod 4). It is anti-symmetric under time-reversal for $q \not\equiv 0$ (mod 4). This will be later important when we discuss the Gaussian Matrix Ensembles. In addition, the parameter q which controls the number of fermions interacting at a time is even, typically taken to be 4. Note for $q = 2$, the SYK "interaction" becomes just a random mass matrix, and the theory is free. The couplings $j_{i_1 \ldots i_q}$ are known as the disorder, and are completely anti-symmetric in their indices. The couplings unrelated by symmetry are independent and identically distributed random variables, drawn from a Gaussian distribution with mean $\langle j_{i_1 \ldots i_q} \rangle = 0$ and variance

$$\langle j^2_{i_1 \ldots i_q} \rangle = \frac{J^2 (q-1)!}{N^{q-1}}. \tag{3}$$

The numerical factors are conventional to simplify the ladder formulas we will encounter later. Note we first draw the couplings from the random distribution, then we define the SYK model for a fixed realization of the couplings. Thus, in a system, the couplings are fixed once-and-for-all.

One of the reasons the SYK model is so interesting is that it can be attacked in many different ways. For example, a brute force approach would be to explicitly construct the matrices ψ_i, and then use them to calculate the Hamiltonian via equation (2). This is because at finite N the Hilbert space is finite-dimensional. H_{SYK} can then be exactly diagonalized to find the eigenvalues and plot a histogram to get the density of states. To construct the exact matrices, we implement the Jodran-Wigner procedure, found in [3]. Let $k = 1, \ldots, n \equiv N/2$, then:

$$\begin{aligned} \psi_{2k-1} &= 1 \otimes 1 \otimes \cdots \otimes 1 \otimes \sigma_1 \otimes \sigma_3 \otimes \cdots \otimes \sigma_3, \\ \psi_{2k} &= 1 \otimes 1 \otimes \cdots \otimes 1 \otimes \sigma_2 \otimes \sigma_3 \otimes \cdots \otimes \sigma_3, \end{aligned} \tag{4}$$

where σ_i are the 2×2 Pauli sigma matrices, satisfying $\{\sigma_i, \sigma_j\} = 2\ \delta_{ij}$, and 1 is the 2×2 identity.

The amenability of the SYK model to an exact diagonalization approach enables us to address questions regarding late-time dynamics, random matrix universality, *e.g.* the slope, dip, ramp and plateau structure of the spectral function, see [4] for a complete discussion, and regarding the low temperature, non-perturbative regime. Although this approach furnishes an exact answer, it does not take us far along the way of physical, conceptual and intuitive understanding. For that, another approach is necessary.

2.1 Iterated Melons Structure of Large-N Feynman Diagrams

One line of attack that is more sophisticated than the exact diagonalization approach and has proved extremely fruitful is re-summing Feynman diagrams in the large-N limit. This is possible because in the large-N limit, a family of Feynman diagrams, that has come to be known as iterated melons, dominates. For example, consider the Euclidean time-ordered two-point function, defined by

$$G(\tau) = \langle T\psi(\tau)\psi(0)\rangle = \theta(\tau)\,\langle\psi(\tau)\psi(0)\rangle - \theta(-\tau)\,\langle\psi(0)\psi(\tau)\rangle\,, \tag{5}$$

where $\psi(\tau) = \exp(-H_{SYK}\ \tau)\psi(0)\exp(H_{SYK}\ \tau)$ is time-evolved in Euclidean time and $\theta(\tau)$ is the Heaviside step function. The minus sign is to account for the anti-commutativity of fermions. It is a familiar fact from Quantum Field Theory (QFT) that the momentum-space free fermion propagator is

$$\begin{aligned} G(p) &= \frac{1}{\not{p} + m} \\ &= \frac{1}{-i\omega} \text{ in Euclidean 1d with } m = 0, \\ &= \int \mathrm{d}\tau \;\; \exp(i\omega\tau)\; G(\tau), \\ \Rightarrow G(\tau) &= \int \mathrm{d}\omega \;\; \exp(-i\omega\tau)\; \frac{-1}{i\omega} \\ &= \frac{1}{2}\; \mathrm{sgn}(\tau). \end{aligned} \tag{6}$$

In the second line, the i appears because we Wick rotate into Euclidean time, so energy also gets Wick-rotated. To determine $G(\tau)$, take $\tau > 0$, so that $G(\tau) = \langle \psi(\tau)\psi(0)\rangle$. For a free, massless fermion $H = 0$, and thus $G(\tau) = \langle \psi(0)\psi(0)\rangle = \langle 0|\psi(0)\psi(0)|0\rangle = \frac{1}{2}$, by the Clifford algebra. If $\tau < 0$, the second term would contribute and we would get $-1/2$. We can use this free propagator to build the propagator in the interacting SYK model via Feynman diagrams. The Feynman expansion for the full, interacting propagator is shown in Figure 1

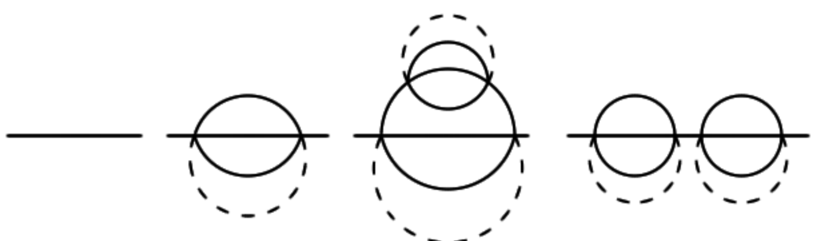

Figure 1: The Feynman diagram expansion of the two-point function is dominated by iterated melon diagrams in the large-N limit in SYK.

above. The dashed arcs denote disorder average. The theory is asymptotically free in the UV, and we are performing a perturbative expansion in $\beta J \ll 1$. A natural question is: Why are we allowed to ignore crossing diagrams like Figure (2)? In order to answer that question, let us firstly focus on the first non-trivial diagram in Figure (1). Let us label the incoming / outgoing line by a, and the loop lines by $b,\ c,\ d$, then that diagram is proportional to

$$\text{two loop diagram } \sim N \cdot N \cdot N \cdot \langle J_{abcd}\; J_{abcd}\rangle = N^3 \frac{J^2\; 3!}{N^{4-1}} = \; O(1), \tag{7}$$

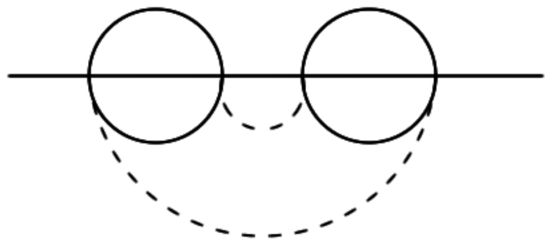

Figure 2: A sub-leading diagram in the $1/N$ expansion.

where there are two factors of N, one for each of the indices $j,\ k,\ l$ over which we are summing. Note the indices match correctly on the two ends linked by the disorder average, and no restrictions are imposed on the internal indices: they are independent. The higher order diagrams in Figure (1) also contribute at $O(1)$ in the $1/N$ expansion. However, the crossed diagram in Figure (2) has additional restrictions coming from the disorder average. If we label the incoming line a, and the lines in the two bubbles or sunsets $b,\ c,\ d$ and $j,\ k,\ l$, then the disorder average sets $b = j,\ c = k,\ d = l$, leaving us with three independent indices. However, there is an additional index on the line connecting the two bubbles, leading to $N^4/N^6 \sim 1/N^2$, and thus the diagram is sub-leading and negligible at large-N. This leads to the solvability of the model, and allows us to simply iterate melons without having to cross them.

Let us define the self-energy $\Sigma(\omega)$ (this would be $\Sigma(p)$ in higher dimensions) by the Feynman diagram in the left panel in Figure 3. Looking at this Feynman diagram, we immediately deduce that

$$\begin{aligned}\Sigma(\tau,\tau') &= \frac{J^2(q-1)!}{N^{q-1}} G(\tau,\tau')^{q-1} \times \frac{N^{q-1}}{(q-1)!} \\ &= J^2 G(\tau,\tau')^{q-1}.\end{aligned} \tag{8}$$

The factor of N^{q-1} is due to N fermions running in $q-1$ loops, and the $1/(q-1)!$ is due to the symmetry of permuting $q-1$ lines. This is the first Schwinger-Dyson equation (SDE). The second self-consistency equation we get by expanding the two-point function as a geometric series in the self-energy $\Sigma(\omega)$

$$\begin{aligned} G(\omega) &= \frac{-1}{i\omega} + \left(\frac{-1}{i\omega}\right)^2 \Sigma(\omega) + \left(\frac{-1}{i\omega}\right)^3 \times \Sigma(\omega)^2 + \left(\frac{-1}{i\omega}\right)^4 \Sigma(\omega)^3 + \dots \\ &= \frac{-1}{i\omega}\frac{1}{1+\Sigma(\omega)/i\omega} \\ &= \frac{1}{-i\omega - \Sigma(\omega)}. \end{aligned} \tag{9}$$

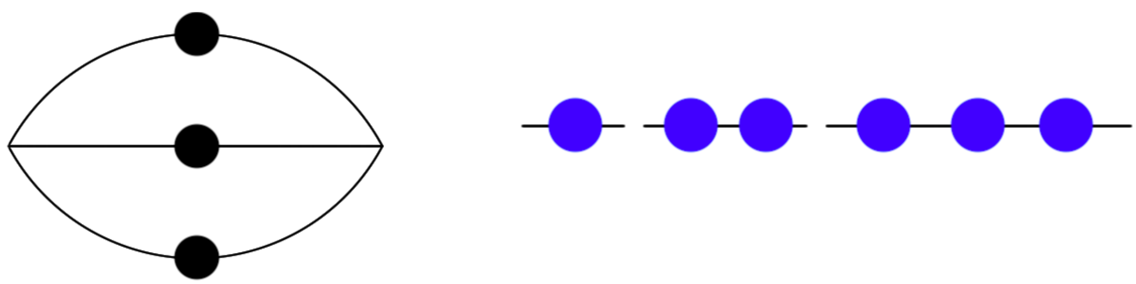

Figure 3: The left figure writes the self-energy $\Sigma(\tau)$ in terms of the dressed, fully interacting propagator $G(\tau)$. The right re-sums $G(\omega)$ as a geometric series in $\Sigma(\omega)$.

This is second of the two Schwinger-Dyson equations. The ability to explicitly write down the self-consistency Schwinger-Dyson equations in the large-N limit of SYK underlies the statement that the SYK model is solvable at large-N. Several comments are in order: Generically, G and Σ are functions of two time arguments, but we are interested in solutions to the SDE that are time-translation invariant. This is a basic restriction we impose. In addition, one of the SDE is in momentum space and the other is in position space. The SDE can be solved either numerically iteratively, or they can be solved analytically in some certain special limits. Two notable limits are the UV limit $\omega \gg \Sigma(\omega)$, and this reduces to the free fermion theory, in which $G(\tau) = \frac{1}{2}\,\mathrm{sgn}(\tau)$, and the second, more interesting limit is the IR, conformal limit $\omega \ll \Sigma(\omega)$, in which case the momentum-space SDE becomes $G(\omega)\Sigma(\omega) = -1$. Fourier-transforming into position space, we get a convolution product on the left-hand side, and a delta function on the right-hand side:

$$\int \mathrm{d}\tau'\, G(\tau - \tau')\Sigma(\tau' - \tau'') = -\delta(\tau - \tau''). \tag{10}$$

On a solution of the SDE, equations (8) and (10) both hold. These two equations are invariant under the *conformal* symmetry:

$$\begin{aligned} G(\tau,\tau') &\to (f'(\tau)f'(\tau'))^{\Delta} G(f(\tau),f(\tau')), \\ \Sigma(\tau,\tau') &\to (f'(\tau)f'(\tau'))^{\Delta(q-1)} \Sigma(f(\tau),f(\tau')), \end{aligned} \tag{11}$$

provided that $\Delta q = 1$. It is clear equation (8) is invariant under the conformal transformations (11), as for equation (10) we have

$$\begin{aligned} &\int \mathrm{d}\tau' (f'(\tau)f'(\tau'))^{\Delta\ q}\ G(f(\tau),f(\tau'))\ \Sigma(f(\tau),f(\tau'')) = -\delta(\tau-\tau'') \\ &\Rightarrow \int \mathrm{d}\tau' f'(\tau')\ G(f(\tau),f(\tau'))\ \Sigma(f(\tau'),f(\tau'') = -\frac{1}{f'(\tau)}\delta(\tau-\tau''), \\ &\Rightarrow \int \mathrm{d}f(\tau')\ G(f(\tau),f(\tau'))\ \Sigma(f(\tau'),f(\tau'')) = -\delta(f(\tau)-f(\tau'')), \\ &\Rightarrow \int \mathrm{d}T\ G(T,T')\ \Sigma(T',T'') = -\delta(T-T''). \end{aligned} \tag{12}$$

In the second line we used $\Delta\ q = 1$, and in the fourth we define a new variable $T = f(\tau)$.

We have shown that the IR limit of the SYK model develops a $Diff(S^1)$ conformal symmetry, and that in the language of Conformal Field Theory (CFT), the fields G and Σ are primary with conformal dimensions Δ and $\Delta(1-q)$, respectively (see [5] for primer on CFT). $Diff(S^1)$ is defined as the group of increasing diffeomorphisms on the thermal circle $f: S^1 \to S^1,\ f'(\tau) > 0$ [1] that wrap precisely once around the circle $f(\tau+\beta) = f(\tau) + 2\pi$. They are monotonically increasing to ensure that they are invertible.

2.2 Breaking of Full Conformal Symmetry

The equations of the strict IR limit of SYK $\omega \to 0$ develop full conformal invariance. The natural question to ask, though is: Is this full conformal symmetry exhibited by the solution to the equations of motion? Or is it spontaneously broken? To answer this question, we

[1] A comment on notation: $Diff(S^1)$ appears at finite temperature, whereas in the zero-temperature $\beta \to \infty$ limit, we simply have $diff(\mathbb{R})$. We can however use unified language, if we agree that $\mathbb{R}$ is the unfolding of the thermal circle for $\beta \to \infty$.

start with an ansatz for the two-point function

$$\begin{aligned} G_c(\tau) &= \frac{b}{|\tau|^{2\Delta}} \operatorname{sgn}(\tau) \text{ (zero temperature)}, \\ G_c^{(\beta)} &= b \left(\frac{\pi}{\beta \sin \frac{\pi\tau}{\beta}} \right)^{2\Delta} \operatorname{sgn}(\tau) \quad \text{(finite temperature)}. \end{aligned} \tag{13}$$

The zero-temperature ansatz is the natural ansatz suggested by CFT for a conformal primary of dimension Δ, and the sgn function imposes anti-commutativity of fermions. The finite-temperature ansatz is the conformal transformation of the zero-temperature one with $f(\tau) = \tan\frac{\pi\tau}{\beta}$, which is correctly periodic under $\tau \to \tau + \beta$. We can determine b using the SDE, but we will not need it in what follows. It is a familiar statement from CFT literature that the thermal two-point function in (13) is no longer invariant not under the full re-parameterization symmetry, but only an SL$(2,\mathbb{R})$ subgroup thereof which acts via

$$\begin{aligned} &\tan\frac{\pi\tau}{\beta} = f(\tau) \to \frac{a\ f(\tau) + b}{c\ f(\tau) + d}, \\ &ad - bc = 1, \\ &a,\ b,\ c,\ d \in \mathbb{R}. \end{aligned} \tag{14}$$

In the strict IR, $\beta J \ll 1$ limit of SYK, the $Diff(S^1)$ symmetry is thus spontaneously broken by the finite-temperature solution to the SDE down to an SL$(2,\mathbb{R})$ subgroup. A corollary is the emergence of Nambu-Goldstone (NG) bosons that live in the coset manifold $Diff(S^1)/\mathrm{SL}(2,\mathbb{R})$:

$$\text{space of NG bosons } = \frac{\text{full group}}{\text{preserved subgroup}} = \frac{Diff(S^1)}{\mathrm{SL}(2,\mathbb{R})}. \tag{15}$$

Since this manifold is infinite-dimensional, we have infinitely many NG bosons. An immediate problem arises: These NG bosons correspond to symmetries of the equations of motion or the action; they are zero modes of the action. Moving along these directions does not incur any cost or action, and integrating over them gives rise to an IR divergence. This divergence is not sensible, since we started out with a finite-dimensional, discrete Hilbert space. This indicates we were not careful in taking the IR limit, i.e. setting $\omega = 0$. We have to reconsider and move away from the IR limit. In order to proceed, however, we will need to present an action formulation of the SYK model. This path integral language will be necessary in what follows.

