A NONLINEAR SLIP-RING PRIMITIVE FOR GENERALISED ELECTRIC MACHINE THEORY

by

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A thesis submitted to the Faculty of Graduate Studies and Research in partial fulfilment of the requirements for the degree of Doctor of Philosophy

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Montreal, P. Q.

1964

ACKNOWLEDGEMENTS

The author wishes to express his gratitude to Dr. T. H. Barton who supervised the research and rendered invaluable assistance and encouragement during the course of the work. To no lesser a degree, the author is indebted to his colleagues and to the technical staff of the Department of Electrical Engineering for their continuing readiness to assist.

The National Research Council, Ottawa, provided the funds without which the project would not have been feasible.

CLAIM OF ORIGINALITY

To the best of the author's knowledge, the following contributions are original. The order of listing is the order of their appearance in the body of the thesis.

(1) Proof that nonlinear as well as linear electric machines may be represented by a unique, symmetric inductance matrix.

(2) Interpretation of Weber's 1931 eddy current theory in the complex frequency domain and consequent experimental verification.

(3) Derivation of an expression for the shaft torque of a nonlinear electric machine from the flux linkage vector and the inductance matrix.

(4) A method of synthesizing stationary circuits and corresponding new canonic forms of two-element-kind networks (especially R-L networks).

All of these, taken together, constitute the establishment of a quite general primitive machine that is not in any way restricted to linear behaviour and that is believed original.

CONTENTS

INTRODUCTION

Since the invention of calculating machinery, the attention of mathematicians has been turning from the search for methods of integrating differential equations to the investigation of theorems of existence. Recent advances in stored-program digital computer technology have contributed to the acceleration of this trend and quickened interest in numerical analysis as well. Numerical techniques now exist for the integration of all ordinary and many partial differential equations; but it is clearly imperative to know beforehand whether the sought solution exists at all. The need for existence proofs is frequently neglected by engineers on the supposition that any physical problem must surely have an answer, most likely even a unique one. As any physical problem must first be described by a conceptual model and only the model transformed into a mathematical problem, this simple view is unfortunately not valid. There exist techniques in many fields for checking the correctness of the mathematical description of models; but the creation of the models themselves is at best an uncertain art.

For about a generation, electric machines have been modelled by most analysts in the manner proposed by Kron. As is set forth in greater detail in the following chapters, this method first of all supposes the machine in question to possess linear iron. Although this assumption is obviously an inaccurate one, it has nevertheless been widely accepted as a necessary evil; the mathematical complexity of nonlinear models is a high price to pay. While the superstructure arising on this basic assumption has been examined in detail by numerous authors, both with a view to methods of solution and to the philosophical validity of the model (Jones' work on commutation is a brilliant example), the effect of nonlinearity on the theoretical validity of the model has not to date been examined. That is, no guarantee has been shown to exist of uniqueness or even existence of solutions of the describing equations of machines without linear iron. In order to remove this fundamental doubt, the present thesis seeks to establish that Kron's linear slip-ring primitive machine is in fact a particularly simple special case of a more general, nonlinear slip-ring primitive. Since Jones and Kurbasov have independently and in different ways shown that a commutator machine may (at least in principle) be thought to consist of a slip-ring machine plus a switching circuit, the slip-ring machine problem is surely the one to solve first.

The problem of stationary nonlinear networks is dealt with in numerous works. As a model for an electric machine differs from such networks primarily in being nonstationary, the major goal in the development of such a model must be the discovery of an expression of the torque produced by the machine. Assuming a slip-ring machine with arbitrary single-valued magnetic characteristics, such an expression is developed and verified. Kron's linear primitive is shown to be a special case of this more general machine. As the usual treatment of machines assumes that any electric machine may be modelled by means of a finite number of equivalent coils, an assumption at variance with the known behaviour of solid iron parts, a new representation of the eddy current behaviour of unlaminated machine portions is proposed and both theoretically and experimentally justified. As a byproduct of the latter investigation, an original method of network synthesis is found to arise. This synthesis method leads to a new set of canonic forms of two-element-kind one-port immittance functions, as well as making some four-terminal network functions easy to realise.

An attempt is also made to express the fundamental quantities associated with a machine in terms of its co-energy; an apparently novel physical interpretation of co-energy results.

I

ANALYSIS OF ELECTRIC MACHINES: THE STATE OF THE ART

1. Classical Methods. Electric machines are not claimed as the exclusive invention of any one man, but are linked with a long list of names including the greatest scientific minds of the nineteenth century. The first rotating machine that may be described as being of fundamentally the same design as a modern commutator machine, however, appears to have been constructed by William Sturgeon in 1832, and displayed and described to the Royal Society in 1836, according to Sturgeon's own account[1]. Joule[2] corroborates this story; Sylvanus Thompson[3] gives the date as 1835. Apparently the paper was not published by the Royal Society but subsequently appeared in Sturgeon's own journal, the *Annals of Electricity*[4]. The ring armature, used until the end of the century, was first developed by Pacinotti in 1860 and used in its final form by Gramme in 1871[5]. In a series of articles in 1885-6 Kapp, followed by Drs. J. and E. Hopkinson in 1886, applied the then rapidly developing theory of magnetic circuits to the design of electric dynamo machines[6]. They may thus be said to have founded the magnetic-circuit-plus-electric-circuit school of machine analysis. This method of analysis held uncontested sway for nearly half a century. During this time naturally a large number of contributions from both practising engineers and research scientists was added to the original theoretical structure. About the end of the nineteenth century, acceptance of alternating current by electric utilities broadened. As a result, the theory had to be widened to accommodate the new a-c machines. Many of the innovations of the early twentieth century are associated with the names of Steinmetz[7], Blondel[8], and others, who may be said to have laid the foundations upon which Park[9], Doherty and Nickle[10], and their followers built the elegant and

intricate structure that machine analysis had become by 1930. Most of the techniques used by practising designers today were developed during that period.

In view of the very large body of experience with the numerous classical techniques, that approach will probably remain the preferred one for design for some time. Classical works, such as those by Langsdorf[11] or Concordia[12], are as useful as ever for gaining a physical insight into machines; Still and Siskind [13] is still the authority for the student of design. Perhaps even more important is the fact that many contemporary research papers and monographs employ the classical approach. The brief but violent flurry of interest about a decade ago in d-c machine performance under solidly short-circuited conditions[14, 15, 16], for example, viewed the problem in its classical formulation. The commutation studies of Sketch, Shaw, and Splatt[17] or the synchronous machine investigations of Sudan, Manohar and Adkins[18] and of Nikiforovskii[19] are more recent examples from other fields.

It might be said that classical theory has consisted of two phases, supplanting but not replacing each other. The first phase involved d-c machine analysis, based on actual magnetic circuit calculations; the second, growing in response to the introduction of a-c machinery, substituted the notions of reactance and reflected load resistance. D-c machine design to this day is based on the concepts of the first phase. The substitution of reactances for fluxes as analytic entities, during the quarter century from Blondel to Park, obviated the need for magnetic circuit analysis and created the notion of an equivalent circuit. It is worth noting that the notion of reactance requires magnetic linearity, or equivalent linearity; the price paid for elegance in analysis is then a loss in accuracy of representation.

2. The Second Century. In 1930, nearly a hundred years after Sturgeon first operated his electrical engine, Kron published[20] the next logical step after Park's machine reactances. Giving the sets of reactances geometrical instead of circuit significance, he viewed the reactances as dyadic operators in a vector space defined by the machine currents as axes. An interesting parallel is that the critics of Kron felt much the same about his work as the critics of Sturgeon a century earlier: both were felt to misuse established terminology and techniques in the crassest manner. While Sturgeon was berated for his apparently unclear grasp of scientific terminology,[21] Kron ran into difficulties with mathematical usage and conventions[22]. Both were recognised by their colleagues to possess genius, and both produced vast quantities of printed materials; Kron, of course, is still doing so. Bewley[23] feels that much of the work of the nineteen-thirties must in fact be credited to Pen-Tung Sah; while this may be a just point, few will deny the magnitude of Kron's contribution.

For nearly a decade, Kron remained the prophet at home; engineers did not understand his mathematics, and mathematicians were not prepared to accept his methodology. A line of more than two dozen articles and two books, all published in that first decade, however, won a few converts; and by the end of another ten years, numerous authors had espoused Kron's ideas. Some, it would appear, were willing to court the wrath of the mathematical world by employing a terminology resembling Kron's, like Adkins[24]; others flirted with it more coyly, like Tustin[25]. Yet others, for example Fitzgerald and Kingsley[26], adopted what might be called a semiclassical approach, stressing the unity of physical phenomena but adhering to the classical mathematical descriptions. Research work of the nineteen-fifties tended to follow suit. For example, Koenig[27] pictures transient processes in machines as being explainable by way of machine inductances, but he dismantles the inductance matrix in order to solve his equations in the classical manner. Similarly, Litman[28] and Riaz[29], each in his own way, formulate what might be called semiprimitive machines.

By the mid-fifties a sufficient number of disciples had been won to the new cause to generate a substantial volume of purely theoretical contributions. Of interest in the present context are primarily the papers of Lynn[30], Tang and Cosgriff[31], and Yu[32, 33], as well as the books by Gibbs[34], Bewley[23], and White and Woodson[35]. As the middle sixties approach, this output naturally enough continues to grow; an excellent selective bibliography illustrating present trends is given by Shepherd in his discussion of Saunders' paper[36].

<u>3. Troublesome Techniques: Classical Theory Today.</u> The difficulties encountered in analysing machines by classical means are much the same today as in the past; the theory has grown to maturity and the work currently being done, apart from experimentation, is for the most part in refinement of existing technique or re-interpretation for pedagogic purposes. As the mathematics involved can take care of a considerable amount of nonlinearity, steady-state operation tends to be stressed. A fairly active field of late has been that of d-c machine transient analysis; a review of this subject is given by Hindmarsh in a series of articles[37], and examples of applications are furnished by Smith[38] and Ageev[39]. Other areas of activity are, at the present time, induction machines with controlled-rectifier feed (especially in the USSR), and doubly-fed induction motors. The initiative in these areas is readily seen to originate in the field of industrial automatic control.

The first and major difficulty encountered in the classical approach is of course the great complexity of modern machines. This complexity is reduced by various transformations in Kron's approach, but persists in classical theory. On the other hand, the latter preserves to some extent the nonlinearity of the real world and even exhibits its effects usefully; the latter case is particularly true if analysis is conducted graphically, as advised by Ahlquist[40]. Unfortunately the complexity of graphical analysis becomes enormous for, say, a doubly-fed induction machine, an amplidyne, or a Schräge motor.

Secondly, as Hindmarsh notes[37], classical theory is ill equipped to deal with the problem of eddy currents in the solid portions of machines. The quite considerable concern with this problem is reflected in the large number of articles devoted to the subject. The papers by Pohl[41, 42] are useful examples, as they first develop an a-c eddy current theory and then extend it to the case of stepfunction magnetisation. Kesavamurthy and Rajagopalan[43] employ a somewhat similar analysis but assume a sharply saturating magnetic material. More recently, Karasev[44] has given a solution for a specific useful case, that of a lifting electromagnet.

The article that could well be termed fundamental on the linear eddy-current problem is that published 1931 by Weber[45]. Subsequently, some observations were given by Wagner[46] and others. No remarkable progress in applying the analysis to nonlinear materials was made until twenty years later, when Dunaevskii[47] applied a piecewise-linear approach to systems with solid iron components. As Dunaevskii did not attempt proper matching of the eddy currents at the segment boundaries, his method is crude but possesses the virtue of simplicity. Experimental results are given by Brockman and Linkous[48] and compared with predictions by Dunaevskii's technique. Some very recent work on this problem has been reported by Karasev[44].

Commutation, finally, is the third major stumbling block. In steady state operation, it has traditionally been treated as a separate problem, quite distinct from other aspects of machines. With treatment of transient problems required, however, it has had to be brought into the general framework of machine analysis. Alger and Bewley[50] treat it on a quite conventional basis, but clarify much that had previously been obscure; Sketch, Shaw and Splatt[17] have provided experimental means for observing the actual commutation process waveforms. A new, and perhaps most illuminating, view of commutation is that proposed by Nacke[51]: he views it as a problem in handling armature leakage flux. The most significant recent contribution to classical commutation theory, however, is undoubtedly the pair of articles by Kurbasov. In the first article[52], the view is advanced that commutator action is an energy exchange process whereby a d-c machine transfers energy from one winding to another, storing it in the rotor inertia in the interim. The second article[53] employs this theory to make up a design procedure for interpoles. Both the viewpoints of Nacke and Kurbasov give a physically comprehensible explanation of a complicated phenomenon, without obscuring the process by mathematical complexity or chains of dubious approximations. They must therefore be considered important additions to the electric machine art.

4. <u>Compromises with Reality: The Modern View.</u> Much like the older approach, the new theory of Kron has its shortcomings. Until very recently, the most important of Kron's transformations, that between slip-ring and commutator primitives, was justifiable only as a mathematical device, i. e. an application of Floquet's theorem regarding the representation of matrix equations with periodically time-varying coefficients[54,55] and their solutions. In other words: although the rotation transformation of Kron's slip-ring primitive did yield an impedance matrix indistinguishable from that of his commutator machine, and vice versa, there was no physical reason to believe that such a transformation did in fact correspond to replacement of slip rings by a commutator. The coincidence of the impedance matrices might thus well be fortuitous. This difficulty was removed by Jones[56] in a brilliant paper, in

which he showed that a slip-ring machine whose rotor connections were periodically altered did indeed possess the mathematical properties required. This theory was further refined and re-presented two years later in collaboration with Barton[57]. It must be noted, however, that this theory of commutation shares the linearity postulate with the remainder of the unified machine theory, as superposition is required for the proof. Since Jones' proof relies on the cancellation of certain transformer voltages by generated voltage components to achieve zero elements in the impedance matrix, the extent to which the approximations involved in his development are valid becomes a critical question. This is so especially because the voltages whose cancellation the proof depends on may easily be of the order of ten times normal machine terminal voltage, making the entire proof hinge on the small difference of large numbers. Because of the linearity assumption, these large numbers can at best be known approximately for real machines, for the concept of an inductance matrix is not applicable to nonlinear systems.

At least equally troublesome is the production of numbers suitable for analysing real machines by Kron's methods. These require the analyst, as is well known, to be in possession of an inductance matrix for the machine in question; hence methods of finding equivalent (in some sense) linear inductance values for machine windings must be developed. The first such measurements to be performed on modern machines were proposed and carried out by Snively and Robinson[58] with mixed success. Saunders[59] subsequently performed similar measurements and claimed good results, although his data would appear to be unusual, and not representative, in the light of further measurements by Koenig[27] and Prescott and El-Kharashi[60], as well as Jones[56]. The latter experimenters, independently and concurrently, came upon a ballistic bridge technique for measuring the flux linkages of a circuit directly; subsequent work by Barton[61] and others has been based on this method of measurement. The influence of eddy currents on measurements made in this way disappears, and no linearity assumption is involved. Even if linearity is assumed, Kron's theory does not take account of the space harmonics of mmf and flux around the periphery of a real machine. Amendments to the usual theory have, however, been proposed by Dunfield and Barton[62], with excellent results reported.

At the root of all the trouble with the new theory, now that the commutator-transformation difficulty seems taken care of, lies the linearity assumption. It is demonstrable that the unified theory provides a rapid and elegant method of analysis, but it cannot at the present provide useful numerical answers; an inductance matrix is required for Kron's analysis, but has not to date been defined for nonlinear machines. In the strictest sense, then, the new theory is not in error in analysing real machines; it is rather not defined at all.

It is the object of the following chapter to demonstrate that the linearity assumption underlying Kron's slip-ring primitive is in fact not required, although it does make the mathematics of analysis manageably simple. That is, it is proposed to show that there always exists a unique describing matrix for any slip-ring machine which may be regarded as the matrix of machine inductances. As this matrix will then be shown to possess those critical properties which unified machine theory requires of an inductance matrix, Kron's linear machine is justified as merely a numerically simple approximation to the more general nonlinear machine. It is hoped that this development will sweep away the remaining major philosophical objection to generalised machine theory, and establish beyond doubt the validity of that extremely powerful analytic tool.

II

INDUCTANCE: A FUNCTION MATRIX

1. The Unified Theory of Machines. To appreciate the importance of the inductance matrix in Kron's theory of machines, a very brief review of the theory itself is probably in order. The details of the theory as applied to specific machines or problems may be found in books and papers already referred to. The present concern is not so much with the technique of analysis as with the nature of the necessary fundamental quantities.

A word on notation is first necessary. It is usual since the first papers of Kron were published to employ the two notations used in these pioneer works: the usual notation of matrix analysis whenever it is desired to write out the numbers or functions making up a quantity, and the conventional notation of tensor analysis whenever it is not necessary to specify the detailed functions. Thus, the statement that the flux linkages λ of the coils of a two-coil system are linearly related to the coil currents by the inductance coefficients, may be written in the matrix manner

$$\begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix} = \begin{bmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{bmatrix} \cdot \begin{bmatrix} i_1 \\ i_2 \end{bmatrix} \qquad (201)$$

or, more briefly, in the Einstein notation

$$\lambda_\alpha = L_{\alpha\beta}\, i^\beta \qquad (202)$$

where the summations, according to conventional usage, are understood in the repetition of the dummy index β. In the following, the essentially geometrical nature of the argument is occasionally underscored by employing the vector-and-dyad notation used by Wills[63]. In this notation, the above would be written

$$\overline{\lambda} = \overline{\overline{L}} \cdot \overline{i} \qquad (203)$$

The advantage of this latter representation lies in its stress on the geometrical significance of the quantities; however, it is less convenient to manipulate than the tensorial notation. In the latter, the order of multiplications is specified by the indices, and the operation of multiplication is formally commutative; in the vector notation, it is not.

In terms of the tensor notation, the machine theory of Kron begins with equation (202). That is to say, Kron views any machine as a collection of coils possessing inductance and resistance; the resistance, in most cases, is merely a physically necessary nuisance. The quantities of primary interest are the coil flux linkages and the coil currents, since all terminal quantities are intimately related to them. Thus, all the machine terminal voltages must be given by

$$e_\alpha = \frac{d}{dt}\left[L_{\alpha\beta} i^\beta\right] \tag{204}$$

or, rewriting,

$$e_\alpha = L_{\alpha\beta}\frac{di^\beta}{dt} + \frac{dL_{\alpha\beta}}{dt} i^\beta \tag{205}$$

Similarly, machine stored energy is easily shown to be

$$U = \frac{1}{2} i^\alpha i^\beta L_{\alpha\beta} \tag{206}$$

and, assuming the system to have one mechanical degree of freedom, a rotational one, the torque may be shown to be given by

$$\Gamma = \frac{1}{2}\frac{dL_{\alpha\beta}}{d\theta} i^\alpha i^\beta \tag{207}$$

The above equations completely specify the electrical behaviour, under any set of circumstances, of a linear slip-ring machine. The machine is assumed to consist of an unrestricted but finite number of coils n, each with arbitrary source currents or voltages. Any real linear slip-ring machine may be constructed out of this primitive by suitably interconnecting its windings. Geometrically speaking, the equations define one vector and two scalars in terms of the currents: the flux linkage set forms an n-vector and the torque and stored energy are scalar quantities. Imagining a cartesian coordinate system with each axis representing one coil current, i. e.

a cartesian n-space, the flux linkages are a vector position function defined by the inductance dyadic operating on the position vector. The two scalars are seen to be quadratic forms also related to the inductance dyad. Interconnection of the machine windings restricts some of the currents to have specified relationships to others, and thus amounts to choosing a subspace out of the general n-space.

In steady-state operation, the behaviour of the machine may be thought of as the tracing of a closed path by a point in currents space--or, more precisely, in the subspace defined by the winding interconnection. By adopting a coordinate system rotating at some velocity about each of the axes embedded in the subspace, the point of operation may be thought to describe a less complicated path in the moving coordinate system. If the path should be such as to trace out a circle or other figure on the surface of a sphere (or, more generally, hypersphere), the path may often be reduced to a point by a quite simple coordinate transformation in which axes rotating at constant velocity are introduced. Simply stated, then, a large variety of machines problems may be transformed into problems in d-c machines by adopting movable coordinate systems in the current n-space. As a practical application of more than routine interest, transient problems in synchronous alternators have been solved by adopting accelerating coordinate systems[64].

Clearly most of the transformations involved must first of all transform the inductance matrix. It is therefore essential to possess a usable inductance matrix to solve any problems at all. In order to use the rich mathematical resources of differential geometry in electric machine analysis, the inductance matrix should preferably be a tensor under a broad class of coordinate transformations.

2. Flux Linkages In Conservative Systems. The foregoing theory is quite clearly of no use in cases in which the inductance matrix cannot be written down as a set of constants. As pointed out in the foregoing, this is unfortunately true for nearly all real problems. It is thus of interest to try to extend this theory to the nonlinear case.

In view of the known behaviour of real machines, it is not unreasonable to try to rewrite Kron's theory for the case of a conservative system in which the flux linkage vector must be a single-valued position function. It is of course realised that of the two common nonlinear phenomena, saturation is therewith taken into account, while hysteresis is not. The omission is not quite so serious as might at first glance seem, however. In many situations, the observable effects of hysteresis can be accounted for by introducing an equivalent resistive loss; in many others, hysteresis can afford to be ignored. It is necessary to keep in mind that while it is in the final analysis saturation that limits the possible design values of a machine, hysteresis is more nearly of nuisance value in most problems. An extension of the theory able to deal with hysteresis will be proposed later, although the complexity of the method is such as to render it impractical.

Under the assumption, then, that the flux linkage vector is single-valued, the basic statement

$$\lambda_\alpha = L_{\alpha\beta}\, i^\beta \tag{202}$$

may be taken as the definition of inductance. For an n-coil system with n independent currents and n flux linkage components, the inductance is clearly an n-by-n square matrix. Other definitions of course might be chosen instead; for example, equation (204) might be rewritten

$$e_\alpha = \left(L_{\alpha\beta} + \frac{\partial L_{\alpha\gamma}}{\partial i^\beta}\, i^\gamma \right) \frac{d\, i^\beta}{dt} \tag{208}$$

and the quantity in parentheses, the coefficient of the time derivative of current, regarded as inductance. For a linear system, these two definitions are equivalent; for a nonlinear one, they are not. Thus, (208) shall be written

$$e_\alpha = \mathcal{L}_{\alpha\beta} \frac{di^\beta}{dt} \tag{209}$$

where a distinctive symbol is employed to distinguish from the inductance defined by equation (202). Both of these definitions are useful; that of equation (209) is taken by some writers as being fundamental[65, 66]. As long as all networks are required to be static, there is little to choose; for dynamic circuit analysis, however, it is desirable to exhibit the system flux linkages explicitly and equation (209) is at a disadvantage. Adoption of this definition would obscure the physical mechanism of voltage generation, as well as complicating the mathematics a little. In consequence, the word <u>inductance</u> as used in this thesis shall always mean that implicitly defined by equation (202). The quantity defined by equation (209) shall be referred to as the <u>dynamic inductance</u> or the <u>incremental inductance</u>. The latter name is derived from the rearrangement of equation (209)

$$d\lambda_\alpha = \mathcal{L}_{\alpha\beta}\, di^\beta \tag{210}$$

which is of the same form as equation (202) except that it relates incremental rather than total quantities. In a similar manner and for similar reasons, the inductance defined by equation (202) will occasionally for emphasis be termed the <u>total inductance</u>. Other definitions yet may be created; the basic properties of linear systems given by equations (206) or (207) may be regarded as definitions of inductance. Of course, of all these possibilities only one may be chosen as the definition; the remaining three will clearly not be true in the general case.

It is now necessary to investigate whether the definition adopted--or, for that matter, any definition--will yield a unique inductance matrix in the general nonlinear case. Single-valuedness, as discussed, will however be assumed.

3. The Continuity Requirement. The flux linkage vector may be taken to be a continuous vector function of position as well as a single-valued one; discontinuities may hardly be allowed, in view of the requirement of conservative system behaviour. It does not, however, immediately follow that the components of the inductance matrix need also be symmetrical and continuous. As a matter of fact, an inductance matrix neither symmetrical nor continuous is readily constructed.

