Projective Invariance and Visual Perception

by

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Abstract

Six experiments tested the assumption that, in visual perception, observers have reliable and direct access to the equivalence of shapes in projective geometry (I call this "the invariance hypothesis in the theory of shape constancy"). This assumption has been made in the study of vision since Helmholtz's time. Two experiments tested recognition of the projective equivalence of planar shapes. In another four experiments, subjects estimated the apparent shape of a solid object from different perspectives. Departure from projective equivalence was assessed in each study by measuring the cross ratio for the plane. This measure of projective invariance is new to perceptual research. Projective equivalence was not found to be perceived uniformly in any of the studies. A significant effect of change in perspective was found in each study. These results were construed as supporting the classical theory of depth cues against the invariance hypothesis.

Sommaire

La présomption selon laquelle l'équivalence de figures relative à la projection plane est accédée par la perception visuelle fut examinée à l'aide de six expériences. (Je la désigne sous le nom d'hypothèse de l'invariance dans la théorie de la constance visuelle.) Depuis Helmholtz, la psychologie traditionelle a soutenu ce point de doctrine dans la théorie de la vision. Six expériences ont examiné l'identification de l'équivalence conservée par toute projection plane. Dans les quatres autres expériences, les observateurs estimaient la forme apparente d'un objet solide, à partir de points de vue obliques. Dans chaque expérience, la disparité des figures par rapport à la géométrie projective fut évaluée par la mesure plane qu'on appelle "cross ratio". Cette mesure particulière d'invariance projective est une nouveauté dans le domaine de la recherche en vision. Il fut montré que, dans aucune des conditions expérimentales, on ne perçoit l'équivalence projective uniformément. Un effet significatif de changement de perspective fut trouvé dans chacune des expériences. Ces données furent interpretées comme évidence en faveur de la théorie classique des indices de profondeurs par opposition à l'hypothèse d'invariance.

Preface and Acknowledgements

There are many theories of shape constancy; their catalogue would be of unreasonable length. Yet the greater number of these theories share one characteristic. They assume or imply that projective congruence is seen directly, or that the magnitude of visual angle is seen directly. I hope that the following experiments and arguments will promote scrutiny of those assumptions. It is not my present intent to construct a theory of shape constancy. Rather, my intent is to spotlight a problem for contemporary theories. The problem is that perceivers may not be exquisitely sensitive to projective congruence. I am of the opinion that this seems a problem only when the consequences of wrong thinking about vision are accepted. A resolution of the problem is expressed by Wittgenstein, though I am at pains to explain what he could mean when he says: "Im Gesichtsraum gibt es keine Messung (1981, p. 266).".

The thesis contains an original contribution from my part, though I have received generous help from others. The six experiments are original contributions. The re-evaluation of the experiment reported by Attneave and Frost (1969) is new, also. That re-evaluation can be found in Chapter Two. Of those who helped, one deserves special mention. Professor John Macnamara has my heartfelt thanks and admiration. He has helped me more than I could say in several chapters. He helped to condense a cloud of ideas into a thesis topic, and any lucid passage that may be found in my turbid prose has been subject to his criticism. Under his supervision, I came to feel that "the supreme vice is shallowness. Whatever is realized, is right". I would like to thank the other members of my thesis committee -- Drs. Don Donderi, Yoshio Takane, and Steve Zucker -- for their poignant direction and helpful comments. Karen Wynn was a faultless research assistant. She conducted the fourth experiment, and she typed data for the experiments into computer files. Veronica Horn conducted the fifth and sixth experiments, and coded those data. Professor Vikram Bhatt showed his kindness by recommending participation in the fourth experiment to members of his architecture class. Professor Peter Dodwell deserves my thanks for his patience and support since my arrival at Queen's University. I thank Rhonda Amsel and Dr. Justine Sergent for their comments on a draft of the experimental section. Minerva Kühlenschmidt and Betty Loggie were endlessly patient in typing the manuscript. I thank Andy, Edward, Ernest, Floyd, Gaétan, Glenn, and Lorine. Without any one of them this thesis would not have been completed. Except for Brian Chamberlain's photograph of the greenhouse, photographic work was done by Robert Lamarche. Spelling and usage in the thesis are meant to conform to the Oxford English Dictionary. Any error in fact, emphasis, or presentation that may be found is my sole responsibility.

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Table of Contents

Page

CHAPTER 3

CHAPTER 4

CHAPTER 5

Page

CHAPTER 6

CHAPTER 7

Page

CHAPTER 8

CHAPTER 9

Page

List of Tables

Table of Figures

Page

Page

CHAPTER 7: Experiment 4

CHAPTER 8: Experiment 5

INTRODUCTION

> In the first place it's objected that in the beginning of the essay I argue either against all use of lines and angles in optics, and then what I say is false; or against those writers only who will have it that we can perceive by sense the optic axes, angles, etc., and then it's insignificant, this being an absurdity which no one ever held. To which I answer that I argue only against those who are of the opinion that we perceive the distance of objects by lines and angles or, as they term it, by a kind of innate geometry.
>
> Berkeley, An Appendix to the Essay on Vision, p.237.

Solid forms look unchanged as they are seen from different viewpoints. One may be tempted to explore the possibility that the explanation for this perceptual constancy is the constant shape of the real object. Why not hypothesize that we notice no change in the real object just because there is no change? To rule out obvious counterexamples, we might formulate a more conservative hypothesis: we notice no geometrical change in an object whenever the laws of optics would allow that the light that reaches our eyes emanates from an object of constant shape. We might add that the laws of optics so allow when lens images of the object, such as images of the object on the retina, are projectively equivalent. It is a simple matter to extend the hypothesis to the perception of a pair of objects: two objects will be seen to have the same shape whenever their images on the retina are projectively equivalent. From this we should conclude that if two surfaces are equivalent in Euclidean geometry, they will necessarily look the same, since the Euclidean equivalence of real objects implies the equivalence in projective geometry of their lens images.

One can approach the issue from a slightly different angle. One might ask: what is the geometrical condition that two objects should appear to have the same form? There seems a simple answer: objects look to have the same form whey they do have the same form. This answer is a psychological claim. The claim is that the physical congruence of two objects is a sufficient geometrical condition for the objects to be seen as congruent. Since large objects sometimes look smaller than small objects, the claim has to be modified. It is possible, too, that a large and a small object should look the same size. The Empire State Building looks larger than the Chrysler Building when seen from the observation deck of the Rockefeller Center in Manhattan. The Empire State Building _is_ larger. Yet the Empire State Building looks larger than the World Trade Center from that standpoint, though the Trade Center is larger. Sometimes these buildings don't look to be the sizes they are. Hence the simple answer to the question is inadequate — for size at least, which is one aspect of shape.

What, then, is the basic geometrical condition that two objects should appear to have the same form? The way that objects look is influenced by many factors. The laws of optics, such as the familiar inverse-square law of the propagation of light, play a central role in the description of the basic condition under which objects can appear to be the same shape. Could it be that forms appear the same when the laws of optics would allow that their appearances had been occasioned by objects of the same shape? In other words, could it be that the relevant projective information is preserved and accessible to viewers in the appearances of objects? It might seem plausible that it is and, certainly, this is a more interesting approach to shape constancy than the first.

Unfortunately, this simple idea cannot be the whole story. A flat object and the shadow it casts on a plane surface are projectively equivalent, and so the retinal images of object and shadow must be projectively equivalent. Yet an object and its shadow may be strikingly different in shape, as seen from any angle. Why, then, should anyone want to hold on to the principle? One might want to make the empirical claim that projective equivalence is accessed and used as the basis or the necessary condition for shape constancy. Alternatively, one might hypothesize that the constancy in shape which does exist in that situation poses no crucial condition for shapes to be seen as congruent. The characteristic geometrical quantity that is preserved in the propagation of light may not be accessible to observers. Call that geometrical quantity "the cross ratio". The proposition that the projective congruences which are preserved in the propagation of light form the basis of the geometrical condition that objects should appear the same shape has been put forward in the literature on shape constancy in vision (eg., in Gibson, 1979, and in Johansson, 1977). Call this contention "the invariance hypothesis" for purposes of identification, as in Boring (1952). The invariance hypothesis has a clear antithesis: that normal observers are not reliably and constantly sensitive to projective equivalence. The statement that is underlined is the thesis that will be defended. There are a number of terms and concepts that will be explained so that the meaning and import of the thesis may become clear. The place and importance of projective congruence in theories of shape constancy will require elaboration, too, after the notion itself is introduced.

The breadth of the invariance hypothesis — that shapes appear the same when the laws of optics would allow that their appearance had been occasioned by objects of the same shape — can be surveyed only when the notion of projective congruence is explained. In geometry, congruence is fundamental. Two shapes are congruent when they count as the same shape in some geometry. Historically and logically, projective congruence is fundamental to geometry. Projective congruence is basic to the laws of optics, too. One must appreciate the relation of projective geometry to other kinds of geometric congruence if one is to evaluate the psychological claim, placing it in the context of theories of shape constancy and size constancy. The reader must, then, pardon an excursus into geometry, so that what is meant by projective congruence and what is meant by an ordering of invariants may be explained. The latter is needed to explain the way in which projective congruence is basic.

A Word About Geometry

Two figures are congruent when their shapes count as the same. To count as the same shape, the figures must have a common measure. For example, two squares are the same when the lengths of their sides are the same. Length is the measure preserved. Such measures are called invariants. Invariants are quantities whose value does not change among figures of the same shape. Common measures are said to be invariant "under an operation" or "under transformation". To extend the example, two squares of the same size remain so regardless of where they are in a plane. In other words, length is invariant under the operation of translation. Measures other than length can be invariant too. The internal angles of two squares of unlike sizes are equal, but the lengths of the corresponding sides differ. In other words, angle is an invariant measure over the operation of magnification, but length is not. The operation of magnification or minification is also known as the similarity transformation. There is an ordering on such invariants which results in an ordering of the geometries that they characterize (Jäger, 1980, pp. 19-21). This ordering could be used to draw particular consequences from the claim that a condition for two objects to look the same in form is that there should exist a congruence between their projections that is implied by the laws of optics.

The notion of an ordering of invariants can be presented simply. Consider two rigid squares on a plane surface, each with its diagonals drawn. The diagonals ensure the rigidity of the figures. Call these figures "crossed squares". If their sides are the same in length, one can be laid exactly on top of the other. It can be superimposed on the other, as if by the overlay of transparencies. Their internal relations of connection, order, parallelism, congruence of lines, congruence of angles, and continuity are the same (these terms name groups of axioms in Hilbert, 1899/1950. "Congruence of angles" was added in 1939). A transformation of one of the crossed squares might be imagined as a regular deformation of one of the transparencies. More accurately, a transformation can be considered a "mapping of the plane onto itself". The sheet might be moved laterally, rotated, or even lifted and the reverse side placed over the static figure. The transparency film might expand, shrink, or stretch in one direction. For example, the film might stretch to change the length of the figures in one direction without increase in their area. The sheet could be stretched regularly in two directions at once. What measures will be invariant over these transformations? How will the invariant measures relate to one another?

The answer to this question supplies us with a hierarchy among geometries. Suppose that one transparency expands to form a larger crossed square; it will have the internal angles of the original, and its sides will remain parallel. The lengths of its sides and diagonals change. It is impossible, on the other hand, that it should retain the lengths of its sides and diagonals, but not also retain its internal angles. Invariance of length, then, implies invariance of angle and of parallels, but invariance of angle and parallels does not imply invariance of length.

Suppose the transparency overlay were stretched in one direction so that one of the crossed squares became a parallelogram. A parallelogram that is not a rectangle still has its opposite sides parallel, as does a square, but the corresponding crossed parallelogram retains neither the lengths of diagonals nor the internal angles of a crossed square. Moreover, its sides need no longer be of the same length as the original. Invariance of length or angle or both implies invariance of parallelism, but invariance of parallelism implies neither invariance of length nor invariance of angle nor both.

Now suppose the transparency is stretched in two directions to produce a trapezium with diagonals. There is a measure that exhibits the invariance between a square and a trapezium. This measure is called the cross ratio. I shall leave the name of the measure unexplained for the moment (but see Chapter 3, entitled "An Introduction to the Stimuli"). Our trapezoid with diagonals that is not a parallelogram has not the lengths of sides, nor the angles, nor the parallels of a crossed square, but its cross ratio is the same as that of the original. The projective invariance measured by the cross ratio is implied by invariance of parallelism, invariance of parallelism and angle, invariance of parallelism, angle, and length, or by any of these considered separately. As before, the converse implications are false. Length, angle, parallelism, and a projective invariant: these invariants in this order form a chain of implication.

The pattern of implications might suggest that invariance of length is, in some sense, the primary or most important invariant. Not so. Although length is the measure most important to a carpenter, it is not the most important one to a mathematician. An insight of the prior place of projective or "descriptive" geometry was expressed by the mathematician Cayley: "The more systematic course in the present introductory memoir...would have been to ignore altogether the notions of distance [i.e., length] and metrical geometry...Metrical geometry is a part of descriptive geometry, and descriptive geometry is all geometry" (1859, cited in Coxeter, 1969, p. 230). His exclamation is an exaggeration, but a slight one, and not fundamentally misleading.

The ordering of invariants is familiar to psychologists. Klymenko and Weisstein (1983) suppose that their readers are familiar with the notion, since, without further explanation, they mention that "Klein, in his Erlangen program, categorized various geometries in terms of the structural properties that remain invariant under transformation". Dissemination of these ideas can be attributed to followers of Gibson - as witness Shaw, McIntyre and Mace (1981) and Michaels and Carello (1981). Besides, the ordering of invariants is often cited by Piaget (1968). The Erlangen program just mentioned provides the organization of chapters for the The Child's Conception of Space (1948/1963). Although Piaget's exposition may be flawed, as Kapadia (1974) notes, the size of Piaget's audience assures that many psychologists have heard of the idea. There are even a few studies in the development of the perception of invariants. Day and McKenzie (1973), Shaw, Roder, and Bushell (1986), and Gibson, Owsley, and Johnston (1978) are among the best examples. The best recent discussion of Klein's ordering of invariants and its application to the psychology of vision can be found in Cutting (1983). A substantiation of the explanatory power of the ordering of invariants can be found in Kramer's (1982) chapter, entitled: "The Unification of Geometry". A more sophisticated overview is found in Torretti (1978), and also in an article by Jasińska and Kucharzewski (1974). There are more basic properties that are properly called geometrical, such as continuity, but their place in applied geometry is abstruse. Certainly they have been little mentioned in the study of vision. The invariants of projective geometry are vital to us, then, since they are basic geometrical properties, and they serve to characterize projective geometry (all the quantities and theorems of a geometric system are derivable from its invariants and their relations, Klein, 1925/1967, pp. 156, 159). Physical optics is a model for projective geometry. A psychological claim that will be shown to be standard in the literature on shape constancy is that "the congruences that are preserved in the propagation of light form the basis of the geometrical condition that objects should appear the same shape". The cross ratio is the simplest measure of projective congruence and all other measures of such congruence are a function of it. Hence it is the simplest measure of the congruence that is preserved in the propagation of light, and the measure appropriate to a test of the claim.

CHAPTER 1

Projective Invariance and Visual Perception: Some Representative Theories

> **We saw that projective Geometry is necessarily true of any form of externality. Its three axioms...were all deduced from the conception of a form of externality, and, since some form is necessary to experience, were all declared a priori.**
>
> **Russell, An Essay on the Foundations of Geometry,** p. 200.

The importance of projective congruence to the study of vision may not be apparent immediately, though claims about sensitivity to projective congruence proceed from reasonable suppositions about the nature of vision. It is surprising that these claims are sometimes held to be inevitable. There are several traditions in the psychology of perception that share a concern with projective properties, despite great differences among these traditions. I will sketch the theories held by four noted authors to show how diverse theories have this connection. Helmholtz represents the tradition of Empiricism within epistemology; he forged the doctrine of unconscious inference. Gibson's theory can be considered a naïve realism; he made the term "direct perception" popular within psychology. Rock represents the Gestalt tradition, though his theory diverges from traditional Gestalt theory. Ullman represents a computational approach to vision. Each of these authors stands for a theoretical tradition as well as being important in his own right. Each makes claims about the role of projective properties in vision. My intent is to show that such claims are common, and that they are working parts of the respective theories, if not essential parts.

Helmholtz

Helmholtz, who was a psychologist as well as a physicist, conceives the human observer as a naive astronomer - an humble Galileo (von Helmholtz, 1867/1925, pp. 4, 164, 282, 297). Once astronomers sought to discover the distances and relations of celestial objects by studying the appearance of the night sky. Similarly, he thinks, ordinary observers discover the distances and relations of middle-sized objects from their appearances. Other psychologists have conceived of vision differently, and the differences in their theories are marked by differences in their conceits. For example, Gibson's (1950a) exposition has the vitality and immediacy of his theme: an aviator who makes a landing without instruments. His attention is on the onrush of the ground below. So, Gibson's theory stresses the importance of motion, ground, and horizon. A recurring theme will be that both Helmholtz and Gibson assume that projective invariance is the basis of shape constancy.

Helmholtz believes that the difference between an astronomer and an ordinary observer is that the ordinary observer is led to conclusions about the geometry of objects automatically, while the astronomer struggles long and hard with theory to achieve his knowledge. Helmholtz speaks of "unconscious conclusions" in this respect, but means only to distinguish the results of vision from conclusions of deductive logic (1867/1925, p. 4). He abandoned the use of the term because it confused his readers (1879/1968, p. 220). The basis of the analogy with a natural astronomer is that an astronomer may compute the relative distances of celestial objects from their perspective images. In the night sky, all lights appear indefinitely distant, as if they were projected onto a surface of some shape. This "surface" was once known as the celestial sphere. Early astronomers had several means at their disposal to estimate the distances of the planets from their projected images. One example is the relative motion of the planets with respect to the "fixed stars"; another is triangulation of the position of the planets, when the diameter of the earth's orbit about the sun is used as a base. Helmholtz draws an analogy to the cues of relative motion and disparity on the scale of the objects seen in everyday situations. He thinks of the perceiver as applying the science of optics unconsciously, whereas an astronomer applies it consciously (1867/1925, p. 4).

The celestial sphere is a fiction that arose because the stars and planets appear indefinitely distant. Helmholtz uses a similar fiction in his theory: the "visual globe". He distinguishes the "geometrical place" of a visual image from the "apparent place" of a visual image. The "geometrical place" of an image is not the position of the physicial object that causes the image; it is a place on the visual field, whose description begins with that of the retinal surface. The geometrical place of an image is determined by the direction of fixation, as well as by the internal optics of the eye. An "apparent place" is where a thing of indefinite proximity to the observer appears to be. The apparent positions of "entoptic" images illustrate such as indefinite character. When one stares at a featureless blue sky, if one focusses "on infinity", faint shadows that cross the field of vision may be seen. They are shadows of epithelial cells that have been shed into the vitreous humor. The point is that these appear to be objects external to the observer, and they do not appear to be at a definite distance from the observer.

To return to the discussion: the totality of apparent places is called the "visual globe". Positions on the globe are not necessarily predicted by projections outward from a retinal image. The apparent place of a point in an image might not be predicted by its geometrical place. In short, Helmholtz recognizes a distinction between the apparent positions things ought to have if they depended solely on the laws of physical optics and their actually apparent positions (1867/1925, p. 164). The distinction is maintained even when the two coincide. The apparent positions of things depend on the laws of optics, too, but in addition they depend on an observer's belief about the real positions of things.

Helmholtz makes the provisional hypothesis that the visual globe reflects knowledge (one might say implicit knowledge) of a projective relation between retinal images and the real shapes of things. The hypothesis is a corollary of the doctrine of unconscious inference. Helmholtz believes that all visual illusions can be explained by a mechanism of shape constancy; namely, that "we always believe that we see such objects as would, under conditions of normal vision, produce the retinal image of which we are actually conscious" (1868/1968, p. 130). Retinal images are produced from solid objects by laws of physical optics that govern the propagation of light. These are projective laws which Helmholtz supposes to be operative in the unconscious process by which shapes are estimated. He stresses that the implicit use of projective relations reflects cognizance of natural law; it is not "hard-wired" - to use a modern idiom. If the form of optical laws were different, new relations could be learned and used in the interpretation of the retinal image (see Helmholtz, 1876, 1878). Helmholtz uses geometry to deduce positions and directions on the visual globe by projection on to a specific surface. The specific description of that visual globe is fixed by the provisional assumption that it is a projection of the visual field. Nevertheless, "we are at liberty to assume any arbitrary form for this surface, as soon as there are any new factors of perception tending to throw light on it" (Helmholtz, 1867/1925, p. 281). All that is necessary to the visual globe, Helmholtz (1867/1925, p. 189) claims, is that it be two dimensional.

Helmholtz thus argues that projective relations are not a necessary part of unconscious inference. He criticizes Hering for postulating such a necessary geometric relation between the real and perceived shapes of objects. Helmholtz sees no more than a "certain pedagogic value" in any geometric description of the relation between retinal images and perceived shapes (1879/1967, p. 223). He thinks that sensation offers an array of symbols or natural signs which have no necessary connection with positions on the retina. He even allows that like signs might correspond to positions distributed randomly across the retina, so that there would be no relevant contiguity between the visual signs of adjacent points. However, "from analogy with other organic contrivances, as well as on other grounds" (1867/1925, p. 536), he makes the assumption that the laws of optics are used to interpret the retinal image when shape is estimated. The stress that Helmholtz places on those laws indicates that projective invariance is a working part of his theory, even though not a necessary one, as it is of the next theory to be mentioned, Gibson's.

Gibson

James Gibson's legacy is difficult to evaluate; his ideas were vital and fluid. In his last book he abandons claims he made in earlier writings, yet there is a significant continuity to his study of vision. Parts of his early work seem a reaction to the theories of space perception that were contemporary with his first researches (for example, that of Luneburg, 1947). With time, his approach emphasizes four themes: one, that perception is direct, and not mediated or the result of a process of inference; two, that complex variables of the viewing situation play a large part in the theory of perception; three, that the normal case for the study of perception is a changing visual array for a moving observer; and four, that the observer should be considered in his environmental niche. Gibson stresses projective invariance in all his researches. I shall discuss the relevant changes in his theory in the chronological order of the three books in which they appear.

Gibson calls the theory contained in The Perception of the Visual World (1950a) a "ground theory" of vision, which he contrasts with "air theories". By "air theories" he means those monolithic theories of space perception, of which the Blank and Luneburg proposal (of hyperbolic geometry as a model for visual space) is only one. In the "ground theory", Gibson supposes that it is meaningless to describe the perception of spatial relations without reference to specific surfaces and edges. Surfaces and edges in the world are related to their images on the retina by a lawful transformation. An observer may attend either to the solidity of surfaces, or the foreshortening of surfaces. Gibson contrasts an observer's normal attitude (the visual world), in which he attends to the solid shapes of things, to a painter-like attitude (the visual field) in which the observer attends to the perspectives of things. The practical difference is that "the visual world contains depth shapes, whereas the visual field contains projected shapes" (1950a, pp. 34-35).

In the theory, the retinal image is important as a projection of the world, and movement is important in that it changes the pattern of shapes projected on the retina. Gibson thinks that variations in the retinal image underlie the classic cues for depth: "the whole problem of pattern perception might be conceived in terms of the geometries of certain invariants and transformations" (1950a, p. 153). He identifies projective geometry as primary among these. In a footnote he mentions the cross ratio of points on a line range as an example of a projective invariant. There may be some doubt that Gibson recognized the hierarchical relations among geometrical invariants at this point. In fact he states, erroneously, that membership is mutually exclusive among the equivalence classes determined by various familiar geometries (1950a, p. 193). His subsequent discussions are free of the confusion. He comes to recognize that projective invariants are fundamental. Consideration of the projective invariants of textured surfaces led Gibson to consider his "ground theory" as a gradient theory. Perspective gradients in texture, as illustrated in pictures of terrain, can lead to the impression of a surface that recedes in depth. Cutting and Millard (1984) show that perspective variation is indeed the most reliable source of information in texture gradients. The gradient theory is meant to explain how observers perceive certain variations in distance over a surface. Two themes important in Gibson's later work are present as asides in this first book. At the time of The Perception of the Visual World, Gibson considers the claims that pictures yield only presumptions of real shape, and that the slant of a surface is an important factor in the perception of its form.

The Senses Considered as Perceptual Systems (1966) has a different emphasis. No longer do the retinal projections of surfaces have primary importance. Now Gibson claims that sense-impressions are the incidental occasions of perception, not the basis of perception. Yet there is evidence that sensations affect estimates of shape, since shape constancy is not always complete. Gibson's emphasis is that properties of the image obtrude on what is normally perceived: the real properties of surfaces. "Putting it in another way, sometimes we attend to the pictorial projections in the visual field instead of exclusively to the ratios and other invariants in the optic array" (1966, p. 306). Thus, he considers projective descriptions of surfaces in the environment to be the elements of a theory of perceived form. Surfaces are called faces or facets, depending on their size. Faces specify form in the visual array, while facets specify texture. No qualitative distinction is made between faces and facets (1966, p. 208).

A theme of the book, and the notion with which Gibson introduces the study of ecological optics, is information. Information is contained in ambient light, but it is distinguished from energy. The process of photosynthesis requires light energy, but not information from light. The function of information is to identify its source. If an observer has information about a tree in this sense, then that tree is thereby specified. The source is not conveyed as if by a literal copy, though the information for an object is contained in the perspective projection of that object. Information resides in the structure of the visual array, that is, in the structure of ambient light. Two origins of the structure in light are the geometric relations of the objects seen, and the geometrical conditions under which those objects are viewed (1966, p. 221). Gibson distinguishes two sorts of information: specification by convention, and specification by projection. His examples of convention and projection are these: a license plate specifies a car by convention, while the car's shadow on a driveway specifies it by projection (1966, p. 235). Projection is the normal means of specification in vision. The basic variables of visual information are, he claims, the invariants of projective geometry (1966, p. 313). In fact, what it means for environmental information to be conveyed is that a property of the stimulus is unambiguously related to properties of surfaces by physical laws (1966, p. 187).

Presumably, the laws governing the propagation of light are important examples of such laws. The invariances expressed in laws of physical optics are, of course, projective. He says: "the relational invariants of perspective or projective geometry...I argue, carry most of the information about the world" (1966, pp. 312-313). Gibson sums up his general account of information in one sentence: "The same stimulus array coming to the eye will always afford the same perceptual experience insofar as it carries the same variables of structural information" (1966, p. 248). Structural information is the invariance that is preserved in ambient light, and the basic dimension of structural information is projective invariance.

The Ecological Approach to Visual Perception (1979) develops the idea of ecological optics as a study parallel to physical optics. The difference between pure geometry and the application of geometry in ecological optics is stressed. Yet the two have more features in common than differences. For example, ecological optics is concerned with "surfaces" and "the medium" instead of "planes" and "space". The difference is akin to that between Euclid's constructions and the Cartesian method of coordinates in analytic geometry, where ecological optics has the greater similarity to Euclid (Gibson, 1979, p. 132). The natural perspective of ecological optics is still described by the "elegant trigonometrical relations" of solid angles in the projection of solid forms onto a surface (1979, p. 70). The study of the perspective of objects in motion, or as seen by a moving observer, takes pride of place in ecological optics. Still, the relevance of the static case, the "pause in locomotion...a temporarily fixed position relative to the environment" is not lost (1979, p. 75). Layout is specified by the perspective structure of light. A perspective structure is present when the observer moves, and also when he pauses. Locomotion itself is specified by the invariant structure of light. The terms "invariant structure" and "perspective structure" connote the difference between "what specifies locomotion" and "what specifies layout", and are not meant to designate different kinds of structures.

The importance of motion in Gibson's last book is a reflection of the originality of his thought about form perception. The classical formulation of the problem of shape constancy may be introduced this way: consider a square surface, such as a tabletop, and the projections of this shape onto arbitrary planes. From any position where the tabletop is visible, except from positions directly above, the plane projections of the tabletop will not be square. A condition of visual perception is that images of surfaces are projected onto the retina. It is entirely probable that one can tell the tabletop is square by the way it looks, without ever having been suspended above it. This example, and the question "how is the shape of the tabletop known to be square"? are a standard form of the problem (1979, pp. 74, 168). The example and the question tempt one to make dubious claims about computations on retinal images as projections. They also tempt one to make a background assumption about the manner in which knowledge is gained from the way things look, in order to be able to connect the example and the question.

Gibson makes neither these claims nor this assumption; he construes the problem in a different way. He speaks of the invariants that are seen by a moving observer as "formless". He means that they are not attached to any shape; that is, they are not attached to any particular static projection of the object. If one is walking about an object, it is hard to imagine what a still picture of the object would be like. Gibson claims that what a moving observer sees is not like a series of snapshots, but the invariants themselves, as abstracted (though Gibson would not use the word) from particular retinal sensations or phenomenal impressions. If the observer pauses in his motion, he can notice the effects of perspective. Those effects can obtrude on perception of the invariants. If an observer resumes his motion, then he cannot notice the effects of perspective, so Gibson claims (1979, p. 197). The possibility of a "painterly attitude" does not arise for a moving observer, since a greater sample of invariants is available (one might say afforded) to him. Gibson's verdict is not that the perception of invariants is different in kind for a moving observer than for a stationary one, but that perception by a stationary observer is secondary in the study of vision. The notion of a "formless" invariant may clarify Gibson's statement on picture perception:

> "All along I have maintained that a picture is a surface so treated that it makes available a limited optic array of some sort at a

point of observation. But an array of what? My first answer was, an array of pencils of light rays. My second was, an array of visual solid angles, which becomes nested solid angles after a little thought. My third answer was, an array considered as a structure. And the final answer was, an arrangement of invariants of structure" (1979, p. 270).

Though its proper place changes from the visual field to formless invariant, projective invariance is essential throughout Gibsonian theory. At first Gibson believes that a description of the environment in terms of certain invariants and transformations can encompass the whole of pattern perception. Later he recognizes that some problems, like occlusion by the advancing edge of a surface, are not easily accommodated in such a description. For him, the word "invariant" does not always have a mathematical force. Nevertheless, the motif of the theory is clear. In large part, Gibsonian theory deals with the projective invariants of real surfaces. For him, the projective laws of physical optics, or their ecological counterparts, are general descriptions of the conditions of perception. Specification by projection is important for vision; the basic variables of information in vision are projective invariants. Though motion is crucial, the theory has implications for picture perception and for the stationary observer, as well. Gibson understands that the perspective structure of pictures specifies the layout of the environment, though it may not specify locomotion. Projective invariance, then, is a keystone of the Gibsonian theory of form perception.

Rock

Irvin Rock's theory of form perception is different from Gibson's later theory in two important respects. First, Rock's goal is the study of phenomenal qualities. He takes "perceived" to be synonymous with "phenomenal" (1975, p. 9). Secondly, Rock makes reference to a "proximal mode" of vision. The proximal mode has many of the characteristics of Gibson's "visual field". In the proximal mode, some phenomenal qualities have an odd characteristic; they are quantitatively identical to properties of proximal stimuli. Proximal stimuli are sensations by another name.

Rock postulates that the projective geometry of light is internalized as a system of rules known implicitly. In this context, he uses the term "central projection", which is an application of projective geometry. The supposition that such knowledge is implicit is neither a vague nor an unfamiliar notion. After all, we seem to be able to determine with ease the angle at which we must incline a bicycle as we round a bend. We do that unconsciously. Tacit knowledge of projective geometry is knowledge of the form of some optical laws; namely, those that describe the geometrical conditions of perception. Rock says that the following rules reduce to a projective principle: that "the visual angle of objects of the same size is inversely proportional to distance (law of visual angle); the size of objects subtending the same visual angle varies directly with distance (corollary or Emmert's law); surface extents yield visual angles that are reduced as a direct function of the angle of their plane with respect to the frontoparallel plane (foreshortening); parallel lines in planes other than the frontoparallel yield converging image lines (linear perspective)" (1983, p. 325). He makes the further claim that we have a quantitative knowledge of these relations; for example, that implicit knowledge of the law of visual angle is exact knowledge of the inverse-square law for the sections of solid angles. And, he says, we must know this (1983 p. 278). The same knowledge is said to account for the kinetic depth effect and the perception of objects in motion (1983, p. 325).

If such knowledge is implicit in perception, how is it used? Rock presents an example from size constancy. He believes that an observer perceives the visual angle of an object (see Rock and McDermott 1964). Somehow, and independently, the observer knows the object's proximity. The observer forms a percept from these data.

The datum of the visual angle, data about the object's proximity, and the resulting percept *are related* as if by an inference performed unconsciously, in which the data would take the place of premises, and the percept would be the conclusion. This unconscious inference is a deductive inference of predicate logic (1983 p. 273), though it may not be instantiated in such a way as to be recognizable as such. A statement of the inverse-square law forms the major premise of the inference (the inverse-square law is a projective regularity). The particular visual angle and distance are described in the minor premise; they are substitution instances of the general law contained in the major premise (1983, p. 279). Yet the conclusion of a deductive argument has the same form as the premises in the sense that both have logical form - both affirm or deny something (Łukasiewicz, 1951, p: 3).

Rock does not make it clear if the percept itself is the conclusion of the argument, as he suggests once (1983, p. 273), or if the conclusion of the deductive argument initiates an automatic procedure that produces an appropriate percept. If the percept is the conclusion of the argument, then the percept has logical form, like a declarative sentence. The point may be elaborated: If observers have knowledge of optical laws, that knowledge should consist of propositions, expressed by symbols. Such symbols alone can provide the premises for deduction, or for any inference. The "output" of deduction is another string of symbols with an interpretation. I assume Rock does not mean to claim that a percept is a string of symbols or that percepts have logical form. On the other hand, if the conclusion initiates a procedure, why is an inference needed? There are more plausible input conditions for a procedure. At any rate, the point to be emphasized is that, for Rock, unconscious inferences are deductive inferences of predicate logic that employ the inverse-square law and perceptual data as premises.

The reader may wonder how, on Rock's account, seeing is different from thinking about physical optics. There are two differences. Both are restrictions on the domain of operations. First, "perception differs from thought primarily because it is rooted in and constrained by the necessity of accounting for the proximal stimulus" (1983, pp. 339-340). That is, the data are different. Secondly, "the other major difference is that perception is based on a rather narrow range of internalized knowledge" (1983, p. 340). Perception is like thought about projective geometry in application to dioptrics.

Rock's theory of form perception has an important similarity to Gibson's later theory. Both assert the obvious: that the physical conditions of perception influence the way things look. They emphasize what is called the "relativity of perception", meaning that the way things look depends on the conditions of observation. People look odd when seen in funhouse mirrors, and the apparent colours of things undersea tend to blue-green with increasing depth in water. The two theories use the relativity of perception in the same way. They present it as a problem. Yet unless one begins with the assumption that things cannot appear to be otherwise than they are, why should one be put out by the obvious fact that they can and do? (Warnock, 1953, p. 148). The strategy of both theories is to suggest that things cannot appear to be otherwise than they are, when viewing conditions are taken into account. Gibson takes viewing conditions into account in his theory when he searches for the invariants in light and in the environment. Rock attributes a process of inference to the observer so that he (the observer) may compensate for the effects of viewing conditions.

The centrality of the conditions of observation to Rock's theory is evident by the examples that count as exceptions to the theory. Some perceptual effects remain unexplained when compensation is made for the condition of observation. These are the geometrical illusions. Rock has no explanation for them. If the strategy — that things cannot appear to be otherwise than they are, when viewing conditions are taken into account — were good strategy, there would be an easy explanation for visual illusions. In their ignorance of the conditions of perception, observers would make errors just insofar as they failed to compensate for the physical conditions of observation. The supposed compensation is automatic, and errors in compensation would be reflected in perceived properties that do not correspond to the real properties of objects. These are illusions. A puzzle remains, for Rock says: "the one area where it is obvious that there is a problem requiring explanation is that of the so-called geometrical illusions" (1975, p. 389).

What precisely is the problem posed by the geometrical illusions? It is that they do not conform to the desired explanation. They are important as exceptions to a general rule that is meant to relate perceived shape and real shape. Rock rejects the explanation that geometrical illusions result from misapplication of the mechanisms that underlie constancy (1983, p. 262). Many figures produce illusions whether those figures are seen in the proximal mode or not. It doesn't help to adopt a "painterly attitude". An artist who sketches a perspective drawing of the Zöllner or café-wall illusions will depict the principal lines as skew, not parallel. He will not have the "right" percept in proximal mode to depict them as they are, that is, parallel. In such cases, the proximal stimuli do not provide "good" data for an inference.

The data are supposed unfit because the percept is illusory, which is as much as to say that the conclusion of the unconscious inference is false. One might say that the form of the proximal stimulus has been mistaken in such cases (1983, p. 262), but that would be to admit that there is more to this story than dioptrics. In effect, Rock claims that there exists a lawful relation between real shapes and perceived shapes that is governed by the projective nature of physical optics. If there were no such regularity or law, the desired inference would not be possible. The geometric illusions constitute a class of exceptions to the law; hence their occurrence is not covered by the inference. The importance of the geometrical illusions as exceptions reflects the centrality of the postulated rule within Rock's theory. Let us pursue the matter further.

One of Rock's rules for the dependence of perceived shape on the geometrical conditions of vision is Emmert's law. At least, the supposed rule has the form of Emmert's law, though it is a peculiar interpretation of Emmert's law. Emmert's law describes how the areas apparently covered by an afterimage are scaled as the afterimage is trained on frontal surfaces of varying distance from the observer. The law generalizes to flat surfaces of different slants, as well. Suppose an observer fixates the center of a medium-sized and brightly coloured square that is painted on a wall one meter distant from him. After several minutes, he moves slightly to the side, so that his afterimage is trained on a blank portion of the same wall. He estimates the area that the afterimage appears to cover. Then he fixes his gaze on a wall five meters distant. The area that the afterimage of the square appears to cover will be larger than before. The estimated area, or that which Rock calls the perceived size, can be expressed as the numerical product of the distance of the relevant wall and the visual angle that was subtended by the painted square (1975, p.33). The peculiar interpretation, which is meant to generalize to other problems of size constancy, is this: "If the perceptual system works like a computer and effectively multiplies the visual angle by the distance of the object in arriving at perceived size, then if visual angle decreases as distance increases, the product may remain constant (size constancy); if visual angle remains constant as distance increases, the product will increase (Emmert's law)" (Rock, 1975, p. 34).

There is an analogue of Emmert's law for flat surfaces of varying slant. The area on a slanted surface that will appear to be demarcated by an afterimage can be predicted when Emmert's law is extended to surfaces whose distance from the observer increases uniformly in one direction. The resulting formula expresses the area over which the afterimage seems projected, as a function of: the distance of the surface from the observer, the orientation of the surface to the observer, and the visual angle that was subtended by the shape that created the afterimage. One might interpret the formula differently. Rock says that the equation relates the perceived shape of planar surfaces which present a constant visual angle, to their distance and orientation from the observer (1975 p. 70). The corresponding general claim is that a law of shape constancy can be derived from Emmert's law, since the law can be extended to make predictions about shape. "Thus for every constancy there is an analogue of Emmert's law that is perfectly comprehensible in terms of the same law that explains that constancy" (1975, p. 561). This indicates the general rule of perceived shape that makes the geometrical illusions seem so important as exceptions, when the rule is assumed.

What does the area apparently covered by an afterimage have to do with perceived shape? Rock's assumption is that, in perception, implicit knowledge of projective relations is used to estimate which real and present object could cause the lingering impression that is the afterimage (1983, p. 325). The same implicit knowledge is thought to be exercised whenever there is constancy of perceived shape. The similarity to Helmholtz's doctrine should be obvious, though Helmholtz does not insist that the unconscious process must apply projective relations. It seems so evident to some that "projective equivalence" is the central problem of form perception that it is passed over as common knowledge in some treatments - for example, as in the first paragraph of Hildreth (1984). The assumption that projective laws are at work seems to follow simply from the dependence of perceived shape on the geometrical conditions of vision. What basis has the putative relation between real shape and perceived shape?

