Submodularity, matroids, and the common colouring problem

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ABSTRACT

We review the theory of submodular functions and matroids in Chapters 1 and 2 respectively. In Chapter 3 we consider the problem of given two matroids $\mathcal{M}_1$ and $\mathcal{M}_2$, find the smallest integer k such that there is a common partition $X_1, ..., X_k$ of $\mathcal{M}_1$ and $\mathcal{M}_2$. We are interested in both the complexity of the problem as well as in finding good approximations (if not the exact value). We propose different approaches and present some partial results. One of the approaches leads naturally to the study of the following problem: Given a digraph $D = (V, A)$ with a source node s, a sink t, and a partition $A_1, ..., A_k$ of $V - \{s, t\}$, find an s-t dipath that contains at most one node from each set A_i. We show that this problem is NP-Complete and present some applications.

ABRÉGÉ

Nous examinons la théorie des fonctions sous-modulaires et des matroïdes dans les Chapitres 1 et 2 respectivement. Dans le troisième chapitre nous considérons le problème suivant: Étant donné deux matroïdes $\mathcal{M}_1$ et $\mathcal{M}_2$ quel est le plus petit entier k tel qu'il existe une partition commune $X_1, ..., X_k$ de $\mathcal{M}_1$ et $\mathcal{M}_2$? Nous sommes intéressés à la fois par la complexité du problème ainsi que par la recherche de bonnes approximations (si ce n'est la solution exacte). Nous proposons de différentes approches ainsi que des résultats partiels. L'une des approches nous amène à étudier le problème suivant: Étant donné un graphe orienté $D = (V, A)$ avec source s, puits t et une partition $A_1, ..., A_k$ de $V - \{s, t\}$, comment trouver un chemin de s à t qui contient au plus un sommet de chaque ensemble A_i ? Nous démontrons ensuite que ce problème est NP-complet et présentons quelques applications.

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Notation

We use $S \subset T$ to mean that $S \subseteq T$ but $S \neq T$.

For sets $A, B \subseteq V$ we use $A + B$ to denote the set union $A \cup B$, and $A - B$ to the denote the set minus $A \backslash B$. For a set function $f : 2^V \to \mathbb{R}$ we denote $f(\{e\})$ by $f(e)$.

Given a graph $G = (V, E)$, for any $S \subseteq V$ we denote by $\delta(S)$ the set of edges in E that have exactly one endpoint in S. For a directed graph $G = (V, A)$ and a set $S \subseteq V$ we denote by $\delta^+(S)$ the set of arcs leaving S, and by $\delta^-(S)$ the set of arcs incoming S . We use subscripts (e.g. $\delta_G(S)$) to specify the graph G if necessary.

We use $\uplus$ to denote a disjoint union of sets.

We use $[k]$ to denote the set $\{1, 2, ..., k\}$.

Given a graph $G = (V, E)$ and a path P, we denote by $V(P)$ the set of nodes of the path P.

We denote by $\log n$ the logarithm to the base 2 of n, i.e., $\log_2 n$, unless otherwise explicitly specified.

Given a finite ground set V and a set K, we denote by K^V the space of vectors whose components belong to K and are indexed by the elements of V. The component of x indexed by v is denoted by x_v. For instance, if $V = \{v_1, v_2\}$, then $\mathbb{R}^V = \{(x_{v_1}, x_{v_2}) : x_{v_1}, x_{v_2} \in \mathbb{R}\}$. Moreover, given a vector $x \in K^V$ and a set $U \subseteq V$, we denote by $x(U)$ the sum $\sum_{v \in U} x_v$.

CHAPTER 1
Submodular Functions

Submodularity is a property of set functions with deep theoretical consequences and far-reaching applications. Optimizing submodular functions is a central subject in operations research and combinatorial optimization [22]. It appears in many important optimization frameworks including cuts in graphs, rank functions of matroids, set covering problems, plant location problems, certain satisfiability problems, combinatorial auctions, and maximum entropy sampling. In computer science it has recently been identified and utilized in domains such as viral marketing [14], information gathering [16], image segmentation [2, 15, 13], document summarization [21], and speeding up satisfiability solvers [25].

However, the interest for submodular functions is not limited to discrete optimization problems. The rich structure of submodular functions and their link with convex analysis through the Lovasz extension and the various associated polytopes makes them particularly adapted to problems beyond combinatorial optimization, namely as regularizers in signal processing and machine learning problems. Indeed, many continuous optimization problems exhibit an underlying discrete structure (e.g., based on chains, trees or more general graphs), and submodular functions provide an efficient and versatile tool to capture such combinatorial structures.

1.1 Introduction

Submodularity is a property of set functions, i.e., functions $f : 2^V \to \mathbb{R}$ that assign each subset $S \subseteq V$ a value $f(S)$. Hereby V is a finite set, commonly called the ground set. We assume $|V| = n$ unless otherwise specified. Submodular functions can be defined in different ways. We start this section by presenting three equivalent definitions of submodularity.

Definition 1.1.1. *(Submodularity) Let V be a finite ground set and $f : 2^V \to \mathbb{R}$ be a real-valued set function. Then f is submodular if for all $A, B \subseteq V$,*

$$f(A) + f(B) \geq f(A \cup B) + f(A \cap B). \tag{1.1}$$

Equivalently, a submodular function can be defined as a set function exhibiting the diminishing marginal returns property.

Definition 1.1.2. *A function $f : 2^V \to \mathbb{R}$ is submodular if for all $A \subseteq B \subseteq V$ and $e \in V - B$,*

$$f(A + e) - f(A) \geq f(B + e) - f(B). \tag{1.2}$$

This diminishing marginal return property makes submodular functions suitable for many applications, in areas such as economics, game theory, electrical networks, and very recently, in machine learning, artificial intelligence, and computer vision. We now show that the above two definitions of submodularity are equivalent.

Lemma 1.1.1. *A set function $f : 2^V \to \mathbb{R}$ satisfies (1.1) if and only if it satisfies (1.2).*

Proof. First, we show that (1.1) implies (1.2). Let $S \subseteq T \subseteq V$ and $e \in V - T$. Then, by applying (1.1) to $A = S + e$ and $B = T$, we get

$$\begin{aligned} f(S+e) + f(T) &\geq f((S+e) \cup T) + f((S+e) \cap T) \\ &= f(T+e) + f(S). \end{aligned}$$

Re-arranging the terms we obtain

$$f(S+e) - f(S) \geq f(T+e) - f(T).$$

Next, let us show that (1.2) implies (1.1). First, notice that (1.2) can be rewritten as

$$f(S) - f(T) \leq f(S+e) - f(T+e) \tag{1.3}$$

for $S \subseteq T \subset T + e$. Let X and Y be two subsets of V. If $Y \subseteq X$, then $X \cap Y = Y$ and $X \cup Y = X$, and hence (1.1) is trivially satisfied for the sets X, Y. So assume $Y \nsubseteq X$, and enumerate the elements of $Y - X$ as $\{e_1, ..., e_k\}$. Then, note that for $i < k$ we have

$$(X \cap Y) \cup \{e_1, ..., e_i\} \subset X \cup \{e_1, ..., e_i\} \subset X \cup \{e_1, ..., e_{i+1}\}.$$

Hence, by taking $S = X \cap Y$, $T = X$, and repeatedly apply (1.3) with the elements $e_1, ..., e_k$ we get

$$\begin{aligned} f(X \cap Y) - f(X) &\leq f((X \cap Y) + e_1) - f(X + e_1) \\ &\leq f((X \cap Y) \cup \{e_1, e_2\}) - f(X \cup \{e_1, e_2\}) \\ &\ldots \\ &\leq f((X \cap Y) \cup \{e_1, ..., e_k\}) - f(X \cup \{e_1, ..., e_k\}) \\ &= f(Y) - f(X \cup Y). \end{aligned}$$

Finally, this can be re-arranged as

$$f(X) + f(Y) \geq f(X \cup Y) + f(X \cap Y).$$

$\square$

Checking directly whether a function satisfies condition (1.1) or (1.2) is clearly impractical. Our next definition of submodularity gives a condition that is often easier to verify. It applies the same diminishing returns condition (1.2), but only considers a restricted family of set pairs.

Definition 1.1.3. *A function $f : 2^V \to \mathbb{R}$ is submodular if for all $A \subseteq V$ and $i, j \in V - A$ we have*

$$f(A + i) - f(A) \geq f(A + i + j) - f(A + j). \tag{1.4}$$

We now show that our three definitions of submodularity are equivalent.

Lemma 1.1.2. *A set function $f : 2^V \to \mathbb{R}$ satisfies (1.4) if and only if it satisfies (1.2).*

Proof. The conditions of Definition 1.1.2 clearly imply those of Definition 1.1.3.

To show the other direction, let $A \subseteq B \subseteq V$ and $j \in V - B$. Enumerate the elements of $B - A$ as $\{b_1, ..., b_s\}$. Now, by repeatedly applying (1.4) we have

$$\begin{aligned} f(A + j) - f(A) &\geq f(A + j + b_1) - f(A + b_1) \\ &\geq f(A \cup \{j, b_1, b_2\}) - f(A \cup \{b_1, b_2\}) \\ &\ldots \\ &\geq f(A \cup \{j, b_1, ..., b_s\}) - f(A \cup \{b_1, ..., b_s\}) \end{aligned}$$

$$= \quad f(B+j) - f(B).$$

Hence condition (1.2) follows. $\square$

A function f is called *supermodular* if $-f$ is submodular. A function is called *modular* if f is both submodular and supermodular, i.e., if f satisfies (1.1) with equality. We now define other properties of set functions that will be useful in this thesis. Let $f : 2^V \to \mathbb{R}$, we say that f is *monotone* if $f(A) \leq f(B)$ whenever $A \subseteq B \subseteq V$. A function that is not monotone is called *non-monotone*. The function f is called *non-negative* if $f(S) \geq 0$ for all $S \subseteq V$, and *symmetric* if $f(S) = f(V - S)$ for all $S \subseteq V$. We say that f is *normalized* if $f(\emptyset) = 0$.

1.2 Examples

Submodular functions arise in many applications. In this section we present some classical examples of submodular functions.

Example 1.2.1. ***Modular functions:*** *A function $2^V \to \mathbb{R}$ is modular if $f(A) + f(B) = f(A \cup B) + f(A \cap B)$ for all $A, B \subseteq V$. Modular functions can always be expressed in the form $f(S) = \sum_{e \in S} w(e)$ for some weight function $w : V \to \mathbb{R}$. If $w(e) \geq 0$ for all $e \in V$, then f is also monotone.*

Example 1.2.2. ***Cut functions in graphs:*** *Given an undirected graph $G = (V, E)$ and a non-negative capacity function $c : E \to \mathbb{R}_+$, we define the cut function $f : 2^V \to \mathbb{R}$ of the graph G as $f(S) = c(\delta(S)) = \sum_{e \in \delta(S)} c(e)$. The cut function of an undirected graph is submodular. If $G = (V, A)$ is a directed graph and $c : A \to \mathbb{R}_+$, the cut function $f : 2^V \to \mathbb{R}$ defined as*

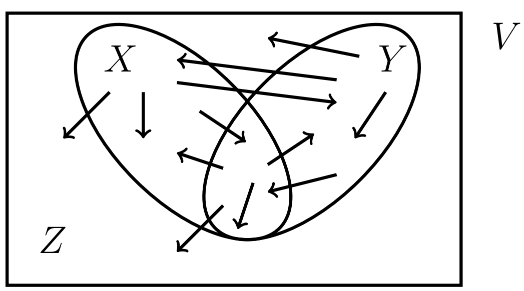
$f(S) = c(\delta^+(S))$ is submodular. In general, cut functions are not monotone.

Lemma 1.2.1. *(a) The cut function of a directed graph is submodular.*

(b) The cut function of an undirected graph is submodular.

Proof. For a directed graph $G = (V, A)$ and $X, Y \subseteq V$, let $E^+(X, Y) := \{(x, y) \in E : x \in X - Y,\, y \in Y - X\}$. We use a counting argument to show the result for directed graphs. Let $Z := V - (X \cup Y)$, and observe that

$$\begin{aligned}
& f(X) + f(Y) \\
= \;& c(\delta^+(X)) + c(\delta^+(Y)) \\
= \;& c(E^+(X, Z)) + c(E^+(X, Y - X)) + c(E^+(Y, Z)) + c(E^+(Y, X - Y)) \\
= \;& c(E^+(X \cup Y, Z)) + c(E^+(X \cap Y, Z)) + c(E^+(X, Y - X)) + c(E^+(Y, X - Y)) \\
= \;& c(\delta^+(X \cup Y)) + c(\delta^+(X \cap Y)) + c(E^+(X, Y)) + c(E^+(Y, X)) \\
\geq \;& c(\delta^+(X \cup Y)) + c(\delta^+(X \cap Y)) \\
= \;& f(X \cup Y) + f(X \cap Y).
\end{aligned}$$



Hence, the cut function of a directed graph is submodular and (a) holds. Finally, (b) follows from (a) by replacing each undirected edge (u, v) by a pair of oppositely directed edges (u, v) and (v, u). $\square$

Example 1.2.3. ***Coverage functions:*** *Given a universe V and sets $S_1, \ldots, S_m \subseteq V$, the coverage function $f : [m] \to \mathbb{Z}_+$ given by $f(A) = |\cup_{i \in A} S_i|$ is monotone* *submodular.*

Example 1.2.4. *Let V be the set of columns of a matrix A. For each $S \subseteq V$, let $r(S)$ be the rank of the matrix formed by the columns in S. Then r is monotone submodular. We will see in Chapter 2 that this is a special case of Example 1.2.5.*

An even more general class is the class of matroids, which we discuss in-depth in Chapter 2.

Example 1.2.5. ***Rank functions of matroids:*** *The rank function of a matroid $\mathcal{M} = (E, \mathcal{I})$, defined as $r_{\mathcal{M}}(A) = \max\{|U| : U \subseteq A, U \in \mathcal{I}\}$, is monotone submodular [1].*

Example 1.2.6. *Let $G = (A \cup B, E)$ be a bipartite graph. For each $S \subseteq A$, let $N(S)$ denote the set of neighbours of S. Then $f(S) := |N(S)|$ is a monotone submodular function on the subsets of A.*

Example 1.2.7. ***Entropy:*** *Let $\Omega = \{X_1, ..., X_n\}$ be a set of random variables, and $H : 2^{\Omega} \to \mathbb{R}$ where $H(S)$ denotes the entropy of the joint distribution of the random variables in S. Then H is monotone submodular [10].*

Example 1.2.8. ***Social Influence:*** *Let V denote the family of nodes in a social network and assume an idea or product is adopted at a set of nodes $S \subseteq V$. The idea propagates through the network following some random diffusion process. Several different diffusion models have been investigated in the literature. The Linear Threshold and Independent Cascade are two of the most basic and widely-studied such models. Let $f(S)$ denote the expected number of* *nodes in the network which end up adopting the idea. Tardos et al. [14] show that for both the Linear Threshold and Independent Cascade models the function f is monotone submodular.*

Example 1.2.9. ***Facility location:*** *Let $V = \{1, ..., n\}$ denote a set of n different locations. We are interested in selecting some of the locations to open up facilities in order to serve a collection of m customers. There is an associated matrix $A \in \mathbb{R}^{m \times n}$ where $A_{i,j}$ denotes the service or value that opening up a facility at location j provides for customer i. If we assume that each customer chooses the facility with highest value, the total value provided to all customers is modeled by the function*

$$f(S) = \sum_{i=1}^{m} \max_{j \in S} A_{i,j}.$$

If the entries of the matrix A are non-negative, f is a monotone submodular function [9].

1.3 Operations that preserve submodularity

There are many natural operations which build new submodular functions from existing ones. In this section we present some of those operations.

Proposition 1.3.1. *Submodularity is preserved under non-negative linear combinations. In other words, if $f_1, ..., f_k : 2^V \to \mathbb{R}$ are submodular functions, and $\alpha_1, ..., \alpha_k \geq 0$, then $f(S) := \sum_{i=1}^{k} \alpha_i f_i(S)$ is submodular.*

Proof. The proof follows straightforward from the definition of submodularity. Let $X, Y \subseteq V$, then

$$\begin{aligned} f(X) + f(Y) &= \sum_{i=1}^{k} \alpha_i f_i(X) + \sum_{i=1}^{k} \alpha_i f_i(Y) \\ &= \sum_{i=1}^{k} \alpha_i [f_i(X) + f_i(Y)] \\ &\geq \sum_{i=1}^{k} \alpha_i [f_i(X \cup Y) + f_i(X \cap Y)] \\ &= \sum_{i=1}^{k} \alpha_i f_i(X \cup Y) + \sum_{i=1}^{k} \alpha_i f_i(X \cap Y) \\ &= f(X \cup Y) + f(X \cap Y), \end{aligned}$$

where the inequality in the third line follows from the fact that $\alpha_1, ..., \alpha_k \geq 0$ and $f_1, ..., f_k : 2^V \to \mathbb{R}$ are submodular functions. $\square$

Definition 1.3.1. *Let $f : 2^V \to \mathbb{R}$ and $A \subseteq V$. The restriction of f to A, denoted by f_A, is a set function defined as $f_A(S) := f(A \cap S)$ for $S \subseteq V$.*

Proposition 1.3.2. *Let $f : 2^V \to \mathbb{R}$ be a submodular function. Then, for any $A \subseteq V$ the restriction f_A is submodular.*

Proof. Fix $A \subseteq V$, and let $X \subseteq Y \subseteq V$ and $e \in V - Y$. Then, either $e \in A$ or $e \notin A$. We show that in both cases condition (1.2) holds for f_A, and hence f_A is submodular. First assume that $e \in A$, then

$$\begin{aligned} f_A(X+e) - f_A(X) &= f((X+e) \cap A) - f(X \cap A) \\ &= f((X \cap A) + e) - f(X \cap A) \\ &\geq f((Y \cap A) + e) - f(Y \cap A) \\ &= f((Y+e) \cap A) - f(Y \cap A) \\ &= f_A(Y+e) - f_A(Y), \end{aligned}$$

where the inequality in the third line follows from the fact that $X \cap A \subseteq Y \cap A$ and that condition (1.2) holds for f.

Now assume that $e \notin A$. In this case the inequality follows trivially since

$$\begin{aligned}
f_A(X+e) - f_A(X) &= f((X+e)\cap A) - f(X \cap A) \\
&= f(X \cap A) - f(X \cap A) \\
&= 0 \\
&= f(Y \cap A) - f(Y \cap A) \\
&= f((Y+e)\cap A) - f(Y \cap A) \\
&= f_A(Y+e) - f_A(Y).
\end{aligned}$$

□

Proposition 1.3.3. *Let $g : 2^V \to \mathbb{R}$ be a monotone submodular function. Then the truncation $f(S) := \min\{g(S), c\}$ remains monotone submodular for any constant $c \in \mathbb{R}$.*

Proof. Let $g : 2^V \to \mathbb{R}$ be a monotone submodular function and fix $c \in \mathbb{R}$. It is clear that $f(S) := \min\{g(S), c\}$ is also monotone. Take an arbitrary set $S \subset V$ and elements $i, j \notin S$. We show that (1.4) holds for f, i.e.,

$$\min\{g(S+i), c\} - \min\{g(S), c\} \geq \min\{g(S+i+j), c\} - \min\{g(S+j), c\}.$$

Note that we have six different cases to consider:

(1) $g(S) \geq c$.

(2) $\min\{g(S+j), g(S+i)\} \geq c \geq g(S)$.

(3) $g(S+j) \geq c \geq g(S+i)$.

(4) $g(S+i) \geq c \geq g(S+j)$.

(5) $g(S+i+j) \geq c \geq \max\{g(S+i), g(S+j)\}$.

(6) $c \geq g(S+i+j)$.

