Results in Categorical Proof Theory

by

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Abstract

We prove a completeness result for the equivalence of proofs in the positive fragment ($\top$, $\wedge$, $\rightarrow$) of intuitionistic propositional logic with respect to sets. We also show that proofs in the full intuitionistic propositional logic factor through interpolants - in this way we prove a stronger interpolation property than the usual one which gives only the existence of interpolants.

Translating that to categorical terms, we give a representation theorem for free cartesian closed categories (Theorem 3.16) in the category of sets and we show that Pushouts of bicartesian closed categories have the interpolation property (Theorem 4.47).

Resumé

Nous montrons un résultat concernant la complétude de l'équivalence des preuves dans le fragment positif ($\top$, $\wedge$, $\rightarrow$) de la logique intuitionniste et propositionnelle par rapport aux ensembles. Nous montrons aussi que les preuves de l'ensemble de la logique intuitionniste et propositionnelle se décomposent par les interpolants - en fait, nous prouvons une propriété d'interpolation plus forte que la propriété habituelle qui donne seulement l'existence des interpolants.

Transférant ces resultat dans le contexte des catégories, nous donnons un théorème de représentation pour les catégories cartésiennes fermées et libres (Théorème 3.16) dans la catégories des ensembles. Nous montrons aussi que les Sommes fibrées des catégories bicartesiennes et fermées ont la propriété d'interpolation (Théorème 4.47).

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1 Introduction

A very successful approach to category theory is the one by Lambek and Lawvere in which they consider certain categories "coming from nature" as certain formal systems coming from logic. The whole approach one may call categorical logic. There is an important characteristic in Lambek's approach which is less emphasized in the approach of Lawvere, namely for Lambek these formal systems have not only formulas and the notion of provability, but also they have the equality among proofs - the notion which appeared in classical proof theory as well. For us, this is an essential feature, and we like to call this part of categorical logic categorical proof theory.

While for most of the proof theorists the notion of equality was just a by-product of proof reduction which in turn was used "just" to investigate provability - the very notion of the equality of the proofs was also under consideration most explicitly by Prawitz. It turns out that the two equalities (of Prawitz and Lambek) are almost the same (for certain fragments of logic exactly the same). Therefore, Lambek's conclusion is that formulas in the formal systems are objects in the corresponding categories and that proofs (or rather their equivalence classes) are arrows in these categories. It was also noted that in that manner proofs become "real mathematical objects" - and perhaps some nonintended mathematical techniques could be applied to investigate them. We believe that even the fact that the proofs became more "real" is a step forward in the understanding what a "general" theory of proofs is [Göd65].

The formal systems investigated in this thesis are the ones coming from intuitionistic propositional logic. In the presence of the equality of proofs that is the same as to investigate bicartesian (or cartesian) closed categories - these are categories "quite often appearing in nature" e.g. toposes are like that. The general goal is to formulate and to prove for these formal systems (in the presence of equality of the proofs) some of the well known properties which hold in the presence of provability only. In this thesis we shall prove a completeness result for the equality of proofs in the positive fragment $\{\top, \wedge, \rightarrow\}$ of intuitionistic propositional logic with respect to sets and we shall show that interpolation property holds for the full intuitionistic propositional logic considering not only provability but also equality of proofs. There is a common "two-step strategy" for proving the above results: first prove the right property of the reductions associated to the formal system and second, show that in categorical terms it gives what you want. Let us be a bit more precise about these two results.

The first result says that for every free cartesian closed category there exists a faithful structure preserving functor into the category of sets.

Informally, a free cartesian closed category is a cartesian closed category freely generated by objects and arrows between generated objects.

Some consequences of the above result are that various extensions of cartesian closed structure do not impose additional equalities among arrows. E.g. let $I : \mathcal{C} \rightarrow \mathcal{B}(\mathcal{C})$ be the canonical map from a free cartesian closed category $\mathcal{C}$ to the free Boolean topos $\mathcal{B}(\mathcal{C})$ generated by $\mathcal{C}$; then I is faithful. But perhaps more important is that it confirms our intuition that cartesian closed categories indeed axiomatize the cartesian closed structure of sets. (In "everyday practice" it means that a diagram commutes in every cartesian closed category if and only if it commutes in Set.)

A key technical step in the proof of the above theorem is that in a free cartesian closed category one can faithfully adjoin infinitely many maps $1 \rightarrow C$ for every object C. This is shown with the help of a system of reductions suggested by G. Mints. Unfortunately the original paper contains some mistakes, as V. Harnik pointed out to us, see remark 3.43; since we think that these reductions are very interesting on their own right, we repair Mints' proof (of confluence as well as normalization). Also, an important ingredient in the proof is a variant of H. Friedman's completeness result for (a variant of) typed λ-calculus.

I would like to add that Michael Makkai told me that the above result should be true and suggested that the Friedman result should be used in the proof. More on the history of the theorem one can find in remarks 3.23, 3.44.

Now, let us say couple of words about the second result. In logic, by interpolation we usually mean a statement as follows: suppose we have a proof of a statement C from hypothesis B (i.e. $B \to C$) where B is in a language L_1 and C is in a language L_2 , then there exists a statement A in the language $L_1 \cap L_2$ so that we can prove $B \to A$ and $A \to C$. There are many proofs of statements of this type for different formal systems. Some of them are purely syntactic and they are obtained as corollaries to cut elimination or normalization.

In our setting (or better to say: in Lambek's approach) the statements of cut elimination and normalization are less elegant but interpolation remains (almost) as elegant as in the basic case. The above statement of interpolation in this setting has to have a form as follows: suppose again B is in the language L_1 and C in a language L_2 and suppose that there exists a proof $B \xrightarrow{t} C$ in the language $L_1 \cup L_2$, then there exists A in $L_1 \cap L_2$ and there are $B \xrightarrow{r} A$ in L_1 and $A \xrightarrow{s} C$ in L_2 such that $t = sr$. One can see that this kind of interpolation is a genuine improvement over the usual interpolation. We may also add that we allow the presence of axioms and even the presence of theories.

We also want to obtain a categorical reformulation of the above statement (independent of such notions as language and theory); therefore we have to formulate the interpolation property in the appropriate form, i.e. as a statement about Pushouts. It turns out that this again generalizes even further the statement of interpolation - to be really precise about that would require some definitions which we prefer to give later. Let us just say that our second main result says that Pushouts (sometimes called bipushouts) in the 2-category of bicartesian closed categories satisfy the naturally formulated interpolation property. The same holds for the 2-category of cartesian closed categories.

This is not the first time that interpolation is investigated from categorical viewpoint - perhaps the best known work is the one by Pitts [Pit83a, Pit83b, Pit87, Pit88, Pit92] there, as well as in almost all the other references, the interpolation "happens" in a poset (usually in the lattice of subobjects of an object) so again we can say that these variants of interpolation concern the provability only - and not the equality of proofs. There is, however, an exception: Pavlović in [Pav92] considers interpolation in a fibrational context and the fibration do not have to be posetal - the results there are of a general nature and they do not give answer whether a particular doctrine e.g. of bicartesian closed categories has the interpolation property or not. Another categorical formulation of interpolation is given in [KP86] for the category of Banach spaces.

Let us now briefly describe the contents of the thesis:

Following the Introduction is the second part called Basics of Bicartesian Closed Categories in which we give basic definitions and the relation between three versions of typed lambda calculus and corresponding categories (bicartesian closed, cartesian closed and (elementary) distributive). Although the connections of this type are well known (cf. [LS86]), we give slightly different presentation; in particular our notion of internal language is different from the existing ones. Also, we think that we give the most explicit connection between bicartesian closed categories and the corresponding language. We are more precise about that in remark 2.15. This basically finishes the common part needed for the proof of both main results. The second part also contains a section on free (bi)cartesian closed categories which is needed for the proof of the first main result.

In the third part we give the proof of the completeness result. It starts with a section 3.1 on Friedman's completeness result for typed lambda calculus; although it is essentially Friedman's proof, certain things had to be prepared and some new cases had to be included. Additional information is in remark 3.26.

In section 3.2 we finish the proof of the first result by proving some properties of the proofs in the positive fragment of intuitionistic propositional logic. We give the first correct proof that the Mints' (Prawitz') reductions for typed lambda calculus with surjective pairing and terminal object are confluent and weakly normalizing. Actually when we started our work no other work was finished which would treat even the same set of equations. For more on that see remarks 3.43, 3.44.

In the fourth part we start the proof of the interpolation result and in section 4.1 we present the right set of reductions.

The proof continues in section 4.2 where we prove that the proofs in intuitionistic propositional logic enjoy a stronger interpolation property than required by the ordinary Craig interpolation. We use ideas from Prawitz' proof of the statement - the differences and similarities are explained at the beginning of the section.

Section 4.3 is just the restatement of the previous fact in categorical terms.

In section 4.4 we prove the "right" categorical statement of the interpolation. It contains subsection 4.4.1 where we explain the relation between strict and nonstrict doctrines of bicartesian closed categories. This relation is contained in [BKP89], we learned about that from Michael Makkai who also formulated theorem 4.69 in the present form (having in mind non-tripleable doctrines). However, the actual proof of this theorem and use of it in this thesis are ours. In the subsection 4.4.2 on

Pushouts we give a construction of Pushouts in the nonstrict doctrine using ordinary 2-pushouts in the strict doctrine and in that way we finish the proof of the second main result. We could probably prove our interpolation result more directly without so detailed exposition of the above relation but since we think (together with Michael Makkai) that our construction can be a sign of a more general phenomenon we give this section in its present length.

In section 4.5 we give couple of applications of the interpolation. Among other things we show that both of the main theorems on interpolation in Heyting algebras from [Pit83a] easily follow from our interpolation result. This section is not really finished but we feel that there is nothing wrong with not finishing a section on applications.

And finally the appendix, which can be considered as a part of section 3.2, in which we give a not original but complete proof of the weak normalization for the set of reductions given in this section. We give this proof because it is often omitted or at least not given with all the details.

2 Basics of Bicartesian Closed Categories

In this section we shall give the definitions of bicartesian closed category, cartesian closed category and (elementary) distributive category and we shall explain (again) the Lambek-type connection between these categories and the appropriate typed λ-calculi. Some characteristics of our approach to the connection are given in remark 2.15. There are many papers where various variants of typed lambda calculi with finite coproducts are dealt with, but we are not aware of the existence of the explicit comparison as done below; however, we have to admit, the comparison is direct. After this, we give the definition of free categories in the appropriate sense and prove some elementary facts about them.

2.1 Categories vs. languages

Common thing about the above categories is that their definitions are based on the existence of certain adjoint functors. Briefly, we can say that bicartesian closed categories are the ones with finite products, finite coproducts and exponents; cartesian closed categories are the ones with finite products and exponents; and distributive categories are the ones with finite products and finite coproducts such that the canonical map $(A \times B) + (A \times C) \to A \times (B + C)$ (for all the objects A, B and C) is an iso. Let us now give precise definitions of these notions:

Definition 2.1 A category $\mathcal{B}$ is (strict) **bicartesian closed** if it has objects 1 and 0, and for every two objects $A, B \in \mathcal{B}$ there are objects - denoted $A \times B$, $A + B$ and

A^B; let us write it in a tabular form as follows:

$\times$	1 $A \times B$
$+$	0 $A + B$
$\rightarrow$	A^B

The category also has to have the following arrows:

$\times$	$0_A \in \hom(A, 1)$ $\pi_{A,B} \in \hom(A \times B, A)$ $\pi'_{A,B} \in \hom(A \times B, B)$
$+$	$\square_A \in \hom(0, A)$ $\iota_{A,B} \in \hom(A, A + B)$ $\iota'_{A,B} \in \hom(B, A + B)$ $d_{A,B,C} \in \hom(A \times (B + C), A \times B + A \times C)$
$\rightarrow$	$\varepsilon_{A,B} \in \hom(A^B \times B, A)$

and the following operations on homsets:

$\times$	$\hom(C, A) \times \hom(C, B) \xrightarrow{\langle\,,\rangle} \hom(C, A \times B)$
$+$	$\hom(A, C) \times \hom(B, C) \xrightarrow{[\,,]} \hom(A + B, C)$
$\rightarrow$	$\hom(A \times B, C) \xrightarrow{*} \hom(A, C^B)$

(the operations should have indexes, but since they are uniquely determined by their arguments we omit them). These (constants and) operations have to satisfy the following equations:

	(T)	$f = 0_A$
	(Pr_1)	$\pi_{A_1,A_2}\langle f_1, f_2\rangle = f_1$
$\times$	(Pr_2)	$\pi'_{A_1,A_2}\langle f_1, f_2\rangle = f_2$
	(SP)	$\langle \pi_{A,B}g, \pi'_{A,B}g\rangle = g$
	(I)	$s = \Box_A$
	(In_1)	$[s_1, s_2]\iota_{A_1,A_2} = s_1$
	(In_2)	$[s_1, s_2]\iota'_{A_1,A_2} = s_2$
$+$	(IC)	$[r\iota_{A,B}, r\iota'_{A,B}] = r$
	(Δ)	$d_{A,B,C}[\langle \pi_1, \iota_1\pi_2\rangle, \langle \pi_1, \iota_2\pi_2\rangle] = 1_{A\times B+A\times C}$
	(Δ^{-1})	$[\langle \pi_1, \iota_1\pi_2\rangle, \langle \pi_1, \iota_2\pi_2\rangle]d_{A,B,C} = 1_{A\times(B+C)}$
	(B)	$\varepsilon_{A,B}\langle h^*\pi_{C,B}, \pi'_{C,B}\rangle = h$
$\to$	(H)	$(\varepsilon_{A,B}\langle k\pi_{C,B}, \pi'_{C,B}\rangle)^* = k$

for every arrow $f \in \hom(A, 1)$, $f_i \in \hom(C, A_i)$, $g \in \hom(C, A \times B)$, $s \in \hom(0, A)$, $s_i \in \hom(A_i, C)$, $r \in \hom(A + B, C)$, $h \in \hom(C \times B, A)$ and $k \in \hom(C, A^B)$.

If a category has only finite products (i.e. satisfies "$\times$" parts of the tables) we call it **cartesian.** If it has finite products and exponents ("$\times$, $\to$" parts) we call it **cartesian closed.** If it has finite products and finite coproducts such that products distribute over coproducts ("$\times$, $+$" parts) we call it an (elementary) **distributive** category. And as we said earlier - a category with finite products, coproducts and exponents ("$\times$, $+$, $\to$" parts) is called a bicartesian closed category. It is well known that the distributivity (the equations Δ and Δ^{-1}) follows from the rest of the axioms.

In the definition (as it stands) we allow nonuniqueness of objects 1, 0, $A \times B$, $A + B$ and A^B (for every A and B). When we want to stress this we call the category **nonstrict.** In the case that we choose only one object to represent the above constructs we call such a category **strict.**

We will use the following abbreviations: if $A_i \xrightarrow{f_i} A_i'$, $i = 1,2$ then $f_1 \times f_2 = \langle f_1\pi_1, f_2\pi_2\rangle : A_1 \times A_2 \longrightarrow A_1' \times A_2'$, $f + g = [\iota_1 f_1, \iota_2 f_2] : A_1 + A_2 \longrightarrow A_1' + A_2'$. Also, $\vec{A} = A_1 \times \cdots \times A_n$ is used to denote products when the brackets are nested on the left; and if $\vec{B}$ is a subsequence of $\vec{A}$ then $\pi_{\vec{A}}^{\vec{B}} : \vec{A} \longrightarrow \vec{B}$ denotes the canonical projection, and similarly for coproducts.

Definition 2.2 The 2-category $\mathcal{BCC}$ of bicartesian closed categories has as 0-cells (small) bicartesian closed categories, as 1-cells functors preserving bicartesian closed structure (bc-functors), and as 2-cells natural isomorphisms. We will also work in the 2-category - the "strict" version of the doctrine $\mathcal{BCC}$ - that is the 0-cells in $\mathcal{BCC}_s$ are strict bicartesian closed categories, 1-cells are strict bc-functors - that is functors which preserve the chosen structure "on the nose" e.g. $F(A \times B) = F(A) \times F(B)$. The 2-cells in $\mathcal{BCC}_s$ are natural isomorphisms. Similarly, $\mathcal{CCC}$ will denote the 2-category of the cartesian closed categories and $\mathcal{CCC}_s$ its strict variant. Often we refer to these 2-categories as (strict) **doctrines**.

Let us just add that the consideration of natural isomorphisms as 2-cells is not too strong a restriction. By now, it is part of the 2-categorical folklore that the doctrines with similar kind of closed structure require natural isomorphisms as 2-cells - otherwise they are not tripleable over the 2-category of categories. For a discussion see [BKP89]. We can also add that in our case these doctrines with all the natural transformations as 2-cells do not have Pushouts - a central object of study in our thesis.

Definition 2.3 (Typed $\lambda\delta$, λ, δ-calculi) A typed $\lambda\delta$-calculus is a formal system which consists of three classes: Types, Terms and Equations. They have to satisfy the following conditions:

Types Types are freely generated from a set of basic types - sorts and the following rules: $1, 0 \in$Types; if $A, B \in$Types then $A \times B, A + B, A^B \in$Types. Again using the tables we can write it as

$\times$	1 $A \times B$
$+$	0 $A + B$
$\to$	A^B

Terms For each type A we have countably many variables of type A (we denote them as x_i^A or $x_i : A$) and they are terms, also if $s : 0$, $s_i : C$, $r : A + B$ $a : A_1 \times A_2$, $a_i : A_i$ (i=1,2), $f : A^B$, $b : B$ are terms then

$\times$	$* : 1$ $\pi(a) : A_1, \pi'(a) : A_2$ $\langle a_1, a_2 \rangle : A_1 \times A_2$
$+$	$\epsilon^C(s) : C$ $\iota_{B,A}(b) : B + A$ $\iota'_{A,B}(b) : A + B$ $\delta x^A.s_1^C, x^B.s_2^C; r^{A+B} : C$
$\to$	$(f`b) : A$ $\lambda x^A.b : B^A$

are terms. (The notions of free and bounded variables in a term t are standard - let us just be explicit about the δ-form: $FV(\delta x^A.s_1^C, x^B.s_2^C; r^{A+B} : C) = (FV(s_1) - \{x^A\}) \cup (FV(s_2) - \{x^B\}) \cup FV(r)$ ($FV(t)$ denotes the set of the free variables in t.))

Let us just illustrate where ϵ and δ come from. For that recall the rules for elimination of the connectives $\perp$ and $\vee$ (in natural deduction):

$$
\begin{array}{c}
\Gamma \\ \vdots \\ s \\ \vdots \\ \perp \quad \epsilon^C(s) \\ \hline C
\end{array}
\qquad\qquad
\begin{array}{ccc}
\Gamma_1 & \Gamma_2 \;\not{A} & \Gamma_3 \;\not{B} \\
\vdots & \vdots & \vdots \\
r & s_1 & s_2 \\
\vdots & \vdots & \vdots \\
A \vee B & C & C \\ \hline
 & C &
\end{array}
\; \delta x^A.s_1^C, x^B.s_2^C; r^{A+B}
$$

As usualy we allow cancellation of some (none or all) of the hypothesis A and B. In our notation these would be denoted by x^A respectively x^B.

Equations They always have the following form $s =_X t$ where $s, t \in$Terms and X is a set of (typed) variables such that $FV(s) \cup FV(t) \subseteq X$.

<u>Convention:</u> when $FV(s) \cup FV(t) = X$ we often omit X in $s =_X t$. Also, typing is omitted whenever convenient.

The following expressions are equations (we call them axioms of $\lambda\delta$-calculus):

	(T)	$f^1 = *$	
$\times$	(Pr_i)	$\pi_i(\langle f_1, f_2 \rangle) = f_i$	$i = 1,2$
	(SP)	$\langle \pi_1(g), \pi_2(g) \rangle = g$	
	(I)	$s^C = \epsilon^C(x^0)$	$x^0 \in FV(s^C)$
$+$	(In_i)	$\delta x_1^{A_1}.s_1, x_2^{A_2}.s_2; \iota_i(r) = s_i(r/x_i)$	$i = 1,2$
	(γ)	$\delta x^A.v(\iota_1(x^A)/z^{A+B}), y^B.v(\iota_2(y^B)/z^{A+B}); w = v(w/z)$	$x^A, y^B \notin FV(v)$
	(β)	$(\lambda x^A.h`r) = h(r/x^A)$	
$\rightarrow$	(η)	$\lambda x^B.(k`x^B) = k$	$x^B \notin FV(k)$

for every term $f : 1$, $f_i : A_i$, $g : A \times B$, $s : C$, $s_i : C$, v^C, $w : A + B$, $h : B$, $r : A$ and $k : A^B$ such that s, v, k satisfy the conditions on the free variables as stated above also, the notation $h(r/x)$ denotes the substitution of r instead of all free occurrences of x in h but first taking care of clashes of variables - so we are all the time working under α-congruence since it is possible to do that naively as in untyped λ-calculus and it is safe for our purposes).

Equations are obtained also by the following rules (we also say that proofs are formed from the axioms and the following rules):

$$(R)\ \ \frac{}{t =_X t} \quad (S)\ \ \frac{s =_X t}{t =_X s} \quad (Tran)\ \ \frac{r =_X s \quad s =_Y t}{r =_{X \cup Y} t}$$

$$(\xi)\ \ \frac{t =_{X \cup \{x\}} s}{\lambda x.t =_X \lambda x.s} \quad (Sub')\ \ \frac{a^B =_X b^B \quad s^{A^B} =_Y t^{A^B}}{(s`a) =_{X \cup Y} (t`b)}$$

$$(\xi\delta)\ \ \frac{s_1^C =_{X \cup \{x^A\}} t_1^C \quad s_2^C =_{Y \cup \{y^B\}} t_2^C \quad r^{A+B} =_Z u^{A+B}}{\delta x^A.s_1, y^B.s_2; r =_{X \cup Y \cup Z} \delta x^A.t_1, y^B.t_2; u}$$

The need for having indexed equations - contexts will be explained later. We can have some other basic types (sorts) and some other basic terms (constants). The part of the calculus denoted by "$\times$" we would call π-calculus but the name is already taken, since we will not work with this calculus only, we shall leave it nameless. The part denoted by "$\times$, $\rightarrow$" we shall call **λ-calculus**, the part denoted by "$\times$, $+$" we shall call **δ-calculus** and as we said earlier all the parts together we call **$\lambda\delta$-calculus**; let us also add that δ-calculus needs an additional rule: (Sub) which we give later. All types and terms of a certain calculus we call the **language**; sometimes we are less precise and we call only the set of basic types and basic constants the language. A set of equations added to the above system we will call a theory of the calculus e.g. **$\lambda\delta$-theory** or just theory.

And one more piece of terminology: sometimes we will speak about **type-terms** i.e. when we want to be specific about the basic types used to built a complex type (using the operations as in the first table of the current definition) then $\mathcal{T}(X_1, \ldots X_n)$

denotes a type built out of the basic types $X_1, \ldots, X_n$. As usually done, we will overuse slghtly the notation and we will write sometimes $\mathcal{T}(A_1, \ldots, A_n)$ to denote the object in a strict bicartesian closed category build out of the objects $A_1, \ldots A_n$ and the operations on objects as in the first table of definition 2.1.

In the presence of (Pr_i) and $(Tran)$ one can see that the reflexivity (rule (R)) is not needed. Also, it is a simple exercise to see that the following rules are derivable (the usual care about clashes of variables is needed for the second rule):

$$(W)\ \frac{t =_X s}{t =_{X \cup Y} s} \quad (Sub)\ \frac{a^B =_X b^B \quad s =_{Y \cup \{x^B\}} t}{s(a/x) =_{X \cup Y} t(b/x)}$$

Also, one can show that any two terms t^C, r^C are equal over a context which contain a variable of the type 0 (use $t^C = \pi_1(\langle t^C, x^0 \rangle) = \epsilon^C(x^0)$). The following lemma is going to be used:

Lemma 2.4 *For any term $F(Z^C)$ such that $x_1, x_2 \notin FV(F)$ and any $u_1, u_2 : C$ (and w of the appropriate type)*

$$F((\delta x_1.u_1, x_2.u_2; w)/Z) = \delta x_1.F(u_1/Z), x_2.F(u_2/Z); w.$$

(Hint: take $v \equiv F((\delta x_1.u_1, x_2.u_2; z^{A+B})/Z)$ and use (γ)).

Also, for a term t^0 and $F : D$ one can show that:

$$F(\epsilon^C(t)/Z) = \epsilon^D(t)$$

The following expression $(x_1^{A_1}, \ldots, x_n^{A_n} \rhd t)$ called term with context is going to be often used, it denotes a term t and a sequence of variables such that $FV(t) \subseteq x_1^{A_1}, \ldots, x_n^{A_n}$.

Definition 2.5 An **interpretation** M of a language L in a bicartesian closed category $\mathcal{B}$ is a function which assigns objects to basic types (sorts), and satisfies

$M(A\#B) = M(A)\#M(B)$, where $\#$ is product, coproduct or exponent; also $M(\#) = \#$ where $\#$ is 1 or 0 (hence, $M(\mathcal{T}(X_1, \dots, X_n)) = \mathcal{T}(M(X_1), \dots M(X_n))$ where $\mathcal{T}(X_1, \dots, X_n)$ is a type-term as at the end of definition 2.3). If the language L has some basic constants it is assumed that the category $\mathcal{C}$ had them prescribed in advance, more precisely if $c : \mathcal{T}(X_1, \dots, X_n)$ is a basic constant in the language L we assume that there exists an arrow in $\hom(1, \mathcal{T}(M(X_1), \dots M(X_n)))$ - such an arrow we will often also denote by c). Then the interpretation assigns arrows to terms as follows (using induction on complexity of terms):

- $M(x_1^{A_1}, \dots, x_n^{A_n} \triangleright x_i) = \pi_{\vec{A}}^{A_i}$
- $M(\vec{x} : \vec{A} \triangleright *) = 0_{\vec{A}}$. If the context were empty then we would have $M(\ \triangleright *) = 1_1$.
- $M(\vec{x} : \vec{A} \triangleright c) = c0_{\vec{A}}$ (here c is a constant). Also we could have empty context, then $M(\ \triangleright c) = c$.
- $M(\vec{x} : \vec{A} \triangleright \pi_i(t)) = \pi_i M(\vec{x} : \vec{A} \triangleright t)$ $i = 1, 2$.
- $M(\vec{x} : \vec{A} \triangleright \langle t_1, t_2 \rangle) = \langle M(\vec{x} : \vec{A} \triangleright t_1), M(\vec{x} : \vec{A} \triangleright t_2) \rangle$.
- $M(\vec{x} : \vec{A} \triangleright \epsilon^C(t)) = \Box_C M(\vec{x} : \vec{A} \triangleright t)$.
- $M(\vec{x} : \vec{A} \triangleright \iota_i(t)) = \iota_i M(\vec{x} : \vec{A} \triangleright t)$.
- $M(\vec{x} : \vec{A} \triangleright \delta y_1^{B_1}.u, y_2^{B_2}.v; w) =$ $[M(\vec{x} : \vec{A}, y_1 : B_1 \triangleright u), M(\vec{x} : \vec{A}, y_2 : B_2 \triangleright v)]d\langle 1_{\vec{A}}, M(\vec{x} : \vec{A} \triangleright w) \rangle$
- $M(\vec{x} : \vec{A} \triangleright (t_1 \text{`} t_2)) = \varepsilon \langle M(\vec{x} : \vec{A} \triangleright t_1), M(\vec{x} : \vec{A} \triangleright t_2) \rangle$.
- $M(\vec{x} : \vec{A} \triangleright \lambda y^B.t) = (M(\vec{x} : \vec{A}, y : B \triangleright t))^*$, if $\vec{x} : \vec{A}$ were not there we would have $M(\ \triangleright \lambda y^B.t) = (M(y^B \triangleright t)\pi'_{1,M(B)})^*$.

The map $d : \vec{A} \times (B_1 + B_2) \to \vec{A} \times B_1 + \vec{A} \times B_2$ mentioned above is the canonical iso which exists in any bicartesian closed category (as well as in any distributive category).

Let I_1, I_2 be two interpretations of a theory T in a category $\mathcal{B}$. Then a **morphism** ψ from I_1 to I_2 is a family of arrows in $\mathcal{B}$ indexed by the set of types from T such that they commute with the basic symbols from the language i.e. $\pi^{I_2(A_i)}_{I_2(A_1) \times I_2(A_2)} \psi_{A_1 \times A_2} = \psi_{A_i} \pi^{I_1(A_i)}_{I_1(A_1) \times I_1(A_2)}$, $\psi_{A_1 + A_2} \iota^{I_1(A_1)+I_1(A_2)}_{I_1(A_i)} = \iota^{I_1(A_1)+I_1(A_2)}_{I_1(A_i)} \psi_{A_i}$, $\varepsilon_{I_2(A_1), I_2(A_2)}(\psi_{A_1^{A_2}} \times \psi_{A_2}) = \psi_{A_1} \varepsilon_{I_1(A_1), I_1(A_2)}$ and for every basic constant ξ^C, $\psi_C I_1(\rhd \xi^C) = I_2(\rhd \xi^C)$. It is interesting to notice that these conditions alone are enough to establish that for every term with context $(\vec{x} : \vec{A} \rhd t^C)$ the following holds:

$$I_2(\vec{x} : \vec{A} \rhd t^C) \psi_{\vec{A}} = \psi_C I_1(\vec{x} : \vec{A} \rhd t^C).$$

A **model** of a $\lambda\delta$-theory T is an interpretation such that all the equations from T are preserved. A morphism between two models we will call a **homomorphism**. A homomorphism $M_1 \overset{\psi}{\Rightarrow} M_2 \in Mod_T \mathcal{B}$ is an **isomorphism** iff all the components of the family are iso in $\mathcal{B}$.

