THE RATIONAL INTEGRAL SOLUTIONS OF THE EQUATION $y^2 - k = x^3$

by

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I AM DEEPLY INDEBTED TO

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C O N T E N T S

CHAPTER ONE

INTRODUCTION.

One of the best known Diophantine equations that mathematicians have worked on for a long time is the equation $y^2-k = x^3$. Although many mathematicians have attempted to solve this equation, it remains unsolved except for certain classes of k. Nagell[1] gives the values of k such that $|k| < 100$ for which the solutions have not been found. The values are: k = 8,9,10,12, 15,18,19,22,24,26,28,30,31,33,35,36,37,38,40,41,43,44,48,50,52,54, 55,56,57,63,64,65,68,71,72,73,74,76,79,80,81,82,89,91,92,94,97,98, 99,-7,-15,-18,-20,-23,-25,-26,-28,-31,-39,-40,-45,-47,-48,-53,-54, -55,-56,-60,-61,-63,-71,-72,-79,-83,-84,-87,-89,-95.

The great mathematician, Fermat, investigated this equation. He claimed to have solved the equation $y^2+2 = x^3$ but it is not known if Fermat had a rigorous proof of the result. It is interesting to note that the only integral solutions of this equation are $y = \pm 5$, $x = 3$.

Euler also did some work on this equation. He solved the equation $y^2- 1 = x^3$ by the method of infinite descent. He also gave a method for solving the equation $y^2+ 2 = x^3$, but the method was fallacious. He wrote the equation in the form $(y + \sqrt{-2})(y- \sqrt{-2})=x^3$ He then stated that there exist integers a and b such that $y + \sqrt{-2} = (a + b\sqrt{-2})^3$. This statement is now known to be correct, but since a rigorous theory of algebraic numbers did not exist at that time, Euler had no justification in stating this result. In fact, it is now known that the equation $y + \sqrt{-K} = (a + b\sqrt{-k})^3$

will give the complete solution of $y^2-k = x^3$ only for certain k.

One of the greatest single steps made in the direction of solving the equation was the development of algebraic number theory. It was found that with the introduction of a new type of element, an ideal, unique factorization holds in any algebraic number field. Mordell[2] applied this in his attempt to solve the equation $y^2- k = x^3$. He succeeded in solving this equation for certain classes of k in his paper of 1912.

A few years later Landau and Ostrowski[3] published a general result proving that the equation $y^2- k = x^3$ has only a finite number of solutions. In fact, the result was proved for the more general equation $ay^2+ by + c = dx^n n\geq 3$ where a, $b^2- 4ac$, d are unequal to zero. The result also gives a method of reducing the equation $y^2- k = x^3$ to a finite number of equations of the form $f(x,y) = N$ where $f(x,y)$ is a binary homogeneous cubic.

The reduction of the equation $y^2- k = x^3$ to the equation $f(x,y) = N$ was an important step forward. Various results have been discovered in connection with the equation $f(x,y) = N$ although the equation is still far from solved in general. I will show later how certain information about the solutions of the equation $y^2- k = x^3$ can be obtained by an application of information about the solutions of the equation $f(x,y) = N$.

In this thesis we shall solve the equation $y^2- k = x^3$ for certain special values of k. It will be seen that by means of the theory of congruence and of quadratic residues, certain cases can be dealt with. We will also, by means of ideal theory, solve

the equation for certain classes of k. In chapter 5 certain properties of the equation $ax^3 + bx^2y + cxy^2 + dy^3 = 1$ will be discussed. A theorem proved at the end of chapter five, which so far as the author knows is original gives an intimate connection between the equation $y^2 - k = x^3$, and the cubics derived from it. This will be applied in chapter 6 to obtain an upper bound to the number of solutions of the equation $y^2 - k = x^3$ for certain classes of k. Before discussing the equation $y^2 - k = x^3$, a few properties of ideal classes will be mentioned, because the concept of ideal classes plays an important part in the development of the theory for this equation.

To begin with, two ideals a and b are said to be equivalent if there exist principal ideals x and y such that ax = by. This relation can easily be seen to be reflexive, symmetric, and transitive; and therefore the ideals are thus broken up into classes. With an obvious definition for the multiplication of classes, it is found that the classes form an Abelian group. One of these classes is the set consisting of all principal ideals, and this class is obviously the identity element of the group. An important result is that for a given field there are only a finite number of ideal classes. Thus, we have a finite group, and many of the properties of ideal classes follow directly from the theory of finite groups. The number of ideal classes in a field is commonly called the class number of the field.

If b is any ideal of an algebraic number field and h is the class number of the field, it follows by the theory of groups that b^h is a principal ideal. If b^m is also known to be a principal

ideal, and if furthermore m is prime to h, then it follows that b itself is also a principal ideal. This result will be used in this thesis to solve the equation $y^2 - k = x^3$ for some classes of k.

CHAPTER TWO

THE FINITE NUMBER THEOREM

In this chapter it will be proved that the equation $ay^2 + by + c = dx^n$ where $a, 4ac - b, d \neq o$ has only a finite number of solutions in integers if $n \geq 3$.

Let $ay^2 + by + c = dx^n$,

therefore $4a^2y^2 + 4aby + 4ac = 4adx^n$,

or $(2ay + b)^2 + 4ac - b^2 = 4adx^n$.

Let $2ay + b = y$, $4ac-b^2 = m$, $4ad = k$.

Then $y^2 + m = kx^n$ where $m, k \neq o$.

If $ay^2 + by + c$ (= dx^n) had an infinite number of solutions so would $y^2 + m = kx^n$, since for every value of y a different value of y is obtained. It therefore suffices to prove that an equation of the type $y^2 - k = mx^n$ has only a finite number of integral solutions for $k, m \neq o$.

CASE 1

Let $k = p^2$ where p is an integer,

therefore $y^2 - p^2 = mx^n$,

or $(y-p)(y+p) = mx^n$.

Let S be a prime factor of $y + p$ which is prime to 2mp. It follows that S is prime to 2p and therefore to y-p. Also S is prime to m.

