Moduli spaces of vector bundles on a Hopf surface, and their stability properties

BY RUXANDRA MORARU,
DEPARTMENT OF MATHEMATICS AND STATISTICS,
McGILL UNIVERSITY, MONTREAL

A thesis submitted to the faculty of Graduate Studies and Research
in partial fulfillment of the requirements for the degree of Ph.D.

August, 2000

0-612-70103-4

Acknowledgments

During the preparation of this thesis, I received funding from NSERC, and from my supervisor, Dr. Jacques Hurtubise. I am pleased to have this opportunity to offer him my deepest gratitude for his generosity, invaluable insight and infallible support. I am also grateful to other members of the mathematics department, notably Dr. Georg Schmidt and Dr. Peter Russell. The typesetting of the thesis was performed using the $\mathcal{A}_{\mathcal{M}}\mathcal{S} - T_EX$ macro package. Finally, I wish to dedicate this thesis to my parents Angela and Alexandru Moraru.

Abstract

We study the moduli spaces $\mathcal{M}_n$ of rank two stable holomorphic $\mathrm{SL}(2,\mathbb{C})$-bundles E over Hopf surfaces $\mathcal{H}$, with $c_2(E) = n$, and their stabilisation properties. We show that one cannot construct stabilisation maps $\mathcal{M}_n \to \mathcal{M}_{n+1}$ that are a natural holomorphic counterpart to Taubes's subtraction procedure that is used to construct such maps in the topological case of moduli spaces of connections. We also study the fiber of a map that associates to any holomorphic bundle a graph, and show that, in certain cases, the fiber is the Jacobian of a Riemann surface. We then show that this map is a Lagrangian fibration, with respect to a Poisson structure that we will define on $\mathcal{M}_n$. Finally, we generalize the notion of graph to connections, and show that the graph map thus obtained is not topologically trivial.

Résumé

Nous étudions les espaces de modules $\mathcal{M}_n$ de fibrés stables holomorphes E de groupe de structure $\mathrm{SL}(2,\mathbb{C})$ sur la surface de Hopf $\mathcal{H}$, avec $c_2(E) = n$, et leurs propriétés de stabilisation. Nous montrons que nous ne pouvons pas utiliser la version holomorphe de la procédure de soustraction de Taubes pour définir des applications de stabilisation $\mathcal{M}_n \to \mathcal{M}_{n+1}$. Nous étudions aussi la fibre d'une application qui associe à tout fibré holomorphe un graphe, et montrons que, dans certains cas, la fibre de cette application est la Jacobienne d'une surface de Riemann. Nous montrons ensuite que cette application est une fibration Lagrangienne, par rapport à une structure de Poisson que nous allons définir sur $\mathcal{M}_n$. Finalement, nous généralisons la notion de graphe dans le cas de connexions, et puis montrons que l'application graphe ainsi obtenue n'est pas topologiquement triviale.

Contents

Introduction

The moduli spaces of holomorphic vector bundles have been extensively studied over the past fifty years, starting with the classification of vector bundles over the Riemann sphere by Grothendieck [Gr], and over an elliptic curve by Atiyah [At]. The moduli space of all vector bundles over a space X is however not, in general, a Hausdorff space. One can get around this problem by only considering stable holomorphic bundles, as was first remarked by Mumford in the case where X is an algebraic curve. The notion of stability is intimately linked with the notion of degree, and therefore requires the existence of a Kähler metric on X. More recently, Hitchin [Bh] extended the definition of stability to any compact complex hermitian manifold. This was done by using a Gauduchon metric to define the degree of a vector bundle. The existence of such metrics on any compact complex manifold was proven by Gauduchon [G].

On a surface, holomorphic vector bundles also correspond to solutions to the Yang-Mills equations, i.e. instantons. This was first proven for any Hodge surface by Donaldson [D], and any compact complex surface by Buchdahl [Bh]. From the topological point of view, Atiyah and Jones [AtJo] studied the global topology of instanton moduli spaces. They conjectured that the inclusion of the space $\mathcal{M}_n$ of framed instantons of charge n into the space $\mathcal{B}_n$ of all connections should induce isomorphisms of homotopy groups H_i and homotopy groups π_i for sufficiently large k. This was first proven for $SU(2)$-instantons on the 4-sphere by Boyer, Hurtubise, Mann, and Milgram [BHMM], and later on ruled surfaces by Hurtubise and Milgram [HM].

Moduli spaces have also been studied from the point of view of symplectic geometry. Mukai proved in [Mu] that the moduli space of simple sheaves on an abelian or $K3$ surface has a natural symplectic structure. Abelian and $K3$ surfaces have trivial canonical bundles, and are therefore symplectic surfaces. Mukai showed that the choice of a symplectic structure on such a surface induces a symplectic structure on the moduli space. This was then generalised to Poisson structures and Poisson surfaces by Bottacin [Bot]: a Poisson structure on a Poisson surface determines in a canonical way a Poisson structure on the moduli space of stable sheaves. Moduli spaces have also given rise to algebraically completely integrable Hamiltonian systems. Among others, Hitchin [H] has shown that the cotangent bundles of the moduli spaces of stable vector bundles over a Riemann surface, endowed with their natural symplectic structures, support algebraically integrable systems. Beauville [Be] has also shown that, with the symplectic structure defined by Mukai [Mu], the moduli space of line bundles over $K3$ surfaces gives an algebraically integrable system.

There have been numerous explicit descriptions of moduli spaces on Kähler manifolds. The case of a non Kähler manifold was first studied by Braam and Hurtubise. In [BH], they considered instantons on Hopf surfaces. The Hopf surface is one of the simplest elliptic surfaces and it has a homogeneous fibre. It does not however possess a cross-section. Let us note that the general case of moduli spaces on elliptic surfaces with a cross-section has been studied by Friedman, Morgan and Witten [FMW]. In this thesis, we propose to generalise some of the results found in [BH] in regards to the stabilisation and the topology of spaces of connections, and also in regards to integrable systems and spectral curves.

The first chapter provides a review of some of the theory of sheaves that will prove useful in the study of holomorphic vector bundles. We begin by discussing extensions of sheaves, and give an explicit description of the transition matrices of extensions of vector bundles. We then turn to deformations of sheaves, and describe how this relates to moduli spaces of stable sheaves and holomorphic vector bundles. We finally give a brief account of how moduli spaces are constructed on any compact complex manifold.

In the second chapter we begin by introducing the Hopf surface $\mathcal{H}$ and the results found in [BH]. An important point is the fact that $\mathcal{H}$ fibres over $\mathbb{P}^1$, with fibre an elliptic curve T. A rank two holomorphic $\mathrm{SL}(2,\mathbb{C})$-bundle E on $\mathcal{H}$ can then be considered as a family of bundles over T parametrised by $\mathbb{P}^1$, by restricting E to the fibres T. Rank two $\mathrm{SL}(2,\mathbb{C})$-bundles over an elliptic curve have however been completely classified by Atiyah [At]. One can then associate to E a divisor in $\mathbb{P}^1 \times \mathbb{P}^1 = \mathbb{P}^1 \times \mathrm{Pic}^0(\mathrm{T})/\pm$ which gives the isomorphism type of E over each fibre T. This divisor will be called the graph of E, and will prove to be a useful tool in the study of holomorphic $\mathrm{SL}(2,\mathbb{C})$-bundles on $\mathcal{H}$. As the Hopf surface $\mathcal{H}$ can be covered by two copies of $\mathbb{C} \times T$, bundles on $\mathcal{H}$ can be constructed by glueing two bundles over $\mathbb{C} \times T$. We then finish the chapter by classifying bundles over $D \times T$, where D is a disc in $\mathbb{P}^1$.

In the third chapter, we study stabilisation maps on the moduli spaces $\mathcal{M}_n^0$ of framed stable holomorphic $\mathrm{SL}(2,\mathbb{C})$-bundles E on $\mathcal{H}$, with $c_2(E) = n$. These maps always exist in the case of moduli spaces of instantons. One can indeed use Taubes' subtraction procedure to "glue in" an instanton at a fixed base point p_0 of $\mathcal{H}$. We would like to know whether such a map can be defined in the holomorphic setting. Such maps can be realised by "glueing in a jumping line" in the case of bundles over S^4, and ruled surfaces (see [BHMM] and [BM]). The holomorphic analogue of the subtraction procedure in the case of bundles over $\mathcal{H}$ seems to be to construct a sheaf by glueing in a copy of $\mathcal{O} \oplus I$ at p_0, and deform the new sheaf to obtain one that is locally free. Even though this can done locally in $\mathcal{M}_n^0$, we will see that the deformation cannot be globally extended to obtain a well defined stabilisation map $\mathcal{M}_n^0 \to \mathcal{M}_{n+1}^0$.

One can define a map $G : \mathcal{M}_n \to \mathbb{P}^{2n+1}$ which associates to each bundle in $\mathcal{M}_n$ its graph. This map is surjective for $n \geq 2$. A natural question to ask is, given a graph g, how many holomorphic bundles correspond to it? In other words, what is the fibre of G? In the case where $n = 1$, it was shown in [BH] that the fibre is always an elliptic curve, and that $G : \mathcal{M}_1 \to Im(G)$ is a principal T-bundle. Let us note that every bundle E in $\mathcal{M}_1$ satisfies a condition that we will denote by $(*)$: there are no points $x \in \mathbb{P}^1$ where $E|_{\pi^{-1}(x)} = L_0 \oplus L_0, L_0^2 \cong \mathcal{O}$. If the graph of the bundle is the graph of a holomorphic map $F : \mathbb{P}^1 \to \mathbb{P}^1$, this is equivalent to requiring that the differential dF does not vanish at certain points. In the fourth chapter, we generalise this result in the case of graphs that are holomorphic maps of degree n which satisfy condition $(*)$. We will prove that, given such graph g, the fibre $G^{-1}(g)$ is the Jacobian of the spectral curve associated to g. This will be done by using two different methods. The first one was used in [BH]. Given that the restrictions of such graphs to $D \times \mathbb{P}^1$ completely determine the isomorphism class of the bundle on $D \times T$, the first method consists in finding the different ways of glueing them together. The spectral curve $\bar{S}$ of a bundle E is obtained from the support of a skyscraper sheaf L. If the graph of E satisfies $(*)$, this sheaf has fibre $\mathbb{C}$ on S. In the second method, we then show that there is a one-to-one correspondence between $G^{-1}(g)$ and the set of holomorphic line bundles on $\bar{S}$.

In chapter five, we use the construction in [Bot] to define a Poisson structure on the moduli spaces $\mathcal{M}_n$. Let Δ be the set of graphs which do not satisfy $(*)$. Given a Poisson structure on $\mathcal{M}_n$, we then show that, over the complement of Δ, the graph map $G : \mathcal{M}_n \to \mathbb{P}^{2n+1}$ is a Lagrangian fibration whose fibres are isomorphic to abelian varieties. In the case of $n = 1$, G is proper, and we actually have an algebraically completely integrable system. In the sixth chapter, we give a partial classification of $\mathcal{M}_2$.

In the final chapter, we consider the topological side of the problem by studying the moduli spaces of connections. Given a $\mathcal{C}^\infty$ bundle E over $\mathcal{H}$ with $c_1(E) = 0$ and $c_2(E) = k$, we denote by $\mathcal{B}_k$ the moduli space of gauge equivalent connections on E. We will see that the notion of graph can be extended to connections. Not every connection A on E induces a global holomorphic structure. However, as every fibre of $\pi : \mathcal{H} \to \mathbb{P}^1$ is an elliptic curve, the restriction of A to any fibre $\pi^{-1}(x)$ defines a holomorphic structure on the restriction of E to $\pi^{-1}(x)$. It would then seem natural to think that one can associate to A a graph, as in the holomorphic case. This will done by constructing a family of Dirac operators $\{\bar{\partial}_A\}$ associated to A, and considering the determinant line bundle $\mathcal{L}$ of the family $\{\bar{\partial}_A\}$. The graph of A will then be defined as the zero set of a section of $\mathcal{L}$, and will correspond to a divisor in $\mathbb{P}^\infty$. If A defines a holomorphic structure on E, this graph coincides with the graph defined in the holomophic case. Furthermore, we will again be able to define a graph map $G : \mathcal{B}_k \to \mathbb{P}^\infty$ which associates to each connection A a graph g. We will show that G is not a homotopically trivial map, and that the fibre of G

can be considered as the total space of an S^1-bundle over $\mathcal{B}_k$.

Chapter 1

Coherent sheaves.

In this chapter, we give a review of some of the theory of sheaves that we will use to study holomorphic vector bundles. The first two sections provide the necessary background for the remaining ones. In the third section, we discuss extensions of sheaves, and give an explicit expression of the transition matrices of extensions of vector bundles. The Serre Construction for holomorphic bundles is then presented in section four. We turn, in section five, to the deformation theory of sheaves, and describe how it relates to moduli spaces of sheaves and vector bundles. The final section gives a brief account of the construction of moduli spaces on any compact complex hermitian manifold. We define stability and give Buchdahl's theorem relating stable holomorphic vector bundles to instantons.

1.1 Commutative and homological algebra.

We begin by recalling certain definitions from commutative and homological algebra, and establish certain results, some of them very well known, that will be of great use in the rest of the thesis. Let us remark that we follow the presentation of [GH] and will therefore use their notation. We will not give proofs for all results, and, unless otherwise stated, we refer the reader to [GH] for them.

1.1.1 Homological algebra.

We start by establishing some notation that will be used throughout the following section. $\mathcal{O} = \lim_{0 \in U} \mathcal{O}(U)$ will represent the germ of analytic functions defined in some neighbourhood U of the origin in $\mathbb{C}^n$. It is clearly the ring $\mathcal{O} = \mathbb{C}\{z_1, \ldots, z_n\}$ of convergent power series, which is a local ring. It therefore has a unique maximal ideal $m = \{z_1, \ldots, z_n\}$, which is the ideal of functions $f \in \mathcal{O}$ with $f(0) = 0$. The units are just $\mathcal{O}^* = \mathcal{O} - m$.

We will mainly be using $\mathcal{O}$-modules, usually denoted by $M, N, E, \ldots$, and *we will always assume that they are finitely generated.* Important examples of $\mathcal{O}$-modules that will often come up are: free $\mathcal{O}$-modules; and if $f_1, \ldots, f_k$ are functions in $\mathcal{O}$, we will often consider

$$\begin{cases} I = \{f_1, \ldots, f_k\} & \text{an ideal in } I \text{ generated by } f_1, \ldots, f_k, \\ \\ M = \mathcal{O}/\{f_1, \ldots, f_k\}. & \end{cases}$$

Let us note that since all $\mathcal{O}$-modules are assumed to be finitely generated, free modules will be identified with projective modules.

A *complex* is given by either

$$(K_.) \qquad \longrightarrow K_n \xrightarrow{\partial} K_{n-1} \xrightarrow{\partial} \ldots, \qquad \partial^2 = 0,$$

or

$$(K^\cdot) \qquad \longrightarrow K^n \xrightarrow{\delta} K^{n-1} \xrightarrow{\delta} \ldots, \qquad \delta^2 = 0.$$

Here the K's will always be finitely generated $\mathcal{O}$-modules and the maps $\mathcal{O}$-module homomorphisms. $H_*(K_.) = \oplus H_n(K_.)$ and $H^*(K^\cdot) = \oplus H^n(K^\cdot)$ are the *homology* and *cohomology*, respectively, that one obtains by taking (co)cycles/(co)boundaries.

Definition 1.1 A *projective resolution* $E_.(M)$ of an $\mathcal{O}$-module M is given by an exact sequence

$$E_.(M) : \ldots \longrightarrow E_m \xrightarrow{\partial} E_{m-1} \xrightarrow{\partial} \ldots \xrightarrow{\partial} E_0 \longrightarrow M \longrightarrow 0,$$

where the E_m are projective(=free) $\mathcal{O}$-modules.

Let us note that a projective resolution is obviously a complex, and that the exactness of the sequence implies that $H_n(E_.(M)) = 0$ for $n > 0$ and $H_0(E_.(M)) \cong M$. Let us also remark that projective resolutions exist for any finitely generated $\mathcal{O}$-module M.

Now suppose that M and N are finitely generated $\mathcal{O}$-modules, and that we have a projective resolution of M

$$E_.(M) : \ldots \longrightarrow E_m \xrightarrow{\partial} E_{m-1} \xrightarrow{\partial} \ldots \xrightarrow{\partial} E_0 \longrightarrow M \longrightarrow 0,$$

This resolution will then induce the two complexes $Hom_{\mathcal{O}}(E_.(M), N)$ and $E_.(M) \otimes_{\mathcal{O}} N$. Let us note that even though the first part

$$0 \longrightarrow Hom_{\mathcal{O}}(M, N) \longrightarrow Hom_{\mathcal{O}}(E_0, N) \xrightarrow{\partial} Hom_{\mathcal{O}}(E_1, N)$$

of $Hom_{\mathcal{O}}(E_.(M), N)$ is exact, as is the last part

$$E_1 \otimes_{\mathcal{O}} N \xrightarrow{\partial} E_0 \otimes_{\mathcal{O}} N \longrightarrow M \otimes_{\mathcal{O}} N \longrightarrow 0$$

of $E_.(M) \otimes_{\mathcal{O}} N$, the complete sequences are in general not exact. But it is easy to see that they are complexes. We then have the following

Definition 1.2 Given finitely generated $\mathcal{O}$-modules M and N,

$$\begin{cases} Ext^n_{\mathcal{O}}(M, N) = H^n(Hom_{\mathcal{O}}(E_.(M), N)), \\ Tor^{\mathcal{O}}_n(M, N) = H_n(E_.(M) \otimes_{\mathcal{O}} N). \end{cases}$$

In the remainder, we will not use Tor. We will therefore only give the properties of Ext. Let us start by noting that the Ext groups are well-defined and independent of the projective resolution $E_.(M)$. Let us also remark that since, as we have seen above,

$$0 \longrightarrow Hom_{\mathcal{O}}(M, N) \longrightarrow Hom_{\mathcal{O}}(E_0, N) \xrightarrow{\partial} Hom_{\mathcal{O}}(E_1, N)$$

is exact, we have

$$Ext^0_{\mathcal{O}}(M, N) \cong Hom_{\mathcal{O}}(M, N).$$

The main property of the *Ext* functor is:

Short exact sequences of $\mathcal{O}$-modules

$$\begin{cases} 0 \to M' \to M \to M'' \to 0, \\ 0 \to N' \to N \to N'' \to 0, \end{cases}$$

induce long exact sequences

$$\begin{cases} \ldots \to Ext^n_{\mathcal{O}}(M,N) \to Ext^n_{\mathcal{O}}(M',N) \to Ext^{n+1}_{\mathcal{O}}(M'',N) \to \ldots, \\ \ldots \to Ext^n_{\mathcal{O}}(M,N) \to Ext^n_{\mathcal{O}}(M,N'') \to Ext^{n+1}_{\mathcal{O}}(M,N') \to \ldots, \end{cases}$$

of Ext's.

For example, given $0 \to N' \to N \to N'' \to 0$, we obtain

$$0 \to Hom_{\mathcal{O}}(M,N') \to Hom_{\mathcal{O}}(M,N) \to Hom_{\mathcal{O}}(M,N'') \to Ext^1_{\mathcal{O}}(M,N'),$$

so that $Ext^1_{\mathcal{O}}(M,\ -\ )$ measures the extent to which $Hom_{\mathcal{O}}(M,\ -\ )$ fails to be right exact. We also have the following

Theorem 1.1 *$Ext^q_{\mathcal{O}}(M,N) = 0$, for $q > 0$ and every $\mathcal{O}$-module N $\Leftrightarrow$ M is projective.*

One can actually refine this to

Corollary 1.1 *$Ext^1_{\mathcal{O}}(M,E) = 0$, for all projective $\mathcal{O}$-modules M $\Leftrightarrow$ M is projective.*

1.1.2 The Koszul complex and some applications.

Koszul complex. Suppose $f_1,\ldots,f_r \in \mathcal{O}$; denote by $I_k = \{f_1,\ldots,f_k\}$ the ideal generated by the first k functions, and set $I = I_r$.

Definition 1.3 $(f_1,\ldots,f_r)$ is a *regular sequence* if, for all $k = 1,\ldots,r$, f_k is not a zero divisor in $\mathcal{O}/I_{k-1}$.

In particular, if $r = 2$, we see that f_2 is not a zero divisor in $\mathcal{O}/I_1$ if and only if f_1 and f_2 are relatively prime. Therefore,

$$(f_1,f_2) \text{ is a regular sequence} \Leftrightarrow f_1 \text{ and } f_2 \text{ are relatively prime.}$$

Given a regular sequence, the Koszul complex gives a projective resolution of the $\mathcal{O}$-module I. Even though it is well-defined for all regular sequences, we shall present it in the case where $r = 2$.

Let (f_1, f_2) be a regular sequence. The *Koszul complex* is therefore defined to be the sequence

$$0 \longrightarrow \mathcal{O} \xrightarrow{\lambda} \mathcal{O} \oplus \mathcal{O} \xrightarrow{\eta} I \longrightarrow 0,$$

where $\lambda : \mathcal{O} \to \mathcal{O} \oplus \mathcal{O}$ is given by $1 \mapsto (-f_2, f_1)$, and $\eta : \mathcal{O} \oplus \mathcal{O} \to I$ is given by $(g_1, g_2) \mapsto f_1 g_1 + f_2 g_2$. It is then clear that $\eta\lambda = 0$. It actually turns out that the Koszul complex is exact as shown in

Lemma 1.1 *If f_1 and f_2 are relatively prime, the sequence*

$$0 \longrightarrow \mathcal{O} \xrightarrow{\lambda} \mathcal{O} \oplus \mathcal{O} \xrightarrow{\eta} I \longrightarrow 0$$

is exact.

Proof: Clearly, λ is injective and $\mathrm{Im}\eta = \{\mathrm{f}_1, \mathrm{f}_2\} = \mathrm{I}$. Moreover, as $\eta\lambda = 0$, we just have to verify that $\mathrm{Ker}\eta \subset \mathrm{Im}\lambda$. If $(g_1, g_2) \in \mathrm{Ker}\eta$, then $f_1 g_1 = -f_2 g_2$. As f_1 and f_2 are relatively prime, there must exist an $a \in \mathcal{O}$ such that $g_1 = -a f_2$ and $g_2 = a f_1$, therefore proving that $(g_1, g_2) = \lambda(a) \in \mathrm{Im}\lambda$. $\square$

Let us now use the Kozsul complex to compute some *Ext* groups involving the local ring $\mathcal{O}$, an ideal I generated by a two relatively prime functions, and the quotient sheaf $\mathcal{O}/I$. These will often be used in the sequel. Locally, we have the following lemma:

Lemma 1.2 *Suppose that I is an ideal generated by two relatively prime functions $x, z \in \mathcal{O}$, then:*

(i) $Ext^i_{\mathcal{O}}(\mathcal{O}, \mathcal{O}) = Ext^i_{\mathcal{O}}(\mathcal{O}, I) = 0$, *for all i;*

(ii) $Hom_{\mathcal{O}}(I, \mathcal{O}) \cong \mathcal{O}$ *and the isomorphism is generated by the natural restriction map* $Hom_{\mathcal{O}}(\mathcal{O}, \mathcal{O}) \to Hom_{\mathcal{O}}(I, \mathcal{O})$*;*

(iii) $Hom_{\mathcal{O}}(I, I) \cong \mathcal{O}$ *and the isomorphism is also generated by the natural restriction map* $Hom_{\mathcal{O}}(\mathcal{O}, \mathcal{O}) \to Hom_{\mathcal{O}}(I, \mathcal{O})$*;*

(iv) $Ext^1_{\mathcal{O}}(I,\mathcal{O}) = \mathbb{C}$;

(v) $Ext^1_{\mathcal{O}}(I,I) = \mathbb{C}^2$; *and*

(vi) $Ext^i_{\mathcal{O}}(I,\mathcal{O}) = Ext^i_{\mathcal{O}}(I,I) = 0$, *for all* $i \geq 2$.

Proof: (i) As $\mathcal{O}$ is free, $Ext^1_{\mathcal{O}}(\mathcal{O},M) = 0$, for any $\mathcal{O}$-module M.
(ii), (iii) Any $\mathcal{O}$-homomorphism $I \xrightarrow{\alpha} \mathcal{O}$ is given by two elements $a, b \in \mathcal{O}$ such that $x \mapsto a$ and $z \mapsto b$. Moreover, by $\mathcal{O}$-linearity, $az - bx = 0$. As x and z are relatively prime, we must have $x|a$ and $z|b$. Thus, $a = hx$ and $b = hz$ for some $h \in \mathcal{O}$. α then extends to a $\mathcal{O}$-homomorphism $\mathcal{O} \to \mathcal{O}$ given by $1 \mapsto h$, thus proving that $Hom_{\mathcal{O}}(I,\mathcal{O})$ is isomorphic to $Hom_{\mathcal{O}}(\mathcal{O},\mathcal{O}) \cong \mathcal{O}$. But this also tells us that α is given by $x \mapsto hx$ and $z \mapsto hz$. Thus, $Hom_{\mathcal{O}}(I,\mathcal{O}) \subset Hom_{\mathcal{O}}(I,I)$, and $Hom_{\mathcal{O}}(I,I) = Hom_{\mathcal{O}}(I,\mathcal{O}) \cong \mathcal{O}$.

If $E_\cdot(I)$ is any projective resolution of I, and N is any finitely generated $\mathcal{O}$-module, we have by definition

$$Ext^n_{\mathcal{O}}(I,N) = H^n(Hom_{\mathcal{O}}(E_\cdot(I),N)).$$

We choose the Kozsul complex as a projective resolution of I

$$E_\cdot(I) : 0 \longrightarrow E_1 \xrightarrow{\partial} E_0 \longrightarrow I \longrightarrow 0,$$

where $E_0 = \mathcal{O} \oplus \mathcal{O}$, $E_1 = \mathcal{O}$, and $\partial : 1 \mapsto (-z, x)$. To compute $Ext^1_{\mathcal{O}}(I,\mathcal{O})$, we shall find the first cohomology group of the complex $Hom_{\mathcal{O}}(E_\cdot(I),\mathcal{O})$. As $Hom_{\mathcal{O}}(E_0,\mathcal{O}) = \mathcal{O} \oplus \mathcal{O}$, $Hom_{\mathcal{O}}(E_1,\mathcal{O}) = \mathcal{O}$, and $Ext^1_{\mathcal{O}}(E_0,\mathcal{O}) = 0$, it is obvious that this complex is simply

$$0 \longrightarrow Hom_{\mathcal{O}}(I,\mathcal{O}) \longrightarrow \mathcal{O} \oplus \mathcal{O} \xrightarrow{\partial^*} \mathcal{O} \longrightarrow Ext^1_{\mathcal{O}}(I,\mathcal{O}) \longrightarrow 0,$$

where $\partial^* : \mathcal{O} \oplus \mathcal{O} \xrightarrow{[-z \ x]} \mathcal{O}$ is the transpose of ∂. We then have

$$Im(\partial^*) = \{-fz + gx : f, g \in \mathcal{O}\} = I,$$

and

$$H^1(Hom_{\mathcal{O}}(E_\cdot(I),\mathcal{O})) = \mathcal{O}/I \simeq \mathbb{C}.$$

The proof of (v) is similar. We now use the complex $Hom_{\mathcal{O}}(E_.(I), I)$. As $Hom_{\mathcal{O}}(E_0, I) = I \oplus I$, $Hom_{\mathcal{O}}(E_1, I) = I$, and $Ext^1_{\mathcal{O}}(E_0, I) = 0$, it is given by

$$0 \longrightarrow Hom_{\mathcal{O}}(I, I) \longrightarrow I \oplus I \xrightarrow{\partial^*} I \longrightarrow Ext^1_{\mathcal{O}}(I, I) \longrightarrow 0,$$

where ∂^* is as above. This time,

$$Im(\partial^*) = \{fx^2 + gxz + hz^2 : f, g, h \in \mathcal{O}\} = I^2;$$

and

$$H^1(Hom_{\mathcal{O}}(E_.(I), \mathcal{O})) = I/I^2 \simeq \mathbb{C}^2,$$

since $I/I^2 = \{ax + bz : a, b \in \mathbb{C}\}$.

(vi) follows from the fact that I has a short projective resolution. $\square$

The Kozsul complex induces the projective resolution

$$0 \longrightarrow \mathcal{O} \xrightarrow{(-z,x)} \mathcal{O} \oplus \mathcal{O} \xrightarrow{[x\ z]} \mathcal{O} \longrightarrow \mathcal{O}/I \longrightarrow 0,$$

of $\mathcal{O}/I$. The latter can be used to show that

Lemma 1.3 *Suppose that I is an ideal generated by two relatively prime functions $x, z \in \mathcal{O}$, then*

(i) $Ext^i_{\mathcal{O}}(\mathcal{O}/I, \mathcal{O}) = 0$, *for* $i \neq 2$;

(ii) $Ext^2_{\mathcal{O}}(\mathcal{O}/I, \mathcal{O}) \simeq \mathcal{O}/I$.

Proof: The computations are similar to the ones in Lemma 1.2. For a detailed proof see [GH] p.690. $\square$

1.2 Coherent sheaves.

We now consider sheaves that are defined globally on a complex manifold X. $\mathcal{O} = \mathcal{O}_X$ will then be the structure sheaf of X. We start by giving the following

Definition 1.4 Let X be a complex manifold with structure sheaf $\mathcal{O}$ and $\mathcal{F}$ be a sheaf of $\mathcal{O}$-modules. Then $\mathcal{F}$ is said to be *coherent* if locally there always is some exact sequence of sheaves of $\mathcal{O}$-modules

$$\mathcal{O}^{(p)} \longrightarrow \mathcal{O}^{(q)} \longrightarrow \mathcal{F} \longrightarrow 0.$$

We shall often use the following properties:

1. *Coherent sheaves admit local syzygies*

$$0 \longrightarrow \mathcal{O}^{(k_n)} \longrightarrow \mathcal{O}^{(k_{n-1})} \longrightarrow \ldots \longrightarrow \mathcal{O}^{(k_0)} \longrightarrow \mathcal{F} \longrightarrow 0.$$

2. *If X is n-dimensional, the Čech cohomology groups $H^i(X, \mathcal{F})$ vanish for $i > n$. Moreover, if X is compact, they are finite-dimensional complex vector spaces.*

3. *Given an exact sequence*

$$0 \longrightarrow \mathcal{F}' \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}'' \longrightarrow 0,$$

of sheaves of $\mathcal{O}$-modules in which two of the three are coherent, then the remaining one is also.

For proofs of these we refer the reader to [GH] or [Hi].

Let us give some examples of coherent sheaves. The simplest are of course locally free sheaves. As example of a coherent sheaf that is not locally free, let us introduce sheaves of ideals.

A subsheaf $I \subset \mathcal{O}$ that is locally finitely generated is called an *ideal sheaf* or *sheaf of ideals*. These are always coherent. They induce the exact sequence

$$0 \longrightarrow I \longrightarrow \mathcal{O} \longrightarrow \mathcal{O}/I \longrightarrow 0,$$

which implies, by property 3, that $\mathcal{O}/I$ is also coherent.

If I is locally generated by functions $f_1, \ldots, f_m$, i.e. $I = \{f_1, \ldots, f_m\}$, then the *support* of $\mathcal{O}/I$ is defined as

$$\begin{aligned} Z &= \operatorname{supp}(\mathcal{O}/\mathrm{I}) \\ &= \{z \in X : I_z \neq O_z\} \\ &= \{z \in X : f_1(z) = \ldots = f_m(z) = 0\}. \end{aligned}$$

Z is an analytic variety, whose *structure sheaf* $\mathcal{O}_Z = \mathcal{O}/I$ is a sheaf of rings with possible nilpotent elements.

Actually, as hinted by the above, most properties of $\mathcal{O}$-modules carry over to coherent sheaves, i.e. projective resolutions exist becomes local syzygies exist, etc.... We can also *sheafify* Ext and Tor: *Given coherent sheaves* $\mathcal{F}$ *and* $\mathcal{G}$, *we may define sheaves* $Ext^k_{\mathcal{O}}(\mathcal{F},\mathcal{G})$ *and* $Tor^{\mathcal{O}}_k(\mathcal{F},\mathcal{G})$ *with the properties:*

1. $$\begin{cases} Ext^k_{\mathcal{O}}(\mathcal{F},\mathcal{G})_x \cong Ext^k_{\mathcal{O}}(\mathcal{F}_x,\mathcal{G}_x), \\ \\ Tor^{\mathcal{O}}_k(\mathcal{F},\mathcal{G})_x \cong Tor^{\mathcal{O}}_k(\mathcal{F}_x,\mathcal{G}_x); \end{cases}$$

2. $$\begin{cases} Ext^0_{\mathcal{O}}(\mathcal{F},\mathcal{G}) \cong Hom_{\mathcal{O}}(\mathcal{F},\mathcal{G}), \\ \\ Tor^{\mathcal{O}}_0(\mathcal{F},\mathcal{G})_x \cong \mathcal{F}\otimes_{\mathcal{O}}\mathcal{G}; \end{cases}$$

3. *The exact sequences of* Ext *and* Tor *are valid; and*

4. $Ext^*_{\mathcal{O}}(\mathcal{F},\mathcal{G})$ *and* $Tor^{\mathcal{O}}_*(\mathcal{F},\mathcal{G})$ *are coherent sheaves.*

These will be used to define global syzygies and global Ext. If X is compact, then for any coherent sheaf $\mathcal{F}$, there exists a global syzygy

$$0 \longrightarrow \mathcal{E}_n \longrightarrow \mathcal{E}_{n-1} \longrightarrow \dots \longrightarrow \mathcal{E}_1 \longrightarrow \mathcal{E}_0 \longrightarrow \mathcal{F} \longrightarrow 0,$$

where all the $\mathcal{E}_i$ are locally free sheaves.

Global Ext.

Let $\mathcal{F}$ and $\mathcal{G}$ be two coherent sheaves on a compact manifold X. One can therefore find a global syzygy

$$0 \longrightarrow \mathcal{E}_n \longrightarrow \mathcal{E}_{n-1} \longrightarrow \dots \longrightarrow \mathcal{E}_1 \longrightarrow \mathcal{E}_0 \longrightarrow \mathcal{F} \longrightarrow 0$$

for $\mathcal{F}$. We then define global Ext as the hypercohomology associated to the complex of sheaves $Hom_{\mathcal{O}}(\mathcal{E}_.(\mathcal{F}),\mathcal{G})$. It is denoted

$$\mathrm{Ext}(\mathrm{X};\mathcal{F},\mathcal{G}) = \mathbb{H}^{*}(\mathrm{X},\mathrm{Hom}_{\mathcal{O}}(\mathcal{E}_.(\mathcal{F}),\mathcal{G}).$$

Global Ext has functorial properties similar to those of local Ext. To calculate global Ext, we use the spectral cohomology sequence $\{'E_r\}$ with

$$'E_2^{p,q} = H^p(X, Ext^q_{\mathcal{O}}(\mathcal{F}, \mathcal{G})),$$
$$'E_\infty^{p,q} \Rightarrow \mathrm{Ext}^{p+q}(X; \mathcal{F}, \mathcal{G}).$$

This spectral sequence has two very useful properties:

1. *For $\mathcal{E}$ a locally free sheaf on X,*

$$\mathrm{Ext}^q(X; \mathcal{E}, \mathcal{G}) \cong \mathrm{H}^q(X, \mathcal{E}^* \otimes_{\mathcal{O}} \mathcal{G}).$$

In particular, for any coherent sheaf $\mathcal{G}$,

$$\mathrm{Ext}^q(X; \mathcal{O}, \mathcal{G}) \cong \mathrm{H}^q(X, \mathcal{G}).$$

2. *Suppose that $Ext^q_{\mathcal{O}}(\mathcal{F}, \mathcal{G}) = 0$ for $0 \leq q < k$. Then*

$$\mathrm{Ext}^k(X; \mathcal{F}, \mathcal{G}) \cong \mathrm{H}^0(X, \mathrm{Ext}^k_{\mathcal{O}}(\mathcal{F}, \mathcal{G})).$$

Chern classes.

On a smooth quasiprojective variety, we can define the Chern classes of any coherent sheaf. This is because, by a theorem of Serre, every coherent sheaf $\mathcal{F}$ on a quasiprojective possesses a global syzygy

$$0 \longrightarrow \mathcal{E}_n \longrightarrow \mathcal{E}_{n-1} \longrightarrow \ldots \longrightarrow \mathcal{E}_1 \longrightarrow \mathcal{E}_0 \longrightarrow \mathcal{F} \longrightarrow 0,$$

where the $\mathcal{E}_i$ are locally free. We can define the total Chern class of $\mathcal{F}$ by the formula

$$c(\mathcal{F}) = \prod_i c(\mathcal{E}_i)^{(-1)^i}.$$

This definition is independent of the global syzygy and it satisfies the Whitney product formula. An important example is the following:

Proposition 1.1 *Let X be a compact complex manifold and let E be a holomorphic rank 2 vector bundle over X for which there are line bundles L, L' and an exact sequence*

$$0 \longrightarrow L \longrightarrow E \longrightarrow L' \otimes I_Z \longrightarrow 0,$$

where Z is a a subvariety of codimension 2. Then we have

$$c_1(E) = c_1(L) + c_1(L'),$$
$$c_2(E) = c_1(L) \cdot c_1(L') + [Z],$$

where $[Z] \in H^4(X, \mathbb{Z})$ is the cycle defined by Z.

Proof: See [F] p.29. $\square$

1.3 Extensions.

1.3.1 Ext^1 and Extensions - Local case.

We start by considering finitely generated modules over the local ring $\mathcal{O} = \mathbb{C}\{z_1, \ldots, z_n\}$. Let us remark that we are following the presentation of [GH].

Definition 1.5 If M and N are $\mathcal{O}$-modules, we define an *extension of M by N* to be a short exact sequence

$$0 \longrightarrow N \longrightarrow E \longrightarrow M \longrightarrow 0. \tag{1.1}$$

For brevity this will be referred to as "the extension E". Two extensions E and E' are said to be *equivalent* if there exists an isomorphism $E \to E'$ such that the diagram

$$\begin{tikzcd}
0 \arrow[r] & N \arrow[r] \arrow[d, equal] & E \arrow[r] \arrow[d] & M \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & N \arrow[r] & E' \arrow[r] & M \arrow[r] & 0
\end{tikzcd}$$

is commutative.

Extensions of M by N do exist; the simplest example being the *trivial* or *split extension* $M \oplus N$:

$$0 \longrightarrow N \xrightarrow{i} M \oplus N \xrightarrow{p} M \longrightarrow 0,$$

where i and p are the usual inclusion and projection. As a description of all other possible extensions, we have

Lemma 1.4 *There is a bijective correspondence between equivalence classes of extensions and $Ext^1_{\mathcal{O}}(M, N)$, with zero corresponding to the trivial extension.*

Proof: Let E be an extension of M by N. The sequence (1.1) then induces the long exact sequence on Ext

$$\ldots \longrightarrow Hom_{\mathcal{O}}(M, E) \longrightarrow Hom_{\mathcal{O}}(M, M) \xrightarrow{\partial} Ext^1_{\mathcal{O}}(M, N) \ldots .$$

Let 1_M be the identity map from M to itself. We then associate to E the class $\partial(1_M) \in Ext^1_{\mathcal{O}}(M, N)$, thus defining a map from extensions to $Ext^1_{\mathcal{O}}(M, N)$. Let us note that $\partial(1_M)$ represents the obstruction to splitting the sequence (1.1); in particular, if $\partial(1_M) = 0$, it splits, and E is the trivial extension.

Conversely, given a projective resolution of M

$$E_2 \longrightarrow E_1 \longrightarrow E_0 \longrightarrow M \longrightarrow 0,$$

a class in $Ext^1_{\mathcal{O}}(M, N)$ is represented by a map $E_1/E_2 \xrightarrow{j} N$. The data

$$\begin{cases} 0 \longrightarrow E_1/E_2 \xrightarrow{i} E_0 \xrightarrow{\pi} M \longrightarrow 0, \\ \\ \qquad\qquad E_1/E_2 \xrightarrow{j} N, \end{cases}$$

then allows us to construct an extension

$$0 \longrightarrow N \longrightarrow F \longrightarrow M \longrightarrow 0$$

as follows: We define $F = (N \oplus E_0)/\mu(E_1/E_2)$, where $\mu = i \oplus i : E_1/E_2 \to (N \oplus E_0)$. The inclusion $n \mapsto n \oplus (0)$ and the projection $n \oplus e_0 \mapsto \pi(e_0)$ then give us the exact sequence defining F. $\square$

Before giving an example, we would like to introduce the global case.

1.3.2 Ext^1 and Extensions - Global case.

We would now like to consider the case of global extensions. We are again following the presentation in [GH]. In this case, let $\mathcal{F}, \mathcal{G}$, be coherent sheaves on a complex manifold X. An exact sequence

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{E} \longrightarrow \mathcal{G} \longrightarrow 0$$

of sheaves of $\mathcal{O}$-modules - here $\mathcal{G}$ must be coherent – is called a *global extension* of $\mathcal{G}$ by $\mathcal{F}$. The equivalence relation and trivial extension are as in the local case. One might again think that the set of equivalence classes of extensions would be in bijective correspondence with $H^0(X, Ext^1_{\mathcal{O}}(\mathcal{G}, \mathcal{F}))$. But this is not the case for the following reason:

Suppose that we have a global section of $H^0(X, Ext^1_{\mathcal{O}}(\mathcal{G}, \mathcal{F}))$. We then choose an open covering $\mathcal{U} = \{\mathcal{U}_\alpha\}$ that is sufficiently fine and consider the local extensions

$$0 \longrightarrow \mathcal{F}|_{\mathcal{U}_\alpha} \longrightarrow \mathcal{E}_\alpha \longrightarrow \mathcal{G}|_{\mathcal{U}_\alpha} \longrightarrow 0.$$

On double covers, $\mathcal{U}_\alpha \cap \mathcal{U}_\beta$, the extensions $\mathcal{E}_\alpha$ and $\mathcal{E}_\beta$ must be equivalent. Hence, there exits an isomorphism $\varphi_{\alpha\beta}$ making the diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{F}|_{\mathcal{U}_\alpha \cap \mathcal{U}_\beta} \arrow[r] \arrow[d, equal] & \mathcal{E}_\alpha|_{\mathcal{U}_\alpha \cap \mathcal{U}_\beta} \arrow[r] \arrow[d, "\varphi_{\alpha\beta}"] & \mathcal{G}|_{\mathcal{U}_\alpha \cap \mathcal{U}_\beta} \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \mathcal{F}|_{\mathcal{U}_\alpha \cap \mathcal{U}_\beta} \arrow[r] & \mathcal{E}_\beta|_{\mathcal{U}_\alpha \cap \mathcal{U}_\beta} \arrow[r] & \mathcal{G}|_{\mathcal{U}_\alpha \cap \mathcal{U}_\beta} \arrow[r] & 0
\end{tikzcd}$$

commute. But the $\varphi_{\alpha\beta}$ may not satisfy the cocycle relations. Indeed, on triple intersections $\mathcal{U}_\alpha \cap \mathcal{U}_\beta \cap \mathcal{U}_\gamma$, the triangle

$$\begin{tikzcd}
& \mathcal{E}_\alpha \arrow[dl, "\varphi_{\alpha\beta}"'] \arrow[dr, "\varphi_{\gamma\beta}"] & \\
\mathcal{E}_\beta \arrow[rr, "\varphi_{\beta\gamma}"'] & & \mathcal{E}_\gamma
\end{tikzcd}$$

may not be commutative. The isomorphisms $\varphi_{\alpha\beta}$ may therefore not patch up to give a global extension. What is true is

Lemma 1.5 *The equivalence classes of global extensions of $\mathcal{G}$ by $\mathcal{F}$ are in bijective correspondence with* $\mathrm{Ext}^1(\mathrm{X}; \mathcal{G}, \mathcal{F})$.

Proof: For a complete proof, we refer the reader to [GH]. We shall however give an outline in order to describe this bijection. We proceed similarly to the local case, only, this time using the exact sequences induced by global Ext's. Let $1_{\mathcal{G}}$ be the identity map from $\mathcal{G}$ to itself. We therefore associate to each extension its image $\partial(1_{\mathcal{G}})$ by the coboundary map induced by the exact sequence

$$\mathrm{Ext}^0(\mathrm{X};\mathcal{G},\mathcal{F}) \longrightarrow \mathrm{Ext}^0(\mathrm{X};\mathcal{G},\mathcal{F}) \xrightarrow{\partial} \mathrm{Ext}^1(\mathrm{X};\mathcal{G},\mathcal{F}) \longrightarrow \ldots.$$

As in the local case, $\partial(1_{\mathcal{G}})$ is the obstruction to splitting the sequence

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{E} \longrightarrow \mathcal{G} \longrightarrow 0.$$

For the converse, we again start with a locally free resolution of the coherent sheaf $\mathcal{G}$:

$$\mathcal{E}_.(\mathcal{G}):\ \ldots \longrightarrow \mathcal{E}_2 \longrightarrow \mathcal{E}_1 \longrightarrow \mathcal{E}_0 \longrightarrow \mathcal{G} \longrightarrow 0;$$

and we choose a covering $\mathcal{U} = \{\mathcal{U}_\alpha\}$ that is sufficiently fine so that each element $e \in \mathrm{Ext}^1(\mathrm{X};\mathcal{G},\mathcal{F})$ is given by a cocycle in the hypercohomology group

$$\mathbb{H}^1(\mathcal{U}, Hom_{\mathcal{O}}(\mathcal{E}_.(\mathcal{G}),\mathcal{F})).$$

By studying the long exact sequence on hypercohomology, we see that e is given by $\varphi \oplus \eta$, where

$$\varphi = \{\varphi_\alpha\} \quad \text{with } \varphi_\alpha \in \mathrm{H}^0(\mathcal{U}_\alpha, \mathrm{Hom}_{\mathcal{O}}(\mathcal{E}_1,\mathcal{F})).$$
$$\eta = \{\eta_{\alpha\beta}\} \quad \text{with } \eta_{\alpha\beta} \in \mathrm{H}^0(\mathcal{U}_\alpha \cap \mathcal{U}_\beta, \mathrm{Hom}_{\mathcal{O}}(\mathcal{E}_0,\mathcal{F})).$$

Now the φ_α define extensions

$$0 \longrightarrow \mathcal{F}|_{\mathcal{U}_\alpha} \longrightarrow \mathcal{E}_\alpha \longrightarrow \mathcal{G}|_{\mathcal{U}_\alpha} \longrightarrow 0,$$

and the $\eta_{\alpha\beta}$ give a rule for patching them up over double intersections that satisfies the cocycle relations. This then defines a global extension. □

We will now present two very important examples that we will often use in the upcoming sections.