Consider the linear two-coil system of equation (201). By the word linear, it is here meant that all components of the inductance matrix are constants:

$$\begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix} = \begin{bmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{bmatrix} \cdot \begin{bmatrix} i_1 \\ i_2 \end{bmatrix} \tag{201}$$

It is seen on brief examination that the description

$$\begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix} = \begin{bmatrix} L_{11} + \frac{1}{3} L_{12} \frac{i_2}{i_1} & \frac{2}{3} L_{12} \\ L_{21} & L_{22} \end{bmatrix} \cdot \begin{bmatrix} i_1 \\ i_2 \end{bmatrix} \tag{211}$$

yields precisely the same two algebraic equations as are incorporated in equation (201) and therefore describes the same system (although with needless complexity). It also satisfies the definition of equation (202) and is thus an equation with a valid inductance matrix on the right. The "inductance" components are not constant nor continuous, nor is the matrix symmetrical. From the point of view here adopted the constancy or position-dependence of any inductance component is a fortuitous matter, since any of the components must be permitted to vary with system currents in order to describe nonlinear phenomena. The matter of continuity, on the other hand, is something else again.

Quite clearly, there is a close interrelationship between continuity, symmetry, and uniqueness of the "inductance" matrix of any linear system. It is possible to create an infinite number of describing inductance matrices for the two-coil system discussed, merely by using the same technique as for the creation of equation (211). Only one of these matrices, however, will be continuous and symmetrical, the linear (i. e. constant) one of equation (201). In other words, unless continuity is insisted upon, a linear system may be thought to have no unique inductance matrix. In a similar manner, it may be argued that whatever the inductance matrix of a nonlinear system may turn out to be, another may always be created by the technique shown unless this method is prohibited by a continuity restriction. On the other hand, insistence on continuity of the inductance components is not in any way objectionable, in that the flux linkage vector has already been restricted to be continuous. No discontinuous inductance components can therefore be shown to be necessary.

It shall thus be assumed in the following that the flux linkage vector is a continuous, single-valued position function, and that the word <u>inductance</u> means a matrix of continuous functions that satisfies equation (202).

<u>4. A Theorem on Inductance.</u> Using the arguments laid out above, the following theorem shall now be shown to hold:

> If $\bar{\lambda}$ is a continuous, irrotational vector position function, and $\bar{\dot{\iota}}$ the position vector of any point in an n-dimensional Cartesian space, then there always exists a square matrix $\bar{\bar{L}}$ whose components are continuous position functions , and which will satisfy
>
> $$\bar{\lambda} = \bar{\bar{L}} \cdot \bar{\iota}$$
>
> as well as be symmetrical and unique.

The proof necessarily consists of several parts, since not only the existence, but also symmetry and uniqueness must be proven.

The conservative character of the system being represented requires that the flux linkage vector be irrotational. Although the proof is simple, it does deal with physical concepts later encountered. It is therefore introduced here as a lemma.

<u>Lemma.</u> The flux linkage vector of a conservative system is irrotational.

Consider a system being moved from a state S to a state S'. During the motion, the vector of system voltages (the time derivative of the flux linkage vector) is

$$\bar{e} = \frac{d}{dt}\bar{\lambda} \tag{212}$$

The system input power must therefore be

$$P = \bar{e}\cdot\bar{i} = \bar{i}\cdot\frac{d\bar{\lambda}}{dt} \tag{213}$$

so that the stored energy change between the two states is

$$\Delta U = \int P\,dt = \int_S^{S'} \bar{i}\cdot d\bar{\lambda} \tag{214}$$

This latter expression may be integrated by parts:

$$\Delta U = \left[\bar{\lambda}\cdot\bar{i}\right]_{S'} - \left[\bar{\lambda}\cdot\bar{i}\right]_S - \int_S^{S'}\bar{\lambda}\cdot d\bar{i} \tag{215}$$

Let

$$\Delta T = \int_S^{S'}\bar{\lambda}\cdot d\bar{i} \tag{216}$$

Then equation (215) becomes

$$\Delta U = \Delta\left[\bar{\lambda}\cdot\bar{i}\right] - \Delta T \tag{217}$$

or, in the special case of state S being the origin,

$$U + T = \bar{\lambda}\cdot\bar{i} \tag{218}$$

The quantity T may be recognised as an n-space generalisation of the quantity Cherry[67, 68] and Millar[69] refer to as the co-energy of a system. As their arguments are largely restricted to one-dimensional current spaces, their definitions and views on the nature of the co-energy are not the same as those presented here. However, examination of the several theorems on co-energy, especially by Cherry[67], indicates that their validity is not based on any particular number of dimensions, and is therefore unaffected. Equations (216) and (218), however, suggest an interpretation of the physical nature of co-energy radically different from Cherry's or Millar's; they may be rewritten to yield

$$\overline{\lambda} = \nabla T \tag{219}$$

From this, the irrotationality of the flux linkage vector immediately follows. In equation (218), there appear two unique scalar position functions, and the co-energy; the co-energy must therefore also be a uniquely defined scalar position function. Equations (216) and (219) identify it as a true state function. From the latter,

$$\nabla \times \overline{\lambda} = \nabla \times \nabla T \equiv 0 \tag{220}$$

completing proof of the lemma.

To proceed with the proof of the theorem itself, equation (220) may be rewritten

$$\frac{\partial \lambda_\alpha}{\partial i^\beta} = \frac{\partial \lambda_\beta}{\partial i^\alpha} \tag{221}$$

regardless of the dimensionality of the space[64, 70]. Note that the incremental inductance of equation (209) may be written

$$\mathcal{L}_{\alpha\beta} = \frac{\partial \lambda_\alpha}{\partial i^\beta} \tag{222}$$

so equation (221) amounts to a statement of symmetry of the incremental inductance matrix. As the reciprocity of incremental quantities is a well known experimental fact, and in fact underlies the use of linear analysis for physical problems, this is not new.

Substitution of the original defining equation (202) for inductance into equation (221) results in

$$L_{\alpha\beta} - L_{\beta\alpha} = i^{\gamma}\left[\frac{\partial L_{\beta\gamma}}{\partial i^{\alpha}} - \frac{\partial L_{\alpha\gamma}}{\partial i^{\beta}}\right] \qquad (223)$$

It will now be shown that this equation cannot be satisfied by any asymmetric inductance matrix, i. e. that there cannot in fact exist any such matrices. As the inductances form a square matrix, they may be written as the sum of a symmetric plus an antisymmetric (skewsymmetric) matrix. Let the symmetric matrix be denoted by S and the skewsymmetric one by A:

$$L_{\alpha\beta} = S_{\alpha\beta} + A_{\alpha\beta} \qquad (224)$$

It is immediately seen that the symmetric portion satisfies equation (223) identically. Thus, on substitution of equation (224) into equation (223) there results

$$2A_{\alpha\beta} = i^{\gamma}\left(\frac{\partial A_{\beta\gamma}}{\partial i^{\alpha}} - \frac{\partial A_{\alpha\gamma}}{\partial i^{\beta}}\right) \qquad (225)$$

The antisymmetric matrix may next be expressed in terms of an associated vector or column matrix. Let

$$A_{\alpha\beta} = \frac{\partial V_{\alpha}}{\partial i^{\beta}} - \frac{\partial V_{\beta}}{\partial i^{\alpha}} \qquad (226)$$

This equation is satisfied if the associated matrix is defined so that

$$\frac{\partial V_{\alpha}}{\partial i^{\beta}} = \frac{1}{2}A_{\alpha\beta} \qquad\qquad \frac{\partial V_{\beta}}{\partial i^{\alpha}} = -\frac{1}{2}A_{\alpha\beta} \qquad (227)$$

which can always be done. In this way, we may write down n^2 differential equations defining the n^2 partial derivatives of the vector (or column matrix) v. As all of these functions are restricted to be single-valued and continuous, the components of v may be solved for. One possible way of making them up is to assume each component to be a product of several factors, each a function of only one coordinate, and carry out the solution by separation of variables. As this procedure is perfectly possible (although there is no need actually to perform it here), equation (226) is verified. Using this way of representing the antisymmetric part of the inductance matrix, equation (225) may be written

$$2\left(\frac{\partial V_\alpha}{\partial i^\beta} - \frac{\partial V_\beta}{\partial i^\alpha}\right) = \left(\frac{\partial^2 V_\beta}{\partial i^\alpha \partial i^\gamma} - \frac{\partial^2 V_\alpha}{\partial i^\beta \partial i^\gamma}\right) i^\gamma \tag{228}$$

Recombining terms, this equation in its turn may be written

$$\left(1 + i^\gamma \frac{\partial}{\partial i^\gamma}\right) A_{\alpha\beta} = -A_{\alpha\beta} \tag{229}$$

For physical, or rather geometrical, interpretation, a pure vector notation is helpful. The vector equivalent of equation (229) is seen to be

$$(\bar{i} \cdot \nabla)\, \bar{A} = -2\bar{A} \tag{230}$$

which may be rewritten

$$\frac{d A_{\alpha\beta}}{d|i|} = -2\frac{A_{\alpha\beta}}{|i|} \tag{231}$$

where the differentiation is carried out in the radial direction. The legitimacy of this operation of course rests on the possibility of converting a Cartesian n-space into one described by hyperspherical coordinates (a radius vector and n-1 angles); a method of so doing for any dimensionality n is shown in Appendix B.

In words, equation (231) requires as its solution a set of functions each of which is continuous in the coordinates. Now the equation is satisfied by functions of the form

$$A_{\alpha\beta} = \frac{f(\theta_1, \theta_2, \ldots, \theta_{n-1})}{|i|^2} \tag{232}$$

which are the only possible nonzero solutions; but these are discontinuous at the origin. Discontinuity is, however, not allowed ex hypothesi. Hence it is necessary that

$$A_{\alpha\beta} = 0 \tag{233}$$

It is worth noting that in equation (232) the skewsymmetric portion of the inductance matrix turned out to be discontinuous, while in equation (211) the discontinuous quantity is one of the self-inductances. There is no contradiction involved here. In the development of equation (232) the self-inductances were all required to be continuous as part of the symmetric portion S of the inductance matrix (a skewsymmetric matrix cannot have any nonzero elements on the principal diagonal). Continuity is thus seen to be a necessary and sufficient condition of symmetry.

5. Uniqueness of the Inductance Matrix. Neither existence nor uniqueness of the inductance matrix have as yet been proven. However, these present no great difficulty; now that symmetry is proven, both existence and uniqueness may be shown to hold by solving the equations already developed for the inductance matrix components.

The precise method of solution is slightly complicated, but not difficult to follow. To begin, equations (224) and (233) assure the symmetry of the inductance matrix. As all the components are both continuous and differentiable, a scalar point function may always be constructed such that the inductance components form the set of second derivatives of such a function. Upon differentiation, the initial defining equation for inductance, equation (202), becomes

$$\frac{\partial \lambda_\alpha}{\partial i^\gamma} = L_{\alpha\gamma} + \frac{\partial L_{\alpha\beta}}{\partial i^\gamma} i^\beta \tag{234}$$

or, if the symmetry of the inductance matrix and the consequent complete symmetry of the triadic (3-way matrix) of its derivatives, is used to permit any rearrangement of indices of the second derivatives,

$$\frac{\partial \lambda_\alpha}{\partial i^\gamma} = \left(1 + i^\beta \frac{\partial}{\partial i^\beta}\right) L_{\alpha\gamma} \tag{235}$$

Again, in order to emphasize the geometric meaning, the same equation may be presented in a purely vector-and-dyadic notation:

$$\frac{\partial \bar{\lambda}}{\partial \bar{i}} = \bar{\bar{\mathcal{L}}} = \left(1 + \bar{i} \cdot \nabla \right) \bar{\bar{L}} \tag{236}$$

The differential operator operating on the inductance matrix,

$$\left(1 + \bar{i} \cdot \nabla \right)$$

will be referred to here as the radial increment operator. In Appendix C to this thesis, a proof is furnished that

$$\left(1 + \bar{i} \cdot \nabla \right) \frac{1}{|i|} \int_0^i \phi \, di = \frac{1}{|i|} \int_0^i \left(1 + \bar{i} \cdot \nabla \right) \phi \, di = 1 \tag{237}$$

that is, it is shown that the radial increment operator is the inverse of the radial averaging integral operator

$$\frac{1}{|i|} \int_0^i [\quad] \, di$$

Since this is so, the differential equations (235) and (236) have as their solutions

$$L_{\alpha\beta} = \frac{1}{|i|} \int_0^i \frac{\partial \lambda_\alpha}{\partial i^\beta} \, di \tag{238}$$

Physically, equation (238) says that the value of the total inductance at any point in current space is equal to the average of the incremental inductance values along a radial line drawn from the origin to the point under consideration.

As the inductance matrix is therewith shown to be calculable from experimental data for any point in the current space, it follows that it must exist. Since the calculation is the result of solving a family of differential equations with given boundary conditions and driving functions, the solution must be unique[71, 72]. Thus, the inductance matrix must always exist, and must have uniquely defined, symmetric components. The theorem is thus proved.

<u>6. Inductances of Realisable Machines.</u> Given a skillful experimenter with very large amounts of patience, the fluxes linking any coil of any electric machine are measurable for all points of a current space. Although a complete n-dimensional flux map of this sort is an arduous task, several versions of the Maxwell-Rayleigh ballistic bridge[56, 60, 62] have been used by experimenters to make measurements of this type. In principle at any rate, there is no objection to construction of such a multidimensional map. All possible current variables may be taken into account using such bridge techniques, except of course eddy currents flowing in the machine iron. It is at least in theory possible, though of course practically out of the question with present techniques, to reproduce these measurements for various yoke eddy current configurations, by use of one or another dynamic measuring method (e. g. Koenig's alternating-current bridges[27]). As shown in the next chapter and Appendix A, any solid iron may always be represented to an arbitrary accuracy by means of a multiplicity of short-circuited coupled coils on an incremental basis. By using equation (238), the corresponding varying total inductances may then be calculated, yielding the complete inductance matrix of the machine.

The experimental labour involved clearly precludes using the above method as a way of obtaining high-accuracy descriptions of machines. However, since it involves a model of arbitrarily high accuracy, it permits trading of labour for accuracy of description, and further assures that an inductance matrix, however inaccurate, exists.

III

EDDY CURRENTS IN SOLID IRON

1. Representations in Time and Frequency. It is evident from the foregoing that representation of nonlinear inductive systems in terms of an inductance matrix is legitimate--if not in all cases, then certainly in those of immediate practical interest, i. e. those in which hysteresis may be neglected and discontinuities in flux-current plots are not encountered. The development, however, relies on all windings being accessible at least in a hypothetical sense, and is thus not directly applicable to portions of machines not describable by lumped-circuit representations. The problem of flux establishment or change in a solid conducting medium of finite dimensions, i. e. the eddy current problem, has long been recognised to be a field rather than circuit problem. It is therefore not immediately amenable to the simple treatment above. The object of the following is to show that it is in fact possible to represent the fields involved as circuit elements, to any desired accuracy.

Considerable interest was shown in this problem in the late nineteen-twenties and early thirties. Important theoretical work was published by both radio engineers interested in high-frequency performance of inductors, for example Scott[73], and power engineers concerned with alternator performance under transient conditions, for example Weber[45]. Perhaps as a result of regarding transform calculus primarily as a mathematical tool without much physical significance, or perhaps because of overriding concern with immediate practical problems, neither group appear to have published any further interpretations of their work. The great difficulty of corroborative experimental work is of course likely to have been a contributing factor.

Weber's work, as already mentioned, is of special interest. In his June 1931 paper[45], he assumes a toroidal magnetic structure partly of solid iron, partly laminated, and part air gap. The exciting winding upon this structure is permitted leakage flux, the toroidal geometry serving merely as a convenient way of setting up a coordinate system. Weber assumes iron to be magnetically linear and analyses the field problem resulting from application of a stepfunction voltage to the exciting winding. The current and core flux are both solved for by first writing the Maxwell field equations, then Laplace-transforming them, and subsequently finding inverse transforms with great mathematical finesse. For the coil current, the solution given is

$$i = \frac{V}{R}\left[1 - \sum_{j=1}^{\infty} C_j\, e^{-p_j t}\right] \tag{301}$$

where i is the current, V the amplitude of the voltage step, R the coil resistance; C_j are dimensionless geometrical constants, and the p_j real positive numbers dependent on the dimensions of the system. Because Weber deals with a toroidal geometry, the constants are expressed in terms of Bessel functions; as pointed out in the discussion by Poritsky, however, as well as by Scott[73], the series of Bessel functions are replaced by series in circular or hyperbolic functions if the problem is solved in rectangular coordinates instead. It is in any case true that

$$\sum_{1}^{\infty} C_j = 1 \tag{302}$$

and that the p_j are monotonic increasing in j. Thus, equation (301) may alternatively be written as

$$i = \frac{V}{R}\sum_{1}^{\infty} C_j \left(1 - e^{-p_j t}\right) \tag{303}$$

To appreciate the significance of this result, at any rate for purposes of the present thesis, it is necessary to find the Laplace transform of equation (303). This is easily done:

$$I(s) = \frac{V}{s}\frac{1}{R}\sum_{1}^{\infty}\frac{C_j}{s+p_j} \tag{304}$$

Now the factor $\frac{V}{s}$ may be recognised as the transform of the exciting step voltage. Thus, dividing by this factor, the complex frequency admittance of the winding is given by

$$Y(s) = \frac{1}{R}\sum_{1}^{\infty}\frac{C_j}{s+p_j} \tag{305}$$

In view of the fact that Weber's assumed system is linear, this admittance cannot depend in any way on the system currents and voltages. It may therefore be regarded, as is the case for all linear networks[74], as a network function not in any way related to the fact that it was derived by means of a step excitation.

<u>2. Realisability of Terminal Admittance.</u> The problem of a circuit representation of the field problem of eddy currents, it is seen, is thus reduced to the problem of synthesizing a network whose driving point admittance conforms to equation (305). Such a synthesis is not difficult, as becomes evident upon examination of the properties of the admittance function Y(s). The most obvious property of this function is that the series defining it converges for all values of s except $s = -p_j$; for these values, Y(s) must diverge. Secondly, it is clear that Y(s) must converge to pure real values for all pure real values of s for which it converges, since the p_j are positive reals and the C_j real.

Consider now the behaviour of Y(s) in the neighbourhood of those values of s for which it does not converge. As s approaches one of these points, say $s = -p_k$, the limiting value of Y(s) (should there be one) is

$$\lim_{s\to -p_k} Y(s) = \frac{1}{R}\sum_{j=1}^{k-1}\frac{C_j}{p_j-p_k} + \frac{1}{R}\sum_{j=k+1}^{\infty}\frac{C_j}{p_j-p_k} + \lim_{s\to -p_k}\frac{C_k}{R}\left(\frac{1}{s+p_k}\right) \tag{306}$$

The limit of course does not exist. However, the first of the series on the right side of equation (306) is a finite series of finite terms anywhere in the neighbourhood of the singularity, and will thus always sum to a finite number. The second summation represents all the terms of order higher than the one being investigated. It may be shown to converge by considering a new summation (representing a new admittance function) identical to the one under consideration, except for omission of the k^{th} term. Since $s = -p_k$ is not a singular point of this new series, the new admittance function must clearly converge there. The first two terms of the right side of equation (306), then, are finite everywhere in the neighbourhood of the singularity under consideration; it is only the last term which does not have a limit. As the singularity is approached from the left along the real axis, this term grows in a negative direction, whereas on being approached from the right, the growth is in the positive direction.

Exactly the same argument may of course be applied to each and every singular point of the admittance Y(s). It follows then that, along the real axis, Y(s) is always proceeding from a large positive value towards a large negative one, as s moves towards the right from one singularity to the next. As Y(s) converges uniformly in each interval between two adjacent singularities, it must have a zero value somewhere in the interval between every pair of adjacent p_j, and this zero must lie on the axis of reals. In other words, the admittance function Y(s) must have interlaced poles and zeros along the real axis. It cannot have any poles in the right half s plane, since all of the p_j are positive. Examination of Y(s) shows it to approach zero for large positive real values of s. From the nonexistence of poles in the right half plane and the interlacing of poles and zeros, it follows that there cannot be any zeros in the right half plane either, except for the zero at infinity. Furthermore, existence of zeros not on the real axis requires the existence of additional poles somewhere[74, 75]; but as all poles must lie on the axis of reals, and have zeros between them, this is impossible.

The value of Y(s) at the origin must thus be either a pole or a positive real number. Application of the final value theorem of Laplace transforms to equations (304) and (304) yields

$$\lim_{t \to \infty} i(t) = \lim_{s \to 0} sI(s) = VY(0) \tag{307}$$

whence

$$Y(0) = \frac{1}{R} \tag{308}$$

which corroborates the expectation noted above, and implies that the first critical point to the left of the origin must be a pole.

Summarising, Y(s) may be shown to possess interlaced poles and zeros, all along the negative real axis, with the first critical point to the left of the origin a pole. It is seen to decrease monotonically with increasing real s. According to the criteria of Hazony[76] and Van Valkenburg[77], it is always possible to synthesize such a function using only resistive and inductive elements. It remains to show an example of such synthesis. Clearly, many possible ways may be devised of synthesizing any realisable network function, so it will not be possible to claim uniqueness for any representation.

<u>3. A Linear Synthesis Problem.</u> In view of its conveniently simple properties, synthesis of a network of suitable input admittance is not hard. The simplest possibility is suggested by the fact that Y(s) as given by equation (305) is al-

ready in partial-fraction form. This suggests a Foster synthesis by means of the second canonic form, yielding a network structure as in Fig. 301. However, this representation is not entirely satisfactory in the sense that it has solely mathematical significance. The physical significance of any one branch of the network in terms of the eddy currents in solid iron is at best obscure.

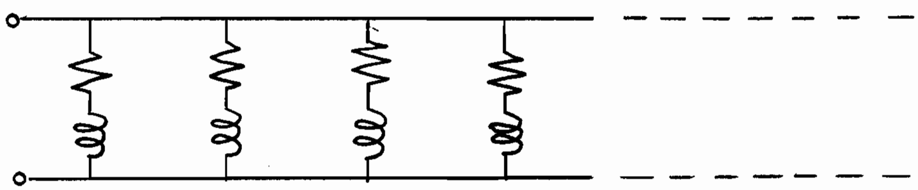


Fig. 301

Another, physically more fruitful, representation is easily constructed. Y(s) may be inverted to yield the corresponding impedance function Z(s). The value of Z(s) at the origin will be Z(0) = R, and the poles and zeros of Z(s) will coincide with the zeros and poles of Y(s) respectively. To synthesize this function, its zero immediately to the left of the origin may be removed to the origin, by defining a new impedance function Z'(s) by

$$Z'(s) + R = Z(s) \tag{309}$$

Now this new impedance function will still have interlaced real-axis poles and zeros, but will differ from Z(s) by having a zero at the origin. It is equally well a network function realisable by means of resistive-inductive networks. The corresponding admittance function Y'(s) has a pole at the origin and may, by the arguments of Van Valkenburg[77], be written as

$$Y'(s) = \frac{K_0}{s} + \sum_{j=1}^{\infty} \frac{K_j}{s+k_j} \tag{310}$$

Consider now the simple network shown in Fig. 302. Assuming the left-hand inductor to be of zero resistance, and the coupling coefficient between windings to be k, the impedance looking into the terminals is readily shown to be

$$Z_n(s) = \frac{R\,Ls}{R_2 + L_2 s} + (1-k^2)\,\frac{L_1 L_2 s^2}{R_2 + L_2 s} \tag{311}$$

and the admittance

$$Y_n(s) = \frac{1}{L_1 s} + \frac{\frac{1}{L_1}\,\frac{k^2}{1-k^2}}{\frac{R_2}{(1-k^2)L_2} + s} \tag{312}$$

It follows immediately that the new admittance function may be synthesized using a shunt combination of an infinite sequence of networks of the type of Fig. 302, as shown in Fig. 303.

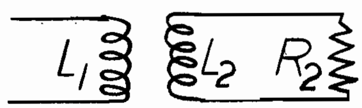


Fig. 302

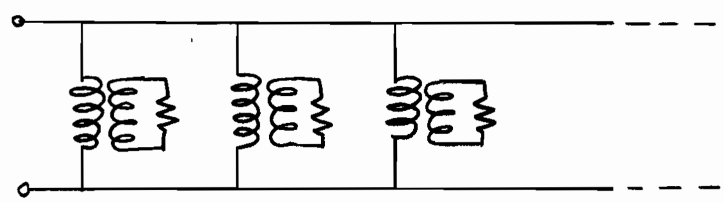


Fig. 303

Such a coupled-circuit representation of the field winding is much more satisfactory from a physical point of view than the Foster form of Fig. 301 because in the case of an electric machine, not only terminal voltage and current but also the flux should be capable of visualisation. The circuits of Fig. 303, unfortunately, are still not entirely satisfactory, for no single inductor anywhere in these networks represents the actual field winding.