From the expression $(y-p)(y+p) = mx^n$, it follows that the power of S contained in $y + p$ equals the power of S contained in x^n. That is, $y + p$ contains S^{ln} for some integral l. By similar reasoning for all other prime factors of $y + p$ which are prime to 2mp, it follows that $y + p$ is an exact nth power of all these factors. Im

other words, the only prime factors of $y + p$ of which $y + p$ may not contain an exact nth power are those that divide 2mp.

That is:-

$y + p = S_1^{r_1} S_2^{r_2} S_3^{r_3} \ldots\ldots S_n^{r_n} X^n$ where $S_1, S_2, S_3, \ldots S_n$ divide 2mp. The $r's$ may each be chosen less than n by incorporating all nth powers into the X. Then, there are only a finite number of possible factors $S_1^{r_1} S_2^{r_2} S_3^{r_3} \ldots\ldots S_n^{r_n}$ since 2mp contains only a finite number of prime factors.

Therefore $y + p = cx^n$, where c runs through a finite number of values.

By exactly the same reasoning $y - p = bz^n$ where b runs through a finite number of values.

It is sufficient to prove that the simultaneous equations $y + p = cx^n$, $y - p = bz^n$ have only a finite number of solutions x,y,z, for fixed b and c, for a finite sum of finite numbers is still finite. Subtracting, we obtain, $2p = cx^n - bz^n$. If the equation $cx^n - bz^n = 0$ is considered, the values of $\frac{x}{z}$ are the nth roots of $\frac{b}{c}$, and the roots are therefore all distinct. Since $n \geq 3$, it follows by Thue's theorem that the equation has only a finite number of solutions. Thus, there are only a finite number of solutions for x,y,z, and the theorem is proved for this case.

CASE 2

If k is not an exact square, let $k = p^2 d$ where p is integral and d has no square factors. We will work in the quadratic field $R(\sqrt{d})$.

We have $y^2 - p^2 d = mx^n$,

or $(y + p\sqrt{d})(y - p\sqrt{d}) = mx^n$.

Let b be a prime ideal factor of $y + p\sqrt{d}$ which is prime to $2pm\sqrt{d}$. b is therefore prime to m and prime to $y + p\sqrt{d} - 2p\sqrt{d}$ or $y - p\sqrt{d}$. b is therefore contained as often in $y + p\sqrt{d}$ as it is contained in x^n. It follows that $y + p\sqrt{d}$ contains b to an exponent divisible by n. As before, it can be seen that the only prime ideal factors of $y + p\sqrt{d}$ which do not necessarily occur with an exponent divisible by n are those that divide $2pm\sqrt{d}$.

$y + p\sqrt{d} = b_1^{r_1} b_2^{r_2} b_3^{r_3} \ldots\ldots b_a^{r_a} X^n$ where b_1, b_2, b_3.....b_n divide $2pm\sqrt{d}$. By incorporation into the x all the r's can be made less than n, and since an ideal has only a finite number of prime ideal factors, $y + p\sqrt{d} = cx^n$ where c may have only a finite number of values. It suffices to show that $y + p\sqrt{d} = cx^n$ has only a finite number of solutions for fixed c.

Now x is an ideal which can lie in any one of a finite number of ideal classes. It is therefore sufficient to prove this result for a fixed ideal class B.

Let W be an ideal of the reciprocal ideal class. Using the symbol $\sim$ for equivalent we have, $W^n \sim W^n cx^n$ (cx^n is a principal ideal). We have $w^n \sim w^n x^n c \sim c$ since $wx \sim 1$. It follows that $g(w^n) = f(c)$ where f and g are principal ideals.

From the relation $y + p\sqrt{d} = cx^n$ we have $f(y + p\sqrt{d}) = fcx^n = g(w^n)x^n = gz^n$ where z is a principal ideal.

Thus, we have a relation involving principal ideals only. Transforming this into an equation involving ordinary numbers only, we obtain $f(y + p\sqrt{d}) = egz^n$ where e is a unit of the field $R(\sqrt{d})$.

This may be expressed in the form $f(y + p\sqrt{d}) = e^m g z^n$ where e is the fundamental unit and m an integer which is positive, negative, or zero. m may be so chosen that $0 \leq m < n$ by incorporating a suitable amount into the z^n. It therefore suffices to prove that the equation $f(y + p\sqrt{d}) = egz^n$ has a finite number of solutions for fixed e,f,g. Multiplying by $\bar{f}$ the conjugate of f we have,

$f\bar{f}(y + p\sqrt{d}) = \bar{f}egz^n$ which gives,

$c(y + p\sqrt{d}) = mz^n$ where c is a rational integer, and m is an algebraic integer of the field $\sqrt{d}$.

Let $z = a + bw$ where (1,w) is a basis of the field $R(\sqrt{d})$. We then obtain,

$c(y + p\sqrt{d}) = m(a + bw)^n$. By taking conjugates $c(y - p\sqrt{d}) = \bar{m}(a + b\bar{w})^n$.

Therefore $$\frac{2pc\sqrt{d}}{w-\bar{w}} = \frac{m(a + bw)^n - \bar{m}(a + b\bar{w})^n}{w - \bar{w}} .$$

The right hand side is easily seen to be a rational integral homogeneous function of a and b, since $\frac{A-\bar{A}}{W-\bar{W}}$ is always real for all A. We thus have an equation of the form $f(a,b) = k$. To apply Thue's theorem, it is sufficient to show that no two linear factors of f(a,b) are alike.

The factors of f(a,b) are $(m)^{\frac{1}{n}}(a + bw) - (\bar{m})^{\frac{1}{n}}(a + b\bar{w})Si$ where Si is one of the nth roots of unity.

Suppose two factors are alike, Then the determinant of the coefficients of two of the factors must vanish.

