DISTRIBUTION PROBLEMS CONNECTED WITH THE MULTIVARIATE LINEAR FUNCTIONAL RELATIONSHIP MODELS

by

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A thesis submitted to the Faculty of Graduate Studies and Research, in partial fulfillment of the requirements for the degree of Doctor of Philosophy.

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Montreal.

January 1984

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Abstract

This thesis deals with a linear functional relationship model in which the unobserved true values satisfy multiple linear restrictions. New test statistics for some of the structural parameters of this model are derived under the assumptions that the observation vectors are normally distributed and that an estimator of the covariance matrix of measurement error is available from independent experiments or replicated observations. Exact null distributions for some test statistics proposed by other authors are also given. In order to obtain computable representations of the densities of these test statistics, the exact densities of many algebraic functions of independent gamma variates are derived in computable forms using the technique of inverse Mellin transform. Alternate representations of some of these densities are also expressed in terms of the density of the product of independent beta type-2 random variables. Finally applications to some econometric errors-in-variables functional models are pointed out.

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SUR LA DISTRIBUTION DE STATISTIQUES RELIÉES

À DES MODÈLES FONCTIONNELS MULTILINÉAIRES

Résumé

Cette thèse a pour objet l'étude de modèles fonctionnels pour lesquels les espérances de vecteurs d'observations distribués selon une loi multinormale satisfont de multiples contraintes linéaires. Supposant qu'un estimateur non biaisé de la matrice de covariance provenant soit d'observations répétées ou encore d'expériences antérieures soit disponible, nous proposons de nouvelles statistiques permettant de tester des hypothèses portant sur les coefficients associés à ces modèles multilinéaires. Afin de déterminer les densités respectives de ces statistiques, nous obtenons sous des formes se prêtant à l'évaluation numérique, les densités exactes de plusieurs fonctions algébriques de variables aléatoires gamma à l'aide de la transformée inverse de Mellin. Nous représentons également certaines de ces densités au moyen d'un produit de variables beta du second type. Nous proposons finalement des applications à quelques modèles économétriques pour lesquels les variables sont toutes sujettes à des erreurs d'évaluation.

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Université McGill,
Montréal Janvier 1984

What is new in this thesis ?

Most of the results given in Chapters 3,4 and 5 are believed to be new and not available anywhere in the literature except the basic materials and the discussions of the techniques used in these chapters.

(1) - The test statistics for the coefficients of multivariate linear functional relationship models given in (3.14) , (3.19) , (3.31) , (3.46) , (3.49) , (3.67) , (3.70) , (3.75) and (3.77) and their respective distributions obtained in computable forms. (Chapters 3 , 4 and 5)

(2) - The exact distribution of two test statistics ((3.2) and (3.8)) proposed by A.P. Basu given in (4.107) and (3.81) respectively.

(3) - The exact density of the k-th root of a product of independent chi-square variates as the density of a product of independent generalized gamma variates. (Section 4.2)

(4) - A representation of the density of a linear combination of independent gamma variates obtained with the technique of inverse Mellin transform. (Section 4.3)

(5) - The exact distribution of a statistic denoted by R, whose numerator is a linear combination of independent gamma variates and whose denominator is the k-th root of a product of independent gamma variates.(Section 4.4)

(6) - The derivation of the h-th moment of R for the case where h is a positive integer allowing one to select the Pearson curve that will best fit its distribution. (Section 4.5)

(7) - An identity expressing Meijer's G-functions of the type $G^{k,k}_{k,k}(\cdot)$ in terms of the densities associated with the product of independent beta type-2 random variables which provides us with another representation of the density of R. (Chapter 5)

Besides these new results,

(1') - the multivariate errors-in-variables functional models and the multivariate linear functional relationship models are presented as generalizations of the multivariate general linear hypothesis model; (Chapter 1)

(2') - a more detailed derivation of Villegas' (1964) test statistic for the coefficients of a single linear functional relationship model is given; (Section 2.1)

(3') - a more systematic derivation of Basu's (1969) test statistic for the coefficients of a multivariate linear functional relationship model is proposed; (Section 2.2)

(4') - an outline of possible applications of the results to the econometric multivariate errors-in-variables functional model is given. (Chapter 6)

<u>Papers resulting from this thesis</u>

Provost, Serge (1983). On Some Tests for the Parameters of a Multivariate Linear Functional Relationship Model. Submitted to The Journal of the Royal Statistical Society, Series B.

Provost, Serge (1983). On the exact distribution of the ratio of a linear combination of chi-square variates over the root of a product of chi-square variates. Submitted to The Canadian Journal of Statistics.

Provost, Serge (1984). Some test statistics for the multivariate errors-in-variables functional models. (In preparation; applications of the results in Chapters 3 and 4 to econometric problems.)

Provost, Serge (1984). A new representation of a test statistic connected with the multivariate linear functional relationship model. (In preparation; expansion of the material in Chapter 5.)

ACKNOWLEDGEMENTS

First and foremost, I would like to express my sincere gratitude to my thesis supervisor, Professor A.M. MATHAI ; for his genuine concern and his expert guidance. Thanks are also due to the professorial and administrative staff of McGill University, who have assisted in my education .

Financial assistance has been extended to the author by the Natural Sciences and Engineering Research Concil of Canada and the provincial F.C.A.C. funds. This support is gratefully acknowledged.

Finally, I must express my appreciation to all the people who have encouraged me in the course of my studies. Of these, I wish to single out my parents for the values they have instilled in me and for the unfailing support they have given me.

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SYMBOLS

Note: In this thesis, all vectors and all matrices will be denoted by underlined lower-case and capital letters respectively, the only exception being I which will represent the identity matrix of the appropriate order.

Σ	- summation sign
Π	- product sign
$E(\cdot)$	- mathematical expectation
$var(\cdot)$	- variance
$cov(\cdot)$	- covariance
$Re(\cdot)$	- real part of $(\cdot)$
$\Gamma(\alpha)$	- $\int_0^\infty x^{\alpha-1} e^{-x} dx$, $Re(\alpha) > 0$
$\sim$	- " is distributed as "
$\overset{ind}{\sim}$	- " are independently distributed as "
$N(\mu,\sigma^2)$	- univariate normal distribution with mean value μ and variance σ^2
$N_p(\underset{\sim}{\mu},\underset{\sim}{V})$	- p-variate normal distribution with mean vector $\underset{\sim}{\mu}$ and covariance matrix $\underset{\sim}{V}$
χ_r^2	- chi-square distribution with r degrees of freedom
$W_p(\nu,\underset{\sim}{\Sigma})$	- central Wishart distribution of dimensionality p, with ν degrees of freedom and associated parameter matrix $\underset{\sim}{\Sigma}$.

$\underset{\sim}{a}'$ - transpose of the vector $\underset{\sim}{a}$

$\underset{\sim}{B}'$ - transpose of the matrix $\underset{\sim}{B}$

ch $[\cdot]$ - vector whose components are the characteristic roots of $[\cdot]$

$|\cdot|$ - determinant; absolute value

$H^{m,n}_{p,q}\{ x \,|\, {(a_1,A_1),\ldots,(a_p,A_p) \atop (b_1,B_1),\ldots,(b_q,B_q)} \}$ - the H-function defined in (4.17)

$G^{m,n}_{p,q}\{ x \,|\, {a_1,\ldots,a_p \atop b_1,\ldots,b_q} \}$ - Meijer's G-function defined in (4.25)

$\psi(\cdot)$ - the psi function defined in (4.49)

$\rho(\cdot,\cdot)$ - the generalized zeta function defined in (4.50)

$F_D(a;\ b_1,\ldots,b_n;\ c\ ;\ x_1,\ldots,x_n)$

- a type D Lauricella's hypergeometric function defined in (4.60)

Chapter 1

THE MULTIVARIATE LINEAR FUNCTIONAL RELATIONSHIP MODEL

1.0 Introduction

This thesis deals with various aspects connected with what is known in the literature as the "multivariate linear functional relationship model".

The problem of inference concerning the coefficients of a single linear relation among several unobserved "true" variables, when the observed vectors are contaminated with errors or fluctuations has a long history. The early writers on this functional model, notably Adcock (1878), Kummel (1879), Pearson (1901) and van Uven (1930) were mainly concerned with the derivation of least squares estimators. Modern statistical methods were used for the first time by Wald (1940). Particular aspect of this problem has been studied by Creasy (1956) assuming the ratio of error variances to be known a priori; by Geary (1942) using product-cumulants and by Theil (1950) resorting to nonparametric methods.

In experimental work, it is usually possible to replicate the observations. Data coming from replicated experiments can be analyzed without much difficulty, because we can easily obtain from them estimators of the experimental errors which can be assumed to have known distributions. The case in which replicated observations are available was considered by Tukey (1951), who showed how estimators of the linear relation could be easily derived from a variance component analysis.

In a series of relatively recent papers, Villegas showed that an estimator previously proposed by Acton (1959) was the maximum likelihood estimator (1961); obtained invariant least squares estimators (1963) and confidence regions (1964); proved that the least squares estimators are asymptotically efficient (1966); considered the case of non-linear relations (1969); studied the problem in connection with time series models (1976) and derived maximum likelihood and least squares estimators in linear and affine models (1982). The problems of testing the linearity of a relation and testing for simultaneous linear relations have been considered, respectively, by Rao (1965) and Basu (1969). A more complex model with strongly correlated observations was studied by Sprent (1966).

The case where no replications are available has been considered by Lindley and El-Sayyad (1968) using Bayesian methods and by Kalbfleisch and Sprott (1970) using likelihood methods.

The maximum likelihood estimation of general linear models was considered in the general multivariate case by Anderson (1951) and Nussbaum (1976), using differential calculus. An algebraic derivation for a general model has been given by

Healy (1980).

Least squares estimation of functional models has been considered by Eckart and Young (1936), Rao (1964) and Höschel(1978).

For a more detailed survey of the extensive literature available on the analysis of linear relations, the interested reader is referred to Madansky (1959) and Moran (1971). The problem where the true vectors are random (said to satisfy a linear structural relationship) is also treated in these references.

In Chapter 2, we present the main results appearing in the papers of Villegas (1964) and Basu (1969) with some modifications and simplifications of the derivations.

Under the assumptions that the observation vectors are normally distributed and that an estimator of the covariance matrix of the measurement error is available from independent experiments or replicated observations new test statistics for the parameters of functional models which satisfy multiple linear restrictions are derived in Chapter 3.

Different representations of the exact densities of these statistics are provided in computable forms using the technique of inverse Mellin transform (Chapter 4) and in terms of the densities of the product of beta type-2 variables (Chapter 5).

Some econometric applications are also pointed out in Chapter 6.

Most of the results given in chapters 3,4 and 5 are believed to be new and not available anywhere in the literature, except the basic materials and the discussions of the techniques used in these chapters.

1.1 Errors of Measurement

The increase in data-gathering projects in the social and medical sciences is producing large bodies of data containing variables obviously difficult to measure, such as people's behaviour, opinions and motivations. Concurrently, there are signs of a rise in research interest stimulated by problems in econometrics where satisfactory methods for investigating relationships among variables that are subject to errors or fluctuations are needed and in sample surveys where the errors of measurement may be very important.

Standard techniques of analysis become erroneous and misleading if certain types of errors are present in the variables. They often result in unsuspected biases and reduced precision. Hence a variety of mathematical models may be needed to describe realistically the types of measurement errors relevant to different measurement problems.

As early as 1902, Karl Pearson wrote a paper on the mathematical theory of errors of measurement. He was interested in the nature of error of measurement when the quantity being measured was fixed and definite, while the measurement was made by a human being. The model where the variables under consideration are connected by a single linear relationship is the basis of Mandel's (1959), theory of error of measurement as applied to the analysis of interlaboratory tests.

Apart from these theories of errors of measurements, there are many articles connected with a topic, in econometric applications, known as "errors-in-variables linear models". We will discuss in Chapter 6 these types of functional relationships which are presented here as generalizations of general linear models.

1.2 General Linear Models

Suppose $Y_1,\ldots,Y_n$ is a sequence of independent random variables with

$$E(Y_i) = \sum_{j=1}^{p} \beta_j x_{ij}, \quad i=1,\ldots,n \; , \tag{1.1}$$

and

$$\operatorname{var}(Y_i) = \sigma^2 \; , \; i=1,\ldots,n \; , \tag{1.2}$$

where $\beta_1,\ldots,\beta_p$, σ^2, are unknown paramaters and the x_{ij}'s, $i=1,\ldots,n$; $j=1,\ldots,p$, are regarded as fixed. Since $E(Y_i)$ is a linear function of the parameters $\beta_1,\ldots,\beta_p$, for $i=1,\ldots,n$, the models specified by (1.1) and (1.2) are called general linear hypothesis models. They are also referred to as linear models for the expectations with independent covariance structure.

Let

$$\underset{\sim}{Y} = \begin{bmatrix} Y_1 \\ \vdots \\ Y_n \end{bmatrix} , \; \underset{\sim}{\beta} = \begin{bmatrix} \beta_1 \\ \vdots \\ \beta_p \end{bmatrix} \text{ and } \underset{\sim}{X} = \begin{bmatrix} x_{11} & \cdots & x_{1p} \\ \vdots & \cdots & \vdots \\ x_{n1} & \cdots & x_{np} \end{bmatrix} . \tag{1.3}$$

We can now rewrite the equations (1.1) and (1.2) as

$$E(\underset{\sim}{Y}) = \underset{\sim}{X}\underset{\sim}{\beta} \tag{1.4}$$

and

$$\text{Cov}(\underset{\sim}{Y}) = E(\underset{\sim}{Y}-\underset{\sim}{X}\underset{\sim}{\beta})(\underset{\sim}{Y}-\underset{\sim}{X}\underset{\sim}{\beta})' = \sigma^2 I \tag{1.5}$$

where I is the $n \times n$ identity matrix. It is assumed that $n \geqslant p$. If X has full rank, that is to say, if the rank of X is equal to p, then these models are called general linear hypotheses or linear models of full rank.

When the distributions of $Y_1,\ldots,Y_n$ are not specified except for their means and variances, the most commonly used distribution-free method for finding the estimates of the parameters is the method of least squares whereas when $Y_1,\ldots,Y_n$ are assumed to have normal distributions, the most commonly used method is the method of maximum likelihood.

In both cases, the estimator of $\underset{\sim}{\beta}$ is obtained by solving the following nonhomogeneous system of linear equations

$$(\underset{\sim}{X}'\underset{\sim}{X})\underset{\sim}{\beta} = \underset{\sim}{X}'\underset{\sim}{Y} \tag{1.6}$$

which is called the normal equation for the general linear hypothesis model. If $\underset{\sim}{X}$ has full rank p, then $\underset{\sim}{X}'\underset{\sim}{X}$ is of rank p and has a unique inverse $(\underset{\sim}{X}'\underset{\sim}{X})^{-1}$ and the least squares estimate as well as the maximum likelihood estimate of $\underset{\sim}{\beta}$ are given by

$$(\underset{\sim}{X}'\underset{\sim}{X})^{-1}\underset{\sim}{X}'\underset{\sim}{Y} \quad .$$

In Anderson (1958), the following generalization of the general linear hypothesis model is discussed.

Let $\underset{\sim}{Y}_1,\ldots,\underset{\sim}{Y}_n$ be a sequence of independent random vectors having a multivariate normal distribution with

$$E(\underset{\sim}{Y}_i) = \underset{\sim}{B}\underset{\sim}{X}_i \qquad (1.7)$$

and

$$\underset{q\times 1}{\text{cov}(\underset{\sim}{Y}_i)} = \underset{\sim}{\Sigma} \ , \ i=1,\ldots,n \qquad (1.8)$$

where

$$\underset{\sim}{X}_i = (X_{i1},\ldots,X_{ip})', \ i=1,\ldots,n,$$

are known, and the $q\times q$ matrix $\underset{\sim}{\Sigma}$ and the $q\times p$ matrix $\underset{\sim}{B}$ are unknown.

It is further assumed that

$$n \geq p+q$$

and that the rank of

$$\underset{p\times n}{\underset{\sim}{X}} = (\underset{\sim}{X}_1,\ldots,\underset{\sim}{X}_n)$$

is p. Estimators for $\underset{\sim}{B}$ and $\underset{\sim}{\Sigma}$ are obtained by the method of maximum likelihood. We will refer to the model specified by (1.7) and (1.8) as the multivariate general linear hypothesis model.

1.3 Functional Relationships in General Linear Models

We now modify the general linear hypothesis model specified by (1.1) and (1.2) as follows.

Let

$$Y_i \overset{ind}{\sim} N(\mu_{i1},\sigma^2) \ , \ i=1,\ldots,n \ ,$$

$$x_{i1} = 1 \ , \quad i=1,\ldots,n \ ,$$

and let

$$\underset{\sim}{x}_i = (x_{i2},\ldots,x_{ip})' \ , \ i=1,\ldots,n \ , \qquad (1.9)$$

be random vectors such that

$$\underset{\sim}{x}_i \sim N(\underset{\sim}{\mu}^{(i)},\underset{\sim}{\Sigma}) \qquad (1.10)$$

where

$$\underset{\sim}{\mu}^{(i)} = (\mu_{i2},\ldots,\mu_{ip})' \ . \qquad (1.11)$$

Then the relationship

$$\mu_{i1} = \beta_1 + \sum_{j=2}^{p} \beta_j \mu_{ij} , \ i=1,\ldots,n \ , \qquad (1.12)$$

characterizes the univariate errors-in-variables functional model where it is assumed that the $\underset{\sim}{\mu}^{(i)}$'s are constant vectors.

Let

$$\text{cov}((Y_i,\underset{\sim}{x}_i')') = \underset{\sim}{V} \ , \qquad (1.13)$$

then rewriting (1.12) as

$$\sum_{j=1}^{p} b_j \mu_{ij} + a = 0 \qquad (1.14)$$

we obtain the single linear functional relationship model.

Similarly the multivariate general linear hypothesis model specified by (1.7) and (1.8) may be modified as follows.

Let

$$\underset{\sim}{Y}_i \sim N_q(\underset{\sim}{\mu}_{i1},\underset{\sim}{\Sigma}_Y) \ , \qquad (1.15)$$

$$X_{i1} = 1 \ , \ i=1,\ldots,n \ ,$$

$$\underset{q \times p}{\underset{\sim}{B}} = (\underset{\sim}{\beta}_1,\ldots,\underset{\sim}{\beta}_p) \tag{1.16}$$

and let

$$\underset{(p-1)\times 1}{\tilde{\underset{\sim}{X}}_i} = (X_{i2},\ldots,X_{ip})' \ , \ i=1,\ldots,n \ ,$$

be random vectors such that

$$\tilde{\underset{\sim}{X}}_i \sim N_{p-1}(\underset{\sim}{\mu}_i,\underset{\sim}{\Sigma}) \tag{1.18}$$

where

$$\underset{\sim}{\mu}_i = (\mu_{i2},\ldots,\mu_{ip})' \ , \ i=1,\ldots,n \ . \tag{1.19}$$

Then the relationship

$$\underset{\sim}{\mu}_{i1} = \underset{\sim}{\beta}_1 + (\underset{\sim}{\beta}_2,\ldots,\underset{\sim}{\beta}_p)\underset{\sim}{\mu}_i \ , \ i=1,\ldots,n \ , \tag{1.20}$$

characterizes the multivariate errors-in-variables functional model where the $\underset{\sim}{\mu}_i$'s are assumed to be constant vectors. If the $\underset{\sim}{\mu}_i$'s are independently and identically distributed random vectors, the model is called the multivariate errors-in-variables structural model.

Inference is based on the observed vectors $\underset{\sim}{Y}_i$ and $\tilde{\underset{\sim}{X}}_i$ which are the sum of the true vectors $\underset{\sim}{\mu}_{i1}$ and $\underset{\sim}{\mu}_i$ respectively, and errors of measurement.

Gleser and Watson (1973) derived the maximum likelihood estimators for the functional model with $q=(p-1)$, $\underset{\sim}{\beta}_1=\underset{\sim}{0}$ and $\underset{\sim}{\Sigma}_Y=\underset{\sim}{\Sigma}=\sigma^2\underset{\sim}{\Sigma}^o$ where $\underset{\sim}{\Sigma}^o$ is known. Bhargava (1979) obtained the maximum likelihood estimators of the functional model with $\underset{\sim}{\beta}_1=\underset{\sim}{0}$, $\underset{\sim}{\Sigma}_Y=\sigma^2 I$ and $\underset{\sim}{\Sigma} = \sigma^2 I$ where σ^2 is unknown. Gleser (1981) gave the limiting distribution of the maximum likelihood estimators for the functional model with $\Sigma_{\tilde{Y}}=\sigma^2 I$ and $\tilde{\Sigma}=\sigma^2 I$ where σ^2 is unknown. Dahm and Fuller (1981) applied the generalized least squares method to the functional model. The maximum likelihood estimators are derived in Amemiya and Fuller (1984) for the structural model and their limiting properties are obtained under a wide range of assumptions.

We may also express (1.20) as follows

$$(-I,\tilde{\beta}_2,\ldots,\tilde{\beta}_p)\begin{pmatrix}\tilde{\mu}_{i1}\\ \tilde{\mu}_i\end{pmatrix} + \tilde{\beta}_1 = \tilde{0},\ i=1,\ldots,n\ . \tag{1.21}$$

Under the assumption that

$$\operatorname{cov}((\tilde{Y}_i,\tilde{X}_i')') = \tilde{V}\ , \tag{1.22}$$

we obtain the multivariate functional relationship model as the following generalization of (1.21):

$$\tilde{B}\begin{pmatrix}\tilde{\mu}_{i1}\\ \tilde{\mu}_i\end{pmatrix} + \tilde{\beta}_1 = \tilde{0}\ , \tag{1.23}$$

where B is a q × (q+p-1) matrix of rank q.

Chapter 2

ANALYSIS OF LINEAR RELATIONS

2.0 Introduction

In this chapter, we present the main results appearing in Villegas (1964) and Basu (1969). Villegas' paper dealt with the problem of inference concerning the parameters of a single linear relation where all the variables involved were subject to normally distributed errors or fluctuations and where the unknown error variances were estimated from replicated observations. His discussion covered the case of correlated errors.

For this single linear functional relationship model, a test based on the F distribution was derived for testing the hypothesis that the unknown relation was a given linear relation. Particular aspect of this problem had been studied by Wald (1940) and Bartlett (1949) using the method of grouping; by Creasy (1956) assuming the ratio of error variances to be known a priori; by Geary (1949) and Halperin (1961) using instrumental variables or a priori weights; and by Hemelrijk (1949) and Theil (1950) resorting to nonparametric methods.