1.3.3 Extensions of I by $\mathcal{O}$.

Suppose that $\mathcal{O} = \mathbb{C}\{z_1, z_2\}$ is the local ring in two variables and $I = \{f_1, f_2\}$ is a regular ideal. As we have seen in the previous section, we have the exact sequence

$$0 \longrightarrow I \longrightarrow \mathcal{O} \longrightarrow \mathcal{O}/I \longrightarrow 0,$$

which, along with the computations of Ext's gave us

$$Ext^1_{\mathcal{O}}(I, \mathcal{O}) \cong Ext^2_{\mathcal{O}}(\mathcal{O}/I, \mathcal{O}) \cong \mathcal{O}/I.$$

Let us give a complete description of the equivalence classes of extensions of I by $\mathcal{O}$. We begin by stating a theorem due to Serre [Sr]:

Theorem 1.2 *Let F be the extension given by $t \in \mathbb{C}$. Then*

$$F \textit{ is locally free} \Leftrightarrow t \neq 0.$$

If $t = 0$, then F is the trivial extension. And if $t \neq 0$, it is the extension

$$0 \longrightarrow \mathcal{O} \xrightarrow{(-f_2, f_1)} \mathcal{O} \oplus \mathcal{O} \xrightarrow{t^{-1}[f_1\ f_2]} I \longrightarrow 0.$$

In particular, if $t = 1$, we get the Koszul complex.

These extensions can also be described as the cokernels of the maps

$$\begin{array}{ccc} \mathcal{O} & \rightarrow & \mathcal{O} \oplus \mathcal{O} \oplus \mathcal{O} \\ 1 & \mapsto & (t, -f_2, f_1) \end{array}$$

More explicitly, the trivial extension is given by

$$0 \longrightarrow \mathcal{O} \xrightarrow{(0, -f_2, f_1)} \mathcal{O} \oplus \mathcal{O} \oplus \mathcal{O} \xrightarrow{\begin{bmatrix} 1 & 0 & 0 \\ 0 & f_1 & f_2 \end{bmatrix}} \mathcal{O} \oplus I \longrightarrow 0;$$

and the extension corresponding to $t \neq 0$ comes from

$$0 \longrightarrow \mathcal{O} \xrightarrow{(t, -f_2, f_1)} \mathcal{O} \oplus \mathcal{O} \oplus \mathcal{O} \xrightarrow{\begin{bmatrix} f_2 & t & 0 \\ -f_1 & 0 & t \end{bmatrix}} \mathcal{O} \oplus \mathcal{O} \longrightarrow 0.$$

1.3.4 Extensions of vector bundles.

We will study in detail the special case of vector bundles. This presentation follows that of [DK]. The definition and properties that we stated in the general case can be written as

Definition 1.6 Let X be a complex manifold and

$$0 \longrightarrow E' \xrightarrow{i} E \xrightarrow{p} E'' \longrightarrow 0, \tag{1.2}$$

be an exact sequence of vector bundles on X. E is then said to be an *extension* of E'' by E'. The *trivial* or *split extension* is $E' \oplus E''$. Moreover, if the ends stay fixed, any two such sequences are said to be *equivalent* if there is a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & E' \arrow[r, "i_1"] \arrow[d, equal] & E_1 \arrow[r, "p_1"] \arrow[d] & E'' \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & E' \arrow[r, "i_2"] & E_2 \arrow[r, "p_2"] & E'' \arrow[r] & 0. \end{tikzcd} \tag{1.3}$$

In this case, since E'' is locally free,

$$\mathrm{Ext}^1(\mathcal{H}; \mathrm{E}'', \mathrm{E}') = \mathrm{H}^1(\mathcal{H}, (\mathrm{E}'')^* \otimes \mathrm{E}'),$$

and by lemma 1.5 we know that

There is a one-to-one correspondence $(E, i, p) \mapsto \partial(1)$ between equivalence classes of extensions of E'' by E' and the cohomology group $H^1(X, Hom_{\mathcal{O}}(E'', E'))$.

In this case, it is very useful to us illustrate this using Čech cohomology. We choose a cover $X = \bigcup U_\alpha$ by open sets over each of which the sequence

$$0 \longrightarrow E' \xrightarrow{i} E \xrightarrow{p} E'' \longrightarrow 0$$

splits. Let

$$j_\alpha : E|_{U_\alpha} \longrightarrow E'|_{U_\alpha} \oplus E''|_{U_\alpha}$$

be the isomorphisms which split the sequence and are compatible with i and p. By this we mean that, over each U_α, the diagram

$$\begin{tikzcd} 0 \arrow[r] & E'|_{U_\alpha} \arrow[r, "i"] \arrow[d, equal] & E|_{U_\alpha} \arrow[r, "p"] \arrow[d, "j_\alpha"] & E''|_{U_\alpha} \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & E'|_{U_\alpha} \arrow[r, "\iota"] & E'|_{U_\alpha} \oplus E''|_{U_\alpha} \arrow[r, "\pi"] & E''|_{U_\alpha} \arrow[r] & 0 \end{tikzcd}$$

commutes, where ι and π are the usual inclusion into the first factor and projection onto the second factor.

On each overlap $U_\alpha \cap U_\beta$, we can then write

$$j_\alpha = j_\beta a_{\alpha\beta},$$

where $a_{\alpha\beta}$ is an automorphism of $E' \oplus E''$ over $U_\alpha \cap U_\beta$ of the form

$$a_{\alpha\beta} = 1 + \begin{pmatrix} 0 & \chi_{\alpha\beta} \\ 0 & 0 \end{pmatrix}.$$

$\chi_{\alpha\beta}$ can then be interpreted as being a holomorphic map from $E''|_{U_\alpha \cap U_\beta}$ to $E'|_{U_\alpha \cap U_\beta}$. A straightforward computation shows that the cocycle relation $\chi_{\gamma\beta} = \chi_{\alpha\beta} + \chi_{\gamma\alpha}$ holds on triple intersections $U_\alpha \cap U_\beta \cap U_\gamma$. The extension class $\partial(1)$ is therefore represented by the Čech cocycle $(\chi_{\alpha\beta})$ on this cover.

Remarks: (i) Suppose that $\{\lambda_\alpha\}$ and $\{\mu_\alpha\}$ are trivialisations of E' and E'', respectively, on the open cover $\{U_\alpha\}$. They induce transition functions $g'_{\alpha\beta}$ and $g''_{\alpha\beta}$. It is then not difficult to verify that E has transition matrices

$$G_{\alpha\beta} = \begin{bmatrix} g'_{\alpha\beta} & \lambda_\alpha \chi_{\alpha\beta} \mu_\beta^{-1} \\ 0 & g''_{\alpha\beta} \end{bmatrix}. \tag{1.4}$$

(ii) Let L_1, L_2 be line bundles on X. Suppose that E,E' are both extensions of L_2 by L_1, given by the classes $\chi, \chi' \in H^1(T, L_2^* \otimes L_1)$, respectively. Since $H^0(X, Aut(L_1)) = H^0(X, Aut(L_2)) = \mathbb{C}^*$, it follows from the definitions that

$$E \text{ and } E' \text{ are isomorphic} \Leftrightarrow \chi' = b\chi \text{ for some } b \in \mathbb{C}^*.$$

Example: Let us give an illustration of the above discussion in the case of $X = \mathbb{P}^1$. Let $\mathcal{O}(1) = \mathcal{O}_{\mathbb{P}^1}(p)$ be the hyperplane bundle, where p is simply a point in $\mathbb{P}^1$. Its dual is then $\mathcal{O}(-1)$. We would like to find an extension of $\mathcal{O}(1)$ by $\mathcal{O}(-1)$. One has

$$H^1(\mathbb{P}^1, Hom_{\mathcal{O}_{\mathbb{P}^1}}(\mathcal{O}(1), \mathcal{O}(-1))) = H^1(\mathbb{P}^1, \mathcal{O}(-2)) = \mathbb{C}.$$

There then exists at least one non-trivial extension of $\mathcal{O}(1)$ by $\mathcal{O}(-1)$. Furthermore, $H^0(\mathbb{P}^1, \mathcal{O}(1)) = \mathbb{C}^2$. As $\mathcal{H}om_{\mathcal{O}}(\mathcal{O}(-1), \mathcal{O}) \cong \mathcal{O}(1)$, the trivial bundle $\mathcal{O} \oplus \mathcal{O}$ can therefore be expressed as such an extension:

$$0 \longrightarrow \mathcal{O}(-1) \longrightarrow \mathcal{O} \oplus \mathcal{O} \longrightarrow \mathcal{O}(1) \longrightarrow 0. \tag{1.5}$$

Let us cover $\mathbb{P}^1$ by the two coordinate patches $U_0 =$ (disc about the origin $z = 0$), and $U_1 = \mathbb{P}^1 -$ (origin). The sequence (1.5) therefore splits on each U_i. We can assume, without loss of generality, that $p =$ (origin). As the divisor p is given on U_0 by z and on U_1 by 1, the transition matrix of $\mathcal{O}(1)$ from U_0 to U_1 is $g_{10}(z) = 1/z$. Let us choose $\{1, 1/z\}$ as a basis for $H^0(\mathbb{P}^1, \mathcal{O}(1)) = \mathbb{C}^2$. This basis then gives us the following two global sections of $\mathcal{H}om_{\mathcal{O}}(\mathcal{O}(-1), \mathcal{O})$:

$$r(z) = \begin{cases} r_0(z) = z, \text{ on } U_0, \\ r_1(z) = 1, \text{ on } U_1; \end{cases} \quad \text{and } \mathrm{s(z)} = \begin{cases} s_0(z) = 1, \text{ on } U_0, \\ s_1(z) = 1/z, \text{ on } U_1. \end{cases}$$

As r and s have no common zeroes, $1 \mapsto (r(z), s(z))$ defines an injective bundle map from $\mathcal{O}(-1)$ to $\mathcal{O} \oplus \mathcal{O}$, and this induces the following commutative diagram on $U_0 \cap U_1$

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{O}(-1) \arrow[r, "{(z,1)}"] \arrow[d, "z"'] & \mathcal{O} \oplus \mathcal{O} \arrow[r, "{[1\ \ -z]}"] \arrow[d, "{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}"] & \mathcal{O}(1) \arrow[r] \arrow[d, "1/z"] & 0 \\
0 \arrow[r] & \mathcal{O}(-1) \arrow[r, "{(1,1/z)}"'] & \mathcal{O} \oplus \mathcal{O} \arrow[r, "{[1/z\ \ -1]}"'] & \mathcal{O}(1) \arrow[r] & 0
\end{tikzcd} \tag{1.6}$$

Method 1. Suppose that the extension (1.5) is given in $H^1(\mathbb{P}^1, \mathcal{O}(-2))$ by the cocycle χ. Let us then find a representative for χ. Referring to (1.6), it is easy to see that the map $\phi_0 : \mathcal{O}(1) \to \mathcal{O} \oplus \mathcal{O}$ defined by $1 \mapsto (1, 0)$ gives a splitting of (1.5) on U_0. Similarly, the map $\phi_1 : \mathcal{O}(1) \to \mathcal{O} \oplus \mathcal{O}$ defined by $1 \mapsto (0, -1)$ gives a splitting on U_1. If we consider $U_0 \cap U_1$ as a subset of U_0, we have

$$\chi(1, z) = (1/z)\phi_1 - \phi_0 \quad \Leftrightarrow \quad \chi = -1/z.$$

Let us fix the trivialisation of $\mathcal{O}(1)$ on U_0 to be 1. Referring to (1.4), we then get the transition matrix

$$G_{10} = \begin{bmatrix} z & z\,\chi \\ 0 & 1/z \end{bmatrix} = \begin{bmatrix} z & -1 \\ 0 & 1/z \end{bmatrix}.$$

Method 2. There is another way of finding the transition matrix of (1.5). On $U_0 \cap U_1$, (1.6) induces the following commutative diagram

$$\begin{tikzcd}
\mathcal{O}(-1) \arrow[r, "{(1,z,1)}"] \arrow[d, "z"'] & \mathcal{O}(-1)\oplus(\mathcal{O}\oplus\mathcal{O}) \arrow[rr, "{\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -z \end{bmatrix}}"] \arrow[d, "{\begin{pmatrix} z & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}}"'] & & (\mathcal{O}\oplus\mathcal{O}) \arrow[d, "{\begin{pmatrix} z & \gamma \\ 0 & 1/z \end{pmatrix}}"'] \\
\mathcal{O}(-1) \arrow[r, "{(1,1,1/z)}"'] & \mathcal{O}(-1)\oplus(\mathcal{O}\oplus\mathcal{O}) \arrow[rr, "{\begin{bmatrix} 1 & -1 & 0 \\ 0 & 1/z & -1 \end{bmatrix}}"'] & & (\mathcal{O}\oplus\mathcal{O})
\end{tikzcd}$$

The rows are exact, and γ must satisfy the constrains of commutativity. Solving, we find that $\gamma = -1$, thus verifying that the transition matrix is

$$G_{10} = \begin{bmatrix} z & -1 \\ 0 & 1/z \end{bmatrix}.$$

Let us note that this second construction has the advantage of not requiring fixed trivialisations. We will use it in the next chapter to find the transition matrices of extensions over $D \times T$, where D is a disc and T an elliptic curve.

1.4 The Serre Construction.

In this subsection, we will assume that X is a complex manifold, and E is a holomorphic rank 2 vector bundle on X. Let L be a line bundle on X, and ϕ be a non zero holomorphic bundle map from L to E. We do not assume that this map has constant rank. There may exist points p where $\phi_p = 0$, and ϕ is then not necessarily an injective bundle map. ϕ can also be considered as a $\mathcal{O}$-homomorphism of sheaves from $\mathcal{O}(L)$ to $\mathcal{O}(E)$, and $\phi \neq 0$ if and only if it is an injective map of coherent sheaves. If we denote $\mathcal{O}(F), \mathcal{O}(E)$ also by F, E, we then have the following well known result

Proposition 1.2 *Let X be a complex manifold and let $\phi : L \to E$ be a holomorphic bundle map from the holomorphic line bundle L to the holomorphic rank two bundle E. Then*

(i) There exists a largest effective divisor D on X such that the map $L \to E$ factors through the inclusion $L \subseteq L \otimes \mathcal{O}_X(D)$.

(ii) Suppose that the divisor D above is zero. Then there is an exact sequence of coherent sheaves

$$0 \to L \to E \to L' \otimes I_Z \to 0,$$

where $L' = \det E \otimes L^{-1}$ is a holomorphic line bundle and Z is a codimension two local complete intersection subscheme of X. Moreover, if X is compact,

$$c_2(E) = c_1(L) \cdot c_1(L') + [Z],$$

where $[Z]$ is the cohomology class Poincaré dual to the algebraic cycle associated to Z.

(iii) In the notation of (i)*, the divisor $D = 0$ if and only if the set of p where $\phi_p = 0$ has codimension at least two in X, or equivalently if and only if the coherent sheaf E/L is torsion free.*

Proof: See [FM].

The Serre construction.

The Serre method consists in reversing this construction. If we start with line bundles L, L' and a locally complete intersection 2-codimensional analytic subspace Z of X, we may ask whether there exist extensions of $L' \otimes I_Z$ by L

$$0 \longrightarrow L \longrightarrow E \longrightarrow L' \otimes I_Z \longrightarrow 0,$$

such that E is locally free. This method works in the case of any complex manifold. But we shall restrict ourselves to the case of surfaces. For a more general discussion, we refer the reader to [Br].

As we have seen in our discussion of global extensions, the answer to our question lies in $\mathrm{Ext}^1(X; L' \otimes I_Z, L)$. To compute this group, we use the spectral sequence with E_2 term

$$E_2^{p,q} = H^p(X; Ext_{\mathcal{O}}(L' \otimes I_Z, L)) \Rightarrow \mathrm{Ext}^{p+q}(X; L' \otimes I_Z, L).$$

By lemma 1.2, we locally have the following situation:

(i) $Hom_{\mathcal{O}}(I_Z, \mathcal{O}) \cong \mathcal{O}$;

(ii) $Hom_{\mathcal{O}}(I_Z, I_Z) \cong \mathcal{O}$;

(iii) $Ext^1_{\mathcal{O}}(I_Z, \mathcal{O}) \cong Ext^2_{\mathcal{O}}(\mathcal{O}/I_Z, \mathcal{O}) \cong \mathcal{O}/I_Z$;

(iv) $Ext^k_{\mathcal{O}}(I_Z, \mathcal{O}) = 0$ for $k \geq 2$.

Globally, the inclusion $I_Z \hookrightarrow \mathcal{O}$ induces

$$Hom_{\mathcal{O}}(L' \otimes I_Z, L) \cong Hom_{\mathcal{O}}(L', L) = (L')^{-1} \otimes L.$$

Moreover, as $Ext^1_{\mathcal{O}}(\mathcal{O}, \mathcal{O}) = 0$ and $Ext^1_{\mathcal{O}}(I_Z, \mathcal{O}) \cong \mathcal{O}/I_Z$, we see that $Ext^1_{\mathcal{O}}(L' \otimes I_Z, L)$ is a skyscraper sheaf with fibre $\mathcal{O}/I_Z$ supported on Z. We can now replace the Ext spectral sequence by

$$0 \longrightarrow H^1(X, (L')^{-1} \otimes L) \longrightarrow \mathrm{Ext}^1(X; L' \otimes I_Z, L) \longrightarrow$$

$$\longrightarrow H^0(X, Ext^1_{\mathcal{O}}(L' \otimes I_Z, L)) \longrightarrow H^2(X, (L')^{-1} \otimes L).$$

Let us consider the extension E corresponding to the element $e \in \mathrm{Ext}^1(X; L' \otimes I_Z, L)$. Its image in $H^0(X, Ext^1_{\mathcal{O}}(L' \otimes I_Z, L))$ is also denoted by e. Serre's Theorem (1.2) then tells us that E is locally free if and only if e is an invertible element of $Ext^1_{\mathcal{O}}(X, L' \otimes I_Z, L)$, i.e. if it generates the sheaf $Ext^1_{\mathcal{O}}(X, L' \otimes I_Z, L)$ or if the natural map

$$\mathcal{O} \longrightarrow Ext^1_{\mathcal{O}}(L' \otimes I_Z, L)$$

defined by e is onto. If $H^2(X, (L')^{-1} \otimes L) = 0$, then there must exist at least one such element. This is why we have

Theorem 1.3 *Suppose in the above situation that X is a surface and that $H^2(X, (L')^{-1} \otimes L) = 0$. Then there exist locally free extensions E of $L' \otimes I_Z$ by L.*

1.5 Deformation of sheaves.

As we shall see in the next section, deformation theory is a very useful tool in the study of the local properties of moduli spaces. We give a detailed description in the cases of sheaves over local rings, and vector bundles. The general case is then simply a combination of those two. We start with

Definition 1.7 Let X be a complex space and $\mathcal{F}$ be any coherent sheaf on X. We can then define the following:

1. A *deformation* of $\mathcal{F}$ is a quadruple $\mathcal{G}_T = (\mathcal{G}, T, t_0, \alpha)$ where
   (i) (T, t_0) is the germ of a complex space with representative T,
   (ii) $\mathcal{G}$ is a coherent T-flat sheaf on $X \times T$, such that $\alpha : \mathcal{G}(t_0) \to \mathcal{F}$ is an isomorphism of $\mathcal{O}_X$-modules on X.
2. If $\mathcal{G}'_{T'} = (\mathcal{G}', T', t'_0, \alpha')$ is another deformation of $\mathcal{F}$, then a *morphism of deformations* $\mathcal{G}_T \to \mathcal{G}'_{T'}$ is a pair (ϕ, f) where $f : (T', t'_0) \to (T, t_0)$ is a morphism of germs and $\phi : (\mathrm{id} \times \mathrm{f})^* \mathcal{G} \to \mathcal{G}'$ is an isomorphism on $X \times T'$ such that $\alpha = \alpha' \circ j^*_{t'_0}(\phi)$.
3. The deformation $\mathcal{G}_T$ is called *complete* if for any other deformation $\mathcal{G}'_{T'}$ there exists a morphism $(\phi, f) : \mathcal{G}_T \to \mathcal{G}'_{T'}$. If in addition the tangential map $Tf : T_{t'_0}T' \to T_{t_0}T$ is the same for all such morphisms, the deformation $\mathcal{G}_T$ is called *semi-universal* or *versal*.

   Deformation theory basically gives us the following result:
   For any coherent sheaf $\mathcal{F}$ on a variety X, the global deformations of $\mathcal{F}$ are

given by $\mathrm{Ext}^1(\mathrm{X}; \mathcal{F}, \mathcal{F})$. *The obstruction to extending the deformations to any order is an element of* $\mathrm{Ext}^2(\mathrm{X}; \mathcal{F}, \mathcal{F})$.

1.5.1 Local case.

We begin by examining the local case. The results of this section are due to Trautmann, and were presented in the more general setting where X is a Stein space.

We restrict ourselves to the Stein space $X = \mathbb{C}^n$. In the remainder of this section, X will always denote $\mathbb{C}^n$. Let us start by giving some notations.

1. Let $\mathcal{O}^p \xrightarrow{\alpha} \mathcal{O}^q$ be an $\mathcal{O}$- homomorphism of $\mathcal{O}$-modules. We can then identify α with a $p \times q$-matrix with entries in $\mathcal{O}$. Moreover, the space of holomorphic $p \times q$-matrices is identified with $\mathcal{O}^{pq}$.

2. Let M, N be $\mathcal{O}$-modules, and suppose that they have the following projective resolutions:

$$\ldots \longrightarrow \mathcal{O}^{p_n} \xrightarrow[A_{n-1}]{} \mathcal{O}^{p_{n-1}} \longrightarrow \ldots \xrightarrow[A_0]{} \mathcal{O}^{p_0} \longrightarrow M \longrightarrow 0,$$

and

$$\ldots \longrightarrow \mathcal{O}^{q_1} \xrightarrow[B]{} \mathcal{O}^{q_0} \xrightarrow[S]{} N \longrightarrow 0.$$

Using the projective resolution of M, we see that $Ext^i_{\mathcal{O}}(M, N)$ is given by the quotient Z^i/B^i, where

$$Z^i = \{\alpha : \mathcal{O}^{p_i} \to N \mid \alpha \circ A_i = 0\},$$

and

$$B^i = \{\alpha : \mathcal{O}^{p_i} \to N \mid \alpha = \beta \circ A_{i-1} \text{ forsome } \beta : \mathcal{O}^{p_{i-1}} \to \mathrm{N}\}.$$

Let us remark that every homomorphism $\mathcal{O}^p \xrightarrow{\alpha} N$ factors through S, i.e. there exists an $F \in \mathcal{O}^{pq_0}$, such that the diagram

$$\begin{tikzcd} & \mathcal{O}^{q_0} \arrow[d, "S"] \\ \mathcal{O}^p \arrow[ur, "\exists F"] \arrow[r, "\alpha"'] & N \end{tikzcd}$$

commutes.

Let us note that projective resolutions always exit, in the case of $\mathcal{O}$-modules. It turns out that deformations of $\mathcal{O}$-modules actually arise from deformations of their resolutions. We have the following lemma:

Lemma 1.6 *Let*

$$\ldots \longrightarrow \mathcal{O}_X^r \xrightarrow[B_0]{} \mathcal{O}_X^q \xrightarrow[A_0]{} \mathcal{O}_X^p \longrightarrow \mathcal{F} \longrightarrow 0$$

be a resolution of M. If N_T is any deformation, then N_T has a resolution

$$\ldots \longrightarrow \mathcal{O}_{X\times T}^r \xrightarrow[B_0+B]{} \mathcal{O}_{X\times T}^q \xrightarrow[A_0+A]{} \mathcal{O}_{X\times T}^p \longrightarrow N_T \longrightarrow 0,$$

where B, A are holomorphic and vanish on $X \times \{t_0\}$.

Proof: It follows from the flatness of N_T and Nakayama's Lemma. For details see [Tr]. $\square$

The converse is also true, as shown by the following result, due to Douady:

Lemma 1.7 *Let A and B be holomorphic matrices on $X \times T$ with $B \circ A = 0$ and let $N :=$ Coker(A), such that we have the complex*

$$\mathcal{O}_{X\times T}^r \xrightarrow[B]{} \mathcal{O}_{X\times T}^q \xrightarrow[A]{} \mathcal{O}_{X\times T}^p \longrightarrow N \longrightarrow 0, \tag{1.7}$$

on $X \times T$. If the induced sequence

$$\ldots \longrightarrow \mathcal{O}_X^r \xrightarrow[B]{} \mathcal{O}_X^q \xrightarrow[A]{} \mathcal{O}_X^p$$

is exact on X, then (1.7) is exact and N defines a flat deformation of $N(t_0)|_X$.

Deformations of order 1.

Let $\mathbb{C}[\epsilon]$ be the ring $\mathbb{C}[t]/(t)^2$, where ϵ is the class of t and let $0[\epsilon]$ be Spec$\mathbb{C}[\epsilon]$. The space $X[\epsilon] = X \times 0[\epsilon]$ has the structure sheaf $\mathcal{O}_X[\epsilon] = \mathcal{O}_X \otimes_{\mathbb{C}} \mathbb{C}[\epsilon]$, and any $f \in \Gamma(X, \mathcal{O}_X[\epsilon])$ has a unique decomposition $f = f_0 + \epsilon f_\epsilon$ with $f_0, f_\epsilon \in \mathcal{O}$.

Again, let

$$\ldots \longrightarrow \mathcal{O}_X^r \xrightarrow[B_0]{} \mathcal{O}_X^q \xrightarrow[A_0]{} \mathcal{O}_X^p \xrightarrow[S_0]{} M \longrightarrow 0$$

be a resolution of M. If N is a deformation of M over $0[\epsilon]$, by lemma 1.6, it has a resolution

$$\cdots \longrightarrow \mathcal{O}^r_{X[\epsilon]} \xrightarrow[B_0+\epsilon B_1]{} \mathcal{O}^q_{X[\epsilon]} \xrightarrow[A_0+\epsilon A_1]{} \mathcal{O}^p_{X[\epsilon]} \longrightarrow N \longrightarrow 0, \tag{1.8}$$

where the entries of B_1, A_1 are in $\mathcal{O}$.

Let us start by showing that the matrix A_1 defines an element of $Ext^1_{\mathcal{O}}(M, M)$. Let us note that in this case

$$Z^1 = \{\alpha : \mathcal{O}^q \to M \mid \alpha \circ B_0 = 0\},$$

and

$$B^1 = \{\alpha : \mathcal{O}^q \to M \mid \alpha = \beta \circ A_0 \text{ forsome } \beta : \mathcal{O}^p \to \mathrm{M}\}.$$

By exactness of (1.8), $(A_0 + \epsilon A_1)(B_0 + \epsilon B_1) = 0$, and, as $\epsilon^2 = 0$, this is equivalent to $A_0B_1 + A_1B_0 = 0$. It then follows that S_0A_1 defines a homomorphism $\alpha : \mathcal{O}^q_X \to M$, such that $(S_0A_1)B_0 = -(S_0A_0)B_1 = 0$. Therefore S_0A_1 defines a class $[\alpha] \in Ext^1_{\mathcal{O}}(M, M)$. Conversely, since any map $\mathcal{O}^q_X \to M$ factors through S_0, if $[\alpha] \in Ext^1_{\mathcal{O}}(M, M)$, it is represented by S_0A_1, where A_1 is a suitable holomorphic matrix. Similarly, since $S_0A_1B_0 = 0$ and $S_0A_0 = 0$, we have that $S_0(A_0Q + A_1B_0) = 0$ for any $Q \in \mathcal{O}^{rq}$, implying that there is a B_1 with $A_0B_1 + A_1B_0 = 0$. Thus $(A_0 + \epsilon A_1)(B_0 + \epsilon B_1) = 0$, and by lemma 1.7, the matrices $B_0 + \epsilon B_1, A_0 + \epsilon A_1$ define a flat deformation of M. Let us note that if we choose a different representative of the class $[\alpha]$, we obtain a deformation that is equivalent to the one given by S_0A_1. This can be shown by arguments similar to the ones above.

Obstructions for extending deformations.

Let us now suppose that we want to extend this to a deformation of order 2, i.e., a deformation on $X \times T'$, where T' is $\mathrm{Spec}(\mathbb{C}[t]/(t)^3)$. We will see that the obstruction for such an extension is an element in $Ext^2_{\mathcal{O}}(M, M)$. In order to describe this element as a 2-cocycle, we need to specify a third map C_0 in the resolution of M:

$$\cdots \xrightarrow[C_0]{} \mathcal{O}^r_X \xrightarrow[B_0]{} \mathcal{O}^q_X \xrightarrow[A_0]{} \mathcal{O}^p_X \longrightarrow M \longrightarrow 0.$$

The set of 2-cocycles is then $Z^2 = \{\alpha : \mathcal{O}^r \to M \mid \alpha \circ C_0 = 0\}$, and the 2-coboundaries are

$$B^2 = \{\alpha : \mathcal{O}^r \to M \mid \alpha = \beta \circ B_0 \text{ forsome } \beta : \mathcal{O}^q \to \mathrm{M}\}.$$

As above, any deformation N_T of order 1 has a resolution

$$\cdots \xrightarrow[C]{} \mathcal{O}^r_{X[\epsilon]} \xrightarrow[B]{} \mathcal{O}^q_{X[\epsilon]} \xrightarrow[A]{} \mathcal{O}^p_{X[\epsilon]} \longrightarrow N_T \longrightarrow 0, \tag{1.9}$$

where $C = C_0 + \epsilon C_1$, $B = B_0 + \epsilon B_1$, and $A = A_0 + \epsilon A_1$.

Let C', B', A' be holomorphic matrices on $X \times T'$ inducing C, B, A on $X \times T$. In the following C', B', A' will be fixed. For suitable choices of holomorphic matrices C_2, B_2, A_2, we can then express C', B', A' as $C' = C_0 + \epsilon C_1 + \epsilon^2 C_2$, $B' = B_0 + \epsilon B_1 + \epsilon^2 B_2$, and $A' = A_0 + \epsilon A_1 + \epsilon^2 A_2$. If these matrices are the result of a deformation of order 2, then, by exactness of the resolution they correspond to, $B' \circ C' = A' \circ B' = 0$. Conversely, by lemma 1.7 (Douady), if $A' \circ B' = 0$, then they define a deformation $N_{T'} := \mathrm{Coker(A')}$ of order 2. We therefore have to find out whether or not $A' \circ B' = 0$.

As $\epsilon^i = 0$ for $i \geq 3$, and, by exactness of (1.9), $(A_0 + \epsilon A_1)(B_0 + \epsilon B_1) = 0$, we have that $A' \circ B' = \epsilon^2 R$, where $R = A_0 B_2 + A_1 B_1 + A_2 B_0$. It is easy to verify that $(S_0 R) C_0 = 0$. We then have a homomorphism $\rho : \mathcal{O}^r \to M$ defined by $S_0 R$, such that $\rho \circ C_0 = 0$, thus defining a class $[\rho] \in Ext^2_{\mathcal{O}}(M, M)$. We then have

Lemma 1.8 *There exists an extension $N_{T'}$ of N_T over $X \times T'$ if and only if $[\rho] = 0$. Therefore $Ext^2_{\mathcal{O}}(M, M)$ is the group of obstructions for extending N_T to T'.*

Proof: If the matrices C', B', A' are the result of a deformation of order 2, then, by the above discussion, $A' \circ B' = 0$. As $A' \circ B' = \epsilon^2 R$, we then have $[\rho] = 0$. Conversely, if $[\rho] = 0$, then $\rho = \beta \circ B_0$, for some $\beta : \mathcal{O}^q \to M$. As we have seen in the previous section, this implies that $R = A_0 B_3 + A_3 B_0$, for some holomorphic matrices B_3, A_3. If we set $\tilde{B}' = B' + \epsilon^2 B_3$, and $\tilde{A}' = A' + \epsilon^2 A_3$, a simple calculation gives $\tilde{A}' \circ \tilde{B}' = 0$, and, by Douady, $N_{T'} := \mathrm{Coker}(\tilde{\mathrm{A}}')$ defines an extension $N_{T'}$ of N_T over $X \times T'$. $\square$

We can then repeat this process as many times as we want. At each stage, assuming that we have a 1-cocycle of order n, the obstruction to extending it to order $n+1$ is an element of $Ext^2_{\mathcal{O}}(M,M)$. Thus, for example if $Ext^2_{\mathcal{O}}(M,M)=0$, extensions to any order always exist.

Example: Let $p_0 \in \mathbb{C}^2$ and (x,z) be coordinates at p_0. The ideal I generated by x and z is then the ideal sheaf of p_0. The set of all deformations of the sheaf $\mathcal{O}\oplus I$ is $Ext^1_{\mathcal{O}}(\mathcal{O}\oplus I,\mathcal{O}\oplus I)$. We have an exact sequence

$$Ext^1_{\mathcal{O}}(I,I) \to Ext^1_{\mathcal{O}}(\mathcal{O}\oplus I,\mathcal{O}\oplus I) \to Ext^1_{\mathcal{O}}(I,\mathcal{O}).$$

The deformations of I are parametrised by $Ext^1_{\mathcal{O}}(I,I) \cong \mathbb{C}^2$. They simply correspond to changing the point p_0 in $\mathbb{C}^2$. We have seen in section 1.3.3, that the elements of $Ext^1_{\mathcal{O}}(I,\mathcal{O}) \cong \mathbb{C}$ give extensions $0 \to \mathcal{O} \to V \to I \to 0$. The deformations of $\mathcal{O}\oplus I$ are then the cokernels of maps of the form

$$\begin{array}{ccc} \mathcal{O} & \to & \mathcal{O}\oplus\mathcal{O}\oplus\mathcal{O} \\ 1 & \mapsto & (t,-z+c,x-b) \end{array},$$

where $t \in Ext^1_{\mathcal{O}}(I,\mathcal{O})$, and $(b,c) \in Ext^1_{\mathcal{O}}(I,I)$.

1.5.2 Global case.

We first describe deformations of vector bundles, and then state how these extend to coherent sheaves.

Deformations of bundles.

The ideas of the local case extend to deformations of vector bundles. Only this time it is the transition functions that are being deformed. We follow the presentation of [F], and therefore use their notation. Let E be a vector bundle on X. A deformation of E is then a vector bundle $\mathcal{E}$ over $X\times T$ such that the restriction of $\mathcal{E}$ to $X\times\{t_0\}$ is isomorphic to E. The vector bundle $\mathcal{E}$ must be given by transition functions, which we will now describe. Suppose that T is Spec$\mathbb{C}[\epsilon]$, where $\mathbb{C}[\epsilon]$ is the ring $\mathbb{C}[t]/(t^2)$ and ϵ is the class of t. Let A_{ij} be transition functions for E with respect to some open cover $\{U_i\}$ of X. We may assume that $\mathcal{E}$ can be trivialised on the open cover $\{U_i \times T\}$ of $X \times T$, and we can choose transition functions for $\mathcal{E}$ of the form

$$A_{ij}(t) = A_{ij} + B_{ij} \cdot t + O(t^2).$$

Let us consider the linear term $B_{ij} \cdot t$. The transition functions $A_{ij}(t)$ must satisfy the cocycle relations, thus implying that on triple intersections $U_i \cap U_j \cap U_k$ we have:

$$B_{ij}A_{jk} + A_{ij}B_{jk} = B_{ik}.$$

Since A_{ij} is a 1-cocycle, this can be rewritten as

$$B_{ij}A_{ij}^{-1} + A_{ij}(B_{jk}A_{jk}^{-1})A_{ij}^{-1} = B_{ik}A_{ik}^{-1},$$

and it follows that $B_{ij}A_{ij}^{-1}$ is a 1-cocycle for $Hom(E, E)$. Furthermore, any two choices of B_{ij} differ by a 1-coboundary for $Hom(E, E)$. Deformations of E thus define elements in $H^1(X, Hom(E, E))$. Conversely, an element $C_{ij} \in H^1(X, Hom(E, E))$ defines first-order terms in a power series expansion for $A_{ij}(t)$, by the rule $B_{ij} = C_{ij}A_{ij}$.

Let us now suppose that we want to extend this to a deformation of order 2, i.e., a deformation on $X \times \mathrm{Spec}(\mathbb{C}[t]/(t)^3)$. This is equivalent to finding B'_{ij} so that $A'_{ij}(t) = A_{ij} + B_{ij} \cdot t + B'_{ij} \cdot t^2$ is a 1-cocycle mod t^3. Given a choice of B'_{ij}, we must then compute the t^2 term in $A'_{ij}(t)A'_{jk}(t)A'^{-1}_{ik}(t)$. After a rather technical computation, one finds that $A'_{ij}(t)A'_{jk}(t)A'^{-1}_{ik}(t) = \mathrm{Id} + \Theta$, where Θ is a 2-cocycle. The obstruction to $A'_{ij}(t)$ being a 1-cocycle mod t^3 is then given by $\Theta \in H^2(X, Hom(E, E))$: in order for the extension to be possible, this cohomology class must vanish. The converse is also true. It can indeed be verified that

$$\exists B_{ij} \text{ such that } A'_{ij}(t) \text{ is a 1-cocycle } \Leftrightarrow \Theta = 0.$$

In general, if we have a 1-cocycle to order n, and we want to lift it to order $n+1$, the obstruction for such an extension is an element in $H^2(X, Hom(E, E))$.

Deformations of coherent sheaves.

Let $\mathcal{F}$ be a coherent sheaf on a complex space X. There exists an open cover

$\mathcal{V} = \{V_i\}$ of X such that each V_i is a Stein space. Therefore $\mathcal{F}|_{V_i}$ has a projective resolution, for all i. $\mathcal{F}$ is then given by:

(i) projective resolutions $\ldots \to \mathcal{O}^{q_i} \to \mathcal{O}^{p_i} \to \mathcal{F}|_{V_i}$ on V_i;

(ii) transition homomorphisms $\tau_{ij} : \mathcal{F}|_{V_i} \to \mathcal{F}|_{V_i}$ on $V_i \cap V_j$, that satisfy the cocycle relation, i.e. $\tau_{ij} = \tau_{ik} \cdot \tau_{kj}$.

A deformation of $\mathcal{F}$ consists in a deformation of these two things. Combining both the local and locally free descriptions, we have

For any coherent sheaf $\mathcal{F}$ on a variety X, the global deformations of $\mathcal{F}$ are given by $\mathrm{Ext}^1(\mathrm{X}; \mathcal{F}, \mathcal{F})$. *The obstruction to extending the deformation to any order is an element of* $\mathrm{Ext}^2(\mathrm{X}; \mathcal{F}, \mathcal{F})$.

1.6 Moduli spaces.

We finish this chapter by giving a review of some of the results involving moduli spaces of stable vector bundles and sheaves over any compact complex manifold.

1.6.1 Degree and stability.

Stability was first introduced by Mumford for holomorphic vector bundles over algebraic curves. Takemoto then generalised it to sheaves over a projective variety. Let X be a smooth projective variety of dimension d and let H be an ample line bundle on X. Let $\mathcal{E}$ be a torsion-free coherent sheaf on X. The *degree* $\deg_H(\mathcal{E})$ *of $\mathcal{E}$ relative to H* is then defined to be the number $c_1(\mathcal{E}) \cdot H^{d-1}$. The *slope* $\mu_H(\mathcal{E})$ *of $\mathcal{E}$ with respect to H* is the rational number $\mu_H(\mathcal{E}) = \deg_H(\mathcal{E})/\mathrm{rk}(\mathcal{E})$. (The slope is also called the *normalised degree of $\mathcal{E}$ with respect to H*.)

Definition 1.8 (Mumford-Takemoto) A torsion-free coherent sheaf $\mathcal{E}$ on X is *H-stable* (resp. *H-semistable*) if, for all coherent subsheaves $\mathcal{S}$ of $\mathcal{E}$ with $0 < \mathrm{rk}(\mathcal{S}) < \mathrm{rk}(\mathcal{E})$, we have

$$\mu_H(\mathcal{S}) < \mu_H(\mathcal{E}) \quad (\mathrm{resp.}\, \mu_{\mathrm{H}}(\mathcal{S}) \leq \mu_{\mathrm{H}}(\mathcal{E})).$$

We call $\mathcal{E}$ *unstable* if it is not semistable and *strictly semistable* if it is semistable but not stable. Finally, a subsheaf $\mathcal{S}$ of $\mathcal{E}$ with $0 < \mathrm{rk}(\mathcal{S}) < \mathrm{rk}(\mathcal{E})$ is *destabilising* if $\mu_H(\mathcal{S}) \geq \mu_H(\mathcal{E})$.

Stability can in fact be generalised to vector bundles over any compact complex manifold X of dimension d. One just has to define the notion of degree on X. A theorem by Gauduchon [G] states that any hermitian metric on a complex manifold has a conformal rescaling so that its associated (1,1) form ω satisfies $\partial\bar{\partial}\omega^{d-1} = 0$. Such a metric is called a *Gauduchon metric.* Let us assume that X is endowed with such a metric. Hitchin then suggested the following notion of degree (see [Bh]): if L is a holomorphic line bundle over X, the *degree of L with respect to ω* is defined, up to a constant factor, by

$$\deg(L) := \int_X F \wedge \omega^{d-1},$$

where F is the curvature of a hermitian connection on L compatible with $\bar{\partial}_L$. Any two such forms F differ by a $\partial\bar{\partial}$-exact form. Since $\partial\bar{\partial}\omega^{d-1} = 0$, we see that the degree is independent of the choice of connection, and is therefore well defined. This notion of degree extends that of the Kähler case. Indeed, if X is Kähler, we get the usual topological degree defined on Kähler manifolds. In general, this degree is however not necessarily a topological invariant. We shall see in the next chapter that in the case where X is a Hopf surface, the degree of a line bundle can take values in a continuum.

Having defined the degree of holomorphic line bundles, we define the *degree* of a torsion-free coherent sheaf $\mathcal{E}$ on X by

$$\deg(\mathcal{E}) := \deg(\det(\mathcal{E})),$$

where $\det(\mathcal{E})$ is the determinant line bundle of $\mathcal{E}$; and the *slope of $\mathcal{E}$* by

$$\mu(\mathcal{E}) := \deg(\mathcal{E})/\mathrm{rk}(\mathcal{E}).$$

The definition of stability therefore extends to this notion of degree, and it is defined exactly as in definition 1.8.

Remarks: It follows from the definition that:

(i) Line bundles are always stable.

(ii) In the case of rank 2 vector bundles on a surface, we need only to check stability with respect to rank 1 torsion free sheaves $\mathcal{S}$.

For the elementary properties of stable (resp. semistable) sheaves, we refer the reader to [OSS] and [Br]. We shall however state, without proof, the ones that will be useful to us in the sequel. Let X be a compact complex hermitian manifold, and let $\mathcal{O} = \mathcal{O}_X$ be its structure sheaf. For any sheaf $\mathcal{E}$, we denote its dual sheaf by $\mathcal{E}^* = Hom_{\mathcal{O}}(\mathcal{E}, \mathcal{O})$.

If $\mathcal{E}$ is any torsion-free sheaf on X, then

Lemma 1.9 *$\mathcal{E}$ is stable if and only if $\mathcal{E}^{**}$ is stable.*

A very important consequence of stability is

Proposition 1.3 *If $\mathcal{E}$ is a stable torsion-free sheaf, then $\mathcal{E}$ is simple, i.e., $End(\mathcal{E}) = \{\lambda \cdot \mathrm{Id} : \lambda \in \mathbb{C}\}$.*

1.6.2 Moduli spaces.

Let X be a compact complex hermitian manifold of dimension d endowed with a Gauduchon metric. We then have a well defined notion of degree and stability. Let $\mathcal{M}_n$ be the moduli space of stable rank 2 SL$(2,\mathbb{C})$ vector bundles E on X with $c_2(E) = n$. It is a well defined space. As E is an SL$(2,\mathbb{C})$-bundle, $\det(E) = \mathcal{O}$. We then have a natural splitting $Hom(E,E) = sl(E) \oplus \mathcal{O}$, where sl are the traceless endomorphisms. The discussion on deformation of sheaves then gives us the following

Theorem 1.4 *Suppose that $x \in \mathcal{M}_n$ is a point corresponding to a stable bundle E. If $H^2(X, sl(E) = 0$, then $\mathcal{M}_n$ is smooth at x of dimension $h^1(X, sl(E))$. In general, there is an analytic neighbourhood of x in $\mathcal{M}_n$ which is isomorphic to the zero set of h holomorphic functions $f_1, \ldots, f_h$ defined in a neighbourhood of the origin in $H^1(X, sl(E))$, where $h = \dim H^2(X, sl(E))$. Moreover, the f_i have no constant or* *linear terms and thus the Zariski tangent space $\mathcal{M}_n$ at x may be identified with $H^1(X, sl(E))$.*

Proof: See [F].

Remark: The moduli space $\bar{\mathcal{M}}_n$ of stable torsion-free coherent simple sheaves on X with $c_1(\mathcal{E}) = 0$ and $c_2(\mathcal{E}) = n$ is again a well defined space. Moreover, a theorem analogous to theorem 1.4 also applies in this case.

Buchdahl's theorem.

Let E be a holomorphic vector bundle on X. The notion of stability of E is intimately related to that of a Hermitian-Einstein connection on E. This is a Hermitian connection A on E, compatible with $\bar{\partial}_E$, whose curvature F_A satisfies the equation

$$\hat{F}_A := * \cdot \frac{1}{(d-1)!}(F_A \wedge \omega^{d-1}) = ik \tag{1.10}$$

for $k \in \mathbb{R}$. k is then constrained by

$$k = \frac{-2\pi}{(d-1)!\mathrm{Vol(X)}} \cdot \mu(E).$$

We then have the following very important

Theorem 1.5 *(Buchdahl [Bh]) An indecomposable holomorphic bundle E on X is stable if and only if E admits an irreducible Hermitian-Einstein connection. This connection is unique.*

This theorem generalises the result of Donaldson [D] for the Kähler case. Uhlenbeck and Yau [UY] have also generalised the result to a Kähler manifold of arbitrary dimension.

