CALCULUS OF VARIATIONS FOR DISCONTINUOUS FIELDS AND ITS APPLICATIONS TO SELECTED TOPICS IN CONTINUUM MECHANICS

by

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SUMMARY

The proposed thesis consists of two parts. In the first part the calculus of variations for discontinuous fields is established. In order to express the first variation of the action integral in the class of discontinuous fields, in terms of arbitrary variations, and in particular, in terms of arbitrary variations on a singular hypersurface, variational conditions of compatibility are formulated. These conditions generalize Thomas' kinematical conditions of compatibility. The necessary and sufficient conditions for the action integral to be stationary in the considered class of fields are derived. In particular, the jump conditions of field quantities across the singular hypersurface, which generalize the Weierstrass-Erdmann conditions to multiple integral problems are obtained. Next, the relations between transformations leaving the action integral invariant (symmetry transformations) and conservation laws are established for the case of discontinuous fields. Finally, in this part, the complementary conservation laws on the singular hypersurface are obtained directly from the jump conditions.

The second part of this thesis deals with applications. In particular, the following topics are investigated. The dualism in the description of the undeformed state and the family of deformed states of 2 and 3-dimensional media set into motion during which a singular hypersurface propagates through these media. Balance laws which admit discontinuities carried by wave fronts for the simple hyperelastic materials and materials of grade 2, in the material and spatial descriptions. Finally, the problem of wave propagation in a plate is investigated

using a model based on a fourth order differential equation for transverse vibrations, including the shear caused by transverse stresses. In this latter topic, the speed of propagation and the decay law for the third order wave are derived. The meaning of such waves within the plate model is studied, and some general observations are stated. Also, the relation of our approach to those studied by other researchers is indicated.

RESUME

La présente thèse consiste de deux parties. Dans la première partie nos établissons le calcul des variations pour les champs discontinus. De faççon à exprimer la première variation de l'intégrale d'action dans la classe des champs discontinus, en terme de variations arbitraires et, en particulier, en termes de variations arbitraires sue une hypersurface singulière, des conditions variationnelles de compatibilité sont formulées. Ces conditions généralisent les conditions cinétiques de compatibilité de Thomas. Les conditions suffisantes et necessaires pour que l'intégral d'action soit stationnaire dans la classe de champ considérés sont calculées. En particulier, sont obtenus les conditions de saut des quantités de champ sur l'hypersurface singulière, qui généralisent les conditions de Weierstrass-Erdmann à des problèmes d'integrales multiples.

Ensuite, nous établissons les relations entre les transformations qui laissent invariante l'intégral d'action (transformations de symmétrie) et les lois de conservations dans le cas des champs discontinus. Finalement, dans cette partie, les lois de conservation complémentaires pour l'hypersurface singulière sont obtenues directement des contiions de saut.

La seconde partie de cette thèse traite d'applications. En particulier sont étudiés les sujets suivants. Le dualisme dans la description de l'état non déformé et la famille d'etat déformés de milieu de 2 ou 3 dimensions mises en motion pendant qu'une hypersurface se propage à travers ces milieux. Les lois de la Balance qui admettent

des discontinuités portées par fronts d'onde pour des matériaux hyperélastiques simples et materiaux de grade 2 dans les descriptions matérielles et spatiales.

Finalement, nous étudions le problème de propagation d'onde dans un modèle de plaque basé sur un système d'équations différentielles de quatrième ordre pour vibrations transverses, incluant le terme d'inertie rotationnelle. Dans ce dernier sujet, la vitesse de propagation et la loi de la décadance de l'onde de troisième ordre sont derivées. La signification de telles ondes dans le contexte du modèle de la plaque est étudiée et quelques observations générales sont formulées. Aussi, sont montrées les relations des nos approches enver les autres chercheurs.

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PART I

CALCULUS OF VARIATIONS FOR DISCONTINUOUS FIELDS

CHAPTER 1

INTRODUCTION

First order variational problems defined by multiple integrals, that is to say, variational problems whose integrands depend on independent variables, state variables and only their first order partial derivatives, find significant applications in the formulation of field theories in physics and engineering. Problems in which the integrand contains higher-order derivatives have also received considerable attention, ever since the origin of the calculus of variations in the early eighteenth century. These variational problems are referred to as higher-order problems. The interest in second-order problems in particular lies in the fact that they can be applied, with varying degrees of success, to various branches of mathematics and to physics, for example, in relativity and continuum mechanics. In general relativity the integrand, called the Lagrangian density function, which gives rise to the Einstein gravitational field equations is $L = R\sqrt{-g}$ where R is the scalar curvature and g is the determinant of the metric tensor. The scalar curvature R inherently contains second order derivatives of the components of the metric tensor. In continuum mechanics the governing differential equations are often of fourth order, for example the oscillations of rigid plates and shells. The inverse problem of the calculus of variations then leads to an associated Lagrangian density function dependent also on second-order partial derivatives of displacement. Efforts have been made to establish a generalized mechanics [1,2,3] and a generalized electrodynamics [4,5] by including higher-order derivatives in the Lagrangian function.

The importance of variational formulations of the laws of continuum physics, whenever they exist, lies in the fact that they are the best way to express such laws, as has been pointed out by Oden and Reddy in [6] and by Marsden and Hughes in [7], for example. This is because the fundamental principles of continuum physics are global in character. Their local forms can be derived from their global forms only if the involved fields are endowed with suitable smoothness properties. However, this smoothness is very often unnatural. In continuum mechanics, for example, it rules out discontinuities which are carried by wave fronts propagating through a material. In the variational formulation of the fundamental principles of continuum mechanics, a simple functional, called the action integral, accounts for all the intrinsic features of the problem: the differential equation of motion (the Euler-Lagrange equations), the natural boundary conditions and the jump conditions associated with propagating discontinuities. Moreover, the variational approach allows a systematic connection to be made between symmetries and conservation laws as well as constituting a natural means for approximating and finding the solution.

The significance of variational principles for discontinuous fields has recently been recognized, see Nemat-Nasser [8]. For a historical account on this subject in continuum mechanics we refer to the Introduction to the second part of this thesis. However, to the knowledge of the author, general variational theorems which admit discontinuities in the field quantities and their partial derivatives have not been elaborated, especially in their relation to propagating discontinuities.

The purpose of this thesis is to establish a general approach to the calculus of variations for discontinuous fields (PART I) and to apply it to some problems in continuum mechanics (PART II). Our approach, which has been proposed by the author in [9] and extended in [10], is based on the theory of singular surfaces. This theory has been developed by Thomas. His results are summarized in his book [11]. An extensive treatment of the basic mathematics of singular surfaces and a historical account on contributions to the field has been given by Truesdell and Toupin in [12]. Thomas' theory has recently been generalized by Cohen and Wang [13]. Their investigation included also a treatment of singular curves propagating through a material surface.

In the next two chapters (Chapter 2 and 3) we shall give a simple account on the basic definitions and results of the theory of singular hypersurfaces. In Chapter 3 we shall recall the geometrical and kinematical conditions of compatibility in a form that will be useful for this study.

In Chapter 4 we shall extend the notion of kinematical conditions of compatibility. We shall derive new conditions which we shall call variational conditions of compatibility. They are associated with the virtual deformation of discontinuous fields. These conditions are expressions for the jumps in the variation of partial derivatives of a tensor field, in terms, in general, of jumps in the tangential derivatives of the displacement variation of the tensor field and in the normal derivative of this field at its singular hypersurface and in the normal variation of this hypersurface. For the special case of an imposed virtual deformation, the displacement variation and the normal variation of a singular hypersurface are reduced to the displacement derivative (called also Thomas' derivative) and the speed of propagation of this hypersurface, respectively. In this case the variational conditions of compatibility are reduced to Thomas' kinematical conditions of compatibility.

Using variational conditions of compatibility, we can express the first variation of the action integral for discontinuous field in terms of arbitrary variations on a singular hypersurface. These arbitrary variations are the displacement variations and the normal variations of the hypersurface. Before this expression for the first variation of action integral is derived in Chapter 6, first we shall review in Chapter 5 single integral problems for discontinuous functions, and in particular we shall recall the Weierstrass-Erdmann (corner) conditions for such problems (cf. Gelfand and Fomin [14] and Oden and Reddy [6]). Also in Chapter 6, we shall extend the Fundamental Lemma of the calculus of variations to include additional integrals induced by the singular hypersurface of discontinuous fields. This lemma leads to necessary and sufficient conditions for the action integral to be stationary in the considered class of fields. They are given in Theorem 6.1. In particular, we shall obtain in this theorem jump conditions of field quantities across a singular surface which generalize the Weierstrass-Erdmann conditions to multiple integral problems.

In the next chapters of this part, we shall establish for the case of discontinuous fields, the relation between transformations leaving the action integral invariant (symmetry transformations) and conservation laws. A historical account on this subject for smooth fields, as well as, the list of standard references will be given in Section 7.1 of Chapter 7. In Section 7.2 of this chapter, the definition of invariance and the discussion of this definition in relation to the considered discontinuous fields will be given. In the last section of Chapter 7, we shall prove the fundamental invariance theorems. For the considered class of fields perhaps the most important is Theorem 7.3 in which the integral identities implied by invariance are given.

In Chapter 8 the conservation theorems for discontinuous fields are formulated and proved. Again, perhaps the most important results are given in Theorem 8.1 in which we shall establish the relation between (integral) conservation laws and symmetry transformations, mentioned above. For higher-order variational problems, and in particular for second-order problems, from this theorem will be seen the conservation laws for an arbitrary subsystem contain terms describing "flux" concentrated on the boundary of the singular hypersurface intersected by this subsystem. Finally in this Chapter, we shall establish in Remark 8.1 the complementary conservation laws on the singular hypersurface that are directly implied by the jump conditions obtained in Theorem 6.1.

Remark. Throught this thesis, we shall employ the summation convention with which a repeated index in a term is understood to be summed over the possible values of this index.

CHAPTER 2

SINGULAR HYPERSURFACES

In order to fix the notation we shall let $(X, t) = (X^A, t)$ $(A = 1,\ldots,N)$ denote a point of R^{N+1} where $N = 2$ or 3.

In the study of singular hypersurfaces we deal only with hypersurfaces given by the following definition. A (smooth) hypersurface in R^N is a set $\Sigma \subset R^N$ such that for each point $X \in \Sigma$ there is a neighbourhood V of X in R^N and a mapping $\chi: U \to V \cap \Sigma$ of an open set $U \subset R^{N-1}$ into $V \cap \Sigma \subset R^N$ subject to the following conditions:

(i) χ is a smooth mapping

(ii) χ is a homeomorphism

(iii) χ is regular at each point $u \in U$.

The mapping χ is called a local parametrization or a (local) coordinate system at X and the neighbourhood $V \cap \Sigma$ of X in Σ a coordinate neighbourhood. Roughly speaking hypersurfaces given by the above definition have no sharp points, edges or self-intersections. Moreover, we always assume that hypersurfaces are orientable $(N = 3)$ and that we have made a choice of unit normal vector N for each $X \in \Sigma$, which is perpendicular to Σ at X.

Let us consider a family $\{\Sigma_t\}_{t\in\mathcal{T}}$ of hypersurfaces $\Sigma_t \subset R^N$ where $\mathcal{T} \subset R^1$ is an interval (time interval). For a given open set $\Omega \subset R^N$ we assume that for each $t \in \mathcal{T}$, Σ_t divides Ω into two non-empty domains, denoted by Ω_t^+ and Ω_t^- and forms a common boundary between them. The unit normal $\mathbf{N}$ on Σ_t is directed toward the set Ω_t^+. It is assumed that a "space-time" representation of the family $\{\Sigma_t\}_{t\in\mathcal{T}}$ is given by a smooth hypersurface

$$\Gamma = \{(X,t) : X \in \Sigma_t,\ t \in \mathcal{T}\}$$

in R^{N+1}. This hypersurface divides $\pi = \Omega \times \mathcal{T}$ into two subsets π^+ and π^- where $\pi^\pm = \{(X,t),\ X \in \Omega_t^\pm,\ t \in \mathcal{T}\}$ (see Fig. 1).

Let $\{\phi_t\}_{t\in\mathcal{T}}$ be a family of mappings such that $\phi_t(X) = \phi(X,t)$ is a scalar-valued, vector-valued or tensor-valued mapping defined and continuously differentiable on π^+ and π^-. If Γ is a hypersurface in π given by $f(X,t) = 0$, where f is real-valued differentiable function defined on π, $\{\Gamma(c)\}$ are the neighbouring hypersurfaces in π given by $f = c$. We shall assume that ϕ and its partial derivatives on $\Gamma(c)$ converge uniformly to bounded limits on Γ as c tends to zero through positive and negative values. Let $A^+(X,t)$ and $A^-(X,t)$ denote the limits of a field A at any point (X,t) on the hypersurface Γ which is approached from π^+ and π^-, respectively. The hypersurface Γ (the moving hypersurface Σ_t) is said to be a singular hypersurface relative to the field ϕ (ϕ_t) if the jump

$$[\![A]\!]\ (X,t) := A^+(X,t) - A^-(X,t)$$

does not vanish for some $(X,t) \in \Gamma$ $(X \in \Sigma_t)$ where A denotes the field ϕ or some of its partial derivatives. In this case ϕ (ϕ_t) will be denoted by a pair (Γ,ϕ) $((\Sigma_t, \phi_t))$ where Γ (Σ_t) is its singular hypersurface. The order of the singular surface corresponds to the lowest order partial derivative of ϕ (ϕ_t) which is discontinuous across Γ (Σ_t).

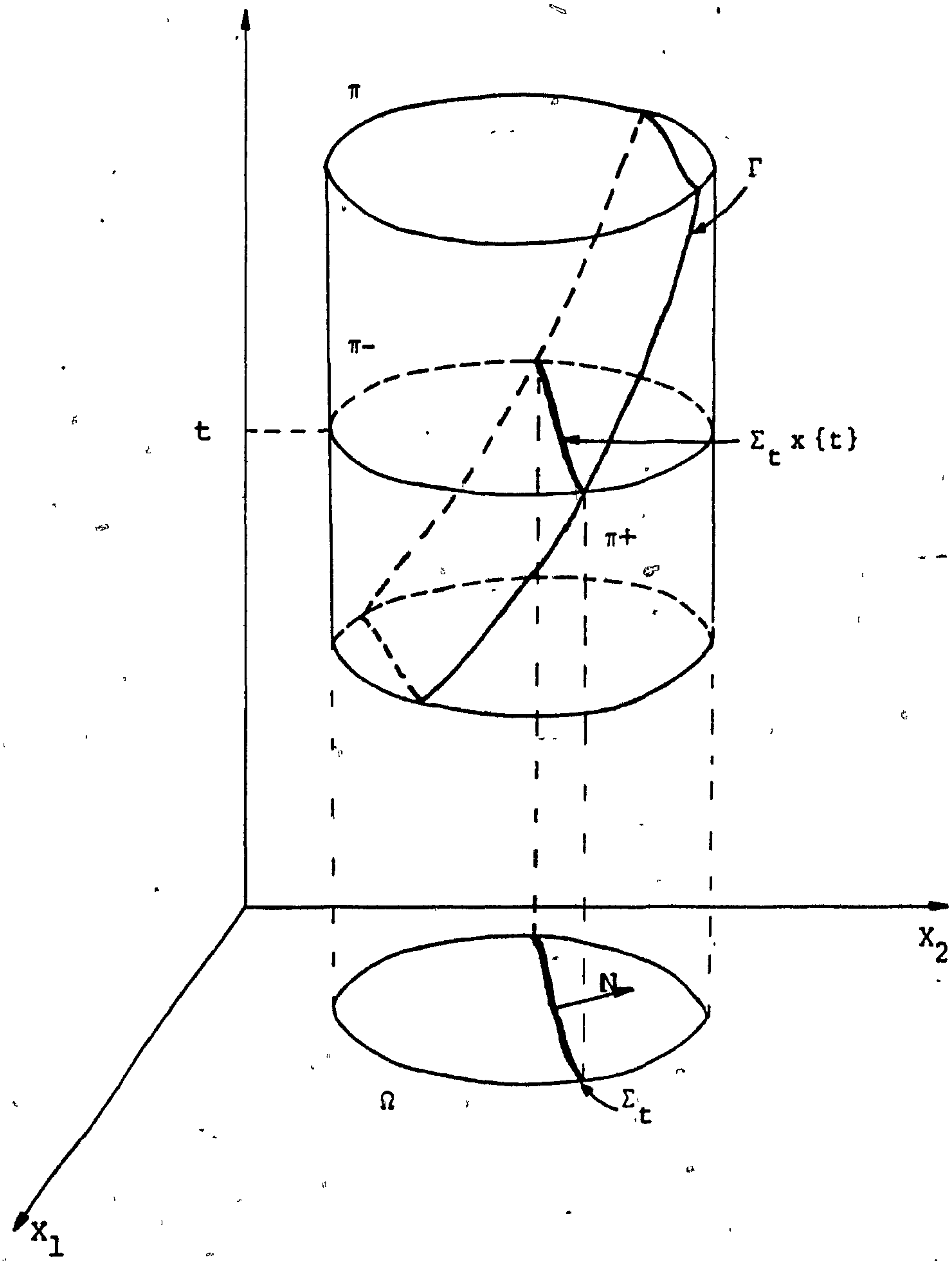


Figure 1

CHAPTER 3

GEOMETRICAL AND KINEMATICAL CONDITIONS OF COMPATIBILITY

For a given (smooth) mapping ϕ on R^N, let $\phi_{,A} \equiv \partial\phi/\partial X^A$, $\phi_{,AB} \equiv \partial^2\phi/\partial X^A \partial X^B$, ... etc. We shall adopt the following notation for the derivatives of a function ϕ at any point on the hypersurface $\Sigma \subset R^N$:

$$\phi_{,A} = \mathcal{D}_A\phi + N_A\, \partial_{(N)}\, \phi$$

$$\partial^2_{(N)}\phi = N^A N^B\, \phi_{,AB}$$

in the above N_A are the components of the unit normal N on Σ,

$$\mathcal{D}_A\phi \equiv (\delta^B_A - N_A N^B)\, \phi_{,B}$$

are the components of the tangential derivative of ϕ and Σ and

$$\partial_{(N)}\phi \equiv N^A \phi_{,A}$$

is the normal derivative of ϕ at the hypersurface Σ.

Let us consider a pair (Σ_t, ϕ_t) as defined in Chapter 2. By Hadamard's Lemma [15,12] we have

$$(\mathcal{D}_A\phi)^{\pm} = \mathcal{D}_A\phi^{\pm}$$

which implies the following limit conditions .

$$(\phi_{,A})^{\pm} = \mathcal{D}_A\phi^{\pm} + N_A(\partial_{(N)}\phi)^{\pm} \tag{3.1}$$

$$(\phi_{,AB})^{\pm} = \frac{1}{2}\,(\mathcal{D}_A\mathcal{D}_B\phi^{\pm} + \mathcal{D}_B\mathcal{D}_A\phi^{\pm})$$

$$+ \frac{1}{2}\,(N_A\Omega^C_B\,\mathcal{D}_C\phi^{\pm} + N_B\Omega^C_A\,\mathcal{D}_C\phi^{\pm})$$

$$+ N_A\mathcal{D}_B(\partial_{(N)}\phi)^{\pm} + N_B\mathcal{D}_A(\partial_{(N)}\phi)^{\pm} - \Omega_{AB}(\partial_{(N)}\phi)^{\pm}$$

$$+ N_AN_B\ (\partial^2_{(N)}\phi)^{\pm} \qquad (3.2)$$

where $\Omega_{AB} = -\mathcal{D}_AN_B = -\mathcal{D}_BN_A$ are the components of the second fundamental form of the hypersurface Σ_t.

From (3.1) and (3.2) we can formulate conditions

$$[\![\phi_{,A}]\!] = \mathcal{D}_A[\![\phi]\!] + N_A[\![\partial_{(N)}\phi]\!] \qquad (3.3)$$

$$[\![\phi_{,AB}]\!] = \mathcal{D}_{(A}\mathcal{D}_{B)}\ [\![\phi]\!] + N_{(A}\Omega^C_{B)}\ \mathcal{D}_C\ [\![\phi]\!]$$

$$+ 2\ N_{(A}\mathcal{D}_{B)}\ [\![\partial_{(N)}\phi]\!] - \Omega_{AB}[\![\partial_{(N)}\phi]\!]$$

$$+ N_AN_B\ [\![\partial^2_{(N)}\phi]\!] \qquad (3.4)$$

where parantheses enclosing indices A and B indicate symmetrization with respect to these indices.

The formulae for jumps of the partial derivatives of ϕ given by (3.3) and (3.4) are the geometrical conditions of compatibility of first order and of second order, respectively.