It is shown in Appendix A of this thesis that any immittance function realisable with R-L elements is realisable also as a single inductor (possibly with series and/or shunt resistance added) to which are coupled numerous short-circuited windings. Such a system is shown in Fig. 304. The mutual inductances between the shorted windings may have zero or non-zero values; the realisation of an admittance function requires one fewer shorted windings than the admittance function has poles. In other words, for an admittance function as defined by equation (305), the realisation must consist of an infinite

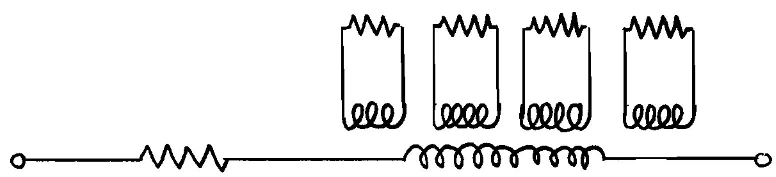


Fig. 304

set of inductors, one corresponding to each term of the infinite series except the first. It is to be noted that this lends a physical significance to the mathematical operation of truncating the infinite series. In a qualitative sense, the multiplicity of coupled coils may conveniently be viewed as representing the multiplicity of existing eddy-current paths in a real solid iron machine part.

4. Realisation of Transfer Functions. As a practical matter, not only the terminal quantities of any given winding of an electric machine are of interest but also the coupling terms that link the winding with others, i. e. the transfer immittances that tie together the various windings of a machine. After all, unified machine theory is in the final analysis merely the investigation of the behaviour of these transfer immittances under various coordinate transformations. The transfer immittances will of course be correctly represented if the fluxes shared by the several windings are correctly given by the relevant inductances. For the case of a magnetic circuit composed of part iron, part air, Weber[45] gives

$$\phi = \phi_0 \left[1 - \sum_{1}^{\infty} D_j \, e^{-P_j t} \right] \qquad (313)$$

The realisation sought is evidently that of Fig. 305. It is clear, then, that the achievement of the prescribed two-port network function is a matter of adjusting the mutual inductances between the newly introduced winding associated with the second port, and the already existing inductors, so that the required coefficients D_j are obtained. All of the exponential terms already exist, one being associated with each of the existing shorted windings. In terms of the frequency domain, all of the required poles already exist; the residues at them must be adjusted by adjusting mutual inductances.

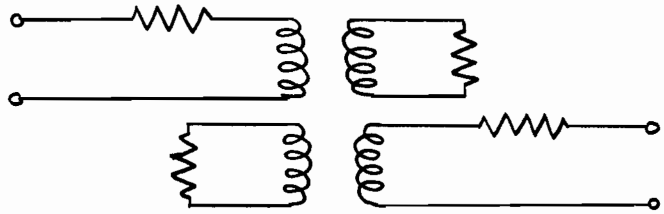


Fig. 305

It should be emphasized once again that although the paper by Weber, on whose results the present argument is based, employs the assumption of toroidal geometry, provision is made for a leakage flux of unrestricted magnitude. Thus the introduction of the additional coil into Fig. 304 to arrive at the two-port of Fig. 305 does not require the use of any perfectly coupled (Brune) coils. The representation is therefore reasonable from a physical point of view. Also, although the examples shown in Appendix A restrict some of the coupling coefficients to be zero, such a restriction is made for arithmetical simplicity and is not in general required.

5. Experimental Methods. Weber's paper, as already mentioned, included some sample calculations but no experimental results. The experiments reported by Scott, on the other hand, are not unlike those of other investigators in being of a specialised nature. In general, experiments dealing with the eddy-current

problem fall mainly into two categories: those of radio engineers, whose main concern has been with the effects on high-frequency inductors of thin, generally nonmagnetic, shields; and the machines engineers, whose main concern has been with time-domain response, and whose experiments consequently have run into well-nigh prohibitive difficulties.

The troubles involved in attempting to perform time-domain experiments are readily appreciated. Consider again equation (301) which gives the time-function current in an inductor with solid iron core portions. The infinite series involved is, in practice, a fairly rapidly converging one. A reasonable experiment might be the application of a step-function voltage, as required, to the winding, and measurement of the current resulting. Such experiments were in fact performed. The windings used for experimental purposes were, as might be expected, the windings of real machines. Figs. 306 to 309 show, respectively, the step-function response of a d-c machine armature and interpoles, of the same armature without interpoles, and the field windings of the same machine under two different conditions of excitation. In Figs. 308 and 309 the response of the field flux linking the armature (as indicated by armature open-circuit voltage) is also shown. It is reasonably evident from these pictures that the machine armature circuit behaves very nearly linearly, and extracting even the second term of the infinite series is a hopeless task (the responses in Figs. 306 and 307 are describable by single exponential expressions to within about 5%). In the case of the field winding, where a much greater amount of solid iron is coupled to the winding much more intimately than is the case for the armature, the second term can be extracted by careful graphical work, and the presence of the third is just discernible. The inherent nonlinearity of the situation can to some extent be taken into account by solving the transient problem by phase-plane techniques, ignoring eddy currents altogether. The difference between the prediction neglecting eddy currents and the experimental measurement must then be the sum of error plus the eddy current terms. Figs. 310 and 311 show the predicted curves and the measurements of Figs. 308 and 309 plotted on the same set of axes. It may be seen that the Weber theory holds at least qualitatively. However, it is equally evident that this approach is unlikely to lead to useful quantitative verification.

Instead of attempting to verify equation (301), however, the verification of equation (305), its frequency-domain counterpart, may be attempted. The advantage here is the disappearance of the restriction of step-function drive; a sinusoid may as well be substituted. Such quantitatively useful experimental verification of the multiple-coupled-coil theory must by implication also verify the original time-domain development.

Since the above theory requires linearity as a presupposition yet iron is inherently magnetically nonlinear, the experiment makes sense only on an incremental basis. Equipment laid out as in Fig. 314 was employed for the measurements. The winding to be investigated (the same d-c machine field as used for the time-domain measurements of Figs. 308-313) was fed by an amplidyne furnishing a direct voltage plus variable-frequency alternating voltage. In turn, the amplidyne was driven by a d-c source and a very low frequency generator connected to separate input windings. Measurements were made over a frequency range of about three and a half decades; at the high-frequency end of the range, i. e. above roughly five cycles per second, the amplidyne became quite useless. An alternator driven at low speeds and connected to the winding under investigation via large series capacitors was employed instead. In view of the low frequencies involved, the only practical method of admittance measurement was the plotting of actual voltage-current traces by means of an oscilloscope

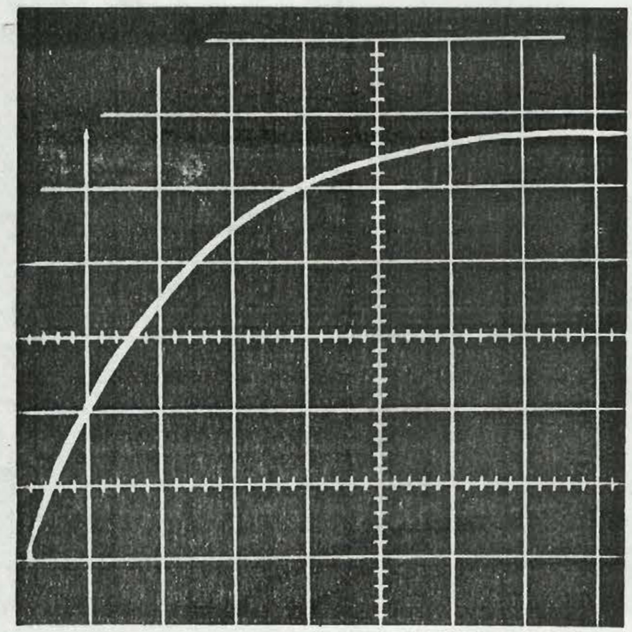


Fig. 306. Current through d-c machine armature and interpoles in response to a voltage step. 5 amp/square, 5 msec/square.

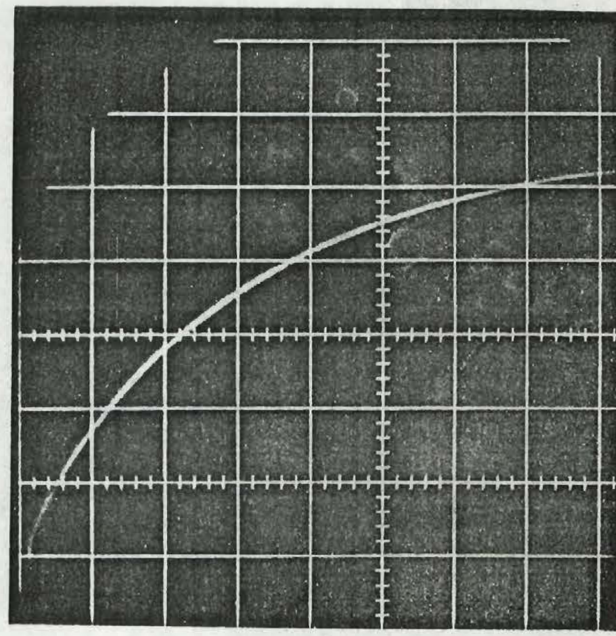


Fig. 307. Current through d-c machine armature alone, in response to a voltage stepfunction. 2.5 amp/square, 2 msec/square.

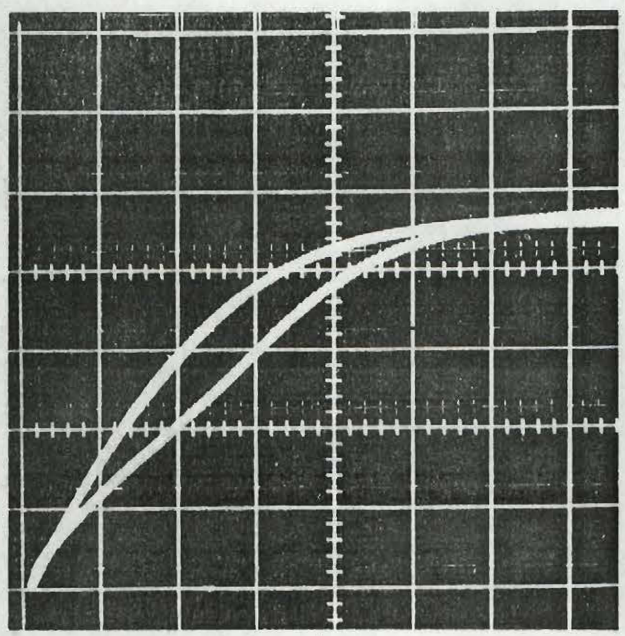


Fig. 308

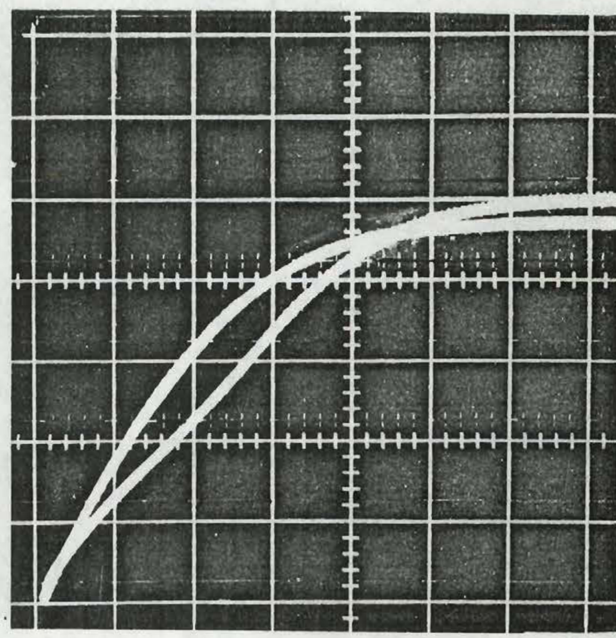


Fig. 309

Field current and generated voltage of a d-c machine in response to a stepfunction field voltage. The more nearly exponential trace represents generated voltage. Scale 55v/square, 0.2 amp/square, 0.2 sec/square.

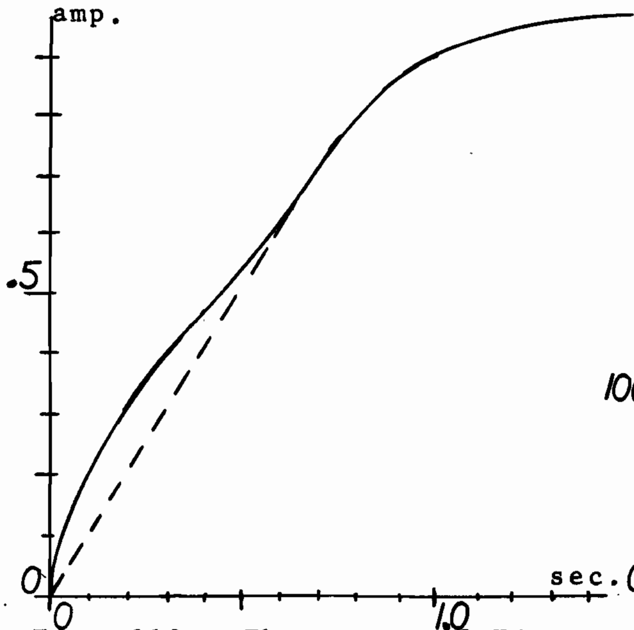


Fig. 310. The curve of Fig. 308, showing predicted and measured current.

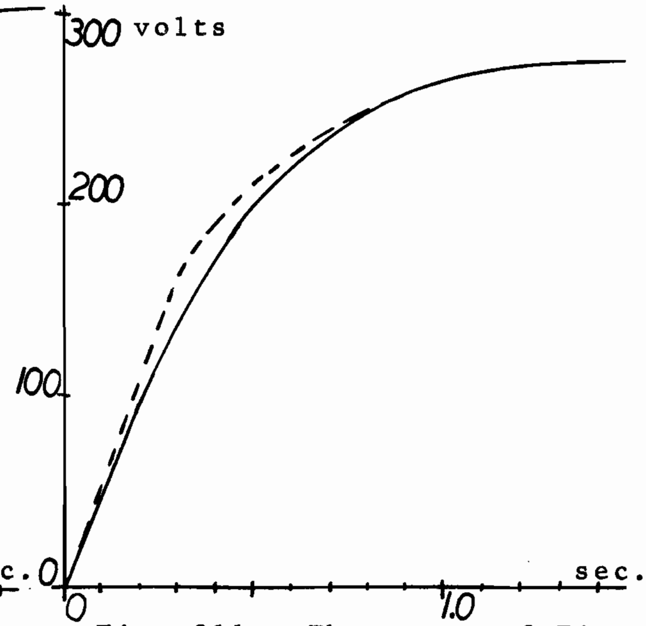


Fig. 311. The curve of Fig. 308, showing predicted and measured open-circuit voltage.

— — — — predicted ———————— experimental

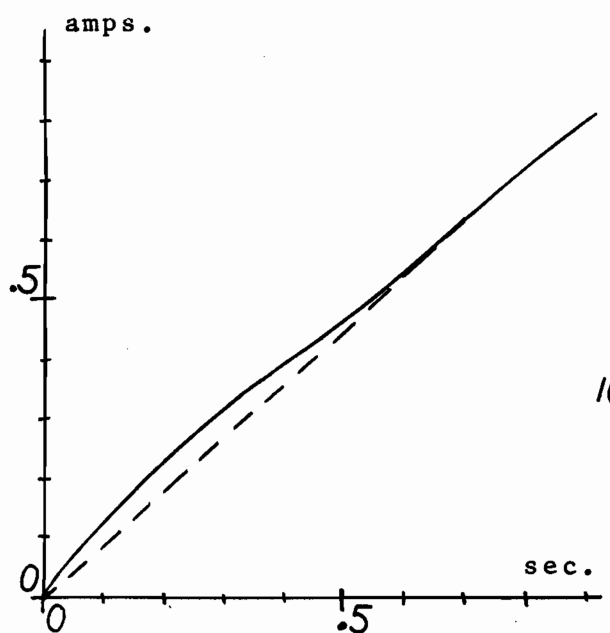


Fig. 312. The curve of Fig. 309, showing predicted and measured current.

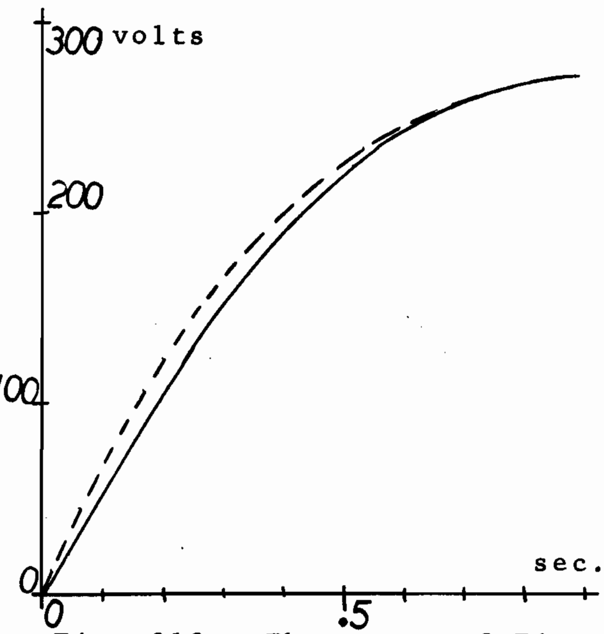


Fig. 313. The curve of Fig. 309, showing predicted and measured open-circuit voltage.

Note: All predicted values above neglect eddy currents.

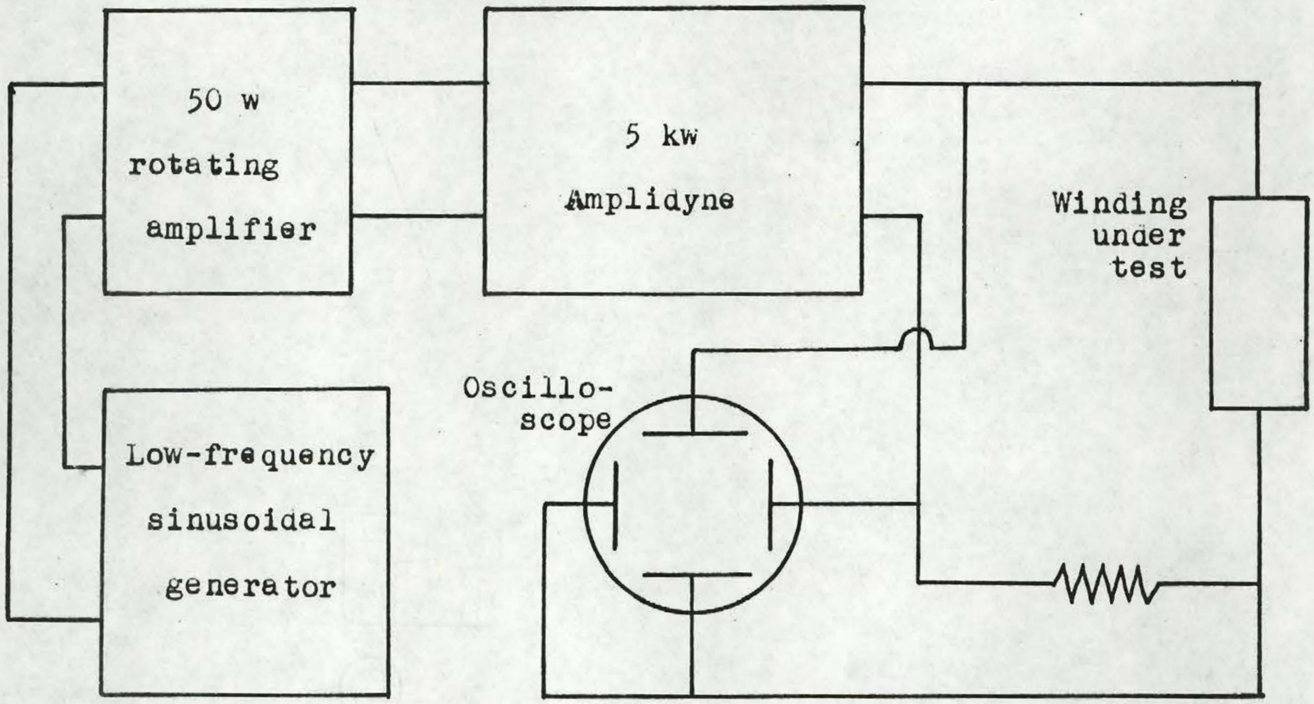


Fig. 314. Equipment layout for measurement of incremental admittance magnitude and angle at low frequencies. The d-c source exciting a second input winding of the amplidyne is not shown.

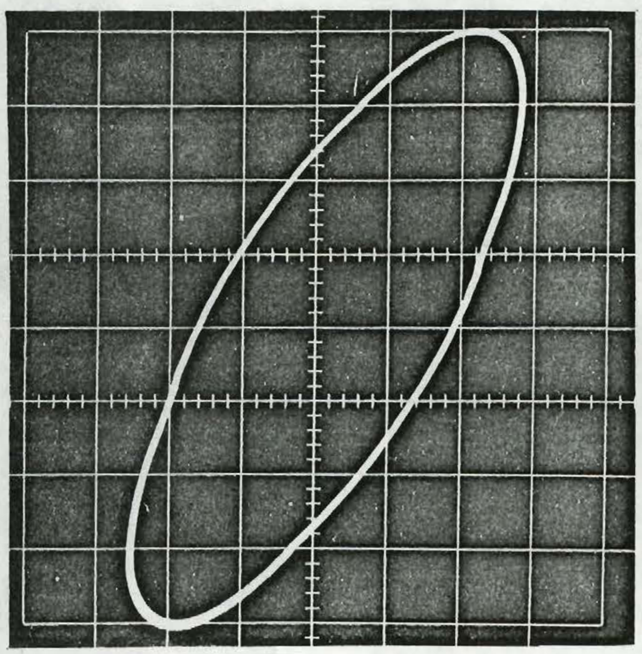


Fig. 315. Terminal voltage vs. current at 60 cps. Scale 10 v/square, 7 ma/square. No d-c. Terminal admittance 0.465 millimhos, phase angle -40°.

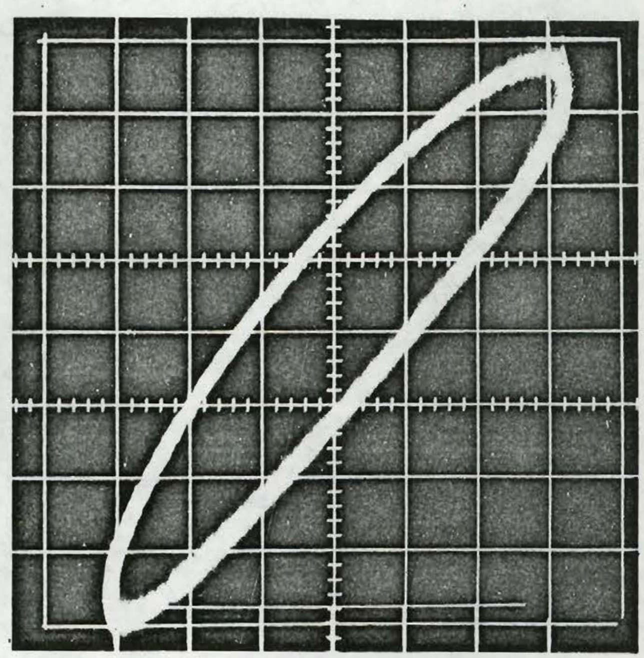


Fig. 316. As Fig. 315, but at 0.7 cps and a d-c current component of 0.9 amp. Note asymmetry arising from nonlinearity. Current scale 38 ma/square, admittance 7.08/ -22° millimhos.

used as an x-y plotter. Because the tracing is slow at low frequencies, the traces were all photographed so as to permit examination. From the inclination of the resulting elliptical traces and from the semiaxis lengths, the magnitude and phase of admittance are readily found. Two such photographs are reproduced in Figs. 315 and 316, representing two points on the admittance-frequency curves.