For an interpretation $I : L \to \mathcal{B}$ (model $I : T \to \mathcal{B}$) and for a bicartesian closed functor $F : \mathcal{B} \to \mathcal{D}$ by $F \circ I$ we denote the interpretation $F \circ I : L \to \mathcal{D}$ (model $F \circ I : T \to \mathcal{D}$) defined as follows: on basic types $F \circ I(A) = F(I(A))$ and on basic constants $F \circ I(c) = F(I(c))$. Now it is easy to see that the first equation is actually true for all types and that the second equations generalize to the terms with contexts i.e. $F \circ I(x : A \rhd t) = F(I(x : A \rhd t))$. So indeed $F \circ I$ is an interpretation of L. That $F \circ I$ is also a model (if I is one) will follow from the soundness below.

Similarly, if we are given a (homo)morphism $\psi : I_1 \Rightarrow I_2$ between two interpretations (models) of a language L (theory T) in a bc-category $\mathcal{B}$ and if $F : \mathcal{B} \to \mathcal{D}$ is a bc-functor then $F \circ \psi$ will denote the (homo)morphism between $F \circ I_1$ and $F \circ I_2$

defined as follows: $F \circ \psi_A = F(\psi_A)$; it is not hard to check that this is indeed a homomorphism. Also, of course, if ψ was an isomorphism $F \circ \psi$ remains one too.

And finally, if $\mathcal{A} \underset{G}{\overset{F}{\underset{\Downarrow\theta}{\rightrightarrows}}} \mathcal{B}$ is a natural isomorphism in $\mathcal{BCC}$ and if $T \xrightarrow{M} \mathcal{A}$ is a model, then $T \underset{G \circ M}{\overset{F \circ M}{\underset{\Downarrow\theta \circ M}{\rightrightarrows}}} \mathcal{B}$ is an isomorphism of models defined as expected i.e. $\theta \circ M_A = \theta_{M(A)}$.

Remark 2.6 Suppose that $N_1, N_2 \in Mod_T\mathcal{B}$. It is easy to see that a family of isos $N_1(X) \overset{\psi_X}{\rightleftarrows} N_2(X)$ (X is a basic type (sort) from the language) extends in at most one way (if any) to an isomorphism $N_1 \overset{\psi}{\Rightarrow} N_2$. (By induction on the complexity of types one can show that the isomorphisms ψ^C must satisfy the following: $\psi_1 = 1_1 = \psi_1^{-1}$, $\psi_0 = 1_0 = \psi_0^{-1}$, if X is a basic type ψ_X and ψ_X^{-1} are given above, $\psi_{A \times B} = \psi_A \times \psi_B$ and $\psi_{A \times B}^{-1} = \psi_A^{-1} \times \psi_B^{-1}$, $\psi_{A+B} = \psi_A + \psi_B$ and $\psi_{A+B}^{-1} = \psi_A^{-1} + \psi_B^{-1}$, and $\psi_{A^B} = (\psi_A \varepsilon(1 \times \psi_B^{-1}))^*$ and $\psi_{A^B}^{-1} = (\psi_A^{-1}\varepsilon(1 \times \psi_B))^*$.)

To guarantee also the existence of an isomorphism $N_1 \overset{\psi}{\Rightarrow} N_2$ which extends the given family, the family has to satisfy the following: for every basic constant ξ^C from the language, $\psi_C N_1(\rhd \xi^C) = N_2(\rhd \xi^C)$. The isomorphisms ψ^C are defined as above.

Now we can show soundness of our interpretation but before that we have to give a useful technical lemma which can be proved by induction on the complexity of terms.

Lemma 2.7 . *Every interpretation M satisfies the following:*

1. $M(z^{A_1 \times \cdots \times A_n} \rhd f(\pi_1(z), \ldots, \pi_n(z))) = M(x_1^{A_1}, \ldots, x_n^{A_n} \rhd f(x_1, \ldots, x_n))$.

If $\vec{y} : \vec{B}$ is not free in t then

2. $M(\vec{x} : \vec{A}, \vec{y} : \vec{B} \rhd t) = M(\vec{x} : \vec{A} \rhd t)\pi^{M(\vec{A})}_{M(\vec{A}),M(\vec{B})}$

3. $M(\vec{x} : \vec{A} \rhd f(\vec{x}, (g(\vec{x}))/y^B)) = M(\vec{x} : \vec{A}, y : B \rhd f(\vec{x}, y^B))\langle 1_{\vec{A}}, M(\vec{x} : \vec{A} \rhd g(\vec{x}))\rangle$.

Proposition 2.8 (Soundness) *Let T be a $\lambda\delta$-theory. Let M be a model of T in a bicartesian closed category. Then*

$$\text{If } T \vdash f =_X g \text{ then } M(X \triangleright f) = M(X \triangleright g).$$

Proof: As usually, this can be proved by induction on the complexity of proofs. However, to check even the base of induction requires some work - that the axiom (γ) is preserved the following argument is needed (here we use $\llbracket\ \rrbracket$ instead of M):

$$\llbracket \vec{x} : \vec{A} \triangleright \delta x^A.v(\iota_1(x^A)/z^{A+B}), y^B.v(\iota_2(y^B)/z^{A+B}); w \rrbracket =$$

$$[\llbracket \vec{x}, x \triangleright v(\iota_1(x)/z) \rrbracket, \llbracket \vec{x}, y \triangleright v(\iota_2(y)/z) \rrbracket] d \langle 1_{\vec{A}}, \llbracket \vec{x} \triangleright w \rrbracket \rangle =$$

$$[\llbracket \vec{x}, x, z \triangleright v \rrbracket \langle 1_{\vec{A}\times A}, \llbracket \vec{x}, x \triangleright \iota_1(x) \rrbracket \rangle, \llbracket \vec{x}, y, z \triangleright v \rrbracket \langle 1_{\vec{A}\times B}, \llbracket \vec{x}, y \triangleright \iota_2(y) \rrbracket \rangle] d \langle 1_{\vec{A}}, \llbracket \vec{x} \triangleright w \rrbracket \rangle =$$

$$[\llbracket \vec{x}, z \triangleright v \rrbracket \pi^{\vec{A}\times(A+B)}_{\vec{A}\times A\times(A+B)} \langle 1_{\vec{A}\times A}, \llbracket \vec{x}, x \triangleright \iota_1(x) \rrbracket \rangle, \llbracket \vec{x}, z \triangleright v \rrbracket \pi^{\vec{A}\times(A+B)}_{\vec{A}\times B\times(A+B)} \langle 1_{\vec{A}\times B}, \llbracket \vec{x}, y \triangleright \iota_2(y) \rrbracket \rangle] d \langle 1_{\vec{A}}, \llbracket \vec{x} \triangleright w \rrbracket \rangle =$$

$$[\llbracket \vec{x}, z \triangleright v \rrbracket \langle 1_{\vec{A}}, \llbracket \vec{x}, x \triangleright \iota_1(x) \rrbracket \rangle, \llbracket \vec{x}, z \triangleright v \rrbracket \langle 1_{\vec{A}}, \llbracket \vec{x}, y \triangleright \iota_2(y) \rrbracket \rangle] d \langle 1_{\vec{A}}, \llbracket \vec{x} \triangleright w \rrbracket \rangle =$$

$$\llbracket \vec{x}, z \triangleright v \rrbracket [\langle \pi^{\vec{A}}_{\vec{A},A}, \llbracket \vec{x}, x \triangleright \iota_1(x) \rrbracket \rangle, \langle \pi^{\vec{A}}_{\vec{A},B}, \llbracket \vec{x}, y \triangleright \iota_2(y) \rrbracket \rangle] d \langle 1_{\vec{A}}, \llbracket \vec{x} \triangleright w \rrbracket \rangle =$$

$$\llbracket \vec{x}, z \triangleright v \rrbracket [\langle \pi^{\vec{A}}_{\vec{A},A}, \iota_1 \pi^{A}_{\vec{A},A} \rangle, \langle \pi^{\vec{A}}_{\vec{A},B}, \iota_2 \pi^{B}_{\vec{A},B} \rangle] d \langle 1_{\vec{A}}, \llbracket \vec{x} \triangleright w \rrbracket \rangle =$$

$$\llbracket \vec{x}, z \triangleright v \rrbracket \langle 1_{\vec{A}}, \llbracket \vec{x} \triangleright w \rrbracket \rangle = \llbracket \vec{x} : \vec{A} \triangleright v(w/z) \rrbracket$$

. (We used that the map d is inverse of $[\langle \pi^{\vec{A}}_{\vec{A}\times A}, \iota_1 \pi^{A}_{\vec{A},A} \rangle, \langle \pi^{\vec{A}}_{\vec{A}\times B}, \iota_2 \pi^{B}_{\vec{A},B} \rangle]$.) □

Remark 2.9 Without contexts we wouldn't have soundness - it would be provable (using (Sub'), β and $(Tran)$):

$$\lambda x^X.f = \lambda x^X.g \vdash f = g$$

and every interpretation in Set which maps X to empty set is a model of the left side but doesn't have to be of the right side. However using the rules with contexts we get "only"

$$\lambda x^X.f = \lambda x^X.g \vdash f =_{FV(f,g)\cup\{x\}} g$$

and the above interpretation is a model for both sides.

Definition 2.10 To every bicartesian closed category $\mathcal{C}$ we can associate a $\lambda\delta$-language $L_{\mathcal{C}}$, called the **internal language**, as follows:

- The objects become the set of basic types. When we want to be precise, the basic type corresponding to an object A we will denote by X_A (this is required when we want to make distinction between types such as $X_A \times X_B$ and $X_{A\times B}$).

- The arrows from the specified terminal object 1 become the basic constants - but in several different ways! More precisely: the basic constants of type $\mathcal{T}(X_{A_1}, \ldots X_{A_n})$ are the arrows $\hom_{\mathcal{C}}(1, \mathcal{T}(A_1, \ldots A_n))$. (Thus, we have (at least) two different constants $c_f : X_{A_1\times A_2}$ and $c_f : X_{A_1} \times X_{A_2}$ corresponding to the same $(1 \xrightarrow{f} A_1 \times A_2) \in \mathcal{C}$.)

The **standard interpretation** M is the interpretation which to every symbol of the internal language assigns the intended meaning: $X_A \mapsto A$ and c_f : $\mathcal{T}(X_{A_1}, \ldots X_{A_n}) \mapsto f : 1 \to \mathcal{T}(A_1, \ldots A_n)$.

The corresponding λ-theory $T_{\mathcal{C}}$ contains all equations satisfied by the standard interpretation: $t^A =_X s^A \in T_{\mathcal{C}}$ iff $M(X \triangleright t) = M(X \triangleright s)$.

(We could have included 'term constructors' (unary functions)- every arrow $A \xrightarrow{f} B$ becomes a term constructor: if $t : A$ is a term then $f(t) : B$ is a new term. However, it wouldn't give anything new in the presence of exponents since among the equations of the theory we would have to include $f(t) = (\hat{f}`t)$ where $\hat{f}$ is the name of the constant corresponding to the transpose of f i.e. $\hat{f} = (f\pi'_{1,A})^*$. In the case of distributive categories, though, this is not redundant and we have to include the unary functions as well.)

The above notions make sense in case of a nonstrict ("ordinary") bicartesian closed category $\mathcal{B}$, not only in case of strict bcc, except that in the nonstrict case we first have to choose a strict structure on $\mathcal{B}$ and then the interpretation of the complex types e.g. $M(X \times Y)$ is the chosen product of $M(X)$ and $M(Y)$ in $\mathcal{B}$.

Proposition 2.11 (Completeness) *For a given $\lambda\delta$-theory T there exists a canonical model $M : T \to \mathcal{C}_T$ such that $M(X \triangleright u) = M(X \triangleright v)$ only if $T \vdash u =_X v$.*

Proof: This is a standard construction and it is given as follows.

Objects Objects are types.

Arrows They are classes of equivalent terms with contexts. To compare $(x_1 : A_1, \ldots, x_n : A_n \triangleright f^D(x_1^{A_1}, \ldots, x_n^{A_n}))$ with $(y_1 : B_1, \ldots, y_m : B_m \triangleright g^D(y_1^{B_1}, \ldots, y_m^{B_m}))$ we first have to have $(\ldots(A_1 \times A_2) \times \ldots) \times A_n \equiv (\ldots(B_1 \times B_2) \times \ldots) \times B_m$, call it C. (So, assuming $m \leq n$ it says that $B_m \equiv A_n, \ldots, B_2 \equiv A_{n-m+2}$ and $B_1 \equiv (\ldots(A_1 \times A_2) \times \ldots) \times A_{n-m+1}$.) Then we say that they are equivalent iff

$$T \vdash f(\pi_1(z), \ldots, \pi_n(z)) =_z g(\pi_1(z), \ldots, \pi_m(z)).$$

The class above gives an arrow $C \to D$.

Composition $(y^{\vec{B}} \triangleright g)(x^{\vec{A}} \triangleright f) = (x^{\vec{A}} \triangleright g(\pi_1(f)/y_1, \ldots, \pi_m(f)/y_m))$. Here f is of the type $\vec{B}$.

Units $1_A = (x : A \triangleright x)$.

Cartesian structure This is going to be defined on the representatives of arrows which have one free variable.

- $0_A = (x : A \triangleright *)$.
- $\pi_{A,B} = (x : A \times B \triangleright \pi(x))$.

- $\langle (x : A \rhd f(x)), (y : A \rhd g(y)) \rangle = (x : A \rhd \langle f(x), g(x) \rangle)$. (Sic!)

Closed structure .

- $\varepsilon_{A,B} = (x : A^B \times B \rhd (\pi_1(x)`\pi_2(x)))$.
- $(x : A \times B \rhd f(x))^* = (x_1 : A \rhd \lambda x_2.f(\langle x_1, x_2 \rangle))$.

Coproducts .

- $\Box_A = (x^0 \rhd \epsilon^A(x^0))$
- $\iota_i = (x^{A_i} \rhd \iota_i(x^{A_i}))$
- $[(x^A \rhd f^C), (y^B \rhd g^C)] = (z^{A+B} \rhd \delta x.f, y.g; z)$

The equivalence classes which correspond to ($\rhd c$), where c is a constant from the language we will denote also by c.

As usual the first thing to check is independence on representatives. But this is true because of the substitution rule (Sub) for typed $\lambda\delta$-calculus.

Second, we have to show that this is a bicartesian closed category. We will show only the equations (B) and (IC):

$$\varepsilon\langle f^*\pi, \pi' \rangle = (x \rhd (\pi(\langle \lambda y.f(\langle \pi(x), y \rangle), \pi'(x) \rangle)`\pi'(\langle \lambda y.f(\langle \pi(x), y \rangle), \pi'(x) \rangle))) =$$

$$(x \rhd (\lambda y.f(\langle \pi(x), y \rangle)`\pi'(x))) \stackrel{\beta}{=} (x \rhd f(\langle \pi(x), \pi'(x) \rangle)) = (x \rhd f(x)) = f.$$

$$[r\iota_1, r\iota_2] = (z^{A+B} \rhd \delta x^A.r(\iota_1(x^A)/Z), y^B.r(\iota_2(y^B)/Z); z^{A+B}) \stackrel{\eta}{=} (z \rhd r(z/Z)) = r.$$

The canonical interpretation which assigns types to the same-name-objects, constants to the same-name-arrows is obviously a model of T. The whole construction is such that 'by definition' completeness follows. □

Corollary 2.12 *The canonical model $M : T \to \mathcal{B}_T$ classifies all models of T in the following sense: the map $\mathcal{BCC}_s(\mathcal{B}_T, \mathcal{D}) \stackrel{-\circ M}{\to} Mod_T\mathcal{D}$ is an isomorphism of categories.*

Let us be more explicit about the 2-dimensional property of the canonical model: suppose that $N_1, N_2 \in Mod_T\mathcal{D}$ are two isomorphic models (i.e. for every type A in T there exists an isomorphism $N_1(A) \stackrel{\psi_A}{\to} N_2(A)$ in $\mathcal{D}$ such that for every term $(x^A \rhd t^B) \in T$, $\psi_B N_1(x^A \rhd t^B) = N_2(x^A \rhd t^B)\psi_A$) then there exists unique natural isomorphism $F_1 \stackrel{\Psi}{\Rightarrow} F_2$ such that $\Psi \circ M = \psi$, in other words: for every type A $\Psi_A = \psi_A$.

Having in mind the remark 2.6, we can require less in the above statement about the 2-dimensional property of $M : T \to \mathcal{B}_T$, that is, we could give the isomorphism between N_1, N_2 giving only a family of isomorphisms $N_1(X) \stackrel{\psi_X}{\to} N_2(X)$, now X is just a basic type, satisfying the following: for every basic constant ξ^C from the language, $\psi_C N_1(\rhd \xi^C) = N_2(\rhd \xi^C)$.

Proof: For the 1-dimensional part, let us just prove surjectivity of the above map $- \circ M$. Take a model $N : T \to \mathcal{D}$ we have to find a bc-functor $F : \mathcal{B}_T \to \mathcal{D}$ such that $N = F \circ M$. F on $Ob(\mathcal{B}_T)$ is easily defined since $Ob(\mathcal{B}_T)$ are types of T so $F(A) = N(A)$. Since the arrows of $\mathcal{B}_T$ are classes of equivalent terms with contexts we are going to define $F = N$ on arrows (also) (recall the definition of interpretation). Now we have to show that F does not depend on the choice of representatives and that F is indeed bc-functor. The first part follows from the completeness, and the second from the definition of the bc-structure on $\mathcal{B}_T$.

As for the 2-dimensional property, let us just say that naturality of $\Psi : F_1 \Rightarrow F_2$ is "the same thing" as the homomorphism property of $\psi : N_1 \Rightarrow N_2$. □

2.2 Free categories

All the statements in this subsection, as well as in the previous one, could be proved for cartesian closed categories or distributive categories instead of bicartesian closed categories.

Definition 2.13 (Free BCC) *Let L be a $\lambda\delta$-language and T be the theory on this language with no additional axioms (empty theory). To this T one can associate the bicartesian closed category C_T as in proposition 2.11. C_T is then called* **free** *bicartesian closed category. (In the non-strict doctrine any category equivalent to C_T is called free.)*

Its universal property is given in corollary 2.12. In picture:

$$\begin{tikzcd} & T \arrow[dl, "M"'] \arrow[dr, "\varphi"] & \\ C_T \arrow[rr, "\exists! F"'] & & \mathcal{D} \end{tikzcd}$$

where M is the canonical model and $\varphi \in Mod_T\mathcal{D}$ but since the theory T doesn't have additional equations we can say that φ is just an interpretation of symbols.

This is a generalization of the notion "category generated by graph" since the free arrows can be of arbitrary type. This is required if we want to consider these categories as categories of proofs, but also if we want to avoid identification of types in the definition of an internal language. We don't have to define a more general notion where the free arrows have arbitrary domain (this is included by the definition of exponents) if we are to define free bicartesian closed or cartesian closed categories, however we would have to do this in case of distributive categories.

When we analyze the above diagram we will obtain exactly the definition of a free (bi)cartesian closed category given in [Mak89] and [HM92]. Before we do that, let us recall the notion of "free arrow".

Proposition 2.14 (Free arrow) *For every bicartesian closed category $\mathcal{C}$ and for every object C in $\mathcal{C}$ there exists a bicartesian closed category $\mathcal{C}[1 \xrightarrow{\xi} I(C)]$ and a bc-functor $I : \mathcal{C} \longrightarrow \mathcal{C}[1 \xrightarrow{\xi} I(C)]$, such that for every bc-functor $F : \mathcal{C} \longrightarrow \mathcal{D}$ and every arrow $F(1) \xrightarrow{a} F(C)$ there exists unique bc-functor $\mathcal{C}[1 \xrightarrow{\xi} I(C)] \xrightarrow{G} \mathcal{D}$ such that $G \circ I = F$ and $G(\xi) = a$. This ξ is called the free arrow.*

Proof: First form the slice category $\mathcal{C}/C$ (in general it does not have to be bicartesian closed) and consider the canonical functor $\mathcal{C} \xrightarrow{I} \mathcal{C}/C$ which maps an object A to $A \times C \xrightarrow{\pi_2} C$ and an arrow $(A \xrightarrow{f} B)$ to $(A \times C \xrightarrow{\langle f\pi_1, \pi_2\rangle} B \times C)$. Now, form the full subcategory of $\mathcal{C}/C$ spanned by the objects from the image of I i.e. all the objects are of the form $A \times C \xrightarrow{\pi_2} C$. Denote this full subcategory by $\mathcal{C}//C$. This is easy to see that $\mathcal{C}//C$ is a bicartesian closed category, that the functor I is a bc-functor and that the whole construct $\mathcal{C} \xrightarrow{I} \mathcal{C}//C$ satisfies the universal property from the proposition - the role of the free arrow $1 \xrightarrow{\xi} I(C)$ is played by the arrow $1 \times C \xrightarrow{\langle \pi_2, \pi_2\rangle} C \times C$. This construction is described in more detail in [Mak89] (for an equivalent construction see [LS86]).

In the special case when the category $\mathcal{C}$ is a free bicartesian closed category the construction can be equivalently described as follows. The category $\mathcal{C}$ is obtained from a "free $\lambda\delta$-theory" T as in the definition 2.13 i.e. $\mathcal{C} = \mathcal{C}_T$. Now add to the language of T a new constant $\xi : C$. Then the new theory, which we denote simply by $T \cup \{\xi\}$, has no additional axioms. Now form the category $\mathcal{C}_{T\cup\{\xi\}}$. I is the obvious functor which maps things to the same name things. $\square$

Going back to the definition of free bicartesian closed category generated by the free theory T (which has only a set of basic types, denoted by $\mathcal{O}$, and a set of basic constants (free arrows), denoted by $\mathcal{A}$, and no additional equations) we can see that the universal property of it is expressed with respect to the notion of interpretation.

To give such an interpretation we first give a map

$$\phi|_{\mathcal{O}} : \mathcal{O} \to \mathcal{T}$$

and than we interpret the free arrows - let us be more precise about that: the free bicartesian closed category generated by the set $\mathcal{O}$ (denoted $\mathcal{C}_{\mathcal{O}}$) has the universal property which applied to $\phi|_{\mathcal{O}}$ gives a unique (in the strict doctrine) structure preserving functor $G : \mathcal{C}_{\mathcal{O}} \to \mathcal{D}$ such that the following diagram commutes:

$$\begin{tikzcd} & \mathcal{O} \arrow[dl, "I"'] \arrow[dr, "\varphi|_{\mathcal{O}}"] & \\ \mathcal{C}_{\mathcal{O}} \arrow[rr, "\exists! G"'] & & \mathcal{D} \end{tikzcd}$$

(I is the canonical inclusion). Now we have $\mathcal{A} \overset{s}{\underset{t}{\rightrightarrows}} \mathcal{C}_{\mathcal{O}}$ where s and t are the maps which give domain and codomain of the basic arrows $\mathcal{A}$ (in the presence of exponents we can assume that s is constant (the terminal object) but in case of distributive categories we have to have this more general possibility). The second part of the notion of interpretation is interpretation of basic constants and here it means that we choose $\varphi|_{\mathcal{A}}(a) \in \hom(I(s(a)), I(t(a)))$ for every $a \in \mathcal{A}$ (if such a choice does not exist, there is no interpretation). The universal property of $\mathcal{C}_{\mathcal{T}}$ now says that there exists unique structure preserving functor $F : \mathcal{C}_{\mathcal{T}} \to \mathcal{D}$ such that

$$\begin{tikzcd} & \mathcal{C}_{\mathcal{O}} \arrow[dl, "J"'] \arrow[dr, "G"] & \\ \mathcal{C}_{\mathcal{T}} \arrow[rr, "\exists! F"'] & & \mathcal{D} \end{tikzcd}$$

commutes (J is the canonical inclusion) and $F(a) = \varphi|_{\mathcal{A}}(a)$ for every a. This is exactly as required in the definition of free (bi)cartesian closed category given in [HM92] or [Mak89].

Remark 2.15 In [LS86] interpretation of terms of a λ-calculus in a cartesian closed category $\mathcal{C}$ uses the previous notion of free arrow. Similarly the notions of the internal language L_C and the theory T_C associated to a cartesian closed category $\mathcal{C}$ use not only the notion of free arrow but also identifications of types in the theory which we avoid. It is not hard to show that this theory is essentially the same as ours - the categories associated to them are equivalent. Let us just add that the interpretation of λ-terms in [MS88] is the same as ours, but for definition of the theory associated to a cartesian closed category they quote [LS86]. Also in both of the (most standard) references the 2-dimensional part of the connection is absent but it is present here (as well as in e.g. [Mak89]).

3 The Completeness Result

In this part we formulate and prove the first main result.

Theorem 3.16 *Let $\mathcal{C}$ be a free cartesian closed category. Then there exists a faithful structure preserving functor $F : \mathcal{C} \to Set$.*

The proof of this result roughly goes as follows: take a free ccc, add infinitely many free arrows for every object, show that this is safe and then use a variant of the Friedman completeness result for typed λ-calculus.

Let us explain what we mean by adding infinitely many arrows to a free ccc and it being safe.

We want to enrich the free cartesian closed category with a lot of free arrows so that in this enriched category 1 generates. But first we have to show that these new arrows don't spoil anything. However obvious it may look, one has to be careful, bearing in mind that in a nonfree case it does not have to be true (e.g., adding a free arrow from 1 to the empty set in the category of sets "spoils the thing": the canonical functor from Set to the new category is not faithful; moreover, the new category is equivalent to a point). In a sense this is the only case when something like that may happen as the following easy lemma describes - "nonempty can be inhabited":

Lemma 3.17 *The following two statements are equivalent for any cartesian closed category $\mathcal{C}$:*

- *The canonical functor $I : \mathcal{C} \to \mathcal{C}[\xi]$ is faithful (where $\mathcal{C}[\xi]$ denotes the category $\mathcal{C}$ with the freely added arrow $\xi : 1 \to C$).*
- *The terminal arrow $0_C : C \to 1$ is epi in $\mathcal{C}$.*

Proof: Assume that I is faithful. If $f0_C = g0_C$ in $\mathcal{C}[\xi]$ then multiply by ξ and use faithfulness. The other direction is also easy. It is enough to prove faithfulness of I on arrows from 1. Take two such arrows $f, g \in \mathcal{C}$. Suppose $I(f) = I(g)$ in $\mathcal{C}[\xi]$, that is $1 \times C \stackrel{\langle f\pi_1,\pi_2\rangle}{\rightarrow} C \times C = 1 \times C \stackrel{\langle g\pi_1,\pi_2\rangle}{\rightarrow} C \times C$ in $\mathcal{C}$ (see proposition 2.14). This is the same as $f\pi_1 = g\pi_1$. Multiplying from the right by $\langle 0_C, 1_C\rangle$ we get $f0_C = g0_C$. Since we assumed that 0_C was epi, we have $f = g$. $\square$

The point which we want to make is that in a free ccc adding of a free arrow is safe. For that we use the following proposition which is going to be proved in section 3.2.

Proposition 3.18 (Free types are nonempty) *If $f =_x g$ in a free λ-calculus and x does not occur as a free variable in either f or g then we also have $f = g$.*

Now, we can establish the following.

Proposition 3.19 (Key proposition) *In a free cartesian closed category $\mathcal{C}$ every 0_C is epi.*

Proof: Let f and g be two arrows in $\mathcal{C}$ such that $f0_C = g0_C$. In the corresponding free λ-calculus it gives $f =_{x^C} g$ (see lemma 2.7.2), here x^C does not appear in f, g. By the previous proposition it means $f = g$ in the λ-calculus. Therefore $f = g$ in $\mathcal{C}$ (by soundness). $\square$

Corollary 3.20 *Let $\mathcal{C}$ be a free cartesian closed category and let $\mathcal{D}$ be a free cartesian closed category obtained from $\mathcal{C}$ by adding infinitely many free arrows $1 \stackrel{c_{ij}}{\rightarrow} C_j$ for every object $C_j \in \mathcal{C}$. Then the canonical functor $I : \mathcal{C} \rightarrow \mathcal{D}$ is a faithful cc-functor.*

Proof: Adding one free arrow is faithful by lemma 3.17 and proposition 3.19. Adding finitely many follows by induction. To add infinitely many free arrows consider the constructions of a free cartesian closed category: let $\mathcal{C} = \mathcal{C}_T$ for a free λ-theory (as in the definition of free cartesian closed category). Then $\mathcal{D}$ can be constructed as $\mathcal{C}_{T'}$ where $T' = T \cup \{\xi_{ij}^{C_j} | C_j \in \mathcal{C}, i \in I\}$ (see the end of the proof of proposition 2.14). The functor I is the unique cc-functor which classifies the model $T \xrightarrow{M'|_T} \mathcal{D}$ where M' is the canonical model $T' \xrightarrow{M'} \mathcal{D}$ and $M'|_T$ is the reduct of it on T; so we have $I \circ M = M'|_T$. If I were not faithful it would mean that there are two closed terms t and s in T such that $T \not\vdash t = s$ and yet $T' \vdash t = s$. Since every proof uses finitely many symbols we would have $T'' = T \cup \{\xi_1^{C_1}, \ldots, \xi_n^{C_n}\}$ - a finite extension of T such that $T'' \vdash t = s$. Since T'' is a finite extension we know (by the above induction) that it has to be faithful and therefore $T \vdash t = s$ contrary to the assumption.