The kinematical conditions of compatibility will be derived using local parametrizations of the hypersurface Σ_t in the following way:

We shall assume that the following restrictions of ϕ_t, $\phi_t|\Omega^+_t$ and $\phi_t|\Omega^-_t$, have in the neighbourhood of each $X \in \Sigma_t$ differentiable (smooth) extensions, denoted by $\tilde{\phi}^{(+)}_t$ and $\tilde{\phi}^{(-)}_t$, respectively. For each $X \in \Sigma_t$ in a chosen $(\chi_t, \mathcal{O}_t)$ where χ_t is a coordinate system at $X \in \Sigma_t$ and $\mathcal{O}_t = V \cap \Sigma_t$ is a coordinate neighbourhood, the mappings $\tilde{\phi}^{(+)}$ and $\tilde{\phi}^{(-)}$ can be represented by the following "trace" formulae

$$\hat{\phi}^{+}_{t}(u) = \hat{\phi}^{+}(u,t) := \tilde{\phi}^{(+)}(\chi(u,t),t)$$

$$\hat{\phi}^{-}_{t}(u) = \hat{\phi}^{-}(u,t) := \tilde{\phi}^{(-)}(\chi(u,t),t)$$

Then by the chain rule for differentiation we have

$$\frac{\partial\hat{\phi}^{\pm}}{\partial t} = \tilde{\phi}^{\substack{(+)\\(-)}}_{,A}\ \frac{\partial\chi^{A}}{\partial t} + \frac{\partial\tilde{\phi}^{\substack{(+)\\(-)}}}{\partial t}$$

$$= (\phi_{,A})^{\pm}\ \frac{\partial\chi^{A}}{\partial t} + \left(\frac{\partial\phi}{\partial t}\right)^{\pm} \tag{3.5}$$

Using in (3.5) the conditions (3.1) we obtain

$$\frac{\partial\hat{\phi}^{\pm}}{\partial t} = D_{A}\phi^{\pm}\frac{\partial\chi^{A}}{\partial t} + (\partial_{(N)}\phi)^{\pm} N_{A}\frac{\partial\chi^{A}}{\partial t}$$

$$+ \left(\frac{\partial\phi}{\partial t}\right)^{\pm} \tag{3.6}$$

It is well known (cf., for example [12]) that the quantity

$$N_{A}\frac{\partial\chi^{A}}{\partial t}$$

is independent of the choice of local parametrization. Hence it follows from (3.6) that the quantities

$$\frac{\partial\hat{\phi}^{\pm}}{\partial t} - D_{A}\phi^{\pm}\frac{\partial\chi^{A}}{\partial t}$$

are also independent of the choice of parametrization.

In the standard notation and terminology

$$U_{(N)} \equiv N_{A}\ \frac{\partial\chi^{A}}{\partial t} \tag{3.7}$$

is the speed of propagation of the hypersurface Σ_t, and

$$\frac{\delta\phi^{\pm}}{\delta t} \equiv \frac{\partial\hat{\phi}^{\pm}}{\partial t} - D_A\phi^{\pm}\frac{\partial\chi^A}{\partial t} \tag{3.8}$$

are called the displacement derivatives of $\phi^{\pm}$ at the singular hypersurface Σ_t.

For a more comprehensive study and a historical development of the concept of the displacement derivative we refer to Bowen and Wang [16]. Introducing the notation (3.7) and (3.8) into (3.6) we obtain

$$\frac{\delta\phi^{\pm}}{\delta t} = U_{(N)}\ (\partial_{(N)}\phi)^{\pm} + (\dot{\phi})^{\pm} \tag{3.9}$$

where $\dot{\phi} \equiv \partial\phi/\partial t$. From (3.9) we get conditions

$$[\![\dot{\phi}]\!] = \frac{\delta[\![\phi]\!]}{\partial t} - U_{(N)}\ [\![\partial_{(N)}\phi]\!] \tag{3.10}$$

which are called the kinematical conditions of compatibility of first order.

The second order conditions are obtained in the following way. First, if we replace ϕ in (3.1) and (3.9) by $\dot{\phi}$ we have

$$(\dot{\phi}_{,A})^{\pm} = N_A(\partial_{(N)}\dot{\phi})^{\pm} + D_A(\dot{\phi})^{\pm} \tag{3.11}$$

$$(\ddot{\phi})^{\pm} = \frac{\delta}{\delta t}(\dot{\phi})^{\pm} - U_{(N)}\ (\partial_{(N)}\dot{\phi})^{\pm} \tag{3.12}$$

Next, let us note that

$$\frac{\delta}{\delta t}(\partial_{(N)}\phi)^{\pm} = \frac{\delta N^A}{\delta t}\ (\phi_{,A})^{\pm} + N^A\frac{\delta}{\delta t}(\phi_{,A})^{\pm}$$

$$= D_A U_{(N)}\ D_A\phi^{\pm} + U_{(N)}\ (\partial^2_{(N)}\phi)^{\pm} + (\partial_{(N)}\dot{\phi})^{\pm} \tag{3.13}$$

where we have applied identity $\delta N^A/\delta t = -D_A U_{(N)}$ (cf. [16]) and conditions

(3.9) in which we have replaced ϕ by $\phi_{,A}$. Now, by introducing (3.9) and (3.13) into (3.10) and (3.12) we derive

$$(\dot{\phi}_{,A})^{\pm} = \mathcal{D}_A\{\frac{\delta\phi^{\pm}}{\delta t} - U_{(N)} \ (\partial_{(N)}\phi)^{\pm}\} + N_A \mathcal{D}_B U_{(N)} \mathcal{D}_B \phi^{\pm}$$

$$+ N_A \frac{\delta}{\delta t} (\partial_{(N)}\phi)^{\pm} - U_{(N)} \ N_A \ (\partial^2_{(N)}\phi)^{\pm} \tag{3.14}$$

$$(\ddot{\phi})^{\pm} = \frac{\delta}{\delta t}\{\frac{\delta\phi^{\pm}}{\delta t} - U_{(N)} \ (\partial_{(N)}\phi)^{\pm}\} - U_{(N)} \mathcal{D}_A U_{(N)} \mathcal{D}_A \phi^{\pm}$$

$$- U_{(N)} \ \frac{\delta}{\delta t} (\partial_{(N)}\phi)^{\pm} + U^2_{(N)} \ (\partial^2_{(N)}\phi)^{\pm} \tag{3.15}$$

The above conditions lead to the following expressions

$$[\![\dot{\phi}_{,A}]\!] = \mathcal{D}_A\{\frac{\delta[\![\phi]\!]}{\delta t} - U_{(N)} \ [\![\partial_{(N)}\phi]\!]\} + N_A \mathcal{D}_B U_{(N)} \mathcal{D}_B [\![\phi]\!]$$

$$+ N_A \frac{\delta}{\delta t} [\![\partial_{(N)}\phi]\!] - N_A U_{(N)} \ [\![\partial^2_{(N)}\phi]\!] \tag{3.16}$$

$$[\![\ddot{\phi}]\!] = \frac{\delta}{\delta t}\{\frac{\delta[\![\phi]\!]}{\delta t} - U_{(N)} \ [\![\partial_{(N)}\phi]\!] - U_{(N)} \mathcal{D}_A U_{(N)} \mathcal{D}_A [\![\phi]\!]$$

$$- U_{(N)} \frac{\delta}{\delta t} [\![\partial_{(N)}\phi]\!] + U^2_{(N)} \ [\![\partial^2_{(N)}\phi]\!] \tag{3.17}$$

which are called the kinematical conditions of compatibility of second order.

In a similar way the higher-order geometrical and kinematical conditions of compatibility can be obtained. For a discussion of that subject we refer to Wang and Truesdell [17].

CHAPTER 4

VARIATIONAL CONDITIONS OF COMPATIBILITY

In this section we shall extend the notion of the kinematical conditions of compatibility associated with a singular hypersurface, as they have been derived in Chapter 3. We shall derive new conditions which we shall call the variational conditions of compatibility. As presented here they form a part of an extension of the variational formulation of the fundamental principles of continuum physics for discontinuous fields by including the theory of singular hypersurface into the calculus of variations.

Let us consider a pair (Σ_t, ϕ_t) as it has been defined in Chapter 2. A (infinitesimal) virtual deformation of (Σ_t, ϕ_t) is the following one-parameter family of pairs

$$(-\varepsilon,\varepsilon) \ni s \mapsto (\Sigma_t(s), \phi_t(s)) \tag{4.1}$$

where $\varepsilon > 0$. In (4.1) $\phi_t(s)(X) = \phi(X, t(s), s)$ and for each s, $\Sigma_t(s)$ is a singular hypersurface (the carrier of a simple discontinuity of ϕ or some of its partial derivatives) relative to $\phi_t(s)$ such that $\Sigma_t(0) \equiv \Sigma_t$ and $\phi_t(0) \equiv \phi_t$.

A normal variation $\delta\Sigma$ of the hypersurface Σ associated with a virtual deformation (4.1) is defined in the following way. For each $X \in \Sigma_t$ we set

$$\delta\Sigma(X',t) = N_A \left.\frac{dX_t^A(s)}{ds}\right|_{s=0} \tag{4.2}$$

where N_A are the components of the unit normal N at $X' = \chi(u,t) \in V \cap \Sigma_t$, $u \in U \subset R^{N-1}$ (c.f. definition of a hypersurface given in Chapter 2) and $\chi_t(s)(u) = \chi(u, t(s), s)$ is a local parametrization of $\Sigma_t(s)$ such that $\chi_t(0) \equiv \chi_t$ is a local parametrization at $X \in \Sigma_t$. In a way similar to that followed for the formula of speed of propagation (3.7) we have that the normal variation (4.2) is independent of the choice of local parametrizations of the hypersurface. The proof of this statement is presented in the Appendix I.

In order to formulate variational conditions of compatibility we shall follow closely the procedure that has been outlined in Chapter 3 for the derivation of the kinematical conditions of compatibility. First let us write

$$\frac{d\hat{\phi}_t^{\pm}(s)}{ds}\Big|_{s=0} = \partial_A\phi^{\pm} \frac{d\chi_t^A(s)}{ds}\Big|_{s=0} + N_A(\partial_{(N)}\phi)^{\pm} \frac{d\chi_t^A(s)}{ds}\Big|_{s=0} + \left(\frac{d\phi_t(s)}{ds}\Big|_{s=0}\right)^{\pm} \qquad (4.3)$$

where $\hat{\phi}_t^+(s)$ and $\hat{\phi}_t^-(s)$ are the representations of the "trace" on $\Sigma_t(s)$ of smooth extensions $\tilde{\phi}_t^{(+)}(s)$ and $\tilde{\phi}_t^{(-)}(s)$ of $\phi_t(s)|\Omega_t^+(s)$ and $\phi_t(s)|\Omega_t^-(s)$, respectively, in a local parametrization $\chi_t(s)$ of the hypersurface $\Sigma_t(s)$ such that $(\Sigma_t(s), \phi_t(s))$ is given by (4.1).

Accordingly, if we define quantities $\delta\phi^{\pm}$ by

$$\delta\phi^{\pm} = \frac{d\hat{\phi}_t^{\pm}(s)}{ds}\Big|_{s=0} - \mathcal{D}_A\phi^{\pm} \frac{d\chi_t^A(s)}{ds}\Big|_{s=0} \qquad (4.4)$$

and recall that $\delta\Sigma$ given by (4.2) is independent of a choice of local parametrization of Σ_t, then we can conclude from (4.3) that $\delta\phi^{\pm}$ are also independent of the choice of local parameterization of Σ_t. We shall call the quantity $\delta\phi$ defined above, the displacement variation of ϕ at the hypersurface Σ_t. Introducing (4.2) and (4.4) into (4.3) we obtain

$$\delta\phi^{\pm} = (\delta\phi)^{\pm} + (\partial_{(N)}\phi)^{\pm} \ \delta\Sigma \tag{4.5}$$

where $\delta\phi = d\Phi_t(s)/ds \,\big|_{s=0}$. In a similar way to the derivation of equation (3.14) in Chapter 3 now we have

$$(\delta\phi_{,A})^{\pm} = D_A\{ \delta\phi^{\pm} - (\partial_{(N)}\phi)^{\pm} \delta\Sigma\} + N_A \ \delta(\partial_{(N)}\phi)^{\pm}$$

$$+ N_A D_B(\delta\Sigma) D_B\phi^{\pm} - N_A \ (\partial^2_{(N)}\phi)^{\pm} \delta\Sigma \tag{4.6}$$

where $\delta\phi_{,A} \equiv d\phi_{t,A}(s)/ds \,\big|_{s=0}$ and we have applied the following identity $\delta N_A = - D_A\delta\Sigma$ which corresponds to the identity $\frac{\delta}{\delta t} N_A = - D_A U_{(N)}$ used in Chapter 3.

Finally, on the basis of (4.5) and (4.6) we establish the relations

$$[\![\delta\phi]\!] = \delta[\![\phi]\!] - [\![\partial_{(N)}\phi]\!] \ \delta\Sigma \tag{4.7}$$

$$[\![\delta\phi_{,A}]\!] = D_A \ \{\delta[\![\phi]\!] - [\![\partial_{(N)}\phi]\!]\delta\Sigma \ \} + N_A \delta[\![\partial_{(N)}\phi]\!] + N_A \ D_B(\delta\Sigma) \ D_B[\![\phi]\!]$$

$$- N_A \ [\![\partial^2_{(N)}\phi]\!] \ \delta\Sigma \tag{4.8}$$

which we call the variational conditions of compatibility of first order. Other such conditions are those for $[\![\delta\dot{\phi}]\!]$, $[\![\delta\dot{\phi}_{,A}]\!]$, ..., where $\dot{\phi} \equiv \partial\phi(X,t(s),s)/\partial t(s)$, and the higher order variational conditions of compatibility are those for $[\![\delta^2\phi]\!]$, $[\![\delta^2\phi_{,A}]\!], \ldots, [\![\delta^{(n)}\phi]\!]$, ... They can be obtained in a similar way. However, in this work we deal with the variational problems for which we need only conditions (4.7) and (4.8), hence these other variational conditions of compatibility are not considered here.

Moreover, if we assume that the virtual deformation (4.1) has the form

$$(-\varepsilon,\varepsilon) \ni s \mapsto (\Sigma_{t+s}, \phi_{t+s})$$

then the conditions (4.7) and (4.8) become the kinematical conditions of compatibility as they have been considered in Chapter 3, i.e. conditions (3.10) and (3.16), respectively.

As has been mentioned at the beginning of this section, the results which we have obtained here are important in the construction of variational problems for discontinuous fields. Such variational problems lead to field equations in which jump conditions must be imposed on the hypersurface of discontinuity. Towards this end, we must now review the Weierstrass-Erdmann (corner) conditions from the classical calculus of variations.

CHAPTER 5

SINGLE INTEGRAL PROBLEMS FOR DISCONTINUOUS FUNCTIONS

In this chapter we review a single integral variational problem for discontinuous functions. We follow the exposition of this problem as it has been given in Oden and Reddy [6]. However, our exposition is given in the form which later will be extended for multiple integral variational problems. Another useful reference for this single integral problem is Gelfand and Fomin [14].

Consider the integral

$$J = \int_a^b F(X, \phi(X), \frac{d\phi}{dX})\, dX \tag{5.1}$$

defined over the interval [a,b]. Suppose that the stationary value of the integral (5.1) is obtained for the function $\phi(X)$ which has a simple discontinuity in its first derivative at an arbitrary internal point X_o in [a,b], i.e. $a < X_o < b$. We can express the integral in (5.1) as the sum

$$J = \int_a^{X_o} F(X, \phi(X), \frac{d\phi}{dX})\, dX + \int_{X_o}^b F(X, \phi(X), \frac{d\phi}{dX})\, dx$$

$$\equiv J_1 + J_2 \tag{5.2}$$

In accordance with the previously introduced notation, we consider a pair (X_o, ϕ) where X_o is a singular point of ϕ as it has been defined above. We embed an assumed stationary "point" (X_o, ϕ) in a one-parameter family

$$(-\varepsilon,\varepsilon) \ni s \mapsto (X_o(s), \phi(s)) \tag{5.3}$$

where $X_o(s)$ is a singular point (as defined above) of the function $\phi(s)(X) = \phi(X,s)$, such that $(X_o(0), \phi(0)) \equiv (X_o,\phi)$. Moreover we assume that $\phi(s)(a) = \phi(s)(b) = 0$ for every $s \in (-\varepsilon,\varepsilon)$.

The variation $\delta\phi$ of ϕ in $[a,X_o) \cup (X_o,b]$ is given by

$$\delta\bar{\phi} = \frac{d\phi(s)}{ds}\Big|_{s=0}$$

Then, the results of Chapter 4 imply

$$(\delta\phi)^{\pm}(X_o) = \hat{\delta}\phi(X_o) - (\phi_X)^{\pm}\hat{\delta}X_o \tag{5.4}$$

where the superscripts "-" and "+" denote limits approaching X_o from left and right, respectively. For infinitesimal deformation (5.3) the quantities $\hat{\delta}\phi$ and $\hat{\delta}X_o$ are schematically indicated on Fig. 2.

The first variation of the integral (5.2) is

$$\delta J = \frac{d}{ds} J_1(s)\Big|_{s=0} + \frac{d}{dt} J_2(s)\Big|_{s=0} = \delta J_1 + \delta J_2$$

where

$$\delta J_1 = \int_a^{X_o} \left[\frac{\partial F}{\partial \phi} - \frac{d}{dX}\frac{\partial F}{\partial \phi_X}\right] \delta\phi \, dX + F^- \hat{\delta}X_o + \left(\frac{\partial F}{\partial \phi_X}\right)^- (\hat{\delta}\phi - (\phi_X)^- \hat{\delta}X_o) \tag{5.5}$$

$$\delta J_2 = \int_{X_o}^{b} \left[\frac{\partial F}{\partial \phi} - \frac{d}{dX}\frac{\partial F}{\partial \phi_X}\right] \delta\phi \, dX - F^+ \hat{\delta}X_o - \left(\frac{\partial F}{\partial \phi_X}\right)^+ (\hat{\delta}\phi - (\phi_X)^+ \hat{\delta}X_o) \tag{5.6}$$

Since each of the integrals J_1 and J_2 has a stationary value for ϕ, the integral terms in (5.5) and (5.6) must vanish for arbitrary $\delta\phi$. This implies that

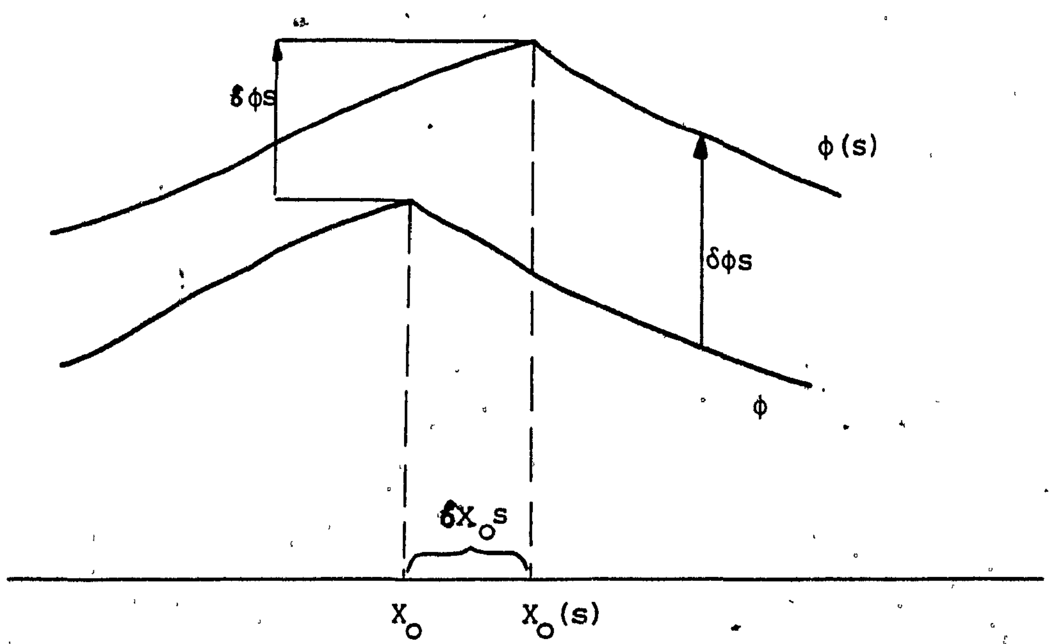


Figure 2

$$\frac{\partial F}{\partial \phi} - \frac{d}{dX}\frac{\partial F}{\partial \phi_X} = 0$$

and we have

$$\delta J_1 = \left(\frac{\partial F}{\partial \phi_X}\right)^- \hat{\delta}\phi + \left(F - \phi_X \frac{\partial F}{\partial \phi_X}\right)^- \hat{\delta}X_o \quad (5.7)$$

$$\delta J_2 = -\left(\frac{\partial F}{\partial \phi_X}\right)^+ \hat{\delta}\phi - \left(F - \phi_X \frac{\partial F}{\partial \phi_X}\right)^+ \hat{\delta}X_o \quad (5.8)$$

By definition, the integral (5.2) has a stationary value for (X_o, ϕ) if the first variation vanishes ($\delta J = \delta J_1 + \delta J_2 = 0$) and this leads to the condition

$$[\![\frac{\partial F}{\partial \phi_X}]\!]\hat{\delta}\phi + [\![F - \phi_X \frac{\partial F}{\partial \phi_X}]\!]\,\hat{\delta}X_o = 0 \quad (5.9)$$

It follows, because of the arbitrariness of $\hat{\delta}\phi$ (X_o) and $\hat{\delta}X_o$, that

$$[\![\frac{\partial F}{\partial \phi_X}]\!] = 0 \quad \text{and} \quad [\![F - \phi_X \frac{\partial F}{\partial \phi_X}]\!] = 0 \quad (5.10)$$

Conditions (5.10) are known as the Weierstrass-Erdmann (corner) conditions for functions with finite jumps in their derivative .