For a detailed account of the various approaches that were used in solving inference problems related to functional and structural linear models prior to 1964, the reader is referred to Kendall and Stuart (1966), Chapter 29.

Following Villegas' approach, Basu (1969) extended the results to the case of several linear relations. For the coefficients of this multivariate linear functional relationship model, he proposed a test statistic, R, without giving its exact distribution in closed form.

In this chapter, we shall provide a more complete derivation of Villegas' statistic and a more systematic derivation of Basu's statistic. The exact distribution of the latter will be given in computable forms in Chapters 4 and 5. From these representations of the exact density of R, percentage points can be computed.

2.1 Villegas' Approach

2.1a Notation and Model

Let

$$a + b_1 g_1 + \ldots + b_p g_p = 0 \tag{2.1}$$

be an unknown linear relation among the p variables $g_1, \ldots, g_p$. We shall assume that the g_i's are not observable in the sense that they are all subject to errors or fluctuations. Rewriting (2.1) in matrix notation, we have

$$a + \underset{\sim}{b}'\underset{\sim}{g} = 0 , \tag{2.2}$$

where

$$\underset{\sim}{b}' = (b_1, \ldots, b_p) \tag{2.3}$$

and

$$\underset{\sim}{g} = (g_1,\ldots,g_p)' \ . \tag{2.4}$$

Now let us consider r points, $\underset{\sim}{g}_1,\ldots,\underset{\sim}{g}_r$, on the hyperplane defined by (2.2). That is,

$$a + \underset{\sim}{b}'\underset{\sim}{g}_i = 0 \ , \ i=1,\ldots,r \ .$$

It is also assumed that n_i measurements $\underset{\sim}{x}_{ij}$, $j=1,\ldots,n_i$, provided by replicated experiments are available for the i-th point, $i=1,\ldots,r$. Let

$$\underset{\sim}{x}_{ij} = \underset{\sim}{g}_i + \underset{\sim}{e}_{ij} \ , \tag{2.5}$$

where

$$\underset{\sim}{e}_{ij} \overset{ind}{\sim} N_p(\underset{\sim}{0},\underset{\sim}{V}) \ , \ j=1,\ldots,n_i \ , \tag{2.6}$$

that is, the $\underset{\sim}{e}_{ij}$'s are all independently and identically distributed (i.i.d.) random vectors, each having a p-variate normal distribution with mean vector $\underset{\sim}{0}$ and unknown positive definite covariance matrix $\underset{\sim}{V}$,

$$n = n_1 + \ldots + n_r \ ,$$

where n denote the total number of observed vectors and

$$\underset{\sim}{\bar{x}}_i = \sum_{j=1}^{n_i} \underset{\sim}{x}_{ij}/n_i \ , \tag{2.7}$$

where $\underset{\sim}{\bar{x}}_i$ denotes the average of all the observed vectors corresponding to the point $\underset{\sim}{g}_i$, $i=1,\ldots,r$. Let also

$$\underset{\sim}{S} = \sum_{i=1}^{r} \sum_{j=1}^{n_i} (\underset{\sim}{x}_{ij} - \underset{\sim}{\bar{x}}_i)(\underset{\sim}{x}_{ij} - \underset{\sim}{\bar{x}}_i)'/(n-r) \ . \tag{2.8}$$

The additional assumption,

$$n - r \geq p ,$$

assures us that $(n-r)\underset{\sim}{S}$ is, with probability one, a positive definite matrix and has the Wishart distribution with parameter matrix $\underset{\sim}{V}$ and $(n-r)$ degrees of freedom, that is,

$$(n-r)\underset{\sim}{S} \sim W_p((n-r),\underset{\sim}{V}) . \qquad (2.9)$$

We will refer to the model specified by (2.2), (2.5) and (2.6) as the single linear functional relationship (SLFR) model although it is generally simply called the linear functional relationship model.

2.1b A Test Statistic

Villegas considered the hypothesis

$$H_o^V : a = a_o \ , \ \underset{\sim}{b} = \underset{\sim}{b}_o \qquad (2.10)$$

for the SLFR model where $\underset{\sim}{V}$ is unknown, a_o is a given scalar quantity and $\underset{\sim}{b}_o$ is a given vector. Under H_o^V , the true hyperplane is

$$a_o + \underset{\sim}{b}_o' \underset{\sim}{g} = 0 \qquad (2.11)$$

and the distances from the points $\bar{\underset{\sim}{x}}_i$, $i=1,\ldots,r$, to the hyperplane (2.11) are respectively given by

$$\delta_i = (a_o + \underset{\sim}{b}_o' \bar{\underset{\sim}{x}}_i)/(\underset{\sim}{b}_o' \underset{\sim}{b}_o)^{1/2} . \qquad (2.12)$$

It is seen from (2.5) and (2.7) that the vectors

$$\bar{\underset{\sim}{x}}_i \sim N_p(\underset{\sim}{g}_i, \underset{\sim}{V}/n_i) \ , \ i=1,\ldots,r. \qquad (2.13)$$

Moreover, since each $\underset{\sim}{g}_i$ lies on the hyperplane (2.11) under H_o^V,

$$\delta_i \sim N(0, v_o^2/n_i), \ i=1,\ldots,r \ , \qquad (2.14)$$

where

$$v_o^2 = (\underset{\sim}{b}_o' \underset{\sim}{V} \underset{\sim}{b}_o)/(\underset{\sim}{b}_o' \underset{\sim}{b}_o) \qquad (2.15)$$

and therefore

$$\sum_{i=1}^{r} n_i \delta_i^2 \sim v_o^2 \ \chi_r^2 \ , \qquad (2.16)$$

that is,

$$\sum_{i=1}^{r} n_i \delta_i^2$$

is distributed as a constant, namely v_o^2, times a chi-square variate having r degrees of freedom.

Furthermore

$$s^2 = (\underset{\sim}{b}_o' \underset{\sim}{S} \underset{\sim}{b}_o)/(\underset{\sim}{b}_o' \underset{\sim}{b}_o) \qquad (2.17)$$

is an unbiased estimator of v_o^2 where $\underset{\sim}{S}$ is defined in (2.8). Also it follows from (2.9) that

$$s^2 \sim W_1((n-r), (\underset{\sim}{b}_o' \underset{\sim}{V} \underset{\sim}{b}_o)/(\underset{\sim}{b}_o' \underset{\sim}{b}_o))/(n-r) \ , \qquad (2.18)$$

that is

$$s^2 \sim v_o^2 \ \chi_{n-r}^2 \ /(n-r) \ . \qquad (2.19)$$

Villegas' statistic being

$$\sum_{i=1}^{r} n_i \delta_i^2 / s^2 \tag{2.20}$$

is distributed as r times a Fisher's F with r and (n-r) degrees of freedom in view of (2.16) and (2.19). It is clear that the numerator is independent from the denominator since the former is a function of $\bar{\underset{\sim}{x}}_i$, $i=1,\ldots,r$, and the latter is a function of S which is independent of $\bar{\underset{\sim}{x}}_i$, $i=1,\ldots,r$, due to the basic assumption of normality. Hence S is independent of

$$\underset{\sim}{\delta} = (n_1^{1/2}\delta_1, \ldots, n_r^{1/2}\delta_r)$$

and thus s^2 and $\underset{\sim}{\delta}'\underset{\sim}{\delta}$ are independently distributed. It is worth mentioning that the distribution of the test statistic (2.20) is free of V allowing us to test H_o^V as defined in (2.10).

2.2 Basu's Generalization of Villegas' Model

2.2a Notation and Model

Let

$$\begin{array}{l} a_1 + b_{11}g_1 + \ldots + b_{1p}g_p = 0 \;, \\ a_2 + b_{21}g_1 + \ldots + b_{2p}g_p = 0 \;, \\ \vdots \qquad \vdots \qquad \ldots \qquad \vdots \qquad \vdots \\ a_k + b_{k1}g_1 + \ldots + b_{kp}g_p = 0 \;, \end{array} \tag{2.21}$$

be k linear relations among the p variables $g_1, g_2, \ldots, g_p$ where $p > k$. As before, it is assumed that the observed values do not satisfy (2.21) because all of them are subject to errors or fluctuations. In matrix notation, (2.21) may be represented by

$$\underset{k\times 1}{\underset{\sim}{a}} + \underset{k\times p\times 1}{\underset{\sim}{B}'\underset{\sim}{g}} = \underset{\sim}{0} \tag{2.22}$$

where

$$\underset{\sim}{g} = (g_1, g_2, \ldots, g_p)' ,$$

$$\underset{\sim}{B} = \begin{pmatrix} b_{11} & b_{21} & \cdots & b_{k1} \\ b_{12} & b_{22} & \cdots & b_{k2} \\ \vdots & \vdots & \cdots & \vdots \\ b_{1p} & b_{2p} & \cdots & b_{kp} \end{pmatrix} = (\underset{\sim}{b}_1, \underset{\sim}{b}_2, \ldots, \underset{\sim}{b}_k), \tag{2.23}$$

and

$$\underset{\sim}{a}' = (a_1, a_2, \ldots, a_k) . \tag{2.24}$$

We assume that $\underset{\sim}{B}$ is of full rank. Then we consider r points, $\underset{\sim}{g}_1, \underset{\sim}{g}_2, \ldots, \underset{\sim}{g}_r$, in the (p-k)-dimensional convex set defined by (2.22). That is,

$$\underset{\sim}{a} + \underset{\sim}{B}'\underset{\sim}{g}_i = \underset{\sim}{0} , \; i=1,\ldots,r .$$

The assumptions regarding the n_i replicated measurements for each of the points $\underset{\sim}{g}_i$, $i=1,\ldots,r$, are identical to those purporting to the SLFR model and consequently (2.5), (2.6), (2.7) , (2.8), (2.9) and (2.13) still hold for the next section.

The model specified by (2.22), (2.5) and (2.6) is called the multivariate linear functional relationship (MLFR) model.

2.2b A Test Statistic

In order to test the hypothesis

$$H_o : \; \underset{\sim}{a} = \underset{\sim}{a}_o \; , \; \underset{\sim}{B} = \underset{\sim}{B}_o \; , \tag{2.25}$$

where V is unknown, Basu (1969) proved the following result.

Lemma 2.1. Let

$$d_i^2 = (\underset{\sim}{a}_o + \underset{\sim}{B}_o'\bar{\underset{\sim}{x}}_i)'(\underset{\sim}{B}_o'\underset{\sim}{B}_o)^{-1}(\underset{\sim}{a}_o + \underset{\sim}{B}_o'\bar{\underset{\sim}{x}}_i), \quad i=1,\ldots,r \ , \tag{2.26}$$

then d_i^2 represents the square of the distance between $\bar{\underset{\sim}{x}}_i$ and the (p-k)-dimensional convex set

$$M_o = \{\underset{\sim}{g} : \underset{\sim}{a}_o + \underset{\sim}{B}_o'\underset{\sim}{g} = \underset{\sim}{0}\} \ , \tag{2.27}$$

where the distance is measured with respect to the Euclidean metric $\{(\bar{\underset{\sim}{x}}_i - \underset{\sim}{g})'(\bar{\underset{\sim}{x}}_i - \underset{\sim}{g})\}^{1/2}$.

In his proof, he first notes that, because of the invariance property of the distance function, minimizing the distance between $\bar{\underset{\sim}{x}}_i$ and M_o is equivalent to minimizing the distance between $\underset{\sim}{0}$ and $M_o - \bar{\underset{\sim}{x}}_i$ where

$$M_o - \bar{\underset{\sim}{x}}_i = \{\underset{\sim}{g} : \underset{\sim}{a}_o + \underset{\sim}{B}_o'(\underset{\sim}{g} + \bar{\underset{\sim}{x}}_i) = \underset{\sim}{0}\} \ ,$$

and that

$$\underset{\sim}{a}_o + \underset{\sim}{B}_o'(\underset{\sim}{g}^* + \bar{\underset{\sim}{x}}_i) = \underset{\sim}{0} \tag{2.27a}$$

where $\underset{\sim}{g}^*$ denotes the point belonging to $M_o - \bar{\underset{\sim}{x}}_i$ which will minimize the distance between $\underset{\sim}{0}$ and $M_o - \bar{\underset{\sim}{x}}_i$. Then using the orthogonality of $(\underset{\sim}{g}-\underset{\sim}{g}^*)$ and $\underset{\sim}{g}^*$ which implies that

$$(\underset{\sim}{g} - \underset{\sim}{g}^*)'\underset{\sim}{g}^* = \underset{\sim}{0}$$

for all $\underset{\sim}{g} \in M_o - \bar{\underset{\sim}{x}}_i$, he shows that $\underset{\sim}{g}^*$ is equal to $\underset{\sim}{B}_o\underset{\sim}{\gamma}$ where $\underset{\sim}{\gamma}$ is a suitable vector of coefficients. At this point, it is seen from (2.27a) that

$$\underset{\sim}{a}_o + \underset{\sim}{B}_o'\overline{\underset{\sim}{x}}_i = -\underset{\sim}{B}_o' \underset{\sim}{g}^* = -\underset{\sim}{B}_o'\underset{\sim}{B}_o\gamma \ .$$

Therefore $\underset{\sim}{\gamma}$ is given by

$$\underset{\sim}{\gamma} = -(\underset{\sim}{B}_o'\underset{\sim}{B}_o)^{-1}(\underset{\sim}{a}_o+\underset{\sim}{B}_o'\overline{\underset{\sim}{x}}_i) \ ,$$

and hence

$$\underset{\sim}{g}^* = -\underset{\sim}{B}_o(\underset{\sim}{B}_o'\underset{\sim}{B}_o)^{-1}(\underset{\sim}{a}_o+\underset{\sim}{B}_o'\overline{\underset{\sim}{x}}_i)$$

so that

$$d_i^2 = \underset{\sim}{g}^{*\prime}\underset{\sim}{g}^* = (\underset{\sim}{a}_o+\underset{\sim}{B}_o'\overline{\underset{\sim}{x}}_i)'(\underset{\sim}{B}_o'\underset{\sim}{B}_o)^{-1}(\underset{\sim}{a}_o+\underset{\sim}{B}_o'\overline{\underset{\sim}{x}}_i), \ i=1,\ldots,r \ .$$

Considering (2.13), we see that

$$\underset{\sim}{a}_o + \underset{\sim}{B}_o'\overline{\underset{\sim}{x}}_i \overset{ind}{\sim} N_k(\underset{\sim}{0},\underset{\sim}{B}_o'\underset{\sim}{V}\underset{\sim}{B}_o/n_i) \ , \ i=1,\ldots,r \ , \qquad (2.28)$$

since $\underset{\sim}{g}_i \in M_o$ under H_o .

Since $\underset{\sim}{B}_o'\underset{\sim}{V}\underset{\sim}{B}_o$ is symmetric and positive definite, there exists a nonsingular matrix

$$\underset{\sim}{K} = (\underset{\sim}{B}_o'\underset{\sim}{V}\underset{\sim}{B}_o)^{-1/2} \qquad (2.29)$$

such that

$$\underset{\sim}{K}'\underset{\sim}{B}_o'\underset{\sim}{V}\underset{\sim}{B}_o\underset{\sim}{K} = I \ . \qquad (2.30)$$

Let

$$\underset{\sim}{z}_i = n^{-1/2}\underset{\sim}{K}'(\underset{\sim}{a}_o + \underset{\sim}{B}_o'\overline{\underset{\sim}{x}}_i), \ i=1,\ldots,r \ ;$$

then it is seen from (2.28) that

$$\underset{\sim}{z}_i \overset{ind}{\sim} N_k(\underset{\sim}{0},I) \qquad (2.31)$$

and from (2.26) that

$$n_i d_i^2 = \underset{\sim}{z}_i'\underset{\sim}{K}^{-1}(\underset{\sim}{B}_o'\underset{\sim}{B}_o)^{-1}\underset{\sim}{K}'^{-1}\underset{\sim}{z}_i \ . \qquad (2.32)$$

Since $(\underset{\sim}{B}_o'\underset{\sim}{B}_o)^{-1}$ is symmetric and positive definite, so is

$$\underset{\sim}{K}^{-1}(\underset{\sim}{B}_o'\underset{\sim}{B}_o)^{-1}\underset{\sim}{K}'^{-1} ,$$

and hence there exists an orthogonal matrix $\underset{\sim}{U}$ such that

$$\underset{\sim}{U}'\underset{\sim}{K}^{-1}(\underset{\sim}{B}_o'\underset{\sim}{B}_o)^{-1}\underset{\sim}{K}'^{-1}\underset{\sim}{U} = \underset{\sim}{M} = \mathrm{diag}(m_1,\ldots,m_k) , \tag{2.33}$$

where $m_j>0$, $j=1,\ldots,k$. Therefore

$$(m_1,\ldots,m_k) = \mathrm{ch}[(\underset{\sim}{B}_o'\underset{\sim}{V}\underset{\sim}{B}_o)^{1/2}(\underset{\sim}{B}_o'\underset{\sim}{B}_o)^{-1}\{(\underset{\sim}{B}_o'\underset{\sim}{V}\underset{\sim}{B}_o)^{1/2}\}'] \tag{2.34}$$

where ch[.] denotes a vector whose components are the characteristic roots (eigenvalues) of [.] .

Now let

$$(f_{i1},\ldots,f_{ik})' = \underset{\sim}{f}_i = \underset{\sim}{U}'\underset{\sim}{z}_i , \quad i=1,\ldots,r ,$$

then the vectors $\underset{\sim}{f}_i$ are i.i.d. $N_k(\underset{\sim}{0},I)$ and

$$n_i d_i^2 = \underset{\sim}{f}_i'\underset{\sim}{M}\underset{\sim}{f}_i = \sum_{j=1}^{k} m_j f_{ij} \tag{2.35}$$

where the f_{ij}'s are i.i.d. $N(0,1)$, $i=1,\ldots,r$; $j=1,\ldots,k$. Hence,

$$n_i d_i^2 \overset{\text{ind}}{\sim} \sum_{j=1}^{k} m_j \chi_j^2(1) , \quad i=1,\ldots,r , \tag{2.36}$$

and

$$d^2 = \sum_{i=1}^{r} n_i d_i^2 \sim \sum_{j=1}^{k} m_j \chi_j^2(r) , \tag{2.37}$$

where $\chi_j^2(r)$ are independent chi-square variates each having r degrees of freedom and the m_j's are defined in (2.34).

Let

$$\underset{\sim}{A}_i = \sum_{j=1}^{n_i} (\underset{\sim}{x}_{ij} - \bar{\underset{\sim}{x}}_i)(\underset{\sim}{x}_{ij} - \bar{\underset{\sim}{x}}_i)', \quad i=1,\ldots,r . \tag{2.38}$$

Then, assuming that

$$n_i > p , \tag{2.38a}$$

$$\underset{\sim}{A}_i \overset{ind}{\sim} W_p(n_i-1,\underset{\sim}{V}) \quad , \; i=1,\ldots,r , \tag{2.39}$$

so that

$$\underset{\sim}{A} = \sum_{i=1}^{r} \underset{\sim}{A}_i \sim W_p(n-r,\underset{\sim}{V}) . \tag{2.40}$$

It is well known that $\underset{\sim}{A}_i$ can be expressed as $(\underset{\sim}{u}_1\underset{\sim}{u}_1'+\underset{\sim}{u}_2\underset{\sim}{u}_2'+\ldots+\underset{\sim}{u}_{n_i-1}\underset{\sim}{u}_{n_i-1}')$ where $\underset{\sim}{u}_1,\underset{\sim}{u}_2,\ldots,\underset{\sim}{u}_{n_i-1}$ are i.i.d. $N_p(0,\underset{\sim}{V})$ and that $\underset{\sim}{A}_i$ is distributed independently of $\bar{\underset{\sim}{x}}_i$, $i=1,\ldots,r$. So $\underset{\sim}{A}$ can be expressed as

$$\underset{\sim}{u}_1\underset{\sim}{u}_1' + \underset{\sim}{u}_2\underset{\sim}{u}_2' + \ldots + \underset{\sim}{u}_{n-r}\underset{\sim}{u}_{n-r}' ,$$

where the $\underset{\sim}{u}_j$'s, $j=1,2,\ldots,n-r$, are i.i.d. $N_p(\underset{\sim}{0},\underset{\sim}{V})$. Thus

$$\text{cov}(n_i^{1/2}(\underset{\sim}{a}_o + \underset{\sim}{B}_o'\bar{\underset{\sim}{x}}_i)) = \underset{\sim}{B}_o'\underset{\sim}{V}\underset{\sim}{B}_o$$

can be estimated by

$$\underset{\sim}{B}_o'\underset{\sim}{S}\underset{\sim}{B}_o = \underset{\sim}{B}_o'\Big(\sum_{i=1}^{r} \underset{\sim}{A}_i\Big)\underset{\sim}{B}_o/(n-r) = \sum_{j=1}^{n-r} \underset{\sim}{w}_j\underset{\sim}{w}_j'/(n-r) = \underset{\sim}{W}/(n-r) , \tag{2.41}$$

where $\underset{\sim}{w}_j = \underset{\sim}{B}_o'\underset{\sim}{u}_j$, $j=1,\ldots,n-r$, and, the $\underset{\sim}{w}_j$'s being i.i.d. $N_k(\underset{\sim}{0},\underset{\sim}{B}_o'\underset{\sim}{V}\underset{\sim}{B}_o)$,

$$\underset{\sim}{W} \sim W_k((n-r),\underset{\sim}{B}_o'\underset{\sim}{V}\underset{\sim}{B}_o) \tag{2.42}$$

independently of d^2 since $\underset{\sim}{W}$ is a function of $(\underset{\sim}{A}_1,\ldots,\underset{\sim}{A}_r)$ where the $\underset{\sim}{A}_i$'s are independent and d^2 is a function of $(\bar{\underset{\sim}{x}}_1,\ldots,\bar{\underset{\sim}{x}}_r)$ where the $\bar{\underset{\sim}{x}}_i$'s are independent and each $\underset{\sim}{A}_i$ is independent of $\bar{\underset{\sim}{x}}_i$, $i=1,\ldots,r$, due to the basic assumption of normality.

Basu (1969) proposed the following statistic to test the hypothesis H_o given in (2.25):

$$R = \frac{(n-r)d^2}{|\underset{\sim}{W}|^{1/k}} \qquad (2.43)$$

where d^2 is defined in (2.37) and (2.26), $\underset{\sim}{W}$ is defined in (2.41) and (2.38) and $|\underset{\sim}{W}|$ denotes the determinant of $\underset{\sim}{W}$.