In the sequel, we will only be interested in $\mathrm{SL}(2, \mathbb{C})$ vector bundles. Let E be such a bundle. Then $\det(E) = \mathcal{O}$ and $\mu(E) = 0$. By (1.10), we see that the Hermitian-Einstein connections on E are precisely the anti-self dual ones, i.e. instantons. Buchdahl's theorem then gives

Corollary 1.2 *An indecomposable holomorphic* $\mathrm{SL}(2,\mathbb{C})$*-bundle* E *on* X *is stable if and only if* E *admits an irreducible anti-self dual connection. This connection is unique.*

We therefore see that there is a correspondence between indecomposable stable $\mathrm{SL}(2,\mathbb{C})$-bundles and anti-self dual connections, i.e. instantons.

Chapter 2

Holomorphic bundles over Hopf surfaces.

In this chapter, we consider holomorphic $\mathrm{SL}(2,\mathbb{C})$ vector bundles over Hopf surfaces. We begin by giving a review of the results found in [BH]. An important fact is that a Hopf surface $\mathcal{H}$ is an elliptic fibration over $\mathbb{P}^1$. A rank two $\mathrm{SL}(2,\mathbb{C})$ vector bundle E over $\mathcal{H}$ can therefore be thought of as a family of bundles over T parametrised by $\mathbb{P}^1$, where the family is obtained by restricting E to each fibre T. Rank two $\mathrm{SL}(2,\mathbb{C})$ vector bundles over an elliptic curve have however been completely classified by Atiyah [At]. It is then possible to associate to E a divisor in $\mathbb{P}^1 \times \mathbb{P}^1 = \mathbb{P}^1 \times \mathrm{Pic}^0(\mathrm{T})/\pm$ which gives the $\mathrm{SL}(2,\mathbb{C})$-isomorphism type of E over each fibre T. This divisor will be called the graph of E. This will be done in the first section.

The Hopf surface $\mathcal{H}$ can be covered by two copies of $\mathbb{C} \times T$. Bundles over $\mathcal{H}$ can therefore be obtained by glueing two bundles over $\mathbb{C} \times T$. Before discussing bundles over $\mathbb{C} \times T$, we must first consider bundles over T. In the second section, we give a brief presentation of line bundles and extensions of line bundles over T, thus giving us the chance to fix some of the notation that will be used in later chapters. In the third section, we classify bundles over $D \times T$, where D is a simply connected subset of $\mathbb{C}$. It was shown in [BH] that in two specific cases, the $\mathrm{SL}(2,\mathbb{C})$-isomorphism class of a bundle on $D \times T$ is completely determined by its graph on $D \times \mathbb{P}^1$. We will show that this can be generalised to two more cases that will be useful to us in chapter 6. We finish by giving explicit transition matrices for certain rank two $\mathrm{SL}(2,\mathbb{C})$-bundles on $D \times T$ that we will use in chapter 3.

2.1 Holomorphic bundles on Hopf surfaces.

Let $\lambda \in \mathbb{C}, |\lambda| > 1$ be a fixed complex number. An action of $\mathbb{Z}$ on $\mathbb{C}^{2*} = \mathbb{C}^2\backslash\{(0,0)\}$ can be given by

$$\begin{array}{ccc} \mathbb{C}^{2*} \times \mathbb{Z} & \longrightarrow & \mathbb{C}^{2*} \\ ((z_1, z_2), n) & \longmapsto & (\lambda^n z_1, \lambda^n z_2) \end{array} . \tag{2.1}$$

The *Hopf surface* $\mathcal{H} = \mathcal{H}_\lambda$ is then defined as $\mathbb{C}^{2*}/\mathbb{Z}$. $\mathcal{H}$ is a complex surface, diffeomorphic to $S^3 \times S^1$; and, as $b_1(\mathcal{H}) = 1$, it cannot be Kähler. Its Dolbeault cohomology is:

$$H^{p,q}(\mathcal{H}) = \begin{cases} \mathbb{C} & \text{if } (p,q) = (0,0), (0,1), (2,1) \text{ or } (2,2), \\ 0 & \text{otherwise.} \end{cases}$$

$\mathcal{H}$ can be expressed as a holomorphic fibration

$$\pi : \mathcal{H} \longrightarrow \mathbb{P}^1, \tag{2.2}$$

which is simply the map $\mathcal{H} \to \mathbb{C}^{2*}/\mathbb{C}^*$. Its fibre is the elliptic curve

$$T \cong \mathbb{C}^*/\lambda^n \cong \mathbb{C}/(2\pi i\mathbb{Z} + \log(\lambda)\mathbb{Z}).$$

It is useful to have coordinate systems which reflect this fibration. Then $z = z_1/z_2, z' = 1/z$ are affine coordinates on $\mathbb{P}^1$; and $t = \log z_1$, $t' = \log z_2$ give linear coordinates on T. $\mathcal{H}$ is then covered by $U_0 = \mathbb{C} \times T$, $U_1 = \mathbb{C} \times T$, with the identification

$$(z', t') = (1/z, t + \log z) \tag{2.3}$$

on the overlap.

We give $\mathcal{H}$ the hermitian metric whose associated $(1,1)$-form on the cover $\mathbb{C}^{2*}$ is

$$\omega = \frac{i}{2}(\mathrm{Vol}(\mathrm{S}^3) \times \ln|\lambda|)^{-\frac{1}{2}} \frac{\mathrm{d}z_1 \wedge \mathrm{d}\bar{z}_1 + \mathrm{d}z_2 \wedge \mathrm{d}\bar{z}_2}{|z_1|^2 + |z_2|^2}. \tag{2.4}$$

The constant is chosen to make the total volume 1. We have $\bar{\partial}\partial\omega = 0$: it is a Gauduchon metric (see section 1.6.1).

2.1.1 Line bundles.

The set of all holomorphic line bundles on $\mathcal{H}$ is given by the Picard group $\mathrm{Pic}(\mathcal{H})$ of $\mathcal{H}$. We begin by giving a description of this group. From the exponential sheaf sequence

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathcal{O} \longrightarrow \mathcal{O}^* \longrightarrow 0,$$

one obtains

$$\mathrm{Pic}(\mathcal{H}) \cong \mathrm{H}^1(\mathcal{H}, \mathcal{O})/\mathrm{H}^1(\mathcal{H}, \mathbb{Z}) \cong \mathbb{C}/\mathbb{Z} \cong \mathbb{C}^*.$$

Furthermore, as $H^2(\mathcal{H}, \mathbb{Z}) = 0$, all line bundles on $\mathcal{H}$ have a trivial Chern class, and $\mathrm{Pic}(\mathcal{H}) \cong \mathrm{Pic}^0(\mathcal{H})$. Line bundles on $\mathcal{H}$ can therefore be realised by constant automorphy factors $k \in \mathbb{C}^*$.

Let us illustrate the above by constructing the universal (Poincaré) line bundle V over $\mathcal{H} \times \mathbb{C}^*$ by using automorphy factors. One starts with a trivial line bundle L on $\mathbb{C}^{2*} \times \mathbb{C}^*$. One then has the following $\mathbb{Z}$-action

$$\begin{array}{ccc} \mathbb{C}^{2*} \times \mathbb{C}^* \times \mathbb{Z} & \longrightarrow & \mathbb{C}^{2*} \times \mathbb{C}^* \\ ((z_1, z_2), \alpha, n) & \longmapsto & ((\lambda^n z_1, \lambda^n z_2), \alpha) \end{array},$$

which is simply induced by (2.1). This action is trivial on $\mathbb{C}^*$. V is then obtained by taking the quotient of L with respect to this action: $s \in L_{(x,\alpha)}$ is identified with $ks \in L_{(\lambda x,\alpha)}$. We then see that, for any $m \in \mathbb{Z}$, the automorphy factor $\alpha = \lambda^m$ gives the line bundle $\pi^*(\mathcal{O}(m))$ on $\mathcal{H}$. From now on, we will denote $\pi^*(\mathcal{O}(m))$ by $\mathcal{O}(m)$, and, if L is any bundle, $L \otimes \pi^*(\mathcal{O}(m))$ by $L(m)$.

For any line bundle L on $\mathcal{H}$, restriction of L to a fibre is a natural operation. As L is given by a constant automorphy factor, its restriction to any fibre must be an element of $\mathrm{Pic}^0(\mathrm{T})$. We then have a map $\mathbb{P}^1 \to \mathrm{Pic}^0(\mathrm{T}) \cong \mathrm{T}$ given by associating to x the bundle $L|_{\pi^{-1}(x)}$. This map must be constant. Restriction to a fibre then induces an exact sequence

$$\begin{gathered} 0 \to \mathrm{Pic}(\mathbb{P}^1) \to \mathrm{Pic}^0(\mathcal{H}) \to \mathrm{Pic}^0(\mathrm{T}) \to 0, \\ 0 \to \mathbb{Z} \to \mathbb{C}^* \to T \to 0. \end{gathered} \tag{2.5}$$

Let us note that $L|_{\pi^{-1}(x)} = L(m)|_{\pi^{-1}(x)}$, for any line bundle L and any $m \in \mathbb{Z}$. Moreover, since the canonical bundle on T is trivial, the canonical bundle of $\mathcal{H}$ is $K(\mathcal{H}) \cong \mathcal{O}(-2)$.

We can also compute the cohomology groups $H^i(\mathcal{H}, L)$ of the sheaf of holomorphic sections of L over $\mathcal{H}$. The line bundle L has a section over $\pi^{-1}(x)$ if and only if L is trivial on $\pi^{-1}(x)$. The sections of L are then constant along $\pi^{-1}(x)$, and must be lifted from $\mathbb{P}^1$. In other words, L has a global section if and only if $L = \mathcal{O}(m)$, for some $m \in \mathbb{Z}$. Therefore

$$h^0(\mathcal{H}, L) = \begin{cases} m+1 & \text{if } V \cong \mathcal{O}(m),\ m \geq 0, \\ 0 & \text{otherwise;} \end{cases}$$

and the only divisors on $\mathcal{H}$ are sums of fibres of π. Furthermore, as $K(\mathcal{H}) \cong \mathcal{O}(-2)$,

$$h^2(\mathcal{H}, L) = \begin{cases} -m-1 & \text{if } L \cong \mathcal{O}(m),\ m \leq -2, \\ 0 & \text{otherwise.} \end{cases}$$

The basic fibration (2.2) induces a topological splitting of the complex tangent bundle of $\mathcal{H}$. We then have $\mathrm{Td}(\mathcal{H}) = 1$, and, as a consequence of the Riemann-Roch theorem,

$$h^1(\mathcal{H}, L) = h^0(\mathcal{H}, L) + h^2(\mathcal{H}, L).$$

2.1.2 Rank 2 bundles.

We now consider the case of $\mathrm{SL}(2, \mathbb{C})$-bundles over $\mathcal{H}$: rank two bundles E with $\Lambda^2 E \cong \mathcal{O}$. We fix $c_2(E) = n$. In our study of such bundles, one of our main tools will be restriction to the fibres $\pi^{-1}(x)$. Atiyah has given a complete classification of holomorphic vector bundles over T [At]. We begin by recalling the case of rank two bundles E with $\Lambda^2 E \cong \mathcal{O}$.

Proposition 2.1 *The* $\mathrm{SL}(2,\mathbb{C})$*-bundles over an elliptic curve* T *are of the following types :*

(i) $L_0 \oplus L_0^*$, $L_0 \in \mathrm{Pic}^0(\mathrm{T})$.

(ii) Non trivial extensions $0 \to L_0 \to E \to L_0 \to 0, L_0^2 \cong \mathcal{O}$.

(iii) $L \oplus L^*$, $L \in \mathrm{Pic}^{\mathrm{k}}(\mathrm{T}), \mathrm{k} < 0$.

Proof: Details can be found in [At].

If we now turn our attention to rank two bundles over $\mathcal{H}$, it has been proven in [BH] that their restrictions to fibres $\pi^{-1}(x)$ are generically of type (i) or (ii). More precisely, we have the following result:

Proposition 2.2 *Let* E *be an* $\mathrm{SL}(2,\mathbb{C})$*-bundle over* $\mathcal{H}$*; then* $E|_{\pi^{-1}(x)}$ *is of type iii) on at most an isolated set of points* $x \in \mathbb{P}^1$.

Proof: See proposition 3.2.2 in [BH].

The bundle E can be described further: we can show that there are at most n points x for which $E|_{\pi^{-1}(x)}$ is of type iii). We begin by fixing a generic bundle V on $\mathcal{H}$ such that $h^0(\pi^{-1}(x), V^*E) = 0$ for at least one x, and so for generic x. This forces $\pi_*(V^*E) = 0$. However, proposition 2.2 implies that $R^1\pi_*(V^*E)$ is a skyscraper sheaf supported on isolated points. These are points x for which one of the following holds:

- $E|_{\pi^{-1}(x)}$ has $V|_{\pi^{-1}(x)}$ as a subline bundle and is of type i) or ii), or
- $E|_{\pi^{-1}(x)}$ is of type iii).

By the Grothendieck-Riemann-Roch theorem, [Ha],

$$ch(R^1\pi_*(V^*E)) = -ch(\pi_!(V^*E)) = nh,$$

where h is the positive generator of $H^2(\mathbb{P}^1,\mathbb{Z})$. The skyscraper sheaf $R^1\pi_*(V^*E)$ is therefore supported on at most n points.

Given a local resolution

$$0 \longrightarrow \mathcal{O}^{\oplus m} \xrightarrow{f(z)} \mathcal{O}^{\oplus m} \longrightarrow R^1\pi_*(V^*E) \longrightarrow 0$$

of $R^1\pi_*(V^*E)$ around a point $z=0$, the multiplicity of the point $z=0$ is defined to be the multiplicity of the zero of $\det(f(z))$ at $z=0$. It is essentially just the complex dimension of $R^1\pi_*(V^*E)$ in a neighbourhood of $z=0$, and is thus a local version of the definition of $c_1(R^1\pi_*(V^*E))$. Let us note that there are different possibilities for a point of a given multiplicity. For example, suppose that the point $z=0$ has multiplicity 2. Locally, the sheaf $R^1\pi_*(V^*E)$ could then be the cokernel of a map $z^2 : \mathcal{O} \to \mathcal{O}$, or of a map $z : \mathcal{O}^{\oplus 2} \to \mathcal{O}^{\oplus 2}$. Taking the above into account, we have that $R^1\pi_*(V^*E)$ is supported on n points, counting multiplicity.

To obtain a complete description of the restriction on E to the fibres, we have to repeat this construction for every line bundle on $\mathcal{H}$. One can actually take the direct image $R^1\pi_*$ for all line bundles simultaneously, as follows: if V is the universal (Poincaré) line bundle over $\mathcal{H} \times \mathbb{C}^*$, and π is the projection $\mathcal{H} \times \mathbb{C}^* \to \mathbb{P}^1 \times \mathbb{C}^*$, consider $R^1\pi_*(E \otimes V)$. As above, this sheaf is supported on a divisor $\tilde{D}$, which is also defined with multiplicity. We can however remark the following:

- Tensoring V by $\mathcal{O}(1)$ does not change the support of $R^1\pi_*$. The divisor $\tilde{D}$ is then invariant under the $\mathbb{Z}$-action on $\mathbb{C}^*$ generated by multiplication by λ.
- As $\Lambda^2 E \cong \mathcal{O}, E \cong E^*$. Substituting V^* for V therefore does not change the support of $R^1\pi_*$, implying that $\tilde{D}$ is also invariant under the involution on $\mathbb{C}^*$ defined by $z \mapsto 1/z$.

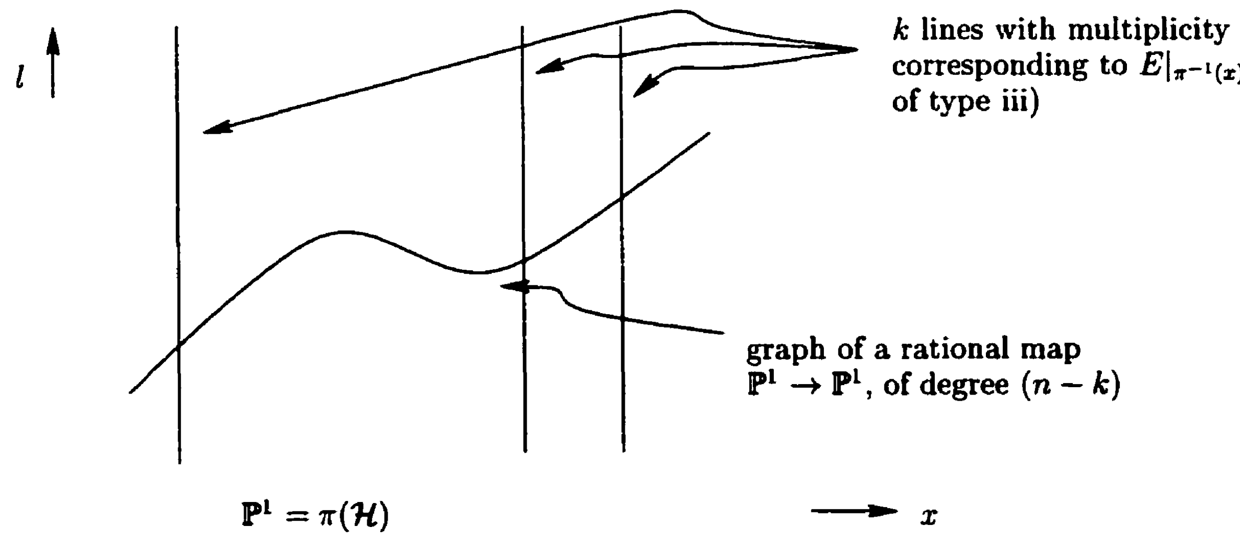
If we quotient $\mathbb{C}^*$ by the $\mathbb{Z}$-action, we get $\mathrm{Pic}^0(\mathrm{T}) \cong \mathrm{T}$. If we also quotient by the involution, we obtain a two sheeted map $\mathrm{Pic}^0(\mathrm{T}) \to \mathbb{P}^1$ whose branch points are the half periods of T. By the above remarks, $\tilde{D}$ then descends to a divisor D on $\mathbb{P}^1 \times \mathbb{P}^1 = \pi(\mathcal{H}) \times \mathrm{Pic}^0(\mathrm{T})/\pm$.

Such divisors are in a linear system $|\mathcal{O}(i,j)|$ on $\mathbb{P}^1 \times \mathbb{P}^1 = \pi(\mathcal{H}) \times \mathrm{Pic}^0(\mathrm{T})/\pm$. We can easily see that $i=n, j=1$. The above computation tells us that $\mathbb{P}^1 \times \{l\}$ intersects D in n points, thus implying that $i=n$. Moreover, if we fix $x \in \mathbb{P}^1$, then

- $\{x\} \times \mathbb{P}^1$ intersects D in only one point (counting multiplicity), corresponding to the pair of bundles $\{L_0, L_0^*\}$ in $\mathrm{Pic}^0(\mathrm{T})/\pm$ that are subbundles of $E|_{\pi^{-1}(x)}$, or

– $\{x\} \times \mathbb{P}^1$ is included in D, if $E|_{\pi^{-1}(x)}$ is of type iii).

Hence, $j = 1$. The divisor D has the following picture in $\mathbb{P}^1 \times \mathbb{P}^1$:



The above description can be summarised in the following proposition:

Proposition 2.3 *To each* $\mathrm{SL}(2,\mathbb{C})$*-bundle* E *on* $\mathcal{H}$ *with* $c_2(E) = n$*, there is associated a divisor* D *in the linear system* $|\mathcal{O}(n,1)|$ *over* $\mathbb{P}^1 \times \mathbb{P}^1$*. This divisor is called the* graph *of* E*.*

2.1.3 Degree and stability.

The metric that we have chosen for $\mathcal{H}$ is a Gauduchon metric. We therefore have a well defined notion of degree. In section 1.6.1, the degree of any line bundle L was defined, up to a constant factor, as

$$\deg(L) = \int_{\mathcal{H}} F \wedge \omega,$$

where F is the curvature of any hermitian connection on L compatible with $\bar{\partial}_L$. Furthermore, if $\mathcal{E}$ is any torsion-free coherent sheaf on $\mathcal{H}$, its degree is defined by

$$\deg(\mathcal{E}) := \deg(\det(\mathcal{E})),$$

and its normalised degree by

$$\mu(\mathcal{E}) := \deg(\mathcal{E})/\mathrm{rk}(\mathcal{E}).$$

Let us also recall that stability was defined as follows:
A torsion-free coherent sheaf $\mathcal{E}$ on $\mathcal{H}$ is stable if and only if for every coherent subsheaf $\mathcal{S} \subset \mathcal{E}$ with $0 < \mathrm{rk}(\mathcal{S}) < \mathrm{rk}(\mathrm{E})$, *we have*

$$\mu(\mathcal{S}) < \mu(\mathcal{E}).$$

Let us compute the degrees of line bundles and $\mathrm{SL}(2,\mathbb{C})$-bundles on $\mathcal{H}$, and, in the latter case, determine which ones are stable.

1) *Line bundles.* Let $L \in \mathrm{Pic}^0(\mathcal{H})$ correspond to an automorphy factor α. We define a metric on the trivial bundle over $\mathbb{C}^{2*}$ by

$$\| s(z) \|^2 = |s(z)|^2 \cdot |z|^{-2\ln|\alpha|/\ln|\lambda|},$$

where $|s(z)|^2$ is the standard metric. $\| \ \|$ descends to L over $\mathcal{H}$, and has curvature

$$F = (-\ln|\alpha|/\ln|\lambda|) \cdot (\partial\bar{\partial}\ln|z|^2).$$

Integrating F against ω, one finds that, up to a positive constant factor,

$$\deg(L) = \ln|\alpha|/\ln|\lambda|.$$

For example, $\deg(\mathcal{O}(m)) = m$. We have also seen that line bundles are always stable.

2) $\mathrm{SL}(2,\mathbb{C})$*-bundles.* As $\Lambda^2(E) \cong \mathcal{O}$, we always have $\deg(E) = 0$. We would however like to know when E is stable. As we have seen in section 1.6.1, it suffices to know whether any line bundle of non-negative degree admits a nonzero map into E. The following two propositions were proven in [BH]:

Proposition 2.4 *Let the graph of E include a non-constant map $\mathbb{P}^1 \to \mathbb{P}^1$. Then E is stable.*

The only bundles non stable bundles are therefore among those whose graph is of the form

$$\left(\sum_{i=1}^{n}(\{x_i\} \times \mathbb{P}^1)\right) + (\mathbb{P}^1 \times \{l\}).$$

More specifically, we have

Proposition 2.5 *Let E have graph*

$$\left(\sum_{i=1}^{n}(\{x_i\} \times \mathbb{P}^1)\right) + (\mathbb{P}^1 \times \{l\}).$$

Then there exist line bundles K, K' on $\mathcal{H}$ such that the set of line bundles which map non-trivially to E is $\{K(m), K'(m), m \leq 0\}$. If l is a half period , $K = K'$.

E is then stable if both $\deg(K)$ *and* $\deg(K')$ *are negative.*

2.1.4 Moduli spaces.

Let $\mathcal{M}_n$ be the moduli space of stable rank 2 $\mathrm{SL}(2,\mathbb{C})$-bundles E on $\mathcal{H}$ with $c_2(E) = n$. Before turning to the problem of smoothness of $\mathcal{M}_n$, let us give another description of stable $\mathrm{SL}(2,\mathbb{C})$-bundles on $\mathcal{H}$.

Let E be an $\mathrm{SL}(2,\mathbb{C})$-bundle with $c_2(E) = n$, $n \geq 1$. E must then be indecomposable, and any connection A on E must be irreducible. The corollary 1.2 to Buchdahl's theorem therefore gives

E is stable if and only if E admits an anti-self dual connection.

As the connection given by Buchdahl's theorem is unique, we therefore have a correspondence between stable bundles and anti-self dual connections, i.e. instantons. It was proven in [BH] that

Proposition 2.6 *The moduli space $\mathcal{M}_n$ of stable rank 2* $\mathrm{SL}(2,\mathbb{C})$*-bundles E on $\mathcal{H}$ with $c_2(E) = n$ is a smooth, non-empty complex $4n$-dimensional manifold, diffeomorphic to the moduli space of* $\mathrm{SU}(2)$*-instantons on E.*

Remark: If E is stable, then $H^0(\mathcal{H}, End(E)) = \mathbb{C}$ and $H^2(\mathcal{H}, End(E)) = 0$. Furthermore, the tangent space to $\mathcal{M}_n$ at E is $H^1(\mathcal{H}, sl(E)) = \mathbb{C}^{4n}$, where sl denotes the traceless endomorphisms.

We will also consider the moduli space $\bar{\mathcal{M}}_n$ of stable torsion-free coherent sheaves $\mathcal{E}$ on X with $c_1(\mathcal{E}) = 0$ and $c_2(\mathcal{E}) = n$. This is again a well-defined moduli space that we will study in further detail in chapter 3. Let us note that $\mathcal{M}_n$ is contained in $\bar{\mathcal{M}}_n$, for all $n \geq 1$. Proposition 2.6 therefore ensures us that $\bar{\mathcal{M}}_n$ is never empty.

2.2 Bundles on T.

2.2.1 Line bundles.

Suppose that T is given by the non-degenerate lattice Λ in $\mathbb{C}$, with generators $2\omega_1, 2\omega_2$. We would like to construct transition functions for line bundles on T, using the standard elliptic functions $\sigma(z), \zeta(z)$ with expansions at $z = 0$

$$\sigma(z) = z + O(z^5)$$
$$\zeta(z) = \frac{1}{z} + O(z^3),$$

and periodicity relations

$$\sigma(z + 2\omega_i) = -\sigma(z) exp(2\eta_i(z + 2\omega_i))$$
$$\zeta(z + 2\omega_i) = \zeta(z) + 2\eta_i,$$

with $\eta_i = \zeta(\omega_i)$. $2\eta_i, 2\omega_i$ satisfy the relation

Theorem 2.1 *(Legendre)*

$$2\eta_1 2\omega_2 - 2\eta_2 2\omega_1 = \pi\sqrt{-1}.$$

We begin by showing

Lemma 2.1 *The function $\phi_i(z) = e^{2\omega_i\zeta(z)}e^{-2\eta_i z}$ is doubly periodic for $i = 1, 2$.*

Proof: For $j = 1, 2$,

$$\phi_i(z + 2\omega_j) = \phi_i(z)\alpha_{ij},$$

where $\alpha_{ij} = e^{-(2\eta_i 2\omega_j - 2\omega_i 2\eta_j)}$ is a constant. If $i = j$, we obviously have that $\alpha_{ij} = 1$. And if $i \neq j$, Legendre's relation implies that

$$2\eta_i 2\omega_j - 2\eta_i 2\omega_j = \pm 2\pi\sqrt{-1},$$

and $\alpha_{ij} = e^{\mp 2\pi\sqrt{-1}} = 1$. Therefore, for any $i = 1, 2$ and $j = 1, 2$, $\phi_i(z + 2\omega_j) = \phi_i(z)$, proving double periodicity. □

Let us cover T by $U_1 = T -$ (origin), $U_0 =$ discaroundtheorigin.

Line bundles of degree zero.

From the periodicity relations, we see that the function

$$\rho^1(\mu, z) = \frac{\sigma(z - \mu)}{\sigma(z)} e^{\mu\zeta(z)}$$

is doubly-periodic, and therefore well defined on the elliptic curve with parameter z. It has an essential singularity at the origin, and a single zero at $\mu = z$. If we set

$$\rho^0(\mu, z) = \frac{\sigma(z - \mu)}{\sigma(z)} = e^{-\mu\zeta(z)}\rho^1(\mu, z),$$

we find that ρ^0 has a single pole in z at the origin. ρ^1, ρ^0 define a section of the line bundle L_μ, with transition function $h_{01}(\mu, z) = e^{-\mu\zeta(z)}$. This section has a single zero at the point $\mu = z$, and a single pole at the origin.If we set p_μ to be the point $\mu = z$ on T, L_μ corresponds to the divisor $p_\mu - p_0$.

Line bundles of degree one.

Similarly, if we set

$$\varrho^0(\lambda, z) = \sigma(z - \lambda) = \sigma(z)e^{-\lambda\zeta(z)}\varrho^1(\lambda, z),$$

we find that ϱ^0 is holomorphic in z at the origin. ϱ^1, ϱ^0 define a section of the line bundle L_λ, with transition function $g_{01}(\lambda, z) = \sigma(z)e^{-\lambda\zeta(z)}$. This section has a single zero at the point $\lambda = z$. If we set p_λ to be the point $\lambda = z$ on T, L_λ corresponds to the divisor p_λ.

Remark: For any period $2\omega_i$,

$$g_{10}(\lambda + 2\omega_i, z) = \phi_i(z) g_{10}(\lambda, z) \psi_i^{-1}(z),$$

where $\phi_i(z) = e^{2\omega_i \zeta(z)} e^{-2\eta_i z}$, and $\psi_i(z) = e^{-2\eta_i z}$. By lemma 2.1, $\phi_i(z)$ is doubly periodic. $g_{10}(\lambda + 2\omega_i, z)$ is therefore also a transition function for L_λ.

Line bundles of degree k.

By the above, we see that any bundle over T of degree k can be given by the transition function

$$(g_{01}(\lambda, z))^k = (\sigma(z))^k e^{-k\lambda\zeta(z)},$$

and that the divisor associated to this line bundle will be kp_λ.

2.2.2 Endomorphisms of rank 2 bundles.

Endomorphisms of rank 2 $\mathrm{SL}(2,\mathbb{C})$-bundle over an elliptic curve T are described in the following lemma:

Lemma 2.2 *Let E be an $\mathrm{SL}(2,\mathbb{C})$-bundle over T. Its global endomorphisms and $\mathrm{SL}(2,\mathbb{C})$-automorphisms are:*

1) $E \simeq L_0 \oplus L_0^*$, $L_0^2 \neq \mathcal{O}, c_1(L_0) = 0$	$\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} a, b \in \mathbb{C}$	$\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} a \in \mathbb{C}^*$
2) $E \simeq L_0 \oplus L_0^*$, $L_0^2 = \mathcal{O}, c_1(L_0) = 0$	$gl(2,\mathbb{C})$	$\mathrm{SL}(2,\mathbb{C})$
3) E *type* (*ii*)	$\begin{pmatrix} a & b \\ 0 & a \end{pmatrix} b \in \mathbb{C}$	$\pm\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} a \in \mathbb{C}$
4) $E \simeq L \oplus L^*$ $c_1(L) < 0$	$\begin{pmatrix} a & f \\ 0 & b \end{pmatrix} a, b \in \mathbb{C}$ $f \in \Gamma(L^{*2})$	$2\pi i \begin{pmatrix} a & f \\ 0 & a^{-1} \end{pmatrix}$ $a \in \mathbb{C}^*$ $f \in \Gamma(L^{*2})$

Proof: This lemma is given in [BH] without proof. It is however a straightforward computation of cohomology groups.

As a direct consequence of lemma 2.2, we have:

Lemma 2.3 *Let E be an* $\mathrm{SL}(2,\mathbb{C})$*-bundle over T. Its global traceless endomorphisms and the kernel of the exponential map* exp*: (global traceless endomorphisms) $\to$* $(\mathrm{SL}(2,\mathbb{C})$*-automorphisms) are:*

1) $E \simeq L_0 \oplus L_0^*$, $L_0^2 \neq \mathcal{O}, c_1(L_0) = 0$	$\begin{pmatrix} a & 0 \\ 0 & -a \end{pmatrix} a \in \mathbb{C}$	$2\pi i \begin{pmatrix} m & 0 \\ 0 & -m \end{pmatrix} m \in \mathbb{Z}$
2) $E \simeq L_0 \oplus L_0^*$, $L_0^2 = \mathcal{O}, c_1(L_0) = 0$	$sl(2,\mathbb{C})$	
3) E *type (ii)*	$\begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} b \in \mathbb{C}$	$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$
4) $E \simeq L \oplus L^*$ $c_1(L) < 0$	$\begin{pmatrix} a & f \\ 0 & -a \end{pmatrix} a \in \mathbb{C}$ $f \in \Gamma(L^{*2})$	$2\pi i \begin{pmatrix} m & f \\ 0 & -m \end{pmatrix}$ $f \in \Gamma(L^{*2})$ $m \in \mathbb{Z}\backslash 0$

$\square$

2.2.3 Extensions of line bundles.

Let us fix $L \in \mathrm{Pic}^{-k}(\mathrm{T})$. Then $H^1(T, L^2) = \mathbb{C}^{2k}$, and we know that this is the space of all possible extensions of L by L^*. It is not difficult to see that the extensions of L by L^* are

- any type (i) bundle,
- any type (ii) bundle,
- any type (iii) bundle $L_{-j} \oplus L_{-j}^*$, with

$$L_{-j} \in \mathrm{Pic}^{-j}(\mathrm{T}),\ 0 < \mathrm{j} < \mathrm{k}.$$

All these possibilities occur, with one exception: if $k = 1$, type (i) bundles $L_0 \oplus L_0^*$, with $L_0^2 \cong \mathcal{O}$ do not occur.

Example: Let us illustrate this in the case where $L \in \mathrm{Pic}^{-1}(\mathrm{T})$. If we take any $L_0 \in \mathrm{Pic}^0(\mathrm{T})$, such that $L_0^2 \neq \mathcal{O}$, let us express $L_0 \oplus L_0^*$ as an extension of L^* by L:

$$0 \longrightarrow L \longrightarrow L_0 \oplus L_0^* \longrightarrow L^* \longrightarrow 0.$$

In order to do so, we have to find an injective bundle map $L \to L_0 \oplus L_0^*$. Suppose that L has divisor $-p_\lambda$, and that L_0 is given by the divisor $p_\mu - p_0$. As we have seen in section 2.2.1, $g(z) = \sigma(z)e^{-\lambda\xi(z)}$ and $h(z) = e^{\mu\xi(z)}$ are then transition functions from U_0 to U_1 of L and L_0, respectively.

Let $r(z) = \{r_i(z)\}_{i=0,1}$ be the global section of $L^* \otimes L_0$ given on U_0 by $r_0(z) = \sigma(z - (\lambda + \mu))$. Similarly, $s(z) = \{s_i(z)\}_{i=0,1}$, where $s_0 = \sigma(z - (\lambda - \mu))$ on U_0, is a global section of $L^* \otimes L_0^*$. As L_0 is not a half period, $\lambda + \mu \neq \lambda - \mu$ in T. It is then clear that $r(z)$ and $s(z)$ do not have the same zeroes. These two sections therefore define an injective bundle map $L \to L_0 \oplus L_0^*$.

All the other cases, stated above, are proven similarly: one can always find enough sections so that at least two of them do not have common zeroes, thus defining an injective bundle map.

2.3 Bundles over $D \times T$.

A natural tool for classifying stable bundles is their graph. Fixing the graph, we would like to know to what extent the graph determines the bundle over over $\mathcal{H}$. We will split this problem into two parts:

- A local problem: choosing $D \subset \mathbb{P}^1$, what bundles over $\pi^{-1}(D) \cong D \times T$ have this graph over $D \times \mathrm{Pic}^0(\mathrm{T})/\pm = \mathrm{D} \times \mathbb{P}^1$?

- A global problem: having covered $\mathbb{P}^1$ by, say, two discs D_1 and D_2, and chosen bundles over $\pi^{-1}(D_i)$ that are compatible on the overlap, in how many distinct ways can one glue them together to obtain bundles over $\mathcal{H}$?

In this section, we consider the local problem; i.e. we describe the local geometry of a rank two bundle E over $D \times T$, where D is a disc in $\mathbb{P}^1$. We begin by recalling the situation over the fibres of π. Let $x \in D$.

- If $\{x\} \times \mathbb{P}^1$ is not included in the graph, let its intersection with the graph be (x, l), where l is the unordered pair of bundles $\{L_0, L_0^*\} \subset \mathrm{Pic}^0(\mathrm{T})/\pm$ which map into E over $\pi^{-1}(x)$.
  a) When l is not a half period (i.e., $L_0 \neq L_0^*$) then $E \cong L_0 \oplus L_0^*$ over $\pi^{-1}(x)$. E is determined by the graph.
  b) When l is a half period, either $L = L_0 \oplus L_0$, or E is of type ii).
- If $\{x\} \times \mathbb{P}^1$ is the graph, then E over $\pi^{-1}(x)$ is a sum $L \oplus L^*$, $c_1(L) < 0$ (type iii). If $\{x\} \times \mathbb{P}^1$ has multiplicity n in the graph, then $c_1(L) = -k$, for some $k \in \mathbb{N}$ such that $k \leq n$.

2.3.1 Extensions of line bundles.

Extensions are a very useful tool in the study of rank two vector bundles. We would like to show that given an $SL(2,\mathbb{C})$-bundle E over $\mathcal{H}$, and a disc $D \subset \mathbb{P}^1$, we can always express E over $\pi^{-1}(D) \cong D \times T$ as an extension of line bundles

$$0 \to L \to E \to L^* \to 0,$$

where the choice of the line bundle L will be determined by E.

Let $p \in \mathcal{H}$, and let D be a disc in $\mathbb{P}^1$ which contains $\pi(p)$. Let (x, z) be a coordinate system centered at $p \in D \times T \cong \pi^{-1}(D)$. Let E be an SL$(2,\mathbb{C})$-bundle on $\mathcal{H}$ with $c_2(E) = n$, and graph $g \in |\mathcal{O}(n,1)|$. By the discussion in section 2.1.2, we know that g decomposes into two pieces:

- the graph of a rational map $F : \mathbb{P}^1 \to \mathbb{P}^1$ of degree k,
- a sum of $(n-k)$ "vertical fibres" $\{x_i\} \times \mathbb{P}^1$, counted with multiplicity.

Let us now restrict E and its graph to $D \times T$. We then set

$$n_0 = k + \left(\begin{array}{c} \text{number of vertical fibres in} \\ D \times \mathbb{P}^1\text{, counted with multiplicity} \end{array} \right).$$

The integer n_0 can be thought of as a relative second Chern class of $E|_{D\times T}$ in $H^4(D \times T, (D - B) \times T)$, where $B \subset D$ is a 2-ball around p. Obviously, $n_0 \leq n$; and if $E|_{x_i \times T} = V \oplus V^*$, for some $x_i \in D$ and $V \in \mathrm{Pic}^{-j}(T)(j \geq 1)$, then $j \leq n_0$.

Let $x \in D$. We have seen in section 2.2.3 that $E|_{x\times T}$ can be expressed as an extension of L^* by L, for any line bundle L on $x \times T$ of degree $-l < -n_0$. We would like to show that we can actually fix a line bundle L of degree $< -n_0$, and get a local extension

$$0 \to L \to E \to L^* \to 0$$

over $D \times T$, where L also denotes the pullback of L to $D \times T$.

Remark: Punctually, we have more possibilities for the choice of L. Let $x \in D$. If $E|_{x\times T} = V_{-j} \oplus V^*_{-j}$, it is then an extension of L^* by L for any line bundle L on $x \times T$ of degree $-l < -j$. But, as we ultimately want to have an extension over $D \times T$, we will however consider, in the general case, line bundles of degree $< n_0$.

For the remainder of this section, let $T = 0 \times T$. Let us now fix a line bundle $L \in \mathrm{Pic}^{-l}(T)$, for some $l > n_0$, and let L also denote the pullback of L on $D \times T$. By the discussion in section 2.2.3, in order to show that we have the extension

$$0 \to L \to E \to L^* \to 0$$

over $D \times T$, we just need to show that there exists an injective bundle map $L \to E$ on $D \times T$. Now, this is obviously the case over T. We therefore just have to prove

Proposition 2.7 *An injective bundle map $L \to E$ on T extends to an injective bundle map $L \to E$ on $D' \times T$, for some open set $D' \subset D$.*

Proof: We start by showing that sections of $L^* \otimes E|_{0\times T}$ extend to a neighborhood $W \times T$ of $0 \times T$ in $D \times T$. Let $\mathcal{I}$ be the ideal sheaf defining $\mathcal{O}_T$ in $\mathcal{O} = \mathcal{O}_{D\times T}$.

We then have the following exact sequence

$$0 \longrightarrow \mathcal{I}^{n-1}/\mathcal{I}^n \longrightarrow \mathcal{O}/\mathcal{I}^n \longrightarrow \mathcal{O}/\mathcal{I}^{n-1} \longrightarrow 0.$$

Since the conormal bundle N^* of T in $D \times T$ is holomorphically trivial, $\mathcal{I}^{n-1}/\mathcal{I}^n \cong S^{n-1}(N^*) \cong \mathcal{O}$ and this sequence becomes

$$0 \longrightarrow \mathcal{O} \longrightarrow \mathcal{O}/\mathcal{I}^n \longrightarrow \mathcal{O}/\mathcal{I}^{n-1} \longrightarrow 0.$$

Let $\mathcal{F} = \mathcal{O}(L^* \otimes E|_{0\times T})$, $\mathcal{F}_{n-1} = \mathcal{F} \otimes \mathcal{O}/\mathcal{I}^{n-1}$ and $\mathcal{F}_n = \mathcal{F} \otimes \mathcal{O}/\mathcal{I}^n$. Let us note that in this notation, $\mathcal{F}_0 = \mathcal{F}$. Tensoring the above sequence by $\mathcal{F}$ gives us

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}_n \longrightarrow \mathcal{F}_{n-1} \longrightarrow 0.$$

By the long cohomology exact sequence, since $H^1(T, \mathcal{F}) = 0$,

$$\ldots \longrightarrow H^0(T, \mathcal{F}_n) \longrightarrow H^0(T, \mathcal{F}_{n-1}) \longrightarrow 0,$$

and sections from the (n-1)-th formal neighborhood of T extend to the n-th formal neighborhood of T. Therefore, by Grothendieck's theorem on formal functions [Ha], sections of $\mathcal{F}$ extend to a neighborhood $D' \times T$ of $0 \times T$ in $D \times T$. $\square$

Remarks: i) In specific cases, we can choose a line bundle L of smaller degree. The proofs in these cases are similar.

ii) We are interested in classifying rank 2 vector bundles over $D \times T$. We therefore need to know whether or not one can find a nowhere zero map from L to a fixed E over $D \times T$. Proposition 2.7 implies that there is no local obstruction, but we need to find such a map globally over $D \times T$. Even though a global nowhere zero map should be possible to find, there does not seem to be an obvious proof.

2.3.2 Local isomorphisms.

In section 2.2.2, we gave all the possible isomorphisms of SL$(2, \mathbb{C})$-bundles over T. Let us now describe the situation over $D \times T$. The following two lemmas were proven in [BH].

Lemma 2.4 *Let E, E' be* $\mathrm{SL}(2,\mathbb{C})$*-bundles over $D \times T$, D simply connected $\subset \mathbb{C}$, which have the same graph in $D \times \mathbb{P}^1$. Let this graph be that of a rational map $\varrho : D \to \mathbb{P}^1$. Assume that both E and E' have the property that they do not restrict to $L_0 \oplus L_0, L_0^2 = \mathcal{O}$, over any $\{x\} \times T$. Then $E \cong E'$.*

Lemma 2.5 *Let E, E' be* $\mathrm{SL}(2,\mathbb{C})$*-bundles over $D \times T$, D simply connected $\subset \mathbb{C}$, which have the same graph in $D \times \mathbb{P}^1$. Let this graph be of the form $(\{z_0\} \times \mathbb{P}^1) + (\mathbb{P}^1 \times \{l\})$. Again, suppose that E and E' have the property, if l is a half period, that they do not restrict to $L_0 \oplus L_0, L_0^2 = \mathcal{O}$, over any $\{x\} \times T$. Let $E \cong E'$ over $\{z_0\} \times T$. Then $E \cong E'$.*

Proof: Let us give a sketch of the proof of lemma 2.5 to illustrate how it can be extended to two other cases. Suppose that over $\{z_0\} \times T$ we have $E = L \oplus L^*$, for some L with $c_1(L) = -1$. It was then proven in [BH] that, in this case, one can write E globally as an extension

$$0 \to L \to E \to L^* \to 0.$$

$E_z = E|_{z \times T}$ is determined by the extension class $e(z) \in W = H^1(T, L^2)$. Away from z_0, each E_z is by a nonzero element of W, and therefore corresponds to an element of $\mathbb{P}(W) = \mathbb{P}^1 = \mathrm{Pic}^0(\mathrm{T})/\pm$. Let $L_0 \in \mathrm{Pic}^0(\mathrm{T})$ be such that $l = \{L_0, L_0^*\}$. As the map portion of the graph is constant, $e(z)$ therefore takes its values in a line $V \subset W$; and $e(z)$ can be thought of as a function $D \to \mathbb{C}$. This function only vanishes at $z = z_0$. As the $\mathrm{GL}(2,\mathbb{C})$-isomorphism type is invariant under rescaling of $e(z)$, we can set $e(z) = (z - z_0)$. E is then isomorphic to a standard extension. The same is true of E'. E and E' are thus isomorphic as $\mathrm{GL}(2,\mathbb{C})$-bundles. However, D is simply connected. Therefore, taking an appropriate square root gives an $\mathrm{SL}(2,\mathbb{C})$-isomorphism. $\square$

These two lemmas extend to two cases that will be useful to us in chapter 6:

Lemma 2.6 *Let E, E' be* $\mathrm{SL}(2,\mathbb{C})$*-bundles over $D \times T$, D simply connected $\subset \mathbb{C}$, which have the same graph in $D \times \mathbb{P}^1$. Let this graph be of the form $2(\{z_0\} \times \mathbb{P}^1) +$*

$(\mathbb{P}^1 \times \{l\})$. *Again, suppose that E and E' have the property, if l is a half period, that they do not restrict to $L_0 \oplus L_0, L_0^2 = \mathcal{O}$, over any $\{x\} \times T$. We also suppose that over $\{z_0\} \times T$ we have $E \cong E' = L \oplus L^*$, with $c_1(L) = -1$. Then $E \cong E'$.*

Lemma 2.7 *Let E, E' be $\mathrm{SL}(2, \mathbb{C})$-bundles over $D \times T$, D simply connected $\subset \mathbb{C}$, which have the same graph in $D \times \mathbb{P}^1$. Let this graph consist of the two pieces:*

- *the graph of a rational map $\varrho : D \to \mathbb{P}^1$,*
- *a vertical fibre $\{z_0\} \times \mathbb{P}^1$.*

Again, suppose that E and E' have the property, if $\varrho(x)$ is a half period, that they do not restrict to $L_0 \oplus L_0, L_0^2 = \mathcal{O}$, over $\{x\} \times T$. Let $E \cong E'$ over $\{z_0\} \times T$. Then $E \cong E'$.

