Locating the Zeros of an Analytic Function by Contour Integrals

by

Eugene Kicak

Abstract

With the ready availability of complex arithmetic in present day computers it is now possible to locate the roots of an analytic function $f(z)$ by contour integrals in the complex plane. If n is the number of roots inside the contour C, then

$$n = \frac{1}{2\pi i} \int_C \frac{f'(z)}{f(z)} dz .$$

This formula is used in conjunction with Romberg's integration method to produce practical algorithms.

After this study was started, the theoretical results needed were found in a paper by J.N. Lyness and L.M. Delves: "On Numerical Contour Integration Round a Closed Contour" (Mathematics of Computation, vol. 21, 1967). However the results are clarified and amplified, and the fine details of the practical aspects are checked here. The procedures are put together into practical computer programs and then illustrated by a number of examples. We also evaluate numerically

$$\frac{1}{2\pi i} \int_C z \frac{f'(z)}{f(z)} \, dz$$

to get a reasonable approximation to a root when it is isolated inside C. The problem of multiple roots is also considered and illustrated with an example.

Mathematics Department

M.Sc. Thesis

Complex Zeros by Numerical Integration

LOCATING THE ZEROS OF AN ANALYTIC FUNCTION BY CONTOUR INTEGRALS

by

Eugene Kicak

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CHAPTER 1

INTRODUCTION

Many iterative methods are currently available for an accurate determination of the zeros of an analytic function provided a reasonable first approximation is given. One of the best known and simplest methods is Newton's iteration:

$$(1.1.1) \qquad z_{n+1} = z_n - \frac{f(z_n)}{f'(z_n)} .$$

However it is a much more difficult problem to find the rough location of the zeros in order to apply these iterative methods as the final stage in the calculation of the roots.

Here our aim is to find such approximations by a method which depends on the explicit use of Cauchy's Theorem: if $f(z)$ is analytic in a simply connected region R and C is any regular closed curve in R such that no zeros of $f(z)$ occur on C, then

$$(1.1.2) \qquad n = \frac{1}{2\pi i} \int_C \frac{f'(z)}{f(z)} \, dz$$

where n is the number of zeros of $f(z)$ that occur in the interior of C, a zero of order k being counted k times.

We choose two different ways of moving along the contour C. In the first approach we consider a function $f(z)$ which is analytic in a region containing a rectangle (or square) with sides parallel to the coordinate axes. Let (X_c, Y_c) be the center of the rectangle of side lengths A and B respectively. We begin by dividing each side into N equal parts and let $(X(I), Y(J))$ be the lower left-hand corner and $(C(I), D(J))$

the center of each sub-rectangle thus formed. This is illustrated in fig.1.1.1 with N = 2.

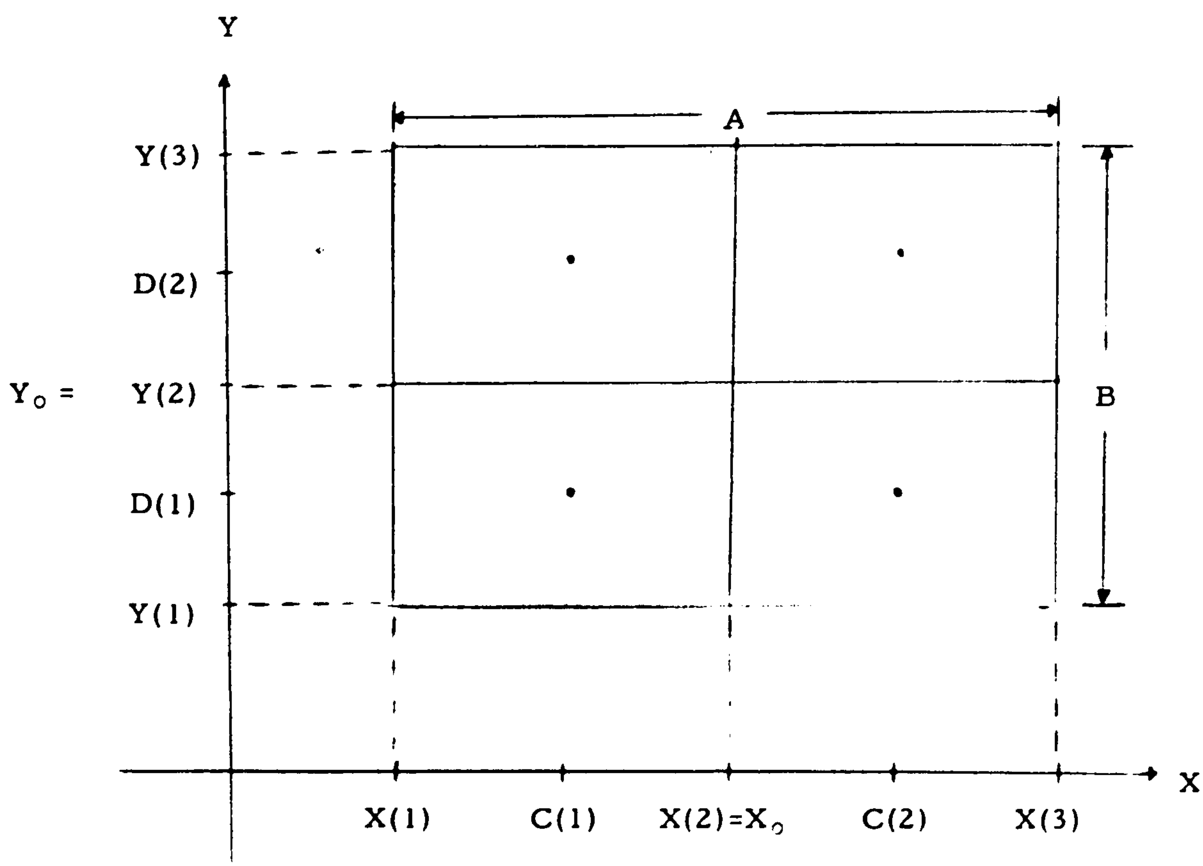


fig.1.1.1 N = 2

With respect to the subrectangle shown in the following figure,

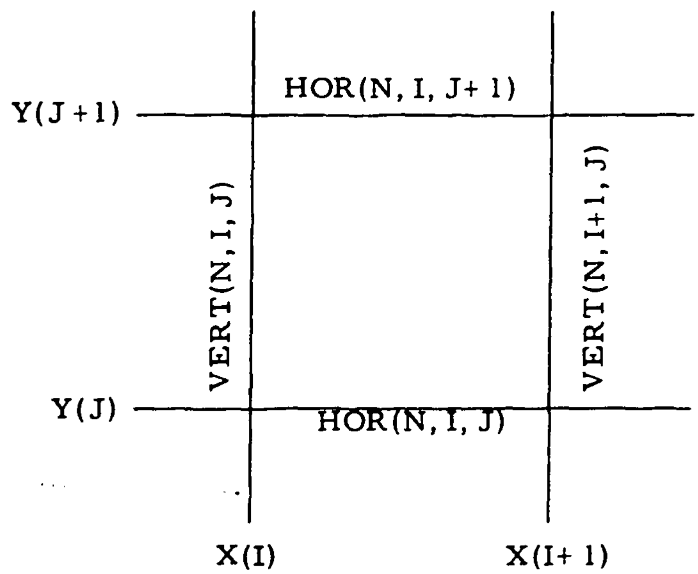


fig. 1.1.2

where $1 \leq I,\ J \leq N$, we use the notation HOR(N, I, J) to denote the value of the contour integral of $f'(z)/f(z)$ along the horizontal line joining the points (X(I), Y(J)) and (X(I+ 1), Y(J)).

Similarly, VERT(N, I, J) denotes the value of the contour integral of $f'(z)/f(z)$ along the vertical line joining the points (X(I), Y(J)) and (X(I), Y(J+ 1)).

ROOTS(N, I, J) then denotes the number of roots inside the subrectangle in fig. 1.1.2. Thus, from eq. (1.1.2) we have

$$\text{ROOTS}(N, I, J) = \frac{1}{2\pi i}[\text{HOR}(N, I, J) - \text{HOR}(N, I, J+1) - \text{VERT}(N, I, J) + \text{VERT}(N, I+1, J)]. \tag{1.1.3}$$

Each integral is evaluated by computing trapezoidal sums on the given line with interval halving, and results are combined by the standard Romberg method until the difference between successive values is less than the prescribed tolerance as explained in detail in chapter 2.

We begin by carrying out numerically contour integrals of $f'(z)/f(z)$ on each segment on the <u>outer</u> sides of the original rectangle only, i.e. we evaluate first

$$\text{HOR}(N, I, J) \quad \text{for} \quad 1 \le I \le N, \;\; J = 1, \;\; J = N+1$$

and

$$\text{VERT}(N, I, J) \quad \text{for} \quad 1 \le J \le N, \;\; I = 1, \;\; I = N+1.$$

Adding the N integrals evaluated on each side of the original rectangle, we combine these sums using eq.(1.1.2) and get the number of zeros inside the original rectangle.

Note that the number of roots being necessarily an integer, we do not have to determine it to high accuracy. If our error is less than 0.5 we will have the exact number of roots. The computed value will be a complex number with the real part being near an integer while the imaginary part, being near zero, will give a rough check on the accuracy of the numerical integration.

If the real part is less than 0.5, we conclude that the original rectangle contains no zeros of $f(z)$ and we stop the procedure. However if there are roots, we then evaluate contour integrals on the remaining edges inside the original rectangle. Using eq.(1.1.3) we now find the number of zeros in each sub-rectangle.

Thus in principle we can isolate the distinct zeros, and find their multiplicities by taking sufficiently small rectangles. The location problem of zeros is thus solved.

In the second approach we do not divide the sides of the original region to integrate along the segments individually. We rather evaluate (1.1.2) round the entire region i.e. trapezoidal sums are computed along the whole boundary with interval halving and results are combined by Romberg's method. In this approach the region is restricted to a square since we need equal sub-divisions on all sides when evaluating the trapezoidal sums above in order to apply the modified Romberg method developed in chapter 2. The original square is divided into subsquares if and only if it contains roots and the procedure is repeated for each subsquare.

In each of the two approaches, as contour integrals are evaluated, we check whether a root is too close to the contour, or possibly on the contour. We perform the following test: if

(1.1.4) $$\left|\frac{f'(z)}{f(z)}\right| \geq K$$

for a preassigned real number K, then a root is close to the path of integration and a warning is printed along with the value of z where the test has failed thus giving us a rough approximation to a root of f(z).

After a first pass we may discard all subrectangles containing no roots and re-enter the program using each of the other subrectangles as an original rectangle. If a subrectangle is found to contain only one root, say

z_0, then an estimate of z_0 can easily be obtained by replacing $f'(z)$ by $z \cdot f'(z)$ in the program since, in this case,

(1.1.5) $$z_0 = \frac{1}{2\pi i} \int_C z \frac{f'(z)}{f(z)} dz$$

where C is the contour of the subrectangle containing z_0. However the accuracy of integration will have to be increased.

If we find a zero sufficiently close to the boundary to give a warning we can re-enter the program after giving the center of the appropriate rectangle a shift of one third the length of the appropriate side, for example, so as to get the root well into the interior of the new rectangle. We use the factor of one third to prevent the root from re-appearing on the boundary of a sub-rectangle in the shifted rectangle (if we take N=2 for example).

If after several subdivisions we find a subrectangle with k roots (normally k will be small i.e. $k \leq 4$), then we can find these roots $\alpha_1, \ldots, \alpha_k$ by noting that

(1.1.6) $$K_r = \sum_{j=1}^{k} \alpha_j^r = \frac{1}{2\pi i} \int_C z^r \frac{f'(z)}{f(z)} dz \qquad r=1,2,\ldots,k$$

where C is the boundary of the subrectangle. Again by replacing $f'(z)$ by $z^r f'(z)$ in the program we can calculate each K_r. From these values, by use of symmetric functions, we can set up an algebraic equation of degree k which has the α_j's as roots. To be specific if k=3 we find

(1.1.7) $$\alpha_1 + \alpha_2 + \alpha_3 = \frac{1}{2\pi i} \int_C z \frac{f'(z)}{f(z)} dz = K_1$$

(1.1.8) $$\alpha_1^2+\alpha_2^2+\alpha_3^2 = \frac{1}{2\pi i}\int_C z^2 \frac{f'(z)}{f(z)}\,dz = K_2$$

(1.1.9) $$\alpha_1^3+\alpha_2^3+\alpha_3^3 = \frac{1}{2\pi i}\int_C z^3 \frac{f'(z)}{f(z)}\,dz = K_3$$

and set the polynomial

(1.1.10) $$z^3+az^2+bz+c = (z-\alpha_1)(z-\alpha_2)(z-\alpha_3)$$

$$= z^3-(\alpha_1+\alpha_2+\alpha_3)z^2+(\alpha_1\alpha_2+\alpha_1\alpha_3+\alpha_2\alpha_3)z-\alpha_1\alpha_2\alpha_3$$

we immediately have

(1.1.11) $$a = -K_1$$

and

(1.1.12) $$b = \tfrac{1}{2}(K_1^2-K_2)$$

by squaring (1.1.7) and subtracting (1.1.8). To find c we use

(1.1.13) $$(\alpha_1+\alpha_2+\alpha_3)^3 = \alpha_1^3+\alpha_2^3+\alpha_3^3+3(\alpha_1\alpha_2^2+\alpha_1\alpha_3^3+\alpha_2\alpha_1^2$$

$$+\alpha_2\alpha_3^2+\alpha_3\alpha_1^2+\alpha_3\alpha_2^2)+6\alpha_1\alpha_2\alpha_3$$

and

(1.1.14) $$(\alpha_1+\alpha_2+\alpha_3)(\alpha_1^2+\alpha_2^2+\alpha_3^2) = \alpha_1^3+\alpha_2^3+\alpha_3^3$$

$$+(\alpha_1\alpha_2^2+\alpha_1\alpha_3^2+\alpha_2\alpha_1^2+\alpha_2\alpha_3^2+\alpha_3\alpha_1^2+\alpha_3\alpha_2^2)$$

which yield

(1.1.15) $$K_1^3 = K_3 + 3(K_1 K_2 - K_3) + 6\alpha_1 \alpha_2 \alpha_3$$

or

(1.1.16) $$c = -\alpha_1 \alpha_2 \alpha_3 = -\tfrac{1}{6}[K_1^3 - 3K_1K_2 + 2K_3].$$

Hence $\alpha_1, \alpha_2, \alpha_3$ are the roots of the cubic:

(1.1.17) $$z^3 - Kz^2 + \tfrac{1}{2}(K_1^2 - K_2)z - \tfrac{1}{6}[K_1^3 - 3K_1K_2 + 2K_3].$$

By solving this polynomial we can find estimates of $\alpha_1, \alpha_2, \alpha_3$ depending on the accuracy of K_1, K_2, K_3.

In this way we can in principle deal with any multiple root situation, although in practice the work gets somewhat out of hand for $k > 4$. In this thesis we will consider in detail the use of Romberg integration for the contour integrals leading up to the location of zeros by (1.1.2) and (1.1.5).

CHAPTER 2

METHODS OF INTEGRATION

2.1 The Euler-Maclaurin Formula

A detailed derivation of the E-M formula can be found in Davis and Rabinowitz, [3], pp.51-55. We merely state it here, modify it slightly, then extend it to contour integration along straight lines in the complex plane.

Let $f(x) \in C^{2q+1}[a,b]$ and let

(2.1.1) $$I = \int_a^b f(x)\,dx$$

(2.1.2) $$T_n = h[\tfrac{1}{2}f(a) + f(a+h) + \ldots + f(a+(n-1)h) + \tfrac{1}{2}f(b)]$$

where $h = (b-a)/n$. T_n is the approximation to I, found by the n-point trapezoidal rule and h is the step length. The notations

(2.1.3) $$T_n f(x), \qquad h \sum_{j=0}^{n}{}'' f(a+jh)$$

will also be used instead of T_n, the primes on the summation denoting that the first and last terms are assigned a weighting factor $\frac{1}{2}$. The E-M formula is thus the following relation between T_n and I:

Proposition 2.1.1

(2.1.4) $$T_n - I = \sum_{k=1}^{q} c_{2k} h^{2k} [f^{(2k-1)}(b) - f^{(2k-1)}(a)] + R$$

with

$$(2.1.5) \qquad R = h^{2q+1} \int_a^b P_{2q+1}\left(\frac{x-a}{h}\right) f^{(2q+1)}(x)\ dx$$

where

$$(2.1.6) \qquad P_{2j}(x) = (-1)^{j-1} \sum_{n=1}^{\infty} \frac{2\cos 2\pi nx}{(2\pi n)^{2j}}$$

$$(2.1.7) \qquad P_{2j+1}(x) = (-1)^{j-1} \sum_{n=1}^{\infty} \frac{2\sin 2\pi nx}{(2\pi n)^{2j+1}} .$$

In particular

$$(2.1.8) \qquad P_{2j+1}(0) = P_{2j+1}(1) = 0 \qquad j > 1$$

$$(2.1.9) \qquad P_{2j}(0) = P_{2j}(1) = (-1)^{j-1} \sum_{n=1}^{\infty} \frac{2}{(2n\pi)^{2j}} \qquad j > 1 .$$

We use (2.1.9) to define constants B_{2j}, called the Bernouilli numbers. By definition,

$$(2.1.10) \qquad \frac{B_{2j}}{(2j)!} = (-1)^{j-1} \sum_{n=1}^{\infty} \frac{2}{(2n\pi)^{2j}} .$$

In (2.1.4) we simply set

$$(2.1.11) \qquad c_{2j} = \frac{B_{2j}}{(2j)!} .$$

Note that in [4], the constants B_{2j} are defined by the formula

$$\text{(2.1.12)} \qquad \frac{B_{2j}}{(2j)!} = \sum_{n=1}^{\infty} \frac{2}{(2n\pi)^{2j}}$$

which is just the absolute value of the B_{2j} defined by (2.1.10), as in [3]. Unless stated otherwise, we shall use (2.1.10) to define B_{2j}.

In [2], Lyness and Delves use a slightly different form for the term R. It can easily be obtained by integrating (2.1.5) by parts. We get

$$\text{(2.1.13)} \qquad R = h^{2q+1}\left[hP_{2q+2}\left(\frac{x-a}{h}\right) f^{(2q+1)}(x)\Big|_a^b - \int_a^b hP_{2q+2}\left(\frac{x-a}{h}\right) f^{(2q+2)}(x)\,dx \right]$$

$$\text{(2.1.14)} \qquad R = h^{2q+2}\left[c_{2q+2}\left(f^{(2q+1)}(b) - f^{(2q+1)}(a)\right) - \int_a^b P_{2q+2}\left(\frac{x-a}{h}\right) f^{(2q+2)}(x)\,dx \right]$$

$$\text{(2.1.15)} \qquad R = h^{2q+2}\left[c_{2q+2}\int_a^b f^{(2q+2)}(x)\,dx - \int_a^b P_{2q+2}\left(\frac{x-a}{h}\right) f^{(2q+1)}(x)\,dx \right]$$

$$\text{(2.1.16)} \qquad R = h^{2q+2}\int_a^b \left[c_{2q+2} - P_{2q+2}\left(\frac{x-a}{h}\right)\right] f^{(2q+2)}(x)\,dx$$

$$\text{(2.1.17)} \qquad R = \int_a^b f^{(2q+2)}(x)\,\omega_{2q+2}(x)\,dx$$

where

$$\text{(2.1.18)} \qquad \omega_{2q+2}(x) = h^{2q+2}\left[c_{2q+2} - P_{2q+2}\left(\frac{x-a}{h}\right)\right].$$

The Euler-Maclaurin formula can now be written in the following form.

Proposition 2.1.2

$$(2.1.19) \qquad T_n - I = \sum_{k=1}^{q} c_{2k}\, h^{2k}\left[f^{(2k-1)}(b) - f^{(2k-1)}(a)\right] + R$$

with

$$(2.1.20) \qquad R = \int_a^b f^{(2q+2)}(x)\, \varphi_{2q+2}(x)\, dx$$

$$(2.1.21) \qquad \varphi_{2q+2}(x) = h^{2q+2}\left[c_{2q+2} - P_{2q+2}\left(\frac{x-a}{h}\right)\right]$$

Corollary 2.1.3

$$(2.1.22) \qquad T_n - I = \sum_{k=1}^{p-1} c_{2k}\, h^{2k}\left[f^{(2k-1)}(b) - f^{(2k-1)}(a)\right] + R$$

and

$$(2.1.23) \qquad R = \int_a^b f^{(2p)}(x)\, \varphi_{2p}(x)\, dx$$

$$(2.1.24) \qquad \varphi_{2p}(x) = \frac{2h^{2p}(-1)^{p-1}}{(2\pi)^{2p}} \sum_{r=1}^{\infty} \frac{1-\cos 2\pi r\left(\frac{x-a}{h}\right)}{r^{2p}}$$

Proof:

Let $q = p-1$ in (2.1.19), then (2.1.20) becomes (2.1.23) and (2.1.21) gives (using (2.1.10), (2.1.6))

(2.1.25) $$\varphi_{2p}(x) = h^{2p}\left[c_{2p} - P_{2p}\left(\frac{x-a}{h}\right)\right]$$

(2.1.26) $$\varphi_{2p}(x) = h^{2p}\left[(-1)^{p-1}\sum_{r=1}^{\infty}\frac{2}{(2r\pi)^{2p}} - (-1)^{p-1}\sum_{r=1}^{\infty}\frac{2\cos 2\pi r\left(\frac{x-a}{h}\right)}{(2\pi r)^{2p}}\right]$$

(2.1.27) $$\varphi_{2p}(x) = \frac{2h^{2p}(-1)^{p-1}}{(2\pi)^{2p}}\sum_{r=1}^{\infty}\frac{1-\cos 2\pi r\left(\frac{x-a}{h}\right)}{r^{2p}}\ .$$

Note:

If, in particular, we consider the interval [0, 1], i.e.

$$a = 0,\quad b = 1,\quad h = 1/n$$

then in Cor. 2.1.3

(2.1.28) $$\varphi_{2p}(x) = \frac{-2(-1)^{p}}{(2\pi n)^{2p}}\sum_{r=1}^{\infty}\frac{1-\cos 2\pi rnx}{r^{2p}}$$

which justifies eq.4.1 p.566 in Lyness and Delves, [2].

We are primarily interested in evaluating integrals of the type

$$L_{\square} = \int_{\square} f(z)\,dz$$

on the contour of a square in the complex plane. To approximate $L_{\square}$, we first consider the trapezoidal rule with complex step length on each side of the square. However the E-M formula has been established so far for real functions of a real variable only. We now wish to show that it still applies in the case of contour integration along straight lines in the complex plane, and find its explicit form.

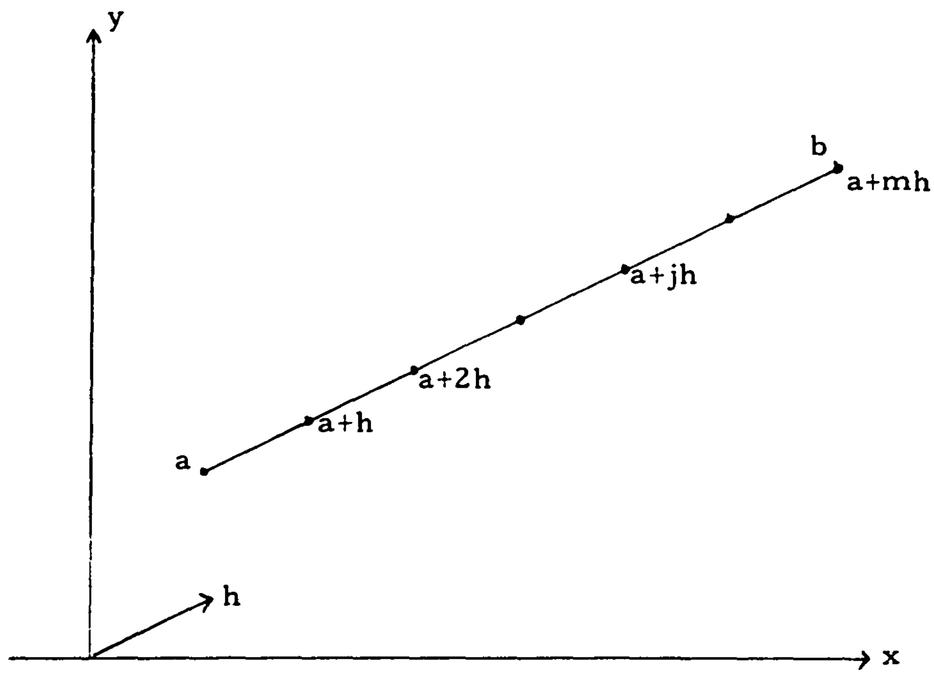


fig. 2.1.1

Let f(z) be analytic in a region containing the straight line joining the complex numbers a and b (see fig. 2.1.1). Divide the line into m equal elements. Let

(2.1.30) $\quad h = (b-a)/m \in \mathbb{C}$

be the complex step length and

(2.1.31) $\quad T_m = h \sum_{j=0}^{m}{}'' f(a+jh) \in \mathbb{C}$

(2.1.32) $\quad I = \int_a^b f(z)\,dz \qquad \in \mathbb{C}$

the trapezoidal value T_m and the exact integral I being evaluated along the line ab. Then the E-M formula still holds, i.e.

Proposition 2.1.4

$$T_m - I = \sum_{k=1}^{q} c_{2k}\, h^{2k} [f^{(2k-1)}(b) - f^{(2k-1)}(a)] + R \tag{2.1.33}$$

where

$$R = \int_a^b f^{(2q+2)}(z)\ \varphi_{2q+2}(z)\ dz \tag{2.1.34}$$

and

$$\varphi_{2q+2}(z) = h^{2q+2}[c_{2q+2} - P_{2q+2}\left(\frac{z-a}{h}\right)] \tag{2.1.35}$$

Proof:

Let

$$z = a + (b-a)t = a + mht \qquad 0 \le t \le 1 \tag{2.1.36}$$

be the parametric representation of the straight line joining a and b. Let

$$z_j = a + jh = a + mht_j \tag{2.1.37}$$

i.e. $t_j = j/m$. Then

$$f(z) = U(t) + iV(t) \qquad 0 \le t \le 1 \tag{2.1.38}$$

where U(t) and V(t) are real-valued. Furthermore, to simplify the notation, let

$$U_j = U(t_j) \qquad V_j = V(t_j). \tag{2.1.39}$$

Now

$$(2.1.40) \qquad T_m = h \sum_{j=0}^{m}{}'' U_j + ih \sum_{j=0}^{m}{}'' V_j$$

$$(2.1.41) \qquad T_m = mh \left[\tfrac{1}{m} \sum_{j=0}^{m}{}'' U_j - \int_0^1 U(t)\,dt \right]$$

$$+ imh\left[\tfrac{1}{m} \sum_{j=0}^{m}{}'' V_j - \int_0^1 V(t)\,dt \right]$$

$$+ mh \int_0^1 [U(t) + iV(t)]\,dt.$$

Each expression in brackets can now be expanded by the E-M formula in Prop.2.1.2. Bringing the integral in (2.1.41) to the left-hand side we get

$$(2.1.42) \qquad T_m - I = mh \left[\sum_{k=1}^{q} c_{2k} \left(\tfrac{1}{m}\right)^{2k} [U^{(2k-1)}(1) - U^{(2k-1)}(0)] + \int_0^1 U^{(2q+2)}(t)\,\psi_{2q+2}(t)\,dt \right]$$

$$+ imh \left[\sum_{k=1}^{q} c_{2k} \left(\tfrac{1}{m}\right)^{2k} [V^{(2k-1)}(1) - V^{(2k-1)}(0)] + \int_0^1 V^{(2q+2)}(t)\,\psi_{2q+2}(t)\,dt \right]$$

where

$$(2.1.43) \qquad \psi_{2q+2}(t) = \left(\tfrac{1}{m}\right)^{2q+2} [c_{2q+2} - P_{2q+2}(mt)]$$

regrouping the terms in (2.1.42) we obtain

$$(2.1.44)\qquad T_s - I = \sum_{k=1}^{q} c_{2k}\, mh\left(\tfrac{1}{s}\right)^{2k} \{[U^{(2k-1)}(1) + iV^{(2k-1)}(1)] - [U^{(2k-1)}(0) + iV^{(2k-1)}(0)]\}$$

$$+ mh\int_0^1 [U^{(2q+2)}(t) + iV^{(2q+2)}(t)]\, \psi_{2q+2}(t)\, dt .$$

Now, using the fact that

$$(2.1.45)\qquad (mh)^r \cdot f^{(r)}(z) = U^{(r)}(t) + iV^{(r)}(t)$$

(2.1.44) becomes

$$(2.1.46)\qquad T_s - I = \sum_{k=1}^{q} c_{2k}\, h^{2k} [f^{(2k-1)}(b) - f^{(2k-1)}(a)]$$

$$+ \int_a^b \varphi_{2q+2}(z)\, f^{(2q+2)}(z)\, dz$$

where

$$(2.1.47)\qquad \varphi_{2q+2}(z) = h^{2q+2} [c_{2q+2} - P_{2q+2}\left(\frac{z-a}{h}\right)$$

which concludes the proof. Note that the argument of P_{2q+2} is always real for any z on the line since

$$(2.1.48)\qquad \frac{z-a}{h} = mt \in \mathbb{R} .$$

Also

$$(2.1.49)\qquad \varphi_{2q+2}(z) \in \mathbb{C} \;\Leftrightarrow\; h^{2q+2} \in \mathbb{C} .$$

2.2 The Standard Romberg Method

The standard treatment is given in Bauer, Rutishauser and Stiefel, [4], to which we shall refer as BRS. It is based on the E-M formula. From Prop.(2.1.2) we can write

(2.2.1) $$T_n - I = \sum_{k=1}^{q} c_{2k} h^{2k} [f^{(2k-1)}(b) - f^{(2k-1)}(a)] + O(h^{2k+2}).$$

If we let

(2.2.2) $$a_{2k} = c_{2k} [f^{(2k-1)}(b) - f^{(2k-1)}(a)]$$

(2.2.3) $$T_0^{(k)} \equiv T_n \quad \text{when} \quad n = 2^k$$

(2.2.4) $$E(h) = T_0^{(k)} - I$$

then (2.2.1) becomes

(2.2.5) $$E(h) = a_2 h^2 + a_4 h^4 + a_6 h^6 + \ldots + a_{2q} h^{2q} + O(h^{2q+2}).$$

$E(h)$ is the discretization error and has what is called an h^2 expansion, i.e.

(2.2.6) $$E(h) \sim a_2 h^2 + a_4 h^4 + a_6 h^6 + \ldots$$

For $q = 1$, (2.2.4) and (2.2.5) give

(2.2.7) $$T_0^{(k)} - I = a_2 h^2 + O(h^4).$$

If we double the number of subintervals, keeping $h=(b-a)2^{-k}$, we get

$$(2.2.8) \qquad T_0^{(k+1)} - I = a_2 \frac{h^2}{4} + O(h^4).$$

Eliminating the term in h^2 in (2.2.7) and (2.2.8) we have

$$(2.2.9) \qquad I = \left[\frac{4T_0^{(k+1)} - T_0^{(k)}}{3}\right] + O(h^4).$$

Define now $T_1^{(k)}$ by the expression in brackets in (2.2.9) i.e.

$$(2.2.10) \qquad T_1^{(k)} = \frac{4T_0^{(k+1)} - T_0^{(k)}}{3} .$$

(2.2.9) now gives

$$(2.2.11) \qquad T_1^{(k)} - I = O(h^4)$$

i.e. $T_1^{(k)}$ eliminates the term in h^2 and we may write

$$(2.2.12) \qquad T_1^{(k)} - I = Ah^4 + O(h^6)$$

where A is some constant. Repeating the process used to obtain (2.2.8), we have

$$(2.2.13) \qquad T_1^{(k+1)} - I = A \frac{h^4}{16} + O(h^6).$$

As before, eliminating the term in h^4 in (2.2.12) and (2.2.13) gives

$$(2.2.14) \qquad I = \left[\frac{16\,T_1^{(k+1)} - T_1^{(k)}}{15}\right] + O(h^6)$$

and we define

(2.2.15) $$T_2^{(k)} = \frac{16\,T_1^{(k+1)} - T_1^{(k)}}{15}$$

hence

(2.2.16) $$T_2^{(k)} - I = O(h^6)$$

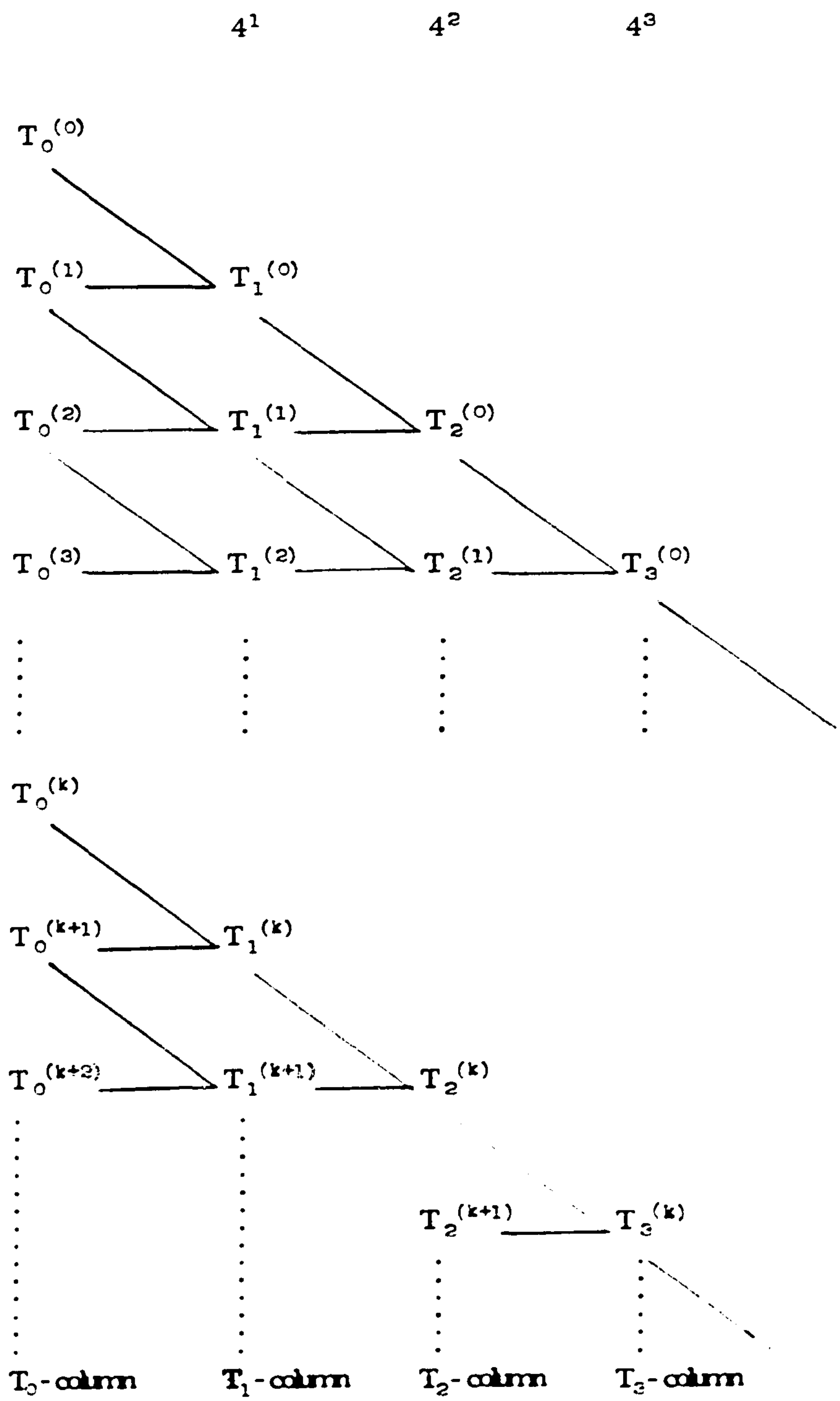
i.e. $T_2^{(k)}$ eliminates the term in h^4 and we may write

(2.2.17) $$T_2^{(k)} - I = Bh^6 + O(h^8).$$

Thus Romberg's method to approximate I, starts with the computation of the trapezoidal values for subdivisions of the full interval into 1, 2, 4, 8,... equal parts. By continuing the process above we then find the values $T_m^{(k)}$, m = 1, 2, 3, . . . which are computed, for $m \neq 0$, with the formula

(2.2.18) $$T_m^{(k)} = \frac{4^m\,T_{m-1}^{(k+1)} - T_{m-1}^{(k)}}{4^m - 1}.$$

These values are then arranged vertically to form a triangular array called the T-table.



