Invariants of Lie Algebras

General and Specific Properties.

by

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Abstact

This thesis consists of three chapters. Chapter I serves as an introduction to the subject of invariants in Lie algebra theory; it also constitutes a bridge between the two remaining chapters. In chapter II, a theorem on the number of independent missing label operators along with its proof is discussed. It is proven that, if there are n missing labels, then there are 2n missing label operators for semi-simple Lie algebras. The proof is supplemented by examples which extend the validity of the theorem to the non semi-simple case. Chapter III is devoted to the classification of the conjugacy classes of the subalgebras of the similitude algebra into isomorphism classes, and to the determination of the invariants for each representative algebra of the conjugacy classes. These results are summarized in tables.

Résumé

Cette thèse est divisée en trois chapitres. Le premier introduit la notion d' invariant dans la théorie d' algèbre de Lie, et relie le continu des deux dernier chapitres. Un théoreme sur le nombre d' operateurs étiquettes manquants indépendants et sa preuve font l' objet du deuxième chapitre où est prouvé que s' il existe un nombre "n" d' étiquettes manquants celui de leurs operateurs est "2n" pour les algèbre de Lie semi-simple; en plus quelque examples montrent la validité du théoreme pour le cas non semi-simple. Enfin le troisième chapitre se consacre à la classification des classes de conjugaison des sous algèbre de l' algèbre de similitude en classes isomorphiques, et à la déterminaison des invariants pour chaque algèbre représentatif des classes de conjugaison. Ces résultats sont tabulés a la fin du troisième chapitre.

Statement of Original Contribution to Knowledge.

1. The theorem on the number of independent missing label operators, and its proof are original, and have already appeared in the Journal of Mathematical Physics Vol. 17 (July) ; in joint authorship with R. T. Sharp.
2. The classification of the conjugacy classes of the subalgebras of the similitude algebra into isomorphism classes, and the determination of the invariants also represent new knowledge. These, too, were prepared for publication in a research journal.

Acknowledgement.

I would like to express my gratitude to Prof. R. T. Sharp, who in being my academic advisor, has acted as my guide, collaborator, and inspirer for further research.

Besides background knowledge, and problem solving experience, productive research also requires a supporting enviroment. For this, I am in debt to J. Patera and P. Winternitz of the Centre de Recherches Mathématique, Université de Montreal. Many ideas were conceived and clarified as a result of participating in discussions with these three researchers.

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CHAPTER I

FOREWORD

The concept of an invariant, in the context of a Lie algebra, first arose as what is now known as the centralizer of the universal enveloping algebra of the Lie algebra. That is, it was defined as a homogeneous polynomial on the algebra, whose Lie product with any element of the algebra is zero. This concept proved extremely fruitful in the representation theory of semi-simple Lie algebras. These polynomials are referred to as Casimir[1] operators, after the man who first used them to demonstrate that any representation of a semi-simple Lie algebra which is reducible is also fully reducible. Racah[2], later showed that the eigenvalues of the Casimir operators label uniquely the irreducible representations (irreps) of the algebra. This notion is most familiar to the physicist via angular momentum theory, or equivalently through the algebra SO(3). Let J_1,J_2, and J_3 be the three components of the angular momentum operator; or the basis elements of SO(3). It is well known, that the abstract representation vectors for a matrix representation of SO(3) are given by the ket $|j,j_3\rangle$ where $j(j_3)$ is the eigenvalue of $J^2 \equiv J_1^2 + J_2^2 + J_3^2(J_3)$; J^2 being the Casimir operator in this case. The specific irreducible subspaces of the representation space is selected by the value of j, while j_3 distinguishes among the different basis vectors within the subspace. This example illustrates the substance of representation theory; at the same time, it provides an example of Racah's theorem, referred to in the above. However, due to its simplicity, the complications that arise in the more general situations do not appear here.

Proceeding directly to the general case, let L be an arbitrary semi-simple Lie algebra of dimension r and of rank ℓ . Then, L has a basis consisting of the ℓ basis elements of the Cartan subalgebra denoted by H_i, $i=1,2,3,\ldots,\ell$ and $(r-\ell)/2$ pairs of ladder operators $E_{\pm\alpha}$. The commutators (Lie products) with respect to this basis are

$$[H_i,H_j] = 0 \qquad [H_i,E_{\alpha_j}] = \alpha_{ij} E_{\alpha_j}$$

$$[E_\alpha,E_B] = N_{\alpha B}\, E_{\alpha+B} \text{ where } \alpha + B \text{ is a non zero root} \tag{1}$$

$$[E_\alpha,E_{-\alpha}] = \sum_{i=1}^{\ell} \alpha_j^i H_i$$

and all others are zero. Since L has rank ℓ it has ℓ algebraically independent Casimir invariants[3]. A matrix representation of the algebra is obtained when the action of the r basis elements of the algebra on the representation vectors is distinctly specified. Let J denote the eigenvalues of the ℓ Casimir operators and M denote the eigenvalues of the ℓ elements of the basis of the Cartan subalgebra. Then consider the matrix element

$$\langle JM|A|J'M'\rangle \qquad \forall\, A \in L$$

It is evident from (1) that the matrix elements of the Cartan subalgebra basis elements and 2 ℓ ladder operators[4] are uniquely determined by the ℓ components of M. Hence an additional $(r-3\ell)/2$ labels are required to distinguish the remaining $(r-3\ell)/2$ pairs of ladder operators. This is commonly referred to as the internal missing label problem.

More generally, it occurs whenever the irreps of an algebra are reduced to the irreps of its subalgebras, and the subalgebras do not provide enough labels to specify the irreps of the algebra unambiguously.
In the above case, the Cartan subalgebra was used as the reduction subalgebra (denoted by $L \supset H$). A canonical reduction $L \supset L_1 \supset L_2 \supset L_3 \supset \ldots \supset L_n$ is

accomplished whenever the Casimir operators of L along with the Casimir operators of all the subalgebras featured in the reduction chain form a complete set. In other words, the eigenvalues of all the Casimir operators are sufficient to label uniquely the basis vectors for the irreps of the algebra. The Gel'fand-Tseitlin patterns[5] explicitly show the existence of such a reduction for all classical Lie algebras, with the exception of Sp(2n) where no such reduction exists. In particular, for SU(n), the reduction is

$$SU(N) \supset S[U(n-1) \otimes U(1)] \supset S[U(n-2) \otimes U(1)] \supset \ldots$$

$$\supset S[U(1) \otimes U(1) \ldots U(1)].$$

However, these reductions far from eliminate the missing label problem. In the application of group-theoretic methods to physical problems, the subalgebras appearing in the reduction are often specified by the particular problem under consideration. For example, consider the algebra SU(3). If it is applied to hadron physics, then the most convenient reduction is $SU(3) \supset S[U(2) \otimes U(1)]$ where the SU(2) irreps are used to label the isospin states, and the U(1) labels the hypercharge states. The reduction is canonical and is fully established in the arsenal of particle physicists. While, in the application of SU(3) to nuclear physics, as the symmetry algebra of the three dimensional isotropic oscillator, the appropriate reduction is $SU(3) \supset SO(3)$, where SO(3) provides the angular momentum label for the states. This reduction is not canonical, and in fact has one missing label. More will be said on the number of missing labels later.

The decomposition of direct products of the irreps of an algebra into the irreps of that algebra, known as the external labeling problem, can be reformulated into an internal labeling problem via the reduction $L \otimes L \supset L$.

(As in the case of the internal labeling problem for SO(3), the external labeling problem is straightforward in this case. However, $SU(3) \otimes SU(3) \supset SU(3)$ has one missing label.) Whenever an irrep of the algebra occurs more than once for a particular direct product decomposition, a missing label problem results.

These examples should have well stressed the importance of the missing label problem in both practical applications and theoretical considerations. The missing label problem can be restated as a degeneracy problem for the vectors of the irreps; a formulation that has been quite fruitful.

Several methods have been proposed to remove this degeneracy for specific cases. These methods can be divided into two categories. The first category consists of adjoining to the other labels integer-valued labels chosen in prescribed ways[6,7,8,9]. These solutions, however, do not produce orthogonal basis vectors; nor can the labels be associated with the eigenvalues of any linear operators acting on the representation space. This type of solution suffers from the disadvantage, that the missing labels cannot be made to correspond to observables as the remaining labels can.

The other more physical approach is based on completing the set of algebra and subalgebra invariants with additional operators, so that the representation vectors will appear unambiguously as the eigenvectors of this completed set[6,10,11,12]. A minor inconvenience results from Racah's[13] proof that the missing label operators cannot have integer eigenvalues. These missing label operators must be subalgebra invariants, since all subalgebra irreducible subspaces must all correspond to the same eigenvalues of the missing label operators; if they are in fact to lift the degeneracies. Each of these operators must also provide an independent label, and so must themselves be functionally independent from the rest of the invariants. The missing label problem has been solved in detail for various semi-simple algebra-subalgebras reductions[7-12]. In particular SU(3) ⊃ SO(3) has been studied in detail by Moshinsky et al[6], in which both approaches are considered.

Tho number of missing labels is

$$(r_g - \ell_g - r_h - \ell_h)/2$$

Where $r_g(r_h)$ is the dimension of the algebra (subalgebra), and while $\ell_g(\ell_h)$ is the rank of the algebra (subalgebra). This formula is discussed in more detail in chapter II..

A survey of the literature, on the missing label operators, will quickly reveal that the number of available such operators is always twice the number required. It turns out that this conjecture is indeed a theorem and constitutes the subject matter of chapter II, which was originally written as a research paper in collaboration with R.T. Sharp. The paper has already appeared in the Journal of Mathematical Physics Vol. 17 (July).

Thus far, only one aspect of the many uses of invariant operators has been considered. Invariant operators play an illuminating and often essential role in special function theory[14,15,16]. Many, and possibly all of the special functions occur as eigenfunctions of invariants. This approach to special function theory, not only facilitates calculations, but also unifies and edifies the entire theory into a consistent and elegant structure. In physics, invariant operators of symmetry algebras can be used to yield appropriate quantum numbers for the system. In many instances, the eigenvectors and eigenvalues of the invariants will completely characterize the system i.e.: the invariants of O(4) will yield the energy and angular momentum quantum numbers for the hydrogen atom, while the invariants of the Poincaré algebra will yield the mass and spin quantum numbers, used to define a relativistic particle. In the case of dynamical symmetries, invariant operators provide mass formulae as in hadron physics and energy spectra as in the hydrogen atom, and etc. Subalgebra invariants can be used to describe specific features of the system, such as angular momentum, isospin, spin, charge states and etc. Invariant operators can also be useful in the study of symmetry breaking. The idealized or first order approximation might suggest a particular symmetry algebra and may allow for the representation of a physical quantity by an algebra invariant. However, second order corrections will usually break or reduce the symmetry algebra to one of its subalgebras. This effect can be incorporated by adding to the representation of the physical quantity a term dependent on the subalgebra invariants.

It is appropriate to notice here, that although the concept of invariant was first introduced as a Casimir operator, it is inadequate as it stands. For there are non semi-simple Lie algebras which do not possess any Casimir operators, but do, however, permit the existence of an invariant. The invariant is defined as any operator function on the Lie algebra such that its commutator with any element of the algebra is zero. The name is well suited, because the function will be left invariant under the action of the group which corresponds to the algebra in question, and because they share many of the important properties of the Casimir operators. A fuller discussion of these invariants appears in chapter II and chapter III.

A systematic study of algebra-subalgebra structures, along with their respective invariants will furnish useful information which can be incorporated in the construction of physical theories. Such a study has been undertaken by J. Patera, R.T. Sharp, P. Winternitz and H. Zassenhaus at the University of Montreal. They developed a method for obtaining all the conjugacy classes of subalgebras for a given Lie algebra with non trivial ideal[17]. (The case of simple Lie algebras had already been treated[18]). Their researches are summarized in a string of papers[19,20]; in which they applied the method to various algebras of interest in physics. One of these papers[20] was devoted to the study of the similitude algebra of space-time. The similitude group is defined as the set of all non-singular constant transformations on space-time which leave the Lorentz metric form invariant. Let SG denote the similitude group, then

$$SG \equiv \{ \Lambda \in \mathbb{R}^{4\times 4},\ \bar{a} \in \mathbb{R}^{1\times 4} \ /\ \text{if}\ \ \bar{x}' = h(\Lambda\bar{x}) + \bar{a}\ \ \text{then}\ \ (ds')^2 = h^2(ds)^2,\ h^2 \in \mathbb{R}^+\}$$

where $(ds)^2 = (dx_1)^2 + (dx_2)^2 + (dx_3)^2 - (dx_0)^2$. Alternatively, the group can be defined componentwise by

$$x_\mu = h\,\Lambda_{\mu\nu} x_\nu + a_\mu$$

where h is the dilatation (scale) transformation, $\Lambda_{\mu\nu}$ represent the matrix components of the homogeneous Lorentz transformations, and the a_μ's represent the space-time translations. It is clear then, that the similitude algebra is of dimension 11 and has the structure

$$D \,\square\, (SL(2,C) \,\square\, T_4)$$

where D is the scale operator, SL(2,C) is a realization of the Lorentz algebra, and T_4 is the four dimensional abelian algebra of the translations; by $\square$ the semi-direct product of algebras is to be understood.

The similitude algebra is of considerable physical interest in elementary particle physics and in gravitational interactions. In high energy physics, it enters through scaling phenomena observed in deep inelastic scattering. It should, therefore, influence the development of short distance behaviour in particle and/or field theory[20-25]. This group also underlies Weyl's unified field theory[26], which explains why it is sometimes referred to as the Weyl group. The similitude algebra, being the largest non-trivial subalgebra of the conformal algebra[22,25] (SO(4,2)), figures prominently in the wide applications of the conformal group in diversified fields of physics[26,27].

Chapter III is devoted to the determination of the invariants for all the conjugate classes of subalgebras of the similitude algebra. These conjugacy classes are, then, assembled into isomorphism classes. This chapter was also written as a research paper.

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CHAPTER II

NUMBER OF INDEPENDENT MISSING LABEL OPERATORS.