In the first case the inequality holds trivially since

$$\min\{g(S+i+j), c\} = \min\{g(S+i), c\} = \min\{g(S+j), c\} = \min\{g(S), c\} = c.$$

For the second case we have

$$\min\{g(S+i+j), c\} = \min\{g(S+i), c\} = \min\{g(S+j), c\} = c \geq \min\{g(S), c\} = g(S)$$

and after substituting we obtain

$$c - g(S) \geq c - c = 0,$$

which is true since we are assuming $c \geq g(S)$.

For case (3) we get

$$g(S+i) - g(S) \geq c - c = 0,$$

which holds from the monotonicity of g.

After subbing in case (4) we have

$$c - g(S) \geq c - g(S+j),$$

which follows again from the monotonicity of g.

For case (5) we get

$$g(S+i) - g(S) \geq g(S+i+j) - g(S+j) \geq c - g(S+j),$$

where the first inequality follows from the submodularity of the function g.

In the last case the inequality follows straightforward from the submodularity of g, i.e.,

$$g(S+i) - g(S) \geq g(S+i+j) - g(S+j).$$

$\square$

Proposition 1.3.4. *Submodularity is preserved under reflection: Let $g : 2^V \to \mathbb{R}$ be a submodular function. Then, the function $f : 2^V \to \mathbb{R}$ defined as $f(S) = g(V - S)$ for $S \subseteq V$ is submodular.*

Proof. The proof follows straightforward from the submodularity of g. Let $X, Y \subseteq V$, then

$$\begin{aligned} f(X) + f(Y) &= g(V - X) + g(V - Y) \\ &\geq g((V - X) \cap (V - Y)) + g((V - X) \cup (V - Y)) \\ &= g(V - (X \cup Y)) + g(V - (X \cap Y)) \\ &= f(X \cup Y) + f(X \cap Y). \end{aligned}$$

$\square$

Definition 1.3.2. *Let $f : 2^V \to \mathbb{R}$ and $A \subseteq V$. The contraction of f on A, denoted by f^A, is a set function defined as $f^A(S) := f(S \cup A)$ for $S \subseteq V$.*

Proposition 1.3.5. *Let $f : 2^V \to \mathbb{R}$ be a submodular function. Then, for any $A \subseteq V$ the contraction f^A is submodular.*

Proof. Fix $A \subseteq V$ and let f^A be the contraction of f on A. Let $X \subseteq Y \subseteq V$ and $e \notin Y$. Then, either $e \in A$ or $e \notin A$. We show that in either case f^A satisfies condition (1.2), and hence f^A is submodular. First assume that $e \notin A$, then we have

$$\begin{aligned}
f^A(X+e) - f^A(X) &= f((X+e)\cup A) - f(X\cup A) \\
&= f((X\cup A)+e) - f(X\cup A) \\
&\geq f((Y\cup A)+e) - f(Y\cup A) \\
&= f((Y+e)\cup A) - f(Y\cup A) \\
&= f^A(Y+e) - f^A(Y),
\end{aligned}$$

where the inequality in the third line follows from the fact that $X \cup A \subseteq Y \cup A$ and that condition (1.2) holds for f. Notice that here we are using that $e \notin A$ since in order to apply condition (1.2) we need to have $e \notin Y \cup A$.

Now assume that $e \in A$. In this case the inequality follows trivially since

$$\begin{aligned}
f^A(X+e) - f^A(X) &= f((X+e)\cup A) - f(X\cup A) \\
&= f(X\cup A) - f(X\cup A) \\
&= 0 \\
&= f(Y\cup A) - f(Y\cup A) \\
&= f((Y+e)\cup A) - f(Y\cup A) \\
&= f^A(Y+e) - f^A(Y).
\end{aligned}$$

□

Proposition 1.3.6. *Let $g : 2^V \to \mathbb{R}$ be a submodular function and $A, B \subseteq V$ any two disjoint sets. Then, the function $f : 2^A \to \mathbb{R}$ defined via $f(S) = g(S \cup B) - g(B)$ is submodular.*

Proof. Fix any two disjoint sets $A, B \subseteq V$. We show that f satisfies condition (1.2). Let $X \subseteq Y \subseteq A$ and $e \in A - Y$. Then

$$\begin{aligned}
f(X+e) - f(X) &= [g((X+e) \cup B) - g(B)] - [g(X \cup B) - g(B)] \\
&= g((X+e) \cup B) - g(X \cup B) \\
&= g((X \cup B) + e) - g(X \cup B) \\
&\geq g((Y \cup B) + e) - g(Y \cup B) \\
&= g((Y+e) \cup B) - g(Y \cup B) \\
&= [g((Y+e) \cup B) - g(B)] - [g(Y \cup B) - g(B)] \\
&= f(Y+e) - f(Y),
\end{aligned}$$

where the inequality in the fourth line follows from the fact that $X \cup B \subseteq Y \cup B$ and that condition (1.2) holds for g. □

Proposition 1.3.7. *The convolution of a submodular function and a modular function is submodular: Let $g : 2^V \to \mathbb{R}$ be a submodular function, and $z : 2^V \to \mathbb{R}$ be a modular function. Then, $f(S) := \min_{U \subseteq S} g(U) + z(S - U)$ for $S \subseteq V$ is submodular.*

Proof. Let $A, A' \subseteq V$, and B, B' the corresponding minimizers defining $f(A)$ and $f(A')$. We show that f satisfies condition (1.1),

$$\begin{aligned}
& f(A) + f(A') \\
= \; & [g(B) + z(A - B)] + [g(B') + z(A' - B')] \\
= \; & g(B) + g(B') + z(A - B) + z(A' - B') \\
\geq \; & g(B \cup B') + g(B \cap B') + z(A - B) + z(A' - B') \\
= \; & g(B \cup B') + g(B \cap B') + z((A - B) \cup (A' - B')) + z((A - B) \cap (A' - B')) \\
= \; & g(B \cup B') + g(B \cap B') + z((A \cup A') - (B \cup B')) + z((A \cap A') - (B \cap B'))
\end{aligned}$$

$$
\begin{aligned}
&= [g(B \cup B') + z((A \cup A') - (B \cup B'))] + [g(B \cap B') + z((A \cap A') - (B \cap B'))] \\
&\geq f(A \cup A') + f(A \cap A'),
\end{aligned}
$$

where the first inequality follows from the submodularity of g, and the second inequality from the definition of f. The equality signs in the fourth and fifth lines follow from the fact that z is modular and from the relations

$$(A - B) \cup (A' - B') = [(A \cap A') - (B \cap B')] \cup [(A \cup A') - (B \cup B')]$$

and

$$(A - B) \cap (A' - B') = [(A \cap A') - (B \cap B')] \cap [(A \cup A') - (B \cup B')].$$

$\square$

A function $f : \mathbb{R}^n \to \mathbb{R}$ is *concave* if the line segment joining any two points on the graph is never above the graph. More precisely,

Definition 1.3.3. *(Concavity) A function $h : K \subseteq \mathbb{R}^n \to \mathbb{R}$ defined on a convex set K is concave if for any two points $x, y \in K$ and $\lambda \in [0, 1]$ we have*

$$\lambda f(x) + (1 - \lambda) f(y) \leq f(\lambda x + (1 - \lambda) y).$$

The following lemma is a property of concave functions that will be useful to prove some of the relationships between concavity and submodularity.

Lemma 1.3.1. *Let $g : \mathbb{R}_+ \to \mathbb{R}$ be a concave function and $a \geq 0$. Then, the function $f(t) := g(a + t) - g(t)$ is non-increasing.*

Proposition 1.3.8. *Let $z : 2^V \to \mathbb{R}_+$ be a modular function given by non-negative weights, i.e., there exists $w : V \to \mathbb{R}_+$ such that $z(S) = \sum_{e \in S} w(e)$ for all $S \subseteq V$. Let $g : \mathbb{R}_+ \to \mathbb{R}$ be a concave function. Then, $f : 2^V \to \mathbb{R}$ defined as $f(S) = g(z(S))$ for $S \subseteq V$ is submodular.*

Proof. Let $A \subseteq V$ and $i, j \notin A$. We use Lemma 1.3.1 to show that f satisfies condition (1.4).

Let $a = w(i)$, $t_1 = z(A) = \sum_{e\in A} w(e)$, and $t_2 = z(A+j) = w(j) + \sum_{e\in A} w(e)$. Since w is a non-negative function, we have that $w(i), w(j) \geq 0$. In particular, $a \geq 0$ and $t_1 \leq t_2$. Hence, by Lemma 1.3.1 we have

$$g(a + t_1) - g(t_1) \geq g(a + t_2) - g(t_2).$$

But notice that

$$\begin{aligned} g(a + t_1) - g(t_1) &= g(w(i) + \sum_{e\in A} w(e)) - g(\sum_{e\in A} w(e)) \\ &= g(z(A + i)) - g(z(A)) \end{aligned}$$

and

$$\begin{aligned} g(a + t_2) - g(t_2) &= g(w(i) + w(j) + \sum_{e\in A} w(e)) - g(w(j) + \sum_{e\in A} w(e)) \\ &= g(z(A + i + j)) - g(z(A + j)). \end{aligned}$$

Thus, condition (1.4) follows and f is submodular. □

Proposition 1.3.9. *The composition of a monotone concave function and a monotone submodular function is submodular. Let $g : 2^V \to \mathbb{R}$ be a monotone submodular function with values in a convex set $K \subseteq \mathbb{R}$, and $h : K \to \mathbb{R}$ a monotone concave function. Then, the composition $f(S) := h(g(S))$ for $S \subseteq V$ is submodular.*

Proof. We show that f satisfies condition (1.4). Let $A \subseteq V$ and $i, j \notin A$. Then, since g is submodular, we have that

$$g(A + i) - g(A) \geq g(A + i + j) - g(A + j),$$

and hence

$$g(A+i+j) \leq g(A+i) - g(A) + g(A+j).$$

Now, it follows from the monotonicity of h,

$$\begin{aligned}
& h(g(A+i+j)) \\
\leq \ & h(g(A+i) - g(A) + g(A+j)) \\
= \ & h(g(A+i) - g(A) + g(A+j)) - h(g(A+j)) + h(g(A+j)).
\end{aligned}$$

And by Lemma 1.3.1 we get

$$\begin{aligned}
& h(g(A+i) - g(A) + g(A+j)) - h(g(A+j)) \\
\leq \ & h(g(A+i) - g(A) + g(A)) - h(g(A)) \\
= \ & h(g(A+i)) - h(g(A)).
\end{aligned}$$

Thus,

$$\begin{aligned}
& f(A+i+j) \\
= \ & h(g(A+i+j)) \\
\leq \ & h(g(A+i) - g(A) + g(A+j)) - h(g(A+j)) + h(g(A+j)) \\
\leq \ & h(g(A+i)) - h(g(A)) + h(g(A+j)) \\
= \ & f(A+i) - f(A) + f(A+j),
\end{aligned}$$

and the proposition follows. □

It is not true in general that the minimum or the maximum of two submodular functions is also submodular. However, in the case when the difference of the two functions is an increasing or decreasing function we have the following result [22]:

Proposition 1.3.10. *Let f and g be two submodular functions such that $f-g$ is either increasing or drecreasing. Then* $\min\{f,g\}$ *is also submodular.*

CHAPTER 2
Matroids and Submodularity

2.1 Definitions and examples

One of the most studied class of submodular functions are the so-called rank functions of matroids. Matroids were introduced by Whitney [26] in 1935 to provide a unifying abstract treatment of dependence in linear algebra and graph theory. Whitney's definition embraces a surprising diversity of combinatorial structures. Moreover, matroids arise naturally in combinatorial optimization since they are precisely the structures for which the greedy algorithm works. There is a strong relationship between matroids and submodularity. In fact, we will see in this chapter that matroids can be defined in terms of submodular functions.

Definition 2.1.1. *A matroid is a pair $(E, \mathcal{I})$ consisting of a finite set E with a non-empty collection $\mathcal{I}$ of subsets of E, called independent sets, such that*

(I1) If $X \subseteq Y$ and $Y \in \mathcal{I}$ then $X \in \mathcal{I}$.

(I2) If $X, Y \in \mathcal{I}$ and $|Y| > |X|$ then $\exists e \in Y - X$ such that $X + e \in \mathcal{I}$.

Axiom (I1) is usually called the *hereditary property*, while (I2) is usually called the *greedy property*. The sets in 2^E that are not independent are called *dependent*. Minimal dependent sets are called *circuits*, and maximal independent sets are called *bases*.

We now present several examples of matroids, in some cases also describing their bases and circuits.

Example 2.1.1. ***Uniform matroid:*** *A very simple class of matroids is given by the uniform matroids. A uniform matroid is a pair* $\mathcal{M} = (E, \mathcal{I})$ *in which*

$$\mathcal{I} = \{X \subseteq E : |X| \leq k\}$$

for some $1 \leq k \leq |E|$*. This is usually denoted by* $U_{k,n}$ *where* $|E| = n$*. It is easy to see that a base of* $U_{k,n}$ *is any set of size* k*, while a circuit is any set of size* $k+1$*.*

Example 2.1.2. ***Matroid restriction:*** *Let* $\mathcal{M} = (E, \mathcal{I})$ *be a matroid, and let* $U \subseteq E$*. Then* $\mathcal{M}' = (U, \mathcal{I}')$ *where* $\mathcal{I}' = \{I \in \mathcal{I} : I \subseteq U\}$ *is a matroid.* $\mathcal{M}'$ *is usually denoted by* $\mathcal{M}|_U$ *and it is called the restriction of* $\mathcal{M}$ *to* U*. The bases of* $\mathcal{M}|_U$ *are the maximum independent sets of* $\mathcal{M}$ *contained in* U*, and its circuits are the circuits of* $\mathcal{M}$ *that are contained in* U*.*

Let us check that $\mathcal{M}|_U$ is indeed a matroid. Let $Y \in \mathcal{I}'$ and $X \subseteq Y$. Since $Y \in \mathcal{I}'$, we have that $Y \in \mathcal{I}$ and $Y \subseteq U$. It follows from the hereditary property of $\mathcal{M}$ that $X \in \mathcal{I}$. Also, since $X \subseteq Y$ we have that $X \subseteq U$. Thus $X \in \mathcal{I}'$ and (I1) holds. To see (I2) let $X, Y \in \mathcal{I}'$ with $|Y| > |X|$. Then $X, Y \in \mathcal{I}$ and $X, Y \subseteq U$. It follows from the greedy property of $\mathcal{M}$ that $\exists e \in Y - X$ such that $X + e \in \mathcal{I}$. Moreover, since $Y \subseteq U$ we have that $e \in U$, and hence $X + e \in \mathcal{I}'$.

Example 2.1.3. ***Partition matroid:*** *Let $E_1, ..., E_m$ be a partition of the ground set E, and*

$$\mathcal{I} = \{X \subseteq E : |X \cap E_i| \leq k_i, \ \forall i \in [m]\}$$

for some numbers $1 \leq k_i \leq |E_i|$ with $i \in [m]$. Then $\mathcal{M} = (E, \mathcal{I})$ is called a partition matroid. Notice that this generalizes uniform matroids. It is not hard to see that the bases in this case are sets $X \subseteq E$ satisfying the cardinality constraints with equality, i.e., sets $X \subseteq E$ such that $|X \cap E_i| = k_i$ for all $i \in [m]$. The circuits are sets $X \subseteq E_i$ such that $|X| = k_i + 1$ for some $i \in [m]$.

Let us check that $\mathcal{M}$, as defined above, is indeed a matroid, i.e., that it satisfies axioms (I1) and (I2) from Definition 2.1.1. The hereditary property holds almost trivially, since if $X \subseteq Y$ and $|Y \cap E_i| \leq k_i$ for all $i \in [m]$, it is clear that $|X \cap E_i| \leq k_i$ for all $i \in [m]$ as well. To see that the greedy property holds, let $X, Y \in \mathcal{I}$ with $|Y| > |X|$. Then, since $E_1, ..., E_m$ is a partition of E we must have that $|X \cap E_{i_0}| < |Y \cap E_{i_0}| \leq k_{i_0}$ for some $i_0 \in [m]$. Let $e \in (Y \cap E_{i_0}) - (X \cap E_{i_0})$. Then, we have

$$|(X + e) \cap E_i| = |X \cap E_i| \leq k_i \quad \forall i \neq i_0$$

and

$$|(X + e) \cap E_{i_0}| = |X \cap E_{i_0}| + 1 \leq |Y \cap E_{i_0}| \leq k_{i_0}.$$

Hence, $X + e \in \mathcal{I}$. Moreover, since $e \in (Y \cap E_{i_0}) - (X \cap E_{i_0})$, we have that $e \in Y - X$, and this concludes the proof.

Notice that the disjointness condition on the sets E_i is key and cannot be relaxed. A simple example to see why this could fail is the following: Let $E = \{1, 2, 3\}$, $E_1 = \{1, 2\}$, $E_2 = \{2, 3\}$, and $k_1 = 1$, $k_2 = 1$. Define $\mathcal{I}$ as above. Then the sets $Y = \{1, 3\}$ and $X = \{2\}$ are independent and $|Y| > |X|$.

However, for each $e \in Y - X$ we have $X + e \notin \mathcal{I}$. Hence, axiom (I2) does not hold and $\mathcal{M} = (E, \mathcal{I})$ is not a matroid.

Example 2.1.4. ***Graphic matroid:*** *A very important class of matroids in combinatorial optimization is the class of graphic matroids. Given an undirected graph $G = (V, E)$ with $|V| = n$, a graphic matroid is a pair $\mathcal{M} = (E, \mathcal{I})$ where*

$$\mathcal{I} = \{F \subseteq E : F \text{ is a forest in } G\}.$$

It is usually denoted by $\mathcal{M}(G)$. The bases of a graphic matroid correspond to the spanning forests of the graph G. If the graph is connected, the bases are the spanning trees of G, i.e., forests of size $n-1$. The circuits correpond to the cycles of G.

To see that a pair $\mathcal{M} = (E, \mathcal{I})$ defined as above is indeed a matroid, first notice that the hereditary property is trivially satisfied. Indeed, if F_2 is a forest in G (i.e., $F_2 \in \mathcal{I}$), any set of edges $F_1 \subseteq F_2$ will also be a forest since removing edges does not create cycles. Now, let us show that the greedy property holds. Let $F_1, F_2 \in \mathcal{I}$ with $|F_2| > |F_1|$. Then, $n - 1 \geq |F_2| > |F_1|$, and hence F_1 does not induce a connected component. Let $C_1, ..., C_k$ be its connected components. Then F_1 induces a spanning tree on each component C_i, and thus $|F_1| = \sum_{i=1}^{k}(|C_i| - 1) = n - k$. Also, since $|F_2| > |F_1|$, F_2 cannot only use edges with both endpoints in some component C_i; if it did, F_2 would also have at least k connected components, and then $|F_2| \leq n - k = |F_1|$ contradicting the fact that $|F_2| > |F_1|$. Thus, there exists $e \in F_2$ with endpoints in distinct components C_i, i.e., $e \in F_2 - F_1$. It follows that $F_1 + e$ is again a forest and so $F_1 + e \in \mathcal{I}$ as we wanted to show.

Although graphic matroids are generally defined for undirected graphs, they can also be associated with directed graphs by disregarding the orientations of the arcs. That is, given a digraph $D = (V, A)$ we can define the graphic matroid associated to D, denoted by $\mathcal{M}(D)$, as $\mathcal{M} = (A, \mathcal{I})$ where

$$\mathcal{I} = \{F \subseteq A : F \text{ when viewed as undirected edges induces a forest on } V\}.$$

Example 2.1.5. ***Linear matroid:*** *Given a matrix A, let E denote the index set of the columns of A. For a subset $X \subseteq E$, let A_X denote the submatrix of A consisting only of those columns indexed by X. Then, define*

$$\mathcal{I} = \{X \subseteq E : rank(A_X) = |X|\},$$

i.e., a set X is independent in $\mathcal{M} = (E, \mathcal{I})$ if and only if the corresponding columns in the matrix A are linearly independent. Notice that in this case a basis B of $\mathcal{M}$ correspond to linearly independent set of columns of cardinality $rank(A)$.