Runs over the entire frequency range were taken at d-c field currents of zero, 0.75, 0.90, and 1.00 ampere. The rated field current of the machine in question was 1.0 ampere, so 0.75 ampere represents operation a little above the knee of the field saturation curve. The results of these runs were plotted on logarithmic paper (log-log for magnitude, semilog for angle) and curves fitted to them. The results for zero d-c field current appear in Figs. 317 and 318; it is evident that the curve fit is very good indeed. Note that in both of these drawings, the solid lines represent not the estimated average experimental curve, but rather the values given by the fitted function in each case. Similarly, parts of the magnitude plots for the cases of 0.75 ampere and 1.00 ampere d-c appear in Figs. 319 and 320. In the latter two cases, the infinite-product expressions for field admittance were truncated below the highest break frequency available in the data; this fact is clearly visible in both graphs as a sudden upward deviation of measured points from the fitted curve, in the region above 50 cps or thereabouts.

The excellent agreement between measured data and synthesizable polynomial-ratio admittance functions (especially since it holds both for magnitude and for angle) is held to demonstrate conclusively the validity of Weber's theory and hence also the legitimacy of representing solid iron as an infinite series of coupled equivalent windings. It remains to note the fact that the experiment here performed dealt

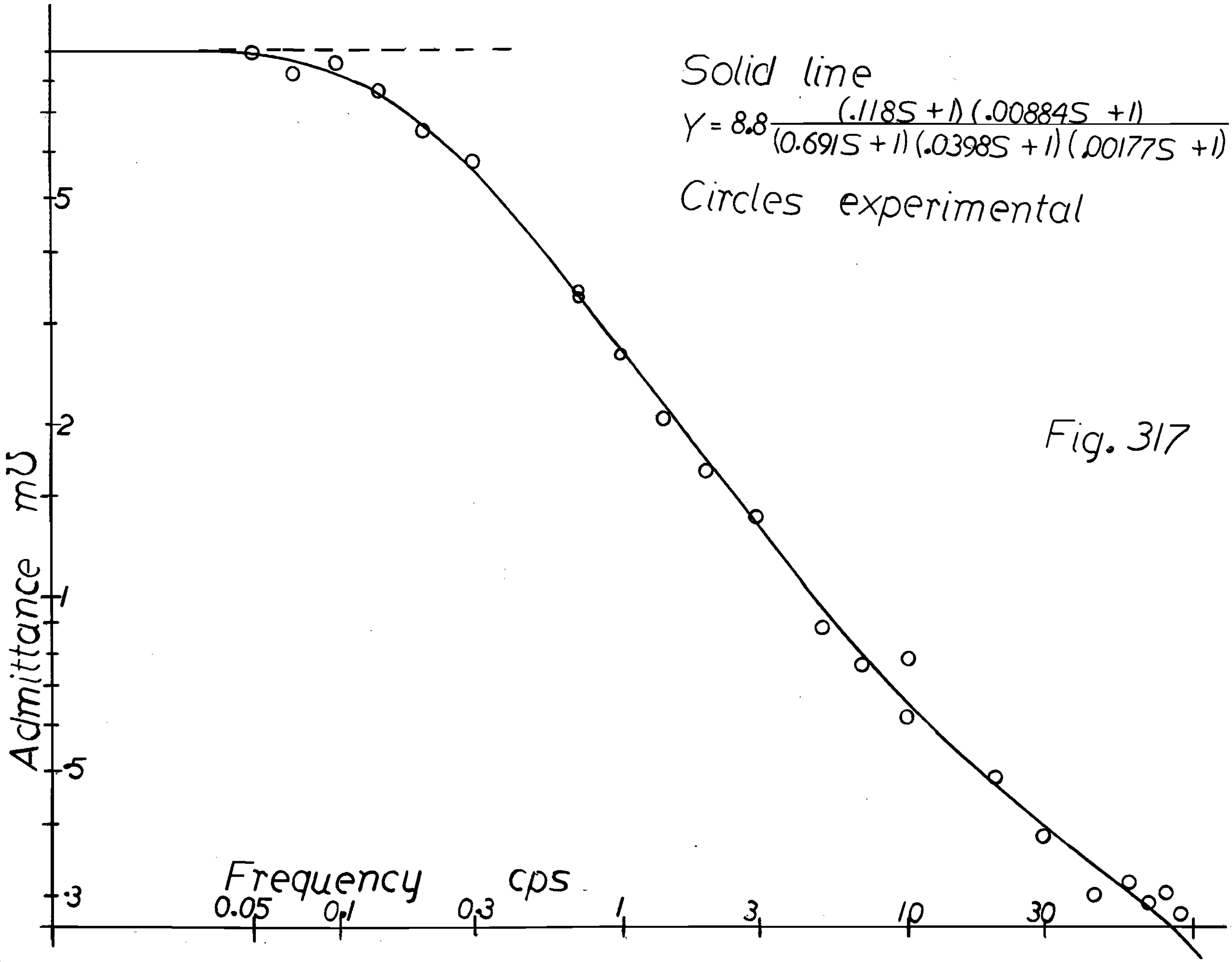

Solid line
Y = 8.8 (.118S + 1) (.00884S + 1) / (0.691S + 1) (.0398S + 1) (.00177S + 1)
Circles experimental
Admittance m℧
5
2
1
.5
.3
Frequency cps
0.05
0.1
0.3
1
3
10
30

Fig. 317

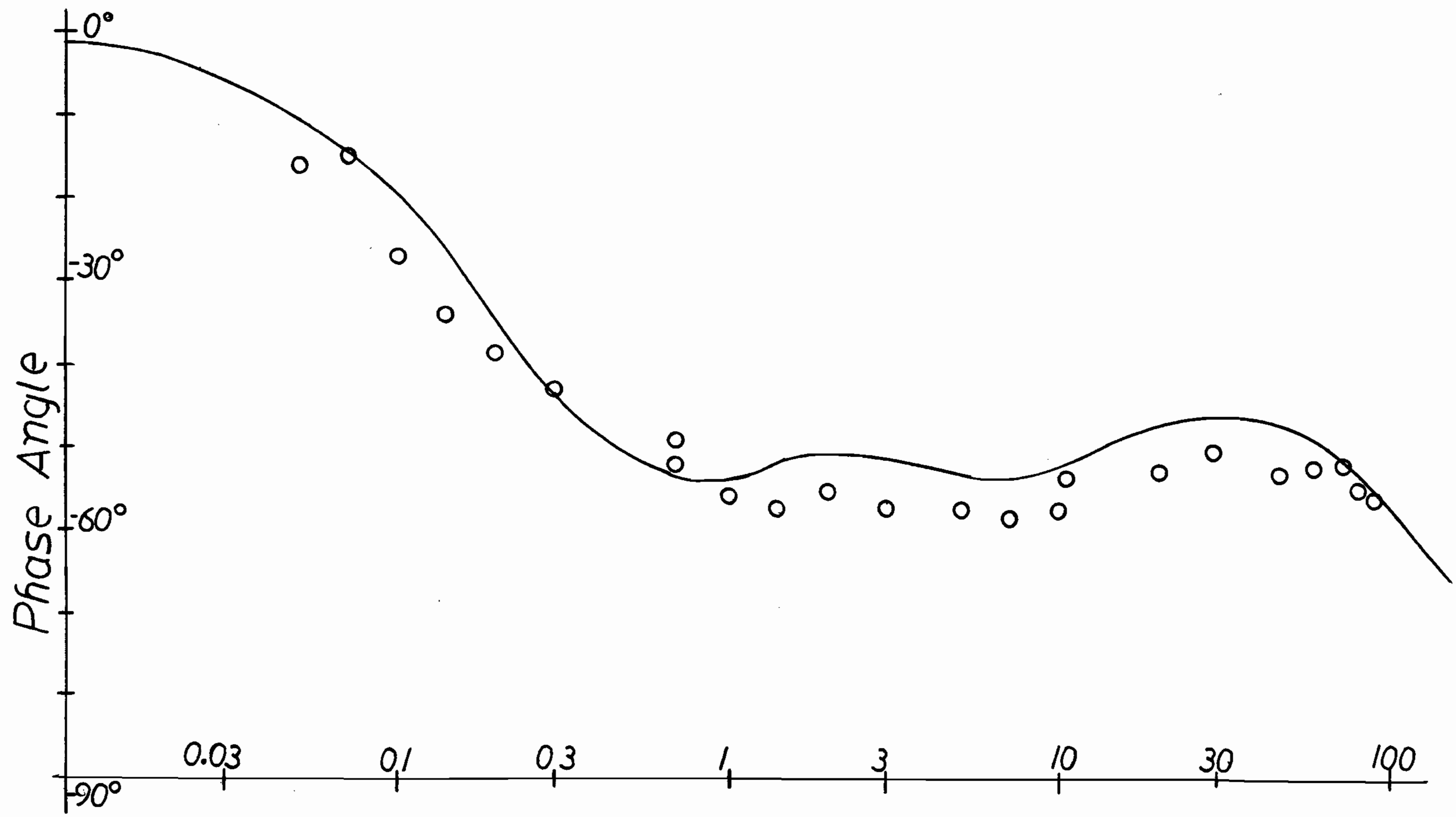


Fig. 318

Circles experimental

Solid line from Fig. 317

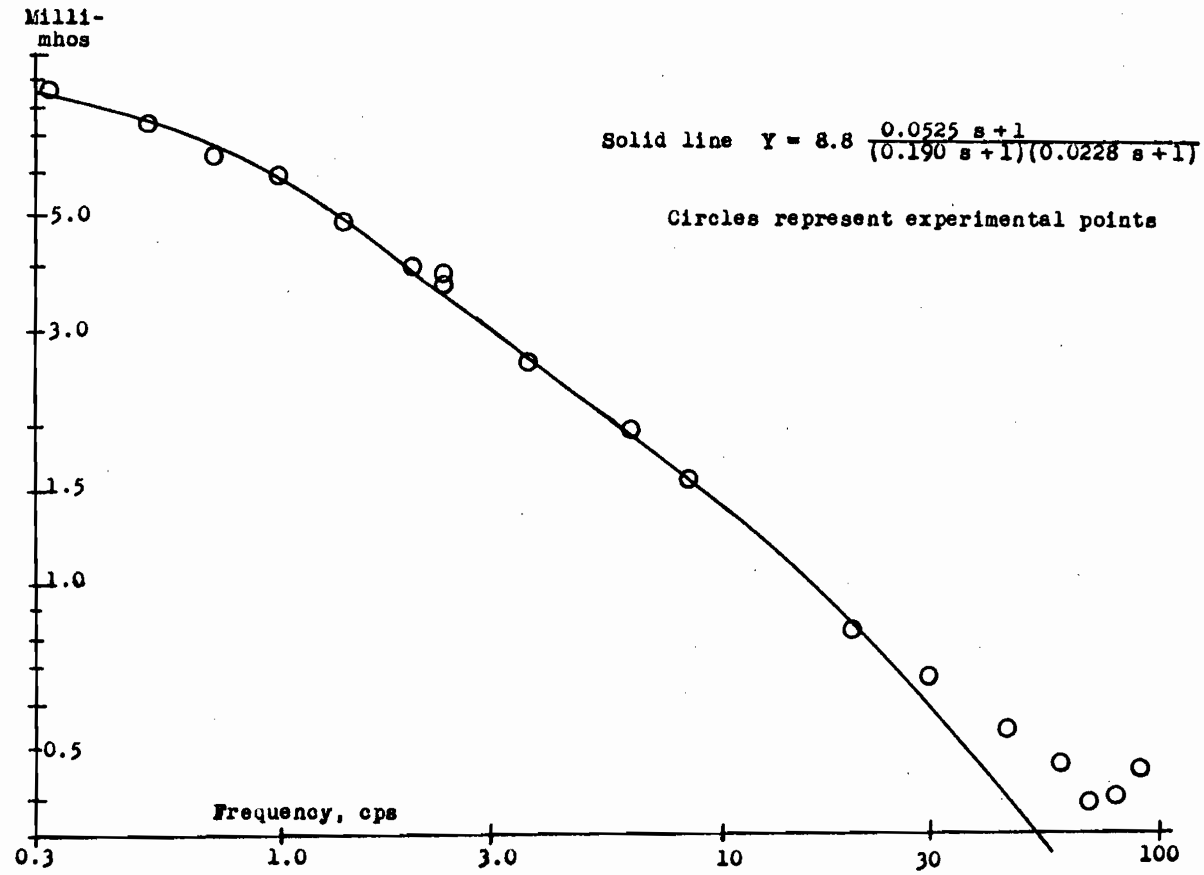


Fig. 319. Behaviour of incremental field admittance with frequency, with d-c field current of 0.75 ampere.

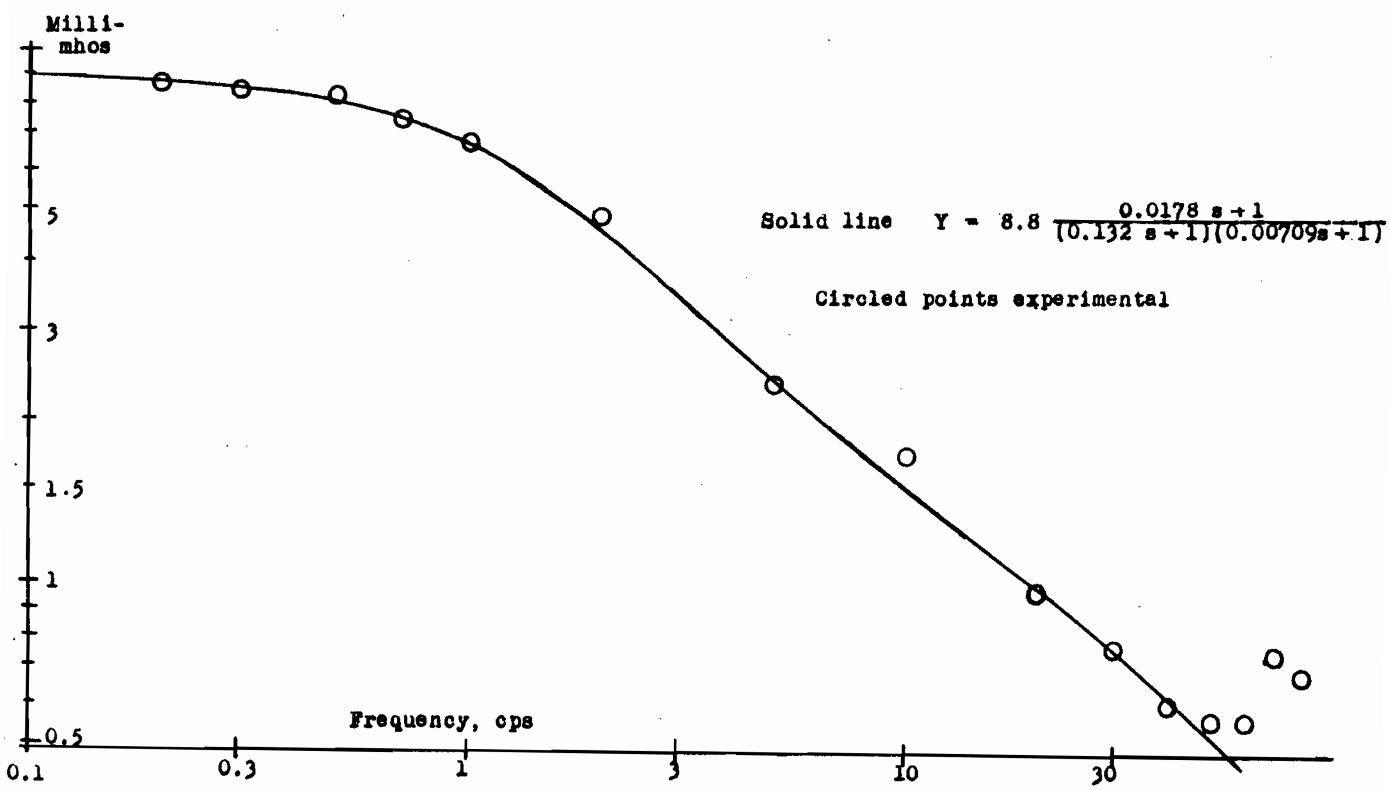


Fig. 320. Behaviour of incremental field admittance with frequency, at the rated value of d-c field current (1.0 ampere).

with incremental rather than total admittance because of the inherent nonlinearity of the problem. As was shown in the preceding chapter, the availability of incremental data is, at least in principle, sufficient for total description of an inductive system. The total inductance matrix can always be derived from measured incremental inductances.

A complete description of a machine involving solid iron is thus in principle possible only by means of an infinitely-extending inductance matrix, since the solid iron requires an infinity of coupled equivalent windings for exact representation. As a practical matter, the infinite series may be truncated rather early, so that the inductance matrix of a real machine may be approximated by one of finite size.

IV

ENERGY CONVERSION BY THE NONLINEAR MACHINE

1. The Nonlinear Machine. In the preceding it has been shown that any purely conservative system of inductive energy storage elements (that is, any system of windings possessing an irrotational flux linkage vector) may be represented by means of an inductance matrix possessing the properties of continuity and symmetry. The derivatives of any order of this matrix with respect to the current coordinates are also seen to form completely symmetric multidimensional matrices. In order to represent an electric machine by such means, the machine must first of all be considered to be a memory-free device (i. e. hysteresis must be removed). Next, the proportional terms in any representation must be removed as resistances. The remaining portion will then be a system of inductive elements possessing the required properties. Representation by an inductance matrix is then possible:

$$e_\alpha = R_{\alpha\beta} i^\beta + \left[L_{\alpha\beta} + \frac{\partial L_{\alpha\beta}}{\partial i^\gamma} i^\gamma \right] \frac{di^\beta}{dt} \qquad (401)$$

But this equation is only valid if the flux linkage vector is irrotational. If the coils described by equation (401) are permitted to move with respect to each other, electromechanical energy conversion will of course take place; if no additional energy is introduced, the stored energy must then change. Alternatively, if the stored energy is kept constant, a net input or output of electrical energy must occur, equal to the mechanical energy involved. Even excluding the resistors R, it is then clear that the resulting electromechanical system is conservative on an overall basis, but not on an electrical basis alone. The possibility of representing a machine, as distinct from a stationary network, by means of an inductance matrix is therefore by no means certain yet.

Much as was done by Kron for linear systems, the static-network ideas contained in equation (401) may be extended to motional networks by the principle of virtual work. As the ultimate goal for the present is representation of electric machines, one dimensionless degree of mechanical freedom is a convenient assumption to adopt. If necessary, the development is easily enough adapted to contain a dimensional mechanical variable. (In the case of linear stroke-motors, this may be of interest, for example.) The development is also capable of extension to more degrees of freedom, if for some reason that should be necessary.

2. Energy Balance for Small Displacements. Let a system such as the above be at rest in a state described by its vector of currents, say $\overline{I}$, and its mechanical position, say θ. As discussed and set forth in equations (212) to (215), the vector of system voltages is always expressible in terms of the flux linkage vector, and the stored energy may be written in terms of the flux linkage and current vectors:

$$U = \int_{0,\theta}^{\overline{I},\theta} \overline{i} \cdot \overline{d\lambda} = \int_{0}^{\Lambda_\alpha} i^\alpha d\lambda_\alpha \qquad (402)$$

where the symbol θ in both limits of integration implies that the integration is to be performed without permitting mechanical motion of the system. If this restriction is not adhered to, the integral will merely yield the input energy, not the energy stored.

Suppose now that the system is moved from the state described to some other neighbouring state, say $(\overline{I} + \overline{\delta I})$, $(\theta + \delta\theta)$. The flux linkage vector $\overline{\lambda}$ will during this process also assume new values; since the motion is restricted to the neighbourhood of the initial state, and all functions required to be continuous to any order, the new values of flux linkages must also be neighbouring ones, say $\bar{\Lambda} + \delta\bar{\Lambda}$. The energy increment during the motion (note that this is the input energy increment, not the stored energy increase) may then be written

$$\delta U = \int_{\Lambda}^{\Lambda+\delta\Lambda} i^{\alpha} d\lambda_{\alpha} \tag{403}$$

The motion here is assumed to be small. But because of the assumed continuity properties, for large as well as small motions the first mean value theorem for integrals may be applied[78, 79] with the result

$$\delta U = (I^{\alpha} + \xi\delta I^{\alpha})\,\delta\Lambda_{\alpha} \tag{404}$$

where $0 \leqq \xi \leqq 1$.

In turn, the mean value theorem for derivatives (Rolle's theorem) may be applied to rewrite equation (404) as

$$\delta U = (I^{\alpha} + \xi\delta I^{\alpha})\,\delta\theta \left(\frac{d\lambda_{\alpha}}{d\theta}\right)' \tag{405}$$

where

$$\left(\frac{d\lambda_{\alpha}}{d\theta}\right)' = \left[\frac{d\lambda_{\alpha}}{d\theta}\right]_{(\theta+\eta\delta\theta),(I^{\alpha}+\xi\delta I^{\alpha})} \qquad 0 \leqq \eta \leqq 1$$

for the sake of brevity.

Now as already pointed out, the energy increment represents additional energy put into the system electrically, not the stored energy increase. The latter quantity may be found by applying the integral of equation (402) to the terminal state. Letting this new stored energy be U',

$$U' = \int_{(0)}^{(I^{\alpha}+\delta I^{\alpha})} i^{\alpha} d\lambda_{\alpha} \tag{406}$$

where the integral is evaluated with the parameter representing the mechanical degree of freedom remaining at its terminal value ($\theta + \delta\theta$) throughout.