CHAPTER 6

MULTIPLE INTEGRAL VARIATIONAL PROBLEMS FOR DISCONTINUOUS FIELDS

Variational problems given by single integrals find significant applications in the expression of the fundamental principles of mechanical systems and geometry. In the same way, problems defined by multiple integrals are important in the expression of principles in the theory of physical fields and in particular in the theory of classical continuum mechanics, see Goldstein [18] and Lanczos [19], for example. This latter theory is our main area for the application of ideas developed here.

Accordingly, now we consider a certain integral

$$A = \int L \, dV_N \, dt \tag{6.1}$$

In order to fix the notation of the region of integration, we shall let $\pi = D \times \mathcal{T}$ denote a closed cylinder in R^{N+1}. A point of π will be denoted by $(X,t) = (X^1, \ldots, X^N, t)$ and then $dV_N \, dt = dX^1 \ldots dX^N \, dt$ is the volume element of R^{N+1}. The history of a system is given by a continuous one-parameter family

$$R^1 \ni t \mapsto \phi_t$$

of a continuous map $\phi_t\colon R^N \to R^n$ ($N = 2$ or 3, n-natural number) such that

$$\phi\colon R^{N+1} \to R^n$$

defined by $\phi^i(X^A,t) = \phi^i_t(X^A)$ $(A = 1,\ldots,N;\ i=1,\ldots,n)$ and $\phi(X,t) = (\phi^1(X^A,t),\ldots,\phi^n(X^A,t))$ is a continuous mapping.

We assume that the integrand L in (6.1), called the Lagrangian density function, is in general of the following form

$$L: R^{N+1} \times R^{n} \times R^{3(N+1)n} \times R^{3N(N+1)n} \to R^{1}$$

$$L = L(X^{A}, t, \phi^{i}(X^{A}, t), \phi^{i}_{,A}, \dot{\phi}^{i}, \phi^{i}_{,AB}, \dot{\phi}^{i}_{,A})$$

where $\phi^{i}_{,A} \equiv \partial\phi^{i}/\partial X^{A}$, $\dot{\phi} \equiv \partial\phi^{i}/\partial t$, ... etc.

The integral (6.1) will be referred to as the action integral or action functional.

6.1 The First Variation of the Action Functional for Discontinuous Fields

Let us consider a system

$$(D; \mathcal{T}; \Sigma_t, \phi_t) \tag{6.2}$$

where (Σ_t, ϕ_t), $t \in \mathcal{T}$ is a pair defined in Chapter 2, for which now $\Omega = R^N$ and $\phi_t(X) = \phi(X,t)$, $X \in R^N$, $t \in \mathcal{T}$, has been defined above. The closed cylinder $D \times \mathcal{T}$, $D \subset R^N$, $\mathcal{T} \subset R^1$ is such that if for some $t \in \mathcal{T}$ $\partial(D \cap \Sigma_t) \neq \phi$ then for each $t \in \mathcal{T}$ $\partial(D \cap \Sigma_t) \neq \phi$ is a regular smooth curve for $N = 3$ or a two-point boundary for $N = 2$, and Σ_t intersects D 'transversally'. Such a $D \times \mathcal{T}$ will be referred to as a "good" X-t cylinder. The action functional for the system (6.2) is given by

$$A_{D\times\mathcal{T}} = \int_{\mathcal{T}} \int_{D\setminus\Sigma_t} L(X^{A}, t, \phi_i, \phi_{i,A}, \dot{\phi}_i, \phi_{i,AB}, \dot{\phi}_{i,A})\, dV_N\, dt \tag{6.3}$$

A (infinitesimal) virtual deformation of the system (6.2) is the following one-parameter family

$$(-\varepsilon, \varepsilon) \ni s \mapsto (D; \mathcal{T}(s); \Sigma_t(s), \phi_t(s)) \tag{6.4}$$

such that for each s the same assumptions as those for the definition of the system (6.2) hold true.

In (6.4) $\mathcal{T}(s) = [t_1(s), t_2(s)]$ and $(\Sigma_t(s), \phi_t(s))$ is defined by (4.1) with $\Omega = R^N$.

The deformation (6.4) induces the deformation of the action functional (6.3)

$$(-\varepsilon,\varepsilon) \ni s \mapsto A_{D\times\mathcal{T}}(s)$$

where $A_{D\times\mathcal{T}}(s)$ is given by

$$A_{D\times\mathcal{T}}(s) = \int_{\mathcal{T}(s)} \int_{D\setminus\Sigma_t(s)} L(X^A, t(s), \phi_i(s), \phi_{i,A}(s), \dot{\phi}_i(s),$$

$$\phi_{i,AB}(s), \dot{\phi}_{i,A}(s))\ dV_N\ dt(s) \qquad (6.5)$$

in which $\dot{\phi}_i(s) \equiv \partial\phi_i(s)/\partial t(s)$, $\phi_{i,A}(s) \equiv \partial\phi_i(s)/\partial X^A$, ... etc.

It is convenient to reduce the integral over the time interval $\mathcal{T}(s)$ in (6.5) to an integral over the original time interval $\mathcal{T} = [t_1,t_2]$ by a change of variables. The transformation of the time element from $\mathcal{T}(s)$ to $\mathcal{T}$ is accomplished by means of the formula

$$dt(s) = \frac{dt(s)}{dt}\ dt$$

$$= [\ 1 + \frac{d}{dt}\ (\frac{dt(s)}{ds}\Big|_{s=0})\ s]\ dt$$

where the last expression is accurate to first order in s.

Now, for the first variation $\delta A_{D\times\mathcal{T}}$ of the functional (6.3) which is defined by

$$\delta A_{D\times\mathcal{T}} = \frac{d}{ds}\ A_{D\times\mathcal{T}}(s)\Big|_{s=0}$$

one can write

$$\delta A_{D\times\mathcal{T}} = \int_{\mathcal{T}} \int_{D\setminus\Sigma_t} \frac{d}{ds} L(s)\Big|_{s=0} dV\, dt - \int_{\mathcal{T}} \int_{\Sigma_t \cap D} [\![L]\!]\,\delta\Sigma\, d\Sigma_{N-1}\, dt$$

$$+ \int_{\mathcal{T}} \int_{D\setminus\Sigma_t} L \frac{d}{dt} \delta t\, dV\, dt \qquad (6.6)$$

where $d\Sigma_{N-1}$ is the Euclidean area element for N=3 and the arc length element for N=2, $\delta\Sigma$ is the normal variation of the hypersurface Σ_t defined by (4.2), $\delta t = \frac{dt(s)}{ds}\Big|_{s=0}$. Also

$$[\![L]\!] \equiv L^+ - L^-$$

is the jump of the Lagrangian density function L across the singular hypersurface Σ_t. The formula (6.6) is given here without proof. However, some proof of (6.6) can be obtained along the same lines as that of the formula for the time derivative of integrals appearing in balance laws in continuum mechanics for discontinuous motions [12] (compare formulae (6.6) and (6.14) given later in this section).

The ideas of a nice proof of formulae such as (6.6) can be found in Baddeley [20]. His approach follows the basic notions of differential geometry, such as differential forms and also, smoothly changing compact differentiable manifolds, possibly, with boundary The proof of formula (6.6) for some particular case, using the notion of generalized functions and differential forms, will be presented in Appendix II.

If the derivative $dL(s)/ds|_{s=0}$ is carried out in (6.6) we obtain

$$\delta A_{D\times\mathcal{T}} = \int_{\mathcal{T}} \int_{D} \frac{\partial L}{\partial t}\delta t + \frac{\partial L}{\partial \phi_i}\delta\phi_i + \frac{\partial L}{\partial \phi_{i,A}}\delta\phi_{i,A} + \frac{\partial L}{\partial \dot{\phi}_i}\delta\dot{\phi}_i + \frac{\partial L}{\partial \phi_{i,AB}}\delta\phi_{i,AB}$$

$$+ \frac{\partial L}{\partial \dot{\phi}_{i,A}}\delta\dot{\phi}_{i,A} + L\frac{d}{dt}\delta t)\, dV_N\, dt - \int_{\mathcal{T}} \int_{\Sigma_t \cap D} [\![L]\!]\,\delta\Sigma\, d\Sigma_{N-1}\, dt \qquad (6.7)$$

Before the analysis of (6.7) can be carried further, we must recognize that from the definition of variation

$$\delta\phi_t(X) = \frac{d}{ds}\,\phi\,(X,t(s),s)\,|_{s=0}$$

we have in general

$$\delta\dot{\phi} \neq \frac{\partial}{\partial t}\,\delta\phi \tag{6.8}$$

It will be convenient to define a new variation $\hat{\delta}\phi$ for which the relation (6.8) will be turned into an equality. To do this, we note that

$$\begin{aligned}\delta\phi &= \delta\phi(X,t)\\ &= \frac{\partial}{\partial t}\,\phi(X,t)\,\delta t + \frac{\partial}{\partial s}\,\phi(X,t(s),s)\,|_{s=0}\\ &= \dot{\phi}\delta t + \hat{\delta}\phi \end{aligned}\tag{6.9}$$

where we have set $\hat{\delta}\phi \equiv \frac{\partial}{\partial s}\,\phi(X,t(s),s)\,|_{s=0}$.

Then, it is clear that

$$\hat{\delta}\dot{\phi} = \frac{\partial}{\partial t}\,\hat{\delta}\phi.$$

Furthermore, recalling the kinematical and variational conditions of compatibility given in Chapter 3 and Chapter 4, respectively, we have from (6.9) the following simple

LEMMA 6.1. The following relations

$$\delta\Sigma = U_{(N)}\,\delta t + \hat{\delta}\Sigma \tag{6.10}$$

$$\delta\phi^{\pm} = \frac{\delta\phi^{\pm}}{\delta t}\,\delta t + \hat{\delta}\phi^{\pm} \tag{6.11}$$

are satisfied for a given pair (Σ_t, ϕ_t).

PROOF: From (6.9) we have that

$$(\hat{\delta}\phi)^{\pm} = (\delta\phi)^{\pm} - (\dot{\phi})^{\pm}\,\delta t \tag{6.12}$$

On introducing conditions (4.5) and (3.9) into (6.12) we obtain

$$(\hat{\delta}\phi)^{\pm} = \delta\phi^{\pm} - \frac{\delta\phi^{\pm}}{\delta t}\,\delta t - (\partial_{(N)}\phi)^{\pm}\,\{\hat{\boldsymbol{\delta}}\Sigma - U_{(N)}\,\delta t\}$$

$$= \hat{\boldsymbol{\delta}}\phi^{\pm} - (\partial_{(N)}\phi)^{\pm}\,\hat{\boldsymbol{\delta}}\Sigma$$

The last equality implies (6.10) and (6.11). □

Let us note that the operators $\hat{\delta}$ and $\hat{\boldsymbol{\delta}}$ are connected only with the form change of a pair (Σ_t, ϕ_t) i.e. they are variations without the variation of time.

On making use of relations (6.9) and (6.10) we can write (6.7) after obvious manipulations in the form

$$\delta A_{D\times\mathcal{T}} = \int_{\mathcal{T}}\int_{D}\{\frac{\partial L}{\partial\phi_i}\,\hat{\delta}\phi_i + \frac{\partial L}{\partial\phi_{i,A}}\,\hat{\delta}\phi_{i,A} + \frac{\partial L}{\partial\dot{\phi}_i}\,\dot{\overline{\hat{\delta}\phi_i}}$$

$$+ \frac{\partial L}{\partial\phi_{i,AB}}\,\hat{\delta}\phi_{i,AB} + \frac{\partial L}{\partial\dot{\phi}_{i,A}}\,\dot{\overline{\hat{\delta}\phi_{i,A}}} + \frac{d}{dt}(L\,\delta t)\}\;dV_N\,dt$$

$$-\int_{\mathcal{T}}\int_{\Sigma\cap D} [\![L]\!]\,(\hat{\boldsymbol{\delta}}\Sigma + U_{(N)}\,\delta t)\;d\Sigma_{N-1}\,dt \tag{6.13}$$

Now, bearing in mind the following identities (cf. [12])

$$\frac{d}{dt}\int_{D}\Phi\,dV_N = \int_{D}\dot{\Phi}\,dV_N - \int_{\Sigma_t\cap D}[\![\Phi]\!]\,U_{(N)}\,d\Sigma_{N-1} \tag{6.14}$$

$$\int_{D}\Phi_{,A}\,dV_N = \int_{\partial D}N_A\Phi\,d\Sigma_{N-1} - \int_{\Sigma_t\cap D}[\![\Phi]\!]\,N_A\,d\Sigma_{N-1} \tag{6.15}$$

and then performing the standard manipulations (including also integration by parts by means of formulae (6.14) and (6.15)) in (6.13) we obtain

$$\delta A_{D\times\mathcal{T}} = \int_{\mathcal{T}}\int_{D} \{-(\dot{\overline{\frac{\partial L}{\partial \dot\phi_i}}}) + [-\frac{\partial L}{\partial \phi_{i,A}} + (\frac{\partial L}{\partial \phi_{i,AB}})_{,B} + (\dot{\overline{\frac{\partial L}{\partial \dot\phi_{i,A}}}})]_{,A} + \frac{\partial L}{\partial \phi_i}\} \hat{\delta}\phi_i \, dV_N dt$$

$$- \int_{\mathcal{T}}\int_{\partial D} \{[-\frac{\partial L}{\partial \phi_{i,A}} + (\frac{\partial L}{\partial \phi_{i,AB}})_{,B} + (\dot{\overline{\frac{\partial L}{\partial \dot\phi_{i,A}}}})]N_A \hat{\delta}\phi_i - \frac{\partial L}{\partial \phi_{i,AB}} N_A \hat{\delta}\phi_{i,B}\} d\Sigma_{N-1} dt$$

$$+ \int_{\mathcal{T}}\int_{\Sigma_t \cap D} [\![\{[-\frac{\partial L}{\partial \phi_{i,A}} + (\frac{\partial L}{\partial \phi_{i,AB}})_{,B} + (\dot{\overline{\frac{\partial L}{\partial \dot\phi_{i,A}}}})]N_A + \frac{\partial L}{\partial \dot\phi_i} U_{(N)}\} \hat{\delta}\phi_i]\!] d\Sigma_{N-1} dt$$

$$+ \int_{\mathcal{T}}\int_{\Sigma_t \cap D} [\![(-\frac{\partial L}{\partial \phi_{i,AB}} N_A + \frac{\partial L}{\partial \dot\phi_{i,A}} U_{(N)}) \hat{\delta}\phi_{i,B}]\!] d\Sigma_{N-1} dt$$

$$- \int_{\mathcal{T}}\int_{\Sigma_t \cap D} [\![L]\!] \hat{\delta}\Sigma \, d\Sigma_{N-1} dt$$

$$+ \int_{D} (\frac{\partial L}{\partial \dot\phi_i} \hat{\delta}\phi_i + \frac{\partial L}{\partial \dot\phi_{i,A}} \hat{\delta}\phi_{i,A} + L\,\delta t)\, dV_N \Big|_{t_2}^{t_1} \tag{6.16}$$

Let us introduce the following notation

$$p^i = \frac{\partial L}{\partial \dot\phi_i} \tag{6.17}$$

$$\pi^{iA} = \frac{\partial L}{\partial \dot\phi_{i,A}} \tag{6.18}$$

$$\Sigma^{iA} = -\frac{\partial L}{\partial \phi_{i,A}} \tag{6.19}$$

$$H^{iAB} = -\frac{\partial L}{\partial \phi_{i,AB}} \tag{6.20}$$

$$T^{iA} = \Sigma^{iA} - H^{iAB}{}_{,B} + \dot{\pi}^{iA} \qquad (6.21)$$

$$E^{i}(L) = -\dot{P}^{i} + T^{iA}{}_{,A} + \frac{\partial L}{\partial \phi_i} \qquad (6.22)$$

where the last expression is referred to as the Euler operator. On introducing the above notation into (6.16) we have the compact form

$$\begin{aligned}
\delta A_{D\times\mathcal{T}} = & \int_{\mathcal{T}} \int_{D} E^{i}(L)\,\hat{\delta}\phi_i \; dV_N dt - \int_{\mathcal{T}} \int_{\partial D} (T^{iA}N_A \;\hat{\delta}\phi_i + H^{iAB}N_A \;\hat{\delta}\phi_{i,B})\, d\Sigma_{N-1}\, dt \\
& + \int_{\mathcal{T}} \int_{\Sigma_t \, D} [\![(T^{iA}N_A + P^{i}\, U_{(N)})\,\hat{\delta}\phi_i]\!] \, d\Sigma_{N-1}\, dt \\
& + \int_{\mathcal{T}} \int_{\Sigma_t \, D} [\![H^{iAB}N_B + \pi^{iA}\, U_{(N)}) \;\hat{\delta}\phi_{i,A}]\!] \, d\Sigma_{N-1}\, dt \\
& - \int_{\mathcal{T}} \int_{\Sigma_t \, D} [\![L]\!] \,\hat{\delta}\Sigma \; d\Sigma_{N-1}\, dt \\
& + \int_{D} (P^{i}\hat{\delta}\phi_i + \pi^{iA}\hat{\delta}\phi_{i,A} + L\,\delta t)\, dV_N \Big|_{t_1}^{t_2}
\end{aligned} \qquad (6.23)$$

In order to express the formula for $\delta A_{D\times\mathcal{T}}$ given by (6.23) in terms of arbitrary variations on the singular hypersurface Σ_t and on the boundary ∂D we need Toupin type integral identities. First, let us recall that on the singular hypersurface Σ_t arbitrary variations are the displacement variations and the normal variation of this hypersurface. On the boundary ∂D the arbitrary variations are the variations of field and their normal derivative.

One of the above mentioned identities on a surface in R^3 has been introduced by Toupin [21] in his early work on the theory of elastic materials with couple-stresses. This identity has also been frequently used in many works related to this subject [22,23]. A more comprehensive treatment of such integral identities has been given in Cheverton and Beatty [24] and in references cited there.

Let us first consider the case N=3. Now, (Σ,Φ) is a pair such that Σ is a singular surface in R^3 relative to field Φ defined on R^3. We assume that for a compact set $D \subset R^3$ with a smooth two-dimensional boundary ∂D, the surface Σ marks out on ∂D a regular closed curve C. The unit normal vectors on ∂D and Σ are denoted by $\mathbf{N}^{\partial D}$ and $\mathbf{N}^{\Sigma}$, respectively. This notation will be used wherever any confusion might arise. Let us define the following unit vectors

$$\mathbf{M}^{\wedge} = -\boldsymbol{\Sigma} \times \mathbf{N}^{\partial D}$$

$$\mathbf{M}^{\vee} = \boldsymbol{\Sigma} \times \mathbf{N}^{\Sigma}$$

in which Σ is the usual left-oriented tangent unit vector on the curve (see Fig. 3).

The integral identities (N=3) we need are the following

$$\int_{\partial D} \boldsymbol{D}\cdot\Phi \, dA = -\int_{\partial D} \Phi\cdot\mathbf{N}^{\partial D}\,\Omega \, dA - \int_{C} [\![\Phi\cdot\mathbf{M}^{\wedge}]\!] \, dL \qquad (6.24)$$

$$\int_{\Sigma\cap D} \boldsymbol{D}\cdot[\![\Phi]\!] \, dA = -\int_{\Sigma\cap D} [\![\Phi\cdot\mathbf{N}^{\Sigma}]\!]\,\Omega \, dA + \int_{C} [\![\Phi\cdot\mathbf{M}^{\vee}]\!] \, dL \qquad (6.25)$$

where dA is the Euclidean area element, dL is the arc length element and $\Omega = \mathrm{tr}\left[\Omega_A^B\right]$, Ω_A^B being the components of the second fundamental form of the surface Σ or the boundary ∂D. Note that the curvature of a surface is defined as $(1/2)\,\Omega$.

In the case of N=2 the boundary ∂D and the hypersurface Σ are smooth regular curves. The orientation of the vectors $\mathbf{M}^{\wedge}$ and $\mathbf{M}^{\vee}$, in this case, are indicated on Fig. 4. Also, the points $X_{(1)}$ and $X_{(2)}$ form

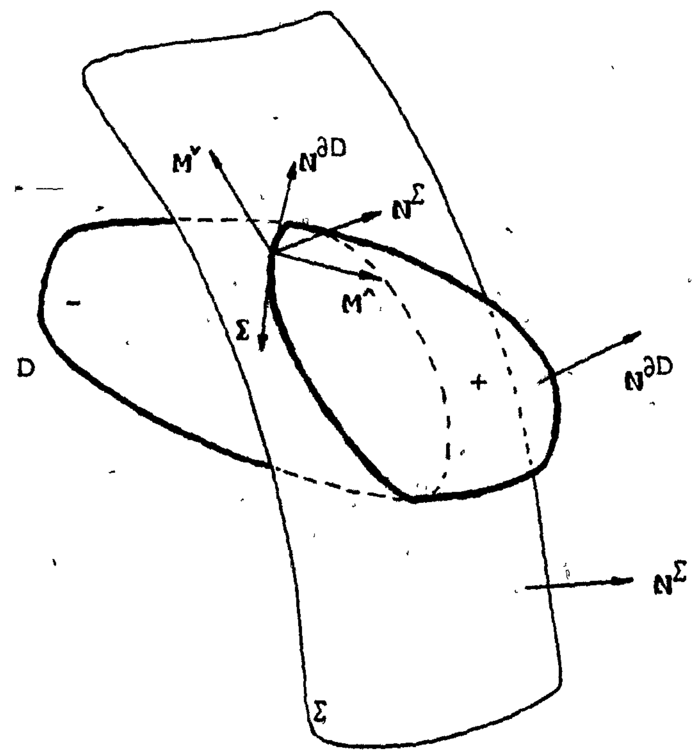


Figure 3

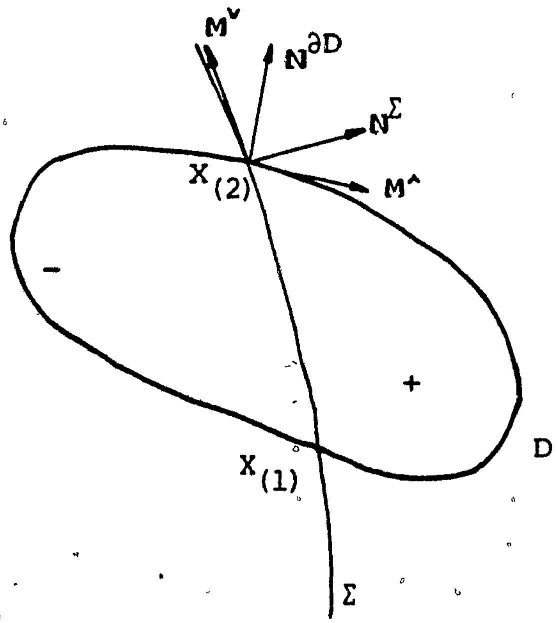


Figure 4

the boundary $\partial(\Sigma \cap D)$ which in the assumed positive orientation is given by $X_{(2)} - X_{(1)}$.