As we are only considering graphs where the vertical bar corresponds to $E_{z_0 \times T} = L \oplus L^*$, where $c_1(L) = -1$, away from z_0, these extensions take values in $\mathbb{P}(W) = \mathbb{P}^1 = \mathrm{Pic}^0(\mathrm{T})/\pm$. For similar reasons to the ones above, one can show that they are given by standard extensions, and we thus have $E \cong E'$ over $D \times T$.

Remark: In lemma 2.6, we cannot consider bundles E, E' such that over $\{z_0\} \times T$ we have $E \cong E' = L \oplus L^*$, with $c_1(L) = -2$. This stems from the fact that, for such an L, the extensions of L^* by L are in $\mathbb{P}(H^1(T, L^2)) = \mathbb{P}^3$. However, there are divisors in $\mathbb{P}^3$ which are cones corresponding to singular quadrics (see [BH]), and by the remark following proposition 2.7, this may prevent us from expressing E as a global extension of L^* by L on $D \times T$.

2.3.3 Transition functions for rank two bundles.

We would like to give an explicit expression of the transition functions of rank two bundles, that have a given graph, over $D \times T$. These constructions can be generalised to any bundle of $D \times T$. The notation of sections 2.2.1 and 2.2.3 will be used in the following.

Let E have graph $n(z_0 \times \mathbb{P}^1) + (\mathbb{P}^1 \times \{l\})$, where $\{l\} = (L_0, L_0^*)$, for $L_0 \in$

$\mathrm{Pic}^0(\mathrm{T})$. We assume that L_0 is not a half period, i.e. $L_0^2 \neq \mathcal{O}$. Suppose that $E|_{z_0 \times T} = L \oplus L^*$, where $\deg(L) = -1$. On $D \times T$, we can express E as an extension

$$0 \longrightarrow L \longrightarrow E \longrightarrow L^* \longrightarrow 0.$$

This extension is given by a class (χ) in $\mathrm{H}^1(\mathrm{D} \times \mathrm{T}, \mathrm{L}^2) = \mathbb{C}^2$. Let us cover $D \times T$ by the two open set $V_0 = D \times U_0$ and $V_1 = D \times U_1$. If L is given by the divisor $-p_\lambda$,

$$M(\lambda) = M_{10}(\lambda, z) = \begin{bmatrix} g & \gamma \\ 0 & g^{-1} \end{bmatrix}$$

is a transition function for E on $D \times T$, where $g = \sigma(z)e^{-\lambda\zeta(z)}$ and γ is a multiple of χ.

We would like to give an explicit expression for $M(\lambda)$. Suppose that L_0 is given by the divisor $D_0 = p_\mu - p_0$. Then, as we have seen in the example of section 2.2.3, there are sections $r(z)$ and $s(z)$ which define an injective bundle map $L \to L_0 \oplus L_0^*$. Let $i = 0$, or 1. Since $r_i(z), s_i(z)$ do not have the same zeroes, we have the following exact sequence on V_i

$$\mathcal{O}(L) \xrightarrow{(r_i, -s_i)} \mathcal{O}(L_0) \oplus \mathcal{O}(L_0^*) \xrightarrow{\begin{bmatrix} s_i & r_i \end{bmatrix}} \mathcal{O}(L^*).$$

As U_i is a non-compact Riemann surface, $H^1(V_i, \mathcal{O}) = 0$, and there exist holomorphic functions $\alpha_i, \beta_i \in \mathcal{O}(V_i)$, such that $\alpha_i r_i + \beta_i s_i = 1$. This then induces the exact sequence

$$\mathcal{O}(L) \xrightarrow{(x^n, r_i, -s_i)} \mathcal{O}(L) \oplus \mathcal{O}(L_0) \oplus \mathcal{O}(L_0^*) \xrightarrow{\begin{bmatrix} 1 & -x^n\alpha_i & x^n\beta_i \\ 0 & s_i & r_i \end{bmatrix}} \mathcal{O}(E).$$

On $V_0 \cap V_1$, these sequences then give us the commutative diagram

$$\begin{tikzcd}[column sep=large]
\mathcal{O}(L) \arrow[r, "{(x^n, r_0, -s_0)}"] \arrow[d, "g"'] & \mathcal{O}(L) \oplus \mathcal{O}(L_0) \oplus \mathcal{O}(L_0^*) \arrow[rr, "{\begin{bmatrix} 1 & -x^n\alpha_0 & x^n\beta_0 \\ 0 & s_0 & r_0 \end{bmatrix}}"] \arrow[d, "{\begin{pmatrix} g & 0 & 0 \\ 0 & h & 0 \\ 0 & 0 & h^{-1} \end{pmatrix}}"] & & \mathcal{O}(E) \arrow[d, "M(\lambda)"'] \\
\mathcal{O}(L) \arrow[r, "{(x^n, r_1, -s_1)}"'] & \mathcal{O}(L) \oplus \mathcal{O}(L_0) \oplus \mathcal{O}(L_0^*) \arrow[rr, "{\begin{bmatrix} 1 & -x^n\alpha_1 & x^n\beta_1 \\ 0 & s_1 & r_1 \end{bmatrix}}"'] & & \mathcal{O}(E)
\end{tikzcd}$$

By commutativity, we obtain the two equations

$$\begin{cases} x^n\beta_1 h^{-1} = x^n g\beta_0 + r_0\gamma, \\ -x^n\alpha_1 h = -x^n g\alpha_0 + s_0\gamma. \end{cases}$$

Multiplying the first equation by α_0 and the second one by β_0, and then adding, we get

$$x^n(\alpha_0\beta_1 h^{-1} - \beta_0\alpha_1 h) = (\alpha_0 r_0 + \beta_0 s_0)\gamma.$$

And, as $\alpha_0 r_0 + \beta_0 s_0 = 1$,

$$\gamma = x^n(\alpha_0\beta_1 h^{-1} - \beta_0\alpha_1 h), \tag{2.6}$$

giving us an explicit description of $M(\lambda)$.

The matrices $M(\lambda)$ will be used in the next chapter, at which point we will be interested in knowing how they are transformed when we move λ by a period $2\omega_i$, for $i = 1, 2$. Let us first look at what happens to γ. If we consider γ as a function of λ, we have

Lemma 2.8 *For $i = 1, 2$,*

$$\gamma(\lambda + 2\omega_i) = \gamma(\lambda)\phi_i^{-1}(z)C,$$

where $C = e^{-4\eta_i(\lambda + 2\omega_i)}$.

Proof: To simplify notation, let $a = \lambda + \mu$, and $b = \lambda - \mu$. Adding $2\omega_i$ to λ then corresponds to adding it to a and b. If we replace λ by $\lambda + 2\omega_i$ in the expressions of $r(z)$ and $s(z)$, we get

$$\begin{aligned} r_0(\lambda + 2\omega_i) &= r_0(\lambda)e^{-2\eta_i(z-(a+2\omega_i))}, \text{ and} \\ s_0(\lambda + 2\omega_i) &= s_0(\lambda)e^{-2\eta_i(z-(b+2\omega_i))}. \end{aligned} \tag{2.7}$$

As $\alpha_0 r_0 + \beta_0 s_0 = 1$ for any λ, we must have,

$$\begin{aligned} \alpha_0(\lambda + 2\omega_i)r_0(\lambda + 2\omega_i) + \beta_0(\lambda + 2\omega_i)s_0(\lambda + 2\omega_i) \\ = \alpha_0(\lambda)r_0(\lambda) + \beta_0(\lambda)s_0(\lambda). \end{aligned}$$

By using equation (2.7), we can rewrite this as

$$\begin{aligned} \left[\alpha_0(\lambda) - \alpha_0(\lambda + 2\omega_i)e^{-2\eta_i(z-(a+2\omega_i))}\right] r_0(\lambda) \\ + \left[\beta_0(\lambda) - \beta_0(\lambda + 2\omega_i)e^{-2\eta_i(z-(b+2\omega_i))}\right] s_0(\lambda) = 0. \end{aligned}$$

As r_0 and s_0 have no common zeroes, it implies that

$$\begin{aligned} \alpha_0(\lambda + 2\omega_i, z) &= \alpha_0(\lambda, z)e^{2\eta_i(z-(a+2\omega_i))}, \text{ and} \\ \beta_0(\lambda + 2\omega_i, z) &= \beta_0(\lambda, z)e^{2\eta_i(z-(b+2\omega_i))}. \end{aligned}$$

Similarly computations give us

$$\begin{aligned} \alpha_1(\lambda + 2\omega_i, z) &= \alpha_1(\lambda, z)e^{2\eta_i(z-(a+2\omega_i))}e^{-2\omega_i\zeta(z)}, \text{ and} \\ \beta_1(\lambda + 2\omega_i, z) &= \beta_1(\lambda, z)e^{2\eta_i(z-(b+2\omega_i))}e^{-2\omega_i\zeta(z)}. \end{aligned}$$

Inserting these into the expression of γ, we obtain

$$\gamma(\lambda + 2\omega_i, z) = \gamma(\lambda)e^{-2\omega_i\zeta(z)}e^{2\eta_i z}e^{-2\eta_i(a+b+4\omega_i)}.$$

In section 2.2.1, we defined $\phi_i(z) = e^{2\omega_i\zeta(z)}e^{-2\eta_i z}$. And, as $a + b = 2\lambda$, this can be written

$$\gamma(\lambda + 2\omega_i, z) = \gamma(\lambda)\phi_i^{-1}(z)C,$$

where $C = e^{-4\eta_i(\lambda+2\omega_i)}$. $\square$

As $g(\lambda + 2\omega_i) = \phi_i^{-1}g(\lambda)\psi_i$, lemma 2.8 then implies that

$$M(\lambda + 2\omega_i) = \begin{bmatrix} \phi_i^{-1} & 0 \\ 0 & \phi_i \end{bmatrix} \begin{bmatrix} g(\lambda) & \gamma(\lambda)C\psi_i \\ 0 & g^{-1}(\lambda) \end{bmatrix} \begin{bmatrix} \psi_i & 0 \\ 0 & \psi_i^{-1} \end{bmatrix}.$$

This can rewritten as

$$M(\lambda+2\omega_i)=\begin{bmatrix}\phi_i^{-1}C & 0\\ 0 & \phi_i C^{-1}\end{bmatrix}\begin{bmatrix}g & \gamma C^{-1}\psi_i\\ 0 & g^{-1}\end{bmatrix}\begin{bmatrix}\psi_i C^{-1} & 0\\ 0 & \psi_i^{-1}C\end{bmatrix}.$$

Remark: Let us note that, since $\phi_i C^{-1}$ is doubly periodic on V_1, and $\psi_i C^{-1} = e^{-2\eta_i z}$ is holomorphic everywhere on V_0, then the transition matrices $M(\lambda+2\omega_i)$ and $\begin{bmatrix}g & \gamma C^{-1}\psi_i\\ 0 & g^{-1}\end{bmatrix}$ define the same vector bundle.

Chapter 3

Stabilisation maps.

Let $\mathcal{E}$ be a fixed rank two C^∞ vector bundle on $\mathcal{H}$ with $c_1(\mathcal{E}) = 0$, $c_2(\mathcal{E}) = n$. Let us also fix a base point p_0 in $\mathcal{H}$. $\mathcal{M}_n^0$ is then the moduli space of equivalence classes of pairs (E, ψ) where E is a stable holomorphic structure on $\mathcal{E}$ with $\Lambda^2 E \cong \mathcal{O}$, and ψ is a trivialisation of E at p_0, i.e. $\mathcal{M}_n^0$ is the moduli space of *framed instantons* of charge n on $\mathcal{H}$. In this chapter, we study stabilisation maps $\mathcal{M}_n^0 \to \mathcal{M}_{n+1}^0$ on these spaces.

If $\tilde{\mathcal{B}}_n$ is the moduli space of *framed connections* on $\mathcal{E}$, where a framed connection is now a pair (A, ψ), where A is any connection and ψ is a framing at p_0, we shall see in chapter 7 that stabilisation maps

$$f_{n,n+1} : \tilde{\mathcal{B}}_n \to \tilde{\mathcal{B}}_{n+1}$$

always exist. They are constructed by glueing an instanton of charge 1 at p_0. We would like to know if it is possible to define an analogous map $g_{n,n+1} : \mathcal{M}_n^0 \to \mathcal{M}_{n+1}^0$ in the holomorphic setting.

The holomorphic version of the maps $f_{n,n+1}$ is constructed by glueing in a copy of $\mathcal{O}\oplus I$ at p_0, and deforming the new sheaf to obtain a locally free sheaf. Let us recall that $\bar{\mathcal{M}}_{n+1}$ is the moduli space of stable simple coherent sheaves $\mathcal{F}$ such that $c_2(\mathcal{F}) = n$. If one forgets the framing, the set of sheaves obtained by glueing in $\mathcal{O}\oplus I$ is a stratum of $\bar{\mathcal{M}}_{n+1}$ that we will denote by $\mathcal{M}_{n+1,n}$. We will study the tangent bundle of $\bar{\mathcal{M}}_{n+1}$ along $\mathcal{M}_{n+1,n}$, which corresponds to the space of deformations of the sheaves in $\mathcal{M}_{n+1,n}$. We will see that the locally free deformations are elements of the normal bundle $\mathcal{N}_{\bar{\mathcal{M}}_{n+1}/\mathcal{M}_{n+1,n}}$, and that the normal bundle is a nontrivial line bundle. This will then lead us to conclude that this holomorphic stabilisation map cannot be defined globally on $\mathcal{M}_n^0$.

3.1 Glueings of $\mathcal{O} \oplus I$.

Throughout this chapter, we assume that I is the ideal sheaf of a point $p_0 \in \mathcal{H}$. If (x, z) are "standard" coordinates centered at p_0, we have that I is generated by $\{x, \sigma(z)\}$, where $\sigma(z)$ is the sigma function defined in section 2.2.1.

3.1.1 Glueings.

To glue in a copy of $\mathcal{O}\oplus I$, it is necessary to specify an inclusion of $\mathcal{O}\oplus I$ into $\mathcal{O}\oplus\mathcal{O}$. We begin by giving a description of the set of inclusions of $\mathcal{O} \oplus I$ into $\mathcal{O} \oplus \mathcal{O}$ as the kernel of a surjective $\mathcal{O}$-homomorphism $\mathcal{O} \oplus \mathcal{O} \to \mathcal{O}/I$. As $Hom_{\mathcal{O}}(\mathcal{O}, \mathcal{O}/I) \cong \mathcal{O}/I \cong \mathbb{C}$, any such map is given by

$$\mathcal{O} \oplus \mathcal{O} \xrightarrow{[-c \;\; a]} \mathcal{O}/I, \tag{3.1}$$

where $[-c \;\; a] \neq [0 \;\; 0]$ and $a, c \in \mathbb{C}$. We consider two such maps to be equivalent if their kernels are equal as subsets of $\mathcal{O} \oplus \mathcal{O}$. If l is the line through the origin in $\mathbb{C}^2$ passing through (a, c), then

$$\mathcal{O} \oplus I = \{s \in \mathcal{O} \oplus \mathcal{O} \mid s(p_0) \in l\}.$$

Thus, for any $k \neq 0$ in $\mathbb{C}$, $\mathcal{O} \oplus \mathcal{O} \xrightarrow{k[-c \;\; a]} \mathcal{O}/I$ has the same kernel as (3.1). The set of equivalence classes of these maps is therefore

$$(\{[-c \;\; a] \mid a, c \in \mathbb{C}\} - [0 \;\; 0])/\mathbb{C}^* \cong \mathbb{P}^1.$$

Let us denote this set by X_0. We can cover X_0 by the two open sets

$$U_0 = \{[-\tilde{c} \;\; 1] \in X_0 \mid \tilde{c} \in \mathbb{C}\} = \mathbb{C},$$

and

$$U_1 = \{[1 \;\; -\tilde{a}] \in X_0 \mid \tilde{a} \in \mathbb{C}\} = \mathbb{C}.$$

3.1.2 Tangent space to X_0.

The tangent bundle of $\mathbb{P}^1$ is $T\mathbb{P}^1 = \mathcal{O}(2) = \mathcal{H}om_{\mathcal{O}}(\mathcal{O}(-1), \mathcal{O}(1))$. Consider a line $l \in \mathbb{P}^1$. The tangent space to $\mathbb{P}^1$ at l can then be described as

$$T\mathbb{P}^1|_l = Hom_{\mathbb{C}}(l, \mathbb{C}^2/l).$$

Let $\alpha \in X_0$. The line l in $\mathbb{P}^1 \cong X_0$ corresponding to α is therefore $l = (\ker\alpha)_{p_0}$, and $\mathcal{O}/I = \mathbb{C}^2/l$. By the above, the tangent space to X_0 at α is given by

$$T_\alpha X_0 = Hom_{\mathcal{O}_{p_0}}((\ker\alpha)_{p_0}, \mathbb{C}^2/(\ker\alpha)_{p_0}).$$

3.1.3 X_0 as projective resolutions.

When working with projective resolutions of $\mathcal{O}\oplus I$, it is more convenient to describe X_0 as a set of equivalence classes of maps $\mathcal{O}\oplus I \to \mathcal{O}\oplus\mathcal{O}$. Two such maps are now considered to be equivalent if their images are equal as subsets of $\mathcal{O}\oplus\mathcal{O}$. Suppose that we start with

$$\mathcal{O}\oplus\mathcal{O} \xrightarrow{[-c\ \ a]} \mathcal{O}/I,$$

and complete it to an exact sequence

$$\mathcal{O}\oplus I \xrightarrow{T} \mathcal{O}\oplus\mathcal{O} \xrightarrow{[-c\ \ a]} \mathcal{O}/I,$$

where T is a matrix of the form $T = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, with $\det T \neq 0$. The image of $\mathcal{O}\oplus I$ in $\mathcal{O}\oplus\mathcal{O}$ is then independent of the choice of b and d.

As we have seen, $\mathcal{O}\oplus I$ can be described as the cokernel of the map

$$\mathcal{O} \xrightarrow{(0, -\sigma(z), x)} \mathcal{O}\oplus\mathcal{O}\oplus\mathcal{O}.$$

If $\mathcal{O}\oplus I$ is the image of T in $\mathcal{O}\oplus\mathcal{O}$, it will then have the following projective resolution

$$\mathcal{O} \xrightarrow{(0, -\sigma(z), x)} \mathcal{O}\oplus\mathcal{O}\oplus\mathcal{O} \xrightarrow{\begin{bmatrix} a & bx & b\sigma(z) \\ c & dz & d\sigma(z) \end{bmatrix}} \mathcal{O}\oplus I.$$

Keeping in mind the above discussion, we choose to cover X_0 by the following open sets U_0, U_1:

$$U_0 = \{T_{\tilde{c}} = \begin{bmatrix} 1 & 0 \\ \tilde{c} & 1 \end{bmatrix}; \tilde{c} \in \mathbb{C}\} = \mathbb{C},$$

and

$$U_1 = \{T'_{\tilde{a}} = \begin{bmatrix} \tilde{a} & -1 \\ 1 & 0 \end{bmatrix}; \tilde{a} \in \mathbb{C}\} = \mathbb{C},$$

with the identification

$$\begin{aligned} T'_{\tilde{a}} &= T'_{1/\tilde{c}} \\ &= T_{\tilde{c}} \begin{bmatrix} 1/\tilde{c} & -1 \\ 0 & \tilde{c} \end{bmatrix} \end{aligned}$$

on the overlap.

3.2 The singular stratum $\mathcal{M}_{n+1,n}$.

In this section we consider the moduli space $\mathcal{M}_n$ of stable SL(2, $\mathbb{C}$) vector bundles on $\mathcal{H}$, with $c_2 = n$, as well as the moduli space $\bar{\mathcal{M}}_n$ of stable simple sheaves of rank 2 on $\mathcal{H}$, with $c_2 = n$. Theses spaces were introduced in section 1.6.2. For every n, $\mathcal{M}_n$ is contained in $\bar{\mathcal{M}}_n$.

3.2.1 Definition of $\mathcal{M}_{n+1,n}$.

If $\mathcal{F}$ is any sheaf in $\bar{\mathcal{M}}_n$, let sing($\mathcal{F}$) be the singular set of $\mathcal{F}$:

$$\text{sing}(\mathcal{F}) = \{\text{p} \in \mathcal{H} \mid \mathcal{F}_{\text{p}} \text{ isnotfreeover } \mathcal{O}_{\text{p}}\}.$$

Let us consider the subset of $\bar{\mathcal{M}}_{n+1}$ consisting of sheaves which are locally free everywhere except at a single point, where they are isomorphic to $\mathcal{O} \oplus I$:

$$\mathcal{M}_{n+1,n} = \left\{ \mathcal{F} \in \bar{\mathcal{M}}_{n+1} \mid \begin{array}{c} \text{sing}(\mathcal{F}) = \text{a single point } p_0, \\ \text{and } \mathcal{F} \cong \mathcal{O} \oplus I \text{ around } p_0 \end{array} \right\}.$$

In this subsection, we would like to show that $\mathcal{M}_{n+1,n}$ fibers over $\mathcal{H} \times \mathcal{M}_n$ with fiber X_0, i.e. there exists a fibration

$$\begin{array}{ccc} X_0 & \longrightarrow & \mathcal{M}_{n+1,n} \\ & & \Big\downarrow P \\ & & \mathcal{H} \times \mathcal{M}_n \end{array} \tag{3.2}$$

For any coherent sheaf $\mathcal{G}$, let $\mathcal{G}^* = Hom_{\mathcal{O}}(\mathcal{G}, \mathcal{O})$ be its the dual sheaf. The projection P of this fibration is then defined to be the map

$$\begin{array}{rcl} P : \mathcal{M}_{n+1,n} & \longrightarrow & \mathcal{H} \times \mathcal{M}_n \\ \mathcal{F} & \longmapsto & (p_0, \mathcal{F}^{**}) \end{array},$$

where $p_0 = \text{sing}(\mathcal{F})$, for all $\mathcal{F} \in \mathcal{M}_{n+1,n}$. That this map well defined is proven by

Lemma 3.1 *If $\mathcal{F} \in \mathcal{M}_{n+1,n}$, then*

(i) $\mathcal{F}^*$ is locally free;

(ii) we have the exact sequence

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}^{**} \longrightarrow \mathcal{O}/I \longrightarrow 0;$$

(iii) $c_1(\mathcal{F}^{**}) = 0$, and $c_2(\mathcal{F}^{**}) = n$; and

(iv) $\mathcal{F}^{**}$ is stable.

Proof: Let $\mathcal{F} \in \mathcal{M}_{n+1,n}$. Let p_0 is the unique singular point of $\mathcal{F}$. $\mathcal{F}$ is then locally free away from p_0, and $\mathcal{F} \cong \mathcal{O} \oplus I$ around p_0. Therefore, as $Hom_{\mathcal{O}}(I, \mathcal{O}) \cong \mathcal{O}$, (see lemma 1.2), $\mathcal{F}^*$ must be locally free, proving (i).

(ii) follows from the fact that, away from p_0, $\mathcal{F}$ is isomorphic to $\mathcal{F}^{**}$.

(iii) The exact sequence in (ii) gives us

$$ch(\mathcal{F}^{**}) = ch(\mathcal{F}) + ch(\mathcal{O}/I).$$

If $[\mathcal{H}]$ is the positive generator of $H^4(\mathcal{H}, \mathbb{Z})$, $ch(\mathcal{F}) = 2 + (n+1)[\mathcal{H}]$, and $ch(\mathcal{O}/I) = -[\mathcal{H}]$. Therefore $ch(\mathcal{F}^{**}) = 2 + n[\mathcal{H}]$.

(iv) As $\mathcal{F}$ is torsion free and stable, (iv) follows from lemma 1.9. $\square$

In order to show that P is a surjective fibration, and that its fiber is X_0, let us now give a more explicit description of $\mathcal{M}_{n+1,n}$. Let $E \in \mathcal{M}_n$. Let us also fix a point $p_0 \in \mathcal{H}$, and a trivialisation of E at p_0. For any line $l \in X_0 \cong \mathbb{P}^1$, we then define the following subsheaf $\tilde{E}_{l,p_0}$ of E:

$$\tilde{E}_{l,p_0} = \{s \in \mathcal{O}(E) | s(p_0) \in l\}.$$

$\tilde{E}_{l,p_0}$ is simply E away from p_0. Moreover, if U is an open set containing p_0 on which E is trivialised, then $\tilde{E}_{l,p_0}|_U = \mathcal{O} \oplus I$, and the inclusion

$$\mathcal{O} \oplus I \xhookrightarrow{T} \mathcal{O} \oplus \mathcal{O},$$

is given by the matrix T corresponding to l. Therefore, $(\tilde{E}_{l,p_0})^{**} \cong E$, and we have the exact sequence

$$0 \longrightarrow \tilde{E}_{l,p_0} \longrightarrow E \longrightarrow \mathcal{O}/I \longrightarrow 0.$$

By similar arguments to the ones used in lemma 3.1, we can show that

Lemma 3.2 *For any $E \in \mathcal{M}_n$, $\tilde{E}_{l,p_0}$ is a stable sheaf of rank 2 with $c_1(\tilde{E}_{l,p_0}) = 0$ and $c_2(\tilde{E}_{l,p_0}) = n + 1$.*

$\square$

Remark: Let us note that for any $\mathcal{F} \in \mathcal{M}_{n+1,n}$, if $p_0 = \mathrm{sing}(\mathcal{F})$ and $\mathcal{F}$ is given by $l \in X_0$, then $\mathcal{F} \cong \tilde{\mathcal{F}}^{**}_{l,p_0}$.

We can now define a local section of the map P. Let D be an open disc in $\mathcal{H}$, and W be an open subset of $\mathcal{M}_n$, such that any bundle $E \in W$ is trivialised on D. Let us then fix a local trivialisation t on D: for any $E \in W$, we have $t : E|_D \cong \mathcal{O} \oplus \mathcal{O}|_D$. Let us fix a line l in $\mathbb{P}^1$. If we restrict ourselves to $D \times W \subset \mathcal{H} \times \mathcal{M}_n$, we then have the following well-defined local section of P

$$\begin{array}{rcl} D \times W & \to & \mathcal{M}_{n+1,n} \\ (p_0, E) & \mapsto & \tilde{E}_{l,p_0} \end{array}$$

This local section is of course independent of the choice of l. P is clearly a surjective fibration with fiber X_0:

$$\begin{tikzcd} X_0 \arrow[r] & \mathcal{M}_{n+1,n} \arrow[d, "P"] \\ & \mathcal{H} \times \mathcal{M}_n \end{tikzcd}$$

Any element $\mathcal{F}$ of $\mathcal{M}_{n+1,n}$ is therefore determined by three things:

(i) its point of singularity $p_0 \in \mathcal{H}$;

(ii) the map $\mathcal{O} \oplus \mathcal{O} \xrightarrow{\alpha} \mathcal{O}/I$ in X_0 giving $\mathcal{O} \oplus I = \ker\alpha$ around p_0;

(iii) the transition functions A_{ij} of $\mathcal{F}^{**}$.

3.2.2 Tangent space of $\mathcal{M}_{n+1,n}$.

Let $\mathcal{F} \in \mathcal{M}_{n+1,n}$. Suppose that it is given by

$$\text{(i) } p_0 \in \mathcal{H}; \quad \text{(ii) } \alpha \in X_0; \quad \text{and} \quad \text{(iii) } \mathcal{F}^{**} \in \mathcal{M}_n.$$

The fibration (3.2) then gives us the following exact sequence of tangent spaces:

$$0 \longrightarrow (TX_0)_\alpha \longrightarrow T_{\mathcal{F}}\mathcal{M}_{n+1,n} \longrightarrow T_{p_0}\mathcal{H} \oplus T_{\mathcal{F}^{**}}\mathcal{M}_n \longrightarrow 0.$$

Each component can be described in terms of first order deformations as follows:

- Since changing the point $p_0 \in \mathcal{H}$ is equivalent to deforming I, we have

$$T_{p_0}\mathcal{H} = H^0(\mathcal{H}, \mathcal{E}xt^1_{\mathcal{O}}(I,I)) = (\text{Deformationsof}I),$$

- $$T_{\mathcal{F}^{**}}\mathcal{M}_n = H^1(\mathcal{H}, sl_{\mathcal{O}}(\mathcal{F}^{**})) = \begin{pmatrix} \text{Deformations of the} \\ \text{equivalence class} \\ \text{of the transition} \\ \text{functions of } \mathcal{F}^{**}. \end{pmatrix}.$$

- If $l = (\ker\alpha)_{p_0}$ is the line in $\mathbb{P}^1 \cong X_0$ corresponding to α, then $\mathcal{O}/I = \mathbb{C}^2/l$, and we have seen in section 3.1.2 that

$$T_\alpha X_0 = Hom_{\mathbb{C}}(l, \mathbb{C}^2/l) = \begin{pmatrix} \text{Deformations of the} \\ \text{map } \mathcal{O} \oplus \mathcal{O} \xrightarrow{\alpha} \mathcal{O}/I \end{pmatrix}.$$

3.2.3 The space $\bar{\mathcal{M}}_{n+1}$ at $\mathcal{M}_{n+1,n}$.

We would like to show

Theorem 3.1 *$\bar{\mathcal{M}}_{n+1}$ is a smooth complex manifold of dimension $4n+4$ along $\mathcal{M}_{n+1,n}$.*

3.2.4 Proof of Theorem 3.1.

By deformation theory, for any $\tilde{E} \in \mathcal{M}_{n+1,n}$, the tangent space to $\bar{\mathcal{M}}_{n+1}$ at $\tilde{E}$ is

$$T_{\tilde{E}}\bar{\mathcal{M}}_{n+1} = \mathrm{Ext}^1(\mathcal{H}; \tilde{\mathrm{E}}, \tilde{\mathrm{E}})_0,$$

where $\mathrm{Ext}^1(\mathcal{H}; \tilde{\mathrm{E}}, \tilde{\mathrm{E}})_0$ is the traceless component of $\mathrm{Ext}^1(\mathcal{H}; \tilde{\mathrm{E}}, \tilde{\mathrm{E}})$. Furthermore, the obstruction to the smoothness of $\bar{\mathcal{M}}_{n+1}$ is in $\mathrm{Ext}^2(\mathcal{H}; \tilde{\mathrm{E}}, \tilde{\mathrm{E}})$. In order to prove the theorem, we need to show that

a) $\mathrm{Ext}^1(\mathcal{H}; \tilde{E}, \tilde{E})_0 = \mathbb{C}^{4n+4}$; and

b) $\mathrm{Ext}^2(\mathcal{H}; \tilde{E}, \tilde{E}) = 0$.

These groups can be computed via the cohomology spectral sequence described in section 1.2. One of the main tools, for this computation, will be the exact sequence

$$0 \longrightarrow End_{\mathcal{O}}(\tilde{E}) \longrightarrow End_{\mathcal{O}}(E) \longrightarrow Hom_{\mathcal{O}}(\mathcal{O}, \mathcal{O}/I) \longrightarrow 0.$$

As $\tilde{E}$ is isomorphic to E, away from p_0, the exactness of this sequence only needs to be checked locally around p_0. This will be given by

Lemma 3.3 *There is an exact sequence*

$$0 \to Hom_{\mathcal{O}}(\mathcal{O} \oplus I, \mathcal{O} \oplus I) \longrightarrow Hom_{\mathcal{O}}(\mathcal{O} \oplus I, \mathcal{O} \oplus \mathcal{O}) \longrightarrow Hom_{\mathcal{O}}(\mathcal{O}, \mathcal{O}/I) \to 0.$$

Proof: The exact sequence

$$0 \longrightarrow \mathcal{O} \oplus I \xrightarrow{Id} \mathcal{O} \oplus \mathcal{O} \xrightarrow{[0\ \ 1]} \mathcal{O}/I \longrightarrow 0,$$

where Id is the identity, induces

$$0 \longrightarrow Hom_{\mathcal{O}}(\mathcal{O} \oplus I, \mathcal{O} \oplus I) \longrightarrow Hom_{\mathcal{O}}(\mathcal{O} \oplus I, \mathcal{O} \oplus \mathcal{O}) \xrightarrow{\beta}$$

$$\xrightarrow{\beta} Hom_{\mathcal{O}}(\mathcal{O} \oplus I, \mathcal{O}/I) = Hom_{\mathcal{O}}(\mathcal{O}, \mathcal{O}/I) \oplus Hom_{\mathcal{O}}(I, \mathcal{O}/I),$$

which is also exact. We have to show that $\mathrm{Im}(\beta) = \mathrm{Hom}_{\mathcal{O}}(\mathcal{O}, \mathcal{O}/\mathrm{I})$. As $Hom_{\mathcal{O}}(\mathcal{O} \oplus I, \mathcal{O} \oplus \mathcal{O}) = Hom_{\mathcal{O}}(\mathcal{O}, \mathcal{O} \oplus \mathcal{O}) \oplus Hom_{\mathcal{O}}(I, \mathcal{O} \oplus \mathcal{O})$, we simply need to verify that

i) $\beta(Hom_{\mathcal{O}}(\mathcal{O}, \mathcal{O} \oplus \mathcal{O})) = Hom_{\mathcal{O}}(\mathcal{O}, \mathcal{O}/I)$, and

ii) $Hom_{\mathcal{O}}(I, \mathcal{O} \oplus \mathcal{O}) \subset \mathrm{Ker}(\beta)$.

For any $(g_1, g_2) \in Hom_{\mathcal{O}}(\mathcal{O}, \mathcal{O} \oplus \mathcal{O})$, $\beta(g_1, g_2)$ is the $\mathcal{O}$-homomorphism in $Hom_{\mathcal{O}}(\mathcal{O}, \mathcal{O}/I)$ given by $1 \mapsto (-\tilde{c}g_1 + g_2)(p_0)$. Any element of $Hom_{\mathcal{O}}(\mathcal{O}, \mathcal{O}/I)$ can obviously be obtained this way, thus showing that $\beta(Hom_{\mathcal{O}}(\mathcal{O}, \mathcal{O} \oplus \mathcal{O})) = Hom_{\mathcal{O}}(\mathcal{O}, \mathcal{O}/I)$, and giving i).

Let us note that $Hom_{\mathcal{O}}(I, \mathcal{O} \oplus \mathcal{O}) \subset Hom_{\mathcal{O}}(I, I \oplus I)$. Indeed, as we saw in the proof of lemma 1.2 (iii), for any $\mathcal{O}$-homomorphism $f : I \to \mathcal{O}$, there exists an $h \in \mathcal{O}$ such that f is given by $x \mapsto hx$ and $z \mapsto hz$. We can therefore view any element of $Hom_{\mathcal{O}}(I, \mathcal{O} \oplus \mathcal{O})$ as a map given by $s \mapsto (h_1, h_2)s$, where $s \in I$, and $h_1, h_2 \in \mathcal{O}$. Hence, β maps every element (h_1, h_2) of $Hom_{\mathcal{O}}(I, \mathcal{O} \oplus \mathcal{O})$ to the map $I \to \mathcal{O}/I$ given by $s \mapsto [(-\tilde{c}h_1 + h_2)s](p_0)$. As $s(p_0) = 0$, for any $s \in I$, $\beta(h_1, h_2) = 0$. Thus $Hom_{\mathcal{O}}(I, \mathcal{O} \oplus \mathcal{O}) \subset \mathrm{Ker}(\beta)$, proving ii). $\square$

Remark: The proof of the above lemma actually shows that the maps in $Hom_{\mathcal{O}}(\mathcal{O}, \mathcal{O}/I) = \beta(Hom_{\mathcal{O}}(\mathcal{O} \oplus I, \mathcal{O} \oplus \mathcal{O}))$ can be thought of as maps $(\mathcal{O} \oplus I)_{p_0} \to \mathcal{O}/I$. Therefore, if $\mathcal{O} \oplus I$ is the kernel of the map $\mathcal{O} \oplus \mathcal{O} \xrightarrow{\alpha} \mathcal{O}/I$ in X_0, we see that

$$Hom_{\mathcal{O}}(\mathcal{O}, \mathcal{O}/I) = Hom_{\mathcal{O}_{p_0}}((\ker \alpha)_{p_0}, \mathbb{C}^2/(\ker \alpha)_{p_0}) = T_\alpha X_0.$$

Corollary 3.1 *(i) We have the exact sequence*

$$0 \to End_{\mathcal{O}}(\mathcal{O} \oplus I) \xrightarrow{\mathrm{Id}} End_{\mathcal{O}}(\mathcal{O} \oplus \mathcal{O}) \xrightarrow{[0\ \ 1]} Hom_{\mathcal{O}}(\mathcal{O}, \mathcal{O}/I) \to 0,$$

(ii) and, as a direct consequence,

$$0 \longrightarrow sl_{\mathcal{O}}(\mathcal{O} \oplus I) \xrightarrow{\mathrm{Id}} sl_{\mathcal{O}}(\mathcal{O} \oplus \mathcal{O}) \xrightarrow{[0\ \ 1]} Hom_{\mathcal{O}}(\mathcal{O}, \mathcal{O}/I) \longrightarrow 0$$

is also exact, where Id *is the identity.*

Proof: (i) As we saw the proof of lemma 1.2 (iii), $Hom_{\mathcal{O}}(I, \mathcal{O}) \cong Hom_{\mathcal{O}}(\mathcal{O}, \mathcal{O})$, where the one-to-one correspondence is given by

$$\left\{ \begin{array}{l} x \mapsto hx \\ y \mapsto hy \end{array} \right\} \leftrightarrow \{1 \mapsto h\}.$$

Thus, $End_{\mathcal{O}}(\mathcal{O} \oplus \mathcal{O}) \cong Hom_{\mathcal{O}}(\mathcal{O} \oplus I, \mathcal{O} \oplus \mathcal{O})$. Furthermore, (ii) is a direct consequence of (i). □

Globally, we have the short exact sequence

$$0 \longrightarrow \tilde{E} \longrightarrow E \longrightarrow \mathcal{O}/I \longrightarrow 0.$$

This sequence, with the above results, gives

Corollary 3.2 *(i)* $Hom_{\mathcal{O}}(\tilde{E}, E) \simeq Hom_{\mathcal{O}}(E, E)$;

(ii) There is an exact sequence

$$0 \longrightarrow End_{\mathcal{O}}(\tilde{E}) \longrightarrow End_{\mathcal{O}}(E) \longrightarrow Hom_{\mathcal{O}}(\mathcal{O}, \mathcal{O}/I) \longrightarrow 0;$$

(iii) There is an exact sequence

$$0 \longrightarrow sl_{\mathcal{O}}(\tilde{E}) \longrightarrow sl_{\mathcal{O}}(E) \longrightarrow Hom_{\mathcal{O}}(\mathcal{O}, \mathcal{O}/I) \longrightarrow 0.$$

□

Furthermore, we have

Lemma 3.4 *(i) There is an exact sequence*

$$H^1(\mathcal{H}, sl_{\mathcal{O}}(E)) \longrightarrow H^1(\mathcal{H}, sl_{\mathcal{O}}(\tilde{E})) \longrightarrow H^0(\mathcal{H}, Hom_{\mathcal{O}}(\mathcal{O}, \mathcal{O}/I)),$$

which gives $H^1(\mathcal{H}, sl_{\mathcal{O}}(\tilde{E})) = \mathbb{C}^{4n+1}$;

(ii) $H^2(\mathcal{H}, Hom_{\mathcal{O}}(\tilde{E}, \tilde{E})) = 0$*;*

(iii) $Ext^1_{\mathcal{O}}(\tilde{E}, \tilde{E})$ *is a skyscraper sheaf supported at* p_0 *with fiber* $\mathbb{C}^3$*; and*

(iv) $Ext^k_{\mathcal{O}}(\tilde{E}, \tilde{E}) = 0$*, for* $k \geq 2$*.*

Proof: (iii) and (iv) are a direct consequence of lemma 3.3.

(i) The exact sequence follows from corollary 3.2 (iii) induces the long exact sequence on cohomology

$$\ldots \to H^i(\mathcal{H}, sl_{\mathcal{O}}(\tilde{E})) \to H^i(\mathcal{H}, sl_{\mathcal{O}}(E)) \to H^i(\mathcal{H}, Hom_{\mathcal{O}}(\mathcal{O}, \mathcal{O}/I)) \to \ldots$$

Since $Hom_{\mathcal{O}}(\mathcal{O}, \mathcal{O}/I)$ is a skyscraper sheaf with fiber $\mathcal{O}/I$ supported at p_0, $H^i(\mathcal{H}, Hom_{\mathcal{O}}(\mathcal{O}, \mathcal{O}/I)) = 0$, for $i \geq 1$. Moreover the stability of E implies that $H^0(\mathcal{H}, sl_{\mathcal{O}}(E)) = 0$. Therefore $H^0(\mathcal{H}, sl_{\mathcal{O}}(\tilde{E})) = 0$, and the above sequence reduces to

$$0 \to H^0(\mathcal{H}, Hom_{\mathcal{O}}(\mathcal{O}, \mathcal{O}/I)) \to H^1(\mathcal{H}, sl_{\mathcal{O}}(\tilde{E})) \to H^1(\mathcal{H}, sl_{\mathcal{O}}(E)) \to 0.$$

Furthermore, since $\dim(\mathcal{M}_{\mathrm{n}}) = 4\mathrm{n}$, $H^1(\mathcal{H}, sl_{\mathcal{O}}(E)) = T_E\mathcal{M}_n = \mathbb{C}^{4n}$.

(ii) As $H^i(\mathcal{H}, Hom_{\mathcal{O}}(\mathcal{O}, \mathcal{O}/I)) = 0$, for $i = 1, 2$, the long exact sequence on cohomology induced by the sequence of corollary 3.2 (ii) gives us that $H^2(\mathcal{H}, End_{\mathcal{O}}(\tilde{E})) \cong H^2(\mathcal{H}, End_{\mathcal{O}}(E))$. By stability of E, $H^2(\mathcal{H}, End_{\mathcal{O}}(E)) = 0$, thus implying (ii). $\square$

Proof of Theorem 1:

We use the cohomology spectral sequence $\{E_r\}$ with

$$'E_2^{p,q} = H^p(\mathcal{H}, Ext^q_{\mathcal{O}}(\tilde{E}, \tilde{E})),$$
$$'E_\infty^{p,q} \Rightarrow \mathrm{Ext}^{p+q}(\mathcal{H}; \tilde{\mathrm{E}}, \tilde{\mathrm{E}}).$$

(i) $\mathrm{Ext}^1(\mathcal{H}; \tilde{E}, \tilde{E}) = E_\infty^{1,0} \oplus E_\infty^{0,1}$.

- $E_\infty^{1,0} = E_2^{1,0} = H^1(\mathcal{H}, Hom_{\mathcal{O}}(\tilde{E}, \tilde{E}))$; and by lemma 3.4 (i), its traceless part is $H^1(\mathcal{H}, sl_{\mathcal{O}}(\tilde{E})) = \mathbb{C}^{4n+1}$.
- $E_\infty^{0,1} = E_3^{0,1}$. As $E_2^{-2,2} = 0$, and $E_2^{2,0} = H^2(\mathcal{H}, Hom_{\mathcal{O}}(\tilde{E}, \tilde{E})) = 0$, by lemma 3.4 (ii), we see that $E_3^{0,1} = Ker(d_2 : E_2^{0,1} \to E_2^{2,0}) = E_2^{0,1}$.

Therefore, since $\mathcal{E}xt^1_{\mathcal{O}}(\tilde{E},\tilde{E})$ is a skyscraper sheaf supported at p with fiber $\mathbb{C}^3$,

$$E_\infty^{0,1} = H^0(\mathcal{H}, \mathcal{E}xt^1_{\mathcal{O}}(\tilde{E},\tilde{E})) = \mathbb{C}^3.$$

This then proves (i).

(ii) $\mathrm{Ext}^2(\mathcal{H};\tilde{E},\tilde{E}) = E_\infty^{2,0} \oplus E_\infty^{1,1} \oplus E_\infty^{0,2}$.

- By above, $E_\infty^{2,0} = E_2^{2,0} = 0$.
- By Lemma 3.4 (iv), $\mathcal{E}xt^2_{\mathcal{O}}(\tilde{E},\tilde{E}) = 0$, implying that $E_2^{0,2} = 0$, and $E_\infty^{0,2} = 0$.
- And since $\mathcal{E}xt^1_{\mathcal{O}}(\tilde{E},\tilde{E})$ is a skyscraper sheaf supported at p,

$$E_2^{1,1} = H^1(\mathcal{H}, \mathcal{E}xt^1_{\mathcal{O}}(\tilde{E},\tilde{E})) = 0,$$

proving that $E_\infty^{1,1} = 0$.

□

Description of the tangent space.

Let us consider the exact sequence

$$T\mathcal{M}_{n+1,n} \longrightarrow T\bar{\mathcal{M}}_{n+1}|_{\mathcal{M}_{n+1,n}} \longrightarrow \mathcal{N}_{\bar{\mathcal{M}}_{n+1}/\mathcal{M}_{n+1,n}}. \tag{3.3}$$

Let $\tilde{E} \in \mathcal{M}_{n+1,n}$. (3.3) then gives us

$$T_{\tilde{E}}\mathcal{M}_{n+1,n} \longrightarrow T_{\tilde{E}}\bar{\mathcal{M}}_{n+1} \longrightarrow \mathcal{N}_{\bar{\mathcal{M}}_{n+1}/\mathcal{M}_{n+1,n},\tilde{E}}.$$

Let us give a geometric description of $T_{\tilde{E}}\bar{\mathcal{M}}_{n+1} = \mathrm{Ext}^1(\mathcal{H};\tilde{\mathrm{E}},\tilde{\mathrm{E}})_0$. We suppose that $\tilde{E}$ is given by the data:

(i) p_0 is its singular point;

(ii) a map $\mathcal{O}\oplus\mathcal{O} \xrightarrow{\alpha} \mathcal{O}/I$ in X_0 giving $\mathcal{O}\oplus I = \ker\alpha$ around p_0; and

(iii) $\{A_{ij}\}$ are transition functions for $E = (\tilde{E})^{**}$ relative to an open cover $\mathcal{V} = \{V_i\}$ of $\mathcal{H}$.

$T_{\tilde{E}}\bar{\mathcal{M}}_{n+1}$ is then the set of all first order deformations of these things.