Alternatively, to add infinitely many free arrows we could form the filtered colimit of all the finite extensions. Then use that two arrows are equal in the colimit if they were already equal in a finite extension. □

To continue the proof of theorem 3.16 we need the following result which is a corollary of the variant of Friedman completeness - this corollary is going to be proved in the next section.

Corollary 3.21 *Let $\mathcal{D}$ be a free cartesian closed category which has infinitely many free arrows for every object. Then there exists a faithful, structure preserving functor $\mathcal{D} \xrightarrow{F} Set$.*

We can recapitulate as follows:

Proof of the first main result - Theorem 3.16: Take a free cartesian closed category $\mathcal{C}$, add infinitely many free arrows to every object in $\mathcal{C}$. Call the new category $\mathcal{D}$. The canonical functor $I : \mathcal{C} \to \mathcal{D}$ is cc and faithful by corollary 3.20. Also the previous functor $F : \mathcal{D} \to Set$ is cc and faithful. So $F \circ I : \mathcal{C} \to Set$ is the faithful cc-functor. □

As we can see the only things which remain to be proved are proposition 3.18 and corollary 3.21. As we said, their proofs are given in following two sections.

Remark 3.22 It is easy to show that we can't get fullness (and even some weaker properties) in the above theorem. Also we could mention that not every cartesian closed category can be faithfully mapped in Set, as a matter of fact not even in a Boolean topos as observed in [Sco80] ($U^U \cong U$ in a Boolean topos implies $U \cong 1$). However by an easy argument one can show that every small cartesian closed category $\mathcal{C}$ can be mapped in a De Morgan topos by a full and faithful structure preserving functor.

Let us just add a remark on the work of others.

Remark 3.23 Concerning the history of Theorem 3.16, we note that the problem whether it was true was raised, along with analogous problem involving monoidal closed categories, by M. Barr and others, many years ago. In [Sol83] is outlined a proof that for every two arrows in a free cartesian closed category without free arrows there is a structure preserving functor into the category of finite sets which distinguishes these two arrows; in the proof the Mints' reductions (then unrepaired) are used.

Theorem 3.16 is formally analogous to results in [Sco80] and [HM92], each of which give representations, in the form of structure-preserving functors, of cartesian-closed and richer structures in certain toposes; in place of faithfulness, other conditions are imposed on the representation. The methods in this paper are quite different from these of [Sco80] or [HM92]. More on the history of the above result one can find in remark 3.44.

3.1 On Friedman completeness for typed lambda calculus

As was promised before, in this section we give the proof of corollary 3.21.

Let us first mention the following obvious fact about λ-calculus (without additional equalities).

Lemma 3.24 *Let* $({\varphi_1}^{B^C}`\xi^C) = ({\varphi_2}^{B^C}`\xi^C)$ *and assume* ξ_C *is a constant which does not appear in* φ_1 *and* φ_2*. Then* $\varphi_1 = \varphi_2$*.*

Proof: In the proof of $({\varphi_1}^{B^C}`\xi^C) = ({\varphi_2}^{B^C}`\xi^C)$ replace all the occurrences of ξ^C with a brand new variable x^C and then use (η). $\square$

Theorem 3.25 (Essentially Friedman[Fri75]) *Let L be a free typed λ-calculus which has infinitely many basic constants for every type. Then there exists a model* $L \xrightarrow{N} Set$ *such that* $N(X \triangleright t_1) = N(X \triangleright t_2)$ *implies* $L \vdash t_1 =_X t_2$*.*

Proof: It is enough to specify N on the basic (free) types and the constants. To do that we introduce an auxiliary map - premodel $\Gamma : L \longrightarrow Set$ which maps a type A to $\{[(\ \triangleright t)] : t : A\}$, $[-]$ denotes an equivalence class (under provable equality), also notice that since the context is empty the terms have to be closed. To simplify notation a bit we will denote a term with a context only by the name of the term if it does not cause confusion. Now if X is a free type (or 1) then $N(X) \overset{def}{=} \Gamma(X)$. To give N on the arrows we need a family of partially defined surjective maps $s_A : N(A) \longrightarrow \Gamma(A)$, $A \in Types(L)$.

Claim 1. Let the family of partial maps $s = \{s_D : D \in Types(L)\}$ be defined as follows:

- $s_X = 1_{\Gamma(X)}$, X is a free object or 1;

- $s_{A\times B}(a,b) = [\langle t_1, t_2\rangle]$, where $t_1 \in s_A(a)$ and $t_2 \in s_B(b)$;
- Let $f \in N(C)^{N(B)}$. Then $s_{C^B}(f)$ is defined and equal to $[\varphi] \in \Gamma(C^B)$ if for every $b \in Dom(s_B)$ $f(b) \in Dom(s_C)$ and

$$s_C(f(b)) = [(\varphi`r)], \;\; r \in s_B(b). \tag{1}$$

Then the family is well defined and all components are surjective.

Proof of claim 1. The proof is by the induction on the complexity of types. Obviously for the free types and 1 the statement is true. Also for the product types. For the exponent type C^B lemma 3.24 insures that there is only one such $[\varphi]$ if any. (Assume that there are two: $[\varphi_1]$ and $[\varphi_2]$, by induction hypothesis s_B is surjective so $[\xi] \in Im(s_B)$ where ξ is not in φ_1, φ_2. Then from (1) follows $(\varphi_1`\xi) = (\varphi_2`\xi)$ and therefore $\varphi_1 = \varphi_2$.) To show that s_{C^B} is surjective take an arbitrary $[\varphi] \in \Gamma(C^B)$ then the witness $f \in N(C)^{N(B)}$ is chosen so that $f(b) \in s_C^{-1}([\varphi`r])$ if $b \in Doms_B$ (take any $r \in s_B(b)$) and arbitrarily otherwise. $\Box$ Claim 1.

Now we can define $N(\xi)$ for ξ^D a basic constant. $N(\xi) = d$ such that $s_D(d) = [\xi]$ (if there are several such $d \in N(D)$ we choose one of them).

Claim 2. For every $(x_1^{A_1}, \ldots, x_n^{A_n} \triangleright f^B)$ in L and every $a_i \in Dom(s_{A_i})$

$$N(f)(a_1, \ldots, a_n) \in Dom(s_B) \quad \text{and}$$

$$s_B(N(f)(a_1, \ldots, a_n)) = [f(t_1/x_1, \ldots, t_n/x_n)], \;\; t_i \in s_{A_i}(a_i) \tag{2}$$

Proof of claim 2. is by induction on the complexity of f. If $f \equiv \xi^D$ then by the definition of $N(\xi)$ we have $s_D(N(\xi)) = [\xi]$ and this is indeed (2) since $s_1(1_1) = 1_1$. Let us check only one case more: $f^{C^B} \equiv \lambda y^B.h^C$. Take $a_i \in Dom(s_{A_i})$. We must show that $N(\lambda y.h)(a_1, \ldots, a_n) \in Dom(s_{C^B})$ and $s_{C^B}(N(\lambda y.h)(a_1, \ldots, a_n) =$

$[\lambda y.h(t_1/x_1, \ldots, t_n/x_n)]$, $(t_i \in s_{A_i}(a_i))$. It is enough to show that $\lambda y.h(t_1/x_1, \ldots, t_n/x_n)$ satisfies (1) in place of φ i.e. for every $b \in Dom(s_B)$ it holds that

$$s_C(N(\lambda y.h)(a_1, \ldots, a_n)(b)) = [(\lambda y.h(t_1/x_1, \ldots, t_n/x_n)`r)]$$

$r \in s_B(b)$, because by the uniqueness of $[\varphi]$ it will follow $s_{C^B}(N(\lambda y.h)(a_1, \ldots, a_n)) = [\lambda y.h(t_1/x_1, \ldots, t_n/x_n)]$. But first we have to check that $(N(\lambda y.h)(a_1, \ldots, a_n))(b) \in Dom(s_C)$; this is so by the induction hypothesis since $N(\lambda y.h)(a_1, \ldots, a_n)(b) = N(h)(a_1, \ldots, a_n, b)$ (and $a_i \in Dom(s_{A_i}), b \in Dom(s_B)$). Again by the induction hypothesis $s_C(N(h)(a_1, \ldots, a_n, b)) = [h(t_1/x_1, \ldots, t_n/x_n, r/y)]$, so indeed

$$s_C(N(\lambda y.h)(a_1, \ldots, a_n)(b)) = [h(t_1/x_1, \ldots, t_n/x_n, r/y)] = [(\lambda y.h(t_1/x_1, \ldots, t_n/x_n)`r)]$$

(recall $t_i \in s_{A_i}(a_i)$ and $r \in s_B(b)$). □ Claim 2.

Now it is clear that N reflects equality: let $N(x^A \triangleright f^B) = N(x^A \triangleright g^B)$, then for every $a \in Dom(s_A)$, $s_B(N(f)a) = s_B(N(g)a)$ and so by (2) we have $f(\xi/x) = g(\xi/x)$ (take $a \in s_A^{-1}(\xi)$, $\xi \notin f, g$). By lemma 3.24 we have $f =_x g$. □

Remark 3.26 The typed λ-calculus for which Friedman proved the theorem didn't have product types nor the terminal type nor additional ("functional") constants. Also the equations didn't have contexts. In his case $\Gamma(A) = \{[t] : t : A\}$ (t not necessarily closed). Obviously for $A \equiv 1$ it wouldn't work in our case. So we had to take only closed terms and therefore we had to introduce "many" constants.

Let us also add that the above theorem was proved independently (and later) by John Kennison [Ken92].

And finally, we restate and give the proof of:

Corollary 3.21 (bis!) Let $\mathcal{D}$ be a free cartesian closed category which has infinitely many free arrows for every object. Then there exists a faithful, structure preserving functor $\mathcal{D} \xrightarrow{F} Set$.

Proof: Let L be the free λ-calculus such that $\mathcal{C}_L = \mathcal{D}$. Then by the previous theorem there is a model N of L in Set which reflects equality. Then, by corollary 2.12 there exists a cc-functor $\mathcal{D} \xrightarrow{F} Set$ such that $N = F \circ M$ (M is the canonical model $M : L \to \mathcal{C}_L$). F is faithful by the construction of $\mathcal{C}_L$ and faithfulness of N. □

3.2 Mints' reductions

To finish off the proof of theorem 3.16 we need to prove proposition 3.18. For that we need a confluent system of reductions for (a free) typed λ-calculus as given above, which does not introduce new variables. So not only products but also the terminal object are included and not all types are inhabited. There are only two references (that we are aware of) where such a system is given: [Min80] and [CD91]. We prefer the system given by Mints and we are going to use that one. The main reason for our choice is that these reductions are closer to Prawitz' reductions for natural deduction and they are simpler than the ones in [CD91].

The reductions in [CD91] are $\mathcal{R}_1$ (see below) but in the opposite direction (and no restrictions), $\mathcal{R}_2$ and in addition infinitely many reductions which are introduced to take care of "Obtulowitz' pairs" e.g. $x^{1\times A} \stackrel{SP^{-1}}{\longleftarrow} \langle \pi_1(x^{1\times A}), \pi_2(x^{1\times A})\rangle \stackrel{T}{\rightarrow} \langle *, \pi_2(x^{1\times A})\rangle$. Because of these pairs they have to add new reductions and by a kind of Knuth-Bendix procedure they add infinitely many reductions but neatly classified in four groups. The above pair they "connect" by an SP_{top} reduction: $\langle *, \pi_2(x^{1\times A})\rangle \rightarrow x^{1\times A}$. (Let us just mention that their remark that Mints' reductions are "only" up to an equivalence relation is unjustified since the equivalence is α-congruence used also by them and almost everybody else.)

Let us briefly introduce some terminology related to the notion of reduction. A binary relation $\mathcal{R}$ on a set of terms is called a reduction; traditionally $(t, s) \in \mathcal{R}$ is denoted $t \stackrel{\mathcal{R}}{\rightarrow} s$. A term t is $\mathcal{R}$-normal if there is no term s such that $t \stackrel{\mathcal{R}}{\rightarrow} s$. A term t is weakly normalizing if there is a finite sequence $t \equiv t_0 \stackrel{\mathcal{R}}{\rightarrow} \cdots \stackrel{\mathcal{R}}{\rightarrow} t_n$ such that t_n is $\mathcal{R}$-normal. A term t is strongly normalizing if every sequence $t \equiv t_0 \stackrel{\mathcal{R}}{\rightarrow} \cdots \stackrel{\mathcal{R}}{\rightarrow} t_n \stackrel{\mathcal{R}}{\rightarrow} \cdots$ is finite. We say that $\mathcal{R}$ is weakly (strongly) normalizing if every term t is weakly (strongly) normalizing. The transitive and reflexive closure of $\mathcal{R}$ we will denote by $\mathcal{R}^*$.

A diagram such as:

$$\begin{tikzcd} a \arrow[r, "\alpha"] \arrow[d, "\beta"'] & b \arrow[d, dashed, "\gamma"] \\ c \arrow[r, dashed, "\delta"'] & d \end{tikzcd}$$

is actually a statement which says: if $a \xrightarrow{\alpha} b$ and $a \xrightarrow{\beta} c$ then there exists d such that $b \xrightarrow{\gamma} d$ and $c \xrightarrow{\delta} d$, where α, β, γ and δ are possibly different reductions.

We say that $\mathcal{R}$ is locally confluent/locally Church-Rosser if

$$\begin{tikzcd} a \arrow[r, "\mathcal{R}"] \arrow[d, "\mathcal{R}"'] & b \arrow[d, dashed, "\mathcal{R}^*"] \\ c \arrow[r, dashed, "\mathcal{R}^*"'] & d \end{tikzcd}$$

also we say that $\mathcal{R}$ is confluent/Church-Rosser if

$$\begin{tikzcd} a \arrow[r, "\mathcal{R}^*"] \arrow[d, "\mathcal{R}^*"'] & b \arrow[d, dashed, "\mathcal{R}^*"] \\ c \arrow[r, dashed, "\mathcal{R}^*"'] & d \end{tikzcd}$$

Notation: We will write $t[x]$ when we refer to a particular occurrence of the variable x (free or bound - but of course not in $\lambda x.$ position); $t[s/x]$ denotes a term equal to t except that instead of x is written s (so it means that we don't care about clashes of variables here). Example: let $t[x] \equiv \lambda x.\langle x, x\rangle$ where we are pointing to the left occurrence of x in $\langle x, x\rangle$. Then $t[f(x)/x] \equiv \lambda x.\langle f(x), x\rangle$. We can see that also $t[f(x)/x] \equiv t[f(x)/y]$ where $t[y] \equiv \lambda x.\langle y, x\rangle$. The same thing is true in general, namely writing $t[s/x]$ we can always assume that the variable x occurred only once in $t[x]$ (again not counting the occurrences in $\lambda x.$). We will try to use just $t[s]$ instead of $t[s/x]$ as often as convenient. (We just defined the notion of "context", but since we used this word earlier for a different thing, here we won't give a particular name to it.)

Mints' system of reductions $\mathcal{R}$ is the following:

$$\mathcal{R}_1 \begin{cases} C[t^{B^A}] \xrightarrow{\eta} C[\lambda x.(t`x)] \;\; x \notin FV(t) & \text{provided neither } \; t \equiv \lambda y.s \\ & \text{nor } \; C[t] \equiv D[(t`s)] \\ C[t^{A\times B}] \xrightarrow{SP} C[\langle \pi_1(t), \pi_2(t)\rangle] & \text{provided neither } \; t \equiv \langle s_1, s_2\rangle \\ & \text{nor } \; C[t] \equiv D[\pi_i(t)] \end{cases}$$

$$\mathcal{R}_2 \begin{cases} C[t^1] \xrightarrow{T} C[*] & \text{if } t^1 \not\equiv * \\ C[(\lambda x.t`s)] \xrightarrow{\beta} C[t(s/x)] & \\ C[\pi_i(\langle t_1, t_2\rangle)] \xrightarrow{Pr_i} C[t_i] & i = 1, 2. \end{cases}$$

To be more precise, we should have said that $C[z]$ has exactly one occurrence of the variable z and then the above reductions would have looked e.g. as follows:

$$C[t^{B^A}/z^{B^A}] \xrightarrow{\eta} C[(\lambda x.(t`x))/z] \;\; x \notin FV(t)$$

provided neither $t \equiv \lambda y.s$ nor $C[z] \equiv D[(z`s)/w]$ for any two terms $D[w]$, s.

The terms in the brackets on the left we call *redexes*. The positions above which are excluded we call *restricted positions*. If t is a redex of a reduction γ (γ-redex) and if $t \xrightarrow{\gamma} s$ is a γ-reduction on t then $\gamma(t)$ will denote the term s. We also write $t \xrightarrow{\mathcal{R}} s$ if there is a reduction $\gamma \in \mathcal{R}$ such that $t \xrightarrow{\gamma} s$ or $t \equiv s$. (So again we are abusing notation a bit: $\mathcal{R}$ denotes (at the same time) its reflexive closure). The smallest equivalence relation containing $\mathcal{R}$ we will denote $\cong^{\mathcal{R}}$, so $t \cong^{\mathcal{R}} s$ if and only if there exists a sequence of terms $t \equiv t_0, t_1, \ldots, t_n \equiv s$ such that for every $0 \leq i < n$ $t_i \xrightarrow{\mathcal{R}} t_{i+1}$ or $t_{i+1} \xrightarrow{\mathcal{R}} t_i$. Often, we want to be precise and to write $t \cong^{\mathcal{R}}_X s$ if there is a sequence as above so that $X = FV(t_0, \ldots, t_n)$. The system of reductions in which the restrictions (on the position as well as on the shape of terms) are omitted, we call *unrestricted reductions* and we denote it by $\mathcal{R}^u$.

The restrictions in the above system are the obvious ones to prevent nontermination - it is interesting that this is "the right" choice i.e. with these restrictions the system is strongly normalizing and also sufficient for the λ-calculus in the following sense:

Proposition 3.27 *For every set of variables X, $\vdash t =_X s$ iff $t \cong_X^{\mathcal{R}} s$*

Proof: To prove that we need a very simple fact which is going to be used once more:

Lemma 3.28 *For every two terms t and s, $t \cong_X^{\mathcal{R}} s$ iff $t \cong_X^{\mathcal{R}^u} s$.*

Proof: In both directions, the proof is by induction on the length of the chain which witness the appropriate relation. The only thing which has to be checked is the base of induction in the proof from right to left, and the only four cases worth checking are the applications of unrestricted reductions when the subterm on which we act is in the restricted position or of restricted shape (or both). Let's check just two cases: suppose that a term $\langle t_1, t_2 \rangle$ appears as a subterm of a term r, we can write this as $r[\langle t_1, t_2 \rangle]$, and suppose that the unrestricted SP was applied on t i.e.

$$r[\langle t_1, t_2 \rangle] \xrightarrow{SP^u} r[\langle \pi_1(\langle t_1, t_2 \rangle), \pi_2(\langle t_1, t_2 \rangle) \rangle].$$

In the restricted case these two terms can be connected as follows:

$$r[\langle t_1, t_2 \rangle] \xleftarrow{Pr_1^\bullet} r[\langle \pi_1(\langle t_1, t_2 \rangle), \pi_2(\langle t_1, t_2 \rangle) \rangle].$$

(Notice that we don't have to separate the case when the term $\langle t_1, t_2 \rangle$ appears in the restricted position.) For the second case we choose the following: suppose

$$r[(t^{AB} {`} s^B)] \xrightarrow{\eta^u} r[(\lambda x^B.(t^{AB} {`} x^B) {`} s^B)].$$

These two terms van be connected in the restricted case as follows:

$$r[(t^{A^B}`s^B)] \xleftarrow{\beta} r[(\lambda x^B.(t^{A^B}`x^B)`s^B)].$$

(Again we did not have to separate the case when $t \equiv \lambda y.u$.) The other cases are equally easy. □

To prove the above proposition we just have to prove that $\vdash t =_X s$ iff $t \cong_X^{\mathcal{R}''} s$ but this is standard; for a simpler situation see, for example, proposition 3.2.1. in [Bar85]. □

The key observation is that $\mathcal{R}_1^*$ and $\mathcal{R}_2^*$ commute. More precisely we have the following proposition:

Proposition 3.29

$$\begin{tikzcd} a \arrow[r, "\mathcal{R}_2^*"] \arrow[d, "\mathcal{R}_1^*"'] & b \arrow[d, dashed, "\mathcal{R}_1^*"] \\ c \arrow[r, dashed, "\mathcal{R}_2^*"'] & d \end{tikzcd}$$

From this proposition, using some more or less obvious properties of the above system of reductions, we can establish several interesting corollaries e.g. confluence, strong normalization (giving also a particular, nice normalization strategy) and also confluence of the system same as the above one but without restrictions.

The proof is going to be divided in several lemmas but before we need to introduce some notation and some definitions.

The following notion makes sense in general: if $t[s/x] \xrightarrow{\rho} t'$ then the ρ-residual of s is whatever remains in t' of s. We are going to use that notion only when ρ is one of the $\mathcal{R}_1$-reductions and s is not the redex on which we apply ρ. Let us just add that the notion of residual as well as the concept of minimal development are standard in literature, see for example [HS86].

Definition 3.30 ($\mathcal{R}_1$-residual) *Let γ be one of the $\mathcal{R}_1$-reductions and let $t[s/x] \xrightarrow{\gamma} t'$ be on a γ-redex R such that $R \not\equiv s$. The γ-residual of s is defined as follows: first, if R is disjoint from s i.e. $t[s/x] \equiv T[s/x, R/y]$ for some term $T[x,y]$ then $t' \equiv T[s/x, \gamma(R)/y]$ and in this case s is the residual of s. Second, if R is a proper subterm of s i.e. $s \equiv S[R/y]$ for some term $S[y] \not\equiv y$, then $t' \equiv t[S[\gamma(R)/y]/x]$ and the residual of s is $S[\gamma(R)/y]$. Third, if s is a proper subterm of R i.e. $R \equiv r[s/x]$ for some term $r[x] \not\equiv x$ and $t[x] \equiv T[r[x]/y]$ for some term $T[y]$. Then we have two cases depending on γ: if $\gamma = \eta$ then $t' \equiv T[(\lambda z.r[s/x]`z)/y]$ and this s is the residual of s; if $\gamma = SP$ then $t' \equiv T[\langle \pi(r[s/x]), \pi'(r[s/x])\rangle/y]$ and these two occurrences of s are the residuals of s.*

The residual of a residual of some term s we will call again the residual of s.

Notice that every residual of a redex remains a redex. Also that residuals of disjoint terms remain disjoint. The only case when a term t can have more then one residual is when we perform an SP reduction on a term that contains t.

Definition 3.31 ($\mathcal{R}_1$-minimal development) *Let $R_1, \ldots, R_n$ be a set of γ-redexes in a term t ($\gamma \in \mathcal{R}_1$ or $\gamma = \mathcal{R}_1$). Then $t \xrightarrow{\gamma^*} s$ is a minimal development (denoted γ^m) on $R_1, \ldots, R_n$ if in each step we reduce a redex which is a residual of one of $R_1, \ldots, R_n$ (one of them at the first step) and minimal among them (with respect to the subterm relation). When we write a set of redexes for a minimal development as above we assume that if $i < j$ then $R_j \not\preceq R_i$ (R_j is not a subterm of R_i).*

Two minimal developments performed one after another don't have to make a minimal development but if the redexes of the second one don't contain any of the redexes of the first then they do make one minimal development on the union of the two sets of redexes. Although we are not going to use it we can notice that the above remarks on residuals tell that every minimal development on $R_1, \ldots, R_n$ ends in n steps (since we never apply SP-reduction on a redex containing a redex from the prescribed list).

Lemma 3.32 *A set of redexes determines the result of minimal development in the following sense: if $t \overset{\gamma'^m}{\to} s'$ and $t \overset{\gamma''^m}{\to} s''$ on the same set of redexes then $s' \equiv s''$.*

Proof: Induction on the number of redexes. Zero redexes don't make a problem. Neither does one. Since the order of reductions for the disjoint redexes is irrelevant we can assume that all the maximal redexes are reduced at the end. Suppose now that we omit all the maximal redexes. By the induction hypothesis without them both minimal developments give the same result (new minimal developments are "initial segments" of the old ones). Moreover (again by the induction hypothesis) the residual of the maximal redexes are the same in both cases and, as observed earlier they are disjoint (and they didn't multiply). Reducing them in whatever order gives the same result. □

The main use of minimal development is in the following lemma:

Lemma 3.33 *If every reduction $\rho \in \mathcal{R}_2$, and every $\gamma \in \mathcal{R}_1$, satisfy the following condition:*

$$\begin{tikzcd} a \arrow[r, "\rho"] \arrow[d, "\gamma^m"'] & b \arrow[d, dashed, "\gamma^m"] \\ c \arrow[r, dashed, "\mathcal{R}_2^*"'] & d \end{tikzcd}$$

then $\mathcal{R}_2^*$ and $\mathcal{R}_1^*$ commute i.e.

$$\begin{tikzcd} a \arrow[r, "\mathcal{R}_2^*"] \arrow[d, "\mathcal{R}_1^*"'] & b \arrow[d, dashed, "\mathcal{R}_1^*"] \\ c \arrow[r, dashed, "\mathcal{R}_2^*"'] & d \end{tikzcd}$$

Proof: Induction on the length of $\mathcal{R}_1^*$. When the length is 1 notice that every one step reduction γ is a minimal development and use the assumption plus induction on the length of $\mathcal{R}_2^*$. The above argument is also used when passing from "n-1" to "n". □

Notation: Let $\gamma \in \mathcal{R}_1$. Then $\gamma^{op} = \beta^*$ if $\gamma = \eta$ and $\gamma^{op} = Pr^*$ if $\gamma = SP$ (notice that $\gamma^{op*} = \gamma^{op}$). Also

$$\gamma^u(t) = \begin{cases} t & \text{if } u = 0 \\ \gamma(t) & \text{if } u = 1 \end{cases}$$

for example

$$\eta^u(t) = \begin{cases} t & \text{if } u = 0 \\ \lambda z.(t`z) & \text{if } u = 1 \end{cases}$$

(of course $z \notin FV(t)$).

From now on we will write just $t[a]$ instead of $t[a/x]$ whenever possible.

Lemma 3.34 *Let $a[b] \xrightarrow{\gamma^m} c$ be a minimal development on redexes $R_1, \ldots, R_i, \ldots, R_{i+j}$, $R_{i+j+1}[b], \ldots, R_{i+j+k}[b]$, where the redexes $R_1, \ldots, R_i$ are proper subterms of b and the term b appears exactly where shown. Then: $c \equiv a'[\gamma^u(b'), \ldots, \gamma^u(b')]$ so that $a[x] \xrightarrow{\gamma^m} a'[x, \ldots, x]$ on the redexes $R_{i+1}, \ldots, R_{i+j}$, $R_{i+j+1+u}[x], \ldots, R_{i+j+k}[x]$ and $b \xrightarrow{\gamma^m} b'$ on $R_1, \ldots, R_i$; here $u = 0$ if $R_{i+j+1}[b] \not\equiv b$ and $u = 1$ if $R_{i+j+1}[b] \equiv b$.*

Our assumption on the order of writing of redexes for a minimal development gives $R_{i+j+1}[b] \prec \cdots \prec R_{i+j+k}[b]$ (the relation $\prec$ stands for "proper subterm").

(Sometimes we will use the following form of the lemma: let $a[b] \xrightarrow{\gamma^m} c$ be a minimal development on redexes $R_1, \ldots, R_i, \ldots, R_{i+j}, R_{i+j+1}[b], \ldots, R_{i+j+k}[b]$, where the redexes $R_1, \ldots, R_i$ are subterms of b and the term b appears exactly where shown

as a proper subterm and maybe $R_i \equiv b$. *Then:* $c \equiv a'[\gamma^u(b'), \ldots, \gamma^u(b')]$ *so that* $a[x] \overset{\gamma^m}{\to} a'[x, \ldots, x]$ *on the redexes* $R_{i+1}, \ldots, R_{i+j}, R_{i+j+1}[x], \ldots, R_{i+j+k}[x]$ *and* $b \overset{\gamma^m}{\to} b'$ *on* $R_1, \ldots, R_{i-u}$*; here* $u = 0$ *if* $R_i \not\equiv b$ *and* $u = 1$ *if* $R_i \equiv b$.*)*

Proof: If all the redexes are disjoint from b then the statement is almost a tautology. There are two other cases - the first one is when there is a maximal redex properly contained in b. By lemma 3.32 we can assume that we first do the reductions in b i.e. $a[b] \overset{\gamma^m}{\to} a[b']$ and then the reductions on the redexes disjoint from b'. Since the reduction $a[b'] \overset{\gamma^m}{\to} d$ on a redex R_d disjoint from b' satisfies the statement i.e. $d \equiv a'[b']$ so that $a[x] \overset{\gamma^m}{\to} a'[x]$ on R_d (it can be proved by induction on the complexity of $a[x]$) we have proved the lemma in this case. The proof now continues by induction on the index k; the previous part is just the base of induction $k = 0$ i.e. there is no redex containing b. So let R denote the maximal redex containing b (it can be b itself) i.e. in the notation above $R = R_{i+j+k+1}[b]$. Our minimal development is $a[b] \overset{\gamma^m}{\to} c$ on the set of redexes as in the statement of the lemma plus R. By lemma 3.32 we can assume that R is the last one reduced. Consider now the minimal development without the last step. Since $a[b] \equiv A[R/y]$ (for an appropriate term A) we can apply the induction hypothesis and conclude that

$$A[R/y] \overset{\gamma^m}{\to} A'[R'/y] \tag{1}$$

on the redexes without R so that

$$A[y] \overset{\gamma^m}{\to} A'[y] \tag{2}$$

on the redexes outside of R - these are some of $R_{i+1}, \ldots, R_{i+j}$ and

$$R \overset{\gamma^m}{\to} R' \tag{3}$$

on the rest of the redexes - they are $R_1, \ldots, R_i$, the redexes from $R_{i+1}, \ldots, R_{i+j}$ which are in R and $R_{i+j+1}[b], \ldots, R_{i+j+k}[b]$. Applying the induction hypothesis to (3) (actually just the base of induction) we have $a' \equiv R'[\gamma^u(b'), \ldots, \gamma^u(b')]$ so that

$$R[x] \xrightarrow{\gamma^m} R'[x, \ldots, x] \tag{4}$$

on the redexes from $R_{i+1}, \ldots, R_{i+j}$ which are in R and $R_{i+j+1+u}[x], \ldots, R_{i+j+k}[x]$, and also

$$b \xrightarrow{\gamma^m} b' \tag{5}$$

on $R_1, \ldots, R_i$. Taking (2) and (4) we get

$$A[R[x]/y] \xrightarrow{\gamma^m} A'[R'[x, \ldots, x]/y] \tag{6}$$

on $R_{i+1}, \ldots, R_{i+j}, R_{i+j+1+u}[x], \ldots, R_{i+j+k}[x]$. Now, if we reduce $R'[x, \ldots, x]$ (which is indeed a redex) we have

$$A[R[x]/y] \xrightarrow{\gamma^m} A'[\gamma(R'[x, \ldots, x])/y] \tag{7}$$

on $R_{i+1}, \ldots, R_{i+j}, R_{i+j+1+1}[x], \ldots, R_{i+j+k+1}[x]$. Since $A[R[x]/y] \equiv a[x]$ we can use $a'[x, \ldots, x]$ to denote $A'[\gamma(R'[x, \ldots, x])/y]$. This together with (5) finishes the proof.