The kernel of the assumption is that constancies of perceived shape are based upon, or derived from, knowledge of just one of the properties of the conditions under which objects are seen. Huygen's principle for the propagation of light describes that property (Rock, 1983, p.324). The principle may be posed thus: "every point of a wavefront may be considered the source of small secondary wavelets, which spread out in all directions with a speed equal to the speed of propagation of the waves" (Sears & Zemansky, 1970, p. 544). Many laws of physical optics, such as the law of reflection and Snell's law, can be derived from the principle. The propagation of light from nearby sources to a surface, say a flat coronal section of the lens of the eye, is a prototypal example of a projective relation. The constancies, or in other words, the invariants that are preserved in the propagation of light are thought to underlie both shape constancy and size constancy. That these are projective invariants is shown in elementary physics (Wyzecki & Stiles, 1982, pp. 2-3). Rock considers the perceptual constancies of shape and size to be consequences of implicit knowledge of the projective invariance derivable from laws of physical optics. The property of projective invariance is meant to provide a foundation on which psychological laws of shape constancy can be based.

Actually, Rock's claim that projective invariance is the basis of shape constancy can be extended to an account of the geometrical illusions, though he does not do so. Gregory's theory of misapplied constancy scaling is such an account.

Gregory supposes that persistent and incorrect estimates of geometric quantities -- the mark of the visual illusions — are attributable to a mistaken application of the ordinary mechanisms of scaling which produce shape constancy. His assumption is that a complete theory of shape constancy explains just how the real shapes of things are seen despite the effects of linear perspective. Gregory considers figures that produce illusions to be impoverished sketches in perspective (1970, p. 90).

Perspective drawings suggest differences in depth, and observers tend to attribute depth to portions of figures in accordance with pictorial cues. When depth is attributed to parts of a figure, there will be concomitant variation in the apparent size or tilt of other elements in the picture. For example, a rectangle that is pictured as distant will look larger than an identical rectangle that is pictured as near. The persistent changes concomitant with pictured depth are the aspects of the figure that identify it as a geometrical illusion. Gregory cites the Ponzo illusion as a perspective drawing analogous to a picture of two objects placed at different distances (to the observer) between receding railway tracks. Similarly, the Müller-Lyer illusion is like a picture of the internal and external corners of a rectilinear building, where the arrowheads represent the join of two right angles. (Perspective line drawings are known to provide cues that indicate pictured depth as effectively as do the cues provided by shaded diagrams or photographs, e.g., Smith, Smith, & Hubbard, 1958).

Gregory supposes that there are textural cues that indicate the flat shape of the surface on which the illusory figure is drawn. These "contradict" the cues that are offered by the figure as a perspective drawing. To eliminate these textural cues, he uses an arrangement of polarizers and a half-silvered mirror - an arrangement of the kind that was used by Attneave and Frost (1969) in their researches. Gregory calls the ~~apparatus~~ "Pandora's Box" (it is described in Gregory, 1973, pp. 158, 159). When illusory figures are presented in the apparatus, binocularly guided judgments of pictured depth can be made on parts of the flat figure. Different estimates of pictured location in depth are made for the main lines of the Müller-Lyer illusion. The estimates depend upon the steepness of the arrowhead that terminates each line. "The prediction is that distortion illusions should reduce to zero when scaling features of the figure are the same as for the object of which the illusion figure is a representation" (1978, p. 349).

It has been noted already that Rock's theory of shape constancy depends on projective invariance and perspective rules. Gregory goes beyond Rock's claim to say that projective invariance and perspective rules provide a complete theory of shape constancy that subsumes the geometric illusions. There is an anomaly in the theory similar to one that will be noted in Attneave's evidence for the use of projective rules in perception: Gregory's theory applies only if the pictured objects can be assumed to have an extremely simple form. Not all pictured corners are right angles, nor are all lines pictured as converging actually parallel in reality. The observer is supposed to make an unjustified assumption that is far more powerful than any that is required to resolve the projective equivalence of pictures. The important point for the present discussion is this: Gregory assumes that visual illusions are equivalent to perspective drawings that are projectively congruent to the objects they depict. The illusions are thought to be produced by an implicit knowledge of that projective equivalence. His theory can be thought of as a logical extension of Rock's account of shape constancy, though one that Rock does not accept.

Ullman

Ullman (1979) also believes that some geometrical laws play the role of assumptions about the world which are implicit in vision. He provides a scheme that interprets a series of still views or pictures as different views of the same solid object. The object might move and rotate with respect to the picture plane. Geometric laws are used to predict which solid object could have given rise to the series of pictures. Different views would correspond to snapshots of the object at instants of its motion and rotation. Ullman's scheme will determine the object that gave rise to the series of pictures, if the series of pictures can be given such a geometric interpretation consistently. The scheme will be discussed as it relates to successive static views of a single object.

Ullman divides the problem of form perception into two nearly independent topics. He conceives of the derivation of figural unities from still views as one problem, and the interpretation of those figures as projections of a solid object as a second problem. He proposes independent processes to carry out these tasks. The figures that result from the first process are data for the second process. Ullman calls the first "the correspondence process", and the second "the interpretation process". The correspondence process establishes relations among the elements within a single picture; it proposes a solution to the problem of figural unity or Gestalt formation. Ullman (1978) makes it explicit that, though figural unity is achieved in two dimensions, the locus of the process is not the retina. The second process determines that two or more pictures have arisen from a unique solid object in rotation or translation (or both). The second process addresses the problem of "projective ambiguity" in vision. Here "projective ambiguity" is a name for the theorem that a single picture may be a projection of an infinite number of planar or solid forms.

The interpretation process is the one of interest here, because it uses geometrical rules to "reconstruct" the object from which the pictures are projected. The object is not actually constructed; rather, a description of the object is recovered from the forms of the pictures. Unlike his description of the correspondence process, Ullman's description of the interpretation process seems more geometrical than psychological in nature. He assumes that any series of still views that can be interpreted consistently by geometry as successive projections of an object in rigid motion will be so interpreted by an observer. He sees it as a psychological problem that there are many objects (differing in shape and orientation) that could give rise to the same two-dimensional projection on the frontal plane. He says that the interpretation scheme for vision must have constraints that determine a unique solution, or at least identify a small number of alternatives. The relevant constraints, he claims, constitute "a set of implicit assumptions about the physical world which, when satisfied, imply the correct solution" (1979, p. 142).

Ullman has an exact conception of these constraints. They are geometric in nature, and the rules of interpretation are geometric rules. The end to which he applies geometric rules is described succinctly in his "structure-from-motion theorem". The structure-from-motion theorem states that from Euclidean measurement on at least three distinct affine mappings of four noncoplanar points onto a plane, the relative positions of the rigid group of four points and their directions of translation and rotation can be recovered. In his words: "Given three distinct orthographic views of four noncoplanar points in a rigid configuration, the structure and motion compatible with the three views are uniquely determined" (1979, p. 148). Given four coplanar points, their structure and motion are simpler to recover than those of four noncoplanar points. The structure and motion compatible with several views of these are derivable by "straightforward trigonometric considerations". Ullman goes on to make the strong psychological claim that the procedure he has constructed to determine the structure of a rigid object (in accordance with the theorem) is a complete account of the interpretation process in vision. That he considers the procedure a complete account is shown in his "rigidity assumption": "Any set of elements undergoing a two-dimensional transformation which has a unique interpretation [by the theorem] as a rigid body moving in space should be interpreted as such a body in motion" (1979, p. 146).

Other researchers accept propositions that are equivalent to the rigidity assumption. Research on the kinetic depth effect assumes that objects are perceived as rigid just when their projections could have been occasioned by a rigid object, also. Johansson, von Hofsten, and Jansson (1980) cite such an assumption as motivation for their proposal of projective geometry as a primary tool in the study of spatial constancy. Oddly, the results of an experiment by Jansson and Johansson (1973) are inconsistent with Ullman's rigidity assumption. In the sentence that follows their paraphrase of the principle, they say: "the outcome of the experiment is in good correspondence with such a ... principle if the latter is extended to include also the partially rigid motion of bending..." About this one need only ask: is a bending object a rigid object? Either objects are rigid, or they are flexible, but they are not both. However, let us continue the discussion at hand.

Orthographic projections (affine mappings onto a plane) are the data for Ullman's procedure that recovers the structure and motion of a rigid configuration. These operations are more constrained than projective mappings onto the plane. Orthographic projections preserve parallelism as well. Ullman asserts that the projection of light from solid objects to the eye is not an affine mapping, but a projective mapping (1979, p. 153). Yet he claims that the differences between the two types of picture that are produced by perspective projection and orthographic projection are not psychologically salient (Attneave and Frost, 1969, provide some relevant evidence). In other words, he claims that affine and projective mappings are not psychologically different. He proposes a method to approximate projective mapping by a combination of the results of several affine mappings. This method he calls the "polar-parallel" method. His "polar-parallel" method supplies an approximation to the geometry of perspective projection by means of orthographic projections. The method uses orthographic projections whose axis of projection changes from region to region. Four closely grouped points constitute a region. Ullman makes the argument that the polar-parallel scheme is about as accurate an approximation to perspective projection as is evident in human judgments (1979, p. 153). He says that the polar-parallel scheme "resembles human performance in both its capacity and its limitations" (1979, p. 154). Recent research, however, has shown that human performance exceeds even the best performance that is expected under the limitations supposed by Ullman's theory. (Braunstein et al., 1986).

The two parts of Ullman's theory, namely the correspondence and interpretation processes, differ markedly in the extent of the psychological support he offers for them. Ullman's psychological theory in the first part is complicated, and makes continual reference to psychophysical researches. In contrast, the psychological claims that he presents for the geometric rules that describe the interpretation process can be expressed in a single sentence:

> "The polar-parallel interpretation scheme is advanced as a competence theory for the perception of structure from motion by humans" (1979, p. 154).

This statement is decidedly odd. A competence theory is an idealization. If it is a mathematical structure, it is not meant to approximate another mathematical structure, as the polar-parallel scheme approximates projective geometry (Pylyshyn, 1972, p. 548).

Nor should a competence theory be designed to model the limitations of performance, as is the polar-parallel scheme.

Ullman's claim can be carried further in one of two ways. The first is to take affine geometry as the competence theory. Ullman says that affine mappings are psychologically indistinguishable from projective mappings (though parallel lines are psychologically salient, and affine mappings do keep parallel lines parallel while projective mappings may not). Nevertheless the constraints on the interpretation process, the constraints that are supposed to be taken as assumptions about the physical world, are <u>projective</u> constraints. The function of the polar-parallel scheme is bound by these constraints. Affine geometry is thus an unpromising candidate for the role of a psychological theory of visual competence. The other alternative is to take as competence theory projective geometry, which is what the polar-parallel scheme is meant to approximate. In face of the desiderata for a competence theory, it is easier to take the second alternative. Then Ullman's statement amounts to the strong assumption that projective geometry is part of the competence theory for visual perception. The latter interpretation of Ullman's claim for a competence theory of vision coincides with the claim made by Rock and Gibson: that projective invariance provides a basis for laws of form perception.

The four theories that have been discussed assume that a sufficient description of the way things look uses the same terms as a description of the conditions under which light reaches the eyes. The projective congruences that are preserved in the propagation of light are the basic elements in this description. These are claimed to ground the phenomenon that an object viewed from different positions and distances appears to have the same shape - to the extent that it does in fact. The explanation offered is a geometrical one. The claim is made by Helmholtz in its most general form, because he believes that if the laws of physics were different, then the interpretation of retinal impressions would proceed according to those altered laws. The interpretation would involve knowledge of different geometric rules. Helmholtz assumes that whatever may be the description of the way light reaches the eyes, that is, whatever optics and kinematic geometry might have to offer, that is what the perceptual system will employ. He claims that our working beliefs about geometry would change so that we would come to know the true shapes of objects under many conditions. Helmholtz's theory is not simply about beliefs, however; it poses a geometrical constraint on what count as perceptual data. Helmholtz (1878) describes these geometric constraints, which imply perception of projective congruence. He holds that, as things stand, we actually perceive projective congruence.

Rock makes the same assumption in different words, that the causal locus of the "proximal stimulus" is deduced from implicit and quantitative knowledge of projective relations. The proximal stimulus is a sensation of visual angle; Rock believes that "visual angle per se is available to phenomenal experience as a sensation of pure extensity, quite apart from [an object's] apparent objective size" (Rock & McDermott, 1964, p. 134). Again, the geometrical form of useful perceptual data is fixed by the use that is supposed to be made of implicit geometrical knowledge. (For a thoroughgoing discussion of the relation between perceptual data and cognitive processing, see Pylyshyn, 1984, Chapter 6). Ullman concurs that some geometrical laws play the role of assumptions about the world that are implicit in visual processing, and that these laws determine projective constraints. Gibson makes much the same point when he says that what it means for environmental information to be conveyed is that a property of the stimulus is unambiguously related to properties of surfaces through physical laws. Gibson says that in vision this is normally a projective relation.

Helmholtz, Rock, and Ullman's theories differ from Gibson's in many ways, but the most salient difference is that the first three postulate a role for belief and inference in shape constancy. By way of contrast, Gibson specifies that projective relations should bear a univocal significance. Nonetheless, I should like to say all four claim that observers are sensitive to projective equivalence. There is little doubt about Gibson's claim. The others presuppose that projective properties are basic to form perception. They suppose projective equivalence to be a datum for inferential processes. This datum has several names. Helmholtz speaks of the pattern of retinal stimulation; Rock speaks of the measure of the visual angle; and Ullman speaks of the dimensions of planar perspective views of objects. Each takes projective equivalence as a datum, as something that we can reason from, and something that does not itself have to be reasoned to. Then the obvious difference between Gibson's theory and the others does not impinge upon the discussion at hand. We are studying an assumption that all four make - an assumption that is logically prior to belief and inference in shape constancy. The four theories share this tenet: that the geometric properties of useful data for perceptual beliefs should include projective properties. The theories take projective congruence as ~~a basic datum~~ for vision. Are these data accessible to the observer? There have been a number of experiments intended to show that they are. Still the answer is unclear, as will be seen in the next chapter.

CHAPTER 2

Projective Invariance and Visual Perception: Recent Research

> **Some things there are which at first sight incline one to think geometry conversant about visible extension.**
>
> **Berkeley, A New Theory of Vision, p. 232.**

An experiment may not be relevant to the theoretical claim it is meant to test. "The existence of the experimental method [in psychology] makes us think we have the means of solving the problems which trouble us; though problem and method pass one another by" (Wittgenstein, 1958/1976, p. 232). I will argue that the tests that have been made to ascertain the place of projective invariance and projective rules in vision have not been the right ones for the task. Projective relations are often misrepresented in the study of vision; many textbooks in perception misrepresent perspective both in their diagrams and in their text. (For a brief assessment see Gillam, 1981.) Recent experiments on static figures involve the identification of pictures or the estimation of certain geometric quantities from pictures. Most often, the pictures are perspective projections of squares or boxes. Two methodologies will be described, to illustrate the kinds of test that have been made. The first kind includes some experiments by Perkins, whose results were achieved independently by Shepard. The second kind is a series of experiments by Attneave and his associates. The two kinds of test will be shown to be similar not only in stimuli, but also in the way they fail to solve the problem.

Perkins and Shepard

Both Perkins (1972, 1982) and Shepard (1981) claim that observers can distinguish perspective projections of some objects, and that the ability of observers to discriminate these projections is evidence that geometric rules for projective operations are used in vision (e.g. Perkins, 1982, p. 73, 87). Yet if, in certain circumstances, an observer succeeds in selecting projectively equivalent pairs, it follows neither that he knows the underlying projective principle nor that he normally has access to projective equivalence. A practical example may serve to clarify the point. In October, an observer is stationed in an Ontario forest, and he is asked to point out maple trees. There is an easy perceptual test: the maples are the red ones. Under the right conditions, maples can be distinguished at a distance of several miles, though there will be some confusion with sumacs. It does not follow that the observer can discriminate maples from oaks or birches in a reliable way. Marked decrements in performance may occur if the observer is tested in June or January. Nor does it follow that the observer knows anything about trees, except that maples are the red ones. The rough and ready test does not reflect the general standards by which maples are identified. There are botanical procedures to identify maples. A particular leaf structure or a type of arborescence might be an identifying mark of maples. An observer is said to discriminate maples reliably if and when he uses such criteria. Analogously, a number of tasks have been developed for which the application of a superficial rule suffices to discriminate perspective projections. Each of the tasks can be performed by the application of a simple rule that does not involve projective invariance.

Perkins and Shepard use tasks that are well-nigh identical in form. In both, the correct choice involves right solid angles crucially. (The corners of a cube are right solid angles.) Both researchers employ other tests, but similar points can be made about each. In the tasks that involve right solid angles, they present a number of projective drawings. These drawings are projectively ambiguous: they can be interpreted as projections of any number of solid objects. Yet they cannot be projections of an arbitrary three-dimensional figure. A number of pictures are presented, of which some can be the perspective projections of rectangular boxes, that is, parallelepipeds all of whose corners are right solid angles. Other pictures are presented that cannot be interpreted so by the rules of perspective projection. Perkins (1972) asked subjects "whether each box appeared rectangular". Shepard devised a mechanism by which subjects could vary the shape of a Necker cube figure. This shape could vary continuously in two ways. He also had subjects select which arrangements of three line segments connected at a single vertex could be perspective drawings of a solid right angle. These tasks, and similar ones, can be solved with a superficial rule. Three points will be made about the solution to these problems:

1. Only quantities in the picture plane need to be judged.
2. The quantities are not judged reliably except in two special cases.
3. The explicit rule is not formally equivalent to a general test for the projective equivalence of a three-dimensional object with a two-dimensional one.

What is the rule that subjects may use to select possible projections of those figures that have right solid angles as corners? It is helpful to know that, apart from rotation of the entire figure, the task has two degrees of freedom corresponding to the measures of two of the angles (Shepard, 1981, p. 303). The third angle is fixed, since the three angles of the picture must sum to 360°. Consider angles internal to the figure of the pictures. There is a simple rule to predict which of the drawings are projections of rectangular boxes. It is this: two angles may not subtend more than 270^{0} or less than 90^{0}. In Shepard's task, this rule can help one to select the region that is not internal to the figure. Why should this rule be simple to learn? The rule reflects a qualitative knowledge of the way that cubes (or other boxes) look. If the subject were not informed that the depicted figure is a box, his decisions would not be justified. Given the information, he can invoke some familiar facts about boxes, and an ability to differentiate a right from an acute angle. It may also be added that most people have practice at drawing boxes.

Here are the familiar facts: when a box is viewed face-on, it looks like a rectangle. All the corners look to be right angles. When the box is seen from some other positions, the perimeter of the object is a salient hexagonal shape, like the drawing of an oblique view of a cube. Abutted pair of angles on this shape look as if they form obtuse angles. The pairs can't look as if they are acute. In general, angles are easily classified as obtuse or acute, and this applies to abutted angles in the diagram of a box. The angles of the diagram are the ones to be estimated. One half of the rule is that two of the angles can't look greater than a certain value. The limit is the value at which the third would have to look acute. Abutted pairs of angles on the shape can't look as if they are acute. If the two together looked less than a right angle, they would be acute. If they were acute, they would look acute. So these jointed angles must never look acute, if they are to represent a right solid angle, and the second half of the rule is that the two can't look less than a right angle.

The finding that observers can use such a test is interesting, much as the finding that observers can use aerial haze as a cue for depth. However, the finding does not prove the main claim. Shepard's claim is that "the implicit rules of formation of internal representations of three-dimensional objects...correspond to the objectively correct rules of projection" (1981, p. 299). The application of the test ignores the general problem of projective equivalence, even for the limited class of cases envisaged. To make the point by analogy, it would be absurd to claim that Euclidean principles are followed if equality of distances is not respected. The recognition of equality of length is a sure test that Euclidean principles are being followed. The use of a simple test that involves the estimation of angles does not imply sensitivity to projective equivalence. The evidence that Perkins and Shepard present does not suggest that an observer is a naive renderer of perspective projections. It is one thing to notice the perspectives of things, and another to do projective geometry. It is still another to demonstrate that a "knowledge of projective geometry...has been incorporated into our perceptual machinery" (Shepard, 1981, p. 298).

The simple observer in the analogy with which we began will not have shown himself to be a botanist if he discriminates not only red leaves from green leaves, but also red leaves from yellow leaves. Moreover, he is liable to be led astray by sumacs, since they too are red in the fall. Neither does an observer show himself to be a geometer if he discriminates not only right angles, but also straight angles (180° angles) from smaller angles. Nor will he know much about the projective geometry of tetrahedra, if he identifies perspective drawings of regular tetrahedra by a rule similar to the one cited for boxes. The rest of Shepard and Perkins' evidence is beset by the same or similar difficulties. Shepard asked observers to identify perspective drawings as drawings of the corners of a tetrahedron, or of a planar "Mercedes-Benz symbol". Perkins devised another task that could be accomplished by an application of the rule that identifies the perspective drawings of cubic corners. Each of these tasks has a solution that involves the sum of two angles in the picture. There is a relation between sums of angles and the projective invariants in a picture -- a loose relation that depends on a host of conditions. If the conditions are unspecified, so too is the relation. If the pictured object is a cube, a simple inequality of angles can be used to categorize perspective drawings. If the pictured object can be some other shape, the relation between correct projection and the sums of angles is unspecified. This ambiguity is particularly evident in Shepard's experiment. He presented pictures and asked subjects to say if they were pictures of a cube, a tetrahedron, or a Mercedes symbol. Some of the pictures represent all three objects.

Something more than an all-or-none reaction to the sums of angles in a picture is required to reveal sensitivity to projective invariance or rules of projection. Nothing need be known about projection to categorize perspective drawings, _if_ the pictured solid is known to be rectangular and the observer has plenty of experience with boxes (see Perkins and Deregowski, 1982, for cross-cultural differences on this same task). The sum of angles in a perspective drawing provides a _cue_ which can indicate whether or not the drawing could depict a specified object. What knowledge of projective geometry does one need to use this visual cue? None at all, _or_ no more than one needs to know principles of the scatter of light to appreciate that buildings rendered indistinct by smog are farther away than buildings that are seen distinctly.

What the experiments do not support is Shepard's (1981, p. 305) claim that "the rules of formation [for visual representation] include some approximation to an inverse of the rules governing the optical projection of three-dimensional objects onto a two-dimensional surface". What they do show is subjects' ability to judge the magnitude of a pair of angles as being greater or less than 90° or as being greater or less than 180°. Shepard's subjects were reliable only when they judged if a figure could depict a right solid angle. They identified perspectives of 120° angles (the Mercedes symbol) less accurately than they identified projections of right solid angles, and they identified perspectives of a tetrahedral corner less accurately still.

Perkins contrived another three tasks. In the first, observers judged the bilateral symmetry of solid wedge-shaped forms from pictures of those forms. The wedges were bilaterally symmetric just when their bases were rectangular. In the second, Perkins presented pictures of medium-sized quadrilaterals in which were enclosed smaller quadrilaterals. Subjects judged if the figures in the picture could represent rectangles that lie in the same tilted plane in space. In another experiment, Perkins presented a variety of single quadrilaterals. Subjects were asked of each if it could represent a planar figure in which two nonadjacent internal angles are both right angles. Perkins is explicit about the nature of his tasks. He says that the rule to identify cubic corners and that used to judge the bilateral symmetry of the wedge-shaped figures are "mathematically equivalent" (Perkins, 1982, p. 82), but he does not state the relevant inequality. The similarity of the two tasks can be seen if all the base vertices of Perkin's wedge figures are joined, and the plane of symmetry of the wedge is drawn. The other two tasks differ from those just mentioned. The other two tasks no longer require subjects to sum nonadjacent angles and compare them to 90°; they require subjects to sum angles and compare them to 180°.

Consider Perkins' task in which subjects judge whether a quadrilateral can be the projection of a figure whose two nonadjacent angles are constrained to be right angles. The four angles of the quadrilateral must sum to 360°. Suppose that the angles of the quadrilateral are labelled a, b, c, and d in the order they are encountered as the perimeter of the figure is traversed. Then "there will be a common plane in which angles a and c are both right angles if and only if either angle a + b is less than 180 degrees, and angle a + d greater than 180 degrees or alternatively, angle a + b is greater than 180 degrees and angle a + d less than 180 degrees" (Perkins, 1982, p. 84). The sum of the four angles is 360°. Two of these are constrained to be 90°. The rules follow as a consequence, since two angles vary. The rule has a form similar to that of the rule for cubic corners. Perkins makes it clear that the solution is the same for the task in which nested quadrilaterals may be coplanar rectangles. In fact, all Perkin's and Shepard's tasks can be solved by a rule of the following form:

$$\theta_1 + \theta_2 < 360 - x$$
$$\theta_1 + \theta_2 \geq x$$

,where θ stands for the magnitude of an angle. The rule seems useful to observers only when the constant x takes on a value of 90 or 180. The angles θ_1 and θ_2 can be measured on the picture. This seems a weak test of the claim that the visual system is sensitive to projective invariance. The three previous objections can be made precise. The experiments are not cogent demonstrations of the perception of projective relations because:

1. The estimation of angles in the picture plane does not disambiguate perspective drawings.
2. The magnitudes of such angles are not judged accurately in the context of a figure, except for right angles and straight angles.

3. The rule by which the problem can be solved is not a general one. The variables of the inequalities are two angles. These do not provide a general criterion for projective invariance; they do not even determine the relative invariant of area.

The said experiments cannot provide appropriate support for the claim that "the human perceiver must be considered a 'sloppy geometer'" (Perkins, 1982, p. 84) _if_, as geometers, they are thought to resolve problems in projection.

Attneave

Some remarkably different experiments provide the same type of evidence that Shepard and Perkins do. The studies were conducted by Attneave and his students, who made estimates of the slopes of lines their dependent variable. The stimulus objects were either rectangles or boxes, as in the previous studies. In one classic experiment Attneave & Frost (1969) used perspective drawings of cubes. They sought to verify a principle of simplicity or "least action" in perceptual organization. They do not define "simplicity", but some analogy may be found in the study of minimal surfaces. (A minimal surface is the surface of _minimum_ area that is bounded by a three-dimensional frame. The film that is formed when a wire frame is dipped in a soap solution is a good model of a minimal surface.) Percepts, Attneave felt, are organized according to this minimum principle: a picture will be perceived as representing the simplest form among those which have that projection in the picture plane; a solid form will be perceived as the simplest of those solids that project the same profile in the picture plane - when there are no specific cues to the real shape. Attneave relates the principle to the Gestalt principle of simplicity, which he says "assumes that the rules of perspective (or some approximation thereto) are implicit in an analog medium representing physical space, within which the representation of an object moves toward a stable state characterized by 'figural goodness' or minimum complexity" (Attneave & Frost, 1969, p. 395).

Attneave and Frost claim, then, that rules of perspective are operative in vision. Their evidence is that subjects' judgments of slope reflect a complex trigonometric function that describes part of the projective relation between cubes and their perspective drawings. At first glance, the similarities between Attneave and Frost (1969) on one hand, and Shepard and Perkins' work on the other, appear incidental; all use boxes or rectangles and all ask subjects to judge drawings that could depict those forms (though they could depict a host of other solid forms as well). But the similarity is more subtle and runs deeper. Although Attneave and Frost claim that subjects judge the three-dimensional slant of depicted line segments, yet their judgments reflect estimates of angle in the pictures themselves. Shepard and Perkins devised tasks that can be solved by a rule about the magnitudes of angles, also. Some of the comments I made on their work apply also to Attneave and Frost's evidence. As before:

1. Only quantities in the picture plane need be judged, and
2. The rule that subjects seem to use is unlike a projective rule.

The minimum principle is meant to resolve the "projective ambiguity" of pictures. A picture should be seen to represent the solid object that possesses the greatest symmetry among those solids that could be depicted by that perspective picture. Conversely, unless a picture has an interpretation as a simple solid, the geometrical characteristics of the object depicted will not be estimated with accuracy. Attneave and Frost focus on the geometrical properties of one simple form - boxes. Since perspective drawings of boxes have a simple and familiar interpretation, their geometric characteristics ought to be judged accurately. The simplest among boxes is the cube. What could be of psychological interest in the geometric properties of a cube? Its orientations in space provide interesting measures. The picture plane was taken as a plane of reference for estimates of orientation. Three adjacent sides of the depicted boxes were judged for their orientation with respect to the picture plane.

Attneave and Frost employed an ingenious apparatus to obtain estimates of slope in three dimensions. With a combination of polarizers, fluorescent diagrams, and a half-silvered mirror, they contrived that a stick might appear aligned with segments of perspective drawings. Though both picture and stick were viewed binocularly, light from the picture to one eye was occluded by crossed polarizers. While the picture did not appear at a definite depth from the observer, it appeared adjacent to the stick. If such a device were not used, textural cues would make the surface of the picture apparent. Then the real orientation of the picture (it is bounded in the picture plane) would have influenced apparent alignments. The obtrusive textural cues were eliminated by the apparatus.

Imagine a box somewhere near the picture plane. Each of the three dihedral edges that meet in the vertex nearest the picture plane forms an angle with respect to the picture plane. One leg of the angle is the edge whose slant is to be judged. The other leg is on the picture plane; it is the intersection of the picture plane with a perpendicular plane that contains the edge whose slant is to be judged. The angle is the slant of the edge with respect to the picture plane. Since the object is box-shaped, the measure of that slant can be derived from the measures of the angles that are projected by right angles of the cubic corner nearest the picture plane. Two of those angles determine the slant of the edge whose projection lies between them. The relation is:

$$\phi_1 = \mathrm{SIN}^{-1}\sqrt{\mathrm{COT}\,\alpha \;\; \mathrm{COT}\,\beta}$$

where $\underline{\phi_1}$ is the angle to be estimated, and $\underline{\alpha}$ and $\underline{\beta}$ are the relevant angles of the perspective drawing (1969, p. 391). Attneave and Frost hypothesized that judgments of apparent slant would be predicted by this rule. Their apparatus enabled them to make a simple test of the hypothesis, since subjects could be asked to align a stick with the apparent slant of a depicted edge. What evidence is there that apparent slant is predicted by the perspective rule?

Recall that Shepard and Perkins' experiments seem to indicate that the magnitudes of angles in the picture plane are cues for the identification of correct perspective projections of boxes. Perhaps they are cues to the orientations of boxes, as well. Attneave realized that the data of his article with Frost could be explained just as well by estimates of angles in the picture plane as by use of a projective rule. Attneave(1972), when faced with the same problem later, wrote: "I must admit, however, to have got myself in a bit of a trap here...This is not to say that the issue is inherently unresolvable, but merely that we have not, up to the present time, designed an experiment likely to resolve it" (p. 300). Some discussion of sampling technique and the form of results may tip the balance of doubt in this matter. Still, there is a prior question. How are angles in the picture plane important to estimates of slant? The trigometric relation given before is equivalent to:

$$\phi_1 = \mathrm{SIN}^{-1}\sqrt{\frac{\cos(\alpha+\beta) + \cos(\alpha-\beta)}{\cos(\alpha+\beta) - \cos(\alpha-\beta)}}$$

where $\underline{\phi_1}$ is the slope of the dihedral edge, and $\underline{\alpha}$ and $\underline{\beta}$ are angles in the picture, as before. The quantity $\underline{\alpha+\beta}$ is mentioned twice in the equation. Let $\underline{\gamma}$ denote the third angle that is attached to the vertex. That angle is opposite to the projection of the edge whose slant is estimated. The value of $\underline{\alpha + \beta}$ is a function of $\underline{\gamma}$, since the three angles sum to 360°. Perhaps the rule that predicts estimates of slant is simply that the apparent slant of an edge increases as the projection of the angle opposite to it decreases (the restrictions on projected angles that can represent cubic corners apply here as elsewhere). Attneave himself poses the question: "What if the observer were merely basing his judgment on a mean or sum of these two angles, or, what would be exactly the same thing, on the complementary angle opposite the line being judged?" (1972, p. 298). If $\underline{\alpha + \beta}$ predicts estimates of slant as well as the formula does, then the angle $\underline{\gamma}$ would seem a better predictor than actual slant. Angle $\underline{\gamma}$ is both a germane and a simple quantity. There is some indication that $\underline{\alpha + \beta}$ may be a better predictor than the projective rule, even when its simplicity is not taken into account.

The experiment fails to distinguish the simple rule based on angle from the projective rule, because the sampling of stimuli was biased. It appears that Attneave and Frost used samples of pictures that represent equal intervals of the angles $\underline{\alpha}$ and $\underline{\beta}$. A sample based on equal intervals of slant would have been a better choice. If Attneave and Frost's data points are grouped into categories of slant from 0° to 29°, 30° to 59°, and 60° to 90°, then of 27 observations, 14, 9, and 4 observations fall into the respective categories. If slant is computed by the formula, and equal ten-degree intervals of $\underline{\alpha}$ and $\underline{\beta}$ are used, but the extremes are excluded, then of 28 observations, 13, 12, and 3 fall into the respective categories. The correlation of $\underline{\alpha + \beta}$ with $\underline{\phi_1}$: is $\underline{r} = 0.943$ in that case. The magnitude of this correlation is as high as the regression coefficient that was found between estimated slope and predicted slope. A persistent curvilinear trend in these data as well as in the data of other experiments indicate that $\underline{\alpha + \beta}$ may be a better predictor. The measure $\underline{\alpha + \beta}$ exhibits such a curvilinear trend where the more complex function predicts linearity. The plot of $\underline{\phi_1}$ versus $\underline{\alpha + \beta}$

shows a marked curvilinear trend, as might be expected from the last formula. The trend is most pronounced when the interval from ϕ_1 =80° to ϕ_1 = 90° is included. Not one of Attneave's stimuli had slopes in that interval. Estimates of slant for stimuli with predicted slants in that range would have distinguished the two hypotheses.

The importance of the curvilinear trend may be seen if equal five-degree intervals of α and β are taken. One hundred and thirty-six values of ϕ_1 are thus produced. When $\alpha + \beta$ is plotted against ϕ_1, the regression coefficient for the 26 values of ϕ_1 that fall within the interval 10° to 20° is 0.23. The regression coefficient of those 23 values of ϕ_1 within the interval 50° to 70° is 1.29. The change of slope in the function is some indication of the function's nonlinearity. A curvilinear trend of this kind is a persistent feature of the data in a series of experiments designed to demonstrate the same point (see Attneave, 1972). The stimuli for the other experiments include rectangular shapes.

Two comments about Attneave and Frost's data are warranted. First, as the boxes tended to cubic shape, the quantity was better estimated, in the sense that the slope of the regression line tended to one. (The quantity estimated may have been $\alpha + \beta$.) Secondly, no appreciable differences were found for estimation of slope between affine projections of cubes and perspective projections. Attneave's evidence is equivocal; it allows the interpretation that:

1. Only one angle in the picture plane need be judged to perform the task, and
2. The rule that subjects seem to use is unlike a projective rule.

Moreover, subjects were asked to judge boxes only. What if less familiar shapes had been used, or any shapes for which it cannot be assumed that their corners are all right angles? Although Attneave and Frost's experiment seems different from those of

Shepard and Perkins, all their tasks can be solved with simple information about angles in perspective drawings.

Previous research does not seem to have provided an adequate assessment of sensitivity to projective invariance. Since the topic is central to a number of theories of shape constancy, there is reason to perform a new series of experiments. There will be two types of experiments. In the first, subjects will be required to judge if two figures are projectively equivalent. In these experiments there are two important manipulanda: (1) the cross ratio, which guages how far two figures are projectively alike; and (2) the degree of transformation to which a figure is subjected. The second is irrelevant to projective equivalence, though it may yield a figure that is very different from the original in the terms of Euclidean, similarity or affine geometry. These manipulanda are independent; when the degree of transformation is null, the magnitude of the cross ratio is left unconstrained.It will not be possible for subjects to achieve success by any such strategies as might have helped them in the studies we have just surveyed. The interest of these experiments is whether subjects will be able to distinguish projectively neutral transformations from projectively relevant changes. The second line of experiment will involve subjects in the completion of projective drawings of large outdoor objects. The objects are irregular quadrilaterals. Subjects will be required to indicate the position of a missing corner in a sketch of these quadrilaterals. It will be interesting to discover whether or not subjects tend to choose a point that preserves in the picture the projective equivalence of the object.

An historical analogy may suggest another reason that observers may be insensitive to projective invariance. At the turn of the century, some psychologists wondered how the world could be seen as upright, when the retinal image is inverted.

One theory was devised to explain how the retinal image is reinverted in the visual system. A number of theories claimed that the inverted position of the retinal image is necessary if the world is to appear upright. Some psychologists still recognize this as a legitimate problem (Rock 1975, p. 495). It's a resolved problem. Once, the ocular-movement and the projection theory explained the "necessity" of an inverted image. A concise statement of the projection theory is that "the eye projects images or objects into space in the direction which the rays of light enter the eye or are thrown upon the retina" (Hyslop, 1897, p. 153). This projection was supposed to follow the laws of physical optics. The orientation of the retinal image is no longer in the center of controversy in psychology. That is not to say discussion of the topic was in vain. The debate prompted Stratton (1897) to begin research on visual adaptation to the effects of distorting goggles.

Today it is generally agreed that an inverted image is not necessary for the world to appear upright (Rock 1966, p. 17). A better explanation is that directions of orientation are assigned to objects. The recognition of disoriented forms often depends on the correct reassignment of orientation (Rock, 1973, pp. 126-7). Therefore, "it should not matter whether the entire image is inverted, upright, or tilted" (Rock, 1966, p. 61). The orientation of the retinal image is a global property which does not contribute to judgments of difference made with the aid of vision. Objects have orientation within a frame of reference (Stratton, 1897, p. 184). The orientation of objects within the frame is known in vision. The ground, horizon, or direction of gravity may be used to determine such a frame of reference in vision. A standing observer will see the ground below his head, even if he stands at the antipodes. The inversion of the retinal image is not necessary for upright vision because it is a <u>global</u> property. Another reason can be presented. The reason will not be accepted generally, but it is consonant with the Gibsonian approach to perception. If one assumes that the retinal image is not seen, then the orientation of the retinal image may be unimportant (Gibson, 1951, p. 404). At least, the retinal image is not important for its details, but for the differences and invariants it reveals (Gibson, 1966, p. 319). To put the matter briefly, there are two points that may account for theoretical neglect of the retinal image's inversion:

1. The relevant geometrical property is a global property.
2. The properties of external objects are seen, not the properties of the retina.

In the second half of the century, some psychologists are concerned with a similar problem: how real objects appear solid, since observers see only certain perspectives of the surfaces of things. A number of theories have claimed that perception or knowledge of projective invariance is necessary if the real and solid shapes of objects are to be seen. Ecological optics and the claim of implicit knowledge of the laws of perspective recapitulate the supposed necessity of fundamental projective laws in vision. The latter theory assumes, like the theory of projection that accounted for the orientation of the retinal image, that an inference following the laws of optics compensates for the projection. Perhaps there is an historical lesson to be learned from the study of the orientation of the retinal image. This lesson might be applied to the study of projective invariance in visual perception. There are two points of similarity:

1. The relevant geometrical property is a global property (e.g., all the perspectives of a planar object have the same projective invariants).

2. The properties of external objects are seen, not the properties of the retina.

Geometrical rules might not need to be applied to projective equivalences when objects are seen as constant in shape, for the same reason that geometrical rules do not need to be applied to compensate for the inverted position of the retinal image. If projective equivalence is a global property to which observers are insensitive, just as they are insensitive to the inversion of the retinal image, then there may be no need to postulate an inference on the laws of optics to compensate for projective ambiguity, just as there is no need to postulate an inference to compensate for the orientation of the retinal image. An alternative account of the way shape constancy is achieved might mention any number of visual cues that are unrelated to projective quantities. The cues that appear to have been used in the studies described in this chapter provide ready examples. Consider the possibility, as Helmholtz did, that a description of projective invariance is not a necessary feature of a theory of form perception.

On the other hand, the claim that projective geometry is a competence theory for visual perception might seem compelling, since projective geometry characterizes relevant invariances in the dioptric conditions of vision. Yet does the way things look need a foundation in the invariants of projective geometry? The hypothesis that projective equivalence is not perceived directly in a reliable way is worth testing, at least.