Let us show that $\mathcal{M}$ is indeed a matroid: Assume that $Y \in \mathcal{I}$ and $X \subseteq Y$. If the set of columns corresponding to Y are linearly independent, any subset of them will also be linearly independent. Hence, $X \in \mathcal{I}$. To show that (I2) holds, let $X, Y \in \mathcal{I}$ and $|Y| > |X|$. Since the columns corresponding to both X and Y are linearly independent, it follows that the column space of the matrix A_X has dimension $|X|$ and the column space of the matrix A_Y has dimension $|Y|$. Moreover, since $|Y| > |X|$, by a fundamental linear algebra property we know that there exists a column of A_Y that is not in the column space of A_X. Adding this column to A_X increases the rank by 1, and thus property (I2) follows.

Example 2.1.6. ***Matching matroid:*** *Let $G = (V, E)$ be a graph and consider $\mathcal{M} = (V, \mathcal{I})$ where*

$$\mathcal{I} = \{S \subseteq V : S \text{ is covered by some matching } M\}.$$

Notice that in the definition the matching M does not need to cover exactly S, other nodes can be also covered. Then $\mathcal{M}$ is a matroid, usually called the matching matroid.

Axiom (I1) is trivial since if $X \subseteq Y$ and $Y \in \mathcal{I}$, there exists a matching M covering Y, but then M also covers X and hence $X \in \mathcal{I}$. To show that (I2) holds let $X, Y \in \mathcal{I}$ with $|Y| > |X|$, and let M be a matching covering X and M' a matching covering Y. Consider $M \triangle M'$, and notice that each induced connected component alternates between M and M' edges. Moreover, since $|M'| > |M|$, some component must have more edges from M' than from M. We perform an augmentation on such a component, and we get a new matching M'' that covers Z, where $Z \supset X$ and $Z \subseteq Y$. Hence, $\exists e \in Y - X$ such that $X + e \in \mathcal{I}$ and axiom (I2) follows.

A matching matroid can also be defined for a ground set $J \subset V$, i.e., the pair $\mathcal{M} = (J, \mathcal{I})$ where

$$\mathcal{I} = \{S \subseteq J : S \text{ is covered by some matching } M\}$$

is still a matching matroid. Here the matchings are allowed to involve nodes from outside J. A special case of this is our next example.

Example 2.1.7. ***Transversal matroid:*** *Let $G = (A \cup B, E)$ be a bipartite graph with bipartition A and B. Consider $\mathcal{M} = (A, \mathcal{I})$ where*

$$\mathcal{I} = \{I \subseteq A : I \text{ is covered by some matching } M\}.$$

Then M is a matroid, usually called a transversal matroid.

Example 2.1.8. ***Laminar matroid:*** *Let $\mathcal{F}$ be a laminar family of subsets over E, i.e., if $A, B \in \mathcal{F}$ then, either $A \cap B = \emptyset$, or $A \subseteq B$, or $B \subseteq A$. Assume that each $x \in E$ is in some set $A \in \mathcal{F}$, and let $k : \mathcal{F} \to \mathbb{Z}_+$ be non-negative weights on the sets of the family $\mathcal{F}$. Then $\mathcal{M} = (E, \mathcal{I})$ is a laminar matroid, where*

$$\mathcal{I} = \{X \subseteq E : |X \cap A| \leq k(A),\ \forall A \in \mathcal{F}\}.$$

Notice that laminar matroids generalize partition matroids, which in turn (as mentioned above) generalize uniform matroids.

Example 2.1.9. ***Gammoid:*** *Let $G = (V, E)$ be a graph and let $S, T \subseteq V$. Then, the pair $\mathcal{M} = (E, \mathcal{I})$ where*

$$\mathcal{I} = \{X \subseteq S : \textit{ there exist } |X| \textit{ node-disjoint paths from } X \textit{ to } T\}$$

is a matroid, and it is called a gammoid.

Transversal matroids can be seen as a particular case of gammoids. Indeed, given a bipartite graph $G = (A \cup B, E)$, consider the gammoid defined on G by taking $S = A$ and $T = B$, i.e., $\mathcal{M} = (A, \mathcal{I})$ where

$$\mathcal{I} = \{X \subseteq A : \text{ there exist } |X| \text{ node-disjoint paths from } X \text{ to } B\}.$$

Given that G is a bipartite graph with bipartition $A \cup B$ we can rewrite the independent sets as

$$\begin{aligned} \mathcal{I} &= \{X \subseteq A : \text{ there exist } |X| \text{ node-disjoint paths from } X \text{ to } B\} \\ &= \{X \subseteq A : \text{ there exist } |X| \text{ node-disjoint edges from } X \text{ to } B\} \end{aligned}$$

$$= \{X \subseteq A : X \text{ is covered by some matching } M\}.$$

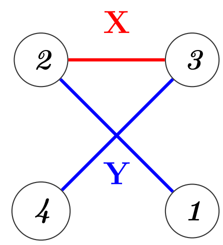
It follows that the gammoid $\mathcal{M}$ defined this way is just the transversal matroid on the bipartite graph $G = (A \cup B, E)$ with ground set A.

We have seen so far that transversal matroids are particular examples of both gammoids and matching matroids. Also, that laminar matroids generalize partition matroids, which in turn generalize uniform matroids. We summarize these relations in the diagram below, where an arrow from A to B means that A generalizes B (or equivalently, that B is a particular case of A).

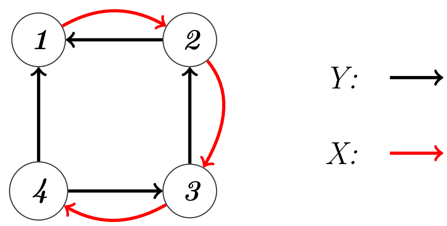
$$\begin{tikzcd}
\text{Gammoid} \arrow[dr] & & \text{Matching} \arrow[dl] \\
& \text{Transversal} &
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
\text{Laminar} \arrow[d] \\
\text{Partition} \arrow[d] \\
\text{Uniform}
\end{tikzcd}$$

We conclude this section by showing a couple of examples of pairs $(E, \mathcal{I})$ that are not matroids.

Example 2.1.10. *Let $G = (V, E)$ be a graph and consider the pair $\mathcal{M} = (E, \mathcal{I})$ where $\mathcal{I} = \{F \subseteq E : F$ is a matching in $G\}$. It is easy to see that axiom (I1) is satisfied since any subset of a matching is also a matching. However, $\mathcal{M}$ is not a matroid in general since axiom (I2) does not necessarily hold. To see this take a graph on four nodes, i.e., $V = \{1, 2, 3, 4\}$, and let $X = \{(2, 3)\}$ and $Y = \{(1, 2), (3, 4)\}$ (see picture below). It is clear that both X and Y are matchings, but there is no edge of Y that can be added to X and still have a matching.*



Example 2.1.11. *Let $D = (V, A)$ be a directed graph and consider $\mathcal{I} = \{F \subseteq A : F$ has no directed cycles$\}$. Then, $\mathcal{M} = (A, \mathcal{I})$ is not a matroid in general. For instance, let G be the complete graph on four nodes, and let $X = \{(1,2),(2,3),(3,4)\}$ and $Y = \{(2,1),(3,2),(4,3),(4,1)\}$ (see picture below). It is easy to check that there is no arc e in $Y - X$ such that $X + e \in \mathcal{I}$, i.e., such that $X + e$ has no directed cycles.*



2.2 Rank function

The *rank function* of a matroid $\mathcal{M} = (E, \mathcal{I})$, denoted by either $r(\cdot)$ or $r_{\mathcal{M}}(\cdot)$, is defined as a function $r : 2^E \to \mathbb{Z}_+$ such that $r(X) = \max\{|U| : U \subseteq X,\ U \in \mathcal{I}\}$. This is, the rank of a set of elements $X \subseteq E$, denoted by $r(X)$, is the size of a maximal independent subset of X. The rank of the matroid $\mathcal{M}$ is defined to be the rank of all of E. We now describe the rank function of some of the examples of matroids mentioned in Section 2.1.

Matroid restriction: Let $\mathcal{M} = (E, \mathcal{I})$ be a matroid and consider the restriction $\mathcal{M}|_U$ for some set $U \subseteq E$. Then the rank function of $\mathcal{M}|_U$ is that of $\mathcal{M}$ restricted to subsets of U, i.e., $r_{\mathcal{M}|_U}(S) = r_{\mathcal{M}}(S)$ for all $S \subseteq U$.

Linear Matroid: Let $\mathcal{M}$ be a linear matroid defined from a matrix A. Then the rank of a set $X \subseteq E$ corresponds to the rank of the matrix A_X in the linear algebra sense (i.e., number of linearly independent columns).

Partition Matroid: The rank function of a partition matroid $\mathcal{M} = (E, \mathcal{I})$ with $\mathcal{I} = \{X \subseteq E : |X \cap E_i| \leq k_i,\ \forall i \in [m]\}$ is given by

$$r(X) = \sum_{i=1}^{m} \min\{|X \cap E_i|, k_i\}.$$

Graphic matroid: Let $\mathcal{M} = (E, \mathcal{I})$ be a graphic matroid defined on a graph $G = (V, E)$ with $|V| = n$. Then, the rank function of $\mathcal{M}$ is given by

$$r(X) = n - k(V, X),$$

where $k(V, X)$ denotes the number of connected components of the graph induced by X.

There is a strong relationship between matroids and submodularity. The next two propositions show that the rank function of a matroid is submodular, and that in fact, matroids can be defined in terms of rank functions.

Proposition 2.2.1. *Let r be the rank function of some matroid $M = (E, \mathcal{I})$. Then r satisfies the following:*

1. $\forall X \subseteq E,\ 0 \leq r(X) \leq |X|$.

2. r is monotone, i.e., $\forall X \subseteq Y,\ r(X) \leq r(Y)$.

3. r is submodular, i.e., $\forall X, Y \subseteq E,\ r(X) + r(Y) \geq r(X \cup Y) + r(X \cap Y)$.

Proof. The first two properties follow straightforward from the definition of rank function. Proving submodularity requires some work.

Consider $X, Y \subseteq E$, and let J be a maximum independent set in $X \cap Y$, i.e., $|J| = r(X \cap Y)$. Extend J to a maximum independent set in X and call it J_X. Finally, extend J_X to a maximum independent set J_{XY} in $X \cup Y$.

Now, notice that since J was a maximum independent set in $X \cap Y$, we can only add elements of $X - Y$ to grow J_X. Hence, $J_X - Y = J_X - J$. In the same way, given that J_X is a maximum independent set in X, we can only add elements of $Y - X$ to grow J_{XY}. It follows that $J_{XY} \cap Y = J_{XY} - (J_X - Y)$, and using the fact that $J_X - Y = J_X - J$ we get $J_{XY} \cap Y = J_{XY} - (J_X - J)$. Thus,

$$\begin{aligned}
r(X) + r(Y) &\geq |J_X| + |J_{XY} \cap Y| && (r(Y) \geq |J_{XY} \cap Y|) \\
&= |J_X| + |J_{XY} - (J_X - J)| \\
&= |J_X| + |J_{XY}| - |J_X| + |J| && (J \subseteq J_X \subseteq J_{XY}) \\
&= |J_{XY}| + |J| \\
&= r(X \cup Y) + r(X \cap Y)
\end{aligned}$$

and the submodularity of r follows. □

Proposition 2.2.2. *Let $r : 2^E \to \mathbb{Z}_+$ be a monotone submodular function satisfying $0 \leq r(X) \leq |X|$, $\forall X \subseteq E$. Then r defines a matroid on E by setting*

$$\mathcal{I} = \{I \subseteq E : r(I) = |I|\}.$$

Proof. We check that axioms (I1) and (I2) hold for the set family $\mathcal{I}$. Let $Y \in \mathcal{I}$ and $X \subseteq Y$. Since $Y \in \mathcal{I}$, we know that $r(Y) = |Y|$. Also, by assumption we have that $r(X) \leq |X|$. Assume $r(X) < |X|$, i.e., $X \notin \mathcal{I}$. Then, from the submodularity of r we have

$$
\begin{aligned}
r(Y) &= r(X \cup (Y - X)) \\
&\leq r(X) + r(Y - X) - r(X \cap (Y - X)) \\
&< |X| + |Y - X| - r(\emptyset) \\
&= |Y|,
\end{aligned}
$$

contradicting the fact that $r(Y) = |Y|$. Hence $r(X) = |X|$ and $X \in \mathcal{I}$.

Now, consider $X, Y \in \mathcal{I}$ with $|Y| > |X|$. Since $X, Y \in \mathcal{I}$, we know that $r(X) = |X|$ and $r(Y) = |Y|$. Assume that there is no element $e \in Y - X$ such that $X + e \in \mathcal{I}$, i.e., $\forall e \in Y - X$ we have $r(X + e) < |X + e| = |X| + 1$. Enumerate the elements of $Y - X$ as $\{e_1, ..., e_k\}$, and notice that by the submodularity of r we have

$$
\begin{aligned}
r(X + e_1) - r(X) &\geq r(X + e_1 + e_2) - r(X + e_1) \\
&\geq r((X \cup \{e_1, e_2\}) + e_3) - r(X \cup \{e_1, e_2\}) \\
&\geq ... \\
&\geq r((X \cup \{e_1, ..., e_{k-1}\}) + e_k) - r(X \cup \{e_1, ..., e_{k-1}\}) \\
&\geq r(Y) - r(X \cup \{e_1, ..., e_{k-1}\}).
\end{aligned}
$$

Since $|X| = r(X) \leq r(X + e_1) < |X| + 1$, we have that $r(X + e_1) = |X| = r(X)$, and hence $r(X + e_1) - r(X) = 0$. Moreover, from the monotonicity of r we know that $r(Y) - r(X \cup \{e_1, ..., e_{k-1}\}) \geq 0$. It follows that

$$
\begin{aligned}
0 &= r(X + e_1) - r(X) \\
&= r(X + e_1 + e_2) - r(X + e_1) \\
&= ... \\
&= r(Y) - r(X \cup \{e_1, ..., e_{k-1}\}).
\end{aligned}
$$

Hence,

$$r(Y) = r(X \cup \{e_1, ..., e_{k-1}\}) = ... = r(X + e_1) = r(X) = |X|,$$

contradicting the fact that $r(Y) = |Y| > |X|$.

Thus, there must exist some element $e \in Y - X$ such that $r(X + e) = |X + e|$, i.e., $X + e \in \mathcal{I}$. It follows that the set family $\mathcal{I}$ satisfies axiom (I2) and this completes the proof. $\square$

The above two propositions imply our next theorem [26], which gives a full characterization of rank functions.

Theorem 2.2.1. *A function $r : 2^E \to \mathbb{Z}_+$ is the rank function of a matroid if and only if r is a monotone submodular function satisfying $0 \leq r(X) \leq |X|$, $\forall X \subseteq E$.*

2.3 Geometric representation of matroids of small rank

Matroids of small rank (e.g., 2,3 or 4) can be represented geometrically. In such diagrams the elements of the ground set E are represented as points following these basic rules:

(1) A dependent element is marked in a single inset.

(2) Two elements that form a circuit are represented by touching points.

(3) If three elements form a circuit, the corresponding points are collinear.

(4) If four elements form a circuit, the corresponding points are coplanar.

In such representations, the lines need not be straight and the planes may be twisted. Let us start with a simple example to show how rules (1)-(2)-(3) from above are applied.

Example 2.3.1. *Let $\mathcal{M} = (E, \mathcal{I})$ be the linear matroid induced by the matrix*

$$A = \begin{bmatrix} 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 \end{bmatrix}.$$

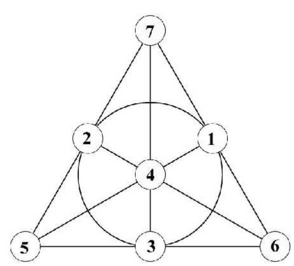
In this case we can explicitly write all the independent sets of $\mathcal{M}$. Indeed, given $E = \{1, 2, 3, 4, 5\}$ we have that

$$\mathcal{I} = \{\emptyset, \{1\}, \{2\}, \{4\}, \{5\}, \{1, 2\}, \{1, 5\}, \{2, 4\}, \{2, 5\}, \{4, 5\}\}.$$

Notice that $\mathcal{M}$ is a matroid of rank 2. Following the above rules we get that the geometric representation of $\mathcal{M}$ is given by the following diagram,

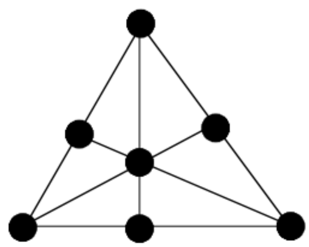
Now we present some examples of important matroids of rank 3 that are usually described through their geometric representations.

Example 2.3.2. *The **Fano matroid**, usually denoted by F_7, is a matroid on a ground set of 7 elements which has the following representation*

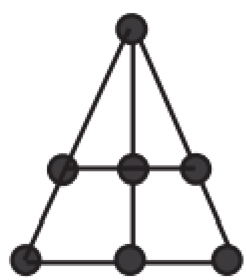


The line $\{1, 2, 3\}$ is represented in the diagram by a circle. All sets of cardinality 3 are independent, except those corresponding to a line in the diagram. The Fano plane has several nice properties. For instance, every point is on *exactly three 3-point lines and every line has exactly three points. The rank of the matroid is* 3*, since the set* $\{5, 6, 7\}$ *is independent (i.e.,* $5, 6,$ *and* 7 *are not collinear), but all sets of cardinality* 4 *are dependent (all the points in the diagram are coplanar). The Fano matroid is a special case of matroids arising from projective planes.*

Example 2.3.3. *The* ***non-Fano matroid*** *is obtained from the Fano matroid by deleting the line* $\{1, 2, 3\}$*. It is usually denoted by* F_7^- *and its geometric representation is given by*



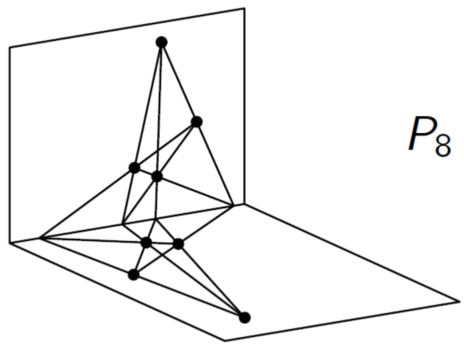
Example 2.3.4. *Another important matroid that arises from the Fano plane is* $\mathbf{P_7}$*, which can be represented as*



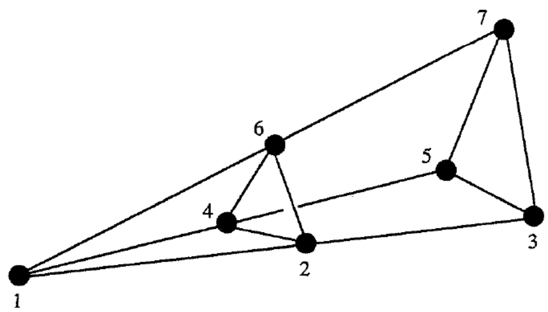
P_7

The next examples are geometric representations of matroids of rank 4.

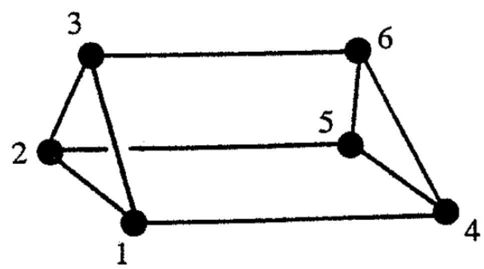
Example 2.3.5. *The matroid* $\mathbf{P_8}$ *can be represented geometrically as*



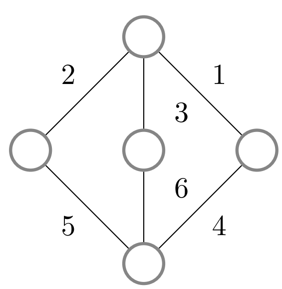
Example 2.3.6. *The **Escher matroid** has the following geometric representation [4],*



Example 2.3.7. *The following representation*



corresponds to the graphic matroid $\mathcal{M}(G)$, where the graph G is given by



Indeed, from the diagram we see that the only dependent subsets of E with fewer than five elements are the three planes $\{1,2,4,5\}$, $\{1,3,4,6\}$ and $\{2,3,5,6\}$.