2.3 Path Integral Formulation

Let us consider the path integral for a single realization of the disorder

$$Z = \int \mathrm{d}\psi(\tau)\ \exp\left(\int_0^\beta \mathrm{d}\tau\ \psi(\tau)\ \frac{\mathrm{d}}{\mathrm{d}\tau}\psi(\tau) + \int_0^\beta \mathrm{d}\tau \sum_{i_1<\cdots<i_q} J_{i_1\ldots i_q}\ \psi_{i_1}(\tau)\ldots\psi_{i_q}(\tau)\right). \quad (16)$$

The path integral formulation is commonly used to study the thermodynamics: the free energy, the entropy and the density of states, and is thus computed at finite temperature, which is implemented as imaginary periodic time coordinate. This does not look encouraging, but progress can be made by computing a disorder-averaged partition function Z, which again, does not correspond to the partition function of any one quantum system. We will use multi-index notation to simplify, so that $\psi_{i_1}\ldots\psi_{i_q} = \Psi_I$ for indices $i_1 \ldots i_q$. We also ignore the zeroth order J-independent term, since it is unchanged in the calculation.

$$\begin{aligned}
\langle Z\rangle &= \int \mathrm{d}\psi(\tau)\ e^{\int \psi\dot{\psi}} \int_0^\beta \mathrm{d}\tau \sum_I \langle J_I\rangle\, \Psi_I(t) + \int_0^\beta\int_0^\beta \mathrm{d}\tau_1\mathrm{d}\tau_2 \sum_I\sum_K \langle J_I J_K\rangle\ \Psi_I(\tau_1)\Psi_K(\tau_2)\\
&= \int \mathrm{d}\psi(\tau)\ e^{\int \psi\dot{\psi}} \int_0^\beta\int_0^\beta \mathrm{d}\tau_1\mathrm{d}\tau_2 \sum_I \frac{J^2(q-1)!}{N^{q-1}}\Psi_I(\tau_1)\ \Psi_I(\tau_2) + \text{ even terms},\\
&= \int \mathrm{d}\psi(\tau)\ \exp\left(\int_0^\beta \mathrm{d}\tau\ \psi(\tau)\ \frac{\mathrm{d}}{\mathrm{d}\tau}\psi(\tau) + \frac{J^2(q-1)!}{N^{q-1}}\int_0^\beta\int_0^\beta \mathrm{d}\tau_1\mathrm{d}\tau_2 \left(\sum_I \Psi_I(\tau_1)\Psi_I(\tau_2)\right)\right).
\end{aligned} \quad (17)$$

Firstly note that we are led to a bi-local action because our couplings are independent of time. There is a variant of SYK, known as Brownian SYK, for which the couplings are $J_{i_1\ldots i_q}(t)$. They are independently drawn in *time* and for distinct indices $\langle J_I(t)J_K(t')\rangle = C\ \delta_{I,K}\ \delta(t-t')$. Such disorder would lead to a local action. Now, we further massage the bi-local term in the action

$$\begin{aligned}
\sum_I \Psi_I(\tau_1)\Psi_I(\tau_2) &= \sum_{i_1<\cdots<i_q} \psi_{i_1}(\tau_1)\psi_{i_1}(\tau_2)\ldots\psi_{i_q}(\tau_1)\psi_{i_q}(\tau_2),\\
&= \frac{1}{q!}\left(\sum_i \psi_i(\tau_1)\ \psi_i(\tau_2)\right)^q,
\end{aligned} \quad (18)$$

where we have used some standard manipulations of Grassmann variables. The factor of $1/q!$ comes because we are now integrating over all i, not just ordered sequences. Now, define

$$G(\tau_1,\tau_2) = \frac{1}{N}\sum_{i=1}^N \psi_i(\tau_1)\ \psi_i(\tau_2), \quad (19)$$

to be the two-point function of fermion operators. We will substitute the Grassmann path integral with a bosonic path integral. So let us introduce $\Sigma(\tau_1, \tau_2)$, whose goal in life is to be a Lagrange multiplier that enforces $G(\tau_1, \tau_2)$ to be the fermion two-point function. We want to exchange the Grassmann integral over the fermions in favor of a bosonic integral over G and Σ. The bi-local action thus becomes

$$S_{\rm BI} = \int_0^\beta \int_0^\beta \mathrm{d}\tau_1 \mathrm{d}\tau_2 \; \Sigma(\tau_1, \tau_2) \left(\sum_{i=1}^N \psi(\tau_1) \; \psi(\tau_2) - N \; G(\tau_1, \tau_2) \right) + \frac{N \; J^2}{q} G(\tau_1, \tau_2)^q. \qquad (20)$$

Note we have a factor of N in front of G^q term, because that term only had in front it a factor of $1/N^{q-1}$, whereas converting from $\sum_i \psi(\tau_1)\psi(\tau_2)$ to $G(\tau_1, \tau_2)$ gave a factor of N^q, so we ended up with an N. Next, we use the famous integration formula for the Gaussian Grassmann integral $\int d\psi \exp(\psi M \psi) = \sqrt{\det M}$. This power of $+1$ is to be contrasted with the power of $-1/2$ obtained from a bosonic integral.

$$\begin{aligned}
&\int \mathrm{d}^N \psi(\tau) \exp\left(\int_0^\beta \int_0^\beta \mathrm{d}\tau_1 \mathrm{d}\tau_2 \psi(\tau_1) \left(-\delta(\tau_1 - \tau_2) \frac{\mathrm{d}}{\mathrm{d}\tau} + \Sigma(\tau_1, \tau_2) \right) \psi(\tau_2) \right) \\
&\qquad = \det\left(\mathbf{1}\partial_\tau - \Sigma \right)^{N/2}, \\
&\Rightarrow \langle Z \rangle = \int \mathrm{D}G \mathrm{D}\Sigma \exp\left(-\frac{N}{2} \mathrm{Tr}\log(\mathbf{1}\partial_\tau - \Sigma) + \int_0^\beta \mathrm{d}\tau_1 \mathrm{d}\tau_2 \; - N \; \Sigma(\tau_1, \tau_2) G(\tau_1, \tau_2) + \frac{N \; J^2}{q} G(\tau_1, \tau_2)^q \right), \\
&\qquad = \int \mathrm{D}G \mathrm{D}\Sigma \exp(-N \; I(G, \Sigma)), \\
&I(G, \Sigma) = \frac{1}{2} \; \mathrm{Tr}\log(\mathbf{1}\partial_\tau - \Sigma) + \int_0^\beta \mathrm{d}\tau_1 \mathrm{d}\tau_2 \; \Sigma(\tau_1, \tau_2) G(\tau_1, \tau_2) - \frac{J^2}{q} G(\tau_1, \tau_2)^q.
\end{aligned} \qquad (21)$$

In the first line we obtained a factor of N because we integrated an N-dimensional Grassmann integral, one for each ψ_i. We also used the formula $\det M = \exp(\mathrm{Tr}(\log M))$. We finally obtained what we were looking for: An action formulation of the SYK model that admits a solution by saddle-point or steepest descent. This is because the action is proportional to N, so that large-N limit is the semi-classical limit. It is easily seen that the saddle point solution to the above action is nothing but the familiar SDE (8, 9). Note the fields G, Σ are generally off-shell, are allowed to fluctuate wildly, and are only given by their values obeying the SDE on a classical solution, by definition.

2.4 The Schwarzian Action of the Nambu-Goldstone Modes

We saw earlier that taking the strict $\omega \to 0$ or $\partial_\tau \to 0$ IR limit is problematic: it gives rise to an infinite number of NG bosons that have zero action. This suggested that we should instead consider a perturbative expansion in the small parameter $1/\beta J$ about $1/\beta J \to 0$ corresponding to the IR limit. Considerations from the four-point function or from effective field theory yield the action for the NG re-parameterization modes $\phi(\tau) \in Diff(S^1)/\mathrm{SL}(2,\mathbb{R})$:

$$S_{\text{soft}} \sim \frac{N}{\beta J}\int_0^\beta \text{Sch}\left(\tan\frac{\phi}{2}, \tau\right), \tag{22}$$

where Sch is the Schwarzian derivative. The Schwarzian derivative is defined as

$$\text{Sch}(f,\tau) = \frac{f'''}{f'} - \frac{3}{2}\left(\frac{f''}{f'}\right)^2. \tag{23}$$

We will have much more to say about this later, but for now we simply note that this action is proportional to N, as is any action in the G, Σ variables (see equation (21)), but is small because it is suppressed by the large parameter $\beta J \gg 1$. Thus in the near-IR limit, we have spontaneously broken conformal symmetry by the solution to SDE, and it is moreover explicitly broken by the ∂_τ term, which leads to the NG bosons having a small, but non-zero action, and there is some cost to moving along these directions in field space. Note we do not consider the $\mathrm{SL}(2,\mathbb{R})$ to be zero modes, as they are gauge redundancies that we quotient out.

An interesting observation is that the Schwarzian derivative vanishes on $f \in \mathrm{SL}(2,\mathbb{R})$. It is a straightforward computation

$$\begin{aligned} f : \tau &\mapsto \frac{a\tau + b}{c\tau + d}, \\ f' &= \frac{a(c\tau+d) - c(a\tau+b)}{(c\tau+d)^2} = \frac{1}{(c\tau+d)^2}, \\ f'' &= -\frac{2c}{(c\tau+d)^3}, \\ f''' &= \frac{6c^2}{(c\tau+d)^4}, \\ \Rightarrow \text{Sch}(f,\tau) &= \frac{6c^2}{(c\tau+d)^2} - \frac{3}{2}\frac{4}{(c\tau+d)^2}, \\ &= 0. \end{aligned} \tag{24}$$

This observation, the vanishing of the Schwarzian derivative on the $\mathrm{SL}(2,\mathbb{R})$ maps in $Diff(S^1)$, can be used to provide an effective field theory argument for the Schwarzian action: it is the lowest-order action in derivatives that vanishes on the $\mathrm{SL}(2,\mathbb{R})$ gauge symmetry.

3 The Schwarzian Theory

As was discussed in the previous section, the Schwarzian theory governs the pseudo-Nambu-Goldstone bosons for the broken re-parameterization symmetry in the infrared of the SYK model. The path integral of a theory of the Schwarzian action is

$$Z(g) = \int \frac{d\mu[\phi]}{\mathrm{SL}(2,\mathbb{R})} \, \exp\left(-\frac{1}{2g^2}\int_0^{2\pi}\left(\frac{\phi''^2}{\phi'^2} - \phi'^2\right)\right). \tag{25}$$

The path integral is performed over all configurations in the quotient space $Diff(S^1)/\mathrm{SL}(2,\mathbb{R})$, where $Diff(S^1)$ is the space of diffeomorphisms on the circle that wrap the circle once $\phi : S^1 \to S^1$ so that $\phi'(\tau) > 0$ and $\phi(\tau + 2\pi) = \phi(\tau) + 2\pi$. The action is $\mathrm{Sch}(\tan\frac{\phi}{2}, \tau)$. To see this, we use the chain rule for the Schwarzian derivative (23)

$$\begin{aligned}
\mathrm{Sch}(F(\phi), \tau) &= \mathrm{Sch}(\phi, \tau) + \mathrm{Sch}(F)(\phi(\tau)) \cdot \phi'(\tau)^2, \\
\Rightarrow \mathrm{Sch}(\tan\frac{\phi}{2}, \tau) &= \mathrm{Sch}(\phi, \tau) + \frac{1}{2}\phi'^2(\tau), \\
&= \frac{\phi'''}{\phi'} - \frac{3}{2}\left(\frac{\phi''}{\phi'}\right)^2 + \frac{1}{2}\phi'^2(\tau), \\
&= \left(\frac{\phi''}{\phi'}\right)' + \left(\frac{\phi''}{\phi'}\right)^2 - \frac{3}{2}\left(\frac{\phi''}{\phi'}\right)^2 + \frac{1}{2}\phi'^2(\tau), \\
&= \frac{1}{2}\phi'^2(\tau) - \frac{1}{2}\left(\frac{\phi''}{\phi'}\right)^2 .
\end{aligned} \tag{26}$$

In the third line, we used $\mathrm{Sch}(\tan\tau/2, \tau) = 1/2$, as can be verified by direct calculation. In the last line we dropped a total derivative, since the derivatives of $\phi \in Diff(S^1)$ are periodic on the circle. As we saw (22), when this Schwarzian theory appears as a sub-sector in the low-energy description of SYK, $1/g^2 \sim N/\beta J$.

An important natural question that arises concerns the interpretation of the path integral as the partition function of a quantum mechanical theory. Stated more precisely: Is there some mathematically well-defined Hilbert space $\mathcal{H}$ together with a Hamiltonian $H(g)$ such that the path integral (25) is a partition function

$$Z(\beta, g) = \sum_{\text{states in } \mathcal{H}} \exp\left(-\beta H(g)\right), \tag{27}$$

over that Hilbert space? A few important points to keep in mind is that in the SYK model, the Schwarzian action was obtained from the action formulation in the Hubbard-Stratonovich fields G, Σ, which involved an average over many realizations of the disorder, or over many quantum systems. It is not obvious that a theory obtained as disorder average of many theories should constitute a unitary quantum system, and thus admit a Hilbert space interpretation. In addition, in the SYK model the Schwarzian theory is only but one factor in the low-energy, infrared limit. It describes the universal sector dual to Jackiw-Teitelboim (JT) gravity in two-dimensional Anti De-Sitter (AdS_2) spacetime (see section 4.1 for more details on JT gravity). However there are infinitely many primary matter fields. These are fermion bi-linears that have their own non-universal action. Thus in the SYK model, the Schwarzian path integral appears as a factor in the low-energy partition function, and thus it is not obvious that it can stand on its own two feet as a UV-complete quantum theory. These observations can be taken as objections against a Hilbert space interpretation of the path integral. We nonetheless point out the following observation: Define $x \equiv \log \phi'$. x is a real, unbounded variable since $\phi' > 0$. Then the Schwarzian action (26) can be written as

$$\begin{aligned} \frac{1}{2}\left(\frac{\phi''}{\phi'}\right)^2 - \frac{1}{2}\phi'^2(\tau) &= \frac{1}{2}x'^2 - \frac{1}{2}\exp(2x), \\ &= \frac{1}{2}(p^2 - \exp(2x)). \end{aligned} \tag{28}$$

We thus see the putative Hamiltonian (28) furnishing a Hilbert space interpretation has a potential unbounded below. In addition, we have to implement the gauge constraints implied by the quotient structure $Diff(S^1)/\mathrm{SL}(2,\mathbb{R})$. These can be implemented as boundary conditions on $\phi(0)$, $\phi'(0)$, and $\phi''(0)$ to gauge-fix the three generators of $\mathrm{SL}(2,\mathbb{R})$. In spite of these valid conceptual difficulties, one may be still interested to see how far one could take a canonical analysis, to which we now turn.

3.1 An Attempt at a Canonical Analysis

We wish to investigate the spectrum of the Hamiltonian

$$H = p^2 - e^{-2x} + i\lambda e^{-x}. \tag{29}$$

We have extracted an explicit factor of i, so that $\lambda \in \mathbb{R}$. The motivation for studying this family of Hamiltonians is that their real part is equal to (28), up to an unimportant multiplicative factor. This Hamiltonian falls under the general class of Hamiltonians studied in [7]. They are of the form

$$H = p^2 + (A + iB)^2 e^{-2x} - (2C + 1)(A + iB)e^{-x}. \tag{30}$$

To see this, set $A = 0, B = 1$ and $2C + 1 = -\lambda \in \mathbb{R}$. The Schrodinger equation $H\psi = E\psi$ can, via a change of variables to z, be put in the following useful form

$$z^2 \frac{d^2\psi}{dz^2} + z\frac{d\psi}{dz} + \left(E - \frac{z^2}{4} - \frac{\lambda}{2}z\right)\psi = 0, \tag{31}$$

where $z = 2(A + iB)\exp(-x) = 2i\exp(-x)$. We see the only coupling appearing is $\lambda \in \mathbb{R}$, so we can hope the spectrum turns out to be real. Define $\psi(z) = z^\alpha e^{-z/2}F(z)$, where $\alpha = \sqrt{-E} \in \mathbb{R}$ for the bound spectrum since $E < 0$. Also, note the Schrodinger equation (31) is invariant under $\alpha \to -\alpha$, since α only appears as $E = -\alpha^2$. The Schrodinger equation for F becomes

$$zF''(z) + (2\alpha + 1 - z)F'(z) - (\alpha - C)F(z) = 0, \tag{32}$$

which is a confluent hypergeometric equation. The general wave function is therefore a linear combination of the two linearly independent solutions

$$\psi(z) = e^{-z/2}\left(a_\alpha z^\alpha {}_1F_1(\alpha - C, 2\alpha + 1|z) + b_\alpha z^{-\alpha} {}_1F_1(-\alpha - C, -2\alpha + 1|z)\right). \tag{33}$$