CHAPTER 3

An Introduction to the Stimuli and Dependent Measure

How should we notice it, if people could not see depth [directly]? So that they were as Berkeley thought we are.

Wittgenstein, Remarks on the Philosophy of Psychology, vol. 1, p. 15.

The perception of projective equivalence is the topic of the six experiments to be reported. One way that projective equivalence is familiar to us is as the relation between various plane shadows of an object that is illuminated by different sources,for example, the shadows of hands cast on a wall by a lighted chandelier. In the introductory section I discussed what is meant by projective invariance in the transformation of rigid figures. Projective invariance has been well understood in formal terms for over a century. The basic projective invariant is called the 'cross ratio'. That is to say the cross ratio provides the fundamental measure of projective equivalence. All other measures of projective equivalence are functions of it. There are no other independent and reliable measures of projective equivalence even for two dimensional objects - see Klein (1925/1967, p. 151). The present chapter describes how projective invariance is measured for the purpose of the experiments. The intricacies of the classical approach to projective geometry are presented by Coxeter (1974), and the reader is referred to Klein (1925/1967) for an exposition of the analytic formulation of projective geometry.

An understanding of the cross ratio is necessary if one is to grasp the experiments. Consequently an introduction to the notions of the cross ratio and the transformation of projection will be useful. The aim of stimulus construction for the first two experiments is to produce pairs of pictures that are either projectively equivalent or depart from projective equivalence in some known and measureable way. The other four experiments are different. They use perspective drawings made with a Leonardo's window, as well as subjects' estimates of a projectively invariant figure, and some calculations on those drawings and estimates. The characteristics of viewpoints and projecting planes, particular perspective projections, and the computation of the two-dimensional analogue of the cross ratio will be discussed in this section. Attention will be drawn to the numerical equality of the cross ratio for a figure and for its projection.

A caveat is in order. There is a quantity called the cross ratio that is familiar to readers of the psychological literature on shape constancy - see Gibson (1950a), Johansson (1977), Michaels and Carello (1981), Cutting (1982), Térouanne (1983), and Simpson (1983). Their measure is the cross ratio of points on a line. Note 1 describes how Cutting (1986) uses the cross ratio of points on a line, and also some pitfalls of his approach. The measure of present interest is more general; it is the cross ratio of points on a plane. The formula for the cross ratio on a line range can be derived from the same general equation as is the formula for the cross ratio on a plane range. The cross ratio on a plane is the more suitable quantity for investigation of the optics of light that falls on plane surfaces. From here onwards, the cross ratio on a plane will be called "the cross ratio". The term "cross ratio on a line" will be specified when that quantity is intended. The use of the cross ratio on a line has led to misunderstanding. Simpson (1983) concludes from the definition of the measure that projective invariance only exists if at least four points are visible. The inadequacy of the measure in this case does not imply that three points cannot be mapped projectively from one line to another. Simpson supposes that the cross ratio is constant only for collinear points. He fails to mention the extension of the cross ratio to the plane. From these false premises, he concludes that if subjects can determine that three dots collinear on a rotating line are indeed collinear, then the cross ratio cannot be necessary for the derivation of structure from motion. Such confusion is avoided if the more general measure is adopted.

The cross ratio is an invariant between figures that bear the relation of projective equivalence to one another, and constant cross ratio implies the projective equivalence of the figures on which the cross ratio is measured. Unlike distance, the cross ratio bears no unit of measure: it is a scalar quantity. A cross ratio can be of any value on the real number line, whether positive, negative, or zero. The measurement of the cross ratio requires five points. The regular pentagon is used as the basis for the figures of the first two experiments because it is a convenient construction on five points. Actually, the cross ratio is a relation among four points, but the measure of that relation varies as the point from which it is calculated changes in position. The cross ratio can be calculated as an internal geometric property of a figure enclosed by five points. In fact, five cross ratios can be measured when the coordinate origin is taken to be a vertex of a pentagon, (that is, when the coordinate origin is translated to this point), since there are (five choose four) suitable arrangements of the points. It is important that the cross ratio be calculated as an internal geometric property of a figure, since it is difficult to identify the projection of the origin of coordinates once the coordinates of the plane of a figure are altered projectively. If the projection of the origin is part of the transformed figure, it is identified easily.

How is the cross ratio computed? It seems best to proceed by example. If 1, 2, 3, and 5 are four distinct points of an arbitrary figure, and if 4 is another point distinct from the rest, then $\Delta 12$ indicates the area of the triangle bounded by points 1, 2, and 4 in that order. The subscript of the origin, in this example point 4, will be understood. For simplicity, let areas of triangles bear a sign, just as distances may bear a positive or negative sign to describe their direction on a line. In the plane, a triangle will be said to have positive sign if the order in which the points are named is counterclockwise. In Figure 1, $\Delta 12$ is positive, since $1 \rightarrow 2 \rightarrow 4$ is a counterclockwise movement about the triangle. The area $\Delta 32$ is negative. Areas whose explicit subscripts are reversed are opposite in sign (for example, $\Delta 52 = -\Delta 25$).

Computational formulae for cross ratios are derivable from the equation known as the fundamental syzygy. In the terms just described, this equation is:

$$\Delta 12\ \Delta 35 + \Delta 13\ \Delta 52 + \Delta 15\ \Delta 23 = 0$$

When the syzygy is divided by the last term on the left-hand side of the equation, and terms are rearranged, then a formula that relates the values of two cross ratios (left- and right-hand sides of the equation) is the result:

$$\frac{\Delta 12\ \Delta 35}{\Delta 15\ \Delta 32} = 1 - \frac{\Delta 13\ \Delta 25}{\Delta 15\ \Delta 23}$$

A thorough discussion of the significance of the fundamental syzygy can be found in Klein (1925/1967. The equations themselves are given on page 158). Note 1 describes the relations of various forms of the cross ratio.

Projected shapes preserve the cross ratio of the shapes of which they are projections, as can be measured. Figure 3.1 depicts a portion of the regular pentagon, labelled in the manner of the formulae just presented. The areas of the relevant triangles and the value of the cross ratio are given in Table 3.1 under the heading "regular pentagon". Polygon 12345 of Figure 3.2 is a projection of the regular pentagon. The method by which it was constructed will be described in a moment. The areas of the relevant triangles and the value of the cross ratio for this shape are given under the heading "projected pentagon" in Table 3.1. The value of its cross ratio is the same as that computed for the regular pentagon. An analogous measure can be derived from the fundamental syzygy to describe the

Figure 3.1

A Regular Pentagon

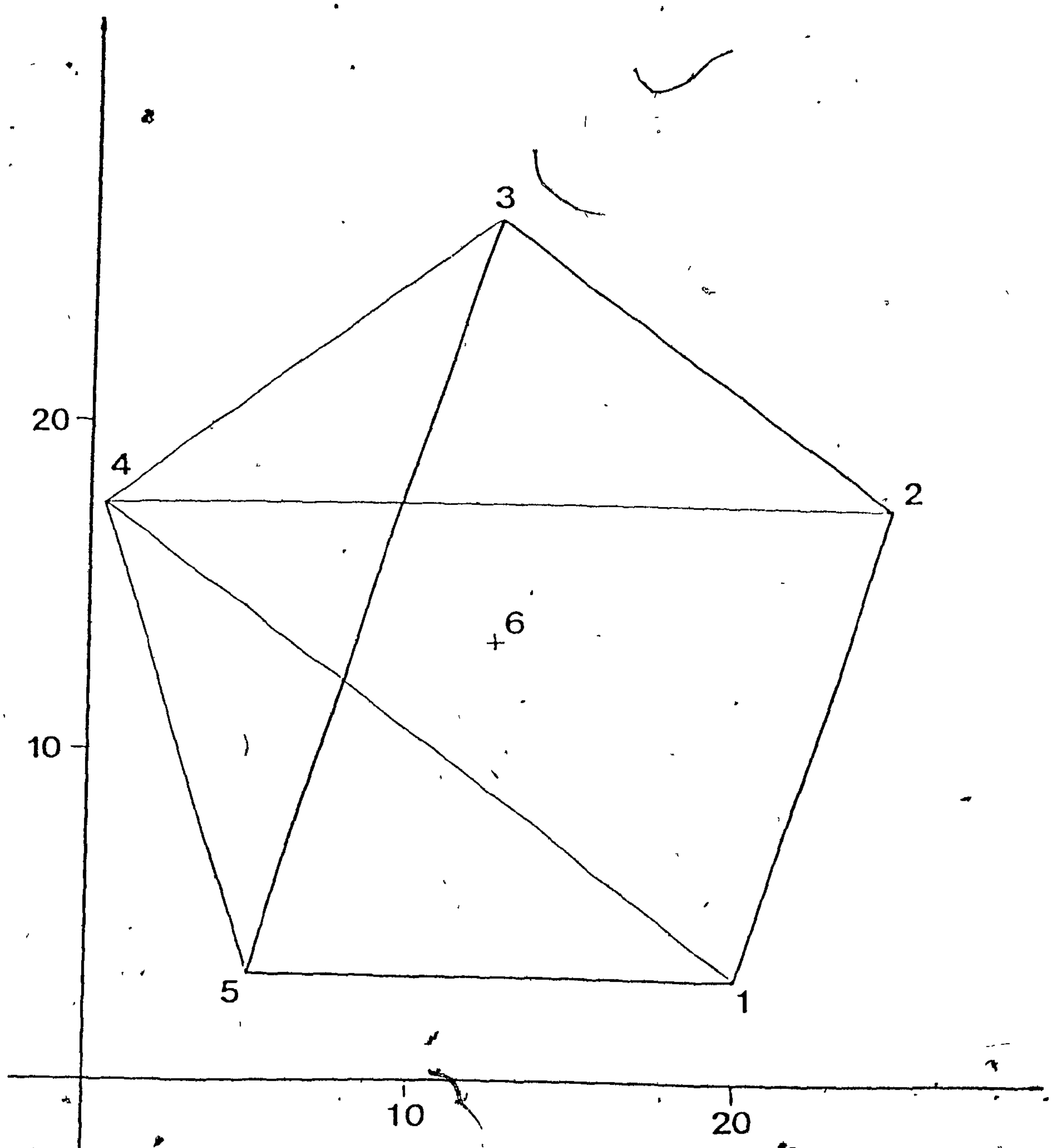


Figure 3.2

A Projectivity of a Regular Pentagon

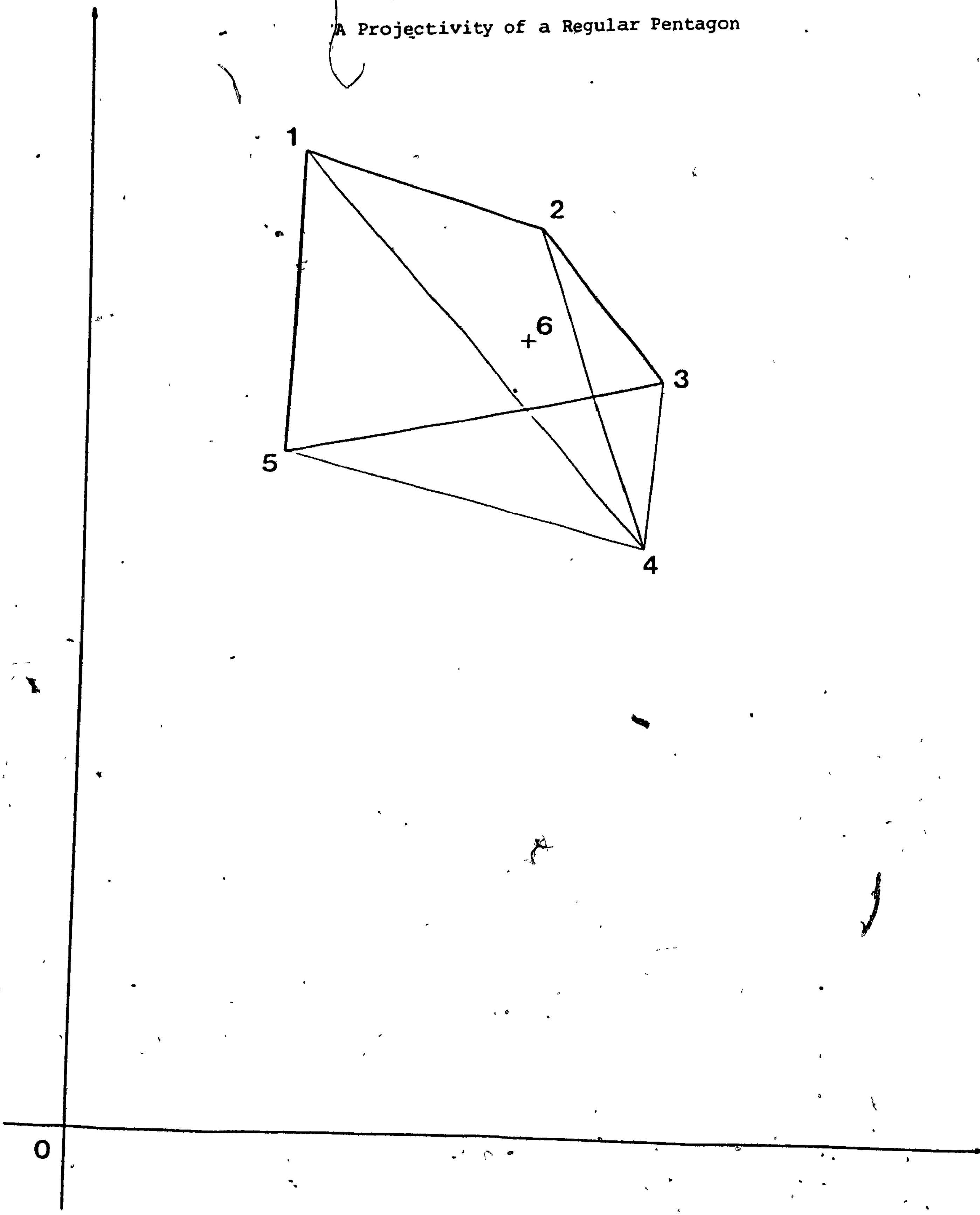


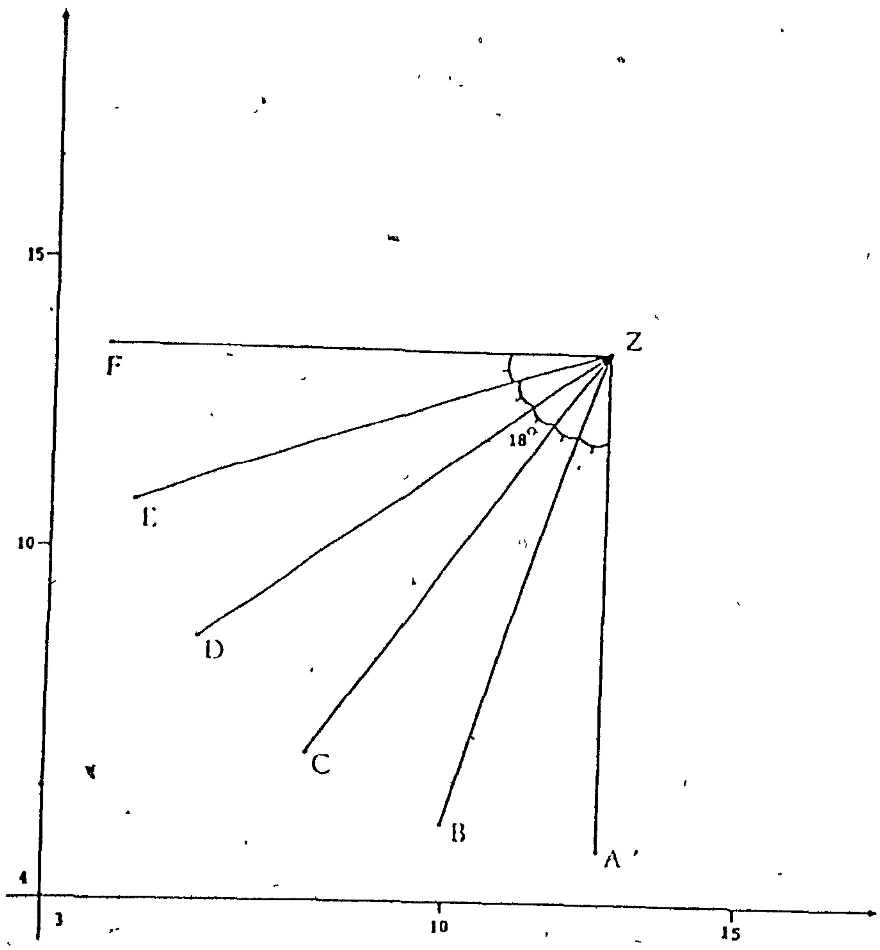
Table 3.1

The Areas of Several Triangles within Two Pentagons and the Computation of a Cross Ratio

Triangle	Regular Pentagon (cm^2)	Projected Pentagon (cm^2)
$\Delta 12$	+173.12	-11.94
$\Delta 35$	-107.00	+10.82
$\Delta 15$	-107.00	+19.12
$\Delta 32$	-107.00	+ 4.16
$\Delta 13$	+173.12	-11.40
$\Delta 25$	-173.12	+18.31
$\frac{\Delta 12 \quad \Delta 35}{\Delta 15 \quad \Delta 32}$	- 1.62 (no unit)	- 1.62
$1 - \frac{\Delta 13 \quad \Delta 25}{\Delta 15 \quad \Delta 25}$	- 1.62	- 1.62

Figure 3.3

Viewpoints for Projections of the Pentagon



This viewpoint map depicts relations among several viewpoints. Six viewpoints A to F specify six projections of the pentagon. The angular separation θ between adjacent viewpoints is eighteen degrees. The letter Z marks the centre to the viewpoint map, and was not itself used as a viewpoint. Point Z has the same X and Y coordinates as the centre of the pentagon that is to be projected. The height of the viewpoint plane is 17.41 cm. above the origin. The axes of the graph are marked in centimeters.

projective equivalence that obtains between solid shapes and their planar projections.

Now that the notion of the cross ratio has been introduced, let us consider the notion of projective transformation. The stimuli of two experiments were constructed from projected pentagons. A viewpoint is necessary to specify a projection of the figure. Figure 3.3 depicts six viewpoints which specify six projections of the regular pentagon. These viewpoints were chosen to be representative of a larger range of viewpoints. Figure 3.3 can be called a viewpoint map.

The coordinates of the projected figure are given by simple equations. Let $\underline{M}$ be the slope in radians of a projecting plane, with respect to axes $\underline{y}$ and $\underline{z}$. Let the viewpoint coordinates be called $\underline{H}$, $\underline{K}$, and $\underline{L}$, respectively, and let the non-zero coordinates of some point of the original regular pentagon on the horizontal coordinate plane be called the unprimed letters $\underline{x}$, and $\underline{y}$. Then the coordinates $\underline{x'}$ and $\underline{y'}$ of other points are given by the equations

$$x' = \frac{(L - MK)\,x + MHy}{My + (L - MK)}$$

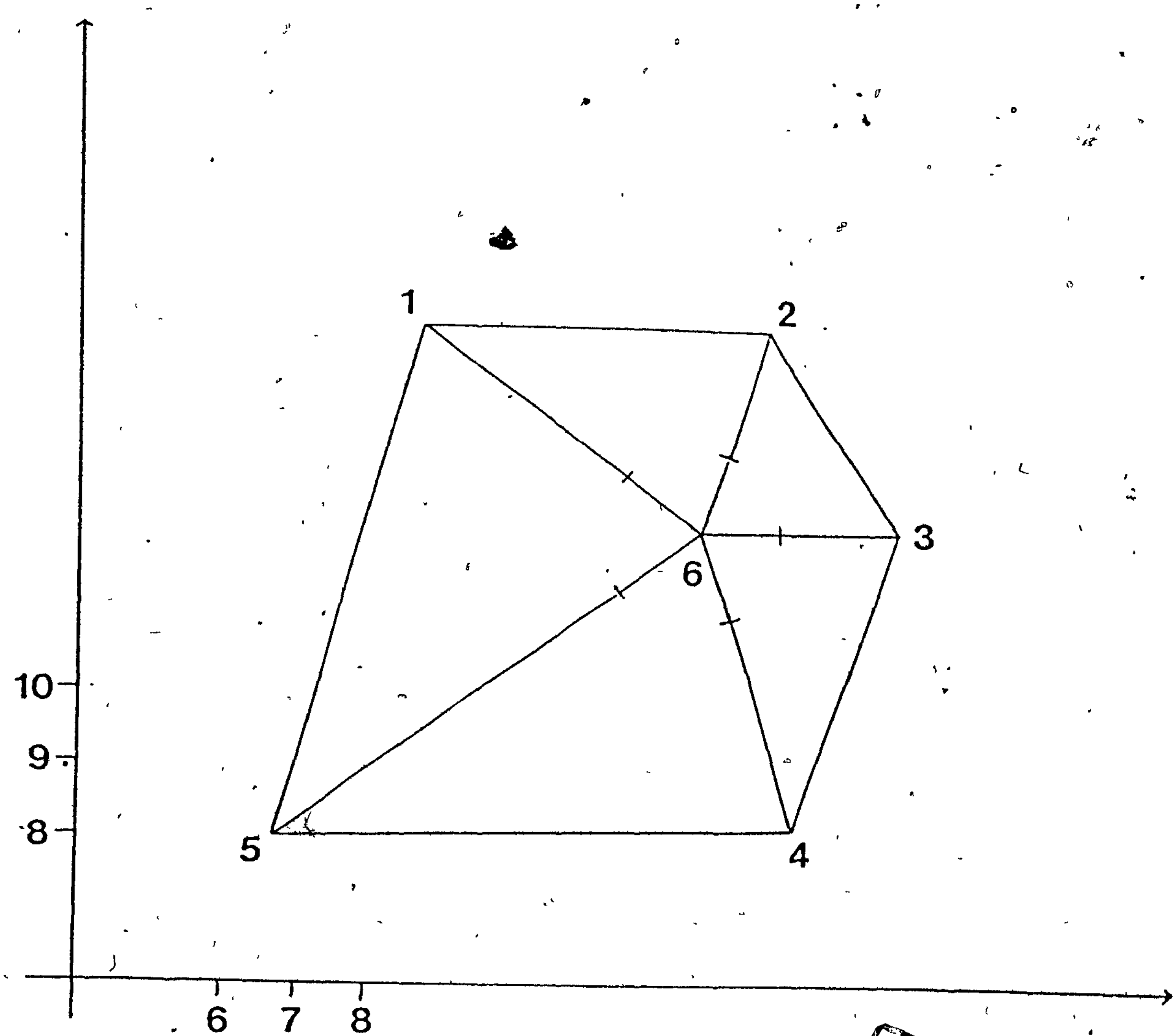
$$y' = \frac{L(\sqrt{1 + M^2})y}{My + (L - MK)}$$

The effect of these equations is to 'stretch' a figure in one direction. If these equations are applied once, the $\underline{x}$ and $\underline{y}$ coordinates of the result reversed, and the formula applied once again, the effect is to 'stretch' the figure in two directions. This is the desired projectivity. These formulae can be found in Chapter Six of Springer (1964). The rationale for these equations is far removed from present purposes. What is important is that the point coordinates so derived preserve the cross ratio. They do, as can be verified. The range of transformed pentagons that was formed by this procedure may be seen by a glance at Figures 3.4 and 3.5. An impression of the geometric means by which these were produced is given by figure 3.6.

Where the fundamentals of projective geometry are applied to the study of vision, some writers are misled, because they fixate on details of a particular axiomatization. They may fixate conventions that serve to introduce the cross ratio. Simpson (1983) makes this error, as I mentioned. Claims about projective equivalence, shape constancy, and the cross ratio have not been concerned with matters of implementation and intricacy of computation. The means that might be used to compute the value of a projective invariant can be superficially different from a given geometrical procedure. (The computation of a cross ratio is different in a number of operations when one uses a ruler than when one uses a polar planimeter.) A geometrical theorem, taken in isolation, may not fit an intended application. For example, there is a theorem of projective geometry to the effect that "there exist four points of which no three are collinear", (Coxeter, 1974, Th. 3.12). It would be a mistake to interpret such a theorem blindly, as if to mean that observers must always see four positions, even when their eyes are closed. Similarly, the projective theorem "any two lines are incident with at least one point" (Coxeter, 1974, Th. 3.11) will not imply that parallels are unobservable, nor that observers are conscient of a "line at infinity". That does not follow even when (following Johansson, 1977), one takes projective geometry as a descriptive system for 'visual space'. In the same way, one might insist that the cross ratio is a measure applicable only to planar pentagons, since it has been presented here by means of planar pentagons. That would be to have a narrow view of the application of geometric concepts. The extension of this same formula to solid noncoplanar shapes, and to shapes on the surface of a sphere, among other classes of shapes, is a straightforward extension. In other words, the difference between the incidental features of a system of axioms and the essential features of its

Figure 3.4

The Projected Pentagon Produced by Viewpoint A



This figure is the projection of the regular pentagon from viewpoint A. Viewpoint A is identified in Figure 3.

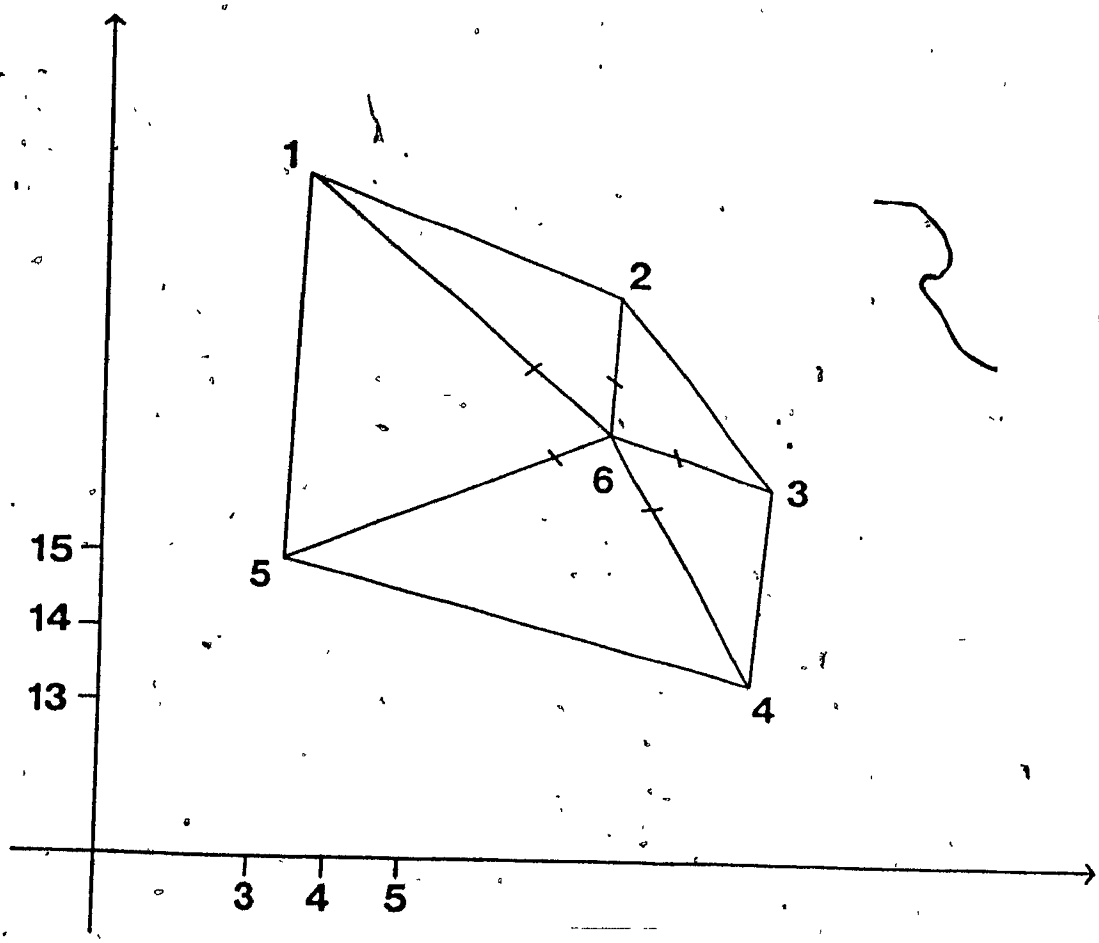
The coordinates of the vertices are:

1:	(8.64, 14.99)
2:	(13.39, 14.93)
3:	(15.27, 12.19)
4:	(13.83, 8.11)
5:	(6.70, 7.97)
6:	(12.50, 12.19)

The axes are marked in centimeters.

Figure 3.5

The Projected Pentagon Produced by Viewpoint F



The coordinates of the vertices are:

1:	(3.62, 19.92)
2:	(7.75, 18.38)
3:	(9.79, 15.90)
4:	(9.55, 13.31)
5:	(3.40, 14.90)
6:	(7.62, 16.61)

The axes are in centimeters.

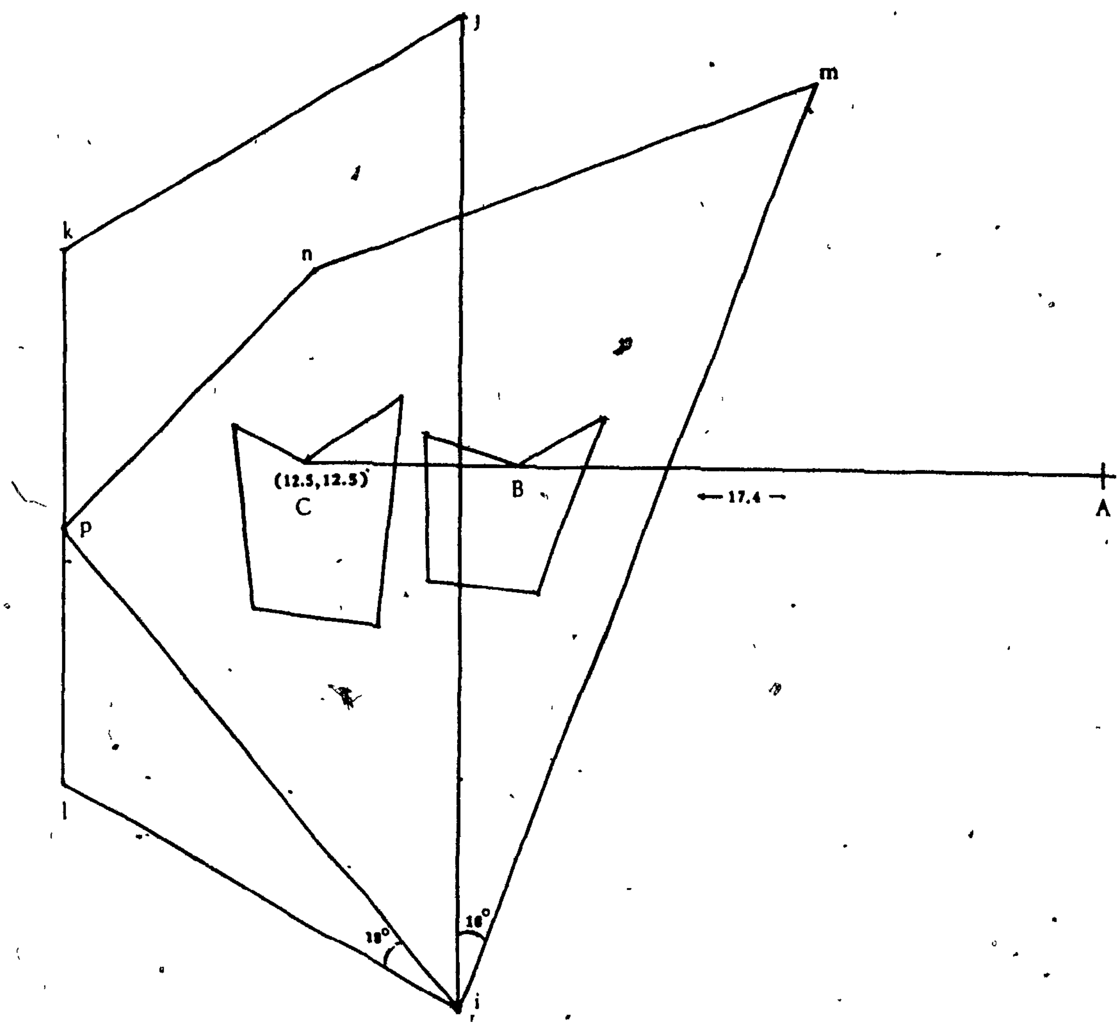
models should be kept firmly in mind in discussion of projective equivalence and the cross ratio.

Incidentally, the feasibility of a computational account of the perception of projective equivalence between pentagonal shapes more severely transformed (in Euclidean terms) than these is outlined in Ullman (1984 b), where the hypothesis is made that just these sorts of equivalences are captured by the human visual system (though it should be noted that his "incremental rigidity scheme" is cast in Euclidean terms rather than in projective terms). We shall look to see if his claim is true.

The stimuli for the first two experiments may be described in turn, now that the notions of projectivity and cross ratio have been explained.

Figure 3.6

An Overview of the
Geometric Situation



The diagram presents a rough approximation to a geometric situation that produces the projective transformations described in this chapter. The projecting plane is tilted 18° in each of two directions, and the perpendicular distance of viewpoint A from the point C of a fragment of the regular pentagon is 17.41 cm. The line between A and C includes the point B' that is the projected image of C. The plane i m n p of the projected figure is hinged on the origin, i. This diagram is neither a scale drawing, nor is it accurate in projective terms.

CHAPTER 4

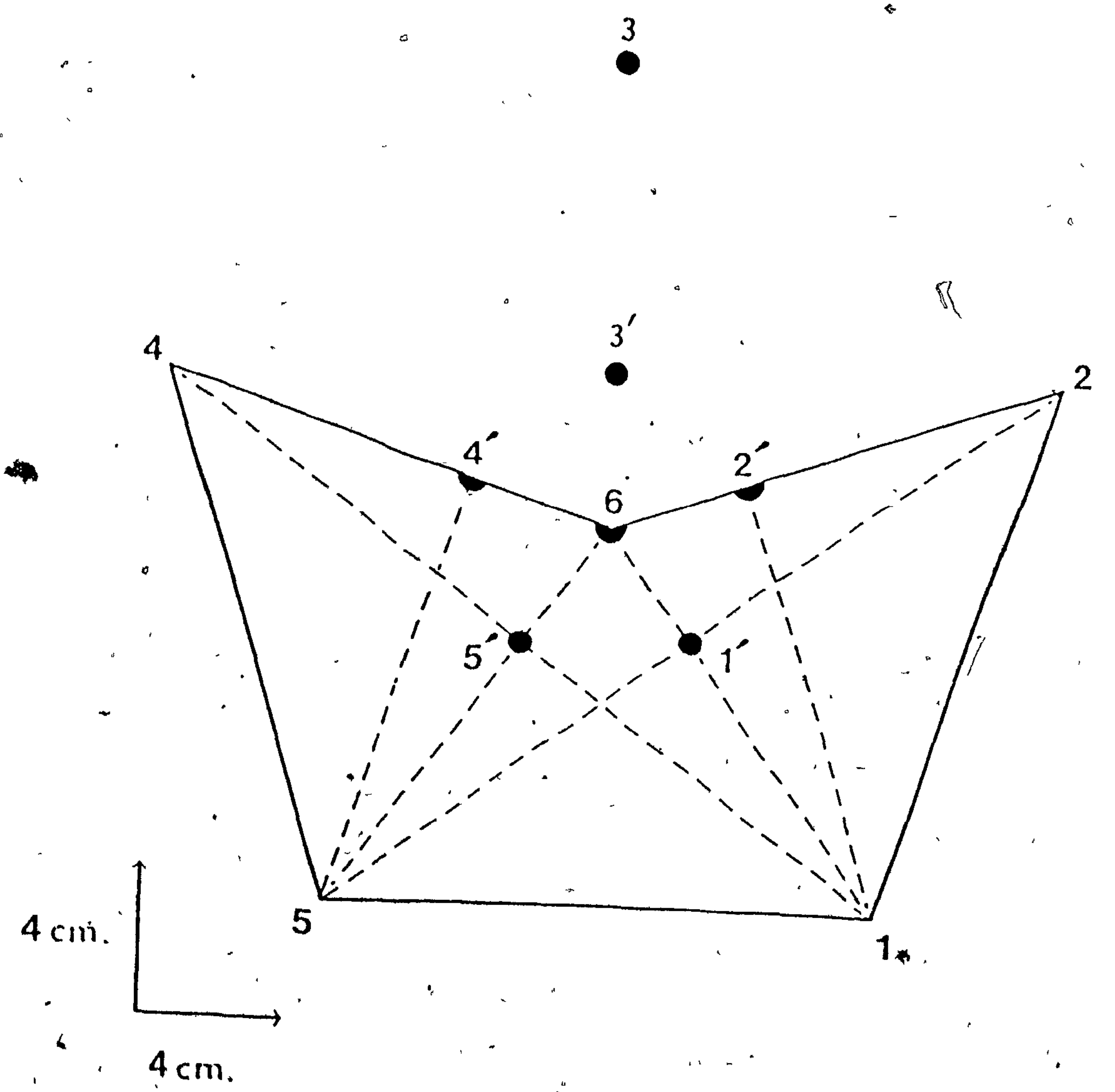
Experiment 1 - Discrimination

> Perhaps you say: "It's quite simple; - if that picture occurs to me and I point to a triangular prism for instance, and say it is a cube, then this use of the word doesn't fit the picture." But doesn't it fit? I have purposely so chosen the example so that it is quite easy to imagine a method of projection according to which the picture does fit after all.
>
> Wittgenstein, Philosophical Investigations, p.54

It is to be shown that perceiver's judgments do not reflect constant sensitivity to projective equivalence. This first experiment assesses perceivers' judgments in an identification task (that is, a forced-choice task). This task does not address the problem of shape constancy through invariance directly. Rather, it examines a prediction that may proceed from the invariance hypothesis. Later studies will face the problem of shape constancy squarely. Subjects are here asked to choose one of three shapes as a match to a standard shape. A match is defined as a projectivity. The formal criteria for projective equivalence were not presented to subjects; rather, several examples of projectivities were presented. The value of the cross ratio for the figure that is chosen is the dependent variable of an analysis of variance. The incorrect alternative shapes in this experiment are interesting ones. Besides differing from the standard figure in cross ratio, they violate the projective constraint of noncollinearity (explained below). If subjects choose such figures over those that are projectively equivalent to the standard, then they have at best so coarse an ability to discriminate projective equivalences, that it is unlikely to be a significant underpinning of the robust phenomena of shape and size constancy.

Figure 4.1

The Standard Shape: A Fragment of the Regular Pentagon



The stimuli for the first experiment are parts of the regular pentagon. The points marked 1, 2, 4 and 5 are fixed vertices of a five-sided figure. One of the primed points can be chosen as fifth vertex for a figure. The cross ratio of the quadrilateral 1245 from the point 6 is 0.62.