1. Introduction

As mentioned in the foreword, in the application of Lie algebras to physical problems, it is often necessary to reduce the irreps of the algebra into the irreps of a subalgebra. In most instances, the reduction will not be canonical. The resolution of this difficulty by the missing label operators will be considered here. The representation vectors will be the common eigenvectors of a complete set of commuting operators. Besides the generalized invariants of the algebra and those of the subalgebra, an appropriate number of missing label operators must be found. These should be subalgebra scalars (invariants) which are functions on the entire algebra. It will be shown that the number of independent missing label operators available, for semi-simple Lie algebras, is just twice the number of labels actually missing. The argument used strongly suggests that the same is true for all Lie algebras.

Numerous examples of this result are known (see foreword). In the case of $SU(3) \supset SO(3)$ there is one missing label. Long ago Racah[1] proposed two independent missing label operators; he surmised and Judd et al[2] later proved that these did constitute the most general solution to the missing label operator problem. Recently a similar result was shown to hold[3,4] for all cases of compact group-subgroup reduction with one missing label. Two cases with two missing labels have been investigated; $SU(4) \supset SU(2) \otimes SU(2)$ (The Wigner supermultiplet model for nuclear states of definite spin and isospin, and $O(5) \supset O(3)$. In each case four missing labels were available.

2. Counting subalgebra scalars.

The invariants of semi-simple Lie algebras were determined long ago[6]. Their number is ℓ, the rank of the algebra, and may be chosen as homogeneous polynomials on the algebra (ie: Casimir operators). Their eigenvalues are necessary and sufficient to label unambiguously the irreps of the algebra (the possibility of using rational functions of Casimir operators is ignored; in any case, this artificial way of reducing the number of labeling operators works only when because of compactness, the eigenvalues are restricted to a discrete set). In order that the following considerations apply to non semi-simple algebras, the plausible assumption[7,8], that the eigenvalues of the independent invariants unambiguously label the irreps of any Lie algebra, is made. The existence of labels which assume only a finite number of values, as the sign of the energy in the case of the Poincaré algebra, is ignored. For non semi-simple algebras, the invariants are not necessarily polynomials, but may be rational or even transcendental functions on the algebra. The invariants are determined[7,8,9] by solving a set of partial differential equations. Each equation corresponds to a row of the commutator table.

Returning to the problem at hand, the method of derivation shows that the number of invariants is[10]

$$\ell = r - R \tag{1}$$

where r is the dimension of the algebra and R is the rank of the commutator table, considered as a matrix. For purposes of computing R, the elements of the algebra are to be considered as c-number variables.

A theorem due to Racah[1] shows that the number of internal labels needed to uniquely label the basis vectors of a general irrep of a Lie algebra is

$$i = (r-\ell)/2 \tag{2}$$

When a subalgebra H is used to label the basis vectors of the irreps of a Lie algebra G, it provides $(r_h + \ell_h)/2 - \ell'$ labels. The appearance of ℓ' allows for the possibility that the invariants of H and G may not be mutualy independent; ℓ' is the number of invariants of G which are functions of the elements of H only. The number of missing labels is thus

$$n = \frac{1}{2}(r_g - \ell_g - r_h - \ell_h) + \ell' \tag{3}$$

(This is a generalization of the equation $n = (r-3\ell)/2$ which was derived in the foreword for the case when H is the Cartan subalgebra.)

Subalgebra scalars which are functions on the algebra may be determined, just as algebra invariants, by solving a set of partial differential equations. The equations are those corresponding to the first r_h rows (i.e. the subalgebra elements) of the algebra commutator table. The method of derivation shows that the number of subalgebra scalars is r_g-R'. The arguments parallel those leading to (1). Here R' is the rank of the first r_h rows of the algebra commutator table. Subtracting the number of algebra and/or subalgebra invariants the number of missing label operators is

$$m = r_g - R' - \ell_g - \ell_h + \ell' \tag{4}$$

In order to prove the conjecture, there remains to show that m = 2n or that

$$\ell' = r_h - R' \tag{5}$$

But (5) follows immediately from the definition of ℓ'. Algebra invariants, which are only functions of the elements of H, are found by solving r_g partial differential equations corresponding to the first r_h columns of the commutator table. Because of anti-symmetry of the table, the first r_h columns have the same rank R' as do the first r_h rows, and so the number of solutions is (5).

Next, it will be shown that for semi-simple Lie algebras $R' = r_h$ (and hence $\ell'=0$). The trivial possibility that $G = G' \otimes G''$ and $H = H' \otimes G''$ is excluded, for then G" plays no role in the labeling problem and $\ell' = \ell_{g''}$. The basis of H are chosen in a canonical fashion, so that the first $r_h - \ell_h$ are Hermitian conjugate pairs (ladder operators) and the last ℓ_h are the weight elements (the Cartan subalgebra basis). The other generators of G may be taken to be irreducible tensor components with respect to H; the ignoring of the trivial case implies that these tensors contain components whose weight elements span all directions in H-weight space.

To show that $R' = r_h$, consider an $r_h \times r_h$ submatrix of the first r_h rows whose determinant does not vanish. The only elements of the first r_h rows which depend on the diagonal (weight) elements of H are those at the intersection of a row and column corresponding to conjugate root elements of H. Such an element (see (1) of foreword) is $\sum_i \alpha_j^i H_i$ where the H_i are the diagonal elements (Cartan basis) and the α_j^i are the weight components of the root in question (the jth). Ignoring the other elements of the first $r_h - \ell_h$ rows and columns since they cannot cancel the ones retained, the value value of this sub-determinant is $\Pi_j^+(\Sigma_i \alpha_j^i H_i) \neq 0$. To complete the proof, ℓ_h more columns whose intersection with rows $r_h - \ell_h$ 1 to r_h have non vanishing determinant are chosen; this is easily done by choosing ℓ_h columns corresponding to tensor components with $\mathring{\ell}_h$ independent H weights.

3. Examples.

As examples involving non semi-simple Lie algebras, two cases are considered; one of dimension three and the other of dimension four. They are taken from Mubarakzyanov's[11] complete list of real Lie algebras of dimension up to five. In reference 7, these algebras are designated $A_{3,1}$ and $A_{4,7}$.

(a) $A_{3,1}$. This is the algebra of the quantum mechanical group in one dimension (the Heisenberg algebra). The only non-zero commutator is $[c_2,c_3] = c_1$. There is one invariant c_1. Hence $r_g = 3$, $\ell_g = 1$. Consider the one dimensional subalgebra c_1. Then $r_h = \ell_h = 1$. According to (3) there is only one missing label, but there are two missing label operators which can be taken as c_2 and c_3.

(b) $A_{4,7}$. The nonzero commutators are $[c_2,c_3] = c_1$, $[c_1,c_4] = 2c_1$ $[c_2,c_4] = c_2$ and $[c_3,c_4] = c_2 + c_3$. There are no invariant operators so that $r_g = 4$, and $\ell_g = 0$. For each of the one dimensional subalgebras c_1, c_2, c_3, and c_4; $r_h = \ell_h = 1$ and $\ell' = 0$. There is one missing label. In each case there are two available missing label operators, as follows:

c_1: c_2 and c_3; c_2: c_1 and $c_1c_4 - c_2c_3$;

c_3: c_1 and $c_2^2 + 2c_1c_4 + 2c_2c_3$; c_4: c_1 and $c_2 \exp[-c_3/c_2]$.

For each of the two dimensional subalgebras c_1, c_4 and c_2, c_4; $r_h = 2$ $\ell_4 = \ell' = 0$ and so there is one missing label. For c_1, c_4 the two missing label operators are c_2^2/c_1 and $c_2 \exp[-c_3/c_2]$; while for c_2, c_4 they are c_2^2/c_1 and $c_2 \times \exp[(c_1c_4 - c_2c_3)/c_2^2]$.

4. Discussion.

The examples of section 3 show that the method of partial differential equations is a practical way of determining the missing label operators for Lie algebras. This method could be used systematically to determine such operators for the algebra–subalgebra combinations of interest in physics. The theorem of section 2 shows that if there are n missing labels, then there are 2n functionally independent missing label operators; in each case for $n > 1$, there remains the problem of choosing a set of n mutually commuting functions of the 2n operators.

In many practical problems, the number of missing labels and missing label operators, is reduced by imposing restrictions on the algebra irreps being considered. Thus $O(5) \supset O(3)$ has two missing labels and four missing label operators for the general irrep of O(5). But if attention is confined to the irreps of the form $(0,\lambda)$ (such irreps are required in connection with nuclear quadrupole vibrations[12] and the Jahn-Teller effect[13]), there is only one missing label.

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CHAPTER III

THE SUBALGEBRAS OF THE SIMILITUDE ALGEBRA AND THEIR INVARIANTS.

1. Introduction

A general method for obtaining the subalgebras of a given Lie algebra was given in reference 1. The method consists of an iterative procedure for reducing the problem of finding the subalgebras of L with non-trivial ideal N, to that of finding the subalgebras of the ideal N and those of the factor algebra. If the algebra L is simple, then matrix realizations of the algebra are used to obtain its subalgebras. In a later paper[2], this method was used to obtain all the subalgebras of the similitude algebra (the semi-direct product of the Poincaré algebra and the dilatation operator; see foreword).

The subalgebras were classified into conjugacy classes under the connected component of the similitude group, and a representative algebra for each class was also listed. Two algebras L and L' are conjugate under a group G if $\exists g \in G$ such that $gLg^{-1} = L'$. The virtue of such a classification is clarified by considering the physical interpretation of such a class.

Identifying the elements of the subalgebras with the infinitesimal transformations on space-time (i.e. rotations, boosts, translations and etc.), the statement that two algebras are conjugate is equivalent to the statement that the two algebras describe the same set of transformations (or observables) as viewed from different coordinate systems. While, unconjugate algebras describe physically distinct operations; for example, the algebra of rotations is physically distinct from the algebra of translations.

Below, the algebras are reclassified into isomorphism classes. Each isomorphism class corresponds to an orbit of GL(n,R) acting on the subalgebras of dimension n. In terms of the structure constants, the statement can be rewritten in a more explicit form. Two algebras L and L' of dimension n and structure constants c_{ij}^{k} and c'^{k}_{ij} respectively, are isomorphic if $\exists g \in GL(n,R)$ such that

$$g_{i\ell} c^{n}_{\ell m} g_{mj} g^{-1}_{nk} = c'^{k}_{ij}$$

Since conjugate algebras have the same structure constants, they are trivially isomorphic, and so only the conjugate classes of subalgebras are classified into isomorphism classes.

The main interest in this type of classification rests on the fact that through isomorphism all algebraic properties obtained for one algebra are immediately transferable to all algebras within the same class. Further, a knowledge of a suitable choice of structure constant within the orbit may greatly reduce the difficulties and complications arising in the actual calculation of a particular result.

For each representative algebra, a basis for the set of invariants is obtained. Here, by invariant is understood a function of the elements of the algebra such that the function commutes with all the elements of the algebra. The function is assumed to be at least first differentiable in all variables, so that any invariant with a finite range of eigenvalues is ignored, like the sign of the energy in the Poincaré algebra.

This chapter constitutes a sequel to a paper by Patera et al [3] in which the conjugacy classes of the subalgebras of the Poincaré algebra were classified into isomorphism classes, and the respective invariants, for each representative of a conjugacy class, were calculated. Since some of the conjugacy classes of the Poincaré algebra are also conjugacy classes of the similitude algebra, parts of their results are contained in this article, providing an independent check for these results.

In section 2, the general method employed in classifying the conjugacy classes of subalgebras into isomorphism classes, and in calculating the invariants is discussed. The calculation of the invariants and the role which these might play in labeling the irreps of an arbitrary Lie algebra has been the subject of considerable investigation recently[3,4,5,6]. From Schur's lemma, it follows, that within the irreps, the invariants are of the form λI where $\lambda \in \mathbb{C}$ and I is the identity operator for the representation; showing that λ is distinct within the different irreps would lead to applications analogous to those of the Casimir operators for these (generalized) invariants. Two special types of invariants, the Casimir operators, and the rational invariants were systematically treated by Abellanos et al[4]. Rational invariants are elements of the quotient field of the enveloping algebra. In other words, they are ratios of homogeneous polynomials on the algebra. Considerations on the more general invariants along with additional references can be found in the above mentioned paper by Patera et al[3]; also a short treatment will appear in the conclusion. For a discussion of operator calculus see V.P. Maslov "Operator Methods" Nauka, Moscow 1973 (in Russian).

Section 3 containing the main results, consists of a list of representative algebras for the conjugacy classes of the similitude algebra and their respective invariants. The algebras are organized first by dimension and then for each dimension, they are grouped into isomorphism classes.

Each isomorphism class is designated by a notation which refers to a standard basis that characterizes that class, whenever such a basis exists. This notation is consistent with the one used by Patera[5], who listed the standard basis of the Lie algebra characterizing each isomorphism class. Comments on the classes and the invariants also appear in section 3. Section 4 is reserved for the conclusion.

2. Methods

All the conjugacy classes of the subalgebras of the similitude algebra, listed in reference 2, are reorganized first by dimension and then for each class the invariants are determined. A knowledge of the invariants, often, aids the mapping of conjugacy classes into isomorphism classes.

An algorithm for obtaining the invariants is achieved by reducing the problem to that of solving a system of linear first order partial differential equations. The method has been discussed in the literature[4,5,6].

In one paper[5], for each of the isomorphism classes of dimension $n \leq 5$, the method was used to obtain the basis for the set of its invariants. (However, here, for each representative algebra the invariants were independently determined). The method consists in identifying the adjoint representation of a Lie algebra with a set of c-number first order linear differential operators. That is, let L be an n dimensional Lie algebra, then

$$\forall X_i \ , \ X_j \in L \ , \ \text{let } \mathrm{ad}(X_i)X_j \equiv [X_i, X_j] = \sum_k c_{ij}^k X_k$$

and replace

$$\mathrm{ad}\ X_i \rightarrow \sum_{j,k} x_k c_{ij}^k \left(\frac{\partial}{\partial x_j} \right)$$

where x_i are c-numbers. The equation for the invariant is

$$[X_i\ ,\ F(X_1,X_2,\ldots,X_n)] = 0 \qquad \forall X_i \in L \tag{1}$$

and reduces to

$$\sum_{k,j}^{n} x_k c^k_{ij} \frac{\partial}{\partial x_j} F(x_1,x_2,\ldots,x_n) = 0 \qquad i=1,2,3,\ldots,n$$

The invariant thus formed must then be converted from a function of c-number variables to an operator function. Simple identification of $X_i \leftrightarrow x_i$ will be sufficient, in the sense that the resulting function will indeed be an operator invariant i.e. satisfy (1), only if all the X_i appearing in the function mutually commute. In the case where the function is a homogeneous polynomial (a Casimir operator) it is well known that the invariant operator must be fully symmetrized in all its variables. The reason being, that if u an v are polynomials in the x_i and

$$\sum_{k,j}^{n} x_k c^k_{ij} \frac{\partial u}{\partial x_j} = v \quad \text{then} \quad [X_i,U] = V \tag{2}$$

provided U and V are the polynomials with the same coefficients as u and v and are fully symmetrized in the corresponding $X_i \in L$.