Notice that we do not have any dependent set of size 3 since there are no three collinear points in the representation.

2.4 Matroid optimization

In this section we consider the fundamental problem of finding an independent set of maximum weight. Notice that this generalizes optimization problems associated with each of our matroid examples, most notably perhaps graphic matroids, where the underlying optimization problem is to find a maximum spanning tree. We show that the classical greedy algorithm solves this problem, and that in fact, the greedy algorithm characterizes matroids.

Assume we are given a matroid $\mathcal{M} = (E, \mathcal{I})$ and weights $w : E \to \mathbb{R}$. The greedy algorithm is defined as follows,

Greedy Algorithm for the maximum weight independent set problem:

Order $E = \{e_1, ..., e_n\}$ so that $w(e_1) \geq w(e_2) \geq ... \geq w(e_m) \geq 0 \geq w(e_{m+1}) \geq ... \geq w(e_n)$

$S := \emptyset$

for $i = 1$ to m do

  If $S + e_i \in \mathcal{I}$ then $S := S + e_i$

end for

output S

Theorem 2.4.1. *Let $\mathcal{M} = (E, \mathcal{I})$ be such that $\mathcal{I}$ is non-empty and it satisfies axiom (I1) from Definition 2.1.1. Then, $\mathcal{M}$ is a matroid if and only if the* *greedy algorithm finds an independent set of maximum weight for each weight function $w : E \to \mathbb{R}$.*

Proof. ($\Rightarrow$) Let $S = \{s_1, ..., s_k\}$ be the output of the greedy algorithm, and assume $T = \{t_1, ..., t_l\}$ is an independent set of larger weight. WLOG, assume $w(s_1) \geq w(s_2) \geq .... \geq w(s_k)$ and $w(t_1) \geq w(t_2) \geq ... \geq w(t_l)$. Then, since $\sum_{i=1}^{k} w(s_i) < \sum_{i=1}^{l} w(t_i)$, there is at least one index i such that $w(t_i) > w(s_i)$, or $w(t_i) > 0$ and $i > k$. Let i_0 be the smallest such index. It follows that $\{s_1, ..., s_{i_0-1}\}$ is a basis of $A = \{s_1, ..., s_{i_0-1}, t_1, ..., t_{i_0}\}$ (i.e., $\{s_1, ..., s_{i_0-1}\} \in \mathcal{I}$ and $r(A) = i_0 - 1$), since otherwise the greedy algorithm would have chosen an element from $t_1, ..., t_{i_0}$ as s_{i_0}. But $\{t_1, ..., t_{i_0}\}$ is a larger independent set contained in A. Thus, we get a contradiction and the implication follows.

($\Leftarrow$) Assume $\mathcal{M}$ is not a matroid, i.e., there exist sets $X, Y \in \mathcal{I}$ such that $|Y| > |X|$ and $\forall e \in Y - X$, $X + e \notin \mathcal{I}$. Let $k = |X|$ and consider the weights given by

$$w(e) = \begin{cases} k+2, & e \in X \\ k+1, & e \in Y - X \\ 0, & \text{otherwise} \end{cases} .$$

Then the greedy algorithm outputs a set $S \supseteq X$ with $w(S) = w(X) = k(k+2)$, while $w(Y) = (k+1) \cdot |Y| \geq (k+1)(k+1) > k(k+2)$. Hence, the greedy algorithm does not output the optimal solution and we get a contradiction. $\square$

2.5 Matroid intersection

In this section we define the intersection of two matroids and we study its properties.

Definition 2.5.1. *Given two matroids $\mathcal{M}_1 = (E, \mathcal{I}_1)$ and $\mathcal{M}_2 = (E, \mathcal{I}_2)$ on a common ground set we define their intersection as $\mathcal{M}_1 \cap \mathcal{M}_2 = (E, \mathcal{I}_1 \cap \mathcal{I}_2)$, where $\mathcal{I}_1 \cap \mathcal{I}_2$ denotes the collection of all sets that are independent in both matroids.*

Matroid intersection captures several optimization problems.

Example 2.5.1. ***Bipartite matching.*** *Let $G = (A \cup B, E)$ be a bipartite undirected graph. Notice that the edges can be partitioned as $E = \biguplus_{v \in A} \delta(v)$ and as $E = \biguplus_{v \in B} \delta(v)$. Hence, we can define partition matroids as follows: Let $\mathcal{M}_1 = (E, \mathcal{I}_1)$ where $\mathcal{I}_1 = \{X \subseteq E : |X \cap \delta(v)| \leq 1, \ \forall v \in A\}$, and $\mathcal{M}_2 = (E, \mathcal{I}_2)$ where $\mathcal{I}_2 = \{X \subseteq E : |X \cap \delta(v)| \leq 1, \ \forall v \in B\}$. Notice that a set $I \subseteq E$ is independent in $\mathcal{M}_1 \cap \mathcal{M}_2$, i.e., $I \in \mathcal{I}_1 \cap \mathcal{I}_2$, if and only if I is a matching.*

The above example shows that the intersection of two matroids is not necessarily a matroid, since in Example 2.1.10 we showed that matchings (in both general graphs and bipartite graphs) are not a matroid in general.

Definition 2.5.2. *Given a digraph $D = (V, A)$ and a special root node $r \in V$, an r-arborescence is a spanning tree (in the underlying undirected graph) directed away from r.*

Example 2.5.2. ***Arborescences.*** *Let $D = (V, A)$ be a digraph and $r \in V$. Consider the matroids $\mathcal{M}_1 = (A, \mathcal{I}_1) = \mathcal{M}(D)$ and $\mathcal{M}_2 = (A, \mathcal{I}_2)$ where*

$$\mathcal{I}_2 = \left\{ X \subseteq A : \begin{array}{c} |X \cap \delta^-(v)| \leq 1, \forall v \neq r \\ |X \cap \delta^-(r)| \leq 0 \end{array} \right\}.$$

Then we have that $I \in \mathcal{I}_1 \cap \mathcal{I}_2$ if and only if I induces a forest on V, $|I \cap \delta^-(r)| = 0$, and $|I \cap \delta^-(v)| \leq 1$ for all $v \neq r$. It follows that $I \subseteq A$ is an r-arborescence if and only if $I \in \mathcal{I}_1 \cap \mathcal{I}_2$ and $|I| = |V| - 1$.

Example 2.5.3. ***Orientations on graphs.*** *Assume we are given an undirected graph $G = (V, E)$ and targets $d(v) \in \mathbb{Z}_+$ for each $v \in V$ such that $\sum_{v \in V} d(v) = |E|$. Can we orient the edges of G to get a digraph where each vertex v has in-degree (i.e., $|\delta^-(v)|$) $d(v)$? We can use matroid intersection to answer this question. First, create a digraph $D = (V, E^2)$ where $E^2 = \biguplus_{uv \in E}\{(u,v),(v,u)\}$. Notice that E^2 can also be partitioned as $E^2 = \biguplus_{v \in V} \delta_D^-(v)$. Consider the partition matroids $\mathcal{M}_1 = (E^2, \mathcal{I}_1)$ and $\mathcal{M}_2 = (E^2, \mathcal{I}_2)$ given by*

$$\mathcal{I}_1 = \{X \subseteq E^2 : |X \cap \{(u,v),(v,u)\}| \leq 1, \forall uv \in E\}$$

and

$$\mathcal{I}_2 = \{X \subseteq E^2 : |X \cap \delta^-(v)| \leq d(v),\ \forall v \in V\}.$$

Then an orientation satisfying the above condition exists if and only if $\mathcal{I}_1 \cap \mathcal{I}_2$ contains an element of size $|E|$.

Example 2.5.4. ***Colourful Spanning Trees.*** *Assume we are given a graph $G = (V, E)$ and every edge has a colour from $[p]$. Let E_i denote the set of edges of colour i. Then we can partition the edges of the graph as $E = \biguplus_{i=1}^p E_i$. We are interested in finding a multi-coloured spanning tree, i.e., one whose edges have all different colours. This problem can be approached using matroid intersection by letting $\mathcal{M}_1 = \mathcal{M}(G)$ and $\mathcal{M}_2$ be a partition matroid with partition $E_1, ..., E_p$ and coefficients $k_j = 1$ for all $j \in [p]$. Then, it is* *clear that a multi-coloured spanning tree exists if and only if $\mathcal{I}_1 \cap \mathcal{I}_2$ contains an element of size $|V| - 1$.*

The above examples motivate the problem of finding a common independent set of maximum cardinality. Since matroid intersection is not necessarily a matroid, the greedy algorithm cannot be used to solve this problem. In this section we discuss an algorithm due to Edmonds which finds such a common independent set in polynomial time. At each step, the algorithm takes a common independent set I, and either ouputs a common independent set J satisfying $|J| = |I| + 1$, or certifies correctly that I is a common independent set of maximum cardinality. Before stating the actual algorithm we need some definitions and partial results.

Definition 2.5.3. *Let $\mathcal{M} = (E, \mathcal{I})$ be a matroid and $I \in \mathcal{I}$. We define the exchange graph of I in $\mathcal{M}$, denoted by $D_{\mathcal{M}}(I)$, as a digraph $D = (E, A)$ where*

$$A = \{(x, y) : x \in I, y \in E - I, I - x + y \in \mathcal{I}\}.$$

Similarly, for two matroids $\mathcal{M}_1 = (E, \mathcal{I}_1)$ and $\mathcal{M}_2 = (E, \mathcal{I}_2)$, and a common independent set $I \in \mathcal{I}_1 \cap \mathcal{I}_2$, we define the *exchange graph of I in $\mathcal{M}_1 \cap \mathcal{M}_2$* as a directed graph $D = (E, A)$ such that

$$A = \{(y, z) : y \in I, z \in E - I, I - y + z \in \mathcal{I}_1\} \cup \{(z, y) : y \in I, z \in E - I, I - y + z \in \mathcal{I}_2\}.$$

This graph is usually denoted by $D_{\mathcal{M}_1, \mathcal{M}_2}(I)$. Notice that both $D_{\mathcal{M}}(I)$ and $D_{\mathcal{M}_1, \mathcal{M}_2}(I)$ are directed bipartite graphs with bipartitions I and $E - I$. The exchange graph of a matroid $\mathcal{M}$ has some interesting properties. For instance, Brualdi [3] showed the following:

Lemma 2.5.1. *Let $\mathcal{M}$ be a matroid, and let I, J be two independent sets in $\mathcal{M}$ such that $|I| = |J|$. Then $D_{\mathcal{M}}(I)$ has a perfect matching on $I \triangle J$.*

Krogdahl [19, 18, 20] gave the following counterpart to Lemma 2.5.1.

Lemma 2.5.2. *Let $\mathcal{M} = (E, \mathcal{I})$ be a matroid. Let $I \in \mathcal{I}$ and $J \subseteq E$ such that $|I| = |J|$. If $D_{\mathcal{M}}(I)$ has a unique perfect matching on $I \triangle J$, then $J \in \mathcal{I}$.*

This implies the following result.

Corollary 2.5.1. *Let $\mathcal{M} = (E, \mathcal{I})$ be a matroid and $I \in \mathcal{I}$. Let $J \subseteq E$ be such that $|J| = |I|$, $r_{\mathcal{M}}(I \cup J) = |I|$, and $D_{\mathcal{M}}(I)$ has a unique perfect matching on $I \triangle J$. Then, for any $e \notin I \cup J$ such that $I + e \in \mathcal{I}$ we have $J + e \in \mathcal{I}$.*

Proof. Let $s \notin I \cup J$ be such that $I + s \in \mathcal{I}$. Denote by N the unique perfect matching on $I \triangle J$ in the digraph $D_{\mathcal{M}}(I)$. Let t be a new element and consider the matroid given by $\mathcal{M}' = (E + t, \mathcal{I}')$ where

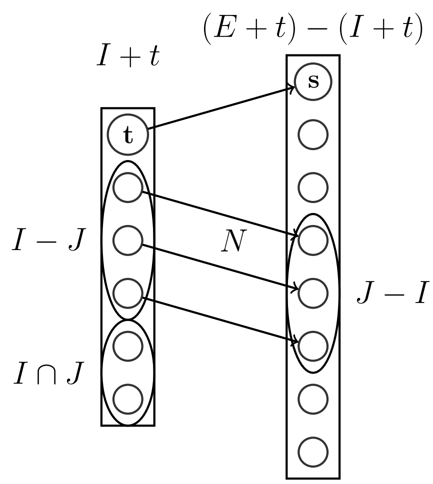
$$\mathcal{I}' = \{I \subseteq E + t : I - t \in \mathcal{I}\}.$$

We claim that $D_{\mathcal{M}'}(I + t)$ has a unique perfect matching on $(I \triangle J) + \{s, t\}$. To see this it is enough to show that (see picture below):

(1) There are no arcs from t to $J - I$.

(2) There is an arc from t to s.

Since then $N + \{(t, s)\}$ is the unique perfect matching on $(I \triangle J) + \{s, t\}$ in $D_{\mathcal{M}'}(I + t)$.



To show (1) assume that there is an arc from t to some $e \in J - I$. Then by definition of exchange graph we have that $(I+t) - t + e \in \mathcal{I}'$, i.e., $I + e \in \mathcal{I}'$. It follows that $I + e - t \in \mathcal{I}$. But $I + e - t$ is just $I + e$. Hence, since $I + e \in \mathcal{I}$ and $I + e \subseteq I \cup J$ we have that $r_{\mathcal{M}}(I \cup J) > |I|$, contradicting our assumption of $r_{\mathcal{M}}(I \cup J) = |I|$. To see condition (2) notice that $I + s \in \mathcal{I}$, and therefore $(I+s)+t \in \mathcal{I}'$. In particular $(I+t)-t+s \in \mathcal{I}'$, so (t, s) is an arc in $D_{\mathcal{M}'}(I+t)$.

Hence $D_{\mathcal{M}'}(I+t)$ has a unique perfect matching on $(I \triangle J) + \{s, t\}$. Moreover, since $I+t \in \mathcal{I}'$ and $|I+t| = |J+s|$, it follows by Lemma 2.5.2 that $J+s \in \mathcal{I}'$. Thus, $J + s - t = J + s \in \mathcal{I}$. $\square$

These two last results are useful when proving the correctness of the matroid intersection algorithm. We now show an easy upper bound on the size of a common independent set.

Claim 2.5.1. *Let $\mathcal{M}_1 = (E, \mathcal{I}_1)$ and $\mathcal{M}_2 = (E, \mathcal{I}_2)$ be two matroids with rank functions r_1 and r_2 respectively. Then*

$$\max_{I \in \mathcal{I}_1 \cap \mathcal{I}_2} |I| \leq \min_{U \subseteq E} r_1(U) + r_2(E - U).$$

Proof. Take an arbitrary $I \in \mathcal{I}_1 \cap \mathcal{I}_2$ and $U \subseteq E$. From the hereditary property of matroids we have that $I \cap U \in \mathcal{I}_1$ and $I - U \in \mathcal{I}_2$. Also, since $I \cap U \subseteq U$ and $I - U \subseteq E - U$, we have that $|I \cap U| \leq r_1(U)$ and $|I - U| \leq r_2(E - U)$ respectively. It follows that

$$|I| = |I \cap U| + |I - U| \leq r_1(U) + r_2(E - U).$$

Since the sets I and U were chosen arbitrarily from $\mathcal{I}_1 \cap \mathcal{I}_2$ and E respectively, the result follows. $\square$

We see at the end of this section that in fact Claim 2.5.1 holds with equality. This result is known as the Matroid Intersection Theorem, and is due to Edmonds [6].

Corollary 2.5.2. *Let $\mathcal{M}_1 = (E, \mathcal{I}_1)$ and $\mathcal{M}_2 = (E, \mathcal{I}_2)$ be two matroids with rank functions r_1 and r_2 respectively. Let $I \in \mathcal{I}_1 \cap \mathcal{I}_2$ and assume $|I| = r_1(U) + r_2(E - U)$ for some $U \subseteq E$. Then I is a common independent set of maximum cardinality.*

Proof. It follows directly from Claim 2.5.1. $\square$

We now present the matroid intersection algorithm and then verify its correctness. We appeal to these arguments in Chapter 3.

<u>Edmonds' Matroid Intersection algorithm</u>

$I \leftarrow \emptyset$

While I not maximal

Construct $D_{\mathcal{M}_1,\mathcal{M}_2}(I)$

$X_1 \leftarrow \{z \in E - I : I + z \in \mathcal{I}_1\}$

$X_2 \leftarrow \{z \in E - I : I + z \in \mathcal{I}_2\}$

Let P be a shortest X_1-X_2 dipath in $D_{\mathcal{M}_1,\mathcal{M}_2}(I)$

if P is not empty then

$I \leftarrow I \triangle V(P)$

end if

end while

output I

We show the correctness of the algorithm in the following two lemmas.

Lemma 2.5.3. *If there is no X_1-X_2 dipath in $D_{\mathcal{M}_1,\mathcal{M}_2}(I) = (E, A)$, then I is a common independent set of maximum cardinality.*

Proof. If $X_1 = \emptyset$ or $X_2 = \emptyset$, then I is already a base of $\mathcal{M}_1$ or $\mathcal{M}_2$, and hence a maximum size common independent set. So assume $X_1 \neq \emptyset$ and $X_2 \neq \emptyset$. Let U be the set of nodes that can reach X_2 in $D_{\mathcal{M}_1,\mathcal{M}_2}(I)$, i.e., a node x belongs to U if there exists a dipath $P = xv_1...v_lt$ in $D_{\mathcal{M}_1,\mathcal{M}_2}(I)$ with $t \in X_2$. By construction of U we have that $X_2 \subseteq U$ and $\delta^-(U) = \emptyset$. Moreover, since there is no X_1-X_2 dipath in $D_{\mathcal{M}_1,\mathcal{M}_2}(I)$ we have that $X_1 \cap U = \emptyset$. We claim that the following two conditions hold:

(1) $r_1(U) \leq |I \cap U|$.

(2) $r_2(E - U) \leq |I - U|$.

To show (1) assume that $r_1(U) > |I \cap U|$. In this case, there exists some element $z \in U - (I \cap U)$ such that $(I \cap U) + z \in \mathcal{I}_1$. Moreover, since $X_1 \cap U = \emptyset$ we have that $z \notin X_1$ and hence $I + z \notin \mathcal{I}_1$. Now, since $(I \cap U) + z \in \mathcal{I}_1$ and $I + z \notin \mathcal{I}_1$, there must exist an element $y \in I - U$ such that $I - y + z \in \mathcal{I}_1$. But then $(y, z) \in A$ with $y \notin U$ and $z \in U$, contradicting the fact that $\delta^-(U) = \emptyset$.