From the spectral sequence, we have the exact sequence

$$H^1(\mathcal{H}, sl_{\mathcal{O}}(\tilde{E})) \longrightarrow \mathrm{Ext}^1(\mathcal{H}; \tilde{\mathrm{E}}, \tilde{\mathrm{E}})_0 \longrightarrow \mathrm{H}^0(\mathcal{H}, \mathrm{Ext}^1_{\mathcal{O}}(\tilde{\mathrm{E}}, \tilde{\mathrm{E}})).$$

1. $\underline{H^1(\mathcal{H}, sl_{\mathcal{O}}(\tilde{E}))}$: As we have seen above, we have the exact sequence

$$H^1(\mathcal{H}, sl_{\mathcal{O}}(E)) \longrightarrow H^1(\mathcal{H}, sl_{\mathcal{O}}(\tilde{E})) \longrightarrow H^0(\mathcal{H}, Hom_{\mathcal{O}}(\mathcal{O}, \mathcal{O}/I)),$$

where

$$H^1(\mathcal{H}, sl_{\mathcal{O}}(E)) = \begin{pmatrix} \text{Deformations of the} \\ \text{equivalence class} \\ \text{of the transition} \\ \text{functions } A_{ij} \text{ of } E \end{pmatrix} = T_E\mathcal{M}_n.$$

Furthermore, if $l = (\ker\alpha)_{p_0}$ is the line in $\mathbb{P}^1 \cong X_0$ corresponding to α, then $\mathcal{O}/I = \mathbb{C}^2/l$, and as we have seen in remark following the proof of lemma 3.3

$$H^0(\mathcal{H}, Hom_{\mathcal{O}}(\mathcal{O}, \mathcal{O}/I)) = \begin{pmatrix} \text{Deformations of the} \\ \text{map } \mathcal{O}\oplus\mathcal{O} \xrightarrow{\alpha} \mathcal{O}/I \end{pmatrix} = Hom_{\mathbb{C}}(l, \mathbb{C}^2/l).$$

2. $\underline{H^0(\mathcal{H}, Ext^1_{\mathcal{O}}(\tilde{E}, \tilde{E}))}$: $Ext^1_{\mathcal{O}}(\tilde{E}, \tilde{E})$ is a skyscraper sheaf supported at p_0 with fiber $Ext^1_{\mathcal{O}}(\mathcal{O}\oplus I, \mathcal{O}\oplus I) = \mathbb{C}^3$. This fiber is the set of all deformations of the sheaf $\mathcal{O}\oplus I$. Any deformation of $\mathcal{O}\oplus I$ is given by a map

$$\begin{array}{rcl} \theta : \mathcal{O} & \to & \mathcal{O}\oplus I \\ 1 & \mapsto & (t, s) \end{array},$$

where $t \in \mathcal{O}$, and $s \in I$. We can actually assume that $t \in \mathcal{O}/I$. Similarly, we can assume that $s = bx + d\sigma(z)$, for $b, d \in \mathcal{O}/I$. The deformation corresponding to θ is then the cokernel of

$$\begin{array}{rcl} \mathcal{O} & \to & \mathcal{O}\oplus\mathcal{O}\oplus\mathcal{O} \\ 1 & \mapsto & (t, -\sigma(z) + b, x - d) \end{array}.$$

We see that $(b,d) \in Ext^1_{\mathcal{O}}(I,I) = T_{p_0}\mathcal{H}$; and $t \in Ext^1_{\mathcal{O}}(I,\mathcal{O})$ corresponds to an extension $0 \to \mathcal{O} \to V \to I \to 0$ which is locally free if and only if $t \neq 0$. Hence, if $t \neq 0$, $\mathcal{O} \oplus I$ is replaced by a copy of $\mathcal{O} \oplus \mathcal{O}$. Let us remark that t is completely determined by the restriction of $\theta : \mathcal{O} \to \mathcal{O} \oplus I$ to $\mathcal{O}_{p_0} = \mathbb{C}^2/l$. Its restricted image is then contained in $(\mathcal{O} \oplus I)_{p_0} = l$. Restriction to $\mathcal{O}_{p_0}$ therefore induces the identification

$$Ext^1_{\mathcal{O}}(I,\mathcal{O}) = Hom_{\mathbb{C}^2/l}(\mathbb{C}^2/l, l).$$

We can then rewrite (3.3) as

$$T_{\tilde{E}}\mathcal{M}_{n+1,n} \longrightarrow T_{\tilde{E}}\mathcal{M}_{n+1} \longrightarrow H^0(\mathcal{H}, Ext^1_{\mathcal{O}}(I,\mathcal{O})).$$

This then implies that $\mathcal{N}_{\tilde{\mathcal{M}}_{n+1}/\mathcal{M}_{n+1,n}}$ is a line bundle with fiber

$$\mathcal{N}_{\tilde{\mathcal{M}}_{n+1}/\mathcal{M}_{n+1,n},\tilde{E}} = H^0(\mathcal{H}, Ext^1_{\mathcal{O}}(I,\mathcal{O})) \cong Hom_{\mathbb{C}^2/l}(\mathbb{C}^2/l, l),$$

where $l = \tilde{E}_{p_0}$.

3.3 Normal bundle.

As we saw in the previous section, the normal bundle $\mathcal{N}_{\tilde{\mathcal{M}}_{n+1}/\mathcal{M}_{n+1,n}}$ is a line bundle. We would like to show that it is not a trivial bundle. In the remainder of this chapter, we will denote the normal bundle $\mathcal{N}_{\tilde{\mathcal{M}}_{n+1}/\mathcal{M}_{n+1,n}}$ by $\mathcal{N}$.

3.3.1 Description of the fiber of $\mathcal{N}$.

Let $\tilde{E} \in \mathcal{M}_{n+1,n}$. We would like to describe the fibre $\mathcal{N}_{\tilde{E}}$. Suppose that $\mathrm{sing}(\tilde{\mathrm{E}}) = \mathrm{p}_0$. Let $\mathcal{V} = \{V_i\}$ be an open cover of $\mathcal{H}$ on which $E = (\tilde{E})^{**}$ is trivialised, and let $\{A_{ij}\}$ be the transition functions of E with respect to this cover. We assume that p_0 is only contained in V_0. Then, if $\alpha \in X_0$ is such that $\tilde{E}|_{V_0} \cong \mathcal{O} \oplus I = \ker \alpha$, and l is the line in $\mathbb{P}^1$ corresponding to α, we know that

$$\tilde{E} = \tilde{E}_{l,p_0} = \{s \in \mathcal{O}(E) \mid s(p_0) \in l\}.$$

Let (x, z) be coordinates centered at p_0 in V_0, so that I is generated by $\{x, \sigma(z)\}$. If α corresponds to the matrix $T = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $\bar{E}$ has the following projective resolution on V_0

$$\mathcal{O} \xrightarrow{(0,-\sigma(z),x)} \mathcal{O}\oplus\mathcal{O}\oplus\mathcal{O} \xrightarrow{\begin{bmatrix} a & bx & b\sigma(z) \\ c & dx & d\sigma(z) \end{bmatrix}} \mathcal{O}\oplus I. \tag{3.4}$$

The fibre $\mathcal{N}_{\bar{E}}$ corresponds to the sheaves obtained by deforming $\mathcal{O} \xrightarrow{(0,-\sigma(z),x)} \mathcal{O}\oplus\mathcal{O}\oplus\mathcal{O}$. These deformations therefore affect $\bar{E}$ only on V_0. Let us describe how these deformations affect the transition matrices A_{i0} on $V_0 \cap V_i$. We start by pulling back A_{i0} through (3.4), and obtain the following commutative diagram on $V_0 \cap V_i$:

$$\begin{tikzcd} \mathcal{O} \arrow[r, "{(0,-\sigma(z),x)}"] \arrow[d, "1"'] & \mathcal{O}\oplus\mathcal{O}\oplus\mathcal{O} \arrow[r, "{\begin{bmatrix} a & bx & b\sigma(z) \\ c & dx & d\sigma(z) \end{bmatrix}}"] \arrow[d, "N_i(T)"] & \mathcal{O}\oplus I \arrow[d, "A_{i0}"] \\ \mathcal{O} \arrow[r, "{(1,0,0)}"'] & \mathcal{O}\oplus\mathcal{O}\oplus\mathcal{O} \arrow[r, "{\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}"'] & \mathcal{O}\oplus\mathcal{O} \end{tikzcd}$$

where

$$N_i(T) = \begin{bmatrix} 0\ 0 \\ \hline A_{i0}T \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & x & \sigma(z) \end{bmatrix} + \begin{bmatrix} 0 & -1/\sigma(z) & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Let us note that $z \neq 0$ on $V_0 \cap V_i$, since $p_0 \notin V_i$ for all $i \neq 0$.

We then deform the lefthand part of this diagram, in such a way as to conserve commutativity:

$$\begin{tikzcd} \mathcal{O} \arrow[r, "{(t,-\sigma(z),x)}"] \arrow[d, "1"'] & \mathcal{O}\oplus\mathcal{O}\oplus\mathcal{O} \arrow[d, "\bar{N}_i(t,T)"] \\ \mathcal{O} \arrow[r, "{(1,0,0)}"] & \mathcal{O}\oplus\mathcal{O}\oplus\mathcal{O} \end{tikzcd}$$

where

$$\bar{N}_i(t,T) = \left[\begin{array}{c} 0\ 0 \\ \hline A_{i0}T \end{array}\right] \begin{bmatrix} 1 & t/\sigma(z) & 0 \\ 0 & x & \sigma(z) \end{bmatrix} + \begin{bmatrix} 0 & -1/\sigma(z) & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Completing the exact sequence of the first row by

$$\mathcal{O} \xrightarrow{(t,-\sigma(z),x)} \mathcal{O}\oplus\mathcal{O}\oplus\mathcal{O} \xrightarrow{\begin{bmatrix} \sigma(z) & t & 0 \\ -x & 0 & t \end{bmatrix}} \mathcal{O}\oplus\mathcal{O},$$

and of the second row by

$$\mathcal{O} \xrightarrow{(1,0,0)} \mathcal{O}\oplus\mathcal{O}\oplus\mathcal{O} \xrightarrow{\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & t \end{bmatrix}} \mathcal{O}\oplus\mathcal{O}, \tag{3.5}$$

we then obtain the new commutative diagram

$$\begin{tikzcd}
\mathcal{O} \arrow[r, "{(t,-\sigma(z),x)}"] \arrow[d, "1"'] & \mathcal{O}\oplus\mathcal{O}\oplus\mathcal{O} \arrow[r, "{\begin{bmatrix} \sigma(z) & t & 0 \\ -x & 0 & t \end{bmatrix}}"] \arrow[d, "\bar{N}_i(t,T)"] & \mathcal{O}\oplus\mathcal{O} \arrow[d, "M_i(t,T)"] \\
\mathcal{O} \arrow[r, "{(1,0,0)}"'] & \mathcal{O}\oplus\mathcal{O}\oplus\mathcal{O} \arrow[r, "{\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & t \end{bmatrix}}"'] & \mathcal{O}\oplus\mathcal{O}
\end{tikzcd}$$

where

$$M_i(t,T) = \begin{bmatrix} t^{-1} & 0 \\ 0 & 1 \end{bmatrix} A_{i0}T \begin{bmatrix} t/\sigma(z) & 0 \\ x & \sigma(z) \end{bmatrix},$$

is obtained by taking the quotient. Let us note that

$$\det(M_i(t,T)) = 1.$$

We then see that if $t \in \mathbb{C} \cong \mathcal{N}_{\tilde{E}}$, then it corresponds to

– $\tilde{E}$, if $t = 0$; and

– the locally free sheaf with transition matrices B_{ij} on the cover $\mathcal{V}$, where $B_{ij} = A_{ij}$ if $i, j \neq 0$, and $B_{i0} = M_i(t, T)$ for all i.

Remarks: (i) If instead of (3.5) we use the exact sequence

$$\mathcal{O} \xrightarrow{(1,0,0)} \mathcal{O} \oplus \mathcal{O} \oplus \mathcal{O} \xrightarrow{\begin{bmatrix} 0 & t & 0 \\ 0 & 0 & 1 \end{bmatrix}} \mathcal{O} \oplus \mathcal{O}$$

in our construction, we obtain the matrices

$$M_i(t,T) = \begin{bmatrix} 1 & 0 \\ 0 & t^{-1} \end{bmatrix} A_{i0} T \begin{bmatrix} t/\sigma(z) & 0 \\ x & \sigma(z) \end{bmatrix},$$

which again have $\det(M_i(t,T)) = 1$. These will be used in 3.4.3.

(ii) Let D be a disk around p_0 which is only contained in V_0, i.e. such that $D \cap V_i = \emptyset$ if $i \neq 0$. We then replace the open set V_0 in the cover $\mathcal{V}$ by the two open sets D and $(V_0 - p_0)$. If we now deform $\tilde{E}$ on D, instead of V_0, the transition matrices A_{ij} are then not affected. What will change is the transition matrix from D to $(V_0 - p_0)$, which is simply the identity Id. Therefore, if we replace A_{i0} by Id in the above construction, we obtain the matrices

$$\bar{M}(t,T) = \begin{bmatrix} t^{-1} & 0 \\ 0 & 1 \end{bmatrix} T \begin{bmatrix} t/\sigma(z) & 0 \\ x & \sigma(z) \end{bmatrix}.$$

These will be used in section 3.4.2.

3.3.2 Non-triviality of the $\mathcal{N}$ over $\mathcal{M}_{n+1,n}$.

Let us start by noting that $H^2(\mathcal{M}_{n+1,n}, \mathbb{Z}) \neq 0$, and that one of its components is $H^2(X_0, \mathbb{Z}) = \mathbb{Z}$. Indeed, by using the cohomology spectral sequence $\{'E_r\}$ associated to the fibration (3.2), with

$$'E_2^{p,q} = H^p(\mathcal{H} \times \mathcal{M}_n, H^q(X_0, \mathbb{Z})),$$
$$'E_\infty^{p,q} \Rightarrow H^{p+q}(\mathcal{M}_{n+1,n}, \mathbb{Z}),$$

we find that $E_\infty^{0,2} = H^0(\mathcal{H} \times \mathcal{M}_n, H^2(X_0, \mathbb{Z}))$. Although $\mathcal{H} \times \mathcal{M}_n$ is connected, it is not simply connected because $\pi_1(\mathcal{H}) = \mathbb{Z}$. We therefore have cohomology with coefficients in the local sheaf $H^2(X_0, \mathbb{Z}) = \mathbb{Z}$. An element in $H^2(X_0, \mathbb{Z})$ however corresponds to a choice of orientation. Since any complex manifold is orientable, going around a loop in $\mathcal{H} \times \mathcal{M}_n$ does not produce monodromy. $H^2(X_0, \mathbb{Z})$ is then the constant sheaf $\mathbb{Z}$, and we see that $E_\infty^{0,2} = H^2(X_0, \mathbb{Z}) = \mathbb{Z}$.

To prove that $\mathcal{N}$ is topologically non-trivial, it is therefore sufficient to show that it is not trivial over X_0.

Let us fix $E \in \mathcal{M}_n$ and $p_0 \in \mathcal{H}$. We can therefore consider X_0 as being the fiber of $P : \mathcal{M}_{n+1,n} \to \mathcal{H} \times \mathcal{M}_n$ at (p_0, E). If $\bar{E}_\alpha$ is the element of this fiber corresponding to $\alpha \in X_0$, we have seen that the fiber of the normal bundle at $\bar{E}_\alpha$ is

$$\mathcal{N}_{\bar{E}_\alpha} = Hom_{\mathcal{O}_{p_0}}(\mathbb{C}^2/(\ker\alpha)_{p_0}, (\ker\alpha)_{p_0}).$$

This then shows that

Lemma 3.5 *The restriction of the normal bundle $\mathcal{N} = \mathcal{N}_{\bar{\mathcal{M}}_{n+1}/\mathcal{M}_{n+1,n}}$ to X_0 is isomorphic to $Hom_{\mathcal{O}}(\mathcal{O}(1), \mathcal{O}(-1)) = \mathcal{O}(-2)$.*

□

We can also give an explicit description of a section s of the normal bundle $\mathcal{N}$ over X_0. If $\bar{M}(t,T)$ are the transition matrices given in remark (i) of section 3.4.1, we define s as follows

$$\begin{aligned} s : X_0 &\longrightarrow \mathcal{N} \\ T &\longmapsto \bar{M}(t,T) \end{aligned}.$$

On the overlap $\tilde{a} = 1/\tilde{c}$, and

$$\bar{M}(t, T'_{\tilde{a}}) = \mathrm{diag}(1/\tilde{c}, \tilde{c})\ \bar{M}(\tilde{c}^{-2}t, T_{\tilde{c}}).$$

$\bar{M}(t, T'_{\tilde{a}})$ and $\bar{M}(\tilde{c}^{-2}t, T_{\tilde{c}})$ therefore define the same vector bundle. This then shows that $\mathcal{N}$ must have the transition function $\tilde{c}^2$, verifying that $\mathcal{N} \simeq \mathcal{O}(-2)$ over $X_0 \cong \mathbb{P}^1$.

3.3.3 Non-triviality of the normal bundle over $\mathcal{M}_n$.

We fix $p_0 \in \mathcal{H}$ and the identity matrix $T = Id$ in X_0. Let us restrict ourselves to the subset $\mathrm{Pic}^{-1}(\mathrm{T})$ of $\mathcal{M}_n$ consisting of vector bundles E such that

– the graph of E is $n(\{p_0\} \times \mathbb{P}^1) + (\mathbb{P}^1 \times \{l\})$, where $l \in \mathrm{Pic}^0(\mathrm{T})$; and

– $E|_{\pi^{-1}(p_0)} = L \oplus L^*$, $L \in \mathrm{Pic}^{-1}(\mathrm{T})$.

We begin by finding transition matrices for E. We know that $\mathcal{H}$ can be covered by two copies of $\mathbb{C} \times T$. We therefore just have to describe the situation on $D \times T$, where D is a disk centered at p_0. We will use the notation of section 2.3.3. $D \times T$ is covered by the two open sets V_0, V_1. $\mathrm{Pic}^{-1}(\mathrm{T})$ has coordinate λ. If L has divisor p_λ, we denote by E_λ the bundle such that $E|_{\pi^{-1}(p_0)} = L \oplus L^*$. We have seen in section 2.3.3 that E_λ has transition matrix

$$A_{10} = \begin{bmatrix} g & \gamma \\ 0 & g^{-1} \end{bmatrix}$$

on $V_0 \cap V_1$, where $g = \sigma(z)e^{-\lambda\zeta(z)}$.

Let us now define the following section $\tilde{s}$ of $\mathcal{N}$ over the cover $\mathbb{C}$ of $\mathrm{Pic}^{-1}(\mathrm{T})$:

$$\begin{array}{rcl} \tilde{s} : \mathbb{C} & \longrightarrow & \mathcal{N} \\ \lambda & \longmapsto & M(t,\lambda) = M_1(t, Id) \end{array},$$

where this time we are working with the following transition matrix

$$M_1(t,T) = \begin{bmatrix} 1 & 0 \\ 0 & t^{-1} \end{bmatrix} A_{10} T \begin{bmatrix} t/\sigma(z) & 0 \\ x & \sigma(z) \end{bmatrix}.$$

If we move λ by the period $2\omega_i$, we obtain the transition matrix

$$M(t, \lambda + 2\omega_i) = \mathrm{diag}(\phi_i^{-1}(z)C, \phi_i(z)C^{-1})M(e^{-2\eta_i z}C^{-1}t, \lambda),$$

where the function $\phi_i(z)$ and the constant $C = e^{-4\eta_i(\lambda+2\omega_i)}$ were defined in sections 2.2.1 and 2.3.3, respectively. Since $\mathrm{diag}(\phi_i^{-1}(z)C, \phi_i(z)C^{-1})$ is doubly-periodic on U_1, the matrices $M(t, \lambda+2\omega_i)$ and $M(e^{-2\eta_i z}C^{-1}t, \lambda)$ define the same vector bundle. $\mathcal{N}$ therefore has $e^{-2\eta_i z}C^{-1}$ as factor of automorphy on $\mathrm{Pic}^{-1}(\mathrm{T})$. The restriction of $\mathcal{N}$ to $\mathrm{Pic}^{-1}(\mathrm{T})$ is therefore a non-trivial line bundle.

3.4 Stabilisation maps.

Let us choose a fixed rank 2 C^∞ vector bundle $\mathcal{E}$ on $\mathcal{H}$ with $c_1(\mathcal{E}) = 0$, $c_2(\mathcal{E}) = n$. We also fix a base point p_0. We consider the moduli space $\mathcal{M}_n^0$ of equivalence classes of pairs (E, ψ) where E is a stable holomorphic structure on $\mathcal{E}$ with $\Lambda^2 E \cong \mathcal{O}$, and ψ is a trivialisation of E at p_0. $\mathcal{M}_n^0$ is then the moduli space of *framed instantons* of charge n on $\mathcal{H}$. Let us also consider the moduli space $\tilde{\mathcal{B}}_n$ of *framed connections*, where a framed connection is now a pair (A, ψ), where A is any connection and ψ is a framing at p_0. As we shall see in chapter 7, stabilisation maps

$$f_{n,n+1} : \tilde{\mathcal{B}}_n \to \tilde{\mathcal{B}}_{n+1}$$

always exist. They are constructed by glueing an instanton of charge 1 at p_0. We would like to know if it is possible to define an analogous map $g_{n,n+1} : \mathcal{M}_n^0 \to \mathcal{M}_{n+1}^0$ in the holomorphic setting.

The holomorphic counterpart of glueing an instanton seems to be the Serre construction. Given a stable holomorphic vector bundle E on $\mathcal{H}$, one finds a stable holomorphic vector bundle E' such that

- it can be expressed as an extension

$$0 \to L \to E' \to L^* \otimes I \to 0, \tag{3.6}$$

  where L is a line bundle, and I is the ideal sheaf of p_0;

- E' is isomorphic to E away from p_0.

Unfortunately, as $H^2(\mathcal{H}, \mathbb{C}) = 0$, we cannot express E' globally as such an extension (or else, E' would always have $c_2(E') = 1$). Let $\mathcal{V} = \{V_i\}$ be an open cover of $\mathcal{H}$ such that p_0 is only contained in V_0. We can then

- apply the idea given above to the restriction of E to V_0;

- glue the bundle thus obtained to $E|_{(\mathcal{H}-p_0)}$, using the transition functions of E.

By what we have seen in the previous sections, this is the same thing as glueing $\mathcal{O} \oplus I$ at p_0, and deforming the new sheaf to obtain a locally free sheaf. In a neighborhood of (E, ψ) in $\mathcal{M}_n^0$, such a map always exists. Defining a such stabilisation map globally

$$\begin{aligned} g_{n,n+1} : \mathcal{M}_n^0 &\rightarrow \mathcal{M}_{n+1}^0 \\ E &\mapsto E' \end{aligned}$$

however implies finding a nowhere zero section of $\mathcal{N}$. And, as $\mathcal{N}$ is a non trivial line bundle, this is impossible to do. The map will always depend on the choice of inclusion of $\mathcal{O} \oplus I$ into $\mathcal{O} \oplus \mathcal{O}$ at p_0. In addition, the results of section 3.4.3 indicate that it is impossible to find a canonical choice of line bundle L giving the above exact sequence (3.6).

Chapter 4

Fibre of the graph map.

The moduli space $\mathcal{M}_1$ of instantons of charge 1 on $\mathcal{H}$ is well understood. It was proven in [BH] that it is the total space of a principal T-bundle given by the graph map $G : \mathcal{M}_1 \to \mathbb{P}^3 \backslash \mathbb{P}^1 \times I$, where T is an elliptic curve and I is a subset of $\mathrm{Pic}^0(\mathrm{T})$ that will be defined in 4.1.1. In this chapter, we study the fibre of the graph map $G : \mathcal{M}_n \longrightarrow \mathbb{P}^{2n+1}$, for $n \geq 2$.

If we restrict ourselves to a certain subset of $\mathbb{P}^{2n+1}$, we can show that the fibre of G is the Jacobian of a Riemann surface of genus $2n-1$. In the first section, this result is obtained by the techniques used in [BH]. In this context, the fibre gives possible glueings for bundles on $D \times T$. In the second section, we approach the problem from the point of view of spectral curves. One can associate to each graph g a spectral curve $\bar{S}$ which is a hyperelliptic curve of genus $2n-1$. We then show that the elements of the fibre $G^{-1}(g)$ are in one-to-one correspondence with line bundles on $\bar{S}$ of a fixed degree. Let us note that our construction also works for $n = 1$, and that in this case $\bar{S}$ is an elliptic curve.

4.1 Fibre of the graph map.

Let

$$G : \mathcal{M}_n \longrightarrow |\mathcal{O}(n, 1)| = \mathbb{P}^{2n+1}$$

be the map that associates to each $\mathrm{SL}(2,\mathbb{C})$-bundle E with $c_2(E) = n$ its graph in $\mathbb{P}^1 \times \mathbb{P}^1$.

4.1.1 The image of the graph map.

As we have seen in 2.1.1, one can describe $\mathrm{Pic}^0(\mathrm{T})$ by constant automorphy factors. If one sets $T = \mathbb{C}^*/\{\lambda^n\}$, the half periods of $\mathrm{Pic}^0(\mathrm{T})$ correspond to $1, -1, \sqrt{\lambda}, -\sqrt{\lambda}$. Let C_1 be the circle in $\mathrm{Pic}^0(\mathrm{T})$ corresponding to factors of norm 1. C_1 projects to an interval I in $\mathrm{Pic}^0(\mathrm{T})/\pm = \mathbb{P}^1$, joining the two half periods $+1$ and -1.

Let $\mathbb{P}^1 \times I$ denote the set of graphs

$$\left\{(\{z\} \times \mathbb{P}^1) + (\mathbb{P}^1 \times \{l\}), z \in \mathbb{P}^1, l \in I\right\}.$$

As proven in [BH], we then have

Theorem 4.1 *For $n > 1$, G is surjective.*
For $n = 1$, the image of G is $\mathbb{P}^3 \backslash \mathbb{P}^1 \times I$.

4.1.2 The fibre of the graph map.

Any bundle on $\mathcal{H}$ can be obtained by glueing two bundles on $D \times T$. We begin by briefly recalling the isomorphism classes of certain bundles on $D \times T$. Let $D \subset \mathbb{C}$ be simply connected and E be an $\mathrm{SL}(2,\mathbb{C})$-bundle on $D \times T$ such that E does not restrict to $L_0 \oplus L_0, L_0^2 = \mathcal{O}$, over any $\{x\} \times T$. Let g be the graph of E. Referring to lemmas 2.4 and 2.5, we have the following possilities

- if g is the graph of a rational map $\varrho : D \to \mathbb{P}^1$, the isomorphism class of E is then uniquely determined by ϱ;
- if g is of the form $(\{z_0\} \times \mathbb{P}^1) + (\mathbb{P}^1 \times \{l\})$, the isomorphism class of E is then uniquely determined by l, and by the choice of line bundle in $\mathrm{Pic}^{-1}(\mathrm{T})$ giving E over $\{z_0\} \times T$.

Let us now suppose that E is a holomorphic $\mathrm{SL}(2,\mathbb{C})$-bundle on $\mathcal{H}$ with $c_2(E) = n$ that has a graph $G(E) = g$ of one of the following two types:

α) g is the graph of a holomorphic map $F : \mathbb{P}^1 \to \mathbb{P}^1$ of degree n, if $n \geq 1$;

β) g is of the form $(\{z_0\} \times \mathbb{P}^1) + (\mathbb{P}^1 \times \{l\})$, if $n = 1$.

We again assume that E does not restrict to $L_0 \oplus L_0$, $L_0^2 = \mathcal{O}$, over any $\pi^{-1}(x)$. Let us note that, in case β), the only possibility for the restriction of E to $\pi^{-1}(z_0)$ is: $E = L \oplus L^*$, for some $L \in \mathrm{Pic}^{-1}(\mathrm{T})$. Furthermore, it was proven in [BH] that, in case β), the fibre of the graph map is $\mathrm{Pic}^{-1}(\mathrm{T})$.

Let D_0, D_∞ be an open cover of $\mathbb{P}^1$ such that z_0 is only contained in D_0. In case α), the above discussion tells us that the restrictions of g to $D_0 \times \mathbb{P}^1, D_\infty \times \mathbb{P}^1$ determine, up to $\mathrm{SL}(2,\mathbb{C})$-isomorphism, unique bundles on $D_0 \times T, D_\infty \times T$. The elements in the fibre $G^{-1}(g)$ then correspond to choices of glueing. By surjectivity of G, we know that at least one glueing will give a stable bundle E. Furthermore, if $\mathrm{Aut} = \mathrm{Aut}_{\mathrm{SL}(2,\mathbb{C})}(\mathrm{E})$, one has

$$G^{-1}(g) \simeq \Gamma(D_0 \times T, \mathrm{Aut}) \setminus \Gamma((\mathrm{D}_0 \cap \mathrm{D}_\infty) \times \mathrm{T}, \mathrm{Aut})/\Gamma(\mathrm{D}_\infty \times \mathrm{T}, \mathrm{Aut}). \qquad (4.1)$$

We will use the pushdown $A = \pi_*(\mathrm{Aut})$ to determine $G^{-1}(g)$. Relative to the cover D_0, D_∞, all the possible glueings will then be given by $H^1(\mathbb{P}^1, A)$. Let $\alpha = \pi_*(\mathrm{sl(E)})$, where sl(E) are the traceless endomorphisms of E, and let K and M be the kernel and the cokernel of the exponential map $\alpha \longrightarrow A$. If we set $L = \alpha/K$, one has sequences

$$0 \longrightarrow K \longrightarrow \alpha \longrightarrow L \longrightarrow 0, \qquad (4.2)$$

$$0 \longrightarrow L \longrightarrow A \longrightarrow M \longrightarrow 0. \qquad (4.3)$$

We will use these sequences to compute $H^1(\mathbb{P}^1, A)$.

4.1.3 Bundles with $c_2 = 1$.

Let E be a stable holomorphic bundle E with $c_2(E) = 1$. The graph $g = G(E)$ of E is then of two possible types:

α) g is the graph of an automorphism $F : \mathbb{P}^1 \to \mathbb{P}^1$,

β) g is a sum $(\{z_0\} \times \mathbb{P}^1) + (\mathbb{P}^1 \times \{l\})$.

In both cases, one must exclude the existence of points x in $\mathbb{P}^1$ such that $E|_{\pi^{-1}(x)} \cong L_0 \oplus L_0, L_0^2 = \mathcal{O}$. In case α), suppose that there is such a point x. $R^1\pi_*(L_0E)$ is then a skyscraper sheaf supported at x, with fibre $\mathbb{C}^n, n \geq 2$. If $l = \{L_0, L_0\}$, this implies that $\mathbb{P}^1 \times \{l\}$ is tangent to the graph of F, and so $dF_x = 0$, which is impossible, as F is an automorphism. In case β), it can be shown that the presence of such points contradicts stability. The argument involves destabilising bundles (see [BH]).

In [BH], the cohomology group $H^1(\mathbb{P}^1, A)$ was computed in both cases, and it was proven:

Theorem 4.2 *(Case α). If g is an automorphism $F : \mathbb{P}^1 \to \mathbb{P}^1$, then the fibre of $G : \mathcal{M}_1 \to \mathbb{P}^3$ at g is T.*
(Case β). If $g \in \mathbb{P}^3 \backslash \mathbb{P}^1 \times I$ is a sum $(\{z_0\} \times \mathbb{P}^1) + (\mathbb{P}^1 \times \{l\})$, then the fibre of $G : \mathcal{M}_1 \to \mathbb{P}^3$ at g is isomorphic to $\mathrm{Pic}^{-1}(\mathrm{T}) \cong \mathrm{T}$.

4.1.4 Bundles with $c_2 = n$.

In this case, we only consider stable holomorphic bundles E with $c_2(E) = n$ and graph $g = G(E)$, such that

- g is the graph of a holomorphic map $F : \mathbb{P}^1 \to \mathbb{P}^1$ of degree n; and
- there are no points x in $\mathbb{P}^1$ where $E|_{\pi^{-1}(x)} \cong L_0 \oplus L_0, L_0^2 = \mathcal{O}$.

We have seen, in the previous section, that if such a point x in $\mathbb{P}^1$ exists, then $dF_x = 0$. If $n \geq 2$, the differential of F must vanish at some points in $\mathbb{P}^1$. We therefore have to exclude the cases where the zeroes of dF correpond to points x such that $E|_{\pi^{-1}(x)} \cong L_0 \oplus L_0, L_0^2 = \mathcal{O}$.

Let us restate this by saying that we only consider bundles that have a graph g which is the graph of holomorphic map $F : \mathbb{P}^1 \to \mathbb{P}^1$ of degree n that satisfies the condition

If $F(x)$ corresponds to a half period, then $dF_x \neq 0$.

Every bundle having such a graph is stable. $G^{-1}(g)$ is then isomorphic to $H^1(\mathbb{P}^1, A)$. Since F is a map of degree n, there are $4n$ points in $\mathbb{P}^1$ that get mapped to half periods. Let $z_i, i = 1, \ldots, 4n$ be these points and $p(z)$ be a polynomial of degree $4n$ that has the z_i as roots. Let $\tilde{S}$ be the Riemann surface associated to $\sqrt{p(z)}$. $\tilde{S}$ is then a of genus $[(4n-1)/2] = 2n-1$ and

$$\omega_i = \frac{z^{i-1}dz}{\sqrt{p(z)}},\ i = 1, \ldots, 2n-1$$

is a basis of $\Gamma(\Omega(\tilde{S}))$. We can then show that $H^1(\mathbb{P}^1, A)$ is the Jacobian of $\tilde{S}$. This is done in

Theorem 4.3 *Let g be the graph of a holomorphic map $F : \mathbb{P}^1 \to \mathbb{P}^1$ of degree n satisfying condition $(*)$. The fibre of the graph map $G : \mathcal{M}_n \to \mathbb{P}^{2n+1}$ at g is then the Jacobian of a Riemann surface of genus $2n-1$.*

Proof: Let $z_i, i = 1, \ldots, 4n$ be points that get mapped by F to half periods, and let $p(z)$ be a polynomial of degree $4n$ that has the z_i as roots. Furthermore, let E be a bundle that has graph g. There are then only two types of restrictions of E to fibres $T = \pi^{-1}(z)$ of $\pi : \mathcal{H} \to \mathbb{P}^1$:

- $L_0 \oplus L_0^*, L_0^2 \neq \mathcal{O}$, with $c_1(L_0) = 0$;

- a nontrivial extension of L_0 by L_0, with $L_0^2 \simeq \mathcal{O}$.

We start by describing α and K. As $\det(E) = \mathcal{O}$, there is a natural splitting $\mathrm{End(E)} = \mathcal{O} \oplus \mathrm{sl(E)}$. Referring to lemma 2.2, we find that $h^i(\pi^{-1}(x), \mathrm{End(E)}) = 2$, for $i = 0, 1$, and $h^0(\pi^{-1}(x), \mathrm{sl(E)}) = 1$, for all x. Grauert's Theorem then implies that the direct image sheaves $R^i\pi_*(\mathrm{End(E)})$, $i = 0, 1$, and $\alpha = \pi_*(\mathrm{sl(E)})$ are locally free. Therefore $\pi_*(\mathrm{End(E)})$ splits as $\mathcal{O} \oplus \alpha$.

1) $\alpha = \mathcal{O}(-2n)$: Let γ is the positive generator of $H^2(\mathbb{P}^1, \mathbb{Z})$. As $\pi_*(\mathrm{End(E)}) \simeq \mathcal{O} \oplus \alpha$, we just have to show that $c_1(\pi_*(\mathrm{End(E)})) = -2n\gamma$. Since the canonical bundle of $\mathcal{H}$ is $K_{\mathcal{H}} \simeq \pi^* K_{\mathbb{P}^1}$, the dualising sheaf of $\pi : \mathcal{H} \to \mathbb{P}^1$ is then holomorphically trivial; and, by relative Serre duality, $\pi_*\mathrm{End(E)} = (\mathrm{R}^1\pi_*(\mathrm{End(E)}))^*$. Therefore

$$c_1(\pi_!(\mathrm{End(E)})) = 2c_1(\pi_*(\mathrm{End(E)})).$$

Let h is the positive generator of $H^4(\mathcal{H},\mathbb{Z})$. We then have the following map on cohomology:

$$\begin{aligned} \pi_*: \ \mathrm{H}^*(\mathcal{H},\mathbb{Z}) &\longrightarrow \mathrm{H}^*(\mathbb{P}^1,\mathbb{Z}) \\ h &\longmapsto \gamma, \\ \sigma &\longmapsto 0, \text{ if } \sigma \neq h. \end{aligned}$$

By Grothendieck-Riemann-Roch,

$$\mathrm{ch}(\pi_!(\mathrm{End(E)})) \cdot \mathrm{td}(\mathbb{P}^1) = \pi_*(\mathrm{ch}(\mathrm{End(E)}) \cdot \mathrm{td}(\mathcal{H})). \tag{4.4}$$

With these generators, $\mathrm{td}(\mathcal{H}) = 1$ and $\mathrm{ch}(\mathrm{End(E)}) = 4-4\mathrm{nh}$. Therefore $\pi_*(\mathrm{ch}(\mathrm{End(E)})\cdot \mathrm{td}(\mathcal{H})) = -4\mathrm{n}\gamma$. Also, since the tangent bundle of $\mathbb{P}^1$ is $\mathcal{O}(2)$, $\mathrm{td}(\mathbb{P}^1) = 1+\gamma$. Inserting these in (4.4), we get $\mathrm{ch}(\pi_!(\mathrm{End(E)})) = -4\mathrm{n}\gamma$, and $c_1(\pi_*(\mathrm{End(E)})) = -2\mathrm{n}\gamma$. We thus obtain the cohomology groups

$$H^0(\mathbb{P}^1,\alpha) = 0, \quad H^1(\mathbb{P}^1,\alpha) = \mathbb{C}^{2n-1}, \quad H^2(\mathbb{P}^1,\alpha) = 0.$$

2) K: Referring to lemma 2.3, we see that K is zero on any open set containing the z_i . Away from the z_i, K is locally the constant sheaf $\mathbb{Z}$: $K = \{2\pi i \ \mathrm{diag}(\mathrm{m},-\mathrm{m}), \mathrm{m} \in \mathbb{Z}\}$. There is a $\mathbb{Z}/2$ monodromy on K that corresponds to branching around z_i. It has the effect of interchanging L_0 and L_0^*. We can also give the following explicit embedding of K into $\alpha \simeq \mathcal{O}(-2n)$:

$$2\pi i \ \mathrm{diag}(\mathrm{m},-\mathrm{m}) \longmapsto \frac{\mathrm{mdz}}{\sqrt{\mathrm{p(z)}}}.$$

To compute the Čech cohomology of K, we use the following Leray covering of the Riemann sphere $\mathbb{P}^1$ described in [BH]. Suppose that each z_i lies on the equator. Let D_i be closed discs along the equator, each containing a z_i and such that $D_i \cap D_j = \emptyset$, if $i \neq j$. The cover is defined by

$$\begin{aligned} U_N &= \text{the points in } \mathbb{P}^1 - (\cup_i D_i) \text{ lying north of a line} \\ &\quad\ \text{which passes below the equator;} \\ U_S &= \text{the points in } \mathbb{P}^1 - (\cup_i D_i) \text{ lying south of a line} \\ &\quad\ \text{which passes above the equator;} \\ V_i &= \text{open disc containing } D_i, \text{ such that } V_i \cap V_j = \emptyset, \text{ if } i \neq j. \end{aligned}$$

Let us fix trivialisations of K on these open sets. The restriction maps are all the identity, except for $\rho^N_{Ni} = -\mathrm{Id}, i = 1, \dots, 4n$, which corresponds to the monodromy about the z_i. The cochains with respect to this covering are $C^0 = K(U_S) \oplus K(U_N) = \mathbb{Z}^2$, as $K(V_i) = 0$ for all $i = 1, \dots, 4n$; $C^1 = K(U_{SN}) \oplus \sum_{i=1}^{4n} K(U_{iS}) \oplus \sum_{i=1}^{4n} K(U_{Ni}) = \mathbb{Z}^{12n}$, as U_{NS} has $4n$ components; and $C^2 = \sum_{i=1}^{4n} K(U_{SNi}) = \mathbb{Z}^{8n}$, since U_{SNi} has 2 components for all $i = 1, \dots, 4n$. Since

$$\begin{array}{rcl} \delta : C^0 & \longrightarrow & C^1 \\ (m, n) & \longmapsto & (n-m, \dots, n-m, m, \dots, m, -n, \dots, -n) \end{array},$$

$Z^0 = ker(\delta : C^0 \longrightarrow C^1) = 0$ and $B^1 = Im(\delta : C^0 \longrightarrow C^1) = \mathbb{Z}^2$. The other coboundary operator is given by

$$\begin{array}{rcl} \delta : C^1 & \longrightarrow & C^2 \\ (n_1, \dots, n_{4n}, m_1, \dots, m_{4n}, k_1, \dots, k_{4n}) & \longmapsto & (a_1, b_1, \dots, a_{4n}, b_{4n}) \end{array},$$

where

$$\begin{array}{rcl} a_1 & = & n_1 + m_1 + k_1 \\ b_1 & = & n_2 + m_1 - k_1 \\ a_2 & = & n_2 + m_2 - k_2 \\ b_2 & = & n_3 + m_2 + k_2 \\ \vdots & \vdots & \vdots \\ a_l & = & n_l + m_l \pm k_l \\ b_l & = & n_{l+1} + m_l \mp k_l \\ \vdots & \vdots & \vdots \\ a_{4n-1} & = & n_{4n-1} + m_{4n-1} + k_{4n-1} \\ b_{4n-1} & = & n_{4n} + m_{4n-1} - k_{4n-1} \\ a_{4n} & = & n_{4n} + m_{4n} - k_{4n} \\ b_{4n} & = & n_1 + m_{4n} + k_{4n}. \end{array}$$

(The alternation between $+$ and $-$ in front of the k_i corresponds to monodromy.) In this case, $Z^1 = ker(\delta : C^1 \longrightarrow C^2) = \mathbb{Z}^{4n}$ and

$$\begin{aligned} B^2 = {} & Im(\delta : C^1 \longrightarrow C^2) \\ = {} & \left\{(a_1, b_1, \dots, a_{4n}, b_{4n}) \in \mathbb{Z}^{8n} \mid \textstyle\sum_{i=1}^{4n}(b_i - a_i) = \text{eveninteger.}\right\} \end{aligned}$$

Combining these, we have

$$H^0(\mathbb{P}^1, K) = 0, \quad H^1(\mathbb{P}^1, K) = \mathbb{Z}^{4n-2}, \quad H^2(\mathbb{P}^1, K) = \mathbb{Z}/2.$$

3) M: Referring to lemma 2.2, $\exp : \alpha \to A$ is surjective away from the z_i,; and it has cokernel $\mathbb{Z}/2$ near the z_i. M is then a skyscaper sheaf supported on the z_i.

$$H^0(\mathbb{P}^1, M) = (\mathbb{Z}/2)^{4n}, \quad H^1(\mathbb{P}^1, M) = H^2(\mathbb{P}^1, M) = 0.$$

4) A, L: Since A is included in $\pi_*\mathrm{End(E)} = \mathcal{O}\oplus\mathcal{O}(-2\mathrm{n})$, the global sections of A are $\pm\mathrm{Id}$ and $H^0(\mathbb{P}^1, A) = \mathbb{Z}/2$. And as $\pm\mathrm{Id}$ is not an exponential at z_i, $H^0(\mathbb{P}^1, L) = 0$.

The long cohomology exact sequences associated to (4.2) and (4.3) are then

$$0 \longrightarrow \mathbb{Z}^{4n-2} \longrightarrow \mathbb{C}^{2n-1} \longrightarrow H^1(\mathbb{P}^1, L) \longrightarrow \mathbb{Z}/2 \longrightarrow 0,$$

and

$$0 \longrightarrow \mathbb{Z}/2 \longrightarrow (\mathbb{Z}/2)^{4n} \longrightarrow H^1(\mathbb{P}^1, L) \longrightarrow H^1(\mathbb{P}^1, A) \longrightarrow 0.$$

The inclusion $\mathbb{Z}^{4n-2} \longrightarrow \mathbb{C}$ is the mapping $H^1(\bar{S}, \mathbb{Z}) \longrightarrow H^1(\bar{S}, \mathcal{O})$ giving the lattice of $\mathrm{Jac}(\bar{\mathrm{S}})$. Therefore, $H^1(\mathbb{P}^1, L) \simeq \mathrm{Jac}(\bar{\mathrm{S}}) \times \mathbb{Z}/2$ and

$$H^1(\mathbb{P}^1, A) \simeq \mathrm{Jac}(\bar{\mathrm{S}})/(\mathbb{Z}/2)^{4n-2} \simeq \mathrm{Jac}(\bar{\mathrm{S}}).$$

□

4.2 Spectral curves and their Jacobian.

4.2.1 Spectral curves.

Let E be an $\mathrm{SL}(2, \mathbb{C})$-bundle with $c_2(E) = n$ over $\mathcal{H}$, and g be its graph. Given the fibration $\pi : \mathcal{H} \to \mathbb{P}^1$, let us also denote by π the projection

$$\begin{aligned} \pi : \mathcal{H} \times \mathbb{C}^* &\longrightarrow \mathbb{P}^1 \times \mathbb{C}^* \\ (x, \alpha) &\longmapsto (\pi(x), \alpha). \end{aligned}$$

If $s : \mathcal{H} \times \mathbb{C}^* \to \mathcal{H}$ is the projection onto the first factor, we then have the commutative diagram

$$\begin{array}{ccccc} s^*E & \longrightarrow & \mathcal{H} \times \mathbb{C}^* & \xrightarrow{\pi} & \mathbb{P}^1 \times \mathbb{C}^* \\ & & \Big\downarrow s & & \Big\downarrow \\ E & \longrightarrow & \mathcal{H} & \xrightarrow[\pi]{} & \mathbb{P}^1 \end{array}$$

Let us briefly recall the construction of the graph g. If V is the Poincaré line bundle over $\mathcal{H} \times \mathbb{C}^*$, $L := R^1\pi_*(s^*E \otimes V)$ is a skyscraper sheaf supported on a divisor $\widetilde{S}$ in $\mathbb{P}^1 \times \mathbb{C}^*$. Furthermore, this divisor is invariant under: the $\mathbb{Z}$-action on $\mathbb{C}^*$ generated by λ, and the involution on $\mathbb{C}^*$ defined by $z \mapsto 1/z$. $\widetilde{S}$ therefore descends to a divisor S on $\mathbb{P}^1 \times \mathbb{P}^1$, that is defined to be the graph g of E. Before descending to S, $\widetilde{S}$ obviously descends to a divisor $\bar{S}$ on $\mathbb{P}^1 \times \mathbb{C}^*/\mathbb{Z}$. $\bar{S}$ is a double cover of S that can be be considered as a spectral curve of g. We would like to know if L also descends to a skyscraper sheaf $\bar{L}$ on $\mathbb{P}^1 \times \mathbb{C}^*/\mathbb{Z}$, with support $\bar{S}$.