That is $$\begin{vmatrix} m^{\frac{1}{n}} - \bar{m}^{\frac{1}{n}} s_i & m^{\frac{1}{n}} w - \bar{m}^{\frac{1}{n}} \bar{w} s_i \\ m^{\frac{1}{n}} - \bar{m}^{\frac{1}{n}} s_j & m^{\frac{1}{n}} w - \bar{m}^{\frac{1}{n}} \bar{w} s_j \end{vmatrix} = 0$$

Let $A_i = \frac{\bar{m}^{\frac{1}{n}} s_i}{m^{\frac{1}{n}}}$ and $A_j = \frac{\bar{m}^{\frac{1}{n}} s_j}{m^{\frac{1}{n}}}$. A_i and A_j are nth roots of $\frac{\bar{m}}{m}$. We have,

$$\begin{vmatrix} 1 - A_i & w - A_i\bar{w} \\ 1 - A_j & w - A_j\bar{w} \end{vmatrix} = 0$$

or $$\begin{vmatrix} 1 - A_i \\ 1 - A_j \end{vmatrix} \begin{vmatrix} 1 & w \\ 1 & \bar{w} \end{vmatrix} = 0$$

Now $\begin{vmatrix} 1 & w \\ 1 & \bar{w} \end{vmatrix}$ is the square root of the discriminant of the field $R(\sqrt{d})$ and is therefore not equal to zero. It follows that $A_i = A_j$ i.e. $i = j$. This proves that no two linear factors can be alike. Thue's theorem applies. There is, therefore, only a finite number of values a and b, and therefore it follows from $c(y + p\sqrt{d}) = m(a + bw)^n$ that there are only a finite number of values y. The theorem is thus completely proved.

This method of proof does not enable us to find the solutions of the equation $y^2 - k = x^3$, but the method does enable us to reduce the solution of this equation to that of a finite number of homogeneous binary cubic equations each equal to a constant.

CHAPTER THREE

SPECIAL METHODS

Before solving the equation $y^2 - k = x^3$ for various special values of k, it is important to note certain restrictions on the possible values of x and y which depend upon the residue of k mod 8.

If $k \equiv 2 \bmod 8$ then y is odd, $x \equiv 3 \bmod 4$;

$k \equiv 3 \bmod 8$ y is even, $x \equiv 1 \bmod 4$;

$k \equiv 5 \bmod 8$ y is even, $x \equiv 3 \bmod 4$;

$k \equiv 6 \bmod 8$ y is odd, $x \equiv 3 \bmod 4$;

$k \equiv 7 \bmod 8$ y is even, $x \equiv 1 \bmod 4$.

This follows from elementary theorems in the theory of congruence. For example, use is made of the fact that if x is odd, then $x^2 \equiv 1 \bmod 8$. These results are useful since they throw out many values of x and y at the outset.

We will begin by proving that the equation $y^2 + 3 = x^3$ has no integral solutions. Since $k \equiv 5 \bmod 8$ we know at the outset that y is even and $x \equiv 3 \bmod 4$.

We have $y^2 + 4 = x^3 + 1 = (x + 1)(x^2 - x + 1)$.

Since $x \equiv 3 \bmod 4$ it follows that $x^2 - x + 1 \equiv 3 \bmod 4$.

Letting $x^2 - x + 1 = 4n + 3$ we have $4n + 3 \mid y^2 + 4$,

$$-4 \equiv y^2 \bmod 4n + 3.$$

The last result is a contradiction of a fundamental theorem of quadratic residues. We thus have a contradiction.

Exactly the same reasoning will apply to the equation $y^2 + 4 - (1 + 8c)^3 = x^3$.

We will now consider the equation $y^2 + 1 - p^3 = x^3$ where $p \equiv 2 \bmod 4$. Therefore we have $y^2 + 1 = x^3 + p^3 = (x + p)(x^2 - px + p^2)$. In this case $k = p^3 - 1 \equiv 7 \bmod 8$, and hence y is even and x is odd.

$\therefore$ $x + p \equiv 3 \bmod 4$, and hence $4n + 3 \mid y^2 + 1$.

This is impossible, and hence this equation has no solutions in rational integers.

This procedure will lead to the same result for the more general equation $y^2 + q^2 - p^3 = x^3$ where q is odd and $p \equiv 2 \bmod 4$ providing it is known that $x + p$ is prime to q.

So far the contradiction has been based on the theorem for the quadratic character of -1. The next result will use the fact that $\left(\frac{2}{p}\right) = (-1)^{\frac{p^2-1}{8}}$.

Consider the equation $y^2 - 45 = x^3$

$\therefore$ y is even and $x \equiv 3 \bmod 4$.

Let $y = 3k$. Therefore $9k^2 - 45 = x^3$.

3 must divide x, and letting $x = 3\ell$, we obtain,

$$9k^2 - 45 = 27\ell^3$$

$$k^2 - 5 = 3\ell^3$$

$$k^2 \equiv 5 \bmod 3.$$

This last result is impossible.

This proves that y is prime to 3.

From the fact that $y^2 - 45 = x^3$ we obtain,

$$y^2 - 72 = x^3 - 27 = (x-3)(x^2+3x+9)$$

$$y^2 - 18 = x^2+27 = (x+3)(x^2-3x+9)$$

x is either congruent to -1 or 3 mod 8.

But if $x \equiv -1 \bmod 8$, $x^2 - 3x + 9 \equiv 5 \bmod 8$; and if

$x \equiv 3 \bmod 8$, $x^2 + 3x + 9 \equiv 3 \bmod 8$.

In either case 2 is obtained as a quadratic residue of a number, which is impossible according to the theory of quadratic residues. Hence $y^2 - 45 = x^3$ has no integral solutions.

Certain cases which make use of algebraic number theory will be discussed at present. A more general theory will be reserved for the following chapter.

Let $y^2 + 2 = x^3$.

$\therefore$ x and y are both odd.

We have $(y + \sqrt{-2})(y - \sqrt{-2}) = x^3$.

Any common factor of $y + \sqrt{-2}$ and $y - \sqrt{-2}$ must divide $2\sqrt{-2}$ and x^3. It is obvious that $y + \sqrt{-2}$ is therefore prime to $y - \sqrt{-2}$.

By a well-known theorem in the theory of numbers $y + \sqrt{-2}$ must be an exact cube.

$$y + \sqrt{-2} = (a + b\sqrt{-2})^3$$

$$y = a^3 - 6ab^2 \qquad 1 = b(3a^2 - 2b^2) \qquad x = a^2 + 2b^2.$$

It follows that $b = \pm 1$ $a = \pm 1$; $y = \pm 5$, $x = 3$

Therefore $y = \pm 5$, $x = 3$ are the only rational integral solutions of the equation $y^2 + 2 = x^3$.

The solution of the equation $y^2 + 4 = x^3$ may be obtained by a special technique.