Since the distribution of $|\underset{\sim}{W}|$ is that of the product of k independent chi-square variates (see Anderson (1958), p.171) times a constant:

$$|\underset{\sim}{W}| \sim |\underset{\sim}{B}_o'\underset{\sim}{V}\underset{\sim}{B}_o| \prod_{i=1}^{k} \chi^2_{n-r-i+1} \qquad (2.44)$$

and that

$$d^2 = \sum_{i=1}^{r} n_i d_i^2 \sim \sum_{j=1}^{k} m_j \chi_j^2(r) ,$$

one can see that R is distributed as the ratio of a linear combination of chi-squares variates over the k-th root of a product of chi-square variates where the chi-squares are all independent.

Chapter 3

TESTS FOR THE PARAMETERS OF A MULTIVARIATE LINEAR FUNCTIONAL RELATIONSHIP MODEL

3.0 Introduction

In this chapter we propose some new statistics for testing hypotheses about the parameters of a MLFR model.

There are two common sources for estimating $\underset{\sim}{V}$, the co-variance matrix of errors. We have seen that replicated observations can provide such estimators which may also be obtained from independent experiments in the past. Moreover in some situations we may assume that V is known.

We are therefore considering the following cases: (*i*) $\underset{\sim}{V}$ is unknown; (*ii*) $\underset{\sim}{V}$ is known up to a constant; (*iii*) $\underset{\sim}{V}$ is completely known; (*iv*) an independent estimate is available for $\underset{\sim}{V}$. The first three cases are discussed in Gleser and Olkin (1972) for the general linear hypothesis model.

For the third case, the distribution of the test statistic turns out to be that of a linear combination of independent chi-square variates. Three representations of its exact density which are suitable for computational purposes are provided. One representation will be given in terms of finite sums, another in terms of a confluent hypergeometric function of several variables and a third in terms of zonal polynomials.

3.1 Summary of Basu's Results

3.1 a One Test Statistic

In Section 2.2a we have provided a different derivation of Basu's test statistic for the parameters of a MLFR model. The statistic proposed by Basu (1969), in our notation of Chapter 2, to test

$$H_o: \underset{\sim}{a} = \underset{\sim}{a}_o \ , \quad \underset{\sim}{B} = \underset{\sim}{B}_o \tag{3.1}$$

where V is unknown, was according to (2.43)

$$R = \sum_{i=1}^{r} n_i d_i^2 / |\underset{\sim}{B}_o' \underset{\sim}{S} \underset{\sim}{B}_o|^{1/k} \tag{3.2}$$

where d_i^2 was defined in (2.26) and $\underset{\sim}{S}$ in (2.8).

Furthemore we proved the independence of the numerator and the denominator of R which were respectively distributed as a linear combination of chi-square variates according to (2.35) and as the k-th root of a constant times a product of chi-square variates according to (2.44). The distributional aspects of R will be treated in Chapters 4 and 5.

3.1 b Another Test Statistic

In order to test H_o, Basu (1969) also considered a second statistic T based on the metric $\{(\underset{\sim}{x}-\underset{\sim}{g})'\underset{\sim}{V}^{-1}(\underset{\sim}{x}-\underset{\sim}{g})\}$ where $\underset{\sim}{V}$ is the covariance matrix defined in (2.6) and $\underset{\sim}{g}$ lies in the set

$$M_o = \{\underset{\sim}{g}: \underset{\sim}{a}_o + \underset{\sim}{B}_o' \underset{\sim}{g} = \underset{\sim}{0}\} \ . \tag{3.3}$$

He showed that according to this metric, the square of the "distance" between $\bar{\underset{\sim}{x}}_i$ the average of all the observed vectors corresponding to a given point in M_o, and M_o was

$$\bar{d}_i^2 = (\underset{\sim}{a}_o + \underset{\sim}{B}_o' \bar{\underset{\sim}{x}}_i)'(\underset{\sim}{B}_o'\underset{\sim}{V}\underset{\sim}{B}_o)^{-1}(\underset{\sim}{a}_o + \underset{\sim}{B}_o' \bar{\underset{\sim}{x}}_i), \ i=1,\ldots,r. \quad (3.4)$$

Moreover, according to (2.28),

$$n_i^{1/2}(\underset{\sim}{a}_o + \underset{\sim}{B}_o' \bar{\underset{\sim}{x}}_i) \overset{ind}{\sim} N_k(\underset{\sim}{0}, \underset{\sim}{B}_o'\underset{\sim}{V}\underset{\sim}{B}_o), \quad i=1,\ldots,r, \quad (3.5)$$

so that

$$n_i\bar{d}_i^2 = (\underset{\sim}{a}_o + \underset{\sim}{B}_o' \bar{\underset{\sim}{x}}_i)'\{(\underset{\sim}{B}_o'\underset{\sim}{V}\underset{\sim}{B}_o)/n_i\}^{-1}(\underset{\sim}{a}_o + \underset{\sim}{B}_o' \bar{\underset{\sim}{x}}_i) \overset{ind}{\sim} \chi_k^2, \quad (3.6)$$

$i = 1,\ldots,r$, and

$$\bar{d}^2 = \sum_{i=1}^{r} n_i\bar{d}_i^2 \sim \chi_{kr}^2, \quad (3.7)$$

that is, $\bar{d}_i^2$ is distributed as a chi-square variate with rk degrees of freedom. We also note that the use of $\bar{d}^2$ would require a prior knowledge of $\underset{\sim}{V}$.

Hence for the case where $\underset{\sim}{V}$ is unknown, a studentized form of the $\bar{d}^2$ statistic was suggested to test H_o. The statistic is the following

$$T^2 = \sum_{i=1}^{r} \bar{T}_i^2 \quad (3.8)$$

where

$$\bar{T}_i^2 = (\underset{\sim}{a}_o + \underset{\sim}{B}_o' \bar{\underset{\sim}{x}}_i)'\{(\underset{\sim}{B}_o'\underset{\sim}{S}\underset{\sim}{B}_o)/n_i\}^{-1}(\underset{\sim}{a}_o + \underset{\sim}{B}_o' \bar{\underset{\sim}{x}}_i). \quad (3.9)$$

Each $\overline{T}_i^2$ has Hotelling's T^2 distribution but they are not independently distributed. If we use the large sample approximation, it is pointed out that we may again approximate the distribution of $\overline{T}^2$ by the chi-square distribution with rk degrees of freedom.

It is also to be noted that, if $\underset{\sim}{V}$ is known, the distribution of $\overline{d}^2$ under H_o is exactly known even if there is no replication (that is, if $n_1=n_2=\ldots=n_r=1$).

3.2 Alternate Test Statistics

3.2 a A Statistic Based on the Distance between M_o and a Weighted Average of the $\overline{\underset{\sim}{x}}_i$'s

Here we will derive a new statistic to test the hypothesis H_o given in (3.1).

Let

$$\underset{\sim}{t}_i = n_i^{1/2}\underset{\sim}{B}_1(\underset{\sim}{a}_o + \underset{\sim}{B}_o' \overline{\underset{\sim}{x}}_i), \quad i=1,\ldots,r, \tag{3.10}$$

where

$$\underset{\sim}{B}_1 = (\underset{\sim}{B}_o'\underset{\sim}{B}_o)^{-1/2} \quad \text{and} \quad \underset{\sim}{B}_1' = \underset{\sim}{B}_1.$$

Then considering (3.5), we see that

$$\underset{\sim}{t}_i \overset{ind}{\sim} N_k(\underset{\sim}{0},\underset{\sim}{V}^*), \quad i=1,\ldots,r, \tag{3.11}$$

where, under the null hypothesis,

$$\underset{\sim}{V}^* = \underset{\sim}{B}_1\underset{\sim}{B}_o'\underset{\sim}{V}\underset{\sim}{B}_o\underset{\sim}{B}_1$$

and $\underset{\sim}{V}$ is unknown.

Let

$$\bar{\underset{\sim}{t}} = \sum_{i=1}^{r} \underset{\sim}{t}_i / r \tag{3.12}$$

and

$$\underset{\sim}{S}^* = \sum_{i=1}^{r} (\underset{\sim}{t}_i - \bar{\underset{\sim}{t}})(\underset{\sim}{t}_i - \bar{\underset{\sim}{t}})'. \tag{3.13}$$

Then for $r > k$, we propose the test statistic

$$Q^* = r(\bar{\underset{\sim}{t}}'(\underset{\sim}{S}^*)^{-1}\bar{\underset{\sim}{t}}). \tag{3.14}$$

In the null case,

$$Q^* \sim \{k/(r-k)\}F_{k,r-k}\ , \tag{3.15}$$

that is, Q^* is distributed as $\{k/(r-k)\}$ times a Fisher's F with k and (r-k) degrees of freedom.

Furthermore, the non-null distribution of Q^* will be that of $\{k/(r-k)\}$ times a noncentral F with the same degrees of freedom and noncentrality parameter $r(\underset{\sim}{u}'(\underset{\sim}{V}^*)^{-1}\underset{\sim}{u})$ where $\underset{\sim}{u}$ is equal to the expectation of $\underset{\sim}{t}_i$ under the non-null hypothesis (see, for instance, Anderson (1958), corollary 5.2.3).

Moreover, we note that since

$$\bar{\underset{\sim}{t}} = \underset{\sim}{B}_1 \sum_{i=1}^{r} \{n_i^{1/2}(\underset{\sim}{a}_o + \underset{\sim}{B}_o'\ \bar{\underset{\sim}{x}}_i)\}/r = \underset{\sim}{B}_1\underset{\sim}{u}^*\{\sum_{i=1}^{r} (n_i^{1/2})/r\} \tag{3.16}$$

where

$$\underset{\sim}{u}^* = \underset{\sim}{a}_o + \{\underset{\sim}{B}_o' \sum_{i=1}^{r} (n_i^{1/2}\ \bar{\underset{\sim}{x}}_i) / \sum_{i=1}^{r} (n_i^{1/2})\}, \tag{3.17}$$

$$\bar{\underset{\sim}{t}}'\bar{\underset{\sim}{t}}\left(\sum_{i=1}^{r}(n_i^{1/2})/r\right)^{-2} = \underset{\sim}{u}^{*\prime}(\underset{\sim}{B}_o'\underset{\sim}{B}_o)^{-1}\underset{\sim}{u}^* \tag{3.18}$$

represents according to (2.26) the square of the Euclidean distance between M_o defined in (3.3) and a weighted average of the $\bar{\underset{\sim}{x}}_i$'s.

3.2 b <u>A Function of the Distance between M_o and the Arithmetic Mean of all the Observations</u>

We may also consider the following statistic to test H_o

$$Q_c = n(\bar{\underset{\sim}{c}}'(\underset{\sim}{S}_c^*)^{-1}\bar{\underset{\sim}{c}}) \sim (k/(n-k))\, F_{k,n-k} \tag{3.19}$$

where

$$n = \sum_{i=1}^{r} n_i \,, \tag{3.20}$$

$$\bar{\underset{\sim}{c}} = \sum_{i=1}^{r}\sum_{j=1}^{n_i} \underset{\sim}{c}_{ij}/n, \tag{3.21}$$

$$\underset{\sim}{S}_c^* = \sum_{i=1}^{r}\sum_{j=1}^{n_i} (\underset{\sim}{c}_{ij}-\bar{\underset{\sim}{c}})(\underset{\sim}{c}_{ij}-\bar{\underset{\sim}{c}})' \tag{3.22}$$

and

$$\underset{\sim}{c}_{ij} = \underset{\sim}{B}_1(\underset{\sim}{a}_o + \underset{\sim}{B}_o'\,\underset{\sim}{x}_{ij}) \overset{ind}{\sim} N_k(\underset{\sim}{0},\underset{\sim}{V}^*), \tag{3.23}$$

$i=1,\ldots,r;\ j=1,\ldots,n_i$, with

$$\underset{\sim}{V}^* = \underset{\sim}{B}_1\underset{\sim}{B}_o'\underset{\sim}{V}\underset{\sim}{B}_o\underset{\sim}{B}_1.$$

Moreover the non-null distribution of Q_c is that of $\{k/(n-k)\}$ times a noncentral F with k and (n-k) degrees of freedom and noncentrality parameter $n(\underset{\sim}{u}_c'(\underset{\sim}{V}^*)^{-1}\underset{\sim}{u}_c)$ where $\underset{\sim}{u}_c$ is equal to the expectation of $\underset{\sim}{c}_{ij}$ under the non-null hypothesis.

Since

$$\bar{\underset{\sim}{c}} = \sum_{i=1}^{r} \sum_{j=1}^{n_i} \underset{\sim}{B}_1(\underset{\sim}{a}_o + \underset{\sim}{B}_o' \underset{\sim}{x}_{ij})/n = \underset{\sim}{B}_1(\underset{\sim}{a}_o + \underset{\sim}{B}_o' \bar{\underset{\sim}{x}}) \qquad (3.24)$$

where

$$\bar{\underset{\sim}{x}} = \sum_{i=1}^{r} \sum_{j=1}^{n_i} \underset{\sim}{x}_{ij}/n ,$$

$$\bar{\underset{\sim}{c}}'\bar{\underset{\sim}{c}} = (\underset{\sim}{a}_o + \underset{\sim}{B}_o' \bar{\underset{\sim}{x}})'(\underset{\sim}{B}_o'\underset{\sim}{B}_o)^{-1}(\underset{\sim}{a}_o + \underset{\sim}{B}_o' \bar{\underset{\sim}{x}}) \qquad (3.25)$$

represents according to (2.26) the square of the Euclidean distance between M_o defined in (3.3) and the arithmetic mean of all the $\underset{\sim}{x}_{ij}$'s.

3.3 A Modification of the Original Hypothesis

3.3 a A Modification Based on the Euclidean Metric

Let us consider a new null hypothesis

$$H_o^p : \underset{\sim}{a} = \underset{\sim}{a}_o,\ \underset{\sim}{B} = \underset{\sim}{B}_o \text{ and } \underset{\sim}{V} \text{ is proportional to } \underset{\sim}{V}_o, \qquad (3.26)$$

that is, $\underset{\sim}{V} = v^2\underset{\sim}{V}_o$, where $\underset{\sim}{a}_o, \underset{\sim}{B}_o, \underset{\sim}{V}_o$ are known and $\underset{\sim}{v}^2$ is a positive scalar quantity which is assumed to be unknown.

Let

$$v\underset{\sim}{K} = \underset{\sim}{K}_1 = (\underset{\sim}{B}_o' \underset{\sim}{V}_o \underset{\sim}{B}_o)^{-1/2},$$

then we can rewrite (2.30) as

$$\underset{\sim}{K}_1' \underset{\sim}{B}_o' \underset{\sim}{V}_o \underset{\sim}{B}_o \underset{\sim}{K}_1 = I$$

and (2.33) as

$$v\underset{\sim}{U}'\underset{\sim}{K}_1^{-1}(\underset{\sim}{B}_o'\underset{\sim}{B}_o)^{-1}(\underset{\sim}{K}_1')^{-1}\underset{\sim}{U}v = \underset{\sim}{M} = v^2\underset{\sim}{M}^* = v^2\mathrm{diag}(m_1^*,\ldots,m_k^*). \quad (3.27)$$

Hence

$$v^2 m_j^* = m_j \ , \ j=1,\ldots,k, \quad (3.28)$$

and

$$(m_1^*,\ldots,m_k^*) = \mathrm{ch}[(\underset{\sim}{B}_o'\underset{\sim}{V}_o\underset{\sim}{B}_o)^{1/2}(\underset{\sim}{B}_o'\underset{\sim}{B}_o)^{-1}\{(\underset{\sim}{B}_o'\underset{\sim}{V}_o\underset{\sim}{B}_o)^{1/2}\}']. \quad (3.29)$$

where ch[.] is defined for (2.34).

Accordingly (2.37) becomes

$$d^2 = \sum_{i=1}^{r} n_i d_i^2 \sim v^2 \sum_{j=1}^{k} m_j^* \chi_j^2(r). \quad (3.30)$$

We propose the following statistic to test H_o^p

$$R^p = d^2/u^2 \quad (3.31)$$

where

$$u^2 = \sum_{i=1}^{r}\sum_{j=1}^{n_i}\{(\underset{\sim}{x}_{ij}-\overline{\underset{\sim}{x}}_i)'\underset{\sim}{V}_o^{-1}(\underset{\sim}{x}_{ij}-\overline{\underset{\sim}{x}}_i)\}/\{p(n-r)\}. \quad (3.32)$$

Let us prove that u^2 is an unbiased estimator for v^2. Letting

$$\underset{\sim}{\Pi} = \underset{\sim}{V}^{-1/2} = (\underset{\sim}{V}_o^{-1/2})/v \ , \quad (3.33)$$

$$\underset{\sim}{y}_{ij} = \underset{\sim}{\Pi}\underset{\sim}{x}_{ij} = (y_{ij1}, y_{ij2},\ldots,y_{ijp})', \quad (3.34)$$

$$i=1,\ldots,r; \ j=1,\ldots,n_i,$$

$$\underset{\sim}{y}_{i.\ell} = \sum_{j=1}^{n_i} y_{ij\ell}/n_i \ , \ \ell=1,\ldots,p \ , \quad (3.35)$$

and

$$\bar{y}_i = (y_{i.1}, y_{i.2}, \ldots, y_{i.p})' = \sum_{j=1}^{n_i} y_{ij}/n_i \ , \qquad (3.36)$$

we have that $\mathrm{cov}(y_{ij}) = I$ and therefore the $y_{ij\ell}$'s are all independent with variance equal to unity. Hence

$$\sum_{j=1}^{n_i} (y_{ij\ell} - y_{i.\ell})^2 \overset{ind}{\sim} \chi^2_{n_i-1} \ , \ i=1,\ldots,r;\ \ell=1,\ldots,p. \qquad (3.37)$$

Thus (3.32) becomes

$$u^2 = v^2 \sum_{i=1}^{r} \sum_{j=1}^{n_i} \{(y_{ij}-\bar{y}_i)'(y_{ij}-\bar{y}_i)\}/\{p(n-r)\}$$

$$= v^2 \sum_{i=1}^{r} \sum_{\ell=1}^{p} \sum_{j=1}^{n_i} (y_{ij\ell} - y_{i.\ell})^2 / \ p(n-r) \ , \qquad (3.38)$$

so that

$$u^2 \sim v^2 \sum_{\ell=1}^{p} \sum_{i=1}^{r} \chi^2_{n_i-1} \ /\{p(n-r)\},$$

that is,

$$u^2 \sim v^2 \ \chi^2_{p(n-r)} \ / \ \{p(n-r)\} \qquad (3.39)$$

since all the chi-squares are independent and therefore

$$E(u^2) = v^2. \qquad (3.40)$$

We will now show that d^2 and u^2 are independently distributed. Let

$$w_{ij} = V_o^{-1/2} x_{ij} = (w_{ij1}, w_{ij2}, \ldots, w_{ijp})' \ , \qquad (3.41)$$

$i=1,\ldots,r;\ j=1,\ldots,n_i$,

$$\bar{\underset{\sim}{w}}_i = (w_{i.1}, \ldots, w_{i.p})' \tag{3.42}$$

where

$$w_{i.\ell} = (w_{i1\ell} + \ldots + w_{in_i\ell})/n_i \ , \ \ell=1,\ldots,p \ ,$$

and

$$s_{i\ell}^2 = \sum_{j=1}^{n_i} (w_{ij\ell} - w_{i.\ell})^2 \ , \ i=1,\ldots,r \ . \tag{3.43}$$

Then

$$d^2 = \sum_{i=1}^{r} n_i(\underset{\sim}{a}_o + \underset{\sim}{B}_o' \bar{\underset{\sim}{x}}_i)'(\underset{\sim}{B}_o'\underset{\sim}{B}_o)^{-1}(\underset{\sim}{a}_o + \underset{\sim}{B}_o' \bar{\underset{\sim}{x}}_i)$$

$$= \sum_{i=1}^{r} n_i(\underset{\sim}{a}_o + \underset{\sim}{B}_o'\underset{\sim}{V}_o^{1/2}\bar{\underset{\sim}{w}}_i)'(\underset{\sim}{B}_o'\underset{\sim}{B}_o)^{-1}(\underset{\sim}{a}_o + \underset{\sim}{B}_o' \underset{\sim}{V}_o^{1/2}\bar{\underset{\sim}{w}}_i) \tag{3.44}$$

is a function of the $w_{i.\ell}$'s which are all independent, $i=1,\ldots,r$; $\ell=1,\ldots,p$, and

$$u^2 = \sum_{i=1}^{r} \sum_{j=1}^{n_i} \{(\underset{\sim}{w}_{ij} - \bar{\underset{\sim}{w}}_i)'(\underset{\sim}{w}_{ij} - \bar{\underset{\sim}{w}}_i)\}/\{p(n-r)\}$$

$$= \sum_{i=1}^{r} \sum_{\ell=1}^{p} s_{i\ell}^2 \ /\{p(n-r)\} \ . \tag{3.45}$$

Due to the basic assumption of normality $s_{i\ell}^2$ and $w_{i.\ell}$ are independent, $i=1,\ldots,r$; $\ell=1,\ldots,p$, and consequently u^2 is independent of $\underset{\sim}{t}_i$, $i = 1,\ldots,r$, and of $d^2 = (\underset{\sim}{t}_1'\underset{\sim}{t}_1 + \ldots + \underset{\sim}{t}_r'\underset{\sim}{t}_r)$ where

$$\underset{\sim}{t}_i = n_i^{1/2}\{\underset{\sim}{B}_1(\underset{\sim}{a}_o + \underset{\sim}{B}_o'\underset{\sim}{V}_o^{-1/2}\bar{\underset{\sim}{w}}_i)\}, \ i=1,\ldots,r \ ,$$

and B_1 is given in (3.10).

Therefore we have that

$$R^p = d^2/u^2 \sim p(n-r) \sum_{j=1}^{k} \{m_j^* \chi_j^2(r)\}/\chi_{p(n-r)}^2 \tag{3.46}$$

where all the chi-squares are independent. Thus, the distribution of R^p is that of the ratio of a linear combination of chi-square variates over a chi-square variate and it will be obtained as corollaries of the results of Chapters 4 and 5.