The integral of equation (406) may be at least partially evaluated by first integrating by parts

$$U' = (I^{\alpha} + \delta I^{\alpha})(\Lambda_{\alpha} + \delta\Lambda_{\alpha}) - \int_{0}^{I+\delta I} \lambda_{\alpha} di^{\alpha} \tag{407}$$

where the integral is again evaluated at the terminal value of the mechanical angle θ. Another application of the mean value theorem (Rolle's theorem) may be made, this time operating on the components of the flux linkage vector:

$$\lambda_{\alpha}(\theta + \delta\theta) = \lambda_{\alpha}(\theta) + \delta\theta \left[\frac{\partial \lambda_{\alpha}}{\partial \theta}\right]_{(\theta+\zeta\delta\theta)} \qquad 0 \leq \zeta \leq 1 \tag{408}$$

This way of writing may now be used to rewrite equation (407) with the integral evaluated at the initial value of the angle θ . The stored energy in the terminal state thus becomes

$$\begin{aligned} U' = {} & I^{\alpha}\Lambda_{\alpha} + \delta I^{\alpha}\Lambda_{\alpha} + \delta\Lambda_{\alpha} I^{\alpha} + \delta\Lambda_{\alpha}\delta I^{\alpha} - \int_{0}^{I} \lambda_{\alpha} di^{\alpha} \\ & - \int_{I}^{I+\delta I} \lambda_{\alpha} di^{\alpha} - \delta\theta \int_{0}^{I} \left(\frac{\partial \lambda_{\alpha}}{\partial \theta}\right)'' di^{\alpha} \\ & - \delta\theta \int_{I}^{I+\delta I} \left(\frac{\partial \lambda_{\alpha}}{\partial \theta}\right)'' di^{\alpha} \qquad \text{where} \qquad \left(\frac{\partial \lambda_{\alpha}}{\partial \theta}\right)'' = \left[\frac{\partial \lambda_{\alpha}}{\partial \theta}\right]_{(\theta+\zeta\delta\theta)} \end{aligned} \tag{409}$$

Now during the change of state from initial to terminal values of the currents and angle, mechanical energy exchange will in general take place. As equations (402), (405), and (409) give the initial stored energy, the energy put in during the state change, and the final stored energy, respectively, all the data necessary for computing the mechanical work done, say δW, is available. Conservation of energy clearly requires that

$$\delta W = U' - U - \delta U \tag{410}$$

Substitution of the values of the right-hand terms from the above equations now yields the value of mechanical work. There obtains

$$\delta W = \delta I^{\alpha}\Lambda_{\alpha} + \delta\Lambda_{\alpha} I^{\alpha} + \delta\Lambda_{\alpha}\delta I^{\alpha} - \int_{I^{\alpha}}^{I+\delta I^{\alpha}} \lambda_{\alpha} d\dot{i}^{\alpha}$$

$$- \delta\theta \int_{0}^{I} \left(\frac{\partial \lambda_{\alpha}}{\partial \theta}\right)'' d\dot{i}^{\alpha} - \delta\theta \int_{I}^{I+\delta I} \left(\frac{\partial \lambda_{\alpha}}{\partial \theta}\right)'' d\dot{i}^{\alpha} \qquad (411)$$

$$- \delta\theta\, I^{\alpha}\left(\frac{d\lambda_{\alpha}}{d\theta}\right)' - \delta\theta\, \xi\, \delta I^{\alpha}\left(\frac{d\lambda_{\alpha}}{d\theta}\right)'$$

The mean value theorem for integrals will next be applied two times, to the first and third integrals in equation (411). These terms become

$$\int_{I}^{I+\delta I} \lambda_{\alpha} d\dot{i}^{\alpha} = \Lambda_{\alpha}\delta I^{\alpha} + \mu\delta\Lambda_{\alpha}\delta I^{\alpha} \quad , \quad 0 \leqq \mu \leqq 1 \qquad (412)$$

and

$$\int_{I}^{I+\delta I} \left(\frac{\partial \lambda_{\alpha}}{\partial \theta}\right)'' d\dot{i}^{\alpha} = \delta I^{\alpha}\left[\frac{\partial \lambda_{\alpha}}{\partial \theta}\right]_{(\theta+\zeta\delta\theta),(I+\epsilon\delta I)} = \delta I^{\alpha}\left(\frac{\partial \lambda_{\alpha}}{\partial \theta}\right)''' \qquad 0 \leqq \epsilon \leqq 1 \qquad (413)$$

so that, on substituting (412) and (413) into equation (411) above, and subsequently dividing both sides by the angle increment $\delta\theta$ (thanks to the repeated applications of the mean value theorems, this increment appears as a factor in all but one of the terms of the right-hand member), there obtains

$$\frac{\delta W}{\delta\theta} = I^{\alpha}\left[\frac{\delta\Lambda_{\alpha}}{\delta\theta} - \left(\frac{d\lambda_{\alpha}}{d\theta}\right)'\right] - \int_{0}^{I}\left(\frac{\partial \lambda_{\alpha}}{\partial \theta}\right)'' d\dot{i}^{\alpha} - \delta I^{\alpha}\left(\frac{\partial \lambda_{\alpha}}{\partial \theta}\right)''' \qquad (414)$$

$$- \xi\,\delta I^{\alpha}\left(\frac{d\lambda_{\alpha}}{d\theta}\right)' + (1-\mu)\,\delta I^{\alpha}\,\frac{\delta\Lambda_{\alpha}}{\delta\theta}$$

Since a physically realisable system is being described, it is not unreasonable to suppose that both the currents and the angle must be continuous functions of time. The sole case in which this assumption may be untenable is that of switching operations in coupled-circuit sets, where instantaneous changes in individual currents may be possible. Such cases must still be subject to the principle of conservation of flux linkages; hence the deficiency may be overcome by carrying out the entire development in terms of another set of current coordinates, e.g. loop instead of branch currents. The proof can thus be made to cover all real cases. As the time taken for the change of state is made smaller and smaller, the increment achieved in each variable must be smaller and smaller also, all variables being continuous in the time parameter. As the time interval approaches zero, the variable increments approach zero also, and the precise values of the parameters less than unity introduced by repeated applications of mean value theorems are unimportant. The two derivatives of the first term of the right-hand member of equation (414) then approach equality, so that the term vanishes. The last three terms, being multiplied by the current increment, also vanish. It may be noted that the remaining incremental quantities in these three terms all appear in quotients, each of which approaches a finite value, so that the vanishing multiplier is the governing factor. In this way,

$$\lim_{\delta t \to 0} \frac{\delta W}{\delta \theta} = \frac{dW}{d\theta} = -\int_0^I \frac{\partial \lambda_\alpha}{\partial \theta} \, di^\alpha \tag{415}$$

The derivative of the mechanical work input with respect to the angle may be recognised readily as the torque exerted upon the system by external mechanical agencies. This quantity must be the negative of the output torque of the machine. The output torque of any electrical machine may therefore be stated as

$$\Gamma = \int_0^I \frac{\partial \lambda_\alpha}{\partial \theta} \, di^\alpha \tag{416}$$

In the case of a slip-ring machine, the current variables are not in any way required to be connected to the angular variable. It follows that the fundamental definition of inductance, equation (202), may be substituted in (416), yielding

$$\Gamma = \int_0^I i^\beta \frac{\partial L_{\alpha\beta}}{\partial \theta} di^\alpha \tag{417}$$

as the output torque of any slip-ring machine. It must be noted that this expression does not hold for commutator machines. The reason is simply that in order to obtain (417) from (416), the currents and angle are assumed to be independent, so that the derivatives of currents with respect to angle vanish; in a commutator machine, this is not true. The function of the commutator is precisely to tie the rotor current distribution to the angular variable in a predetermined manner, and may as a result be viewed merely as a device for making the derivatives of currents with respect to angle nonzero.

3. Torque of a Linear Machine: A Special Case. The first step in verification of the above theory is naturally enough an inquiry into whether it does in fact predict the performance of a linear machine as described by Kron and others.

Only one detail of such an inquiry is in need of clarification, the path of integration. The integral of equation (417) was derived assuming first that the integral of equation (403) was carried out along some chosen path of integration. In that equation, the path of integration was unimportant, since the angle was invariant during the integration and the system in consequence conservative. No alteration, it may be seen upon inspection, is necessary in the course of the development if some other path of integration is initially chosen. It follows that the derived quantities of equations (416) and (417) must be independent of path also. The vector obtained by differentiating the flux linkages with respect to angle must as a result be irrotational, and the dyad obtained by differentiating the inductance dyad with respect to angle must be symmetrical. These are properties already established for the parent quantities.

Consider now the matrix form of equation (417), the defining equation of machine torque:

$$\Gamma = \int_0^I [i]_t \left[\frac{\partial L}{\partial \theta}\right] [di] \tag{418}$$

Now consider that

$$d\left\{[i]_t \left[\frac{\partial L}{\partial \theta}\right][i]\right\} = [i]_t \left[d\frac{\partial L}{\partial \theta}\right][i] + [i]_t \left[\frac{\partial L}{\partial \theta}\right][di] + [di]_t \left[\frac{\partial L}{\partial \theta}\right][i] \tag{419}$$

so that, since the angular derivatives of inductance are a square matrix, and symmetric,

$$[i]_t \left[\frac{\partial L}{\partial \theta}\right][di] = \frac{1}{2} d\left\{[i]_t \left[\frac{\partial L}{\partial \theta}\right][i]\right\} - \frac{1}{2}[i]_t \left[d\frac{\partial L}{\partial \theta}\right][i] \tag{420}$$

and equation (418) may be rewritten

$$\Gamma = \frac{1}{2}[i]_t \left[\frac{\partial L}{\partial \theta}\right][i] - \frac{1}{2}\int [i]_t \left[d\frac{\partial L}{\partial \theta}\right][i] \tag{421}$$

But in a linear machine, the inductances (and hence their angular derivatives) are independent of current. The second term on the right side of equation (421) therefore vanishes and there remains

$$\Gamma = \frac{1}{2}[i]_t \left[\frac{\partial L}{\partial \theta}\right][i] \tag{422}$$

which is precisely the result given by Kron[64], Bewley[23], and others. It is thus clear that the nonlinear theory here developed is a true generalisation of the linear one, including the latter within it.

The development here carried out for nonlinear inductive systems for the most part parallels the work of authors on linear machines insofar as method is concerned. The proof is of necessity more tedious and the result algebraically less explicit. In view of the restrictions placed on the flux linkage vector and the inductance dyad--they are quite weak from an engineering point of view--the complexity is quite naturally to be expected. For example, although many engineering applications employ monotonically nonlinear materials, the development of the torque equation does not require monotonicity. As a compensation for this increased tedium, the resulting equations, (418) for torque and (209) for voltages, are equally valid for ordinary monotonic nonlinearities (such as might be encountered in, say, d-c machines) and for the much more complicated flux-current relationships discussed by Bessonov[80] in connection with self-excited oscillatory systems or by Biringer[81] in dealing with frequency multipliers.

Unfortunately, a correspondingly simple extension of the slip-ring machine theory above to nonlinear commutator machines cannot in general be made. The simple rotation transformation employed in the linear theory[82] is, as mentioned above, based on Floquet's theory of linear systems with time-varying parameters[55] and cannot be employed here. The useful extension of Floquet theory exploited by Hale[54] establishes the existence or nonexistence of certain terms in periodic solutions of nonlinear systems, but does not in its present form provide a ready method of transformation. In other words, commutator machines are at the present not amenable to treatment as a general group, and will have to be tackled individually or in small subgroups.

4. An Anomaly Explained. Since the adoption of unified theory of machines in undergraduate curricula, a need has been felt for suitable supporting laboratory experiments. The first major step in answering this need was made by the M. I. T.

staff in cooperation with the Westinghouse Electric Corporation in designing the MIT-Westinghouse Unified Machine, as it has become known. This electromechanical jack of all trades consists of an unexpectedly large half-horsepower frame and rotor, with an assortment of slip-rings, two-phase windings, brush sets, and commutator. It may be made to operate as a large variety of machines, though it runs well as none. A set of comprehensive experiment descriptions and typical results for this machine has been published[83]. Although the machine has normal iron, it is striking that the correlation between typical results and predictions is surprisingly good. Even with more standard types of machines, agreement between experiment and prediction based on "averaged" measured inductance values is at times quite good. As in all experimentation concerned with nonlinear devices on an average-value basis, the results are open to criticism on the grounds of the choice of averaging method. The seeming fact, then, that linear generalised theory appears to work in the nonlinear case may well be considered anomalous.

The anomaly may be cleared up rapidly using the results derived above. The torque of a general nonlinear slip-ring machine is given by equation (417); in the past, however, equation (422) has been employed, with the values of inductance--or, to be precise, rate of change of inductance--taken as some reasonable average value within the range of operating conditions being considered. An application of the mean value theorem to equation (417) yields

$$\Gamma = \int \bar{i} \cdot \frac{\partial \bar{\bar{L}}}{\partial \theta} \cdot d\bar{i} = \frac{1}{2} [i]_t \left[\frac{\partial L}{\partial \theta} \right]_{aver} [i] \qquad (423)$$

the rate-of-inductance dyadic being evaluated at some point in the interval of integration and for some value of the current vector. As this equation coincides formally with equation (422) and specifies evaluation of the nonlinear dyadic somewhere within the same range of values, it is clear that in cases where the nonlinearity is not very severe, a guessed set of inductance values will come reasonably close to satisfying equation (423).

A similar situation obtains with the voltage vector. The defining equation (204) may be written

$$\bar{e} = \bar{\bar{\mathcal{L}}} \cdot \frac{d\bar{i}}{dt} + \bar{i} \cdot \frac{\partial L}{\partial \theta} \omega \qquad (424)$$

separating the induced (transformer) and speed (generator) voltages. It is evident that the induced voltages depend on the incremental inductance matrix and the speed voltages on the total inductance matrix. It must be noted that the total inductances in this equation are the instantaneous values, not some sort of averages; if the same rate-of-inductance matrix is employed as for the torque calculation, the results will be too low with the usual kind of iron nonlinearity. Conversely, use of the mean total inductance in place of the incremental inductance is likely to lead to a value of induced voltages that is too large. With a reasonably mild nonlinearity, the discrepancies are unlikely to be very great, however. As these two components are usually measured together as a total voltage, the compensatory nature of the errors tends to obscure faulty predictions. Especially in machines with long air gaps, like the MIT-Westinghouse laboratory machine, saturation curves tend to be flat and hysteresis loops negligibly narrow. For such cases, it is perhaps in any case best to ignore the accurate development and rejoice in the simplicity of the linear methods at some tolerable cost in accuracy.

V

EXPERIMENTS WITH A SIMPLE REAL MACHINE

1. Nature of the Experiments. A great mass of experimental data, both acquired by the author and present in the literature, exists on electric machines. It may be seen readily that the formulation of the machine problem in terms of a nonlinear inductance matrix predicts the qualitative nature, or at least does not violate the nature, of these data. For example, the phenomenon of armature reaction, well known for many decades, is seen to be intimately related to the change in apparent armature inductance of a d-c machine when current is made to flow in another winding in space quadrature with it. Both of these observations are seen to be explainable as merely representing nonzero values of armature-field mutual inductance. The change in apparent armature inductance was first reported by Jones[56] but verified by a large number of others. The experiments of Koenig[27] are important as another aspect of the same problem; he first reported the fact that a current change in an armature circuit induced a transient voltage in the field circuit and vice versa. In short, much experimental data is available on commutator machines, both in the literature and newly gathered. These data are only of qualitative use, unfortunately, as there has not yet been any remarkable success in the construction of a general commutator primitive to correspond to the slip-ring machine described. The latter, on the other hand, does offer the possibility of direct experimental verification in quantitative as well as qualitative terms. Because of the great difficulty in amassing sufficient numerical data for salient-pole machines (new sets of measurements are required for every rotor position), a smooth doubly cylindrical structure was employed as the ex-

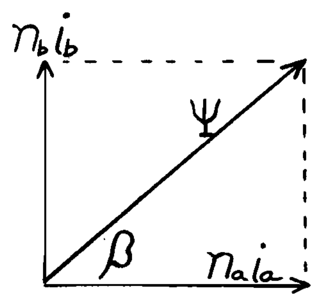
perimental machine. To be precise, a small synchro transmitter with three-phase stator and single-phase rotor windings was tested both for saliency and for sinusoidality of the mmf distribution around the air gap. Near-perfect cylindrical symmetry was found to obtain and the machine therefore put into service, as the number of measurements to be performed could be kept within reasonable bounds. A small torque balance was constructed so as to measure machine shaft torque. Since the balance weights employed were determined to quite high accuracy, but the torque arm length is probably subject to an error of about 1 mm in a total length of 180 millimetres maximum, the accuracy of torque measurement is optimistically estimated to be perhaps 1%.

As a corollary to Koenig's two-coil transient measurements, it was first desired to perform some simple experiment to verify the nature and existence of the truly nonlinear but nevertheless genuine mutual inductance between spatially orthogonal coils. Subsequently, attention was devoted to the torque equation; little data (comparatively speaking) has appeared in the literature on torque measurements performed to sufficient accuracy to display clearly the nonlinearity involved. Since the machine involved was a small one, it was possible to resort to electrical overloads of several hundred per cent. for short periods so as to drive the machine iron deeply into saturation and be assured of truly nonlinear behaviour.

Predictions of performance for all of the experimental work were made by first deriving reasonably general equations for the inductances of a two-coil, doubly cylindrical structure with sinusoidal distributions of the equivalent current sheets. The resulting inductance matrix was then employed to predict performance. Because of the geometrical simplicity of the test machine, this procedure is possible. Whether its extension to large salient-pole machines is practical, is another question.

2. Cylindrical Machine with Orthogonal Windings. Because of its relative simplicity, a machine so arranged as to have its rotor and stator windings at right angles to each other will first be examined. These windings will be referred to as a and b in the following development. The self and mutual inductances are easily derived, as follows.

Let either winding carry current. In view of the sinusoidal distribution of either winding, the total mmf distribution will be sinusoidal also. The total mmf may be written in terms of the individual winding mmf's as follows:



$$\Psi = \sqrt{n_a^2 i_a^2 + n_b^2 i_b^2} \qquad (501)$$

$$n_a i_a = \Psi \cos\beta \qquad (502)$$

$$n_b i_b = \Psi \sin\beta \qquad (503)$$

These relationships may be illustrated by the phasor diagram shown; a phasor diagram may be employed because of the sinusoidal distributions of the windings. The flux distribution will not in general be sinusoidal, but it will always be symmetrical about the same axis as the mmf distribution. Imagining therefore, as Kron does, that the mmf and flux directions coincide, and that flux links coils according to the angle between the coil plane and the axis of flux symmetry, the coil flux linkages may be written as

$$\lambda_a = n_a \phi_a = n_a \cos\beta \;\; \phi(\Psi) \qquad (504)$$

$$\lambda_b = n_b \phi_b = n_b \sin\beta \;\; \phi(\Psi) \qquad (505)$$

where the flux is taken to be a function of the mmf, although of course not a linear function.

From the trigonometry of the phasor diagram of mmf, it next follows that the flux linkages may be rewritten

$$\lambda_a = n_a \frac{n_a i_a}{\Psi} \phi \tag{506}$$

$$\lambda_b = n_b \frac{n_b i_b}{\Psi} \phi \tag{507}$$

By partial differentiation of the flux linkages given by these two equations, it is shown with a little algebraic manipulation that

$$\frac{\partial \lambda_a}{\partial i_b} = \frac{\partial \lambda_b}{\partial i_a} = \frac{n_a n_b}{2} \left| \phi' - \frac{\phi}{\Psi} \right| \sin 2\beta \tag{508}$$

illustrating the reciprocity of incremental mutual inductances as a result of energy conservation independent of the nature of the functional dependence between flux and mmf. In a similar manner

$$\frac{\partial \lambda_a}{\partial i_a} = \frac{n_a^4 i_a^2}{\Psi^2} \left| \phi' - \frac{\phi}{\Psi} \right| - \frac{n_a^2 \phi}{\Psi} \tag{509}$$

and

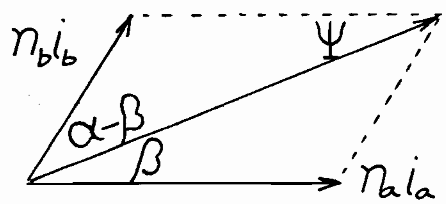
$$\frac{\partial \lambda_b}{\partial i_b} = \frac{n_b^4 i_b^2}{\Psi^2} \left| \phi' - \frac{\phi}{\Psi} \right| - \frac{n_b^2 \phi}{\Psi} \tag{510}$$

Appendix C to this thesis shows that the relationship between total and incremental inductances is a comparatively simple one. Integrating in the manner prescribed, the solution is found, as shown in equation (511). In its turn, this equation clearly illustrates the reciprocity of <u>total</u> mutual inductances at all points in current space (in this case, really a current plane). Examination of this equation also points out what was discussed at some length in Chapter II: in nonlinear systems, reciprocity can only have meaning as a property at a point in current space. The mutual terms are seen to change sign at each quadrant boundary, so that the mutual effect is always to decrease total flux and stored

$$L = \begin{bmatrix} n_a^2 \frac{\Phi}{\Psi}\cos^2\beta + n_a^2\left(\frac{1}{\Psi}\int_0^{\Psi}\frac{\phi}{\psi}\,d\psi\right)\sin^2\beta & \frac{n_a n_b}{2\Psi}\sin 2\beta\left(\Phi - \int_0^{\Psi}\frac{\phi}{\psi}\,d\psi\right) \\ \frac{n_a n_b}{2\Psi}\sin 2\beta\left(\Phi - \int_0^{\Psi}\frac{\phi}{\psi}\,d\psi\right) & n_b^2 \frac{\Phi}{\Psi}\sin^2\beta + n_b^2\left(\frac{1}{\Psi}\int_0^{\Psi}\frac{\phi}{\psi}\,d\psi\right)\cos^2\beta \end{bmatrix} \quad (511)$$

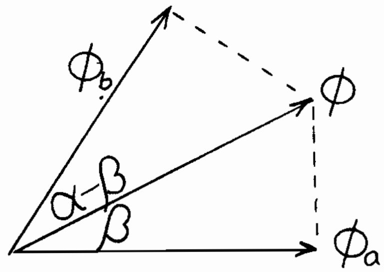
energy. On the other hand, this nonlinear system is seen to share with linear ones the dependence of inductance on the square of turns number.

3. Machine with Windings not Orthogonal. The expressions derived for the inductances of the pair of orthogonal windings may be extended to cover any two non-orthogonal coils as well. As before, the total mmf is easily found with the aid of a phasor diagram:



$$\Psi = \sqrt{n_a^2 i_a^2 + n_b^2 i_b^2 + 2 n_a n_b i_a i_b \cos\alpha} \quad (512)$$

The flux linkages of the coils may be found by first computing the total flux. As previously, it must share the axis of symmetry with the mmf, and depend on it in some functional manner. Resolving the total in the directions of the two winding planes then gives the flux linkages of the coils. A phasor-like diagram may be employed as an aid for the necessary trigonometric calculations; it must be remembered, however, that this diagram is not a true phasor diagram, but merely a sketch of the axes of symmetry. There easily obtains



$$\lambda_a = n_a \phi \cos\beta \qquad (513)$$

$$\lambda_b = n_b \phi \cos(\alpha-\beta) \qquad (514)$$

Although the algebra here is a little more drawn out than in the previous case, there is no difficulty in differentiating to find the incremental mutual inductances

$$\frac{\partial\lambda_a}{\partial i_b} = \frac{\partial\lambda_b}{\partial i_a} = n_a n_b\left[\phi' \cos\beta \cos(\alpha-\beta) - \frac{\phi}{\Psi}\sin\beta \sin(\alpha-\beta)\right] \qquad (515)$$

and the self-inductances

$$\frac{\partial\lambda_a}{\partial i_a} = n_a^2 \frac{\phi}{\Psi} + n^2\left(\phi' - \frac{\phi}{\Psi}\right)\cos^2\beta \qquad (516)$$

$$\frac{\partial\lambda_b}{\partial i_b} = n_b^2 \frac{\phi}{\Psi} + n^2\left(\phi' - \frac{\phi}{\Psi}\right)\cos^2(\alpha-\beta) \qquad (517)$$

Just as in the case of orthogonal windings, the total inductances may next be calculated by integrating the appropriate differential equations. The expressions that result are similar to those already obtained, though a little more complicated. The inductance matrix, equation (518), is shown on page 61.

Although the mutual inductance with zero currents is no longer zero, the other properties enumerated for the orthogonal case may be seen to hold. As a check, it may be verified that the inductance matrix of equation (518) does indeed reduce to that given in (511) in the orthogonal case.

<u>4. Evaluation by Measurements.</u> The general inductance matrix of the machine is now available in terms of the flux-mmf functional relationship and the numbers of turns in the windings. As the turns numbers must be such as to make allownace for the machine leakage fluxes, rather than physical turns numbers,

$$
L = \begin{bmatrix}
n_a^2 \dfrac{\Phi}{\Psi} \cos^2\beta + n_a^2 \dfrac{1}{\Psi}\displaystyle\int_0^{\Psi} \dfrac{\Phi}{\psi}\, d\psi \, \sin^2\beta & n_a n_b \dfrac{\Phi}{\Psi} \cos\beta \cos(\alpha-\beta) + n_a n_b \left(\sin^2\beta \cos\alpha - \dfrac{1}{2}\sin 2\beta \sin\alpha\right) \dfrac{1}{\Psi}\displaystyle\int_0^{\Psi} \dfrac{\Phi}{\psi}\, d\psi \\
n_a n_b \dfrac{\Phi}{\Psi} \cos\beta \cos(\alpha-\beta) + n_a n_b \left(\sin^2\beta \cos\alpha - \dfrac{1}{2}\sin 2\beta \sin\alpha\right) \dfrac{1}{\Psi}\displaystyle\int_0^{\Psi} \dfrac{\Phi}{\psi}\, d\psi & n_b^2 \dfrac{\Phi}{\Psi} \cos^2(\alpha-\beta) + n_b^2 \dfrac{1}{\Psi}\displaystyle\int_0^{\Psi} \dfrac{\Phi}{\psi}\, d\psi \, \sin^2(\alpha-\beta)
\end{bmatrix}
$$

Equation (518)

both the turns numbers and the flux-mmf relationship must be obtained by measurement. Consider the machine first in such a position that the axes of the two windings coincide. The inductance matrix then reduces very simply to

$$L_{|\alpha=\beta=0|} = \begin{bmatrix} n_a^2 & n_a n_b \\ n_b n_a & n_b^2 \end{bmatrix} \frac{\phi}{\Psi} \tag{519}$$

If one winding, say b, is not energised, the flux-mmf ratio is further simplified. The following relationships are then readily deduced:

$$\frac{\phi}{\Psi} = \frac{M_{ab}|i_a,0,0|}{n_a n_b} = \frac{1}{n_a n_b} \frac{\lambda_b}{i_a} \tag{520}$$

$$\phi' = \frac{1}{n_a n_b}\left[\frac{\lambda_b}{i_a} + i_a \frac{d}{d\,i_a}\left(\frac{\lambda_b}{i_a}\right)\right] \tag{521}$$

It is now seen that if the mutual inductance with one winding energised is now measured and the result substituted into equation (518), expressions result that contain no factors that are not analytically known or readily measured. For the very simple case of a completely symmetrical slip-ring machine, then, the components of the inductance matrix are readily enough available.