4^1
4^2
4^3
T_0^(0)
T_0^(1)
T_1^(0)
T_0^(2)
T_1^(1)
T_2^(0)
T_0^(3)
T_1^(2)
T_2^(1)
T_3^(0)
T_0^(k)
T_0^(k+1)
T_1^(k)
T_0^(k+2)
T_1^(k+1)
T_2^(k)
T_2^(k+1)
T_3^(k)
T_0-column
T_1-column
T_2-column
T_3-column

fig.2.2.1

Thus $T_m^{(k)}$ is found from the values in the following positions relative to $T_m^{(k)}$

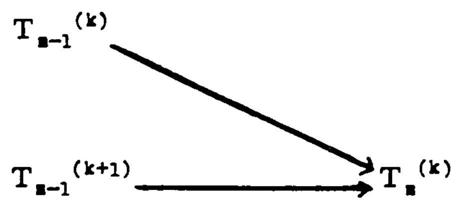


fig. 2.2.2

Since, by construction,

(2.2.19) $$T_m^{(k)} - I = O(h^{2m+2})$$

it follows that, in the T-table, the T_1-column eliminates the term in h^2, the T_2-column eliminates the term in h^4, and so on, any T_m-column eliminates the term in h^{2m}.

2.3 Integration Round a Square Contour

Consider the square of side 2R, whose vertices in order are the complex numbers a_1, a_2, a_3, a_4.

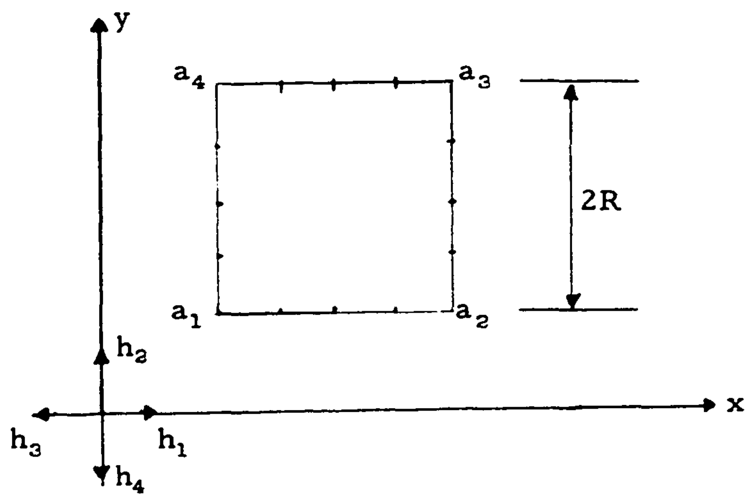


fig.2.3.1

The trapezoidal rule with complex step

$$(2.3.1) \qquad h_j = (a_{j+1} - a_j)/m \qquad\qquad j = 1,2,3,4$$

is used for the section between a_j and a_{j+1}, m being the same on each side. We thus have

$$(2.3.2) \qquad h_k = \begin{cases} 2R/m & k = 1 \\ ih_{k-1} & k = 2,3,4 \end{cases} .$$

This obviously implies that

(2.3.3) $\qquad h_{k-1}^{4r} = h_k^{4r} \qquad r$ an integer $(h_0 \equiv h_4)$.

Applying now the E-M formula in Prop.2.1.4 (with $q=2p$) on each side of the square we find for $k = 1, 2, 3, 4$

(2.3.4)
$$h \sum_{j=0}^{s}{}'' f(a_k+jh_k) - \int_{a_k}^{a_{k+1}} f(z)\,dz =$$
$$\sum_{r=1}^{2p} c_{2r} h^{2r} [f^{(2r-1)}(a_{k+1}) - f^{(2r-1)}(a_k)]$$
$$+ \int_{a_k}^{a_{k+1}} \varphi_k(z)\ f^{(4p+2)}(z)\ dz$$

where

(2.3.5)
$$\varphi_k(z) = h_k^{4p+2} [c_{4p+2} - P_{4p+2}\left(\frac{z-a_k}{h_k}\right)].$$

Adding the four equalities in (2.3.4) we find

(2.3.6)
$$R_s = \sum_{k=1}^{4} [h_k \sum_{j=0}^{s}{}'' f(a_k+jh_k)] - \int_{\underline{\omega}} f(z)\,dz =$$
$$= \sum_{r=1}^{2p} c_{2r} \{ f^{(2r-1)}(a_1)[h_4^{2r} - h_1^{2r}] + f^{(2r-1)}(a_2)[h_1^{2r} - h_2^{2r}]$$
$$+ f^{(2r-1)}(a_3)[h_2^{2r} - h_3^{2r}] + f^{(2r-1)}(a_4)[h_3^{2r} - h_4^{2r}] \}$$
$$+ \int_{\underline{\omega}} \tilde{\varphi}_r(z)\ f^{(4p+2)}(z)\ dz$$

where $\tilde{\varphi}_p(z)$ is respectively equal to $\varphi_1(z)$, $\varphi_2(z)$, $\varphi_3(z)$, $\varphi_4(z)$ along the sides of the square. Thus

$$R_s = \sum_{r=1}^{2p} c_{2r} \sum_{k=1}^{4} f^{(2r-1)}(a_k)[h_{k-1}^{2r} - h_k^{2r}] + \int_{\square} \tilde{\varphi}_p(z) f^{(4p+2)}(z)\, dz \tag{2.3.7}$$

where $h_0 \equiv h_4$.

Now, from (2.3.3),

$$h_{k-1}^{4r} - h_k^{4r} = 0 \qquad r = \text{integer} \tag{2.3.8}$$

which implies that

$$h_{k-1}^{2r} - h_k^{2r} = 0 \qquad r = \text{even integer}. \tag{2.3.9}$$

Hence the terms in $c_4, c_8, c_{12}, \ldots, c_{4p}$ vanish in the right-hand side of (2.3.7) and we obtain

$$\begin{aligned} R_s = {} & c_2 \sum_{k=1}^{4} f'(a_k)[h_{k-1}^2 - h_k^2] \\ & + c_6 \sum_{k=1}^{4} f^{(5)}(a_k)[h_{k-1}^6 - h_k^6] \\ & + c_{10} \sum_{k=1}^{4} f^{(9)}(a_k)[h_{k-1}^{10} - h_k^{10}] \\ & \vdots \\ & + c_{4p-2} \sum_{k=1}^{4} f^{(4p-3)}(a_k)[h_{k-1}^{4p-2} - h_k^{4p-2}] + \int_{\square} \tilde{\varphi}_p(z) f^{(4p+2)}(z)\, dz \end{aligned} \tag{2.3.10}$$

or

$$(2.3.11) \qquad R_m = \sum_{r=1}^{p} \left[c_{4r-2} \sum_{k=1}^{4} f^{(4r-3)}(a_k) \left[h_{k-1}^{\,4r-2} - h_k^{\,4r-2} \right] \right] + \int_{\square} \tilde{\varphi}_p(z)\, f^{(4p+2)}(z)\, dz \,.$$

It is easy to show that

$$(2.3.12) \qquad h_{k-1}^{\,4r-2} - h_k^{\,4r-2} = \begin{cases} -2h_1^{\,4r-2} & k = 1, 3 \\ 2h_1^{\,4r-2} & k = 2, 4 \end{cases}$$

Expanding the inner sum of (2.3.11) and using (2.3.12) above gives

$$(2.3.13) \qquad R_m = \sum_{r=1}^{p} 2c_{4r-2} \left[-f^{(4r-3)}(a_1) + f^{(4r-3)}(a_2) - f^{(4r-3)}(a_3) + f^{(4r-3)}(a_4) \right] h_1^{\,4r-2}$$

$$+ \int_{\square} \tilde{\varphi}_p(z)\, f^{(4p+2)}(z)\, dz \,.$$

If we denote

$$(2.3.14) \qquad d_{4r-2} = 2c_{4r-2} \left[-f^{(4r-3)}(a_1) + f^{(4r-3)}(a_2) - f^{(4r-3)}(a_3) + f^{(4r-3)}(a_4) \right]$$

$$(2.3.15) \qquad h = h_1 = 2R/m$$

(2.3.13) finally becomes

$$(2.3.16) \qquad \sum_{k=1}^{4} \left[h_k \sum_{j=0}^{m}{}'' f(a_k + jh_k) \right] - \int_{\square} f(z)\, dz$$

$$= d_2 h^2 + d_6 h^6 + d_{10} h^{10} + \ldots + d_{4p-2} h^{4p-2} + \int_{\square} \tilde{\varphi}_p(z)\, f^{(4p+2)}(z)\, dz \,.$$

Define, for $i \geq 0$

$$(2.3.17) \qquad S_0^{(i)} = \sum_{k=1}^{4} h_k \sum_{j=0}^{2^i}{}'' f(a_k + jh_k)$$

where $h_k = (a_{k+1} - a_k)/2^i$. Thus $S_0^{(i)}$ is the trapezoidal sum around the <u>whole</u> square obtained with each side being divided into 2^i equal parts. $S_0^{(i)}$ is the analog of $T_0^{(i)}$ used in section 2. If we denote

$$(2.3.18) \qquad I_{\square} = \int_{\square} f(z)\, dz$$

then (2.3.16) gives

$$(2.3.19) \qquad E(h) = S_0^{(i)} - I_{\square} = d_2 h^2 + d_6 h^6 + d_{10} h^{10} + \ldots + d_{4p-2} h^{4p-2}$$

$$+ \int_{\square} \Phi_p(z)\, f^{(4p+2)}(z)\, dz$$

where $h = h_1 = 2R/2^i$. It is clear from (2.3.5) and (2.3.6) that $\Phi_p(z)$ depends on i. Eq. (2.3.19) can also be written in the form

$$(2.3.20) \qquad S_0^{(i)} - I_{\square} = \sum_{r=1}^{p} \frac{d_r^*}{4^{(2r-1)i}} + \int_{\square} \Phi_p(z)\, f^{(4p+2)}(z)\, dz$$

where

$$(2.3.21) \qquad d_r^* = (2R)^{4r-2}\, d_{4r-2}\,.$$

From (2.3.19) we see that the discretization error $E(h)$ contains terms in h^2, h^6, h^{10} but not in h^4, h^8, h^{12}, ...

2.4 The Adapted Romberg Method

In the previous section, we have shown that the error in the trapezoidal rule applied round a square is of the form

(2.4.1) $$E(h) \sim d_2 h^2 + d_6 h^6 + d_{10} h^{10} + \ldots$$

compared with the real application where the error is of the form

(2.4.2) $$E(h) \sim d_2 h^2 + d_4 h^4 + d_6 h^6 + d_8 h^8 + \ldots$$

Because of this we do not use the standard Romberg technique as formulated in section 2 of this chapter.

Beginning with the trapezoidal values $S_0^{(k)}$, defined by (2.3.17), in the S_0-column we apply the usual Romberg procedure to evaluate the S_1-column. This eliminates the term in h^2 and since no term in h^4 appears in (2.4.1) we have

(2.4.3) $$S_1^{(k)} - I_{\square} = O(h^6) .$$

Leaving $h = (b-a)2^{-k}$, (2.4.3) gives

(2.4.4) $$S_1^{(k)} - I_{\square} = Ah^6 + O(h^8)$$

(2.4.5) $$S_1^{(k+1)} - I_{\square} = A\frac{h^6}{64} + O(h^8)$$

eliminating h^6 from (2.4.4) and (2.4.5) we find

(2.4.6) $$\left[\frac{64\,S_1^{(k+1)} - S_1^{(k)}}{63}\right] - I_{\square} = O(h^8).$$

Comparing with the standard technique we see that it all behaves as if we had set the S_2-column identical to the S_1-column i.e.

(2.4.7) $$S_2^{(k)} \equiv S_1^{(k)}, \qquad \forall k$$

and used this S_2-column to evaluate the S_3-column by the usual procedure i.e.

(2.4.8) $$S_3^{(k)} = \frac{4^3 S_2^{(k+1)} - S_2^{(k)}}{4^3 - 1}$$

which, by (2.4.6), gives

(2.4.9) $$S_3^{(k)} - I_{\square} = O(h^8).$$

Continuing this way, for any term in h^{2s} that does not appear in $E(h)$ we set $S_s^{(k)} = S_{s-1}^{(k)}$, otherwise we use the standard relation (2.2.17). The structure of the T-table differs from the normal structure and has the appearance

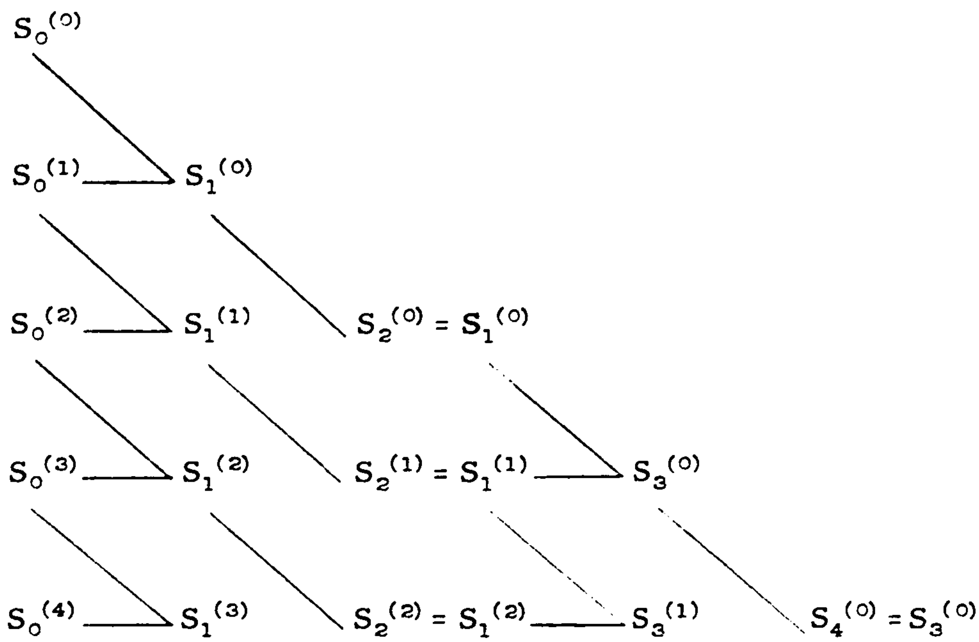


fig. 2.4.1

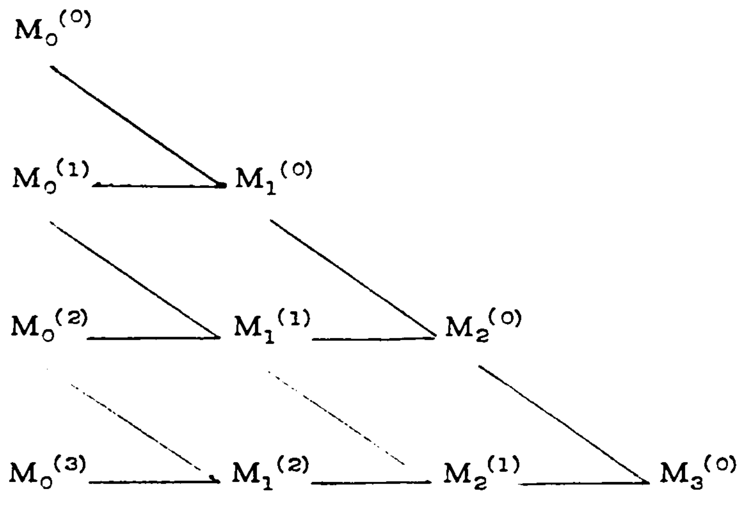
Therefore every even-numbered column can be eliminated from the table without any loss, and in order to avoid odd subscripts only, a new table can be constructed by letting

(2.4.10) $$M_0^{(k)} = S_0^{(k)}$$

and

(2.4.11) $$M_j^{(k)} = \frac{4^{2j-1} M_{j-1}^{(k+1)} - M_{j-1}^{(k)}}{4^{2j-1} - 1} \qquad j \geq 1$$

giving the following table



fig, 2.4.2

Note: The reason why we did not go directly to the table in fig.2.4.2 will be apparent in section 3.1.

Each entry in the adapted table in fig.2.4.1 is given by

$$(2.4.12) \qquad S_m^{(k)} = \begin{cases} S_{m-1}^{(k)} & m \text{ even} \\ \alpha_m S_{m-1}^{(k)} + \beta_m S_{m-1}^{(k+1)} & m \text{ odd} \end{cases}$$

where

$$(2.4.13) \qquad \alpha_m = -\frac{1}{4^m - 1} \qquad \beta_m = \frac{4^m}{4^m - 1} .$$

It is clear that $S_{2p}^{(i)}$ is a well-defined linear combination of values of the first column:

(2.4.14) $$S_{2p}^{(i)} = \sum_{k=0}^{p} \lambda_{pk} S_0^{(k+i)} \qquad (i \geq 0)$$

where the coefficients λ_{pk} are independent of i.

Note: In order to simplify the notation we will sometimes let $\gamma_k = \lambda_{pk}$, for a fixed p. Thus, for i = 0 in particular,

(2.4.15) $$S_{2p}^{(0)} = \sum_{k=0}^{p} \gamma_k S_0^{(k)} \qquad (p \text{ fixed})$$

Lemma 2.4.1

The coefficients λ_{pk} obey the following recurrence formula

(2.4.16) $$\lambda_{pk} = \alpha_{2p-1} \lambda_{p-1,k} + \beta_{2p-1} \lambda_{p-1,k-1}, \qquad 0 \leq k \leq p$$

with

(2.4.17) $$\lambda_{p-1,-1} = \lambda_{p-1,p} = 0.$$

Proof:

From (2.4.14), for i = 0, we have

(2.4.18) $$S_{2p}^{(0)} = \sum_{k=0}^{p} \lambda_{pk} S_0^{(k)}.$$

On the other hand,

$$S_{2p}^{(0)} = S_{2p-1}^{(0)} \qquad \text{by (2.4.12)}$$

$$= \alpha_{2p-1} S_{2p-2}^{(0)} + \beta_{2p-1} S_{2p-2}^{(1)} \qquad \text{by (2.4.12)}$$

$$= \alpha_{2p-1} \sum_{k=0}^{p-1} \lambda_{p-1,k} S_0^{(k)} + \beta_{2p-1} \sum_{k=0}^{p-1} \lambda_{p-1,k} S_0^{(k+1)}$$

$$= \alpha_{2p-1} \sum_{k=0}^{p} \lambda_{p-1,k} S_0^{(k)} + \beta_{2p-1} \sum_{k=-1}^{p-1} \lambda_{p-1,k} S_0^{(k+1)} \qquad \text{by (2.4.17)}$$

$$= \alpha_{2p-1} \sum_{k=0}^{p} \lambda_{p-1,k} S_0^{(k)} + \beta_{2p-1} \sum_{k=0}^{p} \lambda_{p-1,k-1} S_0^{(k)}$$

$$= \sum_{k=0}^{p} [\alpha_{2p-1}\lambda_{p-1,k} + \beta_{2p-1} \lambda_{p-1,k-1}] S_0^{(k)}$$

comparing this with (2.4.18) gives (2.4.16).

For every integer $p \geq 1$ define polynomials $t_{2p}(z)$ as follows

(2.4.19) $$t_{2p}(z) = \sum_{k=0}^{p} \lambda_{p,p-k} z^k .$$

As a consequence of (2.4.16) we have

<u>Lemma</u> 2.4.2

(2.4.20) (i) $$t_{2p}(z) = \left(\frac{4^{2p-1} - z}{4^{2p-1} - 1}\right) t_{2p-2}(z) \qquad (p \geq 1)$$

(2.4.21) (ii) $$t_{2p}(z) = \frac{\left(1 - \frac{z}{4}\right)\left(1 - \frac{z}{4^3}\right)\left(1 - \frac{z}{4^5}\right) \cdots \left(1 - \frac{z}{4^{2p-1}}\right)}{\left(1 - \frac{1}{4}\right)\left(1 - \frac{z}{4^3}\right)\left(1 - \frac{z}{4^5}\right) \cdots \left(1 - \frac{z}{4^{2p-1}}\right)}$$

(2.4.22) (iii) $\sum_{k=0}^{p} \lambda_{pk} = 1$ (p fixed)

(2.4.23) (iv) $\sum_{k=0}^{p} \frac{\lambda_{pk}}{4^{kj}} = 0$ $j=1, 3, 5, \ldots, 2p-1$.

Proof:

(i) $t_{2p}(z) = \sum_{k=0}^{p} [\alpha_{2p-1} \lambda_{p-1,p-k} + \beta_{2p-1} \lambda_{p-1,p-k-1}] z^k$ by (2.4.16)

$= \beta_{2p-1} \sum_{k=0}^{p-1} \lambda_{p-1,p-k-1} z^k + \alpha_{2p-1} z \sum_{k=1}^{p} \lambda_{p-1,p-k} z^{k-1}$ by (2.4.17)

$= (\beta_{2p-1} + \alpha_{2p-1} z) \sum_{k=0}^{p-1} \lambda_{p-1,p-k-1} z^k$

$= (\beta_{2p-1} + \alpha_{2p-1} z) t_{2p-2}(z)$ by (2.4.19)

giving (2.4.20) by (2.4.13).

(ii) From (2.4.20) we infer that

$$t_{2p}(z) = \frac{(4^{2p-1} - z) \ldots\ldots (4^5 - z)(4^3 - z)(4 - z)}{(4^{2p-1} - 1) \ldots\ldots (4^5 - 1)(4^3 - 1)(4 - 1)}$$

giving (2.4.21).

(iii) (2.4.22) is obvious from (2.4.19) and (2.4.21) by letting $z = 1$.

(iv) From (2.4.21) we have

$t_{2p}(4^j) = 0$ $j=1, 3, 5, \ldots, 2p-1$

hence (2.4.19) gives

$$\frac{1}{4^{pj}} \sum_{k=0}^{p} \lambda_{p,p-k} \, 4^{kj} = 0$$

$$\sum_{k=0}^{p} \lambda_{p,p-k} \, 4^{-(p-k)j} = 0$$

$$\sum_{k=0}^{p} \lambda_{pr} \, 4^{-rj} = 0 \qquad (r = p-k) \; .$$

CHAPTER 3

A BOUND ON THE TRUNCATION ERROR

3.1 The Truncation Error in Particular Expansions

We have seen that in the case of integration of a real function over an interval, the discretization error E(h) has the form

(3.1.1) $$E(h) \sim d_2h^2 + d_4h^4 + d_6h^6 + d_8h^8 + d_{10}h^{10} + \ldots$$

while in the case of contour integration round a square, it has the form

(3.1.2) $$E(h) \sim d_2h^2 + d_6h^6 + d_{10}h^{10} + d_{14}h^{14} + \ldots$$

Lyness and Delves in [2] state that in contour integration round triangles or hexagons, the effective discretization error is

(3.1.3) $$E(h) \sim d_2h^2 + d_4h^4 + d_8h^8 + d_{10}h^{10} + d_{14}h^{14} + \ldots$$

while in two-dimensional quadrature over a square of a harmonic function

(3.1.4) $$E(h) \sim d_4h^4 + d_8h^8 + d_{12}h^{12} + \ldots$$

As before, E(h) is defined as the difference between the trapezoidal value and the exact value of the integral, each evaluated on the whole contour under consideration.

When E(h) has the form (3.1.1) the usual Romberg technique is used and it is shown in [4] that for any entry $T_s^{(k)}$ is the standard table, the truncation error is given by

$$(3.1.5) \qquad T_m^{(k)} - \int_0^1 f(x)\,dx = 4^{-(m+1)k} \int_0^1 b_{2m+2}(2^k x)\, f^{(2m+2)}(x)\,dx$$

where the functions b_{2m+2} are defined by the following recurrence relations: for $m=1,2,3,4,\ldots$

$$\text{(a)} \qquad \hat{b}_{2m}(x) = \frac{b_{2m}(2x) - b_{2m}(x)}{4^m - 1}$$

$$(3.1.6) \qquad \text{(b)} \qquad b_{2m+1}(x) = \int_0^x \hat{b}_{2m}(t)\,dt$$

$$\text{(c)} \qquad b_{2m+2}(x) = \int_0^x b_{2m+1}(t)\,dt$$

where

$$(3.1.7) \qquad b_2(x) = \tfrac{1}{2}x(1-x)$$

for $0 \le x \le 1$ and, by definition, periodic with period 1.

We wish to find a similar result for the case where E(h) is of the form (3.1.2), as in square contours. However (3.1.5) can easily be generalized to encompass any form of E(h) involving even powers and to do so, it is convenient to build the adapted table, in each case, as we did in fig.2.4.1 i.e., if a particular term in h^{2m} does not appear in E(h) we set

$$(3.1.8) \qquad T_m^{(k)} \equiv T_{m-1}^{(k)}, \qquad \forall k$$

and call this a non-eliminating step.

Otherwise we use the standard relation (2.2.18).

When the $T_m^{(k)}$-column is constructed from the $T_{m-1}^{(k)}$-column in the standard way and the term in h^{2m} (if there is one in E(h)) is eliminated in the truncation error $T_m^{(k)}$-I. In this case we say that the step from $T_{m-1}^{(k)}$ to $T_m^{(k)}$ is an eliminating step.

Thus, each column in the table can be built in one of two possible ways by either identifying it to the previous column or by using the standard relation (2.2.18). Therefore, there are theoretically, 2^m ways of constructing the first m+1 columns of a T-table. Each of these modifications does not necessarily correspond to a discretization error E(h) used in a practical case. By these modifications we can generalize equation (3.1.5) and the results are given in proposition (3.1.6) below. However we must obtain a generalization of the polynomials defined in equation (3.1.6), which preserves their basic properties.

Definition 3A

A continuous function $\varphi(x)$ defined on $(-\infty, \infty)$ is of class E (even) if

(i) $\varphi(x)$ is even,

(ii) periodic with period 1,

(iii) $\varphi(0) = 0$,

(iv) $\varphi(x) \neq 0$ on $(0, 1)$,

(v) $\varphi'(x)$ exists and does not vanish on $(0, \frac{1}{2})$.

We immediately get the following properties:

(vi) $\varphi(x)$ is symmetric about the line $x = \frac{1}{2}$ since

$$\varphi(x) = \varphi(-x) = \varphi(1-x)$$

by (i) and (ii).

(vii) $\varphi(x) > 0$ on $(0, 1) \Rightarrow \begin{cases} \varphi'(x) > 0 & \text{in } (0, \frac{1}{2}) \\ \varphi'(x) < 0 & \text{in } (\frac{1}{2}, 1) \end{cases}$

by (iii), (v) and (vi).

Definition 3B

A continuous function $\varphi(x)$ defined on $(-\infty, \infty)$ is of class O (odd) if

(i) $\varphi(x)$ is odd

(ii) periodic with period 1

(iii) $\varphi(x) \neq 0$ on $(0, \frac{1}{2})$.

The properties above imply the following ones

(iv) $\varphi(0) = \varphi(\frac{1}{2}) = \varphi(1) = 0$

$\varphi(0) = \varphi(1) = 0$ is obvious while $\varphi(\frac{1}{2}) = 0$ follows from

$$\varphi(\tfrac{1}{2}) = \varphi(1-\tfrac{1}{2}) = \varphi(-\tfrac{1}{2}) = -\varphi(\tfrac{1}{2})$$

(v) $\varphi(x)$ is odd about $x = \frac{1}{2}$ i.e.

$$\varphi(\tfrac{1}{2}+x) = -\varphi(\tfrac{1}{2}-x)$$

since

$$\varphi(\tfrac{1}{2}+x) = \varphi(-\tfrac{1}{2}+x) = -\varphi(\tfrac{1}{2}-x).$$

Lemma 3.1.3

If $\varphi(x)$ is of class O, then

$$\psi(x) = \int_0^x \varphi(t)\, dt$$

is of class E.

Proof:

(i) $\psi(-x) = \int_0^{-x} \varphi(t)\, dt = \int_0^x \varphi(-u)\, du$ (u = -t)

$= \int_0^x \varphi(u)\, du = \psi(x)$ by 3B(i)

(iii) $\psi(0) = 0$ (obvious)

$\psi(1) = \int_0^1 \varphi(t)\, dt = \int_{-\frac{1}{2}}^{\frac{1}{2}} \varphi(t)\, dt = 0$ by 3B(ii), (i)

(ii) $\psi(x+1) = \psi(x) + \int_x^{x+1} \varphi(t)\, dt = \psi(x) + \int_0^1 \varphi(t)\, dt$ by 3B(ii)

$= \psi(x) + \psi(1) = \psi(x)$ by (iii) above

(iv) Suppose $\varphi(t) > 0$ on $(0, \frac{1}{2})$, then $\varphi(t) < 0$ on $(\frac{1}{2}, 1)$ by 3B(v). It follows that $\psi(2) > 0$ on (0, 1) since $\int_0^1 \varphi(t)\, dt = 0$. (Similarly if $\varphi(t) < 0$ on $(0, \frac{1}{2})$).

(v) Obvious since $\psi'(x) = \varphi(x)$ on $(0, \frac{1}{2})$.

Lemma 3.1.4

If $\varphi(x)$ is of class E,

$$\psi(x) = \int_0^x [\varphi(2t) - \varphi(t)]\, dt$$

is of class O.

Proof:

(i) $$\psi(-x) = \int_0^{-x} [\varphi(2t)-\varphi(t)]dt$$

$$= -\int_0^{x} [\varphi(-2u)-\varphi(-u)]du \qquad (u=-t)$$

$$= -\int_0^{x} [\varphi(2u)-\varphi(u)]du \qquad \text{by 3A(i)}$$

$$= \psi(x)$$

(ii) $$\psi(0) = 0 \qquad \text{(obvious)}$$

$$\psi(1) = \int_0^{1} \varphi(2t)dt - \int_0^{1} \varphi(t)dt$$

$$= \tfrac{1}{2}\int_0^{2} \varphi(u)du - \int_0^{1} \varphi(t)dt \qquad (u=2t)$$

$$= \tfrac{1}{2}\cdot 2\int_0^{1} \varphi(u)du - \int_0^{1} \varphi(t)dt \qquad \text{by 3A(ii)}$$

$$= 0$$

$$\psi(x+1) = \psi(x) + \int_x^{x+1} [\varphi(2t)-\varphi(t)]dt$$

$$= \psi(x) + \int_0^{1} [\varphi(2t)-\varphi(t)]dt \qquad \text{by 3A(ii)}$$

$$= \psi(x) + \psi(1)$$

$$= \psi(x)$$

(iii) Assume $\varphi(x) > 0$ on $(0, 1)$. (Similarly if $\varphi(x) < 0$ on $(0, 1)$.)

Then

$$\varphi(2x) - \varphi(x) \begin{cases} > 0 & 0 < x < \frac{1}{3} & \text{(a)} \\ = 0 & x = \frac{1}{3} & \text{(b)} \\ < 0 & \frac{1}{3} < x < \frac{1}{2} & \text{(c)} \end{cases}$$

since

(b) by 3A(vi) we have $\varphi(2x) = \varphi(x)$ if $2x = 1-x$ i.e. if $x = \frac{1}{3}$.

(a) $0 < x < \frac{1}{3} \Rightarrow 0 < 2x < \frac{2}{3}$

$$\Rightarrow \begin{cases} 0 < x < 2x \le \frac{1}{2} \\ \quad \text{or} \\ \frac{1}{2} \le 2x < \frac{2}{3} \end{cases}$$

$$\Rightarrow \begin{cases} \varphi(x) < \varphi(2x) & \text{by 3A(vii)} \\ \quad \text{or} \\ \varphi(2x) > \varphi(\frac{2}{3}) = \varphi(\frac{1}{3}) > \varphi(x) & \text{by 3A(vi)(vii)} \end{cases}$$

(c) $\frac{1}{3} < x < \frac{1}{2} \Rightarrow 0 < 1-2x < \frac{1}{3}$

$$\Rightarrow \varphi(2x) = \varphi(1-2x) < \varphi(\tfrac{1}{3}) < \varphi(x) \qquad \text{by 3A(vi)(vii).}$$

Hence $\psi(x)$ is increasing in $(0, \frac{1}{3})$, decreasing in $(\frac{1}{3}, \frac{1}{2})$.