□

Lemma 3.35 *With the above notation the following hold:*

1. $\gamma^u(\gamma^v(t)) \xrightarrow{\gamma^{op}} \gamma^{u \vee v}(t)$

2. $a[\gamma(b)] \xrightarrow{\gamma^{op}} a[b]$, *providing b is of the 'forbidden shape' (i.e. $b \equiv \langle b_1, b_2 \rangle$ or $b \equiv \lambda x.b_1$) or in a restricted position or both.* □

Lemma 3.36 *The conditions of lemma (3.33) are satisfied when ρ is any $\mathcal{R}_2$ reduction.*

Proof: Case 1. $\rho \equiv Pr$ (and $\gamma \equiv \eta$ or $\gamma \equiv SP$). So suppose we have

$$\begin{tikzcd}
a[\pi(\langle b_1, b_2\rangle)] \arrow[r, "Pr"] \arrow[d, "\gamma^m"'] & a[b_1] \\
a'[\gamma^u(\pi(\langle\gamma^v(b_1'), b_2'\rangle))/x, \cdots, \gamma^u(\pi(\langle\gamma^v(b_1'), b_2'\rangle))/x] &
\end{tikzcd}$$

where the minimal development γ^m is done on the redexes $R_1, \ldots, R_i, \ldots, R_{i+j}$, $R_{i+j+1}[\pi(\langle b_1, b_2\rangle)], \ldots, R_{i+j+k}[\pi(\langle b_1, b_2\rangle)]$, where the redexes $R_1, \ldots, R_i$ are subterms of b_1, redexes which are in b_2 are not even shown and the term $\pi(\langle b_1, b_2\rangle)$ is exactly where shown. By lemma 3.34 the result of the minimal development has to be as above (since we can't apply γ on $\langle b_1, b_2\rangle$ - either the types don't match or the shape is forbidden) where $a[x] \xrightarrow{\gamma^m} a'[x, \ldots, x]$ on $R_{i+1}, \ldots, R_{i+j}, R_{i+j+1+u}[x], \ldots, R_{i+j+k}[x]$ and $b_1 \xrightarrow{\gamma^m} b_1'$ on $R_1, \ldots, R_{i-v}$.

For the sake of simplicity we will write just $a'[x]$ instead of $a'[x, \ldots, x]$ and similarly $a'[\gamma^u(\pi(\langle\gamma^v(b_1'), b_2'\rangle))]$ for $a'[\gamma^u(\pi(\langle\gamma^v(b_1'), b_2'\rangle))/x, \cdots, \gamma^u(\pi(\langle\gamma^v(b_1'), b_2'\rangle))/x]$ and so on. But we don't write Pr instead of Pr^* (e.g.the following diagram). Applying Pr^* we have:

$$\begin{tikzcd}
a[\pi(\langle b_1, b_2\rangle)] \arrow[r, "Pr"] \arrow[d, "\gamma^m"'] & a[b_1] \\
a'[\gamma^u(\pi(\langle\gamma^v(b_1'), b_2'\rangle))] \arrow[r, "Pr^*"'] & a'[\gamma^u(\gamma^v(b_1'))]
\end{tikzcd}$$

By lemma 3.35 we can add one more arrow:

$$\begin{tikzcd}
a[\pi(\langle b_1, b_2\rangle)] \arrow[r, "Pr"] \arrow[d, "\gamma^m"'] & a[b_1] & \\
 & & a'[\gamma^{u\vee v}(b_1')] \\
a'[\gamma^u(\pi(\langle\gamma^v(b_1'), b_2'\rangle))] \arrow[r, "Pr^*"] & a'[\gamma^u(\gamma^v(b_1'))] \arrow[ur, "\gamma^{op}"'] &
\end{tikzcd}$$

Now, if $u \vee v = 0$ it is obvious that $a[b_1] \xrightarrow{\gamma^m} a'[b_1']$ on $R_1, \ldots, R_i, \ldots, R_{i+j}, R_{i+j+1}[b_1], \ldots,$ $R_{i+j+k}[b_1]$ finishes the proof. Therefore suppose $u \vee v = 1$. If b_1 is of 'forbidden shape' or in a restricted position (it couldn't be both because we would have $u \vee v = 0$) then by lemma 3.35 $a'[\gamma(b_1')] \xrightarrow{\gamma^{op}} a'[b_1']$ and again the added γ^m is performed on all the redexes except the one which caused $u \vee v = 1$ i.e. $R_1, \ldots, R_{i-v}$, $R_{i+1}, \ldots, R_{i+j}, R_{i+j+1+u}[b_1], \ldots, R_{i+j+k}[b_1]$.

$$\begin{tikzcd}
a[\pi(\langle b_1, b_2\rangle)] \arrow[rr, "Pr"] \arrow[d, "\gamma^m"'] & & a[b_1] \arrow[dr, "\gamma^m"] & \\
& & & a'[b_1'] \\
a'[\gamma^u(\pi(\langle \gamma^v(b_1'), b_2'\rangle))] \arrow[rr, "Pr^*"'] & & a'[\gamma^u(\gamma^v(b_1'))] \arrow[ur, "\gamma^{op}"'] &
\end{tikzcd}$$

And finally, if b_1 is not of 'forbidden shape' nor in a restricted position (and still $u \vee v = 1$) then $a[b_1'] \xrightarrow{\gamma} a[\gamma(b_1')]$ can be performed so we have:

$$\begin{tikzcd}
a[\pi(\langle b_1, b_2\rangle)] \arrow[rr, "Pr"] \arrow[d, "\gamma^m"'] & & a[b_1] \arrow[dr, "\gamma^m"] & \\
& & & a'[\gamma(b_1')] \\
a'[\gamma^u(\pi(\langle \gamma^v(b_1'), b_2'\rangle))] \arrow[rr, "Pr^*"'] & & a'[\gamma^u(\gamma^v(b_1'))] \arrow[ur, "\gamma^{op}"'] &
\end{tikzcd}$$

where the new γ^m is performed on $R_1, \ldots, R_{i-(u \wedge v)}, R_{i+1}, \ldots, R_{i+j}, R_{i+j+1}[b_1], \ldots,$ $R_{i+j+k}[b_1]$. That finishes the proof of the first case.

<u>Case 2.</u> $\rho \equiv \beta$ (and $\gamma \equiv \eta$ or $\gamma \equiv SP$). So assume we have

$$\begin{tikzcd}
a[(\lambda y.b`c)] \arrow[r, "\beta"] \arrow[d, "\gamma^m"'] & a[b(c/y)] \\
a'[\gamma^u(\lambda y.\gamma^v(b')`\gamma^w(c'))] &
\end{tikzcd}$$

where γ^m is done on the redexes $R_1, \ldots, R_l, \ldots, R_i, \ldots, R_{i+j}, R_{i+j+1}[(\lambda y.b`c)], \ldots,$ $R_{i+j+k}[(\lambda y.b`c)]$, where the redexes $R_1, \ldots, R_l$ are subterms of b, redexes $R_{l+1}, \ldots, R_i$

are in c and the term $(\lambda y.b`c)$ is exactly where shown. (See the simplification in the notation mentioned in the first case.) Again by lemma 3.34 the result of the minimal development has to be as above (since we can't apply γ on $\lambda y.b$) where $a[x] \xrightarrow{\gamma^m} a'[x]$ on the redexes $R_{i+1}, \ldots, R_{i+j}, R_{i+j+1+u}[x], \ldots, R_{i+j+k}[x]$, $b \xrightarrow{\gamma^m} b'$ on $R_1, \ldots, R_{l-v}$ and $c \xrightarrow{\gamma^m} c'$ on $R_{l+1}, \ldots, R_{i-w}$. Without loss of generality we assume that b has at most three occurrences of y in it - so b looks like $b(y, y, y)$ where only the leftmost y is among the redexes (e.g. $R_1 \equiv y$) and only the rightmost y is in the restricted position for this γ. Applying β^* we have

$$a'[\gamma^u(\lambda y.\gamma^v(b'(\gamma(y), y, y))`\gamma^w(c'))] \xrightarrow{\beta^*} a'[\gamma^u(\gamma^v(b'(\gamma(\gamma^w(c')), \gamma^w(c'), \gamma^w(c'))))].$$

The rightmost occurence of $\gamma^w(c')$ is in a restricted position (by the assumption) so applying γ^{op} we get c' at this position (we used lemma 3.35). Also $\gamma(\gamma^w(c')) \xrightarrow{\gamma^{op}} \gamma(c')$ and $\gamma^u(\gamma^v(b')) \xrightarrow{\gamma^{op}} \gamma^{u \vee v}(b')$. So, we can add one more "arrow" to the diagram above and now we have

$$\begin{tikzcd}
a[(\lambda y.b(y,y,y)`c)] \arrow[r, "\beta"] \arrow[d, "\gamma^m"'] & a[b(c/y, c/y, c/y))] \\
a'[\gamma^u(\lambda y.\gamma^v(b'(\gamma(y), y, y))`\gamma^w(c'))] \arrow[r, "\mathcal{R}_2^*"'] & a'[\gamma^{u\vee v}(b'(\gamma(c'), \gamma^w(c'), c'))]
\end{tikzcd}$$

It is easy to see the redexes for the following minimal development: $b(c, c, c) \xrightarrow{\gamma^m} b(c', c', c')$. Now we have two cases: c' of forbidden shape or not (let us just mention that the first case is possible exactly when c is of forbidden shape). In the first case $\gamma(c') \xrightarrow{\gamma^{op}} c'$ and $\gamma^w(c') \xrightarrow{\gamma^{op}} c'$ (in fact $w = 0$ in this case). In the second case $c' \xrightarrow{\gamma} \gamma(c')$ and $c' \xrightarrow{\gamma} \gamma^w(c')$ (recall that first two positions of c' are not restricted in b). In any case the two "branches" of the above diagram are little closer - we have:

$$
\begin{tikzcd}[column sep=tiny]
 & & & a[(\lambda y.b(y,y,y)`c)] \arrow[dlll, "\gamma^m"'] \arrow[drrr, "\beta"] & & & \\
a'[\gamma^u(\lambda y.\gamma^v(b'(\gamma(y),y,y))`\gamma^w(c'))] \arrow[dr, "\mathcal{R}_2^*"'] & & & & & & a[b(c,c,c)] \arrow[ddll, "\gamma^m"] \\
 & a'[\gamma^{u\vee v}(b'(\gamma(c'),\gamma^w(c'),c'))] \arrow[dr, "\gamma^{op}"'] & & & & & \\
 & & a'[\gamma^{u\vee v}(b_0)] & & a[b_0] & &
\end{tikzcd}
$$

where:

$$
b_0 = \begin{cases} b'(c',c',c') & \text{if } c \text{ has forbidden shape} \\ b'(\gamma(c'),\gamma^w(c'),c') & \text{if } c \text{ has allowed shape.} \end{cases}
$$

(passage to b' also doesn't make a problem now). Now if $u \vee v = 0$ solution is obvious so assume $u \vee v = 1$. So the situation is exactly as in the first case - in any case to the above diagram we can add

$$
\begin{tikzcd}[column sep=tiny]
 & & & a[(\lambda y.b(y,y,y)`c)] \arrow[dlll, "\gamma^m"'] \arrow[drrr, "\beta"] & & & \\
a'[\gamma^u(\lambda y.\gamma^v(b'(\gamma(y),y,y))`\gamma^w(c'))] \arrow[dr, "\mathcal{R}_2^*"'] & & & & & & a[b(c,c,c)] \arrow[ddll, "\gamma^m"] \\
 & a'[\gamma^{u\vee v}(b'(\gamma(c'),\gamma^w(c'),c'))] \arrow[dr, "\gamma^{op}"'] & & & & & \\
 & & a'[\gamma^{u\vee v}(b_0)] \arrow[dr, "\gamma^{op}"'] & & a[b_0] \arrow[dl, "\gamma^m"] & & \\
 & & & a'[\gamma^r(b_0)] & & &
\end{tikzcd}
$$

where $r = 0$ if b_0 is of "forbidden shape" or in a restricted position and $r = 1$ otherwise. (Although γ^m is not in general a transitive relation here we took care of that by reducing "from inside" so that these consecutive γ^m's give a minimal development.)

Case 3 $\rho \equiv T$ (and $\gamma \equiv \eta$ or $\gamma \equiv SP$). So suppose we have

$$\begin{tikzcd} a[t^1] \arrow[r, "T"] \arrow[d, "\gamma^m"'] & a[*] \\ a'[t'/x] & \end{tikzcd}$$

(See again the simplified notation from case 1.). The minimal development was done on the redexes $R_1, \ldots, R_i, \ldots, R_{i+j}$, $R_{i+j+1}[t], \ldots, R_{i+j+k}[t]$, where the redexes $R_1, \ldots, R_i$ are subterms of t and the term t is exactly where shown. By lemma 3.34 the result of the minimal development has to be as above (since we can't apply γ on t - the types don't match) where $a[x] \xrightarrow{\gamma^m} a'[x]$ on $R_{i+1}, \ldots, R_{i+j}, R_{i+j+1}[x], \ldots,$ $R_{i+j+k}[x]$ and $t \xrightarrow{\gamma^m} t'$ on $R_1, \ldots, R_i$. Since t' still has type 1 it is obvious that the following holds:

$$\begin{tikzcd} a[t^1] \arrow[r, "T"] \arrow[d, "\gamma^m"'] & a[*] \arrow[d, "\gamma^m"] \\ a'[t'/x] \arrow[r, "T^*"'] & a'[*] \end{tikzcd}$$

where the new γ^m is done on $R_{i+1}, \ldots, R_{i+j}, R_{i+j+1}[*], \ldots, R_{i+j+k}[*]$. □

Proof: of proposition 3.29. Just apply the previous lemma and lemma 3.33. □

Lemma 3.37 $\mathcal{R}_1$ *is canonical (i.e. confluent and strongly normalizing).*

Proof: For details we refer to [Min80] - this part is correct. Let us just say that by Newman's lemma it's enough to show local confluence and strong normalization. Local confluence is easy here. Strong normalization is proved by assigning to each term t a natural number $\#t$ so that

$$t \xrightarrow{\mathcal{R}_1} t' \text{ implies } \#t > \#t'.$$

For that we first define the rank of a type as the number of the type forming operations in it i.e. $\#(A \times B) = \#(A^B) = \#(A) + \#(B) + 1$ and the rank of atomic types and terminal type is zero. Second, we define the degree of a redex

$$d(R) = 2^{\sum \#(A_i)}$$

where $A_1, \ldots, A_n$ are types of redexes in t which contain R. Finally

$$\#t = \sum d(R_j)$$

where R_j are all (occurrences of) redexes in t. □

Lemma 3.38 *The reduction $\mathcal{R}_2$ is canonical.*

Proof: A well known result is that in the typed case Pr_i and β are canonical (see for example [GLT89]). Adding T-contraction won't change much. Local confluence is simple to check; and strong normalization we get by showing that all T-reductions can be postponed after β, Pr-reductions. Let us just show that this is so in case of β. Suppose that before $a[(\lambda x.b`c)] \xrightarrow{\beta} a[(b(c))]$ there was a $T$ reduction. There are only two interesting cases: $C[t^1] \xrightarrow{T} C[*] \equiv c$ and $B[t^1/y] \xrightarrow{T} B[*/y] \equiv b$. In the first case the old reduction looked like $a[(\lambda x.b`C[t^1])] \xrightarrow{T} a[(\lambda x.b`c)] \xrightarrow{\beta} a[(b(c))/y]$, we transform it to $a[(\lambda x.b`C[t^1])] \xrightarrow{\beta} a[(b(C[t^1]))] \xrightarrow{T^*} a[b(c)]$. (By α-congruence we insure that there are no clashes of variables.) In the second case the old reduction looked as $a[(\lambda x.B[t^1/y]`c)] \xrightarrow{T} a[\lambda x.B[*/y]`c] \xrightarrow{\beta} a[b(c)]$. We transform it to $a[(\lambda x.B[t^1/y]`c)] \xrightarrow{\beta}$ $a[(B[t^1/y])(c)] \equiv a[B(c)[(t^1(c))/y]] \xrightarrow{T^*} a[B(c)[*/y]] \equiv a[b(c)]$. (Here we assumed that y was not a free variable in c - it was anyway denoting just a position.) Even simpler is the proof with Pr instead of β. Notice that in those transformations the number of Pr, β reductions remains the same and they "go up". So there is no infinite $\mathcal{R}_2$-reduction, if there were it would have to have infinitely many Pr, β-reductions (no terms have infinitely many consecutive T-reductions); transforming such a reduction we would get arbitrarily long reduction of consecutive Pr, β steps which would contradict strong normalizability of this fragment. □

This (and even less) is enough to show that Mints' reductions are confluent. That is also all what we need to finish the proof of the main theorem. For the record:

Corollary 3.39 *Mints' reductions are confluent.*

Proof: Suppose we have

$$\begin{tikzcd} a \arrow[r, "\mathcal{R}^*"] \arrow[d, "\mathcal{R}^*"'] & b \\ c & \end{tikzcd}$$

(Recall $\mathcal{R} = \mathcal{R}_1 \cup \mathcal{R}_2$.) Then just apply the induction on the number of changes of $\mathcal{R}_1^*$ and $\mathcal{R}_2^*$ in the branches together with lemmas 3.37, 3.38 and proposition 3.29. (That was the pattern of the Hindley-Rossen lemma.) □

Although not needed for the main lemma we can prove that Mints' reductions are not only confluent but also weakly normalizing.

Proposition 3.40 *Terms in $\mathcal{R}_1$ normal form are closed for $\mathcal{R}_2$-reductions, so we have that Mints' reductions are weakly normalizing, the strategy being: first do all $\mathcal{R}_1$-reductions then all $\mathcal{R}_2$-reductions (even more specifically: $\mathcal{R}_2$ can be separated: first all Pr and β and then all T reductions).*

Proof: Just notice that application of Pr and β-reductions on $\mathcal{R}_1$-normal term can't introduce new $\mathcal{R}_1$ redexes. For example if $a[\lambda x.b`c]$ is a $\mathcal{R}_1$-normal term, then $a[b(c)]$ is $\mathcal{R}_1$-normal too - all terms are in even more restricted position then they were before the β-reduction. Also use lemmas 3.37 and 3.38. □

Corollary 3.41 (Akama) *Mints' reductions are strongly normalizing.*

Proof: First observe (examining several cases) that if a term is not in $\mathcal{R}_2$-normal form it can't become $\mathcal{R}_2$-normal after application of $\mathcal{R}_1$-reductions. So assume that we have an infinite chain

$$t_0 \overset{R_{i_1}}{\to} t_1 \overset{R_{i_2}}{\to} \dots \overset{R_{i_n}}{\to} t_n \overset{R_{i_{n+1}}}{\to} \dots$$

($i_j \in \{1,2\}$). Since $\mathcal{R}_1$ is strongly normalizing as proved above, we have that in the above chain infinitely many reductions is of $\mathcal{R}_2$-type. Let $\bar{t}_i$ denote (the unique) $\mathcal{R}_1$-normal form of the term t_i. Then from the above infinite chain we can obtain (by $\mathcal{R}_1$-normalization) the following infinite chain:

$$\bar{t}_0 \overset{\mathcal{R}_2^*}{\to} \bar{t}_1 \overset{\mathcal{R}_2^*}{\to} \dots \overset{\mathcal{R}_2^*}{\to} \bar{t}_n \overset{\mathcal{R}_2^*}{\to} \dots$$

This chain exists by the commutativity of $\mathcal{R}_1^*$ and $\mathcal{R}_2^*$ (proposition 3.29) and the fact that $\mathcal{R}_1$ normal forms are closed for $\mathcal{R}_2$-reductions (proposition 3.40). Also we have that the chain is infinite by the observation from the beginning of the proof. But this contradicts strong normalization of $\mathcal{R}_2$. □

It is obvious that unrestricted Mints' reductions are not normalizing (for example x^{A^B} could be η-expanded and β-reduced infinitely many times); it is interesting, however, that they are confluent.

Corollary 3.42 *Mints' reductions without the restrictions are confluent.*

Proof: Suppose that

$$\begin{array}{ccc} a & \xrightarrow{(\mathcal{R}^u)^*} & b \\ {\scriptstyle (\mathcal{R}^u)^*}\downarrow & & \\ c & & \end{array}$$

That implies $b \cong_{\mathcal{R}^u} c$ and by lemma 3.28, it is the same as $b \cong_{\mathcal{R}} c$ and then from the confluence of $\mathcal{R}$ we have that there exists a term d such that $b \xrightarrow{\mathcal{R}^*} d$ and $c \xrightarrow{\mathcal{R}^*} d$. Since $\mathcal{R} \subset \mathcal{R}^u$ we have:

$$\begin{tikzcd} a \arrow[r, "(\mathcal{R}^u)^*"] \arrow[d, "(\mathcal{R}^u)^*"'] & b \arrow[d, dashed, "(\mathcal{R}^u)^*"] \\ c \arrow[r, dashed, "(\mathcal{R}^u)^*"'] & d \end{tikzcd}$$

□

Remark 3.43 Mints' reductions were given in [Min80]. Unfortunately lemma 7.1 (vi) and theorem 7.3 are not correct. The theorem states that the normalizing strategy is first $\mathcal{R}_2$ then $\mathcal{R}_1$. Applying that on $x^{1\times A}$ we get $\langle \pi(x), \pi'(x)\rangle$. But applying the strategy on $\langle \pi(x), \pi'(x)\rangle$ gives $\langle *, \pi'(x)\rangle$. So two equal terms x and $\langle \pi(x), \pi'(x)\rangle$ don't have the same normal form. If the calculus were without the terminal object (and the appropriate rule) then first $\mathcal{R}_2$ then $\mathcal{R}_1$ would be a normalizing strategy; this was suggested already in [Pra71, 3.5.2 Normalization theorem] (notice however that the uniqueness of the normal form (there called expanded normal form) was not stated c.f. 3.5.3 Strong normalization theorem *loc. cit.*), but also recall that Prawitz considers all first order logical connectives (even *absurdity*) but not the connective *true*.

Let us finally restate and prove

Proposition 3.18(bis!) If $f =_x g$ in a free λ-calculus and x does not occur as a free variable in either f or g then we also have $f = g$.

Proof: Since by corollary 3.39 (free) typed λ-calculus is confluent for a set of reductions which do not introduce new variables, from $f =_x g$ we have that there is a term t such that f and g reduce to it, therefore $f = t$ and $t = g$. □

The above proof concludes the proof of the first main result. Let us just give a remark on the history of the above result.

Remark 3.44 We obtained the theorem 3.16 in spring 1990 and I gave a talk on that on a McGill seminar organized by Prof. Lambek. However, I was using Mints' result without noticing this mistake in it. In December 1991 I corrected these mistakes in Mints' paper and distributed my paper (almost the same as the second and third section of the thesis) in March 1992 to some people at McGill University. Since the end of July, beginning of August 1992 the paper was available from an "ftp-site" as announced on two e-mail lists (under the name "On free CCC"). The only mathematical changes are two additional corollaries about Mints' reductions - corollaries 3.41,3.42 which are immediate consequences of our main result about Mints' reductions i.e. proposition 3.29. The corollary 3.41 is the main result in [Aka93] - a paper which has our paper as a reference. Also, independently, Jay [Jay92] gives a different proof of strong normalization for a system in which every type had a closed normal term - a property not available in general.

4 The Interpolation Result

In this part we formulate and prove the second main result, that is the interpolation property of bicartesian closed categories as well as cartesian closed categories. Also, at the end we give couple of applications which show that our interpolation is indeed a strong generalization of the corresponding result for Heyting algebras.

Let us formulate more precisely what we are after:

Definition 4.45 *A square consisting of categories, functors and a natural transformation*

$$\begin{tikzcd} \mathcal{A} \arrow[r, "F"] \arrow[d, "G"'] & \mathcal{B} \arrow[d, "H"] \arrow[dl, phantom, "\Swarrow\tau"] \\ \mathcal{C} \arrow[r, "K"'] & \mathcal{D} \end{tikzcd}$$

has the interpolation property if for every two objects $C \in \mathcal{C}$ and $B \in \mathcal{B}$ and every arrow $H(B) \xrightarrow{d} K(C)$ in $\mathcal{D}$ there exist an object $A \in \mathcal{A}$ and arrows $B \xrightarrow{b} F(A)$ in $\mathcal{B}$ and $G(A) \xrightarrow{c} C$ in $\mathcal{C}$ such that $d = K(c)\tau_A H(b)$.

We will consider only those squares of the above type with τ a natural isomorphism.

Definition 4.46 *A 2-category of categories has the interpolation property if all the Pushouts have the interpolation property.*

We use the term "Pushout" to specify that we have in mind appropriate version of the 2-categorical weighted bicolimit - the precise definition is coming latter see subsection 4.4.2.

Theorem 4.47 *The 2-category $\mathcal{BCC}$ of bicartesian closed categories has the interpolation property.*

This result holds not only for bicartesian closed categories but also for cartesian closed categories.

The above theorem is proved in two steps. First we prove a stronger interpolation property for intuitionistic propositional logic than known in the literature - namely, as mentioned in the introduction, we not only obtain the interpolant but we also show that the new proofs (of the interpolant and from the interpolant) when composed are actually equal to the proof which we started with. Additional care is needed to handle the presence of axioms; the presence of additional equalities among proofs turns out to be no problem at all. Identifying proofs with terms we can precisely state this as follows.

Proposition 4.48 *Let L_1 and L_2 be two languages, and let T_1 and T_2 be two $\lambda\delta$-theories on the respective languages. Let T_0 be a theory on the language $L_1 \cap L_2$ such that $T_0 \subset T_1 \cap T_2$ (we may as well assume that the theories are deductively closed). Let $(x^B \triangleright t^C)$ be a term in the language $L_1 \cup L_2$ such that the type B is in L_1 and the type C is in L_2. Then, there is a type A in $L_1 \cap L_2$ and terms $(x^B \triangleright r^A)$ in L_1, and $(y^A \triangleright s^C)$ in L_2 such that:*

$$T_1 \cup T_2 \vdash t =_{x^B} s(r/y).$$

The proof of this proposition is given at the end of section 4.2. In the section 4.3 we give the first reformulation of the above proposition in categorical terms and we obtain the interpolation property for "ordinary" 2-pushouts in the 2-category $\mathcal{BCC}_s$ (bicartesian closed categories with the chosen structure, strict bc-functors and natural isomorphisms as 2-cells) - one can notice, though, that the functors associated to the above proposition (i.e. $\mathcal{B}_{T_0} \to \mathcal{B}_{T_i}$) have a particular property - they are inclusions on objects. This is so because L_0 is a subset of L_i - no collapsing of types has occurred. Although the main result - theorem 4.47 does not require any assumption of this type - these somewhat unusual functors will play a substantial role in the proof of it.

The proof of theorem 4.47 then proceeds via connection between the strict and nonstrict doctrines á la [BKP89] and via construction of Pushouts in the nonstrict doctrine $\mathcal{BCC}$ from the 2-pushouts in the strict doctrine $\mathcal{BCC}_s$.

4.1 Prawitz' permutative reductions

As it was mentioned earlier, to obtain the interpolation property for bicartesian closed categories we are going to analyze a syntactic proof of the Craig interpolation property for intuitionistic propositional logic. There, the strategy is to show weak normalization of the appropriate system of proof-reductions, and then to study the normal form. It is hard to attack the whole set of equations as given in definition 2.3, but luckily enough we don't have to do that. We can take a system of equation which is strictly weaker than the one in the definition and this will suffice! It is well known that adding disjunction to the positive fragment $\{\top, \wedge, \rightarrow\}$ of intuitionistic logic brings difficulties of a new kind to the analysis of proofs. E.g. to prove a satisfactory form of normalization of proofs, i.e. the one which will give the subformula property, it is no longer enough to consider just β-like reductions (denoted by R_2 below) but one has to add a substantial part of η-like reductions (or expansions) for disjunctions (and the connective "false") (denoted by C and E below). This is already done in [Pra65], for a "recent" discussion connecting this to linear logic see [GLT89]. We choose to work with Prawitz' original reductions, in the form given in [GLT89].