The corresponding identities can be obtained in the following way. First let us note that

$$\mathcal{D}\cdot\Phi = M^* \cdot \frac{d}{d\ell}\Phi = \frac{d}{d\ell}(M^*\cdot\Phi) - \frac{d}{d\ell}(M^*)\cdot\Phi$$

where $* = \wedge$ for ∂D and $* = \vee$ for Σ and the curve parameter ℓ is the length parameter (i.e. the unit speed parameter). The Frenet formula of a curve (cf. [25], for example) gives

$$\frac{d}{d\ell} M^\wedge = \Omega N^{\partial D}$$

$$\frac{d}{d\ell} M^\vee = \Omega N^{\Sigma}$$

where Ω is the curvature of the appropriate curve.

From the above equations we can easily obtain

$$\int_{\partial D} \mathcal{D}\cdot\Phi \, d\ell = - \int_{\partial D} \Omega\, N^{\partial D}\cdot\Phi \, d\ell - [\![M^\wedge\cdot\Phi]\!] \Big|_{X_{(1)}}^{X_{(2)}} \tag{6.26}$$

$$\int_{\Sigma\cap D} \mathcal{D}\cdot[\![\Phi]\!] \, d\ell = - \int_{\Sigma\cap D} \Omega[\![N^{\Sigma}\cdot\Phi]\!] \, d\ell + [\![M^\vee\cdot\Phi]\!] \Big|_{X_{(1)}}^{X_{(2)}} \tag{6.27}$$

The integral identities (6.24) and (6.25) for N=3 and (6.26) and (6.27) for N-2 we will write in the form

$$\int_{\partial D} \mathcal{D}\cdot\Phi \, d\Sigma_{N-1} = - \int_{\partial D} N^{\partial D}\cdot\Phi \, \Omega \, d\Sigma_{N-1} - \int_{C} [\![M^\wedge\cdot\Phi]\!] d\Sigma_{N-2} \tag{6.28}$$

$$\int_{\Sigma\cap D} \mathcal{D}\cdot[\![\Phi]\!] \, d\Sigma_{N-1} = - \int_{\Sigma\cap D} [\![N^{\Sigma}\cdot\Phi]\!] \, d\Sigma_{N-1} + \int_{C} [\![M^\vee\cdot\Phi]\!] \, d\Sigma_{N-2} \tag{6.29}$$

where $d\Sigma_{N-1}$ and $d\Sigma_{N-2}$ are induced measures on Σ or ∂D and on $C = \partial(\Sigma\cap D)$, respectively, and they are understood in the senses in which they were already presented for each case of N=3 or N=2.

On substituting conditions (4.5) and (4.6) with $\delta = \hat{\delta}$ into (6.23), and after manipulations which include an integration of certain terms by parts using the formulae (6.28) and (6.29), we finally derive the sought expression for $\delta A_{D\times\mathcal{T}}$ as follows

$$\delta A_{D\times\mathcal{T}} = \int_{\mathcal{T}} \int_{D} E^{i}(L)\, \hat{\delta}\phi_i \, dV_N \, dt$$

$$- \int_{\mathcal{T}} \int_{\partial D} \{ [T^{iA}N_A - H^i\Omega - \mathcal{D}_A(H^{iAB}N_B)]\hat{\delta}\phi_i + H^i\partial_{(N)}\hat{\delta}\phi_i \} \, d\Sigma_{N-1} \, dt$$

$$+ \int_{\mathcal{T}} \int_{\Sigma_t \cap D} [\![T^{iA}N_A + P^iU_{(N)} - (H^i + \pi^iU_{(N)})\Omega - \mathcal{D}_A(H^{iAB}N_B + \pi^{iA}U_{(N)})]\!]$$

$$\hat{\delta}\phi_i \, d\Sigma_{N-1} \, dt$$

$$- \int_{\mathcal{T}} \int_{\Sigma_t \cap D} [\![\{ T^{iA}N_A + P^iU_{(N)} - (H^i + \pi^iU_{(N)})\Omega - \mathcal{D}_A(H^{iAB}N_B$$

$$+ \pi^{iA}U_{(N)}) \} \partial_{(N)}\phi_i + (H^i + \pi^iU_{(N)})\partial^2_{(N)}\phi_i$$

$$+ \mathcal{D}_A[(H^i + \pi^iU_{(N)})\ {}_A\phi_i] + L]\!] \hat{\delta}\Sigma \, d\Sigma_{N-1} dt$$

$$+ \int_{\mathcal{T}} \int_{\Sigma_t \cap D} [\![(H^i + \pi^iU_{(N)}) \hat{\delta}\, \partial_{(N)}\phi_i]\!] \, d\Sigma_{N-1} \, dt$$

$$+ \int_{\mathcal{T}} \int_{C_t} \{ [\![H^{iAB}N_A^{\partial D}\hat{M}_B + (H^{iAB}N_B^{\Sigma} + \pi^{iA}U_{(N)})\check{M}_A \} \hat{\delta}\phi_i]\!] \, d\Sigma_{N-2} \, dt$$

$$+ \int_{\mathcal{T}} \int_{C_t} [\![H^i + \pi^iU_{(N)}]\!] \check{M}_A \, \mathcal{D}_A\phi_i \, \hat{\delta}\Sigma \, d\Sigma_{N-2} \, dt$$

$$+ \int_{D} (P^i\hat{\delta}\phi_i + \pi^{iA}\, \hat{\delta}\phi_{i,A} + L\, \delta t) \, dV_N \Big|_{t_1}^{t_2} \qquad (6.30)$$

where $C_t = \partial(\Sigma_t \cap D)$ and the following notation has been introduced

$$H^i = H^{iAB} N_A N_B$$

$$\pi^i = \pi^{iA} N_A$$

Let us note that for N=2 $\partial(\Sigma_t \cap D) = X_{(2)} - X_{(1)}$ and the integrals over C_t are reduced to the difference of jumps of appropriate quantities at the boundary points $X_{(2)}$ and $X_{(1)}$.

In (6.30) the variations $\delta\phi$ and $\partial_{(N)}\delta\phi$ are arbitrary variations on ∂D, also $\delta\phi$, $\delta\Sigma$, $\delta(\partial_{(N)}\phi)^{\pm}$ such that $\delta(\partial_{(N)}\phi)^+ = \delta(\partial_{(N)}\phi)^-$ if $[\![\partial_{(N)}\phi]\!] = 0$, are arbitrary variations on Σ_t.

Before we state the so-called stationary problem in the multiple, second order, calculus of variations for discontinuous fields, first let us recall that (Γ,ϕ) is a pair such that

$$\Gamma = \{ (X,t): X \in \Sigma_t,\ t \in \mathcal{T} \}$$

is a smooth hypersurface in R^{N+1} swept out by a singular hypersurface $\Sigma_t \subset R^N$ relative to the field $\phi_t(X) = \phi(X,t)$

The stationary "point" of the action integral (6.3) is a pair (Γ,ϕ) (or (Σ_t,ϕ_t)) such that the first variation given by (6.30) vanishes for all variations of (Γ,ϕ) $((\Sigma_t,\phi_t))$ obtained by embedding of (Γ,ϕ) $((\Sigma_t\phi_t))$ in a one-parameter family (infinitesimal, virtual deformation)

$$(-\varepsilon,\varepsilon) \ni s \mapsto (\Gamma(s),\ \phi(s))$$

$$((-\varepsilon,\ \varepsilon) \ni s \mapsto (\Sigma_t(s),\ \phi_t(s)))$$

where $\varepsilon > 0$ is small,

$$\Gamma(s) = \{(x,t): X \in \Sigma_t(s),\ t \in \mathcal{T}(s)\}$$

and

$$\phi_t(s)(X) = \phi(s)(X,t) = \phi(X,t(s),s)$$

The variational problem which we now consider is the following stationary problem. For a given action integral (6.3) we wish to find a necessary condition such that $\delta A_{D\times\mathcal{T}} = 0$ for all variations $\delta\phi = \hat{\delta}\phi$ (i.e. $\delta t = 0$) associated with (Γ,ϕ) such that they vanish at the boundary of time interval $\mathcal{T} = [t_1,t_2]$. These variations are referred to as the Lagrange variations. This variational problem leads to the corresponding Euler-Lagrange equations, natural boundary conditions and jump conditions associated with the singular hypersurface Σ_t. These jump conditions generalize the Weierstrass-Erdmann (corner) conditions of the single integral variational calculus which we have reviewed in Chapter 5. To this end we need to extend, the so-called Fundamental Lemma of the calculus of variations (cf. Logan [26], for example), to include the additional integrals induced by the discontinuity of the partial derivatives of the field variables across their singular hypersurface.

6.2 Fundamental Lemma of Calculus of Variations for Discontinuous Fields

Let us assume that Γ is a smooth, oriented and closed hypersurface of dimension m-1 given in R^m. Consider real-valued functions h, h_1, h_2 where h is a continuously differentiable, bounded function on $R^m \setminus \Gamma$ and h_1, h_2 are continuously differentiable bounded functions on Γ.

The sought extension of the Fundamental Lemma of the calculus of variations is the following.

LEMMA 6.2. Let $\pi \subset R^m$ be a compact set with m-1 dimensional smooth boundary $\partial\pi$ such that Γ divides π into two non-empty parts π_1 and π_2, and crosses $\partial\pi$ transversally. If f, g k_a, r_a (a = 1,2) are real-valued continuous functions defined on $\pi\backslash\Gamma$, $\partial\pi\backslash\Gamma$, $\Gamma\cap\pi$ and $\partial(\Gamma\cap\pi)$, respectively, and if

$$\begin{aligned}\ell(\pi) &= \int_{\pi} f(X)\, h(X)\, dV_m + \int_{\partial\pi} g(X)\, h(X)\, d\Gamma_{m-1} \\ &\quad + \int_{\Gamma\cap\pi} k_a(X)\, h_a(X)\, d\Gamma_{m-1} + \int_{\partial(\Gamma\cap\pi)} r_a(X)\, h_a(X)\, d\Gamma_{m-2} \\ &= 0 \end{aligned} \tag{6.31}$$

holds for every h, h_a (a=1,2) as defined above then

$$\begin{aligned} f(X) &= 0 && \text{for all } X \in \pi\backslash\Gamma \\ g(X) &\equiv 0 && \text{for all } X \in \partial\pi\backslash\Gamma \\ k_a(X) &\equiv 0 \quad a=1,2 && \text{for all } X \in \Gamma\cap\pi \\ r_a(X) &\equiv 0 \quad a=1,2 && \text{for all } X \in \partial(\Gamma\cap\pi) \end{aligned}$$

In (6.31) $dV_m = dX^1,\ldots,dX^m$, $(X^1,\ldots,X^m) \in R^m$ and $d\Gamma_k$ (k = m-1 or m-2) are induced measures on appropriate hypersurfaces.

PROOF. On the contrary, assume first that there is X_1 in the interior of $\pi\backslash\Gamma$ for which $f(X_1) > 0$. Then by continuity of f there is an open ball $B_1 = \{X: |X-X_1| < \rho_1\}$ in the interior of $\pi\backslash\Gamma$, of radius $\rho_1 > 0$ for which $f(X) > 0$ for $X \in B_1$. Now, define the function h(X) by

$$h(X) = \begin{cases} 0 & \text{outside of } B_1 \\ [(X-X_1)^2 - \rho_1^2\,]^2 & \text{inside of } B_1 \end{cases}$$

Then, h(X) is differentiable in $\pi\backslash\Gamma$ and

$$\ell(\pi) = \int_{B_1} f(X)\, h(X)\, dV_M > 0$$

This fact provides the contradiction, hence $f(X) = 0$ on $\pi\backslash\Gamma$. Next, assume that there is $X_2 \in \partial\pi\backslash\Gamma$ for which $g(X_2) > 0$. Then, by continuity of g on $\partial\pi\backslash\Gamma$ there is an open ball $B_2 = \{(X: |X-X_2| < \rho_2\}$ in $R^m\backslash\Gamma$ of radius $\rho_2 > 0$ such that $\phi \neq B_2 \cap \partial\pi \subset \partial\pi\backslash\Gamma$ and $g(X) > 0$ for $X \in B_2 \cap \partial\pi$. Now, define the function $h(X)$ by

$$h(X) = \begin{cases} 0 & \text{outside of } B_2 \\ [(X-X_2)^2 - \rho_2^2]^2 & \text{inside of } B_2 \end{cases}$$

Then it is well defined and ($f(X) \equiv 0$ on $\pi\backslash\Gamma$)

$$\ell(\pi) = \int_{B_2 \cap \partial\pi} g(X)\, h(X)\, d\Gamma_{m-1} > 0$$

Hence $g(X) \equiv 0$ on $\partial\pi\backslash\Gamma$.

Next, assume that there is X_3 in the interior of $\Gamma \cap \pi$ such that $k_1(X_3) > 0$ ($k_2(X_3) > 0$) then by continuity of k_1 (k_2) there is an open ball $B_3 = \{X: |X-X_3| < \rho_3\}$ in the interior of π such that $k_1(X) > 0$ ($k_2(X) > 0$) for $X \in B_3 \cap \Gamma$. Define the function $\tilde{h}$ by

$$\tilde{h}(X) = \begin{cases} 0 & \text{outside of } B_3 \\ [(X-X_3)^2 - \rho_3^2]^2 & \text{inside of } B_3 \end{cases}$$

Now, let h_a (a=1,2) be taken as

$$\begin{aligned} h_1(X) &= \tilde{h}(X) && \text{for } X \in \Gamma \\ h_2(X) &= 0 && \text{for } X \in \Gamma \end{aligned}$$

$$\left(\begin{aligned} h_1(X) &= 0 && \text{for } X \in \Gamma \\ h_2(X) &= \tilde{h}(X) && \text{for } X \in \Gamma \end{aligned}\right)$$

Then h_a (a=1,2) are well defined and

$$\ell(\pi) = \int_{B_3 \cap \Gamma} k_1(X) h_1(X)\, d\Gamma_{m-1} > 0$$

$$(\ell(\pi) = \int_{B_3 \cap \Gamma} k_2(X)\, h_2(X)\, d\Gamma_{m-1} > 0)$$

Hence, by contradiction we have that $k_a(X) \equiv 0$ (a=1,2) on $\Gamma \cap \pi$.

Finally, in a way similar to the derivation of the last contradiction, we obtained that $r_a(X) \equiv 0$ (a=1,2) on $\partial(\Gamma \cap \pi)$. □

6.3 Stationary Principle for Discontinuous Fields

Let us consider the Lagrange problem for the action integral $A_{D\times\mathcal{T}}$ given by (6.3). First, recalling that $\delta\phi$ and $\partial_{(N)}\delta\phi$ are arbitrary variations on ∂D and that $\delta\phi$, $\delta\Sigma$, $\delta(\partial_{(N)}\phi)^{\pm}$ such that $\delta(\partial_{(N)}\phi)^{+} = \delta(\partial_{(N)}\phi)^{-}$ if $[\![\partial_{(N)}\phi]\!] = 0$ are arbitrary variations on Σ_t, we prove the following

THEOREM 6.1. The first variation $\delta A_{D\times\mathcal{T}}$ of the action integral (6.3) vanishes for all Lagrange variations associated with (Γ,ϕ) if and only if for each $t \in (t_1,t_2)$ the Euler-Lagrange equations

$$E^i(L) = 0 \qquad X \in D\backslash\Sigma_t \tag{6.32}$$

and the following jump conditions on $\Sigma_t \cap D$

$$[\![T^{iA}N_A + P^iU_{(N)} - \mathcal{D}_A(H^{iAB}N_B + \pi^{iA}U_{(N)})]\!] = 0 \tag{6.33}$$

$$[\![\{T^{iA}N_A + P^iU_{(N)} - \mathcal{D}_A(H^{iA}N_B + \pi^{iA}U_{(N)})\}\ \partial_{(N)}\phi_i$$
$$+ (H^i + \pi^iU_{(N)})\ \partial^2_{(N)}\ \phi_i + L]\!] = 0 \tag{6.34}$$

$$[\![H^i + \pi^i U_{(N)}]\!] = 0 \quad \text{if} \quad [\![\partial_{(N)}\phi_i]\!] = 0 \tag{6.35}$$

$$(H^i + \pi^i U_{(N)})^{\pm} = 0 \quad \text{if} \quad [\![\partial_{(N)}\phi_i]\!] \neq 0 \tag{6.36}$$

hold true. Moreover we have on ∂D the following natural boundary conditions

$$T^{iA}N_A - H^i\Omega - \mathcal{D}_A(H^{iAB}N_B) = 0 \tag{6.37}$$

$$H^{iAB}N_A N_B = 0 \tag{6.38}$$

and the following jump conditions in $\partial(D \cap \Sigma_t) \subset \partial D$

$$[\![H^{iAB}N_A^{\partial D}\hat{M}_B]\!] + [\![(H^{iAB}N_A^{\Sigma} + \pi^{iB}U_{(N)})\,\check{M}_B]\!] = 0$$

$$\text{if } [\![\partial_{(N)}\phi]\!] = 0 \tag{6.39}$$

$$(H^{iAB}N_A^{\partial D}\hat{M}_B)^{\pm} + \{(H^{iAB}N_A^{\Sigma} + \pi^{iB}U_{(N)})\,\check{M}_B\}^{\pm} = 0$$

$$\text{if } [\![\partial_{(N)}\phi]\!] \neq 0 \tag{6.40}$$

PROOF. It follows from expression (6.30) for $\delta A_{D\times\mathcal{T}}$, and the remarks given after the proof of Lemma (6.2) and finally on applying this lemma to (6.30). □

Let us assume that the singular hypersurface Σ_t is a wave, i.e. that $U_{(N)} \neq 0$ at each point (X,t) of Γ. In this case, if the conditions (6.32) – (6.36) hold then (6.34) can be written in the following equivalent forms

$$[\![\{T^{iA}N_A + P^iU_{(N)} - \mathcal{D}_A(H^{iAB}N_B + \pi^{iA}U_{(N)})\}\,\dot{\phi}_i$$

$$+ (H^i + \pi^iU_{(N)})\,\partial_{(N)}\dot{\phi}_i - LU_{(N)}]\!] = 0 \tag{6.41}$$

and

$$[\![\{T^{iA}N_A + P^i U_{(N)} - D_A(H^{iAB}N_B + \pi^{iA}U_{(N)})\} \dot{\phi}_i$$

$$- \frac{1}{U_{(N)}} (H^i + \pi^i U_{(N)})\ddot{\phi}_i - LU_{(N)}]\!] = 0 \qquad (6.42)$$

The proof follows from the kinematical conditions of compatibility (3.10), (3.16) and (3.17) which for the considered field variables (Γ,ϕ) can be written in the following way

$$[\![\partial_{(N)}\phi_i]\!] = - \frac{1}{U_{(N)}} [\![\dot{\phi}_i]\!]$$

$$[\![\partial^2_{(N)}\phi_i]\!] = - \frac{1}{U_{(N)}} [\![\partial_{(N)}\dot{\phi}_i]\!] + \frac{1}{U_{(N)}} \frac{\delta}{\delta t} [\![\partial_{(N)}\phi_i]\!],$$

$$[\![\partial^2_{(N)}\phi_i]\!] = \frac{1}{U^2_{(N)}} [\![\ddot{\phi}_i]\!] + \frac{1}{U^2_{(N)}} \frac{\delta}{\delta t} (U_{(N)}) [\![\partial_{(N)}\phi_i]\!]$$

$$+ 2\frac{1}{U_{(N)}} \frac{\delta}{\delta t} [\![\partial_{(N)}\phi_i]\!].$$

Applying these conditions, together with (6.33) (6.35) and (6.36) in (6.34), we obtain (6.41) and (6.42).