We note here that Basu's statistic

$$R = d^2/|\underset{\sim}{B}_o'\underset{\sim}{S}\underset{\sim}{B}_o|^{1/k}$$

could be used to test H_o^p since its distribution would be that of

$$(n-r)v^2\{\sum_{j=1}^{k} m_j^* \chi_j^2(r)\}/\{v^2|\underset{\sim}{B}_o'\underset{\sim}{V}_o\underset{\sim}{B}_o|^{1/k}(\prod_{i=1}^{k} \chi^2_{n-r+i-1})^{1/k}\} \qquad (3.47)$$

and would be free of v^2. However, in this case, the procedure would be somewhat inefficient in view of the fact that the matrix $\underset{\sim}{V}$ is estimated in its denominator whereas only v^2 needs to be estimated.

Furthermore, if we want to test the hypothesis

$$H_o^k : \underset{\sim}{a} = \underset{\sim}{a}_o \ , \ \underset{\sim}{B} = \underset{\sim}{B}_o \text{ where } \underset{\sim}{V} = v^2\underset{\sim}{V}_o \text{ is known,} \qquad (3.48)$$

we may use d* where

$$d^* = d^2/v^2 \qquad (3.49)$$

and d^2 is defined in (3.30).

We will provide three representions of the density of d* where

$$d^* \sim \sum_{i=1}^{k} m_i^* \chi_i^2(r)$$

and the chi-squares are independent. These representations can be deduced from the results found in Mathai and Pillai (1982).

For the sake of completeness, these are given here.

The first representation is based on the moment generating function of $d^* \geq 0$, namely,

$$\prod_{j=1}^{k} (1 - 2m_j^* t)^{-r/2} \tag{3.50}$$

which can be expanded as a finite sum by a generalized partial fraction technique as follows when r is even:

$$\prod_{j=1}^{k} (-2m_j^* t)^{-r/2} = \prod_{j=1}^{p} (-2m_j^*)^{-r/2} \sum_{j=1}^{k} \sum_{\ell=1}^{r/2} c_{j\ell} (t - 1/(2m_j^*))^{-\ell} \tag{3.51}$$

where the coefficients $c_{j\ell}$ are determined with an algorithm which is described in Mathai and Rathie (1971). Then term by term inversion allows us to write the density of d* as follows

$$f(d^*) = \prod_{i=1}^{k} \{(1-2m_i^*)^{-r/2}\} \sum_{j=1}^{k} \sum_{\ell=1}^{r/2} \{(-1)^{\ell} c_{j\ell} d^{*\,\ell-1} \exp(-d^*/(2m_j^*))\}/(\ell-1)! \tag{3.52}$$

for $d^* > 0$ and $f(d^*) = 0$ for $d^* \leq 0$, where

$$c_{j\ell} = \{ \sum_{j_1=0}^{r/2-\ell-1} \binom{r/2-\ell-1}{j_1} A_j^{(r/2-\ell-1-j_1)} \sum_{j_2=0}^{j_1-1} \binom{j_1-1}{j_2} A_j^{(j_1-1-j_2)} \sum_{j_3=0}^{j_2-1} \binom{j_2-1}{j_3} A_j^{(j_2-1-j_3)} \cdots \sum_{j_{r/2-\ell-1}=0}^{1} A_j^{(1-j_{r/2-\ell-1})} (A_j^{0} B_j)\}/(r/2-\ell)! \tag{3.53}$$

with

$$B_j = \prod_{\substack{i=1\\i\neq j}}^{k} (1/(2m_j^*) - 1/(2m_i^*))^{-r/2} \tag{3.54}$$

and

$$A_j^{(q)} = (-1)^{q+1}\, q! \sum_{\substack{i=1\\i\neq j}}^{k} (r/2)(1/(2m_j^*) - 1/(2m_i^*))^{-(q+1)}, \tag{3.55}$$

$q = 0,1,2,\ldots$.

In this case we assume that all the m_i^*'s are distinct; if some m_i^*'s are equal, we combine the corresponding factors and the density can again be obtained in the same way.

Another representation of the density of d* which is valid whether r is odd or even is given in term of a confluent hypergeometric function of (k-1) variables denoted by $\phi_2\{.\}$. This function is defined as follows in Mathai and Saxena (1978), p. 163,

$$\phi_2(b_1,\ldots,b_n;c;x_1,\ldots,x_n)$$

$$= \sum_{i=0}^{\infty} \sum_{i_1+\ldots+i_n=i} \{(b_1)_{i_1}\ldots(b_n)_{i_n} x_1^{i_1}\ldots x_n^{i_n}\}/ \{(c)_i\, i_1!\ldots i_n!\}\ , \tag{3.56}$$

the notation $(q)_i$ meaning $q(q+1)\ldots(q+i-1)$. The properties of ϕ_2 are well-known in the theory of special functions. This function behaves like a Kummer's confluent hypergeometric function which is defined in Erdelyi(1953)and the series form is convergent for all values of the arguments.

Let us consider the following representation of the various factors of the moment generating function.

$$(1-2m_j^*t)^{-r/2} = (1-2m_1^*t)^{-r/2}(m_1^*/m_j^*)^{r/2}$$

$$\sum_{i_j=0}^{\infty}(r/2)_{i_j}(1-m_1^*/m_j^*)^{i_j}(1-2m_1^*t)^{-i_j}/i_j! \qquad (3.57)$$

for

$$|(1-m_1^*\ m_j^{*-1})/(1-2m_1^*t)| < 1,\ j=1,\dots,k.$$

A sufficient condition for the expansion is that $t < \min\{(2m_1^*)^{-1}, \dots, (2m_k^*)^{-1}\}$ where m_1^* is the smallest of the m_j^*'s such that

$$(1-m_1^*\ m_j^{*-1}) < (1-2m_j^*t),\ j=1,\dots,k.$$

Hence

$$\prod_{j=1}^{k}(1-2m_j^*t)^{-r/2} = \{m_1^{*(k-1)r/2}/(\prod_{j=2}^{k}m_j^{*r/2})\} \qquad (3.58)$$

$$\sum_{i_2=0}^{\infty}\dots\sum_{i_k=0}^{\infty}(r/2)_{i_2}\dots(r/2)_{i_k}(1-m_1^*/m_2^*)^{i_2}\dots$$

$$(1-m_1^*/m_k^*)^{i_k}(1-2m_1^*t)^{-(kr/2\ +i)}/(i_1!\dots i_k!)$$

where $i=i_2+\dots+i_k$. Term by term inversion is possible in this case and the density of d* may be written as follows

$$g(d^*) = [\{d^{*(rk/2)-1}\exp(-d^*/2m_1^*))\}/\{2^{rk/2}m_1^{*r/2}\dots m_k^{*r/2}\Gamma(rk/2)\}]$$

$$\phi_2\{r/2,\dots,r/2;rk/2;(m_1^{*-1}-m_2^{*-1})d^*/2,\dots,(m_1^{*-1}-m_k^{*-1})d^*/2\}$$

(3.59)

where

$$\phi_2\{\cdot\} = \sum_{i=0}^{\infty}\ \sum_{i_1+\dots+i_{k-1}=i}(r/2)_{i_1}\dots(r/2)_{i_{k-1}}$$

$$[\{(m_1^{*-1}-m_2^{*-1})d^*/2\}^{i_1}\ldots\{(m_1^{*-1}-m_k^{*-1})d^*/2\}^{i_{k-1}}]/$$

$$\{(rk/2)_i \; i_1!\ldots i_{k-1}!\}.$$

The density of d* can also be derived in terms of zonal polynomials. First, we express the moment generating function in terms of a determinant:

$$\prod_{i=1}^{k} (1-2m_i^*t)^{-r/2} = |I - 2t\underset{\sim}{M}^*|^{-r/2}$$

$$= |I - (\underset{\sim}{M}^*/\eta)+\underset{\sim}{M}^*(1-2\eta t)/\eta|^{-r/2} \quad (3.60)$$

where η is an arbitrary constant and $\underset{\sim}{M}^*$ is defined in (3.27) and (3.29).

The series obtained when we expand the determinant in terms of zonal polynomials is valid when the norm of the matrix

$$(I - \eta\underset{\sim}{M}^{*-1})/(1 - 2\eta t)$$

is less than unity and a sufficient condition is that $t<1/(2\eta)$ and $\max_i |1 - (\eta/m_i^*)| < (1-2\eta t)$, $i=1,\ldots,k$. The validity of this expansion is guaranteed due to the presence of η and term by term inversion is possible. Then, the density function of d* can be written as follows

$$h(d^*) = \prod_{i=1}^{k} \{(2m_i^*)^{-r/2}\} d^{*-1+rk/2}\, e^{-d^*/(2\eta)}$$

$$\sum_{j=0}^{\infty} \sum_{J} \{(r/2)_J\, C_J\, (I/\eta - M^{*-1})(d^*/2)^j\}/$$

$$\{j!(rk/2)_j\, \Gamma(rk/2)\} \quad (3.61)$$

where $J=(j_1,\ldots,j_k)$; $j_1+\ldots+j_k=j$; $j_1 \geq \ldots \geq j_k \geq 0$; C_J is a zonal polynomial of order j and

$$(r/2)_J = \prod_{i=1}^{k} (r/2 - (i-1)/2)_{j_i}$$

$$= \prod_{i=1}^{k} \Gamma(r/2-(i-1)/2 + j_i) / \Gamma(r/2-(i-1)/2) \tag{3.62}$$

The function $\Gamma(z)$ may be defined by either of the following equations

$$\Gamma(z) = \int_0^\infty e^{-x} x^{z-1} dx, \quad Re(z) > 0 \tag{3.63}$$

and

$$(\Gamma(z))^{-1} = (2\pi i)^{-1} \{ \int_{c-i\infty}^{c+i\infty} e^t t^{-z} dt \}, c>0 \text{ and } Re(z)>0 \ . \tag{3.64}$$

Hence the denominator (and a fortiori the numerator) in (3.62) will be well-defined provided

$$(r + 1)/2 \geq k/2,$$

that is,

$$r > k.$$

The definition of zonal polynomials requires a few concepts from group representation theory. Let V_k be the vector space of homogeneous polynomials $\Phi(\Lambda)$ of degree k in the $n=m(m+1)/2$ different elements of the $m\times m$ symmetric matrix Λ. The dimension N of V_k is the number

$$N = (n+k-1)!/\{(n-1)!k!\}$$

of monomials

$$\prod_{i\leq j}^{m} \lambda_{ij}^{k_{ij}} \text{ of degree } \sum_{i\leq j}^{m} k_{ij} = k.$$

Consider a congruence transformation $\Lambda \to L\Lambda L'$ by a nonsingular $m \times m$ matrix L. Then we can define a linear transformation of the space V_k of polynomials $\Phi(\Lambda)$, that is,

$$\Phi \to L\ \Phi:\ (L\ \Phi)(\Lambda) = \Phi(L^{-1}\Lambda(L^{-1})').$$

A subspace $V' \subset V$ is called invariant if $LV' \subset V'$ for all nonsingular matrices L. Also V' is called an irreducible invariant subspace if it has no proper invariant subspace. It can be shown that V_k decomposes into a direct sum of irreducible invariant subspaces V_K corresponding to each partition $K = (k_1, \ldots, k_m)$, $k = k_1 + \ldots + k_m$ into not more than m parts

$$V_k = \bigoplus_K V_K .$$

The polynomial $(\mathrm{tr}\Lambda)^k \in V_k$ then has a unique decomposition

$$(\mathrm{tr}\Lambda)^k = \sum_K C_K(\Lambda)$$

into polynomials $C_K(\Lambda) \in V_K$ belonging to the respective subspaces. The zonal polynomial is thus a symmetric homogeneous polynomial of degree k in the latents roots of Λ. Zonal polynomials have been developed by Hua (1959) and James (1961). A more detailed discussion may be found in these papers.

3.3 b <u>Based on a Metric Depending on $\underset{\sim}{V}$</u>

For the metric

$$\{(\bar{\underset{\sim}{x}}_i - \underset{\sim}{g})'\underset{\sim}{V}^{-1}(\bar{\underset{\sim}{x}}_i - \underset{\sim}{g})\} \ ,$$

we can obtain, by going over arguments similar to those used in Chapter 2 for the metric $\{(\bar{\underset{\sim}{x}}_i - \underset{\sim}{g})'(\bar{\underset{\sim}{x}}_i - \underset{\sim}{g})\}$, the following result similar to (3.6)

$$n_i\bar{d}_i^2 = n_i(\underset{\sim}{a}_o + \underset{\sim}{B}_o' \bar{\underset{\sim}{x}}_i)'(\underset{\sim}{B}_o'\underset{\sim}{V}_o\underset{\sim}{B}_o)^{-1}(\underset{\sim}{a}_o + \underset{\sim}{B}_o' \bar{\underset{\sim}{x}}_i)/v^2 \sim \chi_k^2 \ , \qquad (3.65)$$

$i=1,\ldots,r$, where $\bar{d}_i$ represents the "distance" between $\bar{\underset{\sim}{x}}_i$ and M_o given in (3.3), under the hypothesis H_o^p defined in (3.26), that is, where $\underset{\sim}{V} = v^2\underset{\sim}{V}_o$ and $\underset{\sim}{V}_o$ is known.

Let

$$\tilde{d}_i^2 = v^2\bar{d}_i^2 = (\underset{\sim}{a}_o + \underset{\sim}{B}_o' \bar{\underset{\sim}{x}}_i)'(\underset{\sim}{B}_o'\underset{\sim}{V}_o\underset{\sim}{B}_o)^{-1}(\underset{\sim}{a}_o + \underset{\sim}{B}_o' \bar{\underset{\sim}{x}}_i) \qquad (3.66)$$

and let u^2 be defined as in (3.32). Then, the statistic

$$\tilde{d}^2/u^2 = \sum_{i=1}^{r} n_i\tilde{d}_i^2/u^2 \sim p(n-r)\{\chi_{rk}^2 \ / \ \chi_{p(n-r)}^2\} \qquad (3.67)$$

since according to (3.39),

$$u^2 \sim v^2 \ \chi_{p(n-r)}^2/\{p(n-r)\}. \qquad (3.68)$$

Moreover $\tilde{d}^2$ and u^2 are independent for the same reasons that d^2 given in (3.44) and u^2 are independently distributed.

We may thus test the hypothesis (3.26) with $\tilde{d}^2/u^2$, where

$$\tilde{d}^2/u^2 \sim rk\{F_{rk,p(n-r)}\}. \qquad (3.69)$$

3.4 Concluding Remarks

For testing H_o^k defined in (3.48), that is the null hypothesis where $\underset{\sim}{V}$ is known, we may also consider the following statistics:

$$Q^{\bar{t}} = r(\bar{\underset{\sim}{t}}'(\underset{\sim}{V}^*)^{-1}\bar{\underset{\sim}{t}}) \sim \chi_k^2 \tag{3.70}$$

instead of Q* given in (3.14) and

$$Q^{\bar{c}} = n(\bar{\underset{\sim}{c}}'(\underset{\sim}{V}^*)^{-1}\bar{\underset{\sim}{c}}) \sim \chi_k^2 \tag{3.71}$$

instead of Q_c given in (3.19), where $\bar{\underset{\sim}{t}}'$ and $\bar{\underset{\sim}{c}}'$ are respectively given in (3.12) and (3.21) and

$$\underset{\sim}{V}^* = \underset{\sim}{B}_1\underset{\sim}{B}_o'\underset{\sim}{V}\underset{\sim}{B}_o\underset{\sim}{B}_1 = \text{cov}(\underset{\sim}{t}_i) = \text{cov}(\underset{\sim}{c}_{ij}), \tag{3.72}$$

$i=1,\dots,r;\ j=1,\dots,n_i$, according to (3.11) and (3.23).

Now, assuming that we have at our disposal an independent estimate of $\underset{\sim}{V}$ denoted by $\hat{\underset{\sim}{V}}$ and that

$$n^*\hat{\underset{\sim}{V}} \sim W_p(n^*,\underset{\sim}{V}), \tag{3.73}$$

we may use

$$\bar{R} = \{\sum_{i=1}^{r} n_i d_i^2\}/|\underset{\sim}{B}_o'\hat{\underset{\sim}{V}}\underset{\sim}{B}_o|^{1/k}, \tag{3.74}$$

instead of R defined in (3.2), in order to test H_o given in (3.1). In this case, no replications are required and

$$\bar{R} \sim \{n^* \sum_{j=1}^{k} m_j\ \chi_j^2(r)\}/\{|\underset{\sim}{B}_o'\underset{\sim}{V}\ \underset{\sim}{B}_o| \prod_{i=1}^{k} \chi_{n^*-i+1}^2\}^{1/k}, \tag{3.75}$$

where all the chi-squares are independent. The exact density of $\overline{R}$ will be discussed in Chapters 4 and 5.

Similarly, let us suppose that we want to test H_o^p , that is, the null hypothesis where $\underset{\sim}{V} = v^2 \underset{\sim}{V}_o$ and $\underset{\sim}{V}_o$ is known and that an independent estimate of v^2 denoted by $\overline{v}^{*2}$ is available where

$$\overline{v}^{*2} \sim v^2 \; \chi^2_{N^*}/N^* \; . \tag{3.76}$$

Then, instead of R^p defined in (3.31), we may use

$$R^{*p} = d^2/\overline{v}^{*2} \sim N^*\{\sum_{j=1}^{k} m_j^* \; \chi_j^2(r)\}/\chi^2_{N^*} \tag{3.77}$$

where d^2 is given in (3.44), m_j^* in (3.29), $j=1,\ldots,k$, and the chi-squares are independent.

Once again, the exact density of R^{*p} will be obtained as corollaries of the results of Chapters 4 and 5.

Furthemore, if the equations in (2.21) are homogeneous linear equations, that is, $a_i=0$, $i=1,\ldots,k$, we can repeat our arguments taking $\underset{\sim}{a} = \underset{\sim}{0}$ throughout and all the results of this chapter would still hold true.

Finally, Basu (1969) claimed that the distribution of $\overline{T}^2$ given in (3.8) could not be obtained in closed form. We will show that the exact density of $\overline{T}^2$ can be obtained in closed form. First, let us state Theorem 5.2.2. in Anderson (1958): Let $T^2 = \underset{\sim}{Y}'\underset{\sim}{S}^{-1}\underset{\sim}{Y}$ where $\underset{\sim}{Y} \sim N_k(\underset{\sim}{0},\underset{\sim}{\Sigma})$ and $(n-r)\underset{\sim}{S}$ is independently distributed as $\underset{\sim}{Z}_1\underset{\sim}{Z}_1'+\ldots+\underset{\sim}{Z}_{n-r}\underset{\sim}{Z}_{n-r}'$ with $\underset{\sim}{Z}_j$ i.i.d. $N_k(\underset{\sim}{0},\underset{\sim}{\Sigma})$, $j=1,\ldots,n-r$. Then

$$\{T^2/(n-r)\}\{(n-r-k+1)/k\} \sim F_{k,n-r-k+1} .$$

Therefore, according to (3.5) and (2.41), it follows that $\bar{T}_i^2$ defined in (3.9) has the following distribution

$$\bar{T}_i^2 = (\underset{\sim}{a}_o + \underset{\sim}{B}_o' \bar{\underset{\sim}{x}}_i)'(\underset{\sim}{B}_o' \underset{\sim}{S} \underset{\sim}{B}_o)^{-1}(\underset{\sim}{a}_o + \underset{\sim}{B}_o' \bar{\underset{\sim}{x}}_i)$$

$$\sim (n-r)\ \chi_k^2 / \chi_{n-r-k+1}^2 , \tag{3.78}$$

$i=1,\ldots,r$ and that

$$\bar{T}^2 = \sum_{i=1}^{r} \bar{T}_i^2 \sim \{(n-r) \sum_{i=1}^{r} \chi_i^2(k)\} / \chi_{n-r-k+1}^2 . \tag{3.79}$$

But since the common chi-square in the denominator is independent of each chi-square in the numerator, it is also independent of the sum of the chi-squares of the numerator which are all mutually independent. Therefore

$$\bar{T}^2 \sim \{(n-r)\ \chi_{rk}^2\} / \chi_{n-r-k+1}^2 , \tag{3.80}$$

that is, $\bar{T}^2$ is distributed as a constant times a central F with rk and n-r-k+1 degrees of freedom:

$$\bar{T}^2 \sim (n-r)\{(rk)/(n-r-k+1)\}\ F_{rk,n-r-k+1} . \tag{3.81}$$

Hence, we may use $\bar{T}^2$ for testing H_o defined in (3.1).

Chapter 4

THE DISTRIBUTION OF CERTAIN ALGEBRAIC FUNCTIONS OF INDEPENDENT GAMMA VARIATES

4.0 Introduction

In this chapter, we will derive the exact density of certain algebraic functions of gamma variates, namely products, ratios, linear combinations and also the ratio of a linear combination of independent gamma variates over the root of a product of independent gamma variates. Corollaries of these results will enable us to write in computable forms the exact densities of the test statistics given in (3.2), (3.46), (3.49), (3.75) and (3.77).

All these densities are given in terms of the H-function which is the topic of the next section where it is presented as an inverse Mellin transform and some of its properties are discussed. The H-function is applicable in a number of problems arising in physical sciences, engineering and statistics. The importance of this function lies in the fact that nearly all the special functions occuring in applied mathematics and in statistics are its special cases.

The H-function has been studied by Fox (1961), Braaksma (1964), Nair (1973), Buschman (1974), Oliver and Kalla

(1976) and Mathai and Saxena (1978) among others.

Section 4.2 is devoted to the analysis of products and ratios of generalized gamma variates. This type of random variables is of vital importance in the fields of reliability analysis and life testing models. It has also been used by Amoroso (1925) and d'Attario (1932) to study the income distributions and by Rogers (1964) in order to obtain the exact distributions of some multivariate test criteria. Also, computable representations of the H-function are provided for the cases of interest. We also point out that the denominators of the statistics (3.2) and (3.75) are in fact distributed as the product of k independent generalized gamma variates.

In Section 4.3 , a representation of the density of a linear combination of real gamma variates is provided. Such linear combinations are connected to various problems in many areas. For instance, for their connection to random division of intervals and distribution of spacings see Dwass (1961), to content of a frustum of a simplex see Ali (1973), to storage capacities and queues see Prabhu (1965). Linear combinations of gamma variates are also related to test statistics and traces of Wishart matrices as can be seen in Mathai (1980) and in Mathai and Pillai (1982). Their connection to time series problems can be seen from MacNeill (1974). They also appear in the study of probability content of offset ellipsoids in Gaussian hyperspace and of distribution of quadratic forms, see for example, Ruben (1962) and Sheil and O'Muirheartaigh (1977).