We have $y^2 + 4 = x^3$

Therefore $(y + 2i)(y-2i) = x^3$.

Two separate cases will be considered, $y \equiv 1 \bmod 2$,

and $y \equiv 0 \bmod 2$.

If y is odd, x is also odd, and it is easily seen that $y + 2i$ is an exact cube.

$$y + 2i = (a+bi)^3$$

$$y = a^3 - 3ab^2 \qquad 2 = b(3a^2 - b^2) \qquad x = a^2 + b^2$$

We obtain $y = \pm 11$, $x = 5$.

If y is even, $y = 2z$ for some integral z

$4z^2 + 4 = x^3$. x is therefore even.

$x = 2b$. Therefore $z^2 + 1 = 2b^3$. z is odd.

We have $\left(\frac{z+1}{2}\right)^2 + \left(\frac{z-1}{2}\right)^2 = b^3$

$$\left[\left(\frac{z+1}{2}\right) + i\left(\frac{z-1}{2}\right)\right]\left[\left(\frac{z+1}{2}\right) - i\left(\frac{z-1}{2}\right)\right] = b^3$$

$\frac{z+1}{2} + i\left(\frac{z-1}{2}\right)$ is prime to $\frac{z+1}{2} - i\left(\frac{z-1}{2}\right)$

Therefore $\frac{z+1}{2} + \frac{i(z-1)}{2} = (l + mi)^3$

$\frac{z+1}{2} = l^3 - 3lm^2$, $\frac{z-1}{2} = 3l^2m - m^3$

By subtraction $l^3 - 3l^2m - 3lm^2 + m^3 = 1$

or $(l-m)(l^2 - 3lm + m^2) = 1$.

We easily obtain $l = 1$, $m = 0$ or $l = 0$, $m = -1$

$\therefore z = \pm 1$, $y = \pm 2$, $x = 2$.

The solutions are, therefore, $y = \pm 11$, $x = 5$ and

$y = \pm 2$, $x = 2$.

Brauer[4] has given a method for solving the equation $y^2 - 2f^3 = x^3$ where f is positive. His special method applies to the

case where $R(\sqrt{-6f})$ has class number prime to 3 and 6 f contains no square factors.

Uspensky and Heaslet employed Brauer's technique on the special case $y^2 - 2 = x^3$. After reading this, and at that time being unaware of Brauer's paper, the following independent generalization was made. The equation $y^2 - 2f^3 = x^3$ was solved in the case where f is positive, $R(\sqrt{-6f})$ has class number prime to 3, and $x + 2f$ is prime to $6(x + f)f$. Our condition, unfortunately, is expressed in terms of the indeterminants of the given equation; whereas Brauer's condition is entirely in terms of f. However this condition is applicable as is shown in the example that follows. I have attempted to prove that this condition is equivalent to Brauer's condition, but I did not succeed.

We will now solve the equation for the case stated above.

$$y^2 - 2f^3 = x^3$$

Let $x = z - f$

$$\therefore\ y^2 - 2f^3 = z^3 - 3z^2f + 3zf^2 - f^3$$

$$\therefore\ y^2 + 6z^2f = z^3 + 3z^2f + 3zf^2 + f^3 = (z + f)^3$$

$$(y + z\sqrt{-6f})\ (y - z\sqrt{-6f}) = (z + f)^3 .$$

Since $x + 2f$ is prime to $6(x + f)f$

$\therefore$ $z + f$ is prime to $6zf$.

$\therefore$ $y + z\sqrt{-6f}$ is prime to $y - z\sqrt{-6f}$.

$\therefore$ $y + z\sqrt{-6f} = e(a + b\sqrt{-6f})^3$ where e is a unit of the field $R(\sqrt{-6f})$.

~~A unit of the field $R(\sqrt{-6f})$.~~

e can be incorporated into the cube, and hence we have $y + z\sqrt{-6f} = (c + d\sqrt{-6f})^3$.

$y = c^3 - 18\,fcd^2$ and $z = 3c^2d - 6fd^3$. (1)

Also $z + f = c^2 + 6fd^2$

$\therefore\ c^2 + 6fd^2 = f + 3c^2d - 6fd^3$

$\therefore\ c^2(1-3d) = f(1-6d^2 - 6d^3)$

$\therefore\ c^2 = f\left(\frac{1-6d^2 - 6d^3}{1-3d}\right) = f\left(\frac{6d^3 + 6d^2 - 1}{3d-1}\right)$

$\therefore\ c^2 = f\left[2d^2 + \frac{8}{3}d + \frac{8}{9} - \frac{1}{9(3d-1)}\right]$ (2)

$\therefore\ 9c^2 = f\left(18d^2 + 24d + 8 - \frac{1}{3d-1}\right)$;

$\frac{f}{3d-1}$ is integral.

$\therefore$ $3d-1$ divides f.

The values of d can easily be found; and from (2), c can be found.

Now we can find z from (1). Since $x = z-f$, x can be found. Thus the equation can be solved in this case.

As a special case we will consider the equation $y^2 - 2 = x^3$. In this case $f = 1$. All the conditions are satisfied in this case.

Proof

$R(\sqrt{-6})$ has class number prime to 3

$\therefore$ It suffices to prove that $x + 2$ is prime to $6(x+1)$

But $x + 2$ is obviously prime to $x + 1$

$\therefore$ It suffices to prove that $x + 2$ is prime to 6 i.e. prime to both 2 and 3.

From the equation $y^2 - 2 = x^3$ we have, x and y are odd since in

this case $k \equiv 2 \bmod 8$.

$\therefore$ $x + 2$ is odd, and is thus prime to 2.

Suppose 3 divides $x + 2$.

$\therefore$ $x \equiv 1 \bmod 3$. Let $x = 3m + 1$

$x^3 = (3m + 1)^3 \equiv 1 \bmod 9$.

From $y^2 - 2 = x^3$ we have $y^2 \equiv 2 + x^3 \equiv 3 \bmod 9$

This last result is impossible since the equation $x^2 \equiv 3 \bmod 9$ is insolvable.

$\therefore$ $x + 2$ is prime to 3

The result has therefore been proved

Applying the method to this case

We have $3d - 1$ divides $1 \quad (f \neq 1)$

d must therefore be 0.