2. Stimuli

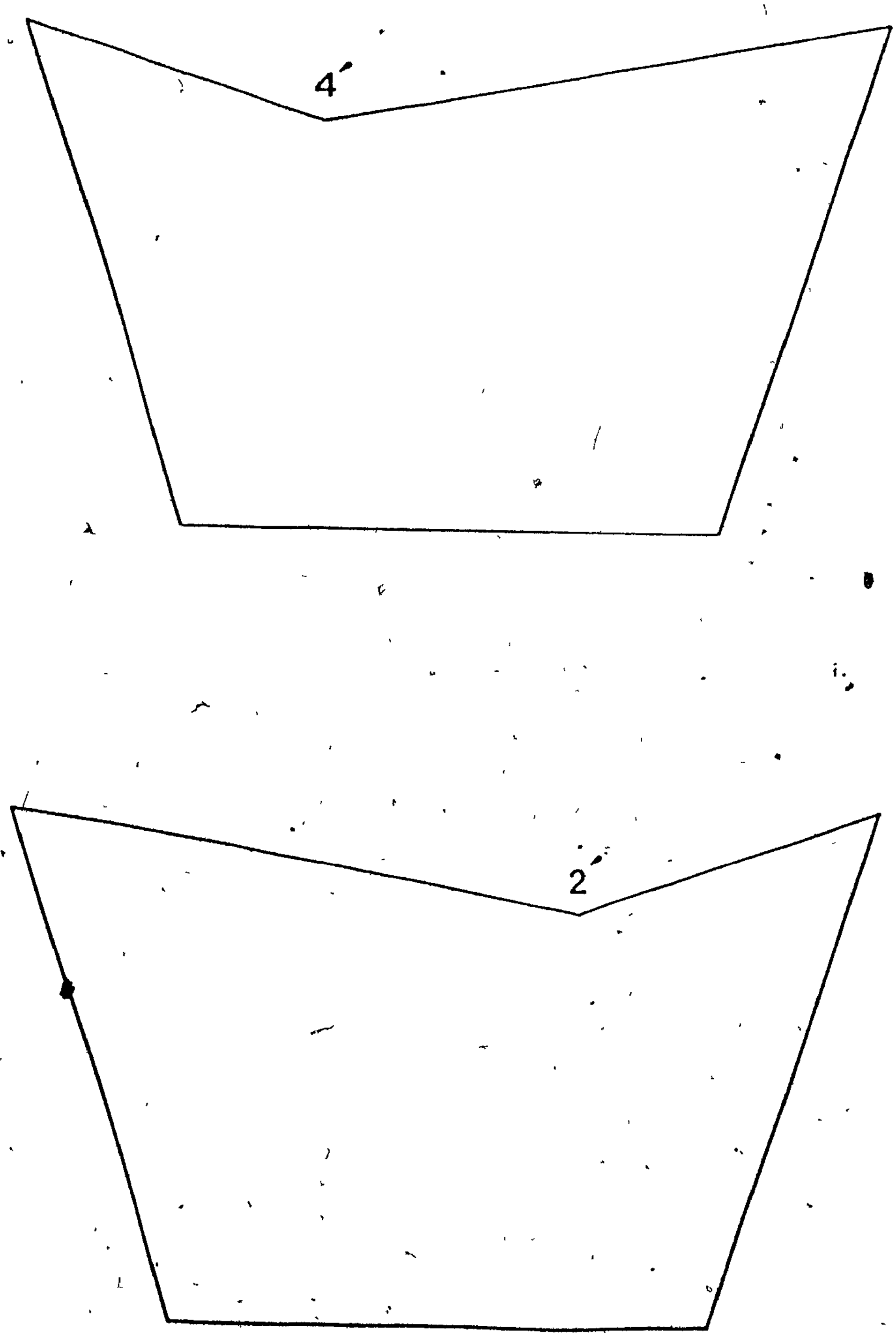
2.1 Standard and comparison figures

Figure 4.1 presents the standard figure. The comparison figures were all based on projections of the regular pentagon. In making the projections, all the six viewpoints, specified in Chapter 3, were employed. One may think of the viewpoint as the position of a point SOURCE of light. This gives us our first factor, with six levels - the SOURCE factor. When the regular pentagon is projected from any of the six viewpoints, (see the last Chapter), it forms an irregular pentagon - see Figures 3.4 and 3.5. By rotating an irregular pentagon, each of its five sides can be used as the base for a segment. Rotation is geometrically neutral for present purposes, but it yields figures that are perceptually different. All 30 of these bases (6 levels of SOURCE X 5 sides) were used. This gives us the second factor, with five levels - the ROTATION factor.

To understand how noncongruent comparison figures were obtained turn again to Figure 4.1. In it there are five points labelled _1_, _2_, _3_, _4_ and _5_. These are the points in which the diagonals of the regular pentagon intersect. The point marked _6_ is the centre of the regular pentagon. Noncongruent comparison figures were obtained by replacing the projection of point _6_ with the projection of one of the points _1'_ -_5'_. Note, however, that the points _3_, _4'_ and _5_ are collinear. In one such figure, if point _6_ were replaced with 3', we would have a quadrilateral. In each combination of SOURCE and ROTATION, one of the points _1'_ - _5'_ gave a rectangle. These were not used, since there is no point asking subjects if they can distinguish quadrilaterals from pentagons. That left four other points to choose from. Of these two (used as origins) always yielded a figure with cross ratio of 2.00 and the other two (used as origins) yielded a figure of cross ratio 1.00. While the second cross ratio is nearer to that of the standard figure, 0.62, a cross ratio of 1.00 signals that three of the five points are collinear. From the standpoint of

Figure 4.2

Two Shapes with a Cross Ratio of 2.00



projective geometry such a figure is a gross departure from the standard figure, in which no three points are collinear. The projective constraint of noncollinearity is that noncollinear points do not become collinear under projective transformation, nor vice versa. Figures of cross ratio 2.00 also represent gross departures from the standard figure, though this is less evident. We now have 30 comparison figures with cross ratios of .62, which is the same as that of the standard figure; we have 60 comparison figures with cross ratios of 1.0 and 60 with cross ratios of 2.0. This gives us our dependent measure. These differences in values of the cross ratio are gross differences.

Since the replacement points are symmetrical about a vertical axis one can divide the figures to be projected into mirror images of each other. Two such figures are presented in Figure 4.2. This gives us a division of comparison figures into those based on left-hand and those based on right-hand points. Let us call this subdivision the REFLECTION factor. It has two levels. The three factors of the experimental design are, then, ROTATION, REFLECTION, and SOURCE. These factors vary between slides.

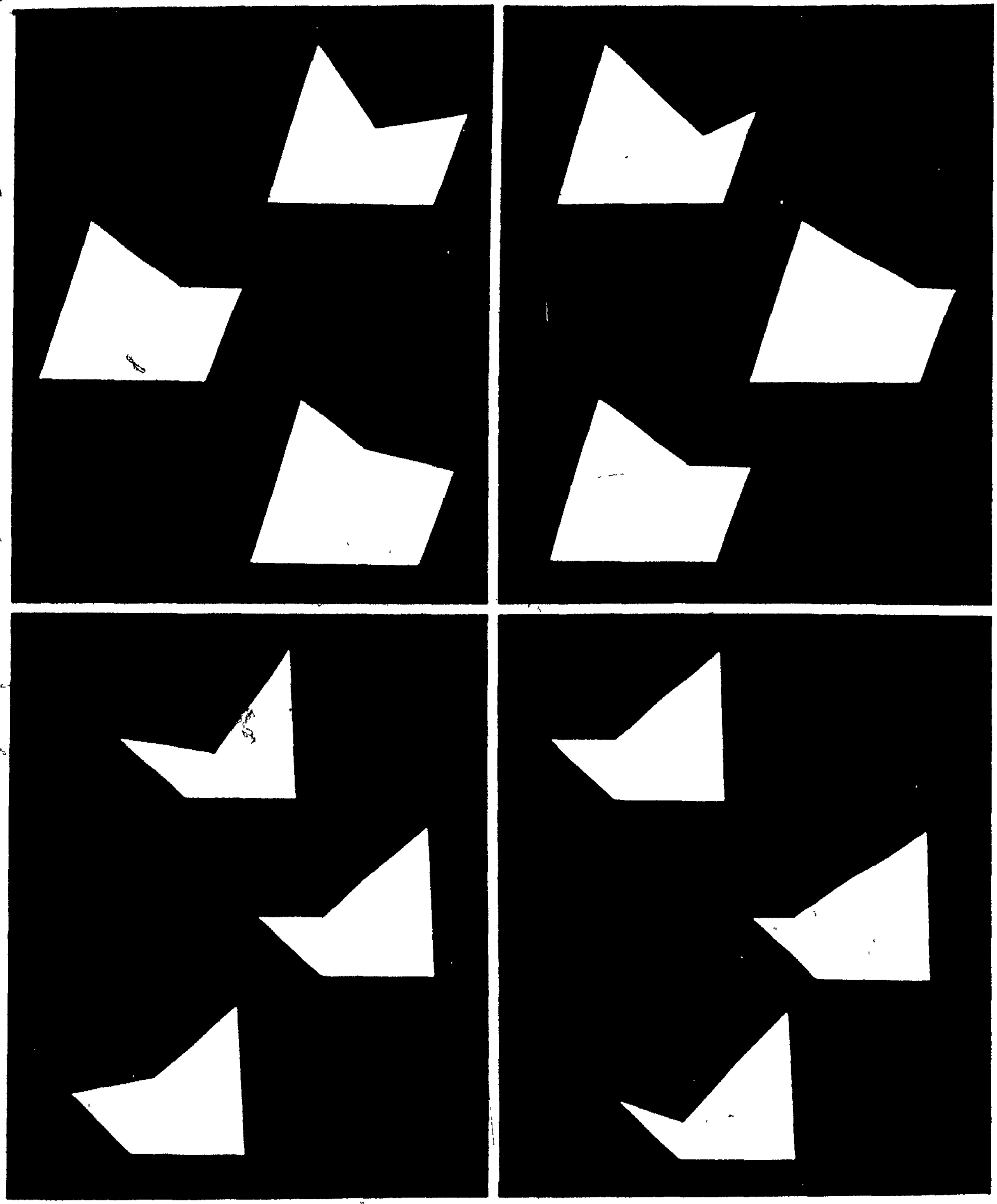
The standard figure was on a card before a subject all during a test session. It was attached to a wall at a distance of 2 m from the subject. Comparison figures were presented three at a time on a slide. Each slide contained one figure that was projectively congruent with the standard, one that had a cross ratio of 1.00 and one that had a cross ratio of 2.00. Moreover, all three comparison figures on a slide had the same base and adjoining sides. In other words the three figures on a slide differed only in two line segments. This means that subjects never had to worry about rotation when they were performing their task.

2.2 The preparation of slides

The shapes were arranged in triads (with some replication) and photographed as such. Each slide shows three shapes that differ only in the

Figure 4.3

Examples of Slides Used in the First Experiment



Four slides that were used in the experiment are depicted. Examine the figures in the lower left-hand box. One of these figures (the top figure) has a cross ratio of one. From top to bottom, the origins of these three shapes are the projections of points 1', 6 , and 5'.

position of a point on their perimeter. Only one shape of the three is projectively equivalent to the standard stimulus shown in Figure 4.1; it has a cross ratio of .62. The other shapes have cross ratios of 1.00 and 2.00. Each condition of the experimental design is represented by one slide. Figure 4.3 gives an impression of several slides that were used in the experiment. As in Figure 4.3, all three comparison figures on a single slide had the same base and adjoining sides - they differed in choice of 'origin'. Sixty slides were photographed on Kodak high-speed duplicating film.

3. Method

3.1 Subjects

Forty people were tested between January 30, 1984, and March 13, 1984. Half the subjects were students or faculty in the Department of Psychology at McGill. Other subjects were recruited by advertisement. There were 17 men and 23 women in the sample, and subjects ranged in age from 19 to 44. Seven wore glasses during the experiment.

3.2 Instructions

The subjects were provided with two means of solving the problem posed in this experiment. A lamp, a pane of glass, and an opaque screen were used to model a projection. Several shapes were fixed on the screen, and a single shape was pasted on to the glass. None of the shapes on the screen were congruent in Euclidean terms with the shape on the glass. The lamp was used to cast a shadow of the figure on the glass onto the screen. The shadow was aligned for length with the base of each of the shapes on the screen in turn. Only one was a precise fit to the shadow. This correspondence was plainly evident. The subjects were informed that the correspondence could be conceived in this way.

The subjects were provided with another means of solving the task, as well. This method was emphasized. A slide of the standard shape was presented, as was a card of the standard shape. The card was fixed to the wall, and the projector was adjusted so that the beam of light from the slide filled the figure on the card exactly. The projector was moved to show how the cross section of the light beam changes when the projector is tilted. Several comparisons were made between the shape of the light patch when the beam was trained on the standard shape and its shape when the beam was trained elsewhere.

The subjects were told that these shapes are projections of one another. They were reminded that, although many shapes are projections of the standard shape, there are many others that are not. The instruction continued in this way:

Now consider this slide. One of these shapes is a projection of just this shape (the card with the standard shape, which was always present, was indicated). The card on which this shape is pasted is flat. It is to just this shape I mean you to compare those shapes. One and only one of these three shapes is a projection of that shape on the card. There is a place in this room to which I could move the projector, so that the light from the projector would fill just this area. As for the other two shapes, one could not illuminate those areas exactly, no matter where one moved the projector. Perhaps the place one would need to put the projector would be hard to reach, or the path or the light beam might be blocked by some object in the room. No matter. For this slide, when I move the projector to your left and down to the floor like this, pointing the projector upwards, the light from just one of these shapes can be made to fill that white area. When I return the projector to the table, and again tilt it at the required angle, you can see which of the shapes becomes the same shape as that one (the standard).

Now, for each slide that I will show you, one and only one of the shapes will be a correct projection of this figure (the standard was indicated). I would like you to say which one it is by saying "top", "middle", or "bottom", for each slide. The figures will be arranged in this fashion, vertically. Here is the first slide. Do you have any questions?"

3.3 Procedure

The room lights were switched off after subjects had been given the instructions. The stimulus patterns for comparison figures were projected on an opaque screen with a Kodak carousel projector. The standard figure, roughly equal in area to the projected comparison figures, remained constantly in view, attached to the screen to the immediate left of where the projected figures appeared. The screen was two meters distant from the subjects, who sat immediately to the left on the projector. The magnified size of the projected images was about three times the size of the original drawings (the originals were drawn on cards of 28 x 35.5 cm.). One extra slide had been photographed which showed a centimeter ruler placed horizontally, and one placed vertically. This slide was used to verify the degree and constancy of the magnification of the image projected on the screen. Yet neither changes in the degree of magnification, nor in the tilt of the projector to the screen, alter the projective qualities of the pattern that is presented. Neither does the angle of viewing the presented figures affect the projective qualities of the retinal images viewers received.

One slide was presented at a time, and the slides were presented in one of two random orders. The subjects were given as much time as they required to make each decision. They were informed that the time they took to make their decision was not a factor in the experiment.

4. Results and Discussion

Subjects chose a shape by saying "top", "middle", or "bottom" on each trial. The real value of the cross ratio of the shape chosen is a datum for analysis. The values represented are 0.62, 1.00, and 2.00. Although the cross ratio can have any real value, only these three are used. These data may be too sparse to satisfy the requirement of the analysis of variance that a dependent measure be a ratio variable. Methods for correcting this paucity of real values are given by Murdock and Ogilvie (1968) and were adapted as follows. Here an assumption of linear scale is made for the cross ratio. Although this assumption is clearly warranted in mathematical terms, the psychological validity of the assumption may be questioned. However, Cutting (1986), at least, is committed to the psychological usefulness of absolute differences of .02 in the cross ratio. A preliminary analysis of variance showed no significant effect related to SOURCE, so each subject's scores (cross ratios) on the six levels of the source variable were averaged, and the resulting averages were submitted to analysis of variance. That is, six values of raw score were combined to make a single data point for analysis of variance. These new data points could have a great many values; and so they meet Murdock and Ogilvie's standards for statistical analysis. The analysis found significant effects of ROTATION and REFLECTION, and a significant interaction of ROTATION and REFLECTION (see Table 4.1). Since projective equivalence is unaffected by REFLECTION and ROTATION, these findings support the notion that the judgments subjects make when asked to match shapes on the basis of projective equivalence are sensitive to factors other than those that have to do with projective equivalence.

A word about the significance tests in this analysis: the univariate analysis of variance of repeated measures data assumes constant covariance of observations. One may obtain an unambiguous test of significance even in violation of the assumption, by using the conservative degrees of freedom specified by Greenhouse and Geisser (1959, p. 102). Conservative degrees of freedom will be used in all subsequent analyses of variance. (The effect of conservative degrees of freedom is that in any F test that involves a repeated factor, both numerator and denominator degrees of freedom are reduced by dividing them by (p-1), where p is the number of conditions in the factor under test.)

4.1 Descriptive Results

Inspection of the data might seem to offer support for the hypothesis that the figures were chosen for their projective equality to the standard figure. The number of times any subject chose any shape of a particular cross ratio was counted. Since there were 40 subjects and 60 slides, the total is 2400. The value of the cross ratio for the standard figure is .62. Shapes with a cross ratio of .62

Table 4.1

Analysis of Variance:
Real Values, Data Collapsed
over the 'Source' Factor

Source	SS	df	MS	F	p
Between Subjects	6.71	39			
Within Subjects	58.44	360			
Rotation	7.03	4	1.75	22.71	$p \leq .001$
Rotation X $Subjects_W$	12.07	156	0.07		
Reflection	3.73	1	3.73	66.57	$p \leq .001$
Reflection X $Subjects_W$	2.19	39	0.05		
Rotation X Reflection	22.38	4	5.59	79.04	$p \leq .001$
Rotation X Reflection X $Subjects_W$	11.04	156	0.07		
Total	65.15	399			

Table 4.2

Numbers of Correct Choices to Individual Slides

SOURCE						ROTATION
1	2	3	4	5	6	(Side of Pentagon)
11	10	12	6	11	10	51
22	29	26	27	30	28	51
31	26	27	26	27	30	12
33	32	31	30	24	25	12
24	23	27	28	25	26	23
14	14	10	12	19	10	23
11	8	6	5	7	4	34
25	22	20	16	17	17	34
20	20	25	20	15	17	45
20	16	18	9	10	11	45

Overall Total of Choices: 1156
(max 2400)

Totals for levels of REFLECTION:
539 617 (max 1200)

Totals for levels of ROTATION:
222 342 232 159 201 (max 480)

Totals for levels of SOURCE:
211 200 202 179 185 178 (max 400)

were chosen 1156 times, which while far from perfect is probably better than random responding. (Since these are repeated measures, it is difficult to apply a statistical test.) Shapes with a cross ratio of 1.00 were chosen 708 times, and those with a cross ratio of 2.00 were chosen 536 times. The impression of sensitivity to projective invariance changes, however, when the number of choices is compared among conditions in which different focus points are used.

Consider all the comparison figures (across SOURCE and ROTATION) that have point 1' as their origin. In each point 1' is at some remove from the projection of point 6. The distances of point 1' from the projection of point 6 were averaged to yield a mean displacement for point 1'. Similar mean displacements were calculated for each of the other points, 2' - 5'. These displacement measures are highly correlated with the frequency with which a projectively nonequivalent comparison figure was chosen: Spearman's rank correlation, rho(5) = 1.00, $p \leq .01$ — see Ferguson (1971, p. 458). It should be noted that differences among the displacement measures in question are projectively irrelevant. This finding coincides with the results of a multidimensional scaling procedure applied by Professor Yoshio Takane to a table of the numbers of choices, contingent on levels of the factor ROTATION and values of the cross ratio. He found, too, that the variability in these data could be accounted for by one factor, that is, in a one-dimensional solution. (He assumed no differences associated with SOURCE and no individual differences.) There may be, then, a single index that guages the influence of nonprojective factors; this is a notion that will be taken up in the next chapter. Such an index is not intended to provide any explanation; rather, it provides a post-hoc description of some results.

The principal reason for mentioning invariance in a theory of vision is to give some account of shape constancy. The principal reason is not to reveal how shadows may be matched to the objects that cast them. Are these two tasks not very different? If so, do the results of the first experiment address the role of projective equivalence in a theory of shape constancy? A first response is that more interesting situations will be employed in later experiments. Besides, we shall want to see the results of many different kinds of tasks, to accumulate evidence about sensitivity to projective equivalence. Such evidence increases by the enumeration of cases. The same question can be asked in several ways, as this question will be. A second response is that this task requires the same sort of geometric comparison as is presupposed in standard theories of shape constancy. It might be true that the sorts of projective transformations that are characteristic of objects and their shadows can be different from the projective transformations that are characteristic of objects and their retinal images. Suppose, though, that the Euclidean properties of shadows could be distinguished from the Euclidean properties of planar objects projected at a slant. Then it should be considered that, in both cases, retinal images may be compared as projections. If one believes that the projective equivalence of retinal images must be compared to achieve shape constancy, then the case of an object and its shadow is not so different. The comparison of their retinal projections must be made in the same way, too, to be consistent.

The significant effects shown by the analysis support the claim that perceivers' judgments in an identification task do not simply reflect a reliable sensitivity to projective constancy. Two aspects of the data reveal the presence of perturbing factors. One is the dependence of frequencies of choice not only on the cross ratio, as indicated by a fair number of correct choices, but also on the projectively irrelevant factors ROTATION and REFLECTION. The other aspect is the greater frequency of choice for projectively nonequivalent figures whose central points depart further from the correctly projected central point. In conclusion, it is noted that subjects chose the wrong comparison figure in more than half the trials. This may be a greater number of correct choices than the one-third expected on the basis of random responding. Yet wrong choices betrayed insensitivity to a gross violation of projective congruence, namely a violation of the collinearity constraint. The impact of experiment 1, then, is that subjects do not seem sensitive to projective equivalence in a consistent and reliable way. Though it is difficult to be definite about it, their sensitivity to projective congruence does not seem strong enough to explain so robust and so pervasive phenomena as shape and size constancy. However, this is only one experiment and the conditions under which subjects were asked to judge projective congruence may seem somewhat artificial. The problem of establishing a base rate of response, against which sensitivity may be guaged, is addressed in the next experiment. The present findings echo Cutting's (1982a) conclusion that projective equivalence is "sometimes perceived, and sometimes not".

CHAPTER 5

Experiment 2 - Identification

> **Suppose however, that ... the method of projection comes before our mind? How am I to imagine this - Perhaps I see before me a scheme shewing the method of projection: say a picture of two cubes connected by lines of projection - But does this really get me any further?**
>
> **Wittgenstein,**
> **Philosophical Investigations, p.55**

In Experiment 1, the angular differences between the planes of the object and its shadow were not indicated to subjects. Now, various theories of shape constancy make the claim that observers are sensitive to projective equivalence; and that they have access to those angles when they make use of projective equivalence. It is indeed true that, given a projectivity and angles between two planes, a geometer can recover the shape of the figure that was projected. We should see if a clue to those angles might help subjects perform the sort of task that they were asked to do in the first experiment. This is still a test of the geometric consequences of the invariance hypothesis rather than a direct examination of the phenomena of shape constancy. Though this task is more 'ecologically valid' than the first, situations even more natural will be presented in experiments to follow.

The idea that slope and projected shape are both necessary to indicate actual shape can be found in Koffka's (1935) construal of the invariance hypothesis. Beck and Gibson (1955) supposed that a form as projected on the retina "determines a family of possible apparent shapes", and that a particular shape becomes apparent only if and when a measure of the slope of a physical object is specified. They hypothesized that "phenomenal shape becomes indeterminate when phenomenal slant is made indeterminate" (p. 126).

The assumption that estimates of shape are allied to familiar tilt is found in the literature on psychophysics, such as in Perrone (1980). Slope has been varied in a number of studies on shape constancy; most of these use planar quadrilaterals as stimuli. The criticism that was applied to Attneave's, Shepard's and Perkins' studies also applies to those experiments, (eg., Olson, 1974, and Stevens, 1983). All their tasks can be solved with information about angles in the perspective drawings. Attneave (1972) had suspected that the curvilinearity in his data could be attributed to subjects' use of some picture-plane variable. However, he decided upon the ratio of the height to the width of rectangles as the important variable, instead of the now familiar variable of angle in the picture plane. Olson (1974), as Attneave's student, used the height-to-width ratio to control for the effects of picture-plane variables. The control may be ineffectual if that variable is not at the bottom of the irregularities in Attneave's data. A better strategy might have been to use other shapes than quadrilaterals as stimuli. Further investigation seems warranted, since previous investigations are not clearly interpretable in favour of the hypothesis that perceivers are sensitive to projective congruence. (My objections are strengthened by Stevens's (1983) data. He found that variation in an angular measure in the picture plane has a significant effect on estimates of tilt, but he did not specify the relation between the two measures.).

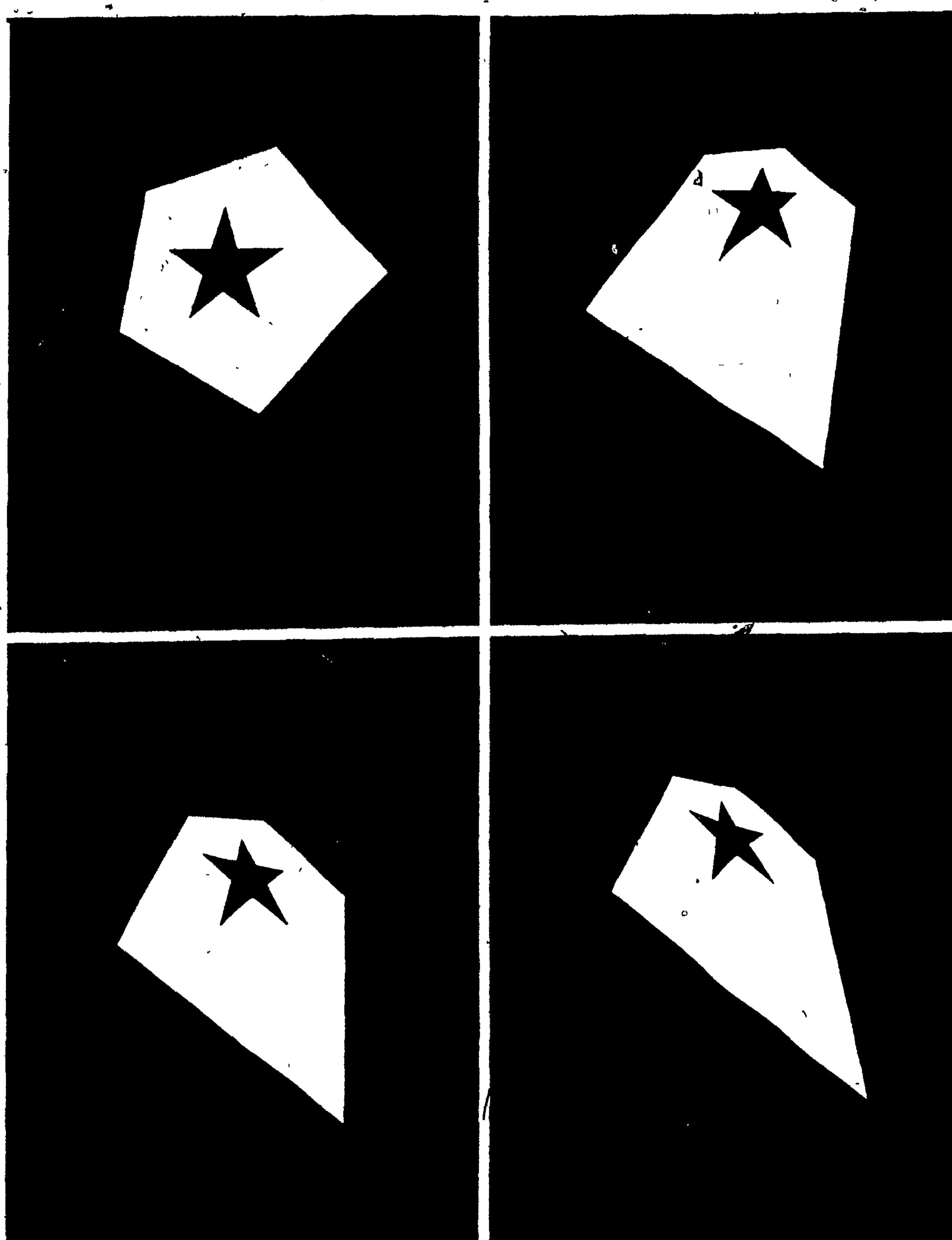
How can slope be indicated on new stimuli? The stimuli of the first experiment are each perspective views of a number of objects which present the same profile, though they are situated differently. Though the stimuli exhibit definite projective relations, each could represent a host of shapes that stand at various orientations. The "projective ambiguity" of these stimuli is shared by all photographs; their "projective ambiguity" is not resolved in more complex pictures, or by a picture more "informative" in some sense. How can the ambiguity be removed? Well, subjects can be told something about the objects that are depicted.

Knowledge influences judgments of shape. Observers use what is called the depth cue of "familiar size". Observers can compare an object of unknown size and proximity with an adjacent object of known size, to estimate both the size and the proximity of the novel object. A cue that would resolve projective ambiguity might by analogy be called "familiar tilt". When an unfamiliar shape in the picture plane is assumed to be the projection of a familiar shape at a tilt, the tilt of the plane in which the familiar shape lies is apparent. There are two independent angles of surface orientation (that is, pitch and yaw, or pitch and roll, see Stevens, 1983, pp. 241-242), but the tilt of a plane away from the observer has been the variable that has attracted most interest. It can be assumed that the estimated slopes of a familiar shape are the best estimates of slope for coplanar shapes. When the dimensions of the frontal projection of a familiar shape are known, the tilt of the familiar shape can be estimated, and then the real dimensions of other shapes that have been tilted or transformed in the same way can be estimated (apart from their absolute size and absolute position). What is necessary to this tale is that an observer assume the dimensions of a familiar figure. This is not hard to arrange. Estimations of tilt normally depend on such assumptions, as do estimates of depth based on texture gradients (e.g., Rock, 1983, p. 250 and Flock, 1964, pp. 382-383).

A familiar and regular figure is used to indicate tilt here. Imagine an equilateral star on a dressing-room door. In the absence of other cues, one can tell how wide the door is open by observing the foreshortening of the star shape, since one knows that the star is equilateral. A star on a plane tilted at an arbitrary angle in space would provide a similar indication, except ~~that~~ the foreshortening of the star could take place in two directions independently. An equilateral star indicates the two independent angles of surface slant in the present experiment. An inspection of a projected star shape provides an immediate impression of the tilt of a figure in space. (Figure 5.1 gives some examples). Once the slopes of the

Figure 5.1

Examples of the Shapes Used in the Second Experiment



Images of the shape at upper left are slanted at progressively increasing angles to the frontal plane. The shape at upper left is a standard figure, the remainder are comparison figures. Variations in position due to the method of construction were nowhere greater than 2 mm. in the originals.

plane and a projective invariant are known, the correspondence of the drawing to a pictured shape that has the same frontal projection is fixed. Do observers supply the missing bit of information, that is, can they identify the projective correspondence of shapes?

Again, the experimental hypothesis is that perceivers' judgments will not reflect a reliable sensitivity to projective congruence. This second experiment assesses perceivers' ability to choose from among six standard shapes the one that is projectively congruent with a comparison figure. That is, the same six standard figures remain before a subject throughout; the comparison figure changes with each trial.

A difference score of cross ratios is the dependent variable. This score is the difference in value between the cross ratio of the comparison shape and that of the standard shape a subject chooses as a match. Unlike the dependent variable of the first analysis, this variable can have many values. If perceivers are not sensitive to projective equivalence, then they should not systematically match shapes that are identical in cross ratio. Hence the difference score of cross ratios should not tend to zero under all conditions.

2. Stimuli

2.1 An indicator of slant

The stimuli of this experiment bear an indication of the plane in which they lie - an equilateral star. The perimeters of the stimulus figures, both transformed and untransformed, are pentagons. The untransformed pentagons are variable in shape, but each contains an equilateral five-pointed star at its center. The projective transformation that is applied to the pentagons is applied also to the stars, regardless of any differences among the untransformed pentagons. It may be worthwhile to foreshadow a result at this point: subjects reported they could estimate the tilt of a figure in space easily by inspecting these stars.

The reader may wonder: if the subjects could use the transformed star shape to determine the slope of a plane, why could they not do the same with the pentagons? The difference is that the untransformed stars are regular shapes, and the subjects know them to be regular. Subjects do not make initial estimates of the pentagons because their dimensions are not known beforehand. Nor are such pentagons reliably recurrent as fixed shapes in all the pictures. In most trials subjects were looking for a projective equivalent to a non-regular pentagon.

2.2 The standard shapes

The standard shapes are described first, because they are simplest. There are six such shapes that define a factor of the experimental design. One of the standard shapes has a regular pentagon as its perimeter. The other five standard shapes have an irregular pentagon as perimeter. The irregular pentagon is the same in each case, except that it is rotated in position about the center of the familiar shape. We call this the ROTATION factor. Each standard pentagon contains an equilateral star so placed that its axes of symmetry do not lie on the diameter of the pentagon. The cross ratio can be computed from the lower left vertex to the other points on the perimeter of each figure. When this computation is performed on the standard figures, six distinct values result. These values range from 1.54 to 1.76, which is also the range for the comparison figures.

2.3 The comparison figures

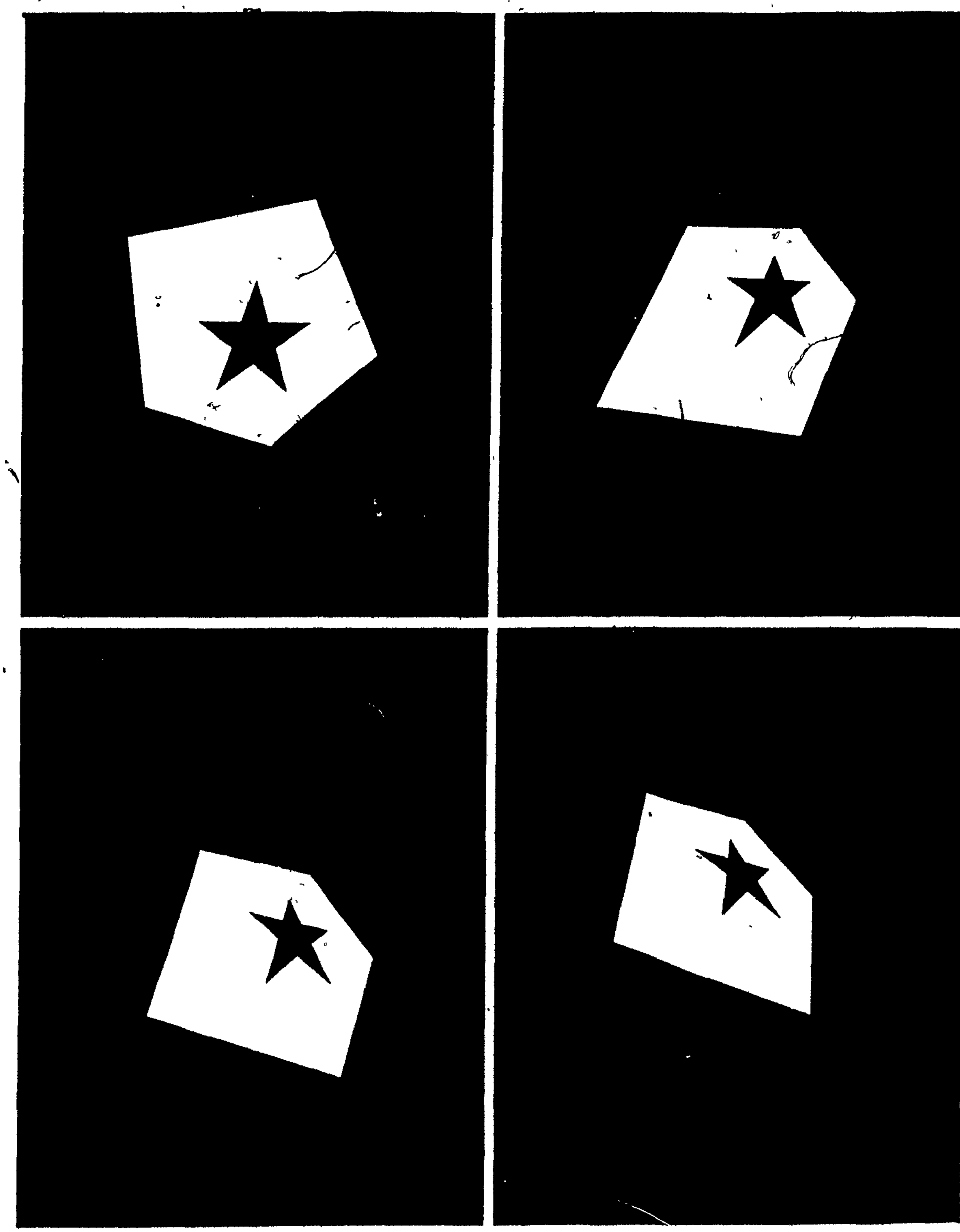
The comparison figures are projections of the standard shapes, as in the last experiment. Again, one may think of the position of a source of light that projects an image of the shape on to a fixed plane. This position is varied to produce shapes that are different in the lengths of their sides, but which are not different in their projective properties. It should be noted that the operations of rotation and projection are not commutative from a Euclidean standpoint. A rotation of a figure followed by a projection of that figure has a different result than a projection followed by a rotation (see Figure 5.2). The corresponding factor in the experimental design is called SOURCE. This factor has three levels. Hence the experiment has a fully-crossed repeated measures design, all conditions repeated. It is a two-factor 6X3 design. Each comparison figure was congruent with one and only one standard figure, when orientation is taken into account. Orientation was emphasized in the instructions and as a matter of fact orientation is accessed in perception. Besides, Rock (1973) has shown that reassignment of orientation is difficult for subjects to achieve with unfamiliar shapes.

The outline of the familiar shape - the equilateral star - varies with the levels of the factor SOURCE, but does not vary across levels of the factor ROTATION. The sides of any star are equal in length to those of other stars contained in other comparison shapes to which the same projective transformation has been applied. It has been said that the star is an indicator of slope, but what slope does it indicate? Let us assume, just for this instant, a fixed viewpoint. The slope of the plane that is depicted in each condition of the SOURCE factor can be expressed by a single number. The ratio of the area of an original figure and the area of the projected figures varies as the cosine of the angle at which that plane intersects the picture plane (the formula is from Bell, 1923). The comparison figures can be projected on to the picture plane by standard figures on planes that intersect the picture plane at angles of 32.1^{o}, 37.9^{o}, and 50.5^{o}.

The slides were prepared and presented as in the first experiment, but this time there was only one comparison figure on each. Six cards with the standard figures were placed in front of a subject. The spatial order of presentation of these cards was random across subjects.

Figure 5.2

More Examples of Shapes



The untransformed figure in the upper left is different from that in Figure 5.1, though its perimeter is a rotation of the other's. The other shapes are comparison shapes derived from that standard shape. The perimeters of the comparison shapes are noticeably different from those of the comparison shapes of Figure 5.1. The shapes of the stars are the same for figures in corresponding positions in Figures 5.1 and 5.2.

3. Method

3.1 Subjects

Forty subjects were tested between March 5 and April 4, 1984. All were students or faculty at McGill. The new sample of subjects included 15 men and 25 women, and subjects ranged in age from 19 to 35. Five wore glasses during the experiment.

3.2 Instructions

Subjects were told that they would have the same six pictures constantly before them (standard), and that they would also be shown slides (comparison) of those pictures. Their task was to match a slide with one of the pictures. The subjects were then shown the six pictures, and some similarities and differences among the pictures were noted. The subjects' attention was called to the orientation of the five-pointed star, which was identical in each picture. The pictures were said to have a proper orientation fixed by the orientation of the star. In particular, differences in the lengths of the segments that comprise the figure in each picture, in the angles of the pentagons that form the perimeter of each figure, in the areas at different locations within the figure, and in the eccentricities of the stars within their pentagonal frames were pointed out.

The comparison slides were said to have been photographed when the camera was held at some odd angle to the paper. Subjects were told that this angle would vary. They were told that the pictures would not be rotated or reversed in the slides, and that the topmost point of the star on the slide would correspond to the uppermost point of the star in the picture.They were shown one of the transformed figures as an example of the result. The subjects were assured that each of the new slides represented one of the pictures. It was stipulated that the topmost point of the star in the picture would match the topmost point of the star on the slide. The idea that shape can be judged from slant was introduced in this way:

> Sometimes the shape of the star in the slide will be altered with respect to the shape of the star in the pictures. You will be asked to identify the slides with the pictures from which they were taken. The question is: to which picture does the slide correspond? The change in the star's shape between the picture and the slide will not help you to answer this question. In each of the pictures before you, the star has the same shape. This altered star in the slide bears the same relation to each of those stars in the pictures. So, the relation will not help you to judge the differences among the pictures. However, the relation between the shape of the star in the picture and the shape of the star as projected by the slide provides a clue to the angle at which the camera was held. Do you see what I mean? One can tell the angle of the camera, or the slant of the plane of the paper in space, by looking at the change in shape of the star between the picture and the slide. Can you imagine how the camera was held to produce this slide?