The density of the statistic given in (3.49) is obtained as a corollary.

In Section 4.4, we derive the density of the ratio of a linear combination of gamma variates over the k-th root of a product of gamma variates and in Section 4.5 we obtain the h-th moment, $(h \in \mathbb{N})$, of this random variable in order to approximate its density with the appropriate Pearson curve.

4.1 The H-Function as an Inverse Mellin Transform

In this chapter, we will obtain the exact densities of the test statistics derived in the preceding chapter. These statistics are essentially functions of summations, products and ratios of real gamma variates and it turns out that the H-function defined in this section as an inverse integral transform will be of great importance for the derivation of their densities.

The basic tools for deriving distributions of sums,differences, products, ratios, powers, and more generally, algebraic functions of continuous random variables are the integral transforms.The most commonly used integral transforms are the Laplace transform, the Fourier transform and the Mellin transform. The aforementioned transforms, each corresponding to a function f(x), are now defined, together with their inverse transforms.

If f(x) is a real piecewise smooth function which is defined and single valued almost everywhere for $x \geq 0$ and which is such that the integral

$$\int_0^\infty |f(x)| \ e^{-kx} \ dx \tag{4.1}$$

converges for some real value k, then

$$L_f(r) = \int_0^\infty e^{-rx} \ f(x) \ dx \tag{4.2}$$

is the Laplace transform of f(x), where r is a complex variable and wherever f(x) is continuous the corresponding inverse Laplace transform is

$$f(x) = (2\pi i)^{-1} \int_{c-i\infty}^{c+i\infty} e^{rx} \ L_f(r) \ dr. \tag{4.3}$$

Equation (4.3) determines f(x) uniquely, if $L_f(r)$ is analytic in a strip consisting of the portion of the plane to the right of (and including) the Bromwich path (c-i∞, c+i∞), the latter denoting the straight line given by

$$\lim_{a\to\infty} (c-ia, \ c+ia) \tag{4.4}$$

where $i = (-1)^{1/2}$ and c is any value greater than k in (4.1).

The Laplace transform provides the means for deriving and analyzing the distribution of sums of nonnegative random variables . On the other hand, if the random variables may take on both positive and negative values, the Fourier transform as defined in (4.6) is an appropriate tool for deriving the probability density function of their sums and their differences.

If f(x) is a real piecewise smooth function which is defined and single valued almost everywhere for $-\infty < x < \infty$, and which is such that

$$\int_{-\infty}^{\infty} f(t)\ dt \tag{4.5}$$

is absolutely convergent where t is a real parameter, then

$$F_f(t) = \int_{-\infty}^{\infty} e^{itx} f(x)\ dx \tag{4.6}$$

is the Fourier transform of f(x) and, wherever f(x) is continuous, the corresponding inverse Fourier transform is

$$f(x) = (2\pi)^{-1} \int_{-\infty}^{\infty} e^{-itx} F_f(t)\ dt\ . \tag{4.7}$$

The expression $F_f(t)$ is also called the characteristic function of f(x) when f(x) is a density function and e^{itx} is called the kernel.

The Mellin transform as defined in (4.9) constitutes the counterpart of the Laplace transform in deriving the distribution of products and ratios of nonnegative random variables.

If f(x) is a real piecewise smooth function which is defined and single valued almost everywhere for $x > 0$ and which is such that

$$\int_{0}^{\infty} x^{k-1} |f(x)|\ dx \tag{4.8}$$

converges for some real value k, then

$$M_f(s) = \int_{0}^{\infty} x^{s-1} f(x)\ dx \tag{4.9}$$

where s is a complex number, is the Mellin transform of f(x) and wherever f(x) is continuous, the corresponding inverse Mellin transform is

$$f(x) = (2\pi i)^{-1} \int_{c-i\infty}^{c+i\infty} x^{-s} M_f(s)\, ds \qquad (4.10)$$

which, together with (4.9), constitutes a transform pair. Equation (4.10) determines f(x) uniquely, if the Mellin transform is an analytic function of the complex variable s for

$$c_1 < Re(s) = c < c_2$$

where c_1 and c_2 are real. This condition is sufficient because the analyticity of the transform ensures that the integrand of the inversion integral is expressible as a Laurent expansion, which expansion is always unique.

Here we give some important properties of these integral transforms. It is assumed that each transform pair exist within the region of convergence.

1. Linearity property

$$\text{Laplace: } L_{c_1 f_1 + c_2 f_2}(r) = c_1\, L_{f_1}(r) + c_2 L_{f_2}(r)$$

$$\text{Fourier: } F_{c_1 f_1 + c_2 f_2}(t) = c_1\, F_{f_1}(t) + c_2 F_{f_2}(t)$$

$$\text{Mellin : } M_{c_1 f_1 + c_2 f_2}(s) = c_1\, M_{f_1}(s) + c_2 M_{f_2}(s) \qquad (4.11)$$

2. Shifting property

$$\text{Laplace: } L_{e^{ax} f}(r) = L_f(r-a)$$

$$\text{Fourier: } F_{e^{ax} f}(t) = F_f(t+ia)$$

$$\text{Mellin: } M_{x^{-a}f}(s) = M_f(s-a) \tag{4.12}$$

3. Scaling

$$\text{Laplace: } L_{f(ax)}(r) = (1/a)\ L_{f(x)}(r/a)$$

$$\text{Fourier: } F_{f(ax)}(t) = (1/a)\ F_{f(x)}(t/a)$$

$$\text{Mellin : } M_{f(ax)}(s) = (1/a)\ M_{f(x)}(s) \tag{4.13}$$

4. Exponentiation

$$\text{Mellin : } M_{f(x^a)}(s) = (1/a)\ M_{f(x)}(s/a) \tag{4.14}$$

Since the great majority of cases, both theoretical and applied, involving the use of Mellin transform in connection with products, ratios and powers of independent random variables, are concerned with Mellin transforms of real variables, we will not discuss Mellin integral transforms for functions f(z) of complex random variables z.

Let

$$M_f(s) = \{\prod_{j=1}^{m} \Gamma(b_j+B_js) \prod_{j=1}^{n} \Gamma(1-a_j-A_js)\}/$$

$$\{\prod_{j=m+1}^{q} \Gamma(1-b_j-B_js) \prod_{j=n+1}^{p} \Gamma(a_j+A_js)\}$$

$$= \quad h(s) \quad \text{, say,} \tag{4.15}$$

where $M_f(s)$ is defined in (4.9); m,n,p,q are nonnegative integers such that $0 \le n \le p$, $1 \le m \le q$; A_j,(j=1,...,p), B_j,(j=1, ...,q) are positive numbers and a_j,(j=1 ,...,p), b_j,(j=1,...,q) are complex numbers such that

$$-A_j(b_h+\nu) \neq B_h(1-a_j+\lambda) \tag{4.16}$$

for $\nu,\lambda = 0,1,2,\ldots;\ h=1,\ldots,m;\ j=1,\ldots,n$. Then the H-function may be defined in terms of the inverse Mellin transform of $M_f(s)$ as follows:

$$f(x) = H^{m,n}_{p,q}(x)$$

$$= H^{m,n}_{p,q}\left\{ x \,\middle|\, \begin{matrix}(a_p, A_p)\\(b_q, B_q)\end{matrix} \right\}$$

$$= H^{m,n}_{p,q}\left\{ x \,\middle|\, \begin{matrix}(a_1,A_1),\ldots,(a_p,A_p)\\(b_1,B_1),\ldots,(b_q,B_q)\end{matrix} \right\}$$

$$= (2\pi i)^{-1} \int_{c-i\infty}^{c+i\infty} h(s)\, x^{-s}\, ds \tag{4.17}$$

where the Bromwich path $(c-i\infty, c+i\infty)$ defined in (4.4), separates the points

$$s = -(b_j+\nu)/B_j,\ j=1,\ldots,m;\ \nu=0,1,2,\ldots\ , \tag{4.18}$$

which are the poles of $\Gamma(b_j+B_js)$, $j=1,\ldots,m$, from the points

$$s = (1-a_j+\lambda)/A_j\ ,\ j=1,\ldots,n;\ \ \lambda=0,1,2,\ldots\ . \tag{4.19}$$

which are the poles of $\Gamma(1-a_j-A_js)$, $s=1,\ldots,n$.

Hence one must have that

$$\max_{1\le j\le m} Re\{-b_j/B_j\} < c < \min_{1\le j\le n} Re\{(1-a_j)/A_j\}. \tag{4.20}$$

The H-function makes sense if the following existence conditions are satisfied:

Case i. For all $x > 0$, when $\mu > 0$ (4.21)

Case ii. For $0 < x < \beta^{-1}$, when $\mu = 0$ (4.22)

where

$$\mu = \sum_{j=1}^{q} B_j - \sum_{j=1}^{p} A_j \tag{4.23}$$

and

$$\beta = \prod_{j=1}^{p} A_j^{A_j} \prod_{j=1}^{q} B_j^{-B_j}. \tag{4.24}$$

It should also be mentioned that when

$$A_j = B_h = 1 \ , \ j=1,\dots,p; \quad h=1,\dots,q \ ,$$

the H-function reduces to a Meijer's G-function. Hence

$$H_{p,q}^{m,n}\left\{ x \left| \begin{matrix} (a_1,1),\dots,(a_p,1) \\ (b_1,1),\dots,(b_q,1) \end{matrix} \right. \right\}$$

$$= G_{p,q}^{m,n}\left\{ x \left| \begin{matrix} a_1,\dots,a_p \\ b_1,\dots,b_q \end{matrix} \right. \right\} \tag{4.25}$$

exists for all $x > 0$ when $q > p$ and for $0 < x < 1$ when $q=p$. A detailed account of Meijer's G-function can be found in Mathai and Saxena (1973).

The behaviour of the H-function for small and large values of the argument has been discussed by Braaksma (1964). The two main results are

$$H_{p,q}^{m,n}(x) = O(|x|^c) \tag{4.26}$$

for small x, where $\mu \geq 0$ and $c = \min(b_j/B_j)$, $j=1,\ldots,m$; and

$$H_{p,q}^{m,n}(x) = O(|x|^d) \tag{4.27}$$

for large x, where $\mu \geq 0$ and $d = \max((a_j-1)/A_j)$, $j=1,\ldots,n$.

Here we give a useful property of the H-function which follows readily from its definition.

$$H_{p,q}^{m,n}\left\{ x \left| \begin{matrix}(a_j,A_j)\\(b_j,B_j)\end{matrix} \right.\right\} = H_{q,p}^{n,m}\left\{ \frac{1}{x} \left| \begin{matrix}(1-b_j,B_j)\\(1-a_j,A_j)\end{matrix} \right.\right\} . \tag{4.28}$$

When normalized with the proper constant, the H- function defined in (4.17) encompasses an entire class of probability density functions provided the parameters are restricted so that the function remains nonnegative and $\int_0^\infty f(x)dx=1$ where f(x) represents the function. We will refer to the random variables belonging to that class as H-function random variables. Because this class includes so many basic distributions as well as the distribution of numerous test statistics in multivariate analysis (see for instance Mathai (1970), (1971), (1972),(1972a), Mathai and Saxena (1969),(1971),(1978), and Mathai and Rathie (1971)), it is important to represent the H-function in computable form.

In fact, $H_{p,q}^{m,n}(x)$ is available as the sum of the residues of $h(s)x^{-s}$ in the points (4.18). The proof of the applicability of the residue theorem as well as a discussion about the contours for evaluating the integrals may be found in Springer (1979). In the next section, we will give some computable representations of the H-function for the cases of interest. Furthermore a computer program has been written by Eldred (1978) for the evaluation of the H-function. This program which is operational evaluates the probability density function (p.d.f.) and the cumulative distribution function (c.d.f.) of an H-function random variable at any value of the random variable and also plots the p.d.f. and the c.d.f. The computer program compiled on an MNF compiler and run on a CDC 6600, is very efficient and poses no precision problem. Should precision problems arise when the program is run on smaller computers, the problem may be solved by compiling the program under IBM's Extended H-compiler.

Now we would like to point out that the c.d.f. may be found by a procedure analogous to that used for the p.d.f. Let f(x) in (4.10) represent a p.d.f.,then the c.d.f. F(y), defined by

$$F(y) = \int_0^y f(x)\,dx \ , \ 0 < y < \infty \ ,$$

can be obtained through use of the Mellin transform of f(x). The following equation has been established in Springer (1979), p.99, where the evaluation of the R.H.S. of (4.29) is discussed in detail

$$F(y) = 1 - (2\pi i)^{-1} \int_{c-i\infty}^{c+i\infty} y^{-s} s^{-1} M_{s+1}(f(y))\,ds \ .$$

Writing F(y) in the form of an H-function inversion integral, one has

$$F(y) = 1 - (2\pi i)^{-1} \int_{c-i\infty}^{c+i\infty} [\{\prod_{j=1}^{m} \Gamma(b_j+B_j(s+1)) \prod_{j=1}^{n} \Gamma(1-a_j-A_j(s+1))\} / \{s \prod_{j=m+1}^{q} \Gamma(1-b_j-B_j(s+1)) \prod_{j=n+1}^{p} \Gamma(a_j+A_j(s+1))\}] y^{-s} ds. \tag{4.29}$$

The definition of the H-function is slightly modified in (4.17) to present it as an inverse Mellin transform. This modification does not affect the results in Braaksma (1964) about its properties and the conditions for its existence, nor the representations in computable forms found in Mathai and Saxena (1978), p.71, where the H-function which will be denoted by $\bar{H}$ is defined as follows.

For

$$\max_{1\le j\le n} \mathrm{Re}((a_j-1)/A_j) < c' < \min_{1\le j\le m} \mathrm{Re}(b_j/B_j) , \tag{4.30}$$

$$\bar{H}^{m,n}_{p,q}\{ x \mid {(a_j,A_j) \atop (b_j,B_j)} \} = (2\pi i)^{-1} \int_{c'-i\infty}^{c'+i\infty} h(-s)\, x^s\, ds \tag{4.31}$$

where h(-s) according to (4.15) is

$$\{\prod_{j=1}^{m} \Gamma(b_j-B_js) \prod_{j=1}^{n} \Gamma(1-a_j+A_js)\} / \{\prod_{j=m+1}^{q} \Gamma(1-b_j+B_js) \prod_{j=n+1}^{p} \Gamma(a_j-A_js)\}$$

The identity

$$\bar{H}^{m,n}_{p,q}\{x \mid {(a_j,A_j) \atop (b_j,B_j)} \} \equiv H^{m,n}_{p,q}\{ x \mid {(a_j,A_j) \atop (b_j,B_j)} \} \tag{4.32}$$

follows by changing direction ; we simply have to replace s by -s in (4.17).

4.2 Products and Ratios of Generalized Gamma Variates

In this section, we will determine the exact density of products and ratios of independent generalized gamma variates which are not necessarily identically distributed. We will also provide computable representations of this density.

The p.d.f. of a generalized real gamma variate, X_j, is

$$g_j(x_j) = \{\beta_j d_j^{\alpha_j/\beta_j} / \Gamma(\alpha_j/\beta_j)\}\, x_j^{\alpha_j - 1}\, e^{-d_j x_j^{\beta_j}} \tag{4.33}$$

for $x_j > 0$, $d_j > 0$, $\alpha_j > 0$, $\beta_j > 0$, and $g_j(x_j) = 0$ elsewhere. Gamma, Weibull, Raleigh, folded normal and negative exponential are special cases of (4.33).

We note here that the denominators of the statistics (3.2) and (3.75) which are distributed, up to a constant, as the k-th root of a product of k independent chi-square variates, can be regarded as having the distribution of the product of k independent generalized gamma variates in view of the fact that the k-th root of a chi-square variate with ν degrees of freedom whose density is

$$\{k\, 2^{-\nu/2}\, \zeta^{(k\nu/2)-1}\, e^{-\zeta^k/2}\}/\Gamma(\nu/2) \quad \text{for } \zeta > 0,$$

is distributed as a generalized gamma variates with parameters

$$\alpha = k\nu/2\ ; \quad \beta = k\ ; \quad d = 1/2\ .$$

In Mathai and Saxena (1978), p. 83, the density of

$$Y = (X_1 \cdots X_m)/(X_{m+1} \cdots X_k) \tag{4.34}$$

where $X_1,\ldots,X_m,X_{m+1},\ldots,X_k$ is a set of mutually independent and identically distributed generalized real gamma variates is obtained in terms of an H-function.

We will derive the density of Y, denoted by g(y), for the case where $X_1,\ldots,X_k$ are not necessarily identically distributed but are independently distributed according to (4.33). If g(y) exists, the (s-1)st moment of Y about the origin is given by

$$E(Y^{s-1}) = \prod_{j=1}^{m} \{\Gamma((\alpha_j+s-1)/\beta_j)/(\Gamma(\alpha_j/\beta_j)d_j^{(s-1)/\beta_j})\}$$

$$\prod_{j=m+1}^{k} \{\Gamma((\alpha_j-s+1)/\beta_j)/(\Gamma(\alpha_j/\beta_j)d_j^{-(s-1)/\beta_j})\}. \tag{4.35}$$

Hence the density function, g(y), is available from the inverse Mellin transform of (4.35), that is,

$$g(y)=(\delta K)(2\pi i)^{-1}\int_{c-i\infty}^{c+i\infty} \prod_{j=1}^{m} \Gamma((\alpha_j-1+s)/\beta_j) \prod_{j=m+1}^{k} \Gamma((\alpha_j+1-s)/\beta_j)(\delta y)^{-s}\, ds$$

$$=(\delta K)H_{n,m}^{m,n}\left\{\delta y\,\middle|\,\begin{matrix}(-\alpha_{m+1}/\beta_{m+1},1/\beta_{m+1}),\ldots,(-\alpha_k/\beta_k,1/\beta_k)\\((\alpha_1-1)/\beta_1,1/\beta_1),\ldots,((\alpha_m-1)/\beta_m,1/\beta_m)\end{matrix}\right\} \tag{4.36}$$

where

$$0 < y < \infty\ , \quad n = k-m\ ,$$

$$\delta = \{\prod_{j=1}^{m} d_j^{1/\beta_j} \Big/ \prod_{j=m+1}^{k} d^{1/\beta_j}\}\ ,$$

$$K = \{ \prod_{j=1}^{k} \Gamma(\alpha_j/\beta_j) \}^{-1}$$

and the H-function in (4.36) exists for all $x>0$ if $\mu>0$ and for $0<x<\beta^{-1}$ if $\mu=0$, where

$$\mu = \sum_{j=1}^{m} \frac{1}{\beta_j} - \sum_{j=m+1}^{k} \frac{1}{\beta_j} \geqslant 0 \qquad (4.37)$$

and

$$\beta = \prod_{j=1}^{m} (1/\beta_j)^{1/\beta_j} \prod_{j=m+1}^{k} (1/\beta_j)^{-1/\beta_j} \qquad (4.38)$$

provided

$$-\beta_h((\alpha_j-1)/\beta_j + \nu) \neq \beta_j((\alpha_h-1)/\beta_h + r) \qquad (4.39)$$

$\nu, r = 0,1,2,\ldots$; $j=1,\ldots,m$ and $h=m+1,\ldots,k$. Moreover the path $(c-i\infty, c+i\infty)$ can be chosen suitably on account of (4.39).

The following representations of $\bar{H}^{m,n}_{n,m}(x)$ which are valid for $H^{m,n}_{n,m}(x)$ in view of (4.32), can be deduced from the results of Mathai and Saxena (1978). These representations will allow us to write $g(y)$ in computable forms and will be useful for the next section.

Case I. When the poles of $\prod_{j=1}^{m} \Gamma(b_j - B_j s)$ are assumed to be simple, that is, when

$$B_h(b_j+\lambda) \neq B_j(b_h+\nu)$$

for $j\neq h$; $j,h=1,\ldots,m$; $\lambda,\nu=0,1,2,\ldots$, we have the following expansion for $\bar{H}^{m,n}_{n,m}(x)$.

$$\bar{H}\,{}^{m,n}_{n,m}\left\{ x \,\middle|\, \begin{matrix} (a_1,A_1),\ldots,(a_n,A_n) \\ (b_1,B_1),\ldots,(b_m,B_m) \end{matrix} \right\} \tag{4.40}$$

$$= \sum_{h=1}^{m} \sum_{\nu=0}^{\infty} \prod_{\substack{j=1\\ j\neq h}}^{m} \Gamma\{b_j - B_j(b_h+\nu)/B_h\} \prod_{j=1}^{n} \Gamma\{ 1 - a_j + A_j(b_h+\nu)/B_j\}$$

$$\{(-1)^{\nu}\, x^{(b_h+\nu)/B_h}\}/\{(\nu!)B_h\}$$

which exists for all $x > 0$ if $\mu>0$ and for $0<x<\beta^{-1}$ if $\mu=0$, where μ and β are defined in (4.23) and (4.24) respectively.

Case II. When the poles of

$$\prod_{j=1}^{n} \Gamma(1 - a_j + A_j s)$$

are assumed to be simple, that is, when

$$A_h(1 - a_j + \nu) \neq A_j(1 - a_j + \lambda),$$

$j\neq h$; $j,h=1,\ldots,n$; $\lambda,\nu =0,1,\ldots$; we have the following expansion for $\bar{H}\,{}^{m,n}_{n,m}(x)$.

$$\bar{H}\,{}^{m,n}_{n,m}\left\{ x \,\middle|\, \begin{matrix} (a_1,A_1),\ldots,(a_n,A_n) \\ (b_1,B_1),\ldots,(b_m,B_m) \end{matrix} \right\} \tag{4.41}$$

$$= \sum_{h=1}^{n} \sum_{\nu=0}^{\infty} \prod_{\substack{j=1\\ j\neq h}}^{n} \Gamma\{1 - a_j - A_j(1-a_h+\nu)/A_h\} \prod_{j=1}^{m} \Gamma\{b_j + B_j(1-a_h+\nu)/A_h\}$$

$$\{(-1)^{\nu}\, x^{-(1-a_h+\nu)/A_h}\}/\{(\nu!)A_h\}$$

which exists for all $x>0$ if $\mu < 0$ and for $x>\beta^{-1}$ if $\mu=0$, where

μ and β are defined in (4.23) and (4.24) respectively.

General expansion in Case I. It is general in the sense that the poles of

$$\prod_{j=1}^{m} \Gamma(b_j - B_j s)$$

are not assumed to be simple.