From (2) we obtain that $c = \pm 1$.

From (1) we have $z = 0 \quad y = \pm 1$.

The only solutions of the equation $y^2 - 2 = x^3$ are therefore $y = \pm 1$, $x = -1$.

CHAPTER FOUR

THE EQUATION $y^2 - k = x^3$ FOR NEGATIVE k

CASE 1

k contains no squares, $k \equiv 2$ or $3 \bmod 4$, h is prime to 3 (h stands for the class number of the field $R(\sqrt{k})$).

It is easily seen that x must be odd in either case. The fact that x is prime to k follows from the assumption that k contains no square factors. $\therefore$ x is prime to 2k.

$$(y + \sqrt{k})\ (y - \sqrt{k}) = x^3.$$

Any common factor of $y + \sqrt{k}$ and $y - \sqrt{k}$ divides $2\sqrt{k}$ and x^3. It follows that $y + \sqrt{k}$ is prime to $y - \sqrt{k}$, and therefore

$$y + \sqrt{k} = b^3 \text{ where b is an ideal of } R(\sqrt{k}).$$

We have:-

b^3 is a principal ideal and b^h is a principal ideal. Also, 3 is prime to h.

It follows that b is a principal ideal.

We therefore obtain $y + \sqrt{k} = e(c+d\sqrt{k})^3$ where e is a unit of $R(\sqrt{k})$. Since k can not equal -3, e can always be incorporated into the cube, and we obtain $y + \sqrt{k} = (U + V\sqrt{k})^3$.

$$\therefore\ y = U^3 + 3UV^2k,\ 1 = V\ (3U^2 + V^2k),\ x = U^2 - V^2k.$$

We obtain $V = \pm 1$, $-k = 3U^2 \mp 1$, $x = 4U^2 \mp 1$.

In this case, there is at most one value of U^2, and therefore one value of x.

This case gives a unique solution except for the sign of y, if $-k$ can be expressed in the form $3U^2 \pm 1$ where U is integral. Otherwise the equation is insoluable.

The result for k = -2 worked out in the preceding chapter is a special case of this more general result. The following cases also follow from this result.

$y^2 + 1 = x^3$	$x = +1,\ y = 0$
$y^2 + 5 = x^3$	no solutions
$y^2 + 6 = x^3$	no solutions
$y^2 + 10 = x^3$	no solutions
$y^2 + 13 = x^3$	$x = +17,\ y = \pm 70$
$y^2 + 14 = x^3$	no solutions
$y^2 + 17 = x^3$	no solutions
$y^2 + 21 = x^3$	no solutions
$y^2 + 22 = x^3$	no solutions
$y^2 + 30 = x^3$	no solutions
$y^2 + 33 = x^3$	no solutions
$y^2 + 34 = x^3$	no solutions
$y^2 + 37 = x^3$	no solutions
$y^2 + 41 = x^3$	no solutions
$y^2 + 42 = x^3$	no solutions
$y^2 + 46 = x^3$	no solutions

CASE 2

k contains no squares, $k \equiv 5 \bmod 8$, h is prime to 3.

It is easily seen that x is odd and y is even. As before, x is prime to 2k. Except for the special case k = -3, we obtain $y + \sqrt{k} = (U + V\sqrt{k})^3$, since h is prime to 3 and $e = \pm 1$.

In this case $(1, \sqrt{k})$ is not a basis of the field $R(\sqrt{k})$, and it is preferable to use the basis (1,w) where $w = \frac{1+\sqrt{k}}{2}$.

We obtain, $y - 1 + 2w = (c + dw)^2$.

$$y - 1 = c^3 - 3cd^2\left(\frac{k-1}{4}\right) + d^3\left(\frac{k-1}{4}\right)$$

$$2 = d\left[3c^2 + 3cd + d^2\left(\frac{k+3}{4}\right)\right]$$

$$x = (c+dw)(c+d\bar{w}) = c^2 + cd - d^2\left(\frac{k-1}{4}\right)$$

It is easily seen that there is a maximum of four solutions in this case.

The following examples illustrate this case.

$y^2 + 11 = x^3$ $\quad x = +3, \quad y = \pm 4 \quad x = +15, \; y = \pm 58$

$y^2 + 19 = x^3$ $\quad x = +7, \quad y = \pm 18$

$y^2 + 35 = x^3$ $\quad x = +11, \quad y = \pm 36$

$y^2 + 43 = x^3$ $\quad$ no solutions

CASE 3 $\quad$ k contains no squares, $k \equiv 1 \bmod 8$, h is prime to 3.

The preceding method can be used to give all the solutions when x is odd, but the case x is even is much more difficult to solve.

To begin with, $y + \sqrt{k}$ is not prime to $y - \sqrt{k}$. However; the only possible prime factors common to $y + \sqrt{k}$ and $y - \sqrt{k}$ are the factors of 2. Since the prime factors of 2 are $(2,w)$ and $(2,\bar{w})$, we obtain $y + \sqrt{k} = (2,w)^r\,(2,\bar{w})^s\,\mathfrak{b}^3$ where b is an ideal of $R(\sqrt{k})$. It is important to note that b is not necessarily a principal ideal in this case. At any rate, b must belong to a fixed ideal class. This is proved as follows. Suppose b_1 and b_2 satisfy this equation. Therefore $b_1^3 \sim b_2^3$. Also, $b_1^h \sim b_2^h$, since each side is a principal

ideal.

$\therefore\ b_1^{3m-hn} \sim b_2^{3m-hn}$ for all integral m and n.

But 3 is prime to h.

$\therefore$ There exist integral m and n such that $3m-hn = 1$

$\therefore\ b_1 \sim b_2$ which was to be proved.

It is obvious that r and s may be chosen so that $0 \leq r < 3$ and $0 \leq s < 3$. By taking conjugates it is easily seen that $r + s = 3$. From this it is not difficult to obtain the homogeneous binary cubics. As an example, the cubic $y^2 + 7 = x^3$ will be reduced to two homogeneous binary cubics.

By the method of case 2, it is seen that $y^2 + 7 = x^3$ has no solutions for odd x. For even x we have the equation,

$y - 1 + 2w = (2,w)^r\ (2,\bar{w})^s\ b^3$ where $r + s = 3$.