Tilt was expressed colloquially as the angle at which a camera must have been held with respect to the paper to produce such a slide. The subjects were not asked to make their estimates of tilt explicit. Subjects identified a standard figure to which the comparison figure was thought to correspond. The standard figure was indicated by its number. Subjects were told that they might take as much time as they desired for their decision, and subjects were asked for their questions about the procedure.

3.3 Procedure

After subjects had been given the instructions, the comparison figures were projected on an opaque screen with a Kodak carousel projector. The eighteen slides were presented in either of two random orders. The room lights were left on. The screen was two meters distant from the subjects, and the magnified size of the projected images was about twice the size of the original drawings. The experimenter recorded the responses.

4. Results and Discussion

Each subject's choice was recorded for each slide. Overall, 29% of choices were correct, which seems higher than chance (16.7%). There was, however, considerable variation about the general mean - see Table 5.2.

To explore the data further, a two-way analysis of variance was performed with, as dependent measure, the difference in cross ratio between pairs of figures judged to match. A significant effect of ROTATION, but none associated with SOURCE, was found (see Table 5.1). Recall that ROTATION is a projectively irrelevant factor. Post-hoc comparisons revealed significant departures from projective equivalence for four of the six levels of ROTATION. It follows that a nonprojective factor is influencing responses. Yet how can the pattern of responses best be summarized and understood?

4.1 Foreshortening

There is a distance measure that exemplifies some of the nonprojective factors that guided subjects's choices. This measure is put forward neither as a new dependent variable, nor as an index of a psychological process. It is merely a gross indicator of geometric properties that, while being nonprojective, are highly correlated with response patterns. Let the vertices of each of the standard and comparison figures be labelled 1 to 5 in counterclockwise order, beginning with the origin. This provides a match of vertices for any pair consisting of a standard and a comparison figure. Let the distance be measured between each vertex and the centre of the star in each figure. Let these distances be labelled in the same way as the vertices, so that they too may be paired. For each pair, consisting of a standard and a comparison figure, let a Pearson correlation coefficient be computed. This coefficient is an index, in Euclidean terms, as to how closely the two figures match in shape. The coefficient is, in fact, an indicator of foreshortening, which is the relative change in distances that is brought about by perspective effects. There is one measure of foreshortening for each pair of a standard and a comparison figure.

A tabulation of the number of times that each comparison figure was matched to each standard figure is given in Table 5.2 for all subjects. These data

Table 5.1

Analysis of Variance: Differences of Cross Ratios

Source	SS	df	MS	F	P
Between Subjects	0.317	39			
Within Subjects	8.284	680			
Source	0.001	2	0.000	0.14	NS
Source X $Subjects_W$	0.374	78	0.004		
Rotation	2.947	5	0.589	63.68	$p \leq .001$
Rotation X $Subjects_W$	1.805	195	0.009		
Source X Rotation	0.198	10	0.019	2.62	NS
Source X Rotation X $Subjects_W$	2.956	390	0.007		
Total	8.601	719			

Post-hoc Comparisons

Rotation Condition	Mean Difference	F for Comparison	P
1	-0.0541	19.00	$P \leq .01$
2	0.0765	37.91	$P \leq .01$
3	-0.1147	85.31	$P \leq .01$
4	0.0250	4.04	
5	0.0199	2.56	
6	-0.0590	22.55	$P \leq .01$

The critical F is 15.60 at $\alpha = .01$, calculated by Scheffé's procedure (Keppel, pp. 97-99).

Table 5.2

Confusion Matrix for the Second Experiment

Here is the contingency table for the data of the second experiment (40 subjects). The null hypothesis is that a greater number of observations will lie along the major diagonal than off it. Columns of the table are labelled by the titles of the transformed pictures that were presented, in order from 1 to 6. Rows are labelled by the titles of the untransformed pictures that subjects indicated to be matches, likewise. Comparison 6 is the regular pentagon.

chosen \ presented	1	2	3	4	5	6
1	72	18	5	6	57	30
2	29	64	51	59	35	63
3	3	13	40	22	3	7
4	3	5	3	3	6	3
5	1	5	6	4	3	2
6	12	15	15	26	16	15
	120	120	120	120	120	120

Correct: 197/720
Incorrect: 523/720

have been collapsed over the levels of the SOURCE factor. If the subjects had judged projective equivalences perfectly, then all their choices would fall along the major diagonal of the table. If subjects chose randomly, all the cells of the table would have equal values in the long run. The pattern of subjects' choices seems different from either of those patterns. Let us, then, see how the indicator of foreshortening correlates with the results.

To this end, new correlation coefficients were calculated. Consider a matrix three times as large as Table 5.2, defined by the presented and chosen figures for each of the three levels of the SOURCE factor. To each cell in the matrix three numbers can be assigned: 1) the number of times that the pair was chosen as a match, 2) a one or a zero to indicate whether or not the pair is a match in projective geometry, and 3) a correlation coefficient (as a measure of foreshortening). The point biserial correlation of 1) and 2) can be computed, as can the Pearson correlation of 1) and 3). The point biserial correlation of 1) and 2), which indicates the degree to which projective equivalence predicts the pattern of subjects' choices, does not account for the pattern of subjects' choices. That correlation is small ($r(108) = 0.110$), and nonsignificant ($\underline{t}(106) = 1.14$, Ferguson, 1971, pp.356-358). The Pearson correlation of 1) and 3), which indicates the degree to which nonprojective factors account for the pattern of subjects' choices, is large ($r(90) = 0.898$) and significant ($t(88) = 19.14$, $p \leq .01$) where it can be computed. If an assumption of average frequency of response is made for those pairs where the correlation could not be computed for lack of variance, then the correlation remains high ($r(108) = 0.882$) and significant. The measure of foreshortening is not significantly correlated with 2). These calculations provide evidence for a post-hoc claim that subjects' response patterns overall are better explained by nonprojective properties than by projective ones. This confirms a tendency that was noticed in Experiment 1. In fact, the tendency is more marked in the second experiment.

I might observe, in passing, that Professor Yoshio Takane has performed a multidimensional scaling of Table 5.2, which contains frequencies of choice to pairs of the standard and comparison figures. In this experiment, as in the last, variability in frequency of choice could be explained adequately by a single dimension. It seems in view of the correlations just reported, that such a dimension can either be identified as foreshortening, or is closely correlated with it.

Though the tasks that were presented in the first and second experiments may seem different, the same projectivities are produced in the two situations that were described to subjects. There is a common geometry to both shadow-casting and the projection of shapes at a slant. There is little except the direction of the propagation of light to differentiate the two situations. One could imagine that relative size differentiates the two situations. When shadows are cast, the result is often larger than the original shape. When outlines are photographed at a slant, the result is often smaller than the original. This might lead one to say that one relation is the "inverse" of the other, in the sense of a transformation and its inverse. Yet the two situations are identical in projective terms, as shown by the equality of the projective invariants. These two situations are best conceived not as a transformation and its inverse, but as projectively similar situations, whose relations are best described by the <u>transitivity of equality</u>. Each standard figure may be a picture of each comparison in <u>either</u> experiment, just as each comparison may be a picture of each standard. Equally well, each standard <u>may</u> be a shadow of each comparison, just as each comparison may be a shadow of each standard. This assertion is no mere theoretical postulate; these shadows and pictures can be produced by mechanical means, with some little trouble. In addition, it should be noted that a projectivity of a projectivity is itself a projectivity.

The main point that can be made about the results is that subjects were not particularly responsive to projective congruence. Subjects identified projective matches in less than a third of the trials. In addition, the significant effect of ROTATION in the analysis supports the claim that subjects' judgments were sensitive to nonprojective influences. This factor is independent of both projective congruence and cues to slant. Thus, even when an indicator of tilt is added to a projected shape, subjects' judgments do not univocally reflect sensitivity to projective equivalence.

Shadows and photographs are mere crutches to a single set of mathematical intuitions. Psychologically, too, the two crutches are similar. The sorts of distortions of an object that can be found in its shadow can all be found in its photograph. One is accustomed to seeing the strange effects of foreshortening in each, for example. Nevertheless, it is of interest that the results of the first experiment are extended when a new metaphor for presenting the task is introduced, as well as an indication of tilt. The evidence begins to mount that viewers are not finely sensitive to projective congruence. The next experiment extends the investigation to ask: are these findings supported by the results of another task in an 'open-air' setting?

CHAPTER 6

Experiment 3 - Production

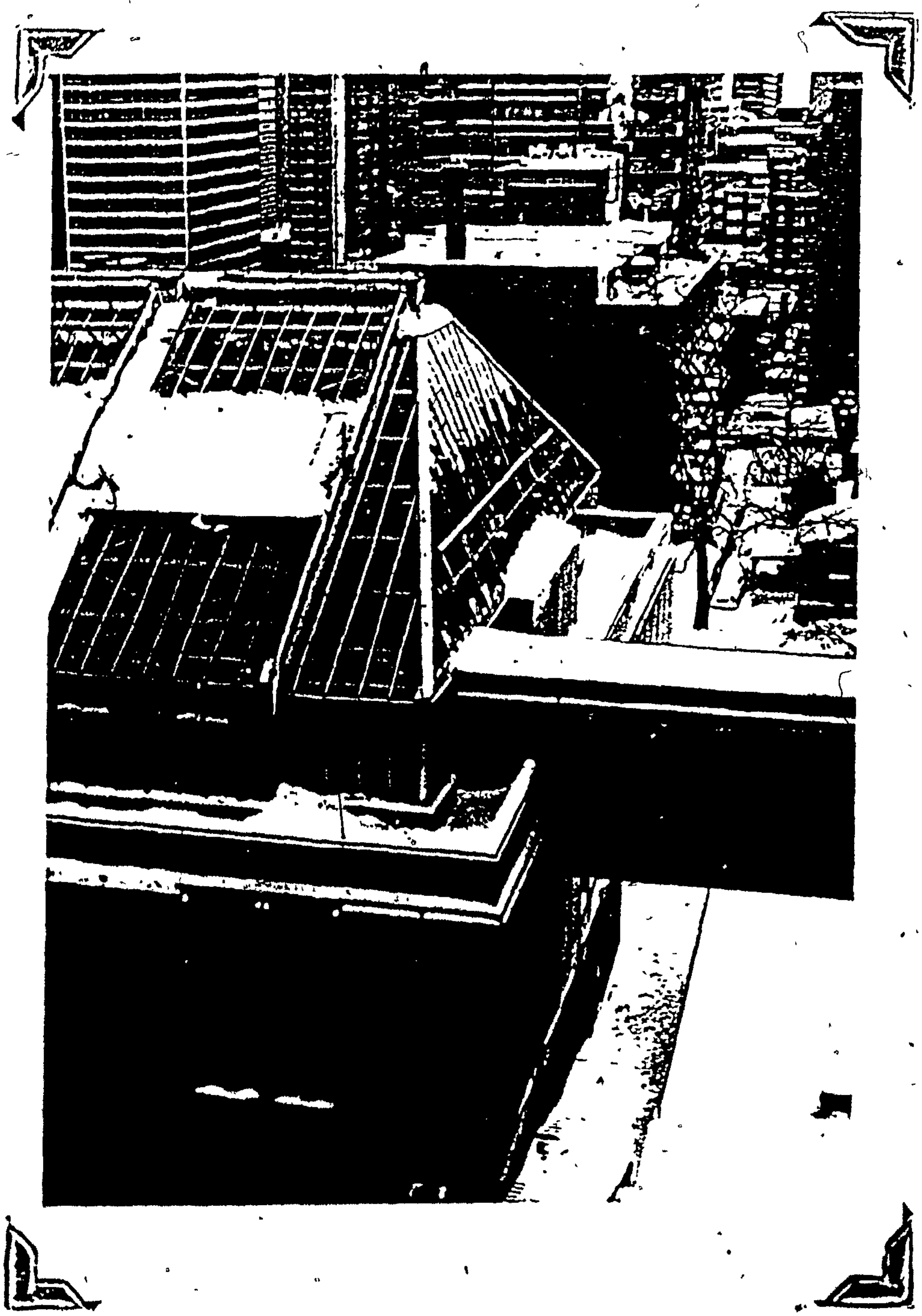
> **Every projection must have something in common with what is projected no matter what is the method of projection. But that only means that I am here extending the concept of 'having in common' and am making it equivalent to the general concept of projection.**
>
> Wittgenstein,
> Philosophical Grammar, p. 163

A different sort of experiment is undertaken next. The stimuli for the first two experiments were photographs of geometric constructions. They are not the usual or natural stimuli for vision. Gibson's research, in particular, stressed that results should have "ecological validity" for large out-of-doors objects in natural viewing situations. The present experiment assesses perceivers' ability to reproduce the projected shape of a large object seen in daylight.

In this experiment subjects were required to estimate the projected shape of two large planar objects. The subjects were given incomplete perspective drawings of the objects, and asked to indicate the apparent position of two features by placing two marks on a drawing. When a subject placed these marks a figure was formed on which two cross ratios could be measured. The quotient of these cross ratios is called the cross ratio measure. The difference of this cross ratio measure from that of a correct perspective drawing is the dependent variable for analysis. If perceivers are not sensitive to projective invariance, then the cross ratio measure of their estimated figure can be significantly different from the cross ratio measure of a correctly drawn perspective sketch.

Figure 6.1

A Photograph of the Object



This is the view from the lobby of the seventh floor of the Stewart Biology Building. The photograph was taken from the eighth window. The experiment was conducted during the summer months.

2. Stimuli

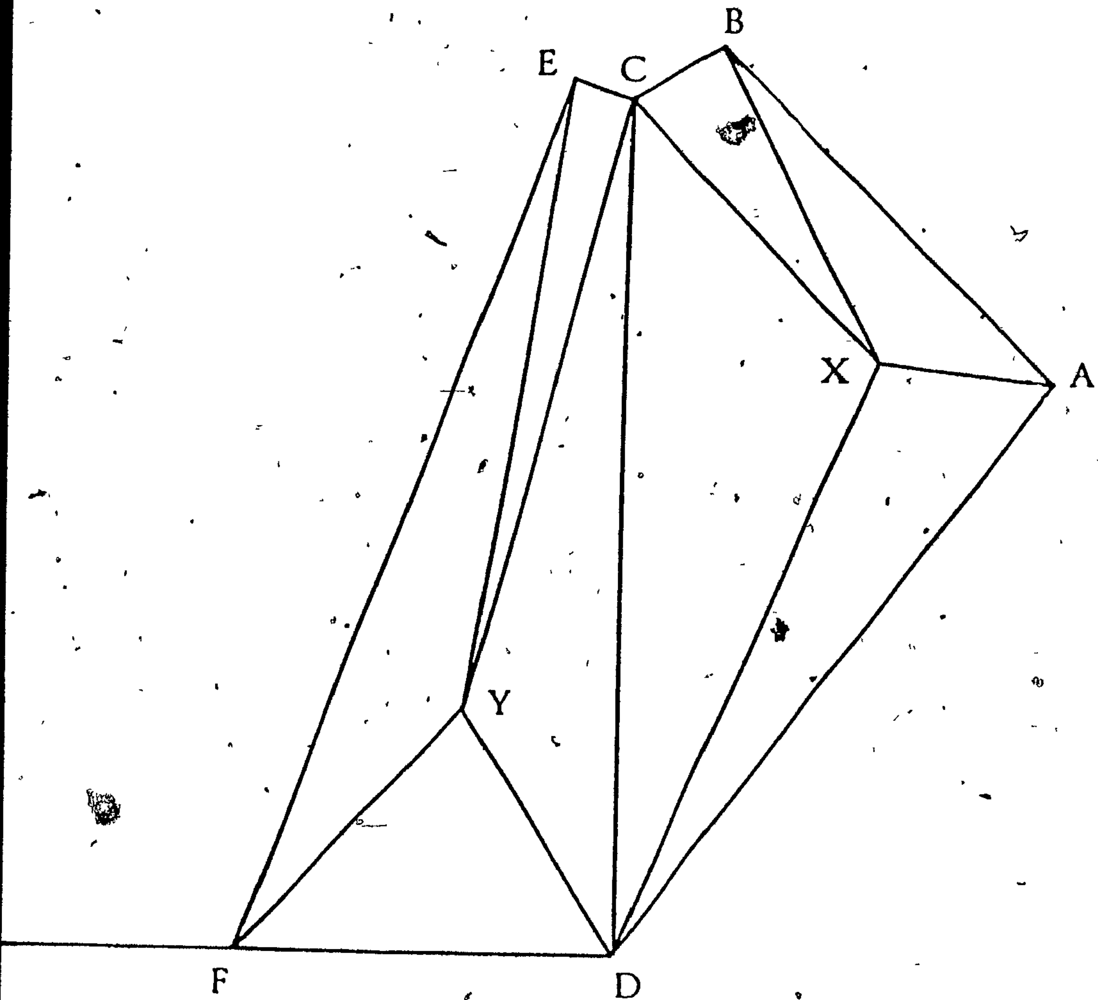
The stimuli for this experiment aren't natural in the way that a geological formation is natural, but they are part of a large greenhouse that was visible from the main part of McGill's Stewart Biology Building. (It was demolished in summer 1985.) Figure 6.1 is a photograph of the greenhouse roof, and Figure 6.2 is a schematic of the same view. The main part of the Biology Building has a bank of windows that faced the greenhouse. The near edge of the greenhouse was some thirty five meters south of the windows.

Six viewing stations were arranged at six windows, three on each of two floors that overlook the greenhouse. The windows are all of the same shape. If the windows are counted beginning from the east, then the stations from which the stimuli were seen are the fourth, eighth, and twelfth windows of the sixth and seventh floors. Figure 6.1 is the view from the seventh floor, eighth window. The horizontal distance between adjacent stations is some 5.0 meters, and the vertical distance between floors is some 5.4 meters as measured by plummet. The floor of the greenhouse roof is on a level with the fourth floor of the main building. The viewing stations correspond to a factor in the experimental design. Accordingly, the experiment has a one factor design with six levels. It is a repeated measures design, all conditions repeated.The factor is called STATION.

Correct perspective drawings were obtained at each viewing station by means of a Leonardo's window. A level was chosen some distance above each window-sill, and the middle of the level was marked at each window. Call these points D. A horizontal line of five centimeters in length was drawn to the left of D. Call this segment DF. A transparent rectangular grid, Leonardo's window by another name, was placed over this

Figure 6.2

A Schematic Diagram of the Object



This diagram is the master projection for the seventh floor and eighth window. The line FD on the object is horizontal in a frontal plane to the observer. The cross ratio was measured for quadrilateral ABCD from point X and for quadrilateral DCEF from point Y. Fixed points X and Y were inserted into the diagrams after subjects had placed two dots on each side of the diagram.

so that the x axis of the grid was level. The experimenter aligned the length of DF with the bottom of a panel of the greenhouse roof, thus fixing a projection. The positions (those apparent to one eye) of all the vertices of the stimuli were marked on the grid. The grid was then removed.

Incomplete copies were made of the correct perspective drawings. Figures that are 'Y'-shaped remain when the lines that include the points A, X, Y, and F are removed from the drawings. These parts of the master drawings were photocopied, then collated into booklets containing one of each picture. The copies were arranged in random order within each booklet. A transparency of the same part of the master drawing was placed on each window. The transparencies were appropriate to the windows; meaning, the figure copied on the transparency would be aligned with the dihedral angle and flanges of the greenhouse just when the segment DF was aligned as in the production of the master drawing.

2.1 Variability in the cross ratio

First of all the quotient of the cross ratio for the two quadrilaterals ABCD and DCEF in the complete correct drawings was calculated, to check on the accuracy of these drawings. The quotient was taken in order that all the data might be used economically and efficiently - in fact, by working with this quotient a great economy of computation was effected. Since the cross ratio of each quadrilateral is a constant across the six viewing windows (STATION), so too is the quotient of cross ratios. To permit the calculation of cross ratios, two points — one interior to each quadrilateral — were noted in each drawing. These points are marked X and Y in Figure 6.2. There was, in fact, a slight variation in the quotient of cross ratios thus calculated. It will come up again when we come to analyse the data.

Since the dependent variable of the analysis depends upon the corresponding cross ratios in subjects' drawings of the projectivities, we should be clear about how it was estimated. A subject was asked to place points A and F on the answer sheets appropriate for a particular window. The experimenter subsequently entered points X and Y in the correct position in the answer sheet and then calculated the quotient of cross ratios for that response. The formula used to make the calculation is

$$\frac{\dfrac{\Delta DCY \;\; \Delta EFY}{\Delta DFY \;\; \Delta FCY}}{\dfrac{\Delta ABX \;\; \Delta CDX}{\Delta ADX \;\; \Delta CBX}}$$

Call the dividend c and the divisor a. The dependent measure was obtained by subtracting from the quotient c ÷ a for each response the value of the 'true' quotient -- the 'true' quotient being that which was obtained from the complete and correct drawings, namely 1.33.

3. Method

3.1 Subjects

Twenty subjects were tested between May 28 and June 12, 1984. The subjects were students or faculty in the Department of Psychology at McGill. There were nine men and eleven women in the sample, and subjects ranged in age from 20 to 32. One of the subjects wore contact lenses. Twelve of the subjects were left-eye dominant, as determined by a sighting task, described by Porac and Coren (1981).

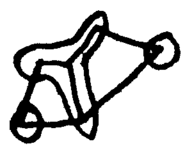
3.2 Instructions

The instructions given to subjects were as follows:

This experiment concerns the shape of the greenhouse you see outside. In particular, it concerns a certain part of the greenhouse; namely, the two sloping planes of glass that form the very end of the greenhouse; the plane of nearly triangular shape that slopes down and forward like this (gesture),and the plane that slopes off to the side like this (gesture). If I were to make a crude drawing of that part of the greenhouse, it might look like this:

As you can see, this is a very crude drawing. The transparency before you on the window is a picture of just this part of the greenhouse.

Namely, the picture is of the two short edges at the top of that part of the greenhouse together with the long edge that slopes down and forward from them (gesture). Do you see the correspondence? Later you'll be asked about the two corners of this part of the greenhouse that are not included in the picture on the window: these corners.



That is, you'll be asked about the position of the lower left-hand corner of this plane of glass (gesture) that slopes down and forward, and the position of the far corner on the other plane of glass, that is, the corner nearest the street. There are other parts of the greenhouse that might be drawn in schematic like this:

You will not be asked about those corners. Stand in front of this picture, then focus on the greenhouse. You may notice that the lines of the picture on the window appear to be doubled. Do they appear to be doubled? (Alternate instructions concerning the apparent doubling were given if the subject responded "no".)

One image appears to be on the left, and the other on the right. I would like you to align the ___________ ("right" or "left", whatever the contralateral side to the dominant eye) image with the corresponding part of the greenhouse. Here's what is meant by "align". This point (a point on the picture was indicated) should appear to coincide with that corner (a point on the greenhouse was indicated) and that point should also correspond to that corner. Both these points on the picture should appear to coincide with points on the greenhouse at the same time.

What is important is that the length of this segment (the vertical segment on the picture was indicated) should have the same apparent length as the long edge of the greenhouse that slopes down and forward. Their lengths should appear to be the same. You may need to move about to discover where the correspondence can be achieved. Tell me when you have aligned the ________ ("right" or "left" as before) image with that part of the greenhouse. (When the subject responded, he was given a pen and clipboard with the appropriate diagram. The diagram was a photocopy of the transparency on the window).

On this picture, I'd like you to mark in the corners I mentioned (their descriptions were repeated). Indicate their position by making a small dot or a cross on the paper. You may need to look back and forth from the window to the paper several times to do this. Try to be as accurate as you can.

After the subject marked in the dots he was led to the second viewpoint.

The task was performed six times, once each at six station points. The order of station points was randomized across subjects. Subjects were allowed to hold the response form at the orientation that was most comfortable for them. This may imply some change in the geometric characterization of the viewing conditions, but that is a matter which will be attended to in later experiments.

4. Results and Discussions

The dependent measure for the first analysis is the difference between 'true' and observed cross ratio quotients. One subject's data were eliminated from the analyses because his scores were a factor of magnitude larger than the scores of other subjects. Variability in the values of quotients of the correct drawings is noticeably less than that of the quotients of subjects' drawings. The comparison is made in Table 6.1. Subjects' quotients were much more variable in two conditions than in other conditions. The former conditions represent the viewing stations farthest removed from the greenhouse. These two conditions are eliminated from this analysis of variance. Though analysis of variance is fairly robust to violation of the assumption of homoscedasticity, particularly where cells of the analysis contain equal numbers of

Table 6.1

Mean Estimates of Cross Ratios for the Master Drawings and for a Typical Subject

Window	$\frac{c}{a}$ for master drawings	$\frac{c}{a}$ for subject A.S.
7,4	1.27	1.32
7,8	1.33	1.09
7,12	1.39	2.04
6,4	1.56	0.83
6,8	1.30	0.74
6,12	1.18	-0.69

The variance of the estimates were significantly different.
{$t(4)$ = 16.44**, Ct (α = .01)(4) = 8.610, McNemar, p.246}

Variance in Estimates of the Quantity $c \div a$

Floor, Window	Standard Deviation of Estimate
7,4	.47
7,8	.34
7,12	1.28
6,4	.31
6,8	.71
6,12	1.25

$$\frac{s^2_{(7,12)}}{s^2_{(6,4)}} = 16.65 \qquad \frac{s^2_{(6,12)}}{s^2_{(6,4)}} = 15.96$$

observations, yet the magnitude of the differences in variance across conditions is large.

An analysis of variance was performed for the four remaining conditions. It showed a significant effect of STATION on the cross ratio measure of the estimated figures. Again, conservative degrees of freedom were used in making the *F* test, to avoid unwarranted assumptions about equal covariance in the repeated measures design (see Table 6.2). The finding indicates that the departures from the true cross ratio measure are not uniform across conditions, as would be expected if, for example, the differences were due to random variability in cross ratio. That is not what is observed. The two largest departures among the four are significantly different from zero, as shown by post-hoc comparisons (see Table 6.2). They represent distortions in projective properties.

Another analysis was done that used the estimates of trapezium ABCD only. Each subject had made a mark to complete drawings of this shape. The distance and orientation of this mark from the position of the corresponding point in a true perspective drawing is taken. All the mean departures are significantly different from the correctly projected point, as shown by Mahalanobis' D^2 computed on the coordinates *X* and *Y*. (See Table 6.3. Mahalanobis' D^2 is equivalent to a familiar statistic, Hotelling's T^2, when T^2 is adjusted for sample size. For Hotelling's T^2, see Scheffé, 1959, p. 417). When scatterplots of the placements for each drawing were produced, the true point was found to be at an extreme of the distribution of the estimated points in each case. Though these findings relate to Euclidean properties, they strongly support the thesis that perceivers' judgments of apparent shape in a natural environment reflect insensitivity to projective equivalence.

Table 6.2

Analysis of Variance: Corrected Ratio of Cross Ratios, Two Conditions Eliminated

Source	SS	df	MS	F	P
Between Subjects	7.88	18			
Within Subjects	11.48	57			
Station	2.52	3	0.842	5.08	$.05 \geq p \geq .01$
Station X Subjects	8.95	54	0.165		
Total	19.36	75			

Post-hoc Comparisons			
Station	Mean Difference	F for Comparison	P
7,4	-0.386	8.57	
7,8	-0.760	33.11	$P \leq .01$
6,4	-0.482	13.35	$P \leq .01$
6,8	-0.265	4.03	

The critical F is 12.60 at $\alpha = .01$, calculated by Scheffé's procedure.

Table 6.3

Departures of Vectors from Zero (mm.)

Condition	Mean of X coordinate	Mean of Y coordinate	Mahalanobis' D^2	F value	p
7,4	3.84	6.97	1.53	13.70	$\leq$.001
7,8	6.87	6.04	2.12	19.03	$\leq$.001
7,12	9.31	5.57	1.54	13.84	$\leq$.001
6,4	3.66	4.74	3.55	31.82	$\leq$.001
6,8	6.81	4.04	13.20	118.45	$\leq$.001
6,12	9.20	3.52	2.36	21.14	$\leq$.001

n = 19

4.1 Areas and ordered distances

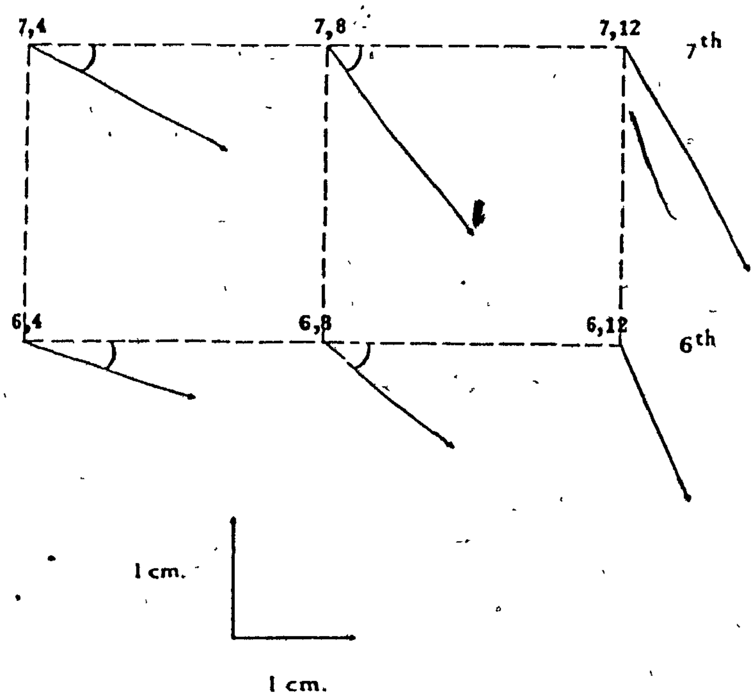
There are measures that may help to show that projective considerations are being shouldered aside in subjects' responses. These measures are meant to indicate perspective or foreshortening effects. One measure is related to a geometric property internal to figure ABCD, while the other is not. The adjusted vector sum of the departures of observed from true points was computed for the figure ABCD at each of the six viewpoints. The pattern of the vector sums runs counter to that which would be expected on the basis of unguided error (for example, as in Heywood and Chessell, 1977). A schematic of the results is shown in Figure 6.3.

Is an internal geometrical property of figure A'BCD associated with the direction and magnitude of the adjusted vector sums? A comprehensive search was made of the variable line lengths, angles, trigometric functions of angles, and areas of individual triangles of the six estimated figures A'BCD to find such a quantity. None provided a medium-sized or significant correlation with the properties of the corresponding figures ABCD of the master projections. One characteristic of figure A'BCD, as estimated by subjects, showed a strong relation to the corresponding property of the projected figure. An ordinal ranking of the area of the figures ABCD of the master drawings correlated strongly and significantly with the area of the figures produced (see Table 6.4). The correlation does not, however, account for both the magnitude and direction of the errors in estimating the position of point A. For a given area of a particular figure, the vector including the estimated point could vary in both direction and magnitude, yet still produce a quadrilateral of the same area.

Perhaps some external geometric variable will capture the departures better. One variable that is of interest needs some introduction. The panes of glass from which the observations were made form part of a wall of the Biology Building. The viewpoints can be assumed roughly coplanar, since each subject stood at arm's length from the windows when they made their judgments. Call the plane of the windows

Figure 6.3

Average Deviations of Estimated Points A' from the Correctly Projected Point A



The mean deviation of estimated points from the projected points of a corner of the object abcd is shown for various viewpoints. The rows are the seventh and sixth floors of the building. The columns are the fourth, eighth, and twelfth windows of the lobbies. Both the direction and the magnitude of the deviations are portrayed.

Table 6.4

Rank Correlations between Measures on the Estimated and Correctly Projected Figures ABCD in Experiment Three

Measure 1	Measure 2	Spearman's Rank Correlation
Projected	Estimated	1.000
Distance	Estimated	1.000
Distance	Projected	1.000
Distance	Angle	-1.000

n = 6, C_{rho} (6)(one-tailed)(α = .01) = .943

Projected: The areas of the quadrilateral ABCD for the master drawings.

Estimated: The areas of the quadrilateral A'BCD as found by construction on the subjects' estimations of the position of point A.

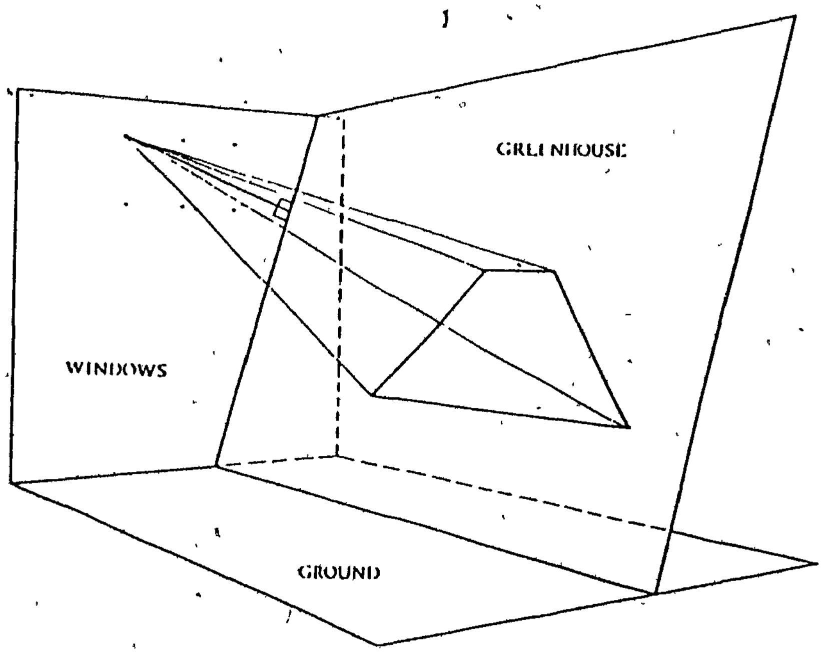
Distance: The perpendicular distance from the viewpoint to the line of intersection of the object plane and the plane of the viewpoints (plane abcd and plane ABCD).

Angle: The deviation in direction of the mean vector from the horizontal. The mean vector is the deviation of subjects' estimates of the position of point A from the position of point A in the master drawing.

Note: Each subject estimated a position for point A. Each of these points deviated from the projected position of point A. The deviation can be represented by a vector. The vector sum over all subjects and for one drawing is a measure of the average deviation of the subjects' choices. The magnitude and direction of this vector sum have several correlates, as displayed.

Figure 6.4

The Pyramid with Object as Base and a Viewpoint as Apex



There is a pyramid that has a viewpoint as its apex and the stimulus object as its base. The variant geometrical quantities of this pyramid change as the viewpoint changes, and, as a consequence, those quantities change as the perpendicular distance from the viewpoint to the object plane changes.

ABCD, just as the quadrilaterals, and call the plane of the greenhouse face abcd. The quadrilateral figures in the drawings are projections of the object abcd. Plane ABCD intersects abcd in a line. This is the line at which surface abcd disappears. One looks "edge on" at the plane abcd from certain positions in the building. Once these positions are known, the disposition of the line can be calculated. A variable that correlates with both the magnitudes and the directions of the discrepancies of estimation is the ordinal ranking of the viewpoints for their perpendicular distance from the line of intersection of the planes abcd and ABCD. The correlation between the ordered distances and the estimated areas was high and significant, as was the correlation between the ordered distances and an ordinal ranking of the projected areas (see Table 6.4). The ordered distances also correlate with the angle of the vectors from the line of intersection, which lies -71° from the horizontal.

The significance of an ordering of the perpendicular distances of viewpoints to plane ABCD is that the distance is the key to the dimensions of the pyramid between the viewpoint and the vertices of the object. Such a pyramid is depicted in Figure 6.4. All the variable quantities in the viewing situation can be expressed roughly as functions of the variability of these distances, since other key variables vary but little in this situation. Book twelve of Euclid is sufficient to demonstrate these propositions. Perrone (1980) found perpendicular distance to be important in the perception of texture gradients. He sought to explain slant underestimation (see Gibson, 1950b). He used the hypothesis that, when confronted with the stimuli ordinarily used in experiments on texture gradients, a subject mistakes his distance to the edge of the aperture through which he sees a gradient for the perpendicular distance. He supposed that this mistaken assumption results in an illusion of slant. In turn, that illusion would result in an illusion of shape, on the hypothesis of shape-slant invariance. Thus, the slant underestimation phenomenon may be related to the departures from projective invariance seen in the present experiment. It should be noted, however, that this does not detract from the significance of the new finding that subjects are not reliably sensitive to projective congruence.

The significant effects shown in the analyses support the claim that perceivers' judgments in a production task do not simply reflect sensitivity to projective constancy. There are real differences between projections and subjects' productions. The cross ratio measure that was computed shows real differences across viewpoint conditions. Relative differences in the measure across conditions are evidenced by the outcome of the analyses of variance. The effect is not explained by a "phenomenal regression to the real object" in terms of area (see Note 2 for an explanation).

Yet the experimental task is so simple. When it is performed, it seems compellingly obvious. What could be more apparent than the position of one clearly visible point close to several others? It might have seemed that the apparent position _must_ be related to projected position. After all, we are told that "the optical information about other bodies available at the sensory surfaces of each organism is governed by the geometrical laws of perspective projection" (Shepard, 1984, p. 422). When these results are considered, it becomes less than obvious that what is _useful_ visual information is governed by the laws of perspective projection, since the basic congruence of perspective projection is not reflected in a variety of judgments guided by vision. Nevertheless, we must look at several factors that might have affected subjects' responses in such a way as to obscure sensitivity to projective equivalence.

CHAPTER 7

Experiment 4: Effects of Training

> **Suppose I explain various methods of projection to someone so that he may go on to apply them; let us ask ourselves when we should say that the method that I intend comes before his mind.**
>
> **Now clearly we accept two different kinds of criteria for this: on the one hand the picture (of whatever kind) that...comes before his mind; on the other, the application which...he makes of what he imagines. (And can't it be clearly seen here that it is absolutely inessential for the picture to exist in his imagination rather than as a drawing or model in front of him...?)**
>
> **Wittgenstein,**
> **Philosophical Investigations, I, p. 55**

One can imagine certain objections to the interpretation of the results of the last experiment. Some of the objections might be inspired by a belief that projective geometry provides a competence theory for the visual perception of form. For example, someone could propose that most of the negative evidence that has been provided is due to performance factors. If competence is remote from performance, particular judgments by subjects may be of small importance to formulation of a competence theory. The negative evidence would then be tangential to the standard theoretical claim that we are sensitive to projective equivalence in vision. Though this objection sounds dogmatic, it does raise interesting possibilities. It suggests an appeal to individual differences, to learning, and to the contrast between novice and expert. What if someone were born with a perfect ability to detect projective congruence, just as some people are born with perfect pitch? Or what if some people could be trained to judge projective congruence, and their performance were general to a variety of situations? Such performance would tell against a radical claim that projective invariants are in fact invisible.

The fourth experiment concerns expert judgments of projected shape. Members of professions that require training in perspective drawing might be more sensitive to projective invariants than most of us. Consequently, students of architecture were recruited for an experiment. All had had instruction in descriptive draughting and systems of projection. It is possible that practice had improved their skill at reproducing the projected shapes of planar forms. In other words, the effects of instruction on sensitivity to projective equivalence will be assessed. Subjects estimated the projected shape of the same planar object as in the last experiment, a quadrilateral pane of the greenhouse that belonged to the McGill Department of Biology. Subjects were given an incomplete perspective drawing of the quadrilateral, and they were asked to indicate the position of a feature by placing a mark on the drawing. Again, if perceivers are not sensitive to projective equivalence then the cross ratio of their estimates of the projected shape can be different from the cross ratio of a correctly drawn perspective sketch (i.e. the "true" cross ratio). The null hypothesis is that the cross ratios of the architects' estimates will be no closer to the true cross ratio than that of untrained subjects' estimates.