All the invariants found for these subalgebras are of the form

$$\exp\left[\frac{U_{m+1}}{U_m}\right] \prod_{i=1}^{m} U_i^{\alpha_i}$$

where the U_i's are relatively prime homogeneous polynomials in the x_i. The exponents as well as the coefficients may be complex numbers. Then replacing x_i by X_i in each polynomial and symmetrizing each individual polynomial will lead to an operator invariant. The proof essentially follows from (2), as for Casimir operators, and is based on the assumption that $[[X_i,U_j], U_j] = 0 \; \forall X_i \in L$, for all polynomials U_j featured in the invariant. This assumption was indeed satisfied for all invariants found here and elsewhere[3,5]. It can be shown that for semi-simple and nilpotent algebras, the Casimir operators are sufficient to define a basis for the set of all invariants. Further, the number of independent invariants for an algebra of dimension n has the same parity as n.

As was mentioned earlier, in order to classify the conjugacy classes into isomorphism classes, the representative subalgebras, for each conjugacy class, are first organized by dimension. Then for each dimension they are further reorganized according to the dimension of the derived algebra; this last step may be repeated as many times as is required.

All isomorphism classes for real Lie algebras of dimension $n \le 5$ have been listed by Mubarakzyanov[7] and later reproduced in a more accessible paper[5]. For algebras of dimension ≤ 5, the classification is reduced to identifying each representative subalgebra with an abstract representative of the isomorphism classes listed in reference 5. The classes are denoted by $A_{n,m}^{\alpha_1\ldots\alpha_i}$, where n identifies the dimension of the subalgebra, m is used to index the different isomorphism classes for the same dimension, and the parameters $\alpha_1\ldots\alpha_n$ are used to select a particular class out of an infinite set of isomorphism classes. The structure constants are functions of the parameters $\alpha_1\ldots\alpha_i$. For example, $A_{3,5}^h$ with standard basis c_1,c_2,c_3 such that $[c_1,c_2] = c_1$, $[c_2,c_3] = hc_2$ with $-1 \le h \le 1$ is such an infinite set of isomorphism classes.

Since no complete classification of isomorphism classes for real Lie algebras of dim ≥ 6 exists[8], a slightly modified approach is used. The algebras are first re-ordered as before, and then for each set consisting of say an algebra of dimension n with a derived algebra of dimension ℓ and second derived algebra of dimension ℓ' and etc., the invariants are used to group the algebra into possible isomorphism classes. At this point, it often is evident which algebras are not isomorphic. Then from the so chosen candidates for a specific class (the number of candidates is not large), their largest abelian ideals are identified and by comparing the respective factor algebras, the algebras are classified into isomorphism classes. As a further test, the explicit transformation, which transforms the structure constants of one algebra into those of the other algebras within the class, is constructed. For the similitude subalgebras, isomorphism classes of dimension ≥ 6 are found to contain at most two sets of conjugacy classes.

3. Isomorphism Classes and Invariants.

The usual basis for the similitude algebra is used, with the L_i's representing the rotation generators, the K_i's representing the boosts generators, D representing the dilatation generator, and the P_μ's represent the space-time translations. Then the commutation relations are

$$[L_i,L_j] = \sum_{k}^{3}\epsilon_{ijk}L_k \qquad [K_i,K_j] = -\sum_{k}^{3}\epsilon_{ijk}L_k$$

$$[L_i,P_j] = \sum_{k}^{3}\epsilon_{ijk}P_k \qquad [L_i,K_j] = \sum_{k}^{3}\epsilon_{ijk}K_k$$

$$[K_i,P_j] = \delta_{ij}P_0 \qquad [K_i,P_0] = P_i$$

$$[K_i,P_j] = \delta_{ij}P_0 \qquad [K_i,P_0] = P_i$$

$$[D,P_u] = 2P_u \qquad [L_i,P_0] = 0$$

$$[D,L_i] = 0 \qquad [D,K_i] = 0$$

$$i,j,k, = 1,2,3$$

The invariants of the Poincaré algebra $S_{1,1}$ are well known

$$m^2 = P_0^2 - P_1^2 - P_2^2 - P_3^2$$

$$W^2 = W_\mu W^\mu$$

where $W_\mu = \epsilon_{\mu\nu\delta\lambda}M_{\nu\delta}P_\lambda$ is the Pauli-Lubanski spin operator with $M_{0i} = K_i$ and $M_{ij} = \epsilon_{ijk}L_k$, while the invariant of the similitude algebra $S_{1,3}$ consists of the ratio of these two operators which reduces to the spin operator in the frame with the eigenvalues of $P_\mu = 0$(the rest frame).

The results of the classification and the basis for the invariants are summarized in tables 1-9. The notation $S_{i,j}$ refers to the conjugacy classes as found in reference 2 . When listing the elements of a subalgebra, the semi-colon is used to indicate that all elements to its right belong to the derived algebra.

All one dimensional subalgebras are isomorphic and appear in table 1. The n dimensional subalgebras appear in table n, except for those of dimension 10 and 11 which are listed in table 9.

If an infinite set of isomorphism classes contains an infinite set of conjugacy classes, then the parameters of the set of isomorphism classes are functions of the parameters of the set of conjugacy classes. The functions need not be one to one, and such a case arises in classifying the set $S^c_{5,4}$ for each c, the corresponding class of $S^c_{5,4}$ is associated with an element of $A^P_{3,7}$ such that P = tan c. The range of c is $0 < c < \Pi$, $c \neq \Pi/2$ which implies that $-\infty < \tan c < \infty$, and $\tan c \neq 0$; however, the range of the parameter P is $P > 0$. This suggests that classes with $|\tan c|$ equal are isomorphic. This can be verified by a slight rearrangement of the basis of the representative algebra of $S^c_{5,4}$. In this case, as for all those of dimension ≤ 5, the range of the parameters for the set of isomorphism classes has been defined[7]. For the higher dimensional case, even though the infinite sets of isomorphism classes have not been constructed yet, the problem can be treated analogously. For example, consider $S^{a,b}_{10,19}$ with $a^2 + b^2 \neq 0$ from table 7, since the invariant has the exponent 1+b, it follows that algebras with distinct b's are not isomorphic. However, $S^{a,b}_{10,19}$ is isomorphic to $S^{-a,b}_{10,19}$; this can be seen by interchanging P_2 with P_1 and $L_2 + K_1$ with $L_1 - K_2$ in the ordered basis of $S^{a,b}_{10,19}$. All other cases can be approached in a similar manner.

4. Conclusion.

As was mentioned earlier, the greatest utility of such a classification of subalgebras lies in its capacity to remove redundancies in computations. The availability of a suitable choice of structure constants for a particular class can reduce the complexities in a particular calculation.

It is expected that the invariants obtained here will provide a useful tool in the representation theory of these algebras; much like the Casimir operators do, in the case of semi-simple Lie algebras. The extension of special function theory, via group theory, in order to include these invariants should produce new and useful results.

On the physical aspect, the eight dimensional algebra $S_{2,1}$ is importantly contained in the "infinite momentum frame"[9,10] calculations, in Dirac's "front frame" dynamics[11], and in the investigations of "Galileian sub-dynamics"[12,13]. One of the invariants of this algebra is

$$L_3 - \frac{P_2}{P_0 - P_3}(L_2 + K_1) - \frac{P_1}{P_0 - P_3}(L_1 - K_2). \tag{3}$$

and because of its somewhat extensive use in physics, it has already been named the light-like helicity or the null-plane helicity[14,15]. The name is appropriate, since for zero mass particles with discrete spin, $L_2 + K_1$ and $L_1 - K_2$ are both zero and so (3) reduces to L_3. The similitude algebra is another example of an algebra with non-polynomial invariant which has reached prominence in physical applications[16].

Further investigations along these lines should reveal the relevent and operational properties of these more general invariants, despite their present rather precarious mathematical status.

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Table 1: One dimensional subalgebras.

Class	Notation	Generators	Range of Parameters	Invariants
A_1	$S_{11,6}$	$L_3 - \tan c\, K_3$;	$0 < c < \pi,\ c \neq \pi/2$	generator
	$S_{12,10}$	L_3;		"
	$S_{12,20}$	$4L_3 + P_0 + P_3$;		"
	$S_{12,21}$	$2L_3 + P_0$;		"
	$S_{12,22}$	$2L_3 - P_3$;		"
	$S_{13,9}$	K_3;		"
	$S_{13,15}$	$2K_3 - P_2$;		"
	$S_{14,9}$	$L_2 + K_1$;		"
	$S_{14,20}$	$2L_2 + 2K_1 - P_0 - P_3$;		"
	$S_{14,21}$	$L_2 + K_1 - P_2$;		"
	$S_{15,8}$	$P_0 - P_3$;		"
	$S_{15,9}$	P_0;		"
	$S_{15,10}$	P_3;		"
	$S_{15,22}$	D;		"
	$S_{15,41}$	$D + 2a\cos c\, L_3 - 2a \sin c\, K_3$;	$0 \leq c < \pi,\ a > 0$	"
	$S_{15,42}$	$D - L_2 - K_1$;		"

Table 2: Two dimensional subalgebras.

Class	Notation	Generators	Range of Parameters	Invariants
$2A_1$	$S_{9,6}$	$L_3, K_3;$		both generators
	$S_{10,5}$	$L_2+K_1,\ L_1-K_2,$		"
	$S_{10,13}$	$L_1-K_2+P_2,\ L_2+K_1;$		"
	$S_{11,12}$	$D,\ \cos c\, L_3-\sin c\, K_3;$	$0<c<\pi,\ c\neq\pi/2$	"
	$S_{11,18}$	$D+2aL_3,\ \cos c\, L_3-\sin c\, K_3;$	" $a>0$	"
	$S_{12,7}$	$L_3,\ P_0-P_3;$		"
	$S_{12,8}$	$L_3,\ P_3\ ;$		"
	$S_{12,9}$	$L_3,\ P_0;$		"
	$S_{12,17}$	$4L_3+P_0+P_3,\ P_0-P_3;$		"
	$S_{12,18}$	$2L_3+P_0,\ P_3;$		"
	$S_{12,19}$	$2L_3-P_3,\ P_0;$		"
	$S_{12,32}$	$D,\ L_3;$		"
	$S_{12,41}$	$D-2aK_3,\ L_3;$	$a>0$	"
	$S_{12,42}$	$2D-4K_3+P_0+P_3,\ L_3;$		"
	$S_{12,46}$	$2D-4K_3+x(P_0+P_3),\ 4L_3+(P_0+P_3);$	$-\infty<x<\infty$	"
	$S_{13,8}$	$K_3,\ P_2;$		"
	$S_{13,11}$	$2K_3-P_2,\ P_1;$		"

Class	Notation	Generators	Range of parameters	Invariants
$2A_1$	$S_{13,24}$	$D, K_3;$		both generators
	$S_{12,30}$	$D+2aL_3, K_3;$	$a>0$	"
	$S_{14,7}$	$L_2+K_1, P_0-P_3;$		"
	$S_{14,8}$	$L_2+K_1, P_2;$		"
	$S_{14,17}$	$2L_2+2K_1-P_0-P_3, P_0-P_3;$		"
	$S_{14,18}$	$L_2+K_1-P_2, P_0-P_3;$		"
	$S_{14,19}$	$2L_2+2K_1-P_0-P_3, P_2;$		"
	$S_{14,30}$	$D, L_2+K_1;$		"
	$S_{14,51}$	$D+L_1-K_2, L_2+K_1;$		"
	$S_{15,5}$	$P_0-P_3, P_2;$		"
	$S_{15,6}$	$P_0, P_3;$		"
	$S_{15,7}$	$P_1, P_2;$		"
	$S_{15,32}$	$D+2a(\cos c\, L_3-\sin c\, K_3), P_0-P_3$	$a\neq 1, c\neq 3\pi/2$	"
A_2	$S_{8,9}$	$K_3; L_2+K_1$		none
	$S_{8,17}$	$2K_3-P_2; L_2+K_1$		"
	$S_{11,5}$	$\cos c\, L_3-\sin c\, K_3; P_0-P_3$	$0<c<\pi, c\neq\pi/2$	"
	$S_{13,7}$	$K_3; P_0-P_3$		"

Class	Notation	Generators	Range of Parameters	Invariants
A_2	$S_{13,13}$	$2K_3 - P_2$; $P_0 - P_3$		none
	$S_{14,49}$	$D - 2aK_3$; $L_2 + K_1$	$a \neq 0$	"
	$S_{14,50}$	$2D + 4K_3 + P_0 - P_3$; $L_2 + K_1$		"
	$S_{14,62}$	$D - K_3$; $2L_2 + 2K_1 - P_0 - P_3$		"
	$S_{14,63}$	$D - 2K_3 + b(L_1 - K_2 + P_1)$; $L_2 + K_1 - P_2$	$b \neq 0$	"
	$S_{15,19}$	D; $P_0 - P_3$		"
	$S_{15,20}$	D; P_0		"
	$S_{15,21}$	D; P_3		"
	$S_{15,35}$	$D + 2a(\cos c\, L_3 - \sin c\, K_3)$; $P_0 - P_3$	$a > 0$, $a \neq 1$ $0 \leq c < 2\pi$, $c \neq 3\pi/2$	"
	$S_{15,36}$	$D - L_2 - K_1$; $P_0 - P_3$		"
	$S_{15,37}$	$D + 2aL_3$; P_0	$a > 0$	"
	$S_{15,38}$	$D + 2aL_3$; P_3	$a > 0$	"
	$S_{15,39}$	$D - 2aK_1$; P_3	$a > 0$	"
	$S_{15,40}$	$D + 2L_3 + 2K_1$; P_3		"

Table 3: Three dimensional subalgebras.