Condition (2) is proved in a similar way. Assume $r_2(E-U) > |I-U|$. Then there exists an element $z \in (E-U)-(I-U)$ such that $(I-U)+z \in \mathcal{I}_2$. Moreover, since $X_2 \subseteq U$ it follows that $z \notin X_2$, and hence $I+z \notin \mathcal{I}_2$. Again, given that $(I-U)+z \in \mathcal{I}_2$ and $I+z \notin \mathcal{I}_2$, there must exist an element $y \in I \cap U$ such that $I-y+z \in \mathcal{I}_2$. But then $(z,y) \in A$ with $z \notin U$ and $y \in U$, contradicting the fact that $\delta^-(U) = \emptyset$. Combining (1) and (2) we get

$$|I| = |I \cap U| + |I-U| \geq r_1(U) + r_2(E-U). \tag{2.1}$$

It follows from Claim 2.5.1 that

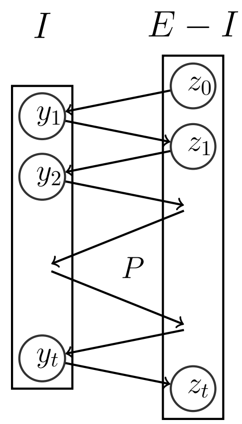
$$|I| = r_1(U) + r_2(E-U),$$

and thus by Corollary 2.5.2 we have that I is a common independent set of maximum cardinality. $\square$

Lemma 2.5.4. *Let P be a shortest X_1-X_2 dipath in $D_{\mathcal{M}_1,\mathcal{M}_2}(I)$. Then, $J = I \triangle V(P)$ belongs to $\mathcal{I}_1 \cap \mathcal{I}_2$, where $V(P)$ is the set of nodes of the dipath P.*

Proof. Let $P = z_0 y_1 z_1 ... y_t z_t$ be a shortest X_1-X_2 dipath in $D_{\mathcal{M}_1,\mathcal{M}_2}(I)$, and let $J = I - \{y_1, ..., y_t\} + \{z_1, ..., z_t\}$ (see picture below). Then $|J| = |I|$ and the arcs from $\{y_1, ..., y_t\}$ to $\{z_1, ..., z_t\}$ form the only perfect matching on $I \triangle J$ in $D_{\mathcal{M}_1}(I)$, since otherwise P would have a shortcut and this would contradict our assumption of P being a shortest dipath. It follows from Lemma 2.5.2 that $J \in \mathcal{I}_1$. Moreover, $z_i \notin X_1$ for all $i \geq 1$, since otherwise $P' = z_i y_{i+1} ... y_t z_t$ would be a shorter X_1-X_2 dipath. Hence, $I + z_i \notin \mathcal{I}_1$ for $i \geq 1$, and this implies $r_1(I \cup J) = r_1(I) = |I| = |J|$. Finally, since $z_0 \in X_1$ we have that $I + z_0 \in \mathcal{I}_1$, and by Corollary 2.5.1 it follows that $J + z_0 \in \mathcal{I}_1$.

Thus, $I - \{y_1, ..., y_t\} + \{z_0, ..., z_t\} = I \triangle V(P) \in \mathcal{I}_1$. By symmetry we have that $I \triangle V(P) \in \mathcal{I}_2$, and the lemma follows.



□

Clearly, the running time of the algorithm is polynomially bounded, since we can construct $D_{\mathcal{M}_1,\mathcal{M}_2}(I)$ and find the path P in polynomial time. Thus:

Theorem 2.5.1. *Given two matroids, a maximum size common independent set can be found in polynomial time.*

To conclude this section, notice that Equation (2.1) in Lemma 2.5.3 proves the other direction for the inequality in Claim 2.5.1. Hence, the min-max result follows with equality.

Theorem 2.5.2. *[Matroid Intersection Theorem] Let* $\mathcal{M}_1 = (E, \mathcal{I}_1)$ *and* $\mathcal{M}_2 = (E, \mathcal{I}_2)$ *be two matroids with rank functions* r_1 *and* r_2 *respectively. Then the size of a largest common independet set is given by*

$$\min_{U \subseteq E} r_1(U) + r_2(E - U).$$

2.6 Matroid union

Definition 2.6.1. *Given matroids $\mathcal{M}_1 = (E_1, \mathcal{I}_1), ..., \mathcal{M}_k = (E_k, \mathcal{I}_k)$, we define their union as $\mathcal{M}_1 \vee ... \vee \mathcal{M}_k = (E_1 \cup ... \cup E_k, \mathcal{I}_1 \vee ... \vee \mathcal{I}_k)$ where*

$$\mathcal{I}_1 \vee ... \vee \mathcal{I}_k = \{I_1 \cup ... \cup I_k : I_1 \in \mathcal{I}_1, ..., I_k \in \mathcal{I}_k\}.$$

We show that unlike matroid intersection, matroid union generates a matroid. We first need the following theorem due to Nash-Williams [23].

Theorem 2.6.1. *Let $\mathcal{M}' = (E', \mathcal{I}')$ be a matroid with rank function r', and let $f : E' \to E$. For any set $U \subseteq E'$ let $f(U) = \{f(s) : s \in U\}$. Then $\mathcal{M} = (E, \mathcal{I})$ is a matroid where $\mathcal{I} = \{f(I') : I' \in \mathcal{I}'\}$. Moreover, the rank function of $\mathcal{M}$ is given by*

$$r(U) = \min_{S \subseteq U}(|U - S| + r'(f^{-1}(S))).$$

Proof. It follows from the definition that $\mathcal{I}$ is non-empty and closed under taking subsets. To see that the greedy property holds let $X, Y \in \mathcal{I}$ with $|Y| > |X|$. Choose sets $X', Y' \in \mathcal{I}'$ such that $f(X') = X$, $f(Y') = Y$, $|X'| = |X|$, $|Y'| = |Y|$, and $|X' \cap Y'|$ is as large as possible. By the greedy property of $\mathcal{M}'$ there exists $e \in Y' - X'$ such that $X' + e \in \mathcal{I}'$. Notice that $f(e) \notin f(X')$, since otherwise we would have that $f(e) = f(s)$ for some $s \in X'$, and we could increase $|X' \cap Y'|$ by replacing X' by $X' - s + e$ (contradicting our assumption). It follows that $f(e) \in f(Y') - f(X')$ and $f(X') + f(e) \in \mathcal{I}$, i.e., $f(e) \in Y - X$ and $X + f(e) \in \mathcal{I}$. Hence $\mathcal{M}$ is a matroid.

To derive the rank function of $\mathcal{M}$, fix $U \subseteq E$ and let $U' = f^{-1}(U)$. Then notice that the rank of U can be written as

$$
\begin{aligned}
r(U) &= \max\{|I| : I \in \mathcal{I}, I \subseteq U\} \qquad (2.2)\\
&= \max\{|I'| : I' \in \mathcal{I}', f(I') \subseteq U, f(I') = |I'|\}\\
&= \max\{|I'| : I' \in \mathcal{I}', I' \subseteq U', f(I') = |I'|\}.
\end{aligned}
$$

Define a partition matroid $\mathcal{P} = (E', \mathcal{I}'')$ with

$$\mathcal{I}'' = \{I' \subseteq E' : |I' \cap f^{-1}(s)| \leq 1 \ \forall s \in U, |I' \cap f^{-1}(s)| = 0 \ \forall s \notin U\}.$$

Observe that the rank function of $\mathcal{P}$ is given by

$$r_{\mathcal{P}}(S') = |\{e \in U : f^{-1}(e) \cap S' \neq \emptyset\}|.$$

In particular, we have that $I' \in \mathcal{I}''$ if and only if $I' \subseteq U'$ and $|f(I')| = |I'|$. It follows from (2.2) that

$$r(U) = \max_{\substack{I' \in \mathcal{I}' \cap \mathcal{I}'' \\ I' \subseteq U'}} |I'| = \max_{I' \in \mathcal{I}' \cap \mathcal{I}''} |I'|.$$

Moreover, from the Matroid Intersection Theorem (Theorem 2.5.2) we have

$$
\begin{aligned}
\max_{\substack{I' \in \mathcal{I}' \cap \mathcal{I}'' \\ I' \subseteq U'}} |I'| &= \min_{S' \subseteq U'} r'(S') + r_{\mathcal{P}}(U' - S')\\
&= \min_{S \subseteq U} r'(f^{-1}(S)) + r_{\mathcal{P}}(f^{-1}(U - S))\\
&= \min_{S \subseteq U} r'(f^{-1}(S)) + |U - S|,
\end{aligned}
$$

where the second equality follows from the fact that it is optimal to have S' of the form $f^{-1}(S)$ for some $S \subseteq U$. Indeed, if $S' \subseteq U'$ is such that $S' \neq f^{-1}(S)$ for all $S \subseteq U$, defining $\bar{S} = U' - f^{-1}(f(U' - S')) = f^{-1}(U - f(U' - S'))$ gives a new set satisfying:

(i) $\bar{S} \subseteq S'$, and hence $r'(\bar{S}) \leq r'(S')$.

(ii) $r_{\mathcal{P}}(U' - \bar{S}) = r_{\mathcal{P}}(U' - S')$. This follows from the definition of $r_{\mathcal{P}}$ and the fact that $U' - \bar{S} = f^{-1}(f(U' - S'))$.

(iii) $\bar{S} = f^{-1}(S)$ for $S = U - f(U' - S')$. $\square$

Theorem 2.6.2. *[Matroid Union Theorem] Let* $\mathcal{M}_1 = (E_1, \mathcal{I}_1), ..., \mathcal{M}_k = (E_k, \mathcal{I}_k)$ *be matroids with rank functions* $r_1, ..., r_k$ *respectively. Then* $\mathcal{M}_1 \vee ... \vee \mathcal{M}_k$ *is again a matroid, with rank function given by*

$$r(U) = \min_{S \subseteq U}[|U - S| + \sum_{i=1}^{k} r_i(S \cap E_i)].$$

Proof. For $i = 1, ..., k$ let $\mathcal{M}'_i = (E'_i, \mathcal{I}'_i)$ be a copy of $\mathcal{M}_i$ with $E'_1, ..., E'_k$ disjoint. Then trivially $\mathcal{M}'_1 \vee ... \vee \mathcal{M}'_k = (E'_1 \cup ... \cup E'_k, \mathcal{I}'_1 \vee ... \vee \mathcal{I}'_k)$ is a matroid, with rank function given by $r'(U) = \sum_{i=1}^{k} r'_i(U \cap E'_i)$, where r'_i denotes the rank function of $\mathcal{M}'_i$. Now consider the function $f : E'_1 \cup ... \cup E'_k \to E_1 \cup ... \cup E_k$ that sends a copy $s \in E'_i$ to its original in E_i. Then, by Theorem 2.6.1 we know that $\mathcal{M} = (E_1 \cup ... \cup E_k, \mathcal{I})$ is a matroid, where

$$\mathcal{I} = \{f(I') : I' \in \mathcal{I}'_1 \vee ... \vee \mathcal{I}'_k\}.$$

Moreover, $\mathcal{M} = \mathcal{M}_1 \vee ... \vee \mathcal{M}_k$, and hence $\mathcal{M}_1 \vee ... \vee \mathcal{M}_k$ is a matroid.

From Theorem 2.6.1 we know that the rank of $\mathcal{M}$ is given by

$$r(U) = \min_{S \subseteq U}[|U - S| + r'(f^{-1}(S))].$$

Notice that the rank of $\mathcal{M}'_1 \vee ... \vee \mathcal{M}'_k$ can be written in terms of f as

$$r'(U) = \sum_{i=1}^{k} r'_i(U \cap E'_i) = \sum_{i=1}^{k} r_i(f(U) \cap E_i).$$

Hence,

$$r'(f^{-1}(S)) = \sum_{i=1}^{k} r_i(f(f^{-1}(S)) \cap E_i) = \sum_{i=1}^{k} r_i(S \cap E_i),$$

and the formula for the rank follows. $\square$

Applying the Matroid Union Theorem to a single matroid gives the following interesting results.

Corollary 2.6.1. *Let $\mathcal{M} = (E, \mathcal{I})$ be a matroid with rank function r, and let $k \in \mathbb{Z}_+$. Then the maximum cardinality of the union of k independent sets of $\mathcal{M}$ is equal to*

$$\min_{U \subseteq E}[|E - U| + k \cdot r(U)].$$

Proof. It follows from Theorem 2.6.2 by taking $\mathcal{M}_1 = ... = \mathcal{M}_k = \mathcal{M}$. $\square$

Given a matroid $\mathcal{M} = (E, \mathcal{I})$ and an integer $k \in \mathbb{Z}_+$, we denote by $\mathcal{M}^k = (E, \mathcal{I}^k)$ the matroid union $\mathcal{M}_1 \vee ... \vee \mathcal{M}_k$ where $\mathcal{M}_1 = ... = \mathcal{M}_k = \mathcal{M}$.

Corollary 2.6.2. *Let $\mathcal{M} = (E, \mathcal{I})$ be a matroid with rank function r, and let $k \in \mathbb{Z}_+$. Then the ground set E can be covered by k independent sets if and only if $k \cdot r(U) \geq |U|$, $\forall U \subseteq E$.*

Proof. The ground set E can be covered by k independent sets if and only if there is a union of k independent sets of size $|E|$, i.e., if there is an independent set in $\mathcal{M}^k$ of size $|E|$. By Corollary 2.6.1 we know that this happens if and only if

$$\min_{U \subseteq E}[|E - U| + k \cdot r(U)] \geq |E|.$$

It follows that

$$\begin{aligned} \min_{U \subseteq E}[|E - U| + k \cdot r(U)] &\geq |E| \\ &\Updownarrow \\ |E - U| + k \cdot r(U) &\geq |E|, \ \forall U \subseteq E \end{aligned}$$

$$\Updownarrow$$

$$k \cdot r(U) \geq |U|, \ \forall U \subseteq E.$$

$\square$

Similarly, we have the following result for the maximum number of disjoint bases in a matroid due to Edmonds [7].

Corollary 2.6.3. *Let $\mathcal{M} = (E, \mathcal{I})$ be a matroid with rank function r, and let $k \in \mathbb{Z}_+$. Then there exist k disjoint bases if and only if*

$$k\,(r(E) - r(U)) \leq |E - U|, \ \forall U \subseteq E.$$

Proof. The matroid $\mathcal{M}$ has k disjoint bases if and only if the maximum size of the union of k independent sets is $k \cdot r(E)$. We know from Corollary 2.6.1 that this happens if and only if

$$\min_{U \subseteq E}[|E - U| + k \cdot r(U)] \geq k \cdot r(E).$$

It follows that

$$\begin{aligned} \min_{U \subseteq E}[|E - U| + k \cdot r(U)] &\geq k \cdot r(E) \\ &\Updownarrow \\ |E - U| + k \cdot r(U) &\geq k \cdot r(E), \ \forall U \subseteq E \\ &\Updownarrow \\ |E - U| &\geq k\,(r(E) - r(U)), \ \forall U \subseteq E. \end{aligned}$$

$\square$

2.7 Matroid intersection polytope

To conclude this chapter we introduce the notions of *independent set polytope* and *matroid intersection polytope*, and we mention two important characterizations due to Edmonds. We first need some concepts from polyhedral combinatorics.

A *convex combination* of $x, y \in \mathbb{R}^n$ is any vector of the form $\lambda x+(1-\lambda)y$, where $\lambda \in [0,1]$. A vector $x \in \mathbb{R}^n$ is a *convex combination* of vectors $x^1, ..., x^k \in \mathbb{R}^n$ if there exist nonnegative $\lambda_1, ..., \lambda_k$ such that $\sum_{i\in[k]} \lambda_i = 1$ and $x = \sum_{i\in[k]} \lambda_i x^i$. A set $X \subseteq \mathbb{R}^n$ is *convex* if it is closed under convex combinations. An *extreme point* of a convex set X is any point $x \in X$ which is not a convex combination of other points in X distinct from x. The *convex hull* of an arbitrary set of points $X \subseteq \mathbb{R}^n$ is the set

$$conv(X) = \{\sum_{x^i\in S} \lambda_i x^i : S \subseteq X, |S| < \infty, \lambda_i \geq 0\, \forall i, \sum_i \lambda_i = 1\}.$$

A *polytope* is any set which is the convex hull of finitely many points, i.e., it is a set of the form $conv(X)$ where X is finite. The extreme points of a polytope are called *vertices*. A *half-space* is any set of the form $\{x : a^T x \leq \gamma\}$ where a is a nonzero vector in $\mathbb{R}^n$ and $\gamma \in \mathbb{R}$. A set P is a *polyhedron* in $\mathbb{R}^n$ if it is the intersection of finitely many half-spaces, i.e., $P = \{x : Ax \leq b\}$ for some $m \times n$ matrix A and $b \in \mathbb{R}^m$. Given $P \subseteq \mathbb{R}^n$, we say that F is a *face* of P if there is $a \in \mathbb{R}^n$ and $\gamma \in \mathbb{R}$ such that

$$F = \{x \in P : a^T x = \gamma\}, \quad \gamma = \max\{a^T x : x \in P\}.$$

We call F a *facet* of P if F is a maximal (inclusion-wise) face of P distinct from P.

Given a ground set E, we define the *incidence vector of a set* $I \subseteq E$ as $\chi^I \in \{0,1\}^E$ where $\chi^I_e = 1$ if $e \in I$ and $\chi^I_e = 0$ if $e \notin I$.

Definition 2.7.1. *Let $\mathcal{M} = (E, \mathcal{I})$ be a matroid and let $X = \{\chi^I : I \in \mathcal{I}\}$. The independent set polytope of $\mathcal{M}$ is defined as $P_{\mathcal{M}} = conv(X)$.*

Edmonds gave a complete characterization of $P_{\mathcal{M}}$ by showing that $P_{\mathcal{M}} = P$ where

$$P = \left\{ x \in \mathbb{R}^E : \begin{array}{c} x(U) \leq r(U), \ \forall U \subseteq E \\ x_e \geq 0, \ \forall e \in E \end{array} \right\}.$$

Moreover, given two matroids $\mathcal{M}_1 = (E, \mathcal{I}_1)$ and $\mathcal{M}_2 = (E, \mathcal{I}_2)$, Edmonds [6] also proved that the intersection of the independent set polytopes of $\mathcal{M}_1$ and $\mathcal{M}_2$ gives exactly the convex hull of the common independent sets, i.e., $P_{\mathcal{M}_1} \cap P_{\mathcal{M}_2} = P_{\mathcal{M}_1 \cap \mathcal{M}_2}$ where

$$P_{\mathcal{M}_1 \cap \mathcal{M}_2} = conv(\{\chi^I : I \in \mathcal{I}_1 \cap \mathcal{I}_2\}).$$

It follows that

$$P_{\mathcal{M}_1 \cap \mathcal{M}_2} = \left\{ x \in \mathbb{R}^E : \begin{array}{c} x(U) \leq r_1(U), \ \forall U \subseteq E \\ x(U) \leq r_2(U), \ \forall U \subseteq E \\ x_e \geq 0, \ \forall e \in E \end{array} \right\}.$$

We refer to $P_{\mathcal{M}_1 \cap \mathcal{M}_2}$ as the *matroid intersection polytope.*

CHAPTER 3
The matroid common colouring problem

3.1 Introduction

We say that a matroid $\mathcal{M} = (E, \mathcal{I})$ is *k-colourable* if there exists a partition $C_1, C_2, ..., C_k$ of E such that $C_i \in \mathcal{I}$ for every $i \in \{1, .., k\}$. We call such a partition a *k-colouring* of $\mathcal{M}$, and we call the sets C_i the *colour classes* of $\mathcal{M}$.

Given two matroids $\mathcal{M}_1 = (E, \mathcal{I}_1)$ and $\mathcal{M}_2 = (E, \mathcal{I}_2)$ with rank functions r_1 and r_2 respectively, we define a function $r_{12} : 2^E \to \mathbb{Z}_+$ where $r_{12}(U) = \max\{|X| : X \subseteq U, X \in \mathcal{I}_1 \cap \mathcal{I}_2\}$. That is, $r_{12}(U)$ is the size of the largest common independent set contained in U. It follows from the definition that $r_{12} \leq \min\{r_1, r_2\}$. For any two matroids $\mathcal{M}_1 = (E, \mathcal{I}_1)$ and $\mathcal{M}_2 = (E, \mathcal{I}_2)$ we also define:

$\chi_i =$ the smallest integer k such that $\mathcal{M}_i$ is k-colourable.

$k_{12} = \max\{\chi_1, \chi_2\}$ (i.e. the smallest integer k such that both matroids $\mathcal{M}_1$ and $\mathcal{M}_2$ are k-colourable).

$\chi_{12} =$ the smallest integer k such that there exists a partition $C_1, C_2, ..., C_k$ of E with $C_i \in \mathcal{I}_1 \cap \mathcal{I}_2$ for every $i \in \{1, .., k\}$.