2. Stimuli

The stimulus was part of the greenhouse used in the previous experiment. The relevant part has been identified as <u>abcd</u> and its projected image was called <u>ABCD</u>. The reader is referred to previous illustrations (Figures 6.1 and 6.2) for an impression of the shape of the stimulus. Nine viewing stations were arranged at nine windows, three on each of three floors that overlook the greenhouse. The stations from which the stimuli were seen are the fourth, eighth, and twelfth windows of the sixth, seventh, and eighth floors. In subsequent discussion, the fourth, eighth, and twelfth windows will be called windows one, two, and three, respectively. Perspective drawings were obtained anew at each viewing station by means of a Leonardo's window, as described in Chapter 6, Section 2.

Incomplete copies were made of the correct perspective drawings. Figures that are shaped like an inverted L remain when the lines that include the points A and x are removed from pictures of the figure ABCD. The figures DCEF were also included as aids to fixation. These parts of the master drawings were photocopied, then collated into booklets containing one of each picture. The copies were arranged into 27 booklets according to the entries in the rows of three different 9 x 9 Latin squares. Transparencies were applied to the windows at each station point as before. For the comfort of the subjects, the heights of the transparencies above the windowsill were changed from the last experiment. Since each subject made a response at each of the nine windows, viewing stations contribute a repeated measures factor. I call that factor STATION; it has nine levels.

3. Method

3.1 Subjects

Twenty-seven subjects were tested between May 9 and June 18, 1985. Nine subjects were final-year students in the School of Architecture at McGill. There were seven men and two women among the architects, and they ranged in age from 20 to 30. Five were left-eye dominant for sighting tasks, and those subjects used their left eye in aligning the transparency with the stimulus object. The other two groups of nine subjects were students or faculty in the Department of Psychology at McGill. There were four men and five women in the second group, and they ranged in age from 21 to 25. Four were left-eye dominant for sighting tasks. There were three men and six women in the third group, and subjects ranged in age from 22 to 29. All nine were left-eye dominant for sighting tasks.

3.2 Instructions and Procedure

The instructions to subjects were identified to those given in the third experiment. The present experiment was conducted by Karen Wynn. The angle at which the clip board was held was not strictly controlled. The effect of such a control will be discussed in the next chapter. The order of the nine station points was counterbalanced by a different Latin square for each group of nine subjects. The three groups of subjects form a between-groups factor called EDUCATION. It has three levels. Recall that the viewing stations correspond to a within-groups factor called STATION. The experimental design, then, is a two-way between-within 3 x 9 design. It may be noticed that both STATION and EDUCATION are random-effects factors. (At least, they may be construed as random-effects factors -- or not — for the generalization of results, though their levels were not randomly assigned. The issues of the "language as a fixed effects" debate may be invoked here. (cf. Clark, 1973). This construal of the factors has no consequence for the direction or significance of the present results.) After the two groups of control subjects had performed the task, they were presented with a set of complete perspective drawings and a corresponding set of incomplete drawings. They were asked to copy from a complete drawing the position of the missing point in each of the nine incomplete drawings. This procedure provides an estimate of the effect due to the copying task itself. These drawings were presented in the same order as the viewing stations themselves.

4. Results and Discussion

The dependent variable for the first analysis is the cross ratio of the shape that a subject produces, minus the true value of the cross ratio. The cross ratio was calculated as follows: a subject was asked to represent point A on the answer sheet appropriate to a given window. Subsequently the experimenter entered a point x (see Figure 6.2) in the correct position on the answer sheet, and calculated the cross ratio of quadrilateral ABCD from point x. The dependent measure was obtained by subtracting the value of the 'true' cross ratio from the cross ratio for the responses. The 'true' cross ratio is that which was obtained from the complete and correct drawings. An analysis of variance showed a significant effect of STATION on the dependent measure (see Table 7.1). Hence cross ratios in at least one of the conditions are significantly different from the mean corrected cross ratio. No gross violations of homoscedasticity were apparent among the conditions. A significant effect of EDUCATION was also found, so that cross ratios in at least one of the groups were different from cross ratios in the other groups. A quasi-F was constructed to test the significance of this effect, because of the presence of random-effects factors in the design. The quasi-F was tested with two and thirty degrees of freedom, following Winer (1971). No significant interaction of the factors EDUCATION and STATION was found. It is important to know if these effects represent departures of the cross ratio from its 'true' value. Departures in the cross ratio from its 'true' value were found among levels of the factor STATION, as witnessed by the test that is tabled for viewing station 6,2. (Table 7.1). The significant effect of STATION substantiates the claim that judgments in this task do not reflect a uniform sensitivity to projective

Table 7.1

Analysis of Variance: Corrected Cross Ratios

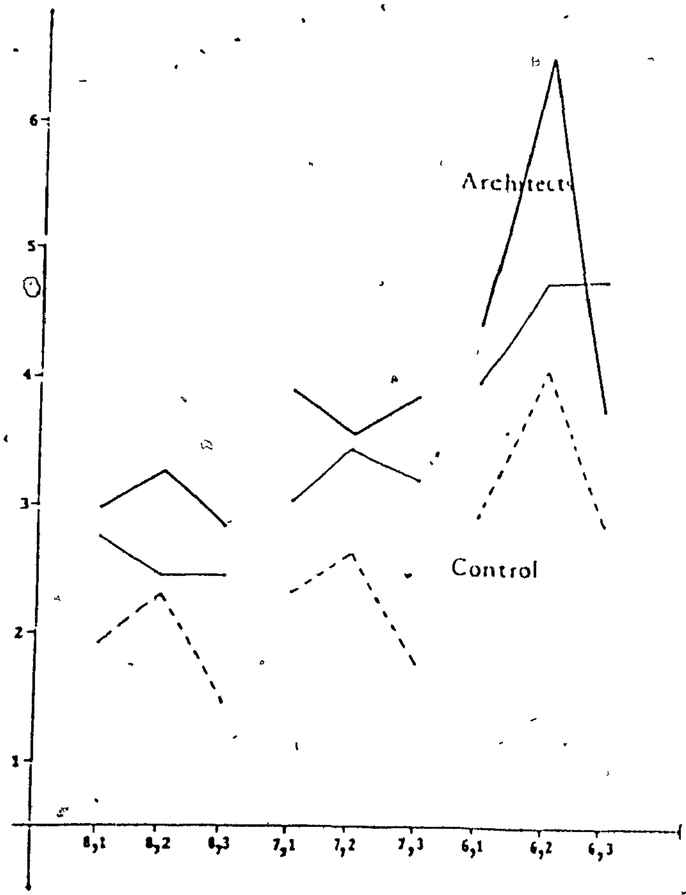
Source	SS	df	MS	F	P
Between Subjects	355.98	26			
Education	84.72	2*	42.36	3.41*	$p \leq .05$
Subjects within Education	271.25	24*	11.30		
Within Subjects	525.53	216			
Station	163.43	8	20.42	11.66	$.01 \leq p \leq .001$
Station X Education	25.82	16	1.61	0.92	NS
Station X Subjects	336.27	192	1.75		
Total	881.52	242			

* A quasi-F was constructed in which the numerator is the sum of mean squares for EDUCATION and STATION X SUBJECTS effects, and the denominator is the sum of mean squares for SUBJECTS WITHIN EDUCATION and STATION X EDUCATION effects. This procedure, as well as the construction of appropriate degrees of freedom for this ratio, is outlined in Winer (1971). A simple ANOVA produces the same trends.

A post hoc comparison was performed on the largest difference from the correct cross ratio among STATIONS. That mean difference is 2.07, and the F value of the corresponding comparison is 32.99. The critical F for that comparison, when $\alpha = .01$, is 16.00, as calculated by Scheffé's procedure.

Figure 7.1

Departures from the True Cross Ratio for Three Groups



The cross ratios of the estimates of shape that were made by three groups are plotted. Mean cross ratios for the group of architects are plotted by solid lines. Station points are plotted on the abscissa and values of the cross ratio on the ordinate. The 'true' value of the cross ratio is 3.06.

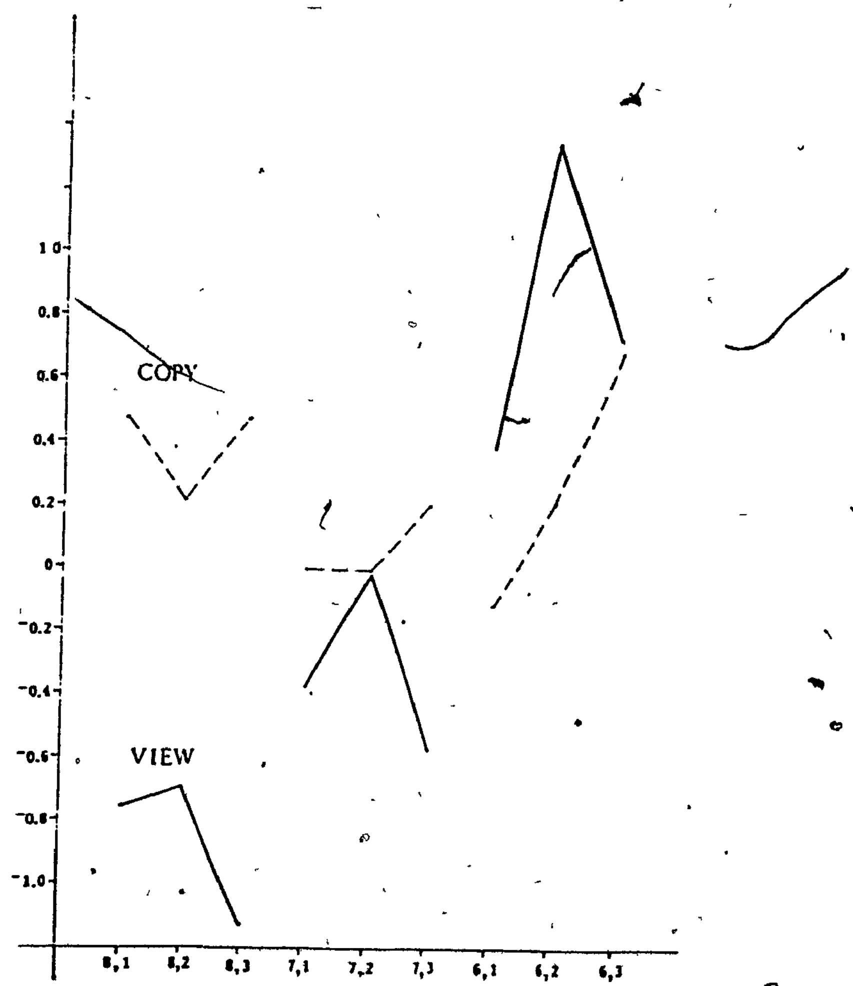
equivalence. Departures of the cross ratio from its true value were also found among levels of the EDUCATION factor. The scores of the group of architects and the scores of the second control group depart significantly from zero ($p \leq .01$), but in opposite directions. The absolute value of the departure in mean cross ratio shown by the group of architects is greater than that shown by the combined control groups. At very least, this substantiates the hypothesis that sensitivity to projective equivalence among trained subjects is no better than among naïve subjects. The differences among the groups are difficult to interpret; for instance, a hypothesis that the differences might be due to the varying number of men and women in each group would be controversial at best (cf. Caplan, MacPherson and Tobin, 1985). This particular hypothesis will be pursued later. Recall too, that no significant interaction was found between the STATION factor and the GROUP factor in the analysis of variance: one effect seems uncomplicated by the other. An illustration of these trends can be seen in Figure 7.1. The 'true' value of the cross ratio is 3.06. The shapes produced by architects show real deviations in cross ratio from that value. The magnitude of the deviation changes with the station point at which the architects stand, in the same way that that magnitude changes for naïve subjects. The main finding is that architects do not produce shapes that are closer in projective terms to correct projections than do subjects who have not had the same training. An education in systems of projection has not made the cross ratio any more evident to these observers.

Perhaps departures from the 'true' cross ratio could be attributed to errors induced by the task of 'copying' a variety of different shapes. At the same time as the subjects saw the greenhouse, they also noted its projected shape on the pane of glass, and their task can be construed as one of copying the projection on the glass on to the answer sheet. To test this idea, I had the eighteen subjects of the two control groups

Figure 7.2

Departures from the True Cross Ratio for Viewing and Copying Tasks

Obtained by 18 Subjects



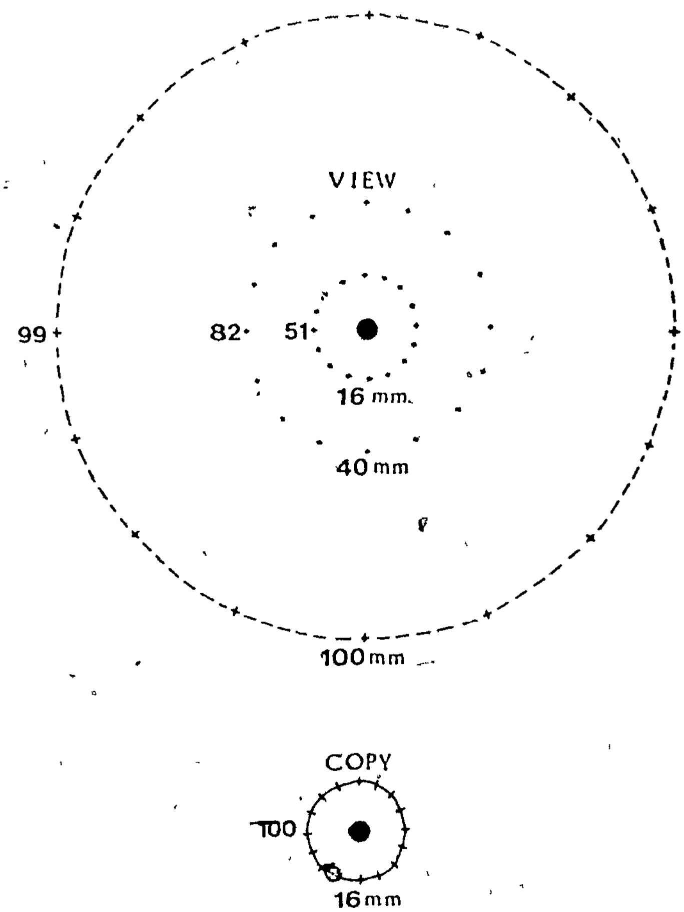
The differences for a task in which those eighteen subjects copied a point by freehand from correct perspective drawings are plotted by dashed lines. Cross ratios obtained in the experimental task vary more than those obtained in the copy task.

estimate the positions of points on fresh answer sheets by copying them from correct perspective drawings. Each station point was represented by a drawing. As before, an analysis of variance was performed on the cross ratios of the shapes the subjects estimated. No significant departure from the mean corrected cross ratio was found (the mean corrected cross ratio for all subjects and all conditions on this task was 0.24). The cross ratios for this task and the cross ratios obtained in the main part of the experiment can be found in Figure 7.2. The magnitude of difference in departures of position between the main task and this copying task can be seen in Figure 7.3. A circle of radius 16 mm., centered on the correctly projected point A, encompasses all the points estimated by all subjects in the copying task. The same circle encompasses only 51% of the points estimated by the same subjects in the other task. To encompass 99% of the scatter of those other points, a circle of radius 100 mm. is required. Most of the points estimated in the experimental task lie in the first quadrant, while the points estimated in the copying task are distributed as if by random scatter about A. The deviations in cross ratio for the main task, then, are not attributable to the mechanical requirements of the task. (It is unlikely too that the difference between the copying task and the main part of the experiment is due to a one-trial practise effect). Rock (1983) claims that subjects can perceive the extensity of objects, that is, the visual angle they subtend. He says subjects can estimate extensity in an exact way, as if they were able to copy their impressions of shape. If he were right, one would be led to attribute subjects' initial performances in this experiment to some irrelevant factor that is characteristic of the task. The copying results make his position less probable.

Nor are subjects' estimates biased by the question they were asked about shape. One might think that the right way to ask subjects about perceived shape is not to ask

Figure 7.3

The Magnitude of Departures of Estimated Positions from Correctly Projected Position for Viewing and Copying Tasks



The magnitude of departures in position from a correctly projected point are plotted. These are the values for all estimates made by eighteen subjects in the control groups. The two conditions that are depicted are the main experimental task (VIEW) and the copy task (COPY) of experiment four. Though a radius of 16 mm. includes the scatter of 100% of points in the COPY condition, it includes only 51% of the scatter of points in the VIEW condition. A circle of radius 100 mm. is required to encompass 99% of those points. While the scatterplot for the COPY condition is nearly circular, most of the points in the scatterplot of the VIEW condition fall within the upper right quadrant of the larger circle.

them about apparent shape, but about real shape. One could say that subjects do not make their best estimates of shape when their attention is drawn away from the real properties of an object. Having viewed the greenhouse roof from several windows, thirteen new subjects were asked to sketch the shape of the object abcd. They did this by positioning four dots on a paper to copy the 'real' shape of that quadrilateral. 'Real shape' was introduced to subjects as the shape an object would present if seen 'face on'. The cross ratios of their sketches were measured, and they were highly variable. The best estimate a subject made to the 'true' cross ratio of 3.06 was 3.88, and the worst estimate was 10.42. The standard deviation of the cross ratios was 4.18. Although subjects reproduced salient features of the object, such as the parallelism of lines bc and ad and the axis of symmetry, the cross ratios that I computed on their estimates were wildly variable. Clearly the question that was posed about apparent shape is not misleading, since, by contrast, questions about objective shape lead to even greater departures in cross ratio. Again, the departures that were found initially are not due to an artifact. Simply, even when trained in systems of projection, subjects do not estimate projective properties well.

4.1 Ordered Distances

On the basis of the data of the last experiment, some hypotheses were made about the following variables: one, the mean direction of departure of estimated points, two, the mean area of the estimated shapes, three, an ordering on perpendicular distances, and four, the correctly projected areas of shapes (the reader may refer back to Table 6.4). The objective quantities three and four, that is, the ordering of areas of correctly projected shapes ABCD and the ordering of perpendicular distances from station points to the intersection of planes, are correlated because they are related

Table 7.2

Rank Correlations between Measures on the Estimated and Correctly Projected Figures ABCD
Experiment Four

Measure 1	Measure 2	Spearman's Rank Correlation
Projected	Estimated	1.000
Distance	Estimated	0.867
Distance	Projected	0.867
Distance	Angle	-0.717

n = 9, C_{rho}(9)(one-tailed)(α = .01) = .783

C_{rho}(9)(one-tailed)(α = .05) = .600

Projected: The areas of the quadrilateral ABCD for the master drawings.

Estimated: The areas of the quadrilateral A'BCD as found by construction on the subjects' estimations of the position of point A.

Distance: The perpendicular distance from the viewpoint to the line of intersection of the object plane and the plane of the viewpoints (plane abcd and plane ABCD).

Angle: The deviation in direction of the mean vector from the horizontal. The mean vector is the deviation of subjects' estimates of the position of point A from the position of point A in the master drawing.

geometrically. Further, the size-distance invariance hypothesis (eg., Kilpatrick and Ittelson, 1953; Rock, 1983) supposes that estimated areas will mirror the exact function of those metric variables of area and distance. Since size-distance invariance seems a reasonable hypothesis, and is one that has had some experimental basis, it is not surprising to find that the ordering of estimated areas does again correlate with those ordered variables in the present experiment (see Table 7.2). Rank correlations were used to emphasize that subjects are not taken to have access to Euclidean measures of the relevant quantities. In fact, contrary to the hypothesis of size-distance invariance, projected areas and mean estimated areas are not related by the same metric function in the Experiments 3 and 4. Though the slopes of both are nearly one, the intercepts of the two regression lines are separated by about 1200 mm^2. Though the correlations may explain how the hypothesis of size-distance invariance has received support, yet this result runs counter to the claim that an exact and constant correspondence exists between visual angle and perceived shape. The data are consistent with the interpretation that subjects achieve nothing more than an ordinal scaling on extensity. This finding reinforces the initial impression that non-projective factors influence judgments of projective equivalence.

In general, the significant effects shown in the analysis support the claims that the cross ratio of the estimate of a projected shape is different from the cross ratio of a correct perspective drawing, and that the cross ratios of estimates made by experienced subjects are not closer to the true cross ratio than those of other groups. These results are free from several artifacts that could be imagined. These results and the results of previous experiments all point in the same direction: that normal subjects' attempts to judge projective equivalence are significantly jostled by nonprojective factors. There is some indication that their performance may reflect an ordinal scaling on areas or magnitudes of visual angles. The results obtain even when subjects are familiar with techniques of perspective draughting. The appropriateness of these architects as subjects may be questioned, if one requires extreme finesse as a criterion for expert performance. (Some mathematicians claim to be able to tell the dimensionality of fractal shapes upon inspection, for instance.) Yet what has been done casts doubt that moderately skilled subjects are more sensitive than naïve ones to projective equivalence. One can reflect, too, that even Canaletto used a camera obscura to produce his perspectives.

CHAPTER 8

Experiment 5: Effect of Viewing Angle

> **To say that a person's seeing a tree is in principle the same sort of affair as a negative in a camera being exposed...will not do at all. But a great deal has been found out about seeing by working on analogies like this. It is, indeed, the good repute of these discoveries which bribes us to try to subjugate our untechnical generalities about seeing ... to the codes that govern so well our technical generalities about cameras ... Nor is there anything to warn us beforehand whether or where the attempted subjugation will fail.**
>
> **Ryle, Perception, p. 110**

The results already obtained raise several questions, of which two will be the focus of this chapter. In the last two experiments subjects placed a dot on a sheet to indicate their response. The spatial orientation of that sheet was not fixed, but rather subjects held the sheet as was comfortable. Now, significant effects of the orientation of the stimulus to the subject have been found. It is natural to ask if there are significant effects of the orientation of the answer sheet. In the next experiment that factor was systematically varied. In addition, the effect of sex differences on responses is explored.

2. Stimuli

The stimulus object for this experiment is part of St. James' Church, that faces the Arch Street windows of Humphrey Hall, which houses the Department of Psychology at Queen's University in Kingston, Ontario. Figure 8.1 represents the stimulus object. The important points are detailed; the back wall and chimney are salient features of the church. Figure 8.2 shows the quadrilateral that can be traced from that perspective. The near edge of the back wall of St. James' Church is some 33 meters

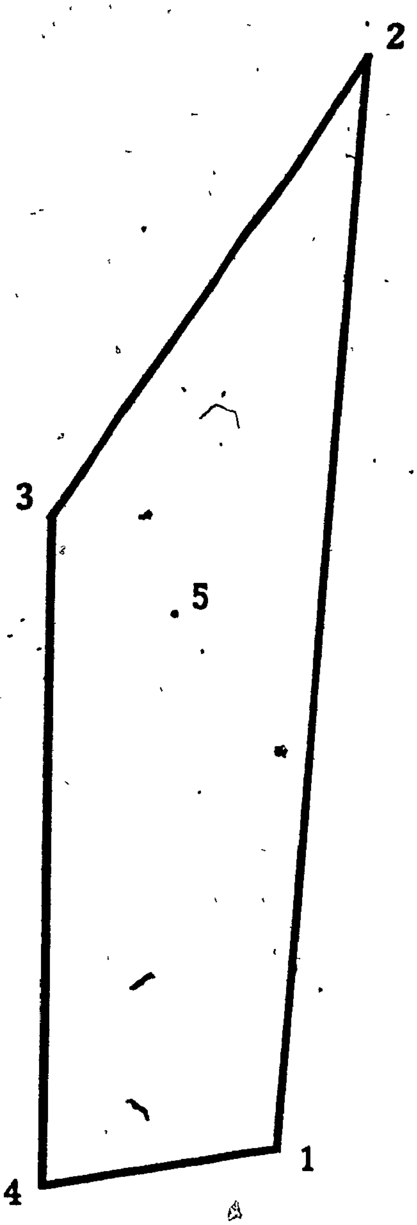
Figure 8.1

A Photograph of the Object

This is the view from one window of the conference room of Humphrey Hall. The photograph was taken from the first window.

Figure 8.2

A Schematic Diagram of the Object



This diagram is the master projection for the first window. The correspondence of these labelled points to features of the object is pictured in Figure 8.2.

distant from the plane of the windows. The nearest viewing station is 22 meters distant from the line of intersection of those planes.

Three viewing stations were arranged at three windows in the conference room of Humphrey Hall. If those windows are counted, beginning with that nearest the church, then the station points will be called stations one, two, and three. The horizontal distance between adjacent stations is some 3.0 metres. The viewing stations correspond to one factor in the experimental design: the factor STATION.

Correct perspective drawings were obtained at a constant height at each viewing station, by means of a Leonardo's window, as before. Incomplete copies were made of the correct perspective drawings. Figures that are 'L' shaped are formed when the lines that include point 2 are removed from the drawings. These parts of the master drawings were photocopied, then collated into booklets containing one of each picture. The copies were randomly ordered within each booklet. A transparency of the same part of the master drawing was placed on each window. The transparencies were appropriate to the windows. The cross ratio that was estimated from the master drawings is 1.20. This is the cross ratio of points 1, 2, 3, and 4 from point 5 (see Figure 8.2). A salient feature of the experiment is that the positioning of the answer sheet was varied. In one condition (call it NATURAL) the answer sheet was placed on a clipboard which subjects held as they wished - as in the two previous experiments. In the other condition (call it FIXED) the clipboard with answer sheet attached was placed over the Leonardo's window. The reason for this was as follows. The answer sheets were exact reproductions of the figures of the Leonardo windows. If the orientation of answer sheets is important, the position that ought to yield the most accurate results is when the answer sheet is superimposed on the Leonardo's window from which it was copied. Hence the FIXED condition.

3. Method

3.1 Subjects

Twenty subjects were tested between January 6 and 21, 1987. The subjects were drawn from the undergraduate subject pool of the Department of Psychology at Queen's University. There were 11 men and 9 women in the sample, and they ranged in age from 18 to 27. Five were left eye dominant for sighting tasks, and 15 were right eye dominant for sighting tasks.

3.2 Instruction and Procedure

The instructions given to subjects were as follows.

This is an experiment in visual perception. You will be asked to judge the apparent position of a part of St. James' Church, which you see outside. In particular, the experiment concerns a certain part of the church, namely, the back wall of the building, and the chimney. If I were to make a crude drawing of that part of the church, it might look like this... .

The transparency before you on the window is a picture of part of the church. Namely, the picture is of the long vertical edge on the left side of the back wall, and of the small horizontal ledge that runs above the ground. (Gesture) Do you see the correspondence between the picture and the parts of the church? Later you'll be asked about the top right hand corner of the chimney. (Gesture) There are other parts of the building that might be drawn in schematic like this ...

You will not be asked about those. Stand in front of this picture; then focus on the church. You may notice that the lines of the picture on the window appear to be doubled. Do they appear to be doubled?

One image appears to be on the left, and the other on the right. I would like you to align the ("right" or "left", whatever the contralateral side to the dominant eye) image with the corresponding part of the church. Here's what is meant by "align". This point (a point on the picture is indicated, and marked on a more complete diagram) and that point should also correspond to that corner. Both these points should appear to coincide with points on the church at the same time.

What is important is that the length of this segment (the vertical segment is indicated on the picture) should have the same apparent length as the long vertical edge of the back wall of the church. Their lengths should appear to be the same. So should the length of this segment and the horizontal ledge. You may need to move about to discover where the correspondence can be achieved. Tell me when you have aligned the ("right" or "left", as before) image with that part of the church.

At this point, the instructions continued in one of two ways depending on whether the condition was NATURAL or FIXED. The instructions for the NATURAL condition, in which subjects held the response board as they liked, continue as follows:

(When the subject responds, he or she is given a pen and clipboard with the appropriate diagram). On this picture, I'd like you to mark in the corner of the chimney that I mentioned (its description is repeated). Indicate its position by marking a small dot or cross on the paper. You may need to look back and forth several times from the window to the paper to do this. Try to be as accurate as you can.

The instructions for the FIXED condition, in which the position of the response sheet was fixed by the experimenter, continue as follows:

(When the subject responds, he or she is given a pen). I would like you to keep your head as still as possible. In a moment, I will ask you to make a mark on a paper that obscures your view of the back wall of the church. I'll ask you to mark in the corner of the chimney (its description is repeated). You should try to keep your head in just the position it is now. Practise raising your pen as if to make a mark on the transparency. Take care to preserve the correspondence of the image with the parts of the church. Now I'd like you to indicate the corner of the chimney that I mentioned. Indicate its position by marking a small dot or a cross on the paper.

Ready? (The response sheet is placed on the window so that it is aligned exactly with the transparency that is used as an aid to fixation). Try to be as accurate as you can.

The task was performed six times by each subject, once under each of the NATURAL and FIXED conditions at each of three viewpoints. The three judgments in the NATURAL condition were made before those of the FIXED condition, or vice versa, and this order was counterbalanced. The present experiment was conducted by Veronica Horn. The two conditions form a within-groups factor called CONDITIONS. The three viewing stations correspond to a within-groups factor called STATIONS. The experimental design, then, is a two-way 3 x 2 within-subjects design.

4. Results and Discussion

The dependent variable for the first analysis is the cross ratio of the shape that a subject produces, minus the true value of the cross ratio. An analysis of variance showed a significant effect of STATION on the dependent measure (see Table 8.1). Hence the different windows yield significantly different results. An effect of CONDITION was found to be significant at the level $p \leq .05$, so that cross ratios in the FIXED and NATURAL conditions differ significantly. No significant interaction of the factors STATION and CONDITION was found.

It is important to know if there are significant departures of the cross ratio from its 'true' value in the factor STATION. There is, as shown by a significant post-hoc comparison (see Table 8.1). In fact, for that particular STATION, no individual esimate falls below the 'true' cross ratio of 1.20. The strong effect of STATION indicates that judgments in this task do not reflect a reliable and uniform sensitivity to the cross ratio. One effect of CONDITION is interesting. In the FIXED condition the range of cross ratio differences (1.53) is twice as large as in the familiar NATURAL condition (0.78). The variability in response is significantly different between the two conditions at both the second window, [$t(18) = 3.091, p \leq .01$. The test for the difference of correlated variances is taken from McNemar, 1962, p. 246] and at the third window ($t(18) = 3.575$, $p \leq .01$). A comparison of Figures 8.3 and 8.4 shows this effect.

These results can be explored further by plotting the mean vector for departure from the 'true' point for each CONDITION at each STATION. Figure 8.5 gives the result. Mahalanobis' D^2 was calculated to see if these vectors are significantly greater than zero. Each of the six is, as Table 8.2 shows. The overall effect is that while a adopting the FIXED position for the answer sheet does affect responses, it does not make them more accurate. In fact, it made them more variable. These results suggest

Table 8.1

Analysis of Variance: Corrected Cross Ratios

Source	SS	df	MS	F	P
Between subjects	2.893	19			
Within subjects	5.109	100			
Condition*	0.436	1	0.436	6.75	$\leq .05$
Condition x subjects	1.227	19	0.064		
Station	1.587	2	0.793	37.06	$\leq .001$
Station x subjects	0.813	38	0.021		
Condition x station	0.119	2	0.059	2.44	$\geq .05$
Condition x station x subjects	0.927	38	0.024		
Total	8.002	119			

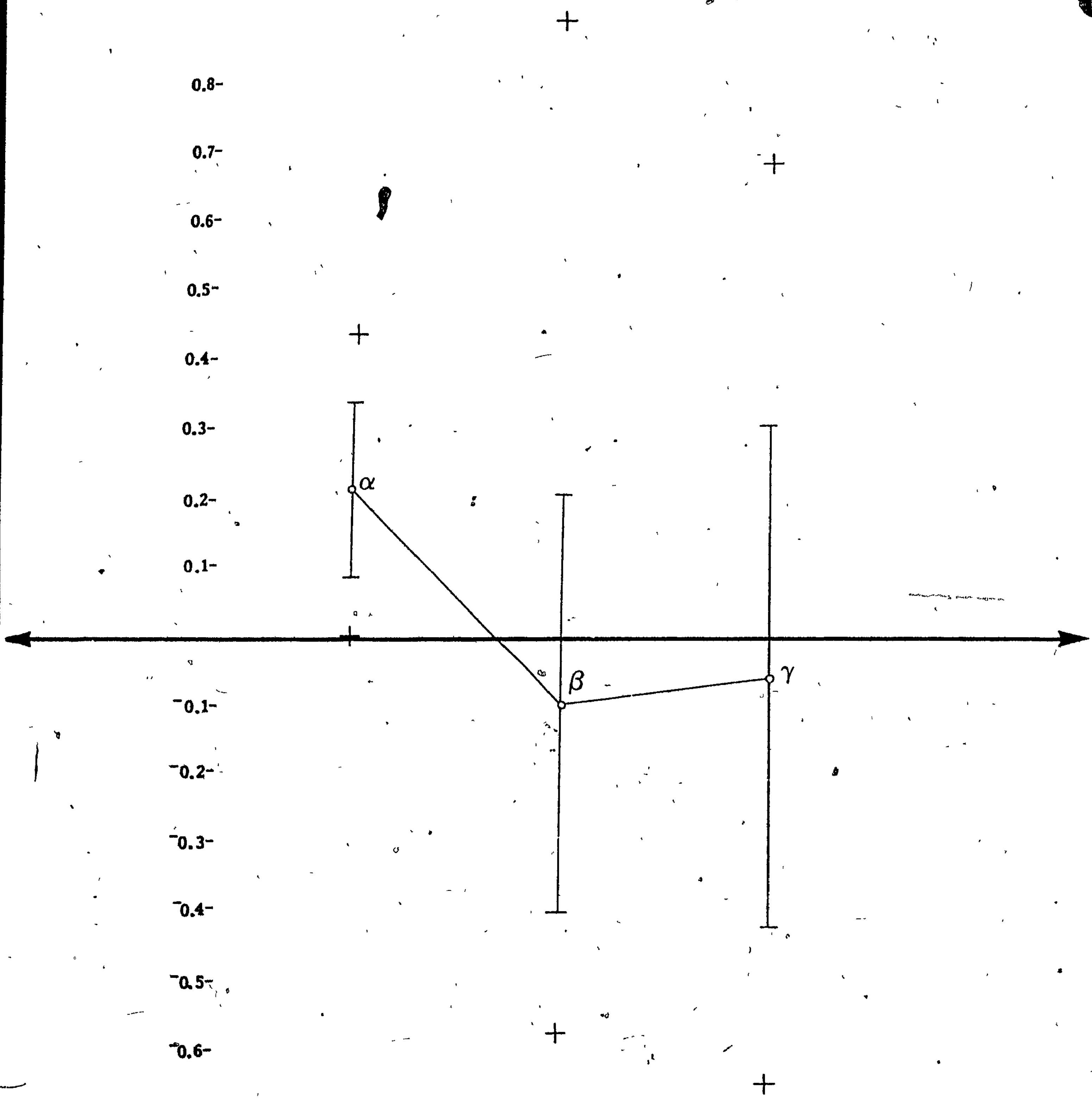
Addendum to Table 8.1

A post hoc comparison was performed on the largest difference from the correct cross ratio among STATIONS. That mean difference is 0.23, and the F value of the corresponding comparison is 46.08. The critical F for that comparison, when $\alpha = .01$, is 10.42, as calculated by Scheffé's procedure.

* Wilcoxon's matched-pairs signed-rank test (Siegel, 1956, p. 76) bears out the effect of CONDITION. Individual parametric ANOVAs for separate CONDITIONS also reveal an effect of STATION. The ANOVA for the FIXED conditions alone was significant ($F(3,57) = 12.213$), as was the ANOVA for the NATURAL conditions alone ($F(3,57) = 7.253$). These may be tested against a critical $F(1,19) = 4.38$ at the level $\alpha = .05$. Post-hoc comparisons on the largest difference from the correct cross ratio were significant in both cases at the level $\alpha = .01$ (Scheffé's method). These additional tests were performed to ensure that the homoscedasticity found in the data does not affect the interpretation of the results.

Figure 8.3

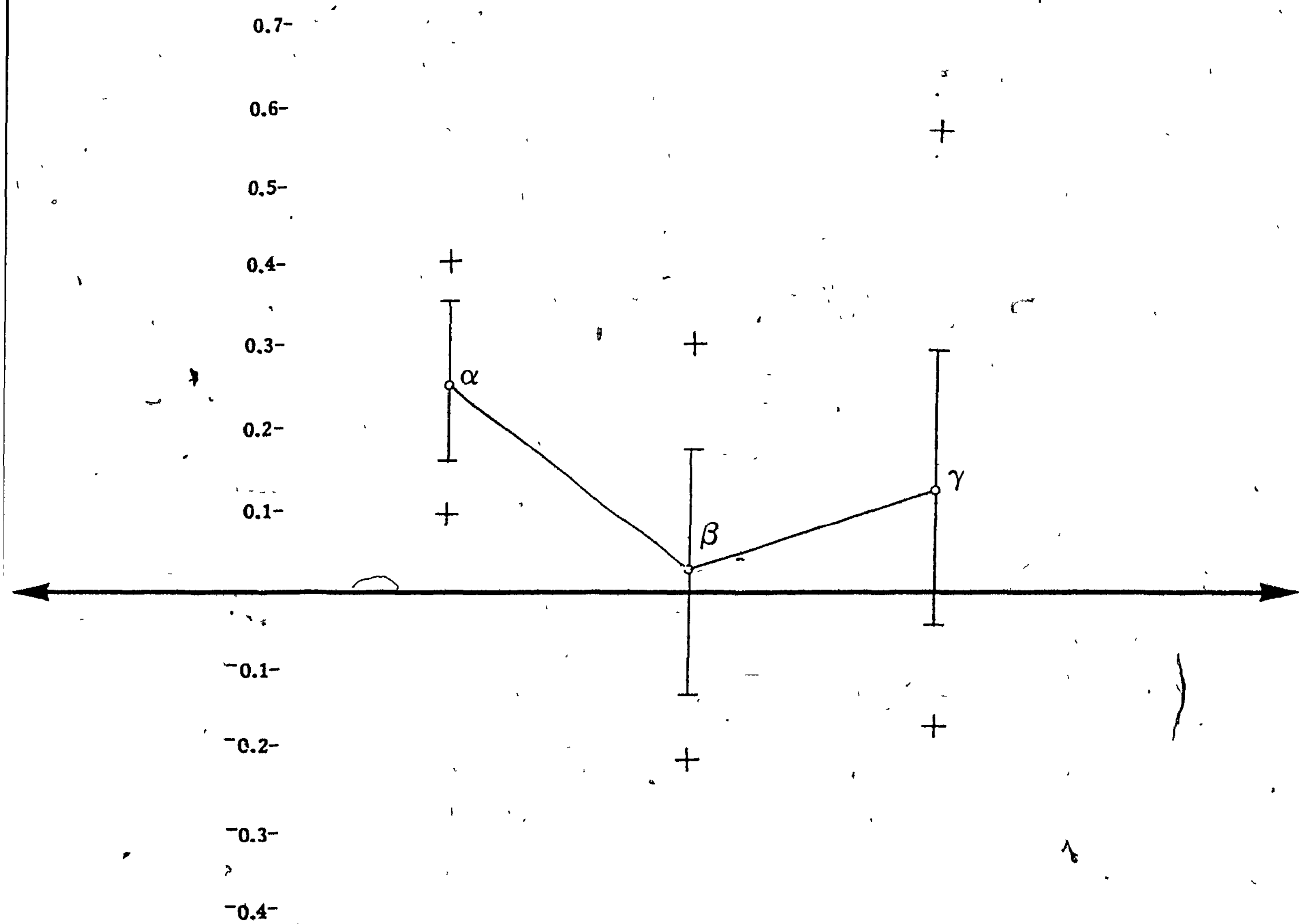
Values of the cross ratio for factor FIXED



Departures from the 'true' cross ratio for the estimates of twenty subjects are shown. The dark line marks the value of the 'true' cross ratio; open circles mark the average values found at three STATION points. Minimum and maximum observations at each STATION are marked by crosses. Response sheets were held FIXED in orientation. STATIONS are marked α, β and γ.