The flux-mmf characteristic of the synchro transmitter was measured in two ways. First, ballistic measurements with a Maxwell-Rayleigh bridge and electronic integrator were made to determine static values. Subsequently, a fast integrating network and oscilloscope were employed to obtain dynamic hysteresis loops. Both of these methods are quite conventional, the former being very similar to Jones' measurements, the latter a standard test of ferromagnetic materials. Since the ballistic measurement does not require exciting current to be left on for long periods, it may be used with considerable electrical overloads (currents several times rated value). The static mean magnetisation curve is shown in Fig. 501, and a dynamic curve exhibiting the hysteresis and eddy currents neglected in the above analysis appears in Fig. 502. Although some error will naturally result, it is evident from Fig. 502 that neglecting hysteresis is not wholly unreasonable.

As already observed, the mutual inductance changes sign at each quadrant boundary if the two windings are orthogonally placed. Hence, as a dynamic verification of Koenig's transient measurements[27], it was attempted to hold the current in one coil at a constant value while impressing an alternating voltage across the other. Qualitatively, the coil carrying direct current only should now have a double-frequency alternating current across its terminals. This voltage will result from the fact that the system is being driven along a straight line in current space that crosses the boundary between two quadrants, so the mutual inductance between the coils must change sign twice each cycle. A series of such tests were performed, the driven coil current being plotted against the induced voltage. One such plot appears in Fig. 503, where the round dots represent points predicted by means of the inductance matrix of equation (518) and the mean magnetisation curve of Fig. 501. The photograph shown is representative of the run of tests. It is evident that in this test difficulty was encountered in holding the coil d-c truly constant; some ripple current did appear, as evidenced by the large area enclosed by the loops of Fig. 503 (these represent the energy being taken out of the machine). It is also to be noted that as an incremental flux variation is being measured, hysteresis does make its presence felt. As the test is a very demanding one, it is felt that the theory is adequately verified by this measurement.

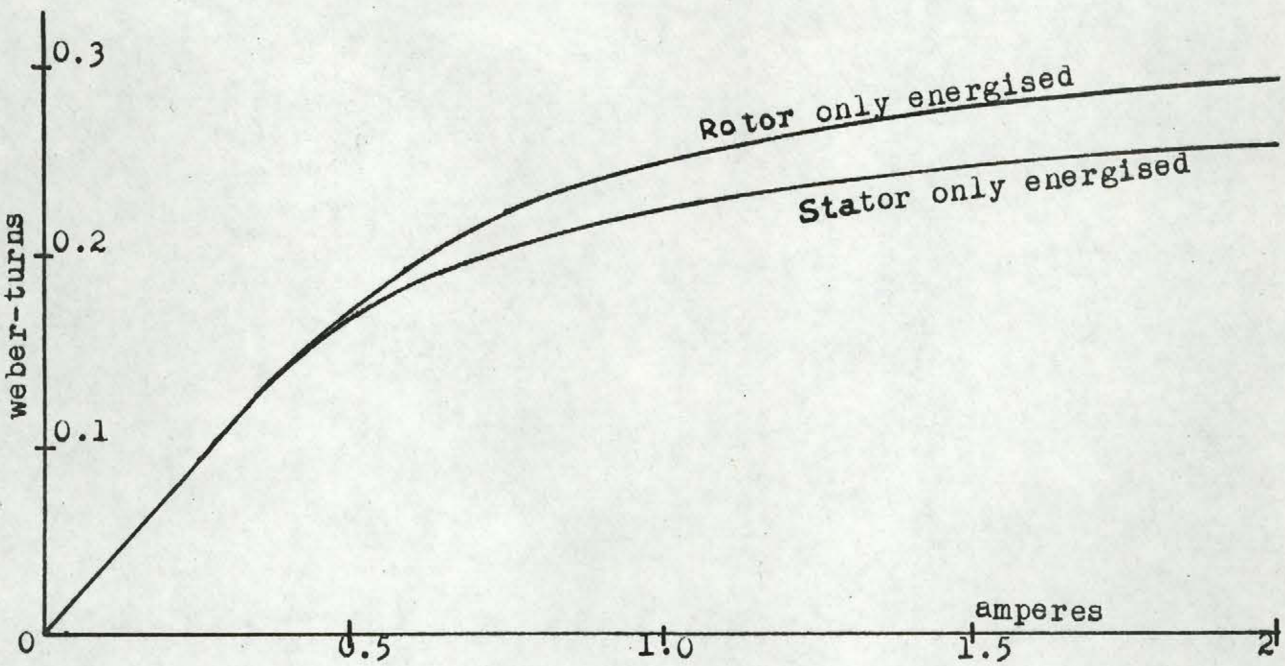


Fig. 501. Static mean magnetisation curve of the test machine, measured by energising only one winding at a time and finding flux linkages of the other.

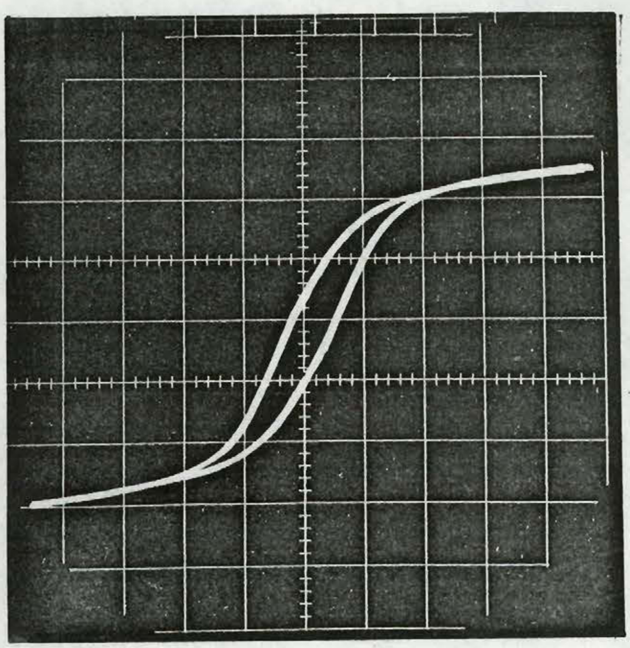


Fig. 502. Dynamic hysteresis loop of test machine, with rotor energised at 30 cps. The rotor peak current is 2.3 amperes.

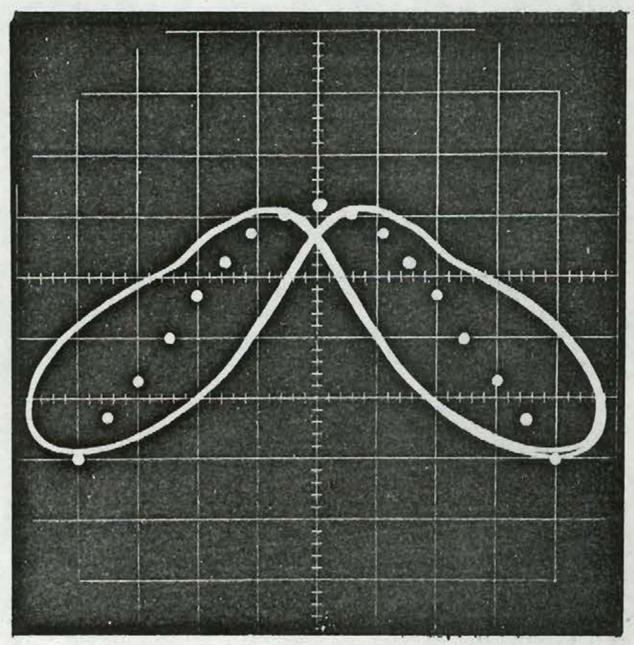


Fig. 503. Stator current (horizontal) vs. rotor terminal voltage (vertical). Rotor direct current 1.5 amperes. Dots represent predicted points.

5. The Torque Equation. A much more interesting problem, and far more demanding of experimental verification than the application of Faraday's law to nonlinear circuits (which, in a sense, is all that can be accomplished with stationary circuits) is of course the torque equation (417). One test of that expression has been made already: in equations (418) to (422) it is shown that it does predict the torque of a linear machine. Since a nonlinear machine with known inductance matrix is now at hand, the torque equation shall next be employed to predict torque of the nonlinear machine.

Substituting the inductance matrix of equation (518) into the torque equation (417), after lengthy algebraic manipulation there obtains

$$\Gamma = \int_0^{i_a} \sin\beta\left[\phi' n_a i_a \cos\beta + \phi \frac{\sin\beta}{\sin\alpha}\cos(\alpha-\beta) n_a\right] di_a + \int_0^{i_b} \sin(\alpha-\beta)\left[\phi' n_b i_b \cos(\alpha-\beta) + \phi\frac{\sin(\alpha-\beta)}{\sin\alpha}\cos\beta\, n_b\right] di_b \tag{522}$$

Unfortunately this expression is somewhat too complicated for convenience. Some considerable simplification is possible upon a change of variables, converting the current plane of this problem into polar instead of Cartesian coordinates. Upon such a transformation, equation (522) becomes

$$\Gamma = \frac{1}{2}\sin\alpha \int_0^{r} r \sin 2\gamma \left[\phi' r\sqrt{1+\sin 2\gamma\cos\alpha} + \frac{\phi}{r\sqrt{1+\sin 2\gamma \cos\alpha}}\right] dr \tag{523}$$

Starting again with the simplest possibility, in the case of orthogonal windings this conveniently reduces to

$$\Gamma\left(\frac{\pi}{2}\right) = \frac{1}{2}\sin 2\gamma \int_0^{r}\left[\phi'(r) + \frac{\phi}{r}\right] r\, dr \tag{524}$$

The latter expression was evaluated by performing the indicated differentiation, multiplication, addition, and integration, in that order, graphically. Unlike in the voltage measurements , graphical calculation was here preferred to computer work because graphical differentiation, continuously visible to the calculator, was considered more reliable than numerical differentiation. One very simple experiment immediately offers itself. If the two windings of the machine are connected in series, the polar angle in current space is fixed, and equation (524) gives the torque of the resulting system directly. Such an experiment was of course performed; the experimental results (shown as circles) and the predicted curve are shown in Fig. 504. It is seen that the prediction is very close indeed, the error in predicted value being less than the stationary-friction dead zone of the torque measuring apparatus (hence in fact unmeasurable). It is interesting to note that the torque increases very nearly proportionally to current for all but the lowest currents, rather than parabolically as linear machine theory would predict. By operating the machine at currents much in excess of its ratings, the maximum current being of the order of 500% of rated, a situation is created in which linear theory is not only unable to predict correct numerical values, but even gives qualitatively wrong answers.

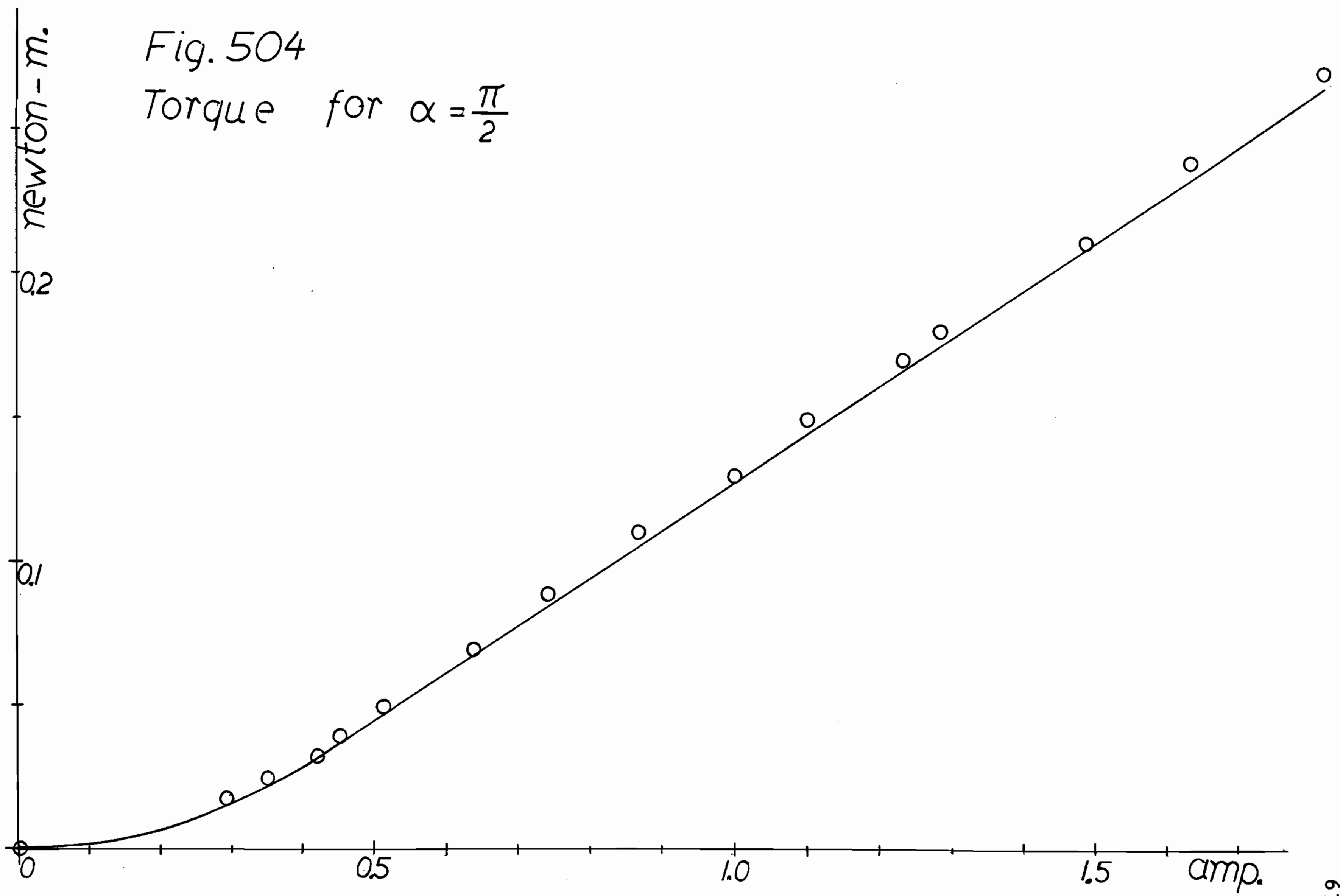
The case in which the windings are oriented at an arbitrary angle to each other may next be dealt with. During any one integration, the angles in equation (523) remain constant so that that expression may be rewritten

$$\Gamma = \frac{\sin\alpha \sin 2\gamma}{1+\sin 2\gamma \cos\alpha} \, \frac{1}{2} \int_0^r \left[\phi'(r) + \frac{\phi}{r}\right] r\,dr \tag{525}$$

and, recognising, the integral from equation (524),

$$\Gamma(\alpha) = \frac{\sin\alpha}{1+\sin 2\gamma \cos\alpha} \, \Gamma\left(\frac{\pi}{2}\right) \tag{526}$$

Fig. 504
Torque for $\alpha = \frac{\pi}{2}$



The next experiment performed was that of investigating the behaviour of machine torque as the rotor position was varied, the winding currents being held constant. In terms of the current plane, the first experiment held the mechanical parameter fixed and investigated the torque variation along a radial line, i.e. the path of integration itself. The second experiment, on the other hand, keeps the system at a fixed point in current space and varies the mechanical parameter. The torque of course is calculated from equation (526). The predicted (solid line) and experimental (circles) values for three different current combinations are shown in Figs. 505, 506, and 507. Again, the machine is quite deliberately strongly saturated in two of these experiments, and moderately in the third. Only the results for 180° of angular rotation are shown, the next half-turn producing a mirror image of these curves.

The remaining curves, Figs. 508 and 509, show similar results, but this time for straight-line traces in current space parallel to one axis. To obtain such a trace, one of the winding currents was held fixed and the other suitably varied. The rotor angle in these runs is held invariant. Again, experiments are seen to yield excellent agreement with the predicted curves.

For a weakly saturated, or nearly linear, system, the torque equation has been shown to be valid analytically. For a strongly saturated system, where linear theory fails even in qualitative terms, the new theory is shown to yield excellent results. It is held that this combination of verifying tests is conclusive.

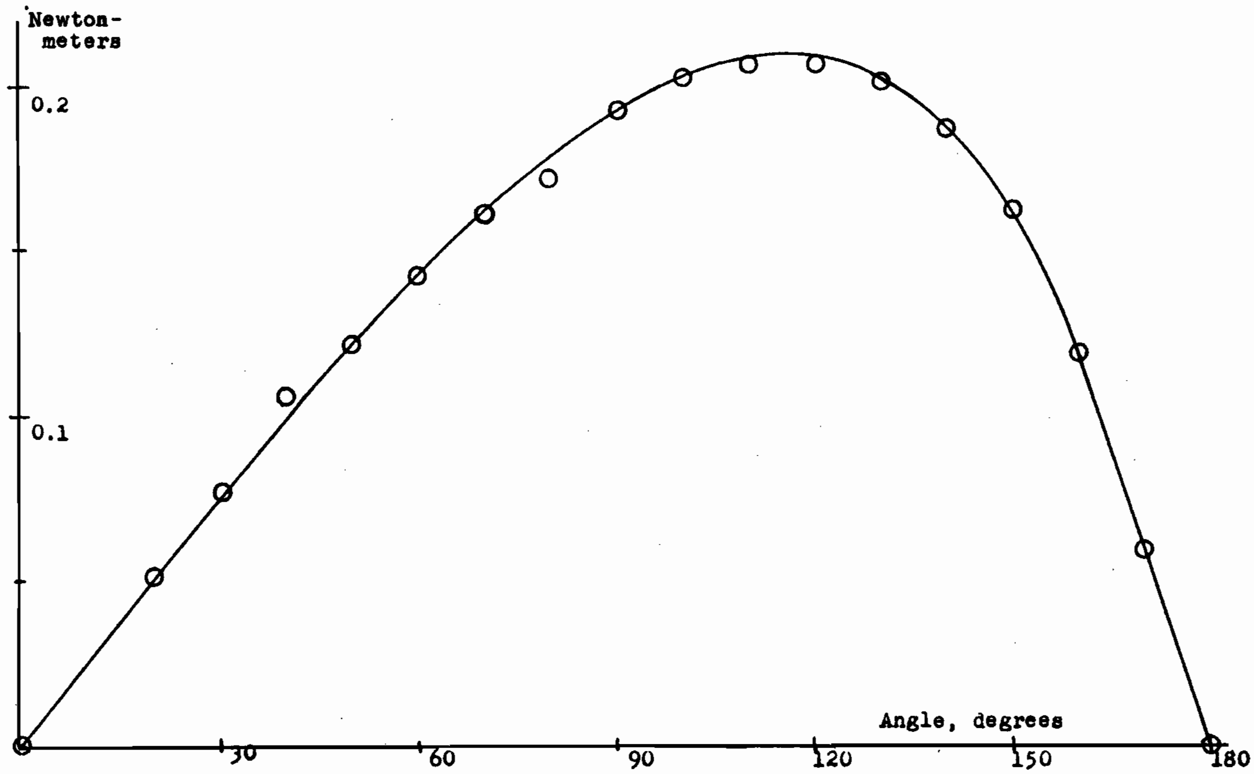


Fig. 505. Shaft torque against rotor position of the test machine with rotor current 1.19 ampere, stator current 0.76 ampere. Solid line represents predicted, circles experimental values.

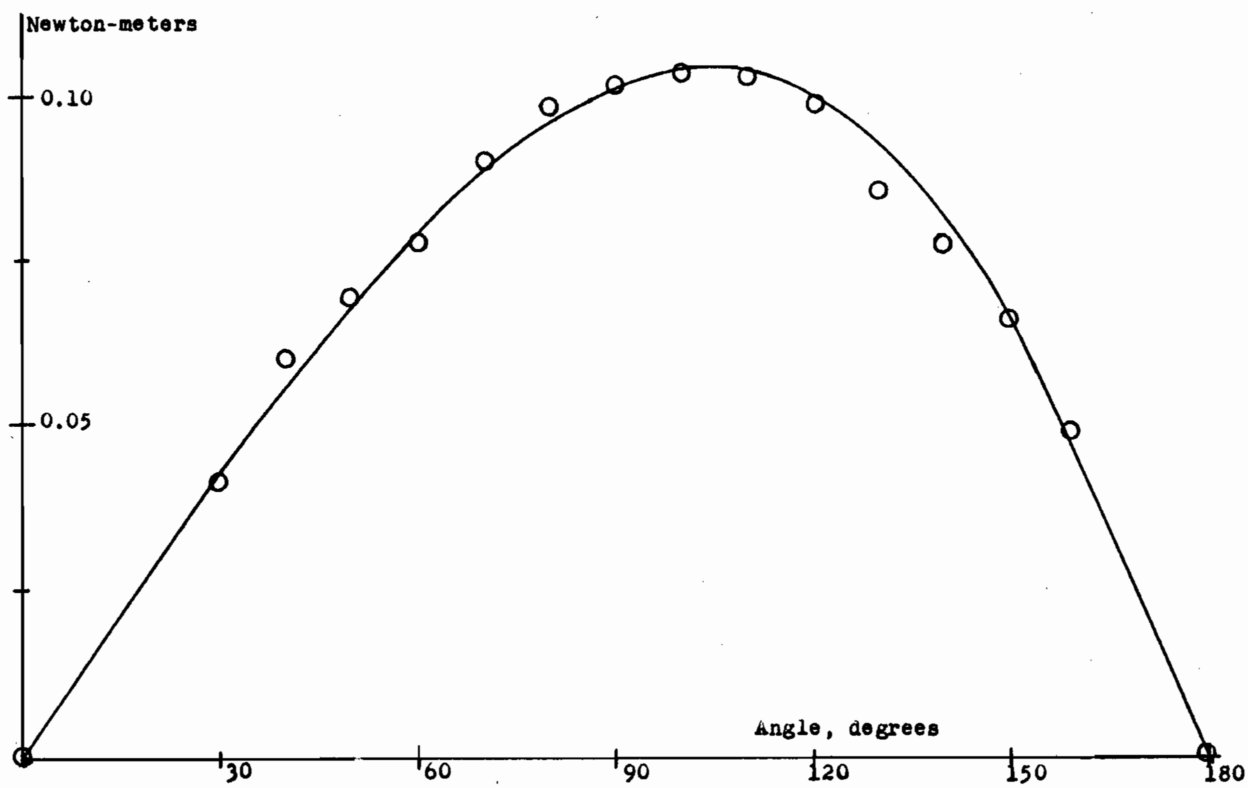


Fig. 506. Shaft torque against rotor position for the test machine with rotor current 1.19 ampere, stator current 0.40 ampere. Solid line represents predicted, circles experimental values.

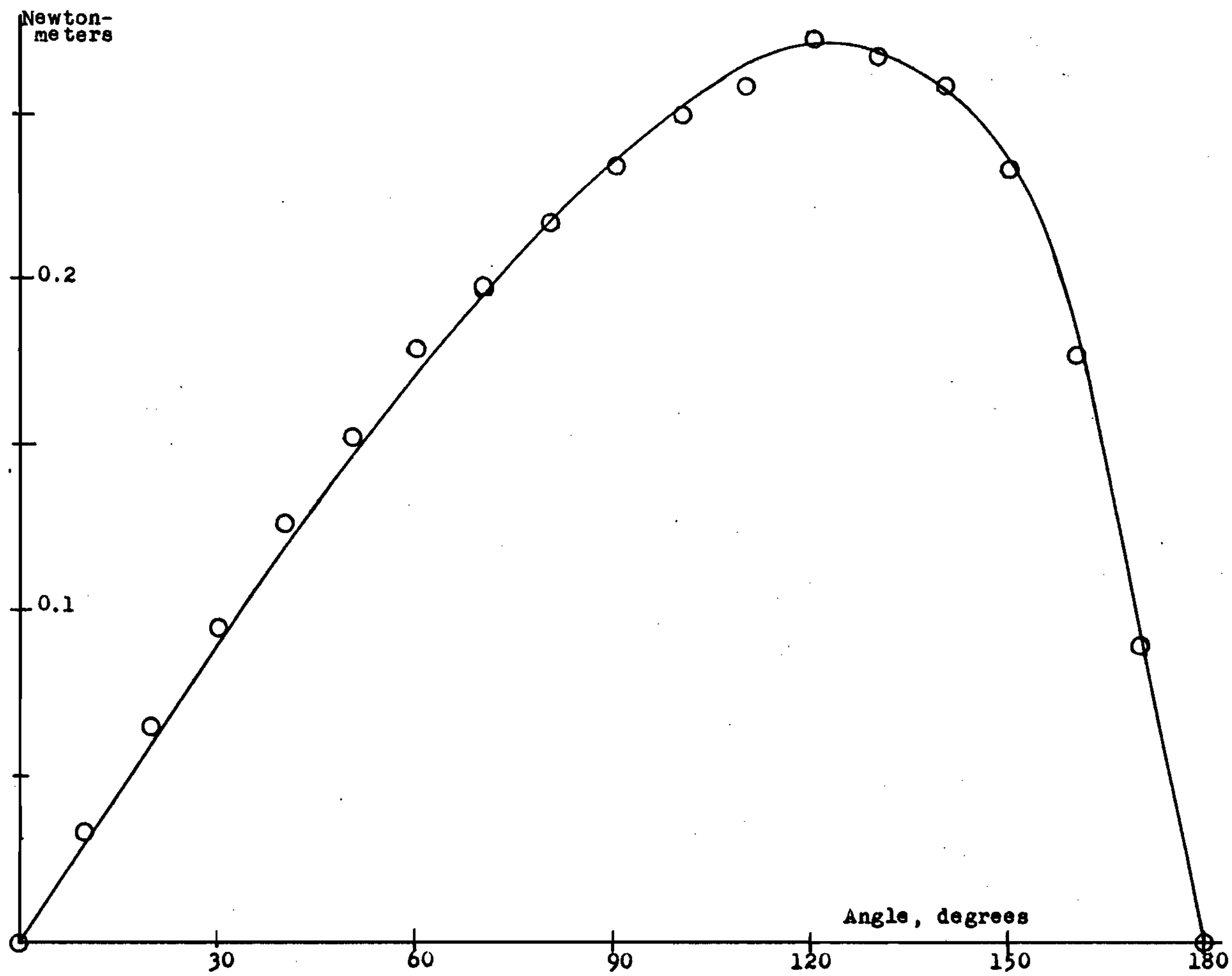


Fig. 507. Shaft torque against rotor position for the test machine. Rotor current 1.19 amperes, stator current 1.19 amperes. Solid line represents predicted, circles experimental points.