The expansion will be obtained with the help of the following equations

$$s = \frac{b_j + \nu}{B_j} \, , \; \nu=0,1,\ldots \, . \tag{4.42}$$

If there exists a pair of values (ν_1, λ_1) such that

$$\frac{b_j + \nu_1}{B_j} = \frac{b_h + \lambda_1}{B_h} \, , \; h \neq j \, ,$$

then the point

$$s = \frac{b_j + \nu_1}{B_j}$$

is a pole of order two if the point does not coincide with the pole of any other gamma of $h(-s)$ given in (4.31).

If the point coincides with the poles coming from (r-1) other gammas of the set $\Gamma(b_j - B_j s)$, $j=1,\ldots,m$, then the point corresponds to a pole of order r.

For a fixed j, we consider the following equations

$$\frac{b_1 + \nu^{(j1)}_{j_1 \cdots j_m}}{B_1} = \frac{b_2 + \nu^{(j2)}_{j_1 \cdots j_m}}{B_2} = \cdots = \frac{b_m + \nu^{(jm)}_{j_1 \cdots j_m}}{B_m} \quad (4.43)$$

for which the following convention is used. For a fixed j, $j_r=0$ or 1, for $r=1,2,\ldots,m$; when $j_r=0$,

$$(b_r + \nu^{(jr)}_{j_1 \cdots j_m})/B_r$$

is to be excluded from the equations in (4.43). Here $\nu^{(jr)}_{j_1 \cdots j_m}$ represents a value of ν in (4.42); clearly, the possible values are 0,1,2,... . Moreover $\nu^{(jr)}_{j_1 \cdots j_m}$ denotes the number corresponding to $\nu^{(jj)}_{j_1 \cdots j_m}$ when the equation

$$\frac{b_j + \nu^{(jj)}_{j_1 \cdots j_m}}{B_j} = \frac{b_r + \nu^{(jr)}_{j_1 \cdots j_m}}{B_r}$$

is satisfied by some values of $\nu^{(jj)}_{j_1 \cdots j_m}$ and $\nu^{(jr)}_{j_1 \cdots j_m}$. Therefore $\nu^{(jr)}_{j_1 \cdots j_m}$ may or may not exist. Under these conditions

$$0 \le j_1 + \cdots + j_m \le m$$

for every fixed j and $j_1+\cdots+j_m$ represents the order of the pole at

$$s = \frac{b_j + \nu^{(jj)}_{j_1 \cdots j_m}}{B_j} .$$

For example, if $j_1+\ldots j_m = r$, then there will be r elements in

(4.43) if

$$\frac{(b_k + \nu^{(jk)}_{j_1 \cdots j_m})}{B_k}, \quad k=1,\ldots,m$$

are called elements in (4.43). In the above notation, a pole of order r is considered r times. In order to avoid duplication it will always be assumed that

$$j_1 = j_2 = \cdots = j_{j-1} = 0$$

while considering the points corresponding to (4.42). When $j_1 + \cdots + j_m = 0$, the corresponding point is not a pole.

If $j_r = 0$ for $r=1,\ldots,m$; $r \neq j$, then

$$s = \frac{b_j + \nu^{(jj)}_{j_1 \cdots j_m}}{B_j}$$

corresponds to a simple pole.

. Letting

$$S^{(jj)}_{j_1 \cdots j_m} = \{\nu^{(jj)}_{j_1 \cdots j_m}\}$$

represent the set of all values $\nu^{(jj)}_{j_1 \cdots j_m}$ takes for given $j_1, \ldots, j_m$, we have the following expansion for $\bar{H}^{m,n}_{n,m}(x)$.

$$\bar{H}^{m,n}_{n,m}\left\{ x \,\middle|\, \begin{matrix}(a_1,A_1),\ldots,(a_n,A_n)\\(b_1,B_1),\ldots,(b_m,B_m)\end{matrix}\right\} = \sum_{j=1}^{m} \sum_{S^{(jj)}_{j_1 \cdots j_m}} R_j \qquad (4.44)$$

where R_j is the residue of $h(-s)x^s$ at the pole

$$s = \frac{b_j + \nu^{(jj)}_{j_1\cdots j_m}}{B_j}$$

and

$\sum_{S^{(jj)}_{j_1\cdots j_m}}$ denotes the summation over all the sets $S^{(jj)}_{j_1\cdots j_m}$;

$$R_j = [\{(j_1+\cdots+j_m)x^{(b_j+\nu^{(jj)}_{j_1\cdots j_m})/B_j}\}/\{(j_1+\cdots+j_m)!\}]$$

$$\sum_{r=0}^{j_1+\cdots+j_m-1} \binom{j_1+\cdots+j_m-1}{r} (-\log(x))^{j_1+\cdots+j_m-1-r}$$

$$\{\sum_{r_1=0}^{r-1} \binom{r-1}{r_1} C_j^{(r-1-r_1)} \sum_{r_2=0}^{r_1-1} \binom{r_1-1}{r_2} C_j^{(r_1-r-r_2)} \cdots\}\ D_j; \qquad (4.45)$$

$$D_j = [\prod_{h=1}^{m} \Gamma\{b_h - B_h(b_j+\nu^{(jj)}_{j_1\cdots j_m})/B_j + j_h(\nu^{(jh)}_{j_1\cdots j_m}+1)\}$$

$$\prod_{h=1}^{n} \Gamma\{1-a_h+A_h(b_j+\nu^{(jj)}_{j_1\cdots j_m})/B_j\}]/$$

$$\{\prod_{h=1}^{m} B_h^{jh}(-1)^{jh}(\nu^{(jh)}_{j_1\cdots j_m})(\nu^{(jh)}_{j_1\cdots j_m}!)^{jh}\} \qquad (4.46)$$

$$C_j^{(0)} = \sum_{h=1}^{m} B_h\ \psi\{b_h - B_h(b_j+\nu^{(jj)}_{j_1\cdots j_m})/B_j + j_h(\nu^{(jh)}_{j_1\cdots j_m}+1)\}$$

$$+ \sum_{h=1}^{m} B_h j_h\{1 + (1/2) + (1/\nu^{(jh)}_{j_1\cdots j_m}\}$$

$$- \sum_{h=1}^{n} A_h \psi\{1 - a_h + A_h(b_j+\nu^{(jj)}_{j_1\cdots j_m})/B_j\} \qquad (4.47)$$

$$C_j^{(t)} = (-1)^{t+1}\, t!\{ \sum_{h=1}^{m} B_h^{t+1}\, \rho(t+1, b_h - B_h(b_j+\nu_{j_1\cdots j_m}^{(jj)})/B_j$$

$$+ j_h(\nu_{j_1\cdots j_m}^{(jj)}+1))$$

$$+ \sum_{h=1}^{m} B_j^{t+1}\, j_h\{(-1)^{-t-1}+(-2)^{-t-1}+\ldots+(-\nu_{j_1\cdots j_m}^{(jh)})^{-t-1}\}$$

$$+ \sum_{h=1}^{n} (-A_h)^{t+1}\, \rho(t+1, 1 - a_h + A_h(b_j+\nu_{j_1\cdots j_m}^{(jj)})/B_j)\} \qquad (4.48)$$

The psi function $\psi(.)$ and the generalized zeta function $\rho(.,.)$ are defined as follows:

$$\psi(z) = \frac{d}{dz}\log(\Gamma(z)) = -\gamma + (z-1)\sum_{n=0}^{\infty}\{(n+1)(z+n)\}^{-1} \qquad (4.49)$$

where γ is the Euler's constant; $\gamma = 0.577\ldots,$

$$\rho(s,\theta) = \sum_{n=0}^{\infty} (n+\theta)^{-s}, \qquad (4.50)$$

$Re(s) > 1$, $\theta \neq 0, -1, -2, \ldots$, $Re(\ .\ )$ denoting the real part of $(\ .\ )$.

We note that the general expansion in case II, that is the case where the poles of

$$\prod_{j=1}^{n} \Gamma(1 - a_j - A_j s)$$

are not assumed to be simple, is available from case I by making the following changes: interchange $m \sim n$, $b_j \sim (1-a_j)$, $B_j \sim A_j$, and $x \sim (1/x)$ in the R.H.S. of (4.44).

4.3 Linear Combinations of Gamma Variates

We will first derive an expression for the exact density of the inverse of a linear combination of independent real gamma variates. As a corollary, we will obtain the exact density of the test statistic d^* in (3.49). General results and other convenient representations of the density of a linear combination of independent chi-square variates are available in Ruben (1962) and Johnson and Kotz (1970).

A real random variable X is said to have a gamma distribution with parameters (a,b), if the density of X, denoted by g(x) is given by

$$g(x) = x^{a-1} e^{-x/b} /\{b^a \Gamma(a)\}, \qquad (4.51)$$

$a>0$, $b>0$, $x>0$ and $g(x) = 0$ elsewhere.

Let $X_1,\ldots,X_n$ be independent real gamma variates with parameters $(\alpha_j, 1/d_j)$, $j=1,\ldots,n$, whose respective densities are given in (4.51) with $a=\alpha_j$, $b= 1/d_j$ where α_j and d_j are positive real numbers, $j=1,\ldots,n$.

The h-th moment of 1/G where

$$G = \sum_{j=1}^{n} m_j X_j \ , \ m_j>0, \ j=1,\ldots,n \ , \qquad (4.52)$$

is

$$E(G^{-h}) = \int_0^\infty \cdots \int_0^\infty G^{-h} \prod_{j=1}^{n} \{x_j^{\alpha_j - 1} e^{-d_j x_j} (d_j^{\alpha_j}/\Gamma(\alpha_j)) dx_j\}.$$

But since

$$G^{-h} = (\Gamma(h))^{-1}\int_0^\infty t^{h-1} e^{-Gt} dt , \qquad (4.53)$$

for Re(h)>0, G>0, where Re(.) denotes the real part of (.), we have

$$E(G^{-h}) = (\Gamma(h))^{-1}\int_0^\infty \cdots \int_0^\infty \int_0^\infty t^{h-1} e^{-t(m_1x_1+\cdots+m_nx_n)}$$

$$\prod_{j=1}^{n} \{x_j^{\alpha_j-1} e^{-d_jx_j}(d_j^{\alpha_j}/\Gamma(\alpha_j)) dx_j\}dt$$

$$= (\Gamma(h))^{-1}\int_0^\infty t^{h-1} \prod_{j=1}^{n}\{\int_0^\infty e^{-x_j(m_jt+d_j)} x_j^{\alpha_j-1}$$

$$(d_j^{\alpha_j}/\Gamma(\alpha_j))dx_j\}dt . \qquad (4.54)$$

Noticing that for j=1,2,...,n,

$$\int_0^\infty (\Gamma(\alpha_j))^{-1} e^{-x_j(m_jt+d_j)} x_j^{\alpha_j-1} dx_j = (m_jt+d_j)^{-\alpha_j} ,$$

(4.54) becomes

$$E(G^{-h}) = (\Gamma(h))^{-1}\int_0^\infty t^{h-1} \prod_{j=1}^{n} \{(m_jt+d_j)/d_j\}^{-\alpha_j} dt$$

$$= (\Gamma(h))^{-1} \int_0^\infty t^{h-1} \prod_{j=1}^{n} (\mu_jt + 1)^{-\alpha_j} dt , \qquad (4.55)$$

where

$$\mu_j = m_j/d_j \quad , \quad j=1,\ldots,n . \qquad (4.56)$$

Letting $u = 1/(1+t)$, that is, $t = (1-u)/u$, one has

$$\left|\frac{dt}{du}\right| = \frac{1}{u^2}$$

and (4.55) becomes

$$E(G^{-h})=(\Gamma(h))^{-1}\int_o^1((1-u)/u)^{h-1}\prod_{j=1}^{n}\{(\mu_j/u)^{-\alpha_j}((1-u)+u/\mu_j)^{-\alpha_j}\}u^{-2}du$$

$$=(\Gamma(h))^{-1}\prod_{j=1}^{n}(\mu_j^{-\alpha_j})\int_o^1 u^{\rho-h-1}(1-u)^{h-1}\prod_{j=1}^{n}\{1-u(\mu_j-1)/\mu_j\}^{-\alpha_j}du \qquad (4.57)$$

where

$$\rho = \alpha_1+\alpha_2+\ldots+\alpha_n . \qquad (4.58)$$

Let

$$\rho - h = q \qquad (4.58a)$$

and

$$\gamma_j = (\mu_j-1)/\mu_j , \qquad (4.58b)$$

then

$$E(G^{-h})=\{(\prod_{j=1}^{n}\mu_j^{-\alpha_j})\Gamma(q)/\Gamma(\rho)\}[\{\Gamma(\rho)/(\Gamma(q)\Gamma(\rho-q))\}$$

$$\int_o^1 u^{q-1}(1-u)^{\rho-q-1}\prod_{j=1}^{n}(1-\gamma_j u)^{-\alpha_j}du] \qquad (4.59)$$

At this point, we give a multiple series representation as well as a single integral representation of a type D Lauricella's hypergeometric function of n variables denoted by $F_D(\quad)$.

$$F_D(a;b_1,\ldots,b_n;c;x_1,\ldots,x_n)$$

$$= \sum_{j=0}^{\infty}\sum_{j_1+\ldots+j_n=j}\{(a)_j/(c)_j\}\{(b_1)_{j_1}\ldots(b_n)_{j_n}$$

$$(x_1^{j_1}\ldots x_n^{j_n})\}/\{(j_1!)\ldots(j_n!)\} \qquad (4.60)$$

where $|x_i| < 1$, $i=1,\ldots,n$, and for instance, $(a)_j = \Gamma(a+j)/\Gamma(a)$, and

$$F_D(a;b_1,\ldots,b_n;c;x_1,\ldots,x_n)$$
$$=\{\Gamma(c)/(\Gamma(a)\Gamma(c-a))\} \int_0^1 u^{a-1}(1-u)^{c-a-1}(1-ux_1)^{-b_1}\cdots (1-ux_n)^{-b_n}\,du\ , \qquad (4.61)$$

where

$$\mathrm{Re}(a) > 0,\ \mathrm{Re}(c-a) > 0\ .$$

These representations may be seen from Mathai and Saxena (1978), p. 162, where other types of Lauricella's hypergeometric functions are also defined. We may now rewrite (4.59) as follows

$$E(G^{-h}) = \{\prod_{j=1}^{n} \mu_j^{-\alpha_j}\Gamma(q)/\Gamma(\rho)\}F_D(q;\alpha_1,\ldots,\alpha_n;\rho;\gamma_1,\ldots,\gamma_n), \qquad (4.62)$$

provided $\mathrm{Re}(q) = \mathrm{Re}(\rho-h) > 0$, $\mathrm{Re}(\rho-q) = \mathrm{Re}(h) > 0$ and

$$|\gamma_j| = |(\mu_j-1)/\mu_j| < 1\ , \qquad (4.63)$$

that is,

$$\mu_j > 1/2 \quad \text{for } j=1,\ldots,n\ .$$

Condition (4.63) allows us to express the h-th moment of $1/G$ as a multiple series. Hence

$$E(G^{-h}) = \{\prod_{j=1}^{n} \mu_j^{-\alpha_j}\Gamma(q)/\Gamma(\rho)\} \sum_{\nu=0}^{\infty} \sum_{\nu_1+\ldots+\nu_n=\nu} \{(q)_\nu/(\rho)_\nu\}$$

$$\times \{(\alpha_1)_{\nu_1} \cdots (\alpha_n)_{\nu_n}\}(\gamma_1^{\nu_1} \cdots \gamma_n^{\nu_n})/(\nu_1! \cdots \nu_n!) = \gamma_h \text{ say,} \quad (4.64)$$

for $|\gamma_j| < 1$, that is, $\mu_j = m_j/d_j > 1/2$, $j=1,\ldots,n$.

We now can state the following theorem.

Theorem 4.1. Let $G = m_1X_1+\ldots+m_nX_n > 0$ where $m_j > 0$, $j=1,\ldots,n$, and $X_1,\ldots,X_n$ are independent gamma variates with parameters $(\alpha_j, 1/d_j)$ and let $\rho = \alpha_1+\ldots+\alpha_n$, then for $0 < \mathrm{Re}(h) < \rho$ and $|\gamma_j| = |(\mu_j-1)/\mu_j| < 1$, that is $\mu_j = m_j/d_j > 1/2$, $j=1,\ldots,n$,

$$E(G^{-h}) = \{\prod_{j=1}^{n} (\mu_j)^{-\alpha_j}\Gamma(q)/\Gamma(\rho)\}F_d(q;\alpha_1,\ldots,\alpha_n;\rho;\gamma_1,\ldots,\gamma_n)$$

$$= \gamma_h$$

where $q = \rho-h$ and γ_h is defined in (4.64).

If the condition $\mu_j > 1/2$ is not satisfied for $j=1,\ldots,n$, then we use the following technique to make the conditions met in the integral (4.59) as well as in the multiple series (4.64). We multiply G by δ/δ where δ is a constant such that $\mu_j' = \delta\mu_j = \delta m_j/d_j > 1/2$ for all j, allowing us to express the h-th moment of 1/G, that is,

$$E(G^{-h}) = E((G'/\delta)^{-h}) = \delta^h E(G'^{-h}),$$

where $G' = \delta G$, as a multiple series. We obtain the density of G' by taking the inverse Mellin transform of $E(G'^{-h})$ and thereby we can easily get the density of G by making an appropriate change of variables.

Noting that

$$\Gamma(q)\ (q)_{\nu} = \Gamma(q)\Gamma(q+\nu)/\Gamma(q) = \Gamma(q+\nu) = \Gamma(\rho+\nu-h) \qquad (4.65)$$

and that

$$\Gamma(\rho)\ (\rho)_{\nu} = \Gamma(\rho+\nu)\ ,$$

(4.64) may be rewritten as

$$E(G^{-h}) = \sum_{\nu=0} K_{\nu}\Gamma(\rho+\nu-h) \qquad (4.66)$$

where

$$K_{\nu} = \sum_{\nu_1+\ldots+\nu_n=\nu} k_{\nu_1\ldots\nu_n}\ , \qquad (4.66a)$$

$$k_{\nu_1\ldots\nu_n} = \left(\prod_{j=1}^{n} \mu_j^{-\alpha_j}\right)\{(\alpha_1)_{\nu_1}\ldots(\alpha_n)_{\nu_n}\}(\gamma_1^{\nu_1}\ldots\gamma_n^{\nu_n})/$$

$$\{\Gamma(\rho+\nu)(\nu_1!\ldots\nu_n!)\}\ ,$$

$$\gamma_j = |(\mu_j-1)/\mu_j| < 1,\ j=1,\ldots,n\ ,$$

$$\mu_j = m_j/d_j\ ,\quad j=1,\ldots,n\ ,$$

$$\rho = \alpha_1+\ldots+\alpha_n$$

and

$$0 < \mathrm{Re}(h) < \rho\ .$$

We obtain the density of $J = 1/G$ denoted $f(J)$ as the inverse Mellin transform of $E(J^h) = E(G^{-h})$.

Hence

$$f(J) = (2\pi i)^{-1} \int_{c-i\infty}^{c+i\infty} E(J^h)\ J^{-(h+1)} dh \tag{4.67}$$

where

$$i = (-1)^{1/2} \quad \text{and} \quad c < \min_{\nu=0,1,\ldots} (\rho + \nu) = \rho .$$

Since the infinite series in (4.66) is uniformly convergent within its region of convergence, $\mu_j > 1/2$, $j=1,\ldots,n$, the density of J may be written as follows

$$f(J) = \sum_{\nu=0}^{\infty} K_\nu J^{-1}\{(2\pi i)^{-1} \int_{c-i\infty}^{c+i\infty} \Gamma(\rho+\nu-h) J^{-h}\ dh\}$$

$$= \sum_{\nu=0}^{\infty} K_\nu J^{-1}\ H_{1,0}^{0,1}\{\ J \left|\begin{matrix}(1-\rho-\nu,\ 1)\\ \overline{}\end{matrix}\right.\}$$

for $c < \rho$. Then according to (4.28), we have

$$f(J) = \sum_{\nu=0}^{\infty} K_\nu J^{-1}\ H_{0,1}^{1,0}\{\ \frac{1}{J} \left|\begin{matrix}\overline{}\\ (\rho+\nu,\ 1)\end{matrix}\right.\} . \tag{4.68}$$

Now since

$$H_{0,1}^{1,0}\{\ x \left|\begin{matrix}\overline{}\\ (b,B)\end{matrix}\right.\} = B^{-1}\ x^{b/B} \exp(-x^{1/B})$$

(equation 1.7.2 in Mathai and Saxena(1978)), we can express the density of J as follows for $J > 0$

$$f(J) = \sum_{\nu=0}^{\infty} K_\nu J^{-1}\ J^{-(\rho+\nu)} \exp(-1/J) . \tag{4.69}$$

Let $g_2(G)$ denote the probabily density function of G. Since

$$J = \frac{1}{G} ,$$

we have

$$\left|\frac{dJ}{dG}\right| = \frac{1}{G^2}$$

and

$$g_2(G) = \sum_{\nu=0}^{\infty} K_\nu \, G^{\rho+\nu-1} \, e^{-G} \qquad (4.70)$$

$$= \sum_{\nu=0}^{\infty} k_\nu \, G^{\rho+\nu-1} \, e^{-G}/\Gamma(\rho+\nu) \ . \qquad (4.71)$$

where

$$k_\nu = K_\nu \Gamma(\rho+\nu) .$$

Therefore the density of G is available as a linear combination of densities of gamma variates with parameters $(\rho+\nu,1)$.

Corollary 4.1 The exact density $g_2(d^*)$ of d^* where d^* is defined in (3.49) and

$$d^* \sim \sum_{i=1}^{k} m_i^* \, \chi_i^2(r)$$

is according to (4.70)

$$g_2(d^*) = (d^*)^{\rho-1} e^{-d^*} \sum_{\nu=0}^{\infty} K_\nu (d^*)^\nu$$

for $d^* > 0$, where

$$\rho = kr/2 \quad ,$$

$$K_\nu = \sum_{\nu_1+\ldots+\nu_k=\nu} k_{\nu_1\ldots\nu_k} \quad ,$$

$$k_{\nu_1\ldots\nu_k} = \{\prod_{j=1}^{k} (2m_j^*)^{-r/2}\}\{(r/2)_{\nu_1} \ldots (r/2)_{\nu_k}\}$$

$$\{\gamma_1^{*\nu_1} \ldots \gamma_k^{*\nu_k}\} / \{\Gamma(\rho+\nu)(\nu_1! \ldots \nu_k!)\}$$

and

$$\gamma_j^* = |(2m_j^*-1)/(2m_j^*)| < 1, j=1,\ldots,k ,$$

that is,

$$m_j^* > 1/4 , j=1,\ldots,k .$$

This representation for the density of a linear combination of chi-square variates is simpler than the representations given in (3.52), (3.59) and (3.61) which have been deduced from the results found in Mathai and Pillai (1982). Moreover, the representation of the density of G given in (4.70) for the case where G is a linear combination of independent exponential variables also differs from the representation given in Mathai (1983).