$\omega^*_{12} = \max_{U \subseteq E} \frac{|U|}{r_{12}(U)}$.

$\omega_{12} = \lceil \omega^*_{12} \rceil$.

We sometimes refer to χ_{12} as the *common colouring number* of $\mathcal{M}_1$ and $\mathcal{M}_2$, or just the common colouring number when there is no ambiguity. In this chapter we consider the problem of finding χ_{12} for two general matroids. We are interested in both the complexity of the problem, as well as in finding good approximations (if not the exact value) for χ_{12}. We start by showing some simple bounds for the common colouring number.

Claim 3.1.1. $k_{12} \leq \chi_{12} \leq k_{12}^2$.

Proof. Observe that $\chi_i \leq \chi_{12}$ for $i = 1, 2$ since any colouring for $\mathcal{M}_1 \cap \mathcal{M}_2$ is also a colouring for $\mathcal{M}_i$. It follows that $\chi_{12} \geq \max\{\chi_1, \chi_2\} = k_{12}$. To see the second inequality, let $X_1, X_2, ..., X_{k_{12}}$ and $Y_1, Y_2, ..., Y_{k_{12}}$ be k-colourings for $\mathcal{M}_1$ and $\mathcal{M}_2$ respectively. Then, $X_1 \cap Y_1, X_1 \cap Y_2, ..., X_1 \cap Y_{k_{12}}, X_2 \cap Y_1, ..., X_{k_{12}} \cap Y_1, ..., X_{k_{12}} \cap Y_{k_{12}}$ is a k_{12}^2-colouring for $\mathcal{M}_1 \cap \mathcal{M}_2$, where $X_i \cap Y_j \in \mathcal{I}_1 \cap \mathcal{I}_2$ follows from the hereditary property of matroids. Hence $\chi_{12} \leq k_{12}^2$. $\square$

Claim 3.1.2. $\omega_{12} \leq \chi_{12}$.

Proof. Let $U \subseteq E$, and denote by χ_{12}^U the common colouring number of the matroids $\mathcal{M}_1|_U$ and $\mathcal{M}_2|_U$. It is clear that $\frac{|U|}{r_{12}(U)} \leq \chi_{12}^U \leq \chi_{12}$. Now the inequality follows by taking the maximum with respect to sets $U \subseteq E$ in the left hand side and the fact that $\chi_{12} \in \mathbb{Z}_+$. $\square$

3.2 The greedy algorithm

In this section we discuss the performance of a greedy algorithm for the common colouring problem. At each step the algorithm packs a largest common colour class available, i.e., a maximum common independent set available. Notice that we can find such colour class in polynomial time thanks to the matroid intersection algorithm due to Edmonds. The following algorithm works in $O(\log n)$ rounds, where each round will use at most ω_{12} common independent sets.

The greedy algorithm for common colouring

$U \leftarrow E$

$V \leftarrow E$

$S \leftarrow \emptyset$

$i \leftarrow 1$

$k \leftarrow 1$

while $U \neq \emptyset$

  find a maximum size common independent set C_i in $(\mathcal{M}_1 \cap \mathcal{M}_2)|_U$

  $U \leftarrow U - C_i$

  if $|\cup_{j=1}^{i} C_j| \geq \frac{|V|}{2}$

  then

    $V \leftarrow V - (\cup_{j=1}^{i} C_j)$

    $S \leftarrow S + \{C_1, C_2, ..., C_i\}$

    $b_k \leftarrow i$

    $i \leftarrow 0$

    $k \leftarrow k + 1$

  end if

  $i \leftarrow i + 1$

end while

output S

Claim 3.2.1. *In the above greedy algorithm we have $b_i \leq \omega_{12} \leq \chi_{12}$ for every $i \geq 1$.*

Proof. The second inequality follows from Claim 3.1.2. To see the first inequality, let $b = b_i$ for some i. If $b = 1$ we are done. Otherwise, we must have that $\sum_{j=1}^{b-1} |C_j| < \frac{|V|}{2}$ and $\sum_{j=1}^{b} |C_j| \geq \frac{|V|}{2}$. Denote by U' the set of remaining uncoloured elements after packing the colour class C_{b-1}, i.e., $U' = V - (\cup_{j=1}^{b-1} C_j)$. Then,

$$|U'| = |V| - \sum_{j=1}^{b-1} |C_j| > 2\sum_{j=1}^{b-1} |C_j| - \sum_{j=1}^{b-1} |C_j| = \sum_{j=1}^{b-1} |C_j|. \tag{3.1}$$

Also, by the greediness of the algorithm we have

$$|C_j| \geq r_{12}(U') \text{ for each } j \in \{1, 2, ..., b-1\}. \tag{3.2}$$

By combining (3.1) and (3.2) we get $|U'| > \sum_{j=1}^{b-1} |C_j| \geq (b-1) \cdot r_{12}(U')$. Hence,

$$b - 1 < \frac{|U'|}{r_{12}(U')} \leq \max_{A \subseteq E} \frac{|A|}{r_{12}(A)} \leq \omega_{12}.$$

Finally, since both b and ω_{12} are integers it follows that $b \leq \omega_{12}$. $\square$

Corollary 3.2.1. *The greedy algorithm finds a common colouring of size $O(\omega_{12} \cdot \log n)$.*

Proof. In each round the algorithm colours at least half of the remaining uncoloured elements. Hence, the algorithm runs in $O(\log n)$ rounds. Moreover, from the above claim we know that the algorithm uses at most ω_{12} common colour classes in each round. It follows that the greedy algorithm outputs a common colouring of size $O(\omega_{12} \cdot \log n)$. $\square$

3.3 A linear programming approach

Note that χ_{12} is the solution to the minimization (covering) problem

$$\begin{array}{ll} \min & \sum_{I\in\mathcal{I}_1\cap\mathcal{I}_2} x(I) \\ \text{s.t.} & \sum_{I\ni e} x(I) \geq 1 \quad \forall e \in E, \\ & x(I) \in \{0,1\} \quad \forall I \in \mathcal{I}_1\cap\mathcal{I}_2. \end{array} \tag{3.3}$$

We define χ^*_{12} to be the solution to the linear program relaxation for the integer program above, i.e.,

$$\begin{array}{rll} \chi^*_{12} = & \min & \sum_{I\in\mathcal{I}_1\cap\mathcal{I}_2} x(I) \\ & \text{s.t.} & \sum_{I\ni e} x(I) \geq 1 \quad \forall e \in E, \\ & & x(I) \geq 0 \quad \forall I \in \mathcal{I}_1\cap\mathcal{I}_2. \end{array} \tag{3.4}$$

Notice that by linear programming duality we have

$$\begin{array}{rll} \chi^*_{12} = & \max & \sum_{e\in E} y(e) \\ & \text{s.t.} & y(I) = \sum_{e\in I} y(e) \leq 1 \quad \forall I \in \mathcal{I}_1\cap\mathcal{I}_2, \\ & & y(e) \geq 0 \quad \forall e \in E. \end{array} \tag{3.5}$$

We refer to χ^*_{12} as the *common fractional colouring number* or just as the *fractional colouring number of* $\mathcal{M}_1 \cap \mathcal{M}_2$. Similarly, we can see χ_1 and χ_2 as the solutions to minimization integer programs, and we can define χ^*_1 and χ^*_2 to be solutions to the respective linear programs relaxations.

Claim 3.3.1. $\max\{\chi^*_1, \chi^*_2\} \leq \chi^*_{12}$.

Proof. Any fractional colouring of $\mathcal{M}_1 \cap \mathcal{M}_2$ is also a fractional colouring of both $\mathcal{M}_1$ and $\mathcal{M}_2$. Thus, $\chi^*_{12} \geq \chi^*_1$ and $\chi^*_{!2} \geq \chi^*_2$. $\square$

We see later that very interestingly, Claim 3.3.1 holds with equality. Before stating the main result of this section we need the following definition.

Definition 3.3.1. *For a polyhedron P of $\mathbb{R}^n$, its antiblocking polyhedron $A(P)$ is defined by*

$$A(P) = \{y \in \mathbb{R}^n : x^T y \leq 1, \ \forall x \in P\}.$$

Theorem 3.3.1. *For any two matroids $\mathcal{M}_1 = (E, \mathcal{I}_1)$ and $\mathcal{M}_2 = (E, \mathcal{I}_2)$ we have $\chi^*_{12} = \omega^*_{12}$.*

Proof. Let us see first that $\chi^*_{12} \geq \omega^*_{12}$. Clearly, it is enough to show that $\chi^*_{12} \geq \frac{|U|}{r_{12}(U)}$ for all $U \subseteq E$. From (3.4) we know that $\chi^*_{12} = \sum_{I \in \mathcal{I}_1 \cap \mathcal{I}_2} y(I)$ for some vector y satisfying $\sum_{I \in \mathcal{I}_1 \cap \mathcal{I}_2} y(I)\chi^I \geq \chi^E$. Therefore,

$$1^T \chi^E \leq 1^T (\sum_{I \in \mathcal{I}_1 \cap \mathcal{I}_2} y(I)\chi^I).$$

But $1^T \chi^E = |E|$ and

$$\begin{aligned} 1^T (\sum_{I \in \mathcal{I}_1 \cap \mathcal{I}_2} y(I)\chi^I) &= \sum_{I \in \mathcal{I}_1 \cap \mathcal{I}_2} y(I)(1^T \chi^I) \\ &= \sum_{I \in \mathcal{I}_1 \cap \mathcal{I}_2} y(I) \cdot |I| \\ &\leq r_{12}(E) \sum_{I \in \mathcal{I}_1 \cap \mathcal{I}_2} y(I) \\ &= r_{12}(E) \cdot \chi^*_{12}. \end{aligned}$$

Hence, $\chi^*_{12} \geq \frac{|E|}{r_{12}(E)}$. Now, for every $U \subseteq E$, let us denote by $\chi^*_{12}|_U$ the fractional colouring number of $(\mathcal{M}_1 \cap \mathcal{M}_2)|_U$ (i.e., $\chi^*_{12}|_U$ is defined in the same way as χ^*_{12} but changing the ground set from E to U). Then

$$\chi^*_{12} \geq \chi^*_{12}|_U \geq \frac{|U|}{r_{12}(U)},$$

where the last inequality follows from the above argument by changing the ground set from E to U. Thus, $\chi^*_{12} \geq \omega^*_{12}$.

To see the other direction let us denote the matroid intersection polytope by $P_{\mathcal{M}_1 \cap \mathcal{M}_2}$. We saw in the previous chapter that the matroid intersection polytope can be written [6] as

$$\begin{aligned}
P_{\mathcal{M}_1 \cap \mathcal{M}_2} &= conv\{\chi^I : I \in \mathcal{I}_1 \cap \mathcal{I}_2\} \\
&= \{x \geq 0 : x(U) \leq r_1(U), x(U) \leq r_2(U), \forall U \subseteq E\} \\
&= \{x \geq 0 : x(U) \leq \min\{r_1(U), r_2(U)\},\ \forall U \subseteq E\} \\
&= \{x \geq 0 : \frac{1}{\min\{r_1(U), r_2(U)\}} x(U) \leq 1,\ \forall U \subseteq E\} \\
&= \{x \geq 0 : (\frac{\chi^U}{\min\{r_1(U), r_2(U)\}})^T x \leq 1,\ \forall U \subseteq E\}.
\end{aligned}$$

Notice that we defined the common fractional colouring number (3.5) as

$$\chi_{12}^* = \begin{array}{ll} \max & 1^T y \\ \text{s.t.} & y \in P \end{array},$$

where $P = \{y \geq 0 : y(I) \leq 1, \forall I \in \mathcal{I}_1 \cap \mathcal{I}_2\}$. Moreover, observe that we can write the polytope P as

$$\begin{aligned}
P &= \{y \geq 0 : y(I) \leq 1, \forall I \in \mathcal{I}_1 \cap \mathcal{I}_2\} \\
&= \{y \geq 0 : \sum_{e \in I} y_e \leq 1, \forall I \in \mathcal{I}_1 \cap \mathcal{I}_2\} \\
&= \{y \geq 0 : y^T \chi^I \leq 1, \forall I \in \mathcal{I}_1 \cap \mathcal{I}_2\} \\
&= \{y \geq 0 : y^T x \leq 1, \forall x \in P_{\mathcal{M}_1 \cap \mathcal{M}_2}\}.
\end{aligned}$$

Hence, $P = P^*_{\mathcal{M}_1 \cap \mathcal{M}_2}$, where $P^*_{\mathcal{M}_1 \cap \mathcal{M}_2}$ denotes the antiblocking polyhedron of $P_{\mathcal{M}_1 \cap \mathcal{M}_2}$. Thus,

$$\chi_{12}^* = \begin{array}{ll} \max & 1^T y \\ \text{s.t.} & y \in P^*_{\mathcal{M}_1 \cap \mathcal{M}_2}. \end{array}$$

It is known [24] that the facets of $P_{\mathcal{M}_1 \cap \mathcal{M}_2}$ correspond to the maximal vertices of $P^*_{\mathcal{M}_1 \cap \mathcal{M}_2}$. It follows that the optimal solution of the above linear program is attained at a vertex corresponding to some facet of $P_{\mathcal{M}_1 \cap \mathcal{M}_2}$. Hence,

$$\chi^*_{12} = \begin{array}{ll} \max & 1^T y \\ \text{s.t.} & y \in P^*_{\mathcal{M}_1 \cap \mathcal{M}_2} \end{array} = \begin{array}{ll} \max & 1^T a \\ \text{s.t.} & a^T x \leq 1 \text{ induces a facet of } P_{\mathcal{M}_1 \cap \mathcal{M}_2} \end{array}.$$

But we know that the facets of $P_{\mathcal{M}_1 \cap \mathcal{M}_2}$ are induced by $a^T x \leq 1$ with $a \in \{\frac{\chi U}{\min\{r_1(U), r_2(U)\}} : U \subseteq E\}$. Hence,

$$\chi^*_{12} = \begin{array}{ll} \max & 1^T(\frac{\chi U}{\min\{r_1(U), r_2(U)\}}) \\ \text{s.t.} & U \subseteq E \end{array} = \begin{array}{ll} \max & \frac{|U|}{\min\{r_1(U), r_2(U)\}} \\ \text{s.t.} & U \subseteq E \end{array}.$$

Given that $r_{12}(U) \leq \min\{r_1(U), r_2(U)\}$, it follows that

$$\chi^*_{12} = \max_{U \subseteq E} \frac{|U|}{\min\{r_1(U), r_2(U)\}} \leq \max_{U \subseteq E} \frac{|U|}{r_{12}(U)} = \omega^*_{12}.$$

□

Theorem 3.3.1 leads to a number of interesting results.

Corollary 3.3.1. $\max_{U \subseteq E} \frac{|U|}{r_{12}(U)} = \max_{U \subseteq E} \frac{|U|}{\min\{r_1(U), r_2(U)\}}$.

Proof. It follows directly from Theorem 3.3.1. □

Corollary 3.3.2. $\chi^*_{12} = \max\{\chi^*_1, \chi^*_2\}$.

Proof. From Theorem 3.3.1 we have

$$\chi^*_{12} = \max_{U \subseteq E} \frac{|U|}{\min\{r_1(U), r_2(U)\}} = \max\{\max_{U \subseteq E} \frac{|U|}{r_1(U)}, \max_{U \subseteq E} \frac{|U|}{r_2(U)}\} = \max\{\chi^*_1, \chi^*_2\}.$$

□

We find this last result very interesting and perhaps surprising, since in principle there is no obvious reason why there should not be any gap between the common fractional colouring and the maximum of the single fractional colourings.

Corollary 3.3.3. *Let $\mathcal{M}$ be a matroid. Then $\chi^*_{\mathcal{M}} = \omega^*_{\mathcal{M}}$.*

Proof. It follows from Theorem 3.3.1 by taking $\mathcal{M} = \mathcal{M}_1 = \mathcal{M}_2$. □

Lemma 3.3.1. *Let $\mathcal{M}$ be a matroid. Then $\chi_{\mathcal{M}} = \lceil \chi^*_{\mathcal{M}} \rceil$.*

Proof. Since $\chi_{\mathcal{M}} \geq \chi^*_{\mathcal{M}}$ we need to show that $\chi_{\mathcal{M}} - \chi^*_{\mathcal{M}} < 1$. In order to do this we will show that for any integer b such that $\mathcal{M}$ is not b-colourable we have $\chi^*_{\mathcal{M}} > b$. Then, since $\chi_{\mathcal{M}}$ is the smallest integer b such that $\mathcal{M}$ is b-colourable, it follows that $\chi^*_{\mathcal{M}} > \chi_{\mathcal{M}} - 1$.

Let $b \in \mathbb{Z}_+$ such that $\mathcal{M}$ is not b-colourable. By matroid union we know that $r_{\mathcal{M}^b}(E) < |E|$. Also, by using the rank formula

$$r_{\mathcal{M}^b}(U) = \min_{S \subseteq U} |E - S| + b \cdot r_{\mathcal{M}}(S),$$

we get $|E| > r_{\mathcal{M}^b}(E) = |E - S| + b \cdot r_{\mathcal{M}}(S)$ for some set $S \subseteq E$. Hence,

$$b < \frac{|E| - |E - S|}{r_{\mathcal{M}}(S)} = \frac{|S|}{r_{\mathcal{M}}(S)} \leq \max_{U \subseteq E} \frac{|U|}{r_{\mathcal{M}}(U)} = \omega^*_{\mathcal{M}} = \chi^*_{\mathcal{M}},$$

where the last equality follows from Corollary 3.3.3. □

Lemma 3.3.2. $\lceil \chi^*_{12} \rceil = k_{12}$.

Proof. Recall that $k_{12} = \max\{\chi_1, \chi_2\}$. Let us show first that $k_{12} \geq \chi^*_{12}$. From Theorem 3.3.1 we know

$$\chi^*_{12} = \max_{U \subseteq E} \frac{|U|}{\min\{r_1(U), r_2(U)\}} = \max\{\max_{U \subseteq E} \frac{|U|}{r_1(U)}, \max_{U \subseteq E} \frac{|U|}{r_2(U)}\}.$$

Moreover, since $\chi_i \geq \chi_i^* = \max_{U \subseteq E} \frac{|U|}{r_i(U)}$, it follows that

$$k_{12} = \max\{\chi_1, \chi_2\} \geq \max\{\max_{U \subseteq E} \frac{|U|}{r_1(U)}, \max_{U \subseteq E} \frac{|U|}{r_2(U)}\} = \chi_{12}^*.$$

Hence, $k_{12} \geq \chi_{12}^*$. Now, WLOG assume that $\chi_1 \geq \chi_2$. It follows that $k_{12} = \chi_1$. Since $k_{12} \geq \chi_{12}^* \geq \chi_1^*$, we have

$$0 \leq k_{12} - \chi_{12}^* \leq k_{12} - \chi_1^* = \chi_1 - \chi_1^* < 1,$$

where the last inequality follows from Lemma 3.3.1. Since k_{12} is an integer, we must have $\lceil \chi_{12}^* \rceil = k_{12}$. $\square$

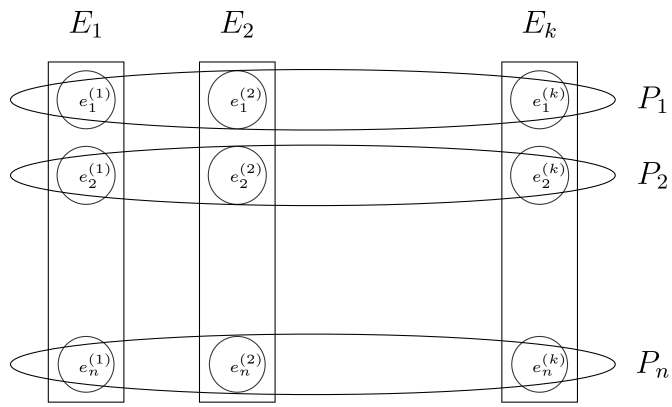
Corollary 3.3.4. $k_{12} = \omega_{12}$.