We recall $C = -\frac{1+\lambda}{2}$. Equation (33) respects the $\alpha \to -\alpha$ symmetry of the Hamiltonian. Let us assume without loss of generality $\alpha \geq 0$. For $\alpha \in \mathbb{R}$, these wave functions describe bound states, which should be normalizable and finite. To obtain the spectrum of bound states, let us give z a small real part and demand the regularity of the bound state wave functions along the contour $\infty(i + \epsilon)$, for $\epsilon > 0$ infinitesimal. This is equivalent to studying the Hamiltonian

$$H = p^2 + (i + \epsilon)^2 e^{-2x} + \lambda(i + \epsilon)e^{-x}, \tag{34}$$

since the corresponding change of variables that takes us to the confluent hypergeometric form is $z = 2(i+\epsilon)e^{-x}$. We discard terms of order $o(\epsilon^2)$.
Let us also impose regularity in the UV limit $z \to 0$. In this limit ${}_1F_1(a,b|z) = 1 + o(z)$ and

$$\psi(z) \sim a_\alpha z^\alpha + b_\alpha z^{-\alpha}, \tag{35}$$

where $\alpha = \sqrt{-E} > 0$ for the bound spectrum. We thus need $b_\alpha = 0$. Now we impose regularity in the IR limit $z \to (i+\epsilon)\infty$. Let us recall the definition of the hypergeometric ${}_1F_1$ function

$$\begin{aligned} {}_1F_1(a,b|z) &= \sum_{n=0}^{\infty} \frac{(a)_n}{(b)_n}\frac{z^n}{n!}, \\ (a)_n &= a(a+1)\dots(a+n-1). \end{aligned} \tag{36}$$

As $n \to \infty$ the ratio of the successive terms is

$$\begin{aligned} \frac{(a)_{n+1}}{(a)_n}\frac{(b)_n}{(b)_{n+1}}\frac{z}{n+1} &= \frac{a+n}{b+n}\frac{z}{n+1}, \\ &\sim \frac{z}{n+1} \text{ as } n \to \infty, \end{aligned} \tag{37}$$

which is the same as the growth of the exponential e^z, independent of the parameters a, b. The wave functions of (33) thus asymptote to $\psi(z) \sim e^{z/2}\left(a_\alpha z^\alpha + b_\alpha z^{-\alpha}\right)$, which for $z \to (i+\epsilon)\infty$, $\epsilon > 0$, is unacceptable. This can be avoided if $a = -n$ for $n \in \{0,1,2,\dots\}$, in which case the ${}_1F_1$ collapses to a polynomial of order n.

$$\psi_n(z) = \eta_n L_n^{2\alpha}(z). \tag{38}$$

We thus have $\alpha - C = -n \Rightarrow \sqrt{-E} = C - n > 0 \Rightarrow E_n = -(C-n)^2,\ n = 0,1,2,\dots < C$. We recalled that $\alpha = \sqrt{-E}$. We have a finite spectrum of bound states. This spectrum is of the form obtained for the Landau problem on the hyperbolic 2-plane $\mathbb{H}_2$ in section (3.1) of [9] with the identification $C = B - \frac{1}{2}$ or $\lambda = -2B$, and up to a constant shift of $B^2 + \frac{1}{4}$. We moreover see for $C < 0$ our analysis finds no spectrum of bound states, but [8] have found an infinite spectrum of bound states for $C = -\frac{1}{2}$ or $\lambda = 0$: $E_n = -(3/2 + 2n)^2,\ n = 0,1,\dots$ which looks qualitatively different and the wave functions in this case reduce to Bessel functions. Thus dealing with $C < 0$ requires considerable care. One logical possibility is that there is no bound spectrum for $C < 0$ except if $C = -1/2$. We can also study the spectrum of scattering states, which have non-negative energies. The wave functions are now given by

$$\psi_k(z) = z^{-1/2}\left(a_k W_{-\lambda/2,ik}(z) + b_k M_{-\lambda/2,ik}(z)\right), \tag{39}$$

where $k^2 = E$, and W and M are the Whittaker W and M functions, respectively. They are two linearly independent solutions. They are both regular in the UV limit $z \to 0$. Now, [10] is able to rule out the Whittaker M as it diverges like $e^{z/2}$ along $z \to \infty$. If we send $z \to (i+\epsilon)\infty$ for $\epsilon > 0$, then the asymptotic expansion for Whittaker M, tells us $M_{\kappa,\mu}(z) \sim e^{z/2} z^{-\kappa}$, which is not acceptable along $(i+\epsilon)\infty$. We thus have

$$\psi_k(z) = z^{-1/2}\sqrt{\frac{k\sinh(2\pi)}{2\pi^3}}\left|\Gamma\left(\frac{1}{2} + ik + \frac{\lambda}{2}\right)\right| W_{-\lambda/2,ik}(z), \tag{40}$$

where the normalization is that of [10].

Now a Hilbert space is not just a set of states, but is naturally endowed with an inner product. We may want to find an inner product with respect to which our Hamiltonian is hermitian.

One natural question to ask is if it is sensible to demand regularity along the contour $(i+\epsilon)\infty$ and then take $\epsilon \downarrow 0$. We would have very different conclusions if we consider the contour $(i-\epsilon)\infty$, and then $\epsilon \downarrow 0$. We believe we need ϵ to be there, otherwise in the $z \to i\infty$ limit $\psi(z) \sim e^{z/2}(a_\alpha z^\alpha + b_\alpha z^{-\alpha})$, where $e^{z/2}$ is just oscillatory, and no precise conclusions or constraints could be reached. In addition we would not be able to exclude Whittaker M in the scattering states. The $i\epsilon$ prescription in QFT is ultimately related to retarded propagation as opposed to advanced propagation, which is acausal. This merits further investigation, along the lines of non-Hermitian Hamiltonians and time-like boundary Liouville theory, or half the the Sine-Gordon model [11].

3.2 Coadjoint Orbits as Symplectic Manifolds

Following [12], we study the symplectic structure of coadjoint orbits, and in particular those of the Virasoro $Diff(S^1)$ group. In the $g \ll 1$ semi-classical regime, the Schwarzian theory is weakly coupled and can be described in terms of the saddle-point classical solution $\phi(\tau) = \tau$ and small fluctuations about it. In the strongly coupled regime, the field $\phi(\tau)$ explores non-perturbative configurations. Quite fortuitously, the theory remains under control even in the non-perturbative regime: The path integral is one-loop exact. This fact is due to the so-called Duistermaat-Heckman (DH) formula [13]. To demonstrate this, we will have to set up some machinery of symplectic manifolds and coadjoint orbits of the Virasoro group. We will initially discuss coadjoint orbits in the abstract setting, then specialize to the Virasoro group to make contact with the Schwarzian theory.

Let G be a Lie group and $\mathfrak{g}$ be its Lie algebra. Elements $u, v, w \in \mathfrak{g}$ are called *adjoint* vectors. Thus the term adjoint vector refers to elements of $\mathfrak{g}$. Coadjoint vectors a, b, c are elements of $\mathfrak{g}^*$, the dual to $\mathfrak{g}$. Thus coadjoint vectors are linear functionals on the Lie algebra. A linear functional is specified by its action on every adjoint vector. If there exists a non-degenerate bi-linear form $[\cdot,\cdot]$ on $\mathfrak{g}$, then the adjoint and coadjoint vector spaces are equivalent, since each adjoint vector v defines a coadjoint vector b_v via $b_v(u) = [v, u]$. This linear functional is unique, since if there are two adjoint vectors v, w such that $b_v = b_w$, then we have that for all $u \in \mathfrak{g}$, $0 = b_v(u) - b_w(u) = [v - w, u]$. The non-degeneracy of the bi-linear form then implies $v = w$. For the Virasoro group, there does not exist such a bi-linear form and the adjoint and coadjoint spaces are in fact inequivalent. On the Lie algebra $\mathfrak{g}$, we define the adjoint representation or the action of the Lie algebra on itself: $u \in \mathfrak{g}$ acts via $u : v \to [u, v]$. We in addition define the coadjoint representation, or the action of the Lie algebra $\mathfrak{g}$ on $\mathfrak{g}^*$. Let $a \in \mathfrak{g}^*$, then define $u(a)$, the action of u on a via

$$(u(a))v = -a([u, v]) = -a(u(v)). \tag{41}$$

These definitions ensure the natural pairing between adjoint and coadjoint vectors, given by $a, v \to a(v) \in \mathbb{R}$ is *invariant* under the action of $\mathfrak{g}$:

$$\begin{aligned} \delta_u a(v) &= (u(a))(v) + a(u(v)), \\ &= -a([u, v]) + a([u, v]), \\ &= 0. \end{aligned} \tag{42}$$

This will be useful later. Fix a coadjoint vector b, and consider the set $W_b = \{u(b) |\ u \in \mathfrak{g}\}$. This set is known as a coadjoint orbit under the action of $\mathfrak{g}$. A remarkable fact we will show is that coadjoint orbits are *always* symplectic manifolds. In the theory of classical mechanics, such manifolds serve as phase spaces. A symplectic manifold is equipped with a symplectic structure, that is a closed, non-degenerate bi-linear 2-form. So if a, a' are vectors tangent to W_b at u [2], we must define naturally $\omega(a, a')$. An important point to realize is that a, a' are not any run-off-the-mill co-adjoint vectors. If we move from $u(b)$ in the direction of a we must still remain on the coadjoint orbit W_b. There thus must be $v \in \mathfrak{g}$ such that $u(b) + a = v(b) \Rightarrow a = (v - u)(b)$. In other words, for a tangent vector a to W_b, we can always find $w \in \mathfrak{g}$ to which it corresponds. This correspondence is not unique, since w can always be shifted by z such that $z(b) = 0$. These two observations together imply that coadjoint orbits are quotient groups: $\mathfrak{g}$/Stab, where Stab is the stabilizer subgroup of $\mathfrak{g}$, i.e. the adjoint vectors z such that $z(b) = 0$. So, for a, a' we can find $w, w' \in \mathfrak{g}$ so that

$$w(b) = a, \ w'(b) = a'. \tag{43}$$

Modulo the stabilizer subgroup. We define the symplectic form via

$$\omega(a, a') = b([w, w']). \tag{44}$$

We have observed that the mapping $a \to w$ is not unique, so we have to check the above symplectic form is indeed well-defined. If we shift $w \to w + z$ such that $z(b) = 0$ we get

$$\begin{aligned} \omega(a, a') &= b([w + z, w']), \\ &= b([w, w'] + [z, w']), \\ &= b([w, w']) + b(z(w')), \\ &= b([w, w']) - z(b)(w'), \\ &= b([w, w']). \end{aligned} \tag{45}$$

So it indeed is well defined. It is obviously anti-symmetric and invariant under $\mathfrak{g}$ action. We have to check it is non-degenerate. Suppose there is a tangent vector a such that for all tangent vectors c, we have $\omega(a, c) = b([w, y]) = 0$, for w fixed (corresponding to a) and for all y (corresponding to c). This implies $b(w(y)) = 0$ for all y and thus $w(b) = 0 = a$. Hence ω is non-degenerate. We finally have to prove that it is closed. Suppose u, v, w are adjoint vectors corresponding to a, a', c respectively. Then

$$\begin{aligned} d\omega(a, a', c) = (u \cdot \partial)b([v, w]) + (v \cdot \partial)b([w, u]) + (w \cdot \partial)b([u, v]) \\ + b([u, [v, w]]) + b([v, [w, u]]) + b([w, [v, u]]). \end{aligned} \tag{46}$$

[2] Here we are using $u \in \mathfrak{g}$ as coordinates on W_b

The elements in the first line constitute the action of adjoint vectors on a natural pairing between an adjoint and a coadjoint, and hence vanish. The elements in the second line vanish by linearity and the Jacobi identity. Hence $d\omega = 0$ and ω is closed. This finishes the demonstration that a coadjoint orbit always is a symplectic manifold. We will now apply the above ideas and language to the Virasoro group.

3.3 Coadjoint Orbits of The Virasoro Group

The Virasoro Group is the central extension of $Diff(S^1)$. The Lie algebra of Vir consists of vector fields $f\partial_\tau$ together with a central element c. The Lie bracket is

$$[f(\tau)\partial_\tau, g(\tau)\partial_\tau] = (f(\tau)g'(\tau) - f'(\tau)g(\tau))\partial_\tau + \frac{ic}{48\pi}\int_0^{2\pi} f'''(\tau)g(\tau) - f(\tau)g'''(\tau). \tag{47}$$

If we take $L_m = ie^{im\tau}\partial_\tau$ to be basis vectors for the Lie algebra, the above commutation relation becomes

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12\pi}n^3\delta_{m+n,0}, \tag{48}$$

which are indeed the commutation relations defining the Virasoro algebra in two-dimensional Conformal Field Theory [14]. A different choice of co-cycle would lead to a slightly different central term. Consult Chapter 2 of reference [6] for more information. We have supplemented $Diff(S^1)$ with a central element c, and thus any element in Vir is $g(\tau)\partial_\tau - iac$. Thus the action of the Lie algebra on itself is

$$\delta_f(g,a) = \left(fg' - f'g, \frac{1}{48\pi}\int_0^{2\pi} f'''g - fg'''\right). \tag{49}$$

In the language of the previous section, this is the adjoint representation of Vir. Note that c is not a number, but an abstract central element. It is the coefficient of c, (a), which takes on numerical values. The coadjoint vectors, or linear functionals on $Diff(S^1)$ are quadratic differentials: $b(\tau)(d\tau)^2 + t\tilde{c}$, where $\tilde{c}$ is dual to c, so that $\tilde{c}(c) = 1$. The coadjoint vectors act on adjoint vectors via

$$\langle (g,a),(b,t)\rangle = -i\left(\int_0^{2\pi} \mathrm{d}\tau\ g(\tau)b(\tau) + at\right). \tag{50}$$

We have defined the adjoint representation in (49). What about the co-adjoint representation? We will verify that the pairing (50) is invariant under

$$\begin{aligned} \delta_f b &= 2f'b + fb' - \frac{f'''t}{24\pi}, \\ \delta_f t &= 0, \end{aligned} \tag{51}$$

where f is an adjoint vector. Let us verify

$$\begin{aligned} \delta_f \langle (g,a),(b,t)\rangle &= \langle \delta_f (g,a),(b,t)\rangle + \langle (g,a), \delta_f (b,t)\rangle, \\ i\langle \delta_f (g,a),(b,t)\rangle &= \int_0^{2\pi} \left(f(\tau)g'(\tau) - f'(\tau)g(\tau)\right) b(\tau) + \frac{t}{48\pi}\int_0^{2\pi} f'''g - fg''', \\ &= \int_0^{2\pi} \left(-2f'(\tau)g(\tau)b(\tau) - f(\tau)g(\tau)b'(\tau)\right) + \frac{t}{24\pi}\int_0^{2\pi} f'''(\tau)g(\tau), \\ i\langle (g,a), \delta_f (b,t)\rangle &= \int_0^{2\pi} g(\tau)\left(2f'(\tau)b(\tau) + f(\tau)b'(\tau) - \frac{tf'''(\tau)}{24\pi}\right), \\ \Rightarrow \delta_f \langle (g,a),(b,t)\rangle &= 0. \end{aligned} \tag{52}$$

In going to the third line, we integrated by parts in both integrals. Thus, the pairing between adjoint and coadjoint vectors is invariant under the group action. This is important to check because the symplectic form inherits this invariance under the group action.[3]

We recall that every point on the coadjoint orbit built on the coadjoint vector $(b(\tau), t)$ is obtained by the coadjoint action of an element of the Lie algebra $Diff(S^1)$. We thus might naively think that the possible coadjoint orbits are all nothing but the Lie algebra. That is almost correct, except we have some equivalence relations: The coadjoint orbit is the Lie algebra $Diff(S^1)$ modulo the stabilizer subgroup. The stabilizer Stab of a coadjoint vector is the subgroup of the Lie algebra whose coadjoint action leaves the coadjoint vector invariant. Thus the coadjoint orbits of the Virasoro group are

$$W_b = \frac{Diff(S^1)}{\text{Stab}}, \tag{53}$$

as we have previously seen. Classifying the coadjoint orbits can thus be precisely formulated as the problem of finding the stabilizer subgroups. We will explore this more thoroughly in the Virasoro case below.

[3] We will interchangeably use the terminology of group G action, and the Lie algebra $\mathfrak{g}$ action.