Therefore $\psi(x) > 0$ in $(0, \frac{1}{2})$ since $\psi(0) = \psi(\frac{1}{2}) = 0$.

<u>Lemma 3.1.5</u>

If $\varphi(x)$ is of class E

$$\psi(x) = \int_0^x [\varphi(t) - K]\,dt$$

is of class O, where $K = \int_0^1 \varphi(t)\,dt$.

Proof:

(i) $$\psi(-x) = \int_0^{-x} [\varphi(t)-K]dt$$

$$= -\int_0^{x} [\varphi(-u)-K]du \qquad (u=-t)$$

$$= -\int_0^{x} [\varphi(u)-K]du \qquad \text{by 3A(i)}$$

$$= -\psi(x)$$

(ii) $$\psi(0) = 0 \qquad \text{(obvious)}$$

$$\psi(1) = \int_0^1 \varphi(t)dt - K = 0$$

$$\psi(x+1) = \psi(x) + \int_0^{x+1} [\varphi(t)-K]dt$$

$$= \psi(x) + \int_0^{1} [\varphi(t)-K]dt \qquad \text{by 3A(ii)}$$

$$= \psi(x) + \psi(1)$$

$$= \psi(x)$$

(iii) Assume $\varphi(t) > 0$ on $(0, 1)$. Then

$$\psi(x) = \int_0^x \varphi(t)dt - x\int_0^1 \varphi(t)dt$$

$$= \int_0^x \varphi(t)dt - 2x\int_0^{\frac{1}{2}} \varphi(t)dt \qquad \text{by 3A(ii)}$$

$$\psi(x) = (1-2x)\int_0^x \varphi(t)dt - 2x\int_x^{\frac{1}{2}} \varphi(t)dt$$

$$= x(1-2x)\varphi(\xi_1) - 2x(\tfrac{1}{2}-x)\varphi(\xi_2) \qquad (0 \le \xi_1 \le x \le \xi_2 \le \tfrac{1}{2})$$

$$= x(1-2x)[\varphi(\xi_1)-\varphi(\xi_2)] < 0 \qquad 0 < x < \tfrac{1}{2}$$

since $0<\xi_1<\xi_2<2 \Rightarrow \varphi(\xi_1)<\varphi(\xi_2)$ by 3A(vii).

(Similarly if $\varphi(t)<0$ on $(0, 1)$.)

Proposition 3.1.6

$$(3.1.9) \qquad T_m^{(k)} - \int_0^1 f(x)\,dx = 4^{-(m+1)k}\int_0^1 b_{2m+2}(2^k x)\, f^{(2m+2)}(x)\,dx + \sum_{j=1}^{m} A_{m,j}^{(k)}\,[f^{(2j-1)}(1) - f^{(2j-1)}(0)]$$

where the definitions of the functions $b_{2m+2}(x)$ and the constants $A_{m,j}^{(k)}$ depend on whether the step from $T_{j-1}^{(k)}$ to $T_j^{(k)}$, $1 \le j \le m$, were eliminating or not. In any case the $b_{2m+2}(x)$ satisfy Def.(3A). The exact definitions are given in the proof below.

Proof:

We establish (3.1.9) by induction. It is shown in [4] that it is true for $m = 1$ where $b_2(x)$ is defined by (3.1.7) and is clearly a function of class E. Assume (3.1.9) is true for $m = 1$ i.e.

$$(3.1.10) \quad T_{s-1}^{(k)} - I = 4^{-ks} \int_0^1 b_{2s}(2^k x)\, f^{(2s)}(x)$$

$$+ \sum_{j=1}^{s-1} A_{s-1,j}^{(k)} \left[f^{(2j-1)}(1) - f^{(2j-1)}(0)\right]$$

where $b_{2s}(x)$ is of class E. There are two cases to consider:

(i) The step from $T_{s-1}^{(k)}$ to $T_s^{(k)}$ is an eliminating step.

(ii) This step is noneliminating.

Case (i):

Since the step is eliminating, the $T_s^{(k)}$-column is constructed by the standard Romberg procedure i.e.

$$(3.1.11) \quad T_s^{(k)} = \frac{4^s\ T_{s-1}^{(k+1)} - T_{s-1}^{(k)}}{4^s - 1}$$

Substituting (3.1.10) into (3.1.11) we obtain

$$(3.1.12) \quad T_s^{(k)} - I = 4^{-ks} \int_0^1 \left[\frac{b_{2s}(2^{k+1}x) - b_{2s}(2^k x)}{4^s - 1}\right] f^{(2s)}(x)\,dx$$

$$+ \sum_{j=1}^{s-1} \left[\frac{4^s A_{s-1,j}^{(k+1)} - A_{s,j}^{(k)}}{4^s - 1}\right] \left[f^{(2j-1)}(1) - f^{(2j-1)}(0)\right].$$

Define $\hat{b}_{2s}(x)$, $b_{2s+1}(x)$ and $b_{2s+2}(x)$ as in (3.1.6). Since $b_{2s}(x)$ is of class E, Lemmas (3.1.3) and (3.1.4) indicate that $b_{2s+1}(x)$ is of class O and $b_{2s+2}(x)$ is of class E.

If we set

$$(3.1.13)\qquad A_{m,j}^{(k)} = \begin{cases} \dfrac{4^m A_{m-1,j}^{(k+1)} - A_{m-1,j}^{(k)}}{4^m - 1} & 1 \le j \le m-1 \quad \text{(a)} \\ 0 & j = m \quad \text{(b)} \end{cases}$$

then integrate by parts (3.1.12) twice we obtain

$$(3.1.14)\qquad T_m^{(k)} - I = 4^{-k(m+1)} \int_0^1 b_{2m+2}(2^k x)\, f^{(2m+2)}(x)\, dx$$

$$+ \sum_{j=1}^{m} A_{m,j}^{(k)} [f^{(2j-1)}(1) - f^{(2j-1)}(0)]$$

which is of the same form as (3.1.10) with $m-1$ replaced by m. This completes the induction proof for case (i).

Case (ii):

Again assume (3.1.10). Since the step from $T_{m-1}^{(k)}$ to $T_m^{(k)}$ is noneliminating we modify the standard procedure and set

$$(3.1.15)\qquad T_m^{(k)} \equiv T_{m-1}^{(k)},\ \forall k$$

hence (3.1.10) becomes

$$(3.1.16)\qquad T_m^{(k)} - I = 4^{-km} \int_0^1 b_{2m}(2^k x)\, f^{(2m)}(x)\, dx$$

$$+ \sum_{j=1}^{m-1} A_{m-1,j}^{(k)} [f^{(2j-1)}(1) - f^{(2j-1)}(0)].$$

The idea now is to bring (3.1.6) to the same form as (3.1.10) with $m-1$ replaced by m. If we try to proceed as in case (i) and tentatively let $\hat{b}_{2m}(x) = b_{2m}(x)$ with $b_{2m+1}(x)$ and $b_{2m+2}(x)$ as in (3.1.6), then we find that $b_{2m+1}(x)$ is no longer of class O as seen from the following result

$$b_{2m+1}(1) = \int_0^1 b_{2m}(x)\,dx = K_{2m} \qquad \text{(by definition)}$$

and $K_{2m} \neq 0$ since $b_{2m}(x)$ is of definite sign over $(0,1)$.

To avoid this difficulty we simply define

$$\hat{b}_{2m}(x) = b_{2m}(x) - K_{2m} \qquad 0 \le x \le 1. \tag{3.1.17}$$

Lemmas (3.1.3) and (3.1.5) then guarantee that $b_{2m+1}(x)$ and $b_{2m+2}(x)$ are of classes O and E respectively. **Eq.** (3.1.16) becomes

$$T_m^{(k)} - I = 4^{-km} \int_0^1 [b_{2m}(2^k x) - K_{2m} + K_{2m}]\, f^{(2m)}(x)\,dx + \sum_{j=1}^{m-1} A_{m-1,j}^{(k)} [f^{(2j-1)}(1) - f^{(2j-1)}(0)] \tag{3.1.18}$$

$$T_m^{(k)} - I = 4^{-km} \int_0^1 \hat{b}_{2m}(2^k x)\,dx + 4^{-km} K_{2m} \int_0^1 f^{(2m)}(x)\,dx + \sum_{j=1}^{m-1} A_{m-1,j}^{(k)} [f^{(2j-1)}(1) - f^{(2j-1)}(0)] \tag{3.1.19}$$

$$T_m^{(k)} - I = 4^{-km} \int_0^1 \hat{b}_{2m}(2^k x)\,dx + 4^{-km} K_{2m} [f^{(2m-1)}(1) - f^{(2m-1)}(0)] + \sum_{j=1}^{m-1} A_{m-1,j}^{(k)} [f^{(2j-1)}(1) - f^{(2j-1)}(0)].$$

If we define now

$$(3.1.20) \qquad A_{m,j}^{(k)} = \begin{cases} A_{m-1,j}^{(k)} & 1 \le j \le m-1 \quad \text{(a)} \\ 4^{-km} K_{2m} & j = m \quad \text{(b)} \end{cases}$$

eq.(3.1.19) becomes

$$(3.1.21) \qquad T_m^{(k)} - I = 4^{-km} \int_0^1 \hat{b}_{2m}(2^k x)\, f^{(2m)}(x)\, dx$$

$$+ \sum_{j=1}^{m} A_{m,j}^{(k)} \left[f^{(2j-1)}(1) - f^{(2j-1)}(0) \right]$$

integrating by parts twice we find that

$$(3.1.22) \qquad \int_0^1 \hat{b}_{2m}(2^k x)\, f^{(2m)}(x)\, dx = 4^{-k} \int_0^1 b_{2m+2}(2^k x)\, f^{(2m+2)}(x)\, dx$$

and

$$(3.1.23) \qquad T_m^{(k)} - I = 4^{-k(m+1)} \int_0^1 b_{2m+2}(2^k x)\, f^{(2m+2)}(x)\, dx$$

$$+ \sum_{j=1}^{m} A_{m,j}^{(k)} \left[f^{(2j-1)}(1) - f^{(2j-1)}(0) \right]$$

which is of the same form as (3.1.10) with $m-1$ replaced by m. This completes the proof.

Corollary 3.1.7

$$(3.1.24) \qquad T_m^{(k)} - \int_0^1 f(x)\, dx = C_m^{(k)} f^{(2m+2)}(\xi) + \sum_{j=1}^{m} A_{m,j}^{(k)} \left[f^{(2j-1)}(1) - f^{(2j-1)}(0) \right]$$

where $0 \le \xi \le 1$ and

$$(3.1.25) \qquad C_s^{(k)} = 4^{-(s+1)k} \int_0^1 b_{2s+2}(x)\, dx.$$

Proof:

We simply apply the integral mean value theorem to (3.1.9) and since $b_{2s+2}(x)$ is of definite sign in $(0, 1)$, we obtain

$$(3.1.26) \qquad T_s^{(k)} - \int_0^1 f(x)\, dx = 4^{-(s+1)k} f^{(2s+2)}(\xi) \int_0^1 b_{2s+2}(2^k x)\, dx$$

$$+ \sum_{j=1}^{s} A_{s,j}^{(k)} [f^{(2j-1)}(1) - f^{(2j-1)}(0)]$$

where $0 \le \xi \le 1$. Since $b_{2s+2}(x)$ is periodic with period 1 we have

$$(3.1.27) \qquad \int_0^1 b_{2s+2}(2^k x) dx = \int_0^1 b_{2s+2}(x)\, dx.$$

This proves the corollary.

Corollary 3.1.8

If the j-th step is eliminating, then

$$(3.1.28) \qquad A_{s,j}^{(k)} = 0.$$

Proof:

Consider the following sketch of the T-table

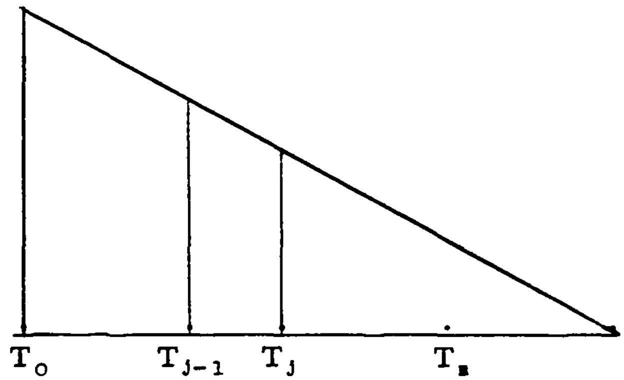


fig. 3.1.1

Let the step from $T_{j-1}^{(k)}$ to $T_j^{(k)}$ be eliminating. There are two possible cases:

(i) $j = m$

(ii) $j < m$.

(i) If $j = m$ then the m-th step is also eliminating. Hence $A_{m,j}^{(k)}$ is defined by (3.1.13)(b) i.e. for $j = m$ we have $A_{m,j}^{(k)} = 0$.

(ii) Let $j<m$. The j-th step is eliminating by assumption; however no restrictions are put on the m-th step. It can therefore be either eliminating or noneliminating. Accordingly, $A_{m,j}^{(k)}$ is defined either by (3.1.13)(a) or (3.1.20)(a).

$$(3.1.29) \qquad A_{m,j}^{(k)} = \begin{cases} \dfrac{4^m A_{m-1,j}^{(k+1)} - A_{m-1,j}^{(k)}}{4^m - 1} & \text{if the m-th is eliminating} \quad (a) \\ A_{m-1,j}^{(k)} & \text{if the m-th is noneliminating} \quad (b) \end{cases}$$

Now $j<m$ implies that $m=j+i$ for some integer $i \geq 1$. If $m=j+1$ then (3.1.29) gives

$$\text{(3.1.30)} \qquad A_{m,j}^{(k)} = \begin{cases} \dfrac{4^m A_{j,j}^{(k+1)} - A_{j,j}^{(k)}}{4^m - 1} = 0 & \text{if the m-th step is eliminating} \quad \text{(a)} \\ A_{j,j}^{(k)} = 0 & \text{if the m-th step is noneliminating} \quad \text{(b)} \end{cases}$$

because $A_{j,j}^{(k)} = 0$, $\forall k$, according to (3.1.13)(a) since the j-th step is eliminating by assumption. By induction, we obviously have

$$A_{j+1,j}^{(k)} = 0 \Rightarrow A_{j+2,j}^{(k)} = 0 \Rightarrow \dots \Rightarrow A_{m,j}^{(k)} = 0$$

completing the proof of the corollary.

Remarks:

(a) In the conventional T-table, every step is eliminating. Hence by Cor.3.1.8, $A_{m,j}^{(k)} = 0$ for every $j = 1, 2, 3, \dots, m$ and (3.1.9) reduces to (3.1.5) as expected.

(b) If we choose all the steps to be noneliminating then every column of the table is identical to the first T_0-column i.e. $T_m^{(k)} = T_0^{(k)}$ $\forall m, k$ and no power of h is eliminated in the truncation error. In this case, (3.1.9) is simply the Euler-Maclaurin summation formula.

3.2 A Bound on the Truncation Error in a Particular Case

The results of the previous section can be applied to any modification of the standard T-table. We now restrict ourselves to the particular case where, for $m > 0$

(3.2.1) $$T_m^{(k)} = \begin{cases} T_{m-1}^{(k)} & m \text{ even} \\ \dfrac{4^m\, T_{m-1}^{(k+1)} - T_{m-1}^{(k)}}{4^m - 1} & m \text{ odd} \end{cases}$$

This is the table which is associated to a discretization error $E(h)$ of the form

$$E(h) \sim d_2 h^2 + d_6 h^6 + d_{10} h^{10} + d_{14} h^{14} + \ldots$$

as in the case of contour integration round squares. (See fig.2.4.1)

Our aim is to find an expression for $C_{2p}^{(k)}$ which is easily bounded. From (3.1.25) it is clear that

(3.2.2) $$C_{2p}^{(k)} = 4^{-(2p+1)k}\, C_{2p}^{(0)}.$$

On setting $m = 2p$ and $k = 0$ in Cor.3.1.7 we get

(3.2.3) $$T_{2p}^{(0)} - I = \sum_{j=1}^{2p} A_{2p,j}^{(0)} [f^{(2j-1)}(1) - f^{(2j-1)}(0)] + C_{2p}^{(0)} f^{(4p+2)}(\xi).$$

However by (2.4.15)

$$T_{2p}^{(0)} = \sum_{k=0}^{p} \gamma_k \, T_0^{(k)}. \tag{3.2.4}$$

Since the Euler-Maclaurin formula gives an expression for $T_0^{(k)}$-I we can therefore obtain another expression for $T_{2p}^{(0)}$ -I which when compared to (3.2.3) will yield a formula for $C_{2p}^{(0)}$ and hence for $C_{2p}^{(k)}$.

<u>Proposition 3.2.1</u>

$$T_{2p}^{(0)}-I\Big(\sum_{k=0}^{p} \gamma_k\Big) = \sum_{j=1}^{2p} c_{2j}\Big(\sum_{k=0}^{p} \frac{\gamma_k}{4^{kj}}\Big)\left[f^{(2j-1)}(1)-f^{(2j-1)}(0)\right] + \int_0^1 f^{(4p+2)}(x)\,\tilde{\varphi}(x)\,dx \tag{3.2.5}$$

where $\tilde{\varphi}(x)$ is defined below, γ_k is defined by (3.2.4), c_{2j} are numbers defined by (2.1.11) and $I = \int_0^1 f(x)\,dx$.

<u>Proof</u>:

Letting $n = 2^k$ and $q = 2p$ in Prop.2.1.2 we have

$$T_0^{(k)}-I = \sum_{j=1}^{2p} \frac{c_{2j}}{4^{kj}} \left[f^{(2j-1)}(1)-f^{(2j-1)}(0)\right] + \int_0^1 f^{(4p+2)}(x)\,\varphi_k(x)\,dx \tag{3.2.6}$$

where

$$\varphi_k(x) = h^{4p+2}\left[c_{4p+2} - P_{4p+2}(2^k x)\right] \qquad (h = 2^{-k}) \tag{3.2.7}$$

for fixed p. Furthermore, from (3.2.4) we get

(3.2.8) $$E = T_{2p}^{(0)} - I(\sum_{k=0}^{p} \gamma_k) = \sum_{k=0}^{p} \gamma_k (T_0^{(k)} - I).$$

Substituting (3.2.6) into (3.2.8) gives

(3.2.9) $$E = \sum_{k=0}^{p} \gamma_k \left(\sum_{j=1}^{2p} \frac{c_{2j}}{4^{kj}} [f^{(2j-1)}(1) - f^{(2j-1)}(0)] + \int_0^1 f^{(4p+2)}(x)\, \varphi_k(x)\, dx \right)$$

or

(3.2.10) $$E - \sum_{j=1}^{2p} c_{2j} \left(\sum_{k=0}^{p} \frac{\gamma_k}{4^{kj}} \right) [f^{(2j-1)}(1) - f^{(2j-1)}(0)]$$

$$= \int_0^1 f^{(4p+2)}(x) \left(\sum_{k=0}^{p} \gamma_k\, \varphi_k(x) \right) dx$$

$$= \int_0^1 f^{(4p+2)}(x)\, \tilde{\varphi}(x)\, dx$$

where

(3.2.11) $$\tilde{\varphi}(x) = \sum_{k=0}^{p} \gamma_k\, \varphi_k(x)$$

$$= \sum_{k=0}^{p} \frac{\gamma_k}{4^{(2p+1)k}} [c_{4p+2} - P_{4p+2}(2^k x)]$$

completing the proof.

<u>Corollary 3.2.2</u>

(3.2.12) $$\sum_{k=0}^{p} \gamma_k = 1.$$

Proof:

Let f(x) = 1 in (3.2.5) and we get

$$1-\left(\int_0^1 1dx\right)\left(\sum_{k=0}^{p} \gamma_k\right) = 0$$

giving (3.2.12). This result was already found in section 2.4.

From (3.2.3) and (3.2.5) it follows that

$$(3.2.13) \qquad T_{2p}^{(0)}-I = \sum_{j=1}^{2p} A_{2p,j}^{(0)} [f^{(2j-1)}(1)-f^{(2j-1)}(0)]+C_{2p}^{(0)} f^{(4p+2)}(\xi)$$

$$= \sum_{j=1}^{2p} c_{2j} \left(\sum_{k=0}^{p} \frac{\gamma_k}{4^{kj}}\right)[f^{(2j-1)}(1)-f^{(2j-1)}(0)]$$

$$+\int_0^1 f^{(4p+2)}(x)\, \tilde{\varphi}(x)\, dx.$$

Comparing the two expressions we expect to have the following equalities

$$(3.2.14) \qquad A_{2p,j}^{(0)} = c_{2j}\left(\sum_{k=0}^{p} \frac{\gamma_k}{4^{kj}}\right) \qquad 1 \le j \le 2p$$

and consequently

$$(3.2.15) \qquad C_{2p}^{(0)} f^{(4p+2)}(\xi) = \int_0^1 f^{(4p+2)}(x)\, \tilde{\varphi}(x)\, dx .$$

We prove this in the following proposition.

Proposition 3.2.3

$$(3.2.16) \qquad c_{2j} \sum_{k=0}^{p} \frac{\gamma_k}{4^{kj}} = A_{2p,j}^{(0)} \qquad 1 \le j \le 2p$$

Proof:

Since (3.2.13) is valid for all functions $f \in C^{4p+2}[0,1]$, we apply it successively to polynomials of degree $2, 4, \ldots, 4p$. Let $f(x) = x^2$. Then (3.2.13) gives

$$(3.2.17) \qquad A_{2p,1}^{(0)}[f'(1)-f'(0)] = c_2\left(\sum_{k=0}^{p} \frac{\gamma_k}{4^k}\right)[f'(1)-f'(0)]$$

$$(3.2.18) \qquad A_{2p,1}^{(0)} = c_2 \sum_{k=0}^{p} \frac{\gamma_k}{4^k}$$

since $f'(1)-f'(0) = 2 \neq 0$. Hence (3.2.16) is valid for $j=1$. Assume it holds for $j=1, 2, \ldots, n-1 < 2p$. We show it remains true for $j=n$. Let $f(x) = x^{2n}$, $(2n \le 4p)$; thus (3.2.16) becomes

$$(3.2.19) \qquad S_1 + A_{2p,n}^{(0)}[f^{(2n-1)}(1)-f^{(2n-1)}(0)] =$$

$$= S_2 + c_{2n}\left(\sum_{k=0}^{p} \frac{\gamma_k}{4^{kn}}\right)[f^{(2n-1)}(1) - f^{(2n-1)}(0)]$$

where

$$(3.2.20) \qquad S_1 = \sum_{j=1}^{n-1} A_{2p,j}^{(0)} [f^{(2j-1)}(1)-f^{(2j-1)}(0)]$$

$$S_2 = \sum_{j=1}^{n-1} c_j \left(\sum_{k=0}^{p} \frac{\gamma_k}{4^{kj}} \right) [f^{(2j-1)}(1)-f^{(2j-1)}(0)].$$

However $S_1 = S_2$ follows from the assumption giving

$$(3.2.21) \qquad A_{2p,n}^{(0)} = c_{2n} \sum_{k=0}^{p} \frac{\gamma_k}{4^{kn}}$$

since $f^{(2n-1)}(1)-f^{(2n-1)}(0) = (2n)! \neq 0$. This completes the proof.

Corollary 3.2.4

$$(3.2.22) \qquad \sum_{k=0}^{p} \frac{\gamma_k}{4^{kj}} = 0$$

for $j=1,3,5,7,\ldots,2p-1$.

Proof:

(3.2.22) follows immediately from (3.1.28) and (3.2.16) since every odd step is eliminating. This formula has already been found in section 2.4 by another method.

Corollary 3.2.5.

$$(3.2.23) \qquad C_{2p}^{(0)} = c_{4p+2} \sum_{k=0}^{p} \frac{\gamma_k}{4^{(2p+1)k}} .$$

Proof:

Let $f(x) = x^{4p+2}$, then

$$f^{(4p+2)}(x) = f^{(4p+2)}(\xi) = (4p+2)! \neq 0$$

and (3.2.15) gives

$$C_{2p}{}^{(0)} = \int_0^1 \tilde{\varphi}(x)\ dx$$

$$= \int_0^1 \left[c_{4p+2}\left(\sum_{k=0}^{p} \frac{\gamma_k}{4^{(2p+1)k}}\right) - \sum_{k=0}^{p} \frac{\gamma_k}{4^{(2p+1)k}} P_{4p+2}(2^k x)\right] dx$$

$$= c_{4p+2} \sum_{k=0}^{p} \frac{\gamma_k}{4^{(2p+1)k}} - \sum_{k=0}^{p} \left[\frac{\gamma_k}{4^{(2p+1)k}} \int_0^1 P_{4p+2}(2^k x)\ dx\right].$$

However, by (2.1.8)

$$\int_0^1 P_{4p+2}(2^k x)dx = \frac{1}{2^k} P_{4p+3}(2^k x)\Big|_0^1 = 0$$

proving (3.2.23).

Corollary 3.2.6

(3.2.24) $$C_{2p}{}^{(k)} = 4^{-(2p+1)k}\, c_{4p+2} \sum_{k=0}^{p} \frac{\gamma_k}{4^{(2p+1)k}} .$$

Proof:

This follows immediately from (3.2.2) and (3.2.23).

Proposition 3.2.7

For any integer $p \geq 1$,

$$(3.2.25) \qquad \frac{2}{(2\pi)^{4p+2}} < c_{4p+2} < \frac{2.2}{(2\pi)^{4p+2}} \quad .$$

Proof:

From (2.1.11) we may write

$$(3.2.26) \qquad c_{2i} = \frac{B_{2i}}{(2i)!} = (-1)^{i-1} \frac{2\zeta(2i)}{(2\pi)^{2i}}$$

where

$$(3.2.27) \qquad \zeta(i) = \sum_{n=1}^{\infty} \frac{1}{n^i} \quad .$$

Thus

$$(3.2.28) \qquad c_{4p+2} = \frac{2\zeta(4p+2)}{(2\pi)^{4p+2}} \quad .$$

However $\zeta(i) > 1$ for any integer i and ζ is a decreasing function of i. Thus for any integer $p \geq 1$ we have

$$(3.2.29) \qquad 1 < \zeta(4p+2) \leq \zeta(6) = \frac{\pi^6}{945} < 1.1$$

proving (3.2.25).

Proposition 3.2.8

$$(3.2.30) \qquad \sum_{k=0}^{p} \frac{\gamma_k}{4^{(2p+1)k}} = (-1)^p \, 4^{-p^2} \prod_{j=1}^{p} \frac{(1-4^{-2j})}{(1-4^{-2j+1})} \quad .$$

Proof:

Let S denote the left-hand side of (3.2.30). Then

(3.2.31) $$S = 4^{-(2p+1)p} \sum_{k=0}^{p} \frac{4^{(2p+1)p}}{4^{(2p+1)k}} \gamma_k$$

(3.2.32) $$= 4^{-(2p+1)p} \sum_{k=0}^{p} 4^{(2p+1)(p-k)} \gamma_k$$

(3.2.33) $$= 4^{-(2p+1)p} t_{2p} (4^{2p+1}) \qquad \text{by (2.4.19)}$$

(3.2.34) $$= 4^{-(2p+1)p} \prod_{k=1}^{p} \frac{(1-4^{2p-2k+2})}{(1-\frac{1}{4^{2k-1}})} \qquad \text{by (2.4.21)}$$

(3.2.35) $$= 4^{-(2p+1)p} \cdot 4^{2+4+\cdots+2p} \prod_{k=1}^{p} \frac{(\frac{1}{4^{2p-2k+2}}-1)}{(1-\frac{1}{4^{2k-1}})}$$

(3.2.36) $$= 4^{-(2p+1)p} \cdot 4^{p(p+1)} (-1)^p \prod_{k=1}^{p} \frac{(1-\frac{1}{4^{2p-2k+2}})}{(1-\frac{1}{4^{2k-1}})}$$

(3.2.37) $$= (-1)^p 4^{-p^2} \frac{(1-2^{-4})(1-2^{-8})\ldots(1-2^{-4p})}{(1-2^{-2})(1-2^{-6})\ldots(1-2^{-(4p-2)})}$$

proving (3.2.30). It can be shown that

(3.2.38) $$\frac{5}{4} < \prod_{j=1}^{p} \frac{(1-4^{-2j})}{(1-4^{-2j+1})} < \frac{4}{3} .$$

Proposition 3.2.9

(3.2.39) $$|C_{2p}^{(k)}| < 4^{-(2p+1)k} \times \frac{3 \times 4^{-p^2}}{(2\pi)^{4p+2}} .$$

Proof:

If we substitute (3.2.25), (3.2.30) and (3.2.38) into (3.2.24) we obtain

$$|C_{2p}^{(k)}| < 4^{-(2p+1)k} \frac{(2.2)}{(2\pi)^{4p+2}} \cdot 4^{-p^2} (\frac{4}{3})$$

$$|C_{2p}^{(k)}| < 4^{-(2p+1)k} \frac{4^{-p^2}}{(2\pi)^{4p+2}} (\frac{8.8}{3})$$

$$|C_{2p}^{(k)}| < 4^{-(2p+1)k} \frac{3.4^{-p^2}}{(2\pi)^{4p+2}} .$$

Proposition 3.2.10

Let f(z) be a complex analytic function in a region containing the interval of integration [0, 1]. If ρ is the shortest distance from this interval to any singularity of f(z) then for any fixed $\epsilon > 0$ there exists a number $K(\epsilon)$ such that

(3.2.40) $$|C_{2p}^{(k)} f^{(4p+2)}(\xi)| < \frac{3K(\epsilon)\, 4^{-(2p+1)k}\, 4^{-p^2}\, (4p+2)!}{(2\pi)^{4p+2}\, (\rho-\epsilon)^{4p+2}}$$

Proof:

It follows immediately from (3.2.39) and the fact that for any $\epsilon > 0$ there exists a number $K(\epsilon)$ such that

$$\left|\frac{f^{(p)}(z)}{p!}\right| < \frac{K(\epsilon)}{(\rho-\epsilon)^p} \ .$$

This comes from the Cauchy formula

$$\frac{f^{(p)}(z)}{p!} = \frac{1}{2\pi i}\int_C \frac{f(\omega)}{(\omega-z)^{p+1}}\,d\omega$$

around a circle C of radius $\rho-\epsilon$ and since f(z) is analytic and hence continuous on the compact set $\{z: |z| \leq \rho-\epsilon\}$ it follows that $|f(z)|$ is bounded by a constant $K(\epsilon)$ on C.

3.3 A Square Contour

As defined in Section 2.3, let $S_0^{(i)}$ be the trapezoidal sum obtained around the whole square when each side is divided into $m = 2^i$ equal parts with $i=0, 1, 2, 3, \ldots$ and let

$$I_{\square} = \int_{\square} f(z)\, dz$$

be the exact value of the integral where $\square$ is a square with sides parallel with the axes. We proceed with the modified Romberg table described in fig. 2.4.1 and apply the results of Sections 3.1 and 3.2 to bound the truncation error for any entry in the adapted table. We find

Proposition 3.3.1

$$S_{2p}^{(0)} - I_{\square} = \int_{\square} \Phi(z)\, f^{(4p+2)}(z)\, dz \qquad (3.3.1)$$

with

$$\Phi(z) = \sum_{i=0}^{p} \gamma_i\, \tilde{\omega}_p(z) \qquad (3.3.2)$$

where γ_i is defined by (2.4.15), $\tilde{\omega}_p(z)$ is defined by (2.3.5) and (2.3.6) with $m = 2^i$.

Proof:

From (2.4.15) and (3.2.12) we have

$$S_{2p}^{(0)} = \sum_{i=0}^{p} \gamma_i\, S_0^{(i)} \qquad (3.3.3)$$

(3.3.4) $$S_{2p}^{(0)} - I_{\square} = \sum_{i=0}^{p} \gamma_i \, [S_0^{(i)} - I_{\square}].$$

Substituting (2.3.20) into (3.3.3) we obtain

(3.3.5) $$S_{2p}^{(0)} - I_{\square} = \sum_{i=0}^{p} \gamma_i \, [\sum_{r=1}^{p} \frac{d^{*}r}{4^{(2r-1)i}} + \int_{\square} \tilde{\varphi}_p(z) \, f^{(4p+2)}(z) \, dz]$$

where $d^{*}r$ and $\tilde{\varphi}_p(z)$ are defined by (2.3.21) and (2.3.6) respectively. (Note that $\tilde{\varphi}_p(z)$ depends on i.)

(3.3.6) $$S_{2p}^{(0)} - I_{\square} = \sum_{r=1}^{p} d^{*}r \, (\sum_{i=0}^{p} \frac{\gamma_i}{4^{(2r-1)i}}) + \sum_{i=0}^{p} \gamma_i \, (\int_{\square} \tilde{\varphi}_p(z) \, f^{(4p+2)}(z) \, dz)$$

(3.3.7) $$= \int_{\square} (\sum_{i=0}^{p} \gamma_i \, \tilde{\varphi}_p(z)) \, f^{(4p+2)}(z) \, dz \qquad \text{by (3.2.22)}$$

(3.3.8) $$= \int_{\square} \Phi(z) \, f^{(4p+2)}(z) \, dz$$

where

(3.3.9) $$\Phi(z) = \sum_{i=0}^{p} \gamma_i \, \tilde{\varphi}_p(z)$$

concluding the proof.

Proposition 3.3.2

For any fixed $\epsilon > 0$ there exists a number $K(\epsilon)$ such that

(3.3.10) $$|S_{2p}^{(0)} - I_{\square}| \leq \frac{2R\,(4p+2)!\;3K(\epsilon)}{4^{p^2}\,(2\pi)^{4p+2}\left[\frac{d}{2R} - \epsilon\right]^{4p+2}}$$

where d is the **shortest** distance from a singularity of f(z) to the boundary of the square with sides of length 2R.