Class	Notation	Generators	Range of Parameters	Invariants
$3A_1$	$S_{9,12}$	$D, L_3, K_3;$		all generators
	$S_{10,4}$	$L_2+K_1, L_1-K_2, P_0-P_3;$		"
	$S_{10,10}$	$L_2+K_1, L_1-K_2+P_2, P_0-P_3;$		"
	$S_{10,14}$	$D, L_2+K_1, L_1-K_2;$		"
	$S_{12,5}$	$L_3, P_0, P_3;$		"
	$S_{12,39}$	$D-2aK_3, L_3, P_0-P_3;$	$a=-1$	"
	$S_{13,6}$	$K_3, P_1, P_2;$		"
	$S_{14,4}$	$L_2+K_1, P_0-P_3, P_2;$		"
	$S_{14,12}$	$2L_2+2K_1-P_0-P_3, P_0-P_3, P_2;$		"
	$S_{15,2}$	$P_0-P_3, P_2, P_1,$		"
	$S_{15,3}$	$P_1, P_2, P_3;$		"
	$S_{15,4}$	$P_0, P_1, P_2;$		"
$A_1 \oplus A_2$	$S_{8,8}$	$(P_2)\oplus(K_3; L_2+K_1)$		P_2
	$S_{8,26}$	$(D)\oplus(K_3; L_2+K_1)$		D
	$S_{9,5}$	$(L_3)\oplus(K_3; P_0-P_3)$		L_3
	$S_{11,11}$	$(D-2\cot c\, L_3+2K_3)\oplus(\cot c\, L_3-K_3; P_0-P_3)$	$0<c<\pi,\ c\neq\pi/2$	$D-2\cot c\, L_3+2K_3$

Class	Notation	Generators	Range of Parameters	Invariants
$A_1 \oplus A_2$	$S_{11,17}$	$(D + 2(a - \cot c)L_3 + 2K_3) \oplus (\cot c\, L_3 - K_3,\ P_0 - P_3)$	$a \geq 0$, $0 < c < \pi$, $c \neq \pi/2$	$D - 2(a - \cot c)L_3 + 2K_3$
	$S_{12,29}$	$(L_3) \oplus (D;\ P_0 - P_3)$		L_3
	$S_{12,30}$	$(L_3) \oplus (D,\ P_3)$		L_3
	$S_{12,31}$	$(L_3) \oplus (D;\ P_0)$		L_3
	$S_{12,39}$	$(L_3) \oplus (D - 2aK_3;\ P_0 - P_3)$	$a \neq 0, -1$	L_3
	$S_{12,40}$	$(L_3) \oplus (2D - 4K_3 + P_0 + P_3,\ P_0 - P_3)$		L_3
	$S_{12,45}$	$(4L_3 + P_0 + P_3) \oplus (2D - 4K_3 + x[P_0 + P_3];\ P_0 - P_3)$	$-\infty < x < \infty$	$4L_3 + P_0 + P_3$
	$S_{13,5}$	$(P_2) \oplus (K_3,\ P_0 - P_3)$		P_2
	$S_{13,11}$	$(P_1) \oplus (2K_3 - P_2;\ P_0 - P_3)$		P_1
	$S_{13,22}$	$(D + 2K_3) \oplus (D - 2K_3;\ P_0 - P_3)$		$D + 2K_3$
	$S_{13,23}$	$(K_3) \oplus (D,\ P_2)$		K_3
	$S_{13,29}$	$(K_3) \oplus (D + 2aL_3;\ P_0 - P_3)$	$a \neq 0$	K_3
	$S_{14,28}$	$(L_2 + K_1) \oplus (D;\ P_0 - P_3)$		$L_2 + K_1$
	$S_{14,29}$	$(L_2 + K_1) \oplus (D;\ P_3)$		$L_2 + K_1$
	$S_{14,45}$	$(P_0 - P_3) \oplus (D - 2aK_3;\ L_2 + K_1)$	$a = -1$	$P_0 - P_3$
	$S_{14,46}$	$(L_2 + K_1) \oplus (D + L_1 - K_2;\ P_0 - P_3)$		$L_2 + K_1$
	$S_{15,31}$	$(P_0 - P_3) \oplus (D - 2aK_3;\ P_2)$	$a = -1$	$P_0 - P_3$

Class	Notation	Generators	Range of Parameters	Invariants
$A_1 \oplus A_2$	$S_{15,33}$	$(P_0-P_3)\oplus(D+2a[\cos c L_3-\sin c K_3], P_0-P_3)$	$a=1,\ c=\pi/2$	P_0-P_3
$A_{3,1}$	$S_{10,11}$	$L_2+K_1-P_2,\ \cancel{L_1}-K_2+bP_2-P_1;\ P_0-P_3$	$b \neq 0$	P_0-P_3
	$S_{10,12}$	$L_2+K_1-P_2,\ L_1-K_2+P_1;\ P_0-P_3$		P_0-P_3
	$S_{14,5}$	$L_2+K_1,\ P_1,\ P_0-P_3$		"
	$S_{14,6}$	$L_2+K_1,\ P_2-bP_1;\ P_0-P_3$	$b \neq 0$	P_0-P_3
	$S_{14,13}$	$2L_2+2K_1-P_0-P_3,\ P_1;\ P_0-P_3$		"
	$S_{14,14}$	$L_2+K_1-P_2,\ P_1;\ P_0-P_3$		P_0-P_3
	$S_{14,15}$	$2L_2+2K_1-P_0-P_3,\ P_2-bP_1;\ P_0-P_3$	$b = 0$	"
	$S_{14,16}$	$L_2+K_1-P_2,\ P_2-bP_1;\ P_0-P_3$	$b \neq 0$	P_0-P_3
$A_{3,2}$	$S_{8,15}$	$2K_3+P_1;\ L_2+K_1,\ P_0-P_3$		$(P_0-P_3)\exp[(L_2+K_1)/(P_3-P_0)]$
	$S_{8,16}$	$2K_3-P_2+bP_1,\ L_2+K_1,\ P_0-P_3$	$b \neq 0$	$(P_0-P_3)\exp[(L_2+K_1)/b(P_3-P_0)]$
	$S_{15,32}$	$D-\cos c(L_2+K_1)+\sin c(L_1-K_2);\ P_2,\ P_0-P_3$	$0 < c < \pi$	$(P_0-P_3)\exp[P_2/(P_3-P_0)]$

Class	Notation	Generators	Range of Parameters	Invariants
$A_{3,3}$	$S_{7,5}$	$K_3;\ L_2+K_1,\ L_1-K_2$		$(L_1-K_2)/(L_2+K_1)$
	$S_{8,7}$	$K_3;\ L_2+K_1,\ P_0-P_3$		$(L_2+K_1)/(P_0-P_3)$
	$S_{8,14}$	$2K_3-P_2;\ L_2+K_1,\ P_0-P_3$		"
	$S_{10,24}$	$D+2aL_3-2bK_3;\ L_2+K_1,\ L_1-K_2$	$a=0,\ b\neq 0$	$(L_2+K_1)/(L_1-K_2)$
	$S_{10,25}$	$D+2aL_3+2K_3+P_0-P_3;\ L_2+K_1,\ L_1-K_2$	$a=0$	"
	$S_{10,32}$	$D-2K_3;\ L_2+K_1,\ L_1-K_2+P_2$		$(L_1-K_2+P_2)/(L_2+K_1)$
	$S_{15,16}$	$D;\ P_2,\ P_0-P_3$		$(P_0-P_3)/P_2$
	$S_{15,17}$	$D;\ P_0,\ P_3$		$(P_0-P_3)/(P_0+P_3)$
	$S_{15,18}$	$D;\ P_1,\ P_2$		P_1/P_2
	$S_{15,32}$	$D-\cos c(L_2+K_1)+\sin c(L_1-K_2);\ P_2,\ P_0-P_3$	$c=0$	$(P_0-P_3)/P_2$
$A_{3,4}$	$S_{11,4}$	$L_3-\tan c\,K_3;\ P_0,\ P_3$	$0<c<\pi,\ c\neq\pi/2$	$P_0^2-P_3^2$
	$S_{13,4}$	$K_3;\ P_0,\ P_3$		"
	$S_{13,12}$	$2K_3-P_2;\ P_0,\ P_3$		$P_0^2-P_3^2$
	$S_{14,48}$	$2D+4K_3+P_0-P_3;\ L_2+K_1,\ P_2$		$P_2(L_2+K_1)$

Class	Notation	Generators	Range of Parameters	Invariants
$A_{3,5}^{h}$				
$h = a/(1+a)$	$S_{14,45}$	$D-2aK_3;\ L_2+K_1,\ P_0-P_3$	$a\neq 0,-1$	$(L_2+K_1)^{1+a}/(P_0-P_3)^{a}$
$= a$	$S_{14,47}$	$D-2aK_3;\ L_2+K_1,\ P_2$	$a\neq 0$	$(L_2+K_1)/P_2^{a}$
$= 1/3$	$S_{14,59}$	$D-K_3;\ 2L_2+2K_1-P_0-P_3;\ P_0-P_3$		$(2L_2+2K_1-P_0-P_3)^3/(P_0-P_3)$
$= 1/2$	$S_{14,60}$	$D-2K_3,\ L_2+K_1-P_2;\ P_0-P_3$		$(L_2+K_1-P_2)^2/(P_0-P_3)$
$= 1/2$	$S_{14,61}$	$D-K_3;\ 2L_2+2K_1-P_0-P_3;\ P_2$		$(2L_2+2K_1-P_0-P_3)^2/P_2$
$= 1+a$	$S_{15,31}$	$D-2aK_3;\ P_2,\ P_0-P_3$	$a\neq 0,-1$	$(P_0-P_3)^{1+a}/P_2$
$= \frac{1-a\sin c}{1+a\sin c}$	$S_{15,33}$	$D+2a\cos c\, L_3-2a\sin c\, K_3;\ P_0+P_3,\ P_0-P_3$	$a>0,\ a\neq 1$; $0\leq c<2\pi,\ c\neq 3\pi/2$	$(P_0+P_3)^{1+a\sin c}/(P_0-P_3)^{1-a\sin c}$
$A_{3,6}$	$S_{6,4}$	$L_3;\ L_2-K_1,\ L_1+K_2$		$(L_2+K_1)^2+(L_1-K_2)^2$
	$S_{6,9}$	$4L_3+P_0-P_3;\ L_2-K_1,\ L_1-K_2$		"
	$S_{10,3}$	$L_3-\tan c\, K_3;\ P_1,\ P_2$	$0<c<\pi$ $c\neq\pi/2$	$P_1^2+P_2^2$
	$S_{10,24}$	$D+2aL_3-2bK_3;\ L_2-K_1,\ L_1-K_2$	$b=0$ or $a\geq 0$	$(L_2+K_1)^2+(L_1-K_2)^2$
	$S_{12,6}$	$L_3;\ P_1,\ P_2$		$P_1^2+P_2^2$
	$S_{12,14}$	$4L_3+P_0+P_3;\ P_1,\ P_2$		"
	$S_{12,15}$	$2L_3+P_0;\ P_1,\ P_2$		$P_1^2+P_2^2$
	$S_{12,16}$	$2L_3-P_3;\ P_1,\ P_2$		"

Class	Notation	Generators	Range of Parameters	Invariants
$A^p_{3,7}$				
$p = \tan c$	$S_{5,4}$	$L_3 - \tan c\, K_3 ;\ L_2 + K_1,\ L_1 - K_2$	$0 < c < \pi,\ c \neq \pi/2$	$\{(L_2+K_1)^2 + (L_1-K_2)^2\} \left[\dfrac{L_1 - K_2 - i(L_2+K_1)}{L_1 - K_2 + i(L_2+K_1)}\right]^{i \tan c}$
$p = b/a$	$S_{11,24}$	$D + 2aL_3 - 2bK_3 ;\ L_2 + K_1,\ L_1 - K_2$	$a \neq 0;\ b \neq 0$	$\{(L_2+K_1)^2 + (L_1-K_2)^2\} \left[\dfrac{L_1 - K_2 - i(L_2+K_1)}{L_1 - K_2 + i(L_2+K_1)}\right]^{ib/a}$
$p = -1/a$	$S_{10,25}$	$2D + 4aL_3 - 4K_3 + P_0 - P_3 ;\ L_1 - K_2,\ L_2 + K_1$	$b = 0$	same as above with $b = -1$
$p = 1/a \csc c$	$S_{15,34}$	$D + 2a \cos c\, L_3 - 2a \sin c\, K_3 ;\ P_1,\ P_2$	$a > 0$ $0 \leq c < \pi,\ c \neq \pi/2$	$(P_1^2 + P_2^2)\{(P_1 - iP_2)/(P_1 + iP_2)\}^{i/a \csc c}$
$A_{3,8}$ $SU(1,1)$	$S_{4,4}$	$;\ L_3,\ K_1,\ K_2$		$K_2^2 + K_1^2 - L_3^2$
$A_{3,9}$ $SU(2)$	$S_{3,4}$	$;\ L_1,\ L_2,\ L_3$		$L_1^2 + L_2^2 + L_3^2$

Table 4: Four dimensional subalgebras.