In (51) we have the coadjoint action of the Lie algebra. This action is the infinitesimal version of the coadjoint action of the Lie group, and thus can be upgraded to it by exponentiation. We will verify that the correct coadjoint action for the Lie group is

$$b_\phi(\tau) = (\phi'(\tau))^2(b(\phi(\tau))) - \frac{t}{24\pi}\mathrm{Sch}(\phi, \tau). \tag{54}$$

Again, $\phi(\tau) \in \mathrm{Diff}(S^1)$ is a monotonic diffeomorphism that wraps the circle once. Let us write $\phi(\tau) = \tau + f(\tau)$, so that we are working in a neighborhood of the identity, or in the Lie algebra. Then

$$\begin{aligned} b_\phi(\tau) &= (1 + f'(\tau))^2(b(\tau) + f(\tau)b'(\tau)) - \frac{t}{24\pi}\left(\frac{f'''(\tau)}{1 + f'(\tau)} - \frac{3}{2}\left(\frac{f''(\tau)}{f'(\tau)}\right)^2\right). \\ &= (1 + 2f'(\tau))(b(\tau) + f(\tau)b'(\tau)) - \frac{t}{24\pi}f'''(\tau) + O(f^2), \\ &= 2f'(\tau)b(\tau) + b'(\tau)f(\tau) - \frac{t}{24}f'''(\tau) + O(f^2). \end{aligned} \tag{55}$$

We thus recover precisely the coadjoint action of the Lie algebra (51). Equation (54) gives us precisely the coadjoint orbit built on the vector $(b(\tau), t)$. To explore the possible coadjoint orbits, we will consider the possible stabilizers: try $b(\tau) = b_0 = -n^2 t/48\pi$, $n = 1, 2, \ldots$. We get

$$\begin{aligned} b_\phi(\tau) &= -\frac{t}{24\pi}\left(\mathrm{Sch}(\phi, \tau) + \frac{1}{2}n^2(\phi'(\tau))^2\right), \\ &= -\frac{t}{24\pi}\mathrm{Sch}\left(\tan\frac{n\ \phi(\tau)}{2}, \tau\right). \end{aligned} \tag{56}$$

We thus see the orbit built on the coadjoint vector $b_n(\tau) = -n^2 t/48\pi$ is invariant under the stabilizer $\mathrm{SL}^{(n)}(2, \mathbb{R})$ defined by

$$\tan\left(\frac{n\tau}{2}\right) = f^{(n)}(\tau) \to \frac{af^{(n)}(\tau) + b}{cf^{(n)}(\tau) + d},\ ad - bc = 1. \tag{57}$$

In particular, for $n = 1$, we have the stabilizer $\mathrm{SL}(2, \mathbb{R})$, leading to a coadjoint orbit $Diff(S^1)/\mathrm{SL}(2, \mathbb{R})$. Thus, the quotient space relevant for the Schwarzian theory is a coadjoint orbit of the Virasoro group, and thus a symplectic manifold. This is the first ingredient in the one-loop exactness of the Schwarzian theory. As a side comment, we could have considered a slightly generic coadjoint vector $b(\tau) = -b_0$. This leads to an orbit

$$b_\phi(\tau) = -\frac{t}{24\pi}\left(\mathrm{Sch}(\phi, \tau) + \frac{24\pi}{c}b_0(\phi'(\tau))^2\right). \tag{58}$$

This does not generically re-sum to the Schwarzian of some tangent function, and thus generically only has a rigid $U(1)$ rotational symmetry $\phi(\tau) \to \phi(\tau) + a$. The coadjoint orbit is thus $Diff(S^1)/U(1)$. A third possibility is that of generic $b(\tau)$ but with $t = 0$. Such an orbit would be

$$b_\phi(\tau) = (\phi'(\tau))^2 b(\phi(\tau)). \tag{59}$$

This generically does not even have a $U(1)$ symmetry, and the coadjoint orbit is $Diff(S^1)$, since it is not stabilized by anything. Interestingly, this state of affairs for a finite-dimensional group would imply that the group $Diff(S^1)$ is even-dimensional, since symplectic manifolds must be even-dimensional.

3.4 $U(1)$ Action of The Schwarzian On The Coadjoint Orbit

The second ingredient necessary for one-loop exactness of the Schwarzian path integral is that the Schwarzian action generates via Poisson brackets a $U(1)$ symmetry of the symplectic manifold. The goal of this section is to show this is indeed satisfied, which will enable us to invoke fermionic localization to prove one-loop exactness. What we need to do is essentially express the symplectic form (44) in a way that enables us to make contact with the Schwarzian action. We will need some mathematical technology for that.

First, recall the field space of the Schwarzian theory is the coadjoint orbit $Diff(S^1)/\mathrm{SL}(2,\mathbb{R})$. A suitable coordinate on it is thus simply a $Diff(S^1)$ map $\phi(\tau)$. Consider for example $\phi(\tau) = \tau + \sum u_n L_n$. Recall that $L_n(\tau) = ie^{in\tau},\ n \in \mathbb{Z}$ constitutes a basis of the Lie algebra or the adjoint vector space. We will now introduce a linear map d that takes elements of $Diff(S^1)$ to 1-forms on the tangent space. This map d will be taken to act independently of τ, and thus commutes with ∂_τ. Let us be more precise: Define

$$d\phi(\tau) = \sum L_n du_n. \tag{60}$$

Since we defined d to commute with ∂_τ, it does not act on $L_n(\tau)$ or τ. Is it precise to say that d as defined above maps an element $\phi \in Diff(S^1)$ to a differential 1-form on the tangent space? Not quite, because it does not map the entire object ϕ, but its individual values. Stated precisely, $d\phi(\tau)$ is not a one-form, but is a one-form-valued function on the circle. It returns a one-form for each value of τ on the circle. Thus, if we really want to map $\phi \in Diff(S^1)$ to a differential one-form mapping tangent vectors to complex numbers, we can integrate $\int d\phi(\tau)$. This and similar integrals produce honest differential one-forms, independent of τ on the circle.

The takeaway from the above is that if we want to use ϕ as a coordinate on the coadjoint orbit, we will need to write the symplectic 2-form ω, which acts on the tangent space, in terms of integrals

$$\omega = \int_0^{2\pi} d\tau \left(G(\phi, \phi', \dots)\ d\phi(\tau) \wedge d\phi'(\tau) + F(\phi, \phi', \dots)\ d\phi(\tau) \wedge d\phi''(\tau) + \text{ other terms } \right). \tag{61}$$

We have introduced a natural anti-commuting wedge product, and the notation $d\phi'(\tau)$. At first glance, this notation does not seem well-defined mathematically. We defined d as a linear map taking elements $\phi \in Diff(S^1)$ to one-forms on the tangent space to the coadjoint orbit. Generally, if $\phi \in Diff(S^1)$, then ϕ' is not an element of $Diff(S^1)$, so the operation $d\phi'$ does not make sense. Recall that we defined d to not act on the τ coordinate, and thus commute with ∂_τ. We thus define $d\phi'(\tau) = \partial_\tau d\phi(\tau)$, which is well-defined. The formula for the symplectic form ω can be found in [12] to be

$$\omega = \int_0^{2\pi} \left(\frac{d\phi'(\tau) \wedge d\phi''(\tau)}{\phi'^2(\tau)} - d\phi(\tau) \wedge d\phi'(\tau) \right). \tag{62}$$

To get more comfortable with this notation-heavy definition, let us carry out a computation with it. This proposed symplectic form is not in fact the symplectic form that we want, because it has zero modes, and these zero modes will be removed by taking a quotient by an $\mathrm{SL}(2,\mathbb{R})$ action. Let us take ϕ in a neighborhood of the identity

$$\phi(\tau) = \tau + \sum_{n \in \mathbb{Z}} u_n L_n(\tau). \tag{63}$$

This gives us access to the tangent space spanned by the tangent vector $\{L_n\}_{n\in\mathbb{Z}}$. Then to evaluate ω, which is a 2-form on the tangent space to the identity, we have to compute $\phi'(\tau), d\phi(\tau), d\phi'(\tau)$ etc at $u_n = 0$, since we are working in the tangent space to the identity

$\phi(\tau) = \tau$. Let us evaluate

$$\begin{aligned} d\phi(\tau) &= \sum_{n\in\mathbb{Z}} L_n(\tau)\ du_n, \\ d\phi'(\tau) &= \sum_{n\in\mathbb{Z}} (in)L_n(\tau)\ du_n, \\ d\phi''(\tau) &= \sum_{n\in\mathbb{Z}} (in)^2 L_n(\tau)\ du_n, \\ \phi'(\tau)|_{u_n=0} &= 1. \end{aligned} \tag{64}$$

Now with these in hand let us evaluate ω as a 2-form on the tangent space to the identity

$$\begin{aligned} \omega &= \sum_{n,m} \int_0^{2\pi} d\tau \left((in)(im)^2 (i)^2 e^{i(m+n)\tau}\ du_n \wedge du_m + (im) e^{i(m+n)\tau}\ du_n \wedge du_m \right), \\ &= 2\pi i \sum_n (n^3 - n)\ du_n \wedge du_{-m}, \end{aligned} \tag{65}$$

where we used $\int_0^{2\pi} e^{i(m+n)\tau} = 2\pi\delta_{n,-m}$. We thus see that the form ω is not a symplectic form yet because it is degenerate. It has three zero modes corresponding to $L_0, L_{\pm 1}$. These generators generate an $SL(2,\mathbb{R})$ subgroup of $Diff(S^1)$. Thus once we mod out by this $\mathrm{SL}(2,\mathbb{R})$, ω becomes non-degenerate. The other requirement for a symplectic form is closure. Let us rewrite (62)

$$\begin{aligned} \partial_\tau \left(\frac{d\phi'(\tau)}{\phi'(\tau)} \right) &= \frac{d\phi''(\tau)}{\phi'(\tau)} - \frac{\phi''(\tau) d\phi'(\tau)}{\phi'(\tau)^2}, \\ \Rightarrow \frac{d\phi'(\tau)}{\phi'(\tau)} \wedge \partial_\tau \left(\frac{d\phi'(\tau)}{\phi'(\tau)} \right) &= \frac{d\phi'(\tau) \wedge d\phi''(\tau)}{\phi'(\tau)^2}, \end{aligned} \tag{66}$$

since $d\phi' \wedge d\phi' = 0$. In addition, we have the familiar-looking identity

$$d \log \phi'(\tau) = \frac{d\phi'(\tau)}{\phi'(\tau)}. \tag{67}$$

Using the above, we can rewrite the symplectic form as

$$\omega = \int_0^{2\pi} d\tau \left(d\log\phi'(\tau) \wedge \partial_\tau \left(d\log\phi'(\tau) \right) - d\phi(\tau) \wedge d\phi'(\tau) \right). \tag{68}$$

We have thus managed to write ω entirely in terms of exact forms, and it is thus closed. So, once we remove the zero modes corresponding to $\mathrm{SL}(2,\mathbb{R})$ generators, we will have a symplectic 2-form: A closed, non-degenerate 2-form. Recall that when we abstractly constructed the symplectic form in (44), we demanded the additional feature of invariance under the group action, or its infinitesimal version, the adjoint action. To be specific, let $\alpha \in Diff(S^1)$. Then α acts on $Diff(S^1)$ via

$$\alpha : \phi(\tau) \to \phi(\alpha(\tau)). \tag{69}$$

This is the so-called left-group action. The infinitesimal version is when α is near the identity, then the group action becomes

$$\begin{aligned} \phi(\tau) &\to \phi(\tau + \alpha(\tau)), \\ &= \phi(\tau) + \alpha(\tau)\phi'(\tau), \\ \Rightarrow \delta\phi(\tau) &= \alpha(\tau)\phi'(\tau). \end{aligned} \tag{70}$$

So each adjoint vector (infinitesimal diffeomorphism) defines a flow on $Diff(S^1)$. The tangent vector field to this flow we call V_α is defined by $V_\alpha\phi = \alpha\phi'$. A Hamiltonian corresponding to this flow satisfies

$$i_{V_\alpha}\omega = dH_\alpha. \tag{71}$$

Let us try to understand this equation. ω is a two-form and thus takes two arguments. Each argument is a tangent vector, which we think of as an adjoint vector. The contraction operation i_{V_α} returns ω with $V_\alpha\phi$ as the first argument, thus giving us $\omega(V_\alpha\phi, \cdot)$. This is a one-form, taking in one adjoint vector. Finding a Hamiltonian function implies that this one-form is exact. Our goal is to show that the $U(1)$ flow corresponding to rigid time-translations on S^1 is generated by the Schwarzian action. Concretely, the contraction $i_{V_\alpha}\omega$ is computed by replacing one copy of $\delta\phi$ in the wedge product by $V_\alpha\phi = \alpha(\tau)\phi'(\tau)$. We can also give it to ω as the second argument, but due to anti-symmetry of ω this comes at the cost of a minus sign. So using (68) let us compute

$$\begin{aligned} i_{V_\alpha}\omega &= \int_0^{2\pi} d\tau \left(\frac{1}{\phi'(\tau)} \left(\alpha(\tau)\phi'(\tau)\right)' \ \partial_\tau \left(d \log \phi'(\tau)\right) - \alpha(\tau)\phi'(\tau) d\phi'(\tau) \right), \\ &= \int_0^{2\pi} d\tau \left(\alpha'(\tau) + \alpha(\tau)\frac{\phi''(\tau)}{\phi'(\tau)} \right) \partial_\tau d \log \phi'(\tau) - \frac{1}{2}\alpha(\tau) d\left(\phi'(\tau)^2\right). \end{aligned} \tag{72}$$

Now, we will use that ∂_τ and d commute to write $\partial_\tau d \log \phi'(\tau) = d\, \partial_\tau \log \phi'(\tau) = d(\phi''(\tau)/\phi'(\tau))$. Let us integrate by parts

$$\int_0^{2\pi} d\tau \ \alpha'(\tau) \ d\left(\frac{\phi''(\tau)}{\phi'(\tau)}\right) = -\int_0^{2\pi} d\tau \ \alpha(\tau) \ d\left(\frac{\phi'''(\tau)}{\phi'(\tau)} - \frac{\phi''(\tau)^2}{\phi'(\tau)^2}\right). \tag{73}$$

$$\int_0^{2\pi} d\tau\ \alpha(\tau) \left(\frac{\phi''(\tau)}{\phi'(\tau)}\right) d\left(\frac{\phi''(\tau)}{\phi'(\tau)}\right) = \int_0^{2\pi} d\tau\ \frac{1}{2}\alpha(\tau) d\left(\frac{\phi''(\tau)}{\phi'(\tau)}\right)^2. \tag{74}$$

Putting the pieces together, we get

$$\begin{aligned} i_{V_\alpha}\omega &= \int_0^{2\pi} d\tau\ \alpha(\tau) d\left(\frac{3}{2}\left(\frac{\phi''(\tau)}{\phi'(\tau)}\right)^2 - \frac{\phi'''(\tau)}{\phi'(\tau)} - \frac{1}{2}\phi'(\tau)^2\right), \\ &= -\int_0^{2\pi} d\tau\ \alpha(\tau)\ d\mathrm{Sch}\left(\tan\frac{\phi(\tau)}{2}, \tau\right), \\ \Rightarrow H_\alpha &= -\int_0^{2\pi} d\tau\ \alpha(\tau)\ \mathrm{Sch}\left(\tan\frac{\phi(\tau)}{2}, \tau\right), \end{aligned} \tag{75}$$

where in the last line we used $i_{V_\alpha}\omega = dH_\alpha$. If we set $\alpha(\tau) = 1$, corresponding to rigid time-translations of the circle, we get nothing but the Schwarzian action. We have thus verified the two conditions required for the application of fermionic localization: the space we are integrating over is a symplectic manifold, and the Schwarzian action generates a time-translation symmetry on this space via Poisson brackets. Incidentally, having shown $i_{V_\alpha}\omega = dH_\alpha$, we immediately conclude

$$d\left(i_{V_\alpha}\omega\right) = 0, \tag{76}$$

which for a closed form such as ω means it is invariant under the flow V_α.

3.5 The Integration Measure

One thing we have thus far swept under the rug, especially in the discussion around (28) is the integration measure. Once we change variables from $\phi \to x = \log\phi'$, the measure picks up a Jacobian factor for the change of coordinates. We expand a bit on that here, following Appendix B of [15]. The integration measure needs to be such that the path integral is invariant under the $Diff(S^1)$ group action, either the left action or the right action. Here we take the right action, where $g = f \cdot h = h^{-1}(f(\tau))$. The inverse is defined as the inverse function, which for $f \in Diff(S^1)$ makes sense, since they are monotonic increasing. Then, writing $g = f \cdot h$ for fixed f and h dependent on each group element g, we get

$$
\begin{aligned}
Z &= \int \mu[g]\mathcal{D}g \; e^{-S[g]}, \\
&= \int \mu[f\cdot h] \|\delta g/\delta f\|_{g=f\cdot h} \mathcal{D}f \; e^{-S[f\cdot h]}, \\
&= \int \mu[f\cdot h] \|\delta g/\delta f\|_{g=f\cdot h} \mathcal{D}f \; e^{-S[f]}, \\
&= \int \mu[f]\mathcal{D}f \; e^{-S[f]}.
\end{aligned} \tag{77}
$$

In going to the third equation we used the invariance of the action under the right action of $Diff(S^1)$, and in the last line, we demanded the entire path integral be invariant under the group action. The double bar notation indicates a determinant, since we have a Jacobian matrix of change of variables. This equation places a restriction on the measure that we now exploit. We have found that $\mu[f] = \mu[f\cdot h]\|\delta g/\delta f\|_{g=f\cdot h}$ for *any* h. Thus, we can set $h = f$, so that $g(\tau) = f\cdot h = h^{-1}(f(\tau)) = \tau$ is the identity function. Then, $\mu[f] = \mu[1] \; \|\delta g/\delta f\|_{g=f\cdot h=1}$. To evaluate the Jacobian matrix, we first recall that $h^{-1}(h(\tau)) = \tau \Rightarrow 1 = (h^{-1}(h(\tau))'h'(\tau)$, so that

$$
\begin{aligned}
\left(\frac{\delta g(\tau)}{\delta f(\tau')}\right)_{g=f\cdot h=1} &= \frac{\delta g(\tau)}{\delta h(\tau)}\frac{\delta h(\tau)}{\delta f(\tau')} \\
&= \frac{\delta g(\tau)}{\delta h(\tau)}\frac{\delta h(\tau)}{\delta f(\tau')} \\
&= \frac{1}{h'(\tau)}\delta(\tau-\tau') \\
&= \frac{1}{f'(\tau)}\delta(\tau-\tau'),
\end{aligned} \tag{78}
$$

where we recall we are evaluating at $h(\tau) = f(\tau)$. The Jacobian matrix is therefore diagonal, and the measure, which is just its determinant up to a constant factor, is thus given by

$$
\mu[f] = \mu[1] \; \prod_f \frac{1}{f'(\tau)}. \tag{79}
$$

Interestingly, if we perform the change of variables $\phi(\tau) = \log f'(\tau)$ or $f(\tau) = \int^{\tau} e^{\phi(\tau')}\, d\tau'$, which again makes sense because f is monotonic increasing, we get an integration measure

$$\begin{aligned} \mu[\phi] &= \mu[f] \|\delta f(\tau)/\delta\phi(\tau')\|, \\ \frac{\delta f(\tau)}{\delta\phi(\tau')} &= e^{\phi(\tau)} = f'(\tau), \\ \Rightarrow \mu[\phi] &= \mu[f] \prod f'(\tau), \\ &= \mu[1]. \end{aligned} \tag{80}$$

So we get a flat measure in the ϕ variable that is simply $\mathcal{D}\phi$. Our discussion thus far has been for $Diff(S^1)$, but for $Diff(S^1)/SL(2,\mathbb{R})$, we gauge away transformations generated by the three generators $L_{\pm}, L_0$, and in the case of $U(1)$, we have to gauge away rigid rotations of S^1.