Figure 8.4

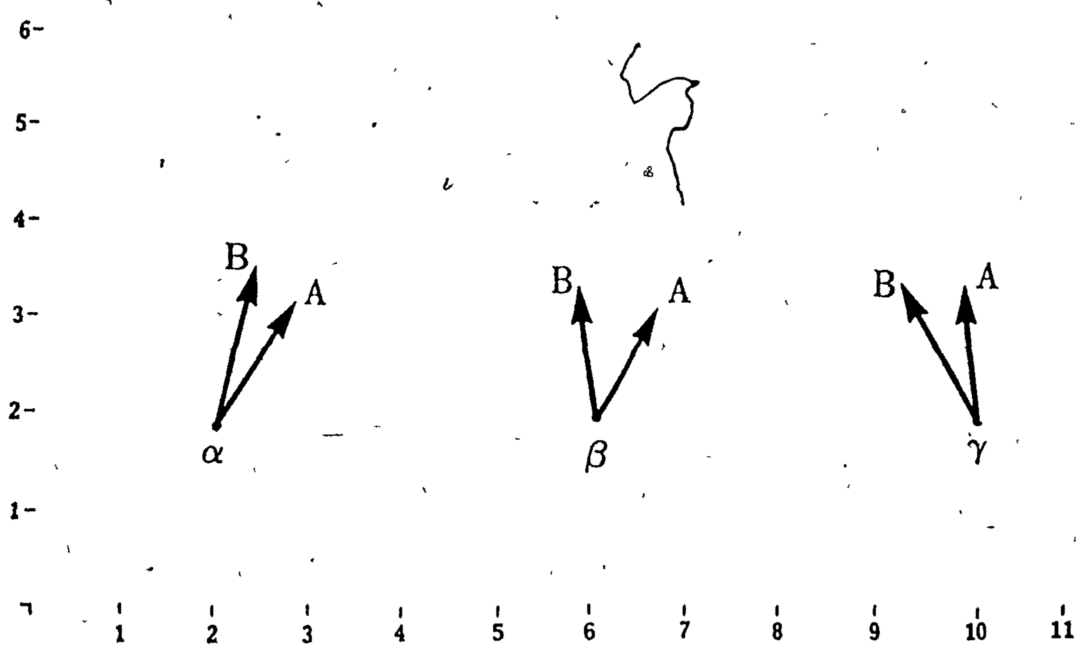
Values of the cross ratio for factor NATURAL



Departures from the 'true' cross ratio (1.2) for the estimates of twenty subjects are shown. Response sheets were held at an orientation NATURAL for each observer. The dark line marks the value of the 'true' cross ratio; open circles mark the average values found at three STATION points. Minimum and maximum observations are marked by crosses. Note the difference in variability between these observations and those shown in Figure 8.1, for which the response sheets were held at fixed orientations.

Figure 8.5

Departures of Estimated Positions from the Correctly Projected Position for FIXED and NATURAL conditions



The average X and Y departures of subject's estimates from a correctly projected point are shown for three STATION points, α, β and γ. The departures that are obtained when the response sheet is held fixed (FIXED condition) are marked as B, and the departures when the orientation of the response sheet is not constrained (NATURAL condition) are marked as A. The axes of the plot are marked in centimeters.

Table 8.2

Departures from Correct Projections				
Station	Condition	Mahalanobis' D^2	F	P
2	Fixed	1.18	11.26	$\leq .01$
3	Fixed	0.71	6.79	$\leq .01$
4	Fixed	0.91	8.66	$\leq .01$
2	Natural	0.72	6.85	$\leq .01$
3	Natural	0.64	6.09	$\leq .01$
4	Natural	0.81	7.76	$\leq .01$

Mahalanobis' D^2 values were computed on X and Y departures from the correctly projected points in each condition in which estimates were made.

that allowing subjects to adopt the NATURAL position for the answer sheet did not bias their responses in a way that was relevant to the experimental hypothesis.

There are other indications that changes in the orientation of the response sheet are not responsible for the magnitude of departures from the true cross ratio that have been observed. A simple demonstration was performed to assess the magnitude of change in cross ratio that is associated with changes in the orientation of the response sheet. Consider again that the experimental task might be like a task of copying; that simple hypothesis was considered in the previous chapter. A more complex hypothesis is considered here: what changes in response may occur, as the slant of the response sheet is increased with respect to the slant of the drawing to be copied? In the present demonstration, two subjects estimated the positions of points by copying them from correct perspective drawings. As before, each station point was represented by a drawing. The bottom of the response sheet rested on a table, and the slope of the response sheet was varied in ten steps between 4° and 50° from the horizontal. The drawing to be copied and the response sheet were side by side on a table in front of the subject, whose head was fixed in a chinrest. Both subjects copied each picture at each orientation four times. The conditions of this copying task were randomized. The departure in cross ratio of the subject's estimate from the cross ratio of the correctly projected drawing was computed for each drawing. Since ten slopes were used four times each, regression statistics can be computed on the magnitude of difference of the cross ratio versus the slope (in degrees) of the response sheet. These statistics present an intriguing pattern. Significant correlations were found between departures in cross ratio and slope for two of the three drawings (Table 8.3). Note that the condition for which the largest departure was observed in the main experiment did not produce a significant correlation. The slopes of the regression lines associated with

Table 8.3

Regression Statistics for Departure of the Cross Ratio Versus Slope of Response Sheet

Subject A.S.

Window	Correlation Coefficient	T value N = 40	Slope of Regression Line (1)	Intercept (2)	Mean (2)	Standard Deviation
1	-.206	-1.30	-1683	-.301	-.316	.042
2	-.472	-3.30*	- 885	-.313	-.343	.035
3	-.422	-2.87*	- 732	-.096	-.131	.047

*P ≤ .01

(1) Degrees per unit of cross ratio

(2) Cross ratio

Subject C.M.

Window	Correlation Coefficient	T value N = 40	Slope of Regression Line (1)	Intercept (2)	Mean (2)	Standard Deviation
1	-.222	-1.40	-1468	-.054	0.036	.044
2	-.552	-4.08*	- 476	-.021	-.075	.056
3	-.438	-3.00*	- 663	-.050	-.089	.050

*p ≤ .01

(1) Degrees per unit of cross ratio.

(2) Cross ratio

these correlations are most important. They indicate a bizarre result: the difference in slope that would be necessary to account for the results that have been observed in this and in previous experiments may be greater than 360^{0}.

The consistency with which the two subjects estimated positions is striking, too. This consistency is indicated in the first place by the standard deviation of the cross ratios of estimates made by the two subjects. Consistent with the results of the copying task as reported in the last chapter, a circle of radius 16 mm. will encompass all the points estimated by subject C.M., and another circle of the same radius will encompass all the points estimated by subject A.S.. It should be noted, that the values of departures in cross ratio that are seen here are unlike those of the main results. The departures for subject A.S. do not represent values like any found in the main experiment. It is clear, then, that this task is different from that of the main experiment both in effect and in kind. Here subjects were asked to copy the real shape of a drawing, and then subjects were asked to reproduce the apparent or projected shape of a distant object. Though the slant of the response board may have some effect on the cross ratio of an estimated position in a figure, that effect is an order of magnitude too small to account for the main results.

4.1 Sex Differences

In Experiment 4 comparisons were made between a group of 9 architecture students and each of two groups of 9 ordinary students each. The numbers of men and women varied from one group to another, so it is of some interest to learn if there are important sex differences that might 'explain' the poor performance of architecture students. Women are often supposed to do less well, or to perform more variably than men, on a variety of 'spatial' tasks. The existence and interpretation of such differences is still a controversial matter. A number of new subjects were culled from the subject pool of undergraduates at Queen's University to test this conjecture. Nine men and nine women completed the experiment, as it is described at the beginning of this chapter. These subjects were given the instructions for the NATURAL level of the CONDITION factor only. An analysis of variance on the cross ratios of the estimated figures did not reveal any significant difference between men and women on this task (see factor SEX, Table 8.4), nor was there any significant interaction of the group means with station points, that is, with the effects of perspective (interaction STATION x SEX). Mean deviations of the cross ratio from its true value were little different between the two groups (Men: $\overline{X}$ = 0.619, S.D. = 0.292, Women: $\overline{X}$ = 0.568, S.D. = 0.293). Again, there is a significant effect of change of perspective viewpoint on the cross ratio of the figures estimated (factor STATION). These means were significantly different from zero at each station point; in fact, the effect is somewhat larger than that found in the main study that is reported earlier in this chapter. Of course the study does not prove the null hypothesis. Nevertheless it allays uneasiness that the results reported earlier may be seriously complicated by sex differences.

The significant effects found in the main experiment reinforce the claim that subjects are not uniformly and reliably sensitive to projective equivalence, in that the cross ratio of the estimate of a projected shape is different from the cross ratio of a correct perspective drawing. These effects were obtained with a new stimulus object, that provided a new value for the cross ratio. These effects were also obtained independent of changes in the orientation of the response sheet. Other results from two subjects also indicate that changes in the slope of the response sheet do not occasion large changes in the estimates; that is, changes of a magnitude that would explain the findings of the main task. These results show remarkable internal

Table 8.4

Analysis of Variance: Cross Ratios

Source	SS	df	MS	F	P
Between Subjects	3.445	17			
Sex	0.035	1	0.035	0.17	P ≥ .05
Subjects Within Sex	3.419	16	0.213		
Within Subjects	1.047	36			
Station *	0.703	2	0.351	32.90	P ≤ .001
Station X Sex	0.002	2	0.001	0.10	P ≥ .05
Station X Subjects	0.342	32	0.010		
Total	4.502	53			

* STATION is treated as a fixed-effects factor here.

Post-Hoc Comparisons

Station	Mean Difference	F for Comparison	P
1	0.594	252.47	≤ .01
2	0.633	286.50	≤ .01
3	0.454	147.14	≤ .01

consistency. A significant sex difference was sought on the main task, but was not found.

Many kinds of objections could be raised to the proposition that observers are not reliably sensitive to projective equivalence; it is useful to remember that some matter of induction is involved in the claim. One might like to know if or when there are ever situations in which observers are reliably sensitive to projective equivalence. There is an obvious example: observers can tell that two squares of the same size are equivalent, in all their geometric properties, when those squares are of the same orientation, and are adjacent in a frontoparallel plane. That case is uninteresting for the theory of shape constancy. There may be other situations: for example, observers can copy a figure accurately, as reported in the last chapter. However, Gibson, Rock, and other proponents of the invariance hypothesis suggest that observers are sensitive to projective equivalence in a wide variety of circumstances - enough to explain the phenomena of shape and size constancy, which are robust over a wide range of viewpoint. Certainly they would predict that subjects are sensitive to projective equivalence in the sorts of situations that have been arranged in these experiments. The last several chapters have assessed the sensitivity observers may have to projective equivalence. Such an effective measurement of planar projective equivalence has not been made before in the study of shape constancy (again, see Note 1). Results that run counter to the predictions of the invariance hypothesis have been found throughout. Perhaps one might not be convinced of the generality of these results, since experimental proof must proceed by induction. Yet here there seems to be a series of counterexamples to the invariance hypothesis. It is conceivable that the invariance hypothesis might be substantiated under other conditions, but the invariance hypothesis itself provides no clue to what such conditions may be.

CHAPTER 9

General Discussion

> **It is natural for mathematicians to regard the visual angle and the apparent magnitude as the sole or principal means of our apprehending the tangible magnitude of objects. But it is plain from what hath been premised, that our apprehension is much more influenced by other things, which have no similitude or necessary connection therewith.**
>
> **Berkeley,**
> **The Theory of Vision Vindicated and Explained,**
> **p. 272.**

The strategy of several theories which incorporate the invariance hypothesis is to suggest that things cannot appear to have other shapes than they do, once viewing conditions are taken into account. This is tantamount to the statement that observers must always be sensitive to projective equivalence to the extent that shape constancy holds. One may be reminded of Rock's (1983) statement that, in vision, we must have a constant, exact, and quantitative knowledge of a projective regularity -- the law of visual angle -- to achieve shape constancy. At the least, these theories claim that observers are reliably sensitive to projective equivalence whenever there is constancy of size and shape. Such broad quantitative claims are rare in psychology; the generality of this claim reflects the hope that projective invariance might provide a privileged and unerring datum for shape constancy. In the context of this programmatic claim, any reliable deviation from sensitivity to projective equivalence that occurs in conditions where shape constancy holds is a meaningful finding. The invariance hypothesis is not an empirically-derived claim about projective equivalence; rather, it has been speculation. This speculation follows as if by logic from specific assumptions about the nature of vision. So far as I am aware, the experiments that are reported here are the first quantitative and effective tests of the hypothesis. What these experiments do not show is that projective equivalences are constantly and reliably perceived; instead, a good number of conditions and tasks have been presented in which observers misjudge projective equivalence grossly.

The experimental verification of any claim involves some matter of induction. What has been demonstrated is that there are exceptions to the invariance hypothesis. All the experiments reveal some such exceptions. Yet consider someone who supports the invariance hypothesis. His interest is not in exceptions, even if the exceptions are important. His interest is in the rule he postulates. He would like to know if there are any conditions under which it can be maintained that projective equivalence is always perceived accurately. These conditions are not in evidence, nor are there strong indications where a researcher of this mind might begin, except that he might use boxes as stimuli. The intent and plan of the experiments has been quite different: to discover exceptions to the invariance hypothesis where they exist. A number of examples have been collected in conditions that have been very easy to set up.

The cross ratio is the dependent measure for analysis in all the experiments (Experiment 3 uses a simple quotient of cross ratios). The cross ratio stands in the same relation to projective geometry as length stands to Euclidean geometry; length and the cross ratio are the fundamental congruences of their respective geometries. We often say that we can judge length by eye; it is no more of a mistake to claim that one might judge cross ratios by eye. Properly, though, one should speak of judgments of Euclidean equivalence or projective equivalence. The proper usage avoids the misleading and improbable claim that observers compute distance or the cross ratio, just as they are found in texts on analytic geometry (but see Rock, 1983).

The close connection between projective equivalence and the cross ratio can be seen by means of the above analogy. Unfamiliarity with both concepts may make the connection seem less close than it really is. The cross ratio is the fundamental congruence of projective geometry: all the quantities and theorems of a projective system can be obtained from these congruences and their syzygies. One might claim that observers could have access to projective congruence, at least over a limited range, without being able to compare cross ratios. There is a sense in which this claim is false outright. One might as well claim that observers have access to Euclidean congruence over a limited range without being able to compare lengths. In fact there are few strong claims about observers as geometers (though see Perkins, 1983); most claims deal squarely with judgments of projective equivalence and the cross ratio.

There is perhaps one sense in which the claim of sensitivity to projective equivalence might seem plausible. Projective competence, it might be thought, could be triggered by incidental conditions of stimulation. Observers might do better or worse at judging the projective equivalence of fuzzy shapes, or curved shapes, or some other thing. I cannot rule out all such possibilities. The only strategy open to someone who contests the general claim is an inductive one. All that can be claimed is that on the showing so far the projective-invariance hypothesis is faring badly. I should like to stress that the cross ratio should be considered a tool, and not an end of investigation, as disconfirmation of the invariance hypothesis may be an end of investigation. Furthermore, the cross ratio does not provide a psychological model, neither for most proponents of the invariance hypothesis, nor for others. The cross ratio is the measure of an ability, not the means of achieving the result, where the result is shape constancy. The uses to which the cross ratio has been put may give some intimation of the applicability of this tool (one may remember, too, that though a test may be fair, its results may be surprising).

Several experiments have assessed the constancy of judgments of projective equivalence. Previous experiments have not made a direct assessment; rather they have dealt with geometrical properties that imply projective properties (for example, as in Foley, 1964). Quantitative experimentation has not been undertaken, with the exception of Cutting's (1986) claims about the psychological efficacy of the cross ratio as a ratio scale variable. It is hard to say what counts as a large or a small difference of the cross ratio in psychological terms, except by appealing to mathematics. 'Large' and 'small' may have no psychological import when applied to values of the cross ratio, as things stand. The experiments that have been conducted here can be thought of as items of a large argument by enumeration. The various empirical results address the issue directly; they do not depend on further geometrical argument or implication for their force. The universal and absolute character of the invariance hypothesis should be kept in mind as a standard when these results are considered. What is demanded under the invariance hypothesis is that there be obvious evidence of fairly precise sensitivity to projective equivalence. It should also be kept in mind that the analyses reported do not admit just any slight variation in values as a significant result. Instead, extremely conservative statistics have been employed consistently. The radically conservative Greenhouse-Geisser degrees of freedom have been used for each analysis of variance test where within-subjects effects are involved. Scheffé tests have been used as post-hoc tests throughout; they are the most conservative of standard criteria for the evaluation of post-hoc comparisons. Nevertheless, marked exceptions from constancy in judgment of projective equivalence have been found in every experiment.

The experiments establish one basic result: there exist departures from constant and reliable judgment of projective equivalence, which occur in a variety of situations. The result is established most convincingly in the experiments that are most relevant to the problem of shape constancy, that is, the experiments reported in chapters 6, 7 and 8. Both the estimated cross ratio and a quotient of cross ratios have been found to vary with changes in perspective viewpoint. The force of the invariance hypothesis, on the other hand, is just that estimates of these quantities should not vary with changes in viewpoint. Nor is it the case that subjects could be using a simple function, say a monotonic function, of cross ratios to assess projective equivalence; in each of these experiments all subjects estimated a single projective property, measured by a single value of the cross ratio, and their judgments vary with perspective viewpoints. Not only do estimated projective properties vary with perspective viewpoint; there are also significant departures of estimates from projective equivalence in the latter four experiments. These have been assessed by post-hoc tests on the dependent measure, the cross ratio. This, then, is the fundamental result. Many of the other results elaborate on this fundamental result, or they anticipate objections that could affect the interpretation of the basic finding.

The measure of projective equivalence, the cross ratio, has been found to depart from its expected value in two ways. It may be different from the expected value on average. It may also be wildly variable, as was found at two station points of experiment three, in the condition in which the orientation of the response board was fixed in experiment five, and when subjects were asked to sketch the shape of an object as it would look from a canonical viewpoint. This variability is large, especially when compared to the small variability in measurement that is incurred by using Leonardo's window. These differences occur in just those situations in which shape constancy would be expected to occur. Such departures from projective equivalence were found in the presence of shapes of different projective properties. Namely, these were a shape that measures 3.06 in cross ratio, another that measures 1.20 in cross ratio, and shapes that measure 1.33 in a ratio of cross ratios. What if some other shape than these had been employed, that is, any other shape but a rectangle? It does not seem outrageous that an irregular quadrilateral could be used to make a test of the invariance hypothesis in a natural setting, nor that such a quadrilateral should be varied in two directions of tilt. Whatever would happen with simple shapes like those used in the first two experiments? Has a claim about the perception of any physical property ever been founded on a range of stimuli as restricted as that which has been used before now to assess the perception of projective equivalence?

Projective properties are relatively unfamiliar to most people. To make things graspable, departures from projective equivalence have also been expressed in more familiar terms, that is, in Euclidean terms. The scatter of points produced by subjects is a perspicuous illustration of the data. Bivariate tests have shown that the average of this scatter is different from the position of the correctly projected point, for most of the station points that were sampled. The magnitudes of the test statistics were large and significant. In the third experiment, the scatter of estimates for one trapezium was distinct from the correctly projected point, and this occurred at each viewpoint. If descriptions of shape are changed from projective terms to Euclidean terms, other statements can be made about the data. The variability of observations can be divided into that due to projective invariance, and that due to other factors, by tautology. Measures of projective transformation (or projective variance) have been used to account for some variability of observation in the third and fourth experiments. These measures are a means of description; they do not constitute an explanation. In particular, these measures do not support the claim that subjects make their judgments on the basis of Euclidean properties. The departures of subjects' estimates have been found to vary with these measures. The estimated areas of the figures that subjects produce vary with the subjects' perpendicular distance from the plane of the object, and also with the area of the correctly projected figure, for example. These correlations are significant, and the results have been replicated. Such findings are not to be explained by a "phenomenal regression" of areas, especially since perceived shape is known to diverge from constancy in the Thouless index for area. Again, these Euclidean descriptions are meant only to present the data in a familiar format.

A number of control conditions ensure that the departures are not due to incidental features of the experimental task. One might imagine that that task is like copying, and that earlier data may be the effects of copying. However, the simple act of copying produces no significant projective differences. Two subjects also copied shapes, when the response board was fixed at a slant. A slight correlation of slant and estimated cross ratio was found. This result reflects such small variability and such a small rate of change in the cross ratio that the basic finding could not be explained as this effect of copying at a slant. Finally, a response form was made to coincide with the Leonardo's window in the experimental task. This manipulation lowered average cross ratios, but it increased the variability of observations dramatically. No significant interaction of this manipulation with the basic effect was found. The observed departures from projective equivalence are not accounted for as the effects of copying, nor as the effect of changes in orientation on the ease of copying.

Individual differences in performance do not seem to affect the interpretation of departures from projective equivalence. A significant difference in performance was found among a group of architects and two groups of control subjects. No significant interaction of this group effect with perspective viewpoint was found, though a significant effect of viewpoint was found. The architects' average estimates differed significantly in cross ratio from the expected value, though it had been anticipated that architects might perform accurately. Yet architects are skilled in the work of descriptive draughting; their strategies have good internal consistency and may produce compelling drawings. A test of sex differences was made, and no significant difference was found. The familiar effect of perspective viewpoint was replicated. There was no significant interaction of performance between groups of men and women, and the effects of viewpoint. Means and standard deviations for the two groups are close in value. In other words, individual differences that might have affected performance on this 'spatial' task did not occur as expected.

The first and second experiments reveal departures from judgment of projective equivalence. The stimuli are pictures, which may not be as telling as real objects for the theory of shape constancy. Each of the projectively irrelevant manipulations in the first experiment had an effect on judgments of projective equivalence. Some pictures were matched to their projective equivalent reliably, while others were reliably mismatched. All the mismatches represent gross violations of projective congruence and collinearity, and yet mismatching pictures were chosen as matches in over half the trials. No base rate of response was set in the first experiment; a claim cannot be made that these observers were always insensitive to projective equivalence. However, their performance falls short of reasonable expectation on the projective-invariance hypothesis of perception. The problem of the base rate is addressed in the second experiment, and there significant post-hoc tests reveal departures from projective matches on average. Those pictures include regular indicators of slant. The first and second experiments contribute additional exceptions to reliable judgment of projective equivalence.

The results of all the experiments serve to make a single point: that normal observers are not constantly and reliably sensitive to projective equivalence in normal vision. I do not claim that projective properties are invisible, nor that there may never be evidence for the recognition of these properties under particular conditions. What has been demonstrated is that there are exceptions to the invariance hypothesis. These exceptions counter the claim that perception of projective equivalence is the constant and reliable basis of all shape constancy. These exceptions occur in a variety of ordinary situations which are germane to the theory of shape constancy.

Now suppose one made a similar series of tests that concerned an obviously invisible property of objects, could the results be any more decisive than those that have been obtained for projective equivalence? Consider an example. Many common objects change the polarization of the light they transmit or reflect, yet this polarizing property is not perceived. Light that passes through a pile of thin glass plates changes in polarization, and the light of a clear blue sky has a distinct pattern of polarization. Unlike radiance or wavelength, the polarization of light has not been of interest in the psychology of vision, since there is no univocally perceived quality that pertains to polarization as wavelength pertains to hue. Polarized light can be substituted for unpolarized light from a scene, and so long as the light has the same composition otherwise, no difference in the scene will be apparent. (The apparatus used by Attneave and by Gregory operates on this principle.)

Sometimes, however, the effects of polarization can be seen. Under some conditions, one can tell that a reflecting surface has polarized the incident light because the surface looks less bright than it ought. Similarly a combination of materials known as a polarizer-analyzer pair can produce colours. In that situation, polarization varies with the wavelength of the transmitted light. It is not polarization that is detected, but concomitant properties. Three characteristics of polarization can be discerned as important to the analogy:

1. The property is essential to the conditions under which objects are seen and yet the property is not itself perceived.

2. The property is not devoid of visible effects. It has a constant effect on perceptible properties in certain situations.

3. The test by which the property is known to be invisible is that different degrees of the property do not have different effects, when concomitant perceived properties are manipulated.

Projective invariance is essential to the conditions under which objects are seen. An object retains its projective properties when it retains its shape, and those projective properties are also retained in many of its optical images. Moreover, projective invariance has a constant effect on perceptible properties in certain situations, because it is implied by some geometrical properties that are perceived in some situations, such as the rectangularity of boxes. Projective properties are not devoid of visible effects, then, since shapes that are congruent in Euclidean geometry (e.g., a pair of triangles in the picture plane) sometimes appear congruent. Their Euclidean congruence implies their projective congruence. The important point to note, however, is that variation in projective properties has little or no perceptual effect, when concomitant properties are systematically manipulated. On the basis of the experiments, I claim that observers are not normally sensitive to projective equivalence in the uniform way that is required by the invariance hypothesis. These data do not point the way to a new theory of shape constancy, but they do indicate that contemporary theories of shape constancy that depend on the perception of projective invariance need to be revised.

Some New Directions

The present research can be carried in several directions. One can consider variations on the experiments, or one can extend their techniques to a new area of study. Already there is other evidence that the projective equivalence of shapes is not a sufficient condition for impressions of rigid motion (cf. Epstein and Park, 1986). Recent studies of the kinetic depth effect have revealed quantitative limitations on the stimuli that occasion perceived rigidity. Caelli, Flanagan, and Green (1982) have found that perceived rigidity depends upon viewing distance and the distance of the projection plane from the object. They construct a measure of linear perspective from these distances. Call the viewing distance from an observer to a point the distance _r_, and call the distance from the projection plane to the same point _f_. (The point is on or near the object.) A measure of the degree of linear perspective is:

$$L_p = \frac{r - f}{r + f}$$

The measure has a value of zero when the observer is at the plane of projection and has a value of one when linear perspective reduces to parallel projection. According to Caelli, Flanagan, and Green, "our abilities to reconstruct 3D objects are limited to a spatiotemporal 'window'. This window corresponds to linear perspectives greater than about .7". In other words, when the projections of some kinds of rotating objects are close to affine projection, only then are their geometric characteristics identified readily. One may note that in these studies, as in those of Attneave, Shepard, Perkins and Stevens, only stimuli such as rectangles, cubes, and line segments were used as stimuli.

Another application of the measure of the cross ratio can be made to the study of texture gradients as visual cues. Perspective gradients of texture are thought to provide the most effective cues to depth (see Cutting and Millard, 1984). Is it sufficient that texture gradients be in correct perspective for them to indicate depth? Rock, Shallo, and Schwartz (1978) have observed that, at least the recognition of regularity in a static texture gradient is necessary if the gradient is to provide an effective cue to relative depth. Even that claim has not been substantiated for moving gradients of texture. Optical flow patterns based on projective relations are still thought to provide sufficient cues to the relative motion of objects on surfaces with respect to an observer. So, for example, Longuet-Higgins and Prazdny (1980) have shown that derivation of the structure of a rigid scene and derivation of the direction of the observer's motion is possible in principle, given the details of a changing perspective image, that is, an "optical flow field" on a hemisphere. The nature of Longuet-Higgins and Prazdnys' claim is noteworthy. Though they do not intend a strong claim for the psychological reality of their proposal, they make a strong presupposition that observers have the ability to solve equations of the necessary type in projective geometry. They say that "to speak of the observer solving such equations is not, of course, to imply that the visual system performs such calculations exactly as a mathematician would: its modes of operation may well be more "geometrical" than "algebraic", and the same applies to the other computations envisaged in this paper. Our main point is that the equations demonstrate the feasibility of calculating both the motion and the structure from the optic flow field alone". To worry about choosing between geometric and algebraic modes of operation presupposes that the observers' task of deriving relative depth from an optical flow field is well described as one in projective geometry. In point of fact, it may not be necessary for an optical flow pattern to conform to the rules of correct projection in order for the pattern to serve as an effective cue to depth. Sometimes the surface of the ground moves in non-rigid ways, and still an observer can discern the motion of the ground relative to himself. A hiker who climbs a shifting sand dune or a skier who traverses the unstable course of a wet-snow avalanche estimates distances under such conditions. Relative depth may be discernible, even when there is a large rate of change in the cross ratio of identifiable textural elements across an optic array. It might be worthwhile to submit this conjecture to the test. I suggest that an appropriate array for transformation is not regular on a local scale, and has an identifiable five-fold symmetry (the Penrose tiling is such a pattern). There does seem to be a niche for measurement of the cross ratio in this context, at least.

Other directions for research are suggested by the literature discussed in the Introduction. Consonant with his extrapolation of Emmert's Law as a projective law for visual perception, Rock (1983, p. 256) claims that observers are sensitive to the visual angle presented by objects. His claim is shared by other proponents of the size-distance invariance hypothesis. On first inspection, his claim seems to have some support in Chapters 6 and 7. The magnitudes of the discrepancies of the estimated figures from the correctly projected figures in those experiments are related to the areas of the correctly projected figures. The areas of the estimated figures are also related to the areas of the correctly projected figures at each viewpoint. Distances measured from the observer to the point whose apparent position is estimated vary in proportion to the perpendicular distance of the observer from the object plane. This perpendicular distance is related to the correctly projected area and the estimated area of the figure at each viewpoint. The correctly projected area of the figure and the distance from the observer to the point on the object itself are geometrically related to components in the equation for solid angle. This solid angle is the visual angle subtended by the stimulus object. If solid angle were seen for itself, then the cross ratio of the estimated figures would have been constant. Given that it was not, another possibility should be considered. Visual angle may not be seen for itself (as is claimed by Rock and McDermott, 1964, and Shallo and Rock, in preparation), but observers' responses may reflect a function of solid angle. It would be useful to determine the form of this function for the perception of tilted surfaces.

A proposal that emerges from other work by Ullman (1979) can be pursued. Remember that one interpretation of the claim that his "polar-parallel" scheme is a competence theory for visual perception is that the theory be grounded in affine geometry. This claim can be explored in several ways. For instance there are several conditions in the fourth experiment in which the cross ratio of estimated figures differs significantly from that of the object. Each subject marked a position on an answer sheet to estimate a shape. The cross ratio of the shape is calculated from a construction on this point. Some of the points in the construction are fixed, not estimated by the subject. What the subject varies, among other things, is the areas of triangles ABX and AXD. Other areas of triangles which enter into the formula of the cross ratio are fixed quantities. The areas of triangles ABX and AXD are a ratio in the formula for the cross ratio. If the cross ratio departs from the correct value, the ratio of the areas of these two triangles must also depart from its correct value. This ratio is an affine invariant (see Klein 1927/1967). When one condition of the experiment is considered, the test of projective equivalence becomes a test for affine equivalence. Departure from projective equivalence and departure from affine equivalence seem to have been demonstrated at one stroke for this particular task in which subjects judge apparent shape. Of course, there are more general ways of investigating the matter. Other studies, similar to the third experiment, can be devised to test for departure from affine invariance in judgments of shape. There is another point that can be made in this context about the proposal of affine geometry as a theory of visual competence. The object that was used in the last two experiments is a relatively simple shape that affords a salient affine property. The top and bottom edges of the quadrilateral abcd are parallel. Parallelism is as important in affine geometry as collinearity is in projective geometry. Subjects didn't seem to use the psychologically salient cue of the position of the base parallel, in that they positioned their points some distance away from the projected image of the parallel, which is a striking feature of the object. That is not to say that the subjects could not have represented the top and bottom lines as parallel. They do so when asked to judge the real shape of the object. Yet in the main task they did not, so they failed to preserve both the cross ratio and an affine invariant in their estimates of a relatively simple shape. The main point here is that Ullman can be seen as making a claim about affine geometry, at least, and there is enough preliminary evidence to warrant a thorough test of his hypothesis.

Another possibility for research would be to go beyond the scope of the present invariance hypothesis. The perception of other invariants still more fundamental than the cross ratio could be tested. The invariants associated with the geometric property of connectedness are a prime example. This property is the analogue of the topological property variously called surroundedness, or adjacency, but in a discrete domain. The significance of this property for the computational approach to vision was described by

Minsky and Papert (1972). The application of such properties to the study of form perception is discussed by Ullman (1984a).

A new series of experiments has been undertaken, in fact. These employ new stimulus objects, new experimental tasks, and a nearly new measure. The projective correspondence of one planar object to another is a central case for the invariance hypothesis. Yet another case is as important: the projective correspondence of a solid, that is, internally noncoplanar, object to a picture or a retinal image. The new experiments assess the sensitivity that observers may have to projective equivalence in the latter situation.

It is not immediately apparent what the analytic form of the equivalence may be between a helicopter and a picture of a helicopter, or between a willow tree and its picture. It may be surprising that this projective equivalence can be assessed by a simple extension of the measure I have used until now. Imagine a set of three coordinate axes, and a solid polyhedral shape which lies in the octant of positive values for x, y and z. A planar image of the solid shape can for formed by projecting it onto the plane z = 1, (or some other plane, except those formed by pairs of the axes. This linear transformation can be achieved by dividing each of the coordinates by 1/z). The projective properties of the planar image can be measured in the familiar way: by the two-dimensional analogue of the cross ratio on a line range. The areas that are the data for the formula of this cross ratio are proportional to certain volumes: the volumes of the pyramids that are enclosed by triplets of ordered points on the plane, and the origin. The ratio of these volumes can be substituted for the ratio of areas that was used before. The result is numerically equivalent, since the volume of a pyramid is one-half the area of its base times its height, and the plane image lies at a constant height above the origin. Let the ratio of volumes be called the three-dimensional analogue of the cross ratio. The conventions by which signs are attributed to these volumes can be found in Klein (1925/1967). The volumes of the pyramids that are enclosed by triplets of ordered points of solid figures, and the origin, may be found. The triplets of points can be labelled in the same way as the points of the planar figure were labelled earlier. Then the value of the cross ratio of the planar figure (either the two- or three-dimensional analogue) is equal to the value of the three-dimensional analogue for the solid figure. This result is a general one; it reflects the generalizability of the cross ratio as a measure. (The generalizability of the measure also shows that the cross ratio is a robust measure; it is ecologically valid in the sense that small deviations out of the plane will not markedly affect the computation of the cross ratio on the plane. A more detailed statement would require an involved simulation, of the type that has been performed by Barron, Jepson, and Tsotsos (1987) for the computation of structure through motion.)

The cross ratios of some solid shapes are markedly different. The solid shapes that will be used in the experiments are models of a ring of carbon atoms (stripped of their nodes for hydrogen). These shapes approximate models of the isomers of cyclohexane, that is, they are the "boat" and "chair" models familiar to students of organic chemistry. One of these solid shapes has a cross ratio of 0, and the other has a cross ratio of about -2. These solid shapes differ in the position of just one point. Some changes in the position of this point change the value of the shapes' cross ratio, while other changes in the position of the same point leave the value of the cross ratio unchanged. In other words, there are some solid shapes that differ in the position of only one point, but which have the same cross ratio. Equal changes in displacement of this point produce different magnitudes of change in the cross ratio, depending on the direction of that displacement with respect to other points. These stimuli allow the value of the cross ratio to the manipulated in a series of experiments, independent of the displacement of a point.

The question that will be posed in these experiments is this: do subjects recognize the equivalence of shapes by their projective equivalence? Three categories of stimuli will be presented: 1) solid shapes that are identical in Euclidean terms, 2) solid shapes that are projectively identical, but which are different in Euclidean terms, and 3) solid shapes that are different in both projective and Euclidean terms (there exists no pair of shapes that are identical in Euclidean terms, but which are different in projective terms). Solid models of these shapes have been constructed with wooden dowels, and slides have been made of their plane projections from various viewpoints. The solid models will be used in an experiment on the kinetic depth effect. (cf. Wallach and Marshall, 1986). Subjects will be asked for their judgments of the identity and similarity of shape of pairs of these solid objects, as they are in rotation behind a translucent backlit screen. The slides will be used in an experiment on the 'mental rotation' effect, where the projective properties of the display will be manipulated. It is expected that changes in projective properties will have an effect on judgments of shape, but that these effects will not be independent of the effects of changes in displacement alone.

Some new directions for research have been discussed, but the aim of that research is ill-defined unless there be a positive statement about the nature of vision. What has been offered is a negative statement: that ordinary observers are not sensitive to projective equivalence. Though the results of the experiments are expressed as significant departures in a quantity, where the null hypothesis predicts no change (for that is what 'invariance' means), yet the impact of the results is that the invariance hypothesis is inappropriate as it stands. It has not been the aim of this dissertation to construct a theory of shape constancy. How to proceed, then? I suggest that a theory of shape constancy is to be found by consideration of more things than projective invariance. (I elaborate this hypothesis in Note 3.) The reader may feel jarred by the result that projective invariance is not a foundation for shape constancy in vision. Anyone, like myself, who has been immured in the tradition that linear perspective is the only real cue to depth, is sure to experience some uneasiness when faced with this proposition. What could such a conclusion imply for a theory of shape constancy? Nothing more difficult or complicated than the invariance hypothesis, at least. One may reflect that the explanation of how knowledge of Euclidean shape might be supposed to emerge from access to projective constancy in vision was not an easy explanation, as witnessed by the depth of mathematical knowledge that is needed to follow some articles on the subject. There are three questions that can give a general direction to the study of shape constancy, and with which a theory of shape constancy should begin. The virtue of the negative conclusion that is presented is that it helps one to face these questions squarely, without prejudice from the invariance hypothesis. They are:

1. How do we come to know the real geometric properties of objects?
2. How much do we know about those geometric properties just by looking?, and
3. What is the form of the representation by which we are acquainted with those properties in vision?

These questions seem impossibly broad, almost philosophical. A hint of an answer to the first can be given, if only to say that such questions can be addressed. Berkeley

(1732/1979, p. 154) puts some original thought about the question in the mouth of one participant in a dialogue when he says:

> "You would have us think, then, that light, shades, and colours, variously combined, answer to the several articulations of sound in language; and that, by means thereof, all sorts of objects are suggested to the mind through the eye, in the same manner as they are suggested by words or sounds through the ear, that is, neither from necessary deduction to the judgment, nor from similitude to the fancy..."

The passage suggests that the connection between concrete properties of shape and perceived properties of shape is neither a relation of isomorphism, nor any other relation that holds of necessity. The passage suggests the possibility of an original response to the first question; indeed, a theory that did not mention primary qualities, nor similarity of form, nor inference, nor invariance, <u>would</u> be an original theory. Such a theory might be called perturbing, since it denies popular assumptions about the perception of shape. It might be perturbing just in that it would bypass one or more psychological delusions about the perception of shape. Berkeley suggests that such a theoretical alternative is viable; I do not seek to advocate or vindicate Berkeley's theory of vision here. Instead, I propose that the apparently simple questions I have listed are more difficult than anyone has supposed. These three questions may seem a simple beginning, but they are questions which deserve serious reconsideration. I recommend these questions to anyone who would tell us the whole story of shape constancy, of which as yet we have scarcely the outline.

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Note 1

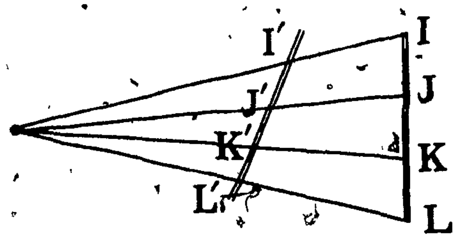
Several decades have passed since J.J. Gibson published The Perception of the Visual World, thus launching an important tradition of research in visual perception. Cutting's (1986) Perception With an Eye for Motion ranks among the best in this tradition of 'ecological optics'; few others combine such flair for experimentation with such breadth of historical knowledge. Cutting has done innovative work in the perception of complex motions (eg. Cutting, 1978) and in the categorical perception of sounds. Cutting seeks to vindicate ecological optics and to establish what has been called "naive realism" in epistemology. An objective of ecological optics is to solve an epistemological problem: how the shapes of things are known through vision. Ecological optics is not meant to supplant modern physical optics; the connection with physical optics is that ecological optics supposes shapes are known by what would once have been called "natural geometry" (eg. Descartes, 1637/1965, p. 106, and see Cutting, Chapter 2). An explanation by natural geometry declares that we perceive the distance and shape of objects by geometric principles which are embodied in the perceptual system. Some psychologists and philosophers, however, have not been convinced by the arguments that physical and ecological optics are linked in the intimate way such an explanation supposes. For example, laws of optics proper would not change, even if the principles of ecological optics turned out to be false; and the principles of visual perception may have only a remote connection with the laws of optics.