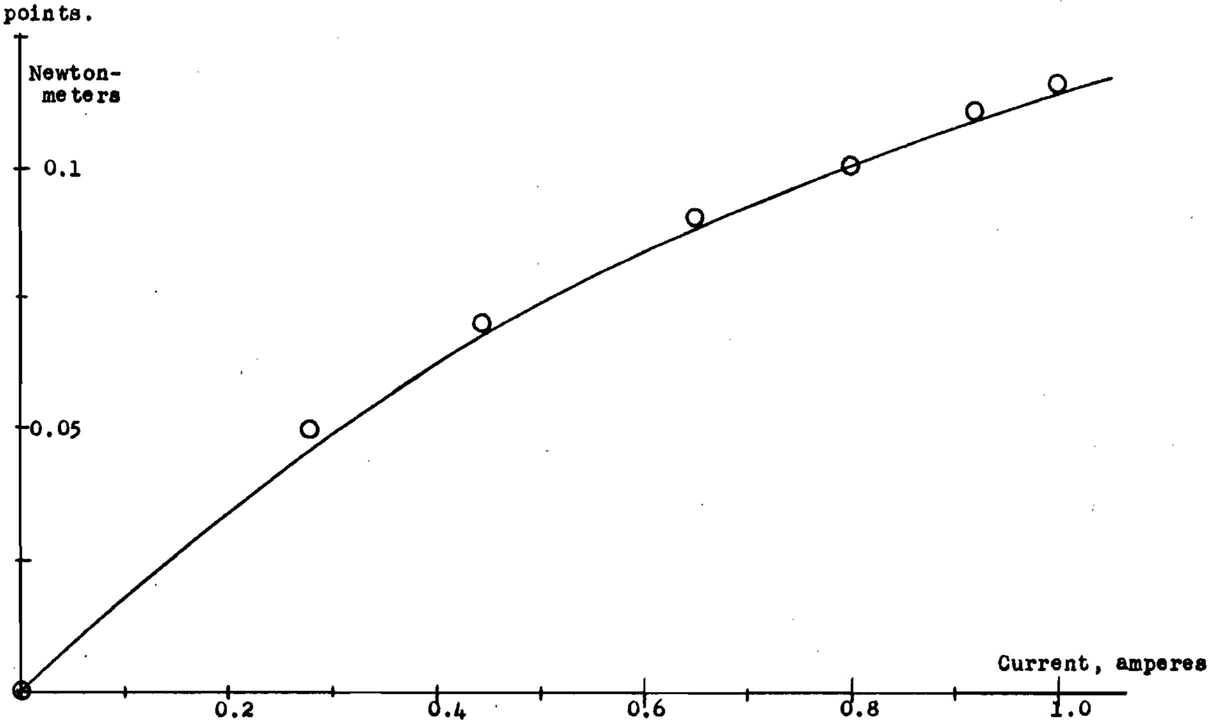


Fig. 508. Shaft torque of the test machine against rotor current, with stator current held constant at 0.70 ampere and rotor position at 58°. Circles represent measured points, solid line predicted values.

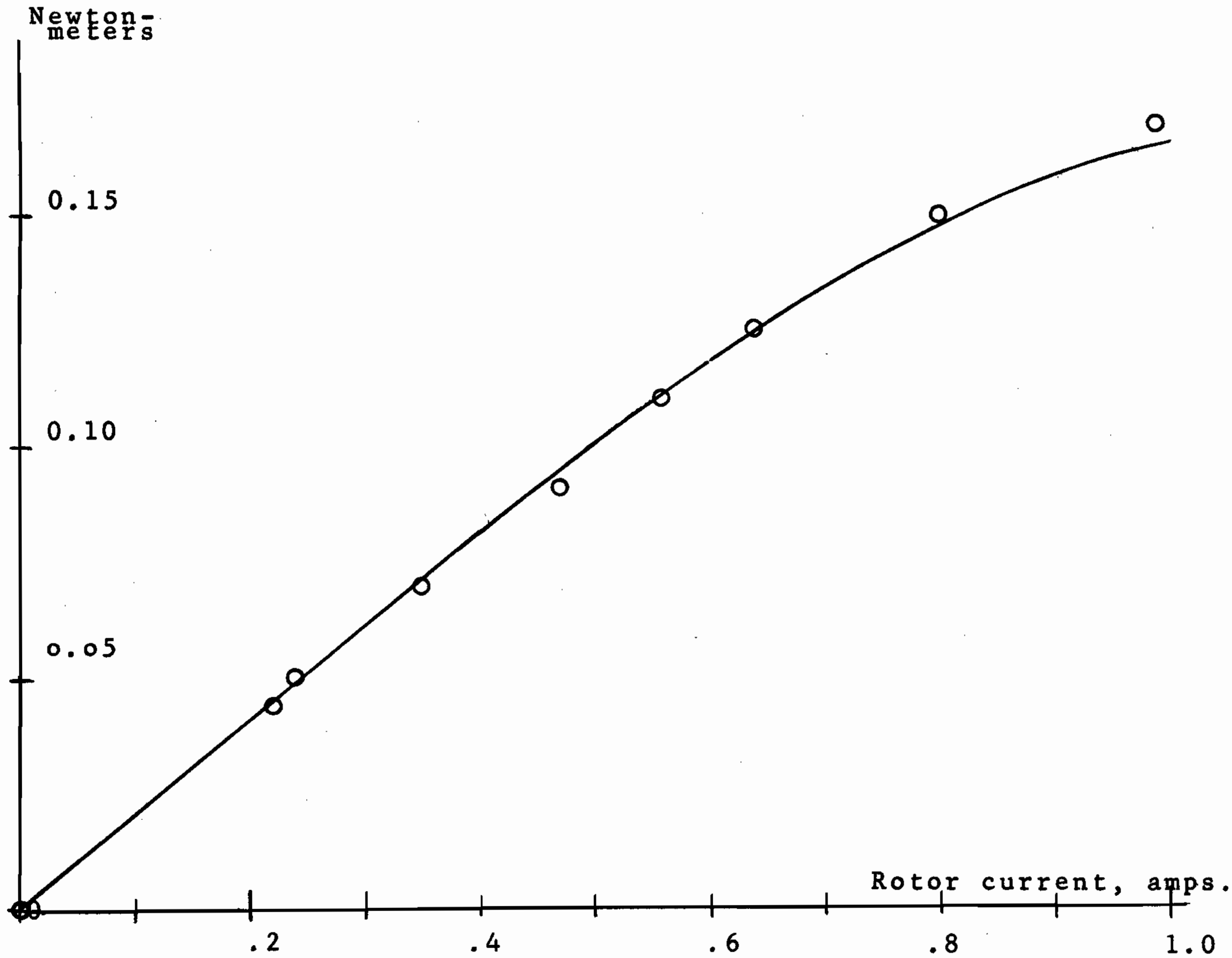


Fig. 509. Shaft torque of the test machine against the rotor current, with stator current held constant at 0.70 ampere and the rotor position at 120°. Solid line represents predicted, circles experimental values.

CONCLUSION

The foregoing development begins by postulating that an electric machine may be thought of as a purely energy-conservative set of coupled coils arranged in two groups free to move relative to each other, plus a set of series resistances. Each of these two groups may contain accessible as well as inaccessible coils (machine windings with available terminals as well as shorted windings and eddy current paths). It is first shown that there always exists a matrix relating the machine flux linkages to the coil currents; by extension of the theory of linear machines, this matrix is termed the inductance matrix. Such a matrix must always be single-valued and symmetrical under the relatively lax restrictions assumed. Subsequently it is shown that all inaccessible coils may be represented as windings coupled to one or more of the accessible coils on an incremental basis. Since an inductance matrix has been shown to exist corresponding to every set of incremental inductances, it follows that any machine may be represented by means of a finite nonlinear inductance matrix to any desired degree of accuracy.

The generalisation given for the inductance matrix is such that it closely resembles the ordinary inductance matrix of linear machine theory. The major difference appears in the fact that the components of the nonlinear inductance matrix are not constants but functions of the machine currents. A natural result of the function-matrix nature of the inductances is that the inductance matrix does not relate voltages to currents directly; it is rather the incremental inductance matrix that does so. The incremental inductance matrix can of course always be constructed from the inductance matrix itself. In the linear case where all derivatives of inductances with respect to currents are zero, the total and incremental inductance matrices coincide; in the general case, they do not.

By considering the energy involved in moving a machine from one position and set of currents to another, an integral expression for shaft torque is derived. In the special case of linearity of the inductance matrix, this expression reduces to the usual slip-ring machine torque equation of Kron; it is also shown to be in accord with the quasi-Lagrangian formulations of Cherry or White and Woodson. An experimental verification of this torque equation in a definitely nonlinear case is shown to result in accurate torque predictions for the test machine.

As byproducts of the development, an experimental verification of Weber's eddy-current theory (1931) is given for the first time, and a new method of synthesizing canonic forms of two-element-kind networks is found.

It is therewith shown that at the price of an increase in complexity, slip-ring machines may now be analysed in terms of a true non-linear slip-ring primitive. As a corollary, the generalised machine theory of Kron is shown to be valid independently of the assumption of linearity. In view of the availability of fast digital computers, wide application of the nonlinear primitive is hindered only by the difficulty of collecting adequate quantities of data about electric machines. It is to be hoped that early development of rapid ways of accumulating data will make possible the fullest exploitation of this most general primitive machine established to date. Equally, further physical investigation of the flux and inductance potential functions may be hoped to yield scalar functions more easily determined experimentally than the matrices used to date.

REFERENCES

1. Historical Sketch of the Rise and Progress of Electromagnetic Engines for Propelling Machinery. Annals of Electricity, vol. III no. 17, March 1838, p. 429. (Editorial article, presumably written by Sturgeon and certainly approved by him.)

2. Biography of William Sturgeon by William Gee. Dictionary of National Biography, vol. LV, pp. 131-5. New York: Macmillan, 1898.

3. Lectures on the Electromagnet, by Silvanus P. Thompson. New York: Johnston, 1891.

4. Researches in Electro-Dynamics, Experimental and Theoretical, by William Sturgeon, Lecturer at the Hon. East India Company's Military Academy, Addiscombe. Annals of Electricity, vol. II, January 1838, no. 7.

5. Flame, Electricity, and the Camera. G. Iles. New York: Hill 1904.

6. The Electrician, 1885-6; Proc. Inst. Civil Engineers, 1885-6; Jour. Soc. Telegr. Engineers, vol. XV, 1886, p. 524; Phil. Trans. 1886 pt. I p. 331. Cited by Thompson, op. cit. p. 13 footnote.

7. C. P. Steinmetz, General Lectures on Electrical Engineering. New York: McGraw-Hill, 1918.

8. Methods of Calculation of Armature Reaction of Alternators. A. Blondel. Transactions of the International Electrical Congress, St. Louis, Mo., 1904, pp. 620-35.

9. Two-Reaction Theory of Synchronous Machines, Parts I and II. R. H. Park. Trans. AIEE, vol. 48, 1929, pp. 716-27; ibid. vol. 52, 1933, pp. 352-5.

10. Synchronous Machines I and II--An Extension of Blondel's Two-Reaction Theory. R. E. Doherty, C. A. Nickle. Trans. AIEE, vol. 45, 1926, pp. 912-42.

11. A. S. Langsdorf, Principles of Direct-Current Machines. Sixth edition. New York: McGraw-Hill, 1959.

12. C. Concordia, Synchronous Machines. New York: Wiley, 1951.

13. A. Still, C. S. Siskind, Elements of Electrical Machine Design. Third edition. New York: Mc-Graw-Hill, 1954.

14. The Short-circuit characteristics of DC Generators. G. E. Frost. Trans. AIEE, vol. 65, 1946, pp. 394-402.

15. Current and Torque of DC Machines on Short Circuit. T. M. Linville. Trans. AIEE, vol. 68, 1949, pp. 956-65.

16. Transient Performance of DC Machinery. J.Cybulski, E. L. Brancato, J. P. O'Connor. Trans. AIEE, vol. 72, 1953, pp. 45-52.

17. A New Method for Observing the Phenomena of Commutation. H. J. H. Sketch, P. A. Shaw, R. J. K. Splatt. IEE paper no. 2955M; June 1959.

18. The Measurement of Transient Torque and Load Angle in Model Synchonous Machines. R. N. Sudan, V. N. Manohar, B. Adkins. IEE paper no. 3151S; February 1960.

19. K voprosu o forsirovannom razvozbuzhdenii sinkhronnykh mashin. N. N. Nikiforovskii. Elektrichestvo, No. 9 (September), 1963, pp. 44-48.

20. Generalized Theory of Electrical Machinery. G. Kron. Trans. AIEE, vol. 49, 1930, pp. 666-85.

21. Dictionary of National Biography, vol. LV, pp. 132-3. New York: Macmillan, 1898.

22. Introduction by Banesh Hoffman to the Dover edition of G. Kron, Tensors for Circuits (Reference 64).

23. L. V. Bewley, Tensor Analysis of Electric Circuits and Machinery. New York: Ronald, 1961.

24. B. Adkins, The General Theory of Electrical Machinery. New York: Wiley, 1951.

25. A. Tustin, Direct Current Machines for Control Systems. London: Spon, 1952.

26. A. E. Fitzgerald, C. Kingsley, Jr., Electric Machinery. New York: McGraw-Hill, 1952. Second edition considerably altered in details, but retains the approach.

27. Transient Response of Direct-Current Dynamos. H. E. Koenig, Trans. AIEE, vol. 69, 1950, pp. 139-145.

28. An Analysis of Rotating Amplifiers. B. Litman. Trans. AIEE, vol. 68, 1949, pt. II, pp. 1111-17.

29. Dynamics of D-C Machine Systems. M. Riaz. Trans. AIEE, vol. 75, 1956, pt. III, pp. 365-9.

30. The Tensor Equations of Electrical Machines. J. W. Lynn. IEE Monograph No. 117S, January 1955.

31. Two-Axis Method of Analysing Electric Machines. K. Y. Tang, R. L. Cosgriff. Trans. AIEE, vol. 74, 1955, pt. III, pp. 1449-55.

32. The Impedance Tensor of the General Machine. Y. N. Yu. Trans. AIEE, vol. 75, pt. I, 1956, pp. 181-7.

33. The Torque Tensor of the General Machine. Y. N. Yu. Transactions Paper 62-118 of the IEEE.

34. W. J. Gibbs, Tensors in Electrical Machine Theory. London: Chapman and Hall, 1952.

35. D. C. White, H. H. Woodson, Electromechanical Energy Conversion. New York: Wiley, 1959.

36. Energy Conversion in Double Cylindrical Structures. R. M. Saunders. IEEE Transactions on Power Apparatus and Systems, Oct. 1963, pp. 631-9.

37. Quick Response of Large D.C. Machines. J. Hindmarsh. Electrical Times, vol. 135, 1959, Jan. 15--Feb. 5.

38. Transient Characteristics of DC Metal-Rolling Motors and Generators. E. P. Smith. Trans. AIEE, vol. 75, pt. II, 1956, pp. 326-31.

39. Vliyaniye nesimmetrii namagnichivayushchikh sil glavnykh polyusov na dobavochnye polyusy i rabochiye kharakteristiki mashin. I. Z. Ageev. Elektrichestvo, 1961, no. 9 (September), pp. 37-41.

40. Equations depicting the Operation of the D.C. Motor. R. W. Ahlquist. Trans. AIEE, vol. 73, 1954, pt. III, pp. 1499-1505.

41. Electromagnetic and Mechanical Effects in Solid Iron due to an Alternating or Rotating Magnetic Field. R. D. Pohl. Journal IEE, vol. 91, 1944, pt. II, p. 239.

42. Rise of Flux due to Impact Excitation. R. D. Pohl. Proc. IEE, vol. 96, 1949, pt. I.

43. Rise of Flux in Solid Iron Core due to Impact Excitation. N. Kesavamurthy, P. K. Rajagopalan. IEE Monograph 339U, 1959

44. Vliyaniye Vikhrevykh tokov na perekhodnye protsessy v elektromagnitakh. V. A. Karasev. Elektrichestvo, 1963, no. 9 (September), pp. 33-37.

45. Field Transients in Magnetic Systems: A Study of Field Transients in Iron Cores Partially Laminated and Partially Solid. E. Weber. Trans. AIEE, December 1931, pp. 1234-46.

46. Transients in Magnetic Systems. C. F. Wagner, Trans. AIEE, vol. 53, 1934, pp. 418-25.

47. Vliyaniye vikhrevykh tokov na protsess ustanovleniya potoka. S. Ya. Dunaevskii, Elektrichestvo, 1951, no. 2 (February), p. 55-63.

48. DC Machines: Response to Impact Excitation. J. J. Brockman, C. E. Linkous. Trans. AIEE, vol. 74, 1955, pt. III, pp. 500-5.

49. The Duality between Interlinked Electric and Magnetic Circuits and the formation of Transformer Equivalent Circuits. E. C. Cherry. Phil. Mag., vol. 40, 1949, pp. 101-11.

50. An Analysis of DC Machine Commutation. J. R. M. Alger, D. T. Bewley, Trans. AIEE, vol. 76, 1957, pt. III, pp. 399-416.

51. Zur Stromwendung von Gleichstrommaschinen. Heinz Nacke. Elektrotechnische Zeitschrift, Ausgabe A, vol. 84, 1963, no. 3, pp. 83-5.

52. Osnovy energeticheskoi teorii kommutstsii mashin postoyannogo toka. A. S. Kurbasov. Elektrichestvo, 1962, no. 7 (July), pp. 23-8.

53. O raschete kommutatsii mashin postoyannogo toka. A. S. Kurbasov. Elektrichestvo, 1962, no. 8 (August), pp. 69-73.

54. J. K. Hale, Oscillations in Nonlinear Systems. New York: McGraw-Hill, 1963.

55. N. Minorsky, Nonlinear Oscillations. Princeton, N. J.: Van Nostrand, 1962.

56. An Analysis of Commutation for the Unified-Machine Theory. C. V. Jones. IEE Monograph No. 302U, April 1958.

57. A Practical Commutator Primitive for Generalized Machine Theory. T. H. Barton, C. V. Jones. AIEE Transactions Paper No. 60-169, 1960.

58. Measurement and Calculation of DC Machine Armature Circuit Inductance. H. D. Snively, P. B. Robinson. Trans. AIEE, vol. 69, 1950, pt. II, pp. 1228-37.

59. Measurement and Calculation of DC Machine Parameters. R. M. Saunders, Trans. AIEE, vol. 70, 1951, pt. I, pp. 700-6.

60. A Method of Measuring Self-Inductances Applicable to Large Electrical Machines. J. C. Prescott, A. K. El-Kharashi. IEE Paper No. 2871M, April 1959.

61. Dynamic Circuit Theory, An Experimental Approach. T. H. Barton, AIEE Conference Paper No. 62-1226, 1962.

62. Private communication. Material to be published soon as part of M. Eng. thesis, McGill University, by J. Dunfield.

63. A. P. Wills, Vector Analysis with an Introduction to Tensor Analysis. New York: Dover, 1961.

64. G. Kron, Tensors for Circuits. New York: Dover, 1959. Particularly Chapter 26 is of interest.

65. H. H. Skilling, Electromechanics. New York: Wiley, 1962.

66. P. C. Shedd, Fundamentals of Electromagnetic Waves. Englewood Cliffs, N. J.: Prentice-Hall, 1954.

67. Some General Theorems for Nonlinear Systems Possessing Reactance. E. C. Cherry. Phil. Mag., vol. 42, 1951, pp. 1161-77.

68. Classes of 4-Pole Networks Having Nonlinear Transfer Characteristics but Linear Iterative Impedances. E. C. Cherry. IEE Paper No. 3114E, January 1960.

69. Some General Theorems for Nonlinear Systems Possessing Resistance. W. Millar. Phil. Mag., vol. 42, 1951, pp. 1150-60.

70. L. A. Pipes, Matrix Methods for Engineering. Englewood Cliffs, N. J.: Prentice-Hall, 1963.

71. E. L. Ince, Ordinary Differential Equations. New York: Dover, 1956.

72. A. G. Webster, Partial Differential Equations of Mathematical Physics. New York: Dover, 1955.

73. Variation of the Inductance of Coils due to the Magnetic Shielding Effect of Eddy Currents in the Cores, K. L. Scott. Proc. IRE, vol. 18, 1930, no. 10, pp. 1750-64.

74. M. E. Van Valkenburg, Network Analysis. Englewood Cliffs, N. J.: Prentice-Hall, 1955.

75. F. M. Reza, S. Seely, Modern Network Analysis. New York: McGraw-Hill, 1959.

76. D. Hazony, Elements of Network Synthesis. New York: Reinhold, 1963.

77. M. E. Van Valkenburg, Modern Network Synthesis. New York: Wiley, 1960.

78. D. V. Widder, Advanced Calculus. New York: Prentice-Hall, 1947.

79. H. S. Carslaw, Introduction to the Theory of Fourier's Series and Integrals. New York: Dover, 1949.

80. L. A. Bessonov, Avtokolebaniya v elektricheskikh tsepyakh so stalyu. Moscow: Gosenergoizdat, 1958.

81. P. P. Biringer, Magnetic Frequency Doublers. University of Toronto, Department of Electrical Engineering: Research Report No. 5. January 1956.

82. T. H. Barton, A Generalized Theory of Electro-Mechanical Energy Conversion. Multigraphed course notes; McGill University, Montreal, 1961.

83. Generalized Machine. Westinghouse Electric Corporation, Motor and Control Division. Buffalo, N. Y., 1960. (Instruction book intended for use with the MIT-Westinghouse laboratory machine.)

APPENDIX A

SYNTHESIS OF R-L NETWORKS

1. Immittance Functions of R-L Networks. The properties of networks realisable by means of resistive and inductive components only, not involving capacitors, have been examined by numerous authors (e.g. Van Valkenburg[A1], Tuttle[A2]). The pertinent properties are readily listed; they are apparently most easily given in terms of the immittance poles and zeros of the network. The critical frequencies (pole and zero locations) of such networks must be pure imaginaries. That is, in a complex frequency plane (s-plane) the poles and zeros must all lie on the negative real axis. Further, all poles and zeros must be simple, and they must be interlaced; any given zero must have a pole to either side of it, and vice versa. The lowest critical frequency must be a zero of impedance which may or may not occur at the origin; the highest critical frequency must be a pole of impedance which may or may not occur at infinite frequency. The immittance must be pure real for all real s; furthermore, it must be a monotonic function of s along the real axis, the impedance always decreasing and the admittance increasing with increasing imaginary frequency.