3.6 One-Loop Answer and Exactness

Recall that we interpret $\psi(\tau) \equiv d\phi(\tau)$ as an anti-commuting, fermionic variable in one dimension. The path integral for the Schwarzian theory is an integral over the symplectic manifold $Diff(S^1)/SL(2,\mathbb{R})$, with coordinates given by $\phi(\tau)$. In general, the measure is not simply $\mathcal{D}\phi(\tau)$, but that times the volume element of the symplectic manifold. The volume element of the symplectic manifold is given by the square root of the determinant of the symplectic form, which is known as the Pfaffian, since it is an anti-symmetric matrix. This Pfaffian is only non-zero in even dimensions and symplectic manifolds are even-dimensional. We recall the following relations:

$$\begin{aligned} \int d^n x \exp\left(-\frac{1}{2}\vec{x}^T M \vec{x}\right) &\sim 1/\sqrt{\det M}, \\ \int d^n \psi \exp\left(\frac{1}{2}\vec{\psi}^T M \vec{\psi}\right) &\sim \det M. \end{aligned} \tag{81}$$

We can use the latter to write

$$\begin{aligned} Z(g) &= \int \frac{d\mu[\phi]}{SL(2,\mathbb{R})} \exp\left(-\frac{1}{2g^2}\int_0^{2\pi} d\tau \,\left(\frac{\phi''^2}{\phi'^2} - \phi'^2\right)\right) \\ &= \int \frac{\mathcal{D}\phi}{SL(2,\mathbb{R})} \sqrt{\det\omega} \exp\left(-\frac{H[\phi]}{2g^2}\right) \\ &= \int \frac{\mathcal{D}\phi\mathcal{D}\psi}{SL(2,\mathbb{R})} \exp\left(\frac{1}{2g^2} H[\phi] + \frac{1}{2}\omega_{ij}\psi^i\psi^j\right). \end{aligned} \tag{82}$$

We now turn back to equation (62) while replacing $d\phi(\tau) \to \psi(\tau)$

$$\begin{aligned} \omega = \omega_{ij}\psi^i\psi^j &= \int_0^{2\pi} d\tau \left(\frac{\psi'\psi''}{\phi'^2} - \psi\psi'\right) \\ \Rightarrow Z(g) &= \int \frac{\mathcal{D}\phi\mathcal{D}\psi}{SL(2,\mathbb{R})}\ \exp\left(-\frac{1}{2}\int_0^{2\pi}\left(\frac{\psi''\psi'}{\phi'^2} - \psi'\psi + \frac{\phi''^2}{g^2\phi'^2} - \frac{\phi'^2}{g^2}\right)\right). \end{aligned} \tag{83}$$

So now, we carry out the calculation of the partition function to first order in perturbation theory, and the Duistermaat-Heckman formula of supersymmetric localization will tell us that this answer is in fact exact. The following computation is based on [16]. First, we write $\phi(\tau) = \tau + g\epsilon(\tau)$. Since ψ wraps once around the circle, under this condition is a periodic variable. Then, we need to impose some conditions to gauge-fix:

$$\begin{aligned} \int d\tau\ \epsilon(\tau) &= \int d\tau\ e^{\pm i\tau}\epsilon(\tau) = 0, \\ \int d\tau\ \psi(\tau) &= \int d\tau\ e^{\pm i\tau}\psi(\tau) = 0, \end{aligned} \tag{84}$$

which is the statement that the Fourier modes $\epsilon_0, \epsilon_{\pm1}$ and $\psi_0, \psi_{\pm1}$ all vanish. Under these gauge conditions, the unique classical solution that wraps once around the unit circle is $\phi(\tau) = \tau$ or $\epsilon = 0$. The action of this classical solution is $I = \pi/g^2$. Then, we expand to compute the one-loop determinant

$$\begin{aligned} I &= \frac{\pi}{g^2} - \frac{1}{2}\int_0^{2\pi}\left(\frac{1}{g^2}\frac{g^2\epsilon''^2}{(1+g\epsilon')^2} - \frac{1}{g^2}(1+g\epsilon')^2 + \frac{\psi''\psi'}{(1+\epsilon' g)^2} - \psi'\psi\right), \\ &= \frac{\pi}{g^2} - \frac{1}{2}\int_0^{2\pi}\left(\epsilon''^2(1-2g\epsilon'+3g^2\epsilon'^2) - \epsilon'^2 + \psi''\psi'(1-2g\epsilon'+3g^2\epsilon'^2) - \psi'\psi\right), \\ &= \frac{\pi}{g^2} - \frac{1}{2}\int_0^{2\pi}\left(\epsilon''^2 - \epsilon'^2 + \psi''\psi' - \psi'\psi + g(-2\epsilon'\epsilon''^2 - 2\psi''\psi') + g^2(3\epsilon'^2 + 3\epsilon''^2\epsilon'^2) + O(g^3)\right), \end{aligned} \tag{85}$$

The quadratic part of the action, which controls the fluctuation determinant, is independent of g. Thus one might expect that the fluctuation determinant be independent of g. However, when performing a change of variables from $\phi \to g\epsilon$, we get $d\phi = gd\epsilon$, so a factor of g for each Fourier mode. However, we are fixing three Fourier modes, $\epsilon_0, \epsilon_{\pm1}$ to be zero, and thus we do not integrate over them, so instead of getting an answer which is g-independent, we get

$$Z(g)_{one-loop} = \frac{C}{g^3}\exp\left(\frac{\pi}{g^2}\right), \tag{86}$$

where the exponent is the classical answer, the pre-factor is the one-loop determinant, and C is some unimportant number. Since we proved the Schwarzian theory satisfies the criteria for the DH theorem, which are

1. The path integral is over a symplectic manifold, which we proved the co-adjoint orbit $Diff(S^1)/SL(2,\mathbb{R})$ to be.

2. The Hamiltonian generates via Poisson brackets a compact $U(1)$ symmetry of the symplectic manifold.

We can invoke the DH theorem and show that the one-loop answer is in fact exact. This answer can be immediately extended to the Schwarzian theory on the symplectic manifold $Diff(S^1)/SL(2,\mathbb{R})$, for which we only gauge-fix $\epsilon_0 = \psi_0 = 0$ so we only exclude one Fourier mode instead of three to get

$$Z(g)_{one-loop} = \frac{C}{g}\exp\left(\frac{\pi}{g^2}\right), \tag{87}$$

where the constant C is unimportant. We make a final remark. In the above, we spoke interchangeably of a functional integral of $d\phi(\tau)$ and an integral over Fourier modes. This is in fact quite common. A functional integral, which is a product of integrals $\prod_\tau \int d\phi(\tau)$ is quite ill-defined and is a product over a continuum. Conceptually, it is clearer to trade it for a product of integrals of Fourier modes $\prod_n \int d\phi_n$ or momentum modes, which are discrete on a compact space, as opposed to forming a continuum.

We saw in equation (22) that for SYK, the coefficient of the Schwarzian action is $1/g^2 \sim N/\beta J$. The partition function $Z(g) \to Z(\beta)$ is computed in the canonical ensemble, in which we fix the temperature. We can instead compute in the micro-canonical ensemble, in which we fix the energy. To translate between them we use the Laplace transform

$$\begin{aligned} Z(\beta) &= \int_0^\infty \rho(E) e^{-\beta E}\, dE, \\ \Rightarrow \rho(E) &= \sinh 2\pi\sqrt{2CE}, \end{aligned} \tag{88}$$

where C is proportional to N/J. So we see that the density of states for the Schwarzian theory seems to be continuous. This could imply that the Schwarzian theory represents a quantum mechanical theory with a continuous spectrum. In physics, we usually encounter continuous spectra when we take a semi-classical limit and coarse grain over the quantum mechanical degrees of freedom. However, due to the one-loop exactness of the Schwarzian path integral, we should not think of this as a semi-classical computation. We should expect a discrete density of states which is a linear combination of Dirac delta spikes with positive coefficients that represent the multiplicity of each energy state. The fact that we end up with a continuous density of states could be because the Schwarzian theory does not represent a viable quantum mechanical system on its own, but could be a sector or a low-energy description, as in the SYK model.

4 Factorization Problem in JT Gravity

In this section, we will study the factorization problem, following [17]. This problem was first articulated in [18]. The problem is that gauge constraints may interfere with a theory in the bulk of a spacetime possessing a dual description on the boundary of the spacetime (via AdS/CFT correspondence [19]), especially on a boundary with disconnected components. An explicit example can be seen in Figure 4. In that example, we consider $U(1)$ electromagnetic gauge theory in AdS and AdS-Schwarzschild spacetimes. The AdS-Schwarzschild geometry, also commonly known as wormhole geometry, corresponds to an eternal black hole solution in AdS_2. This is locally AdS_2 but differs only in global properties. See [20] for details. Gauge theory comes with so-called Wilson line and Wilson loop operators, which are non-local, gauge-invariant operators. In the context of the AdS cylinder, all such gauge-invariant operators can be reconstructed, in principle, in the dual CFT on the boundary in a standard fashion that is well-described in the literature [21][22]. Similarly, in the AdS-Schwarzschild spacetime, gauge-invariant operators that are in a quadrant accessible to an observer at time-like infinity can also be reconstructed in the dual boundary CFT. However, in this spacetime geometry, a new, wormhole-threading Wilson line operator appears and with it a natural question arises: Can this operator be reconstructed in terms of CFT operators in the two boundary theories? This is a quintessential example of the factorization problem.

One stark example in which this problem manifests is in the Jackiw-Teitelboim (JT)

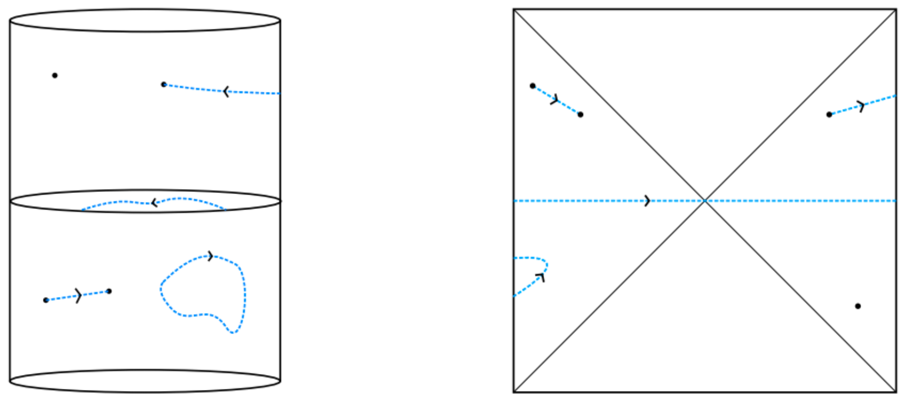

Figure 4: (Left) Electromagnetic, gauge-invariant Wilson line and Wilson loop operators in an AdS geometry. (Right) An illustration of the factorization problem in an AdS-Schwarzschild geometry, caused by the presence of wormhole-threading Wilson line operator. See any standard QFT text for a discussion on Wilson line operators. Figure taken from [23].

theory in two-dimensional spacetime:

$$\mathcal{L} = \Phi_0 R + \Phi(R+2). \tag{89}$$

This is the simplest theory of gravity in two spacetime dimensions, since Einstein-Hilbert action is topological and gives the Euler characteristic. In the following analysis, we will see some features which are quite distinct from our analysis of the Schwarzian theory in the last section. In particular

1. we will work in Lorentzian signature;
2. the Schwarzian which has occupied us for much of the previous section will not appear, due to working primarily in Lorentzian signature;
3. The $SL(2,\mathbb{R})$ subgroup, which was also crucial for analyzing the Schwarzian theory, will not appear; there will be no natural reason to single it out as a subgroup of the gauged diffeomorphism group.

This factorization problem has been encountered in another theory of gravity: 2+1-dimensional Einstein Gravity, and in fact JT gravity and 2+1-dimensional Einstein gravity are similar on a few accounts; including the presence of worm-hole solutions; a two-sided Hilbert space that does not factorize; no CFT dual; black hole entropy which is not computed by the Beckenstein-Hawking formula. Matter can be added to remedy some of these problems, but it is known that AdS_2 is particularly fragile with respect to addition of matter. We will now discuss the classical structure of JT gravity.

4.1 Definition of Jackiw-Teitelboim Gravity

Let M be a 1+1-dimensional spacetime. Then the action of the JT gravity is

$$S = \Phi_0 \left(\int_M d^2x \sqrt{-g} R + 2 \int_{\partial M} \sqrt{|\gamma|} \right) + \int_M d^2x \sqrt{-g}\ \Phi(R+2) + 2 \int_{\partial M} dt \sqrt{|\gamma|}\ \Phi(K-1). \tag{90}$$

K is the trace of the extrinsic curvature of the boundary, γ is the boundary curvature and Φ is the dilaton scalar field. The boundary term $-\int_{\partial M} \sqrt{|\gamma|}\Phi$ is a holographic re-normalization to render the energy of classical solutions obeying the boundary conditions finite. This holographic re-normalization term arises only for asymptotically AdS boundaries. Dilaton theories are often obtained in dimensional reduction, if the higher-dimensional spacetime has a tensor product structure with a compact internal manifold like a sphere. In that context, Φ_0 would be related to the size of that manifold. Upon variation of the action, we get the equations of motion

$$\begin{aligned} R + 2 &= 0, \\ (\nabla_\mu \nabla_\nu - g_{\mu\nu})\Phi &= 0, \end{aligned} \tag{91}$$

and the boundary pieces

$$\delta S_{bnd} = \int_{\partial M} dx \sqrt{|\gamma|} \left(2(K-1)\delta\Phi + (r^\mu \nabla_\mu \Phi - \Phi)\gamma^{\alpha\beta}\delta\gamma_{\alpha\beta} \right), \tag{92}$$

where r^μ is the vector normal to the boundary, which arises because K is the trace of the extrinsic curvature

$$K = \gamma^{\mu\nu} \nabla_\mu r_\nu. \tag{93}$$

So obeying the classical equations of motion makes the bulk contribution to the variation of the action vanish. We also need to impose suitable boundary conditions such that for classical solutions obeying these boundary conditions, the boundary contribution also vanishes. These can be taken to be

$$\begin{aligned} \Phi|_{\partial M} &= \phi_b r_c, \\ \gamma_{tt}|_{\partial M} &= r_c^2. \end{aligned} \tag{94}$$

We have imagined imposing some kind of IR cutoff, whereby the boundary is located at $r = r_c$ and corresponds to a time-like surface, thus having a one-dimensional metric γ_{tt}. These boundary conditions are only preserved by diffeomorphisms that leave the boundary metric invariant $\gamma_{tt} = r_c^2$. Thus, they are (possibly trivial) time translations on the boundary. We can thus look for a boundary Hamiltonian which generates these time translations. Looking at equation (92), we see that the tensor which is sourcing the linear variation $\delta\gamma_{\mu\nu}$ of the boundary metric is

$$\begin{aligned} T^{\mu\nu} &= \frac{2}{\sqrt{|\gamma|}} \frac{\delta S}{\delta \gamma_{\mu\nu}} \\ &= 2\gamma^{\mu\nu}(r^\sigma \nabla_\sigma \Phi - \Phi)|_{\partial M}. \end{aligned} \tag{95}$$

We instead work with $T^{\mu\nu}_{CFT}$ which is the linear response to variations in the "CFT" metric $\gamma^{CFT}_{\mu\nu} = \gamma_{\mu\nu}/r_c^2$, which leads to $T^{\mu\nu}_{CFT} = r_c^3 T^{\mu\nu}$. This is in order to get a finite, $O(1)$ stress tensor in the limit $r_c \to \infty$. If we work on a spacetime with two disconnected components, like the wormhole geometry, we would have such a stress tensor for each boundary component generating time translations on the boundary, and the full Hamiltonian is the sum. These time translations are genuine asymptotic symmetries which act non-trivially on the phase space of classical solutions.