Proof. It follows directly from Theorem 3.3.1 and Lemma 3.3.2. $\square$

3.4 A 3-matroid intersection approach for common colouring

Let $\mathcal{M}_1 = (E, \mathcal{I}_1)$ and $\mathcal{M}_2 = (E, \mathcal{I}_2)$ be two matroids on a common ground set $E = \{e_1, ..., e_n\}$. Let $E_i = \{e_1^{(i)}, e_2^{(i)}, ..., e_n^{(i)}\}$ be disjoint copies of E with $1 \leq i \leq k$, and $E' = E_1 \cup E_2 \cup \cdots \cup E_k$. We define the projection $f : E' \to E$ such that $f(e_j^{(i)}) = e_j$ for $1 \leq j \leq n$ and $1 \leq i \leq k$.

Consider the matroid $\mathcal{M}_1' = (E', \mathcal{I}_1')$, where a set $I \subseteq E'$ is independent if and only if $f(I \cap E_i) \in \mathcal{I}_1$ for $i = 1, ..., k$ (i.e., $\mathcal{M}_1'$ is the matroid union of k disjoint copies of $\mathcal{M}_1$). We define $\mathcal{M}_2' = (E', \mathcal{I}_2')$ in the same way for $\mathcal{M}_2$. Let $\mathcal{T}_k = (E', \mathcal{I}_\mathcal{T})$ be the partition matroid on E' with partitions $P_1 = \{e_1^{(1)}, e_1^{(2)}, ..., e_1^{(k)}\}, ..., P_n = \{e_n^{(1)}, e_n^{(2)}, ..., e_n^{(k)}\}$, and capacities 1, i.e., a set $I \subseteq E'$ is independent in $\mathcal{T}_k$ if and only if $|I \cap P_i| \leq 1$ for $i \in [n]$. Observe that for each set $I \in \mathcal{I}_\mathcal{T}$ we have $|f(I)| = |I|$. The figure below shows this construction.



In this setting, there exists a common k-colouring of $\mathcal{M}_1$ and $\mathcal{M}_2$ if and only if $\max_{I \in \mathcal{I}_1' \cap \mathcal{I}_2' \cap \mathcal{I}_\mathcal{T}} |I| = |E|$. Indeed, if we can find such a set I we immediately have that $f(I \cap E_1), ..., f(I \cap E_k)$ is a common k-colouring. Adding the partition constraints (i.e. $\mathcal{T}_k$) to the intersection is key, otherwise we trivially have that

$$\max_{I \in \mathcal{I}_1' \cap \mathcal{I}_2'} |I| = k \cdot \max_{I \in \mathcal{I}_1 \cap \mathcal{I}_2} |I|,$$

since we can take k disjoint copies of a largest common independent set of $\mathcal{M}_1$ and $\mathcal{M}_2$. However, while the problem of finding the maximum common independent set for two matroids is solvable in polynomial time (matroid intersection algorithm), it is known that it becomes NP-Hard to find such a set when we are dealing with three or more matroids.

Theorem 3.4.1. *It is NP-Hard to find a maximum cardinality common independent set for three matroids.*

Proof. We reduce a general instance of the Hamiltonian path problem (which is known to be NP-Hard) to our problem. Given a digraph $D = (V, A)$ and two nodes $s, t \in V$, the Hamiltonian path problem consists of finding an s-t dipath that visits each node exactly once. We construct three matroids as follows: Let $\mathcal{M}_1 = (A, \mathcal{I}_1)$ be the graphic matroid on D, i.e., $\mathcal{M}_1 = \mathcal{M}(D)$.

Let $\mathcal{M}_2 = (A, \mathcal{I}_2)$ be the partition matroid given by

$$\mathcal{I}_2 = \left\{ F \subseteq A : \begin{array}{c} |\delta^-(v) \cap F| \leq 1, \ \forall v \neq s \\ |\delta^-(s) \cap F| = 0 \end{array} \right\},$$

and $\mathcal{M}_3 = (A, \mathcal{I}_3)$ the partition matroid given by

$$\mathcal{I}_3 = \left\{ F \subseteq A : \begin{array}{c} |\delta^+(v) \cap F| \leq 1, \ \forall v \neq t \\ |\delta^+(t) \cap F| = 0 \end{array} \right\}.$$

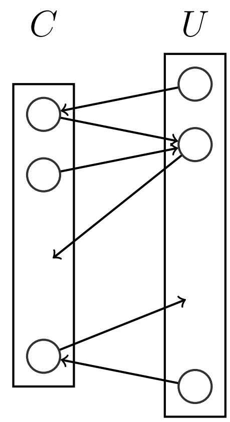
That is, a set $F \subseteq A$ is independent in $\mathcal{M}_2$ if each node has at most one incoming arc in this set (except s that has none), and independent in $\mathcal{M}_3$ if each node has at most one outgoing arc in this set (except t which has none). It follows that $F \subseteq A$ is a common independent set of $\mathcal{M}_1, \mathcal{M}_2$ and $\mathcal{M}_3$ if and only if F is the union of node disjoint dipaths with one of them starting at s and one ending at t. Hence, there exists a Hamiltonian path in D from s to t if and only if

$$\max_{I \in \mathcal{I}_1 \cap \mathcal{I}_2 \cap \mathcal{I}_3} |I| = |V| - 1.$$

$\square$

However, our 3-matroid intersection is a really special case since $\mathcal{T}_k$ is very simple, so it is not immediate that finding a maximum cardinality common independent set for $\mathcal{M}'_1 \cap \mathcal{M}'_2 \cap \mathcal{T}_k$ is NP-Hard. From now on we try to make use of the special structure of $\mathcal{T}_k$, and we try to find a modified version of the matroid intersection algorithm for $\mathcal{M}'_1 \cap \mathcal{M}'_2$ that takes into account the partition constraints from $\mathcal{T}_k$. The setting would be the following: At a given step we have a set of some coloured elements $C \in \mathcal{I}'_1 \cap \mathcal{I}'_2$, and a set of the remaining uncoloured elements that we denote by U (i.e. $U = E - C$). We say that C is *feasible* if it does not contain two copies of the same element of E, i.e., if $|f(C)| = |C|$. Our goal is to run some sort of matroid intersection algorithm with bipartitions C and U in a way that the set C remains feasible at each step of the algorithm. In other words, we want to find a shortest X_1-X_2 augmenting path in $D_{\mathcal{M}'_1 \cap \mathcal{M}'_2}(C)$ such that after doing the augmentation our new set of coloured elements (which has one more element than the previous one) remains feasible.

$D_{\mathcal{M}'_1 \cap \mathcal{M}'_2}(C)$



$$X_1 = \{z \in U : C + z \in \mathcal{I}'_1\}$$
$$X_2 = \{z \in U : C + z \in \mathcal{I}'_2\}$$

We consider the following approach: If $e_i^{(j)} \in C$ for some $i \in \{1, ..., n\}$ and $j \in \{1, ..., k\}$, we say that the elements $e_i^{(l)}$ are *constrained* for all $l \neq j$. Otherwise, we say that the elements $e_i^{(j)}$ are *free* for all $j \in \{1, ..., k\}$. Clearly, we can only add a constrained element to C if we take out its copy in the same step. We can add any free element to C as long as we do not add two copies of the same element at the same time.

Let P be a X_1-X_2 path in the exchange digraph $D_{\mathcal{M}'_1 \cap \mathcal{M}'_2}(C)$. We say that P is *feasible* if it does not have any shortcuts and if it satisfies the following two conditions: (1) If a constrained element belongs to the path, then its copy from C must also belong to the path, and (2) The path cannot contain two free elements that are a copy of the same element (this can also be written as $|f(P \cap U)| = |P \cap U|$). If we can find such a path, the analysis for the matroid intersection algorithm guarantees that the corresponding augmentation would lead to a larger common independent set, while constraints (1) and (2) guarantee that the new set C remains feasible. It follows that the existence of such a path is a sufficient condition for having a valid augmentation of C. We believe it would be very interesting to show whether this condition is necessary as well, i.e., if our current set C is not optimal then such a path must exist.

We are also interested in the complexity of finding such a path. In the next chapter we consider this problem in a more general setting and show that it is NP-Hard in this general setting. However, the nice properties of matroids and the special structure of the exchange digraph $D_{\mathcal{M}'_1 \cap \mathcal{M}'_2}(C)$ do not allow us to determine if the path problem defined above is also NP-Hard.

3.5 SBO matroids and the common colouring problem

In this section we introduce two special types of matroids: strongly base orderable and weakly base orderable matroids. We will see that the common colouring problem can be solved exactly when the two matroids in the intersection are strongly base orderable [5].

Definition 3.5.1. *A matroid is strongly base orderable (abbreviated SBO) if for any two bases B_1 and B_2 there is a bijection $f : B_1 \to B_2$ with the property that $B_1 - X + f(X)$ is a base for any $X \subseteq B_1$.*

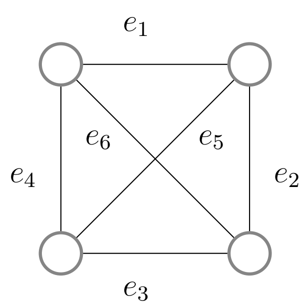
Notice that from the definition it follows that $B_2 + X - f(X)$ is also a base for any $X \subseteq B_1$, since $B_2 + X - f(X) = B_1 - Y + f(Y)$ with $Y = B_1 - X \subseteq B_1$.

Definition 3.5.2. *A matroid is weakly base orderable (abbreviated WBO) if for any two bases there is a bijection $f : B_1 \to B_2$ with the property that $B_1 - e + f(e)$ and $B_2 - f(e) + e$ are bases for any $e \in B_1$.*

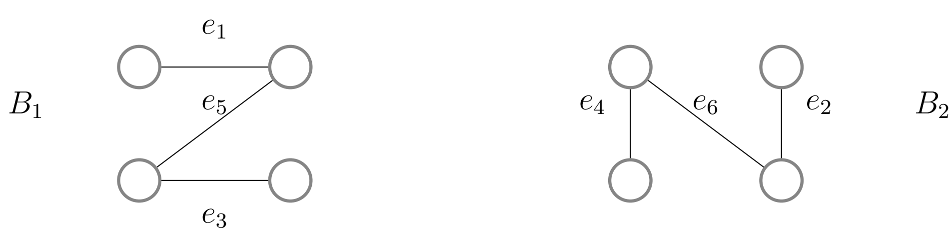
It is clear from the definitions that every strongly base orderable matroid is weakly base orderable. Uniform matroids, gammoids, P_7, and transversal matroids are SBO, while P_8 is WBO but not SBO. $\mathcal{M}(K_4)$, i.e., the graphic matroid on the complete graph on four nodes, is not WBO. We prove this last result since it is going to be useful later in this section.

Claim 3.5.1. $\mathcal{M}(K_4)$ *is not weakly base orderable.*

Proof. Let us denote the ground set E as follows



and let B_1 and B_2 be the following bases of $\mathcal{M}(K_4)$



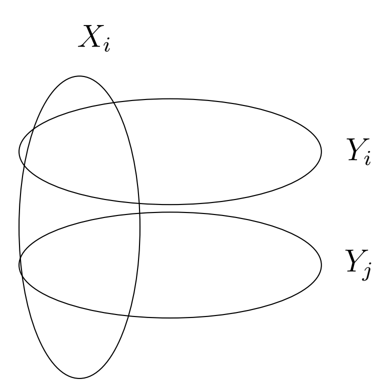
We show that there is no bijection $f : B_1 \to B_2$ such that $B_1 - e + f(e)$ and $B_2 - f(e) + e$ are bases for any $e \in B_1$. Let us start by deciding the image for e_1, i.e., $f(e_1)$. We have three different possibilities for this, either $f(e_1) = e_2$, or $f(e_1) = e_4$, or $f(e_1) = e_6$. Notice that $f(e_1)$ cannot be e_2, since $B_1 - e_1 + e_2$ has a cycle, i.e., $B_1 - e_1 + e_2$ is not independent anymore. In the same way, we cannot define $f(e_1)$ as e_4 since $B_2 - e_4 + e_1$ has a cycle. Hence, the only possible image for e_1 is e_6 (it is easy to check that both $B_1 - e_1 + e_6$ and

$B_2 - e_6 + e_1$ are acyclic). Next, we decide the image for e_3. We have two different possibilites, either $f(e_3) = e_2$ or $f(e_3) = e_4$. But observe that none of these are feasible since both $B_2 - e_2 + e_3$ and $B_1 - e_3 + e_4$ have cycles. Hence, $\mathcal{M}(K_4)$ is not weakly base orderable. $\square$

Davies and McDiarmid [5] showed the following interesting result regarding the colouring number of the intersection of two strongly base orderable matroids.

Theorem 3.5.1. *Let $\mathcal{M}_1 = (E, \mathcal{I}_1)$ and $\mathcal{M}_2 = (E, \mathcal{I}_2)$ be two SBO matroids on a common ground set. Then $\chi_{12} = k_{12}$.*

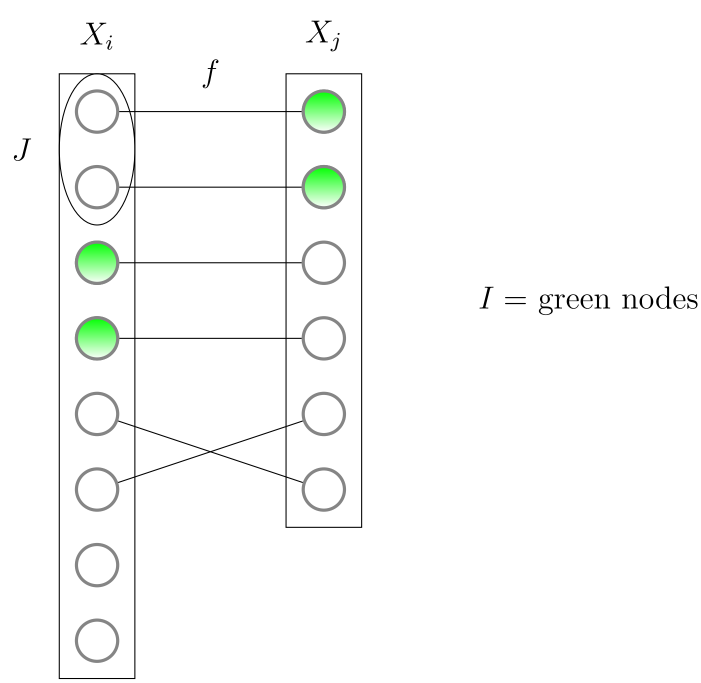
Proof. Let $X_1, ..., X_{k_{12}}$ be a colouring of $\mathcal{M}_1$ and $Y_1, ..., Y_{k_{12}}$ be a colouring of $\mathcal{M}_2$ which maximize $\sum_{i=1}^{k_{12}} |X_i \cap Y_i|$. If this sum equals $|E|$ (i.e., $X_i = Y_i$ for $i \in [k_{12}]$) we are done, since then $X_1, ..., X_{k_{12}}$ is a colouring of $\mathcal{M}_1 \cap \mathcal{M}_2$ and hence $\chi_{12} = k_{12}$; so assume $\sum_{i=1}^{k_{12}} |X_i \cap Y_i| < |E|$. Let $i \in [k_{12}]$ be such that $X_i \neq Y_i$, and WLOG assume that $X_i - Y_i \neq \emptyset$. Then we can find $i \neq j \in [k_{12}]$ such that $Y_j \cap X_i \neq \emptyset$. Extend the colour classes X_i, X_j to bases C_i, C_j of $\mathcal{M}_1$ and the colour classes Y_i, Y_j to bases D_i, D_j of $\mathcal{M}_2$ respectively.



Given that both $\mathcal{M}_1$ and $\mathcal{M}_2$ are strongly base orderable, we can find a bijection $f : C_i \to C_j$ satisfying that $(C_i - X) \cup f(X)$ is a base of $\mathcal{M}_1$ for any

$X \subseteq C_i$, and a bijection $g : D_i \to D_j$ such that $(D_i - X) \cup g(X)$ is a base of $\mathcal{M}_2$ for any $X \subseteq D_i$. Now consider the graph G on the set of nodes E whose edges are given by $\{(e, f(e)) : e \in C_i)\} \cup \{(e, g(e)) : e \in D_i\}$. Notice that the sets of edges $\{(e, f(e)) : e \in C_i)\}$ and $\{(e, g(e)) : e \in D_i\}$ can be seen as two matchings, since f and g are bijections from C_i, D_i to C_j, D_j respectively. Hence, the set of edges of G is given by the union of two matchings and thus G is a bipartite graph. Let $S \cup T$ be its bipartition.

Let I be an independent set in the graph G (i.e., there is no edge between any two nodes in I) that is contained in $X_i \cup X_j$. We claim that $I \in \mathcal{I}_1$, i.e., I is independent in $\mathcal{M}_1$. To see this, let $J := f^{-1}(I \cap X_j)$. Then, by definition of f we have that $C_i - J + f(J) \in \mathcal{I}_1$, and since $I \subseteq X_i - J + f(J) \subseteq C_i - J + f(J)$ it follows that $I \in \mathcal{I}_1$. In the same way, if I is an independent set in the graph G that is contained in $Y_i \cup Y_j$, then I is independent in $\mathcal{M}_2$. The picture below depicts this argument.



Now define $X_i' = S \cap (X_i \cup X_j)$, $X_j' = T \cap (X_i \cup X_j)$, and similarly for Y_i', Y_j'. By the above observation, we know that $X_i', X_j' \in \mathcal{I}_1$ and $Y_i', Y_j' \in \mathcal{I}_2$. Also, notice that $X_i' \cup X_j' = X_i \cup X_j$ and $Y_i' \cup Y_j' = Y_i \cup Y_j$. We claim that

$$|X_i' \cap Y_i'| + |X_j' \cap Y_j'| > |X_i \cap Y_i| + |X_j \cap Y_j|.$$

To see this let $e \in X_i \cap Y_i$, then either $e \in S$ or $e \in T$. If $e \in S$, we have that $e \in X_i'$ and $e \in Y_i'$, and hence $e \in X_i' \cap Y_i'$. If $e \in T$, a similar argument shows that $e \in X_j' \cap Y_j'$. In the same way, for each element $e \in X_j \cap Y_j$ we have that either $e \in X_i' \cap Y_i'$ or $e \in X_j' \cap Y_j'$. Thus,

$$|X_i' \cap Y_i'| + |X_j' \cap Y_j'| \geq |X_i \cap Y_i| + |X_j \cap Y_j|.$$

However, we picked i and j such that $X_i \cap Y_j \neq \emptyset$. Let $e \in X_i \cap Y_j$ and observe that $e \notin (X_i \cap Y_i) \cup (X_j \cap Y_j)$. Moreover, we have that either $e \in X_i' \cap Y_i'$ or $e \in X_j' \cap Y_j'$, depending on whether $e \in S$ or $e \in T$ respectively. It follows that

$$|X_i' \cap Y_i'| + |X_j' \cap Y_j'| > |X_i \cap Y_i| + |X_j \cap Y_j|,$$

but this contradicts the maximality of $\sum_{l \in [k_{12}]} |X_l \cap Y_l|$.

Thus, we can can find colourings $X_1, ..., X_{k_{12}}$ of $\mathcal{M}_1$ and $Y_1, ..., Y_{k_{12}}$ of $\mathcal{M}_2$ such that $\sum_{l \in [k_{12}]} |X_l \cap Y_l| = |E|$ and the theorem follows. $\square$

A natural question is whether the converse holds, i.e., if we have two matroids $\mathcal{M}_1$ and $\mathcal{M}_2$ such that $\chi_{12} = k_{12}$, are the matroids necessarily SBO? The answer is no. In fact, we cannot even guarantee that the matroids are WBO. A simple example is to take $\mathcal{M}(K_4)$ and $U_{3,6}$. It is easy to check that both matroids are 2-colourable, i.e., $k_{12} = 2$. Moreover, notice that $\mathcal{M}(K_4) \cap U_{3,6} = \mathcal{M}(K_4)$, since every subset of cardinality at most 3 is independent in $U_{3,6}$, and $\mathcal{M}(K_4)$ has rank 3 (i.e., every independent set of $\mathcal{M}(K_4)$ is also independent in $U_{3,6}$). It follows that the intersection is 2-colourable, i.e., $\chi_{12} = 2$. However, as shown in Claim 3.5.1, $\mathcal{M}(K_4)$ is not weakly base orderable.