Proof:

By (3.3.4) we have

(3.3.11) $$S_{2p}^{(0)} - I_{\square} = \sum_{l=0}^{p} \gamma_l\,[S_0^{(l)} - I_{\square}].$$

Now, letting $m = 2^l$, we get from (2.3.17)

(3.3.12) $$S_0^{(l)} = \sum_{k=0}^{4} h_k \sum_{j=0}^{m}{}'' f(a_k + jh_k) \qquad (m = 2^l)$$

(3.3.13) $$S_0^{(l)} - I_{\square} = \sum_{k=1}^{4} \left[h_k \sum_{j=0}^{m}{}'' f(a_k + jh_k) - \int_{a_k}^{a_{k+1}} f(z)\,dz\right]$$

where $a_5 = a_1$. A simple change of variable,

(3.3.14) $$z = a_k + (a_{k+1} - a_k)t$$

gives

(3.3.15) $$f(z) = F_k(t) \qquad 0 \leq t \leq 1$$

on the side $a_k\,a_{k+1}$. At times we will consider t as a complex variable and regard (3.3.14) as a complex transformation as well. Applying this to (3.3.13), and substituting into (3.3.11) we get

$$(3.3.16) \qquad S_{2p}^{(0)} - I_{\square} = \sum_{i=0}^{p} \gamma_i \sum_{k=1}^{4} (a_{k+1}-a_k)\left[\frac{1}{m} \sum_{j=0}^{m}{}'' F_k\left(\frac{j}{m}\right) - \int_0^1 F_k(t)\, dt\right]$$

$$(3.3.17) \qquad = \sum_{k=1}^{4} (a_{k+1}-a_k)\left[\sum_{i=0}^{p} \gamma_i \, T_0^{(i)} - \int_0^1 F_k(t)\, dt\right]$$

using (3.2.12) and where

$$(3.3.18) \qquad T_0^{(i)} = \frac{1}{m} \sum_{j=0}^{m}{}'' F_k\left(\frac{j}{m}\right) \qquad (m = 2^i).$$

Thus $T_0^{(i)}$ represents the trapezoidal sums obtained from $F_k(t)$ on the respective sides $a_k\, a_{k+1}$. From (2.4.15) we now get

$$(3.3.19) \qquad S_{2p}^{(0)} - I_{\square} = \sum_{k=1}^{4} (a_{k+1}-a_k)\left[T_{2p}^{(0)} - \int_0^1 F_k(t)\, dt\right].$$

Applying (3.1.24) to each bracket in (3.3.19) we obtain

$$(3.3.20) \qquad S_{2p}^{(0)} - I_{\square} = \sum_{k=1}^{4} (a_{k+1}-a_k)\left[\sum_{j=1}^{2p} A_{2p,j}^{(0)} \left[F_k^{(2j-1)}(1) - F_k^{(2j-1)}(0)\right] + E_k\right]$$

where

$$(3.3.21) \qquad E_k = C_{2p}^{(0)} F_k^{(4p+2)}(\xi_k) \qquad 0 \le \xi_k \le 1.$$

However by (3.1.28), $A_{2p,j}^{(0)} = 0$ for j odd. Hence

$$(3.3.22) \qquad S_{2p}^{(0)} - I_{\square} = \sum_{k=1}^{4} (a_{k+1} - a_k) \sum_{j=1}^{p} A_{2p,2j}^{(0)} [F_k^{(4j-1)}(1) - F_k^{(4j-1)}(0)]$$

$$+ \sum_{k=1}^{4} (a_{k+1} - a_k) E_k .$$

Noting that $a_{k+1} - a_k = mh_k$ and using the relation

$$(3.3.23) \qquad F_k^{(4j-1)}(t) = (mh_k)^{4j-1} f^{(4j-1)}(z)$$

we obtain

$$(3.3.24) \qquad S_{2p}^{(0)} - I_{\square} = \sum_{k=1}^{4} \sum_{j=1}^{p} (mh_k)^{4j} A_{2p,2j}^{(0)} [f^{(4j-1)}(a_{k+1}) - f^{(4j-1)}(a_k)]$$

$$+ \sum_{k=1}^{4} (a_{k+1} - a_k) E_k$$

$$= \sum_{k=1}^{4} (a_{k+1} - a_k) E_k$$

since, for any j, $h_1^{4j} = h_2^{4j} = h_3^{4j} = h_4^{4j}$. By (3.2.40), each E_k is bounded,

$$(3.3.26) \qquad |E_k| < \frac{(4p+2)!\, 3K_k(\epsilon)}{4^{p^2} [2\pi(\rho_k - \epsilon)]^{4p+2}}$$

where ρ_k is the shortest distance from a singularity of $F_k(t)$ to the interval [0, 1] in the complex t-plane and $2R\rho_k$ is the corresponding distance in the z-plane. Thus if we set

$$\begin{cases} \rho = \min \rho_k \\ d = 2R\rho \qquad\qquad 1 \le k \le 4 \\ K(\epsilon) = 4 \max K_k(\epsilon) \end{cases} \tag{3.3.27}$$

we find

$$|S_{2p}{}^{(0)} - I_{\square}| \le 2R \sum_{k=1}^{4} |E_k| \tag{3.3.28}$$

$$\le \frac{2R\,(4p+2)!\;3}{4^{p^2}(2\pi)^{4p+2}} \sum_{k=1}^{4} \frac{K_k(\epsilon)}{[\rho_k - \epsilon]^{4p+2}} \tag{3.3.29}$$

$$\le \frac{2R(4p+2)!\;3K(\epsilon)}{4^{p^2}(2\pi)^{4p+2}[\frac{d}{2R} - \epsilon]^{4p+2}} \tag{3.3.30}$$

proving (3.3.10).

CHAPTER 4

PROGRAMS AND EXAMPLES

4.1 Introduction

In this chapter we apply the principles of the earlier chapters to practical examples.

We are given a function f(z) and its derivative $f'(z)$ and we choose a region which is either a rectangle or a square with sides parallel to the coordinate axes. We then proceed to determine whether this particular region contains any roots of f(z) or not.

Basically we evaluate numerically

$$(4.1.1) \qquad \frac{1}{2\pi i} \int_C \frac{f'(z)}{f(z)}\, dz$$

and this is done in two different ways depending on the manner the contour C is split.

In PROGRAM ONE we begin by dividing each side of the rectangle into N equal segments and evaluate the integral along each segment by the standard Romberg method described in chapter 2 until a specified convergence criterion is satisfied. These results are then added properly to give the total number of roots inside the given rectangle. If there are any roots inside we then continue integration on the inner sides and again the results are combined with the ones found on the outer sides to give the number of roots in each sub-rectangle. This was explained in detail in chapter 1.

In PROGRAM TWO, the path of integration is not a straight line but the whole contour of the region i.e. we begin by evaluating the trapezoidal sums $S_0^{(0)}, S_0^{(1)}, \ldots,$ defined by eq.(2.3.17) and then combine these values by Romberg's method until the difference between successive values is less than the prescribed tolerance. The region is then divided if and only if it contains a root and the procedure above is repeated automatically for each subregion. In PROGRAM TWO the region is restricted to a square since we need equal sub-divisions on all sides in order to apply the modified Romberg technique extensively described in chapters 2 and 3. The standard Romberg scheme is also built in to compare results.

PROGRAM ONE or PROGRAM TWO can be reapplied by the user to each subregion containing more than one root. When a region contains only one root, either program can be made to evaluate

$$(4.1.2) \qquad \frac{1}{2\pi i} \int_C z \frac{f'(z)}{f(z)} dz$$

by assigning the proper value to the parameter NZERO (see the listings of the programs). The integral in (4.1.2) gives the sum of the roots, hence the root itself when it is unique. We can thus find a reasonable approximation to the root.

We illustrate the programs by applying them to the following examples:

Ex. 1: $f(z) = z^5 + 16\sqrt{3} - 16i$

within the square of center (0, 0) and side 4.

Ex. 2: $f(z) = e^z - 2z^2$

within the square of center (0, 1) and side 4.

Ex. 3: $f(z) = \cosh(2z) - 1$

within the square of center (-0.5, 0.5) and side 6.

The first example is used to check our results since all the roots of f(z) can easily be found by de Moivre's theorem; more specifically,

$$f(z) = 0 \Leftrightarrow z \in \{z_0, z_1, z_2, z_3, z_4\}$$

where, for $k = 0, 1, \ldots, 4$,

$$z_k = 2[\cos(\frac{\pi}{6} + k\frac{\pi}{5}) + i\sin(\frac{\pi}{6} + k\frac{\pi}{5})]$$

using a table we find

$$z_0 = 1.7321 + i$$

$$z_1 = -.41582 + 1.95630i$$

$$z_2 = -1.98904 + .20906i$$

$$z_3 = -.81348 - 1.82710i$$

$$z_4 = 1.48628 - 1.33826i$$

as illustrated in the following figure

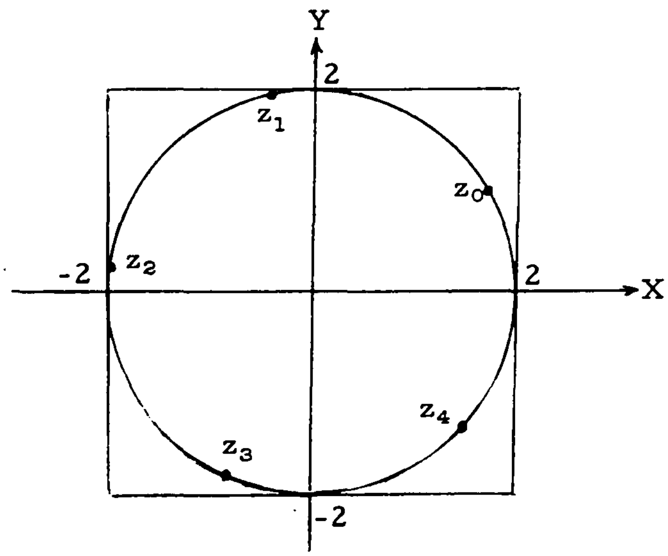


fig.4.1

Example 3 illustrates the problem of multiple roots. Obviously $z = 0$ and $z = i\pi$ are roots of $f(z)$ but successive subdivisions show that the disjoint subsquares obtained around these values keep containing two roots, until a warning is printed to the effect that the path of integration has come too close to a root. As mentioned in chapter 1, in each case, we find these roots α_1, α_2 and β_1, β_2 respectively by evaluating

$$K_1 = \frac{1}{2\pi i} \int_C z \frac{f'(z)}{f(z)} dz = \alpha_1 + \alpha_2$$

and

$$K_2 = \frac{1}{2\pi i} \int_C z^2 \frac{f'(z)}{f(z)} dz = \alpha_1^2 + \alpha_2^2$$

about a suitable contour C (similarly $\beta_1 + \beta_2$ and $\beta_1^2 + \beta_2^2$). Note that

α_1 and α_2 are then the roots of the quadratic

$$x^2 - K_1 x + \frac{K_1^2 - K_2}{2} = 0.$$

From the computed values of K_1, K_2 we easily see that $z = 0$ and $z = i\pi$ are double roots of $f(z)$.

PROGRAM TWO is illustrated with Example 1, showing faster convergence for the modified Romberg technique. When a square contains no roots the successive iterations in computing the contour integral give values converging to zero and because of the very small numbers involved the relative error test (IDEL=-1) will not work. It is therefore necessary to use the absolute error test (IDEL=1).

4.2 PROGRAM ONE

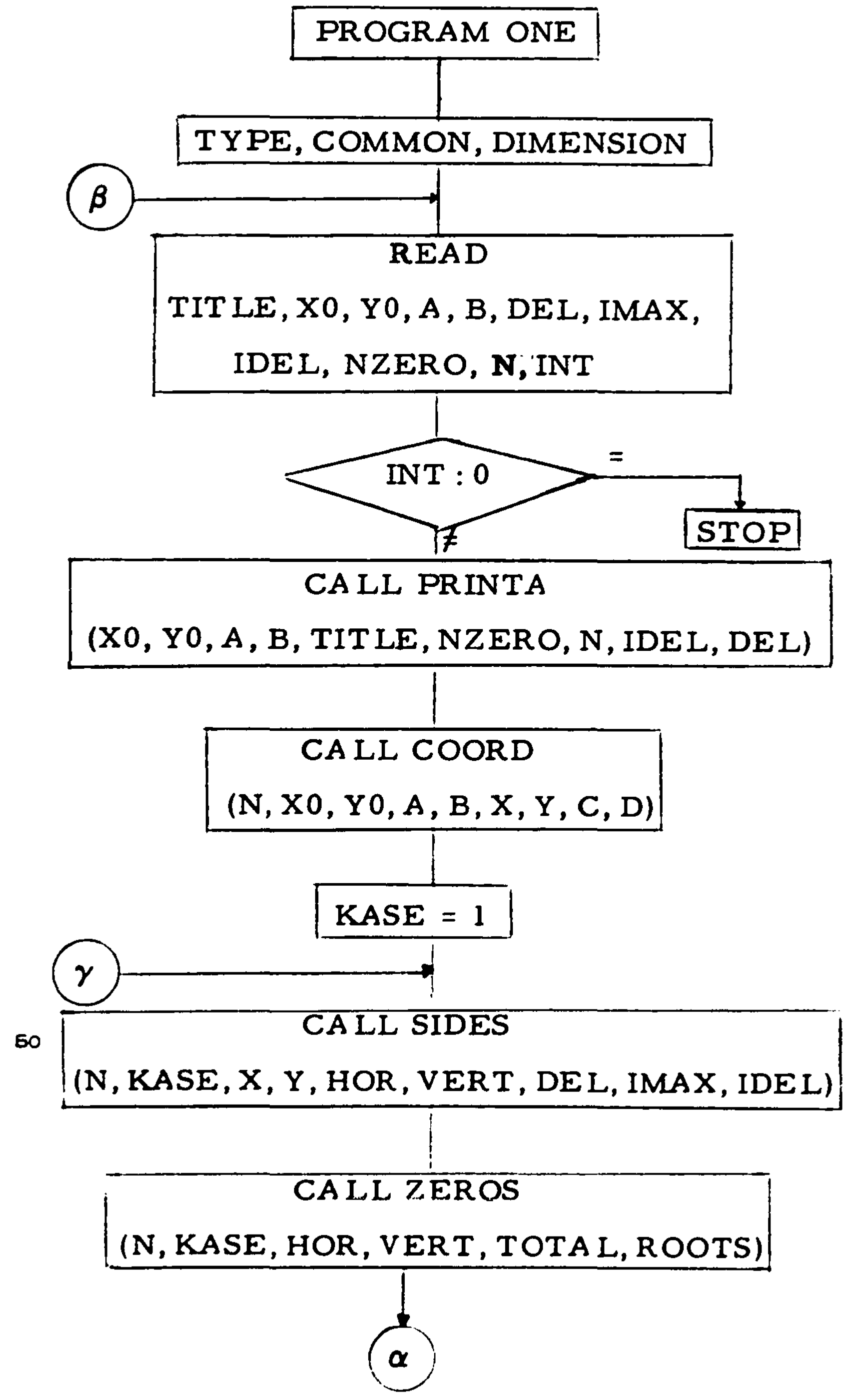


Fig.4.2 Flow chart for PROGRAM ONE

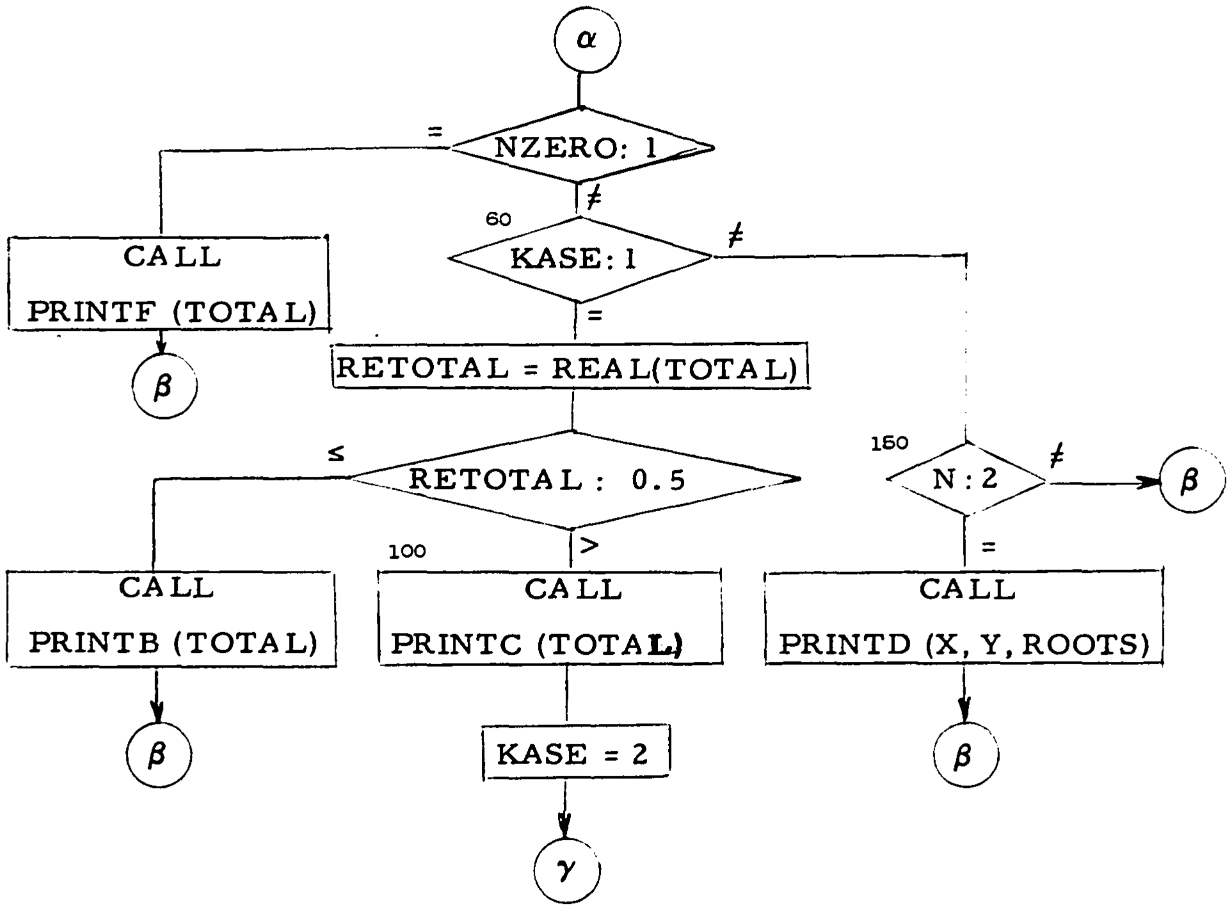

α
NZERO: 1
=
≠
60
KASE: 1
≠
=
CALL
PRINTF (TOTAL)
β
RETOTAL = REAL(TOTAL)
150
N : 2
≠
β
≤
RETOTAL : 0.5
>
100
=
CALL
PRINTB (TOTAL)
CALL
PRINTC (TOTAL)
CALL
PRINTD (X, Y, ROOTS)
β
KASE = 2
β
γ

Fig.4.2 (cont)

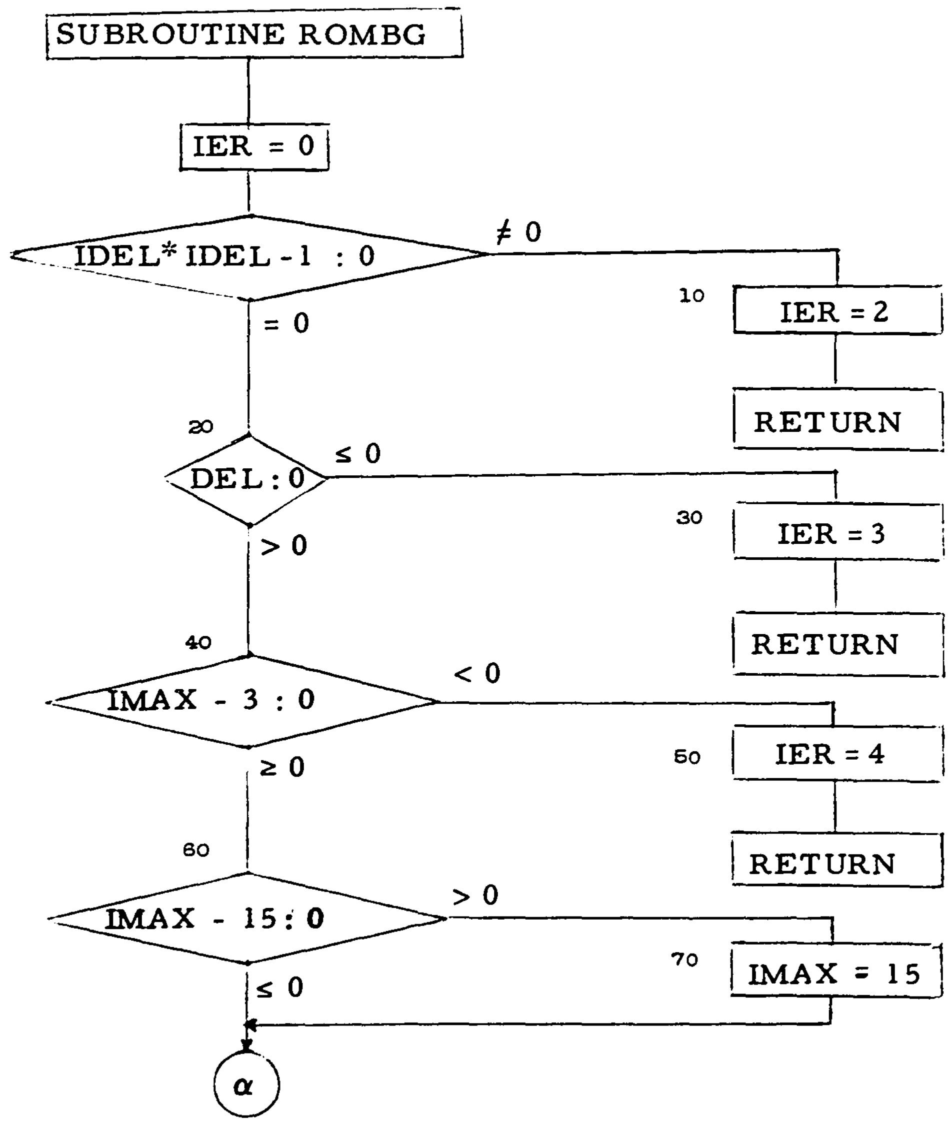


Fig.4.3 Flow chart for SUBROUTINE ROMBG

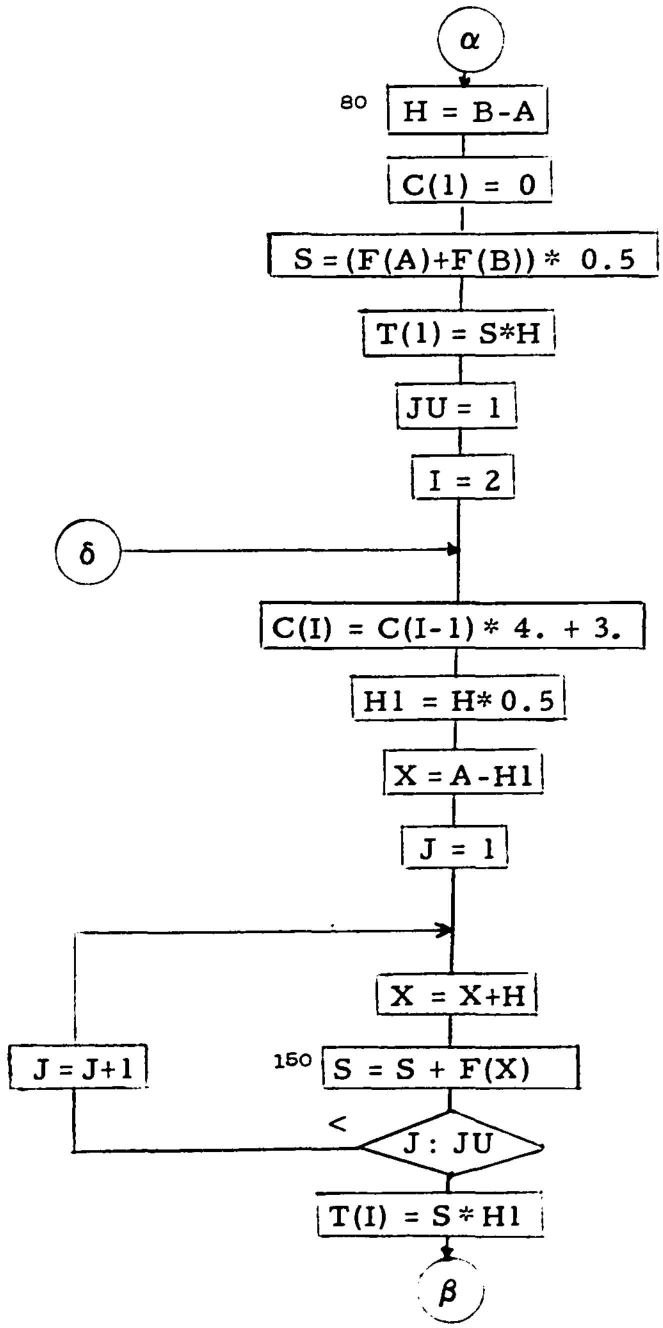

α
80
H = B-A
C(1) = 0
S = (F(A)+F(B)) * 0.5
T(1) = S*H
JU = 1
I = 2
δ
C(I) = C(I-1) * 4. + 3.
H1 = H*0.5
X = A-H1
J = 1
X = X+H
J = J+1
150
S = S + F(X)
<
J : JU
T(I) = S*H1
β

Fig.4.3 (cont)

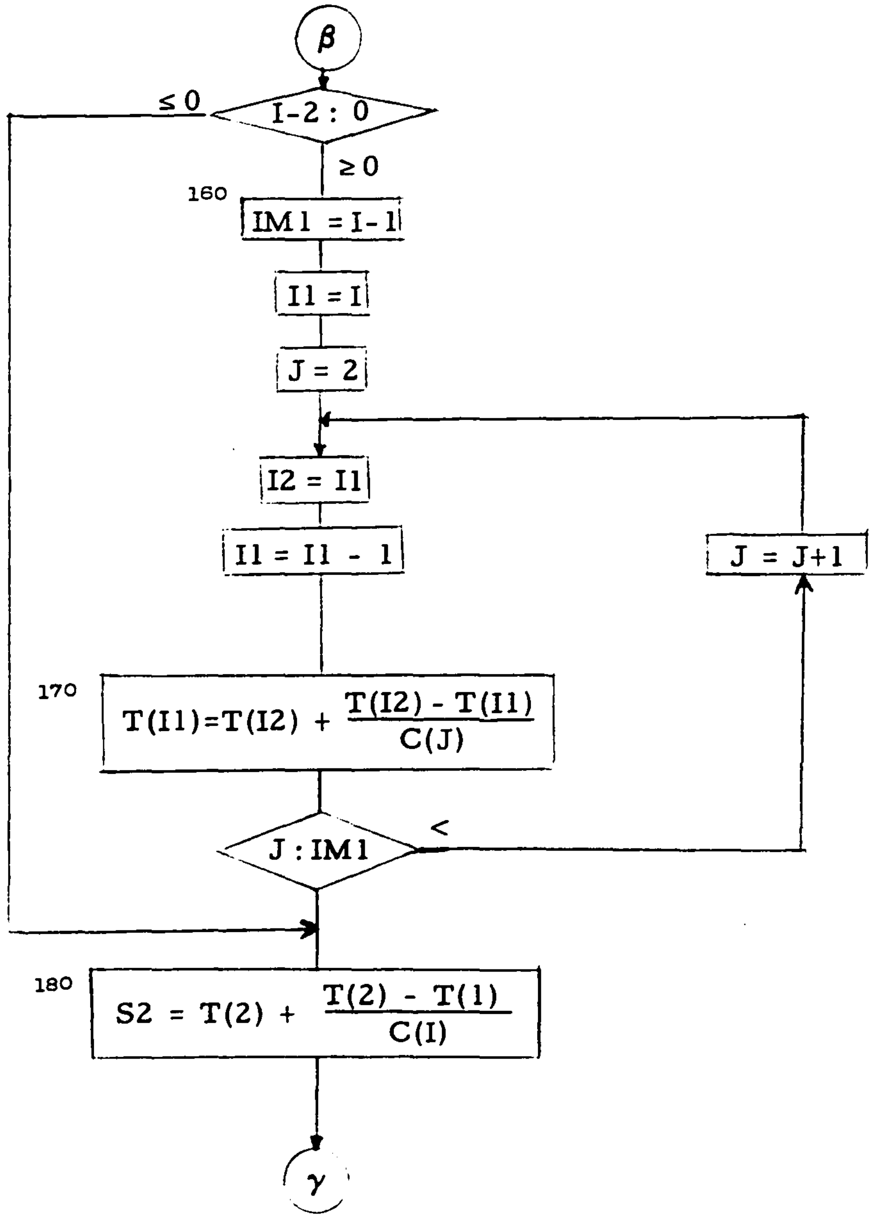

β
≤ 0
I-2 : 0
≥ 0
160
IM1 = I-1
I1 = I
J = 2
I2 = I1
I1 = I1 - 1
J = J+1
170
T(I1)=T(I2) + (T(I2) - T(I1))/C(J)
<
J : IM1
180
S2 = T(2) + (T(2) - T(1))/C(I)
γ

Fig.4.3 (cont)

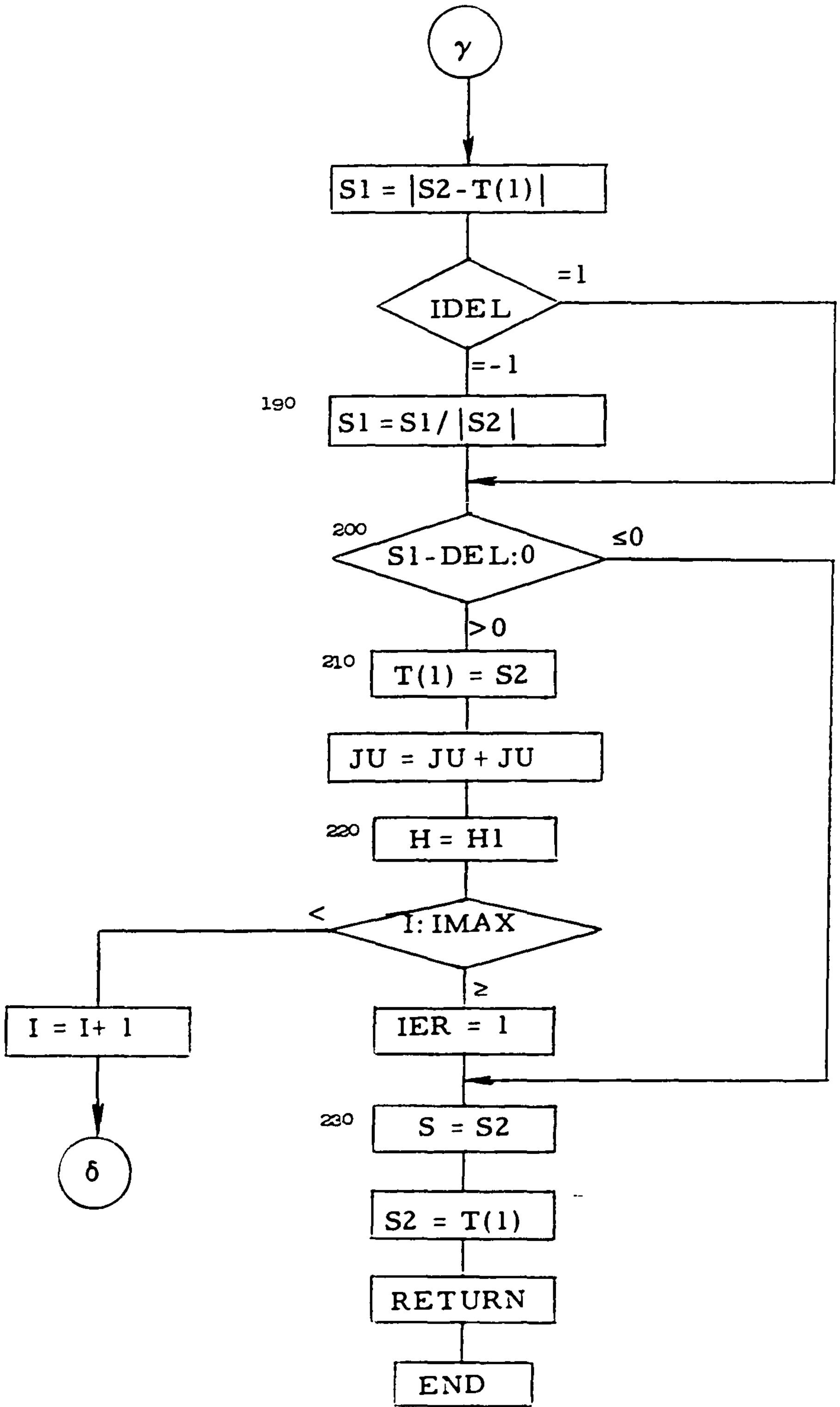

γ
S1 = |S2-T(1)|
IDEL
=1
=-1
190
S1 = S1/ |S2|
200
S1-DEL:0
≤0
>0
210
T(1) = S2
JU = JU + JU
220
H = H1
I: IMAX
<
≥
I = I+ 1
IER = 1
δ
230
S = S2
S2 = T(1)
RETURN
END