Notation: for two terms t and r we write $t \succ r$ if r is an immediate subterm of t. Reflexive and transitive closure of $\succ$ we shall denote $\succ^*$, so that $t \succ^* s$ means that s is a subterm of t.

The following set of reduction we will denote by ρ; we find it convenient to partition ρ as follows:

$$R_2 \begin{cases} (\lambda x.t`s) \xrightarrow{\beta} t(s/x) & \\ \pi_i(\langle t_1, t_2\rangle) \xrightarrow{Pr_i} t_i & i = 1,2 \\ \delta x_1.u_1, x_2.u_2; \iota_i(t) \xrightarrow{in_i} u_i(t/x_i) & i = 1,2. \end{cases}$$

$$C\begin{cases}\pi_i(\delta x_1.u_1, x_2.u_2; t) \xrightarrow{1,2} \delta x_1.\pi_i(u_1), x_2.\pi_i(u_2); t\\ (\delta x_1.u_1, x_2.u_2; t)`r \xrightarrow{3} \delta x_1.u_1`r, x_2.u_2`r; t\\ \delta x_1.u_1, x_2.u_2; (\delta y_1.v_1, y_2.v_2; t) \xrightarrow{4} \delta y_1.(\delta x_1.u_1, x_2.u_2; v_1), y_2.(\delta x_1.u_1, x_2.u_2; v_2); t\\ \epsilon^A(\delta x.u, y.v; t) \xrightarrow{5} \delta x.\epsilon^A(u), y.\epsilon^A(v); t\end{cases}$$

$$E\begin{cases}\pi_i(\epsilon^{A_1\times A_2}(t)) \xrightarrow{1,2} \epsilon^{A_i}(t)\\ \epsilon^{A^B}(t)`r \xrightarrow{3} \epsilon^A(t)\\ \delta x.u^C, y.v^C; \epsilon^{A+B}(t) \xrightarrow{4} \epsilon^C(t)\\ \epsilon^A(\epsilon^0(t)) \xrightarrow{5} \epsilon^A(t)\end{cases}$$

The following "scheme" of reductions in natural deduction corresponds to the C-reductions:

$$\dfrac{\dfrac{\begin{matrix} \\ \vdots \\ A\vee B\end{matrix} \quad \begin{matrix}\cancel{A}\\ \vdots \\ C\end{matrix} \quad \begin{matrix}\cancel{B}\\ \vdots \\ C\end{matrix}}{C}\,\delta \quad \tau}{D} \quad\longrightarrow\quad \dfrac{\begin{matrix}\vdots \\ A\vee B\end{matrix} \quad \dfrac{\begin{matrix}\cancel{A}\\ \vdots \\ C\end{matrix}}{D}\,\tau \quad \dfrac{\begin{matrix}\cancel{B}\\ \vdots \\ C\end{matrix}}{D}\,\tau}{D}\,\delta$$

Here, τ stands for one of the five rules for elimination of connectives - so it can have one, two or three hypotheses; δ stands for the elimination of disjunction. Similarly one can represent the E-reductions.

To see that the system of equations generated by the above set of reductions ρ is weaker than the one in definition 2.3 use lemma 2.4. The following is not really needed for our purposes but let us notice that the equation $\delta x^A.\iota_1(x^A), y^B.\iota_2(y^B); w = w$ is not provable in the above system, but it is equivalent to the equation (γ) relative to the above system.

The above reduction system ρ is strongly normalizing and has the Church-Rosser property (stated and partially proved in [Pra71],[GLT89],[Gir71]) (although I have never checked that), however - here we give (and prove in the appendix A) just what we need and this is weak normalization (cf. [Pra65] p.50).

Theorem 4.49 (Prawitz' weak normalization) *Every term t in the $\lambda\delta$-calculus can be reduced to a normal (reduced) form (with respect to the above system ρ).*

4.2 Interpolation in the λ-calculus setting

Theorem 4.49 is used to prove the Craig interpolation theorem for intuitionistic propositional logic. In this section we give the proof which follows Prawitz' proof which is given as a hint in [Pra65]. There are several differences between our proof and the Prawitz proof. First the minor ones: he works in natural deduction and does not have the connective 1 ("true") in the language - we work with typed lambda calculus and we have all the propositional connectives. The most important difference is that we do more, i.e. we check that not only do we obtain an interpolant but also that the two proofs when composed are equal to the proof which we began with (actually we even get more: the two proofs when composed reduce to the normal form of the proof that we begun with). Also we allow the presence of arbitrary axioms as well as additional equations of proofs and Prawitz does not consider this at all.

A general remark - whenever a variable appears "out of nowhere" it means that this is a brand new variable.

First we need the following lemma (cf. [Pra65], Cor. 3 pp. 54.).

Lemma 4.50 *For a free $\lambda\delta$-calculus (no additional equations but with constants) the following holds: let t^C be a ρ-normal term such that $t \equiv \pi_i(s)$ or $t \equiv s`r$ or $t \equiv \delta x.u, y.v; s$ or $\epsilon^C(s)$ or t is an atomic term (i.e. a variable or a constant (or $*$)). Then there exists a chain $t \equiv t_0 \succ \cdots \succ t_n$ of successive subterms of t such that they are in the following relation: for every $0 \leq i < n$, $t_i \equiv \pi_j(t_{i+1})$, or $t_i \equiv t_{i+1}`u$, or $t_i \equiv \delta x.u, y.v; t_{i+1}$ (for some u, v, x and y), or $t_i \equiv \epsilon^B(t_{i+1})$ and t_n is an atomic term (i.e. $t_n \equiv v^E$ where v is a variable or a constant (or $*$)).*

In particular if the term t is not $*$ then t_n is not $*$. So, if in addition the $\lambda\delta$-calculus doesn't have constants then t_n is a variable.

Furthermore, by C-normality we have: if $t_{n-1} \equiv \delta x.u, y.v; t_n$ then the above chain $t \equiv t_0 \succ \cdots \succ t_n$ is actually just $t \equiv \delta x.u, y.v; t_1$ and t_1 is either $x^{E_1+E_2}$ or a constant (for some ρ-normal terms u, v).

Similarly, by E-normality, if $t_{n-1} \equiv \epsilon^B(t_{n-1})$ then $t \equiv \epsilon^B(t_1)$ and t_1 is a variable x^0 (we don't have constants of type 0).

Proof: Induction on the complexity of t. If the complexity is zero this is no problem by the property of t_n. If $t \equiv \pi_i(s)$ then $s^{C_1 \times C_2} \equiv s_1`r_1$ for some ρ-normal terms r_1 and s_1, or $s \equiv \pi_j(s_1)$ for a ρ-normal term s_1, or s is of the zero complexity (s can't be a δ-form by C-normality, ϵ-form by E-normality nor $\langle u, v\rangle$ by R_2-normality, and the other cases don't type-match). All three cases are all right by the induction hypothesis. So to get the chain for t we just add t on the top of the chain for s. Similarly the case when $t \equiv s`r$ ($s^{C_1^{C_2}} \equiv \pi_j(s_1)$ or $s \equiv s_1`r_1$ or s is atomic). Third case is when $t \equiv \delta x.u, y.v; s$. Then again s can be either $s_1`r$ or $\pi_j(s_1)$ or atomic ; this is handled again by the induction hypothesis. Notice that t_n must have a complex type or type 0 unless $t \equiv *$. □

Definition 4.51 *A^+ (A^-) is the set of atoms which occur positively (negatively) in A. For a context $\Gamma = x_1^{B_1}, \ldots, x_n^{B_n}$ we define $\Gamma^+ = \cup_i B_i^+$ ($\Gamma^- = \cup_i B_i^-$). Also we define $1^+ = 1^- = 0^+ = 0^- = \phi$ (empty set).*

Lemma 4.52 *For a free $\lambda\delta$-calculus without free constant terms we have: let $(\Gamma \triangleright t^C)$ be a ρ-normal term and let $\Gamma_1 \cup \Gamma_2 = \Gamma$ be a partition of the context. Then there are $(\Gamma_1 \triangleright r^A)$ and $(\Gamma_2, y^A \triangleright s^C)$ such that*

1. *$t =_\Gamma s(r/y)$,*

2. *$A^+ \subseteq \Gamma_1^+ \cap (\Gamma_2^- \cup C^+)$,*

3. $A^- \subseteq \Gamma_1^- \cap (\Gamma_2^+ \cup C^-)$.

Actually one proves in 1) that $s(r/y) \xrightarrow{\rho^*} t$.

Proof: First notice that in the case when $\Gamma_1 = \phi$ the result is obvious: just take $A \equiv 1$, $r \equiv *$ and $s \equiv t$. Now we proceed by induction on the complexity of t. We have the following cases:

Case 1. $t^C \equiv x^C$. We have two subcases: $x^C \in \Gamma_1$ then take $A \equiv C$, $r \equiv x^C$ and $s \equiv y^C$; and in the second subcase $x^C \in \Gamma_2$ then take $A \equiv 1$, $s \equiv x^C$ and $r \equiv *$.

Case 2. $t^C \equiv *$, so $C \equiv 1$. Just take $A \equiv 1$, $s \equiv *$ and $r \equiv *$.

Case 3. $t^C \equiv \langle t_1^{C_1}, t_2^{C_2} \rangle$, so $C \equiv C_1 \times C_2$ and $t_i^{C_i}$ are ρ-normal. By the induction hypothesis there are $(\Gamma_1 \triangleright r_i^{A_i})$ and $(\Gamma_2, y_i^{A_i} \triangleright s_i^{C_i})$ $(i = 1,2)$ such that

$$t_i =_\Gamma s_i(r_i/y_i),\ A_i^+ \subseteq \Gamma_1^+ \cap (\Gamma_2^- \cup C_i^+),\ A_i^- \subseteq \Gamma_1^- \cap (\Gamma_2^+ \cup C_i^-).$$

Now take $A \equiv A_1 \times A_2$, $s \equiv \langle s_1(\pi_1(y^{A_1 \times A_2})/y_1), s_2(\pi_2(y^{A_1 \times A_2})/y_2) \rangle$ and $r \equiv \langle r_1, r_2 \rangle$ and see that it satisfies the lemma.

Case 4. $t^C \equiv \iota(t_1^{C_1})$, so $C \equiv C_1 + C_2$ and $t_1^{C_1}$ is ρ-normal. By the induction hypothesis there are $(\Gamma_1 \triangleright r_1^{A_1})$ and $(\Gamma_2, y_1^{A_1} \triangleright s_1^{C_1})$ such that

$$t_1 =_\Gamma s_1(r_1/y_1),\ A_1^+ \subseteq \Gamma_1^+ \cap (\Gamma_2^- \cup C_1^+),\ A_1^- \subseteq \Gamma_1^- \cap (\Gamma_2^+ \cup C_1^-).$$

Now take $A \equiv A_1$, $s \equiv \iota(s_1)$ and $r \equiv r_1$ and see that it satisfies the lemma.

Case 5. $t^C \equiv \lambda x^{C_2}.t_1^{C_1}$, so $C \equiv C_1^{C_2}$, and $t_1^{C_1}$ is ρ-normal. Then by the induction hypothesis applied to the term $(\Gamma, x^{C_2} \triangleright t_1^{C_1})$ and the partition of its context as $\Gamma_1 \cup (\Gamma_2, x^{C_2})$ there are $(\Gamma_1 \triangleright r_1^{A_1})$ and $(\Gamma_2, x^{C_2}, y_1^{A_1} \triangleright s_1^{C_1})$ such that

$$t_1 =_{\Gamma, x} s_1(r_1/y_1),\ A_1^+ \subseteq \Gamma_1^+ \cap ((\Gamma_2^-, C_2^-) \cup C_1^+),\ A_1^- \subseteq \Gamma_1^- \cap ((\Gamma_2^+, C_2^+) \cup C_1^-).$$

Now take $A \equiv A_1$, $r \equiv r_1$ and $s \equiv \lambda x^{C_2}.s_1$ and check that the lemma holds (use $(C_1^{C_2})^- = C_2^+ \cup C_1^-$).

Case 6. (In this case only, we use lemma 4.50.) $t \equiv \pi_i(s)$ or $t \equiv s`r$ or $t \equiv \delta x.u, y.v; s$ or $t \equiv \epsilon^U(s)$ where r, s, u and v are some ρ-normal terms. Then by lemma 4.50 we have that there exists a chain of immediate subterms: $t \equiv t_0 \succ t_1 \succ \cdots \succ t_i \succ \cdots \succ t_n$ such that the following subcases can take place: $t_{n-1}^{E_i} \equiv \pi_i(x^{E_1 \times E_2})$ or $t_{n-1}^{E_1} \equiv x^{E_1^{E_2}} {`} u^{E_2}$ or $t_{n-1} \equiv \delta x.u, y.v; x^{E_1+E_2}$ or $t_{n-1} \equiv \epsilon^B(x^0)$ for some ρ-normal terms u and v since in the calculus we don't have additional constants and $t \not\equiv *$.

Subcase 6.1.1. $t_{n-1}^{E_i} \equiv \pi_i(x^{E_1 \times E_2})$ and $x^{E_1 \times E_2} \in \Gamma_1$.

Let t' be like the term t except that it has x^{E_i} instead of t_{n-1} (notice however, that t' can contain $x^{E_1 \times E_2}$) so $t \equiv_\Gamma t'(\pi_i(x^{E_1 \times E_2})/x^{E_i})$. The complexity of t' is lower then the complexity of t, so we can apply the induction hypothesis on $(\Gamma \cup \{x^{E_i}\} \triangleright t')$ and the partition $\Gamma, x^{E_i} = (\Gamma_1, x^{E_i}) \cup \Gamma_2$. Then by the induction hypothesis then exist $(\Gamma_1, x^{E_i} \triangleright R^A)$ and $(\Gamma_2, y^A \triangleright S^C)$ such that

$$t' =_{\Gamma, x^{E_i}} S(R/y), \quad A^+ \subseteq (\Gamma_1 \cup E_i)^+ \cap (\Gamma_2^- \cup C^+), \quad A^- \subseteq (\Gamma_1 \cup E_i)^- \cap (\Gamma_2^+ \cup C^-).$$

Recall that $x^{E_1 \times E_2}$ is in Γ_1. So we define $r = R(\pi_i(x^{E_1 \times E_2}))$, $s = S$ and A stays the same and we can check that the lemma is satisfied. First:

$$t =_\Gamma t'(\pi_i(x^{E_1 \times E_2})/x^{E_i}) =_\Gamma S((R(\pi_i(x^{E_1 \times E_2})/x^{E_i}))/y) \equiv s(r/y).$$

The second and third part of the conclusion are satisfied since $E_i^s \subset (E_1 \times E_2)^s$ where $s = +, -$. It is obvious that the stronger hypothesis gives the reduction instead of the equality.

Subcase 6.1.2. $t_{n-1}^{E_i} \equiv \pi_i(x^{E_1 \times E_2})$ and $x^{E_1 \times E_2} \in \Gamma_2$. Again consider $(\Gamma,\ x^{E_i}\ \triangleright\ t')$ where t' is the same as t except that x^{E_i} appears instead of $\pi_i(x^{E_1 \times E_2})$, therefore $t'(\pi_i(x^{E_1 \times E_2})/x^{E_i}) \equiv t$. Since t' is less complex then t apply the induction hypothesis on the partition $\Gamma, x^{E_i} = \Gamma_1 \cup (\Gamma_2, x^{E_i})$. Then by the induction hypothesis then exist $(\Gamma_1 \triangleright R^A)$ and $(\Gamma_2, x^{E_i}, y^A \triangleright S^C)$ such that

$$t' =_{\Gamma,x^{E_i}} S(R/y),\ \ A^+ \subseteq \Gamma_1^+ \cap ((\Gamma_2 \cup E_i)^- \cup C^+),\ \ A^- \subseteq \Gamma_1^- \cap ((\Gamma_2 \cup E_i)^+ \cup C^-).$$

Recall that $x^{E_1 \times E_2}$ is in Γ_2. So we define $r = R$, $s = S(\pi_i(x^{E_1 \times E_2}))$ and A stays the same and we can check that the lemma is satisfied. First, $t =_\Gamma t'(\pi_i(x^{E_1\times E_2})/x^{E_i}) =_\Gamma S(\pi_i(x^{E_1\times E_2})/x^{E_i})(R/y) \equiv s(r/y)$. The second and third part of the conclusion are satisfied since $E_i^s \subset (E_1 \times E_2)^s$ where $s = +, -$. As earlier, it is obvious that the stronger hypothesis gives the reduction instead of the equality.

<u>Subcase 6.2.1.</u> $t_{n-1}^{E_1} \equiv x^{E_1^{E_2}}{}^{\text{‘}}u^{E_2}$ and $x^{E_1^{E_2}} \in \Gamma_1$.

Then we apply the induction hypothesis on $(\Gamma \rhd u^{E_2})$ and the "reverse" partition $\Gamma_2 \cup \Gamma_1 = \Gamma$ to get $(\Gamma_2 \rhd R_1^{A_1})$ and $(\Gamma_1, y_1^{A_1} \rhd S_1^{E_2})$ such that

$$u =_\Gamma S_1(R_1/y_1), \qquad A_1^+ \subseteq \Gamma_2^+ \cap (\Gamma_1^- \cup E_2^+), \qquad A_1^- \subseteq \Gamma_2^- \cap (\Gamma_1^+ \cup E_2^-).$$

Applying the induction hypothesis once more on $(\Gamma, z^{E_1} \rhd w^C)$, where $w(t_{n-1}/z) \equiv t$, and the partition $(\Gamma_1, z^{E_1}) \cup \Gamma_2 = \Gamma, z^{E_1}$ to get $(\Gamma_1, z \rhd R_2^{A_2})$ and $(\Gamma_2, y_2^{A_2} \rhd S_2^C)$ such that

$$w =_{\Gamma,z} S_2(R_2/y_2),\ \ A_2^+ \subseteq (\Gamma_1^+ \cup E_1^+) \cap (\Gamma_2^- \cup C^+),\ \ A_2^- \subseteq (\Gamma_1^- \cup E_1^-) \cap (\Gamma_2^+ \cup C^-).$$

Now take $A = A_2^{A_1}$, $r \equiv \lambda y_1.R_2((x\text{‘}S_1)/z)$ and $s \equiv S_2((y^{A_2^{A_1}}\text{‘}R_1)/y_2)$ and check that the lemma is satisfied. Indeed the first conclusion follows from:

$s(r/y) \equiv S_2((\lambda y_1.R_2((x\text{‘}S_1)/z)\text{‘}R_1)/y_2) \xrightarrow{\beta} S_2(R_2((x\text{‘}S_1(R_1/y_1))/z))/y_2)$ (y_1 appears only in S_1) $\equiv S_2(R_2((x\text{‘}u)/z))/y_2) \equiv S_2(R_2(t_{n-1}/z))/y_2) \equiv S_2(R_2/y_2)(t_{n-1}/z)$ (since z appears only in R_2) $= w(t_{n-1}/z) \equiv t$. To get $\xrightarrow{\beta}$ instead of $=$ in the last step use the stronger induction hypothesis: "actually one proves in 1) that $s(r/y) \xrightarrow{\beta^*} t$" not just the equality.

For the other two conclusions use that $x^{E_1^{E_2}} \in \Gamma_1$, so $E_1^s \subseteq \Gamma_1^s$ and $E_2^s \subseteq \Gamma_1^{-s}$ where $s = -, +$.

<u>Subcase 6.2.2.</u> $t_{n-1}^{E_1} \equiv x^{E_1^{E_2}}{}^{\text{‘}}u^{E_2}$ and $x^{E_1^{E_2}} \in \Gamma_2$.

Then we apply the induction hypothesis on $(\Gamma \triangleright u^{E_2})$ and the partition $\Gamma_1 \cup \Gamma_2 = \Gamma$ to get $(\Gamma_1 \triangleright R_1^{A_1})$ and $(\Gamma_2, y_1^{A_1} \triangleright S_1^{E_2})$ such that

$$u =_\Gamma S_1(R_1/y_1), \qquad A_1^+ \subseteq \Gamma_1^+ \cap (\Gamma_2^- \cup E_2^+), \qquad A_1^- \subseteq \Gamma_1^- \cap (\Gamma_2^+ \cup E_2^-).$$

Applying the induction hypothesis once more on $(\Gamma, z^{E_1} \triangleright w^C)$, where $w(t_{n-1}/z) \equiv t$, and the partition $\Gamma_1 \cup (\Gamma_2, z^{E_1}) = \Gamma, z^{E_1}$ to get $(\Gamma_1 \triangleright R_2^{A_2})$ and $(\Gamma_2, z^{E_1}, y_2^{A_2} \triangleright S_2^C)$ such that

$$w =_{\Gamma,z} S_2(R_2/y_2), \quad A_2^+ \subseteq \Gamma_1^+ \cap (\Gamma_2^- \cup E_1^- \cup C^+), \quad A_2^- \subseteq \Gamma_1^- \cap (\Gamma_2^+ \cup E_1^+ \cup C^-).$$

Now take $A = A_1 \times A_2$, $s \equiv S_2(\pi_2((y^{A_1 \times A_2})/y_2))((x`S_1(\pi_1(y^{A_1 \times A_2})/y_1))/z)$ and $r \equiv \langle R_1, R_2 \rangle$ and check that the lemma is satisfied. (We will check only the first conclusion. Indeed:

$$s(r/y) \equiv S_2(\pi_2(\langle R_1, R_2 \rangle)/y_2))((x`S_1(\pi_1(\langle R_1, R_2 \rangle)/y_1))/z) \xrightarrow{\rho}$$

$$S_2(R_2/y_2)((x`S_1(R_1/y_1))/z) \xrightarrow{\rho}$$

$$S_2(R_2/y_2)((x`u)/z) \equiv S_2(R_2/y_2)(t_{n-1}/z) \xrightarrow{\rho} w(t_{n-1}/z) \equiv t.$$

The last two reductions where under the stronger induction hypothesis; otherwise we have equality.)

<u>Subcase 6.3.1.</u> $t_{n-1} \equiv \delta x_1^{E_1}.t^1, x_2^{E_2}.t^2; x^{E_1+E_2}$ and $x^{E_1+E_2} \in \Gamma_1$. First notice that by the end of the lemma 4.50 $t \equiv \delta x_1.t^1, x_2.t^2; x^{E_1+E_2}$ for some ρ-normal terms t^1, t^2. Since the complexity of t^i is smaller the the complexity of t we apply the induction hypothesis on $(\Gamma, x^{E_i} \triangleright t^i)$, $(i = 1, 2)$ and the partition $\Gamma, x^{E_i} = (\Gamma_1, x^{E_i}) \cup \Gamma_2$. So, we have that there are $(\Gamma_1, x^{E_i} \triangleright r_i^{A_i})$ and $(\Gamma_2, y_i^{A_i} \triangleright s_i^C)$ such that (for $i = 1, 2$):

$$t^i =_{\Gamma,x^{E_i}} s_i(r_i/y_i), \quad A_i^+ \subseteq (\Gamma_1, x^{E_i})^+ \cap (\Gamma_2^- \cup C^+), \quad A_i^- \subseteq (\Gamma_1, x^{E_i})^- \cap (\Gamma_2^+ \cup C^-).$$

Now, let $A = A_1 + A_2$, $r \equiv \delta x_1.\iota_1(r_1), x_2.\iota_2(r_2); x^{E_1+E_2}$ and $s \equiv \delta y_1.s_1, y_2.s_2; y^{A_1+A_2}$. Obviously s and r have right context; also the lemma is satisfied: first

$$s(r/y^{A_1+A_2}) \equiv \delta y_1.s_1, y_2.s_2; (\delta x_1.\iota_1(r_1), x_2.\iota_2(r_2); x^{E_1+E_2}) \xrightarrow{C_1}$$

$$\delta x_1.(\delta y_1.s_1, y_2.s_2; t_1(r_1)), x_2.(\delta y_1.s_1, y_2.s_2; t_2(r_2)); x^{E_1+E_2}$$

$$\xrightarrow{\rho} \delta x_1.s_1(r_1/y_1), x_2.s_2(r_2/y_2); x^{E_1+E_2} \overset{ind.hyp.}{\longrightarrow} \delta x_1.t^1, x_2.t^2; x^{E_1+E_2} \equiv t.$$

The second and the third part of the conclusion follow from $E_i^s \subset (E_1 + E_2)^s$ where $s = +, -$.

<u>Subcase 6.3.2.</u> $t_{n-1} \equiv \delta x.u, y.v; x^{E_1+E_2}$ and $x^{E_1+E_2} \in \Gamma_2$. The same as the previous except that the partition of Γ is $\Gamma_1 \cup (\Gamma_2, x^{E_i})$.

<u>Subcase 6.4.</u> Finally assume $t_{n-1} \equiv \epsilon(x^0)$. Again, by the end of the lemma 4.50 $t \equiv \epsilon^C(x^0)$. If $x^0 \in \Gamma_1$ then take $A = 0$, $s \equiv \epsilon^C(x^0))$ and $r \equiv x^0$. Obviously the lemma is satisfied. If $x^0 \in \Gamma_2$ then take $A = 1$, $r \equiv *$ and take $s \equiv \epsilon^C(x^0)$. Indeed $(\Gamma_1 \triangleright *)$ and $(\Gamma_2, y^1 \triangleright \epsilon^C(x^0))$ satisfy the lemma. □

Now we have to prove a similar lemma but in the case when the $\lambda\delta$-calculus contains additional constant terms (but not additional equations). To see what kind of difficulty we have let's give an example: let $(x^B \triangleright c^{C^B}{}^{\iota}x^B)$ then for $\Gamma_1 = x^B$ (and $\Gamma_2 = \phi$) the above lemma (as it is) would be false - A would have to be 1 but this wouldn't do. However the problem really doesn't exist, we just treat the additional constants as variables/additional hypothesis - what they actually are, and state the lemma carefully.

Lemma 4.53 *For a free $\lambda\delta$-calculus (with free constant terms) we have: let $(\Gamma \triangleright t^C)$ be a ρ-normal term and let $\Gamma_1 \cup \Gamma_2 = \Gamma$ be a partition of the context; also let $\Sigma = \xi_1^{D_1}, \ldots, \xi_m^{D_m}$ be the set of the free constants which appear in t and let $\Sigma = \Sigma_1 \cup \Sigma_2$ be a partition of that. Then there are $(\Gamma_1 \triangleright r^A)$ and $(\Gamma_2, y^A \triangleright s^C)$ such that*

1. $t =_\Gamma s(r/y)$,

2. $A^+ \subseteq (\Gamma_1^+ \cup \Sigma_1^+) \cap (\Gamma_2^- \cup \Sigma_2^- \cup C^+)$, *3.* $A^- \subseteq (\Gamma_1^- \cup \Sigma_1^-) \cap (\Gamma_2^+ \cup \Sigma_2^+ \cup C^-)$,

4. r^A *contains only constants from* Σ_1 *and* s^C *contains only constants from* Σ_2.

Actually one proves in 1) that $s(r/y) \xrightarrow{\rho^\bullet} t$.

Proof: We have to take care of two things: the first is that 1) is not stated with $\Gamma \cup \Sigma$ as context but just Γ, similarly the terms r and s are over the smaller contexts, and the second is the additional conclusion 4). We take the term T^C which is the same as t^C except that we put new variables $w_i^{D_i}$ instead of $\xi_i^{D_i}$. Then the above lemma gives $(\Gamma_1, \Sigma_1 \triangleright R^A)$ and $(\Gamma_2, \Sigma_2, y^A \triangleright S^C)$ such that $T =_{\Gamma\cup\Sigma} S(R/y)$ and the rest as in 2) and 3) and they don't have any constants (except maybe $*$) - we used the same name Σ for the set of the variables replacing ξ_i. Now just substitute the constants in the place of the appropriate variables and get the statement (r is the term R but with the constants instead of the variables from Σ_1, similarly s is the "new" S). □

In the previous example the lemma gives two solutions depending what the partition of the $\Sigma = w^{C^B}$ is (and with the given partition of context as $\Gamma_1 = x^B$). When $\Sigma_1 = C^B$ (and $\Sigma_2 = \phi$) then take $A = C$, $r \equiv t$ and $s^C \equiv y^C$. In the second case when $\Sigma_2 = C^B$ take $A = B$, $r \equiv x^B$ and $s^C \equiv (w^{C^B}{}^{\backprime}y^B)$.

The following lemma (cf. [Pra65] Cor.5 pp. 46.) is an immediate consequence of the previous lemma when $\Gamma_2 = \phi$ and the fact that every term has a ρ-normal form.

Lemma 4.54 *For every term* $(\Gamma \triangleright t^C)$ *and for every partition* $\Sigma = \Sigma_1 \cup \Sigma_2$ *of the (free) constants from* t, *there exist* $(\Gamma \triangleright r^A)$ *and* $(y^A \triangleright s^C)$ *such that*

1. $t =_\Gamma s(r/y)$,

2. $A^+ \subseteq (\Gamma^+ \cup \Sigma_1^+) \cap (\Sigma_2^- \cup C^+)$, *3.* $A^- \subseteq (\Gamma^- \cup \Sigma_1^-) \cap (\Sigma_2^+ \cup C^-)$,

4. r^A *contains only constants corresponding to* Σ_1 *and* s^C *contains only constants corresponding to* Σ_2.