Class	Notation	Generators	Range of Parameters	Invariants
$4A_1$	$S_{15,1}$	$P_0, P_1, P_2, P_3;$		all generators
$2A_1 \oplus A_2$	$S_{4,11}$	$(D+2K_3)\oplus(L_3)\oplus(D, P_0-P_3)$		$L_3,\ D+2K_3$
	$S_{19,12}$	$(L_2+K_1)\oplus(L_1-K_2)\oplus(D; P_0-P_3)$		$L_2+K_1,\ L_1-K_2$
	$S_{13,2}$	$(P_1)\oplus(P_2)\oplus(K_3, P_0-P_3)$		P_2, P_1
$2A_2$	$S_{8,24}$	$(D+2K_3;\ L_2+K_1)\oplus(D, P_0-P_3)$		none
	$S_{8,25}$	$(K_3,\ L_2+K_1)\oplus(D, P_2)$		"
	$S_{11,16}$	$(D+2\cot c\, L_3-2K_3;\ P_0-P_3)\oplus(D-2\cot c\, L_3+2K_3;\ P_0+P_3)$	$0<c<\pi,\ c\neq\pi/2$	none
	$S_{11,16}$	$(D+2(a+\cot c)L_3-2K_3;\ P_0-P_3)\oplus(D+2(a-\cot c\, L_3)+2K_3;\ P_0+P_3)$	$a>0$ $0<c<\pi,\ c\neq\pi/2$	"
	$S_{13,19}$	$(D-2K_3,\ P_0-P_3)\oplus(D+2K_3;\ P_0+P_3)$		none
	$S_{13,20}$	$(D-2K_3;\ P_2)\oplus(K_3;\ P_0-P_3)$		"
	$S_{13,27}$	$(D+2aL_3-2K_3;\ P_0-P_3)\oplus(D+2aL_3+2K_3;\ P_0+P_3)$	$a>0$	none
$A_1\oplus A_{3,1}$	$S_{10,3}$	$(L_2+K_1)\oplus(L_1-K_2, P_2;\ P_0-P_3)$		$L_2+K_1,\ P_0-P_3$
	$S_{10,9}$	$(L_2+K_1-P_2)\oplus(L_2+K_1,\ L_1-K_2-P_1;\ P_0-P_3)$		$L_2+K_1-P_2,\ P_0-P_3$

Class	Notation	Generators	Range of Parameters	Invariants
$A_1 \oplus A_{3,1}$	$S_{14,2}$	$(P_2) \oplus (L_2+K_1, P_1; P_0-P_3)$		$P_2;\ P_0-P_3$
	$S_{14,10}$	$(P_2) \oplus (2L_2+2K_1-P_0-P_3, P_1; P_0-P_3)$		"
$A_1 \oplus A_{3,2}$	$S_{8,11}$	$(P_2) \oplus (2K_3+P_1, L_2+K_1, P_0-P_3)$		$P_2;\ (P_0-P_3)\exp(2(L_2+K_1)/(P_3-P_0))$
	$S_{14,39}$	$(L_2+K_1) \oplus (D+L_1-K_2; P_2, P_0-P_3)$		$L_2+K_1;\ (P_0-P_3)\exp(2P_2/(P_0-P_3))$
$A_1 \oplus A_{3,3}$	$S_{7,12}$	$(D) \oplus (K_3; L_2+K_1, L_1-K_2)$		$D;\ (L_1-K_2)/(L_2+K_1)$
	$S_{8,4}$	$(P_2) \oplus (K_3; L_2+K_1, P_0-P_3)$		$P_2;\ (P_0-P_3)/(L_2+K_1)$
	$S_{10,23}$	$(P_0-P_3) \oplus (D+2aL_2-2bK_3; L_2+K_1, L_1-K_2)$	$a=0,\ b=-1$	$P_0-P_3;\ (L_1-K_2)/(L_2+K_1)$
	$S_{12,27}$	$(L_3) \oplus (D; P_3, P_0)$		$L_3;\ P_0/P_3$
	$S_{13,21}$	$(K_3) \oplus (D; P_1, P_2)$		$K_3;\ P_1/P_2$
	$S_{14,25}$	$(L_2+K_1) \oplus (D; P_2, P_0-P_3)$		$L_2+K_1;\ (P_0-P_3)/P_2$
	$S_{15,25}$	$(P_0-P_3) \oplus (D+2a\cos c L_3-2a\sin c K_3; P_2, P_1)$	$a=1,\ c=3\pi/2$	$P_0-P_3;\ P_1/P_2$

Class	Notation	Generators	Range of Parameters	Invariants
$A_1 \oplus A_{3,4}$	$S_{9,4}$	$(L_3) \oplus (K_3, P_0, P_3)$		L_3; $(P_0 - P_3)/(P_0 + P_3)$
	$S_{13,3}$	$(P_1) \oplus (K_3; P_0, P_3)$		P_1; "
	$S_{13,10}$	$(P_1) \oplus (2K_3 - P_2; P_0, P_3)$		P_1; $(P_0 - P_3)/(P_0 + P_3)$
	$S_{14,30}$	$(P_0 - P_3) \oplus (D - 2aK_3; L_2 + K_1, P_2)$	$a = -1$	$P_0 - P_2$, $P_2/(L_2 + K_1)$
$A_1 \oplus A^h_{3,5}$				
$h = \frac{1+a}{1-a}$	$S_{12,36}$	$(L_3) \oplus (D - 2aK_3; P_0, P_3)$	$a > 0$	L_3; $(P_0 - P_3)^{1+a}/(P_0 + P_3)^{1-a}$
$= 1/2$	$S_{15,29}$	$(P_0 + P_2) \oplus (D - 2aK_2, P_1, P_0 - P_2)$	$a = 1$	$P_0 + P_2$; $(P_0 - P_2)^2/P_1$
$A_1 \oplus A_{3,6}$	$S_{6,3}$	$(P_0 - P_3) \oplus (L_3; L_2 + K_1, L_1 - K_2)$		$P_0 - P_3$; $(L_2 + K_1)^2 + (L_1 - K_2)^2$
	$S_{6,11}$	$(D) \oplus (L_3; L_2 + K_1, L_1 - K_2)$		D; "
	$S_{9,3}$	$(K_3) \oplus (L_3; P_1, P_2)$		K_3; $P_1^2 + P_2^2$
	$S_{12,2}$	$(P_0 - P_3) \oplus (L_3, P_1, P_2)$		$P_0 - P_3$, "
	$S_{12,3}$	$(P_3) \oplus (L_3; P_2, P_1)$		P_3; $P_1^2 + P_2^2$
	$S_{12,4}$	$(P_0) \oplus (L_3; P_2\ P_1)$		P_0; "
	$S_{12,11}$	$(P_0 - P_3) \oplus (4L_3 + P_0 + P_3; P_2, P_1)$		$P_0 - P_3$; $P_1^2 + P_2^2$

Class	Notation	Generators	Range of Parameters	Invariants
$A_1 \oplus A_{3,6}$	$S_{12,12}$	$(P_3) \oplus (2L_3 + P_0;\ P_2,\ P_1)$		$P_3\ ;\ P_1^2 + P_2^2$
	$S_{12,13}$	$(P_0) \oplus (2L_3 - P_3\ ;\ P_2,\ P_1)$		P_0 "
$A_1 \oplus A_{3,7}^{p}$				
$p = \tan c$	$S_{5,8}$	$(D) \oplus (L_3 - \tan c\, K_3;\ L_2 + K_1,\ L_1 - K_2)$	$0 < c < \pi,\ c \neq \pi/2$	$D;\ \{(L_1-K_2)^2 + (L_2+K_1)^2\} \left[\dfrac{L_1 - K_2 - i(L_2+K_1)}{L_1 - K_2 + i(L_2+K_1)}\right]^{i \tan c}$
$= 1/a$	$S_{10,23}$	$(P_0 - P_3) \oplus (D + 2aL_3 - 2bK_3;\ L_2 + K_1,\ L_1 - K_2)$	$b = -1,\ a > 0$	$P_0 - P_3$, same with $\tan c \to 1/a$
$= 1/a$	$S_{13,28}$	$(K_3) \oplus (D + 2aL_3;\ P_1,\ P_2)$	$a > 0$	$K_3\ ;\ (P_1^2 + P_2^2)\{(P_1 - iP_2)/(P_1 + iP_2)\}^{i/a}$
$A_1 \oplus A_{3,8}$	$S_{4,3}$	$(P_3) \oplus (;L_3,\ K_1,\ K_2)$		$P_3\ ;\ L_3^2 - K_1^2 - K_2^2$
	$S_{4,8}$	$(D) \oplus (;L_3,\ K_1,\ K_2)$		$D\ ;\ L_3^2 - K_1^2 - K_2^2$
$A_1 \oplus A_{3,9}$	$S_{3,3}$	$(P_0) \oplus (;L_3,\ L_1,\ L_2)$		$P_0\ ;\ L_1^2 + L_2^2 + L_3^2$
	$S_{3,8}$	$(D) \oplus (;L_3,\ L_1,\ L_2)$		$D;\ L_1^2 + L_2^2 + L_3^2$

Class	Notation	Generators	Range of Parameters	Invariants
$A_{4,1}$	$S_{10,7}$	$L_1-K_2-P_1,\ 2L_2+2K_1-P_0-P_3;\ P_2,\ P_0-P_2$		$P_0-P_3;\ P_2^2+(P_0-P_3)(2L_2+2K_1-P_0-P_3+2P_2)$
	$S_{10,8}$	$L_1-K_2,\ 2L_2+2K_1-P_0-P_3;\ P_2,\ P_0-P_3$		$P_0-P_3;\ P_2^2+(P_0-P_3)(2L_2+2K_1-P_0-P_3)$
	$S_{14,3}$	$L_2+K_1,\ P_0+P_3;\ P_1,\ P_0-P_3$		$P_0-P_3;\ P_1^2+P_3^2-P_0^2$
	$S_{14,11}$	$L_2+K_1-P_2,\ P_0+P_3;\ P_1,\ P_0-P_3$		" ; "
$A_{4,2}^{\alpha}$				
$\alpha=1$	$S_{9,7}$	$2K_3-P_2;\ L_2+K_1;\ L_1-K_2,\ P_0-P_3$		$(L_1-K_2)/(P_0-P_3);\ (P_0-P_3)\exp((2L_2+2K_1)/(P_0-P_3))$
$=1$	$S_{15,26}$	$D-L_2-K_1;\ P_1,\ P_0-P_3,\ P_2$		$P_2/(P_0-P_3),\ (P_0-P_3)\exp(2P_1/(P_0-P_3))$
$A_{4,4}$	$S_{15,30}$	$D+2L_3-2K_1;\ P_0,\ P_1,\ P_2$		$(P_0-P_2)\exp(P_1/(P_0-P_2));\ (P_0^2-P_2^2-P_1^2)/(P_0-P_2)^2$
$A_{4,5}^{\beta,\gamma}$				
$\beta=1\ \ \gamma=1$	$S_{9,4}$	$K_3;\ L_2+K_1,\ L_1-K_2,\ P_0-P_3$		$(P_0-P_3)/(L_1-K_2);\ (P_0-P_3)/(L_2+K_1)$
$=\frac{b}{1+b}\ =\frac{b}{1+b}$	$S_{10,23}$	$D+2aL_3-2bK_3;\ L_2+K_1,\ L_1-K_2,\ P_0-P_3$	$a=0,\ b\neq 0,-1$	$(P_0-P_3)^b/(L_1-K_2)^{1+b};\ (P_0-P_3)^b/(L_2+K_1)^{1+b}$
$=\frac{1}{2}\ =\frac{1}{2}$	$S_{10,29}$	$D-2K_3;\ L_2+K_1,\ L_1-K_2,\ P_0-P_3$		$(P_0-P_3)/(L_1-K_2)^2;\ (P_0-P_3)/(L_2+K_1)^2$

Class	Notation	Generators	Range of Parameters	Invariants
$A_{4,5}^{\beta,\gamma}$				
$\beta=\frac{1}{1+a}$ $\gamma=\frac{a}{1+a}$	$S_{14,38}$	$D-2aK_3;\ L_2+K_1,\ P_2,\ P_0-P_3$	$a\neq 0,-1$	$(P_0-P_3)/P_2^{1+a}$; $(P_0-P_3)^a/(L_2+K_1)^{1+a}$
$=2/3$ $=1/3$	$S_{14,54}$	$D-K_3;\ 2L_2+2K_1-P_0-P_3,\ P_2,\ P_0-P_3$		$(P_0-P_3)^2/P_2^3$, $(P_0-P_3)/(2L_2+2K_1-P_0-P_3)^3$
$=1$ $=1$	$S_{15,13}$	$D;\ P_1,\ P_2,\ P_3-P_0$		$(P_0-P_3)/P_2$; $(P_0-P_3)/P_1$
$=1$ $=1$	$S_{15,14}$	$D;\ P_1,\ P_2,\ P_3$		P_3/P_2; P_3/P_1
$=1$, $=1$	$S_{15,15}$	$D;\ P_1,\ P_2,\ P_0$		P_0/P_2; P_0/P_1
$=\frac{1}{1+a}$ $=\frac{1}{1+a}$	$S_{15,25}$	$D-2a\cos c\,L_3+2a\sin c\,K_3;\ P_1,\ P_2,\ P_0-P_3$	$c=\pi/2$ $a>0$	$(P_0-P_3)/P_2^{1+a}$; $(P_0-P_3)/P_1^{1+a}$
$=\frac{1}{1-a}$ $=\frac{1}{1-a}$	$S_{15,25}$	$D-2a\cos c\,L_3+2a\sin c\,K_3;\ P_1,\ P_2,\ P_0-P_3$	$c=3\pi/2$ $a>0$ $a\neq 1$	$(P_0-P_3)/P_2^{1-a}$; $(P_0-P_3)/P_1^{1-a}$
$=\frac{1+a}{1-a}$ $=\frac{1}{1-a}$	$S_{15,21}$	$D-2aK_2;\ P_1,\ P_2,\ P_0$	$a>0$	$(P_0+P_2)^{1+a}/(P_0-P_2)^{1-a}$; $(P_0+P_2)/P_1^{1-a}$
$A_{4,6}^{\alpha,\rho}$				
α, $\rho=\tan c$	$S_{5,3}$	$L_3-\tan c\,K_3;\ L_2+K_1,\ L_1-K_2,\ P_0-P_3$	$0<c<\pi,\ c\neq\pi/2$	$(P_0-P_3)^2/\{(L_2+K_1)^2+(L_1-K_2)^2\}$; $\{(L_2+K_1)^2+(L_1-K_2)^2\}\left[\frac{L_1-K_2-i(L_2+K_1)}{L_1-K_2+i(L_2+K_1)}\right]^{i\tan c}$
$\alpha=\frac{1+b}{a}$ $\rho=\frac{b}{a}$	$S_{10,23}$	$D+2aL_3-2bK_3;\ L_2+K_1,\ L_1-K_2,\ P_0-P_3$	$a\neq 0$, $b\neq -1$	$\frac{(P_0-P_3)^{2b/(1+b)}}{(L_1-K_2)^2+(L_2+K_1)^2}$; $\{(L_1-K_2)^2+(L_2+K_1)^2\}\left[\frac{L_1-K_2-i(L_2+K_1)}{L_1-K_2+i(L_2+K_1)}\right]^{\frac{ib}{a}}$