The classical solutions obey the equations (91). The first equation tells us our space is constantly negatively curved, so that is it a piece of AdS_2. AdS_2 can be embedded in Minkowski$(1,2)$ via

$$1 = T_1^2 + T_2^2 - X^2. \tag{96}$$

To write down the classical solutions to JT gravity explicitly, we introduce coordinates

$$\begin{aligned} T_1 &= \sqrt{1+x^2}\cos\tau, \\ T_2 &= \sqrt{1+x^2}\sin\tau, \\ X &= x, \end{aligned} \tag{97}$$

so that

$$\begin{aligned} dT_1 &= \frac{x}{\sqrt{1+x^2}} \cos\tau \ dx - \sqrt{1+x^2} \sin\tau \ d\tau + dx^2, \\ dT_2 &= \frac{x}{\sqrt{1+x^2}} \sin\tau \ dx + \sqrt{1+x^2} \cos\tau \ d\tau, \\ dX &= dx. \end{aligned} \tag{98}$$

The AdS_2 metric in these coordinates becomes

$$ds^2 = -(1+x^2)d\tau^2 + \frac{1}{1+x^2}dx^2, \tag{99}$$

and the solution for the dilaton is

$$\Phi = \Phi_h \sqrt{1+x^2} \cos\tau = \Phi_h T_1, \tag{100}$$

where Φ_h is a constant. We can alternatively work in Schwarzschild coordinates

$$\begin{aligned} T_1 &= \frac{r}{r_s}, \\ T_2 &= \frac{1}{r_s}\sqrt{r^2 - r_s^2} \ \sinh t, \\ X &= \frac{1}{r_s}\sqrt{r^2 - r_s^2} \ \cosh t, \end{aligned} \tag{101}$$

so that

$$\begin{aligned} dT_1 &= \frac{dr}{r_s}, \\ dT_2 &= \frac{r dr}{r_s\sqrt{r^2 - r_s^2}} \sinh t + \frac{1}{r_s}\sqrt{r^2 - r_s^2} \ \cosh t \ dt, \\ dX &= \frac{r dr}{r_s\sqrt{r^2 - r_s^2}} \cosh t + \frac{1}{r_s}\sqrt{r^2 - r_s^2} \ \sinh t \ dt. \end{aligned} \tag{102}$$

The AdS_2 metric and the dilaton become in these coordinates

$$\begin{aligned} ds^2 &= -\frac{dr^2}{r_s^2} + \frac{r^2 dr^2}{r_s^2(r^2 - r_s^2)} - \frac{1}{r_s^2}(r^2 - r_s^2)dt^2 \\ &= \frac{dr^2}{r^2 - r_s^2} - \frac{r^2 - r_s^2}{r_s^2}dt^2, \\ \Phi &= \frac{\Phi_h}{r_s} r. \end{aligned} \tag{103}$$

In Schwarzschild coordinates, constant r slices correspond to constant Φ slices, and thus the IR-boundary simply sits at $r = r_c$, so that

$$\begin{aligned} \Phi|_{\partial M} &= \phi_b r_c, \\ \Rightarrow r_s &= \frac{\Phi_h}{\phi_b}. \end{aligned} \tag{104}$$

We can easily compute the boundary stress tensor at $r = r_c$, keeping in mind the boundary conditions (94)

$$
\begin{aligned}
T^{tt} &= 2\gamma^{tt} r_c^3 (r^\sigma \nabla_\sigma \Phi - \Phi), \\
r^\mu &= (\sqrt{r^2 - r_s^2}, 0)^\mu, \\
\nabla_\mu \Phi &= \partial_\mu \Phi = (\Phi_h / r_s, 0)^\mu, \\
T^{tt} &= 2\gamma^{tt} r_c^3 \left(r_c \sqrt{1 - (r_s/r_c)^2} \frac{\Phi_h}{r_s} - \frac{\Phi_h}{r_s} r_c \right) \\
&= -2 \frac{r_c^3}{r_c^2} \frac{\Phi_h}{r_s} \left(r_c \left(1 - \frac{r_s^2}{2r_c^2} + \dots \right) - r_c \right) \\
&= \Phi_h r_s \\
&= \frac{\Phi_h^2}{\phi_b},
\end{aligned}
\tag{105}
$$

so that the boundary Hamiltonians on the left and right boundaries are $H_L = H_R = \Phi_h^2/\phi_b$. These are the left and right time-like boundaries of the asymptotically AdS regions of the maximally extended AdS-Schwarzschild geometry, and H_L (resp. H_R) generate time translations on the left (resp. right) boundary. Any physical, non-gauge diffeomorphism of JT gravity has to asymptote to a non-trivial time translation on the boundary for it to (a) be an allowable diffeomorphism that respects the boundary conditions (b) generate a non-trivial action on the phase space of classical solutions.

4.2 The Phase Space of Classical Solutions

We have obtained a one-parameter family of classical solutions for JT gravity, parameterized by Φ_h, which is the value of the dilaton at the horizon where $r = r_s$. So, it looks like our phase space of classical solutions is one-dimensional. This is problematic. For example, when solving Newtonian dynamics, a classical solution is uniquely specified by $x(0)$ and $p(0)$, so that setting initial conditions is nothing but picking a specific classical solution in the phase space. Thus, phase space is even-dimensional. We have thus missed some other parameters. Recall that H_L and H_R generate a non-trivial action on the classical phase space. Thus, given a classical solution, acting with boundary time translations generated by H_L or

H_R gives us another *physically distinct* classical solution, *without changing* the value of the dilaton at the horizon, Φ_H. Thus, H_L and H_R represent two other dimensions in our classical phase space that we were seeking, but in fact they are not independent, because $H_L = H_R$. Thus, we get only one additional parameter, leading to a two-dimensional phase space, and conventionally, we generate time translations with $H = H_L + H_R$. Let us call the canonically conjugate coordinate δ, so our phase space is parameterized by (Φ_h, δ). Since $H_L - H_R = 0$, if we take a time-slice with endpoints on the two disconnected boundaries of the wormhole solution, and we translate each endpoint by the same amount but in opposite directions, we get an equivalent time-slice, and thus an equivalent solution to the JT gravity classical equations of motion. However, translating both endpoints of the time-slice by the same amount in the same direction generates a new, distinct solution. So, by gauge symmetry we can always start with a time-slice with $t_L = t_R$, and we easily see that $\delta = (t_L + t_R)/2$, since we start at the same boundary time, and time-evolve in the same amount on each side.

Our Hamiltonian system is quite simple

$$\begin{aligned} \dot{\delta} &= 1, \\ H &= \frac{2\Phi_h^2}{\phi_b}, \\ \dot{\Phi}_h &= 0. \end{aligned} \tag{106}$$

The range of these phase space coordinates is $0 < \Phi_h$ and $\delta \in \mathbb{R}$. The former is imposed because in string-theoretic and dimensional reduction constructions, the dilaton controls the size of an internal compact manifold, so it has to be positive [24]. We can calculate the symplectic form via

$$\dot{x}^a = (\omega^{-1})^{ba} \partial_b H. \tag{107}$$

Because of anti-symmetry, there is a single independent component of the symplectic form that we need to compute

$$
\begin{aligned}
\dot{\delta} &= 1 = (\omega^{-1})^{\Phi\delta}\partial_{\Phi_h} H, \\
&= (\omega^{-1})^{\Phi\delta}\frac{4\Phi_h}{\phi_b}, \\
\Rightarrow 1 &= (\omega^{\delta\Phi})^{-1}\frac{4\Phi_h}{\phi_b}, \\
\Rightarrow \omega^{\delta\Phi} &= \frac{4\Phi_h}{\phi_b}, \\
\Rightarrow \omega &= \frac{4\Phi_h}{\phi_b} d\delta \wedge d\Phi_h, \\
&= d\delta \wedge dH,
\end{aligned}
\tag{108}
$$

where in the last line we recognized $dH = 4(\Phi_h/\phi_b)d\Phi_h$. So, as we had recognized earlier, δ and H are canonically conjugate, which makes sense given we have defined δ as the time evolved by the full canonical Hamiltonian $H = H_L + H_R$.

Alternatively, we can introduce another pair of canonically conjugate coordinates. We can take the re-normalized geodesic distance between the two endpoints of a time-slice, where prior to computing the distance, we use the symmetry generated by the trivial operator on phase space $H_L - H_R$ to translate its endpoints so they are at $t_L - t_R = 0$. We need to re-normalize the distance because the asymptotically AdS_2 is infinitely far away in geodesic distance, so it would give an infinite answer. Instead we work with

$$
L = L_{bare} - 2\log(2\Phi|_{\partial M}), \tag{109}
$$

where L_{bare} is computed along a constant-time slice

$$
\begin{aligned}
L_{bare} &= \int_{-x_b}^{x_b} \frac{dx}{\sqrt{1+x^2}}, \\
&= 2\ln(x + \sqrt{1+x^2}),
\end{aligned}
\tag{110}
$$

and x_b (b for boundary) is the point at which $\Phi|_{\partial M} = \Phi_h\sqrt{1+x_b^2}\cos\tau(t_L)$. Notice that we wrote $\tau(t_L)$ and not t_L, because we recall that it is only in global coordinates that surfaces of constant r are surfaces of constant Φ, and thus the boundary is at constant r and t is the boundary time. It is t_L and t_R that are canonically conjugate to H_L and H_R, not τ. A surface of constant τ does not have constant Φ, and thus τ is not the boundary time. This raises the question of how to express τ in terms of t at the boundary. If we take $x \to \infty$ and

$r \to \infty$ in global and Schwarzschild coordinates, respectively, which take us outward to the boundary, we get:

$$\begin{aligned} \frac{T_1}{X} &= \cos\tau, \\ \frac{T_1}{X} &= \frac{1}{\cosh t}, \\ \Rightarrow \cos\tau &= \frac{1}{\cosh t_L} = \frac{1}{\cosh t_R}. \text{ (asymptotic boundary only)} \end{aligned} \tag{111}$$

This relation is not general, but only holds at the boundary, which is all we need to compute the re-normalized geodesic distance. So

$$\begin{aligned} \sqrt{1+x_b^2} &= \frac{\phi_b r_c}{\Phi_h}\cosh(\delta), \\ \lim_{r_c\to\infty} x_b &= \frac{\phi_b r_c}{\Phi_h}\cosh\delta. \end{aligned} \tag{112}$$

Substituting into L_{bare} while taking $r_c \to \infty$

$$\begin{aligned} L &= 2\ln\left(2\frac{\phi_b r_c}{\Phi_h}\cosh\delta\right) - 2\log(\phi_b r_c), \\ &= 2\ln\left(2\frac{\cosh\delta}{\Phi_h}\right). \end{aligned} \tag{113}$$

The canonically conjugate variable to the re-normalized geodesic length is given by

$$P = \frac{\Phi_h^2}{\phi_b}\left(\tanh\delta + \frac{\delta}{\cosh^2\delta}\right). \tag{114}$$

It is straightforward to verify that

$$\begin{aligned} dL &= 2\tanh\delta\; d\delta - 2\frac{d\Phi_h}{\Phi_h}, \\ dP &= \frac{2\Phi_h\; d\Phi_h}{\phi_b}\left(\tanh\delta + \frac{\delta}{\cosh^2\delta}\right) + \frac{\Phi_h^2}{\phi_b}\left(1-\delta\tanh\delta\right)(2\;\mathrm{sech}^2\delta)d\delta, \\ dL\wedge dP &= \frac{4\Phi_h}{\phi_b}\left(\tanh^2\delta + \delta\tanh\delta\;\mathrm{sech}^2\delta\right)d\delta\wedge d\Phi_h + \frac{4\Phi_h}{\phi_b}(1-\delta\tanh\delta)\;\mathrm{sech}^2\delta\; d\delta\wedge d\Phi_h, \\ &= \frac{4\Phi_h}{\phi_b}(\tanh^2\delta + \;\mathrm{sech}^2\delta)d\delta\wedge d\Phi_h, \\ &= \frac{4\Phi_h}{\phi_b}d\delta\wedge d\Phi_h, \\ &= \omega, \end{aligned} \tag{115}$$

so these are indeed a canonically conjugate pair. L_{bare} is positive definite, but the re-normalized length is in general not, $L \in \mathbb{R}$, and so is $P \in \mathbb{R}$. This pair of coordinates, although considerably more complicated, can be easier to use since their flows are not bounded; recall that Φ_h was bounded to $(0, \infty)$. In these coordinates, the Hamiltonian becomes

$$\begin{aligned} H &= \frac{2e^{-L}}{\phi_b} \cosh^2(f^{-1}(P\phi_b e^{-L})), \\ f(x) &= x + \sinh x \cosh x, \end{aligned} \tag{116}$$

and f^{-1} is the inverse function, defined by $f^{-1}(f(x)) = x$, which we know exists because f is odd-parity. We now move from the world of phase space of classical solutions to the quantum Hilbert space

4.3 Quantum Hilbert Space

We can do the naive thing and define the quantum Hilbert space as spanned by delta-normalized energy eigenstates, so that $H|E\rangle = E|E\rangle$, with $E > 0$. The requirement $E > 0$ can obviously be shifted arbitrarily, so long as the energy is bounded below. We then define the time-shift operator

$$\delta = i\frac{\partial}{\partial E}. \tag{117}$$

By the usual arguments, similar to writing $p = -i\partial_x$, this time-shift is Hermitian. However, this *cannot* be self-adjoint. If δ were self-adjoint, then $e^{ia\delta}$ is a 1-parameter family of unitary operators, and thus $e^{ia\delta}|E\rangle$ are delta-normalized energy eigenstates with shifted energy $E \to E + a$. We would thus contradict the lower bound on energy. Thus δ is Hermitian, but not self-adjoint. The spectral theorem, which asserts the existence of an orthonormal basis for a self-adjoint operator, thus does not apply. We can instead define the quantum re-normalized geodesic length operator. It is sufficiently complicated that a complete analysis is prohibitive, but classically at least, it is better behaved than δ, because its flows do not terminate. It is thus plausible that this operator is self-adjoint. We can use the quantum version of (113) in the energy representation

$$2(\log\cosh i\partial_E)\psi_L(E) = \left(L + \ln\frac{\phi_b E}{2}\right)\psi_L(E), \tag{118}$$

where we used $\delta \to i\partial_E$ and $\langle E|\psi_L\rangle = \psi_L(E)$. This looks extremely hard to work with, given the $\cosh i\delta$. We can instead of working with energy eigenfunctions, work with eigenfunctions of the time-shift operator, defined by

$$\psi_L(\delta) = \int_{-\infty}^{\infty} \frac{dE'}{2\pi} e^{i\delta E'} \psi_L(E'). \tag{119}$$

So that in this representation, using the inverse Fourier transform, the hopeless left-hand side becomes

$$2\int_{-\infty}^{\infty} d\delta \; e^{-i\delta E} \log\cosh\delta \;\psi_L(\delta). \tag{120}$$

The integral runs from $-\infty$, and that as $\delta \to \infty$, $\log\cosh\delta \sim O(\delta)$, so we need $\psi_L(\delta) \sim O(1/\delta)$ at large δ. From Fourier analysis, we recall that smoothness and decay are dual to each other. So, if $\psi_L(E)$ is smooth at $E = 0$, then $\psi_L(\delta)$ would decay rapidly at large $|\delta|$. So, $\psi_L(E)$ must vanish smoothly at $E = 0$, since a discontinuous vanishing would lead to decay slower than $1/\delta$ for $\psi_L(\delta)$. This is the smoothness-decay duality. Our Hilbert space thus consists of L_2-normalizable functions of L.