Fig.4.3 (cont.)

```
      PROGRAM ONE
C
C     ------------------------------------------------------------------
C
C        PURPOSE
C           (1) TO EVALUATE THE NUMBER OF ROOTS OF THE GIVEN FUNCTION
C               INSIDE THE PRESCRIBED RECTANGLE AND INSIDE EACH
C               SUB-RECTANGLE WHEN THE SIDES ARE DIVIDED INTO N
C               EQUAL PARTS,SET NZERO DIFFERENT FROM 1
C           (2) TO EVALUATE THE VALUE OF THE ROOT (IF UNIQUE) OF
C               THE GIVEN FUNCTION INSIDE THE PRESCRIBED RECTANGLE
C               SET NZERO=1 AND N=1
C
C        PARAMETERS
C           (X0,Y0)  -CENTER OF RECTANGLE
C           A,B      -SIDE LENTGHS
C           N        -NO. OF SEGMENTS ON EACH SIDE
C           NZERO    -PURPOSE CODE (SEE PURPOSE)
C           IDEL     -ERROR CODE (SEE SUBROUTINE ROMBG)
C           DEL      -ACCURACY (SEE SUBROUTINE ROMBG)
C
C        METHOD
C           INTEGRATION IS PERFORMED ON EACH OUTER SEGMENT BY ROMBERGS
C           STANDARD METHOD AND RESULTS ARE COMBINED TO GIVE THE
C           NUMBER OF ROOTS IN THE ORIGINAL RECTANGLE OR TO GIVE (2)
C           (SEE PURPOSE). IN CASE (1),IF THE ORIGINAL REGION CONTAINS
C           A ROOT,INTEGRATION IS CONTINUED ON INNER SIDES AND RESULTS
C           ARE COMBINED WITH THE ONES FOUND ON OUTER SIDES TO GIVE
C           THE NUMBER OF ROOTS IN EACH SUB-RECTANGLE
C
C     ------------------------------------------------------------------
      TYPE COMPLEX(4) HOR,VERT,TOTAL,ROOTS
      COMMON NZERO
      DIMENSION X(5),Y(5),C(4),D(4)
      DIMENSION HOR(4,4,5),VERT(4,5,4),ROOTS(4,4,4),TITLE(70)
C     ------------------------------------------------------------------
      READ 15,(TITLE(K),K=1,70)
   15 FORMAT (70(A1))
   20 READ 25,X0,Y0,A,B,DEL,IMAX,IDEL,NZERO,N,INT
   25 FORMAT(4F10.3,F10.6,5I5)
      IF(INT.EQ.0) STOP
      CALL PRINTA(X0,Y0,A,B,TITLE,NZERO,N,IDEL,DEL)
      CALL COORD(N,X0,Y0,A,B,X,Y,C,D)
      KASE=1
   50 CALL SIDES(N,KASE,X,Y,HOR,VERT,DEL,IMAX,IDEL)
      CALL ZEROS(N,KASE,HOR,VERT,TOTAL,ROOTS)
      IF(NZERO.NE.1) GO TO 60
      CALL PRINTF(TOTAL)
      GO TO 20
   60 IF(KASE.NE.1)GO TO 150
      RETOTAL=REAL(TOTAL)
      IF(RETOTAL.GT.0.5)GO TO 100
      CALL PRINTB(TOTAL)
      GO TO 20
  100 CALL PRINTC(TOTAL)
      KASE=2
      GO TO 50
  150 CONTINUE
```

```
      IF(N.NE.2) GO TO 20
      CALL PRINTD(X,Y,ROOTS)
      GO TO 20
      END
```

FORTRAN DIAGNOSTIC RESULTS FOR ONE

NO ERRORS

```
      SUBROUTINE COORD(N,X0,Y0,A,B,X,Y,C,D)
C     --------------------------------------------------------
C     EVALUATES COORDINATES OF CORNERS AND CENTERS OF ALL
C     SUBRECTANGLES WHEN THE SIDES OF THE ORIGINAL RECTANGLE
C     ARE DIVIDED INTO N EQUAL PARTS.
C     --------------------------------------------------------
      DIMENSION X(5),Y(5),C(4),D(4)
C     --------------------------------------------------------
      X(1) = X0 - A/2.
      Y(1) = Y0 - B/2.
      RN=N
      H1 = A/RN
      H2 = B/RN
      M  = N+1
C     --------------------------------------------------------
      DO 4I = 2,M
      RI =I
      X(I)=X(1)+(RI-1.)*H1
    4 Y(I)=Y(1)+(RI-1.)*H2
C     --------------------------------------------------------
      DO 6I = 1,N
      C(I)=(X(I)+X(I+1))/2.
    6 D(I)=(Y(I)+Y(I+1))/2.
C     --------------------------------------------------------
      RETURN
      END
```

FORTRAN DIAGNOSTIC RESULTS FOR COORD

NO ERRORS

```
      SUBROUTINE SIDES(N,KASE,X,Y,HOR,VERT,DEL,IMAX,IDEL)
C     -----------------------------------------------------------
C     DETERMINES ON WHICH EDGES NUMERICAL INTEGRATION IS
C     TO BE PERFORMED AND CALLS INTEGRATION ROUTINE
C             KASE=1  INTEGRATION ON OUTER SIDES
C             KASE=2  INTEGRATION ON INNER SIDES
C             KASE=3  INTEGRATION ON ALL SIDES
C     -----------------------------------------------------------
      TYPE COMPLEX(4) A,B,F,S2,S,HOR,VERT,CMPLX
      COMMON NZERO
      DIMENSION X(5),Y(5),HOR(4,4,5),VERT(4,5,4)
      DIMENSION R(2)
      EQUIVALENCE(R,S)
      EXTERNAL F
C     -----------------------------------------------------------
      GO TO (5,10,15),KASE
C     -----------------------------------------------------------
    5 L=1
      M=N+1
      JUMP=N
      PRINT 7
    7 FORMAT(1H0,2X,25HINTEGRALS ON OUTER SIDES-)
      PRINT 8
    8 FORMAT(1H0,32X,9HREAL PART,5X,14HIMAGINARY PART,5X,
     110HITERATIONS,//)
      GO TO 20
C     -----------------------------------------------------------
   10 L=2
      M=N
      JUMP=1
      PRINT 11
   11 FORMAT(1H0,2X,25HINTEGRALS ON INNER SIDES-,/)
      GO TO 20
C     -----------------------------------------------------------
   15 L=1
      M=N+1
      JUMP=1
   20 CONTINUE
C     -----------------------------------------------------------
      ICASE=1
C     INTEGRATION ON HORIZONTAL SIDES ONLY
      DO 40 J=L,M,JUMP
      DO 40 I=1,N
      A=CMPLX(X(I),Y(J))
      B=CMPLX(X(I+1),Y(J))
      CALL ROMBG(F,A,B,DEL,S2,S,IDEL,IER,IMAX,ITER)
      IF(IER.NE.0)GO TO 150
      HOR(N,I,J)=S
      PRINT 30,N,I,J,R,ITER
   30 FORMAT(11X,6HHOR  (,2(I1,1H,),I1,4H) = ,4X,E12.3,5X,E12.3,
     15X,I5)
   40 CONTINUE
C     -----------------------------------------------------------
      ICASE=2
C     INTEGRATION ON VERTICAL SIDES ONLY
      DO 60 I=L,M,JUMP
      DO 60 J=1,N
      A=CMPLX(X(I),Y(J))
```

```
      B=CMPLX(X(I),Y(J+1))
      CALL ROMBG(F,A,B,DEL,S2,S,IDEL,IER,IMAX,ITER)
      IF(IER.NE.0)GO TO 150
      VERT(N,I,J)=S
      PRINT 50,N,I,J,R,ITER
   50 FORMAT(11X,6HVERT ( ,2(I1,1H,),I1,4H) = ,4X,E12.3,5X,E12.3,
     15X,I5)
   60 CONTINUE
      RETURN
C     ------------------------------------------------------------
  150 CALL PRINTE(ICASE,N,I,J,IER)
      RETURN
      END
```

FORTRAN DIAGNOSTIC RESULTS FOR SIDES

NO ERRORS

```
      SUBROUTINE ZEROS(N,KASE,HOR,VERT,TOTAL,ROOTS)
C     ----------------------------------------------------------
C     COMBINES VALUES OF INTEGRALS
C     ----------------------------------------------------------
      TYPE COMPLEX(4) HOR,VERT,TOTAL,ROOTS,SUM,CMPLX,T,CONSTANT
      DIMENSION HOR(4,4,5),VERT(4,5,4),ROOTS(4,4,4),SUM(4)
      DIMENSION R(2)
      EQUIVALENCE(R,T)
C     ----------------------------------------------------------
      IF(KASE .NE.1) GO TO 135
C     ----------------------------------------------------------
      L=1
      M=N+1
      JUMP=N
      K=1
C     ----------------------------------------------------------
      DO 110 J=L,M,JUMP
      SUM(K)=0.
      DO 100 I=1,N
      SUM(K) = SUM(K)+HOR(N,I,J)
  100 CONTINUE
      K=K+1
  110 CONTINUE
C     ----------------------------------------------------------
      DO 130 I=L,M,JUMP
      SUM(K)=0.
      DO 120 J=1,N
      SUM(K)=SUM(K) + VERT(N,I,J)
  120 CONTINUE
      K=K+1
  130 CONTINUE
C     ----------------------------------------------------------
      C=1.0/6.28318
      CONSTANT = -CMPLX(0.0,C)
      TOTAL=(SUM(1)-SUM(2)-SUM(3)+SUM(4))*CONSTANT
      RETURN
C     ----------------------------------------------------------
  135 PRINT 140
  140 FORMAT(1H0,2X,36HNUMBER OF ROOTS IN EACH SUBRECTANGLE,//)
      DO 160 J=1,N
      DO 160 I=1,N
      ROOTS(N,I,J)=(HOR(N,I,J)-HOR(N,I,J+1)-VERT(N,I,J)
     1+VERT(N,I+1,J))*CONSTANT
      T=ROOTS(N,I,J)
      PRINT 155,N,I,J,R
  155 FORMAT(11X,6HROOTS(,2(I1,1H,),I1,4H) = ,4X,E12.3,5X,E12.3)
  160 CONTINUE
C     ----------------------------------------------------------
      RETURN
      END
```

FORTRAN DIAGNOSTIC RESULTS FOR ZEROS

NO ERRORS

```
      SUBROUTINE ROMBG(F,A,B,DEL,S2,S,IDEL,IER,IMAX,T)
C
C     ............................................................
C
C        PURPOSE
C           INTEGRATES THE GIVEN FUNCTION OVER THE PRESCRIBED
C           LINE SEGMENT IN COMPLEX PLANE
C
C        USAGE
C           CALL ROMBG(F,A,B,DEL,S2,S,IDEL,IER,IMAX,I)
C
C        DESCRIPTION OF PARAMETERS
C           F    -NAME OF USER FUNCTION SUBPROGRAM GIVING F(X)
C           A    -LOWER INTEGRATION LIMIT
C           B    -UPPER INTEGRATION LIMIT
C           IDEL-ACCURACY CODE WHERE
C                  IDEL=-1 RELATIVE ERROR
C                  IDEL= 1 ABSOLUTE ERROR
C           DEL -REQUIRED ACCURACY OR TOLERANCE
C           IMAX-MAXIMUM NUMBER OF ITERATIONS
C           S    -RESULTANT FINAL VALUE OF INTEGRAL
C           C    -WORKING VECTOR OF LENGTH IMAX
C           T    -WORKING VECTOR OF LENGTH IMAX
C           IER -RESULTANT ERROR CODE WHERE
C                  IER=0   REQUIRED ACCURACY MET
C                  IER=1   REQUIRED ACCURACY NOT MET IN IMAX ITERATIONS
C                  IER=2   IDEL NOT 1 OR -1
C                  IER=3   DEL NOT POSITIVE
C                  IER=4   IMAX LESS THAN 3
C
C        SUBROUTINES AND FUNCTION SUBPROGRAMS REQUIRED
C           F- FUNCTION SUBPROGRAM WHICH COMPUTES F(X) FOR X BETWEEN
C           A AND B
C           CALLING PROGRAM MUST HAVE FORTRAN EXTERNAL STATEMENT
C           CONTAINING NAMES OF FUNCTION SUBPROGRAMS LISTED IN CALL TO
C           ROMBG
C
C        METHOD
C           TRAPEZOIDAL SUMS ARE COMPUTED ON GIVEN LINE WITH
C           INTERVAL HALVING AND RESULTS ARE COMBINED BY
C           STANDARD ROMBERG METHOD UNTIL DIFFERENCE BETWEEN
C           SUCCESSIVE VALUES IS LESS THAN DEL
C           FAILURE TO REACH THE TOLERANCE AFTER IMAX ITERATIONS
C           TERMINATES THE SUBROUTINE EXECUTION
C
C     ............................................................
C
      TYPE COMPLEX(4) F,A,B,H,H1,S2,S,X,T
      COMMON NZERO
      DIMENSION C(15),T(15)
C
C     ............................................................
C
C     CHECK FOR PARAMETER ERRORS
      IER=0
      IF(IDEL*IDEL-1)10,20,10
   10 IER=2
      RETURN
```

```
   20 IF(DEL)30,30,40
   30 IER=3
      RETURN
   40 IF(IMAX-3)50,60,60
   50 IER=4
      RETURN
   60 IF(IMAX-15)80,80,70
   70 IMAX=15
   80 H=B-A
      C(1)= 0.
C     ----------------------------------------------------------
C     SIGMA(I)= TRAPEZOIDAL SUM WHEN PRESCRIBED RANGE IS
C                DIVIDED INTO  2**(I-1)  EQUAL PARTS.
C     ----------------------------------------------------------
C     COMPUTE SIGMA(1)
      S=(F(A)+F(B))*0.5
      T(1)=S*H
C     ----------------------------------------------------------
C     COMPUTE SIGMA(I)
      JU=1
      DO 220 I=2,IMAX
      C(I)=C(I-1)*4.+3.
      H1=H*0.5
      X=A-H1
      DO 150 J=1,JU
      X=X+H
  150 S=S+F(X)
      T(I)=S*H1
C     ----------------------------------------------------------
C     COMBINE TRAPEZOIDAL SUMS BY ROMBERGS METHOD
      IF(I-2)180,180,160
  160 IM1=I-1
      I1=I
      DO 170 J=2,IM1
      I2=I1
      I1=I1-1
  170 T(I1)=T(I2)+(T(I2)-T(I1))/C(J)
  180 S2=T(2)+(T(2)-T(1))/C(I)
C     ----------------------------------------------------------
C     TEST FOR CONVERGENCE
      S1= CABS(S2-T(1))
      IF(IDEL) 190,190,200
  190 S1= S1/CABS(S2)
  200 IF(S1-DEL) 230,230,210
C     REQUIRED ACCURACY NOT MET - NEXT ITERATION
  210 T(1)=S2
      JU=JU+JU
  220 H=H1
      IER=1
C     REQUIRED ACCURACY MET
  230 S=S2
      S2=T(1)
      RETURN
      END
```

FORTRAN DIAGNOSTIC RESULTS FOR ROMBG

NO ERRORS

```
      FUNCTION F(Z)
      REAL K,NORM
C     ------------------------------------------------------------
      TYPE COMPLEX(4) F,DERIV,FUNC,Z,T,CI,ARG,CMPLX,CSIN,CCOS
C     ------------------------------------------------------------
      COMMON NZERO
      DIMENSION R(2)
      EQUIVALENCE(R,T)
      T=Z
C     ------------------------------------------------------------
      CI = CMPLX(0.,2.)
      ARG = CI*Z
      FUNC = CCOS(ARG) -1.0
      DERIV = -CI*CSIN(ARG)
C     ------------------------------------------------------------
      F=DERIV/FUNC
      IF(NZERO.NE.1) GO TO 30
      F=F*Z
C     F=F*Z
      GO TO 80
   30 NORM=CABS(F)
      K=100.
      IF(NORM.LT.K)GO TO 80
      PRINT 40
   40 FORMAT(1H0,10X,45HWARNING-A ROOT IS TOO CLOSE TO THE CONTOUR AT)
      PRINT 60,R
   60 FORMAT(1H0,22X,2HZ=,E10.3,4H+J( ,E10.3,1H))
      PRINT 70
   70 FORMAT(1H0,10X,24HINTEGRATION IS ABANDONED/1H1)
      STOP
   80 RETURN
      END
```

FORTRAN DIAGNOSTIC RESULTS FOR F

NO ERRORS

```
      SUBROUTINE PRINTA(X0,Y0,A,B,TITLE,NZERO,N,IDEL,DEL)
      DIMENSION TITLE(70)
      PRINT 5
    5 FORMAT(1H1)
      IF(NZERO.NE.1) GO TO 9
      PRINT 6
    6 FORMAT(11X,25HEVALUATION OF THE ROOT OF)
      GO TO 11
    9 PRINT 10
   10 FORMAT(11X,18HNUMBER OF ROOTS OF)
   11 PRINT 15,(TITLE(K),K=1,70)
   15 FORMAT(1H0,19X,70A1)
      PRINT 20
   20 FORMAT(1H0,10X,40HINSIDE THE FOLLOWING RECTANGLE OR SQUARE)
      PRINT 25, X0,Y0
   25 FORMAT(1H0,19X,13HCENTER     = (,F7.3,1H,,F7.3,1H))
      PRINT 30,A
   30 FORMAT(20X,12HSIDE A     = ,F5.2)
      PRINT 35,B
   35 FORMAT(20X,12HSIDE B     = ,F5.2)
      PRINT 60
   60 FORMAT(1H0,10X,46HUSING THE STANDARD ROMBERG CONTOUR INTEGRATION)
      PRINT 100
  100 FORMAT(1H0,2X,8HRESULTS-/1H0)
      PRINT 110,DEL
  110 FORMAT(1H0,2X,11HPARAMETERS-,12X,6HDEL = ,F10.6,2X,
     110H(ACCURACY))
      IF(IDEL.EQ.1) GO TO 130
      PRINT 120,IDEL
  120 FORMAT(1H ,25X,5HIDEL=,I4,9X,21H(RELATIVE ERROR CODE))
      GO TO 144
  130 PRINT 140,IDEL
  140 FORMAT(1H ,25X,5HIDEL=,I4,9X,21H(ABSOLUTE ERROR CODE))
  144 PRINT 145,N
  145 FORMAT(1H ,25X,5HN    =,I4,9X,21H(NO. OF SUBDIVISIONS))
      RETURN
      END
```

FORTRAN DIAGNOSTIC RESULTS FOR PRINTA

NO ERRORS

```
      SUBROUTINE PRINTB (TOTAL)
      TYPE COMPLEX(4) TOTAL,T
      DIMENSION R(2)
      EQUIVALENCE(R,T)
      T=TOTAL
      PRINT 5
    5 FORMAT(//,3X,24HNO ROOTS IN GIVEN REGION)
      PRINT 10,R
   10 FORMAT(//,3X,14HVALUE FOUND = ,E10.4,4H+J( ,E10.4,2H )/1H1)
      RETURN
      END
```

FORTRAN DIAGNOSTIC RESULTS FOR PRINTB

NO ERRORS

```
      SUBROUTINE PRINTC(TOTAL)
      TYPE COMPLEX(4) TOTAL,T
      DIMENSION R(2)
      EQUIVALENCE(R,T)
      T=TOTAL
      PRINT 5
    5 FORMAT(//,3X,31HNUMBER OF ROOTS IN WHOLE REGION)
      PRINT 10,R
   10 FORMAT(1H0,29X,E12.3,6H   +J   ,E12.3)
      RETURN
      END
```

FORTRAN DIAGNOSTIC RESULTS FOR PRINTC

NO ERRORS

```
      SUBROUTINE PRINTD(X,Y,ROOTS)
      TYPE COMPLEX(4) ROOTS,R212,R222,R211,R221
      DIMENSION X(5),Y(5),ROOTS(4,4,4)
      DIMENSION R(2),U(2),V(2),W(2)
      EQUIVALENCE(R,R212),(U,R222),(V,R211),(W,R221)
      PRINT 100
  100 FORMAT(1H1,2X,36HNUMBER OF ROOTS IN EACH SUBRECTANGLE)
      PRINT 5
    5 FORMAT(/////)
      PRINT 10,Y(3)
   10 FORMAT(2X,F7.3,1X,31(2H* ))
   15 FORMAT(10X,1H*,2(29X,1H*))
      DO 16 I=1,4
   16 PRINT 15
      R212=ROOTS(2,1,2)
      R222=ROOTS(2,2,2)
      PRINT 20,R,U
   20 FORMAT(10X,1H*,2(4X,F7.3,6H  +J  ,F7.3,5X,1H*))
      PRINT 25
   25 FORMAT(10X,1H*,2(12X,5HROOTS,12X,1H*))
      DO 26 I=1,4
   26 PRINT 15
      PRINT 10,Y(2)
      DO 27 I=1,4
   27 PRINT 15
      R211=ROOTS(2,1,1)
      R221=ROOTS(2,2,1)
      PRINT 20,V,W
      PRINT 25
      DO 28 I=1,4
   28 PRINT 15
      PRINT 10,Y(1)
      PRINT 30,X(1),X(2),X(3)
   30 FORMAT(//,8X,3(F7.3,20X)/1H1)
      RETURN
      END
```

FORTRAN DIAGNOSTIC RESULTS FOR PRINTD

NO ERRORS

```
      SUBROUTINE PRINTE(ICASE,N,I,J,IER)
      IF(ICASE.EQ.2)GO TO 20
      PRINT 10,N,I,J
   10 FORMAT(10X,22HERROR MESSAGE FOR HOR(,3I2,1H))
      GO TO 40
   20 PRINT 30,N,I,J
   30 FORMAT(10X,23HERROR MESSAGE FOR VERT(,3I2,1H))
   40 GO TO(50,60,70,80),IER
   50 PRINT 55
   55 FORMAT(10X,46HIER=1  REQUIRED ACCURACY NOT MET IN IMAX STEPS)
      STOP
   60 PRINT 65
   65 FORMAT(10X,23HIER=2  IDEL NOT 1 OR -1/1H1)
      STOP
   70 PRINT 75
   75 FORMAT(10X,23HIER=3  DEL NOT POSITIVE/1H1)
      STOP
   80 PRINT 85
   85 FORMAT(10X,23HIER=4  IMAX LESS THAN 3/1H1)
      STOP
      END
```

FORTRAN DIAGNOSTIC RESULTS FOR PRINTE

NO ERRORS

```
      SUBROUTINE PRINTF(TOTAL)
      TYPE COMPLEX(4) TOTAL,T
      DIMENSION R(2)
      EQUIVALENCE(R,T)
      T=TOTAL
      PRINT 5
    5 FORMAT(//,3X,22HROOT IN GIVEN REGION =)
C   5 FORMAT(//,3X,30HSUM OF ROOTS IN GIVEN REGION =)
C   5 FORMAT(//,3X,41HSUM OF SQUARES OF ROOTS IN GIVEN REGION =)
      PRINT 10,R
   10 FORMAT(///,22X,7HZ =    ,E12.3,6H  +J  ,E12.3/1H1)
      RETURN
      END
```

FORTRAN DIAGNOSTIC RESULTS FOR PRINTF

```
NO ERRORS
LOAD,56,01
RUN,3,NM
```

4.3 PROGRAM TWO

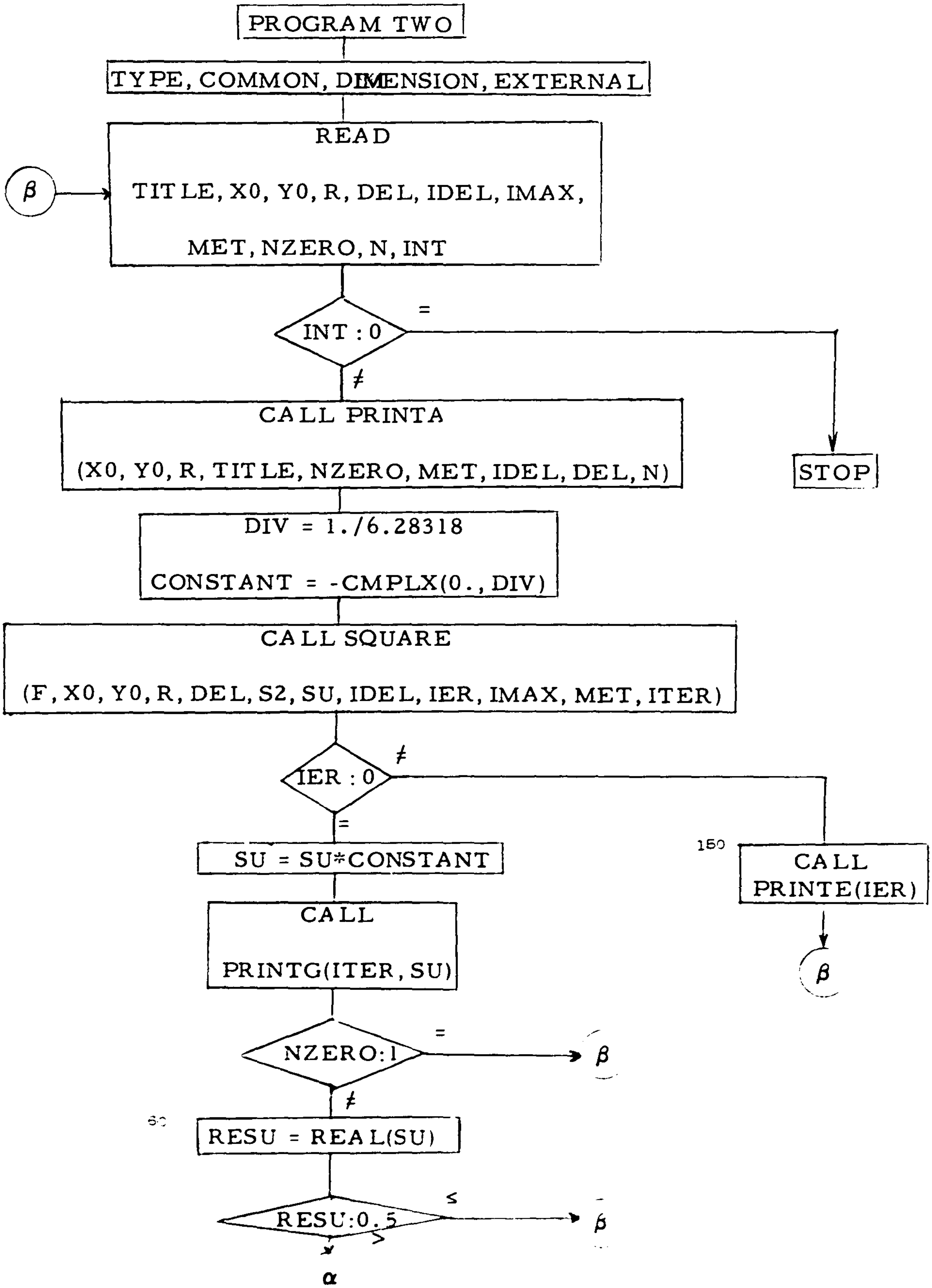


Fig.4.4 Flow chart for PROGRAM TWO

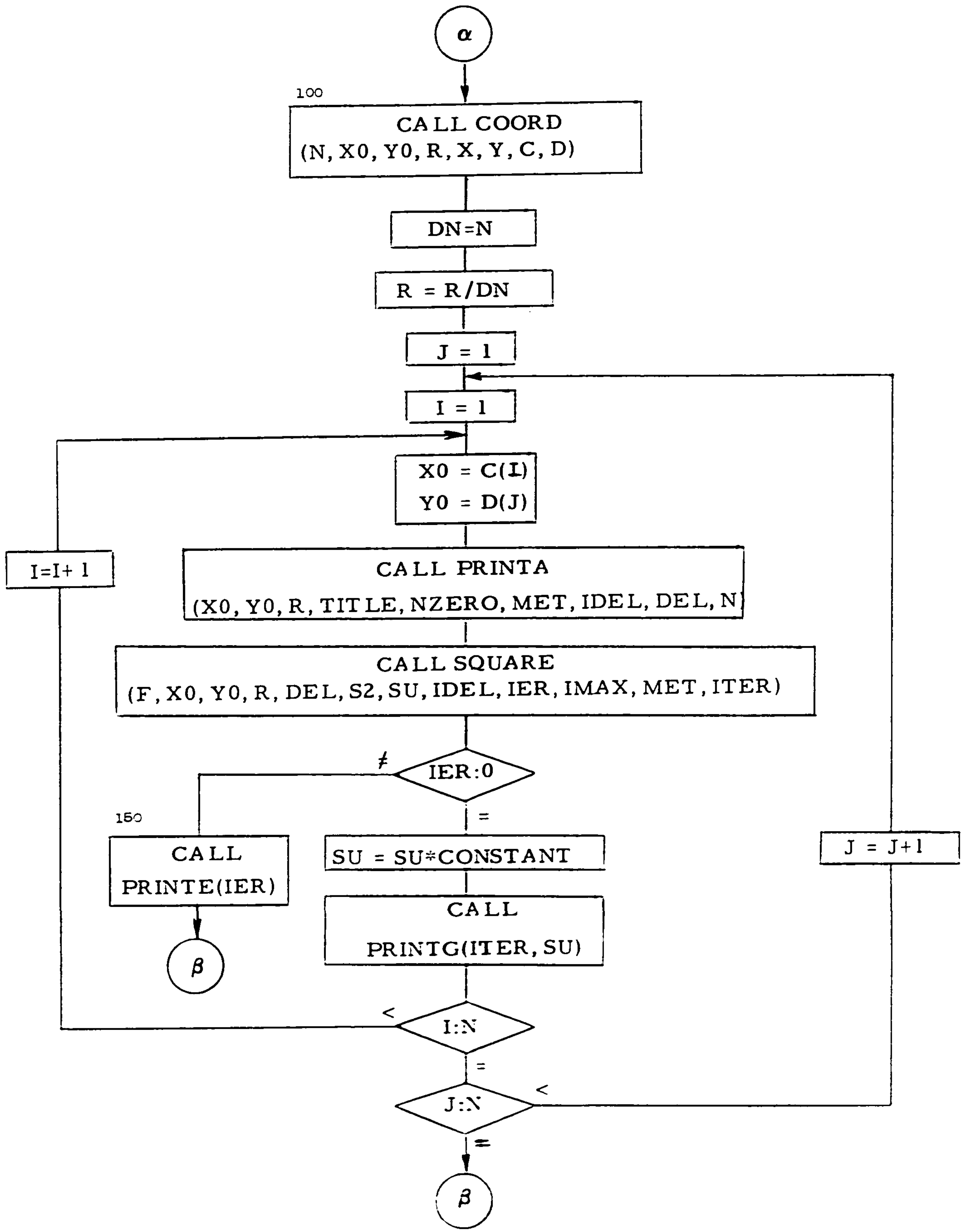

α
100
CALL COORD
(N, X0, Y0, R, X, Y, C, D)
DN=N
R = R/DN
J = 1
I = 1
X0 = C(I)
Y0 = D(J)
I=I+ 1
CALL PRINTA
(X0, Y0, R, TITLE, NZERO, MET, IDEL, DEL, N)
CALL SQUARE
(F, X0, Y0, R, DEL, S2, SU, IDEL, IER, IMAX, MET, ITER)
≠
IER:0
=
150
CALL
PRINTE(IER)
β
SU = SU*CONSTANT
CALL
PRINTG(ITER, SU)
J = J+1
<
I:N
=
<
J:N
=
β

Fig. 4.4 (cont.)

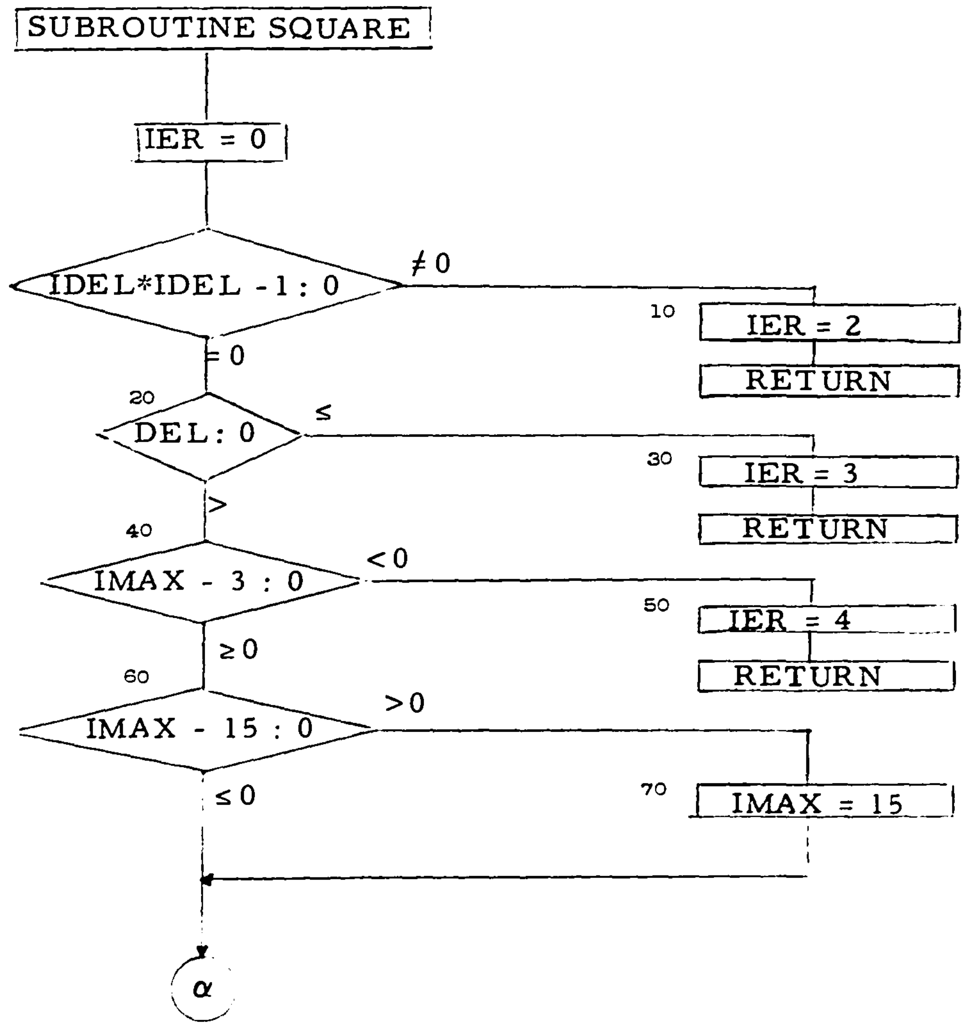


Fig.4.5 Flow chart for SUBROUTINE SQUARE

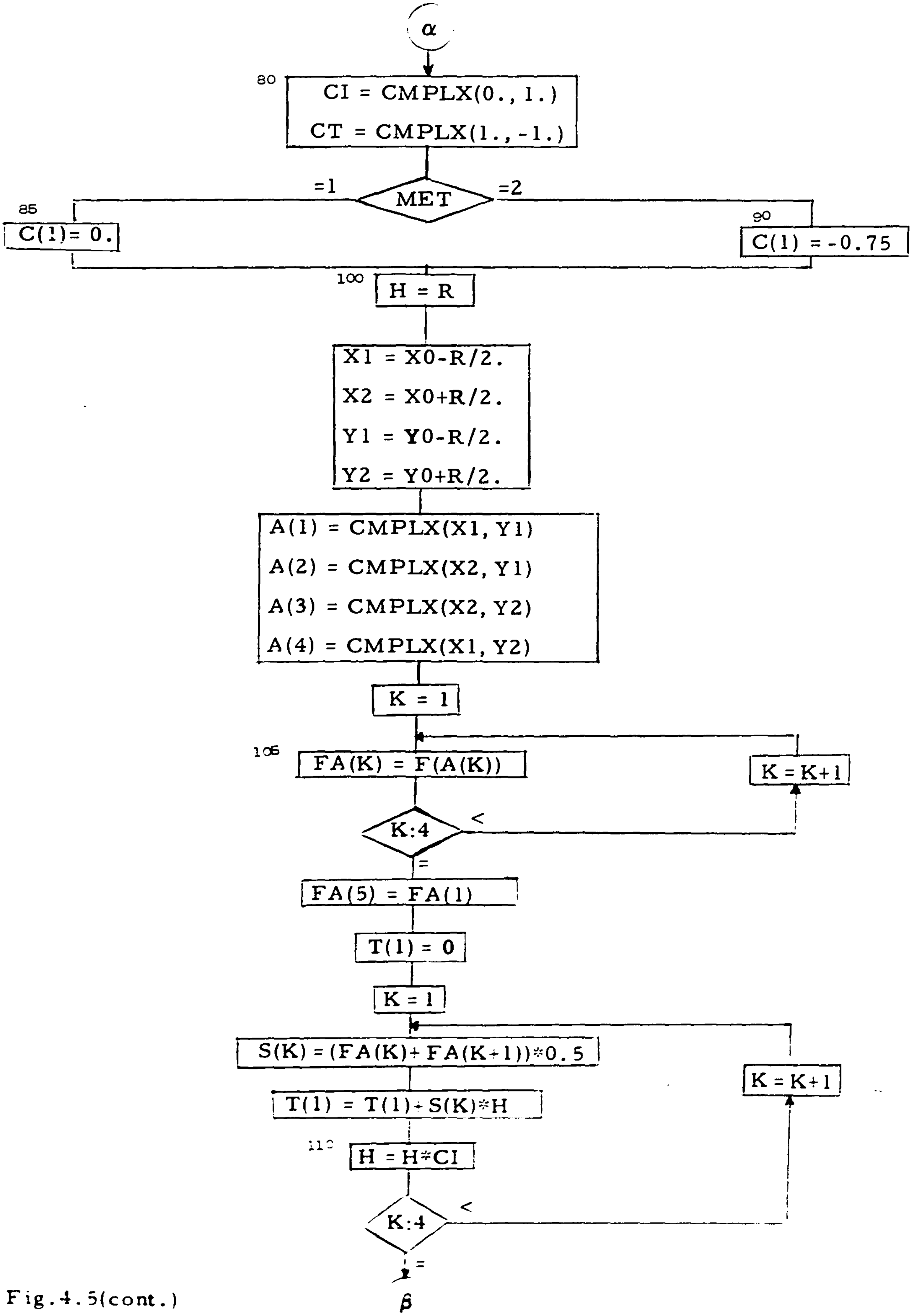

α
80
CI = CMPLX(0., 1.)
CT = CMPLX(1., -1.)
MET
=1
=2
85
C(1)= 0.
90
C(1) = -0.75
100
H = R
X1 = X0-R/2.
X2 = X0+R/2.
Y1 = Y0-R/2.
Y2 = Y0+R/2.
A(1) = CMPLX(X1, Y1)
A(2) = CMPLX(X2, Y1)
A(3) = CMPLX(X2, Y2)
A(4) = CMPLX(X1, Y2)
K = 1
105
FA(K) = F(A(K))
K = K+1
K:4
<
=
FA(5) = FA(1)
T(1) = 0
K = 1
S(K) = (FA(K)+FA(K+1))*0.5
K = K+1
T(1) = T(1)+S(K)*H
H = H*CI
K:4
<
=
β

Fig.4.5(cont.)