As a simple application of this corollary, we see that the density of d* when k=1 is

$$g_2(d^*) = d^{*r/2-1} e^{-d^*} \sum_{\nu=0}^{\infty} (2m_1^*)^{-r/2} \{\Gamma(r/2+\nu)/\Gamma(r/2)\}$$

$$\{(2m_1^*-1)/(2m_1^*)\}^{\nu} d^{*\nu} / \{\Gamma(r/2+\nu)(\nu!)\}$$

$$= \{d^{*r/2-1} e^{-d^*} (2m_1^*)^{-r/2} / \Gamma(r/2)\}$$

$$\sum_{\nu=0}^{\infty} \{(2m_1^*-1) d^* / (2m_1^*)\}^{\nu} / \nu!$$

$$= \{d^{*r/2-1} e^{-d^*} (2m_1^*)^{-r/2} / \Gamma(r/2)\} e^{d^* - d^*/(2m_1^*)}$$

$$= d^{*r/2-1} e^{-d^*/(2m_1^*)} / (2m_1^*)^{r/2} \Gamma(r/2) ,$$

which is the density of a gamma random variable with parameters $(r/2, m_1^*/2)$ as it should be.

There is a vast literature on the distributions of linear combinations of random variables where the individual variables are assumed to have particular types of distributions. In order to unify as well as to generalize these results, Mathai and Saxena (1973) considered linear combinations of random variables where each component variable is assumed to have a density associated with an H-function. Here we list one result for the sake of illustration.

Let $X_1, \ldots, X_n$ be n independent random variables where X_i has the density function

$$f_i(x_i) = (x_i^{\gamma_i - 1}/C_i) e^{-ap_i x_i} H^{k_i, \ell_i}_{r_i, s_i} \{ z_i x_i^{\mu_i} \mid {}^{(a_{r_i}^{(i)}, A_{r_i}^{(i)})}_{(b_{s_i}^{(i)}, B_{s_i}^{(i)})} \} \quad (4.72)$$

for $0 < x_i < \infty$ and $f_i(x_i) = 0$ elsewhere, $i = 1, \ldots, n$; where

$$C_i = (ap_i)^{-\gamma_i} H^{k_i, \ell_i + 1}_{r_i + 1, s_i} \{ \frac{z_i}{(ap_i)^{\mu_i}} \mid {}^{(\gamma_i + 1, \mu_i), \ (a_{r_i}^{(i)}, A_{r_i}^{(i)})}_{(b_{s_i}^{(i)}, B_{s_i}^{(i)})} \} \quad (4.73)$$

$\mu_i > 0$, $p_i > 0$, $\{\gamma_i + 1 + \mu_i \min(b_j/B_j)\} > 0$, $j = 1, \ldots, k_i$, $i = 1, \ldots, n$. Since there exists at least one set of parameters for which $f_i(x_i)$ in (4.72) is nonnegative and $\int_0^\infty f_i(x_i) dx_i = 1$, it is assumed that the parameters are such that $f_i(x_i) \geq 0$ for $0 < x_i < \infty$ and $\int_0^\infty f_i(x_i) dx_i = 1$.

Let

$$U = p_1X_1+p_2X_2+\ldots+p_nX_n \quad , \tag{4.74}$$

then the density h(u) of U is given as follows

$$h(u) = \{\prod_{j=1}^{n} p_i^{-\gamma_i}/C_i\} \prod_{i=1}^{n} \sum_{h=1}^{k_i} \sum_{\nu_i=0}^{\infty} \prod_{j=1}^{k_i} \Gamma(b_j^{(i)}-B_h^{(i)}(b_h^{(i)}+\nu_i)/B_j^{(i)})$$

$$[\{\prod_{j=1}^{\ell_i} \Gamma(1-a_j^{(i)}+A_j^{(i)}(b_h^{(i)}+\nu_i)/B_h^{(i)})\Gamma(\gamma_i+\mu_i(b_h^{(i)}+\nu_i)/B_h^{(i)})$$

$$(z_i p_i^{\mu_i})^{(b_h^{(i)}+\nu_i)/B_h^{(i)}}\}/\{\prod_{j=k_i+1}^{s_i}\Gamma(1-b_j^{(i)}+B_j^{(i)}(b_h^{(i)}+\nu_i)/B_h^{(i)})$$

$$\prod_{j=\ell_i+1}^{r_i} \Gamma(a_j^{(i)}-A_j^{(i)}(b_h^{(i)}+\nu_i)/B_h^{(i)})\}]\ \{(-1)^{\nu_i}/(\nu_i!B_h^{(i)})\}$$

$$[e^{-au}\ u\{\sum_{j=1}^{n}(\mu_j(b_h^{(j)}+\nu_j)/B_h^{(j)}+\nu_j)-1\}/$$

$$\Gamma(\sum_{j=1}^{n}\{\mu_j(b_h^{(j)}+\nu_j)/B_h^{(j)}+\gamma_j\})] \quad . \tag{4.75}$$

The density function in (4.75) is derived by finding the Laplace transform of U which is given by

$$L(r) = \prod_{i=1}^{n} L_{f_i}(p_i r) \tag{4.76}$$

due to the stochastic independence of $X_1,\ldots,X_n$ where $L_f(r)$ is defined in (4.2). We get the density of U by collecting the appropriate terms and by taking the inverse Laplace transform of L(r) where the inverse Laplace transform is defined in (4.3) .

We would like to point out that series representations of the distributions of quadratic forms in normal variables were considered by many authors for the central as well as for the noncentral cases, see for example, Kotz, Johnson and Boyd (1967). Tables of distributions of positive definite quadratic forms in central normal variables are available in Johnson and Kotz(1968).

Moreover, various approximations are available for the distribution of a linear combination of independent chi-square variates; see for instance, Oman and Zacks (1981), Jensen and Solomon (1972) for a Gaussian approximation, or Solomon and Stephens (1977) where the distribution is to be fitted by $A\omega^D$, where ω has a chi-square distribution with s degrees of freedom and the constants A, s and D are found by matching moments.

The multivariate analog of this problem has been considered recently by Tan and Gupta (1983) who proposed an approximation to the distribution of a linear combination of central Wishart matrices.

4.4 The Ratio of a Linear Combination of Gamma Variates over the k-th Root of a Product of Gamma Variates

In this section, we will determine the exact distribution of a statistic of the type

$$R = G/Q \ , \tag{4.77}$$

where G is the linear combination of gamma variates defined in

(4.52) and

$$Q = (Y_1 Y_2 \cdots Y_k)^{1/k} , \qquad (4.78)$$

where $Y_1, \ldots, Y_k$ are independent real gamma variates with parameters $(q_j, 1/D_j)$, $j=1,\ldots,k$, their respective densities being

$$g_j(y_j) = (D_j^{q_j}/\Gamma(q_j))\; y_j^{q_j-1}\; e^{-D_j y_j} , \qquad (4.79)$$

$y_j > 0$, $D_j > 0$, $q_j > 0$, $j=1,\ldots,k$, and $g_j(y_j) = 0$ elsewhere. It is also assumed that G and Q are independently distributed.

We are considering a ratio of the type $R = G/Q$ in order to obtain the densities of the test statistics (3.2), (3.46), (3.75) and (3.77) which are all structurally equivalent to R.

Let

$$L = 1/R . \qquad (4.80)$$

Since Q and G are independent, the h-th moment of L is

$$E(L^h) = \prod_{\ell=1}^{k} E(Y_\ell^{h/k}) \quad E\{(\sum_{j=1}^{n} m_j X_j)^{-h}\} . \qquad (4.81)$$

Hence, according to (4.66) and (4.79),

$$E(L^h) = \prod_{\ell=1}^{k} \{D_\ell^{-h/k} \Gamma(q_\ell + h/k)/\Gamma(q_\ell)\} \sum_{\nu=0}^{\infty} K_\nu \Gamma(\rho+\nu-h) \qquad (4.82)$$

for $\mu_j = m_j/d_j > 1/2$, $j=1,\ldots n$, and $0 < \text{Re}(h) < \rho$, where ρ is given in (4.58).

If the condition $\mu_j > 1/2$ is not satisfied for $j=1,\ldots,n$, we use the following technique. We multiply both numerator and denominator of R by B ; then we absorb the numerator B in μ_j so that the new μ_j is $B\mu_j$ such that $B\mu_j > 1/2$ for all j. We keep the denominator B with $(Y_1 Y_2 \cdots Y_k)^{1/k}$ so that the h-th moment of the denominator becomes

$$B^h \prod_{\ell=1}^{k} D_\ell^{-h/k} \Gamma(q_\ell + h/k)/\Gamma(q_\ell).$$

From (4.82) we have

$$E(L^h) = \sum_{\nu=0}^{\infty} K_\nu \prod_{\ell=1}^{k} \{(D_\ell^{-h/k}/\Gamma(q_\ell)\} \prod_{\ell=1}^{k} \Gamma(q_\ell + h/k)\Gamma(\rho+\nu-h). \quad (4.83)$$

The procedure for obtaining the h-th moment of $L = Q/G$ could have been used even if Q and G were not independent but a product and a linear combination of some independent gamma variates $X_1, X_2, \ldots, X_p$. In that case we would integrate out over the joint density of $X_1, X_2, \ldots, X_p$ after replacing G^{-h} by using the technique in equation (4.53).

From the uniqueness of the inverse Mellin transform, it is seen that this moment expression uniquely determines $g(L)$, the density of L, where $L = 1/R$. Hence

$$g(L) = (2\pi i)^{-1} \int_{c-i\infty}^{c+i\infty} E(L^h)\, L^{-(h+1)}\, dh$$

where $i = (-1)^{1/2}$ and

$$\max_{1 \le \ell \le n} (-kq_\ell) < c < \min_{\nu=0,1,\ldots} (\rho+\nu) = \rho \ .$$

Hence c can be taken to be zero since $q_\ell > 0$ and $\rho > 0$.

Since the infinite series in (4.83) is uniformly convergent within its region of convergence, $\mu_j > 1/2$, $j=1,\ldots,n$, the density of L may be written as follows

$$g(L) = \sum_{\nu=0}^{\infty} L^{-1} K'_\nu \{(2\pi i)^{-1} \int_{-i\infty}^{+i\infty} \prod_{\ell=1}^{k} \Gamma(q_\ell + h/k)\Gamma(\rho+\nu-h)(\xi L)^{-h}\, dh\} \quad (4.84)$$

where

$$K'_\nu = K_\nu / \prod_{\ell=1}^{k} \Gamma(q_\ell) , \quad (4.85)$$

$$\xi = \prod_{\ell=1}^{k} D_\ell^{1/k} , \quad (4.86)$$

K_ν is defined in (4.66) and the path in the complex plane running from $-i\infty$ to $+i\infty$, separates the poles of $\Gamma(q_\ell + h/k)$ which are

$$-k(q_\ell + \eta),\ \ell=1,\ldots,n;\ \eta=0,1,\ldots ,$$

from the poles of $\Gamma(\rho+\nu-h)$ which are

$$(\rho+\nu+\lambda) ,\ \nu=0,1,\ldots ;\ \lambda=0,1,\ldots .$$

The density of L can thus be expressed in terms of H-functions as follows:

$$g(L) = \sum_{\nu=0}^{\infty} L^{-1} K'_\nu H_{1,k}^{k,1}\{\xi L \mid {(1-\rho-\nu,1) \atop (q_1,1/k),\ldots,(q_k,1/k)}\} \quad (4.87)$$

and the density of

$$R = 1/L \quad (4.88)$$

denoted by h(R) is therefore

$$h(R) = g(1/R)\left|\frac{dL}{dR}\right| = \{g(1/R)\}/R^2$$

$$= \sum_{\nu=0}^{\infty} R^{-1} K'_\nu \, H^{k,1}_{1,k}\left\{\frac{\xi}{R} \,\middle|\, \begin{matrix}(1-\rho-\nu,1)\\(q_1,1/k),\ldots,(q_k,1/k)\end{matrix}\right\} . \qquad (4.89)$$

For the H-function appearing in (4.89), we have according to (4.23) and (4.24) that

$$\mu = \sum_{\ell=1}^{k} (1/k) - 1 = (k/k) - 1 = 0 \qquad (4.90)$$

and

$$\beta = \prod_{j=1}^{k} (1/k)^{-1/k} = (1/k)^{-k/k} = k . \qquad (4.91)$$

Hence we may use (4.40) where m, n, x, a_j, A_j, b_1 and B_1 are respectively replaced by k, 1, (ξ/R), $(1-\rho-\nu)$, 1, q_j and $(1/k)$, in order to obtain h(R) in a computable form for

$$(\xi/R) < k \quad . \qquad (4.92)$$

This last representation is valid provided the poles of

$$\prod_{\ell=1}^{k} (q_\ell + h/k)$$

are all simple. If this is not the case, that is, if

$$-k(q_\ell - \nu_1) = -k(q_j - \lambda_1)$$

for some pairs of values (ν_1,λ_1) and $\ell \neq j$, the general representation for Case 1 discussed in Section 4.2 ought to be used.

Then noting that according to (4.28)

$$H^{k,1}_{1,k}\left\{\frac{\xi}{R}\,\middle|\, \begin{matrix}(1-\rho-\nu,1)\\ (q_1,1/k),\ldots,(q_k,1/k)\end{matrix}\right\} \tag{4.93}$$

$$= H^{1,k}_{k,1}\left\{\frac{R}{\xi}\,\middle|\, \begin{matrix}(1-q_1,1/k),\ldots,(1-q_k,1/k)\\ (\rho+\nu,1)\end{matrix}\right\}, \tag{4.94}$$

we may once again use (4.40) where m, n, x, a_j, A_j, b_1 and B_1 are respectively replaced by 1, k, R/ξ, $(1-q_j)$, $1/k$, $(\rho+\nu)$ and 1 in order to express h(R) in a closed form for

$$(R/\xi) < k^{-1} ,$$

that is, for

$$(\xi/R) > k \quad . \tag{4.95}$$

We also notice that for ν fixed, the poles of $\Gamma(\rho+\nu+h)$ which are

$$-(\rho+\nu+\lambda) = h \ , \ \lambda=0,1,\ldots \ , \tag{4.96}$$

are all simple.

Therefore h(R) is available in computable forms for all $R > 0$.

The density of $L = 1/R$ may also be expressed in terms of Meijer's G-function which is defined in (4.25). Let

$$T = L^k \ , \tag{4.97}$$

then

$$E(T^h) = E\{(L^k)^h\} = E(L^{kh}) \ . \tag{4.98}$$

Hence we can obtain the h-th moment of

$$(1/R)^k = L^k \tag{4.99}$$

upon substituting hk for h in (4.83). Hence

$$E(T^h) = \sum_{\nu=0}^{\infty} K'_\nu \prod_{\ell=1}^{k} \Gamma(q_\ell+h)\Gamma(\rho+\nu-hk)\xi^{-hk} \tag{4.100}$$

for $\mu_j > 1/2$ and $0 < \mathrm{Re}(hk) < \rho$, where K'_ν and ξ are given in (4.85) and (4.86) respectively.

Noticing that

$$\Gamma(\rho+\nu-hk) = \Gamma(k\{(\rho+\nu)/k \;-h\})$$

$$=(2\pi)^{(1-k)/2}\, k^{\rho+\nu-hk-1/2} \prod_{\ell=0}^{k-1} \Gamma((\rho+\nu+\ell)/k-h) \tag{4.101}$$

by the Gauss-Legendre multiplication formula:

$$\Gamma(mz) = (2\pi)^{(1-m)/2}\, m^{mz-1/2} \prod_{j=0}^{m-1} \Gamma(z+j/m) \;, \tag{4.102}$$

(4.100) becomes

$$E(T^h) = \sum_{\nu=0}^{\infty} K''_\nu \prod_{j=1}^{k} \Gamma(q_j+h) \prod_{\ell=1}^{k} \Gamma((\rho+\nu+\ell-1)/k-h)(k\xi)^{-kh}, \tag{4.103}$$

where

$$K''_\nu = (2\pi)^{(1-k)/2}\, k^{\rho+\nu-1/2}\, K'_\nu \quad . \tag{4.104}$$

The inverse Mellin transform of $E(T^h)$ will give us the density of T, g(T), as a linear combination of Meijer's G-functions times T^{-1}:

$$g(T) = \sum_{\nu=0}^{\infty} T^{-1} K''_\nu \{(2\pi i)^{-1} \int_{-i\infty}^{+i\infty} \prod_{j=1}^{k} \Gamma(q_j+h) \prod_{\ell=1}^{k} \Gamma((\rho+\nu+\ell-1)/k-h)(k\xi)^{-kh}\, T^{-h}\, dh\} \tag{4.105}$$

$$= \sum_{\nu=0}^{\infty} T^{-1} K_\nu'' G_{k,k}^{k,k}\{(k\xi)^k T \mid {a_1,\ldots,a_k \atop q_1,\ldots,q_k}\} \quad (4.106)$$

where

$$a_\ell = 1 - (\rho+\nu+\ell-1)/k \ , \ \ell=1,\ldots,k \ .$$

Representations of Meijer's G-function may be found in Mathai and Saxena (1973a). We may also use the representations of the H-function given in (4.40) and (4.44) with $A_j = 1$, $B_j = 1$ and $m = n = k$.

Since a chi-square variate with N degrees of freedom is in fact a gamma variate with parameters (N/2,2), the densities of the statistics (3.2) and (3.75) can be directly obtained in terms of H-functions of the type $H_{1,k}^{k,1}(\cdot)$ from (4.89) or in terms of G-functions of the type $G_{k,k}^{k,k}(\cdot)$ from (4.106) after an appropriate change of variables.

Similarly one can get the densities of the statistics (3.46) and (3.77) in terms of H-functions of the type $H_{1,1}^{1,1}(\cdot)$ from (4.89) or in terms of G-functions of the type $G_{1,1}^{1,1}(\cdot)$ from (4.106).

Let us express the density of the test statistic R defined in (3.2) in terms of H-functions. According to (2.37), (2.44) and (4.89) the density of R is

$$h(R) = \sum_{\nu=0}^{\infty} R^{-1} K_\nu' H_{1,k}^{k,1}\{\frac{\xi}{R} \mid {(1-\rho-\nu,1) \atop (q_1,1/k),\ldots,(q_k,1/k)}\} \ , \quad (4.107)$$

where

$$q_j = (n-r-j+1)/2 \ , \ j=1,\ldots,k \ , \quad (4.108)$$

$$\rho = rk/2 \ , \tag{4.109}$$

$$\xi = \prod_{\ell=1}^{k} (1/2)^{1/k} = 1/2$$

and according to (4.85)

$$K' = \sum_{\nu_1+\ldots+\nu_n=\nu} \left(\prod_{j=1}^{k} (2m_j^o)^{-r/2} \right) \{(r/2)_{\nu_1} \ldots (r/2)_{\nu_k}\}$$

$$(\gamma_1^{\nu_1} \ldots \gamma_k^{\nu_k}) / \{\Gamma(\rho+\nu)(\nu_1! \ldots \nu_k!) \prod_{j=1}^{k} \Gamma(q_j)\} \tag{4.110}$$

with

$$m_j^o = m_j / |B_o' V B_o|^{1/k} \ , \ j=1,\ldots,k \ , \tag{4.111}$$

and

$$\gamma_j = |(2m_j^o - 1)/(2m_j^o)| < 1 \ , \ j=1,\ldots,k \ , \tag{4.112}$$

that is,

$$m_j^o > 1/4 \ , \ j=1,\ldots,k \ . \tag{4.113}$$

4.5 Approximations of the Distributions through the Moments

In this section, we will derive an expression for the h-th moment of a statistic of the type R defined in (4.77) for the case where h is a positive integer and then we will obtain an approximation to the density of R by selecting the Pearson curve available by using the first four moments of R. Approximations to the densities of the test statistics given in (3.2), (3.46), (3.75) and (3.77) may be obtained as direct corollaries since these statistics can be expressed in terms of R.

Since the discussion of Pearson curves is readily available from many sources we will not go into the details. Only the very basic ideas will be given here so that the discussion in this section will be self-contained.

Pearson curves are probability densities

$$y = Q(x)$$

which are solutions of the differential equation

$$\frac{dy}{dx} = \frac{(x-a)\,y}{(b_o+b_1x+b_2x^2)} \quad . \tag{4.114}$$

Given the mean value μ and the central moments μ_2, μ_3 and μ_4 of the distribution to be approximated, the selection of a particular Pearson curve is based on the following moment ratios

$$r_1 = \mu_3^2/\mu_2^3 \quad , \tag{4.115}$$

$$r_2 = \mu_4/\mu_2^2 \tag{4.116}$$

and

$$\kappa = r_1(r_2+3)^2/\{4(2r_2-3r_1-6)(4r_2-3r_1)\} \quad . \tag{4.117}$$

There are twelve types of curves and the set of rules for determining which curve best fits a given probability distribution has been developed by Karl Pearson in the late 1880s. A complete development of the curves and the associated rules can be found in many books, see for example Elderton and Johnson (1969). Tables of standardized percentage points are included in Pearson and Hartley (1972) together with examples of their use. The steps in fitting a Pearson curve to a theoretical distribution in order to find percentage points are given in Solomon and Stephens (1978).

Let us compute the first four moments of R. It is seen from (4.80) and (4.81) that

$$E(R^h) = E\{(G/Q)^h\} = \prod_{\ell=1}^{k} E(Y_\ell^{-h/k}) E\{(\sum_{j=1}^{n} M_j)^h\} , \tag{4.118}$$

where Y_ℓ is distributed according to the density given in (4.79) for $\ell=1,\ldots,k$;

$$M_j = m_j X_j$$

is a real gamma variate with parameters $(\alpha_j, m_j/d_j)$, $j=1,\ldots,n$, according to the definition of G given in (4.52) and all Y_ℓ's and M_j's are assumed to be mutually independent.