Ecological Optics and the Cross Ratio

Ecological optics supposes that a description of the geometrical conditions of vision is sufficient basis for a theory of visual shape constancy. Its strategy is to suggest that things cannot appear to be otherwise than they are, when viewing conditions are taken into account. This is in fact a distinctive epistemological premise, though sometimes it is presented as if it were a universally accepted truth (Warnock, 1982, p. 148). Where shapes are concerned, the claim amounts to this: there are geometric properties of objects that are apparent as they are seen from various positions or angles, and these geometric properties provide a basis for shape constancy. This claim is the familiar invariance hypothesis (eg., Boring, 1952). These basic properties turn out to be projective properties; the invariance hypothesis stipulates a place for projective geometry in the theory of shape constancy. Projective equivalences — quantified as projective invariants — are the elements of this geometry. There is a projective equivalence between flat objects and the candle shadows they project onto a planar surface. Projective equivalences are equivalences preserved by projective operations; ideally, projective operations can be effected with an unmarked straight edge alone (unlike Euclidean operations, which require straight edge and compass).

Cutting's Chapter 5 tells how the notion of invariance is applied to problems of shape constancy. This chapter is the best recent account of the invariance hypothesis (it is a revision of Cutting, 1983). One important projective invariant is the cross ratio, concerning which Cutting states: "I do not suggest that the use of the cross ratio in perception solves fundamental and sweeping epistemological problems. It is merely one example, perhaps not even a prototypical one, that can be used to promote realism as a perspective in perception and epistemology" (p. 79). To define the cross ratio, let i, j, k, and l be four points ordered in one direction along a line. Let x_{ij} stand for the directed distance between points i and j, so that x_{ji} is of opposite sign to x_{ij}. Then one cross ratio is expressed by:

$$\frac{x_{ij}\, x_{kl}}{x_{il}\, x_{jk}}$$



For points on a line, the cross ratio is the <u>simplest</u> projective invariant.

(Many "simple" geometric properties of shape and size, such as length and angle, are not projective invariants.) Since it is both simple and basic, the cross ratio is central to the geometric description of the way light is propagated from objects to the retina. The cross ratio is just the sort of robust quantity that might explain perceived constancy of shape and size. Gibson (1950a) thought as much, as have others (eg. Johannson, 1977, and Michaels and Carello, 1981). Yet no one has demonstrated that the cross ratio actually performs the psychological work that ecological optics assigns to it. Ineed, Neisser (1977, p. 24, cited in Cutting, p. 71) says that such claims are ecological optics' "largest outstanding promissory note", and Cutting (1982b, p.214) has called Gibson's (1950a) identification of the cross ratio with the relevant perceptual invariant "a landmark in the study of perception". Cutting aims to perform the wanted demonstration. Nonetheless, Cutting's own judgment in later chapters of the book is that the cross ratio is sometimes effective in specifying form, and sometimes not (pp. 131, 142).

Perception With an Eye for Motion has a natural division into four sections. Section one (Chapter 1 through 5) provides a general and historical background on optics, invariance, and projective relations. Section two (Chapters 6 through 9) describes the application of the cross ratio to the study of vision, and describes some experiments. Section three (Chapters 10 through 13) presents a theoretical review and empirical study of direction-finding by means of 'optical flow'. The last section (Chapters 14 and 15) rounds out the book with a theoretical discussion of 'direct' and 'directed' perception.

The Cross Ratio in Eight Experiments

Twelve of Cutting's own experiments are described in his book. In the first eight experiments, presented in Section two, he seeks to establish the cross ratio as an effective invariant for the visual perception of form. There are positive things to say about these experiments. Psychophysical method and experimental design are used admirably (the Weber fraction condition of the second experiment is a noteworthy example). Yet there are errors in these experiments that arise from confusion, and which make the interpretation of the evidence problematic. The nature of these confusions makes this a difficult book to review, because they represent major flaws in an otherwise impressive attempt to establish ecological optics on a firm experimental basis. Some of these pervasive confusions are broadly theoretical, with implications for general arguments, while others are practical or arithmetical.

A confusion of the first kind occurs when Cutting wrongly identifies claims about projective invariance with claims about rigidity and coplanarity. Some authors (eg., Simpson, 1983) mistakenly hold that a constant cross ratio among points is a sufficient condition for the collinearity of points. Cutting makes the same mistake, and he also makes the mistake of taking a constant cross ratio (as applied in a slightly different way, see p. 105) as a sufficient condition for the coplanarity of parallel lines. Yet, for one example, the measure of a cross ratio can be defined among points that fall on a full circle (cf. Schwerdtfeger, 1979). (There do exist projective criteria for collinearity — see Coxeter, 1969, p. 220 — and for coplanarity — see Veblen and Young, 1910, p. 329). The confusion of collinearity and projective invariance motivates Cutting to produce a "projected-angles" formula for the computation of the cross ratio, so that "the shape of the projection surface becomes irrelevant. Thus the projection surface can be a plane, as in a movie theatre, or a curved surface, such as the retina or cornea of the eye" (p. 81). But this implies that the shape of the <u>projecting</u> surface need not be planar, either, contrary to Cutting's intent. Further, Cutting considers preservation of the cross ratio to be criterial for an object's rigidity (for examples, pp. 97, 105, 115). Yet the constant cross ratio of a moving object is no guarantee that the rigid shape of the object will be maintained as it is moved (that is, under its displacement).

Some authors (eg. Helmholtz, 1878 and Ullman, 1979, p. 146) have claimed that rigid bodies must be <u>postulated prior</u> to any discussion of geometry. If this is true, and the rigidity of objects that undergo displacement is ensured only by a postulate on which geometric matters themselves depend, then geometric quantities are not sure criteria for the rigidity of objects. Some have thought that the bare possibility that geometry can be applied to physical objects at all depends on this postulate (Torretti, 1978, pp. 157, 158). Of course, if one <u>assumes</u> that a body is rigid under displacement, its projections onto a planar surface can be predicted. In the absence of a competing hypothesis, ecological optics might safely have assumed the rigidity of objects in a theory of shape constancy. Yet there exists a strong alternative psychological claim about shape constancy and rigidity under displacement. Helmholtz (1876, 1878) makes a plausible claim that the phenomena of shape constancy would not obtain if objects were not rigid under displacement (that is, as they are, when they are). Moreover, projective invariance could obtain under the conditions Helmholtz specifies. The rigidity postulate is a working part of Helmholtz's theory. Helmholtz evaluates what Cutting has only assumed. In short, Cutting has not established the role of coplanarity or rigidity in shape constancy.

Some errors in arithmetic are apparent in Cutting's presentation of the cross ratio. These undermine the value of the first eight experiments. The errors are brought to light by a general equation that obtains among the various cross ratios that can be computed on the same points. That equation is called 'the fundamental syzygy'. With the notation introduced earlier, the fundamental syzygy can be expressed as:

$$x_{ij}\, x_{kl} + x_{ik}\, x_{lj} + x_{il}\, x_{jk} = 0$$

(Klein, 1925/1967, Vol. 2, pp. 155-158.) (If, instead, _i_, _j_, _k_, and _l_ are four vertices of a plane quadrilateral, none of which coincide with the origin, then a planar projective invariant, the two-dimensional analogue of the cross ratio, can be computed. This analogue of the cross ratio can be obtained by substituting for x_{ij} the areas of triangles formed by the origin and ordered pairs of points.)

Six formulae of cross ratios can be obtained by rearranging the terms of the syzygy. If _A_ stands for

$$\frac{x_{ij}\, x_{kl}}{x_{il}\, x_{jk}}$$, the cross ratio given earlier, then

the other cross ratios can be expressed as:

$$\frac{1}{A},\ 1 - A,\ \frac{1}{1 - A},\ \frac{A - 1}{A},\ \text{and}\ \frac{A}{A - 1}$$

(see Schreier and Sperner, 1961, p. 55).

Cutting seems not to realize that these formulae are determined in this way, for he says: "It is something of an embarrassment to discover that the six classes of cross ratios have different numerical properties" (p. 123). In one table, he lists the amount of change in the cross ratio concomitant with differences in the position of points on a line (Table 6.1, p. 89).

Table 6.1 (p. 89, Cutting)

Six Classes of Cross Ratios

Cross ratio	Range	Value for even distribution	Change with interior shift*	Change with exterior shift*
1. (AD.BC)/(AC.BD)	0.0–1.0	0.75	0.13	0.04
2. (AC.BD)/(AD.BC)	1.0–∞	1.33	0.25	0.07
3. (AD.BC)/(AB.CD)	0.0–∞	3.00	2.24	0.49
4. (AB.CD)/(AD.BC)	0.0–∞	0.33	0.25	0.07
5. (AC.BD)/(AB.CD)	0.0–∞	4.00	0.68	0.68
6. (AB.CD)/(AC.BD)	0.0–∞	0.25	0.04	0.04

pairs of equal values

Table 6.1 Revised

Cross Ratio	Range	Value for even distribution	Change with leftward movement* of point A	Change leftward movement* of point B
1. (AD.BC)/(AC.DB)	−∞ – +∞	-0.75	0.05	-0.05
2. (AC.DB)/(AD.BC)	−∞ – +∞	-1.33	-0.10	0.09
3. (AD.BC)/(AB.CD)	−∞ – +∞	+3.00	-0.72	1.14
4. (AB.CD)/(AD.BC)	−∞ – +∞	+0.33	0.10	-0.09
5. (AC.DB)/(AB.CD)	−∞ – +∞	-4.00	0.72	-1.14
6. (AB.CD)/(AC.DB)	−∞ – +∞	-0.25	-0.05	0.05

A reconstruction of Cutting's Table 6.1 (p. 89) can be used to illustrate some misuses of the cross ratio and a failure to produce a projective mapping. *The movements of points A and B are as illustrated in Cutting's Figure 6.4 (p. 88), and the values of the displacements are given in that Figure. I assume for the revised Table that points A and B are spaced seven units apart.

The values he does list violate the relations implicit in the syzygy; thus, there should be six pairs of equal values along the last two columns of Table 6.1, yet just three are equal. This discrepancy is due to arithmetical error. Other errors occur because Cutting puts arbitrary bounds on the value of the cross ratio. He restricts his principal measure of the cross ratio, so that it varies between one and zero, when in fact it can take any real value. He also disregards the sign of the cross ratio, deeming a difference in cross ratios of -0.69 equal to a difference of +0.69 (see also Figure 6.4, p. 88). Hence, Cutting has an invalid measure of the cross ratio. (Granted, he only wishes to measure relative change in the cross ratio for an independent variable.)

Another serious criticism is that the stimuli of the first eight experiments do not preserve projective properties when the locations of individual lines in the stimuli are changed. Cutting uses four parallel bars as his basic stimulus, like four parallel fingers of a hand. One of these may have a motion independent of the rest, like a finger that waggles while the hand is moving. Cutting measures the cross ratio on a line that intersects these parallel bars at right angles. He computes the cross ratio for those collinear points of intersection. He varies the position of one of the four parallel bars that is, the position of one of four points on the line, in an effort to produce constant values of the cross ratio (eg., Figure 6.4, p. 88). There is an artifact in his procedure; when one cross ratio is constant, other cross ratios on the same points vary (see Table 6.1). One assumes that all eight experiments employ similar materials. That is to say, all eight experiments fail to control projective properties. Cutting makes the claim that observers can be sensitive to changes of 0.05 in any cross ratio of such stimuli, though he uses only one measure of the cross ratio. This he calls the "canonical cross ratio" (pp. 80, 81). Yet the value of other cross ratios on the same points is not constant; those cross ratios can vary more than 1.50 (note rows 3 and 5 of the revised Table). This is no relation of projectivity, even though one measure of the cross ratio remains constant. Cutting is not free to choose one cross ratio measure and ignore the rest; in any projective mapping, all the cross ratio measures are preserved. When some cross ratios are not preserved, no projective mapping has been established. All cross ratios can be controlled at once; <u>any</u> projective mapping from line to line will achieve that end. The point is that the method by which Cutting attempts to change the distances among points or bars, while preserving a single cross ratio, is <u>not</u> a projective mapping. The variation in values seen among these cross ratio measures indicates that, for his purposes, his stimuli are disordered. His experimental conditions do not manipulate projective properties, as he intended (I conjecture that a distance ratio, not the cross ratio, is the quantity that Cutting has manipulated in these experiments. A distance ratio is a simple ratio of two distances.)

Other Evidence

Chapters 8 and 9 seem to exaggerate the importance of the cross ratio as an index of projective invariance. Klein (1925/1967, Vol. 2, p. 158) cautions against treating the quantities of projective geometry as if they were constructions on the cross ratio. He remarks that such an emphasis makes insight into projective geometry difficult.. Perhaps it also makes insight into the applications of projective properties to perception difficult. Cutting measures the cross ratios of points on a line, and complains that "the cross ratio is confined to collinear points or coplanar parallel lines" (p. 115). Yet he continues to use the cross ratio, and fails to explore other invariant measures of planar projections. Instead he proposes to assess planar projective properties by measuring many cross ratios on skew lines, which he models in turn by an index of the proximity of points, that is by a "distance-density function". His motivation is tenuous. He says "the general pattern...is that cross ratio change is a function of proximity of a displaced element with others... Thus changes in cross ratio may have a systematic relation to various measures of density. The pursuit of this idea is the crux of this chapter" (p. 116). Cutting may have lost sight of his goal at this point, since he recognizes that the measure he chooses is not a projective invariant (note 1 p. 273). Why choose thus when there are measures of planar projective invariance? One example is the two-dimensional analogue of the cross ratio, which I mentioned in connection with the fundamental syzygy. Is there any greater motivation to use a distance-density model to approximate cross ratios than there is to approximate visual angle by the same method? The appropriate measures should be applied in the first place. Chapter 8 explores and tests the hypothesis that "if a rigidity-violating element always appears in a region of the same density, it ought to be equally easy to detect" (p. 124). This is a fair hypothesis about constraints on the information that is used in making perceptual judgments, but it is not an hypothesis about projective invariance or the cross ratio.

Cutting's general conclusion about the cross ratio is that it is sometimes effective in form perception, and sometimes not. He concludes that cross ratios are useful to observers as they judge the rigidity of planes in rotation, but not when they judge the rigidity of moving planes that remain parallel to the picture plane. The latter case is often considered to be simpler. The parallel movement of planar figures cannot be distinguished by a projective property from the rotation of planar figures, and Cutting wisely attempts no such distinction. As it is, perhaps the cross ratio is never a psychologically effective invariant. A strong case for its effectiveness has not yet been made.

Optical Flow

Chapters 10 to 13 are devoted to problems of optical flow; they are excellent. The focus of expansion in an optical flow field has been cited as a visual cue by which an observer senses the direction of his own motion. Chapter 10 contains summaries of a good sample of relevant research and criticism. Cutting provides practical evidence about direction finding based on another cue: motion parallax. He makes this topic easy to understand. In these chapters, Cutting defends the hypothesis that an observer can use changes in the optic array to tell the direction of his movement. Koenderink and van Doorn (1981) have provided an analysis which suggests that the focus of expansion does not indicate the direction of an observer's own movement, independent of eye movements. Cutting suggests that changes based on differential motion parallax do indicate the direction in which an observer is moving. He provides two experiments to document the point. He specifies the asymmetries of optical flow that occur as an observer moves in a straight line, or along a circular track. He shows that these asymmetries of optical flow are effective indicators of direction. Here at least, there seems to be evidence for an explanation based on ecological optics.

Directed Perception

Cutting's book neither extends ecological optics, nor does it lend support to naive realism. Yet it does seek to offer a new approach to the theory of perception, which he calls 'directed perception'. Cutting draws conclusions about directed perception from his results. He stresses the variety of stimulus information that is relevant to judgments of the shapes of objects. He contrasts directed perception with both direct perception and perception mediated by inference. He distinguishes directed perception from other theories in the way that this information is "mapped" between environment and observer. Direct perception implies that things cannot appear to be otherwise than they are, that is, the physical descriptions of objects are assumed to be univocal: they map in a one-to-one manner onto the kinds of information available to the observer. Theories that suppose perception to be mediated by inference assume information to be equivocal; many kinds of physical description are specified by many kinds of information. Directed perception is different: many kinds of information are assumed to specify each property visible to an observer. There is a many-to-one mapping of physical descriptions onto information. Cutting's claim is that while *information* about shape is not univocal, neither is it wholly equivocal.

His theme is the immediacy of perception, that is, how things may be said to be "present to the mind". Cutting mentions the history of the notion of immediacy in his "Nine Issues Concerning Direct Perception" (p. 224). He would have done well to refer to a more comprehensive history of the relevant ideas, such as Yolton (1984). Perhaps he mistrusts such philosophic reviews, since he says of Austin's (1962) discussion of direct perception that it "was more about words than about perception" (p. 234). Cutting's own historical discussions occur in the first four and the last two chapters of the book. These are broad and insightful, though misleading in a few points. Here is one: he suggests that the abstraction of primary from secondary qualities was acceptable doctrine for Berkeley (p. 227, but see Grayling, 1986, p. 74). At another point, he attributes to Poincaré the view that projective notions are central to the understanding of visual perception (p. 143, but see Poincaré, 1913). I have not been convinced by Cutting's arguments for the directedness of perception. When Cutting discusses the theory of depth cues, for instance, he says that the number of cues seems to increase without end, that "it seems difficult not to let everything flood in" (P. 245). Here the theory of depth cues may be defended by a paraphrase of Austin's (1961, p. 221) statement on the many uses of language: "I think we should not despair too easily and talk, as people are apt to do, about the infinite [number of depth cues. Psychologists] will do this when they have listed as many, let us say, as seventeen; but even if there were something like ten thousand [depth cues], surely we could list them all in time. This, after all, is no larger than the number of beetles that entomologists have taken the pains to specify." The point is that the number of depth cues or the complexity of the "mapping" between environment and observer is no bar to a theory of form perception that is based on depth cues. There is nothing inherently wrong with theory that is replete with Baconian detail. Cutting does construct an effective argument to the effect that "directed perception" is a better theory of form perception than is "direct perception". But in my view he has not shown that "directed perception" is a satisfactory theory. There are perspectives on this subject that are yet unseen.

Perception With an Eye for Motion has many positive qualities, and it has serious flaws. The format of the book makes it easy to read, and the footnotes are apt and informative. Although I would not recommend this book to an unsophisticated reader, any serious student of perception should have a copy.

Note 2

Phenomenal Regression and the Thouless Index in Experiment Three

Shape constancy is not always complete. If one accepts that shape constancy is achieved as if by operations on projective invariants, one may suppose that when constancy is incomplete, shape is perceived as a compromise between real shape and some projective equivalent such as a perspective image on the retina. Thouless (1931a, 1931b) characterizes apparent shape on a scale between physical shape and a perspective image on the retina. He supposes that a "law of compromise" holds for shape and other perceptible qualities. For instance, apparent brightness may be a compromise between a surface's reflectance and the product of reflectance and luminance upon the retina. Thouless calls the tendency "phenomenal regression to the 'real' object" or, for short, "phenomenal regression". His measure of phenomenal regression, the Thouless index, can be applied to any perceptible quality. Brunswick (1933) develops the same formula, and is explicit about his assumptions. Brunswick imagines that the index reflects congruence relations in the abstract, rather than any single property, since it is unitless and can be applied to many perceptible qualities. Thus he recognizes that the unit of projective congruence is dimensionless. He develops a measure of the degree of projective transformation as a corollary of the index. He sees the use of a logarithmic formulation of the index as a conversion of the scale of perceived shape from an arithmetic series to a geometric series. In this way, he relates the values of the index to Fechner's Law, which makes logarithmic scales omnipresent in psychophysics. Both Thouless and Brunswick use projective congruence as the standard by which they compare perceived shape to real shape. They accept the tenet that projective invariance is the basis of shape constancy.

The reader may wonder whether subjects' estimates could not have been explained as phenomenal regression. Thouless' (1931) formula is an accepted way to estimate the magnitude of this effect, since it is independent of viewing distance (Battro, Reggini & Karts, 1978). The assumption that the Thouless index for area is constant across perspectives can be tested with the present data and that in turn tests whether observed effects are to be attributed to phenomenal regression.

Let s stand for the area of the figure found by correct methods of projection, r for the actual size of the object and p the estimated or "phenomenal" size. Thouless's index of phenomenal regression to the real object is given by the formula:

$$i = \frac{\log p - \log s}{\log r - \log s}$$

The index ranges between zero and one. Mean values of s and p (across subjects) can be supplied for six viewpoints. Assume the index constant; one "variable" remains: r. When the values for s and p are substituted, will r (the real size of the object) vary? Since the real size of the object is constant, variation in r will reflect variation in the Thouless index for area, contrary to expectation. Let subscripts be assigned to p and s to indicate the viewpoints to which they belong, and let $k = \log r$. The quantity to be estimated is r. The assumption that the Thouless index is independent of viewing distance predicts that: $i_1 = i_2 = i_3 = i_4 = i_5 = i_6$, so that the following should be identities:

$$\frac{\log p_1 - \log s_1}{k - \log s_1} = \frac{\log p_2 - \log s_2}{k - \log s_2} = \ldots = \frac{\log p_6 - \log s_6}{k - \log s_6}$$

If a pair of viewpoints is chosen, say the viewpoints with indices $\underline{i}_1$ and $\underline{i}_2$, an estimate of $\underline{k}$ is:

$$k = \frac{\log s_1 \log p_2 - \log s_2 \log p_1}{(\log s_1 - \log s_2 + \log p_2 - \log p_1)}$$

This computation was performed on all distinct pairs of conditions. The values of $\underline{s}$ were taken to be the areas of figures ABCD as found in the master drawings. The values of $\underline{p}$ were the mean estimated areas ABCD. The value of $\underline{r}$ ranged from 874 cm^2 to 15106 cm^2. When these values were reintroduced to the individual equations

$$i = \frac{\log p_j - \log s_j}{\log r_j - \log s_j}$$

the index ranged from meaningless negative scores to .19. A variation of at least one-fifth of the range of the index seems sizeable. The variation in derived estimates of the size of the real object suggest that phenomenal regression as described by the Thouless index does not explain the effect that has been found.

The Thouless index for area has been applied to other experiments on shape constancy. The variability in the index has been examined as it applies to the height-to-width ratio of rectangular stimuli. That ratio is proportional to the area of rectangular stimuli. Experiments have varied the angle which a rectangular stimulus makes with the picture plane. Lichte (1952) reviews several experiments, and finds that the Thouless index varies with angle of rotation from the frontal plane. Koffka (1935) also recognizes such an effect. A quantitative claim about the index was made by Hsia (1943), using data from Thouless's own investigations. He conjectured that the index $\underline{c}$ varies as

$$c = (R - \cos \theta)/(1 - \cos \theta)$$

where R is the observer's match of height-to-width ratio, and θ is angle of tilt from the frontal plane. The present results can be taken as additional evidence for this claim. A consequence of Hsia's claim is that R varies as cos θ, since c is a constant. The ratio R can be construed as an estimate of area. The area of the quadrilaterals that the subjects produce is correlated with the angle their line of sight makes with the object plane. The angle can be estimated, since the stimulus object was about forty meters from the viewing stations. The correlation of estimated area and the cosine of angle of tilt is significant. (Spearman's rank correlation = 0.971, N=6, $p \leq .01$). In other words, the finding that the Thouless index for area varies with the viewpoint of an observer is consonant with established data on shape constancy. This is not a new or surprising finding; estimated area has already been found to be a function of ordered distances, and the cosine of angle of tilt is another function of the ordered distances. Accordingly, values of the angle of tilt are ordered in the same way as the ordered distances, which could explain the high rank correlation of the cosine of angle of tilt with estimated areas.

Note 3

Depth Cues and the Invariance Hypothesis

> **What seems to have misled the writers of optics in this matter is that they imagine men judge of distance as they do of a conclusion in mathematics, betwixt which and the premises it is indeed absolutely requisite there be an apparent, necessary connexion: But it is far otherwise in the sudden judgments men make of distance.**
>
> **Berkeley, An Essay Towards a New Theory of Vision, p. 176.**

The experiments can be considered evidence for the classical theory of depth cues, against the theory of directly perceived invariance. To understand this claim, it is necessary to know how the tenets of the invariance theory and the theory of depth cues differ and what explanation is added by the theory of depth cues. I ask the reader's indulgence to outline this auxiliary claim at some length. Depth cues require no introduction, since they are reviewed in any elementary textbook on perception (e.g., Goldstein, 1980, chap. 7; Rock, 1975, chap. 3; or Schiffman, 1976, chap. 16). In general, the theory of depth cues is a theory of shape constancy, as is the invariance theory. Both theories propose to account for the same phenomena. The theory of depth cues emphasizes distance, while the theory of invariance emphasizes projective quantities.

Depth cues are meant to contribute to impressions of depth and form in varying degree. In contrast, the Gibsonian program incorporates a theory of invariance that can be construed as a search for foundational laws of shape constancy. Gibson thought that impressions of depth and form have one kind of concomitant, not diverse kinds with varying effectiveness. It will become clear that depth cues do not afford such laws. Gibson believed that geometrical laws of stimulation underlie the classical depth cues. "The theory is that they are retinal gradients and steps of ordinal stimulation and that they are geometrically precise" (Gibson, 1950a, p. 138). Depth cues might then be described as the effects of perspective. A proponent of the theory of depth cues might respond that such a reduction of depth cues to geometric laws of perspective is neither plausible, necessary, nor desirable. According to the theory of invariance, such basic geometric laws in turn would relate properties of the effective stimulus for vision to properties of objects, and relate them univocally. The validity of these laws would then depend on optical laws, whose validity is the province of physics. This much is undisputed: the projected image (e.g., the retinal image) of an object is related to the object's geometrical form by laws of physical optics. Equivalently, projective invariants -- which Gibson thinks are seen directly -- are related to an object's form by laws of ecological optics. That is what Gibson (1966, p. 187) means when he claims that perception yields information about form.

The principle implicit in Gibson's program is both general and practical: it is that the basic congruences of the dioptric conditions of observation are also the elements of a psychology of apparent shape. The consequence for his theory is that linear perspective (that is, perspective projection) and the effects of perspective on texture are emphasized, at the expense of other cues to depth. He presents the dependence of perceived shape on the geometrical structure of the light impinging on the eye as the central problem of shape constancy. Yet is there a problem at all? There may be no problem if impressions of shape are reliable cues to real shape. The only problem is that Gibson and others postulate a certain kind of geometrical relationship between impressions of shape and real shape.

The postulate is part of a long intellectual tradition, as old as Locke. According to the tradition, the appearances of things distinguish them insofar as those appearances are one of two things: "constant effects or else exact resemblances of something in the things themselves" (Locke, 1690/1959, vol. 2, p. 498). The way things look distinguishes them when there is a recurrent correspondence between the way they are and the way they look. That is, if there is either a constant effect of actual shape on perceived shape, or else there is an exact resemblance between actual shape and perceived shape, then objects can be distinguished by the way they look. The two grounds for distinguishing objects by their appearance were once marked by the division of perceptible qualities into secondary qualities and primary qualities, respectively. This distinction is foreign to the theory of depth cues. (Gibson implies that he does not make the distinction in just this way, but his distinction between specification by convention and specification by projection implies just this distinction. The resemblance he claims is in terms of projective quantities.) In modern times, the notion of resemblance between object and idea seems to have lost its definite sense (for example, as in the exchange between Henle, 1984 and Pribram, 1984). Modern writers concentrate on the consequences of postulating primary qualities, rather than on the relation of resemblance (that is, on Lockian orthodoxy) itself.

What is the import of the division between primary and secondary qualities? All primary qualities were supposed to be immediately perceived, as were some secondary qualities. Today, one would say they are directly perceived. Some aspects of form are still thought of as primary qualities. If some are, it is important to know which aspects they are, because the descriptions (say, the geometrical descriptions) "designating the primary qualities of things may also be used, without ambiguity or change of meaning, to designate the ways in which some things appear" (Chisholm, 1975, p. 133). This is the importance of primary qualities for contemporary theories of vision, and it is the reason the distinction is made today.

The invariance theory motivates a search for such aspects of form. The reason is that the descriptions of those aspects can be borrowed from dioptrics ready for use in a theory of shape constancy, as is desirable for an ecological optics. If the property of being square exemplified a primary quality, then square things would have to look square; things that look square would have to be square; and "square" could be taken as a basic description of a way things look. It's unlikely that "squareness" describes a primary quality; to some people, other geometric properties seem more plausible candidates. If projective properties were primary qualities, then projective descriptions could be taken as a basic level of description for the way things look. This might lead one (such as Johansson, 1977, p. 403) to make "the principles of central projection, as established in projective geometry, an appropriate model for visual space perception." Where the study of visual imagery develops an analogy with vision, the search for primary aspects of form motivates similar claims for visual imagery. There the mistake of thinking that some qualities are "primary" in that they designate both the appearances and the physical properties of things can be construed as a "seemingly innocent scope slip that takes image of object x with property P to mean (image of object x) with property P instead of the correct image of (object x with property P)" (Pylyshyn, 1981, p. 153).

Now the basic difference between the invariance theory and the theory of depth cues can be stated briefly. The invariance theory postulates primary qualities of form in the sense that there are certain geometrical descriptions that are true both of the objects seen and also of their appearances. The theory of depth cues, on the other hand, does not recognize the distinction between primary and secondary qualities of form. In effect it treats all perceived qualities as secondary qualities. Inasmuch as the experiments show that projective properties are not primary properties, or that they are not directly seen, they support the theory of depth cues against the invariance theory. The distinction between primary and secondary qualities will be elaborated, to indicate why one theory might be accepted.

There is an assay that distinguishes primary from secondary qualities. Primary qualities, if there were any, would license the following inference: "if there exists something which appears to have a certain property, then there exists something which has that property" (Chisholm, 1957, p. 173). Let us understand "there exists something" as "there is something in the visual field of the perceiver." Qualities that license the inference without exception could bear equivalent descriptions in both physical and psychological applications without undue ambiguity. Secondary qualities would not license this inference. If something appears pink and elephantlike, it doesn't follow on that alone that something akin to a pink elephant exists.

When an aspect of shape is supposed primary, then if something appears to have that shape, something of that shape must exist. The inference might seem banal, since it's unusual for medium-sized objects to have shapes other than those they appear to have. Yet the import of the statement is that it is <u>necessary</u> that something of the right shape should exist. About primary qualities, so runs the theory, there is no chance of illusion. (One may be reminded of the importance and place of geometric illusions in Rock's theory of shape constancy). The inference is rooted in the definition of a primary quality. The look of a primary quality is an exact resemblance of that physical quality, though "resembles" has only the force of "is amenable to the same description as" in this context.

An example may help to clarify what it means for projective properties to be primary qualities. The example demonstrates the initial plausibility of the claim, as well. The sides of World Trade Center One are rectangular. One might hypothesize that a necessary condition for me to know they are rectangular is that they appear quadrilateral from any vantage point. Whatever one's vantage point, whether Rockefeller Center, Trinity Church, or the Staten Island Ferry, the walls of World Trade Center One always appear quadrilateral. So do the walls of World Trade Centre Two. (All convex quadrilaterals are equivalent in projective geometry, and quadrilaterality can be taken to be a projective quality). Their constancy of appearance, that is, the perception of a constant projective property, could conceivably be a basis or a necessary condition for my knowledge that World Trade Center One and World Trade Center Two are the same shape.

This example hints at an explanation of the way shapes are known through vision, an explanation based on the invariance hypothesis. The assertion that projective properties are primary qualities, though it involves a claim about resemblance, does not imply any such thing as that brain states are quadrilateral or that cross ratios of ideas can be measured. It is no longer <u>crucial</u> to the theory how or where the relation of resemblance can be found. The assertion does involve the claim that terms drawn from projective geometry are adequate to describe how things appear without metaphor, elision or ambiguity. A description of apparent shape would not be parasitic on a knowledge of objects, in the sense that an understanding of the expression "from here those people look like ants" is dependent on knowledge about ants. The appearance of primary qualities of shape would bear descriptions of shape independently, in their own right.

The overwhelming number of perceptible properties are secondary qualities. Originally, the primary qualities of objects were thought to be "the bulk, figure, number, situation, and motion or rest of their solid parts" (Locke, 1690/1959, vol. 2, p. 178). Not all aspects of figure, that is, not all aspects of shape, are now thought to be primary qualities. Length is not: that is the point of postulating the classical cues for depth, or any theory of shape constancy at all. The way that distance is seen might well be likened to the way that distance is heard. There is a temptation to say one sees distance, but not tempted to say one hears distance. Various signs or cues for distance (such as the regularities of pitch known as the Doppler effect) are used in audition, but distance is not heard directly. A similar account of visually perceived distance is given by the classical depth cues. Various signs or cues for distance are employed in vision, but none of these is described as an exact resemblance of distance. They are constant effects of distance, and they are not described in the way distance is. Properties that provide constant effects, and not exact resemblances, are secondary qualities. A similar account of projective qualities is not hard to imagine. To quote one writer who denies that there are distinct primary qualities of shape, "Can you even separate the ideas of extension and motion, from the ideas of all those qualities which they who make the distinction, term secondary?" (Berkeley, 1713/1979, vol. 2, p. 193).

How is the theory of depth cues different from the invariance hypothesis? The theory stipulates that geometric quantities are made visible by cues that have no necessary connection with the quantities they represent. What suggests some shapes can suggest other shapes, and what suggests colours can indicate shape. These cues have the nature of signs. Though the relation of a sign to its significate is called an arbitrary relation, the cues do not denote haphazardly, nor is their significance established through human convention. Depth cues are natural signs that indicate the shapes of things. The tradition is that these cues have been used to explain the perception of distance from the observer, but there is no reason that their use should not encompass more than length.

What is an explanation that employs secondary qualities like? There is a fine example of a depth cue, one which has been neglected. The example is telling because it seems absurd to postulate a similarity or a resemblance between this sign and the quantity it signifies. Far-away mountains appear blue, and the Hollywood hills look brown from downtown Los Angeles on a smoggy day; this is the effect of aerial perspective. The apparent colours and brightnesses of things change with their distance from an observer. The reader may note that rough terrain is often chosen for these examples, not because aerial perspective cannot be demonstrated elsewhere, but because cues to textural gradients are ineffective in these situations, since the regularity of textural elements cannot be assumed (see Gibson and Flock, 1962).

Aerial perspective once played a greater role in perceptual theory than it does now. Leonardo considers that "perspective is divided into three parts. The first part includes only the outlines of bodies, the second includes the diminution of colours at varying distances, and the third, loss of distinctness of bodies at various distances" (Da Vinci, 1490/1956, p. 5). The modern emphasis on linear perspective scarcely allows discussion of aerial perspective. Rock wonders if it can be considered a perceptual cue at all (Rock, 1975, p. 92). Uncharacteristically, Gibson singles it out as a depth cue that is independent of projective invariance: "it is only reasonable to suppose that it would suggest the impression of distance, as red suggests warmth, without compelling the impression in the way a stimulus is supposed to do (Gibson, 1950a,p.115)". That does not mean that aerial perspective is ineffective as a cue to distance. It can indicate depth in the absence of cues of familiar size, texture gradient, and relative position in the picture plane. Leonardo describes a demonstration to this effect. The importance of Leonardo's example is that he eliminates cues due to linear perspective.

> "There is another perspective, which we call aerial, because through the differences in the air we can perceive the varying distances of various buildings which are cut off at the visual base by a single line. This would be the case if many buildings were seen beyond a wall, so that all of them appeared to be the same size above the edge of that wall...Therefore, paint the first building above the wall its true colour; the next in distance make less sharp in outline and bluer; another which you wish to place an equal distance away, paint corresponding bluer still...By this rule, the buildings which are on one line and seem to be of the same size will clearly be understood, so that it will be known which is the most distant and which is longer than the other."

Da Vinci (1490/1956, p. 101) realizes that the effect depends upon a natural interpretation of this sign for depth. The interpretation of such signs as a means for the perception of depth and form is a current theme in psychological theory. Rock, Shallo, and Schwartz (1978) have found that perspective cues such as texture gradients depend on such an interpretation (see Rock, Wheeler, Shallo, and Rotunda, 1982, as well). The importance of aerial perspective is that it becomes a prototypal depth cue when depth cues are considered as natural signs or indicators of shape. The study of aerial perspective illuminates the possibility of perceptual theory <u>without</u> primary qualities. No geometric invariant may ever be singled out that has a necessary connection with perceived form. There may not be a basic level of description for the way things look, at least in geometrical terms. Perhaps <u>all</u> cues to depth and shape function on the model of aerial perspective. That is the alternative postulated in the theory of depth cues.

The plausibility of such a program may be hard to imagine for those (among whom I am included) who have been immured in the tradition that linear perspective is the only real indicator of depth. There is a subtle connotation to the very terms "cue" and "sign" that indicates the fixedness of this idea. "Depth cue" is a misnomer, because, counter to the purposes of the theory of depth cues, the term presupposes that the interpretation of cues is secondary to the immediate perception of shape. The usage implies a "contrast with inspection of the item itself" (Austin, 1961, p. 75), that is, "depth cues" sound like that with which one makes do when one is not seeing depth directly. Yet the claim made about depth cues is that this is how one sees depth and form. "Sign" has the same connotation as "cue": signs, like traces, are what one has when the object itself is absent. "When we talk of 'signs of a storm', we mean signs of an impending storm, or of a past storm, or of a storm beyond the horizon; we do not mean a storm on top of us" (Austin, 1961, p. 74). A sign for depth is meant to be different; these signs have prior place in the theory.

Linear perspective does not have pride of place in the theory of depth cues. Aerial perspective may serve as a reminder that the study of depth cues can be broadened in other directions. The characteristics of aerial perspective, by which it is a model for visual signs, are these:

1. There is no necessary connection between the sign and the quality it signifies (aerial perspective depends upon the composition of the air, and the seasons of the year), and

2. The physical law that is reflected in the regularity, that is, the sign, need not be invoked to explain perception of the concomitant quality (it has not been proposed that implicit knowledge of gaseous density or of the scattering of light is invoked when depth is seen by aerial perspective).

The theory of depth cues differs from the invariance hypothesis in these two claims. Since the invariance hypothesis can be construed as a foundational program that seeks to establish the basic congruences of optical laws as primary qualities for vision, the difference is crucial.

The experiments provide evidence that projective equivalence is not seen uniformly. The sense in which it can be said to be seen might be that in which we can say that depth is heard. Yet this assertion runs counter to established accounts of shape constancy. For instance, Rock bases his theory on a projective regularity of optics, the inverse-square law. He supposes the inverse-square law to be a sufficient basis for a theory of shape constancy. His supposition is essentially the one with which we began: shapes appear the same when the laws of optics would allow that their appearances had been occasioned by objects of the same shape.

What could be wrong with such a minimally adventurous claim? It is just possible that the wrong sort of explanation has been sought for the phenomena of shape constancy. The invariances given in laws of optics are not, it seems, the basis for a satisfactory account of perception. The laws of optics are part of a description of the causes of perception. Perhaps a description of the causes of perception is insufficient to a theory of shape constancy; perhaps the best explanation of shape constancy is not a causal explanation. I would affirm that there are causal processes in vision, and there are useful causal explanations in the study of vision. "But not all our questions about perception are causal questions; and the proffering of causal answers to non-causal questions leads to inevitable dissatisfaction, which cannot be relieved by promises of yet more advanced causal answers still to be discovered" (Ryle, 1956/1975, p. 201). The invariance hypothesis is a stipulation about the causal conditions of perception. Under the hypothesis, the geometric conditions of vision are stipulated to be a sufficient explanation of phenomena of shape. Various arguments and results have been presented against this hypothesis. I hope that they motivate acceptance of a simple principle for vision research, that is: A description of the geometrical and causal conditions of vision is not a sufficient basis for a theory of visual shape constancy. The results of the experiments do not compel this conclusion. It is hard to imagine what ever would. Still, the results can be taken as one reason to accept the principle.