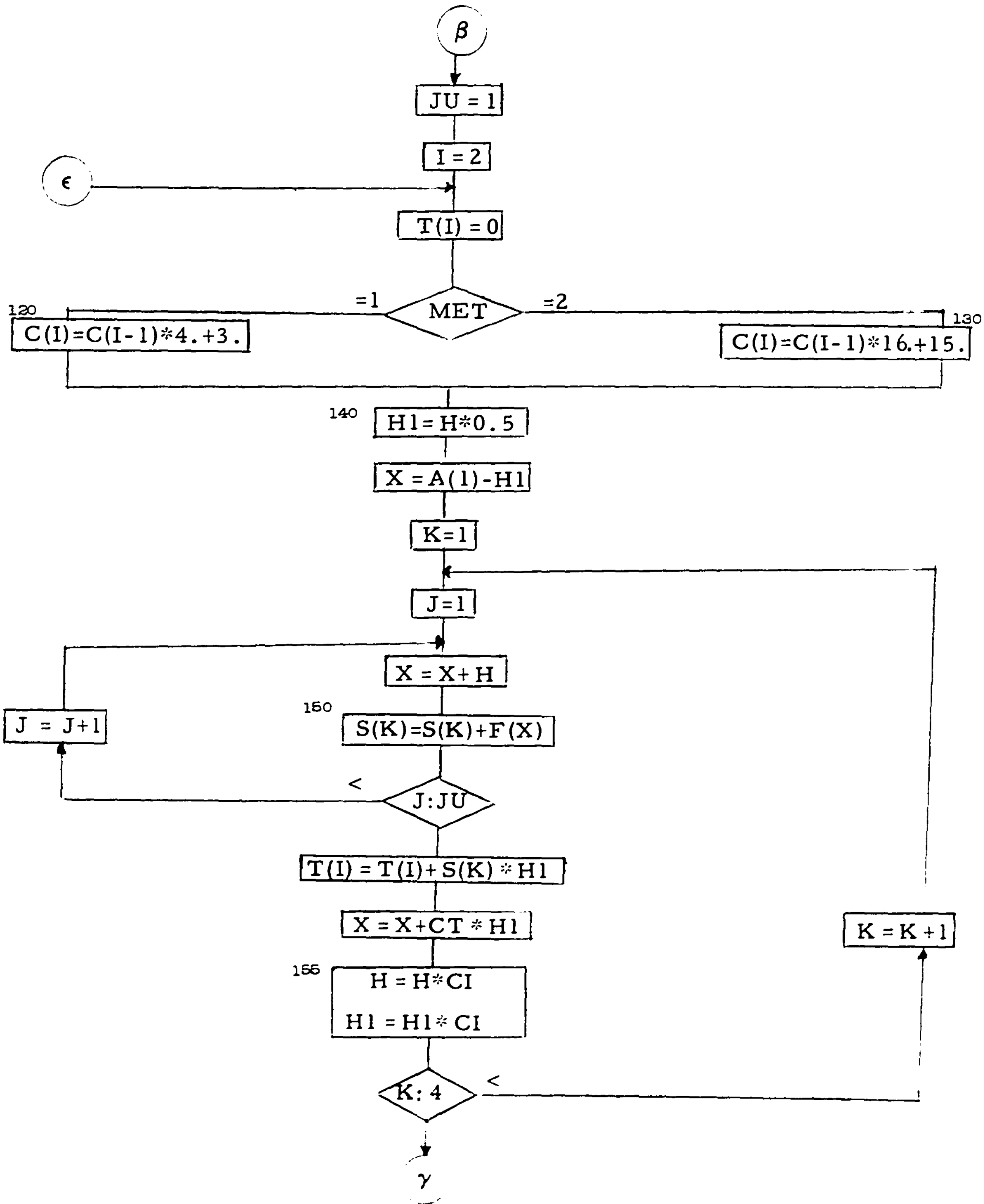

β
JU = 1
I = 2
ε
T(I) = 0
MET
=1
=2
120
C(I)=C(I-1)*4.+3.
130
C(I)=C(I-1)*16.+15.
140
H1=H*0.5
X=A(1)-H1
K=1
J=1
X=X+H
150
S(K)=S(K)+F(X)
J = J+1
J:JU
<
T(I)=T(I)+S(K)*H1
X=X+CT*H1
155
H = H*CI
H1 = H1*CI
K:4
<
K=K+1
γ

Fig.4.5 (cont.)

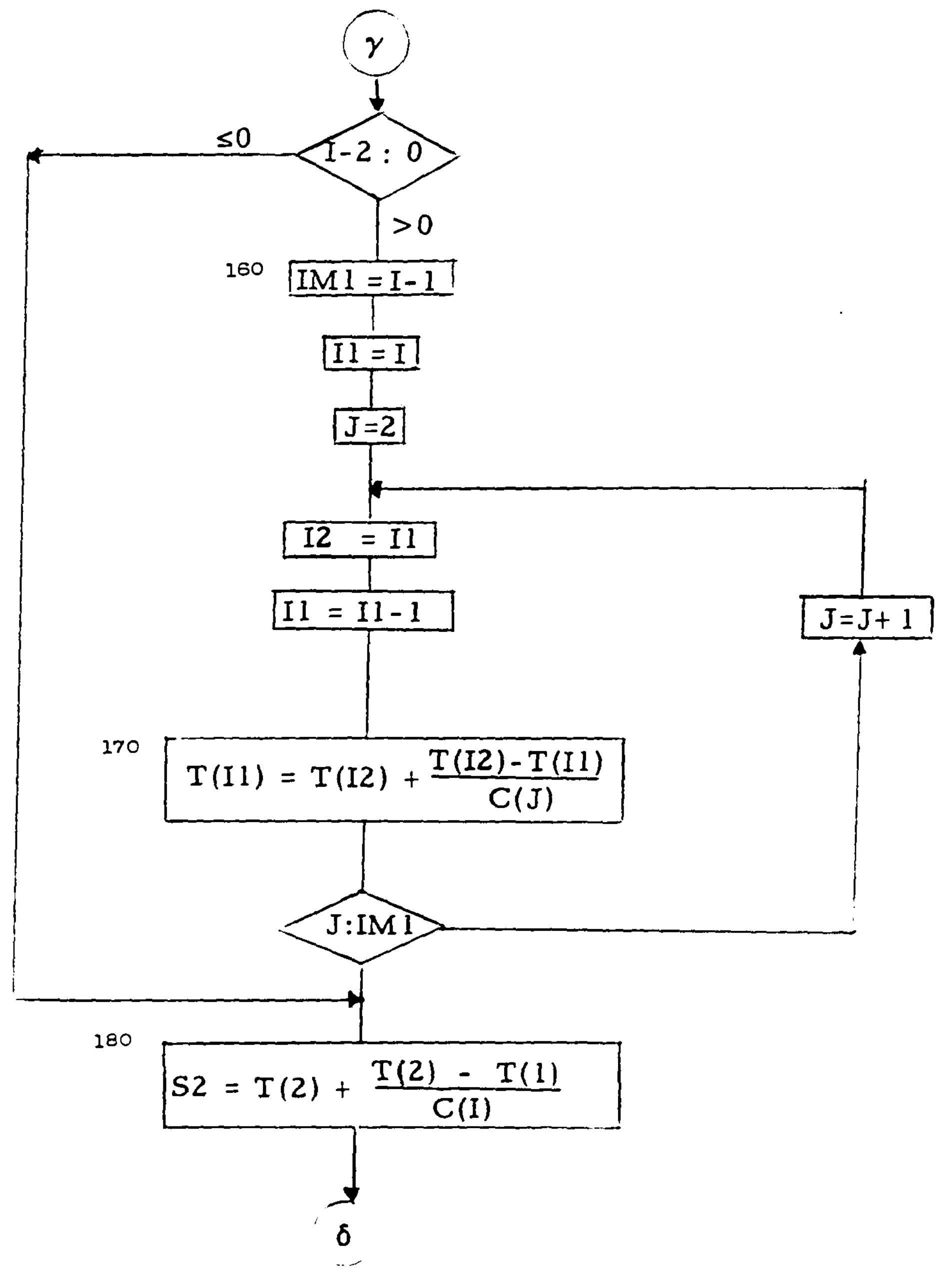

γ
I-2 : 0
≤0
>0
160
IM1 = I-1
I1 = I
J=2
I2 = I1
I1 = I1-1
J=J+1
170
T(I1) = T(I2) + T(I2)-T(I1) / C(J)
J:IM1
180
S2 = T(2) + T(2) - T(1) / C(I)
δ

Fig.4.5 (cont.)

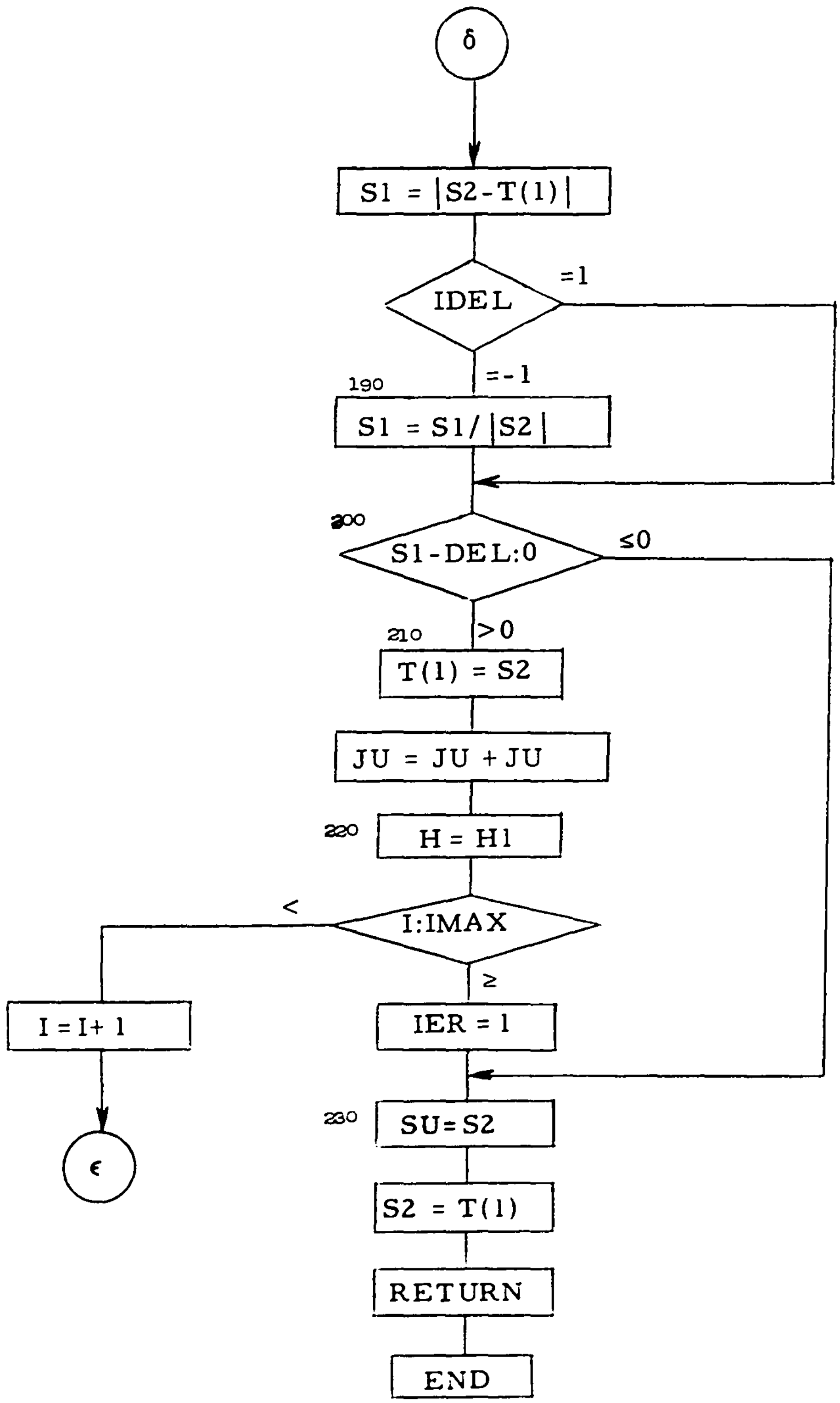

δ
S1 = |S2-T(1)|
IDEL
=1
=-1
190
S1 = S1/|S2|
200
S1-DEL:0
≤0
210
>0
T(1) = S2
JU = JU + JU
220
H = H1
<
I:IMAX
≥
I = I+ 1
IER = 1
ε
230
SU=S2
S2 = T(1)
RETURN
END

Fig.4.5 (cont.)

```
      PROGRAM TWO
C
C     ------------------------------------------------------------------
C
C        PURPOSE
C           (1) TO EVALUATE THE NUMBER OF ROOTS OF THE GIVEN FUNCTION
C               INSIDE THE PRESCRIBED SQUARE AND INSIDE EACH SUB-
C               SQUARE WHEN THE SIDES ARE DIVIDED INTO N EQUAL PARTS
C               SET NZERO DIFFERENT FROM 1
C           (2) TO EVALUATE THE VALUE OF THE ROOT (IF UNIQUE) OF
C               THE GIVEN FUNCTION INSIDE THE PRESCRIBED SQUARE
C               SET NZERO=1
C
C        PARAMETERS
C           (X0,Y0) -CENTER OF SQUARE
C           R       -SIDE LENTGH
C           N       -NO. OF SEGMENTS ON EACH SIDE
C           NZERO   -PURPOSE CODE (SEE PURPOSE)
C           MET     -METHOD CODE WHERE
C                    MET=1  STANDARD ROMBERG METHOD
C                    MET=2  MODIFIED ROMBERG METHOD
C           IDEL    -ERROR CODE (SEE SUBROUTINE SQUARE)
C           DEL     -ACCURACY (SEE SUBROUTINE SQUARE)
C
C        METHOD
C           INTEGRATION IS PERFORMED ROUND THE GIVEN SQUARE BY
C           THE TRAPEZOIDAL RULE WITH INTERVAL HALVING AND RESULTS
C           ARE COMBINED BY ROMBERGS STANDARD METHOD (MET=1) OR BY
C           ROMBERGS MODIFIED METHOD (MET=2) TO GIVE THE NUMBER OF
C           ROOTS IN THE ORIGINAL SQUARE OR TO GIVE (2) (SEE PURPOSE)
C           IN CASE (1),IF THE ORIGINAL SQUARE CONTAINS A ROOT, ITS
C           SIDES ARE DIVIDED INTO N EQUAL PARTS AND INTEGRATION IS
C           CARRIED ROUND EACH ONE TO GIVE THE NUMBER OF ROOTS IN IT
C
C     ------------------------------------------------------------------
      TYPE COMPLEX(4) F,S2,SU,CONSTANT,CMPLX
      COMMON NZERO
      DIMENSION X(5),Y(5),C(4),D(4),TITLE(70)
      EXTERNAL F
C     ------------------------------------------------------------------
      READ 15,(TITLE(K),K=1,70)
   15 FORMAT (70(A1))
   20 READ 25,X0,Y0,R,DEL,IDEL,IMAX,MET,NZERO,N,INT
   25 FORMAT(3F10.3,F10.6,6I5)
      IF(INT.EQ.0) STOP
      CALL PRINTA(X0,Y0,R,TITLE,NZERO,MET,IDEL,DEL,N)
      DIV = 1./6.28318
      CONSTANT = -CMPLX(0.,DIV)
      CALL SQUARE(F,X0,Y0,R,DEL,S2,SU,IDEL,IER,IMAX,MET,ITER)
      IF(IER.NE.0) GO TO 150
      SU = SU*CONSTANT
      CALL PRINTG(ITER,SU)
      IF(NZERO.NE.1) GO TO 60
      GO TO 20
   60 RESU = REAL(SU)
      IF(RESU.GT.0.5) GO TO 100
      GO TO 20
  100 CALL COORD(N,X0,Y0,R,X,Y,C,D)
```

```
      DN = N
      R = R/DN
      DO 120 J=1,N
      DO 120 I=1,N
      X0 = C(I)
      Y0 = D(J)
      CALL PRINTA(X0,Y0,R,TITLE,NZERO,MET,IDEL,DEL,N)
      CALL SQUARE(F,X0,Y0,R,DEL,S2,SU,IDEL,IER,IMAX,MET,ITER)
      IF(IER.NE.0) GO TO 150
      SU = SU*CONSTANT
      CALL PRINTG(ITER,SU)
  120 CONTINUE
      GO TO 20
  150 CALL PRINTE(IER)
      GO TO 20
      END
```

FORTRAN DIAGNOSTIC RESULTS FOR TWO

NO ERRORS

```
      SUBROUTINE COORD(N,X0,Y0,R,X,Y,C,D)
      DIMENSION X(5),Y(5),C(4),D(4)
      X(1) = X0-R/2.
      Y(1) = Y0-R/2.
      RN = N
      H1 = R/RN
      M = N+1
      DO 40 I=2,M
      RI = I
      X(I) = X(1) + (RI-1.)*H1
   40 Y(I) = Y(1) + (RI-1.)*H1
      DO 60 I=1,N
      C(I) = (X(I) + X(I+1))/2.
   60 D(I) = (Y(I) + Y(I+1))/2.
      RETURN
      END
```

FORTRAN DIAGNOSTIC RESULTS FOR COORD

NO ERRORS

```
      SUBROUTINE SQUARE(F,X0,Y0,R,DEL,S2,SU,IDEL,IER,IMAX,MET,I)
C
C     ..........................................................
C
C        PURPOSE
C           INTEGRATES THE GIVEN FUNCTION OVER THE PRESCRIBED
C           SQUARE IN COMPLEX PLANE
C
C        USAGE
C           CALL SQUARE(F,X0,Y0,R,DEL,S2,SU,IDEL,IER,IMAX,MET,I)
C
C        DESCRIPTION OF PARAMETERS
C           F    -NAME OF USER FUNCTION SUBPROGRAM GIVING F(X)
C           X0,Y0-COORDINATES OF CENTER OF SQUARE
C           R    -SIDE LENTGH OF SQUARE
C           IDEL-ACCURACY CODE WHERE
C                IDEL=-1 RELATIVE ERROR
C                IDEL= 1 ABSOLUTE ERROR
C           DEL -REQUIRED ACCURACY OR TOLERANCE
C           IMAX-MAXIMUM NUMBER OF ITERATIONS
C           SU   -RESULTANT FINAL VALUE OF INTEGRAL
C           C    -WORKING VECTOR OF LENGTH IMAX
C           T    -WORKING VECTOR OF LENGTH IMAX
C           IER -RESULTANT ERROR CODE WHERE
C                IER=0    REQUIRED ACCURACY MET
C                IER=1    REQUIRED ACCURACY NOT MET IN IMAX ITERATIONS
C                IER=2    IDEL NOT 1 OR -1
C                IER=3    DEL NOT POSITIVE
C                IER=4    IMAX LESS THAN 3
C           MET -ROMBERG INTEGRATIO CODE WHERE
C                MET=1    STANDARD ROMBERG COEFFICIENTS USED
C                MET=2    ADAPTED ROMBERG COEFFICIENTS USED
C
C        SUBROUTINES AND FUNCTION SUBPROGRAMS REQUIRED
C           F- FUNCTION SUBPROGRAM WHICH COMPUTES F(Z) FOR Z
C           ON CONTOUR OF SQUARE
C           CALLING PROGRAM MUST HAVE FORTRAN EXTERNAL STATEMENT
C           CONTAINING NAMES OF FUNCTION SUBPROGRAMS LISTED IN CALL TO
C           SQUARE
C
C        METHOD
C           TRAPEZOIDAL SUMS ARE COMPUTED AROUND THE WHOLE
C           SQUARE WITH INTERVAL HALVING AND RESULTS ARE
C           COMBINED BY STANDARD OR ADAPTED ROMBERG METHOD
C           UNTIL DIFFERENCE BETWEEN SUCCESSIVE VALUES IS
C           LESS THAN DEL
C           FAILURE TO REACH THE TOLERANCE AFTER IMAX ITERATIONS
C           TERMINATES THE SUBROUTINE EXECUTION
C
C     ..........................................................
C
      TYPE COMPLEX(4) F,H,H1,S2,S,X,T,CI,CT,A,FA,SU,CMPLX
      DIMENSION C(15),T(15),A(4),FA(5),S(4)
C
C     ..........................................................
C
C     CHECK FOR PARAMETER ERRORS
      IER=0
```

```
      IF(IDEL*IDEL-1)10,20,10
   10 IER=2
      RETURN
   20 IF(DEL)30,30,40
   30 IER=3
      RETURN
   40 IF(IMAX-3)50,60,60
   50 IER=4
      RETURN
   60 IF(IMAX-15)80,80,70
   70 IMAX=15
   80 CI=CMPLX(0.,1.)
      CT=CMPLX(1.,-1.)
      GO TO (85,90),MET
   85 C(1)=0.
      GO TO 100
   90 C(1)=-0.75
C     ----------------------------------------------------------------
C     SIGMA(I)= TRAPEZOIDAL SUM ON WHOLE SQUARE WHEN SIDES
C                ARE DIVIDED INTO  2**(I-1) EQUAL PARTS
C     ----------------------------------------------------------------
C     COMPUTE SIGMA(1)
  100 H=R
      X1=X0-R/2.
      X2=X0+R/2.
      Y1=Y0-R/2.
      Y2=Y0+R/2.
      A(1)=CMPLX(X1,Y1)
      A(2)=CMPLX(X2,Y1)
      A(3)=CMPLX(X2,Y2)
      A(4)=CMPLX(X1,Y2)
      DO 105 K=1,4
  105 FA(K) = F(A(K))
      FA(5) = FA(1)
      T(1) = 0.
      DO 110 K=1,4
      S(K) = (FA(K) + FA(K+1))*0.5
      T(1) = T(1) + S(K)*H
  110 H = H*CI
C     ----------------------------------------------------------------
C     COMPUTE SIGMA(I)
      JU=1
      DO 220 I=2,IMAX
      T(I) = 0.
      GO TO(120,130),MET
  120 C(I) = C(I-1)*4.+3.
      GO TO 140
  130 C(I) = C(I-1)*16.+15.
  140 H1 = H*0.5
      X = A(1)-H1
      DO 155 K=1,4
      DO 150 J=1,JU
      X=X+H
  150 S(K) = S(K)+F(X)
      T(I) = T(I)+S(K)*H1
      X=X+CT*H1
      H=CI*H
  155 H1=CI*H1
C     ----------------------------------------------------------------
C     COMBINE TRAPEZOIDAL SUMS BY ROMBERGS METHOD
```

```
      IF(I-2)180,180,160
  160 IM1=I-1
      I1=I
      DO 170 J=2,IM1
      I2=I1
      I1=I1-1
  170 T(I1)=T(I2)+(T(I2)-T(I1))/C(J)
  180 S2=T(2)+(T(2)-T(1))/C(I)
C     ----------------------------------------------------------
C     TEST FOR CONVERGENCE
      S1= CABS(S2-T(1))
      IF(IDEL) 190,190,200
  190 S1= S1/CABS(S2)
  200 IF(S1-DEL) 230,230,210
C     REQUIRED ACCURACY NOT MET - NEXT ITERATION
  210 T(1)=S2
      JU=JU+JU
  220 H=H1
      IER=1
C     REQUIRED ACCURACY MET
  230 SU=S2
      S2=T(1)
      RETURN
      END
```

FORTRAN DIAGNOSTIC RESULTS FOR SQUARE

NO ERRORS

```
      FUNCTION F(Z)
      REAL K,NORM
C     -------------------------------------------------------------
      TYPE COMPLEX(4) F,DERIV,FUNC,CMPLX,Z,T
C     -------------------------------------------------------------
      COMMON NZERO
      DIMENSION R(2)
      EQUIVALENCE(R,T)
      T=Z
C     -------------------------------------------------------------
      FUNC=Z*Z*Z*Z*Z+16.*SQRT(3.)-CMPLX(0.,16.)
      DERIV=5.0*Z*Z*Z*Z
C     -------------------------------------------------------------
      F=DERIV/FUNC
      IF(NZERO.NE.1) GO TO 30
      F=F*Z
      GO TO 80
   30 NORM=CABS(F)
      K=100.0
      IF(NORM.LT.K)GO TO 80
      PRINT 40
   40 FORMAT(1H0,10X,45HWARNING-A ROOT IS TOO CLOSE TO THE CONTOUR AT)
      PRINT 60,R
   60 FORMAT(1H0,22X,2HZ=,E10.3,4H+J( ,E10.3,1H))
      PRINT 70
   70 FORMAT(1H0,10X,24HINTEGRATION IS ABANDONED/1H1)
      STOP
   80 RETURN
      END
```

FORTRAN DIAGNOSTIC RESULTS FOR F

NO ERRORS

```
      SUBROUTINE PRINTA(X0,Y0,R,TITLE,NZERO,MET,IDEL,DEL,N)
      DIMENSION TITLE(70)
      PRINT 5
    5 FORMAT(1H1)
      IF(NZERO.NE.1) GO TO 9
      PRINT 6
    6 FORMAT(11X,25HEVALUATION OF THE ROOT OF)
      GO TO 11
    9 PRINT 10
   10 FORMAT(11X,18HNUMBER OF ROOTS OF)
   11 PRINT 15,(TITLE(K),K=1,70)
   15 FORMAT(1H0,19X,70A1)
      PRINT 20
   20 FORMAT(1H0,10X,27HINSIDE THE FOLLOWING SQUARE)
      PRINT 25, X0,Y0
   25 FORMAT(1H0,19X,13HCENTER     = (,F7.2,1H,,F7.2,1H))
      PRINT 30,R
   30 FORMAT(20X,12HSIDE       = ,F5.2)
      PRINT 100
  100 FORMAT(1H0,5X,8HRESULTS-/1H0)
      PRINT 110,DEL
  110 FORMAT(1H0,5X,11HPARAMETERS-,9X,6HDEL = ,F10.6,2X,
     110H(ACCURACY))
      IF(IDEL.EQ.1) GO TO 130
      PRINT 120,IDEL
  120 FORMAT(1H ,25X,5HIDEL=,I4,9X,21H(RELATIVE ERROR CODE))
      GO TO 144
  130 PRINT 140,IDEL
  140 FORMAT(1H ,25X,5HIDEL=,I4,9X,21H(ABSOLUTE ERROR CODE))
  144 GO TO(150,170),MET
  150 PRINT 160
  160 FORMAT(1H0,5X,11HMETHOD    -,9X,28HSTANDARD ROMBERG INTEGRATION
     1/1H0)
      GO TO 190
  170 PRINT 180
  180 FORMAT(1H0,5X,11HMETHOD    -,9X,28HMODIFIED ROMBERG INTEGRATION
     1/1H0)
  190 RETURN
      END
```

FORTRAN DIAGNOSTIC RESULTS FOR PRINTA

NO ERRORS

```
      SUBROUTINE PRINTE(IER)
      GO TO(50,60,70,80),IER
   50 PRINT 55
   55 FORMAT(10X,46HIER=1  REQUIRED ACCURACY NOT MET IN IMAX STEPS)
      RETURN
   60 PRINT 65
   65 FORMAT(10X,23HIER=2  IDEL NOT 1 OR -1)
      RETURN
   70 PRINT 75
   75 FORMAT(10X,23HIER=3  DEL NOT POSITIVE)
      RETURN
   80 PRINT 85
   85 FORMAT(10X,23HIER=4  IMAX LESS THAN 3)
      RETURN
      END
```

FORTRAN DIAGNOSTIC RESULTS FOR PRINTE

NO ERRORS

```
      SUBROUTINE PRINTG(ITER,SU)
      TYPE COMPLEX(4) SU,T
      COMMON NZERO
      DIMENSION Q(2)
      EQUIVALENCE(Q,T)
      T = SU
      PRINT 50,ITER
   50 FORMAT(1H0,5X,23HNUMBER OF ITERATIONS = ,I5)
      IF(NZERO.EQ.1) GO TO 70
      PRINT 60,Q
   60 FORMAT(1H0,5X,17HNUMBER OF ROOTS = ,E12.3,6H   +J   ,E12.3/1H1)
      RETURN
   70 PRINT 80,Q
   80 FORMAT(1H0,5X,16HVALUE OF ROOT = ,E12.3,6H   +J   ,E12.3/1H1)
      RETURN
      END
```

FORTRAN DIAGNOSTIC RESULTS FOR PRINTG

```
NO ERRORS
LOAD,56,01
RUN,3,NM
```

4.4. EXAMPLES WITH PROGRAM ONE

NUMBER OF ROOTS OF

F(Z) = Z**5 + 16.*SQRT(3.) -16.*I

INSIDE THE FOLLOWING RECTANGLE OR SQUARE

CENTER = (0, 0)
SIDE A = 4.00
SIDE B = 4.00

USING THE STANDARD ROMBERG CONTOUR INTEGRATION

RESULTS-

PARAMETERS- REL = .001000 (ACCURACY)
TDEL= -1 (RELATIVE ERROR CODE)
N = 2 (NO. OF SUBDIVISIONS)

INTEGRALS ON OUTER SIDES-

	REAL PART	IMAGINARY PART	ITERATIONS
HOR (2,1,1) =	-1.241E 00	4.612E 00	7
HOR (2,2,1) =	9.978E-01	3.348E 00	6
HOR (2,1,3) =	-1.891E 00	-5.013E 00	9
HOR (2,2,3) =	1.702E 00	-2.703E 00	5
VERT (2,1,1) =	-2.449E 00	-2.456E 00	6
VERT (2,1,2) =	2.550E 00	-5.197E 00	11
VERT (2,3,1) =	-8.887E-01	3.720E 00	6
VERT (2,3,2) =	1.043E 00	4.366E 00	6