We shall prove later, that the energy density E for the Lagrangian density function

$$L(X^A, t, \phi_i, \phi_{i,A}, \dot{\phi}_i, \phi_{i,AB}, \dot{\phi}_{i,A})$$

is given by

$$E = \frac{\partial L}{\partial \dot{\phi}_i} \dot{\phi}_i + \frac{\partial L}{\partial \dot{\phi}_{i,A}} \dot{\phi}_{i,A} - L$$

$$= P^i \dot{\phi}_i + \pi^{iA} \dot{\phi}_{i,A} - L$$

Introducing E into (6.41), after simple manipulations we obtain

$$[\![U_{(N)} E + T^{iA} N_A \dot{\phi}_i + H^{iAB} N_B \dot{\phi}_{i,A} - D_A \{ H^{iAB} N_B + \pi^{iA} U_{(N)}) \dot{\phi}_i \}]\!] = 0 \tag{6.43}$$

which is another form of (6.34)

Now, let us take an arbitrary "good" cylinder $D^* \times \mathcal{T} \subset D \times \mathcal{T}$ and integrate conditions (6.33) and (6.43) over $\Sigma_t^* = D^* \cap \Sigma_t$, then we obtain

$$\int_{\Sigma_t^*} [\![T^{iA} N_A + P^i U_{(N)}]\!] \, d\Sigma_{N-1} = \int_{C_t^*} [\![(H^{iAB} N_B + \pi^{iA} U_{(N)}) \check{M}_A]\!] d\Sigma_{N-2} \tag{6.44}$$

$$\int_{\Sigma_t^*} [\![T^{iA} N_A \dot{\phi}_i + H^{iAB} N_B \dot{\phi}_{i,A} + E U_{(N)}]\!] \, d\Sigma_{N-1} =$$

$$= \int_{C_t^*} [\![(H^{iAB} N_B + \pi^{iA} U_{(N)}) \check{M}_A \dot{\phi}_i]\!] \, d\Sigma_{N-2} \tag{6.45}$$

where $C_t^* = \partial(D^* \cap \Sigma_t)$. In the derivation of the above identities we have integrated by parts, using formulae (6.25), the last terms in (6.33) and (6.43) and then we have applied conditions (6.35) and (6.36).

CHAPTER 7

INVARIANCE OF ACTION INTEGRALS

7.1 Invariance - A Preview

If the equations of motion of a physical system are derivable from a variational principle a general and systematic procedure for the relation between conservation laws and transformations leaving invariant the action integral can be established. The fundamental work on this problem for smooth fields, was done in the early part of this century by Emmy Noether [27]. Influenced by the work of Klein [28] and of Lie [29] on the transformation properties of differential equations under a continuous group of transformations, Noether proved two fundamental results, now known as Noether theorems; classically, they can be stated as follows:

(I) If the action integral is invariant under an r-parameter continuous group of transformations of the variables, then there result r identities between the Euler operator $E^1(L)$ and quantities which can be written as divergences.

(II) If the action integral is invariant under a group of transformations which depend upon q arbitrary functions and their derivatives up to some order s then there exist identities between the Euler operator $E^i(L)$ and their derivatives up to order s.

The Noether identities have important physical consequences. The invariance of the action integral of the physical system under a r-parameter group of transformations (a group of symmetry transformations) leads directly to conservation laws for the system involved. In the single integral variational problem this means that the symmetry property leads to expressions which are constant along the stationary paths, i.e. first integrals of the equations of motion. For multiple integral problems the conservations laws have the form of a vanishing divergence which is interpreted as conservation of a "flux" quantity. The second Noether theorem is related to parameter invariant variational problems, i.e. action integrals which are invariant under arbitrary transformation of the independent variables, and will not be considered in this work.

Our treatment of multiple integrals of calculus of variations, though extended for discontinuous fields, was limited to special forms of the Lagrange density functions and requires a special form for variations of dependent and independent variables. For example, we shall not consider variations of the independent variable X. Moreover this treatment is exclusively in Euclidean spaces using standard Euclidean coordinates. Accordingly, we shall impose similar limitations for derivation of conservation theorems for discontinuous fields.

The classical variational methods of obtaining the Noether theorems are presented in many monographs, textbooks and papers. Let us mention only some popular and standard references: the books by Courant and Hilbert [30], Funk [31], Gelfand and Fomin [14], Rund [32], Sagan [33], Logan [26] and the paper by Hill [34], the latter of which is a common source quoted by physicists. A modern version of the Noether theorems using the formalism of modern differential geometry (bundles, jets, ... etc.) can be found in papers by Komorowski [35], Trautman [36], Garcia [37], Goldshmidt and Sternberg [38], among others. Also the monograph by Marsden [39] discusses a modern version of the theorem. In the general setting of nonlocal variations the Noether theorem is discussed by Edelèn [40].

The definition of invariance of an action integral under a group of transformations has a strictly local character. Therefore, we can conclude immediately then the so-called (local) fundamental invariance identities and the Noether identities hold, for discontinuous fields as well at each point which is not on a singular hypersurface. However, the above invariance identities we shall obtain directly, starting with the action integral considered in the class of discontinuous fields.

For the classical case, i.e. for the case of smooth fields, our presentation of the invariance and conservation theorems follows perhaps the most closely the exposition of these problems given in the book by Logan [26].

7.2 Invariance of Multiple Integrals

To facilitate the exposition, and in particular to avoid the ambiguities of the infinitesimal language, we shall use exclusively a one-parameter group of transformations.

A family

$$R^1 \supset \mathcal{O} \ni s \mapsto g(s)$$

of maps $g(s): R^m \to R^m$, where an interval $\mathcal{O}$ in R^1 contains the origin as an interior point, is called a one-parameter group of transformations of R^m if $G: R^m \times \mathcal{O} \to R^m$, defined by $G(X,s) = g(s)(X)$ is a mapping and

$$g(0) = id$$

$$g(s_1) \circ g(s_2) = g(s_1+s_2)$$

for any $s_1, s_2 \in \mathcal{O}$ such that $s_1 + s_2 \in \mathcal{O}$.

The curve $s \mapsto G(X,s)$ is called a trajectory of the group. Through any point $X \in R^m$ there passes exactly one trajectory of the group. Denoting $\frac{\partial G}{\partial s}(X,s)$ by $\frac{dg(s)}{ds}(X)$, one can write the generator (i.e. the vector field tangent to the trajectories) as

$$\xi = \frac{dg(s)}{ds}\bigg|_{s=0}$$

The type of transformations that will be considered here are transformations of $(x^1,\ldots,x^n,t)$ - space associated with the action integral (6.3), i.e. physical space-time. To be more precise, we require that the transformations are given by

$$\bar{x}^i = G^i(x,t,s) \qquad (i=1,\ldots,n) \tag{7.1}$$

$$\bar{t} = G^{n+1}(x,t,s). \tag{7.2}$$

The generators of the transformations G^i and G^{n+1} are given by

$$\xi^i(x,t) = \frac{\partial G^i}{\partial s}(x,t,s)\bigg|_{s=0} \tag{7.3}$$

$$\tau(x,t) = \frac{\partial G^{n+1}}{\partial s}(x,t,s)\bigg|_{s=0}. \tag{7.4}$$

Example 7.1. A one-parameter transformation of the (x,t) plane is given by

$$\bar{x} = x \cos \varepsilon - t \sin \varepsilon$$

$$\bar{t} = x \sin \varepsilon + t \cos \varepsilon$$

Geometrically, it is a rotation through an angle ε. In this case the generators are given by $\xi = -t$, $\tau = x$. By expanding $\sin \varepsilon$ and $\cos \varepsilon$ in Taylor series about $\varepsilon = 0$ we obtain the "infinitesimal" rotations

$$\bar{x} = x - \varepsilon t + o(\varepsilon), \quad \bar{t} = t + \varepsilon x + o(\varepsilon)$$

where $o(\varepsilon)$ denote terms which go to zero faster than $|\varepsilon|$, i.e.

$$\frac{o(\varepsilon)}{|\varepsilon|} \underset{|\varepsilon| \to 0}{\to} 0$$

In general, by Taylor's theorem (under suitable smoothness assumptions on G^i, G^{n+1} $(i=1,\ldots,n)$) the right-hand sides of (7.1) and (7.2) can be expanded about $s = 0$ to obtain

$$\bar{x}^i = x^i + \xi^i(x,t)s + o(s) \tag{7.5}$$

$$\bar{t} = t + \tau(x,t)s + o(s) \tag{7.6}$$

where $\xi^i(x,t)$ and $\tau(x,t)$ are generators of the transformation. The transformations (7.5) and (7.6) are the infinitesimal transformations associated with the transformations (7.1) and (7.2).

Let us recall that the first variation of the action integral (6.3) has been derived in the class of fields (Γ,ϕ), where $\Gamma \subset R^{N+1}$ is a singular hypersurface relative to the mapping $\phi : R^{N+1} \to R^n$, with restriction to the "good" closed cylinder $\pi = D \times \mathcal{T} \subset R^{N+1}$ as a domain of integration of this integral. The graph of the mapping ϕ, denoted by $Gr(\phi)$, is a piecewise-smooth manifold and Γ is the projection on R^{N+1} its singular subset.

We assume (it can be proved, however, this proof is very technical and is omitted) that for sufficiently small s, say $|s| < \varepsilon$

a one-parameter group of transformation (7.1) and (7.2) carries a piece-wise-smooth manifold $Gr(\phi)|_{D\times\mathcal{T}}$ into $Gr(\phi(s))|_{D\times\mathcal{T}(s)}$ with the same order as that of singular hypersurface Γ. (The order of a singular hypersurface has been defined in Chapter 2.)

Instead of a proof of this assumption, it will be justified by considering the following important example.

Example 7.2. We subject $Gr(\phi)$ to the following transformations

$$\bar{x}^i = G^i(x,s) \qquad (i=1,\ldots,n) \tag{7.7}$$

$$\bar{t} = G^{n+1}(t,s) \tag{7.8}$$

where G^i is an infinitesimal rotation in R^n around a fixed, arbitrary direction, and G^{n+1} is an infinitesimal translation in R^1. Then we obtain

$$\bar{x}^i = G^i\ (\phi(X,t),s)$$

$$\bar{t} = G^{n+1}(t,s)$$

The second equation may be solved (for a general smooth transformation (7.8) may be solved for sufficiently small s) to obtain

$$t = T(\bar{t},s)$$

Upon substitution of these quantities into the first equation we obtain

$$\bar{x}^i = G^i(\phi(X,T(\bar{t},s),s)$$

$$\equiv \bar{\phi}^i(X,\bar{t})$$

Therefore, the functions $x^i = \phi^i(X,t)$ and the functions $\bar{x}^i = \bar{\phi}^i(X,\bar{t})$ are related by means of transformations (7.7) and (7.8) via the conditions

$$G^i(\phi(X,t),s) = \bar{\phi}^i(X,T(\bar{t},s)).$$

To be more explicit

$$\phi^i(X,t(s),s) \equiv \bar{\phi}^i(X,\bar{t}) = \phi^i(X,t) + \epsilon^i_j \phi^j \ (X,t) \ s$$

$$t(s) \equiv \bar{t} = t + cs$$

where ϵ^i_j are components of the infinitesimal rigid rotation tensor in R^n and c is a constant.

Clearly, in the above example the pair (Γ,ϕ) is carried into a one-parameter family $(\Gamma(s), \phi(s))$ where Γ and $\Gamma(s)$ are singular hypersurfaces relative to ϕ and $\phi(s)$, respectively, with the same order of singularity. Note that under the transformations (7.7) and (7.8) the domain of integration $D \times \mathcal{T}$ is transformed into $D \times \mathcal{T}(s)$, where

$$\mathcal{T}(s) \equiv \bar{\mathcal{T}} = \{G^{n+1}(t,s): t \in \mathcal{T}\}.$$

The example considered above is crucial for applications to classical continuum mechanics. However, the general characterization of the problem of a group of transformations for discontinuous fields (except obvious smoothness assumptions) should be carried out in full detail in order to formulate a general invariance property of action integrals for discontinuous fields. We will not attempt to resolve this problem here.

We can now define, what is implied by stating that the multiple integral (6.3) is invariant under the one-parameter group of transformations (7.1) and (7.2).

The action integral (6.3) is invariant under the one-parameter transformation (7.1) and (7.2) if and only if given any (Γ,ϕ) and any "good" cylinder $\pi = D \times \mathcal{T} \subset R^{N+1}$ we have

$$\int_{\bar{\mathcal{T}}} \int_{D\setminus\bar{\Sigma}_{\bar{t}}} L(X^A, \bar{t}, \bar{\phi}_i, \bar{\phi}_{i,A}, \dot{\bar{\phi}}_i, \bar{\phi}_{i,AB}, \dot{\bar{\phi}}_{i,A})\, dV_N\, d\bar{t}$$

$$-\int_{\mathcal{T}} \int_{D\setminus\Sigma_t} L(X^A, t, \phi_i, \phi_{i,A}, \dot{\phi}_i, \phi_{i,AB}, \dot{\phi}_{i,A})\, dV_N\, dt$$

$$= o(s) \tag{7.9}$$

for every s sufficiently small, say $|s| < \varepsilon$.

The condition (7.9) means that

$$\frac{d}{ds} \int_{\mathcal{T}(s)} \int_{D\setminus\Sigma_t(s)} L(X^A, t(s), \phi_i(s), \phi_{i,A}(s), \dot{\phi}_i(s), \phi_{i,AB}(s),$$

$$\dot{\phi}_{i,A}(s))\, dV_N\, dt(s)\Big|_{s=0}$$

$$= 0 \tag{7.10}$$

where

$$\bar{t} \equiv t(s)$$

$$\bar{\phi}(X,\bar{t}) \equiv \phi(X,t(s),s)$$

and $\mathcal{T}(s) \equiv \bar{\mathcal{T}}$ are related by a one-parameter group of transformations.

Before proceeding with a derivation of invariance identities and conservation laws for the action integral for discontinuous fields we must clarify the variations, induced by symmetry transformation on the hypersurface of discontinuity, of partial derivatives of such fields. First, let us do this for the special case considered in Example 7.2

Example 7.2 (cont.). We have had that

$$\phi^i(X,t(s),s) \equiv \bar{\phi}^i(X,\bar{t}) = \phi^i(X,t) + \varepsilon^i_j\, \phi^j(X,t)\, s$$

$$t(s) \equiv \bar{t} = t + cs$$

follows from the infinitesimal rotation in R^n and the translation in R^1.

From the above we obtain

$$\dot{\phi}^i \delta t + \hat{\delta}\phi^i = \varepsilon^i_j \phi^j \qquad (7.11)$$

$$\delta t = c \qquad (7.12)$$

where $\hat{\delta}\phi \equiv \frac{\partial}{\partial s}\phi(X,t(s),s)|_{s=0}$ is the variation of the form of ϕ. Because ϕ is a continuous mapping, then recalling the kinematical and variational conditions of compatibility given by (3.10) and (4.7), respectively, we obtain from (7.11) that (cf. also Lemma 6.1)

$$U_{(N)}\delta t + \hat{\delta}\Sigma = 0 \qquad (7.13)$$

hold for an arbitrary (Γ,ϕ). In (7.13) $\hat{\delta}\Sigma$ is the normal variation of the hypersurface Σ_t (with $\delta t = 0$) and $U_{(N)}$ is the speed of propagation of Σ_t.

Now, by taking limits of (7.11) on the singular hypersurface Γ, and using conditions (3.9) and (4.5) together with (7.13) we obtain that

$$\frac{\delta\phi^i}{\delta t}\delta t + \hat{\hat{\delta}}\phi^i = \varepsilon^i_j\,\phi^j \qquad (7.14)$$

holds on Γ. In (7.14) $\frac{\delta}{\delta t}\phi^i$ is the displacement derivative and $\hat{\hat{\delta}}\phi^i$ is the displacement variation of ϕ relative to the singular hypersurface Σ_t (with $\delta t = 0$). Recall that

$$\Gamma = \{(X,t) : X \in \Sigma_t,\ t \in R^1\}$$

is swept out in R^{N+1} by a moving singular hypersurface, Σ_t in R^N relative to $\phi_t(X) = \phi(X,t)$.

For an arbitrary one-parameter group of transformations (7.1) and (7.2) the extensions of the formulae (7.13) and (7.14) can be obtained as follows. First, let us introduce the notation

$$\phi^i(X,t(s),s) \equiv \bar{\phi}^i(X,\bar{t}) = G^i(\phi(X,t)t,\ s) \quad (i=1,\dots,n) \tag{7.15}$$

$$t(s) \equiv \bar{t} = G^{n+1}(\phi(X,t),\ t,\ s) \tag{7.16}$$

Then, from these equations we obtain

$$\dot{\phi}^i\tau(\phi,\ t) + \hat{\delta}\phi^i = \xi^i(\phi,\ t) \equiv \delta\phi^i \quad (i=1,\dots,n) \tag{7.17}$$

$$\tau(\phi,\ t) \equiv \delta t \tag{7.18}$$

where $\xi^i(x,t)$ and $\tau(x,t)$ are the generations of the transformations (7.1) and (7.2) given by (7.3) and (7.4) respectively. Following the same remarks as those in Example 7.2 considered before we obtain from (7.17) the condition

$$U_{(N)}\ \tau(\phi,\ t) + \hat{\delta}\Sigma = 0 \tag{7.19}$$

Also, in a similar way to the last example, we have

$$\frac{\delta\phi^i}{\delta t}\ \tau(\phi,\ t) + \hat{\delta}\phi^i = \xi^i(\phi,\ t) \tag{7.20}$$

The conditions (7.19) and (7.20) are satisfied on an arbitrary singular hypersurface. We have just proved the following.

LEMMA 7.1. For each pair (Γ,ϕ) where $\phi: R^{N+1} \to R^n$ is a piecewise-smooth mapping and Γ is its singular hypersurface (i.e. Γ is a projection on R^{N+1} of a singular subset of a piecewise-smooth N+1-dimensional manifold

$Gr(\phi) \subset R^{N+1} \times R^n)$ and for each group of transformations

$$\bar{x}^i = G^i(x,t,s) \qquad (i=1,\ldots,n)$$
$$\bar{t} = G^{n+1}(x,t,s)$$

with the generators

$$\xi^i(x,t) = \frac{\partial G^i}{\partial s}(x,t,0)$$

$$\tau(x,t) = \frac{\partial G^{n+1}}{\partial s}(x,t,0)$$

the following condition

$$U_{(N)}\tau(\phi,\ t) + \hat{\delta}\Sigma = 0$$

$$\frac{\delta\phi^i}{\delta t}\tau(\phi,\ t) + \hat{\delta}\phi^i = \xi^i(\phi,\ t)$$

are satisfied at each point of Γ where $U_{(N)}$ and $\frac{\delta\phi}{\delta t}$ are respectively, the speed of propagation and displacement derivatives, and $\hat{\delta}\Sigma$ and $\hat{\delta}\phi^i$ are respectively, the normal variation of Γ (in R^N) and the displacement variation, both induced by the transformations. □

As an additional observation, we note that Lemma 6.1 from Chapter 6 and Lemma 7.1 imply that the "total" normal variation $\hat{\delta}\Sigma$ is zero and the "total" displacement variation $\hat{\delta}\phi^i$ equals $\xi^i(\phi,\ t)$.

7.3 The Fundamental Invariance Theorems

Now, we shall prove that the invariance of the action integral considered in the class of discontinuous fields under an admissible group of transformations implies the existence of differential and integral identities.

THEOREM 7.1. A necessary condition for the action integral (6.3) to be invariant under the group of transformations (7.1) and (7.2) is that the Lagrangian density function L given by

$$L = L(X^A, t, \phi_i, \phi_{i,A}, \dot{\phi}_i, \phi_{i,AB}, \dot{\phi}_{i,A})$$

and its derivatives satisfy outside of Γ the following identity

$$\frac{\partial L}{\partial t}\tau + \frac{\partial L}{\partial \phi_i}\xi^i + \frac{\partial L}{\partial \phi_{i,A}}(\xi^i_{,A} - \dot{\phi}^i\tau_{,A}) + \frac{\partial L}{\partial \dot{\phi}_i}(\dot{\xi}^i - \dot{\phi}^i\dot{\tau})$$

$$+ \frac{\partial L}{\partial \phi_{i,AB}}(\xi^i_{,AB} - \dot{\phi}^i_{,A}\tau_{,B} - \dot{\phi}^i_{,B}\tau_{,A} - \dot{\phi}^i\tau_{,AB}) + \frac{\partial L}{\partial \dot{\phi}_{i,A}}(\dot{\xi}^i_A -$$

$$- \dot{\phi}^i_{,A}\dot{\tau} - \ddot{\phi}\tau_{,A} - \dot{\phi}^i\dot{\tau}_{,A}) + L\,\dot{\tau} = 0 \qquad (7.21)$$

where, Γ is a singular hypersurface in R^{N+1} relative to ϕ and ξ^i and τ are the generators of the transformation and are given by (7.3) and (7.4), respectively.

PROOF. To prove this theorem, first let us note that equation (7.17) implies that the following set of identities are valid:

$$\hat{\delta}\phi = \xi - \dot{\phi}\tau$$

$$\hat{\delta}\phi_{,A} = \xi_{,A} - \dot{\phi}_{,A}\tau - \dot{\phi}\tau_{,A}$$

$$\dot{\overline{\hat{\delta}\phi}} = \dot{\xi} - \ddot{\phi}\tau - \dot{\phi}\dot{\tau}$$

$$\hat{\delta}\phi_{,AB} = \xi_{,AB} - \dot{\phi}_{,AB}\tau - \dot{\phi}_{,A}\tau_{,B} - \dot{\phi}_{,B}\tau_{,A} - \dot{\phi}\tau_{,AB}$$

$$\dot{\overline{\hat{\delta}\phi_{,A}}} = \dot{\xi}_{,A} - \ddot{\phi}_{,A}\tau - \dot{\phi}_{,A}\dot{\tau} - \ddot{\phi}\tau_{,A} - \dot{\phi}\dot{\tau}_A .$$

Also from Lemma 7.1 we have that

$$U_{(N)}\tau + \hat{\delta}\Sigma = 0$$

on Γ.