Class	Notation	Generators	Range of Parameters	Invariants
$A_{4,6}^{\alpha,\beta}$				
$\alpha = \tan c \quad \beta = 0$	$S_{11,2}$	$L_3 - \tan c\, K_3;\ P_1,\ P_2,\ P_0 - P_3$	$0 < c < \pi,\ c \neq \pi/2$	$P_1^2 + P_2^2,\ (P_1^2 + P_2^2)\left[(P_1 - iP_2)/(P_1 + iP_2)\right]^{i \tan c}$
$\alpha = 1/a\cos c$ $\beta = \frac{1 + a\sin c}{a\cos c}$	$S_{15,25}$	$D + 2a\cos c\, L_3 - 2b\sin c\, K_3;\ P_1,\ P_2,\ P_0 - P_3$	$a > 0 \quad c \neq \pi/2,\ 3\pi/2$ $0 \leq c < 2\pi$	$\frac{(P_0 - P_3)^2}{(P_2^2 + P_1^2)^{1 + a\sin c}},\ (P_1^2 + P_2^2)\left[\frac{P_1 - iP_2}{P_1 + iP_2}\right]^{i/a\cos c}$
$\alpha = 1/a \quad \beta = 1/a$	$S_{15,27}$	$D + 2aL_3;\ P_1,\ P_2,\ P_3$	$a > 0$	$P_3^2/(P_1^2 + P_2^2);\ (P_1^2 + P_2^2)\left[(P_1 - iP_2)/(P_1 + iP_2)\right]^{i/a}$
$\alpha = 1/a \quad \beta = 1/a$	$S_{15,28}$	$D + 2aL_3;\ P_1,\ P_2,\ P_0$	$a > 0$	$P_0^2/(P_1^2 + P_2^2);$ "
$A_{4,7}$	$S_{14,41}$	$2D - 4K_3 + P_0 + P_3;\ L_2 + K_1,\ P_1,\ P_0 - P_3$		none
	$S_{14,36}$	$2D - 4K_3 + x(P_0 - P_3);\ L_2 + K_1 - P_2,\ P_1,\ P_0 - P_3$	$-\infty < x < \infty,\ x \neq 0$	"
$A_{4,8}$	$S_{14,40}$	$D - 2aK_3;\ L_2 + K_1,\ P_1,\ P_0 - P_3$	$a = -1$	$(D + 2K_3)(P_3 - P_0) + 2P_1(L_2 + K_1);\ P_0 - P_3$
	$S_{14,43}$	$D - 2aK_3;\ L_2 + K_1,\ P_2 - cP_1,\ P_0 - P_3$	$a = -1,\ c \neq 0$	$c(D + 2K_3)(P_3 - P_0) + 2(cP_1 - P_2)(L_2 + K_1);\ P_0 - P_3$
	$S_{14,58}$	$D - 2bK_3 + a(b-1)(P_0 - P_3);\ L_2 + K_1 - P_2,\ P_2 - aP_1,\ P_0 - P_3$	$a \neq 0 \quad b = -1$	$a\left[D + 2K_3 - 2a(P_0 + P_3)\right](P_3 - P_0) + 2(aP_1 - P_2)(L_2 + K_1 - aP_1);$ $P_0 - P_3$

Class	notation	Generators	Range of Parameters	Invariants
$A_{4,9}^{h}$				
$h=0$	$S_{8,5}$	$K_3, P_1; L_2+K_1, P_0-P_3$		none
$=0$	$S_{8,6}$	$K_3, P_2-bP_1, L_2+K_1, P_0-P_3$	$b\neq 0$	"
$=0$	$S_{8,12}$	$2K_3-P_2, P_1; L_2+K_1, P_0-P_3$		none
$=0$	$S_{8,13}$	$2K_3-P_2, P_2-bP_1; L_2+K_1, P_0-P_3$	$b\neq 0$	"
$=1$	$S_{10,30}$	$D-2K_3; L_2+K_1-P_2, L_1-K_2+bP_3-P_1, P_0-P_3$	$b\neq 0$	none
$=1$	$S_{10,31}$	$D-2aL_3-2K_3; L_2+K_1-P_2, L_1-K_2-P_1, P_0-P_3$	$a=0$	"
$=0$	$S_{14,26}$	$D, L_2+K_1; P_1, P_0-P_3$		none
$=0$	$S_{14,27}$	$D, L_2+K_1; P_2-bP_1, P_0-P_3$	$b\neq 0$	"
$=a$	$S_{14,40}$	$D-2aK_3; L_2+K_1, P_1, P_0-P_3$	$a\neq 0,-1$	none
$=0$	$S_{14,42}$	$D+L_1-K_2, L_2+K_1; P_1, P_0-P_3$		"
$=a$	$S_{14,43}$	$D-2aK_3; L_2+K_1, P_2-cP_1, P_0-P_3$	$a\neq 0,-1, c\neq 0$	none
$=0$	$S_{14,44}$	$D+L_1-K_2, L_2+K_1; P_2-cP_1, P_0-P_3$	$c\neq 0$	"
$=\frac{1}{2}$	$S_{14,45}$	$D-K_3; 2L_2+2K_1-P_0-P_3, P_1, P_0-P_3$		none
$=1$	$S_{14,56}$	$2D-4K_3+x(P_0-P_3); L_2+K_1-P_2, P_1, P_0-P_3$	$x=0$	"

Class	Notation	Generators	Range of Parameters	Invariants
$A_{4,9}^{\lambda}$				
$\lambda = 1/2$	$S_{14,57}$	$2D-2K_3+2b(L_1-K_2)+ba(P_0-P_3);\ 2L_2+2K_1+P_2+P_3$	$a \neq 0$	none
		$P_2-aP_1,\ P_0-P_3$	$-\infty < b < \infty$	"
$= b$	$S_{14,58}$	$D-2bK_3+a(b-1)(P_0-P_3);\ L_2+K_1-P_2,\ P_2-aP_1,$	$a \neq 0,\ b \neq -1$	none
		P_0-P_3	$-\infty < b < \infty$	"
$A_{4,10}$	$S_{6,6}$	$L_3;\ L_2+K_1-P_2;\ L_1-K_2-P_1,\ P_0-P_3$		$P_0-P_3;\ 2(P_0-P_3)L_3-(L_1+K_2-P_2)^2-(L_1-K_2-P_1)^2$
	$S_{10,31}$	$D-2aL_3-2K_3;\ L_2+K_1-P_2,\ L_1-K_2-P_1,\ P_0-P_3$	$a = 0$	$P_0-P_3,\ 2(P_0-P_3)(D-2K_3)-(L_1+K_2-P_2)^2-(L_1-K_2-P_1)^2$
$A_{4,11}^{p}$				
$p = 1/a$	$S_{10,31}$	$D+2aL_3-2K_3;\ L_2+K_1-P_2,\ L_1-K_2-P_1,\ P_0-P_3$	$a \neq 0$	none
$A_{4,12}$	$S_{2,4}$	$L_3,\ K_3;\ L_2+K_1,\ L_1-K_2$		none
	$S_{5,12}$	$D-2aK_3,\ L_3-\tan c\, K_3;\ L_2+K_1,\ L_1-K_2$	$a \neq 0$ $0<c<\pi,\ c \neq \pi/2$	"
	$S_{6,16}$	$D-2aK_3,\ L_3;\ L_2+K_1,\ L_1-K_2$	$a \neq 0$	none
	$S_{6,17}$	$2D+4K_3+(P_0-P_3),\ L_3;\ L_2+K_1,\ L_1-K_2$		"

Class	Notation	generators	Range of Parametres	Invariants
$A_{4,12}$	$S_{6,20}$	$2D+4K_3+x(P_0-P_3),\ 4L_3+P_0-P_3;\ L_2+K_1,\ L_1-K_2$	$-\infty<x<\infty$	none
	$S_{7,17}$	$D+2aL_3,\ K_3;\ L_2+K_1,\ L_1-K_2$	$a\neq 0$	"
	$S_{11,9}$	$L_3-\tan c\,K_3,\ D;\ P_1,\ P_2$	$0<c<\pi,\ c\neq\pi/2$	none
	$S_{11,15}$	$D+2aL_3,\ L_3-\tan c\,K_3;\ P_1,\ P_2$	$a\neq 0$ $0<c<\pi,\ c\neq\pi/2$	"
	$S_{12,28}$	$D,\ L_3;\ P_1,\ P_2$		none
	$S_{12,37}$	$D-2aK_3,\ L_3;\ P_1,\ P_2$	$a>0$	"
	$S_{12,38}$	$2D-4K_3+P_0+P_3;\ L_3;\ P_1,\ P_2$		none
	$S_{12,44}$	$2D-4K_3+P_0+P_3;\ 4L_3+P_0+P_3,\ P_1,\ P_2$	$-\infty<x<\infty$	"

Table 5: Five dinensional subalgebras.

Class	Notation	Generators	Range of Parameters	Invariants
$A_1\oplus 2A_2$	$S_{9,10}$	$(L_3)\oplus(D+2K_3;\ P_0+P_3)\oplus(D-2K_3;\ P_0-P_3)$		L_3
$A_2\oplus A_{3,3}$	$S_{7,11}$	$(D;\ P_0-P_3)\oplus(D+2K_3;\ L_1-K_2,\ L_2+K_1)$		$(L_2+K_1)/(L_1-K_2)$

Class	Notation	Generators	Range of Parameters	Invariants
$2A_1 \oplus A_{3,4}$	$S_{13,1}$	$(P_1) \oplus (P_2) \oplus (K_3;\ P_0 + P_3,\ P_0 - P_3)$		$(P_0 - P_3)/(P_0 + P_3)$, P_1, P_2
$2A_1 \oplus A_{3,6}$	$S_{12,1}$	$(P_0) \oplus (P_3) \oplus (L_3;\ P_1,\ P_2)$		$P_1^2 + P_2^2$
$A_2 \oplus A_{3,6}$	$S_{6,10}$	$(D,\ P_0 - P_3) \oplus (L_3;\ L_1 - K_2,\ L_2 + K_1)$		$(L_1 - K_2)^2 + (L_2 + K_1)^2$
	$S_{9,2}$	$(K_3;\ P_0 - P_3) \oplus (L_3;\ P_1,\ P_2)$		$P_1^2 + P_2^2$
$A_2 \oplus A_{3,7}^{p}$				
$p = \tan c$	$S_{5,7}$	$(D;\ P_0 - P_3) \oplus (2L_3 - \tan c\,[2K_3 + D],\ L_2 + K_1,\ L_1 - K_2)$	$0 < c < \pi,\ c \neq \pi/2$	$\{(L_1 - K_2)^2 + (L_2 + K_1)^2\} \left[\dfrac{L_1 - K_2 - i(L_2 + K_1)}{L_1 - K_2 + i(L_2 + K_1)}\right]^{i \tan c}$
$p = 1/a$	$S_{13,26}$	$(K_3,\ P_0 - P_3) \oplus (D + 2K_3 + 2aL_3;\ P_1,\ P_2)$	$a \neq 0$	$(P_1^2 + P_2^2)\{(P_1 - iP_2)/(P_1 + iP_2)\}^{i/a}$
$A_2 \oplus A_{3,8}$	$S_{4,7}$	$(D;\ P_3) \oplus (;\ L_3,\ K_1,\ K_2)$		$L_3^2 - K_1^2 - K^2$
$A_2 \oplus A_{3,9}$	$S_{3,7}$	$(D;\ P_0) \oplus (;\ L_1,\ L_2,\ L_3)$		$L_1^2 + L_2^2 + L_3^2$

Class	Notation	Generators	Range of Parameters	Invariants
$A_1 \oplus A_{4,1}$	$S_{14,1}$	$(P_2) \oplus (L_2 + K_1 ;\ P_0 - P_3,\ P_1,\ P_0 + P_3)$		$P_0 - P_3 ;\ P_2 ;\ P_1^2 - (P_0 - P_3)(L_2 + K_1)$
$A_1 \oplus A_{4,5}$	$S_{15,23}$	$(P_0 + P_3) \oplus (D + 2a(\cos c\, L_3 - \sin c\, K_3 ;\ P_0 - P_3,\ P_1,\ P_2)$	$c = \pi/2,\ a = 1$	$P_0 + P_3,\ (P_0 - P_3)/P_2^2,\ (P_0 - P_3)/P_1^2$
$A_1 \oplus A_{4,9}^h$				
$h = 0$	$S_{8,2}$	$(P_2) \oplus (K_3,\ P_1 ;\ L_2 + K_1,\ P_0 - P_3)$		P_2
$h = 0$	$S_{10,16}$	$(L_2 + K_1) \oplus (D,\ L_1 - K_2,\ P_2,\ P_0 - P_3)$		$L_2 + K_1$
$A_1 \oplus A_{4,12}$	$S_{2,8}$	$(0) \oplus (L_3,\ K_3 ;\ L_2 + K_1,\ L_1 - K_2)$		D
	$S_{6,15}$	$(P_0 - P_3) \oplus (L_3,\ D - 2aK_3 ;\ L_2 + K_1,\ L_1 - K_2)$	$a = -1$	$P_0 - P_3$
	$S_{9,9}$	$(K_3) \oplus (L_3,\ D ;\ P_2,\ P_1)$		K_3
	$S_{12,34}$	$(P_0 - P_3) \oplus (L_3,\ D - 2aK_3 ;\ P_2,\ P_1)$	$a = -1$	$P_0 - P_3$
$A_{5,4}$	$S_{10,2}$	$L_2 + K_1,\ L_1 - K_2,\ P_1,\ P_2 ;\ P_0 - P_3$		$P_0 - P_3$
$A_{5,5}$	$S_{19,6}$	$L_1 - K_2,\ 2L_2 + 2K_1 - P_0 - P_3,\ P_1\ P_2 ;\ P_0 - P_3$		$P_0 - P_3$