We begin by describing the $\mathbb{Z}$-action on $\mathbb{P}^1 \times \mathbb{C}^*$. This action is induced from the $\mathbb{Z}$-action on $\mathcal{H} \times \mathbb{C}^*$, the latter being

$$\begin{array}{ccc} {\mathbb{C}^*}^2 \times \mathbb{C}^* & \xrightarrow{\lambda} & {\mathbb{C}^*}^2 \times \mathbb{C}^* \\ (z_1, z_2, \alpha) & \longmapsto & (\lambda z_1, \lambda z_2, \lambda\alpha) \end{array}.$$

We see that this action is trivial on $\mathcal{H}$, and therefore trivial on $\mathbb{P}^1$. Let us however show that the fibres of the skyscraper sheaf L are not preserved by this action.

As $L := R^1\pi_*(s^*E \otimes V)$, we begin by looking at how $\mathbb{Z}$ acts on $s^*E \otimes V$. Let us first note that it acts trivially on s^*E. Let us now show that it does not preserve the fibres of the Poincaré line bundle V on $\mathcal{H} \times \mathbb{C}^*$, i.e. λ does not send $V_{(x,\alpha)}$ to $V_{(x,\lambda\alpha)}$. We recall that V can be constructed by constant automorphy factors: one starts with the trivial line bundle $\bar{\mathbb{C}}$ over ${\mathbb{C}^*}^2 \times \mathbb{C}^*$ and identifies $t \in \bar{\mathbb{C}}_{(z_1,z_2,\alpha)}$ with $\alpha t \in \bar{\mathbb{C}}_{(\lambda z_1,\lambda z_2,\alpha)}$, where the $\mathbb{Z}$-action is trivial on $\mathbb{C}^*$. If we make $\mathbb{Z}$ act on $\mathbb{C}^*$ as well, then

$$\begin{array}{ccc} {\mathbb{C}^*}^2 \times \mathbb{C}^* \times \mathbb{C} & \xrightarrow{\lambda} & {\mathbb{C}^*}^2 \times \mathbb{C}^* \times \mathbb{C} \\ (z_1, z_2, \alpha, t) & \longmapsto & (\lambda z_1, \lambda z_2, \lambda\alpha, \alpha t) \end{array}.$$

In $\mathcal{H}$, (z_1, z_2) and $(\lambda z_1, \lambda z_2)$ of course define the same point, say x. The above however tells us that λ sends $V_{(x,\alpha)}$ to $V_{(x,\lambda\alpha)} \otimes \pi^*(\mathcal{O}(-1))$. Hence, the Poincaré line bundle does not descend on $\mathcal{H} \times \mathbb{C}^*/\mathbb{Z}$.

If we pushdown to $\mathbb{P}^1 \times \mathbb{C}^*$, we encounter the same problem:

$$R^1\pi_*(s^*E \otimes V)_{(z,\alpha)} \xrightarrow{\lambda} R^1\pi_*(s^*E \otimes V \otimes \pi^*(\mathcal{O}(-1)))_{(z,\lambda\alpha)},$$

for any $(z, \alpha) \in \mathbb{P}^1 \times \mathbb{C}^*$. By the Projection formula,

$$R^1\pi_*(s^*E \otimes V \otimes \pi^*(\mathcal{O}(-1))) \cong R^1\pi_*(s^*E \otimes V) \otimes \mathcal{O}(-1).$$

Thus $L_{(z,\alpha)} \xrightarrow{\lambda} L_{(z,\lambda\alpha)} \otimes \mathcal{O}(-1)$, and L does not descend on $\mathbb{P}^1 \times \mathbb{C}^*/\mathbb{Z}$.

We can however get around this problem by constructing a sheaf $\mathcal{L}$ on $\mathbb{P}^1 \times \mathbb{C}^*$ such that $\mathcal{L}_{(z,\alpha)} \xrightarrow{\lambda} \mathcal{L}_{(z,\lambda\alpha)} \otimes \mathcal{O}(1)$. The fibres of $L \otimes \mathcal{L}$ will then be preserved by the action of $\mathbb{Z}$. Let us assume that the graph of E does not have a vertical bar at the origin p_0 of $\mathbb{P}^1$. We can describe the bundle $\mathcal{O}(-1)$ as being given by the divisor $-p_0$. (If the graph has a vertical bar at p_0, we will simply take another point of $\mathbb{P}^1$ to descibe the divisor of $\mathcal{O}(-1)$.) Let $W = (p_0 \times \mathbb{C}^*) \cap \widetilde{S}$ be the set of points on $\widetilde{S}$ which lie above p_0. If (a, b) is a representation of the pair of points that lie above p_0 on S, W is then the set of all translates of this pair by λ in $\widetilde{S}$:

$$W = \bigcup_{i \in \mathbb{Z}} (\lambda^i a, \lambda^i b).$$

Moreover, $a + b$ is a divisor on $\widetilde{S}$. Let us denote $T^i(a + b) := \lambda^i a + \lambda^i b$ the translate of $(a + b)$ by λ^i. We then define a divisor on $\widetilde{S}$ as the locally finite sum

$$\mathcal{D} := \sum_{i \in \mathbb{Z}} i T^{-i}(a + b).$$

Let $\mathcal{L}$ be the line bundle on $\widetilde{S}$ associated to the invertible sheaf $\mathcal{O}(\mathcal{D})$. Let us also denote by $\mathcal{L}$ the line bundle thought of as a sheaf on $\mathbb{P}^1 \times \mathbb{C}^*$.

We fix a section γ of $\mathcal{O}(-1)$. It will then have a zero at p_0. We then define the following $\mathbb{Z}$-action on the sheaf $L \otimes \mathcal{L}$ over $\mathbb{P}^1 \times \mathbb{C}^*$

$$\begin{tikzcd}
L \otimes \mathcal{L}|_{(z,\alpha)} \arrow[r, "{(z,\alpha,s\otimes t) \mapsto (z,\lambda\alpha,\lambda s\otimes\gamma t)}", "\lambda"'] \arrow[d] & L \otimes \mathcal{L}|_{(z,\lambda\alpha)} \arrow[d] \\
\mathbb{P}^1 \times \mathbb{C}^* \arrow[r, "\lambda", "{(z,\alpha)\mapsto(z,\lambda\alpha)}"'] & \mathbb{P}^1 \times \mathbb{C}^*.
\end{tikzcd}$$

$L \otimes \mathcal{L}$ is therefore invariant under this action, and it descends to a sheaf $\bar{L} := (L \otimes \mathcal{L})/\sim$ on $\mathbb{P}^1 \times T^* = (\mathbb{P}^1 \times \mathbb{C}^*)/\sim$, with support $\bar{S}$. Let us note that if we pull back $\bar{L}$ to $\mathbb{P}^1 \times \mathbb{C}^*$ and tensor it by $\mathcal{L}^*$, we get back L.

Let us now assume that g is the graph of a holomorphic map $F : \mathbb{P}^1 \to \mathbb{P}^1$ of degree n which satisfies condition $(*)$, i.e. there are no points x in $\mathbb{P}^1$ such that $E|_{\pi^{-1}(x)} = L_0 \oplus L_0, L_0^2 = \mathcal{O}$. The restriction of the skyscraper sheaf $\bar{L}$ to $\bar{S}$ is then a line bundle. Furthermore, as the first Chern class of $\bar{L}$ is given by $\bar{S}$, the first Chern class of $\bar{L}|_{\bar{S}}$ is completely determined by the graph. Therefore, one can associate to each element of the fibre $G^{-1}(g)$ a line bundle $\bar{L}|_{\bar{S}}$ on $\bar{S}$, and these line bundles all have the same degree. Let us also note that since F has degree n, $\bar{S}$ is a double cover of $\mathbb{P}^1$ with $4n$ branch points, and must therefore be a curve of genus $2n-1$. Hence, if $n = 1$, $\bar{S}$ is an ellipic curve, and if $n \geq 2$, $\bar{S}$ is a hyperelliptic curve.

4.2.2 Description of Jac($\bar{\mathrm{S}}$).

Let E be a holomorphic SL(2, $\mathbb{C}$)-bundle on $\mathcal{H}$ with $c_2(E) = n$. In this section, we would like to prove that, if the graph g of E is the graph of a holomorphic map of degree n which satisfies the condition $(*)$, we can then recover E from the sheaf $L = R^1\pi_{1*}(s^*E \otimes V)$. Keeping in mind the discussion at the end of section 4.2.1, this will then give us a one-to-one correspondence between the fibre $G^{-1}(g)$ of the graph map at g and line bundles on $\bar{S}$ of a fixed degree.

Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be two copies of the Hopf surface, with projections onto $\mathbb{P}^1$ denoted by π_1 and π_2, respectively. Let $\mathcal{H}_1 \times_{\mathbb{P}^1} \mathcal{H}_2$ be the fibred product induced by $\pi_i : \mathcal{H}_i \longrightarrow \mathbb{P}^1$, for $i = 1, 2$, and let $p_i : \mathcal{H}_1 \times_{\mathbb{P}^1} \mathcal{H}_2 \longrightarrow \mathcal{H}_i, i = 1, 2$, be the natural projections associated to this product. For $i = 1, 2$, we will also denote by p_i the following projections

$$\begin{array}{rcccc} p_i & : & \mathcal{H}_1 \times_{\mathbb{P}^1} \mathcal{H}_2 \times \mathbb{C}^* & \longrightarrow & \mathcal{H}_i \times \mathbb{C}^* \\ & & (x_1, x_2, \alpha) & \longmapsto & (p_i(x_1, x_2), \alpha) \end{array},$$

and by π_i the projections

$$\begin{array}{rcccc} \pi_i & : & \mathcal{H}_i \times \mathbb{C}^* & \longrightarrow & \mathbb{P}^1 \times \mathbb{C}^* \\ & & (x_i, \alpha) & \longmapsto & (\pi_i(x_i), \alpha) \end{array}.$$

Let E be a holomorphic SL(2, $\mathbb{C}$)-bundle on $\mathcal{H}_1$ with $c_2(E) = n$. We will assume that the graph of E is a holomorphic map $g : \mathbb{P}^1 \to \mathbb{P}^1$ of degree n which satisfies the condition $(*)$ defined in section 4.1.4, i.e. that there are no points z in $\mathbb{P}^1$ such that $E|_{\pi^{-1}(z)} \cong L_0 \oplus L_0, L_0^2 \cong \mathcal{O}$. We have the following commutative diagram

$$\begin{tikzcd} & \mathcal{H}_1 \times_{\mathbb{P}^1} \mathcal{H}_2 \times \mathbb{C}^* \arrow[r, "p_2"] \arrow[d, "p_1"'] & \mathcal{H}_2 \times \mathbb{C}^* \arrow[d, "\pi_2"] \\ s^*E \arrow[r] & \mathcal{H}_1 \times \mathbb{C}^* \arrow[r, "\pi_1"] \arrow[d, "s"'] & \mathbb{P}^1 \times \mathbb{C}^* \arrow[d] \\ E \arrow[r] & \mathcal{H}_1 \arrow[r, "\pi_1"] & \mathbb{P}^1. \end{tikzcd}$$

Moreover, if $\tilde{s}$ is the canonical identification of $\mathcal{H}_2 \times \mathbb{C}^*$ with $\mathcal{H}_1 \times \mathbb{C}^*$

$$\begin{array}{rcccc} \tilde{s} & : & \mathcal{H}_2 \times \mathbb{C}^* & \longrightarrow & \mathcal{H}_1 \times \mathbb{C}^* \\ & & (x_2, \alpha) & \longmapsto & (x_2, \alpha) \end{array},$$

we also have the following commutative diagram

$$\begin{tikzcd} \mathcal{H}_1 \times \mathbb{C}^* \arrow[d, "s"'] & \mathcal{H}_2 \times \mathbb{C}^* \arrow[l, "\tilde{s}"'] \arrow[dl, "\sigma"] \\ \mathcal{H}_1. & \end{tikzcd}$$

Using the notation of the previous subsection, we have the skyscraper sheaf $L = R^1\pi_{1*}(s^*E \otimes V)$, with support $\widetilde{S} = \text{supp(L)} \subset \mathbb{P}^1 \times \mathbb{C}^*$. We can then obtain E from L in the following three steps:

- we pullback L to $\mathcal{H}_2 \times \mathbb{C}^*$, and then push it down to $\mathcal{H}_1 \times \mathbb{C}^*$ to get the skyscraper sheaf $\bar{s}_*(\pi_2^* L)$ whose support is now $\pi_1^{-1}(\bar{S})$;
- we tensor $\bar{s}_*(\pi_2^* L)$ by V^* to counter the influence of V in L;
- we finally want to push down to $\mathcal{H}_1$. Let us however note that the fibre of s is not proper. Nevertheless, as $\bar{s}_*(\pi_2^* L) \otimes V^* = \bar{s}_*(\pi_2^* R^1\pi_{1*}(s^*E \otimes V)) \otimes V^*$, the presence of both V and V^* implies that the $\mathbb{Z}$-action, induced from $\mathcal{H}_1 \times \mathbb{C}^*$, is fibre preserving. Taking the quotient with respect to this action, we thus obtain a sheaf on the compact manifold $\mathcal{H}_1 \times \mathbb{C}^*/\mathbb{Z}$. We can now push down to $\mathcal{H}_1$, and find that
$$E \cong s^*(\bar{s}_*(\pi_2^* L) \otimes V^*/_\sim).$$

This will be proven in

Proposition 4.1 *Let E be a holomorphic $\mathrm{SL}(2, \mathbb{C})$-bundle on $\mathcal{H}_1$ with $c_2(E) = n$. We assume that the graph of E is a holomorphic map $g : \mathbb{P}^1 \to \mathbb{P}^1$ of degree n, and that there are no points z in $\mathbb{P}^1$ where $E|_{\pi^{-1}(z)} \cong L_0 \oplus L_0, L_0^2 \cong \mathcal{O}$. If $L = R^1\pi_{1*}(s^*E \otimes V)$, then*

$$E \cong s_*(\bar{s}_*(\pi_2^* L) \otimes V^*)/_\sim),$$

where the quotient is taken with respect to the $\mathbb{Z}$-action induced from $\mathcal{H}_1 \times \mathbb{C}$.

Proof: As the proof is very technical, we will state some of the results that we need without proof. Their proofs will be given in the next section.

Let us denote by $\text{diag}(\mathcal{H})$ the diagonal in $\mathcal{H}_1 \times_{\mathbb{P}^1} \mathcal{H}_2$, i.e. if (t_1, t_2, z) are local coordinates at a point in the fibred product, then $\text{diag}(\mathcal{H}) = \{\mathrm{t}_1 = \mathrm{t}_2\}$. If we set $D = \mathrm{diag}(\mathcal{H}) \times \mathbb{C}^* \subset \mathcal{H}_1 \times_{\mathbb{P}^1} \mathcal{H}_2 \times \mathbb{C}^*$, D is then an effective divisor in $\mathcal{H}_1 \times_{\mathbb{P}^1} \mathcal{H}_2 \times \mathbb{C}^*$, and we have the following exact sequence on $\mathcal{H}_1 \times_{\mathbb{P}^1} \mathcal{H}_2 \times \mathbb{C}^*$:

$$0 \to p_1^*(s^*E \otimes V) \to p_1^*(s^*E \otimes V)(D) \to p_1^*(s^*E \otimes V)(D)|_D \to 0. \tag{4.5}$$

Pushing down to $\mathcal{H}_2 \times \mathbb{C}^*$, we obtain the long exact sequence

$$\begin{aligned} 0 \longrightarrow p_{2*}(p_1^*(s^*E \otimes V)) \longrightarrow p_{2*}(p_1^*(s^*E \otimes V)(D)) \longrightarrow \\ \longrightarrow p_{2*}(p_1^*(s^*E \otimes V)(D)|_D) \longrightarrow R^1p_{2*}(p_1^*(s^*E \otimes V)) \longrightarrow \\ \longrightarrow R^1p_{2*}(p_1^*(s^*E \otimes V)(D)) \longrightarrow \ldots . \end{aligned} \tag{4.6}$$

We then have the following:

Lemma 4.1 *i)* $p_{2*}(p_1^*(s^*E \otimes V)) = 0$,
ii) $R^1p_{2*}(p_1^*(s^*E \otimes V)) = \pi_2^*L$,
iii) $p_{2*}(p_1^*(s^*E \otimes V)(D))$ *is a locally free sheaf of rank two,*
iv) $R^1p_{2*}(p_1^*(s^*E \otimes V)(D)) = 0$.

Inserting this into (4.6), we obtain the exact sequence on $\mathcal{H}_2 \times \mathbb{C}^*$

$$0 \to p_{2*}(p_1^*(s^*E \otimes V)(D)) \longrightarrow p_{2*}(p_1^*(s^*E \otimes V)(D)|_D) \longrightarrow \pi_2^*L \to 0.$$

Tensoring by $\bar{s}^*V^*$, and applying the Projection Formula, we have

$$\begin{aligned} 0 \longrightarrow p_{2*}(p_1^*(s^*E \otimes V)(D) \otimes (\bar{s}p_2)^*V^*) \longrightarrow \\ \longrightarrow p_{2*}(p_1^*(s^*E \otimes V)(D) \otimes (\bar{s}p_2)^*V^*|_D) \longrightarrow \pi_2^*L \otimes \bar{s}^*V^* \longrightarrow 0. \end{aligned}$$

As $\bar{s}$ is a finite morphism, pushing down on $\mathcal{H}_1 \times \mathbb{C}^*$ again gives us an exact sequence

$$\begin{aligned} 0 \longrightarrow (\bar{s}p_2)_*(p_1^*(s^*E \otimes V)(D) \otimes (\bar{s}p_2)^*V^*) \longrightarrow \\ \longrightarrow (\bar{s}p_2)_*(p_1^*(s^*E \otimes V)(D) \otimes (\bar{s}p_2)^*V^*|_D) \longrightarrow \bar{s}_*(\pi_2^*L \otimes \bar{s}^*V^*) \longrightarrow 0. \end{aligned} \tag{4.7}$$

Let us note that $\bar{s}_*(\pi_2^*L \otimes \bar{s}^*V^*) \cong \bar{s}_*(\pi_2^*L) \otimes V^*$. (4.7) can be further simplified, given

Lemma 4.2 *On* $\mathcal{H}_1 \times \mathbb{C}^*$, $(\bar{s}p_2)_*(p_1^*(s^*E \otimes V)(D) \otimes (\bar{s}p_2)^*V^*|_D) \cong s^*E$.

We thus obtain the exact sequence on $\mathcal{H}_1 \times \mathbb{C}^*$

$$0 \longrightarrow (\bar{s}p_2)_*(p_1^*(s^*E \otimes V)(D) \otimes (\bar{s}p_2)^*V^*) \longrightarrow s^*E \longrightarrow \bar{s}_*(\pi_2^*L) \otimes V^* \longrightarrow 0.$$

We finally want to take the pushdown to $\mathcal{H}_1$. However, as we have previously remarked, the fibre of s is not proper. But we can get around this problem, because we now have a well-defined $\mathbb{Z}$-action on every sheaf of the sequence. Taking the quotient by $\mathbb{Z}$, we again obtain an exact sequence on $\mathcal{H}_1 \times \mathbb{C}^*/\mathbb{Z}$

$$0 \to (\bar{s}p_2)_*(p_1^*(s^*E \otimes V)(D) \otimes (\bar{s}p_2)^*V^*)/_\sim \longrightarrow s^*E/_\sim \longrightarrow \bar{s}_*(\pi_2^*L) \otimes \bar{s}^*V^*/_\sim \to 0. \tag{4.8}$$

Let us note that the support of $\bar{s}_*(\pi_2^*L) \otimes \bar{s}^*V^*/_\sim$ is now the spectral curve $\bar{S}$ associated to the graph of E. Before taking the pushdown, let us examine the restriction of this sequence to a fibre $T^*_{x_1} = \{x_1\} \times \mathbb{C}^*/\mathbb{Z}$ of s. On $T^*_{x_1}$, $s^*E \cong \mathcal{O} \oplus \mathcal{O}$. Furthemore, $\bar{s}_*(\pi_2^*L) \otimes \bar{s}^*V^*/_\sim$ is now a skyscraper sheaf with fibre $\mathbb{C}$ supported on the two points $p, q \in \mathbb{C}^*/\mathbb{Z}$ corresponding to E over $\pi^{-1}(x_1)$. By lemma 4.1, we know that the sheaf $(\bar{s}p_2)_*(p_1^*(s^*E \otimes V)(D) \otimes (\bar{s}p_2)^*V^*)/_\sim$ is locally free of rank two. Its first Chern class must be equal to -2, and we see, by the construction, that it is actually isomorphic to $\mathcal{O}(-p) \oplus \mathcal{O}(-q)$. On $T^*_{x_1}$, (4.8) becomes

$$0 \longrightarrow \mathcal{O}(-p) \oplus \mathcal{O}(-q) \longrightarrow \mathcal{O} \oplus \mathcal{O} \longrightarrow \mathbb{C}_{p,q} \longrightarrow 0, \tag{4.9}$$

where $\mathbb{C}_{p,q}$ is a skyscraper sheaf supported on $\{p, q\}$ with fibre $\mathbb{C}$. By Riemann-Roch, we see that on each $T^*_{x_1}$, the holomorphic sections of $(\bar{s}p_2)_*(p_1^*(s^*E \otimes V)(D) \otimes (\bar{s}p_2)^*V^*)/_\sim$ are all trivial, and its pushdown to $\mathcal{H}_1$ must then be 0. Moreover, on each $T^*_{x_1}$, the isomorphism on global holomorphic sections of $\mathcal{O} \oplus \mathcal{O}$ and $\mathbb{C}_{p,q}$ given by (4.9), induces an isomorphism on s_*:

$$0 \longrightarrow s_*(s^*E/_\sim) \cong s_*(\bar{s}_*(\pi_2^*L) \otimes \bar{s}^*V^*)/_\sim \longrightarrow 0. \tag{4.10}$$

As s is simply the projection onto the first factor of a direct product, and E is invariant under the $\mathbb{Z}$-action, we have $s_*(s^*E/_\sim) \cong E$, and (4.10) gives us the required isomorphism of $s_*(\bar{s}_*(\pi_2^*L) \otimes \bar{s}^*V^*/_\sim)$ with E. $\square$

4.2.3 Proof of lemmas 4.1 and 4.2.

Before proving the lemmas, we need the following

Lemma 4.3 *If $M = \mathcal{H}_1 \times_{\mathbb{P}^1} \mathcal{H}_2 \times \mathbb{C}^*$, then*

i) $p_{1*}(\mathcal{O}_M) \cong R^1 p_{1*}(\mathcal{O}_M) \cong \mathcal{O}_{\mathcal{H}_1 \times \mathbb{C}^*}$;

ii) For any $(x_1, \alpha) \in \mathcal{H}_1 \times \mathbb{C}^*$, *if* $T_{x_1} = \{x_1\} \times \pi_2^{-1}(\pi_1(x_1)) \times \{\alpha\}$, *and* $p = D \cap T_{x_1} = (x_1, x_1, \alpha)$, *then*

$$\mathcal{O}_M(D)|_{T_{x_1}} \cong \mathcal{O}_{T_{x_1}}(p);$$

iii) $p_{1*}(\mathcal{O}_M(D)) \cong \mathcal{O}_{\mathcal{H}_1 \times \mathbb{C}^*}$;

iv) $R^1 p_{1*}(\mathcal{O}_M(D)) = 0$;

v) $p_{1*}(\mathcal{O}_M(D)|_D) \cong \mathcal{O}_{\mathcal{H}_1 \times \mathbb{C}^*}$.

Proof: i) We will use the commutative diagram

$$\begin{array}{ccc} M = \mathcal{H}_1 \times_{\mathbb{P}^1} \mathcal{H}_2 \times \mathbb{C}^* & \xrightarrow{p_2} & \mathcal{H}_2 \times \mathbb{C}^* \\ \Big\downarrow p_1 & & \Big\downarrow \pi_2 \\ \mathcal{H}_1 \times \mathbb{C}^* & \xrightarrow[\pi_1]{} & \mathbb{P}^1 \times \mathbb{C}^*. \end{array}$$

As $\mathcal{O}_M \cong p_2^*(\mathcal{O}_{\mathcal{H}_2 \times \mathbb{C}^*})$, for $i \geq 0$, $R^i p_{1*}(\mathcal{O}_M) \cong R^i p_{1*}(p_2^*(\mathcal{O}_{\mathcal{H}_2 \times \mathbb{C}^*}))$. Also, $\pi_1 : \mathcal{H}_1 \times \mathbb{C}^* \to \mathbb{P}^1 \times \mathbb{C}^*$ is a flat morphism, thus implying that, for all $i \geq 0$,

$$R^i p_{1*}(p_2^*(\mathcal{O}_{\mathcal{H}_2 \times \mathbb{C}^*})) \cong \pi_1^*(R^i \pi_{2*}(\mathcal{O}_{\mathcal{H}_2 \times \mathbb{C}^*})); \tag{4.11}$$

and we see that we just have to show that, for $i = 0, 1$,

$$R^i \pi_{2*}(\mathcal{O}_{\mathcal{H}_2 \times \mathbb{C}^*}) \cong \mathcal{O}_{\mathbb{P}^1 \times \mathbb{C}^*}.$$

Furthermore, as $\pi_2 : \mathcal{H}_2 \times \mathbb{C}^* \longrightarrow \mathbb{P}^1 \times \mathbb{C}^*$ is simply the identity on $\mathbb{C}^*$, this reduces to proving that, for $\pi_2 : \mathcal{H}_2 \longrightarrow \mathbb{P}^1$, and $i = 0, 1$,

$$R^i \pi_{2*}(\mathcal{O}_{\mathcal{H}_2}) \cong \mathcal{O}_{\mathbb{P}^1}.$$

For any $z \in \mathbb{P}^1$, let $T_z = \pi_2^{-1}(z)$. For $i = 0, 1$, Riemann-Roch then implies

$$R^i\pi_{2*}(\mathcal{O}_{\mathcal{H}_2})_z = H^i(T_z, \mathcal{O}_{\mathcal{H}_2}|_{T_z}) = \mathbb{C}.$$

Thus, by Grauert's theorem, $R^i\pi_{2*}(\mathcal{O}_{\mathcal{H}_2})$ is an invertible sheaf and

$$R^i\pi_{2*}(\mathcal{O}_{\mathcal{H}_2}) \otimes \mathbb{C}(z) \longrightarrow H^i(T_z, \mathcal{O}_{\mathcal{H}_2}|_{T_z})$$

is an isomorphism.

For $i = 0$, the right-hand side of this map is canonically isomorphic to $\mathbb{C}$, and the left-hand side is simply $\pi_{2*}(\mathcal{O}_{\mathcal{H}_2})_z$. The image of the global section 1 of $\mathcal{O}_{\mathbb{P}^1}$ via the structural map $\mathcal{O}_{\mathbb{P}^1} \to \pi_{2*}(\mathcal{O}_{\mathcal{H}_2})$ therefore generates the stalk at every point, showing that $\pi_{2*}(\mathcal{O}_{\mathcal{H}_2}) \cong \mathcal{O}_{\mathbb{P}^1}$; and implying that $ch(\pi_{2*}(\mathcal{O}_{\mathcal{H}_2})) = 1$.

By Grothendieck-Riemann-Roch,

$$ch(\pi_{2!}(\mathcal{O}_{\mathcal{H}_2})) \cdot td(\mathcal{H}_2) = \pi_{2*}(ch(\mathcal{O}_{\mathcal{H}_2})) \cdot td(\mathbb{P}^1).$$

We have $td(\mathcal{H}_2) = 1$ and $ch(\mathcal{O}_{\mathcal{H}_2}) = 1$; moreover, as $\mathcal{H}_2$ has real dimension 4, and $\mathbb{P}^1$ has real dimension 2, π_{2*} is zero on 0-cocycles; implying that $\pi_{2*}(ch(\mathcal{O}_{\mathcal{H}_2})) = 0$. Therefore,

$$ch(\pi_{2!}(\mathcal{O}_{\mathcal{H}_2})) = 0 \implies ch(R^1\pi_{2*}(\mathcal{O}_{\mathcal{H}_2})) = ch(\pi_{2*}(\mathcal{O}_{\mathcal{H}_2})) = 1.$$

$R^1\pi_{2*}(\mathcal{O}_{\mathcal{H}_2})$ is then an invertible sheaf on $\mathbb{P}^1$ of degree zero, proving that $R^1\pi_{2*}(\mathcal{O}_{\mathcal{H}_2}) \cong \mathcal{O}_{\mathbb{P}^1}$.

ii) to iv) $D \cap T_\alpha = p$. Let (t_1, t_2, z, z') be coordinates centered at p. D is then given by $\{t_1 - t_2 = 0\}$, and $T_\alpha = \{t_1 = x_1, z = \pi_1(x_1), z' = \alpha\}$. Therefore, as

$$\begin{aligned}(D \cdot T_\alpha)_p &= l(\mathbb{C}[t_1, t_2, z, z']/(t_1 - t_2, t_1 - x_1, z - \pi_1(x_1), z' - \alpha)) \\ &= \dim_{\mathbb{C}}(\mathbb{C}) = 1,\end{aligned}$$

$D \cdot T_\alpha = p$, and $\mathcal{O}_M(D)|_{T_\alpha} = \mathcal{O}_{T_\alpha}(p)$ is an invertible sheaf of degree one. The inclusion $\mathcal{O}_M \hookrightarrow \mathcal{O}_M(D)$ then induces an isomorphism on holomorphic sections, and $p_{1*}(\mathcal{O}_M(D)) = \mathcal{O}_{\mathcal{H}_1 \times \mathbb{C}^*}$. Furthermore, by Riemann-Roch,

$$R^1p_{1*}(\mathcal{O}_M(D))_{(x_1,\alpha)} = H^1(T_{x_1}, \mathcal{O}_{T_\alpha}(p)) = 0,$$

for all $(x_1, \alpha) \in \mathcal{H}_1 \times \mathbb{C}^*$. Thus $R^1 p_{1*}(\mathcal{O}_M(D)) = 0$.

v) As D is an effective divisor, we have the exact sequence

$$0 \to \mathcal{O}_M \to \mathcal{O}_M(D) \to \mathcal{O}_M(D)|_D \to 0.$$

Pushing down to $\mathcal{H}_1 \times \mathbb{C}^*$ and referring to i) and iv), we obtain

$$0 \to \mathcal{O}_{\mathcal{H}_1 \times \mathbb{C}^*} \to p_{1*}(\mathcal{O}_M(D)) \to p_{1*}(\mathcal{O}_M(D)|_D) \to \mathcal{O}_{\mathcal{H}_1 \times \mathbb{C}^*} \to 0. \tag{4.12}$$

As we have seen, the inclusion $\mathcal{O}_M \hookrightarrow \mathcal{O}_M(D)$ induces an isomorphism $\mathcal{O}_{\mathcal{H}_1 \times \mathbb{C}^*} \cong p_{1*}(\mathcal{O}_M(D))$, thus splitting (4.12) and implying v). $\square$

Proof of Lemma 4.1:

We have to prove the following:

i) $p_{2*}(p_1^*(s^*E \otimes V)) = 0$,

ii) $R^1 p_{2*}(p_1^*(s^*E \otimes V)) = \pi_2^* L$,

iii) $p_{2*}(p_1^*(s^*E \otimes V)(D))$ *is a locally free sheaf of rank two,*

iv) $R^1 p_{2*}(p_1^*(s^*E \otimes V)(D)) = 0$.

Let us again note that, since $\pi_2 : \mathcal{H}_2 \times \mathbb{C}^* \to \mathbb{P}^1 \times \mathbb{C}^*$ is a flat morphism, the commutative diagram

$$\begin{array}{ccc} \mathcal{H}_1 \times_{\mathbb{P}^1} \mathcal{H}_2 \times \mathbb{C}^* & \xrightarrow{p_1} & \mathcal{H}_1 \times \mathbb{C}^* \\ \downarrow{\scriptstyle p_2} & & \downarrow{\scriptstyle \pi_1} \\ \mathcal{H}_2 \times \mathbb{C}^* & \xrightarrow[\pi_2]{} & \mathbb{P}^1 \times \mathbb{C}^* \end{array}$$

implies that, for all $i \geq 0$,

$$R^i p_{2*}(p_1^*(s^*E \otimes V)) \cong \pi_2^*(R^i \pi_{1*}(s^*E \otimes V)). \tag{4.13}$$

Let $(z, \alpha) \in \mathbb{P}^1 \times \mathbb{C}^*$ and $T_{(z,\alpha)} = \pi_1^{-1}(z) \times \{\alpha\}$.

i) For generic (z, α), $(s^*E \otimes V)|_{T_{(z,\alpha)}}$ is a sum of non trivial line bundles of degree zero, and, by Riemann-Roch,

$$\pi_{1*}(s^*E \otimes V)_{(z,\alpha)} = H^0(T_{(z,\alpha)}, (s^*E \otimes V)|_{T_{(z,\alpha)}}) = 0.$$

Hence, $\pi_{1*}(s^*E \otimes V) = 0$ and $p_{2*}(p_1^*(s^*E \otimes V)) \cong \pi_2{}^*(\pi_{1*}(s^*E \otimes V)) = 0$.

ii) As $L = R^1\pi_{1*}(s^*E \otimes V)$, ii) follows from (4.13).

iii) and iv) For any $(x_2, \alpha) \in \mathcal{H}_2 \times \mathbb{C}^*$, let $p = D \cap T_{x_2} = (x_2, x_2, \alpha)$, where $T_{x_2} = p_2{}^{-1}(x_2, \alpha) = \pi_1{}^{-1}(\pi_2(x_2)) \times \{x_2\} \times \{\alpha\}$. By arguments similar to the ones used in the proof of lemma 4.3 ii), we see that $\mathcal{O}(D)|_{T_{x_2}} \cong \mathcal{O}_{T_{x_2}}(p)$ and $\deg(\mathcal{O}(\mathrm{D})|_{\mathrm{T}_{x_2}}) = 1$. Thus, as $p_1^*(s^*E \otimes V)|_{T_{x_2}}$ is a sum of line bundles of degree zero, $p_1^*(s^*E \otimes V)(D)|_{T_{x_2}}$ is a sum of line bundles of degree one. By Riemann-Roch

$$p_{2*}(p_1^*(s^*E \otimes V)(D))_{(x_2,\alpha)} = H^0(T_{x_2}, p_1^*(s^*E \otimes V)(D)|_{T_{x_2}}) = \mathbb{C}^2, \qquad (4.14)$$

and

$$R^1p_{2*}(p_1^*(s^*E \otimes V)(D))_{(x_2,\alpha)} = H^1(T_{x_2}, p_1^*(s^*E \otimes V)(D)|_{T_{x_2}}) = 0.$$

Therefore $R^1p_{2*}(p_1^*(s^*E \otimes V)(D)) = 0$, and by Grauert's theorem, (4.14) implies that $p_{2*}(p_1^*(s^*E \otimes V)(D))$ is a locally free sheaf of rank two. □

Proof of Lemma 4.2:

We have to prove the following:

On $\mathcal{H}_1 \times \mathbb{C}^*$, $(\bar{s}p_2)_*(p_1^*(s^*E \otimes V)(D) \otimes (\bar{s}p_2)^*V^*|_D) \cong s^*E$.

As $\bar{s}p_2 = p_1$ on D,

$$(\bar{s}p_2)_*(p_1^*(s^*E \otimes V)(D) \otimes (\bar{s}p_2)^*V^*|_D)$$

$$\cong p_{1*}(p_1^*(s^*E \otimes V)(D) \otimes p_1^*V^*|_D) \cong s^*E \otimes p_{1*}(\mathcal{O}(D)|_D).$$

By lemma 4.3 v), $p_{1*}(\mathcal{O}(D)|_D) \cong \mathcal{O}$, thus proving the lemma. □

Chapter 5

Poisson structure and integrable systems on $\mathcal{M}_n$.

In this chapter, we begin by using Bottacin's construction [Bot] to define a Poisson structure on the moduli space $\mathcal{M}_n$ of stable SL(2, $\mathbb{C}$)-bundles E with $c_2(E) = n$ on $\mathcal{H}$. This structure will be induced, in natural way, from a Poisson structure on $\mathcal{H}$. Let Δ be the subset of $\mathbb{P}^{2n+1}$ consisting of graphs which do not satisfy condition $(*)$ (see section 4.1.4), or which contain vertical bars. We then compute the dimension of the symplectic leaves of $\mathcal{M}_n$, and show that the graph map $\mathcal{M}_n \to \mathbb{P}^{2n+1}$ is a Lagrangian fibration over the complement of Δ. Let us note that, to prove the latter, we will use arguments similar to those found in [Be].

5.1 Poisson structure on $\mathcal{M}_n$.

We start by recalling some definitions and results of symplectic geometry, that can be found, for example, in [We].

Let X be a smooth algebraic variety over the complex field $\mathbb{C}$. A (holomorphic) *Poisson structure* on X is a Lie algebra structure $\{\cdot,\cdot\}$ on $\mathcal{O}_X$ satisfying the Leibniz identity $\{f, gh\} = \{f, g\}h + g\{f, h\}$. This structure is equivalently given by an antisymmetric contravariant 2-tensor $\theta \in H^0(X, \wedge^2 TX)$, where we set

$\{f,g\} = \langle\theta, df \wedge dg\rangle$. If the bracket defined by θ satisfies the Jacobi identity

$$\{f,\{g,h\}\} + \{g,\{h,f\}\} + \{h,\{f,g\}\} = 0, \tag{5.1}$$

for any $f,g,h \in \Gamma(U,\mathcal{O}_X)$, then θ defines a Poisson structure on X. For any function $f \in \Gamma(U,\mathcal{O}_X)$, the map $g \mapsto \{f,g\}$ is a derivation of $\Gamma(U,\mathcal{O}_X)$. There then exists a vector field H_f on U such that $H_f \cdot g = \{f,g\}$, for all $g \in \Gamma(U,\mathcal{O}_X)$. The vector field H_f is called the *Hamiltonian vector field* associated to f.

Note that giving $\theta \in H^0(X,\wedge^2 TX)$ is equivalent to giving a homomorphism of vector bundles $B : T^*X \longrightarrow TX$, with $\langle\theta,\alpha\wedge\beta\rangle = \langle B(\alpha),\beta\rangle$ (or $\langle\alpha,B(\beta)\rangle$, up to a sign), for 1-forms α,β. If B has maximal rank everywhere, the Poisson structure is said to be *symplectic*. For any even r, let

$$X_r := \{p \in X \mid \text{rk }(B) \ = r\}.$$

A basic result [We] then asserts that the X_r are subvarities, and that they are canonically foliated into *symplectic leaves*, i.e. r-dimensional subvarities which inherit a symplectic structure.

Let us define an operator $\tilde{d} : H^0(X,\wedge^2 TX) \longrightarrow H^0(X,\wedge^3 TX)$ by

$$\begin{aligned}\tilde{d}(\alpha,\beta,\gamma) = \quad & B(\alpha)\theta(\beta,\gamma) - B(\beta)\theta(\alpha,\gamma) + B(\gamma)\theta(\alpha,\beta)\\ & -\langle[B(\alpha),B(\beta)],\gamma\rangle + \langle[B(\alpha),B(\gamma)],\beta\rangle - \langle[B(\beta),B(\gamma)],\alpha\rangle,\end{aligned}$$

for 1-forms α,β,γ, where $[\cdot,\cdot]$ denotes the usual commutator of vector fields. It is then easy to verify that

Proposition 5.1 *The bracket $\{\cdot,\cdot\}$, defined by an element $\theta \in H^0(X,\wedge^2 TX)$, satisfies the Jacobi identity if and only if $\tilde{d}\theta = 0$.*

The Poisson structure on $\mathcal{M}_n$ will be constructed using a Poisson structure on $\mathcal{H}$. The latter are given by the following proposition:

Proposition 5.2 *A Poisson structure on $\mathcal{H}$ is given by a global section s of the anticanonical bundle $\omega_{\mathcal{H}}^{-1}$.*

Proof: An element $s \in H^0(\mathcal{H}, \wedge^2 T\mathcal{H}) = H^0(\mathcal{H}, \omega_{\mathcal{H}}^{-1})$ that satisfies the condition $\tilde{d}s = 0$ is, by definition, a Poisson structure on $\mathcal{H}$. As $\mathcal{H}$ is a surface, the map $\tilde{d}$ must be identically zero. $\square$

Note that $\omega_{\mathcal{H}}^{-1} \cong \mathcal{O}(2)$, thus implying that $H^0(\mathcal{H}, \omega_{\mathcal{H}}^{-1}) = \mathbb{C}^3$. Poisson structures on $\mathcal{H}$ then exist. However, as $H^2(\mathcal{H}, \mathbb{R}) = 0$, a Poisson structure s on $\mathcal{H}$ cannot be symplectic.

5.1.1 Poisson structures on $\mathcal{M}_n$.

We recall that for every $E \in \mathcal{M}_n$, we have

$$T_E\mathcal{M}_n \cong H^1(\mathcal{H}, sl(E)),$$

and

$$T_E^*\mathcal{M}_n \cong H^1(\mathcal{H}, sl(E) \otimes \omega_{\mathcal{H}}).$$

Let us choose a Poisson structure $s \in H^0(\mathcal{H}, \mathcal{O}(2))$ on $\mathcal{H}$. We define an element $\theta = \theta_s \in H^0(\mathcal{M}_n, \otimes^2 T\mathcal{M}_n)$ as follows: for any $E \in \mathcal{M}_n$, $\theta(E) : T_E^*\mathcal{M}_n \times T_E^*\mathcal{M}_n \longrightarrow \mathbb{C}$ is defined by

$$\begin{aligned}\theta(E) : H^1(\mathcal{H}, sl(E) \otimes \omega_{\mathcal{H}}) \times H^1(\mathcal{H}, sl(E) \otimes \omega_{\mathcal{H}}) &\xrightarrow{\circ} \\ H^2(\mathcal{H}, End(E) \otimes \omega_{\mathcal{H}}^2) \xrightarrow{s} H^2(\mathcal{H}, End(E) \otimes \omega_{\mathcal{H}}) &\xrightarrow{\mathrm{Tr}} \mathbb{C}.\end{aligned} \tag{5.2}$$

The first map is the cup-product of two cohomology classes, the second is multiplication by s, and the third is the trace map. Note that the stability of E implies that $H^0(\mathcal{H}, End(E)) = \mathbb{C}$. By Serre duality, it then follows that the trace map $Tr : H^2(\mathcal{H}, End(E) \otimes \omega_{\mathcal{H}}) \to \mathbb{C}$ is an isomorphism.

The graded commutativity of the cup-product makes $\theta(E)$ skew-symmetric. To prove that θ defines a Poisson structure on $\mathcal{M}_n$, we therefore only have to prove that it satisfies the closure condition $\tilde{d}\theta = 0$. The latter is a consequence of the following theorem, due to F.Bottacin [Bot]:

Theorem 5.1 *Let S be a Poisson surface and $s \in H^0(S, \omega_S^{-1})$ a Poisson structure on S. The antisymmetric contravariant 2-tensor $\theta = \theta_s \in H^0(\mathcal{M}^0, \wedge^2 T\mathcal{M}^0)$ defines* *a Poisson structure on the moduli space $\mathcal{M}^0$ of H-stable vector bundles on S. (H is taken to be a very ample divisor on S.)*

In the proof, he shows that $\tilde{d}\theta = 0$. A Poisson surface is a smooth algebraic surface that admits a non-zero Poisson structure. As $\mathcal{H}$ is not an algebraic surface, it is not a Poisson surface. The algebraic hypothesis is however only used in the construction of the moduli spaces of sheaves on S. The arguments proving that $\tilde{d}\theta = 0$ therefore also hold in the case of $\mathcal{H}$, or for any compact surface, and θ defines a Poisson structure on $\mathcal{M}_n$.

As we have seen above, giving θ is equivalent to giving a homomorphism of vectors bundles

$$B : T^*\mathcal{M}_n \longrightarrow T\mathcal{M}_n,$$

with $\theta(\alpha \otimes \beta) = \langle B(\alpha), \beta \rangle$, for 1-forms α, β. From the definition of θ, the homomorphism B is clearly the map induced on cohomology by multiplication by the section s: at point $E \in \mathcal{M}_n$, we have

$$B(E) : H^1(\mathcal{H}, sl(E) \otimes \omega_{\mathcal{H}}) \xrightarrow{s} H^1(\mathcal{H}, sl(E)). \tag{5.3}$$

5.1.2 The rank of θ.

We shall now compute the rank of the Poisson structure θ, i.e., the dimension of the symplectic leaves of the Poisson variety $\mathcal{M}_n$.