Next, by the chain rule for partial differentiation we have the following identity

$$\frac{d}{dt}(L\tau) = \frac{\partial L}{\partial t}\tau + \frac{\partial L}{\partial \phi_i}\dot{\phi}_i\tau + \frac{\partial L}{\partial \phi_{i,A}}\dot{\phi}_{i,A}\tau + \frac{\partial L}{\partial \dot{\phi}_i}\ddot{\phi}_i\tau$$

$$+ \frac{\partial L}{\partial \phi_{i,AB}}\dot{\phi}_{i,AB}\tau + \frac{\partial L}{\partial \dot{\phi}_{i,A}}\ddot{\phi}_{i,A}\tau + L\dot{\tau} .$$

Upon substituting all of the above identities into Equation (6.13) ($\tau \equiv \delta t$) we obtain, after simplification, the expression

$$\int_{\mathcal{T}}\int_{D} \{\frac{\partial L}{\partial t}\tau + \frac{\partial L}{\partial \phi_i}\xi^i + \frac{\partial L}{\partial \phi_{i,A}}(\xi^i_{,A} - \dot{\phi}\tau_{,A}) + \frac{\partial L}{\partial \dot{\phi}_i}(\dot{\xi}^i - \dot{\phi}^i\dot{\tau})$$

$$+ \frac{\partial L}{\partial \phi_{i,AB}}(\xi^i_{,AB} - \dot{\phi}^i_{,A}\tau_{,B} - \dot{\phi}^i_{,B}\tau_{,A} - \dot{\phi}^i\tau_{,AB})$$

$$+ \frac{\partial L}{\partial \dot{\phi}_{i,A}}(\dot{\xi}^i_{,A} - \dot{\phi}^i_{,A}\dot{\tau} - \ddot{\phi}^i\tau_{,A} - \dot{\phi}^i\dot{\tau}_{,A}) + L\dot{\tau}\}\, dV_N\, dt = 0 .$$

$$(A,B=1,\ldots,N;\ i=1,\ldots,n) \qquad (7.22)$$

By the arbitrariness of $D \times \mathcal{T}$ we obtain from (7.22) that the identity (9.21) holds true at each point $(X,t) \in \Gamma$. □

Remark 7.1. The invariance identities (7.21) can be expressed in an equivalent form

$$\frac{d}{dt}(L\tau) + \frac{\partial L}{\partial \phi_i}C^i + \frac{\partial L}{\partial \phi_{i,A}}C^i_{,A} + \frac{\partial L}{\partial \dot{\phi}_i}\dot{C}^i$$

$$+ \frac{\partial L}{\partial \phi_{i,AB}}C^i_{,AB} + \frac{\partial L}{\partial \dot{\phi}_{i,A}}\dot{C}^i_{,A} = 0 \qquad (7.23)$$

where

$$C^i \equiv \xi^i - \dot{\phi}^i\tau. \qquad (7.24)$$

Identities (7.21) (or (7.23)), exactly as in the case of smooth fields, can be interpreted in two ways. If the transformations and an invariant action integral is known, the equations (7.21) (Eq. (7.23)) represent a set of identities in X^A, t and the partial derivatives of ϕ. On the other hand, if the Lagrangian density function L is unknown then the equation (7.21) (Eq. (7.23)) represent quasi-linear partial differential equations for L, and consequently they can serve to characterize the Lagrangian density function, or action integrals, that possess given invariance properties.

Noether's identities, or more precisely their local forms, follow directly from the invariance identities given by (7.23). At any point $(X,t) \notin \Gamma$ we obviously have the following set of identities.

$$\frac{\partial L}{\partial \phi_{i,A}} C^i_{,A} = \left(\frac{\partial L}{\partial \phi_{i,A}} C^i\right)_{,A} - \left(\frac{\partial L}{\partial \phi_{i,A}}\right)_{,A} C^i$$

$$\frac{\partial L}{\partial \phi_i} \dot{C}^i = \dot{\overline{\frac{\partial L}{\partial \phi_i} C^i}} - \dot{\overline{\left(\frac{\partial L}{\partial \phi_i}\right)}} C^i$$

$$\frac{\partial L}{\partial \phi_{i,AB}} C^i_{,AB} = \left[\frac{\partial L}{\partial \phi_{i,AB}} C^i_{,B} - \left(\frac{\partial L}{\partial \phi_{i,AB}}\right)_{,B} C^i\right]_{,A} + \left(\frac{\partial L}{\partial \phi_{i,AB}}\right)_{,AB} C^i$$

$$\frac{\partial L}{\partial \phi_{i,A}} \dot{C}^i_{,A} = \dot{\overline{\left[\frac{\partial L}{\partial \phi_{i,A}} C^i_{,a}\right]}} - \left[\dot{\overline{\left(\frac{\partial L}{\partial \phi_{i,A}}\right)}} C^i\right]_{,A} + \dot{\overline{\left(\frac{\partial L}{\partial \phi_{i,A}}\right)}}_{,A} C^i$$

Substituting these expressions into (7.23) we get

$$E^i(L)\ C_i + \frac{d}{dX^A}\left[-T^{iA}C_i - H^{iAB}C_{i,B}\right] + \frac{d}{dt}\left[L\tau + P^iC_i + \pi^{iB}C_{iB}\right] = 0 \quad (7.25)$$

where we have introduced notation (6.17) - (6.22).

In this way, we have derived the extension of the (local) Noether's theorem for discontinuous fields which we now state.

THEOREM 7.2. A necessary condition for the action integral (6.3), which is considered in the class of discontinuous fields, to be invariant under the group of transformations (7.1) and (7.2) is that the following identities

$$E^i(L)(\xi_i - \dot{\phi}_i\tau) + \frac{\partial}{\partial X^A}\Big[\Big\{\frac{\partial L}{\partial \phi_{i,A}} - \Big(\frac{\partial L}{\partial \phi_{i,AB}}\Big)_{,B} - \Big(\overline{\frac{\partial L}{\partial \dot{\phi}_{i,A}}}\Big)^{\bullet}\Big\}(\xi_i - \dot{\phi}_i\tau)$$

$$+ \frac{\partial L}{\partial \phi_{i,AB}}(\xi_i - \dot{\phi}_i\tau)_{,B}\Big] + \frac{\partial}{\partial t}\Big[L\tau + \frac{\partial L}{\partial \dot{\phi}_i}(\xi_i - \dot{\phi}_i\tau)$$

$$+ \frac{\partial L}{\partial \dot{\phi}_{i,AB}}(\xi_i - \dot{\phi}_i\tau)_{,B}\Big] = 0 \qquad (7.26)$$

hold true at any point which is not on the singular hypersurface. In (7.26) $E^i(L)$ is the Euler operator given by

$$E^i(L) = -\Big(\overline{\frac{\partial L}{\partial \dot{\phi}_i}}\Big)^{\bullet} - \Big(\frac{\partial L}{\partial \phi_{i,A}}\Big)_{,A} + \Big(\frac{\partial L}{\partial \phi_{i,AB}}\Big)_{,BA}$$

$$+ \Big(\overline{\frac{\partial L}{\partial \dot{\phi}_{i,A}}}\Big)^{\bullet}_{,A} + \frac{\partial L}{\partial \phi_i} \qquad (7.27)$$

and ξ^i and τ are the generators of the transformations given by (7.3) and (7.4), respectively. □

The identities (7.26) are known as (local) Noether identities.

The next theorem is an integral version of Theorem 7.2. Using the notation for derivatives of L given by (6.17)–(6.22) we have

THEOREM 7.3. If the assumptions of Theorem 7.2 are satisfied then the following integral identities

$$\int_{\mathcal{T}^*}\int_{D^*} E^i(L)\ (\xi_i - \dot{\phi}_i\tau)dV_N\ dt - \int_{\mathcal{T}^*}\int_{\partial D^*} \{T^{iA}N_A\ (\xi_i - \dot{\phi}_i\tau)$$

$$+ H^{iAB}N_A\ (\xi_i - \dot{\phi}_i\tau)_{,B}\}\ d\Sigma_{N-1}\ dt + \int_{D^*} \{P^i\ (\xi_i - \dot{\phi}_i\tau)$$

$$+ \pi^{iA}\ (\xi_i - \dot{\phi}_i\tau)_{,A} + L\tau\ \}\ dV_N \Big|_{t_1^*}^{t_2^*}$$

$$+ \int_{\mathcal{T}^*}\int_{\Sigma_t\cap D^*} [\![(T^{iA}N_A + P^iU_{(N)})\ (\xi_i - \dot{\phi}_i\tau)]\!]\ d\Sigma_{N-1}\ dt$$

$$+ \int_{\mathcal{T}^*}\int_{\Sigma_t\cap D^*} [\![(H^{iAB}N_A + \pi^{iB}U_{(N)})(\xi_i - \dot{\phi}_i\tau)_{,B} + LU_{(N)}\tau]\!]\ d\Sigma_{N-1}\ dt$$

$$= 0 \tag{7.28}$$

hold true for an arbitrary "good" cylinder $D^* \times \mathcal{T}^* \subset D \times \mathcal{T}$.

PROOF. Theorem 7.3 follows by substituting expressions on $\delta\phi$ and δt given by (7.17) and (7.18), respectively, into $\delta A_{D\times\mathcal{T}} = 0$ where $\delta A_{D\times\mathcal{T}}$ is given by expression (6.16). Also, this theorem follows directly from Theorem 7.2 by integrating (7.26) over $D^* \times \mathcal{T}^*$ and using (6.14) and (6.15). □

CHAPTER 8

CONSERVATION THEOREMS FOR DISCONTINUOUS FIELDS

The conservation theorems in the case of smooth fields (all jumps vanish) are simple corollaries of the fundamental invariance theorems, i.e. Theorem 7.2 and Theorem 7.3 for this case. The Noether identities hold everywhere in the domain of integration of the action integral and they imply the local conservation laws which take the form of a vanishing divergence.

The integral conservation laws are trivially obtained by integration of local conservation laws over an arbitrary subdomain and applying the Divergence Theorem. For discontinuous fields we have the following fundamental conservation theorem. (A very important application of this theorem we shall consider in the second part of this thesis).

THEOREM 8.1. If the action integral defined by (6.3) is invariant under the group of transformations given by (7.1) and (7.2) and if the Euler-Lagrange equations (6.32) and the jump conditions (6.33) - (6.36) of the theorem 6.1 are satisfied, the following integral identities

$$\int_{D^*} \{L\tau + \frac{\partial L}{\partial \dot{\phi}_i} (\xi_i - \dot{\phi}_i \tau) + \frac{\partial L}{\partial \phi_{i,A}} (\xi_i - \dot{\phi}_i \tau)_{,A}\} \, dV_N \;\Big|_{t_1^*}^{t_2^*}$$

$$- \int_{\mathcal{T}^*} \int_{\partial D^*} \{ [- \frac{\partial L}{\partial \phi_{i,A}} + (\frac{\partial L}{\partial \phi_{i,AB}})_{,B} + \overline{(\frac{\partial L}{\partial \phi_{i,A}})}^{\bullet}] N_A (\xi_i - \dot{\phi}_i \tau) - (\frac{\partial L}{\partial \phi_{i,AB}}) N_A$$

$$(\xi_i - \dot{\phi}_i \tau)_{,B}\} \, d\Sigma_{N-1} \, dt$$

$$-\int_{\mathcal{T}^*}\int_{C_t^*} [\![(\frac{\partial L}{\partial \phi_{i,AB}} N_B + \frac{\partial L}{\partial \phi_{i,A}} U_{(N)}) \overset{\vee}{M}_A (\xi_i - \dot{\phi}_i \tau)]\!] \, d\Sigma_{N-2} \, dt = 0 \qquad (8.1)$$

hold true for an arbitrary "good" cylinder $D^* \times \mathcal{T}^* \subset D \times \mathcal{T}$. In (8.1) ξ_i, τ are the generators of the group of transformations given by (7.3) and (7.4), respectively, $C_t^* = \partial(D^* \cap \Sigma_t) \subset \partial D^*$, and $\overset{\vee}{M}_A$ are the components of the vector $\overset{\vee}{M}$ defined in Chapter 6 (c.f. Fig. 3 and Fig. 4)

PROOF. In the notation of (6.17) - (6.22) and (7.24) the identity (8.1) can be written in an equivalent compact form

$$\int_{D^*} (L\tau + P^i C_i + \pi^{iA} C_{i,A}) \, dV_N \Big|_{t_1^*}^{t_2^*} - \int_{\mathcal{T}^*}\int_{\partial D^*} (T^{iA} N_A C_i + H^{iAB} N_B C_{i,A}) d\Sigma_{N-1} \, dt$$

$$- \int_{\mathcal{T}^*}\int_{C_t^*} [\![(H^{iAB} N_B + \pi^{iA} U_{(N)}) \overset{\vee}{M}_A C_i]\!] \, d\Sigma_{N-2} \, dt = 0 \qquad (8.2)$$

From Theorem 7.3 it follows that we only need to prove that

$$\int_{\mathcal{T}^*}\int_{\Sigma_t^*} [\![(T^{iA} N_A + P^i U_{(N)}) C_i + (H^{iAB} N_A + \pi^{iB} U_{(N)}) C_{i,B} + L U_{(N)} \tau]\!] d\Sigma_{N-1} \, dt$$

$$= - \int_{\mathcal{T}^*}\int_{C_t^*} [\![(H^{iAB} N_B + \pi^{iA} U_{(N)}) \overset{\vee}{M}_A C_i]\!] d\Sigma_{N-2} \, dt \qquad (8.3)$$

where $\Sigma_t^* = D^* \cap \Sigma_t$.

From the jump conditions (6.34) - (6.36) and (6.43) we have

$$[\![(T^{iA} N_A + P^i U_{(N)}) C_i + (H^{iAB} N_A + \pi^{iB} U_{(N)}) C_{i,B} + L U_{(N)} \tau]\!]$$

$$= [\![\{ T^{iA} N_A + P^i U_{(N)} - \mathcal{D}_A (H^{iAB} N_B + \pi^{iA} U_{(N)}) \} C_i + \mathcal{D}_A \{ (H^{iAB} N_B + \pi^{iA} U_{(N)}) \bar{C}_i \}$$

$$+ L U_{(N)} \tau]\!]$$

$$= [\![\{T^{iA}N_A + P^iU_{(N)} - D_A(H^{iAB}N_B + \pi^{iA}U_{(N)}\} \xi_i - \{EU_{(N)} + T^{iA}N_A\dot{\phi}_i$$

$$+H^{iAB}N_A\dot{\phi}_{i,B} - D_A[(H^{iAB}N_B + \pi^{iA}U_{(N)})\dot{\phi}_i\}\}\tau + D_A\{(H^{iAB}N_A + \pi^{iA}U_{(N)})C_i\}]\!]$$

$$= D_A[\![(H^{iAB}N_B + \pi^{iA}U_{(N)})C_i]\!].$$

On substituting this into the left side of (8.3) and then using integration by parts by means of the formulae (6.28) and (6.29) we obtain that (8.2) holds true. Thus the proof is completed.□

The "local" conservation theorem is a simple consequence of Theorem 7.2 and Theorem 6.1 as the following corollary shows.

Corollary 8.1. If the assumptions of Theorem 8.1 are satisfied then the following identity

$$\frac{\partial}{\partial X^A}\{-T^{iA}C_i - H^{iAB}C_{i,B}\} + \frac{\partial}{\partial t}\{L\tau + P^iC_i + \pi^{iA}C_{i,A}\} = 0$$

$$(X,t) \notin \Gamma \qquad (8.4)$$

holds true. In (8.4)

$$C_i = \xi_i - \dot{\phi}_i\tau$$

and ξ_i and τ are the generators of the group of transformations given by (7.3) and (7.4), respectively.□

Let us note that the identity (8.2) can be written in the equivalent form

$$\int_{D^*} (L\tau + P^i C_i + \pi^{iA} C_{i,A}) dV_N \Big|_{t_1^*}^{t_2^*} - \int_{\mathcal{T}^*} \int_{\partial D^*} [\{T^{iA} N_A - H^i \Omega - \mathcal{D}_A (H^{iAB} N_B)\} C_i$$

$$+ H^i \partial_{(N)} C_i] \, d\Sigma_{N-1} \, dt - \int_{\mathcal{T}^*} \int_{\mathcal{C}_t^*} [\![\{H^{iAB} N_B^{\partial D} M_A^{\wedge} + (H^{iAB} N_B^{\Sigma} + \pi^{iA} U_{(N)}) M_A^{\vee}\}$$

$$C_i]\!] \, d\Sigma_{N-2} \, dt = 0 \qquad (8.5)$$

where

$$H^i = H^{iAB} N_A N_B$$

and $M_A^{\wedge}$ are the components of the vector $M^{\wedge}$ defined in Chapter 6 (c.f. Figs. 3 and 4).

Remark 8.1. The conservation laws expressed by (8.2) or (8.5) are complemented by the conservation laws for jumps on the singular hypersurface. They are given by

$$\int_{\Sigma_t^*} [\![(T^{iA} N_A + P^i U_{(N)}) \xi_i + (H^{iAB} N_B + \pi^{iA} U_{(N)}) \xi_{i,A}]\!] \, d\Sigma_{N-1}$$

$$- \int_{\mathcal{C}_t^*} [\![(H^{iAB} N_B^{\Sigma} + \pi^{iA} U_{(N)}) M_A^{\vee} \xi_i]\!] d\Sigma_{N-2} = 0 \qquad (8.6)$$

$$\int_{\Sigma_t^*} [\![(T^{iA} N_A \dot{\phi}_i + H^{iAB} N_B \dot{\phi}_{i,A} + EU_{(N)}) \tau + (H^{iAB} N_B + \pi^{iA} U_{(N)}) \dot{\phi}_i \tau_{,A}]\!] d\Sigma_{N-1}$$

$$- \int_{\mathcal{C}_t^*} [\![(H^{iAB} N_B^{\Sigma} + \pi^{iA} U_{(N)}) M_A^{\vee} \dot{\phi}_i \tau]\!] \, d\Sigma_{N-2} = 0 \qquad (8.7)$$

where $\Sigma_t^* = \Sigma_t \cap D^*$, $\mathcal{C}_t^* = \partial(\Sigma_t \cap D^*)$ and ξ_i and τ are the generators of the group of transfomration given by (7.3) and (7.4), respectively.

PROOF. Identities (8.6) and (8.7) are derived directly from jump conditions given in Theorem 6.1 and in Eq. (6.4). □

PART II

APPLICATIONS TO SELECTED TOPICS

IN

CONTINUUM MECHANICS

CHAPTER 9

INTRODUCTION TO PART II

The use of variational principles in continuum mechanics has been established over the years by many outstanding researchers. Some of them we have already cited in Part I of this thesis. Since, the number of published papers, books and monographs on this subject is very large, we shall refer here only to the book by Washizu [41], which deals with the variational principles of elasticity and plasticity in their classical setting, and to the recent book by Marsden and Hughes [7] where the subject is treated within the framework of modern analysis on manifolds. These references contain full bibliographies on the subject. The latter of them also gives an account of the related parts of pure and applied mathematics. For general variational methods in applied science, in particular in finite-element methods, we refer to Oden [42] and Zienkiewicz [43].

The variational theorems for discontinuous fields have recently been recognized in connection with the static and dynamics of composite materials, fracture mechanics and finite-element formulations of problems in continuum mechanics.

Originally, discontinuity conditions were treated as part of a variational theorem in linear elasticity by Prager [44]. More general cases of such theorems in linear and nonlinear elasticity have been developed by Nemat-Nasser [45,46,47] where some of thier applications have been studied also. For a large class of materials in continuum mechanics, using variational methods, the jump conditions for discontinuous fields have been treated in Oden and Reddy [6]. The general theorems of the calculus of variations which admit propagating discontinuities in the field quantities have been elaborated in the first part of this thesis.

In this part of the thesis, we shall deal with applications of the results established in Part I, to selected topics in continuum mechanics. In particular, we shall investigate the following topics:

1. Kinematics of elastic deformation with discontinuities carried by wave fronts and in particular, the dualism in the description of the undeformed state and the family of deformed states of 2 and 3-dimensional media set into motion, during which a singular hypersurface propagates through these media.
2. Balance laws, in the material and spatial descriptions, for simple hyperelastic materials and materials of grade 2, which admit discontinuities carried by wave fronts in these materials.
3. The problem of wave propagation in a plate model based on a fourth order differential equation for transverse vibrations, including the shear caused by transverse stresses.

The first topic will be presented in Chapter 10 (dualism in the description of a singular hypersurface) and in Sections 11.2 and 11.4 of Chapter 11 (dualism in the description of balance laws of 3-dimensional elasticity theories). Our approach (though set in the standard coordinate system on R^n) is based on classical tensor analysis following monographs by Truesdell and Toupin [12] and Truesdell and

Noll [48]. Some definitions and proofs using the notion of differential forms will be presented in Appendix III.