Class	Notation	Generators	Range of Parameters	Invariants
$A_{5,7}^{a,b,c}$				
$a=1, c=1$ $b=1$	$S_{15,12}$	$D; P_1, P_2, P_3, P_0$		$P_2/(P_0-P_3)$, $(P_0-P_3)/P_1$, $(P_0-P_3)/(P_0+P_3)$
$a=1/(1+a)$ $b=1/(1+a)$ $c=(1-a)/(1+a)$	$S_{15,23}$	$D+2a(\cos c L_3 - \sin c K_3); P_1, P_2, P_3, P_0$	$c=\pi/2$ $a>0, a\neq 1$	$(P_0-P_3)/P_2^{1+a}$; $(P_0-P_3)/P_1^{1+a}$; $(P_0-P_3)^{1-a}/(P_0+P_3)^{1+a}$
$A_{5,11}^{c}$ $c=1$	$S_{15,24}$	$D-L_2-K_1; P_2, P_1, P_3, P_0$		$(P_0-P_3)/P_2$; $(P_0-P_3)\exp\{(P_0+P_3)/(P_3-P_0)\}$ $4P_1/(P_0-P_3) + (P_0+P_3)^2/(P_0-P_3)^2$
$A_{5,13}^{b,p,q}$				
$b=-1$ $p=0$ $q=\cot c$	$S_{11,1}$	$\cos c L_3 - \sin c K_3; P_1, P_2, P_3, P_0$	$0<c<\pi$, $c\neq\pi/2$	$P_0^2-P_3^2$; $P_1^2+P_2^2$; $(P_0-P_3)^{2\cot c}\left\{\dfrac{P_1-iP_2}{P_1+iP_2}\right\}^{i}$
$b=\dfrac{1-a\sin c}{1+a\sin c}$ $p=\dfrac{1}{1+a\sin c}$ $q=\dfrac{a\cos c}{1+a\sin c}$	$S_{15,23}$	$D-2a(\cos c L_3 - \sin c K_3); P_1, P_2, P_3, P_0$	$0<c<\pi$, $c\neq\pi/2$ $a>0$	$\dfrac{(P_0-P_3)}{P_0+P_3}^{b}$; $\dfrac{(P_0-P_3)}{P_1^2+P_2^2}^{2p}$; $(P_0-P_3)^{2q}\left\{\dfrac{P_1-iP_2}{P_1+iP_2}\right\}^{i}$

Class	Notation	Generators	Range of Parameters	Invariants
$A_{5,19}^{c,b}$				
$c=1$ $b=1$	$S_{7,3}$	$K_3,\ L_2+K_1,\ L_1-K_2,\ P_2,\ P_0-P_3$		$(P_0-P_3)/(L_2+K_1)$
$=a+1$ $=a$	$S_{10,22}$	$D-2aK_3;\ L_2+K_1,\ L_1-K_2,\ P_2,\ P_0-P_3$	$a\neq 0$	$(P_0-P_3)^a/(L_2+K_1)^{1+a}$
$=2$ $=1$	$S_{10,28}$	$D-2K_3,\ L_2+K_1,\ L_1-K_2-P_1,\ P_2,\ P_0-P_3$		$(P_0-P_3)/(L_2+K_1)^2$
$=1$ $=1$	$S_{14,23}$	$D;\ L_2+K_1,\ P_1, P_2,\ P_0-P_3$		$(P_0-P_3)/P_2$
$=a+1$ $=1$	$S_{14,33}$	$D-2aK_3;\ L_2+K_1,\ P_1,\ P_2,\ P_0-P_3$.	$a\neq 0$	$(P_0-P_3)/P_2^{a+1}$
$=3$ $=2$	$S_{14,52}$	$D-K_3;\ 2L_2+2K_1-P_0-P_3,\ P_1,\ P_2,\ P_0-P_3$		$(P_0-P_3)^2/P_2^3$
$A_{5,20}^{b}$				
$b=1$	S_{76}	$2K_3-P_1;\ L_2+K_1,\ L_1-K_2,\ P_0-P_3,\ P_2$		$(P_0-P_3)\exp\{(L_2+K_1)/(P_3-P_0)\}$
$b=1$	$S_{14,35}$	$D+L_1-K_2;\ L_2+K_1,\ P_1,\ P_2,\ P_3-P_0$		$(P_0-P_3)\exp\{2P_2/(P_0-P_3)\}$
$A_{5,23}^{b}$				
$b=1$	$S_{14,34}$	$2D-4K_3+P_0+P_3;\ L_2+K_1,\ P_1,\ P_2,\ P_0-P_3$		$(P_0-P_3)/P_2^2$

Class	Notation	Generators	Range of Parameters	Invariants
$A^{h}_{5,30}$				
$h=0$	$S_{8,3}$	$K_3, P_1;\ L_2+K_1,\ P_3, P_0$		$P_1^2+P_3^2-P_0^2$
$=0$	$S_{8,10}$	$2K_3-P_2,\ P_1;\ L_2+K_1,\ P_3, P_0$		"
$=2$	$S_{19,29}$	$D-K_3;\ L_1-K_2,\ 2L_2+2K_1-P_0-P_3,\ P_2,\ P_0-P_3$		$(P_2^2+P_2^2-P_0^2-2(P_3-P_0)(L_2+K_1))^3/(P_3-P_0)^4$
$\neq 1/a$	$S_{14,37}$	$D-2aK_3;\ L_2+K_1,\ P_1,\ P_3, P_0$	$a \neq 0$	$(P_1^2+P_3^2-P_0^2)^{1+a}/(P_0-P_3)^2$
$=1$	$S_{14,53}$	$D-2K_3;\ L_2+K_1-P_2,\ P_1,\ P_3, P_0$		$(P_1^2+P_3^2-P_0^2)/(P_0-P_3)$
$A^{h}_{5,32}$				
$h=0$	$S_{14,24}$	$D,\ L_2+K_1;\ P_1,\ P_3,\ P_0$		$(P_1^2+P_3^2-P_0^2)/(P_0-P_3)^2$
$A^{a,b}_{5,33}$				
$a=1,\ b=1$	$S_{8,21}$	$K_3,\ D;\ L_2+K_1,\ P_2,\ P_0-P_3$		$P_2(L_2+K_1)/(P_0-P_3)$
$=0\ \ =1$	$S_{13,17}$	$K_3,\ D;\ P_1,\ P_2,\ P_0-P_3$		P_2/P_1
$=-1\ \ =2$	$S_{13,18}$	$K_3,\ D;\ P_1,\ P_3,\ P_0$		$(P_0-P_3)^2/P_1(P_0+P_3)$

Class	Notation	Generators	Range of Parameters	Invariants
$A_{5,35}^{\gamma,b}$				
$b=1,\ \gamma=0$	$S_{2,3}$	$L_2, K_3;\ L_2+K_1,\ L_1-K_2,\ P_0-P_3$		$(P_0-P_3)^2/\{(L_1-K_2)^2+(L_2+K_1)^2\}$
$b=\frac{1+a}{a}$, $\gamma=-\frac{\tan c}{a}$	$S_{5,11}$	$D-2aK_3,\ \cos c\,L_3-\sin c\,K_3;\ L_2+K_1,\ L_1-K_2,\ P_0-P_3$	$a\neq 0$, $0<c<\pi$, $c\neq\pi/2$	$\frac{(P_0-P_3)^{2a}}{((L_1-K_2)^2+(L_2+K_1)^2)^{1+a}}\left[\frac{L_1-K_2-i\,L_2+K_1}{L_1-K_2+i\,L_2+K_1}\right]^{i\tan c}$
$b=\frac{1+a}{2}$, $\gamma=0$	$S_{6,15}$	$D-2aK_3,\ L_3;\ L_2+K_1,\ L_1-K_2,\ P_0-P_3$	$a\neq 0,-1$	$(P_0-P_3)^{2a}/\{(L_1-K_2)^2+(L_2+K_1)^2\}^{1+a}$
$b=1$, $\gamma=1/a$	$S_{7,16}$	$D-2aL_3,\ K_3;\ L_2+K_1,\ L_1-K_2,\ P_0-P_3$	$a\neq 0$	$\frac{P_0-P_3}{(L_1-K_2)^2+(L_2+K_1)^2}\left[\frac{L_1-K_2-i(L_2+K_1)}{L_1-K_2+i(L_2+K_1)}\right]^{i/a}$
$b=1$, $\gamma=-\tan c$	$S_{11,8}$	$D,\ L_3-\tan c\,K_3;\ P_1,\ P_2,\ P_0-P_3$	$0<c<\pi$, $c\neq\pi/2$	$\frac{(P_0-P_3)^2}{P_1^2+P_2^2}\left[\frac{P_2-iP_1}{P_2+iP_1}\right]^{i\tan c}$
$b=1-a\tan c$, $\gamma=-\tan c$	$S_{11,14}$	$D-2aL_3,\ L_3-\tan c\,K_3;\ P_1,\ P_2,\ P_0-P_3$	$a\neq 0$, $0<c<\pi$, $c\neq\pi/2$	$\frac{(P_0-P_3)^2}{(P_1^2+P_2^2)^{1-a\tan c}}\left(\frac{P_2-iP_1}{P_2+iP_1}\right)^{i\tan c}$
$b=1\ \ \gamma=0$	$S_{12,24}$	$D,\ L_3;\ P_1,\ P_2,\ P_0-P_3$		$(P_0-P_3)^2/(P_1^2+P_2^2)$
$=1\ \ =0$	$S_{12,25}$	$D,\ L_3;\ P_1,\ P_2,\ P_3$		$P_3^2/(P_1^2+P_2^2)$
$=1\ \ =0$	$S_{12,26}$	$D,\ L_3;\ P_1,\ P_2,\ P_0$		$P_0^2/(P_1^2+P_2^2)$

Class	Notation	Generators	Range of Parameters	Invariants
$A_{5,35}^{\gamma,b}$				
$b=1+a,\ \gamma=0$	$S_{12,34}$	$D-2aK_3,\ L_3;\ P_1,\ P_2,\ P_0-P_3$	$a\neq 0,-1$	$(P_0-P_3)^2/(P_1^2+P_2^2)^{1+a}$
$=2\quad =0$	$S_{12,35}$	$2D-4K_3+P_0+P_3,\ L_3;\ P_1,\ P_2,\ P_0-P_3$		$(P_0-P_3)^2/(P_1^2+P_2^2)$
$=2\quad =0$	$S_{12,36}$	$2D-4K_3+x(P_0-P_3),\ 4L_3+P_0+P_3;\ P_1,P_2,\ P_0-P_3$	$-\infty<x<\infty$	"
$A_{5,36}$	$S_{8,22}$	$D,K_3;\ L_2+K_1,\ P_1,\ P_0-P_3$		$D-2K_3-2(L_2+K_1)P_1/(P_0-P_3)$
	$S_{8,23}$	$D,\ K_3;\ L_2+K_1,\ P_2-cP_1,\ P_0-P_3$	$c\neq 0$	$D-2K_3+2(L_2+K_1)(P_2-cP_1)/(cP_0-cP_3)$
$A_{5,37}$	$S_{6,19}$	$D-2K_3,\ L_3;\ L_2+K_1-P_2,\ L_1-K_2-P_1,\ P_0-P_3$	-	$\{(L_2+K_1-P_1)^2+(L_1-K_2-P_1)^2\}/(2P_0-2P_3)+2L_3$

Table 6: Six dimensional subalgebras.

Dim. of derived L	Notation	Generators	Range of Parameters	Invariants
$Sl(2,C)$ 6	$S_{1,2}$	$; L_1, L_2, L_3, K_1, K_2, K_3$		$(\bar{L})^2-(\bar{K})^2$; $\bar{L}\cdot\bar{K}$
$E(3)$ 6	$S_{3,2}$	$; L_1, L_2, L_3, P_1, P_2, P_3$		$P_1^2+P_2^2+P_3^2$, $\bar{P}\cdot\bar{L}$
$E(2,1)$ 6	$S_{4,2}$	$; K_1, K_2, L_3, P_1, P_2, P_0$		$P_1^2+P_2^2-P_0^2$, $P_0L_3+(\bar{P}\times\bar{L})_3$
5	$S_{5,2}$	$L_3-\tan c\,K_3; L_2+K_1, L_1-K_2, P_1, P_2, P_0-P_3$	$0<c<\pi$, $c\neq\pi/2$	none
	$S_{10,20}$	$D+2aL_3-2bK_3; L_2+K_1, L_1-K_2, P_1, P_2, P_0-P_3$	$b=0$, $a\neq 0$	"
5	$S_{6,2}$	$L_3; L_2+K_1, L_1-K_2, P_1, P_2, P_0-P_3$		P_0-P_3; $P_0L_3+(\bar{P}\times\bar{K})_3-\bar{P}\cdot\bar{L}$
5	$S_{6,5}$	$4L_3+P_0+P_3; L_2+K_1, L_1-K_2, P_1, P_2, P_0-P_3$	—	P_0-P_3; $P_0L_3+(\bar{P}\times\bar{K})_3-\bar{P}\cdot\bar{L}+(P_0^2-P_3^2)/4$
5	$S_{10,20}$	$D-2aL_3-2bK_3; L_2+K_1, L_1-K_2, P_1, P_2, P_0-P_3$	$b\neq 0, -1$ $-\infty<a<\infty$	none
5	$S_{10,20}$	$D-2aL_3-2bK_3; L_2+K_1, L_1-K_2, P_1, P_2, P_0-P_3$	$b=-1$ $-\infty<a<\infty$	P_0-P_3, $D-2\{a(P_0L_3-\bar{P}\cdot\bar{L}+(\bar{P}\times\bar{K})_3)+P_0K_3-\bar{P}\cdot\bar{K}-(\bar{P}\times\bar{L})_3\}/(P_0-P_3)$

Dim. of derived L	Notation	Generators	Range of Parameters	Invariants
5	$S_{10,21}$	$2D+4aL_3-4K_3+P_0+P_3\,;\ L_2+K_1,\ L_1-K_2,\ P_1,\ P_2,\ P_0-P_3$	$-\infty<a<\infty$	none
5*	$S_{10,26}$	$D-K_3\,;\ L_1-K_2,\ 2L_2+2K_1-P_0-P_3,\ P_1,\ P_2,\ P_0-P_3$		"
5	$S_{14,31}$	$D-2aK_3\,;\ L_2+K_1,\ P_1,\ P_2,\ P_3,\ P_0$	$a\neq 0$	$P_2^{a+1}/(P_0-P_3)$; $P_2^2/(P_1^2+P_3^2-P_0^2)$
4	$S_{7,10}$	$D,\ K_3\,;\ L_2+K_1,\ L_1-K_2,\ P_2,\ P_0-P_3$		none
$A_1\oplus A_5$ 4	$S_{8,1}$	$(P_2)\oplus(L_3\,;\ L_2+K_1,\ P_1,\ P_3,\ P_0)$		P_2 ; $P_1^2+P_3^2-P_0^2$
4	$S_{8,19}$	$D,\ K_3\,;\ L_2+K_1,\ P_1,\ P_2,\ P_0-P_3$		none
4	$S_{8,20}$	$D,\ K_3\,;\ L_2+K_1,\ P_1,\ P_3,\ P_0$		"
$A_{3,4}\oplus A_{3,6}$ 4	$S_{9,1}$	$(K_3\,;\ P_0,\ P_3)\oplus(L_3\,;\ P_1,\ P_2)$		$P_0^2-P_3^2$; $P_1^2+P_2^2$