And now an important corollary which we find very interesting:

Corollary 4.55 *In a free $\lambda\delta$-calculus on a language $L = L_1 \cup L_2$ (with free constants) for every term $(x^B \triangleright t^C)$ such that $B \in L_1$ and $C \in L_2$ there exist $(x^B \triangleright r^A)$ and $(y^A \triangleright s^C)$ such that:*

1. $t =_{x^B} s(r/y)$,

2. $(x^B \triangleright r^A) \in L_1$,

3. $(y^A \triangleright s^C) \in L_2$.

(Notice that this implies that $A \in L_1 \cap L_2$.)

Before we prove the corollary let us give an example in categorical terminology: suppose that L_1 consists of three free objects/types X, Y, Z and suppose that it has only one free arrow/constant $a : Y \to Z$. Suppose also that L_2 consists of the same types and the only free arrow is $b : X \to Y$. Now suppose that we want to interpolate $ab : X \to Z$. For a moment it may look a bit suprising that there is any "useful" arrow in L_1 from X. But there is! It is quite easy to see that the interpolation is obtained from the following two arrows:

$$(\langle(a\pi')^*, 1_X\rangle : X \to Z^Y \times X) \in L_1 \text{ and } (\varepsilon\langle\pi, b\pi'\rangle : Z^Y \times X \to Z) \in L_2.$$

Now we go back to

Proof of the previous corollary: Given $L_1 \cup L_2$ make Σ_1 to be the set of (free) constants from L_1, and Σ_2 to be the set of (free) constants from $L_2 - L_1$. Also notice that for a term u if all the variables and constants which appear are from a language L then the term u is on the language L i.e. all the types which appear in u are made out of the basic types which appear in the typing of the variables and constants. Now apply the previous lemma to obtain that all the constants from r^A are from L_1 and since x^B was in L_1 we have that $(x^B \triangleright r^A) \in L_1$. To show that $(y^A \triangleright s^C) \in L_2$ we reason similarly and in addition we check that $y^A \in L_2$ - from the previous lemma parts 2) and 3) we have that $A \subset \Sigma_2 \cup C$ and this gives $A \in L_2$ by the definition of Σ_2 and the assumption $C \in L_2$. The first conclusions in the corollary and in the previous lemma are the same. □

Finally we can notice that the above proof actually gives the proof of a stronger result which allows arbitrary $\lambda\delta$-theories, not only free ones - this was already mentioned as proposition 4.48 which we can now restate and prove:

Proposition 4.48 (bis!) Let L_1 and L_2 be two languages, and let T_1 and T_2 be two $\lambda\delta$-theories in the respective languages. Let T_0 be a theory in the language $L_1 \cap L_2$ such that $T_0 \subset T_1 \cap T_2$ (we may as well assume that the theories are deductively closed). Let $(x^B \triangleright t^C)$ be a term in the language $L_1 \cup L_2$ such that the type B is in L_1 and the type C is in L_2. Then, there is a type A in $L_1 \cap L_2$ and terms $(x^B \triangleright r^A)$ in L_1, and $(y^A \triangleright s^C)$ in L_2 such that:

$$T_1 \cup T_2 \vdash t =_{x^B} s(r/y).$$

Proof: In the previous corollary we proved the statement without referring to theories, i.e. $\vdash t =_{x^B} s(r/y)$. From that, of course, follows $T_1 \cup T_2 \vdash t =_{x^B} s(r/y)$. □

It is interesting to notice that not every interpolant in the usual sense is one in our sense e.g. $X \times X \vdash X \times X$ has as an interpolant X but in our case X can't be an interpolant if the above proof is just $1_{X \times X}$ and X is atomic (since there is only one arrow in hom(X, X) - this itself can be proved by the subformula property - it would mean that X is isomorphic to $X \times X$ for every bicartesian closed category).

4.3 Interpolation in the categorical setting

Now, we want to see a categorical rewording of the previous proposition. For that, we will first give an (expected) pushout construction.

Proposition 4.56 *Let L_1 and L_2 be two languages, and let T_1 and T_2 be two $\lambda\delta$–theories in the respective languages. Let T_0 be a theory in the language $L_1 \cap L_2$ such that $T_0 \subset T_1 \cap T_2$ (again we may assume that the theories are deductively closed). To this situation we can associate the following diagram in $\mathcal{BCC}_s$:*

$$\begin{tikzcd} \mathcal{B}_{T_0} \arrow[r, "F_1"] \arrow[d, "F_2"'] & \mathcal{B}_{T_1} \\ \mathcal{B}_{T_2} & \end{tikzcd}$$

where $\mathcal{B}_{T_0} \xrightarrow{F_i} \mathcal{B}_{T_i}$ $i = 1, 2$ are obtained from the respective interpretations $T_0 \xrightarrow{M_i|_{L_0}} \mathcal{B}_{T_i}$ where $T_i \xrightarrow{M_i} \mathcal{B}_{T_i}$ is the canonical model; see corollary 2.12 ($M\,|_L$ means the reduct of the model M in the language L).

Now, the 2-pushout of the above diagram we form as follows:

$$\begin{tikzcd} \mathcal{B}_{T_0} \arrow[r, "F_1"] \arrow[d, "F_2"'] & \mathcal{B}_{T_1} \arrow[d, "H"] \\ \mathcal{B}_{T_2} \arrow[r, "K"'] & \mathcal{B}_{T_1 \cup T_2} \end{tikzcd}$$

where H is the unique functor from corollary 2.12 such that $H \circ M_1 = M_{1\cup 2}\,|_{L_1}$, similarly $K \circ M_2 = M_{1\cup 2}\,|_{L_2}$; here, $M_{1\cup 2}$ is the canonical model $T_1 \cup T_2 \rightarrow \mathcal{B}_{T_1 \cup T_2}$.

Proof: This 2-pushout has the following universal property: for every two functors $\mathcal{B}_{T_1} \xrightarrow{P} E$ and $\mathcal{B}_{T_2} \xrightarrow{Q} E$ such that $PF_1 = QF_2$ there exists a unique functor $\mathcal{B}_{T_1 \cup T_2} \xrightarrow{T}$

E such that $TH = P$ and $TK = Q$ and also, for any other $\mathcal{B}_{T_1 \cup T_2} \xrightarrow{T'} E$ (not necessarily satisfying any of the previous equations) and two natural isomorphisms $TH \overset{\phi_1}{\Rightarrow} T'H$ and $TK \overset{\phi_2}{\Rightarrow} T'K$ which satisfy $\phi_1 F_1 = \phi_2 F_2$ there exists a unique natural isomorphism $T \overset{\psi}{\Rightarrow} T'$ such that $\phi_1 = \psi H$ and $\phi_2 = \psi K$.

To see this, notice that the functors P and Q induce a model of $T_1 \cup T_2$ in E, call it $P \cup Q$ - it is determined by its restrictions in L_i i.e. $P \cup Q \mid_{L_1} = P \circ M_1$ and $P \cup Q \mid_{L_2} = Q \circ M_2$ - there are no conflicts since the two interpretation agree in $L_1 \cap L_2$ i.e. $P \circ (F_1 \circ M_0) = Q \circ (F_2 \circ M_0)$. Then, again by corollary 2.12, we have the unique functor $T : \mathcal{B}_{T_1 \cup T_2} \to E$ such that

$$T \circ M_{1 \cup 2} = P \cup Q.$$

Let us now check that such a T satisfies $TH = P$ and $TK = Q$. We will establish just the first equality, for that is enough to see that $TH \circ M_1 = P \circ M_1$. This follows by restricting the previous equation (4.3) in L_1 since $P \cup Q \mid_{L_1} = P \circ M_1$ and $M_{1 \cup 2} \mid_{L_1} = H \circ M_1$.

To finish the proof, we have to establish the 2-dimensional property of the above construction: suppose that one more functor is given $\mathcal{B}_{T_1 \cup T_2} \xrightarrow{T'} E$ (not necessarily satisfying any of the previous equations) and two natural isomorphisms $TH \overset{\phi_1}{\Rightarrow} T'H$ and $TK \overset{\phi_2}{\Rightarrow} T'K$ which satisfy $\phi_1 F_1 = \phi_2 F_2$, we want a unique natural isomorphism $T \overset{\psi}{\Rightarrow} T'$ such that $\phi_1 = \psi H$ and $\phi_2 = \psi K$. Using the 2-dimensional part of corollary 2.12 we can restate the above paragraph as follows. Given a model $M' : T_1 \cup T_2 \to E$ and two isomorphisms of models $(P \cup Q \mid_{L_1} \overset{\phi_1}{\Rightarrow} M' \mid_{L_1}) \in Mod_{T_1} E$, $(P \cup Q \mid_{L_2} \overset{\phi_2}{\Rightarrow} M' \mid_{L_2}) \in Mod_{T_2} E$ which satisfy $\phi_{1A} = \phi_{2A}$ for every type A in $T_1 \cap T_2$. We want a unique isomorphism $P \cup Q \overset{\psi}{\Rightarrow} M'$ such that $\psi_B = \phi_{1B}$ for every type $B \in L_1$ and $\psi_C = \phi_{2C}$ for every type $C \in L_2$. For the uniqueness it is enough to notice that $P \cup Q \overset{\psi}{\Rightarrow} M'$ is "already" defined on the basic types $\psi_X = \phi_{iX}$ iff $X \in L_i$ (no problem if $X \in L_1 \cap L_2$), so it is unique by the remark 2.6. For the existence use again the remark 2.6 - since every atomic constant ξ^C from $L_1 \cup L_2$ has to be in L_1 or L_2 and the type C therefore is in T_1 or T_2 (the intersection again doesn't make a problem). □

Proposition 4.57 *The pushout square from the above proposition has the interpolation property.*

Proof: Since the above square commutes we take the identity to be the 2-cell from the definition of the interpolation property. Take $B \in \mathcal{B}_{T_1}$ and $C \in \mathcal{B}_{T_2}$ and assume that there exists an arrow $H(B) \xrightarrow{d} K(C) \in \mathcal{B}_{T_1 \cup T_2}$. By the definition of a category associated to a theory there exists a term $(x^B \triangleright t^C)$ in the language $L_1 \cup L_2$ such that $d = [x^B \triangleright t^C]$. Now apply proposition 4.48. □

4.4 Strict vs. non-strict

In this section we shall prove that the previous interpolation result for the strict doctrine $\mathcal{BCC}_s$ (proposition 4.57) implies our second main result, i.e. the interpolation theorem 4.47 for the non-strict doctrine $\mathcal{BCC}$. To do that, as we said in the introduction, we proceed along the lines of [BKP89].

First, in the next subsection, we give a connection in the form of an adjointness, between the strict and non-strict doctrines of bicartesian closed categories. The statement itself is a direct consequence of the theorem [BKP89, Thm. 3.13] however, here we give Makkai's formulation (motivated by possible applications to non-tripleable doctrines) and we give an original and direct (though quite syntactic) proof of it. Doing that we prove some lemmas which are used later.

After establishing the connection between the doctrines, we give a construction of Pushouts (bipushouts) in the non-strict doctrine using the "ordinary" 2-pushouts from the strict doctrine. This is done in order to prove the interpolation result (recall that the pushouts in the strict doctrine were of a special kind - the main characteristic being that they were constructed over the functors which are inclusions on objects cf. proposition 4.57).

This is similar to the work in [BKP89, 5.8 and 5.9] but not the same. The difference is that we use a special variant of their statement (cf. our theorem 4.69) but this is not enough here - we have on our disposal only the ordinary pushouts in the strict doctrine (and even they are of the special kind); they, on the other hand, can afford the "luxury" of pseudo-pushouts in the strict doctrine.

Let us just repeat that all the statements about interpolation apply to cartesian closed categories as well.

4.4.1 Adjointness

To go from a strict to a nonstrict doctrine is quite easy - just forget the chosen structure; to go backwards is a bit more subtle. The naive approach of choosing a strict structure on every bicartesian closed category (possible by the Axiom of Choice) won't work since it may not be possible to choose the structures in such a way that the functors become strict.[1]

Let us now examine more closely the relation between a bc-category $\mathcal{B}$ and the category $\mathcal{B}_{\mathcal{T}_\mathcal{B}}$ (often we will denote $\mathcal{B}_{\mathcal{T}_\mathcal{B}}$ by $\mathcal{B}_s$). First recall that in the case of an "ordinary" (nonstrict) category to talk about the internal language $L_\mathcal{B}$ and the theory $\mathcal{T}_\mathcal{B}$ we have to choose a bc-structure Σ on $\mathcal{B}$ and then these notions are defined as in definition 2.10; however we will use just $L_\mathcal{B}$ for $L_{(\mathcal{B},\Sigma)}$, and $T_\mathcal{B}$ for $T_{(\mathcal{B},\Sigma)}$ when Σ is understood. Notice that although $\mathcal{B} \in \mathcal{BCC}$ was nonstrict $\mathcal{B}_{\mathcal{T}_\mathcal{B}} \in \mathcal{BCC}_s$ is strict.

To proceed further we need additional notation: Let $(f : A \to B) \in \mathcal{B}$ (and $\mathcal{B}$ is a strict bc-category). Then, as earlier, $\hat{f} : 1 \to B^A$ denotes the unique transpose of f, i.e. $\hat{f} = (f\pi'_{1\times A})^*$. Let $\overrightarrow{A_i}$ be a finite sequence of objects from $\mathcal{B}$ and let $\overrightarrow{X_{A_i}}$ be the corresponding sequence of basic types from $L_\mathcal{B}$. Let $\mathcal{T}_i$, $i = 1,2$ be two type-terms satisfying the following: $\mathcal{T}_1(\overrightarrow{A_i}) = A$ and $\mathcal{T}_2(\overrightarrow{A_i}) = B$ (as one may recall from definition 2.3 the type-term simply means a type built out of the basic types which are specified in the parenthesis - since the same operations exist on objects of a (strict) bcc we can use this notation for the objects as well). For a particular kind

[1] As an example consider the functor $\mathcal{A} \xrightarrow{F} \mathcal{B} \in \mathcal{BCC}$ such that $\mathcal{A}$ is the 4 element Boolean algebra, $\mathcal{B}$ the category with just two isomorphic (but different) objects and F functor which maps the three non-bottom elements to one of the objects and the bottom to the other object. It is easy to see that there is no strict structure on the codomain category which would make F strict. Let us remark here that when F is an inclusion on objects we can do that - this is going to be exploited later.

of arrow in $\mathcal{B}_s$ we use the following notation:

$$\mathcal{T}_1(\overrightarrow{X_{A_i}}) \xrightarrow{c_f} \mathcal{T}_2(\overrightarrow{X_{A_i}}) = [x : \mathcal{T}_1(\overrightarrow{X_{A_i}}) \triangleright c_f^{\mathcal{T}_2(\overrightarrow{X_{A_i}})^{\mathcal{T}_1(\overrightarrow{X_{A_i}})}} \text{`} x^{\mathcal{T}_1(\overrightarrow{X_{A_i}})}].$$

Often we will talk about a special kind of the above arrow - those which have basic types for domain and codomain e.g. $X_A \xrightarrow{c_f} X_B$; we shall call them elementary arrows in $\mathcal{B}_s$.

Also recall the fact mentioned earlier that there exists a forgetful 2-functor

$$\mathcal{BCC}_s \xrightarrow{||} \mathcal{BCC}$$

which on objects (0-cells) just forgets the chosen structure (on 1- and 2-cells doesn't do anything).

Lemma 4.58 *For every $\mathcal{B} \in \mathcal{BCC}$ there is an equivalence*

$$\mathcal{B} \xrightarrow{\eta_{\mathcal{B}}} |\mathcal{B}_s|$$

defined as follows: $\eta_{\mathcal{B}}(A \xrightarrow{f} B) = X_A \xrightarrow{c_f} Y_B$.

Proof: Consider the following commutative diagram:

$$\begin{tikzcd}
 & \mathcal{T}_{(\mathcal{B},\Sigma)} \arrow[dl, "M_{\mathcal{B}}"'] \arrow[dr, "M"] & \\
\mathcal{B}_{\mathcal{T}_{(\mathcal{B},\Sigma)}} \arrow[rr, "\mu_{(\mathcal{B},\Sigma)}"'] & & (\mathcal{B},\Sigma)
\end{tikzcd}$$

where M and $M_{\mathcal{B}}$ are the canonical models and $\mu_{\mathcal{B}}$ is defined to be the functor induced by corollary 2.12. Explicitly, $\mu_{\mathcal{B}}(X_A) = A$, X_A a basic type of $L_{\mathcal{B}}$ (corresponding to an object $A \in \mathcal{B}$) and $\mu_{\mathcal{B}}(1 \xrightarrow{c_f} \mathcal{T}(\overrightarrow{X_{A_i}})) = 1 \xrightarrow{f} \mathcal{T}(\overrightarrow{A_i})$. It is easy to see that $\mu_{\mathcal{B}}$ is an equivalence of categories (it is onto on objects - therefore essentially surjective, also it is full since "c_f is mapped on f" and it is faithful by the definition of $\mathcal{T}_{(\mathcal{B},\Sigma)}$

and by the completeness of $T_{(\mathcal{B},\Sigma)}$ with respect to $M_{\mathcal{B}}$ since $u =_x v \in T_{(\mathcal{B},\Sigma)}$ iff $M(x \triangleright u) = M(x \triangleright v)$).

Now notice that $|\mu_{\mathcal{B}}|\eta_{\mathcal{B}} = 1_{\mathcal{B}}$ by definition of $\eta_{\mathcal{B}}$ (and $\mu_{\mathcal{B}}$). Obviously then, $\eta_{\mathcal{B}}$ is an equivalence of categories because $|\mu_{\mathcal{B}}|$ has a pseudo-inverse. $\square$

Lemma 4.59 *Suppose $\mathcal{C}_s \xrightarrow{H_i} \mathcal{A}$ $i = 1,2$ are two strict bc-functors and suppose that on the basic types and elementary arrows they agree, i.e. $H_1(X_A \xrightarrow{c_f} X_B) = H_2(X_A \xrightarrow{c_f} X_B)$. Then $H_1 = H_2$.*

Proof: By corollary 2.12 it is enough to show that they agree on the basic arrows, i.e. that $H_1(1 \xrightarrow{c_g} \mathcal{T}(\overrightarrow{X_{A_i}})) = H_2(1 \xrightarrow{c_g} \mathcal{T}(\overrightarrow{X_{A_i}}))$ for $g \in \hom_{\mathcal{C}}(1, \mathcal{T}(\overrightarrow{A_i}))$. For that, let us establish several claims:

<u>Claim 1.</u> For every two objects $\mathcal{T}(\overrightarrow{X_{A_i}})$ and $X_{\mathcal{T}(\overline{A_i})}$ in $\mathcal{C}_s$ there are (unique coherent) isomorphisms

$$\mathcal{T}(\overrightarrow{X_{A_i}}) \underset{\gamma_{\mathcal{T}}^{-1}}{\overset{\gamma_{\mathcal{T}}}{\rightleftarrows}} X_{\mathcal{T}(\overline{A_i})}$$

in $\mathcal{C}_s$. Moreover they are build inductively out of *elementary* arrows, bc-operations (as π, ε etc.) and inverses of already constructed isomorphisms. Notice also that $\mu_{\mathcal{C}}(\gamma_{\mathcal{T}}) = \mu_{\mathcal{C}}(\gamma_{\mathcal{T}}^{-1}) = 1_{\mathcal{T}(\overline{A_i})}$.

<u>Proof of the Claim 1:</u> is by induction on complexity of $\mathcal{T}$. Let us just check two cases: first $\mathcal{T} = 1$ (the terminal object) then we have $1 \xrightarrow{\gamma_1} X_1 = 1 \xrightarrow{c_{1_1}} X_1$ and $X_1 \xrightarrow{\gamma_1^{-1}} 1 = 0_{X_1}$ (the latter one is a bc-operation (constant actually) and the former one is its inverse). For the second case assume that $\mathcal{T} = \mathcal{T}_2^{\mathcal{T}_1}$ and assume that $\mathcal{T}_1 \underset{\gamma_{\mathcal{T}_1}^{-1}}{\overset{\gamma_{\mathcal{T}_1}}{\rightleftarrows}} X_{\mathcal{T}_1}$ and $\mathcal{T}_2 \underset{\gamma_{\mathcal{T}_2}^{-1}}{\overset{\gamma_{\mathcal{T}_2}}{\rightleftarrows}} X_{\mathcal{T}_2}$ have been constructed as above. Then it is easy to check that $\gamma_{\mathcal{T}} = (\gamma_{\mathcal{T}_2}\varepsilon(1 \times \gamma_{\mathcal{T}_1}^{-1}))^*$ and $\gamma_{\mathcal{T}}^{-1} = (\gamma_{\mathcal{T}_2}^{-1}\varepsilon(1 \times \gamma_{\mathcal{T}_1}))^*$ satisfy the claim. $\square$

From the above claim immediately follows that $H_1(\gamma_T) = H_2(\gamma_T)$ and $H_1(\gamma_T^{-1}) = H_2(\gamma_T^{-1})$.

Claim 2. For every basic arrow in $\mathcal{C}_s$ i.e. $[\,\triangleright c_f^{\mathcal{T}(\overrightarrow{X_{A_i}})}] : 1 \to \mathcal{T}(\overrightarrow{X_{A_i}})$, the following equation is among the axioms of $\mathcal{T}_\mathcal{C}$:

$$[\,\triangleright c_f^{\mathcal{T}(\overrightarrow{X_{A_i}})}]\gamma_T^{-1} = [\,\triangleright c_f^{X_{\mathcal{T}(\overrightarrow{A_i})}}].$$

Proof of the Claim 2: Using the faithfulness of $\mathcal{C}_s \xrightarrow{\mu_\mathcal{C}} \mathcal{C}$ and the fact that the both sides are mapped to f. $\square$

The lemma follows from the two claims. $\square$

Now we want to see what is a natural strict functor between $\mathcal{A}_s$ and $\mathcal{B}_s$ corresponding to a nonstrict bc-functor $\mathcal{A} \xrightarrow{F} \mathcal{B}$. The naive guess is wrong: suppose that a constant $c_f^{X_{A_1} \times X_{A_2}} \in L_\mathcal{A}$ was induced by the arrow $f \in \hom_\mathcal{A}(1, A_1 \times A_2)$. Then (on the level of languages) $c_f^{X_{A_1} \times X_{A_2}} \mapsto c_{F(f)}^{X_{F(A_1)} \times X_{F(A_2)}}$ is wrong because this constant does not even exist in $L_\mathcal{B}$ ($F(f) \notin \hom_\mathcal{B}(1, F(A_1) \times F(A_2))$ - F is not strict). To give the correct definition we need some preparatory lemmas.

Lemma 4.60 *Let $\mathcal{A} \xrightarrow{F} \mathcal{B}$ be a functor in $\mathcal{BCC}$. Let $(\mathcal{A}, \Sigma_1)$, $(\mathcal{B}, \Sigma_2)$ be the two categories with added strict structure. For any finite set of objects $\overrightarrow{A_i} \in \mathcal{A}$ and any type-term $\mathcal{T}(\overrightarrow{A_i})$ there are unique coherent isomorphisms*

$$\mathcal{T}(\overrightarrow{F(A_i)}) \underset{\rho_T^{-1}}{\overset{\rho_T}{\rightleftarrows}} F(\mathcal{T}(\overrightarrow{A_i}))$$

Proof: What we mean by the "coherent isomorphism" is the following: we have the following basic coherent isomorphisms:

$$0 \underset{\rho_0^{-1}}{\overset{\rho_0}{\rightleftarrows}} F(0)\,,\quad 1 \underset{\rho_1^{-1}}{\overset{\rho_1}{\rightleftarrows}} F(1)\,,\ \text{and}\ \ F(A_1)\Box F(A_2) \underset{\rho_{A_1\Box A_2}^{-1}}{\overset{\rho_{A_1\Box A_2}}{\rightleftarrows}} F(A_1\Box A_2)$$

where $X \Box Y$ stands for $X \times Y$ or $X + Y$ or X^Y.

They are called coherent because we require that they commute with appropriate structure e.g. $\pi_i \rho^{-1}_{A_1 \times A_2} = F(\pi_i)$ and $F(\pi_i)\rho_{A_1 \times A_2} = \pi_i$, $i = 1, 2$. Since the functor F preserves bc-structure (not necessarily on the nose) these isomorphisms exist and they are unique.

Using these basic coherent isomorphisms (and by induction on the complexity of $\mathcal{T}$) we can build the above mentioned canonical coherent isomorphisms $\rho_{\mathcal{T}}$ and $\rho^{-1}_{\mathcal{T}}$. Also, the proof of uniqueness of these coherent isomorphisms is by induction on the complexity of $\mathcal{T}$. $\Box$

Lemma 4.61 *Let $\mathcal{A} \xrightarrow{F} \mathcal{B}$ be a functor in $\mathcal{BCC}$ and let $(\mathcal{A}, \Sigma_1)$, $(\mathcal{B}, \Sigma_2)$ be the two categories with added strict structure. Let $T_{\mathcal{A}} \xrightarrow{M_F} \mathcal{B}_{T_{\mathcal{B}}}$ be an interpretation such that $M_F(X_A) = Y_{F(A)}$ where X_A denotes the basic type from the language $L_{\mathcal{A}}$ corresponding to the object $A \in \mathcal{A}$ and Y_B denotes the basic type in $L_{\mathcal{B}}$ corresponding to the object $B \in \mathcal{B}$. On basic constants M_F is defined as follows: let $c_f : \mathcal{T}(\overrightarrow{X_{A_i}})$ be the basic constant in $L_{\mathcal{A}}$ corresponding to $f \in \hom_{\mathcal{A}}(1, \mathcal{T}(\overrightarrow{A_i}))$. We define*

$$M_F(\, \triangleright c_f : \mathcal{T}(\overrightarrow{X_{A_i}})) = [\, \triangleright c_{\rho^{-1}_{\mathcal{T}} F(f) \rho_1} : \mathcal{T}(\overrightarrow{Y_{F(A_i)}})] : 1 \to \mathcal{T}(\overrightarrow{F(A_i)}).$$

Then this interpretation is also a model.

Proof: Obviously, M_F is well defined. Now we have to check that M_F preserves the axioms of $T_{\mathcal{A}}$. Recall that $u =_x v$ is an axiom of $T_{\mathcal{A}}$ iff $M(x \triangleright u) = M(x \triangleright v)$ where $L_{\mathcal{A}} \xrightarrow{M} (\mathcal{A}, \Sigma_1)$ is the canonical interpretation.

Claim: For every term $(x_1 : \mathcal{T}_1(\overrightarrow{X_{A_j}}), \ldots, x_n : \mathcal{T}_n(\overrightarrow{X_{A_j}}) \triangleright t : \mathcal{T}(\overrightarrow{X_{B_k}})) \in L_{\mathcal{A}}$, the following holds:

$$F(M(x_1 : \mathcal{T}_1(\overrightarrow{X_{A_j}}), \ldots, x_n : \mathcal{T}_n(\overrightarrow{X_{A_j}}) \triangleright t : \mathcal{T}(\overrightarrow{X_{B_k}})) =$$

$$\rho_T M_F(x_1 : \mathcal{T}_1(\overrightarrow{X_{A_j}}), \ldots, x_n : \mathcal{T}_n(\overrightarrow{X_{A_j}}) \triangleright t : \mathcal{T}(\overrightarrow{\cdots}))\rho^{-1}_{\mathcal{T}_1 \times \cdots \times \mathcal{T}_n}.$$

<u>Proof of the claim:</u> By induction on the complexity of the term t. □

Using the claim and the fact that the ρ's are isomorphisms it is immediate to see that M_F is a model of $T_{\mathcal{A}}$. □

We can establish the following 2-adjointness between the 2-categories (doctrines) $\mathcal{BCC}$ and $\mathcal{BCC}_s$:

Theorem 4.62 (cf. Thm. 3.13 [BKP89]) *Let $\mathcal{BCC}_s \xrightarrow{||} \mathcal{BCC}$ be a 2-functor which on objects (0-cells) just forgets the chosen structure (on 1- and 2-cells doesn't do anything), let $\mathcal{BCC} \xrightarrow{(\,)_s} \mathcal{BCC}_s$, $(\mathcal{A} \underset{G}{\overset{F}{\underset{\Downarrow\theta}{\rightrightarrows}}} \mathcal{B}) \xmapsto{(\,)_s} (\mathcal{A}_s \underset{G_s}{\overset{F_s}{\underset{\Downarrow\theta_s}{\rightrightarrows}}} \mathcal{B}_s)$ be a 2-functor defined as follows: $\mathcal{A}_s = \mathcal{B}_{T_{\mathcal{A}}}$ (similarly $\mathcal{B}_s$), F_s is obtained from the above defined model $T_{\mathcal{A}} \xrightarrow{M_F} \mathcal{B}_{T_B}$ (see the above lemma), and $F_s \overset{\theta_s}{\Rightarrow} G_s$ is defined to be the unique natural isomorphism corresponding to the isomorphism $T_{\mathcal{A}} \underset{M_G}{\overset{M_F}{\underset{\theta^s\Downarrow}{\rightrightarrows}}} \mathcal{B}_{T_B}$ which on basic types just gives the arrow corresponding to θ_A, more precisely: $\theta^s_{X_A} = [x^{X_{F(A)}} \triangleright c_{\theta_A}^{{X_{G(A)}}^{X_{F(A)}}} \text{`} x^{X_{F(A)}}]$. Then:*

$$(\,)_s \dashv |\,|$$

and the 2-natural transformation $1_{\mathcal{BCC}} \xrightarrow{\eta} |(\,)_s|$, which is the unit of the adjunction satisfies that for every $\mathcal{B} \in \mathcal{BCC}$

$$\mathcal{B} \xrightarrow{\eta_{\mathcal{B}}} |\mathcal{B}_s|$$

is an equivalence of categories (and $\eta_{\mathcal{B}} \in \mathcal{BCC}$).