As $\mathcal{H}$ is not symplectic, s has a divisor. Let D be this divisor. Since $\omega_{\mathcal{H}}^{-1} \cong \mathcal{O}(2)$, $D = T_1 + T_2$, where the T_i are irreducible nonsingular fibres of π. More specifically, as $\mathcal{O}(2) = \pi^*(\mathcal{O}(2))$, if $\mathcal{O}(2)$ is given on $\mathbb{P}^1$ by the divisor $z_1 + z_2$, we see that T_i is the elliptic curve $\pi^{-1}(z_i)$, for $i = 1, 2$. For any vector bundle on $\mathcal{H}$, we have the exact sequence

$$0 \longrightarrow sl(E) \otimes \omega_{\mathcal{H}} \xrightarrow{s} sl(E) \longrightarrow sl(E|_D) \longrightarrow 0, \tag{5.4}$$

which induces the long exact cohomology sequence

$$0 \longrightarrow H^0(\mathcal{H}, sl(E) \otimes \omega_{\mathcal{H}}) \longrightarrow H^0(\mathcal{H}, sl(E)) \longrightarrow H^0(\mathcal{H}, sl(E|_D))$$

$$\longrightarrow H^1(\mathcal{H}, sl(E) \otimes \omega_{\mathcal{H}}) \xrightarrow{B(E)} H^1(\mathcal{H}, sl(E)) \longrightarrow \dots.$$

Since the bundle E is stable, $H^0(\mathcal{H}, End(E)) = \mathbb{C}$ and $H^2(\mathcal{H}, sl(E)) = 0$. Hence, $H^0(\mathcal{H}, sl(E)) = 0$ and, by Serre duality, $H^0(\mathcal{H}, sl(E) \otimes \omega_{\mathcal{H}}) = (H^2(\mathcal{H}, sl(E)))^* = 0$. The above sequence then becomes

$$0 \longrightarrow H^0(\mathcal{H}, sl(E|_D)) \longrightarrow H^1(\mathcal{H}, sl(E) \otimes \omega_{\mathcal{H}}) \xrightarrow{B(E)} H^1(\mathcal{H}, sl(E)) \longrightarrow \dots,$$

and we have the following result:

Proposition 5.3 *The kernel of the Hamiltonian morphism $B(E)$ is given by*

$$\ker B(E) = H^0(D, sl(E|_D)).$$

Hence

$$\begin{aligned} \operatorname{rk} B(E) = &\ \dim \mathcal{M}_n - \dim H^0(D, sl(E|_D)) \\ = &\ 4n - \dim H^0(D, sl(E|_D)). \end{aligned}$$

The rank of the Poisson structure θ at the point $E \in \mathcal{M}_n$ is therefore determined by the restriction of E to the fibres T_1, T_2. For $i = 1, 2$, the restriction $E|_{T_i}$ can be of three possible types:

i) $L_0 \oplus L_0^*$, $L_0 \in \mathrm{Pic}^0(\mathrm{T_i})$;

ii) a nontrivial extension of L_0 by L_0, $L_0^2 \cong \mathcal{O}$;

iii) $L \oplus L^*$, $L \in \mathrm{Pic}^k(\mathrm{T_i})$, $k < 0$.

Referring to lemma 2.3, we obtain the dimension of the cohomology groups:

- If $E|_{T_i}$ is of type i), with $L_0^2 \not\cong \mathcal{O}$, or of type ii), then $h^0(T_i, sl(E|_{T_i})) = 1$.
- If $E|_{T_i}$ is of type i), with $L_0^2 \cong \mathcal{O}$, then $h^0(T_i, sl(E|_{T_i})) = 3$.

– If $E|_{T_i}$ is of type iii), then $h^0(T_i, sl(E|_{T_i})) = 1 - 2k$.

We summarise the above in the corollary:

Corollary 5.1 *Let Δ be the set of graphs in $\mathbb{P}^{2n+1}$ that contain vertical bars, or that correspond to vector bundles E such that, for some $z \in \mathbb{P}^1$, $E|_{\pi^{-1}(z)} \cong L_0 \oplus L_0$, with $L_0^2 \cong \mathcal{O}$. If $E \in \mathcal{M}_n$ is such that*

i) $G(E) \in \Delta$, then $\mathrm{rk}\, B(E) \leq 4n - 4$;

ii) $G(E) \in (\mathbb{P}^{2n+1} - \Delta)$, then $\mathrm{rk}\, B(E) = 4n - 2$.

□

5.2 Integrable systems.

To define algebraically completely integrable systems, we need the following two definitions from symplectic geometry.

Let (X, ω) be a symplectic variety. An irreducible subvariety $Y \subset X$ is *isotropic* if for generic $y \in Y$, the subspace T_yY is an isotropic subspace of ω, i.e., $\omega|_{T_yY} = 0$. It is *Lagrangian* if it is isotropic and $\dim Y = \frac{1}{2}\dim X$.

The above definitions can be extended to Poisson varieties. Let (X, θ) be a Poisson variety. An irreducible subvariety $Y \subset X$ is *isotropic* (respectively *Lagrangian*) if it is generically an isotropic (respectively Lagrangian) subvariety of a symplectic leaf; i.e., Y is contained in the closure $\bar{Z}$ of a symplectic leaf $Z \subset X$ and the intersection $Y \cap Z$ is an isotropic (respectively Lagrangian) subvariety of Z.

We now turn to integrable systems. An algebraically completely integrable Hamiltonian system structure on a family $H : X \to B$ of abelian varieties is a Poisson structure on X with respect to which $H : X \to B$ is a Lagrangian fibration. This can be extended to families of abelian varieties with degenerate fibres:

Definition 5.1 Let X be a smooth algebraic variety (not necessarily complete), B an algebraic variety, Δ a proper closed subvariety of B, and $H : X \to B$ a proper morphism such that the fibres over the complement of Δ are isomorphic to abelian varieties. If the morphism $H : X \to B$ is a Lagrangian fibration over the complement of Δ, a Poisson structure on X is said to be an *algebraically completely integrable system structure* on $H : X \to B$.

Remark: The definition implies that, away from Δ, the Hamiltonian vector fields corresponding to functions on B are tangent to the fibres of H, and are translation invariant.

We have previously seen that, for all n, the graph map

$$G : \mathcal{M}_n \longrightarrow \mathbb{P}^{2n+1}$$

is a fibration whose generic fibre is the Jacobian of a spectral curve $\bar{S}$. Let Δ be the set of graphs in $\mathbb{P}^{2n+1}$ that contain vertical bars, or that correspond to vector bundles E such that, for some $z \in \mathbb{P}^1$, $E|_{\pi^{-1}(z)} \cong L_0 \oplus L_0$, with $L_0^2 \cong \mathcal{O}$. Choose a graph g in the complement of Δ. The spectral curve $\bar{S}$ determined by g is then an elliptic curve if $n = 1$, and a hyperelliptic curve $\bar{S} \xrightarrow{2-1} \mathbb{P}^1$ of genus $2n-1$ if $n \geq 2$. We have seen in chapter 4 that the fibre of the graph map at g is isomorphic to the Jacobian of $\bar{S}$. The fibre $G^{-1}(g)$ is in fact also a Lagrangian subvariety of the symplectic leaf which contains it. This is proven in the following proposition:

Proposition 5.4 *The fibration $G : \mathcal{M}_n \longrightarrow \mathbb{P}^{2n+1}$ is Lagrangian over the complement of Δ.*

Proof: Let us fix a graph $g \in (\mathbb{P}^{2n+1} - \Delta)$, and let $\bar{S}$ be the spectral curve in $\mathbb{P}^1 \times T^*$ which covers g. The curve $\bar{S}$ then has genus $2n-1$. Let E be any vector bundle in $\mathcal{M}_n$ having graph g. Referring to corollary 5.1, the rank of $B(E)$ is $4n-2$. The fibre $G^{-1}(g)$ is therefore contained in a symplectic leaf Z of dimension $4n-2$. Furthermore, the fibre $G^{-1}(g)$ is the Jacobian of the spectral curve $\bar{S}$ of genus $2n-1$. Thus $\dim Jac(\bar{S}) = 2n - 1 = \frac{1}{2} \dim Z$, and we just have to show that $Jac(\bar{S})$ is isotropic, i.e. the Poisson structure θ vanishes on $Jac(\bar{S})$.

Let L be the line bundle corresponding to E in $Jac(\bar{S}) = G^{-1}(g)$. We then have an injection of tangent spaces $T_L Jac(\bar{S}) \longrightarrow T_E \mathcal{M}_n$ (and equivalently a surjection of cotangent spaces $T^*_E\mathcal{M}_n \longrightarrow T^*_L Jac(\bar{S})$.) By deformation theory, the tangent space to $Jac(\bar{S})$ at L is identified with $H^1(\bar{S}, \mathcal{O}_{\bar{S}}) = H^1(\bar{S}, Hom_{\mathcal{O}_{\bar{S}}}(L, L))$. We have also seen that the tangent space to $\mathcal{M}_n$ at E is identified with $H^1(\mathcal{H}, sl(E))$. The injection of tangent spaces therefore induces a morphism of sheaves

$$H^1(\bar{S}, \mathcal{O}_{\bar{S}}) \longrightarrow H^1(\mathcal{H}, sl(E)).$$

This morphism can be described as follows: as in section 1.5, let $\mathbb{C}[\epsilon] = \mathbb{C}[t]/(t)^2$, where ϵ is the class of t. We set $\bar{S}[\epsilon] = \bar{S} \times \mathrm{Spec}\mathbb{C}[\epsilon]$ and $\mathcal{H}[\epsilon] = \mathcal{H} \times \mathrm{Spec}\mathbb{C}[\epsilon]$. Let η be a tangent vector to $Jac(\bar{S})$ at L which corresponds to the infinitesimal deformation L_ϵ on $\bar{S}[\epsilon]$. The locally free sheaf L_ϵ is an extension of L by L on $\bar{S}$:

$$0 \longrightarrow L \longrightarrow L_\epsilon \longrightarrow L \longrightarrow 0. \tag{5.5}$$

Our vector η is therefore given by the extension class of (5.5) in $H^1(\bar{S}, \mathcal{O}_{\bar{S}})$. By using the construction of section 4.2.2 and the exact sequence (5.5), we then obtain a locally free sheaf E_ϵ that is an extension of E by E:

$$0 \longrightarrow E \longrightarrow E_\epsilon \longrightarrow E \longrightarrow 0.$$

The extension class of E_ϵ in $H^1(\mathcal{H}, sl(E))$ is then the image of η in $T_E\mathcal{M}_n$.

Dually, the surjection of cotangent spaces $T^*_E\mathcal{M}_n \to T^*_L Jac(\bar{S})$ induces a surjective morphism of sheaves $H^1(\mathcal{H}, sl(E) \otimes \omega_{\mathcal{H}}) \to H^1(\bar{S}, \mathcal{O}_{\bar{S}} \otimes \omega_{\bar{S}})$. Let us consider the diagram

$$\begin{array}{ccc}
H^1(\mathcal{H}, sl(E) \otimes \omega_{\mathcal{H}}) \times H^1(\mathcal{H}, sl(E) \otimes \omega_{\mathcal{H}}) & \longrightarrow & H^2(\mathcal{H}, End(E) \otimes \omega_{\mathcal{H}}) \\
\downarrow & & \downarrow \\
H^1(\bar{S}, \mathcal{O}_{\bar{S}} \otimes \omega_{\bar{S}}) \times H^1(\bar{S}, \mathcal{O}_{\bar{S}} \otimes \omega_{\bar{S}}) & \longrightarrow & H^2(\bar{S}, \mathcal{O}_{\bar{S}} \otimes \omega_{\bar{S}}).
\end{array}$$

By the Yoneda Pairing, this diagram is commutative. As L is an invertible sheaf on the curve $\bar{S}$, the cohomology group $H^2(\bar{S}, \mathcal{O}_{\bar{S}} \otimes \omega_{\bar{S}})$ must be zero. The Poisson structure θ is therefore vanishes on $T^*_L Jac(\bar{S})$. $\square$

Chapter 6

Stable holomorphic bundles with $c_2 = 2$.

In this chapter, we give a partial classification of $\mathcal{M}_2$. We use the method of [BH] to find the fiber of the graph map in the following three cases:

α) $G(E)$ is the graph of a rational map $F : \mathbb{P}^1 \to \mathbb{P}^1$ of degree 2,

β) $G(E)$ decomposes into two pieces: the graph of an automorphism $F : \mathbb{P}^1 \to \mathbb{P}^1$ and a sum $(\{z_0\} \times \mathbb{P}^1) + (\mathbb{P}^1 \times \{l\})$,

γ) $G(E)$ is a sum $(\{z_0\} \times \mathbb{P}^1) + (\{z_1\} \times \mathbb{P}^1) + (\mathbb{P}^1 \times \{l\})$, where z_0 may equal z_1.

As we have seen in section 4.1.2 of chapter 4, this method consists in taking two bundles over $D \times T$, and finding the number of ways that we can glue them together to obtain distinct $\mathrm{SL}(2, \mathbb{Z})$- isomorphism classes of bundles. The isomorphism class of the bundle however needs to be fixed over each $D \times T$. We were therefore restricting ourselves to graphs that completely determine the bundle over $D \times T$, i.e. graphs satisfying the conditions of lemmas 2.4 and 2.5. In the case were $c_2 = 2$, we can now also consider graphs which satisfy the conditions of lemmas 2.6 and 2.7. After glueing such bundles, we then obtain graphs over $\mathcal{H}$ of type α), β), or γ). As in section 4.1.2, finding the fiber $G^{-1}(g)$ of at $g \in \mathbb{P}^3$ then reduces to computing the cohomology group $H^1(\mathbb{P}^1, A)$, where $A = \pi_*(\mathrm{Aut}_{\mathrm{SL}(2,\mathbb{C})}(\mathrm{E}))$, and E is a bundle with graph equal to g.

Let us note that in the case where E is a bundle on $D \times T$ with a graph of the form $2(\{z_0\} \times \mathbb{P}^1) + (\mathbb{P}^1 \times \{l\})$, and $E = L \oplus L^*$, for $L \in \mathrm{Pic}^{-2}(\mathrm{T})$ on $\pi^{-1}(z_0)$, the graph does not necessarily fix the isomorphism class of E. In this case, we therefore cannot find the fiber of the graph map using this method.

6.1 Global data, $\mathbf{c_2 = 2}$

If E is a vector bundle with $c_2 = 2$, the graph $G(E)$ of E is of three possible types:

α) $G(E)$ is the graph of a rational map $F : \mathbb{P}^1 \to \mathbb{P}^1$ of degree 2,

β) $G(E)$ decomposes into two pieces: the graph of an automorphism $F : \mathbb{P}^1 \to \mathbb{P}^1$ and a sum $(\{z_0\} \times \mathbb{P}^1) + (\mathbb{P}^1 \times \{l\})$,

γ) $G(E)$ is a sum $(\{z_0\} \times \mathbb{P}^1) + (\{z_1\} \times \mathbb{P}^1) + (\mathbb{P}^1 \times \{l\})$.

Let us first discuss the possible existence of points x, in $\mathbb{P}^1$, such that $E|_{\pi^{-1}(x)} = L_0 \oplus L_0, L_0^2 = \mathcal{O}$. In case α, one cannot exclude the existence of such points. As $G(E)$ is the graph of a rational map $F : \mathbb{P}^1 \to \mathbb{P}^1$ of degree 2, the zero x of dF may correspond to a half-period l. The fact that $\mathbb{P}^1 \times \{l\}$ is tangent to the graph of F at x would then imply that $R^1\pi_*(L_0E)$ is a skyscraper sheaf concentrated over x, with fiber $\mathbb{C}^n, n \geq 2$. And it would then be possible for $E|_{\pi^{-1}(x)} = L_0 \oplus L_0$.

In cases β and γ, the existence of such points is impossible. In β, the map part of the graph is an automorphism of $\mathbb{P}^1$. Such points are therefore excluded by the fact that dF can never be zero. In γ, the presence of such points would contradict stability. Indeed, if K and K' are the maximal destabilising bundles of E, then $\deg(\mathrm{K}) + \deg(\mathrm{K}') \geq -2 + (\mathrm{numberofsuchpoints})$, and, either K or K' would have to have positive degree (see [BH]).

Proposition 6.1 *(Case α). If g is a graph of type α, the fibre of $G : \mathcal{I}_2 \to \mathbb{P}^1$ at g is the Jacobian of a hyperellipic curve of genus 3.*

Proof: This is a special case of the above discussion.

Proposition 6.2 *(Case β). If g is a graph of type β, the fibre of $G : \mathcal{I}_2 \to \mathbb{P}^1$ at g is*

(i) $\mathbb{C}^2/\mathbb{Z}^2 \times \mathrm{Pic}^{-1}(\mathrm{T})$ if $F(z_0)$ is a half period;

(ii) $\mathbb{C}^2/\mathbb{Z}^3 \times \mathrm{Pic}^{-1}(\mathrm{T})$ if $F(z_0)$ is not a half period.

and

Proposition 6.3 *(Case γ). If g is a graph of type γ, the fibre of $G : \mathcal{I}_2 \to \mathbb{P}^1$ at g is*

i) If l is a half-period

(A) $\mathbb{C} \times \mathrm{Pic}^{-1}(\mathrm{T}) \times \mathrm{Pic}^{-1}(\mathrm{T})$ if $z_0 \neq z_1$ in the graph of E;

(B) $\mathbb{C} \times \mathrm{Pic}^{-1}(\mathrm{T})$, if $z_0 = z_1$ in the graph of E and $E|_{\pi^{-1}(z_0)} = L \oplus L^*, c_1(L) = -1$.

ii) If l is not a half period

(A) a subset of $\mathbb{C}^* \times \mathbb{Z} \times (\mathrm{Pic}^{-1}(\mathrm{T}))^2$ if $z_0 \neq z_1$ in the graph of E;

(B) a subset of $\mathbb{C} \times \mathbb{Z} \times \mathrm{Pic}^{-1}(\mathrm{T})$, if $z_0 = z_1$ in the graph of E and $E|_{\pi^{-1}(z_0)} = L \oplus L^*, c_1(L) = -1$.

6.2 Proof of Propositions 6.2 and 6.3:

The proof is similar to that of proposition 6.1. We again basically have to compute the cohomology groups of α, K, M, A, and L. Because these involve the same ideas in all cases, we shall discuss each sheaf simultaneously for all cases.

Before we start, let us fix some notation, in each case, that will hold for the remainder.

Vertical bars.

In β and γ, the graphs contain vertical bars.

β) The graph contains one vertical bar at z_0. Let us assume that

$$E|_{\pi^{-1}(z_0)} = L \oplus L^*, c_1(L) = -1.$$

γ) (A) The graph contains two vertical bars, at z_0 and z_1, and $z_0 \neq z_1$. We assume that

$$E|_{\pi^{-1}(z_0)} = L \oplus L^*, c_1(L) = -1$$

and

$$E|_{\pi^{-1}(z_1)} = L' \oplus L'^*, c_1(L') = -1.$$

(B) The graph contains a double vertical bar, i.e., a vertical bar of multiplicity two, at $z_0 = z_1$. We set

$$E|_{\pi^{-1}(z_0)} = L \oplus L^*, c_1(L) = -1.$$

Half periods.

As we have seen in section 4.1.4, half periods play a special role in the computations, because they introduce zeroes in the sheaf K, and also monodromy, which corresponds to replacing L_0 by L_0^*, around the half period. We will therefore have to distinguish further cases, when computing the cohomology of K, M.

β) In this case, F is an automorphism of $\mathbb{P}^1$. F must therefore map four distinct points to the four half periods. Now, is $F(z_0)$ a half period? We then distinguish

i) $F(z_0)$ is a half period,

ii) $F(z_0)$ is not a half period.

γ) Here, the graph contains a horizontal bar corresponding to the constant map $F(z) = l$, where l is a pair of dual line bundles of degree zero. We thus have two possibilities:

i) l is a half period,

ii) l is not a half period.

6.2.1 Computation for α.

Lemma 6.1 *If we follow the notation of section 4.1.4, then*

β) $\alpha \cong \mathcal{O}(-3)$, *and* $H^i(\mathbb{P}^1, \alpha) = \begin{cases} 0 & \textit{for } i = 0, 2, \\ \mathbb{C}^2 & \textit{for } i = 1. \end{cases}$

γ) $\alpha \cong \mathcal{O}(-2)$, *and* $H^i(\mathbb{P}^1, \alpha) = \begin{cases} 0 & \textit{for } i = 0, 2, \\ \mathbb{C} & \textit{for } i = 1. \end{cases}$

Proof: As we have seen in section 4.1.4, for any x which does not correspond to a vertical bar in the graph of E, $h^1(\pi^{-1}(x), \mathrm{EndE}) = 2$. The presence of vertical bars will however make the dimension jump, and $R^1\pi_*(\mathrm{EndE})$ is no longer a locally free sheaf. Indeed, by Riemann-Roch, we find

β) $h^1(\pi^{-1}(z_0), \mathrm{EndE}) = 4$,

γ) (A) $h^1(\pi^{-1}(z_i), \mathrm{EndE}) = 4$, for $i = 0, 1$,

(B) $h^1(\pi^{-1}(z_0), \mathrm{EndE}) = 4$,

Thus, we can write $R^1\pi_*(\mathrm{EndE}) = \mathcal{F} \oplus \mathcal{S}$, where $\mathcal{F}$ is locally free of rank 2, and $\mathcal{S}$ is a skyscraper sheaf. Let $\mathrm{supp}(\mathcal{S})$ be the the support of $\mathcal{S}$. Then by the above,

$$\mathrm{supp}(\mathcal{S}) = \begin{cases} z_0 & \text{in } \beta, \text{ and } \gamma(B) \\ z_0 \text{ and } z_1 & \text{in } \gamma(A). \end{cases}$$

Furthermore, at each point of the support, $\mathcal{S}$ has fiber

$$\begin{cases} \mathbb{C}^2 & \text{in } \beta, \text{ and } \gamma(A), \\ \mathbb{C}^4 & \text{in } \gamma(B), \end{cases}$$

where the $\mathbb{C}^4$ in $\gamma(B)$ is due to the fact that the vertical bar has multiplicity two. Thus, if h is the positive generator of $H^2(\mathbb{P}^1, \mathbb{Z})$, we see that

$$c_1(\mathcal{S}) = \begin{cases} 2h & \text{in } \beta, \\ 4h & \text{in } \gamma. \end{cases}$$

By relative duality, $\pi_*(\mathrm{EndE}) = (\mathrm{R}^1\pi_*(\mathrm{EndE}))^* = \mathcal{F}^*$. Therefore, since $c_1(\alpha) = c_1(\pi_*(\mathrm{EndE}))$, we have $c_1(\alpha) = -c_1(\mathcal{F})$. Furthermore, $c_1(R^1\pi_*(\mathrm{EndE})) = c_1(\mathcal{F}) + c_1(\mathcal{S})$, and $c_1(\pi_!(\mathrm{EndE})) = 2c_1(\alpha) - c_1(\mathcal{S})$. By Grothendieck-Riemann-Roch, $\mathrm{ch}(\pi_!(\mathrm{EndE})) = -8\mathrm{h}$. Therefore, combining the above, we get

$$c_1(\alpha) = \begin{cases} -3h & \text{in } \beta, \\ -2h & \text{in } \gamma. \end{cases}$$

Hence, as $\mathrm{Pic}(\mathbb{P}^1) = \mathbb{Z}$, we see that $\alpha = \mathcal{O}(-3)$, in β, and $\alpha = \mathcal{O}(-2)$, in γ. The cohomology groups stated in the lemma are then a simple consequence of Riemann-Roch. $\square$

Remark: In β, $\alpha \cong \mathcal{O}(-2)$. It will be useful, for the remainder, to associate a divisor to α.

(A) One can think of α as being given by the divisor $D = -z_0 - z_1$. Therefore, any section of α must vanish at z_0 and z_1.

(B) Here, we will assume that α is given by the divisor $D = -2z_0$. Thus, any section of α must vanish at z_0.

Computation of the cohomology groups for K.

Lemma 6.2 *Keeping the notation as above, we have*

$\beta)$ *(i) If $F(z_0)$ is a half period, then*

$$H^0(\mathbb{P}^1, K) = 0, H^1(\mathbb{P}^1, K) = \mathbb{Z}^2, H^2(\mathbb{P}^1, K) = \mathbb{Z}/2.$$

(ii) If $F(z_0)$ is not a half period, then

$$H^0(\mathbb{P}^1, K) = 0, H^1(\mathbb{P}^1, K) = \mathbb{Z}^3, H^2(\mathbb{P}^1, K) = \mathbb{Z}/2.$$

$\gamma)$ *(i) If l is a half period, K is then the zero sheaf.*

(ii) If l is not a half period, then

$$H^i(\mathbb{P}^1, K) = \begin{cases} 0, & \textit{if } i = 0, \\ \mathbb{Z}, & \textit{if } i = 1, \textit{ in } \gamma(A), \\ 0, & \textit{if } i = 1, \textit{ in } \gamma(B), \\ \mathbb{Z}, & \textit{if } i = 2. \end{cases}$$

Proof: The table in lemma 2.3 will be very useful:

Let E be an SL$(2,\mathbb{C})$-bundle over T. Its global traceless endomorphisms and the kernel of the exponential map exp: (global traceless endomorphisms) $\to$ (SL$(2,\mathbb{C})$-automorphisms) are:

1) $E \simeq L_0 \oplus L_0^*$, $L_0^2 \neq \mathcal{O}, c_1(L_0) = 0$	$\begin{pmatrix} a & 0 \\ 0 & -a \end{pmatrix}$ $a \in \mathbb{C}$	$2\pi i \begin{pmatrix} m & 0 \\ 0 & -m \end{pmatrix}$ $m \in \mathbb{Z}$
2) $E \simeq L_0 \oplus L_0^*$, $L_0^2 = \mathcal{O}, c_1(L_0) = 0$	$sl(2,\mathbb{C})$	
3) E *type (ii)*	$\begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix}$ $b \in \mathbb{C}$	$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$
4) $E \simeq L \oplus L^*$ $c_1(L) < 0$	$\begin{pmatrix} a & f \\ 0 & -a \end{pmatrix}$ $a \in \mathbb{C}$ $f \in \Gamma(L^{*2})$	$2\pi i \begin{pmatrix} m & f \\ 0 & -m \end{pmatrix}$ $f \in \Gamma(L^{*2})$ $m \in \mathbb{Z}\backslash 0$

We begin with

β) i) $F(z_0)$ is a half period.

Let $z_i, i = 1, \ldots, 3$ be the points that get mapped to half periods, and let $p(z)$ be a polynomial of degree 4 that vanishes at the $z_i, i = 0, \ldots, 3$. K is then zero on any set containing the z_i. Away from the z_i, K is locally the constant sheaf $\mathbb{Z}$. As branching around the z_i interchanges L_0 and L_0^*, there is a corresponding $(\mathbb{Z}/2)$

monodromy on K. An explicit embedding of K into $\alpha = \mathcal{O}(-3)$ is given by

$$\mathrm{diag} 2\pi\mathrm{i}(\mathrm{m}, -\mathrm{m}) \mapsto \frac{\mathrm{mdz}}{(\mathrm{z}-\mathrm{z}_0)\sqrt{\mathrm{p}(\mathrm{z})}}.$$

Using a similar Leray covering to the one described in section 4.1.4, one finds

$$H^0(\mathbb{P}^1, K) = 0, H^1(\mathbb{P}^1, K) = \mathbb{Z}^2, H^2(\mathbb{P}^1, K) = \mathbb{Z}/2.$$

(The computation is very similar.)

ii) $F(z_0)$ is not a half period.

In this case, there are four points, $z_i, i = 1, \ldots, 4$ say, that get mapped to half periods, and they are all distinct from z_0. Let $p(z)$ be a polynomial of degree four that vanishes at the half periods. As above, K is zero on any open set containing the z_i, $i = 1, \ldots, 4$; and, as it is a subsheaf of α, it must also be zero on any open set containing z_0 (because every section of α vanishes at z_0.) Away from the z_i, K is locally the constant $\mathbb{Z}$ sheaf. K then embeds in α by

$$m \mapsto \frac{mdz}{(z-z_0)\sqrt{p(z)}}.$$

There is again monodromy about the $z_i, i = 1, \ldots, 4$. In order to compute cohomology of K, we shall use the following Leray cover: we choose open sets $U_N, U_S, V_i, i = 1, \ldots, 4$, as above; we add an open set V_0 about z_0. Let us fix trivialisations of K on these open sets. The restriction maps are all the identity, except for $\rho^N_{Ni} = -\mathrm{Id}, \mathrm{i} = 1, \ldots, 4$, which corresponds to the monodromy about the z_i.

- $C^0 = K(U_S) \oplus K(U_N) = \mathbb{Z}^2$, as $K(V_i) = 0$ for all $i = 0, \ldots, 4$;
- $C^1 = K(U_{SN}) \oplus \sum_{i=0}^4 K(U_{iS}) \oplus \sum_{i=0}^4 K(U_{Ni}) = \mathbb{Z}^{15}$, as U_{SN} has 5 components;
- $C^2 = \sum_{i=0}^4 K(U_{SNi}) = \mathbb{Z}^{10}$, as U_{SNi} has 2 components for all $i = 0, \ldots, 4$.

The first coboundary map is given by

$$\begin{aligned} \delta : C^0 &\longrightarrow C^1 \\ (m, n) &\longmapsto (n-m, \ldots, n-m, m, \ldots, m, -n, \ldots, -n) \end{aligned}$$

We then obviously have that $Z^0 = ker(\delta : C^0 \longrightarrow C^1) = 0$ and $B^1 = Im(\delta : C^0 \longrightarrow C^1) = \mathbb{Z}^2$. Furthermore

$$\begin{array}{ccc} \delta : C^1 & \longrightarrow & C^2 \\ (n_0, \ldots, n_4, m_0, \ldots, m_4, k_0, \ldots, k_4) & \longmapsto & (a_0, b_0, \ldots, a_4, b_4) \end{array},$$

where

$$\begin{aligned}
a_0 &= n_0 + m_0 + k_0 \\
b_0 &= n_1 + m_0 + k_0 \\
a_1 &= n_1 + m_1 + k_1 \\
b_1 &= n_2 + m_1 - k_1 \\
a_2 &= n_2 + m_2 - k_2 \\
b_2 &= n_3 + m_2 + k_2 \\
a_3 &= n_3 + m_3 + k_3 \\
b_3 &= n_4 + m_3 - k_3 \\
a_4 &= n_4 + m_4 - k_4 \\
b_4 &= n_0 + m_4 + k_4.
\end{aligned}$$

(The alternation between + and − in front of the k_i corresponds to monodromy, for $i = 1, \ldots, 4$.) In this case, $Z^1 = ker(\delta : C^1 \longrightarrow C^2) = \mathbb{Z}^5$ and $B^2 = Im(\delta : C^1 \longrightarrow C^2) = 2\mathbb{Z}^{10}$. Combining these, we have

$$H^0(K, \mathbb{P}^1) = 0, H^1(K, \mathbb{P}^1) = \mathbb{Z}^3, H^2(K, \mathbb{P}^1) = \mathbb{Z}/2.$$

We now turn to

γ) *i) l is a half period.*

Referring to lemma 2.3, we see that K is the zero sheaf, in this case.

ii) l is not a half period.

We now have to consider the cases A), B) separately:

(A) *$z_0 \neq z_1$ in the graph of E:*

By the above remark, since K is a subsheaf of α, any section of K must be zero at z_0 and z_1. Away from the z_i, K is the constant sheaf $\mathbb{Z}$. (In this case, the sheaf is constant because there are no half periods and,therefore, no monodromy.) K is then zero on any set containing the z_i. An explicit embedding of K into $\alpha = \mathcal{O}(-2)$ is given by

$$\mathrm{diag} 2\pi \mathrm{i}(\mathrm{m}, -\mathrm{m}) \mapsto \frac{\mathrm{m}}{(\mathrm{z}-\mathrm{z}_0)(\mathrm{z}-\mathrm{z}_1)}.$$

We again use a similar Leray covering to the one described above: U_N, U_S, V_0, and V_1, where the last two sets are open neighborhoods of z_0 and z_1. If we fix trivialisations of K, since it is the constant sheaf $\mathbb{Z}$, all restrictions must be the identity.

- $C^0 = K(U_S) \oplus K(U_N) = \mathbb{Z}^2$, as $K(V_i) = 0$ for $i = 0, 1$;
- $C^1 = K(U_{SN}) \oplus \sum_{i=0}^{1} K(U_{iS}) \oplus \sum_{i=0}^{1} K(U_{Ni}) = \mathbb{Z}^6$, as U_{SN} has 2 components;
- $C^2 = K(U_{SN0}) \oplus K(U_{SN1}) = \mathbb{Z}^4$, as U_{SNi} has 2 components for $i = 0, 1$. The first coboundary map is now

$$\begin{array}{rcl} \delta : C^0 & \longrightarrow & C^1 \\ (m, n) & \longmapsto & (n_m, n_m, m, m, -n, -n) \end{array},$$

and we obtain $Z^0 = ker(\delta : C^0 \longrightarrow C^1) = 0$ and $B^1 = Im(\delta : C^0 \longrightarrow C^1) = \mathbb{Z}^2$. Moreover

$$\begin{array}{rcl} \delta : C^1 & \longrightarrow & C^2 \\ (n_0, n_1, m_0, m_1, k_0, k_1) & \longmapsto & (a_0, b_0, a_1, b_1) \end{array}$$

where

$$\begin{aligned} a_0 &= n_0 + m_0 + k_0 \\ b_0 &= n_1 + m_0 + k_0 \\ a_1 &= n_1 + m_1 + k_1 \\ b_1 &= n_0 + m_1 + k_1. \end{aligned}$$

In this case, $Z^1 = ker(\delta : C^1 \longrightarrow C^2) = \mathbb{Z}^3$ and $B^2 = Im(\delta : C^1 \longrightarrow C^2) = \mathbb{Z}^3$, where B^2 is given by the condition $b_1 = a_0 - b_0 + a_1$. Combining these, we have

$$H^0(\mathbb{P}^1, K) = 0, H^1(\mathbb{P}^1, K) = \mathbb{Z}, H^2(\mathbb{P}^1, K) = \mathbb{Z}.$$

(B) $z_0 = z_1$ *in the graph of* E:

By the above remark, since K is a subsheaf of α, any section of K must be zero at

z_0. Away from z_0, K is the constant sheaf $\mathbb{Z}$. (In this case, the sheaf is constant because there are no half periods and,therefore, no monodromy.) K is then zero on any set containing z_0. An explicit embedding of K into $\alpha = \mathcal{O}(-2)$ is given by

$$\mathrm{diag}2\pi\mathrm{i}(\mathrm{m}, -\mathrm{m}) \mapsto \frac{\mathrm{m}}{(\mathrm{z}-\mathrm{z}_0)^2},$$

We again use a similar Leray covering to the one described above: U_N, U_S, V_0, where the last set is an open neighborhood of z_0. If we fix trivialisations of K, since it is the constant sheaf $\mathbb{Z}$, all restrictions must be the identity.

- $C^0 = K(U_S) \oplus K(U_N) = \mathbb{Z}^2$, as $K(V_0) = 0$;
- $C^1 = K(U_{SN}) \oplus K(U_{0S}) \oplus K(U_{N0}) = \mathbb{Z}^3$, as U_{SN} has only one component;
- $C^2 = K(U_{SN0}) \oplus K(U_{SN1}) = \mathbb{Z}^2$, as U_{SN0} has 2 components.

We now have

$$\begin{array}{rcl} \delta : C^0 & \longrightarrow & C^1 \\ (m,n) & \longmapsto & (n-m, m, -n) \end{array},$$

giving us $Z^0 = ker(\delta : C^0 \longrightarrow C^1) = 0$ and $B^1 = Im(\delta : C^0 \longrightarrow C^1) = \mathbb{Z}^2$. Furthermore

$$\begin{array}{rcl} \delta : C^1 & \longrightarrow & C^2 \\ (n,m,k) & \longmapsto & (a,b) \end{array},$$

where

$$\begin{array}{rcl} a & = & n+m+k \\ b & = & n+m+k. \end{array}$$

In this case, $Z^1 = ker(\delta : C^1 \longrightarrow C^2) = \mathbb{Z}^2$ and $B^2 = Im(\delta : C^1 \longrightarrow C^2) = \mathbb{Z}$, where B^2 is given by the condition $b = a$. Combining these, we have

$$H^0(\mathbb{P}^1, K) = H^1(\mathbb{P}^1, K) = 0, H^2(\mathbb{P}^1, K) = \mathbb{Z}.$$

6.2.2 Proof of proposition 6.2.

In this case, lemma 6.1 tells us that $\alpha \cong \mathcal{O}(-3)$. A, L are then subsheaves of $\pi_*(\mathrm{EndE}) = \mathcal{O} \oplus \mathcal{O}(-3)$, and the global sections of A are $\pm\mathrm{Id}$. Furthermore, as $-\mathrm{Id}$ is not an exponential at the z_i, Id is the only global section of L. Thus $H^0(\mathbb{P}^1, A) = \mathbb{Z}/2$ and $H^0(\mathbb{P}^1, L) = 0$.

i) $F(z_0)$ is a half period.

Away from the z_i, as $F(z)$ is never a half period, we have

α	A	
$\begin{pmatrix} b & 0 \\ 0 & -b \end{pmatrix}$	$\begin{pmatrix} e^b & 0 \\ 0 & e^{-b} \end{pmatrix}$	$b \in \mathbb{C}$

$\exp : \alpha \to A$ is then surjective, and $M = 0$. At z_0, $\alpha = \mathcal{O}(-3)$ must be zero, and $\pm\mathrm{Id}$ are the only possible elements of A. Thus $M = \mathbb{Z}/2$ at z_0. As $F(z_i)$ is a half period for $i = 1, \ldots, 3$, at those points

α	A	
$\begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix}$	$\begin{pmatrix} \pm 1 & b \\ 0 & \pm 1 \end{pmatrix}$	$b \in \mathbb{C}$

and $\exp : \alpha \to \mathrm{A}$ has cokernel $M = \mathbb{Z}/2$. M is then a skyscraper sheaf with fiber $\mathbb{Z}/2$ supported at the z_i. The cohomology long exact sequences associated to (4.2) and (4.3) then give

$$0 \to \mathbb{Z}^2 \to \mathbb{C}^2 \to H^1(\mathbb{P}^1, L) \to \mathbb{Z}/2 \to 0,$$

and

$$0 \to \mathbb{Z}/2 \to (\mathbb{Z}/2)^4 \to H^1(\mathbb{P}^1, L) \to H^1(\mathbb{P}^1, A) \to 0.$$

Thus $H^1(\mathbb{P}^1, A) = H^1(\mathbb{P}^1, L)/(\mathbb{Z}/2)^2 = (\mathbb{C}^2/\mathbb{Z}^2 \times \mathbb{Z}/2)/(\mathbb{Z}/2)^2 = \mathbb{C}^2/\mathbb{Z}^2$. Since there are $\mathrm{Pic}^{-1}(\mathrm{T})$ possibilities for the choice of the line bundle L giving E over $\pi^{-1}(z_0)$, the fiber of G at g must be $\mathbb{C}^2/\mathbb{Z}^2 \times \mathrm{Pic}^{-1}(\mathrm{T})$.

ii) $F(z_0)$ is not a half period.

From the above discussion, we see that M is a skyscaper sheaf with fiber $\mathbb{Z}/2$ supported at the z_i, $i = 0, \ldots, 4$. Furthermore, we obtain the exact sequences

$$0 \to \mathbb{Z}^3 \to \mathbb{C}^2 \to H^1(L, \mathbb{P}^1) \to \mathbb{Z}/2 \to 0,$$

and

$$0 \to \mathbb{Z}/2 \to (\mathbb{Z}/2)^5 \to H^1(L, \mathbb{P}^1) \to H^1(A, \mathbb{P}^1) \to 0.$$

Thus $H^1(L, \mathbb{P}^1) = \mathbb{C}^2/\mathbb{Z}^3 \times \mathbb{Z}/2$, and $H^1(A, \mathbb{P}^1) = \mathbb{C}^2/\mathbb{Z}^3$. There are now $\mathrm{Pic}^{-2}(\mathrm{T})$ possibilities for the choice of line bundle L giving E over $\pi^{-1}(z_0)$, and the fiber of $G : \mathcal{M}_2 \to \mathbb{P}^1$ at g is $\mathbb{C}^2/\mathbb{Z}^3 \times \mathrm{Pic}^{-1}(\mathrm{T})$. $\square$

6.2.3 Proof of proposition 6.3.

We have seen that $\alpha \cong \mathcal{O}(-2)$, in case γ. A, L are therefore subsheaves of $\pi_*(\mathrm{End E}) = \mathcal{O} \oplus \mathcal{O}(-2)$, and the global sections of A are $\pm\mathrm{Id}$. Furthermore, as $-\mathrm{Id}$ is not an exponential at the z_i, Id is the only global section of L. Thus $H^0(\mathbb{P}^1, A) = \mathbb{Z}/2$ and $H^0(\mathbb{P}^1, L) = 0$.

i) l is a half period.

In this case $K = 0$, and we are therefore only working with the exact sequence

$$0 \to \alpha \xrightarrow{\exp} A \longrightarrow M \to 0. \tag{6.1}$$

Away from the z_i, we have

α	A	
$\begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix}$	$\begin{pmatrix} \pm 1 & b \\ 0 & \pm 1 \end{pmatrix}$	$b \in \mathbb{C}$

The cokernel of $\exp : \alpha \to \mathrm{A}$ is then $M = \mathbb{Z}/2$. At the z_i, α is zero, and $\pm\mathrm{Id}$ are the only two possible germs of A. We then again have $M = \mathbb{Z}/2$, implying that M is the constant $\mathbb{Z}/2$ sheaf. Inserting all of the above into the long cohomology sequence associated to (6.1), we obtain

$$0 \to \mathbb{Z}/2 \to \mathbb{Z}/2 \to \mathbb{C} \to H^1(A, \mathbb{P}^1) \to 0,$$

and $H^1(A, \mathbb{P}^1) = \mathbb{C}$. The only invariants, apart from the graph, are the isomorphism types over $\pi^{-1}(z_0)$ and $\pi^{-1}(z_1)$. The fiber $G^{-1}(g)$ is therefore

- $\mathbb{C} \times \mathrm{Pic}^{-1}(\mathrm{T}) \times \mathrm{Pic}^{-1}(\mathrm{T})$ in case (A),

- $\mathbb{C} \times \mathrm{Pic}^{-1}(\mathrm{T})$ in case (B).

ii) l is not a half period.

Away from the z_i, we now have

$$\begin{array}{c|cl} \alpha & A & \\ \begin{pmatrix} b & 0 \\ 0 & -b \end{pmatrix} & \begin{pmatrix} e^b & 0 \\ 0 & e^{-b} \end{pmatrix} & b \in \mathbb{C} \\ \hline \end{array}$$

$\exp : \alpha \to \mathrm{A}$ is then surjective, and $M = 0$. At the z_i, we again have $\alpha = 0$, and $A = \pm\mathrm{Id}$. M is therefore a skyscaper sheaf with fiber $\mathbb{Z}/2$ supported at the z_i.

In case (A), as $H^1(\mathbb{P}^1, K) = H^2(\mathbb{P}^1, K) = \mathbb{Z}$, the cohomology long exact sequences associated to (4.2) and (4.3) then give

$$0 \to \mathbb{Z} \to \mathbb{C} \to H^1(\mathbb{P}^1, L) \to \mathbb{Z} \to 0,$$

and

$$0 \to \mathbb{Z}/2 \to (\mathbb{Z}/2)^2 \to H^1(\mathbb{P}^1, L) \to H^1(\mathbb{P}^1, A) \to 0.$$

Therefore $H^1(\mathbb{P}^1, A) = H^1(\mathbb{P}^1, L)/(\mathbb{Z}/2) = (\mathbb{C}/\mathbb{Z} \times \mathbb{Z})/(\mathbb{Z}/2) = \mathbb{C}^* \times \mathbb{Z}$. The only invariants, apart from the graph, are the isomorphism types over $\pi^{-1}(z_0)$ and $\pi^{-1}(z_1)$. The fiber $G^{-1}(g)$ is therefore a subset of $\mathbb{C}^* \times \mathbb{Z} \times \mathrm{Pic}^{-1}(\mathrm{T}) \times \mathrm{Pic}^{-1}(\mathrm{T})$.

In case (B), we have $H^1(\mathbb{P}^1, K) = 0$ and $H^2(\mathbb{P}^1, K) = \mathbb{Z}$. Inserting all of the above into the cohomology long exact sequences associated to (4.2) and (4.3), we obtain

$$0 \to \mathbb{C} \to H^1(\mathbb{P}^1, L) \to \mathbb{Z} \to 0,$$

and

$$0 \to \mathbb{Z}/2 \to (\mathbb{Z}/2)^2 \to H^1(\mathbb{P}^1, L) \to H^1(\mathbb{P}^1, A) \to 0.$$

Therefore $H^1(\mathbb{P}^1, L) = \mathbb{C} \times \mathbb{Z}$, and $H^1(\mathbb{P}^1, A) = (\mathbb{C} \times \mathbb{Z})/(\mathbb{Z}/2) = \mathbb{C} \times \mathbb{Z}$. Apart from the graph, the only invariant is the isomorphism type over $\pi^{-1}(z_0)$. $G^{-1}(g)$ is therefore a subset of

– $\mathbb{C} \times \mathbb{Z} \times \mathrm{Pic}^{-1}(\mathrm{T})$ in case (B). $\square$

Chapter 7

Connections on Hopf surfaces.

In this chapter, we consider the topological side of our problem. We will see that the notion of graph can be extended to connections. We begin by finding the homotopy groups and certain cohomology groups of the moduli spaces of connections and framed connections. We then construct a map that will associate to any connection a graph. This construction generalises the notion of graph in the holomorphic setting. We then show that this map is not homotopically trivial, and find that its generic fibre is the total space of an S^1-bundle.

7.1 Bundles and connections.

Let E be a $\mathcal{C}^\infty$ bundle on $\mathcal{H}$ with $c_1(E) = 0$ and $c_2(E) = k$. We will then denote by $\mathcal{A} = \mathcal{A}_{\mathcal{H},E}$ the space of connections on E, and by $\mathcal{G}$ the gauge group – the group of bundle automorphisms which cover the identity map on $\mathcal{H}$. The moduli space of gauge equivalence classes of connections on E is $\mathcal{B} = \mathcal{A}/\mathcal{G}$. Let us then fix a point p_0 in $\mathcal{H}$. A *framed connection* in E is a pair (A, t), where A is a connection, and t is a trivialisation of E_{p_0} – a linear map $t : E_{p_0} \longrightarrow \mathbb{C}^2$. If we fix the framing t, we define $\mathcal{G}_0 \subset \mathcal{G}$ to be its stabiliser:

$$\mathcal{G}_0 = \{g \in \mathcal{G} | g(p_0) = Id\}.$$

The orbit space $\tilde{\mathcal{B}} = \mathcal{A}/\mathcal{G}_0$ is then the space of equivalence classes of framed connections. Furthermore, we have the equivalence

$$\tilde{\mathcal{B}} \simeq Map^*(\mathcal{H}, BSU(2))_E,$$

where $\simeq$ represents weak homotopy, Map^* denotes base point preserving maps, and $Map^*(\mathcal{H}, BSU(2))_E$ is the homotopy class corresponding to $E \longrightarrow \mathcal{H}$.