In this case $w = \frac{1+\sqrt{-7}}{2}$, and therefore $w\bar{w} = 2$

$\therefore$ w and $\bar{w}$ both divide 2. The ideals $(2,w)$ and $(2,\bar{w})$ are equal to (w) and $(\bar{w})$ respectively.

We have $y-1+2w = (w)^r\ (\bar{w})^s\ b^3$.

Since the class number of $R(\sqrt{-7})$ is prime to 3, and the only units of $R(\sqrt{-7})$ are ± 1, this may be written as a relation involving ordinary numbers.

$$y-1+2w = w^r \bar{w}^s b^3$$

The possible values of r and s are:-

$r = 2,\ s = 1$; and $r = 1,\ s = 2$.

We have $y - 1 + 2w = 2w\,(c+dw)^3$ (1)

or $y - 1 + 2w = 2\bar{w}\,(c+dw)^3$ (2)

Equating the coefficients of w, we obtain

$$c^3 + 3cd^2 - 3cd^2 - 3d^3 = 1$$

$$-c^3 + 6cd^2 + 2d^3 = 1$$

From these cubics the trivial solutions $c = \pm 1$ $d = 0$ may be obtained. These give the solutions $y = \pm 1$, $x = 2$; to the equation $y^2+7 = x^3$. There does exist a tedious process for obtaining all the solutions of these two cubics.(5) A more detailed discussion of the cubics obtained from the equation $y^2-k=x^3$ will be found in the following chapter.

CASE 4

$k \equiv 2$ or $3 \bmod 4$, h is prime to 3.

Let k be of the form kf^2 where k contains no square factors. If it so happens that x is prime to 2kf, this case can be easily dealt with.

We have, $y^2-kf^2 = x^3$

$$(y + f\sqrt{k})(y - f\sqrt{k}) = x^3$$

Any prime common factor of $y + f\sqrt{k}$ and $y - f\sqrt{k}$ must divide $2f\sqrt{k}$ and x.

$\therefore$ $y + f\sqrt{k}$ is prime to $y - f\sqrt{k}$.

The rest of the reasoning is similar to that of the case $y^2-k = x^3$ where k contains no square factors.

We have $y + f\sqrt{k} = (c+d\sqrt{k})^3$

$\therefore\ y = c^3+3cd^2k$

$f = d(3c^2+d^2k)$

$x = c^2-d^2k$

The maximum number of solutions is $4T(f)$ where $T(f)$ is the number of divisors of f(if f is not divisible by 3, $2T(f)$ may be taken as an upper bound).

Let us consider the case $f = 4$.

$y^2 - 16k = x^3$. Suppose 2 divides x

$\therefore$ 2 divides y. Let $x = 2x_1$, $y = 2y_1$.

$\therefore\ y_1^2 - 4k = 2x_1^3$. 2 divides y_1.

Let $y_1 = 2y_2$ $\therefore\ 2y_2^2 - 2k = x_1^3$, 2 divides x_1.

$\therefore\ y_2^2 - k = 4x_2^3$ where $x_1 = 2x_2$.

$\therefore\ y_2^2 \equiv k \bmod 4$; but this is impossible since $k \equiv 2$ or $3 \bmod 4$.

Therefore x is odd.

But x is prime to the odd part of k.

$\therefore$ x is prime to 8k i.e. prime to 2kf.

Hence the condition for solvability by the preceding method is satisfied. As an example, we have the following result.

$y^2 + 32 = x^3$ has no solutions. $(k = -2)$

CASE 5

$k \equiv 5 \bmod 8$, h prime to 3

Let k be of the form kf^2 where k contains no square factors. As before, if x is prime to 2kf, we obtain

$y + f\sqrt{k} = (c + d\sqrt{k})^3$ or in terms of a basis 1, w,

$y - f + 2fw = (u + vw)^3$

$$y - f = u^3 - 3uv^2\left(\frac{k-1}{4}\right) + v^3\left(\frac{k-1}{4}\right)$$

$$2f = v\left[3u^2 + 3uv + v^2\left(\frac{k+3}{4}\right)\right]$$

$$x = u^2 + uv - v^2\left(\frac{k-1}{4}\right).$$

This case can therefore easily be solved.

In this case there is an upper bound of $8T(f)$ solutions; and if f is prime to 3, an upper bound is $4T(f)$,

Let us consider the case f = 2

$y^2 - 4k = x^3$ let 2 divide x.

$\therefore$ 2 divides y, and letting $x = 2x_1$, $y = 2y_1$ we have,

$$y_1^2 - k = 2x_1^3.$$

Since k is odd, y_1 must be odd $\therefore$ $y_1^2 \equiv 1 \bmod 8$

$2x_1^3 \equiv 1 - 5 \equiv 4 \bmod 8$

$\therefore$ $x_1^3 \equiv 2 \bmod 4$.

This is impossible. Therefore x is odd.

But x is prime to the odd part of k.

$\therefore$ x is prime to 4k i.e. 2kf.

Therefore this case can be solved in general.

As an example, we have the following result.

$y^2 + 44 = x^3$ $x = 5$, $y = \pm 9$ $(k = -11)$.

CHAPTER FIVE

THE EQUATION $ax^3 + bx^2y + cxy^2 + dy^3 = f$

I have shown that it is possible to reduce the solution of the equation $y^2-k = x^3$ to that of a finite number of homogeneous binary cubics each equal to a constant.

Thus, if the equation $ax^3 + bx^2y + cxy^2 + dy^3 = f$ can be solved, then the original equation can also be solved. I will now show that any equation of the form $F(x,y) = N$ can be reduced to a finite number of equations $f(x,y) = 1$.

We have, $ax^3 + bx^2y + cxy^2 + dy^3 = N$.

Let us assume at first that x is prime to N. If so, there exists a u such that $xu \equiv y \bmod N$ for all y.

We obtain, $ax^3 + bx^3u + cx^3u^2 + dx^3u^3 \equiv 0 \bmod N$.

Since x is prime to N, $a+bu+cu^2 +du^3 \equiv 0 \bmod N$.

All the possible values of u mod N can be found by trial. (there is obviously a maximum of N values).