NUMBER OF ROOTS IN WHOLE REGION

5.000E 00 +J 2.411E-05

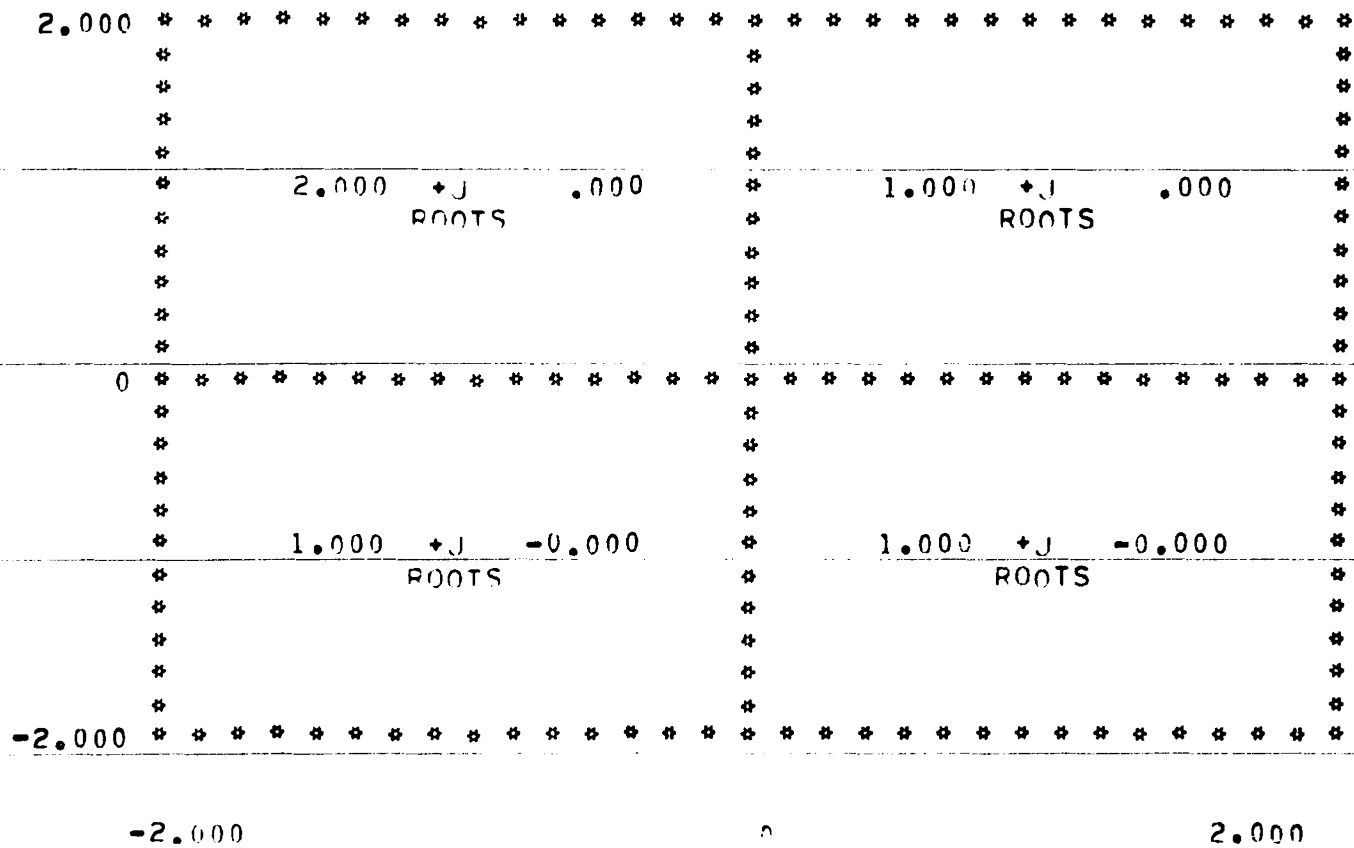
INTEGRALS ON INNER SIDES-

HOR (2,1,2) =	6.585E-01	1.309E 00	7
HOR (2,2,2) =	6.585E-01	2.618E-01	5
VERT (2,2,1) =	-5.493E-01	5.236E-01	5
VERT (2,2,2) =	2.230E-06	1.047E 00	6

NUMBER OF ROOTS IN EACH SUBRECTANGLE

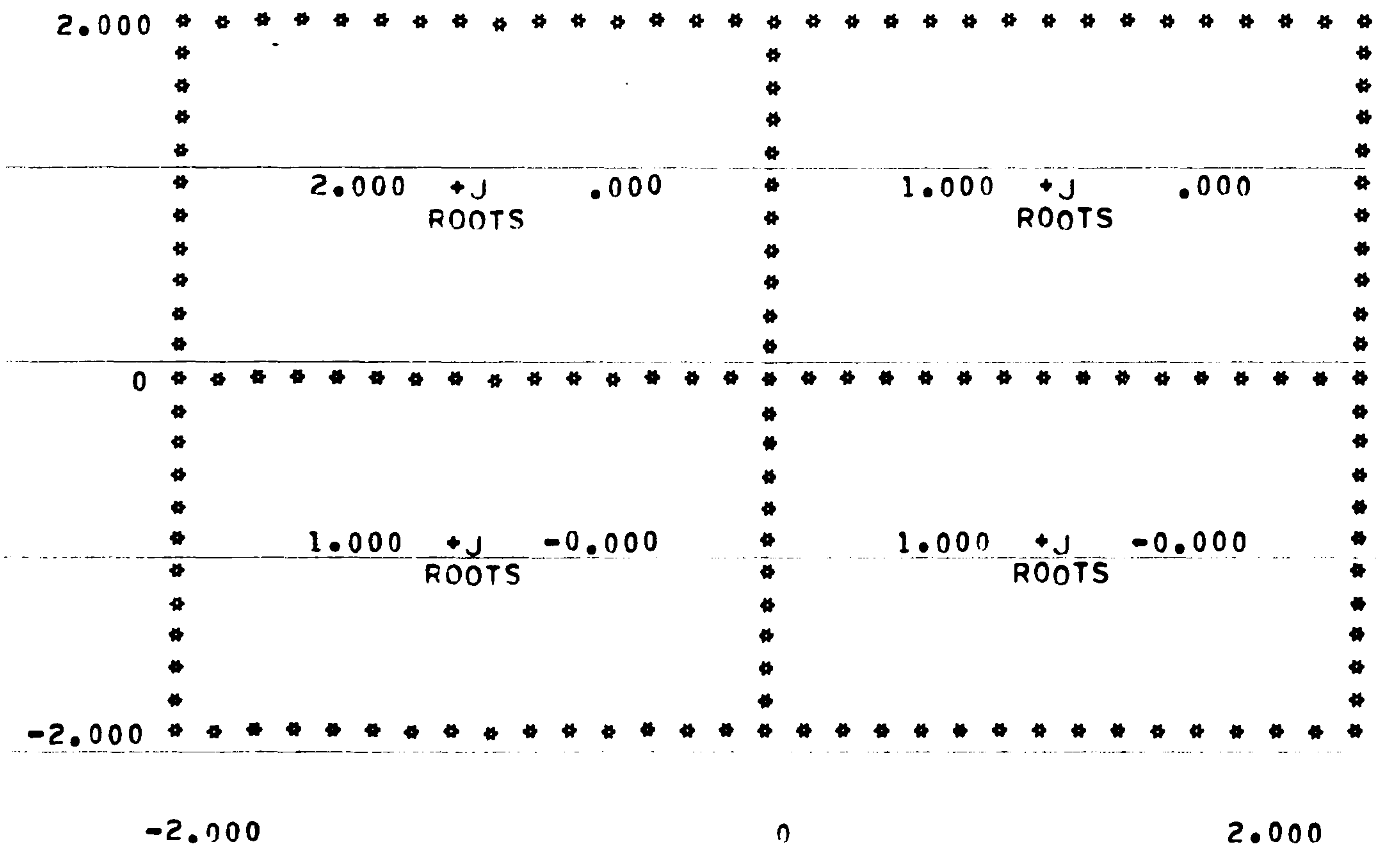
ROOTS(2,1,1) =	1.000E-00	-9.870E-07
ROOTS(2,2,1) =	1.000E 00	-5.654E-07
ROOTS(2,1,2) =	2.000E 00	1.950E-05
ROOTS(2,2,2) =	1.000E 00	6.159E-06

NUMBER OF ROOTS IN EACH SUBRECTANGLE



```
NUMBER OF ROOTS OF

          F(Z) = Z**5 + 16.*SQRT(3.) -16.*I

INSIDE THE FOLLOWING RECTANGLE OR SQUARE

          CENTER    = (        0,        0)
          SIDE A    =  4.00
          SIDE B    =  4.00

USING THE STANDARD ROMBERG CONTOUR INTEGRATION

RESULTS-

PARAMETERS-                DEL =     .001000   (ACCURACY)
                           IDEL=    1          (ABSOLUTE ERROR CODE)
                           N   =    2          (NO. OF SUBDIVISIONS)

INTEGRALS ON OUTER SIDES-

                              REAL PART     IMAGINARY PART     ITERATIONS

          HOR  (2,1,1) =     -1.241E 00       4.612E 00            8
          HOR  (2,2,1) =      9.978E-01       3.348E 00            6
          HOR  (2,1,3) =     -1.891E 00      -5.013E 00           10
          HOR  (2,2,3) =      1.702E 00      -2.703E 00            6
          VERT (2,1,1) =     -2.449E 00      -2.456E 00            7
          VERT (2,1,2) =      2.550E 00      -5.197E 00           12
          VERT (2,3,1) =     -8.887E-01       3.720E 00            6
          VERT (2,3,2) =      1.043E 00       4.366E 00            7

NUMBER OF ROOTS IN WHOLE REGION

                              5.000E 00   +J   -1.034E-06

INTEGRALS ON INNER SIDES-

          HOR  (2,1,2) =      6.585E-01       1.309E 00            7
          HOR  (2,2,2) =      6.585E-01       2.618E-01            5
          VERT (2,2,1) =     -5.493E-01       5.236E-01            5
          VERT (2,2,2) =      2.230E-06       1.047E 00            6

NUMBER OF ROOTS IN EACH SUBRECTANGLE

          ROOTS(2,1,1) =      1.000E-00      -1.783E-06
          ROOTS(2,2,1) =      1.000E 00      -5.654E-07
          ROOTS(2,1,2) =      2.000E 00       1.853E-07
          ROOTS(2,2,2) =      1.000E 00       1.129E-06
```

NUMBER OF ROOTS IN EACH SUBRECTANGLE



NUMBER OF ROOTS OF

F(Z) = Z**5 + 16.*SQRT(3.) -16.*I

INSIDE THE FOLLOWING RECTANGLE OR SQUARE

CENTER = (-1.00, 1.00)
SIDE A = 2.00
SIDE B = 2.00

USING THE STANDARD ROMBERG CONTOUR INTEGRATION

RESULTS-

PARAMETERS- DEL = .001000 (ACCURACY)
IDEL= -1 (RELATIVE ERROR CODE)
N = 2 (NO. OF SUBDIVISIONS)

INTEGRALS ON OUTER SIDES-

	REAL PART	IMAGINARY PART	ITERATIONS
HOR (2,1,1) =	6.312E-01	1.293E 00	6
HOR (2,2,1) =	2.731E-02	1.606E-02	4
HOR (2,1,3) =	-1.339E 00	-1.066E 00	4
HOR (2,2,3) =	-5.526E-01	-3.948E 00	8
VERT (2,1,1) =	1.446E 00	-4.087E 00	10
VERT (2,1,2) =	1.104E 00	-1.110E 00	4
VERT (2,3,1) =	-1.537E-02	2.749E-02	4
VERT (2,3,2) =	1.537E-02	1.020E 00	5

NUMBER OF ROOTS IN WHOLE REGION

2.000E 00 +J 1.849E-05

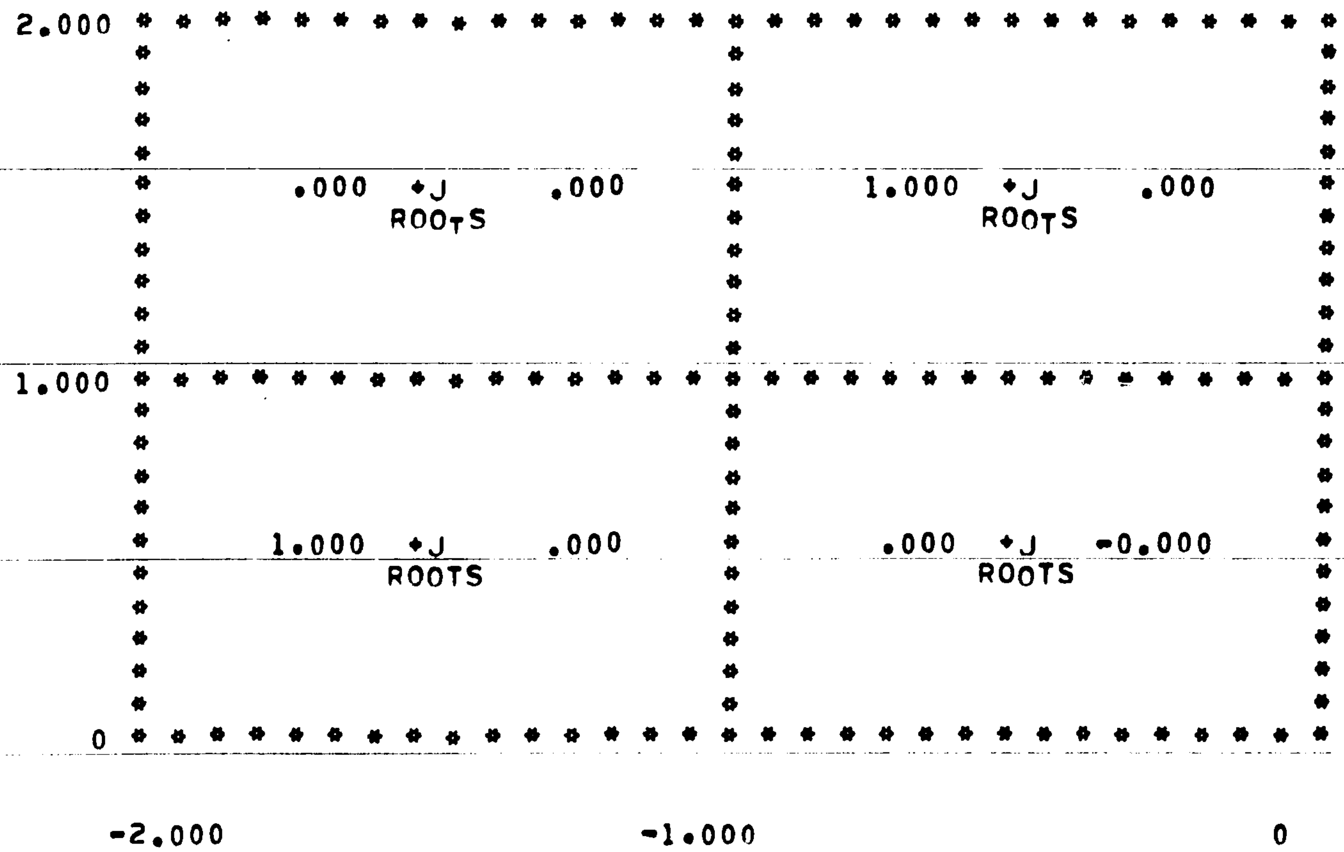
INTEGRALS ON INNER SIDES-

HOR (2,1,2) =	-6.287E-01	-9.262E-01	4
HOR (2,2,2) =	-1.738E-01	6.655E-02	5
VERT (2,2,1) =	1.857E-01	-2.300E-02	5
VERT (2,2,2) =	3.942E-01	-1.249E 00	5

NUMBER OF ROOTS IN EACH SUBRECTANGLE

ROOTS(2,1,1) =	1.000E 00	8.398E-06
ROOTS(2,2,1) =	7.012E-08	-4.515E-08
ROOTS(2,1,2) =	1.319E-06	7.326E-07
ROOTS(2,2,2) =	1.000E 00	9.409E-06

NUMBER OF ROOTS IN EACH SUBRECTANGLE



EVALUATION OF THE ROOT OF

F(Z) = Z**5 + 16.*SQRT(3.) -16.*I

INSIDE THE FOLLOWING RECTANGLE OR SQUARE

CENTER = (-1.00, -1.00)
SIDE A = 2.00
SIDE B = 2.00

USING THE STANDARD ROMBERG CONTOUR INTEGRATION

RESULTS-

PARAMETERS-	DEL =	.001000	(ACCURACY)
	IDEL=	1	(ABSOLUTE ERROR CODE)
	N =	1	(NO. OF SUBDIVISIONS)

INTEGRALS ON OUTER SIDES-

	REAL PART	IMAGINARY PART	ITERATIONS
HOR (1,1,1) =	1.194E 01	-1.637E 00	8
HOR (1,1,2) =	-1.021E 00	-2.317E 00	7
VERT (1,1,1) =	2.312E 00	6.707E 00	7
VERT (1,2,1) =	8.270E-01	9.151E-01	6

ROOT IN GIVEN REGION =

Z = -8.135E-01 +J -1.827E 00

EVALUATION OF THE ROOT OF

F(Z) = Z**5 + 16.*SQRT(3.) -16.*I

INSIDE THE FOLLOWING RECTANGLE OR SQUARE

CENTER = (1.00, -1.00)
SIDE A = 2.00
SIDE B = 2.00

USING THE STANDARD ROMBERG CONTOUR INTEGRATION

RESULTS-

PARAMETERS- DEL = .001000 (ACCURACY)
IDEL= 1 (ABSOLUTE ERROR CODE)
N = 1 (NO. OF SUBDIVISIONS)

INTEGRALS ON OUTER SIDES-

	REAL PART	IMAGINARY PART	ITERATIONS
HOR (1,1,1) =	8.009E 00	1.696E 00	6
HOR (1,1,2) =	1.069E 00	4.083E-01	5
VERT (1,1,1) =	8.270E-01	9.151E-01	6
VERT (1,2,1) =	2.295E 00	8.966E 00	6

ROOT IN GIVEN REGION =

Z = 1.486E 00 +J -1.338E 00

```
EVALUATION OF THE ROOT OF

          F(Z) = Z**5 + 16.*SQRT(3.) -16.*I

INSIDE THE FOLLOWING RECTANGLE OR SQUARE

          CENTER     = (    1.00,     1.00)
          SIDE A     =  2.00
          SIDE B     =  2.00

USING THE STANDARD ROMBERG CONTOUR INTEGRATION

RESULTS-


PARAMETERS-             DEL =     .001000  (ACCURACY)
                        IDEL=    1         (ABSOLUTE ERROR CODE)
                        N   =    1         (NO. OF SUBDIVISIONS)

INTEGRALS ON OUTER SIDES-
```

	REAL PART	IMAGINARY PART	ITERATIONS
HOR (1,1,1) =	1.069E 00	4.083E-01	5
HOR (1,1,2) =	6.882E 00	5.101E-01	6
VERT (1,1,1) =	-1.753E 00	8.363E-02	7
VERT (1,2,1) =	-2.223E 00	1.107E 01	7

```
ROOT IN GIVEN REGION =


          Z =     1.732E 00  +J     1.000E 00
```

EVALUATION OF THE ROOT OF

F(Z) = Z**5 + 16.*SQRT(3.) -16.*I

INSIDE THE FOLLOWING RECTANGLE OR SQUARE

CENTER = (-0.50, 1.50)
SIDE A = 1.00
SIDE B = 1.00

USING THE STANDARD ROMBERG CONTOUR INTEGRATION

RESULTS-

PARAMETERS- DEL = .001000 (ACCURACY)
IDEL= 1 (ABSOLUTE ERROR CODE)
N = 1 (NO. OF SUBDIVISIONS)

INTEGRALS ON OUTER SIDES-

	REAL PART	IMAGINARY PART	ITERATIONS
HOR (1,1,1) =	7.609E-02	-1.760E-01	4
HOR (1,1,2) =	9.016E 00	6.342E-01	9
VERT (1,1,1) =	1.622E 00	1.899E 00	5
VERT (1,2,1) =	-1.730E 00	9.643E-02	5

ROOT IN GIVEN REGION =

Z = -4.158E-01 +J 1.956E 00

EVALUATION OF THE ROOT OF

F(Z) = Z**5 + 16.*SQRT(3.) -16.*I

INSIDE THE FOLLOWING RECTANGLE OR SQUARE

CENTER = (-1.50, .50)
SIDE A = 1.00
SIDE B = 1.00

USING THE STANDARD ROMBERG CONTOUR INTEGRATION

RESULTS-

PARAMETERS- DEL = .001000 (ACCURACY)
IDEL= 1 (ABSOLUTE ERROR CODE)
N = 1 (NO. OF SUBDIVISIONS)

INTEGRALS ON OUTER SIDES-

	REAL PART	IMAGINARY PART	ITERATIONS
HOR (1,1,1) =	-9.984E-01	-2.304E 00	6
HOR (1,1,2) =	1.916E 00	8.634E-01	4
VERT (1,1,1) =	-1.723E 00	9.480E 00	11
VERT (1,2,1) =	-1.218E-01	1.498E-01	4

ROOT IN GIVEN REGION =

Z = -1.989E 00 +J 2.091E-01

NUMBER OF ROOTS OF

F(Z) = CEXP(Z) - 2.*Z**2

INSIDE THE FOLLOWING RECTANGLE OR SQUARE

CENTER = (0, 0)
SIDE A = 4.00
SIDE B = 4.00

USING THE STANDARD ROMBERG CONTOUR INTEGRATION

RESULTS-

PARAMETERS- DEL = .001000 (ACCURACY)
IDEL= -1 (RELATIVE ERROR CODE)
N = 2 (NO. OF SUBDIVISIONS)

INTEGRALS ON OUTER SIDES-

	REAL PART	IMAGINARY PART	ITERATIONS
HOR (2,1,1) =	-7.471E-01	1.455E 00	4
HOR (2,2,1) =	2.469E-01	2.010E 00	4
HOR (2,1,3) =	-7.471E-01	-1.455E 00	4
HOR (2,2,3) =	2.469E-01	-2.010E 00	4
VERT (2,1,1) =	-7.179E-01	-1.567E 00	4
VERT (2,1,2) =	7.179E-01	-1.567E 00	4
VERT (2,3,1) =	-2.773E 00	1.251E 00	6
VERT (2,3,2) =	2.773E 00	1.251E 00	6

NUMBER OF ROOTS IN WHOLE REGION

2.000E 00 +J 0

INTEGRALS ON INNER SIDES-

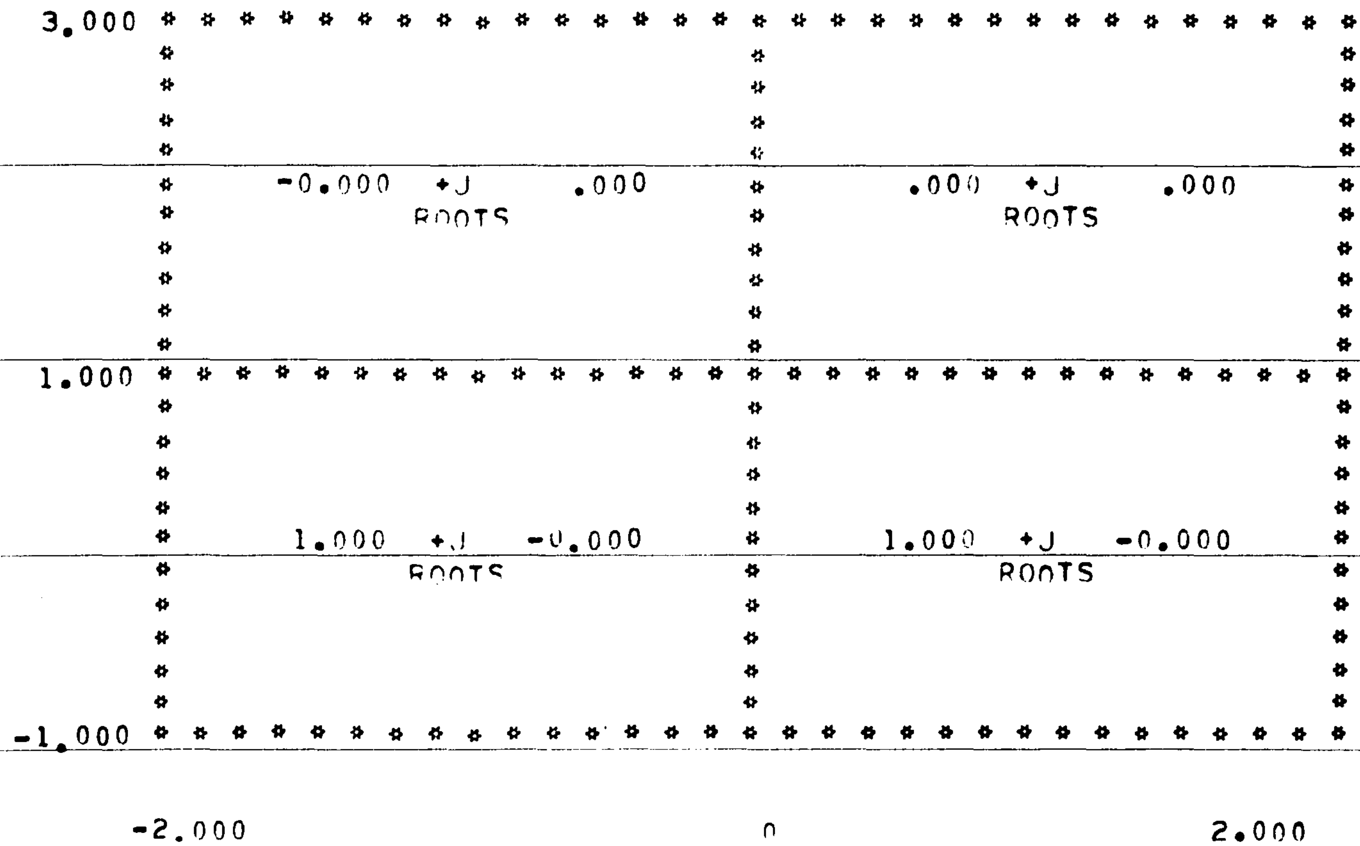
WARNING-A ROOT IS TOO CLOSE TO THE CONTOUR AT

Z=-5.313E-01+J(0)

INTEGRATION IS ABANDONED

```
NUMBER OF ROOTS OF

          F(Z) = CEXP(Z) - 2.*Z**2

INSIDE THE FOLLOWING RECTANGLE OR SQUARE

          CENTER     = (        0,     1.00)
          SIDE A     =  4.00
          SIDE B     =  4.00

USING THE STANDARD ROMBERG CONTOUR INTEGRATION

RESULTS-


PARAMETERS-              DEL  =    .001000   (ACCURACY)
                         IDEL=  -1           (RELATIVE ERROR CODE)
                         N    =   2          (NO. OF SUBDIVISIONS)

INTEGRALS ON OUTER SIDES-

                               REAL PART      IMAGINARY PART      ITERATIONS


      HOR  (2,1,1) =         -1.323E 00        1.882E 00            5
      HOR  (2,2,1) =          3.218E-03        2.735E 00            5
      HOR  (2,1,3) =         -4.230E-01       -1.173E 00            3
      HOR  (2,2,3) =          3.066E-01       -1.463E 00            3
      VERT (2,1,1) =                  0       -1.880E 00            4
      VERT (2,1,2) =          9.494E-01       -1.021E 00            4
      VERT (2,3,1) =          1.410E-11        1.452E 00            5
      VERT (2,3,2) =          2.152E 00        9.614E-01            3

NUMBER OF ROOTS IN WHOLE REGION

                              2.000E 00  +J     7.989E-05

INTEGRALS ON INNER SIDES-

      HOR  (2,1,2) =         -1.323E 00       -1.882E 00            5
      HOR  (2,2,2) =          3.218E-03       -2.735E 00            5
      VERT (2,2,1) =          8.395E-12        6.397E-01            6
      VERT (2,2,2) =          1.849E 00       -3.116E-01            4

NUMBER OF ROOTS IN EACH SUBRECTANGLE


      ROOTS(2,1,1) =          1.000E 00       -1.336E-12
      ROOTS(2,2,1) =          1.000E 00       -9.050E-13
      ROOTS(2,1,2) =         -1.846E-06        2.973E-06
      ROOTS(2,2,2) =          2.293E-05        7.691E-05
```

NUMBER OF ROOTS IN EACH SUBRECTANGLE



EVALUATION OF THE ROOT OF

F(Z) = CEXP(Z) - 2.*Z**2

INSIDE THE FOLLOWING RECTANGLE OR SQUARE

CENTER = (-1.00, 0)
SIDE A = 2.00
SIDE B = 2.00

USING THE STANDARD ROMBERG CONTOUR INTEGRATION

RESULTS-

PARAMETERS- DEL = .001000 (ACCURACY)
IDEL = 1 (ABSOLUTE ERROR CODE)
N = 1 (NO. OF SUBDIVISIONS)

INTEGRALS ON OUTER SIDES-

	REAL PART	IMAGINARY PART	ITERATIONS
HOR (1,1,1) =	3.443E 00	-2.320E-01	5
HOR (1,1,2) =	3.443E 00	2.320E-01	5
VERT (1,1,1) =	0	4.076E 00	4
VERT (1,2,1) =	0	1.148E 00	6

ROOT IN GIVEN REGION =

Z = -5.398E-01 +J 0

```
EVALUATION OF THE ROOT OF

          F(Z) = CEXP(Z) - 2.*Z**2

INSIDE THE FOLLOWING RECTANGLE OR SQUARE

          CENTER     = (    1.00,        0)
          SIDE A     =   2.00
          SIDE B     =   2.00

USING THE STANDARD ROMBERG CONTOUR INTEGRATION
```

```
RESULTS-


PARAMETERS-             DEL =     .001000   (ACCURACY)
                        IDEL=    1          (ABSOLUTE ERROR CODE)
                          N =    1          (NO. OF SUBDIVISIONS)

INTEGRALS ON OUTER SIDES-

                              REAL PART     IMAGINARY PART     ITERATIONS

       HOR  (1,1,1) =        2.973E 00       2.954E 00          5
       HOR  (1,1,2) =        2.973E 00      -2.954E 00          5
       VERT (1,1,1) =                0       1.148E 00          6
       VERT (1,2,1) =        7.513E-12       4.590E 00          6


ROOT IN GIVEN REGION =


              Z =        1.488E 00   +J    -1.196E-12
```

NUMBER OF ROOTS OF

F(Z) = COSH(2.*Z) - 1.0

INSIDE THE FOLLOWING RECTANGLE OR SQUARE

CENTER = (-0.500, .500)
SIDE A = 6.00
SIDE B = 6.00

USING THE STANDARD ROMBERG CONTOUR INTEGRATION

RESULTS-

PARAMETERS- DEL = .001000 (ACCURACY)
IDEL= -1 (RELATIVE ERROR CODE)
N = 2 (NO. OF SUBDIVISIONS)

INTEGRALS ON OUTER SIDES-

	REAL PART	IMAGINARY PART	ITERATIONS
HOR (2,1,1) =	-6.076E 00	-7.486E-01	5
HOR (2,2,1) =	4.072E 00	-2.953E 00	6
HOR (2,1,3) =	-6.542E 00	-6.443E-01	5
HOR (2,2,3) =	4.533E 00	-4.195E 00	7
VERT (2,1,1) =	-2.981E-04	-6.000E 00	3
VERT (2,1,2) =	-2.496E-04	-6.000E 00	3
VERT (2,3,1) =	-3.486E-03	5.998E 00	4
VERT (2,3,2) =	-2.906E-03	5.998E 00	4

NUMBER OF ROOTS IN WHOLE REGION

4.000E 00 +J 8.293E-05

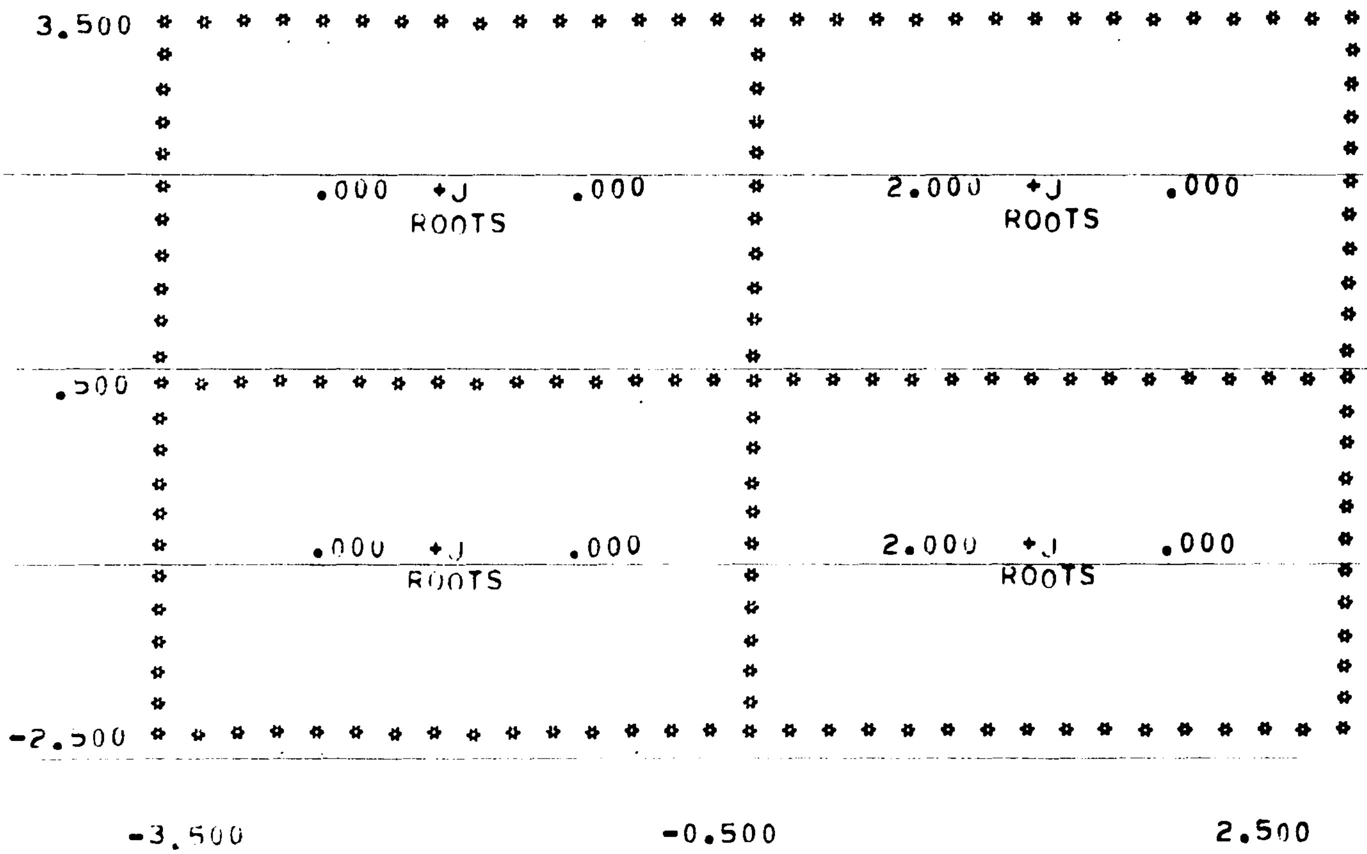
INTEGRALS ON INNER SIDES-

HOR (2,1,2) =	-6.303E 00	-7.358E-01	5
HOR (2,2,2) =	4.297E 00	-3.534E 00	7
VERT (2,2,1) =	-2.279E-01	-5.987E 00	7
VERT (2,2,2) =	-2.395E-01	-5.908E 00	7