We finally remark that there is a natural map $\beta : \tilde{\mathcal{B}} \to \mathcal{B}$ which forgets the framing. Since all connections on the Hopf surface $\mathcal{H}$ are irreducible, the fibre of this map is $SU(2)/C(SU(2))$, where $C(SU(2)) = \pm Id$ is the centre of $SU(2)$. The fibre of β is therefore $SO(3)$, and we have the fibration

$$\begin{array}{ccc} SO(3) & \longrightarrow & \tilde{\mathcal{B}} \\ & & \downarrow \\ & & \mathcal{B}. \end{array}$$

For details of this, see [DK].

7.1.1 Homotopy of $\tilde{\mathcal{B}}$.

We begin by computing the homotopy groups of $\tilde{\mathcal{B}}$.

Proposition 7.1

$$\pi_q(\tilde{\mathcal{B}}) = \pi_{q+3}(S^3) \times \pi_{q+2}(S^3) \times \pi_q(S^3),$$

for all q, and, in particular,

$$\pi_1(\tilde{\mathcal{B}}) = \mathbb{Z}_2 \times \mathbb{Z}.$$

Proof:

$$\begin{aligned} \pi_q(\tilde{\mathcal{B}}) = \;& \pi_q(\mathcal{M}ap^*(S^3 \times S^1, BSU(2))) \\ = \;& [S^q, \mathcal{M}ap^*(S^3 \times S^1, BSU(2))]_\circ \\ = \;& [S^q \wedge (S^3 \times S^1), BSU(2)]_\circ \\ = \;& [\Sigma^q(S^3 \times S^1), BSU(2)]_\circ \end{aligned}$$

As $S^3 \wedge S^1 \approx S^4$,

$$\Sigma(S^3 \times S^1) \approx \Sigma(S^4) \vee \Sigma(S^3) \vee \Sigma(S^1).$$

Since Σ distributes over $\vee$, by iterating the above q times, we get that

$$\begin{aligned}\Sigma^q(S^3 \times S^1) = &\ \Sigma^q(S^4) \vee \Sigma^q(S^3) \vee \Sigma^q(S^1) \\ = &\ S^{q+4} \vee S^{q+3} \vee S^{q+1}.\end{aligned}$$

Inserting this into the above gives

$$\begin{aligned}\pi_q(\tilde{\mathcal{B}}) = &\ [S^{q+4} \vee S^{q+3} \vee S^{q+1}, BSU(2)]_\circ \\ = &\ [S^{q+4}, BSU(2)]_\circ \times [S^{q+3}, BSU(2)]_\circ \times [S^{q+1}, BSU(2)]_\circ \\ = &\ \pi_{q+4}(BSU(2)) \times \pi_{q+3}(BSU(2)) \times \pi_{q+1}(BSU(2)).\end{aligned}$$

Let us remark that, as $\Omega BX = X$, for any group X, and $\pi_i(\Omega X) = \pi_{i-1}(X)$, we have $\pi_i(BX) = \pi_{i-1}(X)$. Thus, as $SU(2) = S^3$, the above becomes

$$\pi_q(\tilde{\mathcal{B}}) = \pi_{q+3}(S^3) \times \pi_{q+2}(S^3) \times \pi_q(S^3),$$

and we are done.

The first four homotopy groups of S^3 are

$$\pi_p(S^3) = \begin{cases} 0 & \text{if } p = 1,2; \\ \mathbb{Z} & \text{if } p = 3; \\ \mathbb{Z}_2 & \text{if } p = 4. \end{cases}$$

The above then gives us $\pi_1(\tilde{\mathcal{B}}) = \mathbb{Z}_2 \times \mathbb{Z}$. $\square$

7.1.2 Cohomology of $\tilde{\mathcal{B}}$.

Proposition 7.2 *$\tilde{\mathcal{B}}$ has the same rational cohomology as $\mathcal{H}$:*

$$H^p(\tilde{\mathcal{B}}, \mathbb{Q}) = \begin{cases} \mathbb{Q}, & \textit{if } p = 0,1,3,4; \\ 0, & \textit{otherwise.} \end{cases}$$

Proof: Since $\mathcal{H} \simeq S^3 \times S^1$, we have the following cofibration

$$S^3 \vee S^1 \xrightarrow{i} \mathcal{H} \xrightarrow{p} S^3 \wedge S^1.$$

Applying the functor $Map^*(\quad, BSU(2))$, we then obtain the fibration

$$\begin{tikzcd} \Omega^3 S^3 \simeq Map^*(S^4, BSU(2)) \arrow[r, "\bar{i}"] & Map^*(\mathcal{H}, BSU(2)) \arrow[d, "\bar{p}"] \\ & Map^*(S^3 \vee S^1, BSU(2)) \simeq S^3 \times \Omega^2 S^3 \end{tikzcd} \tag{7.1}$$

Since $c_2(E) = k$, we restrict ourselves to the kth component $\Omega^3_k S^3$ of the third loop space. $\tilde{\mathcal{B}}$ is then given by the fibration

$$\Omega^3_k S^3 \xrightarrow{\bar{i}} \tilde{\mathcal{B}} \simeq Map^*(\mathcal{H}, BSU(2))_E \xrightarrow{\bar{p}} S^3 \times \Omega^2 S^3. \tag{7.2}$$

And, as $\Omega^3_k S^3$ has trivial rational cohomology,

$$\mathrm{H}^*(\tilde{\mathcal{B}}; \mathbb{Q}) \simeq \mathrm{H}^*(\mathrm{S}^3 \times \Omega^2 \mathrm{S}^3; \mathbb{Q}).$$

The rational cohomology groups can then be computed using the Künneth theorem.

By applying the Leray-Serre spectral sequence to the fibration

$$\Omega^2 S^3 \longrightarrow P(\Omega S^3) \longrightarrow \Omega S^3,$$

we have

$$\mathrm{H}^*(\Omega^2 \mathrm{S}^3; \mathbb{Q}) = \begin{cases} \mathbb{Q} & p = 0, 1 \\ 0 & p \geq 2 \end{cases}.$$

For details, see [DK]. The Künneth theorem therefore gives

$$H^p(\tilde{\mathcal{B}}; \mathbb{Q}) = \begin{cases} \mathbb{Q}, & \text{if } p = 0, 1, 3, 4; \\ 0, & \text{otherwise.} \end{cases}$$

□

Let us now show that

Corollary 7.1

$$H^p(\tilde{\mathcal{B}};\mathbb{Z}) = \begin{cases} \mathbb{Z}, & \text{if } p = 0,1; \\ \mathbb{Z}_2, & \text{if } p = 2; \\ \mathbb{Z}\oplus T_p, & \text{if } p = 3,4; \\ T_p, & \text{if } p \geq 5, \end{cases}$$

where T_p is a torsion module, for $p \geq 3$.

Proof: By proposition 7.1,

$$\pi_1(\tilde{\mathcal{B}}) = \mathbb{Z}_2 \times \mathbb{Z} = \mathbb{Z}_2 \oplus \mathbb{Z}.$$

And, as $\pi_1(\tilde{\mathcal{B}})$ is abelian, this implies that $H_1(\tilde{\mathcal{B}}) = \mathbb{Z}_2 \oplus \mathbb{Z}$.

Let us remark that $\mathrm{Ext}(\mathbb{Z}_\mathrm{m}, \mathbb{Z}) = \mathbb{Z}_\mathrm{m}$; and

$$\mathrm{Ext}(\mathbb{Z},\mathbb{Z}) = \mathrm{Ext}(\mathbb{Z},\mathbb{Q}) = \mathrm{Ext}(\mathbb{Z}_\mathrm{m},\mathbb{Q}) = 0.$$

Therefore, by the Universal Coefficient Theorem,

$$\mathrm{H}^\mathrm{p}(\tilde{\mathcal{B}};\mathbb{Q}) = \mathrm{Hom}(\mathrm{H}_\mathrm{p}(\tilde{\mathcal{B}});\mathbb{Q}), \tag{7.3}$$

and

$$\mathrm{H}^\mathrm{p}(\tilde{\mathcal{B}};\mathbb{Z}) = \mathrm{Hom}(\mathrm{H}_\mathrm{p}(\tilde{\mathcal{B}});\mathbb{Z}) \oplus \mathrm{Ext}(\mathrm{H}_{\mathrm{p}-1}(\tilde{\mathcal{B}});\mathbb{Z}). \tag{7.4}$$

By (7.4), we see that

$$\begin{aligned} H^1(\tilde{\mathcal{B}};\mathbb{Z}) = {} & \mathrm{Hom}(\mathrm{H}_1(\tilde{\mathcal{B}});\mathbb{Z}) \oplus \mathrm{Ext}(\mathrm{H}_0(\tilde{\mathcal{B}});\mathbb{Z}) \\ = {} & \mathrm{Hom}(\mathbb{Z}_2 \oplus \mathbb{Z};\mathbb{Z}) \oplus \mathrm{Ext}(\mathbb{Z};\mathbb{Z}) \\ = {} & \mathbb{Z}. \end{aligned}$$

Furthermore, as $H^2(\tilde{\mathcal{B}},\mathbb{Q}) = 0$, (7.3) implies that $\mathrm{H}_2(\tilde{\mathcal{B}})$ must be a torsion module. Therefore $\mathrm{Hom}(\mathrm{H}_2(\tilde{\mathcal{B}});\mathbb{Z}) = 0$; and, as

$$\mathrm{Ext}(\mathrm{H}_1(\tilde{\mathcal{B}});\mathbb{Z}) = \mathrm{Ext}(\mathbb{Z}_2;\mathbb{Z}) \oplus \mathrm{Ext}(\mathbb{Z};\mathbb{Z}) = \mathbb{Z}_2,$$

(7.4) implies that $\mathrm{H}^2(\tilde{\mathcal{B}};\mathbb{Z}) = \mathbb{Z}_2$.

By using similar arguments, we can show that, for $p = 3, 4$,

$$\mathrm{H}^{\mathrm{p}}(\tilde{\mathcal{B}};\mathbb{Z}) = \mathbb{Z} \oplus \mathrm{T}_{\mathrm{p}},$$

where $T_p = \mathrm{Ext}(\mathrm{H}_{\mathrm{p}-1}(\tilde{\mathcal{B}});\mathbb{Z})$. As $\mathrm{H}_2(\tilde{\mathcal{B}})$ is a torsion module, T_3 must also be a torsion module. Similarly, we see that T_4 is also a torsion module. □

7.2 Stabilisation maps.

From now on, we will use the subscript k if we want to specify the charge of the connection in a given moduli space. In this section, we use the "subtraction procedure" of Taubes to construct stabilisation maps for the moduli spaces $\tilde{\mathcal{B}}_k$ of framed connections on $\mathcal{H}$. Before doing so, let us remark that the space of all connections $\mathcal{A}$ is an affine space which possesses the following L^2-metric: for any two $A, B \in \mathcal{A}$,

$$\|A - B\| = \left(\int_{\mathcal{H}} |A - B|^2 d\mu\right)^{1/2}, \tag{7.5}$$

where $|\cdot|$ is the Killing metric on su(2,$\mathbb{C}$). This metric is preserved by the actions of $\mathcal{G}$ and $\mathcal{G}_0$, and therefore descends to the quotients. The spaces $\mathcal{B}_k$ and $\tilde{\mathcal{B}}_k$ are endowed with the quotient topologies.

Claim 1 *There exists a map $g_{k,k+1} : \tilde{\mathcal{B}}_k \to \tilde{\mathcal{B}}_{k+1}$ that sends a pair (A, t) to a pair $(\tilde{A}, \tilde{t})$, for any k. This map is continuous with respect to the L^2-norm.*

This map is obtained by the "subtraction procedure" of Taubes [T]. Let us give an outline of this construction. We start with an $\mathrm{SL}(2,\mathbb{C})$-bundle $E' \to S^4$, with $c_2(E') = 1$, and choose a connection A' on E' whose curvature is concentrated at the south pole s of S^4. We also fix a trivialisation t' of E' over $S^4\backslash n$, where n is the north pole. Throughout the following, $S^4\backslash n$ will be identified with $\mathbb{R}^4$ via the stereographic projection from n. Let $z : B \longrightarrow \mathbb{R}^4$ be a coordinate system centred at p_0 and defined on a ball B around p_0 in $\mathcal{H}$. We then identify B with $z(B) \subset S^4\backslash n$, and $p_0 \in \mathcal{H}$ with $s \in S^4\backslash n$. We can assume, without loss of generality, that $B = \{z : |z| < 1\} \subset \mathbb{R}^4$.

Given any pair $(A, t) \in \tilde{\mathcal{B}}_k$, we can then define a canonical identification of E with E' over $B \backslash s$, which will depend on the choice of (A, t). We have already fixed a trivialisation t' of E' on $S^4 \backslash n$. By using parallel transport of t by A along paths in B, we can extend t to a trivialisation of E on B that we will also denote by t. By identifying these two trivialisations over $B \backslash s$, we then obtain an SL(2, $\mathbb{C}$)-bundle $\tilde{E}$ over $\mathcal{H}$ such that $c_2(\tilde{E}) = k + 1$. It is important to note that, as every bundle $\tilde{E}$ thus constructed has charge $k + 1$, the isomorphism class of $\tilde{E}$ is independent of the choice of (A, t). The actual bundle we construct does however depend on this choice. This choice will also be used to define connections on $\tilde{E}$.

Before we construct a connection on $\tilde{E}$, let us introduce a bump function $\eta(z)$ on $\mathbb{R}^4$. We require that $\eta = 0$ if $|z| < \frac{1}{2}$ and that $\eta = 1$ if $|z| > \frac{2}{3}$. Let us also cover $\mathcal{H}$ by the following three open sets: $U = B \backslash s$, $\mathcal{H}^- = \{p \in \mathcal{H} : \mathrm{dist(p, s)} > \frac{2}{3}\}$ and $B^- = \{z \in \mathbb{R}^4 : |z| < \frac{1}{2}\}$. As we have chosen A' to have curvature concentrated at s, one can assume that A' is the product connection on $B \backslash B^-$. (If it is not, one can always multiply A' by a bump function supported in B^-.)

$\tilde{E}$ has, by construction, a canonical product structure over U. Let θ denote the induced product connection on $\tilde{E}|_U$. We then define $\tilde{A} = (\theta + \eta A + A')$ on U. On $\mathcal{H}^-$, $\tilde{E}$ is canonically identified with E. $\tilde{A}$ is then given on $\tilde{E}|_{\mathcal{H}^-}$ by $\tilde{A} = A$. Finally, as $\tilde{E}$ is identified with E' on B^-, we set $\tilde{A} = A'$ on B^-. It is easy to see that these connections agree where the domains of definition overlap. We therefore have a well-defined connection $\tilde{A}$ on $\tilde{E}$.

The map $g_{k,k+1} : \tilde{\mathcal{B}}_k \to \tilde{\mathcal{B}}_{k+1}$ is then defined by sending the pair (A, t) in $\tilde{\mathcal{B}}_k$ to the pair $(\tilde{A}, t')$ in $\tilde{\mathcal{B}}_{k+1}$. This map well defined, and it is continuous with respect to the L^2-norm defined above.

7.3 Graph map.

Let E be a fixed SL(2,$\mathbb{C}$)-bundle on $\mathcal{H}$ with $c_2(E) = n$, $\mathcal{A}$ be the space of connections on E and $\mathcal{G}$ be the gauge group. We want to construct a map from $\mathcal{B} = \mathcal{A}/\mathcal{G}$

to $\mathbb{P}^1 \times \mathbb{P}^1$ that will associate to each connection a "graph".

As we have seen in section 2.1.2, one can associate a graph g to every holomorphic SL(2, $\mathbb{C}$)-bundle E on $\mathcal{H}$. This graph basically keeps track of the type of E on each fibre $\pi^{-1}(x)$, for $x \in \mathbb{P}^1$. As we have seen, $E|_{\pi^{-1}(x)}$ must have one of the following three types:

(i) $L_0 \oplus L_0^*, L_0 \in \mathrm{Pic}^0(\mathrm{T})$.

(ii) Non trivial extensions $0 \to L_0 \to E \to L_0 \to 0, L_0^2 \cong \mathcal{O}$.

(iii) $L \oplus L^*, L \in \mathrm{Pic}^{\mathrm{k}}, \mathrm{k} < 0$.

Thus, if $c_2(E) = n$, its graph $g \in |\mathcal{O}(n, 1)|$ decomposes into two pieces:

- the graph of a rational map $F : \mathbb{P}^1 \to \mathbb{P}^1$ of degree k,
- $(n - k)$ vertical fibres $\{x_i\} \times \mathbb{P}^1$.

And we know that $E|_{\pi^{-1}(x)}$ is of type (iii) if and only if it has a vertical bar in its graph.

Can one associate such a graph to a connection on E? Unfortunately, as $\mathcal{H}$ is a surface, not every connection defines a holomorphic structure. Indeed, a connection A induces a holomorphic structure if and only if the (0,2) part of its curvature is zero, i.e., $F_A^{0,2} = 0$. And this is obviously not the case, in general. There are however no $(0, 2)$-forms on the fibres $T_x = \pi^{-1}(x)$ of $\pi : \mathcal{H} \to \mathbb{P}^1$. The restriction to T_x of any connection A on $E \to \mathcal{H}$ therefore always defines a holomorphic structure on $E|_{T_x}$. And $E|_{T_x}$, endowed with this holomorphic structure, must have one of the above three types. It then seems possible to associate to any connection A a graph that will be a generalisation of the graph of holomorphic SL(2, $\mathbb{C}$)-bundles on $\mathcal{H}$.

We will define the graph of a connection A to be the zero set of a section of the determinant bundle of a family of Dirac operators associated to A. The determinant bundle of a family of elliptic operators is usually defined in the context of K-theory by using the index of the family. For brevity, we shall define the determinant bundle without introducing the notion of index of a family of operators. For a general discussion of this construction, we refer the reader to [DK].

7.3.1 Families of Dirac operators.

Let us begin by giving T a spin structure. On a Kähler manifold X, a spin structure is defined to be a choice of square root of the canonical line bundle $K = \Lambda^2 T^* X$, i.e. a line bundle $K^{1/2}$ such that $K^{1/2} \otimes K^{1/2} = K$. The spinors are then the $(0,p)$-forms which take values in $K^{1/2}$. As the canonical bundle of T is holomorphically trivial, we choose $\mathcal{O}$ as our spin structure. The spinors are then $S^+ = \Omega^{0,0}$ and $S^- = \Omega^{0,1}$. With this structure, the Dirac operator on T is simply $\bar{\partial} : \Omega^{0,0} \to \Omega^{0,1}$. Furthermore, given a vector bundle $W \to X$ with a connection A, the partial connection $\bar{\partial}_A : \Omega^{0,0}(W) \to \Omega^{0,1}(W)$ is also a Dirac operator.

We want to construct a family of Dirac operators on T that is parametrised by $\mathbb{P}^1 \times \mathbb{C}^*$. To do this, we start with a family of connections $\{\tilde{A}_{z\alpha}\}$ parametrised by $\mathbb{P}^1 \times \mathbb{C}^*$. The partial connections $\bar{\partial}_{\tilde{A}_{z\alpha}}$ will then give us the desired family of operators. Let V be the Poincaré bundle over $\mathcal{H} \times \mathbb{C}^*$ and π the projection $\mathcal{H} \times \mathbb{C}^* \to \mathbb{P}^1 \times \mathbb{C}^*$. (We use the same notation as in section 2.1.1.) For each $(z, \alpha) \in \mathbb{P}^1 \times \mathbb{C}^*$, $V|_{\pi^{-1}(z,\alpha)}$ is a line bundle in $\mathrm{Pic}^0(\mathrm{T})$, where $T = \pi^{-1}(z)$. (The projection $\mathcal{H} \to \mathbb{P}^1$ is also denoted by π.) Let $\xi_{z\alpha}$ be the flat connection on $V|_{\pi^{-1}(z,\alpha)}$. Then, for any $A \in \mathcal{A}$,

$$\tilde{A}_{z\alpha} = A \otimes \mathbb{I} \oplus \mathbb{I} \otimes \xi_{z\alpha}$$

is a connection on $(E \otimes V)|_{\pi^{-1}(z,\alpha)}$, where $\mathbb{I}$ represents the appropriate identity matrix. Our family of Dirac operators is then $\bar{\partial}_{\tilde{A}} = \{\bar{\partial}_{\tilde{A}_{z\alpha}}\}$, for $(z, \alpha) \in \mathbb{P}^1 \times \mathbb{C}^*$. Let us note that each of these partial connections is compatible with the holomorphic structure of $(E \otimes V)|_{\pi^{-1}(z,\alpha)}$. Indeed, as 2-forms on T have type $(1,1)$, $F^{0,2}_{\tilde{A}_{z\alpha}} = 0$ and $\bar{\partial}_{\tilde{A}_{z\alpha}} = \bar{\partial}_{E\otimes V}$. Furthermore, for each $(z, \alpha) \in \mathbb{P}^1 \times \mathbb{C}^*$, the dual operator of $\bar{\partial}_{\tilde{A}_{z\alpha}}$ is

$$\bar{\partial}^*_{\tilde{A}_{z\alpha}} = - * \bar{\partial}_{\tilde{A}_{z\alpha}} *,$$

where $*$ is the usual Hodge star operator.

7.3.2 Determinant line bundle.

As we have stated above, the determinant line bundle of a family of operators is closely related to the index of the family. The index of any elliptic operator P is

$$\mathrm{ind(P)} = \dim(\ker \mathrm{P}) - \dim(\mathrm{coker\ P}).$$

We therefore have to give a description of $\ker \bar{\partial}_{\tilde{A}_{z\alpha}}$ and coker $\bar{\partial}_{\tilde{\mathrm{A}}_{z\alpha}}$, for all $\mathbb{P}^1 \times \mathbb{C}^*$.

The $\bar{\partial}$-Laplacian is defined to be the operator $\Delta_{\bar{\partial}} = \bar{\partial}\bar{\partial}^* + \bar{\partial}^*\bar{\partial}$. Differential forms satisfying the Laplace equation $\Delta_{\bar{\partial}}\eta = 0$ are called harmonic forms. The space of harmonic forms of type (p, q) on T is denoted $\mathcal{K}^{p,q}(T)$. Let us note that

$$\Delta_{\bar{\partial}}\eta = 0 \Leftrightarrow \bar{\partial}\eta = 0 \text{ and } \bar{\partial}^*\eta = 0.$$

As T is a Kähler manifold, the Hodge decomposition theorem states that $\mathcal{K}^{p,q}(T) = H^q(T, \Omega^p)$, where Ω^p is the sheaf of holomorphic p-forms. Moreover, since T is a curve, $\bar{\partial}$ is zero on all (0,1)-forms, and $\bar{\partial}^*$ is zero on all (0,0)-forms. We therefore have

$$\ker \bar{\partial} = \mathcal{K}^{0,0}(T) = H^0(T, \mathcal{O}),$$

and

$$\mathrm{coker}\ \bar{\partial} = \ker \bar{\partial}^* = \mathcal{K}^{0,1}(\mathrm{T}) = \mathrm{H}^1(\mathrm{T}, \mathcal{O}).$$

For any vector bundle $W \to T$, this decomposition also applies to W-valued forms. Since $\tilde{A}_{z\alpha}$ is compatible with the holomorphic structure of $E \otimes V|_{\pi^{-1}(z,\alpha)}$ for all $(z, \alpha) \in \mathbb{P}^1 \times \mathbb{C}^*$, we then have

$$\ker \bar{\partial}_{\tilde{A}_{z\alpha}} = H^0(T, E \otimes V), \quad \text{and} \quad \mathrm{coker}\ \bar{\partial}_{\tilde{\mathrm{A}}_{z\alpha}} = \mathrm{H}^1(\mathrm{T}, \mathrm{E} \otimes \mathrm{V}).$$

Let us not that the spaces $H^1(T, E \otimes V)$ correspond to the fibres of the skyscraper sheaf $R^1\pi_*(E \otimes V)$ that was used in the holomorphic case to define the graph.

We have seen in section 2.1.2 that we always have

$$\dim H^0(T, E \otimes V) = \dim H^1(T, E \otimes V).$$

If the dimension of $\ker \bar{\partial}_{\tilde{A}_{z\alpha}}$ were constant for all $(z, \alpha) \in \mathbb{P}^1 \times \mathbb{C}^*$, the collection of these vector spaces would then form a locally trivial vector bundle over $\mathbb{P}^1 \times \mathbb{C}^*$.

In our case, $\dim(\ker \bar\partial_{\tilde A_{z\alpha}})$ jumps at the points in $\mathbb{P}^1 \times \mathbb{C}^*$ where V injects in E or where E is of type (iii). We can however still construct a vector bundle with the kernels. There exists a map $\psi : \underline{\mathbb{C}}^N \longrightarrow \Gamma(E \otimes V \otimes \bar S)$, where $\underline{\mathbb{C}}^N$ is the trivial bundle, such that $\bar\partial_{\tilde A} \oplus \psi$ is surjective (see [DK]). And this gives us the line bundle

$$\mathcal{L} = (\Lambda^{max} \ker(\bar\partial_{\tilde A} \oplus \psi)) \otimes (\Lambda^N \mathbb{C}^N)^*$$

on $\mathbb{P}^1 \times \mathbb{C}^*$. It is the *determinant line bundle* of the family $\{\bar\partial_{\tilde A}\}$.

7.3.3 Graph map.

Let A be a connection and $\bar\partial_{\tilde A}$ be the family of Dirac operators associated to it. The graph of A will be defined as the zero set of a section of the determinant bundle of $\bar\partial_{\tilde A}$. We first see that, for all $(z,\alpha) \in \mathbb{P}^1 \times \mathbb{C}^*$, we have an exact sequence

$$0 \to \text{Ker } \bar\partial_{\tilde A_{z\alpha}} \xrightarrow{i} \text{Ker } (\bar\partial_{\tilde A_{z\alpha}} \oplus \psi) \xrightarrow{P} \mathbb{C}^N \xrightarrow{\psi} \text{Coker } (\bar\partial_{\tilde A_{z\alpha}}) \to 0, \tag{7.6}$$

where i and P are the usual inclusion and projection maps, respectively. Furthermore, the projections fit together to define a homomorphism P of vector bundles

$$\begin{array}{ccc} \text{Ker}(\bar\partial_{\tilde A} \oplus \psi) & \xrightarrow{P} & \underline{\mathbb{C}}^N \\ \downarrow & & \downarrow \\ \mathbb{P}^1 \times \mathbb{C}^* & \xrightarrow{Id} & \mathbb{P}^1 \times \mathbb{C}^*. \end{array}$$

By the exact sequence (7.6),

$$\det P_{z\alpha} = 0 \Longleftrightarrow \text{Ker } \bar\partial_{\tilde A_{z\alpha}} = \text{Coker } \bar\partial_{\tilde A_{z\alpha}} \neq 0.$$

The map $\det P$ then gives us a homomorphism from $\Lambda^N\text{Ker}(\bar\partial_{\tilde A}))$ to $\Lambda^N\mathbb{C}^N$, and is therefore a section of $\mathcal{L} = (\Lambda^{max}\text{Ker}(\bar\partial_{\tilde A}))\otimes(\Lambda^N\mathbb{C}^N)^*$. The zero set $\tilde D = (\det P)^{-1}(0)$ will then correspond to the "graph" of A. Let us note that, by construction, if A defines a holomorphic structure on E, $\tilde D$ is equal to the divisor defined in section 2.1.2. Furthermore, as in the holomorphic case, we can show that $\tilde D$ is invariant under

– the $\mathbb{Z}$-action on $\mathbb{C}^*$ generated by multiplying by λ, and

– the involution on $\mathbb{C}^*$ defined by $z \mapsto 1/z$.

Let us first show that tensoring V by $\mathcal{O}(1)$ does not change the zero set of $\det P$. From the exact sequence

$$0 \longrightarrow \mathrm{Pic}(\mathbb{P}^1) \longrightarrow \mathrm{Pic}^0(\mathcal{H}) \longrightarrow \mathrm{Pic}^0(\mathrm{T}) \longrightarrow 0, \tag{7.7}$$

we see that, for all $z \in \mathbb{P}^1$, $\mathcal{O}(1)|_{\pi^{-1}(z)} \simeq \mathcal{O}$. Let χ be a connection on $\mathcal{O}(1)$. On $T = \pi^{-1}(z)$, there then exists a holomorphic gauge transformation u such that $\chi = du \cdot u^{-1}$. Since u is holomorphic, this means that the $(0,1)$-component of $\chi_{\pi^{-1}(z)}$ is zero and $\bar{\partial}_{\chi_{\pi^{-1}(z)}} = \bar{\partial}$. Thus, if we set $\bar{\partial}_{E\otimes V\otimes\mathcal{O}(1)} = \bar{\partial}_{E\otimes V} \otimes \mathbb{I} + \mathbb{I} \otimes \bar{\partial}_{\mathcal{O}(1)}$, we see that $\bar{\partial}_{E\otimes V\otimes\mathcal{O}(1)} = \bar{\partial}_{E\otimes V}$ and the section that we will get by repeating our construction with $E \otimes V \otimes \mathcal{O}(1)$ will have the same zero set as detP. $\tilde{D}$ is therefore invariant under the $\mathbb{Z}$-action on $\mathbb{C}^*$ generated by multiplying by λ.

Also, since $\Lambda^2 E \simeq \mathcal{O}$, $E \simeq E^*$ and, if we substitute V^* for V, $E \otimes V^* \simeq (E\otimes V)^*$. Thus, if we repeat our construction with V^* instead of V, since $\bar{\partial}_{(E\otimes V)^*} = -(\bar{\partial}_{E\otimes V})^t$, the zero set of the section we obtain will again be $\tilde{D}$. $\tilde{D}$ is then invariant under the involution on $\mathbb{C}^*$ defined by $z \mapsto 1/z$.

$\tilde{D}$ therefore descends to a divisor D on $\mathbb{P}^1 \times \mathbb{P}^1$. And D is defined to be the "graph" of A. Again, if A defines a holomorphic structure on E, it coincides with the graph defined in section 2.1.2.

Any connection A on $E \to \mathcal{H}$, where E is an $\mathrm{SL}(2,\mathbb{C})$-bundle with $c_2(E) = n$, has a graph which decomposes into two pieces:

– the graph of a C^∞ map $F : \mathbb{P}^1 \to \mathbb{P}^1$,

– vertical fibres $\{x_i\} \times \mathbb{P}^1$.

This is exactly as in the holomorphic case, except for the fact that the map portion need not be holomorphic, and that there may not be a finite number of vertical bars. As in the holomorphic case, the graph cannot however be made up only of vertical bars (see [BH]). The set of all possible graphs can actually be described as follows

$$\left\{ s \in \Gamma(\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}(n,1)) \;\middle|\; \begin{array}{c} s \text{ holomorphic on } \{x\} \times \mathbb{P}^1, \\ \forall x \in \mathbb{P}^1, \text{ and } s \neq 0 \end{array} \right\} / \mathbb{C}^*,$$

where $s \neq 0$ because the graph cannot be made up only of vertical bars. (We have to quotient by $\mathbb{C}^*$ to take into account the action of the gauge group, i.e., the equivalence classes of connections.) A section $s \in \Gamma(\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}(n,1))$, which is holomorphic on each $\{x\} \times \mathbb{P}^1$, is equivalent to a section $\tilde{s} \in \Gamma(\mathbb{P}^1, p_{1*}(\mathcal{O}(n,1)))$, where $p_1 : \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1$ is projection onto the first factor. What is the fibre of $p_{1*}(\mathcal{O}(n,1))$? For each $x \in \mathbb{P}^1$,

$$\begin{aligned} p_{1*}(\mathcal{O}(n,1))_x &= \mathcal{O}(n) \otimes p_{1*}(\mathcal{O}(1))_x \\ &= \mathcal{O}(n) \otimes H^1(\{x\} \times \mathbb{P}^1, \mathcal{O}(1)) \\ &= \mathcal{O}(n) \otimes \mathbb{C}^2. \end{aligned}$$

The set of all graphs is therefore

$$\{\tilde{s} \in \Gamma(\mathbb{P}^1, \mathcal{O}(n) \otimes \mathbb{C}^2) | \tilde{s} \neq 0\} / \mathbb{C}^*.$$

As $\{\tilde{s} \in \Gamma(\mathbb{P}^1, \mathcal{O}(n) \otimes \mathbb{C}^2) | \tilde{s} \neq 0\}$ is a infinite-dimensional complex vector space, we will denote it $V_{\mathbb{C}}^{\infty}$. The set of graphs is thus

$$(V_{\mathbb{C}}^{\infty} - 0) / \mathbb{C}^* \cong \mathbb{P}^{\infty},$$

and we have a map $G : \mathcal{B} \to \mathbb{P}^{\infty}$. Furthermore, this map extends in a natural way to the space of framed connections $\tilde{\mathcal{B}}$.

7.4 Fibre of the graph map.

7.4.1 Non-triviality of the graph map.

In the previous section, we constructed a map $G : \tilde{\mathcal{B}} \to \mathbb{P}^{\infty}$ which associates to each pair (A, t), where A is a connection and t is a framing, the graph of A. If the connection A induces a holomorphic structure on E, this graph coincides with the graph defined on holomorphic $\mathrm{SL}(2, \mathbb{C})$-bundles. In the holomorphic case, the graph map is surjective for $n \geq 2$. Since the infinite complex projective space $\mathbb{P}^\infty$ is the Eilenberg-MacLane space $K(\mathbb{Z}, 2)$, we have the equivalence $[\tilde{\mathcal{B}}, \mathbb{P}^\infty] = H^2(\tilde{\mathcal{B}}, \mathbb{Z})$. We have seen in section 7.1.2 that $H^2(\tilde{\mathcal{B}}, \mathbb{Z}) = \mathbb{Z}_2$. There then exist maps from $\tilde{\mathcal{B}}$ to $\mathbb{P}^\infty$ which are not homotopic to the constant map. In this section, we show that $G : \tilde{\mathcal{B}} \to \mathbb{P}^\infty$ is such a map, i.e., G is not topologically trivial.

Let $(A, t) \in \tilde{\mathcal{B}}$. We assume that A induces a holomorphic structure on E and that its graph is of the form $2k(\{z_\infty\} \times \mathbb{P}^1) + (\mathbb{P}^1 \times \{l\})$, where z_∞ is the point in $\mathbb{P}^1$ at infinity. Let us note that we must have $c_2(E) = 2k$. We also assume that $l = \{L_0, L_0^*\}$, where L_0 is given by a divisor of the form $D = 2p_\lambda - 2p_0$; and that $E = L \oplus L^*$ on $\pi^{-1}(z_\infty)$, for some $L \in \mathrm{Pic}^{-2}(\mathrm{T})$. We shall see that there is a natural $SU(2)$-action on the pair (A, t), and on its graph $g = G(A)$. The orbits of this action then give the commutative diagram

$$\begin{tikzcd} SU(2) \cdot ([(A,t)]) \arrow[r, hook] \arrow[d, "G"'] & \tilde{\mathcal{B}} \arrow[d, "G"] \\ SU(2) \cdot (g) \arrow[r, hook] & \mathbb{P}^\infty. \end{tikzcd}$$

We begin by describing the action of $SU(2)$ on $\mathcal{H}$. Let us recall that the fibre of $\pi : \mathcal{H} \to \mathbb{P}^1$ is the elliptic curve $T \cong \mathbb{C}/(2\pi i\mathbb{Z} + \ln(\lambda)\mathbb{Z})$. Let $(z_1, z_2) \in \mathbb{C}^2$. $z = z_2/z_1, z' = 1/z$ are affine coordinates on $\mathbb{P}^1$; and $t = \log z_1, t' = \log z_2$ are linear coordinates on T. $\mathcal{H}$ is then covered by the two coordinate patches $U_0 = \mathbb{C} \times T$, $U_1 = \mathbb{C} \times T$, with the identification

$$(z', t') = 1/z, t + \log z)$$

on the overlap. $SU(2)$ can be described as

$$SU(2) = \{ \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix} | a\bar{a} + b\bar{b} = 1, \text{ for a}, \bar{\text{a}}, \text{b}, \bar{\text{b}} \in \mathbb{C}\}.$$

It acts on $\mathbb{C}^{2*}$ by simple matrix multiplication. This action preserves the equivalence classes in $\mathbb{P}^1$. If $U(1) = \mathbb{C}^*$ is considered as the closed subgroup of $SU(2)$ given by $\{ \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} | a \in \mathbb{C}^* \}$, we see that it leaves the point at infinity in $\mathbb{P}^1$ fixed.

Moreover, $U(1)$ acts on T by translation: $t \mapsto t + \log a$. As line bundles in $\mathrm{Pic}^0(\mathrm{T})$ are invariant under translation, we see that the stabiliser of the graph g is $U(1)$. The orbit of g is then $SU(2)/U(1) \cong \mathbb{P}^1$.

The matrices that fix A must be elements of $U(1)$ that also fix z in $\mathbb{P}^1$. $\pm\mathrm{Id}$ are then the only possibilities. The action of $-\mathrm{Id}$ on T is given by $t \mapsto t + \pi i$. We know that this translation acts trivially on L_0. We also see that the divisor $D = 2p_\lambda - 2p_0$ of L_0 is invariant under this action. As L is fixed under the translations

$$\{t \mapsto t + v \mid v \in \mathbb{Z}\{\frac{\ln \lambda}{2}, \pi i\}\},$$

the matrix $-\mathrm{Id}$ therefore fixes A. Moreover, since $-\mathrm{Id}$ is in the centre of $SU(2)$, it also fixes the pair (A, t), and its orbit is $SU(2)/\pm \cong \mathbb{R}P^3$.

The above commutative diagram then becomes

$$\begin{tikzcd} \mathbb{R}P^3 \arrow[r, hook] \arrow[d, "G"'] & \tilde{\mathcal{B}} \arrow[d, "G"] \\ \mathbb{P}^1 \arrow[r, hook] & \mathbb{P}^\infty. \end{tikzcd}$$

We have $H^2(\mathbb{R}P^3, \mathbb{Z}) = H^2(\tilde{\mathcal{B}}, \mathbb{Z}) = \mathbb{Z}_2$ and $H^2(\mathbb{P}^1, \mathbb{Z}) = H^2(\mathbb{P}^\infty, \mathbb{Z}) = \mathbb{Z}$, and the horizontal inclusion maps are not topologically trivial. Furthermore, the restriction of G to $\mathbb{R}P^3$ is equal to the S^1-bundle associated to $\mathcal{O}(2) \to \mathbb{P}^1$, which is not trivial. This then proves that G cannot be homotopically trivial, in the case of framed connections on a bundle with even second Chern class. The stabilisation maps $\tilde{\mathcal{B}}_k \to \tilde{\mathcal{B}}_{k+1}$ then imply that it must be true for any k. The graph map $G : \tilde{\mathcal{B}} \to \mathbb{P}^\infty$ is therefore nontrivial.

7.4.2 Fibre of the graph map.

We have seen that $G : \tilde{\mathcal{B}} \to \mathbb{P}^\infty$ homotopically nontrivial. We can assume that it is a surjective fibration. Let V be its fibre. Up to homotopy, we then have

$$V \longrightarrow \tilde{\mathcal{B}} \xrightarrow{G} \mathbb{P}^\infty.$$

This fibration will then give rise another fibration:

$$\Omega\, \mathbb{P}^{\infty} \longrightarrow V \longrightarrow \tilde{\mathcal{B}}. \tag{7.8}$$

However, as $\mathbb{P}^{\infty} = K(\mathbb{Z}, 2)$,

$$\Omega\, \mathbb{P}^{\infty} = K(\mathbb{Z}, 1) = S^1.$$

(7.8) then becomes

$$S^1 \longrightarrow V \longrightarrow \tilde{\mathcal{B}},$$

and we see that V is the total space of an S^1-bundle on $\tilde{\mathcal{B}}$. One can take the pullback to $\tilde{\mathcal{B}}$ of the universal S^1-bundle on $\mathbb{P}^{\infty}$.

Conclusion

In this thesis, we have studied moduli spaces on Hopf surfaces both from a holomorphic and topological point of view.

For stabilisation maps, we have shown that, in this case, the natural algebro-geometric description of Taubes' subtraction procedure does not enable one to construct global stabilisation maps $\mathcal{M}_n^0 \to \mathcal{M}_{n+1}^0$ in the holomorphic setting.

We have also generalised the notion of graph to connections. If a given connection A on a holomorphic SL$(2, \mathbb{C})$-bundle E, with $c_2(E) = n$, defines a holomorphic structure on E, we have seen that the graph of the connection A coincides with the graph of E. There is therefore a natural inclusion from the space of all holomorphic graphs $\mathbb{P}^{2n+1}$ into the space of all topological graphs $\mathbb{P}^\infty$.

We have also studied the fibre of the graph map. In the holomorphic case, we obtained an explicit description of this fibre as the Jacobian of a Riemann surface, for a certain set of graphs. It would be interesting to know if there exists an analogous description for the fibre in the topological case. Does the fibre, in this case, represent the number of ways of glueing two connections? Furthermore, we have shown that the holomorphic graph map supports a Lagrangian fibration, with respect to a Poisson structure on $\mathcal{M}_n$. Do we have a similar situation for moduli spaces of connections?

Finally, it would be interesting to give a further classification of holomorphic vector bundles over $D \times T$ in terms of their graphs in $D \times \mathbb{P}^1$. This would enable us to obtain a more complete classification of vector bundles on $\mathcal{H}$. We believe that the techniques used in this thesis would be useful in the study of moduli spaces of holomorphic bundles over the more general elliptic fibrations that have been studied by Friedman and Morgan [FM].

Bibliography

[At] M. F. Atiyah: Vector bundles over an elliptic curve, *Proc. Lond. Math. Soc.* (3) **7**, 414-452 (1957).

[AtHS] M. F. Atiyah, N. J. Hitchin and I. M. Singer: Self-duality in four-dimensional Riemannian geometry, *Proc. R. Soc. Lond.* **362**, 425-461 (1978).

[AtJo] M. F. Atiyah and J.D.S. Jones: Topological aspects of Yang-Mills theory, *Commun. Math. Phys.* **61**, 97-118 (1978).

[Be] A. Beauville: Systèmes hamiltoniens complètement intégrables associés aux surfaces $K3$, *Symp. Math.*, **32**, 25-31 (1992).

[BHMM] C. Boyer, J. Hurtubise, B. M. Mann, R. J. Milgram, The Topology of instanton moduli spaces, I: The Atiyah-Jones conjecture, *Ann. Math.* **137**, 561-609 (1993).

[BH] P.J.Braam and J.Hurtubise: Instantons on Hopf surfaces and monopoles on solid tori, *J. reine angew. Math.*, **400**, 146-172, (1989).

[Bot] F. Bottacin: Poisson structures on moduli spaces of sheaves over Poisson surfaces, *Invent. math.*, **121**, 421-436 (1995).

[Br] V. Brînzănescu: Holomorphic vector bundles over compact complex surfaces, Lecture Notes in Mathematics, **1624**, Springer, Berlin Heidelberg, 1996.

[Bh] N. P. Buchdahl: Hermitian-Einstein connections and stable vector bundles over compact complex surfaces, *Math. Ann.*, **280**, 625-648, (1978).

[D] S. K. Donaldson: Anti-self-dual Yang-Mills connections over complex algebraic surfaces and stable vector bundles, *Proc. Lond. Math. Soc.* (3) **50**, 1-26 (1985).

[DK] S. K. Donaldson and P. B. Kronheimer: *The Geometry of Four-Manifolds*, Oxford Mathematical Monographs, Oxford, Clarendon Press 1990.

[F] R. Friedman: *Algebraic Surfaces and Holomorphic Vector Bundles*, UTX, Springer, New York Berlin Heidelberg, 1998.

[FM1] R. Friedman and J. Morgan: On the diffeomorphism type of certain algebraic surfaces, *J. Diff. Geom.* **27** 297-369, (1988).

[FM2] R. Friedman and J. Morgan: *Smooth Four-Manifolds and Complex Surfaces*, Ergebnisse der Mathematik und ihrer Grenzgebiete 3. Folge, **27**, Springer-Verlag, Berlin, 1994.

[FMW] R. Friedman, J. Morgan and E. Witten: Vector bundles over elliptic fibrations, J. Algebr. Geom. 8 **2**, 279-401 (1999).

[G] P. Gauduchon: Le théorème de l'excentricité nulle, *C. R. A. S. Paris* **285**, 387-390 (1977).

[GH] P. Griffiths and J. Harris: *Principles of Algebraic Geometry*, New York, Wiley 1978.

[Gr] A. Grothendieck: Sur la classification des fibrés holomorphes sur la sphère de Riemann, Amer. J. Math. **79**, 121-138 (1975)

[Ha] R. Hartshorne: *Algebraic Geometry*, GMT **52**, Berlin-Heidelberg-New York, Springer-Verlag 1977.

[Hi] F. Hirzebruch: *Topological Methods in Algebraic Geometry*, Grundl. Math. Wiss. **131**, Springer, Berlin Heidelberg 1966.

[HM] J. C. Hurtubise and R. J. Milgram: The Atiyah-Jones conjecture for ruled surfaces, *J. reine angew. Math.* **466**, 111-143 (1995).

[Mu] S. Mukai: Symplectic structure of the moduli space of sheaves on an abelian or $K3$ surface, *Invent. math.*, **77**, 101-116 (1984).

[OSS] C. Okonek, M. Schneider, and H. Spindler: *Vector bundles on complex projective space*, Progress in Mathematics **3**, Boston 1980.

[SiTr] Y. T. Siu and G. Trautmann: Deformations of coherent sheaves with compact support, *Memoirs Amer. Math. Soc.* (238), **29** 1-155, (1981).

[Sr] J. P. Serre: *Sur les modules projectifs*, Sém. Dubreil-Pisot 1960/1961, Exp. 2, Fac. Sci. Paris 1963.

[T] C. H. Taubes: Path-connected Yang-Mills moduli spaces. *J. Differential Geom.* **19**, 337-392 (1984).

[Tr] G. Trautmann: Deformations of coherent analytic sheaves with finite singularities, Séminaire P. Lelong (Analyse) Année 1974/75, Lecture Notes in Mathematics **524**, Springer, Berlin Heidelberg, (1976).

[UY] K. Uhlenbeck and S. T. Yau: On the existence of Hermitian Yang-Mills connections in stable vector bundles, *Commun. Pure App. Math. Suppl.* **39** S.257-S.293, (1986).

[We] A. Weinstein: The local structure of Poisson manifolds. *J. Differential Geom.* **18**, 523-557 (1983).