4.4 Single Boundary Path Integral

We can compute the saddle point approximation to the Euclidean path integral for JT gravity with a single asymptotic boundary, with boundary conditions

$$\begin{aligned} \gamma_{t_E t_E} &= r_c^2, \\ \Phi|_{\partial M} &= \phi_b r_c. \end{aligned} \tag{121}$$

We recall the Euclidean action

$$-S_E = \int_M d^2x\sqrt{g}(\Phi_0 R + \Phi(R+2)) + 2\int_{\partial M} dx\sqrt{\gamma}(\Phi_0 K + \Phi(K-1)). \tag{122}$$

We will sum over geometries with a fixed topology. The Euclidean action receives contributions from many single-boundary geometries with different topologies [25]. However, these contributions are organized or weighted by $e^{(1-2g)S_0}$, where g is the genus and S_0 is the zero-temperature entropy, proportional to Φ_0. Thus, the leading-order contribution is the disk topology, with $g = 0$. We thus sum over these discs, for which the curvature is $R = -2$

constant negative, and the extrinsic curvature is

$$\begin{aligned}
K &= \gamma^{\mu\nu}\nabla_\mu r_\nu, \\
&= \gamma^{t_E t_E}\nabla_{t_E} r_{t_E}|_{\partial M}, \\
&= \gamma^{t_E t_E}(\partial_{t_E} r_{t_E} - \Gamma^{\rho}_{t_E t_E} r_\rho), \\
&= -\gamma^{t_E t_E}\Gamma^{\rho}_{t_E t_E} r_\rho, \\
&= \frac{1}{2}\gamma^{t_E t_E} g^{rr}(\partial_r g_{t_E t_E}) r_r|_{\partial M}, \\
&= \frac{1}{2r_c^2}(r_c^2 - r_s^2)\frac{2r_c}{\sqrt{r_c^2 - r_s^2}}, \\
&= \frac{\sqrt{r_c^2 - r_s^2}}{r_c}, \\
&= 1 - \frac{1}{2}\left(\frac{r_s}{r_c}\right)^2 + \dots .
\end{aligned} \tag{123}$$

The metric is the Euclidean signature metric

$$ds^2 = (r^2 - r_s^2)dt_E^2 + \frac{dr^2}{r^2 - r_s^2}. \tag{124}$$

To avoid a conical singularity at the origin, we have an inverse temperature $t_E \sim t_E + \beta$, with

$$r_s = \frac{2\pi}{\beta}. \tag{125}$$

In the saddle point approximation, we can evaluate the contribution of the disc topology to the Euclidean path integral, with $R + 2 = 0$:

$$\begin{aligned}
-S_{E,bulk} &= \Phi_0(4\pi\chi), \\
-S_{E,bnd} &= 2\Phi_0 r_c\beta + \phi_b r_c^2\beta\left(\frac{r_s}{r_c}\right)^2 = 2\Phi_0 r_c\beta + \phi_b\beta r_s^2,
\end{aligned} \tag{126}$$

where χ is the Euler characteristic of the disc. We obtain the saddle point approximation for the Euclidean path integral.

$$Z(\beta) = \int \mathcal{D}g\mathcal{D}\ \Phi e^{-S_E} = e^{4\pi\Phi_0 + 4\pi^2\phi_b/\beta}. \tag{127}$$

We can also compute some thermodynamic-looking quantities like the free energy $F(\beta) = S - \beta \langle H_L \rangle$

$$\begin{aligned} E = \langle H_L \rangle = -\partial_\beta \log Z(\beta) = \frac{4\pi^2 \phi_b}{\beta^2} = \phi_b r_s^2 = \frac{\Phi_h^2}{\phi_b}, \\ \log Z(\beta) = F(\beta) = S - \beta \langle H_L \rangle , \\ \Rightarrow S = 4\pi(\Phi_h + 4\pi\Phi_0). \end{aligned} \tag{128}$$

In the second equation, we made extensive use of $r_s = \Phi_h/\phi_b$, equation (104). We emphasize these do not yet have the interpretation of thermodynamic quantities, but are simply formulas calculated from a Euclidean path integral that resemble thermodynamic formulas for average energy, entropy and free energy. We do not yet have $Z(\beta) = \sum_{microstates} e^{-\beta H}$ from which one derives the expression for the expectation value of the energy.

4.5 Failure to Factorize

The only immediate Hilbert space interpretation of the Euclidean path integral is as the norm of the un-normalized Hartle-Hawking (HH) state. The HH state arises as follows: In the AdS/CFT correspondence, a time slice of an asymptotically AdS Lorentzian geometry, or an asymptotically AdS Euclidean geometry, is dual to a state in a CFT on the boundary of the asymptotically AdS region. The wormhole geometry was proposed in [26] to be dual to a pure entangled state in the tensor product Hilbert space and is basically an eternal black hole geometry:

$$|\psi_\beta\rangle = \frac{1}{\sqrt{Z(\beta)}} \sum_i e^{-\beta E_i/2} \, |i*\rangle_L \, |i\rangle_R \, , \tag{129}$$

where $|i\rangle_{R(L)}$ is the an energy eigen-state in the CFT on the right (left) asymptotically AdS boundary, and $Z(\beta)$ is the partition function on a boundary CFT. The Euclidean path integral computed above does not have the interpretation of a sum over microstates, each weighted by a Boltzmann factor. This interpretation can actually arise from factorization: if we assume the bulk Hilbert space can be written as a factorized Hilbert space, with one factor from each disconnected boundary component, we get that the Hartle-Hawking (or any) state can be interpreted as a sum of tensor-product states as above. So, HH states are a one-parameter family labeled by β, which generically is not constrained to be $\beta = 2\pi/r_s$, but is instead just a wave function peaked (maximally probable) at that value with some finite spread about it. The one-sided path integral is then the norm of this un-normalized thermofield double state

$$Z[\beta] = \sum_i e^{-\beta E_i}. \tag{130}$$

The question now is: Is this interpretation viable in JT gravity? Can we interpret the Euclidean path integral as a sum over microstates, weighted by a Boltzmann factor? The answer is no. The two-sided quantum Hilbert space is the Hilbert space of a single particle quantum mechanics that does not admit a factorization into left- and right-sided Hilbert spaces. Moreover, a thermofield double does not make sense, since as we have seen $H_L = H_R$ due to the absence of matter. All energy is sourced by the bifurcation horizon, hence $E_L = E_R = \Phi_H^2/\phi_b$, determined purely by the value at the horizon. Another point is that the two-sided Hilbert space, which describes essentially a complicated single-particle quantum mechanics, has a continuous energy spectrum, which would lead to a divergence. The semi-classical answer to the single-boundary Euclidean path integral is perfectly finite, so there is clear tension there. As has been remarked, due to the absence of matter and that energy is completed supplied by the bifurcation horizon $H_L = H_R$, so the addition of matter has been put forward as a possible solution to the factorization problem in JT gravity.

5 Spin Glasses

As we have discussed, the SYK model lives at the crossroads of many interesting areas of physics and mathematics. One particularly interesting area that has long befuddled physicists is spin glasses. At a basic level, the SYK model describes a system of spin-1/2 fermions interacting through their magnetic moments, similarly to the Ising model, for example. Such magnetic systems exhibit different phases, like a ferromagnetic, an anti-ferromagnetic and a paramagnetic phase (consult any standard book on statistical mechanics for reference). The spin glass phase is currently understood to be one such phase, and the purpose of the ensuing discussion is to understand its characteristics and address whether it arises in the SYK model.

We follow [27] in this chapter. To appreciate the idea of a glass one has to first think about thermodynamic equilibrium: Is an ordinary glass in thermodynamic equilibrium? Common sense would lead us to believe it is. It clearly is quiescent, but in a thermodynamic sense, it is *not* in equilibrium. There are a handful of criteria that a system has to satisfy in general in order for it to be in thermodynamic equilibrium. In general a system, such as a gas of atoms, has a micro-state, described in terms of the positions and the momenta of all the atoms. This is neither an obtainable nor a useful description, since a macroscopic system has the order of 10^{24} particles, and the particles are rapidly zipping about and colliding, so these quantities are always in flux. So a system in thermodynamic equilibrium has to be describable in terms of a couple of bulk parameters, like the volume, temperature, pressure and so on. These in addition have to satisfy the following criteria:

1. They have to be unchanging in time (quiescent).
2. They have to be homogeneous in space, so as to maximize the entropy, otherwise they would seek to homogenize to maximize the entropy.
3. The macro-state has to be describable entirely in these thermodynamic variables, and not the additional information of *history* or how the state came to be. This particular *history-independence* requirement is what the glass physical system fails to satisfy.

5.1 Glass Formation and the Glass Transition

To see how glass fails to satisfy history-independence, we have to consider the process by which a glass is formed, and contrast it with another process, crystallization. To form a crystal, we start with the liquid phase, and in an idealized process, cool the liquid or lower its temperature in a quasi-static manner. By a quasi-static manner we mean, we mean an idealization in which we lower the temperature a very small amount, knocking the system out of equilibrium; the system however possesses an ideally short *relaxation* timescale for it to make the necessary adjustments to go back to equilibrium. After having waited for time τ, the relaxation timescale, we repeat the process, perturbing the system infinitesimally.

If we follow this quasi-static cooling procedure, we encounter a phase transition at a sharp and well-defined freezing temperature T_f, as signaled by a discontinuous decrease in the entropy and a breaking of continuous translation symmetry or homogeneity down to a discrete symmetry of translations by lattice vectors. However, suppose we do not follow a quasi-static cooling trajectory but instead choose to cool rapidly. Then, the crystal or solid phase can actually be skipped altogether and the material can remain liquid well below the freezing temperature. This phase is known as the *super-cooled* liquid. This liquid clearly has higher entropy than the crystal at the same temperature, and is more disordered. However, as we continue to lower the temperature, we can extrapolate the entropy *vs.* temperature curve, and find that at sufficiently low temperatures the super-cooled liquid has lower entropy than the crystal. This is clearly paradoxical, and signals a breakdown of our extrapolation. In particular, what happens is that liquid grows more and more sluggish, and its relaxation timescale becomes longer and longer and it can no longer equilibrate. There is a *glass* temperature T_g at which the long range molecular motion ceases and the super-cooled liquid becomes a physical glass.

A glass is thus said to be *arrested* in a disordered phase, since it has become so viscous that the constituent atoms cannot execute long-range motion. So, why do we not call this a phase transition? As we said, a phase transition is characterized by a discontinuity in some physical parameters, which is not exhibited at the glass temperature. In addition, the freezing temperature for a quasi-static process is unique and well-defined. However, for a glass transition, it is dependent on the trajectory and the cooling rate. It should be noted that a glass refers generically to a system of constituents which have been arrested in a disordered phase, rather than just ordinary glass. Note we also cannot apply the usual methods of statistical mechanics to the study of arrested systems, since that usually involves a macro-state exploring many different micro-states, and then we average over these micro-states. However in an arrested system, there are no constant fluttering and collisions to smooth out or average over, so the conventional methods do not apply.

5.2 General Features of Spin Glasses

We have not defined what is meant by spin glasses, but we will discuss their general properties. One interesting property is the Kondo effect, which appeared in dilute magnetic alloys. These are usually noble metals like gold and silver, in which a magnetic substance like iron

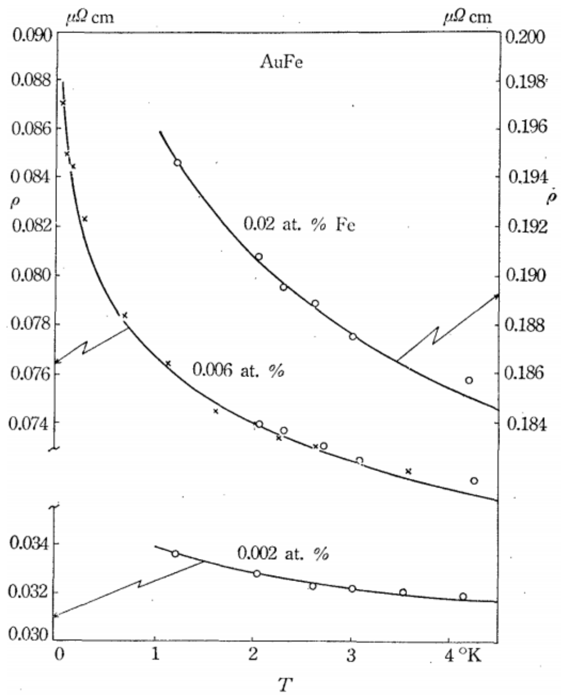

Figure 5: An increase in the resistivity of the dilute magnetic alloy AuFe as the temperature is lowered, for different concentrations of magnetic impurity. Taken from [28].

has been substituted at random locations in very dilute concentrations, usually a few tenths of a percent. They exhibit the *Kondo* effect [28], a peculiar electrical conductivity property. As the temperature of a *conventional* metal is lowered, the conductivity goes up, or the resistance goes down, since the molecular vibrations decrease in amplitude and collide less with conduction electrons. However, for dilute magnetic alloys the resistivity decreases, then at very low temperatures begins to increase again, as seen in Figure (5). At dilute concentrations, like a few tenths of a percent, there would be a few islands of magnetic elements dispersed at random throughout the host crystal. These islands would be sufficiently separated such that they do not interact appreciably. However, at increasing concentrations, the islands can start interacting. We thus expect that increasing concentrations of the magnetic impurity give rise to long-range magnetic ordering, where the different magnetic islands would have some correlated properties.

One thermodynamic property that seems to indicate a phase transition at low temperatures in gold-iron alloys was the magnetic susceptibility χ. χ is a tensor which measures the

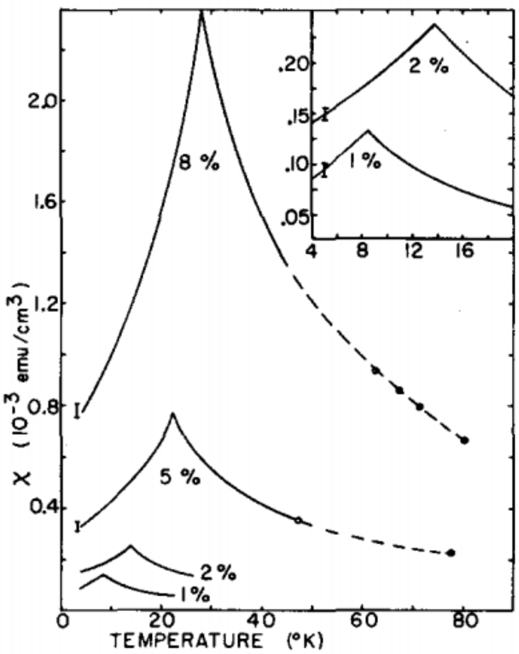

Figure 6: Cusp in the magnetic susceptibility as a function of temperature of the dilute magnetic alloy AuFe at different concentrations of Fe. Taken from [29].

linear response of a solid to small changes in the external magnetic field

$$\chi_{ij} = \lim_{\|h\|\to 0} \left(\frac{\partial h_i}{\partial x_j} \right)_T . \tag{131}$$

The work of [29] reveals a cusp at low temperatures, as in Figure (6). We recall that a hallmark of a phase transition is a singular behaviour of thermodynamic variable. However, the situation is not as clean as one might hope. We expect not a single thermodynamic variable to be singular, but all of them. Singular behaviour can include

1. Divergence to "infinity" above and below the transition temperature

2. Finite value above and below the transition temperature but discontinuous

3. Continuous everywhere but has a cusp at the transition temperature, which corresponds to a discontinuity in the first derivative.

Thus, a measurement of another simple thermodynamic quantity like the heat capacity, should also show singular behaviour at at the same temperature of the gold-iron magnetic susceptibility cusp. However, this does not occur. Instead, the specific heat capacity of

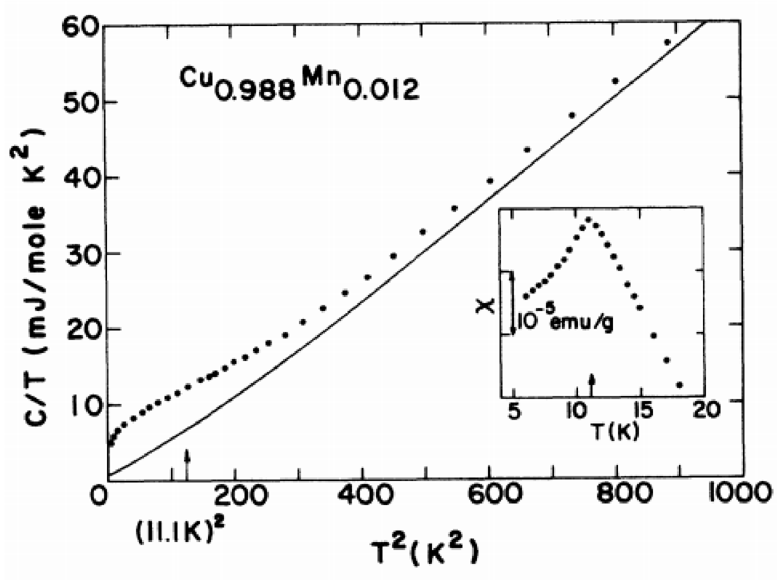

Figure 7: The heat capacity for the dilute magnetic alloy CuMn is smooth and does not display a cusp or otherwise singular behaviour. Taken from [30].

$Cu_{0.988}Mn_{0.012}$ (subscripts refer to concentrations) displays a round, smooth maximum, that does not even correspond to the putative transition temperature at the iron-gold cusp [30], as seen in Figure (7). So, there are two simple thermodynamic functions that seem to have contradictory implications as to the occurrence of a phase transition.