In Chapter 11 we shall derive balance laws (balance of the linear moments, angular momenta and energies) in a 3-dimensional simple hyperelastic material and materials of grade 2, through which a singular surface, i.e. a wave surface across which partial derivatives of motion are discontinuous, is propagating. These balance laws will be given in both material and spatial descriptions. In the above we follow Toupin's terminology, see Toupin [3]. For a review of the generalized elasticity, including materials of grade 2, and their historical developments, we refer to the article by Tiersten and Bleustein [2]. The interest in various generalizations of classical elasticity and their applications has existed since the work by Cosserats [49]. For recent developments we mention the theory of grade consistent micropolar materials by Brulien and Hjalmars [50] or Ieşan [51], among others. Applications of higher order materials can be found in Collet [52] (elastic ferromagnets) or in Sun and Yang [53], Kanatani [54] and Kerr and Accorsi [55] (continuous models for frame-type structures).

It should be clear, that our results from Part I provide a general framework for the investigation of propagating discontinuities in such models. However, such a study is postponed until the future, because, presently there does not exist a general, consistent dynamical continuous model for frame-structures such as gridworks, trusses, etc. For such static models we refer to Kanatani [54].

In this thesis and in particular in Chapter 12 we shall investigate the problem of wave propagation in a 2-dimensional plate model governed by the fourth order differential equation for transverse vibrations, including the shear caused by transverse stresses. Such equations have been studied by Duvaut and Lions [56] and they have recently been derived from 3-dimensional linear elasticity for plates using the method of asymptotic expansion by Gusein-Zade [57] and Raoult [58]. These authors have studied the applicability of the plate models and the latter has also given sharp convergence estimates.

Using the inverse methods of the calculus of variations (c.f. Santilli [59] and Bampi and Morro [60]) we shall associate with the evolution equation for plates a 2-dimensional Lagrangian density function. We shall investigate only third-order waves within the plate model. Recall that the k-order wave front in the plate model corresponds to a wave curve across which the lowest, k-order derivative of the vertical displacement of the middle surface of the plate, with respect to time, is discontinuous. Applying the variational theorem for discontinuous motions (Theorem 6.1 from Chapter 6) we obtain the speed of propagation of such a wave front in the plate model we have chosen. This speed of propagation which is expressed only by material constants, implies that the wave front consists of a parallel family of curves. With this information we can integrate the wave amplitude equation to find the decay law for third order waves in the plate model. Also in this chapter we shall study the meaning of such waves and, as well, we shall state some general observations concerning their decay law. Finally, we shall indicate the relation of our approach to those studied by other researchers.

CHAPTER 10

KINEMATICS OF ELASTIC DEFORMATION

FUNDAMENTAL FORMULAE

The key to understanding finite deformation theory is an appreciation of the dualism in the description of the undeformed and the deformed states of matter. In Classical Field Theory, which employs classical tensor analysis, the key is an appreciation of the two sets of (curvilinear) coordinate systems, material and spatial, that are used to describe deformation processes. For this approach, we refer to Truesdell and Toupin [12] and Truesdell and Noll [48]. Modern Field Theory, employs tensor analysis on manifolds, and the dualism mentioned above is formulated using manifold ideas; the pull-back and push-forward. For this appraoch we refer to Marsden and Hughes [7] in which also the relations and notations of classical tensor analysis and tensor analysis on manifolds are indicated. The references mentioned above contain a full list of references on the subject and its historical development.

In this part of the thesis and in particular, in this chapter we shall employ classical analysis on R^n using standard Cartesian coordinate systems, the same as we did in Part I. This choice, of course, cannot affect the physics involved.

An N-dimensional material body is identified with a compact set $B \subset R^N$ which has a smooth boundary ∂B homeomorphic to a (N-1) sphere.

This identification is referred to as the reference configuration. The points X of B represent the positions of material or mass points of a body in this configuration. We shall consider 2 or 3-dimensional bodies, i.e. we shall assume that N = 2 or 3, respectively.

A motion of the body, as used here, refers to a change in its size, shape, orientation and location in physical space without causing breakage, cracking or slippage, which would destroy the continuity of the process. Accordingly, the motion of a body in physical E^3-space is given by a one-parameter family $\{\psi_t\}$ of homeomorphisms $\psi_t: B \rightarrow \mathcal{B}_t$, $\mathcal{B}_t \subset E^3$ being an N-dimensional manifold with boundary, such that $\psi : B \times R^1 \rightarrow E^3$ defined by $\psi(X,t) = \psi_t(X)$ is a continuous mapping which gives the spatial position

$$x = \psi(X,t), \quad X \in B \tag{10.1}$$

at each time t of each material point X. The image $\psi_t(D)$ of any subset $D \subset B$ is called the configuration of D at time t. The (Cartesian) coordinates X_A (A=1,...,N) of X are called the material coordinates. They are the name of a mass point and as such remain with the mass point in the configuration of the body at each time t. The (Cartesian) coordinates x_i (i=1,2,3) of the position x are called the spatial coordinates of the mass point.

The existence of a homeomorphism $\psi_t: B \rightarrow \mathcal{B}_t$ for each t implies that we can write the inverse relation to (10.1)

$$X = \Psi(x,t), \quad x \in \mathcal{B}_t \tag{10.2}$$

The mapping Ψ for each t maps the manifold $\mathcal{B}_t$ onto B. The relation (10.2) can be interpreted in another way. One can consider (10.2) as a (global) curvilinear coordinate system on $\mathcal{B}_t$. This time dependent coordinate system is called a convected coordinate system on $\mathcal{B}_t$.

The partial derivatives

$$\begin{aligned} \psi_{i,A} &\equiv \frac{\partial\psi_i(X,t)}{\partial X_A} \\ \dot{\psi}_i &\equiv \frac{\partial\psi_i(X,t)}{\partial t} \end{aligned} \qquad (10.3)$$

of the motion ψ are, respectively, the components of the so-called deformation gradient and the velocity of the material point X at time t. Let us note that the assumption that $\mathcal{B}_t$ is an N-dimensional manifold in E^3 is equivalent to the following condition

$$\text{rank } (\psi_{i,A}) = N$$

for each t. For N = 2, the reference configuration of a body and its configuration at time t are depicted in Fig. 5.

We assume that during time interval $\mathcal{T} = [t_1, t_2]$ a singular hypersurface of the motion ψ of B is propagating through the material. To be more precise, we consider a one-parameter family $\{\Sigma_t\}$, $t \in \mathcal{T}$ of a closed subset Σ_t in B. For a given motion $\psi_t : B \to \mathcal{B}_t$, $t \in \mathcal{T}$, let us assume that for each $t \in \mathcal{T}$ there exists an open set U in R^N containing B, a homeomorphism $\tilde{\psi}_t : U \to \mathcal{U}_t$ and a pair $(\tilde{\Sigma}_t, \tilde{\psi}_t)$ as was defined in Chapter 2, such that $B \times \mathcal{T} \subset R^{N+1}$ is a good cylinder (c.f. Chapter 7) and $\tilde{\Sigma}_t \cap B = \Sigma_t$, $\tilde{\psi}_t | B = \psi_t$. If a closed hypersurface $\tilde{\Sigma}_t$ in U is a singular hypersurface such that its speed of propagation $U_{(N)}$, defined by (3.7) does not vanish at each point of $\tilde{\Sigma}_t$ for each $t \in \mathcal{T}$, then Σ_t is called a wave propagating through the body B. Let us recall that the order of wave corresponds to the lowest order of

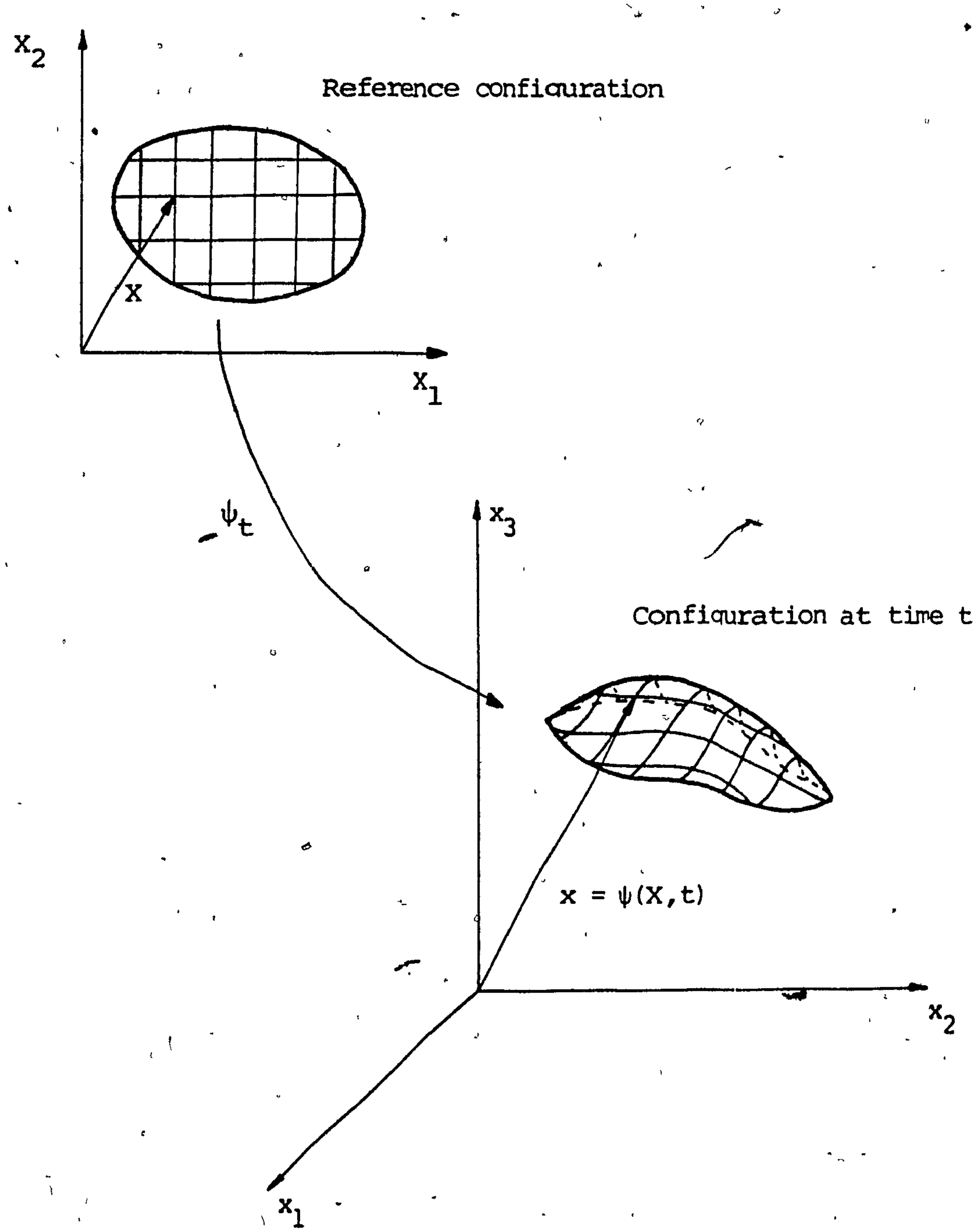

X2
Reference configuration
X
X1
ψt
x3
Configuration at time t
x = ψ(X,t)
x2
x1

Figure 5

derivative of the motion with respect to time which is discontinuous across the wavefront. Also, let us note that if for some $t \in \mathcal{T}$ $\tilde{\Sigma}_t \cap \partial B \neq 0$ then for each $t \in \mathcal{T}$ $C_t = \tilde{\Sigma}_t \cap \partial B$ is an (N-2)-dimensional submanifold in ∂B, and as well Σ_t approaches ∂B transversely. The subset σ_t in B_t given by $\psi_t(\Sigma_t)$ is a spatial representation of a wave at time t and is also referred to as a wave.

In a manner similar to that in Chapter 2, a "space-time" description of the wave Σ_t is given by

$$\Gamma = \{(X,t): X \in \Sigma_t,\ t \in \mathcal{T}\}$$

in $\pi = B \times \mathcal{T}$. Also its space-time image in $\pi = \bigcup_{t\in\mathcal{T}} (B_t \times \{t\})$ is defined by

$$\gamma = \{(x,t): x \in \sigma_t,\ t \in \mathcal{T}\}$$

where $\sigma_t = \psi_t(\Sigma_t)$. Schematically, it is depicted in Fig. 5 where for simplicity N=2 and physical space is represented by E^2.

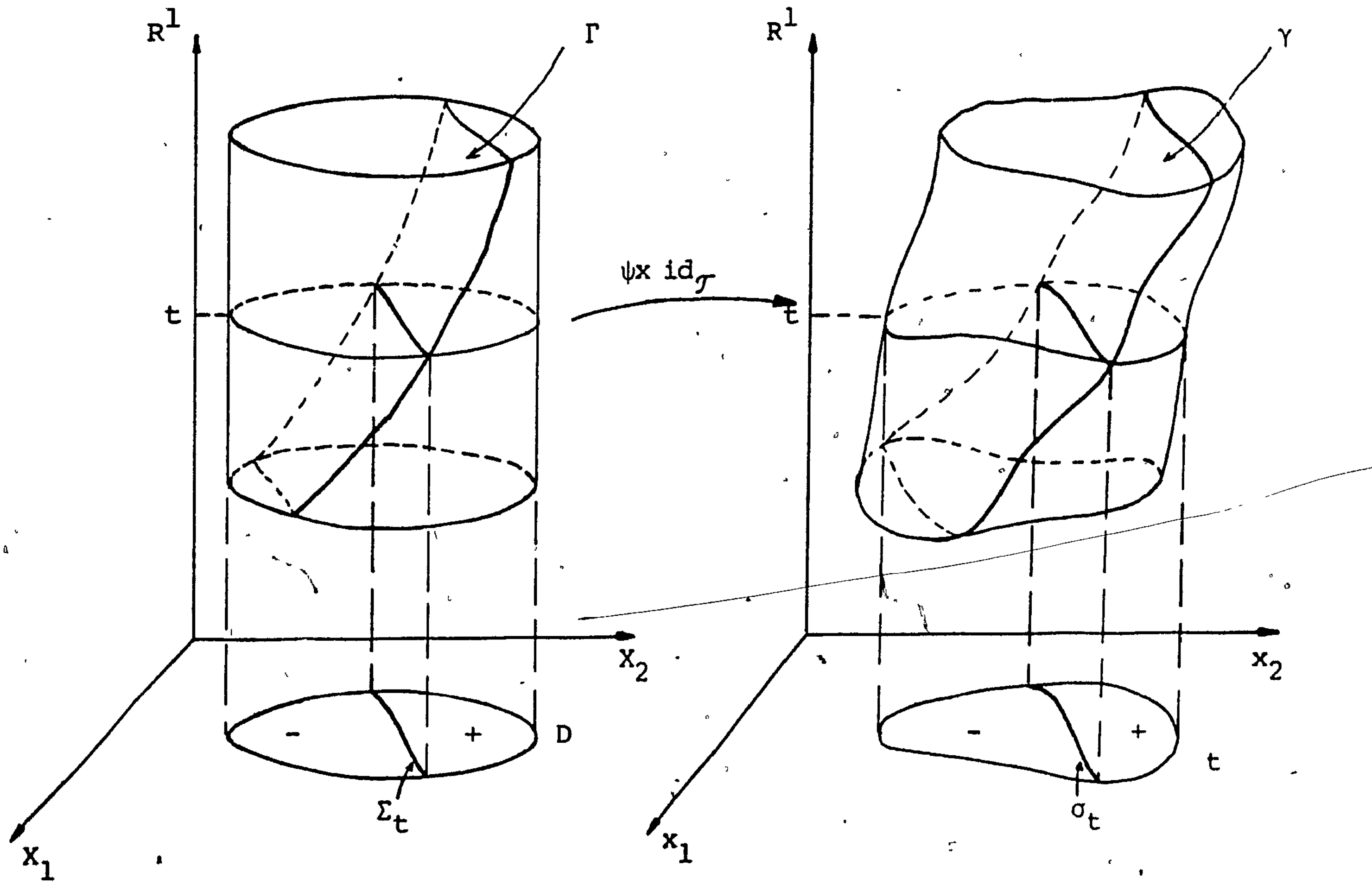
The wave σ_t (or Σ_t) of first order is called a shock wave. The vector field s_i defined on σ_t by

$$s_i = [\![\dot{\psi}_i]\!] \tag{10.4}$$

is called an amplitude of the shock wave. The wave σ_t (or Σ_t) of second order is called an acceleration wave. The amplitude of an acceleration wave is defined by

$$a_i = [\![\ddot{\psi}_i]\!] \tag{10.5}$$

where $\ddot{\psi}_i \equiv \partial^2\psi_i(X,t)/\partial t^2$ is the acceleration of the mass point X at time t. In general, the amplitude of a wave of order k is given by



R^1
Γ
t
X_2
-
+
D
Σ_t
X_1
ψx id$_{\mathcal{T}}$
R^1
γ
t
x_2
-
+
t
σ_t
x_1

Figure 6

$[\![\psi_i^{(k)}]\!]$ where $\psi_i^{(k)} \equiv \partial^k\psi_i(X,t)/\partial t^k$. From the kinematical conditions of compatibility of order k (for k > 3 such conditions can be obtained from those of order 1 and 2 by the iteration process) we have

$$[\![\psi_i^{(k)}]\!] = (-U_{(N)})^k [\![\partial^k_{(N)} \ \psi_i]\!] \tag{10.6}$$

where

$$[\![\partial^k_{(N)} \ \psi_i]\!] \equiv [\![N_{A_1} \ldots N_{A_k} \ \psi_{i,A_1 \ldots A_k}]\!]$$

in which N_{A_i} is a component of the normal vector N on Σ_t.

In many practical cases deformations of elastic bodies are "small". A useful quantity in describing such deformations is the displacement vector which gives the change in position of a mass point, namely its position in the configuration at time t minus its position in the reference configuration. The mathematical definition of the components of a displacement vector must involve the relation of the material and spatial coordinates. We assume that the material coordinates of an N-dimensional body are identical with the first N^{th} coordinates of the spatial coordinate system. The displacement vector is defined by

$$x = (\psi_r(X_s,t), \ \psi_\ell \ (X_s,t))$$

$$= (X_r,0) + (u_r(X_s,t), \ u_\ell(X_s,t)) \tag{10.7}$$

$$(r,s=1,\ldots,N;\ell = N+1,\ldots,3)$$

By a "small" deformation we understand a deformation such that, for each t, $\psi_r(X_s,t)$ prescribes a homeomorphism $\psi^I = (\psi_r,0)$ between B and $\psi^I(B,t)$ and the mapping $\psi^{II} = (0,\psi_\ell)$ is an "infinitesimal" displacement. For a "small" deformation, the spatial representation of a (material) singular hypersurface Σ_t is defined by

$$\sigma_t = \{x_r : x_r = \psi_r(X_s,t),\ X_s \in \Sigma_t\}$$

$$(r,s=1,\ldots,N)$$

i.e. $\sigma_t = \psi_t^I(\Sigma_t)$. From (10.7) we have components of the displacement gradient given by

$$\left.\begin{aligned} u_{r,s} &= \psi_{r,s} - \delta_{rs} \\ u_{\ell,s} &= \psi_{\ell,s} \end{aligned}\right\} \quad (r,s=1,\ldots,N;\ \ell = N+1,\ldots,3) \qquad (10.8)$$

where δ_{rs} is the Kronecker delta, and also that the material time derivative of displacement u is equal to the velocity $\dot{\psi}$. It should be pointed out now, that for N=2 the above description of a deformation will be used in the linear theory of elastic plates in the framework of which we shall investigate propagating discontinuities.

A function $F(X,t)$, $X \in B$ is referred to as a function given in material description. Its spatial description, when the arguments of F are transformed to the position $x_r = \psi_r(X,t)$, $(r=1,\ldots,N)$ is denoted by f and is given by

$$f(x_r,t) = F(X_r,t) \qquad (10.9)$$

The connection between material and spatial descriptions is important because final equations are often expressed in spatial coordinates, since they correspond to a laboratory coordinate system. In other words, most measurements (but not all, for example, in a simple tension experiment) are made relative to spatial position in the laboratory rather than to the material position fixed to and moving with the deformed body. The transformation between the material and spatial descriptions of various tensors of the elasticity theories will be given in the next chapter. Now, we present such a transformation connecting material and spatial representations of the singular hypersurfaces Σ_t and σ_t, respectively.

If the singular hypersurface σ_t is represented by

$$\phi(x_r, t) = 0 \qquad (r=1,\ldots,N) \tag{10.10}$$

then for each t, the material description of this hypersurface is given by

$$\Phi(X_r, t) \equiv \phi(\psi_r(X_s, t), t) = 0 \qquad (r,s=1,\ldots,N) \tag{10.11}$$

Let $d\Sigma_{N-1}$ and $d\sigma_{N-1}$ denote the (induced) Euclidean measure elements on Σ_t and σ_t, respectively. Their definitions, using the notion of differential forms, will be presented in Appendix III. Here, let us note that for N=3, $d\Sigma_2$ and $d\sigma_2$ are the Euclidean area elements dA and da and for N=2 $d\Sigma_1$, $d\sigma_1$ are the arc length elements dL and $d\ell$, in the material and spatial descriptions, respectively. The unit normal vectors on Σ_t and σ_t are given by the following formulae

$$\mathbf{N} = \frac{\text{Grad } \Phi}{|\text{Grad } \Phi|}, \qquad \mathbf{n} = \frac{\text{grad } \phi}{|\text{grad } \phi|} \tag{10.12}$$

We shall prove in Appendix III the following relation

$$N_r d\Sigma_{N-1} = J^{-1} n_s \psi_{s,r} d\sigma_{N-1} \qquad (r,s=1,\ldots,N;\ N=2 \text{ or } 3) \tag{10.13}$$

where $J = \det(\psi_{r,s})$ is the Jacobian of the mapping $x_r = \psi_r(X_s, t)$. For N=3 (10.13) is Nanson's formula (see [12], for example) which relates area elements in the reference configuration and deformed configurations. Now, we shall prove that the formula (10.13) is in fact well defined for discontinuous motions. From the well known Euler-Piola-Jacobi identity (cf. [12])

$$(J^{-1}\,\psi_{r,s})_{,r} = 0 \tag{10.14}$$

we have that

$$(J^{-1}\,n_r\,\psi_{r,s})^+ = (J^{-1}\,n_r\,\psi_{r,s})^- \tag{10.15}$$

on σ_t, which justifies (10.13).