Then noting that

$$E(Y_\ell)^{-h/k} = D_\ell^{h/k}\, \Gamma(q_\ell - h/k)/\Gamma(q_\ell) , \tag{4.119}$$

provided $q_\ell > h/k$, $\ell=1,\ldots,k$, and that

$$E(M_j^h) = (m_j/d_j)^h \Gamma(\alpha_j + h)/\Gamma(\alpha_j) , \tag{4.120}$$

so that, for $h, h_1, \ldots, h_k \in \mathbb{N}$,

$$E\{(\sum_{j=1}^{n} M_j)^h\} = E\{\sum_{h_1+\ldots h_n=h} (M_1^{h_1}\ldots M_n^{h_n}) h!/(h_1!\ldots h_n!)\}$$

$$= \sum_{h_1+\ldots+h_n=h} \{h!/(h_1!\ldots h_n!)\} \prod_{j=1}^{n} (m_j/d_j)^{h_j} \Gamma(\alpha_j+h_j)/\Gamma(\alpha_j)$$

$$= \Phi \quad , \tag{4.121}$$

we get the following expression for the h-moment of R when h is a positive integer

$$\mu_h' = \{ \prod_{\ell=1}^{k} D_\ell^{h/k} \Gamma(q_\ell - h/k)/\Gamma(q_\ell) \} \Phi \; , \tag{4.122}$$

where Φ is given in (4.121), $q_\ell > h/k$, $\ell=1,\ldots,k$, and $h,h_1,\ldots,h_n \in \mathbb{N}$. For instance the mean value of R is

$$\mu = \{ \prod_{\ell=1}^{k} D_\ell^{1/k} \Gamma(q_\ell - 1/k)/\Gamma(q_\ell) \} \{ \sum_{j=1}^{n} (m_j/d_j) \Gamma(\alpha_j+1)/\Gamma(\alpha_j) \} \, . \tag{4.123}$$

From the first four moments of R, namely μ, μ_2', μ_3' and μ_4' readily available from (4.122) from which one can compute μ_2, μ_3 and μ_4, a Pearson curve may be fitted to the theoretical distribution of R.

It is worth mentioning that Carter (1970) has written a programm in FORTRAN language named STOFAN (stochastic function analyzer) which includes procedures to find the moments of the probability density function of an algebraic function of H-function independent random variables and to approximate the probability density function and the cumulative distribution function from the moments.

The resulting approximations to the densities of random variables of the type R should prove very useful considering the lenghty computations that may be required to evaluate the percentage points of the exact distributions of the statistics given in (3.2) , (3.46) , (3.75) and (3.77).

Chapter 5

CONNECTION TO PRODUCT OF BETA TYPE-2 VARIABLES

5.0 Introduction

In Section 4.4, we have expressed the density of a random variable whose numerator is the sum of independent gamma variates and whose denominator is the root of a product of independent gamma variates in terms of H-functions as well as in terms of Meijer's G-functions.

In this chapter, we provide another representation of the density of such a variable based on a new identity expressing Meijer's G-functions of the type $G_{k,k}^{k,k}(\cdot)$ in terms of the densities associated with the product of independent beta type-2 random variables. This simpler representation may also be programmed for computational purposes. The exact densities of the statistics corresponding to (3.2), (3.75), (3.46) and (3.77) may be obtained as corollaries.

5.1 The Exact Density of R in Terms of that of the Product of Independent Beta Type-2 Variables

A random variable T_j is said to have a beta type-2 density with parameters α_j and β_j, if its probability density function is

$$f_j(t_j) = \{\Gamma(\alpha_j+\beta_j)/(\Gamma(\alpha_j)\Gamma(\beta_j))\}t_j^{\alpha_j-1}(1+t_j)^{-(\alpha_j+\beta_j)}, \quad (5.1)$$

for $0<t_j<\infty$, $Re(\alpha_j)>0$, $Re(\beta_j)>0$, and $f_j(t_j)=0$ elsewhere.

The h-th moment of T_j is then

$$E(T_j^h) = \{\Gamma(\alpha_j+h)\Gamma(\beta_j-h)\}/\{\Gamma(\alpha_j)\Gamma(\beta_j)\} \tag{5.2}$$

for $-Re(\alpha_j) < Re(h) < Re(\beta_j)$.

Let

$$W = T_1 T_2 \ldots T_k$$

where $T_1, T_2, \ldots, T_k$ are k independent beta type-2 random variables with the respective densities given in (5.1), then the h-th moment of W is

$$E(W^h) = \prod_{j=1}^{k} E(T_j^h) = \prod_{j=1}^{k} \{\Gamma(\alpha_j+h)\Gamma(\beta_j-h)\}/ \prod_{j=1}^{k}\{\Gamma(\alpha_j)\Gamma(\beta_j)\} , \tag{5.3}$$

Now if we compare the gammas containing h in (4.103) to the expression in (5.3), it is evident that the random variable corresponding to the gamma product in (4.103) can be looked upon as structurally a product of k independent beta type-2 random variables.For such a structure, the exact density is available in Mathai (1984). For the sake of completeness, we state the result here without proof.

If $T_1, \ldots, T_k$ are independent beta type-2 random variables with positive parameters (α_j, β_j), $j=1,\ldots,k$, and if $W=T_1 \cdots T_k$, then $\psi(W)$, the density of W, which is a solution of the integral equation

$$E(W^{s-1}) = \int_0^\infty W^{s-1} \psi(W)\, dW$$

$$= \prod_{j=1}^{k} \left[\{\Gamma(\alpha_j+s-1)\Gamma(\beta_j-s+1)\}/\{\Gamma(\alpha_j)\Gamma(\beta_j)\}\right] \tag{5.4}$$

is given by

$$\psi(W) = C_k W^{\beta_1-1}(1+W)^{-(\alpha_1+\beta_1)} \sum_{j=0}^{\infty}\{(\alpha_1+\beta_1)_j/j!\} \sum_{j_1=0}^{j} \binom{j}{j_1}(-1)^{j_1}$$

$$\sum_{j_2=0}^{j_1} \binom{j_1}{j_2} W^{j_2}(1+W)^{-j_1} \prod_{i=1}^{k-1}\{\Gamma(\beta_1+\alpha_{i+1}+j_2)\Gamma(\alpha_1+\beta_{i+1}+j_1-j_2)/$$

$$\Gamma(\alpha_1+\beta_1+\alpha_{i+1}+\beta_{i+1}+j_1)\}\ , \quad 0 < W < \infty, \tag{5.5}$$

where

$$C_k = \prod_{i=1}^{k}\{\Gamma(\alpha_i+\beta_i)/(\Gamma(\alpha_i)\Gamma(\beta_i))\}\ ,$$

$$\mathrm{Re}(\alpha_i) > 0 \text{ and } \mathrm{Re}(\beta_j) > 0\ , \quad i=1,\ldots,k\ , \tag{5.6}$$

and where for example, $(a)_j = a(a+1)\cdots(a+j-1)$ and

$$\binom{m}{n} = \frac{m!}{n!(m-n)!} \quad \text{with } 0! = 1\ .$$

Furthermore, according to (4.9) and (4.10), the density $\psi(W)$ may be expressed as an inverse Mellin transform as follows

$$\psi(W)=\{\int_{c-i\infty}^{c+i\infty} W^{-s}\, E(W^{s-1})\, ds\}(2\pi i)^{-1} \tag{5.7}$$

that is, according to (5.4),

$$\psi(W) = \{(2\pi i)\prod_{j=1}^{k}(\Gamma(\alpha_j)\Gamma(\beta_j))\}^{-1}\int_{c-i\infty}^{c+i\infty}\prod_{j=1}^{k}\{\Gamma(\alpha_j-1+s)\Gamma(\beta_j+1-s)\}\, W^{-s} dW \tag{5.8}$$

where

$$\max_j \{\mathrm{Re}(-\alpha_j+1)\} < c < \min_j\{\mathrm{Re}(\beta_j+1)\}$$

Hence, in view of (4.15), (4.17) and (4.22), we have for $0 < W < 1$,

$$\psi(W)=\{\prod_{j=1}^{k}(\Gamma(\alpha_j)\Gamma(\beta_j))\}^{-1}\, H_{k,k}^{k,k}\{W\Big|{}^{(-\beta_1,1),\ldots,(-\beta_k,1)}_{(\alpha_1-1,1),\ldots,(\alpha_k-1,1)}\} \quad (5.9)$$

which may be expressed as a Meijer's G-function according to (4.25):

$$\psi(W) = \{\prod_{j=1}^{k}(\Gamma(\alpha_j)\Gamma(\beta_j))\}^{-1} G_{k,k}^{k,k}\{W\Big|{}^{-\beta_1,\ldots,-\beta_k}_{\alpha_1-1,\ldots,\alpha_k-1}\} \ . \quad (5.10)$$

Also, in view of (4.28), we may rewrite (5.9) as follows for $W > 1$

$$\psi(W)= \prod_{j=1}^{k}\{(\Gamma(\alpha_j)\Gamma(\beta_j))\}^{-1} H_{k,k}^{k,k}\{\tfrac{1}{W}\Big|{}^{(2-\alpha_1,1),\ldots,(2-\alpha_k,1)}_{(\beta_1+1,1),\ldots,(\beta_k+1,1)}\} \quad (5.11)$$

$$= \prod_{j=1}^{k}\{(\Gamma(\alpha_j)\Gamma(\beta_j))\}^{-1} G_{k,k}^{k,k}\{\tfrac{1}{W}\Big|{}^{2-\alpha_1,\ldots,2-\alpha_k}_{\beta_1+1,\ldots,\beta_k+1}\} \quad (5.12)$$

$$=\{(2\pi i)\prod_{j=1}^{k}(\Gamma(\alpha_j)\Gamma(\beta_j))\}^{-1}\int_{c-i\infty}^{c+i\infty}\prod_{j=1}^{k}\{\Gamma(\beta_j+1+s)\Gamma(\alpha_j-1-s)\}W^s dW \quad (5.13)$$

where

$$\max_j\{\mathrm{Re}(-\beta_j-1)\} < c < \min_j\{\mathrm{Re}(\alpha_j-1)\}$$

Therefore we have the following identity for a Meijer's G-function of the type $G_{k,k}^{k,k}(W)$ which is valid for $0 < W < \infty$, provided it is understood that for $W > 1$, $G_{k,k}^{k,k}(W)$ appearing in (5.14) is in fact the function $G_{k,k}^{k,k}(\frac{1}{W})$ appearing in (5.12):

$$G_{k,k}^{k,k}\{W|_{\alpha_1-1,\ldots,\alpha_k-1}^{-\beta_1,\ldots,-\beta_k}\}= \prod_{j=1}^{k}(\Gamma(\alpha_j)\Gamma(\beta_j))\ \psi(W) \qquad (5.14)$$

where $Re(\alpha_j) > 0$ and $Re(\beta_j) > 0$, $j=1,\ldots,k$, and $\psi(W)$ is given in (5.5). This new representation which is simpler than the representations obtained in Chapter 4 can be programmed for computational purposes and provides us with a new expression for the density given in (4.106). Hence we have new representations for the densities of the statistics (3.2) and (3.75) which can be obtained in terms of (4.106).

For k=1, (5.14) reduces to

$$H_{1,1}^{1,1}\{W|_{(\alpha_1-1,1)}^{(-\beta_1,1)}\}= G_{1,1}^{1,1}\{W|_{\alpha_1-1}^{-\beta_1}\} = \Gamma(\alpha_1+\beta_1)W^{\alpha_1-1}(1+W)^{-(\alpha_1+\beta_1)} \qquad (5.15)$$

for $W > 0$, $Re(\alpha_1) > 0$ and $Re(\beta_1) > 0$.

Hence the densities of the statistics (3.46) and (3.77) which may be obtained in terms of g(L) defined in (4.87) for $k = 1$, are also available in terms of the densities of beta type-2 random variables according to (5.15).

As a particular case of (5.15), we have the following identity which is corroborated by equation (1.7.3) in Mathai and Saxena (1978)

$$G_{1,1}^{1,1}\{W|_{0}^{-\beta_1}\} = \Gamma(1+\beta_1)\ (1+W)^{-(1+\beta_1)} \qquad (5.16)$$

for $W>0$ and $\beta_1>0$.

We will now rewrite the density of T given in (4.106), where $T=(1/R)^k$ and R is defined in (4.77), in terms of the densities of beta type-2 random variables.

$$g(T) = \sum_{\nu=0}^{\infty} T^{-1} K_\nu'' G_{k,k}^{k,k}\{W|_{\alpha_1-1,\ldots,\alpha_k-1}^{-\beta_1,\ldots,-\beta_k}\} \qquad (5.17)$$

for $0<W<1$ where

$$W=(k\zeta)^k T, \qquad (5.18)$$

$$\beta_\ell = (\rho+\nu+\ell-1)/k \quad -1, \ell=1,\ldots,k; \; \nu=1,2,\ldots \qquad (5.19)$$

$$\alpha_j = q_j+1, \; j=1,\ldots,k, \qquad (5.20)$$

$$\mathrm{Re}(\beta_j)>0, \; \mathrm{Re}(\alpha_j)>0, \; j=1,\ldots,k,$$

these two last conditions being easily met by the parameters.

Also, letting

$$K_\nu''' = K_\nu''\{\prod_{j=1}^{k}(\Gamma(\alpha_j)\Gamma(\beta_j)).\,, \qquad (5.21)$$

we obtain the following expression for g(T) when $0<W<1$

$$g(T) = \sum_{\nu=0}^{\infty} T^{-1} K_\nu'''\{\prod_{j=1}^{k}(\Gamma(\alpha_j)\Gamma(\beta_j))\}^{-1} G_{k,k}^{k,k}\{W|_{\alpha_1-1,\ldots,\alpha_k-1}^{-\beta_1,\ldots,-\beta_k}\} \qquad (5.22)$$

that is, according to (5.10),

$$g(T) = \sum_{\nu=0}^{\infty} T^{-1} K_\nu''' \psi(W) \qquad (5.23)$$

where $\psi(W)$ is given in (5.5).

Similarly, for $W>1$, (4.106) becomes

$$g(T) = \sum_{\nu=0}^{\infty} T^{-1} K_{\nu}''' \{ \prod_{j=1}^{k} (\Gamma(\alpha_j)\Gamma(\beta_j)) \}^{-1} G_{k,k}^{k,k} \{ \frac{1}{W} \Big| \begin{matrix} 2-\alpha_1, \ldots, 2-\alpha_k \\ \beta_1+1, \ldots, \beta_k+1 \end{matrix} \} \tag{5.24}$$

where W, β_ℓ, α_j and K_ν''' are respectively defined in (5.18), (5.19), (5.20) and (5.21).

Hence according to (5.12)

$$g(T) = \sum_{\nu=0}^{\infty} T^{-1} K_{\nu}''' \psi(W) \tag{5.25}$$

where $\psi(W)$ is given in (5.5).

Therefore, the density of $T=(1/R)^k$ is given by (5.25) for $W > 0$, that is, for $T > 0$ in view of (5.18).

Chapter 6

SOME ECONOMETRIC APPLICATIONS

6.0 Introduction

In this chapter, different types of econometric models are discussed and the applicability of the test statistics obtained in Chapter 3 to the econometric pure error model, where there are k linear relationships, is pointed out.

6.1 Types of Econometric Models

Let us consider the simple case where the true values, g_1 and g_2, of two observations, x_1 and x_2, subject to errors of measurement, e_1 and e_2 respectively, are connected by the following linear relation

$$g_1 = \alpha + \beta g_2 \ , \tag{6.1}$$

where

$$x_1 = g_1 + e_1 \tag{6.2}$$

and

$$x_2 = g_2 + e_2 \ . \tag{6.3}$$

In the econometric terminology, this is a pure error model where x_1 is an endogenous variable and x_2 is an exogenous variable. This model corresponds to the errors-in-variables functional model defined in Section 1.3 and to the single linear functional relationship model discussed in Section 2.1, for p=2.

The appellation " functional model " is due to Malinvaud(1970).

If a stochastic component is added to the right-hand side of (6.1), the resulting more general model is referred to in the econometric literature as a shock-error model, see for instance Anderson and Hurwicz (1948). When this stochastic component is attributable to g_2, the model is a particular case of the multivariate errors-in-variables structural model for which maximum likelihood estimators have been derived in Amemiya and Fuller (1984). Models which incorporate both shocks and errors were also considered by Zelner (1970), Golberger (1972), Griliches (1974) and Geraci (1977). Gleser (1981) also obtained large sample results for the multivariate pure error model.

For the pure shocks models, it is assumed that the variables are measured without error but that a stochastic component is present in the equations specifying their relationships. Such models may usually be assimilated to the general linear hypothesis models discussed in Section 1.2. Konijn(1962) showed how the identification rules for the pure shock model carry over to a shock-error model if error variances are known.

6.2 The Multivariate Errors-in-Variables Econometric Models

In Chapter 3, we have obtained new test statistics for the structural coefficients, $\underset{\sim}{a}$ and $\underset{\sim}{B}$, of a multivariate linear functional relationship model (defined in Section 2.2a). We will now show that this MLFR model boils down to the multivariate errors-in-variables functional (MEVF) model when we set

$$\underset{k\times p}{\underset{\sim}{B}'} = (-I \, , \underset{k\times m}{\underset{\sim}{B}^*}) \, , \tag{6.4}$$

where

$$m = p - k \quad . \tag{6.5}$$

Rewriting (2.5), that is,

$$\underset{p\times 1}{\underset{\sim}{x}_{ij}} = \underset{\sim}{g}_i + \underset{\sim}{e}_{ij}$$

as

$$\begin{bmatrix} \underset{\sim}{x}_{ij}^{(1)} \\ \underset{\sim}{x}_{ij}^{(2)} \end{bmatrix} = \begin{bmatrix} \underset{\sim}{g}_i^{(1)} \\ \underset{\sim}{g}_i^{(2)} \end{bmatrix} + \begin{bmatrix} \underset{\sim}{e}_{ij}^{(1)} \\ \underset{\sim}{e}_{ij}^{(2)} \end{bmatrix} \, , \tag{6.6}$$

$i=1,\ldots,r;\ j=1,\ldots,n_i$, where $\underset{\sim}{x}_{ij}^{(1)}$, $\underset{\sim}{g}_i^{(1)}$ and $\underset{\sim}{e}_{ij}^{(1)}$ are k-dimensional vectors, $\underset{\sim}{x}_{ij}^{(2)}$, $\underset{\sim}{g}_i^{(2)}$ and $\underset{\sim}{e}_{ij}^{(2)}$ are m-dimensional vectors and

$$\underset{\sim}{e}_{ij} \overset{ind}{\sim} N_p(\underset{\sim}{0},\underset{\sim}{V}) \, , \ i=1,\ldots,r;\ j=1,\ldots,n_i;\ \underset{\sim}{V}>0, \tag{6.7}$$

$$\underset{\sim}{a} + \underset{\sim}{B}'\underset{\sim}{g} = \underset{\sim}{0}$$

becomes, upon substituting (6.4),

$$\underset{\sim}{a} - I\,\underset{\sim}{g}^{(1)} + \underset{\sim}{B}^*\underset{\sim}{g}^{(2)} = \underset{\sim}{0} \, ,$$

or equivalently

$$\underset{\sim}{g}^{(1)} = \underset{\sim}{a} + \underset{\sim}{B}^*\,\underset{\sim}{g}^{(2)} \quad . \tag{6.8}$$

The MEVF model specified by (6.6), (6.7) and (6.8), is in fact the pure error econometric model where there are k linear relationships. When it is further assumed that the $\underset{\sim}{g}_i^{(2)}$'s are independently and identically distributed random vectors, the resulting model is called the multivariate errors-in-variables structural (MEVS) model.

In addition to the functional and the structural models, Villegas and Rennie (1976) proposed another version in which the true vectors constitute an autoregressive process. It is also stated in Villegas(1982) that from a Bayesian viewpoint the differences among these three versions are not very important.

We would like to mention that the maximum likelihood estimators for the structural coefficients of a single linear functional relationship model coincide with those derived for the structural coefficients of a single linear structural relationship model.This result is proved in Patefield(1981) where these models are inappropriately referred to as multivariate linear relationships since this appellation is reserved to the case where there are more than one linear relationship among several variables.

Similarly, the maximum likelihood estimators of $\underset{\sim}{a}$ and $\underset{\sim}{B}^*$ obtained in Amemiya and Fuller (1984) for the MEVF model are the same as the maximum likelihood estimators derived for the MEVS model; moreover it is stated that the maximum likelihood estimator of $\underset{\sim}{V}$ defined in (6.7) for the MEVF model differs only by a scalar multiple from that for the MEVS model.

We would also like to note that for the shock-error model discussed in Geraci (1976), the author states that it is possible to compensate the stochastic component due to measurement errors by the stochastic component associated with shocks in the equations. This idea of merging these two types of disturbances is also suggested in Intriligator (1978). Therefore, it is conceivable that some shock-error econometric models may be statistically equivalent to pure error models.

Although the results of this thesis deal uniquely with the functional model which can be reduced to the econometric pure error model where there are k linear relationships (the MEVF model), they may perform adequately in some inference problems connected with shock-error econometric models. Moreover many econometric variables are measured with substantial error and, in some cases, the stochastic component associated with shocks in the equations is negligible compared to measurement errors.

As early as 1934, R. Frish considered errors in variables to be an important element in stochastic formulations of econometric behaviour.

Most pure error models that have been discussed so far in the econometric literature deal with the single errors-in-variables functional (SEVF) model. Some examples are the Douglas-type production function defined in Davis (1941) which is linear in the logarithms; the models discussed in Koopmans (1937) and Geary (1949) in connection with econometric time series; Friedman's (1957) theory of the consumption function which is based entirely on a postulated exact proportional relationship between permanent consumption and permanent income where the measured income is the sum of permanent and transient incomes and the measured consumption is the sum of permanent and transient consumptions;and Johnson's (1963) generalization of the two-variable linear case. The SEVF model is also discussed in Tintner (1950), in Sargan (1958), in Kunimoto (1980) in connection with large econometric models and in Anderson (1976),where the model is made equivalent to a model of simultaneous equations in econometrics and where the MLFR model is presented as a generalization.

6.3 Conclusion

In Chapter 3, we considered some tests for the structural coefficients of a MLFR model under various conditions imposed on the covariance matrix of errors.

The exact densities of the corresponding test statistics may be obtained as corollaries of the results of Chapters 4 and 5, where computable representations of the densities of some algebraic functions of independent gamma variates are derived.

Hence, with the help of the new results obtained in Chapters 3-5, we can test many hypotheses about the parameters of a MEVF model which is a particular case of the MLFR model as well as a generalization of the SEVF model discussed in the econometric literature.

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