Selecting an arbitrary u, we have $y = xu + Nz$,

Where z is an integer. By substituting, we obtain:-

$$ax^3 + bx^2(xu + Nz) + cx\,(xu + Nz)^2 + d(xu + Nz)^3 = N$$

or $$x^3(a + bu + cu^2 + du^3) + Nx^2z\,(b + 2cu + 3du^2) + N^2xz^2 (c + 3du) + N^3z^3d = N$$

By construction N divides $a + bu + cu^2 + du^3$.

Let $a + bu + cu^2 + du^3 = MN$

$\therefore$ $Mx^3 + x^2z(b + 2cu + 3du^2) + xz^2(cN + 3duN) + dN^2z^3 = 1.$

Thus the equation is reduced to the form $f(x,y) = 1$.

There remains the possibility that x is not prime to N.

Let x and N have a prime common factor P. Let $x = Px_1$ and $N = PN_1$,

$\therefore$ $P(aP^2x_1^3 + 6Px_1^2y + cx_1y^2) + dy^3 = PN_1$

P must divide d or y. If P divides d, let $d = Pd_1$

$\therefore$ $aP^2x_1 + bPx_1^2y + cx_1y^2 + d_1y^3 = N_1$

Thus the N has been reduced to N_1.

If P divides y we have $y = Py_1$, and we obtain

$$P^3(ax_1^3 + bx_1^2y + cx_1y^2 + dy_1^3) = N$$

This can only happen if N contains P^3 as a factor, and letting $N = P^3N_1$, we have, $ax_1^3 + bx_1^2y + cx_1y^2 + dy_1^3 = N_1$.

In either case N can be reduced, and therefore in a finite number of steps we can reduce the problem to the case where x is prime to N.

A useful corollary is that if d is prime to N and if N contains no cubes, then x is prime to N.

$\therefore$ In all cases the solution of the equation $F(x,y) = N$ can be reduced to that of a finite number of equations of the form $f(x,y) = 1$.

Certain trivial reductions may also be made. If we have; $ax^3 + bx^2y + cxy^2 + dy^3 = N$, we obtain $a^3x^3 + a^2bx^2y + a^2cxy^2 + a^2dy^3 = a^2N$. By letting ax and y be the new unknowns, the equation has been transformed to the form $x^3 + ax^2y + bxy^2 + cy^3 = N$. It is easily seen that the equation can also be reduced to the form $x^3 + bxy^2 + cy^3 = N$.

Any important function associated with a cubic form is the discriminant. The discriminant of $ax^3 + bx^2y + cxy^2 + dy^3$ is $18\,abcd - 4b^3d + b^2c^2 - 4ac^3 - 27a^2d^2$. This is exactly equal to $a^4(x_1 - x_2)^2(x_1 - x_3)^2(x_2 - x_3)^2$ where x_1, x_2, x_3 are the roots of the equation $ax^3 + bx^2 + cx + d = 0$. It follows that the discriminant is zero if and only if the equation has repeated roots, the discriminant is positive if all the roots are real and distinct, and the discriminant is negative if the equation has complex roots.

If the discriminant is zero, it is easily seen that the cubic $ax^3 + bx^2y + cxy^2 + dy^3$ must have a rational factor. The cubic equation then takes the form $(mx-ny)(Px^2 + qx + r) = 1$. This equation can easily be solved.

Wilhelm Ljunggren [5] published a method of solving the equation $x^3 + Px^2y + oxy^2 + ry^3 = 1$ where the discriminant is positive. He was able to reduce the solution of this equation to that of two simultaneous exponential equations in two unknowns. There also exists methods of dealing with exponential equations.[6]

No method exists at present for solving the equation $ax^3 + bx^2y + cxy^2 + dy^3 = 1$ for negative discriminant. However; there are interesting results known in connection with upper bounds

to the number of solutions to the equation.

Nagell[7] discovered a certain theorem of this type. The theorem states that if $f(x,y)$ is an irreducible binary cubic form with negative discriminant, then the Diophantine equation $f(x,y) = 1$ has at most three solutions except in the following cases.

$x^3 + xy^2 + y^3 = 1$ or an equivalent form,

$x^3 - x^2y + xy^2 + y^3 = 1$ or an equivalent form

$x^3 - xy^2 + y^3 = 1$ or an equivalent form.

(Two forms are equivalent if one may be transformed into the other by a transformation whose determinant is ± 1). Equivalent forms have the same number of solutions. Also, the theorem states that the equation $x^3 + xy^2 + y^3 = 1$ has four solutions, the equation $x^3 - x^2y + xy^2 + y^3 = 1$ has four solutions, and the equation $x^3 - xy^2 + y^3 = 1$ has five solutions. It is useful to note that the discriminants are -31, -44, -23 respectively.

If $f(x,y)$ is reducible, there is obviously a maximum of four solutions since we have $(cx + dy)(Px^2 + qxy + ry^2) = 1$ giving $cx + dy = \pm 1$. Combining these two results we obtain a maximum of five solutions in either case.

It is not difficult to extend this result to an equation of the form $f(x,y) = N$ where $f(x,y)$ is a homogeneous cubic with negative discriminant. The equations obtained by reducing this equation to a finite number of equations of the form $f(x,y) = 1$ all have discriminant of the same sign as $f(x,y)$. Therefore if $f(x,y)$ has negative discriminant, an upper bound to the number of solutions is five times the number of auxiliary equations obtained.

If x or y must be prime to N, then 5N is an upper bound to the number of solutions. This is ensured if N contains no cubes, and either the coefficient of x^3 or of y^3 is prime to N. If furthermore, N is a prime, the upper bound can be taken to be 15 since there can be 3 auxiliary equations at the most.

By extending this reasoning, an upper bound may be obtained to the number of solutions of the equation $f(x,y) = N$ for any N. The method, in general, is to find the number of auxiliary equations (or at least an upper bound to that number) and multiply the answer by five.

The following theorem will enable us to use the preceeding results to find an upper bound to the number of solutions of $y^2 - k = x^3$ for certain classes of k.

THEOREM

The auxiliary cubics obtained from the equation $y^2 - k = x^3$ all have discriminant opposite in sign to k.