These properties are in nearly all respects very similar to the properties of purely reactive networks, except that imaginary rather than real frequencies are referred to. It is usual in books on network theory, as a matter of fact, to examine the purely reactive case first, and subsequently treat the R-C and R-L cases by means of a coordinate transformation that maps the positive and negative real frequency semiaxes of the s-plane into the negative imaginary frequency axis. The properties listed above are then automatically obtained. If required, such networks may then be synthesized using either the Foster or Cauer techniques borrowed from the theory of purely reactive networks via the coordinate transformation.

Some difficulty, however, is encountered in such syntheses, because of the nature of the coordinate transformation. In the case of R-L networks the residues at the poles of the impedance function are all real and negative[A1], so the Foster partial-fraction technique leads to negative inductors. The usual prescription, heuristically arrived at, is to work with the function $\frac{Z(s)}{s}$ instead of the impedance Z(s) itself.

Now any technique of network synthesis is essentially an organised method for rewriting impedance or admittance functions in such a way as to permit recognition of complicated functions as agglomerations of resistive, inductive, or capacitive elements. The structure of the agglomeration then yields the mode of interconnection as well as the required element values. In order for the networks synthesized to be real passive networks, the element values must all be positive.

2. Synthesis of a Standard Form. From the point of view of the electric machines engineer, it is felt that the restriction to passive two-terminal elements in the Foster and Cauer canonic forms is excessively restrictive. The requirement of two-terminal inductors, in particular, amounts to the specification of zero mutual inductance--a strikingly needless condition especially in view of the fact that precisely a nonzero mutual inductance can be called upon to furnish the negative sign attached to each residue at the poles of an R-L impedance function. This latter statement is easily verified by examining the network of Fig. A-1. The impedance presented at the terminals of this network is seen to be

$$Z(s) = L_1 s - \frac{M^2 s^2}{L_2 s + Z_2} \tag{A-1}$$

More generally, a network consisting of an inductor and a resistor with a number of other similar networks with zero impressed voltage, all electrically unconnected but magnetically coupled as shown in Fig. A-2, may be described by

$$Z(s) = R_1 + L_1 s - s^2 [M_C]_t [Z_C^{-1}] [M_C] \tag{A-2}$$

where the subscript C refers to all the coils other than the one with accessible terminals. The coefficient of s^2 in this equation clearly represents a set of admittances which are not only realisable by means of R-L networks, but in fact exist in Fig. A-2.

Synthesis is now an easy matter, once the negative sign of each residue has been associated with a mutual inductance. Consider an impedance function with a pole at infinity and a zero at zero. It may be written

$$Z(s) = Hs \frac{P(s)}{Q(s)} \tag{A-3}$$

where H is a constant, and P(s) and Q(s) are polynomials of order n with the coefficients of s^n and s^o nonzero. One may proceed with long division and write

$$Z(s) = HK_0 s + Hs^2 \frac{U(s)}{Q(s)} \tag{A-4}$$

where U(s) is a polynomial one order lower than Q(s), so that a partial-fraction expansion may be made:

$$Z(s) = HK_0 s + Hs^2 \sum_{j=1}^{n} \frac{K_j}{s + \gamma_j} \tag{A-5}$$

All the coefficients K_j, being the residues of Z(s), must be negative. The development of equation (A-5) from (A-3), it may be recognised, is precisely the procedure prescribed by Van Valkenburg for the synthesis of Foster series forms, except for the "mistake" of forgetting to divide by s at the very beginning to enable a partial fraction expansion of $\frac{Z(s)}{s}$ to be made. Because of this "mistake", no Foster form can be extracted from equation (A-5). However, the network is known to be realisable; and comparison of equation (A-4) with (A-2) leads immediately to the conclusion that a coupled-circuit realisation must result.

The case in which the lowest zero is not at the origin is seen to yield the same treatment and result, except that a constant additive term appears on the right side of equation (A-4). This term represents Z(0) and is of necessity always positive so that it may be realised by a simple series resistor. The possibility of the highest pole not at infinity may be dealt with by first extracting $Y(\infty)$ in the form of a shunt resistor. The remainder is synthesizable by Cauer methods, and therefore guaranteed synthesizable by any other technique as well.

It is important to realise that the mutual inductances here dealt with are not Brune ideal transformers arising from the reconstruction of negative inductances; they are physically real mutual terms with less than unity coupling. This aspect of the problem will no doubt be clearly seen in the examples to follow.

3. Some Simple Syntheses. To illustrate the principle involved, let the simple impedance function

$$Z(s) = 2 \frac{(s+1)(s+4)}{(s+2)}$$

be required. As is immediately seen, the expression is realisable.

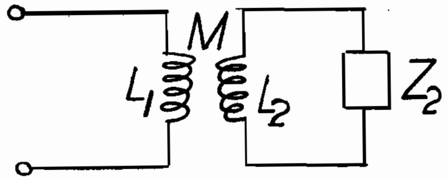


Fig. A-1

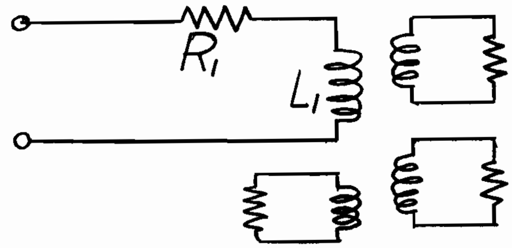


Fig. A-2

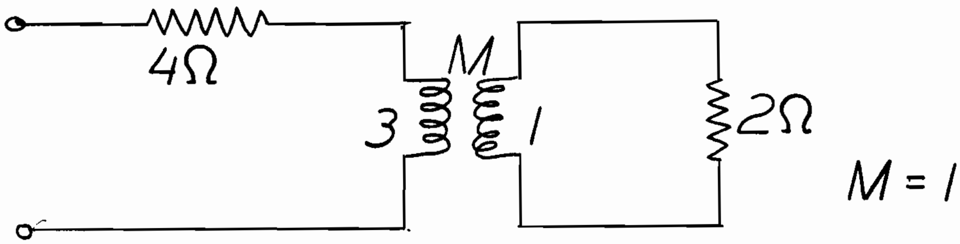


Fig. A-3

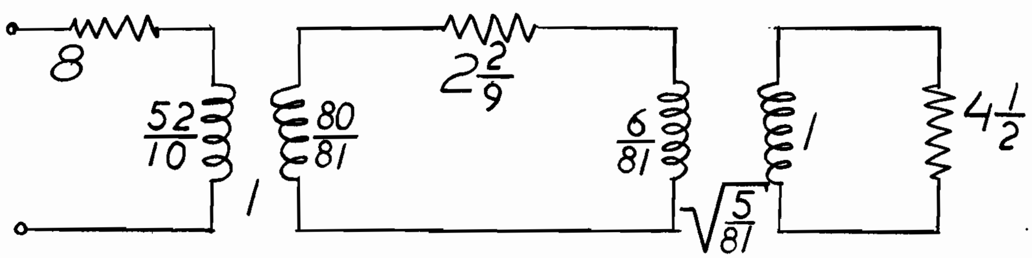


Fig. A-4

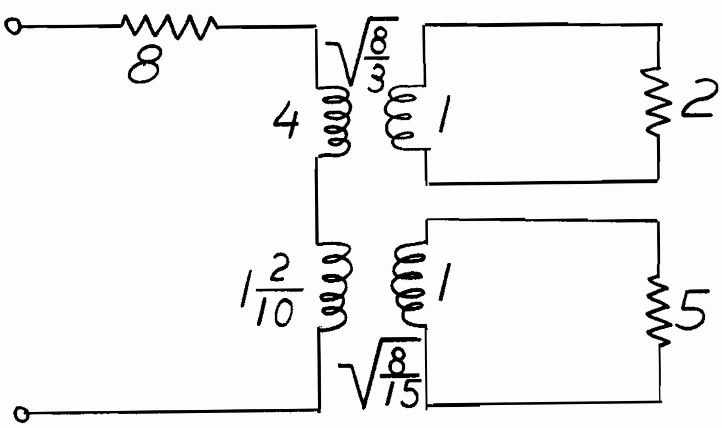


Fig. A-5

The required method of rewriting is easily obtained by long division:

$$
\begin{array}{r|l}
2+s & 8+10s+2s^2 \quad = \quad 4+3s-\dfrac{s^2}{2+s} \\
 & \underline{8+4s} \\
 & \quad 6s+2s^2 \\
 & \quad \underline{6s+3s^2} \\
 & \qquad -s^2
\end{array}
$$

so that

$$Z(s) = 4 + 3s - \frac{s^2}{2+s} \tag{A-7}$$

which may be recognised, term by term, as the network of Fig. A-3.

A more interesting case is one of the next order of complexity, so that more than one secondary winding is required. Consider, for example

$$Z(s) = 2\,\frac{(s+1)(s+4)(s+10)}{(s+2)(s+5)} \tag{A-8}$$

which is again a realisable function. The primary quantities are quickly extracted by one long division

$$
\begin{array}{r|l}
10+7s+s^2 & 80+108s+30s^2+2s^3 \quad = \quad 8+\frac{52}{10}s+\ldots\ldots \\
 & \underline{80+56s+8s^2} \\
 & \quad 52s+22s^2+2s^3 \\
 & \quad \underline{52s+\frac{364}{10}s^2+\frac{52}{10}s^3} \\
 & \qquad -\frac{144}{10}s^2-\frac{32}{10}s^3
\end{array}
$$

so that the impedance function may be rewritten

$$Z(s) = 8 - \frac{52}{10}s - \frac{16}{5}s^2 \frac{(s + 4\frac{1}{2})}{(s+2)(s+5)} \qquad \text{(A-9)}$$

But the synthesis is not now complete. Several avenues exist from this point. As the first mode of procedure, it may be imagined that each coil is coupled only to those with numbers one higher and one lower, so that the nonzero terms in the impedance matrix of the network of Fig. A-2 are those on the principal diagonal as well as the immediately adjoining elements. This restriction leads to comparison of equations (A-9) and (A-1), and consequent realisation by applying the technique a second time to the last term in equation (A-9). The network obtained is that of Fig. A-4.

Alternatively, all the mutual inductances may be imagined zero except those in the first row and column of the impedance matrix so that coupling exists between the input coil and any other, but no coupling between any nonaccessible pair. The network obtained is then that of Fig. A-5; the values are readily obtained by partial-fraction expansion of equation (A-9):

$$Z(s) = 8 + \frac{52}{10}s - \frac{8}{3}s^2 \frac{1}{s+2} - \frac{8}{15}s^2 \frac{1}{s+5} \qquad \text{(A-10)}$$

It is to be noted that the values of circuit elements are not uniquely determined by the above expression; only two numbers, the pole location and the residue, are prescribed for finding the elements R and L, and the coupling M. This fact may lead to interesting possibilities in the practical construction of networks, since coupling coefficient between two coils is frequently easier to adjust than the inductances themselves.

In the most general case, none of the mutual inductances might be taken to be zero. An infinite number of possible network configurations is seen to result, all similar in form but different in element values. Analogously, of course, many possible syntheses of a higher-order network may be produced with zero coupling, by using intermixtures of the Foster and Cauer forms.

4. Two Big Questions. Since the synthesis of resistive-inductive networks for their own sake is of restricted interest at the present time, the foregoing is of limited interest if no extensions from it are possible. Two questions must be answered if the synthesis technique here produced is to be of more general interest: first, is the technique applicable to L-C and R-L-C networks? secondly, are the networks canonic?

While detailed replies to these two questions are not only problems requiring considerable investigation, but also outside the scope of this thesis, the probable replies do appear to be yes. To the first question, it must be replied that any network amenable to this method of synthesis must contain a reasonable number of inductors, so only some R-L-C network functions might be so synthesizable; but some certainly are. As a simple example will demonstrate, and a retracing of the coordinate transformation mentioned will show, the synthesis of purely reactive networks is possible. Let, for example,

$$Z(s) = \frac{s(s^2+4)}{(s^2+1)} \tag{A-11}$$

As before, long division may be applied to yield

$$Z(s) = 4s + 3s^2 \frac{s}{1+s^2} \tag{A-12}$$

which may be recognised immediately as the combination of a one-farad capacitor and a one-henry inductor, coupled to a four henry primary. The coupling coefficient here is 0.866, but may be reduced by increasing the secondary capacitor size. Other similar synthesis examples may be worked out in precisely the same manner.

Again subject to the proviso that further work may turn up unexpected results, the reply to the second question (are they canonic?) seems fairly sure to be yes, at least in the R-L cases. The case in which all mutual inductances are zero save those bordering upon the main diagonal of the inductance matrix has a magnetic circuit topology closely resembling (though not congruent with) the electric circuit topology of the Foster shunt form; conversely, the case of all mutual inductances zero save those of the first row and column is reminiscent of the Foster series form in its topological structure. The new networks are not merely duals of the Foster forms in Cherry's sense[49]. They include, however, the same number of elements as the well-known Foster and Cauer forms, and the same number of constraints. The principal difference is of course the substitution of magnetic for electrical constraint equations.

While the synthesis method here provided is of immediate interest in connection with the representation of solid iron cores, it is easy to see that the resulting networks, especially the L-C forms, have some features that may well render them attractive for communications purposes. Particularly the arbitrariness of turns numbers involved in the magnetic constraints permits the use, for example, of resistors or capacitors of one value only, or of standard values only. Such a possibility may be attractive in certain applications.

APPENDIX B

COORDINATES IN N-SPACE

The spaces dealt with by Kron in his original version of the theory of electric machines, as those employed by later investigators, tend to be Cartesian. They are generally stationary or quasi-stationary and orthogonal, although at times nonstationary ones are encountered (e.g. in acceleration problems). Such Cartesian coordinate systems are easy to visualise and to interpret physically. On the other hand, some other kinds of coordinates lend themselves better to some of the mathematical manipulations that arise. Of particular interest in this thesis is the case of hyperspherical coordinates. To show the possibility of converting any Cartesian space into a corresponding hyperspherical one, the transformation formulas will be developed here.

Consider the very simplest nontrivial space, that is, two-space. If it is described in terms of Cartesian coordinates x_1, x_2, hyperspherical (here usually called polar) coordinates may be introduced by the relations

$$\begin{aligned} x_1 &= r_2 \cos \theta_2 \\ x_2 &= r_2 \sin \theta_2 \end{aligned} \tag{B-1}$$

To construct a similar set of relations for three-space, the radius vector r_2 of the two-space may be thought of as the projection on the x_1-x_2 plane of the three-dimensional radius vector r_3. As that vector in general is at an angle θ_3 to that plane, such a relationship may be expressed by

$$r_2 = r_3 \cos \theta_3 \tag{B-2}$$

The relationships between three-dimensional Cartesians and sphericals may then be stated as

$$x_1 = r_3 \cos\theta_2 \cos\theta_3$$
$$x_2 = r_3 \sin\theta_2 \cos\theta_3 \qquad \text{(B-3)}$$
$$x_3 = r_3 \sin\theta_3$$

Imagining next that the three-space case is merely the projection on the x_1-x_2-x_3 subspace of the four-dimensional radius vector r_4, making an angle θ_4 with this subspace, there next obtains

$$r_3 = r_4 \cos\theta_4 \qquad \text{(B-4)}$$

This process may be continued for any number of dimensions, with the result in the n-dimensional case

$$x_1 = r_n \cos\theta_2 \cos\theta_3 \ldots\ldots \cos\theta_{n-1} \cos\theta_n$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$x_m = r_n \sin\theta_m \cos\theta_{m+1} \cos\theta_{m+2} \ldots\ldots \cos\theta_n \qquad \text{(B-5)}$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$x_n = r_n \sin\theta_n$$

It is easily verified that the radius r_n is in fact the root-sum-square value of the Cartesian coordinates, and thus represents the hyperdistance from the origin of the coordinate system.

As any space of finite or countably infinite dimensionality may this way be described in terms of hyperspherical coordinates, it follows that the differential equations encountered in the body of the thesis may be solved in such a coordinate system.

APPENDIX C

RADIAL INCREMENT AND AVERAGE OPERATORS

1. The two inverse operators. As seen in the body of this thesis, differential equations of the form

$$\left(1 + \bar{i}\cdot\nabla\right) X = Y \qquad \text{(C-1)}$$

or, in terms of Cartesian n-spatial coordinates,

$$X + i_1 \frac{\partial X}{\partial i_1} + i_2 \frac{\partial X}{\partial i_2} + \ldots\ldots + i_n \frac{\partial X}{\partial i_n} = Y \qquad \text{(C-2)}$$

frequently arise in the analysis of nonlinear systems of the kinds discussed. The source of such equations is the fundamental relationship between total and incremental quantities such as inductance. Solutions of differential equations of this form will naturally be required; they will be secured in this Appendix.

The operator operating on the function X of equation (C-1) above will be called the radial increment operator, and the integral operator

$$\frac{1}{r}\int_0^r [\quad] d\rho$$

will be called the radial averaging integral. To begin, it shall be shown that the following theorem holds:

Theorem 1: The radial increment operator and the radial averaging integral are inverse operations:

$$\left(1+\bar{i}\cdot\nabla\right)\left[\frac{1}{|i|}\int_0^i \phi\, d\rho\right] = \frac{1}{|i|}\int_0^i \left(1+\bar{\rho}\cdot\nabla\right)\phi\, d\rho = \phi \qquad \text{(C-3)}$$

The differential operator portion of the increment operator may be regarded as the radius vector multiplied by the radial directional derivative. Equation (C-3) may thus be written

$$\frac{1}{|i|}\int_0^i \phi\, d\rho + |i|\left[-\frac{1}{|i|^2}\int_0^i \phi\, d\rho + \frac{1}{|i|}\,\phi\right] = \phi \qquad \text{(C-4)}$$

or, simplifying,

$$\left(\frac{1}{|i|} - \frac{1}{|i|}\right)\int_0^i \phi\, d\rho + \phi = \phi \qquad \text{(C-5)}$$

which is seen to be an identity. The theorem is thus proven subject to the restriction that ϕ may have an arbitrary radially invariant component. For physical reasons, such arbitrary additional terms are not of interest at the present. The inverse theorem (operators operating in the inverse order) is of course proved in exactly the same manner; it is not subject to the arbitrariness mentioned in precisely the same way.

<u>2. Solution of a differential equation.</u> With the aid of the radial averaging operator, equation (C-1) may now be solved. The solution, for reasons physically evident in the body of the thesis, is here subjected to the boundary conditions of finite value at all finite boundaries and points. As any differential equations problem, this one may be solved by first finding the solution of the homogeneous part and then seeking a particular integral by means of the averaging operator. The homogeneous equation

$$(1 + \bar{i}\cdot\nabla)\, X = 0 \qquad \text{(C-6)}$$

may be solved by separation of variables. Assume the function X to be expressible as

$$X = R(r)\ T_1(\theta_1)\ T_2\ (\theta_2)\ldots\ldots T_{n-1}(\theta_{n-1}) \qquad \text{(C-7)}$$

where hyperspherical coordinates (Appendix B) are employed. Substituting equation (C-7) into (C-6) produces two possibilities: the trivial case of one of the functions T zero, or the requirement

$$R + rR' = 0 \tag{C-8}$$

whose solution is easily found to be

$$\begin{aligned} rR &= F(\theta_1, \theta_2, \ldots, \theta_{n-1}) \\ R &= \frac{1}{r} F \end{aligned} \tag{C-9}$$

The arbitrary function of angles F is easily evaluated from the boundary condition that X and hence R must remain finite at the origin. At the origin, r = 0 so necessarily F = 0 also. The solution of the homogeneous equation must then be zero with the boundary conditions imposed.

The particular integral required is quickly obtained by operating on both sides of equation (C-1) with the radial averaging operator. There results

$$\frac{1}{R}\int_0^R (1+\bar{i}\cdot\nabla) X \, di = \frac{1}{R}\int_0^R Y \, di \tag{C-10}$$

which may by Theorem 1 be written

$$X = \frac{1}{R}\int_0^R Y \, di \tag{C-11}$$

so that the result may be stated as

<u>Theorem 2</u>: Differential equation (C-1), subject to the requirement of finite value at all finite points, has as its complete solution equation (C-11).

APPENDIX D

THE COENERGY APPROACH TO THE TORQUE EQUATION

The development of the torque equation by way of the mean value theorems of Chapter IV may be shortened by deriving the torque equation in the manner suggested by the quasi-Lagrangian formulation of White and Woodson[35]. By taking the coenergy of an inductive system to be the line integral of flux linkages, as suggested in Chapter II, the derivation may be carried out in scalar rather than vector terms. The simplification in the mathematics is substantial.

Consider two neighbouring states of a system, one given by $\overline{I}$, θ_o, the other by $\overline{I} + \delta\overline{I}$, $\theta_o + \delta\theta$. The system may be moved from one state to the other in a large number of ways. There is only one evident way, however, of moving it from the initial state to the final in an energy-conservative manner. Such a path will lead first from the initial state to the origin with the angle invariant, so no mechanical work is done. At the origin, with zero stored energy and coenergy, the angle may be altered to its terminal value without involving mechanical work; then the currents may be built up to their terminal values. Let the change in stored energy along this path be ΔU and the change in coenergy ΔT. In operation the machine will follow some other path; let the change in input energy along that path be δU and the change in co-energy δT. Since the terminal values of the fluxes and currents of the system must be the same regardless of path, the end-state energy and co-energy must also be independent of the path. Thus the extra input energy and co-energy in following the nonconservative path must be associated with mechanical work done along the latter contour. The equality of fluxes and currents at the end-state may be expressed as the equality of the flux-current product, or

$$\Delta U + \Delta T = \delta U + \delta T \tag{D-1}$$

This equation may be written

$$\Delta U - \delta U = -(\Delta T - \delta T) \tag{D-2}$$

The left member of equation (D-2) is the difference in energy put in. Conservation of energy requires that this amount be equal the amount of energy taken out mechanically. Letting the work done be δW,

$$\delta W = -(\Delta T - \delta T) \tag{D-3}$$

The increments of coenergy on the right are not difficult to evaluate. The increment along the conservative path is

$$\Delta T = (\Delta T + T) - T = \int_0^{I+\delta I} \bar{\lambda}(\theta_0 + \delta\theta) \cdot d\bar{i} - \int_0^{I} \bar{\lambda}(\theta_0) \cdot d\bar{i} \tag{D-4}$$

which may be written, for small current and angle changes,

$$\Delta T = \delta\theta \frac{\partial T}{\partial \theta} + \overline{\delta I} \cdot \bar{\lambda}(\theta_0 + \delta\theta) \tag{D-5}$$

The increment in coenergy along the nonconservative path is, by definition,

$$\delta T = \int_{\bar{I}, \theta_0}^{\bar{I}+\delta\bar{I}, \theta_0+\delta\theta} \bar{\lambda} \cdot d\bar{i} \tag{D-6}$$

or, for small increments, using the mean value theorems,

$$\delta T = \overline{\delta I} \cdot \bar{\lambda}(\theta_0 + \zeta\delta\theta) \tag{D-7}$$

where $0 \leqq \zeta \leqq 1$.

In this way, equation (D-3) may be written

$$-\delta W = \delta\theta \frac{\partial T}{\partial \theta} + \overline{\delta I} \cdot \left[\bar{\lambda}(\theta_0 + \delta\theta) - \bar{\lambda}(\theta_0 + \zeta\delta\theta) \right] \tag{D-8}$$

As the angle and current increments are taken small, they may now be permitted to approach zero. In the limit,

$$\frac{dW}{d\theta} = -\frac{\partial T}{\partial \theta} \tag{D-9}$$

Substituting the definition of co-energy and the definition of inductance in turn, there readily obtains

$$\Gamma = \int \frac{\partial \bar{\lambda}}{\partial \theta} \cdot d\bar{i} \tag{417}$$

$$\Gamma = \int \bar{i} \cdot \frac{\partial \overline{\overline{L}}}{\partial \theta} \cdot d\bar{i} \tag{418}$$

These results, it may be noted, bear some resemblance to those cited by White and Woodson. There are, however, differences both in application as well as in interpretation that should not be overlooked. It is important to note that the formulation of co-energy changes given by White and Woodson is in the usual Lagrangian form, requiring that the system of n currents and m mechanical degrees of freedom be treated in a space of dimensionality n+m. In such a space, many of the simplest Cartesian coordinate transformations become meaningless, while an attempt to use coordinates such as the hypersphericals of Appendix B converts the entire space into a dimensionally inhomogeneous entity. In the present development, the restriction of the problem to only one mechanical degree of freedom permits viewing that degree as a parameter, and preserves the Kronian current space as such.

The restrictions applying to the present development, it will be seen, are very much the same as in Chapter IV.