NUMBER OF ROOTS IN EACH SUBRECTANGLE

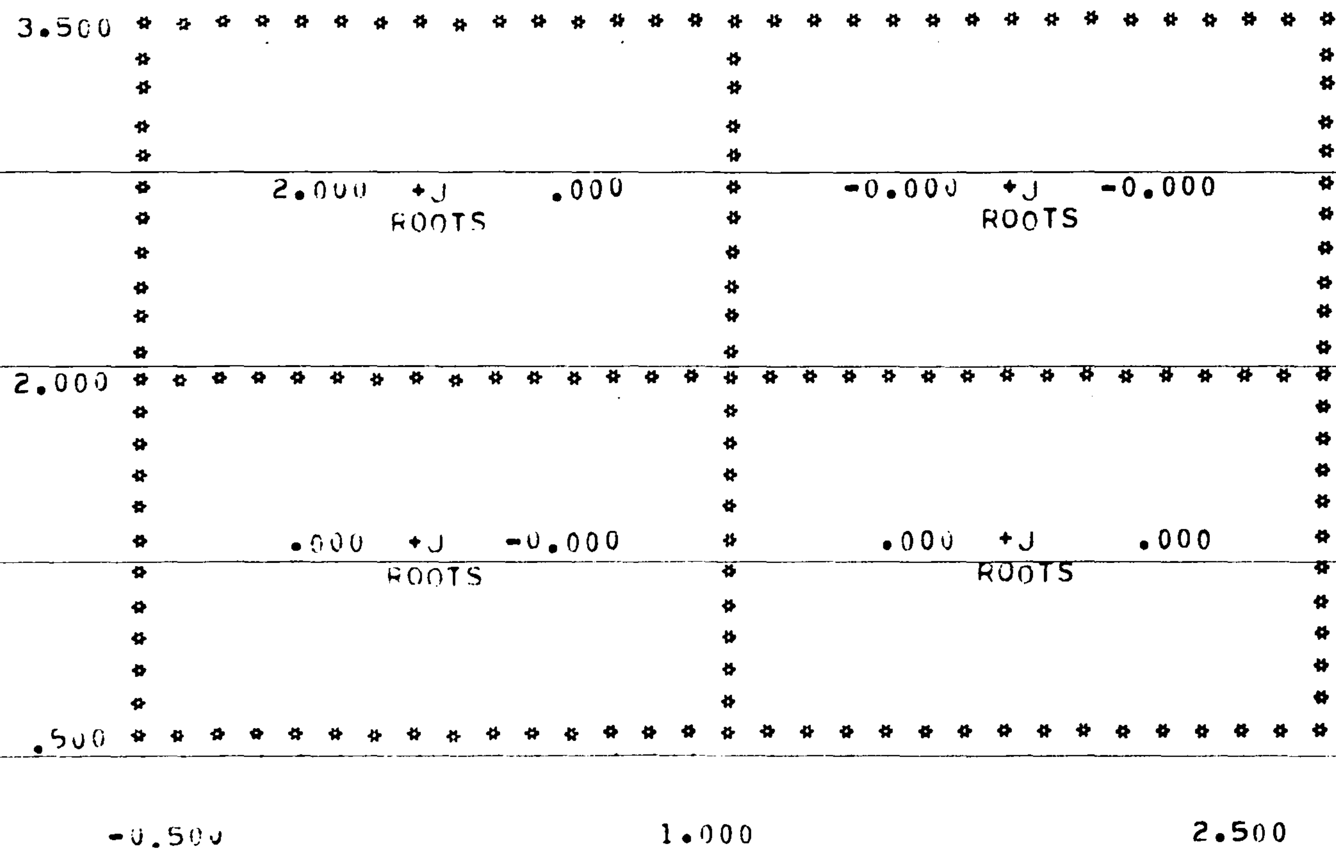
ROOTS(2,1,1) =	8.201E-06	3.570E-05
ROOTS(2,2,1) =	2.000E 00	6.708E-06
ROOTS(2,1,2) =	2.958E-05	3.759E-05
ROOTS(2,2,2) =	2.000E 00	2.929E-06

NUMBER OF ROOTS IN EACH SUBRECTANGLE



```
NUMBER OF ROOTS OF

          F(Z) = COSH(2.*Z) - 1.0

INSIDE THE FOLLOWING RECTANGLE OR SQUARE

          CENTER    = (  1.000,   2.000)
          SIDE A    =  3.00
          SIDE B    =  3.00

USING THE STANDARD ROMBERG CONTOUR INTEGRATION

RESULTS-


PARAMETERS-              DEL  =    .001000   (ACCURACY)
                         IDEL=  -1           (RELATIVE ERROR CODE)
                         N    =   2          (NO. OF SUBDIVISIONS)

INTEGRALS ON OUTER SIDES-

                                REAL PART      IMAGINARY PART      ITERATIONS

      HOR  (2,1,1) =            1.167E 00       -3.301E 00           6
      HOR  (2,2,1) =            3.130E 00       -2.331F-01           4
      HOR  (2,1,3) =            1.338E 00       -4.007E 00           6
      HOR  (2,2,3) =            3.195E 00       -1.885F-01           4
      VERT (2,1,1) =            7.842E-01       -1.821E 00           5
      VERT (2,1,2) =           -1.024E 00       -4.087E 00           6
      VERT (2,3,1) =            1.607E-02        2.978E 00           3
      VERT (2,3,2) =           -1.897E-02        3.019F 00           3


NUMBER OF ROOTS IN WHOLE REGION

                                2.000E 00   +J    1.086E-06

INTEGRALS ON INNER SIDES-

      HOR  (2,1,2) =            6.982E-01        1.088E 00           5
      HOR  (2,2,2) =            2.830E 00        1.775E-01           4
      VERT (2,2,1) =            3.152E-01        2.566E 00           4
      VERT (2,2,2) =           -3.838E-01        3.385E 00           5

NUMBER OF ROOTS IN EACH SUBRECTANGLE


      ROOTS(2,1,1) =            2.652E-07       -1.678E-06
      ROOTS(2,2,1) =            9.108E-07        2.866F-06
      ROOTS(2,1,2) =            2.000E 00        2.743F-06
      ROOTS(2,2,2) =           -1.104E-06       -2.845E-06
```

NUMBER OF ROOTS IN EACH SUBRECTANGLE



NUMBER OF ROOTS OF

F(Z) = COSH(2.*Z) - 1.0

INSIDE THE FOLLOWING RECTANGLE OR SQUARE

CENTER = (1.000, -1.000)
SIDE A = 3.00
SIDE B = 3.00

USING THE STANDARD ROMBERG CONTOUR INTEGRATION

RESULTS-

PARAMETERS-	DEL =	.001000	(ACCURACY)
	IDEL=	-1	(RELATIVE ERROR CODE)
	N =	2	(NO. OF SUBDIVISIONS)

INTEGRALS ON OUTER SIDES-

	REAL PART	IMAGINARY PART	ITERATIONS
HOR (2,1,1) =	1.016E 00	-2.698E 00	5
HOR (2,2,1) =	3.056E 00	-2.553E-01	4
HOR (2,1,3) =	1.167E 00	-3.301E 00	6
HOR (2,2,3) =	3.130E 00	-2.331E-01	4
VERT (2,1,1) =	4.419E-01	-1.685E 00	4
VERT (2,1,2) =	-6.698E-01	-4.302E 00	6
VERT (2,3,1) =	9.420E-03	2.975E 00	3
VERT (2,3,2) =	-1.289E-02	3.024E 00	3

NUMBER OF ROOTS IN WHOLE REGION

2.000E 00 +J -5.381E-07

INTEGRALS ON INNER SIDES-

HOR (2,1,2) =	7.574E-01	1.487E 00	5
HOR (2,2,2) =	2.883E 00	2.197E-01	4
VERT (2,2,1) =	1.833E-01	2.500E 00	4
VERT (2,2,2) =	-2.599E-01	3.476E 00	5

NUMBER OF ROOTS IN EACH SUBRECTANGLE

ROOTS(2,1,1) =	-4.199E-06	-7.867E-09
ROOTS(2,2,1) =	2.269E-06	1.713E-06
ROOTS(2,1,2) =	2.000E 00	2.478E-10
ROOTS(2,2,2) =	-2.430E-06	-2.243E-06

NUMBER OF ROOTS IN EACH SUBRECTANGLE

.500		
	2.000 +J .000 ROOTS	-0.000 +J -0.000 ROOTS
-1.000		
	-0.000 +J -0.000 ROOTS	.000 +J .000 ROOTS
-2.500		

-0.500 1.000 2.500

NUMBER OF ROOTS OF

F(Z) = COSH(2.*Z) - 1.0

INSIDE THE FOLLOWING RECTANGLE OR SQUARE

CENTER = (.250, 2.750)
SIDE A = 1.50
SIDE B = 1.50

USING THE STANDARD ROMBERG CONTOUR INTEGRATION

RESULTS-

PARAMETERS- DEL = .001000 (ACCURACY)
TDEL= -1 (RELATIVE ERROR CODE)
= 2 (NO. OF SUBDIVISIONS)

INTEGRALS ON OUTER SIDES-

	REAL PART	IMAGINARY PART	ITERATIONS
HOR (2,1,1) =	-2.096E-01	6.401E-01	4
HOR (2,2,1) =	9.079E-01	4.475E-01	4
HOR (2,1,3) =	-7.475E-01	-2.937E 00	5
HOR (2,2,3) =	2.086E 00	-1.069E 00	4
VERT (2,1,1) =	-9.680E-01	-1.266E 00	4
VERT (2,1,2) =	-5.573E-02	-2.821E 00	5
VERT (2,3,1) =	-3.689E-01	1.477E 00	3
VERT (2,3,2) =	-1.492E-02	1.908E 00	4

NUMBER OF ROOTS IN WHOLE REGION

2.000E 00 +J 8.462E-06

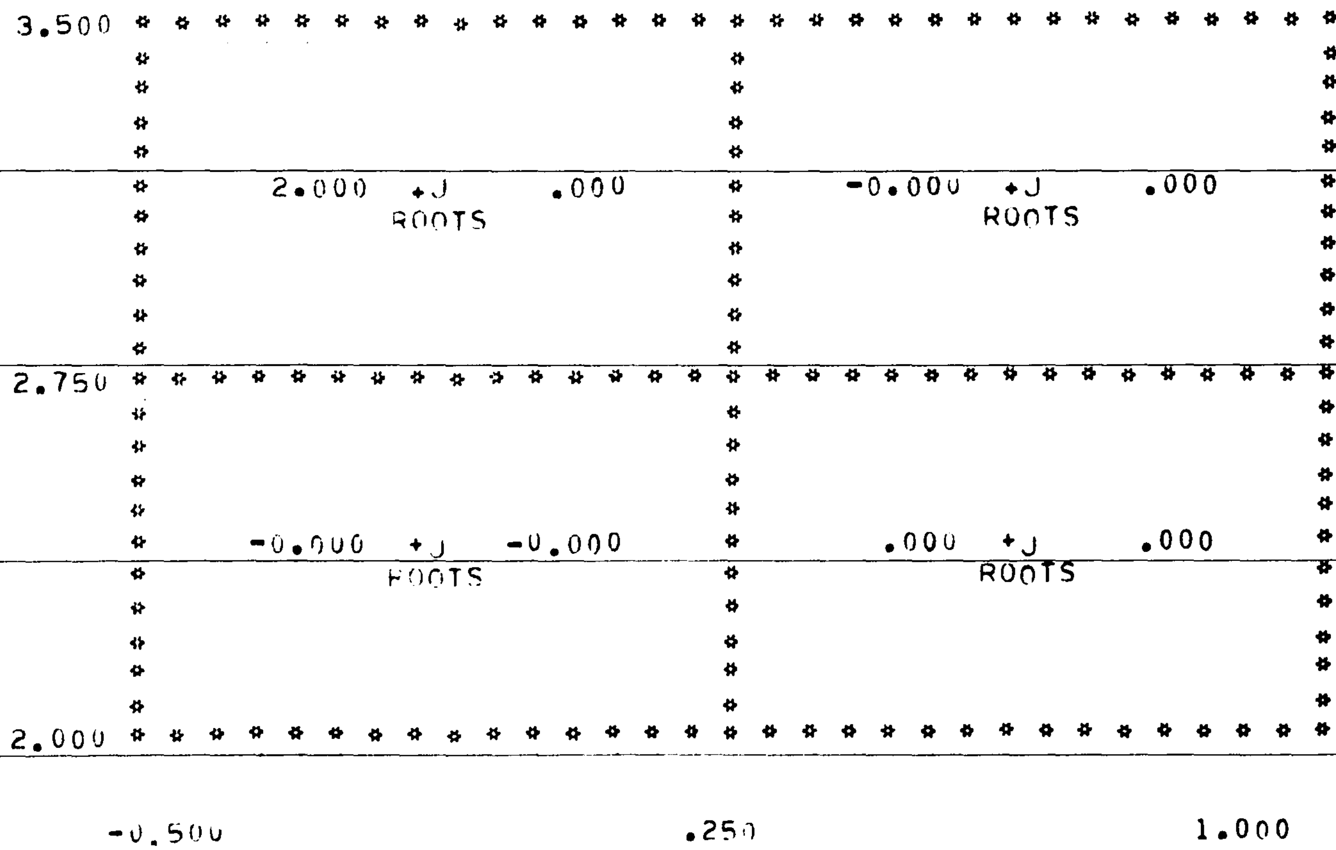
INTEGRALS ON INNER SIDES-

HOR (2,1,2) =	-6.890E-01	2.754E 00	5
HOR (2,2,2) =	1.986E 00	1.077E 00	4
VERT (2,2,1) =	-1.447E 00	8.475E-01	4
VERT (2,2,2) =	-1.142E-01	4.054E 00	6

NUMBER OF ROOTS IN EACH SUBRECTANGLE

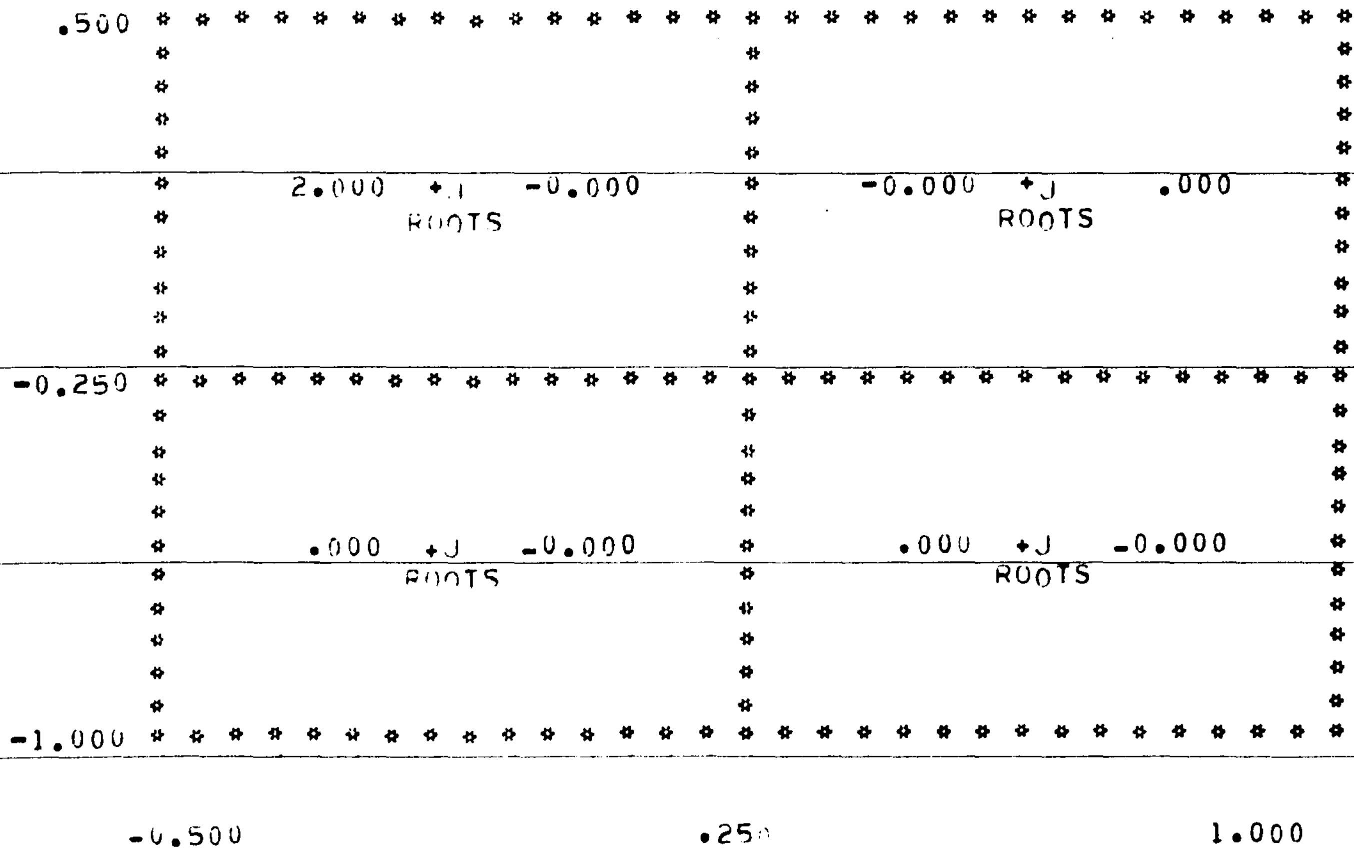
ROOTS(2,1,1) =	-1.065E-06	-1.327E-06
ROOTS(2,2,1) =	3.114E-06	8.599E-06
ROOTS(2,1,2) =	2.000E 00	1.036E-06
ROOTS(2,2,2) =	-5.643E-06	1.534E-07

NUMBER OF ROOTS IN EACH SUBRECTANGLE



```
NUMBER OF ROOTS OF

        F(Z) = COSH(2.*Z) - 1.0

INSIDE THE FOLLOWING RECTANGLE OR SQUARE

        CENTER    = (   .250, -0.250)
        SIDE A    =  1.50
        SIDE B    =  1.50

USING THE STANDARD ROMBERG CONTOUR INTEGRATION

RESULTS-

PARAMETERS-              DEL  =    .001000  (ACCURACY)
                         IDEL=  -1          (RELATIVE ERROR CODE)
                         N    =   2         (NO. OF SUBDIVISIONS)

INTEGRALS ON OUTER SIDES-

                          REAL PART       IMAGINARY PART     ITERATIONS

      HOR  (2,1,1) =     -2.383E-01        8.889E-01            4
      HOR  (2,2,1) =      9.957E-01        5.977E-01            4
      HOR  (2,1,3) =     -5.350E-01       -2.247E 00            5
      HOR  (2,2,3) =      1.702E 00       -1.054E 00            4
      VERT (2,1,1) =     -1.080E 00       -1.555E 00            4
      VERT (2,1,2) =      4.100E-01       -2.747E 00            5
      VERT (2,3,1) =     -3.705E-01        1.585E 00            3
      VERT (2,3,2) =      1.106E-01        1.891E 00            4

NUMBER OF ROOTS IN WHOLE REGION

                          2.000E 00  +J     2.598E-06

INTEGRALS ON INNER SIDES-

      HOR  (2,1,2) =     -9.789E-01        3.661E 00            6
      HOR  (2,2,2) =      2.445E 00        9.654E-01            4
      VERT (2,2,1) =     -1.820E 00        1.217E 00            5
      VERT (2,2,2) =      8.539E-01        3.911E 00            6

NUMBER OF ROOTS IN EACH SUBRECTANGLE

      ROOTS(2,1,1) =      4.762E-07       -1.486E-06
      ROOTS(2,2,1) =      1.510E-05       -4.312E-06
      ROOTS(2,1,2) =      2.000E 00       -1.390E-11
      ROOTS(2,2,2) =     -1.052E-05        8.396E-06
```

NUMBER OF ROOTS IN EACH SUBRECTANGLE



NUMBER OF ROOTS OF

F(Z) = COSH(2.*Z) - 1.0

INSIDE THE FOLLOWING RECTANGLE OR SQUARE

```
CENTER    = ( -0.125,    .125)
SIDE A    =    .75
SIDE B    =    .75
```

USING THE STANDARD ROMBERG CONTOUR INTEGRATION

RESULTS-

```
PARAMETERS-        DEL  =      .001000   (ACCURACY)
                   IDEL=   -1            (RELATIVE ERROR CODE)
                   N    =   2            (NO. OF SUBDIVISIONS)
```

INTEGRALS ON OUTER SIDES-

	REAL PART	IMAGINARY PART	ITERATIONS
HOR (2,1,1) =	-1.465E 00	1.226E 00	4
HOR (2,2,1) =	4.858E-01	2.436E 00	5
HOR (2,1,3) =	-7.139E-01	-9.566E-01	4
HOR (2,2,3) =	1.789E-01	-1.291E 00	4
VERT (2,1,1) =	-1.476E-01	-1.541E 00	4
VERT (2,1,2) =	5.576E-01	-1.206E 00	4
VERT (2,3,1) =	-4.545E-01	2.561E 00	5
VERT (2,3,2) =	1.308E 00	1.351E 00	4

NUMBER OF ROOTS IN WHOLE REGION

2.000E 00 +J -3.011E-11

INTEGRALS ON INNER SIDES-

HOR (2,1,2) =	-2.218E 00	-1.050E 00	4
HOR (2,2,2) =	9.319E-01	-3.754E 00	6
VERT (2,2,1) =	-9.007E-01	-3.816E 00	6
VERT (2,2,2) =	2.061E 00	-1.113E 00	5

NUMBER OF ROOTS IN EACH SUBRECTANGLE

ROOTS(2,1,1) =	-1.052E-05	8.396E-06
ROOTS(2,2,1) =	2.000E 00	-9.843E-11
ROOTS(2,1,2) =	1.133E-05	-6.909E-06
ROOTS(2,2,2) =	4.764E-07	-1.486E-06

NUMBER OF ROOTS IN EACH SUBRECTANGLE

```
.500 * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * *
     *                             *                             *
     *                             *                             *
     *                             *                             *
     *                             *                             *
     *       .000   +J   -0.000    *      .000   +J   -0.000     *
     *            ROOTS            *           ROOTS             *
     *                             *                             *
     *                             *                             *
     *                             *                             *
     *                             *                             *
.125 * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * *
     *                             *                             *
     *                             *                             *
     *                             *                             *
     *                             *                             *
     *     -0.000   +J    .000     *      2.000   +J   -0.000    *
     *            ROOTS            *           ROOTS             *
     *                             *                             *
     *                             *                             *
     *                             *                             *
     *                             *                             *
-0.250 * * * * * * * * * * * * * * * * * * * * * * * * * * * * * *

     -0.500                      -0.125                       .250
```

NUMBER OF ROOTS OF

F(Z) = COSH(2.*Z) - 1.0

INSIDE THE FOLLOWING RECTANGLE OR SQUARE

CENTER = (-0.125, 3.125)
SIDE A = .75
SIDE B = .75

USING THE STANDARD ROMBERG CONTOUR INTEGRATION

RESULTS-

PARAMETERS- DEL = .001000 (ACCURACY)
TDEL= -1 (RELATIVE ERROR CODE)
N = 2 (NO. OF SUBDIVISIONS)

INTEGRALS ON OUTER SIDES-

	REAL PART	IMAGINARY PART	ITERATIONS
HOR (2,1,1) =	-9.499E-01	1.098E 00	4
HOR (2,2,1) =	2.609E-01	1.656E 00	4
HOR (2,1,3) =	-1.045E 00	-1.138E 00	4
HOR (2,2,3) =	2.977E-01	-1.799E 00	4
VERT (2,1,1) =	-4.285E-01	-1.387E 00	4
VERT (2,1,2) =	3.727E-01	-1.434E 00	4
VERT (2,3,1) =	-1.184E 00	1.936E 00	4
VERT (2,3,2) =	1.070E 00	2.119E 00	5

NUMBER OF ROOTS IN WHOLE REGION

2.000E 00 +J -5.639E-06

INTEGRALS ON INNER SIDES-

HOR (2,1,2) =	-2.834E 00	1.935E-01	5

WARNING-A ROOT IS TOO CLOSE TO THE CONTOUR AT

Z=-7.813E-03+J(3.125E 00)

INTEGRATION IS ABANDONED

EVALUATION OF THE ROOT OF

F(Z) = COSH(2.*Z) - 1.0

INSIDE THE FOLLOWING RECTANGLE OR SQUARE

CENTER = (.250, -0.250)
SIDE A = 1.50
SIDE B = 1.50

USING THE STANDARD ROMBERG CONTOUR INTEGRATION

RESULTS-

PARAMETERS- DEL = .001000 (ACCURACY)
IDEL= 1 (ABSOLUTE ERROR CODE)
N = 1 (NO. OF SUBDIVISIONS)

INTEGRALS ON OUTER SIDES-

	REAL PART	IMAGINARY PART	ITERATIONS
HOR (1,1,1) =	2.278E 00	-5.182E-01	4
HOR (1,1,2) =	3.010E 00	2.375E-01	4
VERT (1,1,1) =	-2.623E-01	3.013E 00	4
VERT (1,2,1) =	4.698E-01	3.769E 00	3

SUM OF ROOTS IN GIVEN REGION =

Z = 2.286E-06 +J -9.254E-07

EVALUATION OF THE ROOT OF

F(Z) = COSH(2.*Z) - 1.0

INSIDE THE FOLLOWING RECTANGLE OR SQUARE

CENTER = (.250, -0.250)
SIDE A = 1.50
SIDE B = 1.50

USING THE STANDARD ROMBERG CONTOUR INTEGRATION

RESULTS-

PARAMETERS- DEL = .001000 (ACCURACY)
IDEL= 1 (ABSOLUTE ERROR CODE)
N = 1 (NO. OF SUBDIVISIONS)

INTEGRALS ON OUTER SIDES-

	REAL PART	IMAGINARY PART	ITERATIONS
HOR (1,1,1) =	1.791E-01	-2.804E 00	4
HOR (1,1,2) =	7.312E-01	1.745E 00	4
VERT (1,1,1) =	7.959E-01	-1.247E 00	4
VERT (1,2,1) =	1.348E 00	3.302E 00	4

SUM OF SQUARES OF ROOTS IN GIVEN REGION =

Z = 3.736E-07 +J 4.575E-07

EVALUATION OF THE ROOT OF

F(Z) = COSH(2.*Z) - 1.0

INSIDE THE FOLLOWING RECTANGLE OR SQUARE

CENTER = (.250, 2.750)
SIDE A = 1.50
SIDE B = 1.50

USING THE STANDARD ROMBERG CONTOUR INTEGRATION

RESULTS-

PARAMETERS-	DEL =	.001000	(ACCURACY)
	IDEL=	1	(ABSOLUTE ERROR CODE)
	N =	1	(NO. OF SUBDIVISIONS)

INTEGRALS ON OUTER SIDES-

	REAL PART	IMAGINARY PART	ITERATIONS
HOR (1,1,1) =	-1.453E 00	1.580E 00	5
HOR (1,1,2) =	1.571E 01	4.372E 00	7
VERT (1,1,1) =	1.242E 01	-2.914E-01	6
VERT (1,2,1) =	-9.891E 00	2.501E 00	5

SUM OF ROOTS IN GIVEN REGION =

Z = -3.184E-07 +J 6.283E 00

```
EVALUATION OF THE ROOT OF

        F(Z) = COSH(2.*Z) - 1.0

INSIDE THE FOLLOWING RECTANGLE OR SQUARE

        CENTER    = (    .250,   2.750)
        SIDE A    =  1.50
        SIDE B    =  1.50

USING THE STANDARD ROMBERG CONTOUR INTEGRATION
```

RESULTS-

PARAMETERS-

DEL	=	.001000	(ACCURACY)
IDEL	=	1	(ABSOLUTE ERROR CODE)
N	=	1	(NO. OF SUBDIVISIONS)

INTEGRALS ON OUTER SIDES-

	REAL PART	IMAGINARY PART	ITERATIONS
HOR (1,1,1) =	-3.101E 00	-1.259E 00	5
HOR (1,1,2) =	-1.345E 01	6.046E 01	7
VERT (1,1,1) =	-7.060E 00	3.662E 01	7
VERT (1,2,1) =	-1.741E 01	-2.568E 01	6

SUM OF SQUARES OF ROOTS IN GIVEN REGION =

```
        Z =      -1.974E 01   +J    -6.844E-07
```

4.5 EXAMPLES WITH PROGRAM TWO

```
NUMBER OF ROOTS OF

         F(Z) = Z**5 + 16.*SQRT(3.) -16.*I

INSIDE THE FOLLOWING SQUARE

         CENTER    = (        0,        0)
         SIDE      =  4.00

RESULTS-

PARAMETERS-        DEL =     .010000   (ACCURACY)
                   IDEL=   -1          (RELATIVE ERROR CODE)

METHOD     -       STANDARD ROMBERG INTEGRATION


NUMBER OF ITERATIONS =    11

NUMBER OF ROOTS =   5.000E 00   +J    -3.861E-04
```

```
NUMBER OF ROOTS OF

          F(Z) = Z**5 + 16.*SQRT(3.) -16.*I

INSIDE THE FOLLOWING SQUARE

          CENTER    = (  -1.00,  -1.00)
          SIDE      =  2.00

RESULTS-

PARAMETERS-        DEL =     .010000  (ACCURACY)
                   IDEL=  -1          (RELATIVE ERROR CODE)

METHOD    -        STANDARD ROMBERG INTEGRATION

NUMBER OF ITERATIONS =     7

NUMBER OF ROOTS =   1.000E-00  +J     6.170E-06
```

```
NUMBER OF ROOTS OF

          F(Z) = Z**5 + 16.*SQRT(3.) -16.*I

INSIDE THE FOLLOWING SQUARE

          CENTER    = (   1.00,  -1.00)
          SIDE      =  2.00

RESULTS-

PARAMETERS-          DEL =     .010000  (ACCURACY)
                     IDEL=  -1          (RELATIVE ERROR CODE)

METHOD     -         STANDARD ROMBERG INTEGRATION

NUMBER OF ITERATIONS =      5

NUMBER OF ROOTS =   1.000E 00   +J     5.917E-05
```

```
NUMBER OF ROOTS OF

        F(Z) = Z**5 + 16.*SQRT(3.) -16.*I

INSIDE THE FOLLOWING SQUARE

        CENTER     = (  -1.00,   1.00)
        SIDE       =  2.00

RESULTS-

PARAMETERS-         DEL =     .010000  (ACCURACY)
                    IDEL=  -1          (RELATIVE ERROR CODE)

METHOD     -        STANDARD ROMBERG INTEGRATION

NUMBER OF ITERATIONS =    10

NUMBER OF ROOTS =   2.000E 00  +J   -3.861E-04
```

```
NUMBER OF ROOTS OF

          F(Z) = Z**5 + 16.*SQRT(3.) -16.*I

INSIDE THE FOLLOWING SQUARE

          CENTER    = (    1.00,    1.00)
          SIDE      =  2.00

RESULTS-

PARAMETERS-          DEL =     .010000  (ACCURACY)
                     IDEL=  -1          (RELATIVE ERROR CODE)

METHOD     -         STANDARD ROMBERG INTEGRATION


NUMBER OF ITERATIONS =     5

NUMBER OF ROOTS =   1.000E 00  +J     5.784E-05
```

```
NUMBER OF ROOTS OF

        F(Z) = Z**5 + 16.*SQRT(3.) -16.*I

INSIDE THE FOLLOWING SQUARE

        CENTER     = (       0,       0)
        SIDE       =  4.00

RESULTS-


PARAMETERS-          DEL =    .010000  (ACCURACY)
                     IDEL=  -1         (RELATIVE ERROR CODE)

METHOD     -         MODIFIED ROMBERG INTEGRATION


NUMBER OF ITERATIONS =    10

NUMBER OF ROOTS =   5.003E 00  +J    3.711E-03
```

```
NUMBER OF ROOTS OF

          F(Z) = Z**5 + 16.*SQRT(3.) -16.*I

INSIDE THE FOLLOWING SQUARE

          CENTER     = (  -1.00,  -1.00)
          SIDE       =  2.00

RESULTS-

PARAMETERS-          DEL =    .010000  (ACCURACY)
                     IDEL=  -1         (RELATIVE ERROR CODE)

METHOD    -          MODIFIED ROMBERG INTEGRATION

NUMBER OF ITERATIONS =      6

NUMBER OF ROOTS =    1.000E 00   +J    -6.391E-05
```

```
NUMBER OF ROOTS OF

                F(Z) = Z**5 + 16.*SQRT(3.) -16.*I

INSIDE THE FOLLOWING SQUARE

                CENTER    = (   1.00,  -1.00)
                SIDE      =  2.00

RESULTS-

PARAMETERS-           DEL =    .010000  (ACCURACY)
                      IDEL=  -1         (RELATIVE ERROR CODE)

METHOD     -          MODIFIED ROMBERG INTEGRATION

NUMBER OF ITERATIONS =     5

NUMBER OF ROOTS =   1.000E-00  +J     8.483E-06
```

NUMBER OF ROOTS OF

F(Z) = Z**5 + 16.*SQRT(3.) -16.*I

INSIDE THE FOLLOWING SQUARE

CENTER = (-1.00, 1.00)
SIDE = 2.00

RESULTS-

PARAMETERS- DEL = .010000 (ACCURACY)
IDEL= -1 (RELATIVE ERROR CODE)

METHOD - MODIFIED ROMBERG INTEGRATION

NUMBER OF ITERATIONS = 10

NUMBER OF ROOTS = 2.000E 00 +J -1.010E-04

```
NUMBER OF ROOTS OF

          F(Z) = Z**5 + 16.*SQRT(3.) -16.*I

INSIDE THE FOLLOWING SQUARE

          CENTER    = (    1.00,    1.00)
          SIDE      =  2.00

RESULTS-


PARAMETERS-          DEL =     .010000  (ACCURACY)
                     IDEL=  -1          (RELATIVE ERROR CODE)

METHOD     -         MODIFIED ROMBERG INTEGRATION


NUMBER OF ITERATIONS =      5

NUMBER OF ROOTS =   9.998E-01  +J    -7.812E-06
```

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