Proof: It is not hard to see that the above defined functors are indeed 2-functors (not merely homomorphisms) - we will just check that θ_s is well defined. By corollary 2.12 θ_s is well defined if the isomorphism of models $M_F \overset{\theta^s}{\Rightarrow} M_G$ is well defined. By remark 2.6 the latter exists if for all basic constants $c_f^{\mathcal{T}(\overrightarrow{X_{A_i}})} \in L_{\mathcal{A}}$

the following holds: $\theta^s_{\mathcal{T}(\overrightarrow{X_{A_i}})} M_F(c_f^{\mathcal{T}(\overrightarrow{X_{A_i}})}) = M_G(c_f^{\mathcal{T}(\overrightarrow{X_{A_i}})})$; this is the same as $\theta^s_{\mathcal{T}(\overrightarrow{X_{A_i}})}[\rhd c^{\mathcal{T}(\overrightarrow{F(A_i)})}_{\rho_{\mathcal{T}}^{-1}F(f)\rho_1}] = [\rhd c^{\mathcal{T}(\overrightarrow{G(A_i)})}_{\sigma_{\mathcal{T}}G(f)\sigma_1}]$, and by faithfulness of $\mu_{\mathcal{B}}$ this is the same as $\mu_{\mathcal{B}}(\theta^s_{\mathcal{T}(\overrightarrow{X_{A_i}})})\mu_{\mathcal{B}}([\rhd c^{\mathcal{T}(\overrightarrow{F(A_i)})}_{\rho_{\mathcal{T}}^{-1}F(f)\rho_1}]) = \mu_{\mathcal{B}}([\rhd c^{\mathcal{T}(\overrightarrow{G(A_i)})}_{\sigma_{\mathcal{T}}^{-1}G(f)\sigma_1}])$ (here σ is for G what ρ was for F). By definition of $\mu_{\mathcal{B}}$ this is the same as $\mu_{\mathcal{B}}(\theta^s_{\mathcal{T}(\overrightarrow{X_{A_i}})})\rho_{\mathcal{T}}^{-1}F(f)\rho_1 = \sigma_{\mathcal{T}}^{-1}G(f)\sigma_1$. To show that, we will first establish the following

<u>Claim:</u> For every type $\mathcal{T}(\overrightarrow{X_{A_i}}) \in L_{\mathcal{A}}$ $\mu_{\mathcal{B}}(\theta^s_{\mathcal{T}(\overrightarrow{X_{A_i}})}) = \sigma_{\mathcal{T}}^{-1}\theta_{\mathcal{T}(\overrightarrow{A_i})}\rho_{\mathcal{T}}$, where $\theta^s_{\mathcal{T}(\overrightarrow{X_{A_i}})}$ are the isomorphisms made out of isomorphisms $\theta^s_{X_{A_i}}$ as in remark 2.6.

<u>Proof of the claim:</u> By induction on the complexity of $\mathcal{T}(\overrightarrow{X_{A_i}})$. For the atomic types this is so by definition since for the atomic $\mathcal{T}$ the isomorphisms $\rho_{\mathcal{T}}$ and $\sigma_{\mathcal{T}}$ are identities. Let us just check the most complicated case: $\mathcal{T} = \mathcal{T}_1^{\mathcal{T}_2}$. We have the following chain of equation:

$$
\begin{array}{ll}
\sigma^{-1}_{\mathcal{T}_1^{\mathcal{T}_2}}\theta_{\mathcal{T}_1^{\mathcal{T}_2}}\rho_{\mathcal{T}_1^{\mathcal{T}_2}} & \\
\quad = (\sigma_{\mathcal{T}_1}^{-1}\varepsilon(1 \times \sigma_{\mathcal{T}_2}))^*\theta_{\mathcal{T}_1^{\mathcal{T}_2}}(\rho_{\mathcal{T}_1}\varepsilon(1 \times \rho_{\mathcal{T}_2}^{-1}))^* & \text{by definition of } \rho_{\mathcal{T}},\ \sigma_{\mathcal{T}} \\
\quad = (\sigma_{\mathcal{T}_1}^{-1}\varepsilon(1 \times \sigma_{\mathcal{T}_2}))^*(\theta_{\mathcal{T}_1}\varepsilon(1 \times \theta_{\mathcal{T}_2}^{-1}))^*(\rho_{\mathcal{T}_1}\varepsilon(1 \times \rho_{\mathcal{T}_2}^{-1}))^* & \text{by remark 2.6} \\
\quad = (\sigma_{\mathcal{T}_1}^{-1}\varepsilon(1 \times \sigma_{\mathcal{T}_2})((\theta_{\mathcal{T}_1}\varepsilon(1 \times \theta_{\mathcal{T}_2}^{-1})((\rho_{\mathcal{T}_1}\varepsilon(1 \times \rho_{\mathcal{T}_2}^{-1}))^* \times 1))^* \times 1))^* & u^*v = (u(v \times 1))^* \\
\quad = (\sigma_{\mathcal{T}_1}^{-1}\theta_{\mathcal{T}_1}\varepsilon(1 \times \theta_{\mathcal{T}_2}^{-1})((\rho_{\mathcal{T}_1}\varepsilon(1 \times \rho_{\mathcal{T}_2}^{-1}))^* \times 1)(1 \times \sigma_{\mathcal{T}_2}))* & \left\{\begin{array}{l}(a \times 1)(1 \times b) = (1 \times b)(a \times 1) \\ \text{and } \varepsilon(h^* \times 1) = h\end{array}\right. \\
\quad = (\sigma_{\mathcal{T}_1}^{-1}\theta_{\mathcal{T}_1}\rho_{\mathcal{T}_1}\varepsilon(1 \times \rho_{\mathcal{T}_2}^{-1})(1 \times \theta_{\mathcal{T}_2}^{-1})(1 \times \sigma_{\mathcal{T}_2}))* & \\
\quad = (\sigma_{\mathcal{T}_1}^{-1}\theta_{\mathcal{T}_1}\rho_{\mathcal{T}_1}\varepsilon(1 \times (\sigma_{\mathcal{T}_2}^{-1}\theta_{\mathcal{T}_2}\rho_{\mathcal{T}_2})^{-1}))* & (1 \times a)(1 \times b) = 1 \times ab \\
\quad = (\mu_{\mathcal{B}}(\theta^s_{\mathcal{T}_1})\varepsilon(1 \times (\mu_{\mathcal{B}}(\theta_{\mathcal{T}_2}))^{-1}))^* & \text{by induction hypothesis} \\
\quad = \mu_{\mathcal{B}}(\theta^s_{\mathcal{T}}) & \text{by remark 2.6.}
\end{array}
$$

□

Using the above claim the above equation is the same as $\sigma_{\mathcal{T}}^{-1}\theta_{\mathcal{T}(\overrightarrow{A_i})}\rho_{\mathcal{T}}\rho_{\mathcal{T}}^{-1}F(f)\rho_1 = \sigma_{\mathcal{T}}^{-1}G(f)\sigma_1$, i.e. $\theta_{\mathcal{T}(\overrightarrow{A_i})}F(f)\rho_1 = G(f)\sigma_1$. The last equation is a consequence of the naturality of $F \overset{\theta}{\Rightarrow} G$; namely from $\theta_{\mathcal{T}}(\overrightarrow{A_i})F(f) = G(f)\theta_1$ multiplying both sides from the right by the isomorphism ρ_1 the wanted equation follows provided

$\theta_1\rho_1 = \sigma_1 : 1 \to G(1)$. This is so because $G(1)$ is a terminal object in $\mathcal{B}_s$, although not the chosen one.

Let us now give the 2-natural transformation η as follows: for every bc-category $\mathcal{B}$ we define $\eta_{\mathcal{B}}$ as in lemma 4.58, i.e. $\eta_{\mathcal{B}}(A \xrightarrow{b} B) = X_A \xrightarrow{c_b} Y_B$.

The 1-dimensional part of 2-naturality (i.e. "ordinary naturality") follows by definition of F_s. The 2-dimensional part is the following.

Let $\mathcal{A} \underset{G}{\overset{F}{\underset{\Downarrow\theta}{\rightrightarrows}}} \mathcal{B} \in \mathcal{BCC}$, then we have to check that $|\theta_s|\eta_{\mathcal{A}} = \eta_{\mathcal{B}}\theta : |F_s|\eta_{\mathcal{A}} \Rightarrow \eta_{\mathcal{B}}G$. Take an object $A \in \mathcal{A}$ and calculate: $(|\theta_s|\eta_{\mathcal{A}})_A = \theta^s_{X_A} = [x^{X_{F(A)}} \triangleright c_{\widehat{\theta_A}}^{X_{G(A)}^{X_{F(A)}}} \text{‘} x^{X_{F(A)}}]$. On the other hand $(\eta_{\mathcal{B}}\theta)_A = \eta_{\mathcal{B}}(\theta_A) = [(x^{X_A} \triangleright c_{\widehat{\theta_A}}^{X_{G(A)}^{X_{F(A)}}} \text{‘} x^{X_{F(A)}})]$, so we are done.

Now we are going to show the adjointness. First the 1-dimensional part: we have to prove that to each $\mathcal{C} \xrightarrow{F} |\mathcal{A}| \in \mathcal{BCC}$ there is exactly one $\mathcal{C}_s \xrightarrow{H} \mathcal{A} \in \mathcal{BCC}_s$ such that $F = |H|\eta_{\mathcal{C}}$. (Again we repeat the "old" remark that to talk about $\mathcal{C}_s$ when $\mathcal{C} \in \mathcal{BCC}$ we first had to choose a strict bc-structure Σ so the notation $\mathcal{C}_s$ has this structure hidden.)

The uniqueness of H is easy to establish: first notice that on objects H is uniquely determined since $H(X_A) = |H|(\eta_{\mathcal{C}}(A)) = F(A)$ and since H has to be a strict bc-functor we have $H(\mathcal{T}(\overrightarrow{X_{A_i}})) = \mathcal{T}(\overrightarrow{F(A_i)})$. Also H is uniquely given on elementary arrows by the same reason, i.e. $H(X_A \xrightarrow{c_f} Y_A) = |H|\eta_{\mathcal{C}}(A \xrightarrow{f} B) = F(A) \xrightarrow{F(f)} F(B)$. Now we are going to construct H as follows. We are given a functor $\mathcal{C} \xrightarrow{F} |\mathcal{A}|$. Above we have defined $\mathcal{C}_s \xrightarrow{F_s} |\mathcal{A}|_s$ (recall that we have hidden strict structures and here we can choose the strict structure Σ' on $|\mathcal{A}|$ so that $(|\mathcal{A}|, \Sigma') = \mathcal{A}$). Also above, we gave the construction of $|\mathcal{A}|_s \xrightarrow{\mu_{|\mathcal{A}|}} \mathcal{A}$. Let us define $H = \mu_{|\mathcal{A}|}F_s$. By naturality of η we have $|F_s|\eta_{\mathcal{C}} = \eta_{|\mathcal{A}|}F$. Multiplying both sides on the left by $|\mu_{|\mathcal{A}|}|$ we get $|\mu_{|\mathcal{A}|}|\eta_{|\mathcal{A}|}F = |H|\eta_{\mathcal{C}}$. Since $|\mu_{|\mathcal{A}|}|\eta_{|\mathcal{A}|} = 1_{|\mathcal{A}|}$ we obtain $F = |H|\eta_{\mathcal{C}}$.

The 2-dimensional part of the adjointness says that for every $\mathcal{C} \underset{|H_2|\eta_C}{\overset{|H_1|\eta_C}{\underset{\Downarrow\theta}{\longrightarrow}}} |\mathcal{A}| \in \mathcal{BCC}$ there exists a unique $\mathcal{C}_s \underset{H_2}{\overset{H_1}{\underset{\Downarrow\kappa}{\longrightarrow}}} \mathcal{A} \in \mathcal{BCC}_s$ such that $|\kappa|\eta_C = \theta$. Define $\kappa = \mu_{|\mathcal{A}|}\theta_s$ (domain and codomain of κ are all right by the uniqueness of the H in the 1-dimensional part of the adjunction). Indeed it satisfies the equation: $|\mu_{|\mathcal{A}|}\theta_s|\eta_C = |\mu_{|\mathcal{A}|}||\theta_s|\eta_C = |\mu_{|\mathcal{A}|}|\eta_{|\mathcal{A}|}\theta = \theta$. The uniqueness of κ is also not a problem since by remark 2.6 κ is determined by its behavior on basic types. Since $|\kappa(X_A)| = |\kappa|\eta_C(A) = \theta_A$ and the functor $|\ |$ doesn't do anything, κ is indeed unique. □

Remark 4.63 We actually need a more precise construction of the functor $(\ )_s$ for the following section (it could be avoided if we had a different setting as in e.g. [BKP89] where the nonstrict doctrine has actually strict objects but nonstrict functors).

Sufficiency condition: Take an object $\mathcal{A}$ and take all the possible strict structures on it. For each of these structures there is an isomorphic copy $\mathcal{A}'$ of $\mathcal{A}$ on which we take this particular structure when constructing $\mathcal{A}'_s$.

This is going to be used in the following form. Let $\mathcal{A}$ be an object in the nonstrict doctrine. The above condition then implies that for every possible strict structure Σ on $\mathcal{A}$ there exists an isomorphism $\mathcal{A} \xrightarrow{\sigma} \mathcal{A}'$ in the doctrine such that when constructing $\mathcal{A}'_s$ we choose as the strict structure on $\mathcal{A}'$ the structure induced by Σ (and σ).

4.4.2 Pushouts

The above adjointness can be used for the construction of Colimits in $\mathcal{BCC}$ providing that we have the appropriate colimits in $\mathcal{BCC}_s$ (the terminology "Limits" and "Colimits" is adopted from [MP89] and it means weighted bilimits and weighted bicolimits respectively; see also [Str80] where they were called indexed bi(co)limits). This was already done in [BKP89] where "the appropriate colimits" essentially meant pseudo-colimits. Since our interpolation (so far) was proved for (a particular kind of) 2-pushouts in the strict doctrine we have to use them for the construction of Pushouts in the nonstrict doctrine and not merely pseudo-pushouts. Often we call 2-pushouts just pushouts.

Suppose we are given a diagram $\mathcal{A} \xrightarrow{F} \mathcal{B}$, $\mathcal{A} \xrightarrow{G} \mathcal{C}$ in $\mathcal{BCC}$. We want to construct a Pushout. That is, we want a diagram

$$\begin{tikzcd}
\mathcal{A} \arrow[r, "F"] \arrow[d, "G"'] & \mathcal{B} \arrow[d, "H"] \arrow[dl, phantom, "\Swarrow\tau"] \\
\mathcal{C} \arrow[r, "K"'] & \mathcal{D}
\end{tikzcd}$$

where $\tau : HF \Rightarrow KG$ is a natural isomorphism which satisfies the following: for every two functors $\mathcal{B} \xrightarrow{P} \mathcal{E}$ and $\mathcal{C} \xrightarrow{Q} \mathcal{E}$ and a natural isomorphism $\theta : PF \Rightarrow QG$ there exists a functor $\mathcal{D} \xrightarrow{T} \mathcal{E}$ and there exist two natural isomorphisms $\theta_1 : P \Rightarrow TH$ and $\theta_2 : TK \Rightarrow Q$

$$\begin{tikzcd}
\mathcal{A} \arrow[r, "F"] \arrow[d, "G"'] & \mathcal{B} \arrow[d, "H"] \arrow[dl, phantom, "\Swarrow\tau"] \arrow[ddr, "P"] & \\
\mathcal{C} \arrow[r, "K"] \arrow[drr, "Q"'] & \mathcal{D} \arrow[dr, "T"] \arrow[r, phantom, "\Downarrow\theta_1"] \arrow[d, phantom, "\Leftarrow\theta_2"] & {} \\
& {} & \mathcal{E} \arrow[loop below, phantom, "\theta"]
\end{tikzcd}$$

such that $(\theta_2 G)(T\tau)(\theta_1 F) = \theta$ and such that for every $\mathcal{D} \xrightarrow{T'} \mathcal{E}$ and natural isomorphisms $\phi_1 : TH \Rightarrow T'H$, $\phi_2 : TK \to T'K$ which satisfy $T'\tau \circ \phi_1 F = \phi_2 G \circ T\tau$ there exists unique natural isomorphism $\psi : T \Rightarrow T'$ such that $\phi_1 = \psi H$ and $\phi_2 = \psi K$.

We can notice that the interpolation property is invariant for Pushouts, i.e. if one Pushout over $\mathcal{A} \xrightarrow{F} \mathcal{B}$, $\mathcal{A} \xrightarrow{G} \mathcal{C}$ has the interpolation property than all the other Pushouts over the same diagram have this property. Even more is true, the interpolation property is indeed a 2-categorical (we even may say bicategorical) notion in the following sense.

Lemma 4.64 *Suppose we have the following diagram*

$$
\begin{tikzcd}
\mathcal{A}' \arrow[rrr, "F'"] \arrow[ddd, "G'"'] & & & \mathcal{B}' \arrow[ddd, "H'"] \\
& \mathcal{A} \arrow[ul, "t_{\mathcal{A}}"] \arrow[r, "F"] \arrow[d, "G"'] & \mathcal{B} \arrow[ur, "t_{\mathcal{B}}"'] \arrow[d, "H"] \arrow[ur, phantom, shift left=4, "t_F"] & \\
& \mathcal{C} \arrow[dl, "t_{\mathcal{C}}"] \arrow[r, "K"'] \arrow[dl, phantom, shift left=4, "t_G"] & \mathcal{D} \arrow[dr, "t_{\mathcal{D}}"'] \arrow[ul, phantom, shift right=2, "\tau"] \arrow[dr, phantom, shift left=4, "t_H"] \arrow[dr, phantom, shift right=4, "t_K"'] & \\
\mathcal{C}' \arrow[rrr, "K'"'] & & & \mathcal{D}' \arrow[loop right, "\tau'"]
\end{tikzcd}
$$

where $t_{\mathcal{A}}$, $t_{\mathcal{B}}$, $t_{\mathcal{C}}$ and $t_{\mathcal{D}}$ are equivalences of categories and $t_F : t_{\mathcal{B}}F \Rightarrow F't_{\mathcal{A}}$, $t_G : t_{\mathcal{C}}G \Rightarrow G't_{\mathcal{A}}$, $t_H : t_{\mathcal{D}}H \Rightarrow H't_{\mathcal{B}}$, $t_K : t_{\mathcal{D}}K \Rightarrow K't_{\mathcal{C}}$, $\tau : HF \Rightarrow KG$ and $\tau' : H'F' \Rightarrow K'G'$ are natural isomorphisms such that

$$K't_G \circ t_K G \circ t_{\mathcal{D}}\tau = \tau' t_{\mathcal{A}} \circ H't_F \circ t_H F : t_{\mathcal{D}}HF \Rightarrow K'G't_{\mathcal{A}}. \tag{1}$$

(This is essentially a strong transformation $\mathcal{I}^- \underset{\Phi^{-\prime}}{\overset{\Phi^-}{\underset{\Downarrow t}{\rightrightarrows}}} \mathcal{BCC}$ where $\mathcal{I}^-$ is a commutative square, and t is an equivalence in $Hom(\mathcal{I}^-, \mathcal{BCC})$, see [MP89, Prop.4.1.3].) Then:

1. *If one of the squares has the interpolation property the other one has it too.*
2. *Also, if one of the squares is a Pushout the other one is too.*

Proof: *1.* Assume the inside square has interpolation property and we want to show that the outside one has it too.

Take $B' \in \mathcal{B}'$, $C' \in \mathcal{C}'$ and $(H'(B') \xrightarrow{d'} K'(C')) \in \mathcal{D}'$. We are done if we produce $A' \in \mathcal{A}'$, $(B' \xrightarrow{b'} F'(A')) \in \mathcal{B}'$ and $(G'(A') \xrightarrow{c'} C') \in \mathcal{C}$ such that $d' = K'(c')\tau'_{A'}H'(b')$.

First from the essential surjectivity of t_B and t_C we have two objects $B \in \mathcal{B}$ and $C \in \mathcal{C}$ and two isomorphisms $(t_B(B) \xrightarrow{u} B') \in \mathcal{B}'$ and $(t_C(C) \xrightarrow{v} C') \in \mathcal{C}'$. Also we have the isomorphism: $d'' : t_D(H(B)) \to t_D(K(C))$ which is defined to be:

$$t_D(H(B)) \xrightarrow{t_H(B)} H'(t_B(B)) \xrightarrow{H'(u)} H'(B') \xrightarrow{d'} K'(C') \xrightarrow{K'(v^{-1})} K'(t_C(C)) \xrightarrow{t_K(C)^{-1}} t_D(K(C)).$$

Using faithfulness of t_D we have an arrow $(H(B) \xrightarrow{d} K(C)) \in \mathcal{D}$ such that $t_D(d) = d''$. By the interpolation property of the top square we have an object $A \in \mathcal{A}$ and two morphisms $(B \xrightarrow{b} F(A)) \in \mathcal{B}$ and $(G(A) \xrightarrow{c} C) \in \mathcal{C}$ such that $d = K(c)\tau_A H(b)$. Now take $A' = t_A(A)$, $b' = t_F(A)t_B(b)u^{-1} : B' \to F'(A')$ and $c' = vt_C(c)t_G(A)^{-1} : G'(A') \to C'$ and check that it gives an interpolant for d'. That is check that:

$$d' = K'(vt_C(c)t_G(A)^{-1})\tau'_{A'}H'(t_F(A)t_B(b)u^{-1}).$$

For, use the definition of d'' and the facts that $t_D(d) = d''$ and $d = K(c)\ \tau_A\ H(b)$, so the equation which we have to check now looks like:

$$K'(v)t_K(C)t_D(K(c))t_D(\tau_A)t_D(H(b))t_H(B)^{-1}H'(u)^{-1}$$

$$= K'(vt_C(c)t_G(A)^{-1})\tau'_{A'}H'(t_F(A)t_B(b)u^{-1}).$$

This is the same as:

$$t_K(C)t_D(K(c))t_D(\tau_A)t_D(H(b))t_H(B)^{-1}$$

$$= K'(t_C(c))K'(t_G(A)^{-1})\tau'_{A'}H'(t_F(A))H'(t_B(b)).$$

Since t_H is a natural isomorphism we have

$$t_H(F(A))t_D(H(b)) = H'(t_B(b))t_H(B)$$

and then the above equation is equivalent to:

$$t_K(C)t_D(K(c))t_D(\tau_A) = K'(t_C(c))K'(t_G(A)^{-1})\tau'_{A'}H'(t_F(A))t_H(F(A)).$$

Similarly, since t_K is a natural isomorphism we have

$$K'(t_C(c))t_K(G(A)) = t_K(C)t_D(K(c))$$

and then the above equation is equivalent to:

$$K'(t_G(A))t_K(G(A))t_D(\tau_A) = \tau'_{A'}H'(t_F(A))t_H(F(A)).$$

Fortunately, the last equation is a special case of equation (1).

2. This is a part of the bicategorical folklore. □

Lemma 4.65 *Suppose $I = \{2 \xleftarrow{20} 0 \xrightarrow{01} 1\}$ is a 2-category and suppose $\mathcal{I} \underset{\Phi'}{\overset{\Phi}{\underset{\Downarrow t}{\rightrightarrows}}} \mathcal{BCC}$ where t is an equivalence in $Hom(\mathcal{I}, \mathcal{BCC})$. Suppose also $I : \mathcal{I} \hookrightarrow \mathcal{I}^-$ is an inclusion of the 2-categories (recall that $\mathcal{I}^-$ is just a commutative square), and suppose that there is a homomorphism $\mathcal{I}^- \xrightarrow{\Phi^-} \mathcal{BCC}$ such that $\Phi^- I = \Phi$.*

Then there exists a homomorphism $\mathcal{I}^- \xrightarrow{\Phi^{-\prime}} \mathcal{BCC}$ and an equivalence $\mathcal{I}^- \underset{\Phi^{-\prime}}{\overset{\Phi^-}{\underset{\Downarrow t^-}{\rightrightarrows}}} \mathcal{BCC}$ such that $\Phi^{-\prime} I = \Phi'$ and $t^- I = t$.

Proof: It is easy to construct what is needed. □

Now we can start constructing Pushouts in the doctrine $\mathcal{BCC}$. By the lemmas 4.64 and 4.65 to show that every Pushout over a diagram $\Phi : I \to \mathcal{BCC}$ has the interpolation property it is enough to show that for a Pushout over one equivalent diagram $\Phi' : I \to \mathcal{BCC}$.

For the beginning we can restrict our attention on slightly more special diagrams and for that we need the following easy lemma.

Lemma 4.66 *Every functor $\mathcal{A} \xrightarrow{F} \mathcal{B}$ can be factored (in the doctrine $\mathcal{BCC}$) as*

$$\begin{tikzcd} \mathcal{A} \arrow[rr, "F"] \arrow[dr, "F'"'] & & \mathcal{B} \\ & \mathcal{B}' \arrow[ur, "F''"'] & \end{tikzcd}$$

where F' is an inclusion on objects and F'' is an equivalence of categories.

Proof Let us first define the category $\mathcal{B}'$ as follows: The objects of $\mathcal{B}'$ are objects of $\mathcal{B}$ and also pairs $(F(A), A)$ where A is an object of $\mathcal{A}$. Arrows in $\mathcal{B}'$ are only arrows between first coordinates - more explicitly $\hom_{\mathcal{B}'}((F(A), A), B) = \hom_{\mathcal{B}}(F(A), B)$, the other cases are similar. The composition and identities in $\mathcal{B}'$ are the ones from $\mathcal{B}$. This is indeed in the doctrine because the properties of the doctrine are defined up to a (unique coherent) iso anyway. The definition of F' is the obvious one: $(A_1 \xrightarrow{f} A_2) \mapsto ((F(A_1), A_1) \xrightarrow{F(f)} (F(A_2), A_2))$. This functor is in the doctrine, basically for the same reason that $\mathcal{B}'$ is.

To construct F'' we do the following: $((F(A), A) \xrightarrow{g} B) \mapsto (F(A) \xrightarrow{g} B)$, similarly for the other cases. The pseudo-inverse of F'' (call it F''') is just the inclusion of $\mathcal{B}$ in $\mathcal{B}'$. $\square$

Since $\mathcal{C} \xleftarrow{G} \mathcal{A} \xrightarrow{F} \mathcal{B}$ is an equivalent diagram to $\mathcal{C}' \xleftarrow{G'} \mathcal{A} \xrightarrow{F'} \mathcal{B}'$, by the above lemmas we can assume that the functors in the diagram for which we want to construct Pushout are inclusions on objects.

These "unusual" functors - inclusions on objects - have several interesting features.[2] Recall now the example from section 4.4.1 where we had a functor $\mathcal{A} \xrightarrow{F} \mathcal{B} \in \mathcal{BCC}$ which was not an inclusion on objects; we were not able to assign strict structures to $\mathcal{A}$ and $\mathcal{B}$ such that F becomes a strict functor (the actual example was manufactured out of a 4 element Boolean algebra and a category with isomorphic (but different) initial and terminal object). However, when F is an inclusion on objects we can do that (and even a bit better):

Lemma 4.67 *Let $\mathcal{A} \xrightarrow{F} \mathcal{B} \in \mathcal{BCC}$ be an inclusion on objects. Let Σ_1 be an arbitrary strict structure on $\mathcal{A}$. Then, there exists a strict structure Σ_2 on $\mathcal{B}$ such that $(\mathcal{A}, \Sigma_1) \xrightarrow{F} (\mathcal{B}, \Sigma_2) \in \mathcal{BCC}_s$.*

Proof: Let $\{B_\alpha\}_{\alpha<\kappa}$ be the set of objects of $\mathcal{B}$. Let us just define strict products. $B_\alpha \times B_\beta = F(A \times A')$ if $B_\alpha = F(A)$ and $B_\beta = F(A')$; otherwise choose any product of B_α and B_β to be $B_\alpha \times B_\beta$ (the arrow part of the definition is equally simple). For the terminal object in Σ_2 choose $F(1)$ where 1 is the terminal object in Σ_1. □

After this lemma, using the sufficiency condition on the functor $(\ )_s$ (see remark 4.63) we can assume that the diagram $\mathcal{C} \xleftarrow{G} \mathcal{A} \xrightarrow{F} \mathcal{B}$ has not only inclusions on objects as functors, but also that the strict structures on the categories needed for the definition of the functor $(\ )_s$ are such that F and G are strict functors. The advantage of this situation is that we don't have to worry about the definition of the functor $F_s : \mathcal{A}_s \longrightarrow \mathcal{B}_s$ - the naive guess now works (see the remark after lemma 4.59 and several lemmas after)! Let us write this down as

[2]The day before submission of the thesis (July 6, 1993) we received [JS92] in which the use of these functors is similar to ours - it relates 2-pushouts and pseudo-pushouts. We can say that the doctrines are not the same as ours, that the proofs are different and that interpolation is not mentioned in their work.