CASE 1

Suppose k is not an exact square, and $k \equiv 1 \bmod 4$. From chapter two it may be seen that the auxiliary cubic is the coefficient of $\sqrt{k}$ in $(t + v\sqrt{k})(a + b\sqrt{k})^3$ where $t + v\sqrt{k}$ is an arbitrary number of $R(\sqrt{k})$.

$\therefore$ The auxiliary cubic is of the form

$$va^3 + 3ta^2b + 3kvab^2 + tkb^3$$

By substitution, discriminant is

$$162v^2k^2t^2 - 108\,t^4k + 81\,v^2k^2t^2 - 108\,k^3v^4 - 27\,t^2k^2v^2 =$$

$$= -108k(t^4 - 2t^2kv^2 + k^2v^4)$$

$$= -108k\,(t^2 - kv^2)^2.$$

By hypothesis $T^2 - kv^2 \neq 0$. Hence the result follows.

CASE 2

k is not an exact square, and $k \equiv 1 \bmod 4$. This time it is necessary to consider the coefficient of w in $(t + vw)(a+bw)^3$.

Let f(a,b) be the coefficient of w.

$\therefore$ The coefficient of $\sqrt{k} = \frac{1}{2}f(a,b)$. $\left(w = \frac{1 + \sqrt{k}}{2}\right)$

But $(t + vw)(a+bw)^3 = \left(\frac{2t + v}{2} + \frac{v\sqrt{k}}{2}\right)\left(\frac{2a + b}{2} + \frac{b\sqrt{k}}{2}\right)^3$

$\therefore$ The coefficient of $\sqrt{k} = F\left(\frac{2a + b}{2}, \frac{b}{2}\right)$ where F is a homogeneous binary cubic with discriminant opposite in sign to k by the previous result.

But $\frac{1}{2}f(a,b) = F\left(\frac{2a + b}{2}, \frac{b}{2}\right)$.

$\therefore$ The discriminant of f is of the same sign as the discriminant of F.

Hence the theorem is proved for this case.

CASE 3 k is an exact square.

From chapter two a typical auxiliary cubic is $2p = cx^3 - bz^3$. This has discriminant $-27b^2c^2$. Since k, being an exact square, is positive, the theorem is proved for this case also.

In the equation $y^2 - k = x^3$, if k is positive, the discriminant of all the cubics are negative. This theorem can therefore be used to find an upper bound to the number of solutions of $y^2 - k = x^3$ for positive k. In the following chapter I will obtain upper bounds for a few classes of k.

CHAPTER SIX

THE EQUATION $y^2 - k = x^3$ FOR POSITIVE k

CASE 1

k contains no squares, $k \equiv 2$ or $3 \bmod 4$, h is prime to 3.

As in the case where k is negative, we obtain, $y + \sqrt{k} = e(a + b\sqrt{k})^3$ where e is a unit e may have an infinite number of values. The values of e are of the form $(T + v\sqrt{k})^n$ where $t + v\sqrt{k}$ is the fundamental unit, and n is a positive integer, negative integer, or zero. By incorporation into the cube, n may be chosen satisfying the inequality $-2<n<2$. The following cubics are obtained:

$y + \sqrt{k} = (T + v\sqrt{k})(a + b\sqrt{k})^3$ gives:-

$va^3 + 3Ta^2b + 3kvab^2 + Tkb^3 = 1.$

$y + \sqrt{k} = (T - v\sqrt{k})(a + b\sqrt{k})^3$ gives:-

$-va^3 + 3Ta^2b - 3kvab^2 + Tkb^3 = 1.$

$y + \sqrt{k} = (a + b\sqrt{k})^3$ gives $b(3a^2 + b^2k) = 1.$

The last equation has no solutions.

The discriminants of the first and second equations are negative and not equal to -31, -44, or -23. If the equations are irreducible, this suffices to show that each of them has a maximum of 3 solutions. This gives a maximum of 6 solutions to the equation $y^2 - k = x^3$. If the cubics are reducible, the equations can be solved, and a maximum of 8 solutions is obtained.

Hence in this case there is a maximum of 8 solutions.

CASE 2

k contains no squares, $k \equiv 5 \bmod 8$, h is prime to 3.

As in the case where k is negative, we obtain

$y - 1 + 2w = (s + tw)(c + dw)^3$ where $s + tw$ is a unit of $R(\sqrt{k})$.

We obtain the following three possible cases

$$y - 1 + 2w = (a + bw)(c + dw)^3$$

$$y - 1 + 2w = (a + b\bar{w})(c + dw)^3$$

$$y - 1 + 2w = (c + dw)^3$$

The last equation gives the cubic,

$$2 = d\left[3c^2 - 3cd + d^2\left(\frac{k + 3}{4}\right)\right]$$

Unless k = 5 where this gives the solution $y = \pm 2$, $x = -1$; this equation has no solutions.

The other cubics are of the form $f(c,d) = 2$. Such a cubic has a maximum of 10 solutions where c is odd, and a maximum of 5 where c is even, giving a maximum of 15 solutions.

Since there are two such cubics, there is a maximum of 30 solutions in this case.

If $k \equiv 1 \bmod 8$, this method will give the upper bound of 30 to the odd solutions of x.

CASE 3 $k \equiv 2$ or $3 \bmod 4$, h is prime to 3.

Let k be of the form kf^2. If x is prime to 2 kf we obtain $y + f\sqrt{k} = e(c + d\sqrt{k})^3$. In this case we obtain two cubics of the form $g(c,d) = f$, and a cubic $f = d(3c^2 + d^2k)$.

We will consider the case f is prime. ~~Unless f = 3~~, In general the last equation has a maximum of 2 solutions. ~~In general, the equation has a maximum of 4 solutions~~. A cubic of the form $g(x,y) \equiv f$ where f is prime has a maximum of 15 solutions where x is prime to f, and a maximum of 5 where x is not prime to f. This gives a total of 40 solutions for both cubics. The total number of solutions of the equation $y^2 - k = x^3$ is 42 in this case.

It is possible to obtain an upper bound to the number of solutions for all positive k, but the arithmetic becomes cumbersome for many values of k. The cases that were done here seem to be the most interesting.

It is important to note that the upper bound obtained in many of these cases may be extremely coarse, and a much smaller upper bound may possibly be obtained by using other methods. This method, however, enables us to determine an upper bound for any negative k.

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