Dim. of derived L	Notation	Generators	Range of Parameters	Invariants
4	$S_{11,7}$	$D, L_3-\tan c\, K_3; P_1, P_2, P_3, P_0$	$0<c<\pi$, $c\neq\pi/2$	$(P_2^2+P_1^2)/(P_0^2-P_3^2)$; $(P_1^2+P_2^2)/(P_0-P_3)^2\{(P_2-iP_1)/(P_2+iP_1)\}^{i\tan c}$
4	$S_{11,13}$	$D+2aL_3, L_3-\tan c\, K_3; P_1, P_2, P_3, P_0$	$a>0$ $0<c<\pi$, $c\neq\pi/2$	" ; $\frac{(P_1^2+P_2^2)^{1-a\tan c}}{(P_0-P_3)^2}\left\{\frac{P_2-iP_1}{P_2+iP_1}\right\}^{i\tan c}$
4	$S_{12,23}$	$D, L_3; P_1, P_2, P_3, P_0$		$(P_2^2+P_1^2)/(P_0+P_3)^3$; $(P_2+P_1^2)/(P_0-P_3)^2$
4	$S_{12,33}$	$D-2aK_3; P_1, P_2, P_3, P_0$	$a>0$	$(P_2^2+P_1^2)^{1-a}/(P_0+P_3)^2$; $(P_2^2+P_1^2)^{a+1}/(P_0-P_3)^2$
4	$S_{13,16}$	$D, K_3; P_1, P_2, P_3, P_0$		$P_2^2/(P_0^2-P_3^2)$; P_1/P_2
4	$S_{13,25}$	$D+2aL_3, K_3; P_1, P_2, P_3, P_0$	$a>0$	$(P_0^2-P_3^2)/(P_1^2+P_2^2)$; $(P_2^2+P_1^2)\{(P_2-iP_1)/(P_2+iP_1)\}^{i/a}$
4	$S_{14,22}$	$D, L_2+K_1; P_1, P_2, P_3, P_0$		$(P_1^2+P_3^2-P_0^2)/P_2^2$; $(P_0-P_3)/P_2$
4	$S_{14,32}$	$D+L_1-K_2, L_2+K_1; P_1, P_2, P_3, P_0$		$(P_0-P_3)\exp(2P_2/(P_0-P_3))$; $(P_1^2+P_2^2+P_3^2-P_0^2)/(P_0-P_3)^2$

Dim. of derived L	Notation	Generators	Range of Parameters	Invariants
$A_2 \oplus A_{4,12}$ 3	$S_{2,7}$	$(D, P_0 - P_3) \oplus (L_3, D+2K_3; L_2+K_1, L_1-K_2)$		none
3	$S_{9,2}$	$K_3, P_1, P_2; L_2+K_1, L_1-K_2, P_0-P_3$		none
	$S_{10,15}$	$D, L_1-K_2, L_2+K_1; P_1, P_2, P_0-P_3$		"
3	$S_{9,8}$	$D, K_3, L_3; P_1, P_2, P_0-P_3$		none
3	$S_{10,1}$	$L_2+K_1, L_1-K_2, P_0+P_3; P_1, P_2, P_0-P_3$		P_0-P_3; $P_1^2+P_2^2+P_3^2-P_0^2$

Table 7: Seven dimensional subalgebras.

Dim. of derived L	Notation	Generators	Range of Parameters	Invariants
$D \oplus sl(2,C)$ 6	$S_{1,4}$	$(D) \oplus (; L_1, L_2, L_3, K_1, K_2, K_3)$		D; $\bar{L}^2 - \bar{K}^2$; $\bar{L} \cdot \bar{K}$
$A_1 \oplus E(3)$ 6	$S_{3,1}$	$(P_0) \oplus (; L_1, L_2, L_3, P_1, P_2, P_3)$		P_0, $\bar{P} \cdot \bar{L}$, $\bar{P}^2 - P_0^2$

Dim. of derived L	Notation	Generators	Range of Parameters	Invariants
$D \square E(3)$ 6	$S_{3,6}$	$D;\ P_1, P_2, P_3, L_1, L_2, L_3$		$\bar{P}\cdot\bar{L} / (P_1^2 + P_2^2 + P_3^2)$
$A_1 \oplus E(2,1)$ 6	$S_{4,1}$	$(P_3) \oplus (;\ P_1, P_2, P_0, K_1, K_2, L_3)$		$P_3\ ;\ P_0 L_3 + (\bar{P}\times\bar{K})_3\ ;\ P_1^2 + P_2^2 - P_0^2$
$D \square E(2,1)$ 6	$S_{4,6}$	$D;\ P_1, P_2, P_0, K_1, K_2, K_3$		$(P_0 L_3 + (\bar{P}\times\bar{K})_3)^2 / (P_1^2 + P_2^2 - P_0^2)$
6	$S_{7,1}$	$K_3;\ L_1 - K_2, L_2 + K_1, P_1, P_2, P_3, P_0$		$\bar{P}^2 - P_0^2$
6	$S_{10,19}$	$D - 2aL_3 - 2bK_3;\ L_1 - K_2, L_2 + K_1, P_1, P_2, P_3, P_0$	$a^2 + b^2 \neq 0$	$(P_0^2 - \bar{P}^2)^{1+b} / (P_0 - P_3)^2$
5	$S_{3,2}$	$K_3, L_3;\ L_1 - K_2, L_2 + K_1, P_1, P_2, P_0 - P_3$		$(P_0 L_3 + (\bar{P}\times\bar{K})_3 - \bar{P}\cdot\bar{L}) / (P_0 - P_3)$
	$S_{6,9}$	$D, L_3;\ L_1 - K_2, L_2 + K_1, P_1, P_2, P_0 - P_3$		"
5	$S_{5,6}$	$D, L_3 - \tan c\, K_3;\ L_2 + K_1, L_1 - K_2, P_1, P_2, P_0 - P_3$	$0 < c < \pi,\ c \neq \pi/2$	$D \sin c + 2\{\cos c(\bar{P}\cdot\bar{L} - (\bar{P}\times\bar{K})_3 - P_0 L_3) + \sin c(P_0 K_3 - (\bar{P}\times\bar{L})_3 - \bar{K}\cdot\bar{P})\}/(P_0 - P_3)$
5	$S_{3,14}$	$D - 2aL_3, K_3;\ L_2 + K_1, L_1 - K_2, P_1, P_2, P_0 - P_3$	$a \neq 0$	$D + 2\{a(\bar{P}\cdot\bar{L} - (\bar{P}\times\bar{K})_3 - P_0 L_3) + (P_0 K_3 - (\bar{P}\times\bar{L})_3 - \bar{K}\cdot\bar{P})\}/(P_0 - P_3)$

Dim. of derived L	Notation	Generators	Range of Parameters	Invariants
5	$S_{5,10}$	$D-2aK_3,\ L_3-\tan c\,K_3;\ L_2+K_1,\ L_1-K_2,\ P_1,\ P_2,\ P_0-P_3$	$a\neq 0$, $0<c<\pi$, $c\neq \pi/2$	$D\sin c+2\{(a+D\cos c)(\bar{P}\cdot\bar{L}-(\bar{P}\times\bar{K})_3-P_0L_3)+\sin c\,(P_0K_3-(\bar{P}\times\bar{L})-\bar{P}\cdot\bar{K})\}/(P_0-P_3)$
5	$S_{6,1}$	$L_3;\ P_0+P_3;\ L_2+K_1,\ L_1-K_2,\ P_1,\ P_2,\ P_0-P_3$		$P_0-P_3;\ P_0^2-\bar{P}^2,\ P_0L_3+(\bar{P}\times\bar{K})_3-\bar{P}\cdot\bar{L}$
5	$S_{6,13}$	$D-2aK_3,\ L_3;\ L_1-K_2,\ L_2+K_1,\ P_1,\ P_2,\ P_0-P_3$	$a\neq 0,-1$	$(P_0L_3+(\bar{P}\times\bar{K})_3-\bar{P}\bar{L})/(P_0-P_3)$
5	$S_{6,13}$	$D-2aK_3,\ L_3;\ L_1-K_2,\ L_2+K_1,\ P_1,\ P_2,\ P_0-P_3$	$a=-1$	$P_0-P_3;\ P_0L_3+(\bar{P}\times\bar{K})_3-\bar{P}\cdot\bar{L}$, $2(P_0K_3-(\bar{P}\times\bar{L})_3-\bar{K}\bar{P})+(P_0-P_3)D$
5	$S_{6,14}$	$2D-4K_3+P_0+P_3,\ L_3;\ L_2+K_1,\ L_1-K_2,\ P_1,\ P_2,\ P_0-P_3$		$(P_0L_3+(\bar{P}\times\bar{K})_3-\bar{P}\cdot\bar{L})/(P_0-P_3)$
5	$S_{6,18}$	$2D-4K_3+x(P_0-P_3),\ 4L_3+P_0+P_3;\ L_1-K_2,\ L_2+K_1,\ P_1,\ P_2,\ P_0-P_3$	$-\infty<x<\infty$	$\{P_2^2+P_1^2+P_3^2-P_0^2+4[(P_0-P_3)L_3+P_1(L_1-K_2)+P_2(L_2+K_1)]\}/(P_0-P_3)$
5	$S_{7,9}$	$D,\ K_3;\ L_2+K_1,\ L_1-K_2,\ P_1,\ P_2,\ P_0-P_3$		$2(P_0K_3-(\bar{P}\times\bar{L})_3-\bar{P}\cdot\bar{K})/(P_0-P_3)+D$
5	$S_{8,18}$	$D,\ K_3;\ L_2+K_1,\ P_1,\ P_2,\ P_3,\ P_0$		$(P_1^2+P_3^2-P_0^2)/P_2^2$

Dim. of derived L	Notation	Generators	Range of Parameters	Invariants
4	$S_{9,7}$	$D, K_3, L_3;\ P_1, P_2, P_3, P_0$		$(P_1^2+P_2^2)/(P_0^2-P_3^2)$
4	$S_{10,14}$	$D, L_1-K_2, L_2+K_1;\ P_1, P_2, P_3, P_0$		$(P_0^2-\bar{P}^2)/(P_0-P_3)^2$

Table 8: Eight dimensional subalgebras.

Dim. of derived L	Notation	Generators	Range of Parameters	Invariants
6	$S_{2,1}$	$K_3, L_3;\ L_1-K_2, L_2+K_1, P_0, P_1, P_2, P_3$		$P_0^2-\bar{P}^2$, $(P_0L_3+(\bar{P}\times\bar{K})_3-\bar{P}\cdot\bar{L})/(P_0-P_3)$
6	$S_{3,5}$	$D, L_3;\ L_1, L_2, P_0, P_1, P_2, P_3$		$(\bar{P}\cdot\bar{L})^2/P_0^2$, $\bar{P}^2/P_0^2$
6	$S_{4,5}$	$D, L_3;\ K_1, K_2, P_0, P_1, P_2, P_3$		$(P_0L_3+(\bar{P}\times\bar{K})_3)/P_3$; $(P_1^2+P_2^2-P_0^2)/P_3^2$
6	$S_{5,5}$	$D, L_3-\tan c\,K_3;\ L_2+K_1, L_1-K_2, P_0, P_1, P_2, P_3$	$0<c<\pi$, $c\neq\pi/2$	none
6	$S_{5,9}$	$D-2aK_3, L_3-\tan c\,K_3;\ L_2+K_1, L_1-K_2, P_0, P_1, P_2, P_3$	$a\neq 0$, $0<c<\pi$, $c\neq\pi/2$	none

Dim. of derived L	Notation	Generators	Range of Parameters	Invariants
6	$S_{6,8}$	$D, L_3;\ L_1 - K_2,\ L_2 + K_1,\ P_0,\ P_1,\ P_2,\ P_3$		$(P_0 L_3 + (\bar{P} \times \bar{K})_3 - \bar{P}\cdot\bar{L})/(P_0 - P_3)\ ;\ (P_0^2 - \bar{P}^2)/(P_0 - P_3)^2$
6	$S_{6,12}$	$D - 2aK_3,\ L_3;\ L_1 - K_2,\ L_2 + K_1,\ P_0,\ P_1,\ P_2,\ P_3$	$a \neq 0$	" $;\ (P_0^2 - \bar{P}^2)^{1+a}/(P_0 - P_3)^2$
6	$S_{7,8}$	$D, K_3;\ L_1 - K_2,\ L_2 + K_1,\ P_0,\ P_1,\ P_2,\ P_3$		none
6	$S_{7,13}$	$D + aL_3,\ K_3;\ L_2 + K_1,\ L_1 - K_2,\ P_0,\ P_1,\ P_2,\ P_3$	$a \neq 0$	none
5	$S_{2,6}$	$D, L_3, K_3;\ L_2 + K_1,\ L_1 - K_2,\ P_1,\ P_2,\ P_0 - P_3$		$\dfrac{P_0 L_3 + (\bar{P} \times \bar{K})_3 - \bar{P}\cdot\bar{L}}{P_0 - P_3},\ D - \dfrac{2}{P_0 - P_3}\{\bar{P}\cdot\bar{K} + (\bar{P} \times \bar{L})_3 - P_0 K_3\}$

Table 9: Nine, ten, and eleven dimensional subalgebras.

Dim. of derived L	Notation	Generators	Range of Parameters	Invariants
(6)	$S_{2,5}$	$D, L_3, K_3; L_1 - K_2, L_2 + K_1, P_0, P_1, P_2, P_3$		$(P_0 L_3 + (\bar{P} \times \bar{K})_3 - \bar{P} \cdot \bar{L}) / (P_0 - P_3)$
(10)	$S_{1,1}$	$; L_1, L_2, L_3, K_1, K_2, K_3, P_1, P_2, P_3, P_0$		$\bar{P}^2 - P_0^2; \; W^2$
(10)	$S_{1,3}$	$D; L_1, L_2, L_3, K_1, K_2, K_3, P_1, P_2, P_3, P_0$		$W^2 / (\bar{P}^2 - P_0^2)$