Lemma 4.68 *Suppose that $F : (\mathcal{A}, \Sigma_1) \to (\mathcal{B}, \Sigma_2)$ is a strict bc-functor which is an inclusion on objects. Then $F_s : \mathcal{A}_s \to \mathcal{B}_s$ which is given as in lemma 4.61 becomes simply the following: on basic types $X_A \mapsto Y_F(A)$ and on basic arrows $c_f \mapsto c_{F(f)}$ (the rest is determined since F is strict bc-functor). Up to the renaming of symbols we can assume that we have inclusion of the languages $L_\mathcal{A} \subseteq L_\mathcal{B}$ and of the theories $T_\mathcal{A} \subseteq T_\mathcal{B}$. Then F_s is constructed as in proposition 4.56.*

We are now all set for the proof of our second main result.

Proof of Theorem 4.47: Start from a diagram

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{F} & \mathcal{B} \\ {\scriptstyle G}\downarrow & & \\ \mathcal{C} & & \end{array}$$

in the doctrine $\mathcal{BCC}$ where both functors are inclusions on objects and the strict structures on the categories needed for the construction of the functor $(\ )_s$ are such that F and G are strict with respect to these structures. This is done without loss of generality by the previous lemmas.

The idea of our proof is to construct two squares over this diagram - for one of them it will be easy to establish the interpolation property, for the other one we will have that it is a Pushout. Then we will show that they are equivalent in the manner of lemma 4.64 and the lemma says that both of them are Pushouts which satisfy the interpolation property.

To construct what is needed first apply the functor $(\ )_s$ to the above diagram and get

$$\begin{array}{ccc} \mathcal{A}_s & \xrightarrow{F_s} & \mathcal{B}_s \\ {\scriptstyle G_s}\downarrow & & \\ \mathcal{C}_s & & \end{array} \tag{2}$$

a diagram in $\mathcal{BCC}_s$. By the previous lemma this diagram is of the same type as the one in proposition 4.56.

First we construct the square which has the interpolation property. We construct a 2-pushout in $\mathcal{BCC}_s$ over this diagram, as given in proposition 4.56 (since $T_{\mathcal{A}} \subseteq T_{\mathcal{B}} \cup T_{\mathcal{C}}$ up to the renaming of symbols). We obtain

$$\begin{tikzcd} \mathcal{A}_s \arrow[r, "F_s"] \arrow[d, "G_s"'] & \mathcal{B}_s \arrow[d, "U"] \\ \mathcal{C}_s \arrow[r, "V"'] & D \end{tikzcd}$$

Recall that $D = \mathcal{B}_{T_{\mathcal{B}} \cup T_{\mathcal{C}}}$ (the union is not disjoint, i.e. the symbols coming from $L_{\mathcal{A}}$ are identified) and functors U and V are induced by the inclusions of the languages. This 2-pushout has the interpolation property as shown in proposition 4.57 .

Now, in the doctrine $\mathcal{BCC}$ we have the following commutative diagram:

$$\begin{tikzcd} \mathcal{A} \arrow[r, "F"] \arrow[d, "G"'] & \mathcal{B} \arrow[d, "|U|\eta_{\mathcal{B}}"] \\ \mathcal{C} \arrow[r, "|V|\eta_{\mathcal{C}}"'] & |D| \end{tikzcd}$$

We have to show that this square satisfy the interpolation property. For, notice that this square is equivalent to

$$\begin{tikzcd} |\mathcal{A}_s| \arrow[r, "|F_s|"] \arrow[d, "|G_s|"'] & |\mathcal{B}_s| \arrow[d, "|U|"] \\ |\mathcal{C}_s| \arrow[r, "|V|"'] & |D| \end{tikzcd}$$

and this commutative square is obtained applying the forgetful functor $|\,|$ on a square which had the interpolation property, therefore it has the property itself. Now apply lemma 4.64.

Second, we construct a square over the above diagram 2 which is a Pushout in $\mathcal{BCC}$. To show that we are going to use [BKP89, 5.8 and 5.9]. In the form appropriate for us it says:

Theorem 4.69 ([BKP89]) *The diagram*

$$\begin{tikzcd} \mathcal{A} \arrow[r, "F"] \arrow[d, "G"'] & \mathcal{B} \arrow[d, "|U'|\eta_{\mathcal{B}}"] \arrow[dl, phantom, "\swArrow |\theta|\eta_{\mathcal{A}}"] \\ \mathcal{C} \arrow[r, "|V'|\eta_{\mathcal{C}}"'] & |\mathcal{D}'| \end{tikzcd}$$

is a Pushout in $\mathcal{BCC}$, where

$$\begin{tikzcd} \mathcal{A}_s \arrow[r, "F_s"] \arrow[d, "G_s"'] & \mathcal{B}_s \arrow[d, "U'"] \arrow[dl, phantom, "\swArrow \theta"] \\ \mathcal{C}_s \arrow[r, "V'"'] & \mathcal{D}' \end{tikzcd}$$

is a pseudo-pushout in the strict doctrine $\mathcal{BCC}_s$.

For the proof we refer to the above reference - here we will give only the construction of the pseudo-pushout in our terminology and we will recall its universal property.

First, let us give the construction of $\mathcal{D}'$ this is $\mathcal{B}_{T'}$ where the language of the theory T' is

$$L' = L_{\mathcal{B}} \sqcup L_{\mathcal{C}} \sqcup \{\theta_A : Y_A^{X_A}, \theta_A^{-1} : X_A^{Y_A} | A \text{ object of } \mathcal{A}\}$$

($\sqcup$ denotes disjoint union) and

$$\begin{aligned} T' = T_{\mathcal{B}} \sqcup T_{\mathcal{C}} \sqcup \{\theta_A`(\theta_A^{-1}`y^{Y_A}) = y^{Y_A}, \theta_A^{-1}`(\theta_A`x^{X_A}) = x^{X_A} | A \in \mathcal{A}\} \\ \sqcup\{\theta_A`b_f = c_f | (f : 1 \to A) \in \mathcal{A} \text{ and } b_f \in L_{\mathcal{B}}, c_f \in L_{\mathcal{C}} \text{ corresponding constants}\} \end{aligned}$$

(the last mentioned θ's can be of complex type and they are then defined by induction from the basic θ's as in remark 2.6, we don't have to impose the isomorphism identities for these θ's because they follow as in the mentioned remark). Then, the functors U' and V' are simply defined to be inclusions of "symbols". It is easy to check that they are strict bc-functors. The natural isomorphism $\theta : U'F_s \Rightarrow V'G_s$ is defined in the obvious way using the above constants θ_A.

This diagram has the following universal property in the strict doctrine $\mathcal{BCC}_s$: for every $P : \mathcal{B}'_s \to \mathcal{E}$, $Q : \mathcal{C}'_s \to \mathcal{E}$ and a natural isomorphism $\tau : PF_s \Rightarrow QG_s$ there exists a unique functor $R : D' \to \mathcal{E}$ such that $RU' = P$, $RV' = Q$ and $R\theta = \tau$; and also for any two functors $R, R' : D' \to \mathcal{E}$ and any two natural isomorphisms $\phi_1 : RU' \Rightarrow R'U'$, $\phi_2 : RV' \Rightarrow R'V'$ which satisfy $R'\theta \circ \phi_1 F_s = \phi_2 G_s \circ R\theta$ there exists a unique natural isomorphism $\psi : R \Rightarrow R'$ such that $\phi_1 = \psi U'$ and $\phi_2 = \psi V'$. The proof that this diagram has this universal property is quite similar to the proof of proposition 4.56.

Now we start the third part of the proof of theorem in which we show that the two squares are equivalent as required in lemma 4.64. Let, as above, $\mathcal{B}_s \xrightarrow{U} D \xleftarrow{V} \mathcal{C}_s$ be "the" pushout over $\mathcal{C}_s \xleftarrow{G_s} \mathcal{A}_s \xrightarrow{F_s} \mathcal{B}_s$ and let $\mathcal{B}_s \xrightarrow{U'} D' \xleftarrow{V'} \mathcal{C}_s$ be "the" pseudo-pushout over the same diagram. By the universal property of D' there exists unique strict bc-functor $R : D' \to D$ such that:

$$U = RU' \qquad V = RV' \qquad 1_{VF_s} = R\theta. \tag{3}$$

We want to show that there exists the appropriate strict functor in the other direction which will give the equivalence. For that we have to establish the following "rare"

<u>Claim:</u> There exists a strict bc-functor and a natural isomorphism $\mathcal{B}_s \underset{U''}{\overset{U'}{\Downarrow\theta'}} D'$ such that $\theta' F_s = \theta$ (this implies $U''F_s = V'G$).

Proof of the claim: To give U'' it is enough (by corollary 2.12) to give a model $M'' : T_B \to D'$. On basic types:

$$M''(X_B) = \begin{cases} Y_{G(A)} & \text{if } B = F(A) \\ X_B & \text{otherwise.} \end{cases}$$

The definition is so far correct since F is inclusion on objects. To give M'' on basic constants we have to introduce a family of isomorphisms $\theta'_B : B \to U''(B)$ in D' where B is an arbitrary type in T_B (we have in mind that U' is just "an inclusion of symbols"). This family is defined inductively as in remark 2.6, here we will give just the basis of the induction:

$$\theta'_{X_B} = \begin{cases} \theta_A & \text{if } B = F(A) \\ 1_{X_B} & \text{otherwise.} \end{cases}$$

Now we can define M'' on basic arrows as follows:

$$M''(\triangleright b_f : B) = \theta'_B[\triangleright b_f].$$

Notice the special case of the above definition: if f is in $\mathcal{A}$ then (using naturality of θ) it gives $M''(b_f) = c_f$ (here b_f, c_f are constants in L_B, L_C respectively, corresponding to $F(f)$, $G(f)$).

Now we have to check that M'' is indeed a model: for that is enough to show the following easy

Fact: For every term $(x_1 : B_1, \ldots, x_n : B_n \triangleright t : B)$ in L_B the following holds:

$$M''(x_1 : B_1, \ldots, x_n : B_n \triangleright t : B) = \theta'_B[x_1 : B_1, \ldots, x_n : B_n \triangleright t : B]\theta'^{-1}_{B_1 \times \cdots \times B_n}.$$

This is easily proved by induction on the complexity of t.

Since $U'(x_1 : B_1, \ldots, x_n : B_n \rhd t : B) = [x_1 : B_1, \ldots, x_n : B_n \rhd t : B]$ we have that M'' is a model of $T_{\mathcal{B}}$ since U' is a strict bc-functor. We also can see that $U''F_s = V'G_s$ (compare them on basic symbols from $L_{\mathcal{A}}$).

The above fact is exactly what is needed to show that θ' defines a natural isomorphism $U' \Rightarrow U''$. Also notice that indeed $\theta' F_s = \theta$. We were able to assume that F_s, G_s are inclusions on symbols (up to a renaming) due to the above restrictions on F, G. This ends the proof of the claim. $\square$

Now, by the universal property of

$$\begin{tikzcd} \mathcal{A}_s \arrow[r, "F_s"] \arrow[d, "G_s"'] & \mathcal{B}_s \arrow[d, "U"] \\ \mathcal{C}_s \arrow[r, "V"'] & D \end{tikzcd}$$

we have that in the strict doctrine there exists unique bc-functor $S : D \to D'$ such that:

$$U'' = SU \qquad V' = SV. \tag{4}$$

By equations (3), (4) and the universal property of D it follows that $RS = 1_D$.

Now we want to compare SR and $1_{D'}$. We will use the 2-dimensional part of the universal property of D' since we have that there exist natural isomorphisms $1_{D'}U' \overset{\theta'}{\Rightarrow} U'' = SU = SRU'$ and $1_{V'} : 1_{D'}V' \Rightarrow V' = SV = SRV'$ which satisfy $\theta' F_s = \theta$ and in a fancier form $(SR\theta)(\theta' F_s) = (1_{V'}G_s)(1_{D'}\theta)$ which is exactly needed in the definition of the universal property of D' and therefore there exists a unique natural isomorphism $\psi : 1_{D'} \Rightarrow SR$ such that

$$\psi U' = \theta' \ \text{ and } \ \psi V' = 1_{V'}. \tag{5}$$

This obviously gives the equivalence between D and D'.

Going back to the nonstrict doctrine we can see that we have a strong transformation between the two squares whose components are equivalences exactly as required in lemma 4.64:

$$\begin{tikzcd}
\mathcal{A} \arrow[rrr, "F"] \arrow[ddd, "G"'] & & & \mathcal{B} \arrow[ddd, "|U'|\eta_{\mathcal{B}}"] \\
& \mathcal{A} \arrow[r, "F"] \arrow[d, "G"'] \arrow[ul, "1_{\mathcal{A}}"'] & \mathcal{B} \arrow[d, "|U|\eta_{\mathcal{B}}"] \arrow[ur, "1_{\mathcal{B}}"'] & \\
& \mathcal{C} \arrow[r, "|V|\eta_{\mathcal{C}}"'] \arrow[dl, "1_{\mathcal{C}}"] & |\mathcal{D}| \arrow[dr, "|S|"'] \arrow[dr, phantom, shift left=4, "\Nearrow\, |\theta'^{-1}|\eta_{\mathcal{B}}"] & \\
\mathcal{C} \arrow[rrr, "|U'|\eta_{\mathcal{C}}"'] & & & |\mathcal{D}'| \arrow[loop right, "|\theta|\eta_{\mathcal{A}}"]
\end{tikzcd}$$

(omitted 2-cells are identities). The equation required in the lemma is

$$|V'|\eta_{\mathcal{C}} 1_G \circ 1_{|SV|\eta_{\mathcal{C}} G} \circ 1_{|SU|\eta_{\mathcal{B}}} F = |\theta|\eta_{\mathcal{A}} \circ 1_{|U'|\eta_{\mathcal{B}} F} \circ |\theta'^{-1}|\eta_{\mathcal{B}} F : |SU|\eta_{\mathcal{B}} F \Rightarrow |V'|\eta_{\mathcal{C}} G.$$

This holds since by the naturality of η (and omitting identities) it is equivalent to

$$1_{|V'|\eta_{\mathcal{C}} G} = |\theta|\eta_{\mathcal{A}} \circ |\theta'^{-1} F_s|\eta_{\mathcal{A}}$$

and this follows from $1_{U'' F_s} = \theta \circ \theta'^{-1} F_s$.

Since the bigger square is a Pushout in $\mathcal{BCC}$ by theorem 4.69 the smaller square is too by lemma 4.64. Also we have established that this smaller square has the interpolation property and it finishes the proof of the interpolation Theorem 4.47. □

4.5 Applications of the interpolation

As mentioned in the introduction, this section can be considered as a "work in progress" and these applications are rather easy to obtain from the interpolation. However, since they have real categorical flavor and since we easily get the results well known in the literature we consider them as applications.

The first application is a proof of the well known theorem about the interpolation property of Heyting algebras. The theorem is first proved in the important work by Maksimova [Mak77]. The theorem as stated in [Pit83a, Thm. B.] is the following:

Theorem 4.70 *Every pushout square in the category* **Ha** *of Heyting algebras (and structure preserving morphisms) has the interpolation property.*

Proof: Every Heyting algebra is a bicartesian closed category and homomorphisms of these algebras are bc-functors. We can view **Ha** as a 2-category (2-cells being identities). Therefore there is an inclusion I : **Ha** $\to \mathcal{BCC}$. Also, it is easy to show that a left adjoint to this functor is "posetal collapse" $P : \mathcal{BCC} \to$ **Ha**. To construct a pushout of $C \xleftarrow{g} A \xrightarrow{f} B$ in **Ha** we can do the following: include the diagram in $\mathcal{BCC}$ and construct a Pushout there. Then apply the functor P, in this way we obtain the square

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "g"'] & B \arrow[d, "P(H)"] \\ C \arrow[r, "P(K)"'] & P(\mathcal{D}) \end{tikzcd}$$

Since as a left adjoint P preserves Colimits, this square is a pushout. Also a posetal collapse of a square which has the interpolation property is again a square with the interpolation property. From that the theorem follows. □

We shall come back to Heyting algebras in a moment but before that let us establish another interesting fact.

Proposition 4.71 *The full functors are stable under Pushouts in $\mathcal{BCC}$ and $\mathcal{CCC}$ doctrines.*

Proof: We want to show that in a Pushout square as below if F is a full functor then K must be.

$$\begin{tikzcd} \mathcal{A} \arrow[r, "F"] \arrow[d, "G"'] & \mathcal{B} \arrow[d, "H"] \\ \mathcal{C} \arrow[r, "K"'] & \mathcal{D} \arrow[ul, phantom, "\Swarrow \tau"] \end{tikzcd}$$

Suppose that $x : K(C_1) \to K(C_2)$ is an arrow in $\mathcal{D}$. Since $H(F(1))$ is a terminal object and $K(C_2^{C_1})$ is an exponent of $K(C_2)$ by $K(C_1)$ (both in $\mathcal{D}$) then (as in any ccc) there exists unique arrow $\hat{x} : H(F(1)) \to K(C_2^{C_1})$ in $\mathcal{D}$ such that $K(\varepsilon)u = x$ where $u : K(C_1) \to K(C_2^{C_1} \times C_1)$ is the unique arrow such that $K(\pi)u = \hat{x}0_{K(C_1)}$ and $K(\pi')u = 1_{K(C_1)}$ (and $0_{K(C_1)} : K(C_1) \to H(F(1))$ is a unique arrow).

Now we can apply our interpolation theorem to $\hat{x} : H(F(1)) \to K(C_2^{C_1})$ and we get $A \in \mathcal{A}$, $(b : F(1) \to F(A)) \in \mathcal{B}$ and $(c : G(A) \to C_2^{C_1}) \in \mathcal{C}$ such that $\hat{x} = K(c)\tau_A H(b)$. Since F is full by the assumption there exists an arrow $a : 1 \to A$ in $\mathcal{A}$ such that $F(a) = b$. By naturality of τ we obtain that $K(G(a))\tau_1 = \tau_A H(F(a))$ and by the way a was chosen we have that $\hat{x} = K(cG(a))\tau_1$. Now we can check that $K(\langle cG(a)0_{C_1}, 1_{C_1}\rangle) : K(C_1) \to K(C_1^{C_2} \times C_1)$ satisfies the equations defining the above u and then by the uniqueness of u we have that $u = K(\langle cG(a)0_{C_1}, 1_{C_1}\rangle)$ (here, 0_{C_1} is the unique arrow $C_1 \to G(1)$). So finally, $x = K(\varepsilon\langle cG(a)0_{C_1}, 1_{C_1}\rangle)$, i.e. K is indeed full. $\square$

Coming back to Heyting algebras, let us prove the other main theorem from [Pit83a] - Thm A.

Theorem 4.72 *Monomorphisms are stable under pushout in* **Ha**.

Suppose

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "g"'] & B \arrow[d, "k"] \\ C \arrow[r, "h"'] & D \end{tikzcd}$$

is a pushout in **Ha** and f is mono, we want to show that this h is mono. By the proof of the previous theorem we know that the above square is posetal collapse of a Pushout square from $\mathcal{BCC}$ (i.e. $k = P(K)$ and $h = P(H)$). Since the monomorphism g as a functor is full - it follows by the previous proposition that H is full. Also, posetal collapse of a full functor is a monomorphism i.e. h is a monomorphism. $\square$

Since our interpolation result was valid for cartesian closed categories as well we can say that the same statements hold for their posetal collapse. These are known in the literature as Brouwerian semilattices so we can just conclude that in the category of Brouwerian semilattices pushouts have the interpolation property and that monomorphisms are stable under pushouts as well.

For the other application let us first recall the following fact related to the Beck-Chevalley property. Suppose that for the Pushout square as above there are two functors $F_! : \mathcal{B} \to \mathcal{A}$ left adjoint to F and $K_! : \mathcal{D} \to \mathcal{C}$ left adjoint to K. Then there exists a canonical natural transformation $\rho : K_!H \Rightarrow GF_!$ (this does not depend on the interpolation property; also we are not assuming that $F_!$, $G_!$, ρ are in the doctrine). In picture, the situation is the following:

$$\begin{tikzcd}
\mathcal{A} \arrow[r, shift right, "F"'] \arrow[d, "G"'] & \mathcal{B} \arrow[l, shift right, "F_!"'] \arrow[d, "H"] \\
\mathcal{C} \arrow[r, shift left, "K"] \arrow[ur, phantom, "\rho \quad \tau"] & \mathcal{D} \arrow[l, shift left, "K_!"]
\end{tikzcd}$$

This canonical ρ is defined to be the following natural transformation:

$$\varepsilon_{GF_!} \circ K_!(\tau_{S_!} \circ H(\eta^S)) : K_!H \Rightarrow GF_!$$

.

The following statement is present in the poset variant in [Pit83a] and generalized in [Pav92].

Proposition 4.73 *The above square satisfies the interpolation property iff ρ has a left inverse.*

The proof of the above statement is not hard once when we know that the statement holds. We are going to sketch the proof of the relevant corollary:

Corollary 4.74 *The above ρ has left inverse when the above Pushout square is in the doctrines $\mathcal{BCC}$ or $\mathcal{CCC}$.*

Proof: We have to construct a natural transformation $\sigma : GF_! \Rightarrow K_!H$ such that $\sigma\rho = 1_{K_!H}$. Since the Pushout has the interpolation property we can find the interpolant for $\eta^K_{H(B)} : H(B) \to K(K_!(B))$. That is there are $A \in \mathcal{A}$, $b : B \to F(A)$ and $c : G(A) \to K_!(B)$ such that $\eta^K_{H(B)} = K(c)\tau_A H(b)$. Now define $\sigma_A = cF(\varepsilon^F_A F_!(b))$ and check that σ so defined satisfies the required property. $\square$

A Proof of the Weak Normalization Theorem

In this appendix we prove the theorem 4.49. First some definitions:

Definition A.75 *A maximal chain of immediate subterms in a term t is called a* **thread**. *A* **segment** *in a term t is a chain of immediate subterms $t_n \succ \cdots \succ t_1$ of t such that:*

1. $t_1 \not\equiv \delta x.u, y.v; w$
2. *each t_i $i < n$ is a minor premise of a δ, i.e. is in one of the following positions $\delta x.t_i, y.v; w$ or $\delta x.u, y.t_i; w$,*
3. *t_n is not in a such position.*

So any term not of the "δ-form" nor a minor premise of a "δ-form" is a segment ($n = 1$). Also notice that all t_i are of the same type!

A **maximum segment** *is a segment where t_1 has one of the following forms: $\langle u, v\rangle$, $\lambda x.r$, $\iota_i(r)$ or $\epsilon^A(r)$; and t_n is in one of the following positions: $\pi_i(t_n)$, $(t_n{}^{\text{`}}r)$, $\delta x.u, y.v; t_n$ or $\epsilon^B(t_n)$.*

So we can see that the term t_n in a maximum segment is an immediate subterm of a C-redex when $n > 1$ (because t_n then has a "δ-form", e.g. $t_n \equiv \delta x.t_{n-1}, y.v; w$); and it is an immediate subterm of an R_2-redex or E-redex when $n = 1$, more precisely, this is an immediate subterm of an R_2-redex if t_1 is not an ϵ-form, and this is an immediate subterm of an E-redex otherwise, i.e. when t_1 is an ϵ-form. Also any ρ-redex contains the top of a maximum segment as an immediate subterm.

Since the subterms of a term t make a tree we can define the **depth of a segment** *$S = t_n \succ \cdots \succ t_1$ to be the number k such that $t \equiv r_0 \succ \cdots \succ r_k \equiv t_n$; we write $k = depth(S)$.*

Proof (of theorem 4.49): Let $S = t_n^A \succ \ldots \succ t_1^A$ be a segment. Then the degree of S (denoted $d(S)$) is defined to be the complexity of A, i.e. $d(A^B) = d(A \times B) = d(A + B) = d(A) + d(B)$, $d(0) = 1$ and $d(X) = 0$ when X is a free type or the terminal type; the length of S (denoted $l(S)$) is defined to be n (for the above S). To every term t we assign an ordered pair $\#t = (d, l)$ where d is the highest degree of a maximum segment in t (or 0 if such a segment does not exist) and $l = l(S_1)+\cdots+l(S_k)$ where $S_1, \ldots, S_k$ are all the maximum segments in t with the degree d. Also we say $(d_1, l_1) < (d_2, l_2)$ iff $d_1 < d_2$ or, $d_1 = d_2$ and $l_1 < l_2$.

We assume $d > 0$, if $d = 0$ we are done - there are no ρ-redexes.

Take a maximum segment $S = t_n \succ \cdots \succ t_1$ of the degree d with the largest depth among such maximum segments.

<u>First case:</u> t_n has nor a δ nor a ϵ-form so then $n = 1$ and t_n is in $\pi_j(t_n)$, or in t_n‘r, or in $\delta x.u, y.v; t_n$. So this is an R_2-redex. Then the whole segment S is just the term $t_1^{A_1}$ and $\#t = (d(S), 1 + l(S_2) + \cdots + l(S_k))$, where $S_2, \ldots, S_k$ are the other maximum segments of the degree $d(S)$. Applying the appropriate R_2-reduction on this redex so that $t \xrightarrow{R_2} t'$ we can see that $\#t' < \#t$ because an R_2-reduction performed on a maximum segment/term of the highest degree can't produce a maximum segment of a higher degree. The idea comes from Turing - see [Gan80]. (Check the cases: the "worst" one is $t \equiv \ldots(\lambda x^A.r^{B}$‘$s^A)\ldots$ where $S \equiv \lambda x.r$; so $d(t) = d(B^A)$. The possible new maximum segments can appear in $r(s/x)$ but then their degree is $d(A)$. So the only maximum segments of the degree $d(S)$ are the old ones (if any). Since we reduced "the innermost" maximum segment they didn't multiply.Therefore, $l(t)$ decreased by 1 so indeed $\#t > \#t'$.)

<u>Second case:</u> t_n is an ϵ-form, so again $n = 1$. Therefore this is an immediate subterm of an E-redex. Then again as above the whole segment S is just the term t_1.

Applying the appropriate E-reduction on this redex so that $t \xrightarrow{E} t'$ we can see that $\#t' < \#t$.

Let's check the worst case, i.e. when $t_1 \equiv \epsilon^{E_1+E_2}(r^0)$ and $t \equiv \tau[\delta x.u^C, y.v^C; \epsilon^{E_1+E_2}(r^0)]$ for the appropriate $\tau[z^C], u, v, r \ldots$ (the variable z in $\tau[z]$ denotes the only place where the "substituting" is done, so we don't care about the possible clashes of variables). Then $t' \equiv \tau[\epsilon^C(r^0)]$ and $\#t = (d(E_1+E_2), 1+l(S_2)+\cdots+l(S_k))$ where $S_2, \ldots, S_k$ are the other segments of degree $d(E_1+E_2)$. After the reduction the only new maximum segment would have to contain $\epsilon^C(r)$.

The first subcase is when the term r is the outermost term in the segment i.e. "t_n" or r. Then it would mean that the term r in $t \equiv \tau[\epsilon^{E_1+E_2}(r^0)]$ were also a maximum segment which is properly inside $S \equiv t_1$ - therefore $d(E_1 + E_2) > d(C)$. So since the possible new maximum segment is of a smaller degree notice that $d(t) > d(t')$ if there are no other maximum segments of the degree $d(t) = d(E_1 + E_2)$ and, if there are some other maximum segments in t of the degree $d(t)$ then $l(t) > l(t')$.

The second subcase is when $\epsilon^C(r^0)$ is the innermost term in the chain for the new maximum segment, i.e. "t_1". If the chain were of length 1 then it would mean that the maximum segment is actually an immediate subterm of an E-redex. Comparing C and E reductions one can see that then $\delta x.u^C, y.v^C; \epsilon^{E_1+E_2}(r^0)$ is also an immediate subterm of a C-reduction. The only critical case is when $d(C) = d(E_1 + E_2)$ but then the new maximum segment is actually replacing an old maximum segment of the same degree but since $l(t) = 1 + l(S_2) + \cdots + l(S_k) > l(S_2^{new}) + \cdots + l(S_k) = l(t')$; (because $l(S_2) = l(S_2^{new}) = 1$) we have $d(t) = d(t')$ but $l(t) > l(t')$. Similarly if the chain is of the length greater then 1 then both $\delta x.u^C, y.v^C; \epsilon^{E_1+E_2}(r^0)$ and $\epsilon^C(r^0)$ are minor premises of the same δ-form and they are beginnings of the "same" segment. So the above formula again holds except that now $l(S_2) = l(S_2^{new}) > 1$ and the conclusion is as above. The noncritical cases give $d(E_1 + E_2) > d(C)$.

The other cases are indeed better: applying $E_{1,2,3}$ the degree of the maximum segment gets smaller and applying E_5 we don't get essentially new maximum segments.

<u>Third case:</u> t_n is a δ-form, so $n > 1$ and this is an immediate subterm of a C-redex. All the cases are similar - let's just check the case where we apply C_4. Let $t \equiv \ldots \delta x_1.u_1, x_2.u_2; (\delta y_1.t_{n-1}, y_2.v_2; r) \ldots$ (so $t_n^{A+B} \equiv \delta y_1.t_{n-1}^{A+B}, y_2.v_2^{A+B}; r$). Then

$$t \xrightarrow{C_4} \ldots \delta y_1.(\delta x_1.u_1, x_2.u_2; t_{n-1}), y_2.(\delta x_1.u_1, x_2.u_2; v_2); r \ldots \equiv t'$$

Since t_{n-1} is no longer a minor premise of a δ-form then by the definition of a maximum segment the new segment "reduct of S" has length $n-1$ (and the same degree); since the maximum segments of this degree didn't multiply because we reduced the "innermost" such segment we get $\#t - 1 = \#t'$. $\square$

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