Another natural question is whether we can adapt the above argument to two WBO matroids, or one SBO and one WBO. The answer to this is that we do not know. The SBO condition is key in the above proof to show that X_i', X_j', Y_i', and Y_j' are independent sets of $\mathcal{M}_1$ and $\mathcal{M}_2$ respectively.

CHAPTER 4
The directed path with partition constraints problem and applications

4.1 The problem

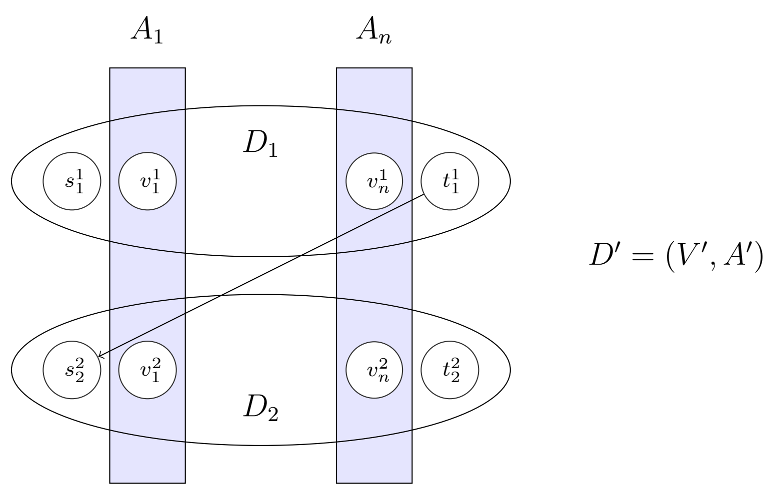
Recall that in the previous chapter we considered the problem of finding what we called a *feasible* path in the exchange graph $D_{\mathcal{M}'_1 \cap \mathcal{M}'_2}(C)$. Such a path would automatically lead to a valid augmentation of the current common independent set $C \in \mathcal{M}'_1 \cap \mathcal{M}'_2$. Moreover, there were some partition constraints (encoded in the matroid $\mathcal{T}_k$) guaranteeing that the sets $f(C \cap E_1), ..., f(C \cap E_k)$ were pairwise disjoint and independent in both $\mathcal{M}_1$ and $\mathcal{M}_2$. Our goal was to get a good approximation of a common k-colouring of $\mathcal{M}_1$ and $\mathcal{M}_2$ by finding a good approximation of $\max_{C \in \mathcal{I}'_1 \cap \mathcal{I}'_2 \cap \mathcal{I}_{\mathcal{T}}} |C|$.

In this chapter we study a problem that arises naturally from the setting discussed above. Given a digraph $D = (V, A)$ with a source node s, a sink t, and a partition $A_1, ..., A_k$ of $V - \{s, t\}$, find an s-t dipath that contains at most one node from each set A_i. We denote this problem by DPPC (Directed Path with Partition Constraints). Note that some of the sets A_i can be singletons. Also, WLOG we can assume that the sets A_i are stable sets. After writing this thesis we found out that this problem generalizes the *path avoiding forbidden pairs* (PAFP) problem, first studied by Krause et al. [17] and shown to be NP-Hard by Gabow et al. [11]. In this chapter we give an alternative hardness proof for the DPPC problem and mention some applications.

Lemma 4.1.1. *The DPPC problem is NP-Complete.*

Proof. Given a digraph D with two (not necessarily distinct) sources s_1, s_2 and two (not necessarily distinct) sinks t_1, t_2, it is known that the problem of finding s_1-t_1 and s_2-t_2 node disjoint dipaths is NP-Complete (Fortune, Hopcroft, and Wyllie [8]). We reduce a general instance of this problem to the DPPC problem.

Given a digraph $D = (V, A)$ with sources s_1, s_2, and sinks t_1, t_2, we create an instance of DPPC as follows: Create two disjoint copies of D, call them D_1 and D_2. Denote by v_i^j the node in D_j which is a copy of $v_i \in V$ in D. Delete s_2^1 and t_2^1 from D_1, and s_1^2 and t_1^2 from D_2. Add an arc from t_1^1 to s_2^2. Call this new digraph $D' = (V', A')$. Finally, for each node $v_i \in V - \{s_1, s_2, t_1, t_2\}$, let $A_i := \{v_i^1, v_i^2\}$. The picture belows depicts this construction.



Now, consider an instance of DPPC with digraph D', source s_1^1, sink t_2^2, and a partition $A_1, ..., A_k$ of $V' - \{s_1^1, t_2^2\}$. Then, notice that any s_1^1-t_2^2 dipath will first follow a dipath P_1 from s_1^1 to t_1^1 in D_1, then use the arc (t_1^1, s_2^2), and finally follow a dipath P_2 from s_2^2 to t_2^2 in D_2. Moreover, our partition constraints guarantee that we use at most one node from each pair $\{v_i^1, v_i^2\}$. Hence, P_1 and P_2 will denote s_1-t_1 and s_2-t_2 node disjoint dipaths in our original digraph D. $\square$

Observe that from a general instance of DPPC we can get a bipartite instance by just subdividing each arc and assigning the new node arbitrarily to any of the two sets A_i that the original arc was visiting. Hence, we have the following result.

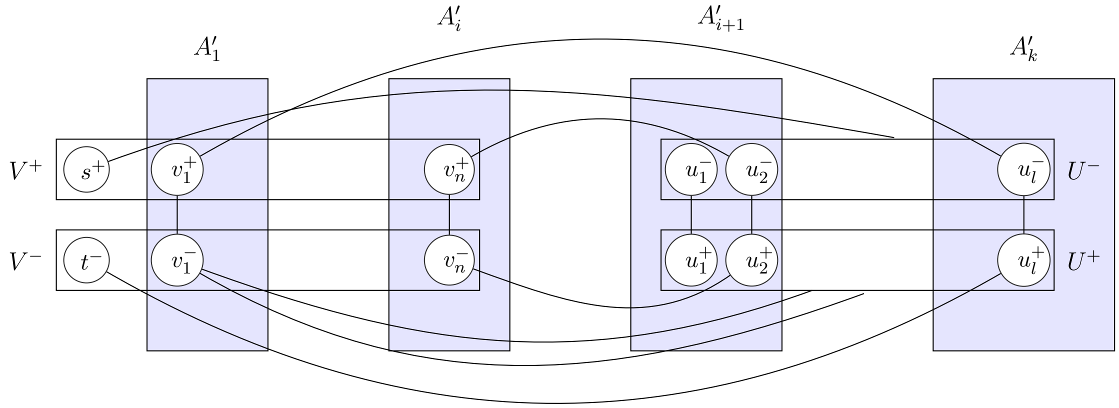
Corollary 4.1.1. *DPPC is NP-Complete for bipartite digraphs.*

4.2 Applications

4.2.1 The perfect matching with partition constraints problem

Given an instance of DPPC with a bipartite digraph $D = (V \cup U, A)$, source $s \in V$, sink $t \in V$, and a partition $A_1, ..., A_k$ of $(V \cup U) - \{s, t\}$, we create an undirected bipartite graph using an idea due to Edmonds as follows:

Let $V^+ := \{s^+, t^+, v_1^+, ..., v_n^+\}$ and $V^- := \{s^-, t^-, v_1^-, ..., v_n^-\}$ be two copies of V, and $U^+ := \{u_1^+, ..., u_l^+\}$ and $U^- := \{u_1^-, ..., u_l^-\}$ be two copies of U. Delete t^+ from V^+ and s^- from V^-. For every arc $uv \in A$ add an (undirected) edge between u^+ and v^-. In addition, we add edges between v_i^+ and v_i^- for all $i \in [n]$, and between u_i^+ and u_i^- for all $i \in [l]$. Denote by E the set of edges of this new graph. Then it is easy to see that the graph $B_D = ((V^+ \cup U^+) \cup (V^- \cup U^-), E)$ is bipartite. Moreover, for each $A_i = \{u_{i_1}, ..., u_{i_p}, v_{j_1}, ..., v_{j_m}\}$ we define $A_i' := \{u_{i_1}^+, u_{i_1}^-, ..., u_{i_p}^+, u_{i_p}^-, v_{j_1}^+, v_{j_1}^-, ..., v_{j_m}^+, v_{j_m}^-\}$. The figure below shows the construction of B_D.



We consider the problem of finding a perfect matching M on B_D such that $|M \cap \delta(A'_i)| \leq 2$ for all $i \in [k]$. We denote this problem by PMPC (Perfect Matching with Partition Constraints).

Lemma 4.2.1. *A perfect matching M as above exists if and only if there is a feasible s-t dipath in D for the DPPC problem.*

Proof. First, assume that such an s-t dipath P exists. Denote this path by $su_{i_1}v_{j_1}u_{i_2}v_{j_2}...v_{j_{p-1}}u_{i_p}t$. Then, we get a perfect matching M on B_D by first taking $\{(s^+, u_{i_1}^-), (u_{i_1}^+, v_{j_1}^-), (v_{j_1}^+, u_{i_2}^-), ..., (v_{j_{p-1}}^+, u_{i_p}^-), (u_{i_p}^+, t^-)\}$. Next, for each node u_i^+ that has not been matched yet take the edge (u_i^+, u_i^-), and for each node v_i^+ that has not been matched yet take the edge (v_i^+, v_i^-). It is easy to check that this indeed gives a perfect matching M. Moreover, since P is a solution to the DPPC problem, we must have that M satisfies $|M \cap \delta(A'_i)| \leq 2$ for all $i \in [k]$.

Now, assume that we have a perfect matching M on B_D satisfying $|M \cap \delta(A'_i)| \leq 2$ for all $i \in [k]$. We get a feasible s-t dipath P for DPPC as follows: First, delete from M the edges of the form (u_i^+, u_i^-) and (v_i^+, v_i^-). After doing this, we are left with a set of edges of the form

$$\{(s^+, u^-_{i_1}), (u^+_{i_1}, v^-_{j_1}), (v^+_{j_1}, u^-_{i_2}), ..., (v^+_{j_{p-1}}, u^-_{i_p}), (u^+_{i_p}, t^-)\}.$$

We claim that $su_{i_1}v_{j_1}u_{i_2}v_{j_2}...v_{j_{p-1}}u_{i_p}t$ is a feasible s-t dipath P for DPPC. The fact that $su_{i_1}v_{j_1}u_{i_2}v_{j_2}...v_{j_{p-1}}u_{i_p}t$ is an s-t dipath in D follows from the construction of B_D and the fact that

$$\{(s^+, u^-_{i_1}), (u^+_{i_1}, v^-_{j_1}), (v^+_{j_1}, u^-_{i_2}), ..., (v^+_{j_{p-1}}, u^-_{i_p}), (u^+_{i_p}, t^-)\} \subseteq E.$$

To see feasibility with respect to the partition sets $A_1, ..., A_k$, notice that $|M \cap \delta(A'_i)| \leq 2$ for all $i \in [k]$ if and only if $su_{i_1}v_{j_1}u_{i_2}v_{j_2}...v_{j_{p-1}}u_{i_p}t$ contains at most one node from each A_i. □

Corollary 4.2.1. *The perfect matching problem with partition constraints is NP-Complete.*

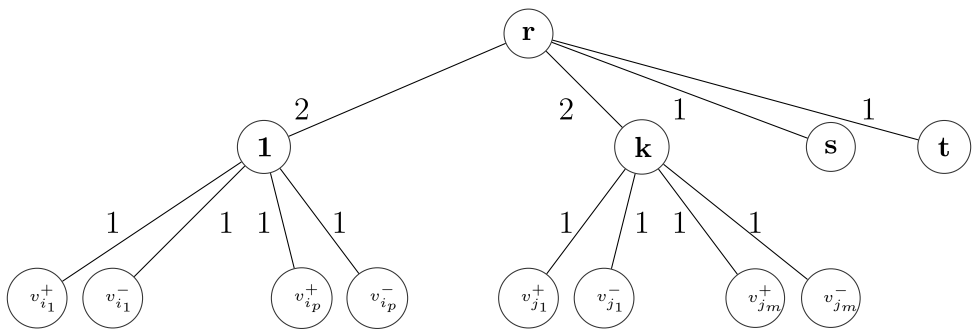
Proof. This follows from Corollary 4.1.1 and Lemma 4.2.1. □

4.2.2 The maximum integral flow problem is NP-Hard for trees with edge capacities 1 and 2

This result was first proved by Garg-Vazirani-Yannakakis in [12], by reducing the three-dimensional matching problem to the maximum integral flow problem. Here we give a different proof of this result by making use of the hardness result for PMPC (Corollary 4.2.1).

Let $B = (V, E)$ be an instance of the perfect matching problem with partition constraints described above, with B a bipartite (undirected) graph. We have seen that this problem is NP-Complete. We construct a tree T_B of height 2 as follows:

The nodes at level 1 are labelled: $1, 2, ..., k, s, t$. Each node $i \in [k]$ has as children the elements of the set A_i. Thus, there are $k+2$ nodes in the first level and $|V|-2$ in the second level. Edges (r,i) have capacity 2 for every $i \in [k]$. All other edges have unit capacity. The figure below depicts the construction of the tree.



For each edge $(u,v) \in E$ where u has a "+" label and v has a "−" label, we add a source-sink pair (u,v). NP-Hardness now follows from:

Claim 4.2.1. *The graph B has a perfect matching M satisfying $|M \cap \delta(A_i)| \leq 2$ for all $i \in [k]$ if and only if T_B has an integral flow of $\frac{|V|}{2}$ units.*

Proof. Assume M is a perfect matching on B satisfying $|M \cap \delta(A_i)| \leq 2$ for all $i \in [k]$. For every edge in M of the form (v_i^+, v_i^-), we route a unit flow for the source-sink pair (v_i^+, v_i^-). After doing this, we must be left with a set of edges in M of the form $\{(s^+, v_{i_1}^-), (v_{i_1}^+, v_{i_2}^-), ..., (v_{i_{p-1}}^+, v_{i_p}^-), (v_{i_p}^+, t^-)\}$. For each of these edges (u,v) we route a unit flow for the source-sink pair (u,v). Now, since M satisfies $|M \cap \delta(A_i)| \leq 2$ for all $i \in [k]$, it follows that the integral flow defined above is feasible with respect to the edge capacities of the tree T_B. Moreover, it is a $\frac{|V|}{2}$ units flow since M is a perfect matching on B, i.e. $|M| = \frac{|V|}{2}$.

Conversely, assume T_B has an integral flow of $\frac{|V|}{2}$ units. We get a perfect matching M on B as follows: for each source-sink pair (u,v) routed by the flow (note that there must be $\frac{|V|}{2}$ of these pairs), add the edge (u, v) to M. To see that the condition $|M \cap \delta(A_i)| \leq 2$ holds for every $i \in [k]$, notice that the edge (r, i) has capacity 2 for each $i \in [k]$. Hence, for each $i_0 \in [k]$ we can have at most two source-sink pairs with exactly one element being a child of i_0. Thus, $|M \cap \delta(A_{i_0})| \leq 2$. $\square$

Bibliography

[1] Garrett Birkhoff. On the combination of subalgebras. In *Mathematical Proceedings of the Cambridge Philosophical Society*, volume 29, pages 441–464. Cambridge Univ Press, 1933.

[2] Yuri Y Boykov and M-P Jolly. Interactive graph cuts for optimal boundary & region segmentation of objects in nd images. In *Computer Vision, 2001. ICCV 2001. Proceedings. Eighth IEEE International Conference on*, volume 1, pages 105–112. IEEE, 2001.

[3] Richard A Brualdi. Comments on bases in dependence structures. *Bulletin of the Australian Mathematical Society*, 1(02):161–167, 1969.

[4] Tom Brylawski and Douglas G Kelly. *Matroids and combinatorial geometries*. Dept. of Mathematics, University of North Carolina at Chapel Hill, 1980.

[5] Joan Davies and Colin McDiarmid. Disjoint common transversals and exchange structures. *Journal of the London Mathematical Society*, 2(1):55–62, 1976.

[6] J. Edmonds. Matroids, submodular functions and certain polyhedra. *Combinatorial Structures and Their Applications*, pages 69–87, 1970.

[7] Jack Edmonds. Lehmans switching game and a theorem of tutte and nash-williams. *J. Res. Nat. Bur. Standards B*, 69:73–77, 1965.

[8] Steven Fortune, John Hopcroft, and James Wyllie. The directed subgraph homeomorphism problem. *Theoretical Computer Science*, 10(2):111–121, 1980.

[9] Alan M Frieze. A cost function property for plant location problems. *Mathematical Programming*, 7(1):245–248, 1974.

[10] Satoru Fujishige. Polymatroidal dependence structure of a set of random variables. *Information and Control*, 39(1):55–72, 1978.

[11] Harold N Gabow, Shachindra N Maheshwari, and Leon J Osterweil. On two problems in the generation of program test paths. *Software Engineering, IEEE Transactions on*, (3):227–231, 1976.

[12] Naveen Garg, Vijay V. Vazirani, and Mihalis Yannakakis. Primal-dual approximation algorithms for integral flow and multicut in trees. *Algorithmica*, 18(1):3–20, 1997.

[13] Stefanie Jegelka and Jeff Bilmes. Submodularity beyond submodular energies: coupling edges in graph cuts. In *Computer Vision and Pattern Recognition (CVPR), 2011 IEEE Conference on*, pages 1897–1904. IEEE, 2011.

[14] David Kempe, Jon Kleinberg, and Éva Tardos. Maximizing the spread of influence through a social network. In *Proceedings of the ninth ACM SIGKDD international conference on Knowledge discovery and data mining*, pages 137–146. ACM, 2003.

[15] Pushmeet Kohli, M Pawan Kumar, and Philip HS Torr. P^3 & beyond: Move making algorithms for solving higher order functions. *Pattern Analysis and Machine Intelligence, IEEE Transactions on*, 31(9):1645–1656, 2009.

[16] Andreas Krause and Carlos Guestrin. Near-optimal observation selection using submodular functions. In *AAAI*, volume 7, pages 1650–1654, 2007.

[17] KW Krause, MA Goodwin, and RW Smith. *Optimal software test planning through automated network analysis.* TRW Systems Group, 1973.

[18] S Krogdahl. A combinatorial proof for a weighted matroid intersection algorithm. *Univ. of*, 1976.

[19] Stein Krogdahl. A combinatorial base for some optimal matroid intersection algorithms. Technical report, DTIC Document, 1974.

[20] Stein Krogdahl. The dependence graph for bases in matroids. *Discrete Mathematics*, 19(1):47–59, 1977.

[21] Hui Lin and Jeff Bilmes. A class of submodular functions for document summarization. In *Proceedings of the 49th Annual Meeting of the Association for Computational Linguistics: Human Language Technologies-Volume 1*, pages 510–520. Association for Computational Linguistics, 2011.

[22] László Lovász. Submodular functions and convexity. In *Mathematical Programming The State of the Art*, pages 235–257. Springer, 1983.

[23] Crispin St John Alvah Nash-Williams. An application of matroids to graph theory. In *Theory of GraphsInternational SymposiumThéorie des graphesJournées internationales détude (Rome*, pages 263–265, 1966.

[24] Alexander Schrijver. *Combinatorial optimization: polyhedra and efficiency*, volume 24. Springer, 2003.

[25] Matthew Streeter and Daniel Golovin. An online algorithm for maximizing submodular functions. In *Advances in Neural Information Processing Systems*, pages 1577–1584, 2009.

[26] Hassler Whitney. On the abstract properties of linear dependence. *American Journal of Mathematics*, pages 509–533, 1935.