One natural question to pose is: to *which phase* is the transition happening? The high temperature phase is a paramagnetic phase, characterized by constantly gyrating and fluctuating spins with zero time-averaged spin

$$\langle \mathbf{m} \rangle = \lim_{\tau \to \infty} \frac{1}{\tau} \int_0^{\tau} \mathbf{m}(\tau)\ \mathrm{d}t = 0. \tag{132}$$

and no long-range magnetic ordering, so no spatial correlation and a vanishing space-averaged spin at each instant in time. The low-temperature phase seems to have no long-range antiferromagnetic order; it also exhibits zero bulk magnetization, ruling out a ferromagnetic phase. Experimental probes seemed to indicates spins *stuck* in random, orientations. This spin-freezing has no-long range ordering: the random frozen spins are not correlated. So the *static* properties that have come to define our understanding of spin-glasses are

1. Cusp at low temperature in the magnetic susceptibility-temperature curve for dilute

magnetic alloys, with lower temperatures for lower concentrations (1-2%) of magnetic impurity.

2. A smooth, rounded maximum in the specific heat capacity.

3. Spin freezing below the glass temperature T_f as defined by the cusp in the magnetic susceptibility.

4. No spatial long-range magnetic ordering.

We have discussed the static properties characterizing a spin glass. We now turn to the extremely interesting dynamical properties.

5.3 Out of Equilibrium Properties

We can ensure a system is out-of-equilibrium if we disturb it in a rather severe way. This enables us to access the system's dynamic behaviour. One way to do that is via a so-called *deep thermal quench*, whereby we rapidly cool the system from well-above the spin-freezing temperature T_f to well below it. This kind of experiment can be used to demonstrate a property called remanence. For example, we can place a spin glass in a strong magnetic field, cool the spin glass in a deep thermal quench, then turn of the magnetic field. When the strong magnetic field is turned on, it comes as no surprise that the spins are aligned with the field. However, when it is turned off, the spin glass maintains residual magnetization termed remanence. Interestingly, the rate at which the magnetization decays depends on the history of the system. Alternatively, one can cool the spin glass in a deep thermal quench in the absence of a magnetic field, and only then turn on a strong magnetic field. After turning it off, there is again a residual magnetization but it now decays at a different rate than the first scenario, and it also depends on the rate of the cooling. So this property of remanence displays interesting history-dependence.

Another interesting dynamical property is "memory", which is exhibited in the following scenario: Cool a spin glass in a deep thermal quench in the presence of a strong magnetic field. Then, leave it in the magnetic field at constant temperature for a waiting time t_w.

After time t_w, turn off the field. The residual magnetization in the spin glass will decay, but the decay rate changes abruptly after time t_w.

The central idea potentially characterizing spin glasses is that they are out-of-equilibrium and have relaxation or equilibration time-scales that are longer than experimental time-scales. This can be owed to a large degree of meta-stability, whereby the energy landscape of the spin glass has many local minima with large energy barriers, so that if the systems falls in one, it requires some energy to overcome the energy barrier and continue in its search or descent toward the true, stable equilibrium or ground state. We thus describe a system with long relaxation timescale as having a "jagged" energy landscape. This idea will be further explored below.

5.4 A Universal Hamiltonian for Spin Glasses

A central idea in physics is modeling; recognizing the central and foremost features characterizing a phenomenon, and throwing away the details. In doing that, you sacrifice precision for clarity. Anderson and Edwards apply this methodology to the study of the spin glass system [31]. Their work identifies a feature they thought of as universal to all spin glass systems: A confluence of ferromagnetic and anti-ferromagnetic interactions. The Edwards-Anderson (EA) Hamiltonian that simply captures this feature is a nearest-neighbor interaction:

$$H_{EA} = -\sum_{\langle xy \rangle} J_{xy} \mathbf{m}_x \cdot \mathbf{m}_y - \sum_x \mathbf{h} \cdot \mathbf{m}_x. \tag{133}$$

The first sum is over all pairs of nearest neighbors. The couplings J_{xy} are drawn from a random distribution, for example the Bernoulli distribution (coin flip) so they would be $\pm J$ or the Gaussian distribution with mean zero. They thus can be of either sign, and represent an interweaving of ferromagnetic and anti-ferromagnetic interactions. The couplings are described in the literature as "quenched". They are drawn from a random distribution, and then they determine the Hamiltonian. A Hamiltonian is thus picked for "each realization of the disorder". This Hamiltonian would be somewhat analogous to the $O(3)$ Ising model. A simpler model would thus be the $\mathbb{Z}_2$ Ising model with $\sigma_x \in \{-1, 1\}$

$$H = \sum_{\langle xy \rangle} J_{xy} \sigma_x \sigma_y - h \sum_x \sigma_x. \tag{134}$$

A feature that can be immediately gleaned from the Hamiltonian (134) is frustration [32]. For a ferromagnetic material, it is easy to find the ground state. You simply align all the spins. However, for frustrated magnetism, energy minimization imposes contradictory or insatiable constraints that cannot be satisfied simultaneously , and finding the ground state becomes more difficult. So this raises the question of which couplings to satisfy and which one to leave unsatisfied. Is there a single unique ground state up to some global symmetry? Or can there be many different ground states depending on which couplings we choose to satisfy?

It has also not escaped our attention that frustration is not exclusive to spin glasses or the EA Hamiltonian. Frustration arises whenever the product of couplings in a closed loop is negative. Thus, an anti-ferromagnet on a $(2n+1)$-gon lattice would exhibit frustration. The EA model of spin glasses exhibits *both* frustration and quenched disorder.

Another aspect we can discuss is whether the spin glass, as described experimentally, represents a genuine novel state of matter that is reachable via some phase transition. If so, it should accompany a breaking of some symmetry and should be described by an order parameter that indicates how this symmetry was in fact broken. EA proposed the following order parameter

$$q_{EA} = \lim_{N\to\infty} \frac{1}{N} \sum_{x} . \langle \sigma_x \rangle^2 \,, \tag{135}$$

where the correlation function is temporal for each given site. If the spin glass is indeed a new phase, then $q_{EA} > 0$. This is because the spins would be arrested and frozen to a single arbitrary direction for each. This would constitute a breaking of spin-flip symmetry in the Ising model. If $q_{EA} = 0$, this indicates the correlation decays in time, and the individual spins are gyrating, but over very long relaxation timescales, and the spin glass would not constitute a new phase and there is no symmetry breaking.

5.5 All-to-all Interaction

We have seen that the EA Hamiltonian, although idealized and captures only the universal aspects presumably characterizing any spin glass, is still marred by frustration and quenched disorder. It is thus extremely hard to “solve”. One can thus fall back to an even simpler system: The mean field theory of the EA Hamiltonian. A mean field theory is a further idealization in which every spin experiences precisely the same environment felt by every other spin. So this is equivalent to homogeneity and discards geometric information. One way of achieving this mean field limit is by taking an all-to-all interaction, otherwise known as the infinite-range model. It is termed infinite-range because every spin can interact equally strongly with every other spin. In this limit, just as in the SYK model, it does not make sense to speak of a spatial lattice, because distances do not mean anything anymore. We have lost all the geometric information since it no longer enters our infinite-range Hamiltonian. We thus substitute the lattice site labels for labels to just distinguish the spins. The same thing is done for the SYK model. The Hamiltonian for the infinite-range model or the all-to-all interaction was first written down by Sherrington and Kirkpatrick [33]

$$\mathcal{H}_{SK} = -\frac{1}{\sqrt{N}} \sum_{1\leq i,j\leq N} J_{ij}\sigma_i\sigma_j - h \sum_{1\leq i\leq N} \sigma_i. \tag{136}$$

The couplings again represent quenched disorder, and the sum is over all-to-all couplings, as opposed to just nearest-neighbor couplings. A new ingredient relative to the EA Hamiltonian is the factor of $1/\sqrt{N}$. The sum contains $N(N-1)$ terms, and to ensure a finite energy density (per spin) in the large N-limit, we need $J_{ij} \sim \mathcal{J}/\sqrt{N}$, so that $\mathcal{H}_{SK}/N \sim O(1)$. Note the energy itself has to diverge, because it is extensive, but the factor of $1/\sqrt{N}$ allows the energy density to remain finite in the large-N limit.

5.6 Replicas and the Breaking of Replica Symmetry

Recall we had the EA order parameter describing the breaking of spin-flip symmetry in the EA-Ising model

$$q_{EA} = \lim_{N\to\infty} \frac{1}{N} \sum_x \langle\sigma_x\rangle^2 , \tag{137}$$

where the average is a long time average. There were two interesting findings regarding the infinite-range model: The first, observed by Thouless and de Almedia, is that there exists a phase transition to an unstable, spin glass phase, for Gaussian quenched disorder [34], as seen in Figure 8. This does not tell us though about a phase transition in the more realistic nearest-neighbhor model. The second, is that if we assume q_{EA} is the *only* order parameter

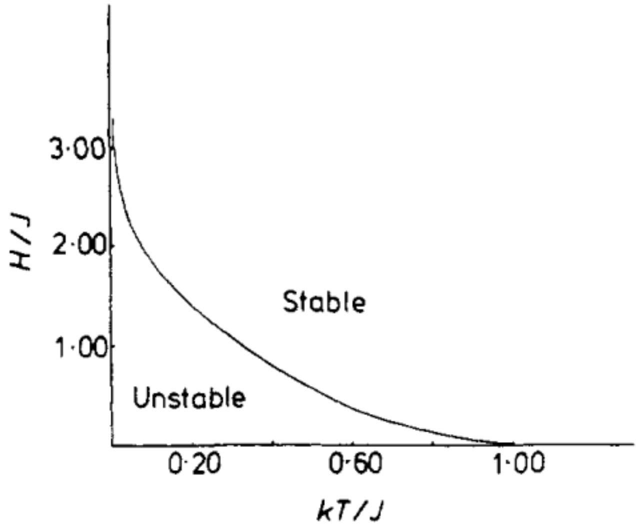

Figure 8: The phase boundary at constant magnetic field between the high temperature paramagnetic phase and the low temperature spin-glass phase in the SK model with Gaussian quenched disorder with mean zero and standard deviation J. The separating boundary is called the de Almeida–Thouless (AT) line. Taken from [34].

for the broken symmetry of the above-mentioned spin glass phase transition, then we end up with negative entropies at low temperatures [35]. The work of Sherrington and Kirkpatrick [33] attempts to attribute the unphysical negative entropy to the replica method. Their work identifies the source of trouble as changing the order of the thermodynamic limit $N \to \infty$ and the limit $n \to 0$, in which the number of replicas is sent to zero. However, Parisi's solution was to instead propose an *infinite* number of order parameters for the spin glass phase transition, describing the breaking of replica symmetry [35].

What does it look like to have infinitely many order parameters describing a phase transition? To wit, the spin-flip symmetry is indeed broken in the SK model at low temperatures. However, unlike the ferromagnet, where there are two states which are spin-flips of each other, in the SK model there are infinitely many pairs of spin-flipped states. So a description in terms of only the q_{EA} order parameter is insufficient. The order parameters discovered by Parisi describing the replica symmetry breaking are in a thermodynamic sense, quite bizarre and novel. The reason being is they describe the *relationships* between the infinitely-many pairs. So these order parameters are not thermodynamic functions of state that refer only to the underlying microscopic state alone. So answering the question of: "How does this thermodynamic state break the symmetry of the Hamiltonian?" requires knowledge of not just the state itself [36]. This situation is quite unique, and represents a big departure from the conventional theory of phase transitions.

5.7 An Inner Product on States

We have been loosely using the term "thermodynamic state", by which we mean a probability measure on all spins, assigning a probability that each Ising spin points up, for example. We can then define a sort of inner product on the space of thermodynamic states [37]

$$q_{\alpha\beta} = \frac{1}{N}\sum_{i=1}^{N} \langle \sigma_i \rangle_\alpha \langle \sigma_i \rangle_\beta \,, \tag{138}$$

where the averages are long-time averages, and the subscript refers to the thermodynamic state, which assigns at each time instant the probability that the Ising spin points up, for example. We note that for any thermodynamic state α, $q_{EA} = q_{\alpha\alpha}$, and for the spin-reversed thermodynamic state $\tilde{\alpha}$, $q_{\alpha\tilde{\alpha}} = -q_{EA}$. For any two arbitrary thermodynamic states, we have $-q_{EA} < q_{\alpha\beta} < q_{EA}$, which is quite believable because it is reminiscent of ordinary inner products.

One should note that all the thermodynamic states actually look the same statistically, in that they

1. Possess no spatial order or correlation.

2. Have zero magnetization.

3. Have the same norm as measured by $q_{\alpha\alpha}$.

However, as we lower the temperature, crossing the Almeida-Thouless (AT) line (defined in Figure 8) and into the spin glass phase, as the spin glass stabilizes, it must settle into one of these many thermodynamic states. To extract more information out of the inner products $q_{\alpha\beta}$, people have tried to study the distribution of their values, which lie in the range $[-q_{EA}, q_{EA}]$. To be precise, the distribution of the inner products is studied for a fixed realization of the disorder, or a fixed Hamiltonian. So, the disorder couplings J_{ij} are drawn from a random distribution, like the normal $\mathcal{N}(0, \mathcal{J}/\sqrt{N})$, and then the distribution of inner products is computed.

For different realizations of the couplings, the distributions of two-point functions on thermodynamic states look considerably different, exhibiting a non-self-averaging property, even in the large-N limit [37]. This is surprising because we expect that in thermodynamic limits for sample-to-sample fluctuations to die out. However, we have to keep in mind that the overlaps between thermodynamic states are not thermodynamic variables.

6 Spin Glass Phase in the SYK Model?

In [39], a low-temperature spin glass phase transition in the SYK model is investigated. As we have discussed, Parisi had proposed replica symmetry-breaking as a signal of a spin-glass phase transition. This could be signaled by the condensation of off-diagonal modes F_{ab}, which are given by

$$F_{ab}(\tau_1, \tau_2) = \frac{J^2}{q} \langle \left(\frac{1}{N} \sum_i \psi_i^a(\tau_1) \psi_i^b(\tau_2) \right)^{q/2} \rangle \tag{139}$$

in the saddle point approximation. The off-diagonal modes condense or become non-zero when their effective potential becomes unstable by generating an imaginary mass term. The squared mass m_{ab} for the off-diagonal mode F_{ab} can be calculated in the nearly-conformal limit to be

$$\begin{aligned} m_{ab}^2 &= \frac{1}{(\beta J)^2} \left(qN - aN^{2-q/2} \log^2 \beta J \right), \\ a &= \frac{4}{\pi} (q/2)! \left(\frac{1}{2} - \frac{1}{q} \right) \tan(\pi/q). \end{aligned} \tag{140}$$

The squared-mass becomes negative at the critical temperature for the putative phase transition

$$T_c = J \exp\left(- \left(\frac{q}{a} \right)^{1/2} N^{(q-2)/4} \right), \tag{141}$$

where $q = 4$. This can be simplified to $\beta_c J = e^{\sqrt{2\pi N}}$, which falls in the regime $1 \ll N \ll \beta J$. This regime is not consistent with the nearly-conformal regime in which we started the computation. The nearly conformal regime is defined by $1 \ll \beta J \ll N$, which is the UV regime of nearly-free Majorana fermions. What this computation implies is that if there exists a spin-glass phase transition in the SYK model, it must occur in the low-temperature, strongly coupled regime of the Schwarzian theory. The computation must thus be repeated in that regime and the instability of the off-diagonal modes must be checked. A computation in the Schwarzian regime gives the mass-squared in the effective potential of the off-diagonal modes:

$$m_{ab}^2 = \frac{4N - c_2 \log^2 N}{(\beta J)^2}, \tag{142}$$

where c_2 is a positive constant. This mass-squared is always positive, and thus the replica off-diagonal modes never condense. This indicates the absence of a spin-glass phase transition.

Another diagnostic that can be examined is deviations from random-matrix theory (RMT) predictions. In the Sherrington-Kirkpatrick (SK) mean field theory model of the spin-glass, the energies of pure states are independent random variables, and the level-spacing statstics follow an exponential distribution [38]. The paramagnetic phase of SYK however, obeys RMT statistics, and exhibits things like level-repulsion and spectral rigidity [4]. The level-spacing statistics and ground state energy distributions were numerically obtained in [39], and no deviation from RMT predictions for the corresponding Gaussian Unitary Ensemble (GUE) were found. Thus, the two criteria of condensation of off-diagonal modes and uncorrelated energies were not detected in low-temperature SYK, and it thus seems to remain in the well-known, paramagnetic, chaotic phase.

7 Conclusion and Future Work

The computation of the Euclidean path integral in the section on the "The Factorization Problem in JT Gravity" was only a saddle point approximation, which led to a continuous, smooth density of states. In [40] a general, brute-force procedure for computing one-loop corrections to thermal partition functions in AdS_3 was given. Can this procedure by applied to JT gravity? We would like to see some evidence of discreteness of spectrum, which simply cannot be probed at the saddle point or semi-classical level. Another interesting line of thought, although somewhat vague, is the time-like boundary Liouville theory of [11], which has a "wrong" sign potential, which is unbounded below, and more generally non-hermitian Hamiltonians that may have some complex coefficients. When do they admit a real spectrum and a clean Hilbert space interpretation?

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