The speed of propagation in the material description is given by

$$U_{(N)} = -\frac{\partial\Phi/\partial t}{|\mathrm{Grad}\,\Phi|} \tag{10.16}$$

and its spatial version, which is denoted by $c_{(n)}$ is given by

$$c_{(n)} = -\frac{\partial\phi/\partial t}{|\mathrm{grad}\,\phi|} \tag{10.17}$$

From (10.12), (10.13), (10.16) and (10.17) we obtain

$$U_{(N)}\,d\Sigma_{N-1} = J^{-1}\,(c_{(n)} - \dot{\psi}_r n^r)\,d\sigma_{N-1} \tag{10.18}$$

Now from (10.13) and (10.18), it follows that

$$[J^{-1}(c_{(n)} - \dot{\psi}_r\,n^r)]^+ = [J^{-1}\,(c_{(n)} - \dot{\psi}_r\,n^r)]^- \tag{10.19}$$

(it can also be proved directly). The quantity

$$\begin{aligned} U &\doteq c_{(n)} - \dot{\psi}_r\,n^r \\ &= c_{(n)} - \dot{u}_r\,n^r \end{aligned} \tag{10.20}$$

is called the local speed of propagation. It gives a measure of the normal speed of the hypersurface σ_t with respect to the material points that are instantaneously situated upon this hypersurface (cf. also Chen [61]). It should be noted that the above transformation formulae have been established for singular hypersurfaces including also shock waves. If a singular hypersurface is an acceleration wave or higher order wave than all quantities such as the local speed of propagation, the Jacobian J, are defined uniquely on σ_t.

Finally, let us stress that the above treatment of the dualism mentioned at the beginning of this chapter, in the case of discontinuous fields has to be elaborated in the language of tensor analysis on a manifold. It is a beautiful, challenging problem.

CHAPTER 11

BALANCE LAWS (CONSERVATION THEOREMS) FOR DISCONTINUOUS MOTIONS IN 3-DIMENSIONAL THEORIES OF ELASTICITY

The dynamical laws of physics are unchanged in form by several transformations, called symmetry transformations, in time and space. A symmetry transformation that consists of a simple displacement in time, $t \rightarrow t + c$ leads to no change in the basic physical laws, since time is homogeneous. Similarly, the homogeneity of space leads to the laws of physics being invariant in form to displacement in space $x = x + a$, while the isotropy of space leads to the physical laws being invariant in form to rotations in space $x \rightarrow \mathbf{R} \cdot x$. All of the symmetry transformations mentioned above are members of the Euclidean group of transformations and they form a basis of pre-relativistic mechanical theories.

In continuum mechanics the invariance properties of the laws of elasticity are expressed by the postulate that the (elastic) action integral is invariant under the group of Euclidean transformations in $E^3 \times R^1$

$$\bar{x} = \mathbf{R} \cdot x + a \qquad (11.1)$$

$$\bar{t} = t + c$$

where $\mathbf{R}$ is a constant orthogonal tensor (rotation in E^3), $\mathbf{a}$ is a constant vector (displacement in E^3) and c is a constant (displacement in R^1).

It will suffice to consider only infinitesimal transformations (cf. Toupin [3]) of the group (11.1) given by

$$\bar{x}_i = x_i + (\varepsilon_{ij}\, x^i + a_i)s$$
$$\bar{t} = t + cs \tag{11.2}$$

where s is an infinitesimal parameter say $s\varepsilon(-\varepsilon,\varepsilon)$, and $\varepsilon > 0$ a small number. Also in (11.2) a_i and c are arbitrary constants and s ε_{ij} are the components of the infinitesimal rigid rotation tensor **R-I**, i.e. except for the antisymmetry conditions $\varepsilon_{ij} = -\varepsilon_{ji}$, they are arbitrary constants.

Under the infinitesimal transformation (11.2) the motion $\psi_i(X_A,t)$ (i=1,2,3; A=1,2,3) is mapped to a one-parameter family of motions

$$(-\varepsilon,\varepsilon) \ni s \mapsto \begin{cases} \psi_i(X_A, t(s), s) = \psi_i(X_A,t) + (\varepsilon_{ij}\, \psi^i(X_A,t) + a_i)s \\ t(s) = t + cs \end{cases} \tag{11.3}$$

Assuming that a singular surface (a wave) Σ_t is propagating through a material we obtain from Lemma 7.1 the following relations

$$U_{(N)}\, c + \hat{\delta}\Sigma = 0 \tag{11.4}$$

$$\frac{\delta\psi_i}{\delta t}\, c + \hat{\delta}\,\psi_i = \varepsilon_{ij}\, \psi^j \tag{11.5}$$

where $U_{(N)}$ is the speed of propagation, $\hat{\delta}\Sigma$ is the normal variaton of Σ_t and $\hat{\delta}\,\psi_i$ is the displacement variation, both of them induced by the group of Eucidean transformations.

11.1 Balance Laws for Simple Elastic Materials. Material Description

The action density function L for a simple hyperelastic material (see Toupin [3] for this terminology) is given by

$$L = \frac{1}{2}\rho_o \dot{\psi}^2 - W(X_A, \psi_{i,A}) \qquad (11.6)$$

where $\psi_i(X_A,t)$ (i=1,2,3, ; A = 1,2,3) is a motion.

In (11.6)

$$\frac{1}{2}\rho_o \dot{\psi}^2$$

is the kinetic energy per unit volume in the reference configuration, and

$$W(X_A, \psi_{i,A})$$

is the energy of deformation per unit volume in the reference configuration. First, let us note that (7.24) now is given by

$$C_i = \varepsilon_{ij}\psi^j + a_i - \dot{\psi}_i c . \qquad (11.7)$$

Then assuming that a singular surface (a wave) Σ_t is propagating through a material during the time interval, we obtain from conservation theorem (Theorem 8.1) written in the form (8.2) that the following integral identities (recall that $\varepsilon_{ij} = -\varepsilon_{ji}$, a_i and c are arbitrary constants as they have been considered above),

$$\int_D P^i \, dV \Big|_{t_1}^{t_2} = \int_{\mathcal{T}} \int_{\partial D} T^{iA} N_a \, dA \, dt \qquad (11.8)$$

$$\int_D P^{[i} \psi^{j]} \Big|_{t_1}^{t_2} = \int_{\mathcal{T}} \int_{\partial D} T^{[iA} N_A \psi^{j]} \, dA \, dt \qquad (11.9)$$

$$\int_D (P^i \dot{\psi}_i - L) \, dV \Big|_{t_1}^{t_2} = \int_{\mathcal{T}} \int_{\partial D} T^{iA} N_A \dot{\psi}_i \, dA \, dt \qquad (11.10)$$

hold true for an arbitrary sub-body $D \subset B$. In the above identities

$$T^{iA} = \frac{\partial W}{\partial \psi_{i,A}} \tag{11.11}$$

is the Piola-Kirchhoff stress tensor

$$P^{i} = \frac{\partial L}{\partial \dot{\psi}_{i}} = \rho_{o}\dot{\psi}_{i} \tag{11.12}$$

is the density of linear momentum,

$$M^{ij} = P^{[i}\,\psi^{j]} = \rho_{o}\dot{\psi}^{[i}\,\psi^{j]} \tag{11.13}$$

is the density of angular momentum (with respect to the origin of E^3) and

$$E = P_{i}\,\dot{\psi}^{i} - L = \frac{1}{2}\,\rho_{o}\,\dot{\psi}^{2} + W \tag{11.14}$$

is the density of energy. In (11.9) the brackets enclosing indicies i and j indicate antisymmetrization with respect to these indicies. The equations (11.8) - (11.10) are balance laws for P^i, M^{ij} and E, respectively in an arbitrary domain of the body through which a singular surface may be propagating.

For simple hyperelastic materials the jump conditions across the wave Σ_t, obtained in Theorem 6.1, take the following form

$$U_{(N)}\,[\![P^{i}]\!] = -\,[\![T^{iA}N_{A}]\!] \tag{11.15}$$

$$U_{(N)}[\![E]\!] = -\,[\![T^{iA}N_{A}\dot{\psi}_{i}]\!] \tag{11.16}$$

where T^{iA}, P^i and E are given by (11.11), (11.12) and (11.14), respectively. The jump conditions (11.15) and (11.16) are well-known dynamical conditions of compatibility for hyperelastic materials, i.e. they are equations for jumps of the momentum and energy densities across a wave. The conditions (11.15) and (11.16) imply the following balance laws

$$\int_{\Sigma_t^*} [\![P^i]\!] \, \overset{\circ}{U}_{(N)} \, dA = - \int_{\Sigma_t^*} [\![T^{iA} N_A]\!] \, dA \tag{11.17}$$

$$\int_{\Sigma_t^*} [\![E]\!] \, U_{(N)} \, dA = - \int_{\Sigma_t^*} [\![T^{iA} N_A \dot{\psi}_i]\!] \, dA \tag{11.18}$$

for an arbitrary part Σ_t^* of the singular surface Σ_t. They can be derived also by applying the Kotchine theorem (for this theorem see Wang and Truesdell [17]) to balance laws (11.8) and (11.10).

11.2 Balance Laws for Simple Elastic Materials. Spatial Description

In this section we give the spatial description for balance laws (11.8) - (11.10) and (11.17) - (11.18). Because for hyperelastic materials it is a standard procedure (see Eringen and Suhubi [62], for example) we shall only state the final results.

The balance laws (11.8) - (11.10) have the following spatial representation

$$\int_{B_t} p^i \, dv \Big|_{t_1}^{t_2} = \int_{T} \int_{\partial B_t} t^{ij} \, n_j \, da \, dt \tag{11.19}$$

$$\int_{B_t} p^{[i} \psi^{j]} \, dv \Big|_{t_1}^{t_2} = \int_{T} \int_{\partial B_t} t^{[ik} n_k \psi^{j]} \, da \, dt \tag{11.20}$$

$$\int_{B_t} e \, dv \Big|_{t_1}^{t_2} = \int_{T} \int_{\partial B_t} t^{ik} n_k \dot{\psi}_i \, da \, dt \tag{11.21}$$

In (11.19) - (11.21), $\mathcal{D}_t = \psi_t(D)$, $D \subset B$, n_k are components of the unit normal vector $\mathbf{n}$ on $\partial \mathcal{D}_t$,

$$t^{ij} = J^{-1} T^{iA} \psi^j_{,A} \tag{11.22}$$

is the Cauchy stress tensor,

$$p^i = J^{-1} P^i \tag{11.23}$$

is the spatial density of linear momentum and

$$\begin{aligned} e &= J^{-1} E \\ &= \frac{1}{2} \rho \dot{\psi}^2 + J^{-1} W \end{aligned} \tag{11.24}$$

is the spatial density of energy in which $\rho = J^{-1} \rho_o$ is the mass density per unit volume in E^3. To obtain (11.19) - (11.21) from the balance laws in the material description given in (11.8) - (11.10) it is enough to note that $dv = J\, dV$ and that

$$t^{ik} n_k \, da = T^{iA} N_A \, dA \tag{11.25}$$

which follows from (10.13) and (11.22).

On substituting (11.22), (11.23), (11.24), (11.25) and (10.18) into (11.17) and (11.18) we obtain

$$\int_{\sigma_t^*} [\![p^i U]\!] \, da = - \int_{\sigma_t^*} [\![t^{ij} n_j]\!] \, da \tag{11.26}$$

$$\int_{\sigma_t^*} [\![e\, U]\!] \, da = - \int_{\sigma_t^*} [\![t^{ij} n_j \dot{\psi}_i]\!] \, da \tag{11.27}$$

where $\sigma_t^* = \psi_t(\Sigma_t^*)$. The identities (11.26) and (11.27) express the corresponding balance laws in the spatial description.

11.3 Balance Laws for Elastic Material of Grade 2. Material Description

We consider elastic materials of grade 2 with dynamical and structural properties defined by the following Lagrange density function (for a nonlinear static theory of elastic materials of grade 2 we refer to Toupin [3]).

$$L = \frac{1}{2}\rho_0 \dot{\psi}^2 + \frac{1}{2} I^{AB} \dot{\psi}^i_{,A} \dot{\psi}_{i,B} - W(X_A, \psi_{i,A}, \psi_{i,AB}) \tag{11.28}$$

where I^{AB} is a positive symmetric tensor called the rotational inertia tensor, and W is the density of elastic deformation. In this case

$$T^{iA} = \frac{\partial W}{\partial \psi_{i,A}} - \left(\frac{\partial W}{\partial \psi_{i,AB}}\right)_{,B} + I^{AB} \dot{\psi}^i_{,B} \tag{11.29}$$

is the generalized Piola-Kirchhoff stress tensor and

$$H^{iAB} = \frac{\partial W}{\partial \psi_{i,AB}} \tag{11.30}$$

is the hyperstress tensor.

Now, the identity (8.2) with C_i given by (11.7) (this identity has been derived in the conservation theorem, i.e. Theorem 8.1) implies the following identities

$$\int_D P^i \, dV \Big|_{t_1}^{t_2} = \int_{\mathcal{T}} \int_{\partial D} T^i \, dA \, dt + \int_{\mathcal{T}} \int_{C_t^*} \{[\![K^i]\!] + [\![\pi^{iA} \check{M}_A U_{(N)}]\!]\} \, dL \, dt \tag{11.31}$$

$$\int_D (P^{[i}\psi^{j]} + \pi^{[iA}\psi^{j]}_{,A}) \, dV \Big|_{t_1}^{t_2} = \int_{\mathcal{T}} \int_{\partial D} (T^{[i}\psi^{j]} + H^{[i}\partial_{(N)}\psi^{j]}) \, dA \, dt$$

$$+ \int_{\mathcal{T}} \int_{C_t^*} [\![K^{[i} + \pi^{[iA} \check{M}_A U_{(N)}]\!] \psi^{j]} \, dL dt \tag{11.32}$$

$$\int_D (P^i\dot{\psi}_i + \pi^{iA}\dot{\psi}_{i,A} - L)\, dA \Big|_{t_1}^{t_2} = \int_{\mathcal{T}} \int_{\partial D} (T^i\dot{\psi}_i + H^i\partial_{(N)}\dot{\psi}_i)\, dA\, dt$$

$$+ \int_{\mathcal{T}} \int_{C_t^*} [\![K^i\dot{\psi}_i + \pi^{iA} \overset{\vee}{M}_A U_{(N)}\dot{\psi}_i]\!]\, dL\, dt \qquad (11.33)$$

where D is an arbitrary sub-body and

$$T^i = T^{iA}N_A - H^i\Omega - D_A\,(H^{iAB}N_B) \qquad (11.34)$$

is the generalized traction,

$$H^i = H^{iAB}N_A N_B \qquad (11.35)$$

is the hypertraction, both are defined on $\partial D_t = \psi_t(\partial D)$ and are related to the geometry of ∂D (i.e. they are given in the material description),

$$K^i = H^{iAB}N_B^{\partial D}\hat{M}_A + H^{iAB}N_B^{\Sigma}\overset{\vee}{M}_A$$

$$= K^i_{\partial D} + K^i_{\Sigma} \qquad (11.36)$$

is the line force on $c_t^* = \psi_t(C_t^*)$ and is related to $C_t^* = \partial(\Sigma_t \cap \partial D) \subset D$. In the above we have also that

$$\pi^{iA} = \frac{\partial L}{\partial \dot{\psi}_{i,A}} = I^{AB}\dot{\psi}^i_{,B} \qquad (11.37)$$

The identities (11.31) – (11.33) are balance laws for the considered material. Thus identifying $P^i = \frac{1}{2}\rho_0\dot{\psi}^i$ as the density of linear momentum, $M^{ij} = P^{[i}\psi^{j]} + \pi^{[iA}\psi^{j]}_{,A}$ as the density of angular momentum (with respect to the origin in E^3) and

$$E = P^i\dot{\psi}_i + \pi^{iA}\dot{\psi}_{i,A} - L$$

$$= \frac{1}{2}\rho_0\dot{\psi}^2 + \frac{1}{2} I^{AB}\dot{\psi}^i_{iA}\dot{\psi}_{i,B} + W \qquad (11.38)$$

as the density of energy, all of them in the material description. The identities (11.31) - (11.33) express the laws of conservation of the (material) linear momentum, angular momentum and energy, respectively for an arbitrary sub-body D through which the singular surface Σ_t is propagating. It is easy to verify, using (11.36), (11.34) and (6.24) that the balance laws (11.31) - (11.33) can be written in the following equivalent form

$$\int_D P^i \, dV \Big|_{t_1}^{t_2} = \int_{\mathcal{T}} \int_{\partial D} T^{iA} N_A \, dA \, dt + \int_{\mathcal{T}} \int_{C_t^*} [\![K_\Sigma^i + \pi^{iA} \check{M}_A U_{(N)}]\!] \, dL \, dt \tag{11.39}$$

$$\int_D M^{ij} \, dV \Big|_{t_1}^{t_2} = \int_{\mathcal{T}} \int_{\partial D} (T^{[iA} N_A \psi^{j]} + H^{[iAB} N_B \psi^{j]}_{,A}) \, dA \, dt$$

$$+ \int_{\mathcal{T}} \int_{C_t^*} [\![K_\Sigma^{[i} + \pi^{[iA} \check{M}_A U_{(N)}]\!] \psi^{j]} \, dL \, dt \tag{11.40}$$

$$\int_D E \, dV \Big|_{t_1}^{t_2} = \int_{\mathcal{T}} \int_{\partial D} (T^{iA} N_A \dot{\psi}_i + H^{iAB} N_B \dot{\psi}_{i,A}) \, dA \, dt$$

$$+ \int_{\mathcal{T}} \int_{C_t^*} [\![K_\Sigma^i \dot{\psi}_i + \pi^{iA} \check{M}_A U_{(N)} \dot{\psi}_i]\!] \, dL \, dt \tag{11.41}$$

where

$$K_\Sigma^i = H^{iAB} N_B^\Sigma \check{M}_A$$

is a part of the (total) line force K^i which is related only to the geometry of the singular surface Σ_t. The remaining part $K_{\partial D}^i$ of K^i is related only to the geometry of ∂D.

The complementary balance laws on the singular surface Σ_t, for elastic materials with the Lagrangian density function (11.28) are obtained from (6.44) and (6.45) as follows

$$\int_{\Sigma_t^*} U_{(N)} [\![P^i]\!] \, dA = - \int_{\Sigma_t^*} [\![T^{iA} N_A]\!] dA + \int_{C_t^*} [\![K_\Sigma^i + \pi^{iA} \check{M}_A U_{(N)}]\!] \, dL \tag{11.42}$$

$$\int_{\Sigma_t^*} U_{(N)} \, [\![E]\!] \, dA = - \int_{\Sigma_t^*} [\![T^{iA} N_A \dot{\psi}_i + H^{iAB} N_B \dot{\psi}_{i,A}]\!] \, dA$$

$$+ \int_{C_t^*} [\![K_\Sigma^i \dot{\psi}_i + \pi^{iA} \check{M}_A U_{(N)} \dot{\psi}_i]\!] \, dL \tag{11.43}$$

where $\Sigma_t^* = \Sigma_t \cap D$ and $C_t^* = \partial(\Sigma_t \cap D) = \partial\Sigma_t^*$.

The physical meaning of the identities (11.42) and (11.43), which are implied directly from the jump conditions derived in Theorem 6.1, is that these identities express the conservation of linear momentum and energy on a singular surface in the considered material. Thus, by applying the Kotchin theorem to balance laws (11.39) and (11.41) one can obtain the identities (11.42) and (11.43).

11.4 Balance Laws for Elastic Materials of Grade 2. Spatial Description

For elastic materials of grade 2, following Toupin [3] we let

$$t^{ij} = J^{-1} T^{iA} \psi^j_{,A} \tag{11.44}$$

$$h^{jik} = J^{-1} H^{iAB} \psi^j_{,A} \psi^k_{,B} \tag{11.45}$$

be the generalized Cauchy stress tensor and the hyperstress tensor, respectively. Let us note that in (11.44) and (11.45) T^{iA} and H^{iAB} are given by (11.29) and (11.30) respectively.

On substituting (10.13) into (11.44) and (11.45) we obtain

$$t^{ij} n_j \, da = T^{iA} N_A \, dA \tag{11.46}$$

$$h^{ji}\,da = h^{jik}n_k\,da$$

$$= H^{iAB}N_B\psi^j_{,A}\,dA \qquad (11.47)$$

Now, let us define p^i, π^{ij} and e by

$$p^i = J^{-1}P^i \qquad (11.48)$$

$$\pi^{ij} = J^{-1}\pi^{iA}\psi^j_{,A} \qquad (11.49)$$

$$e = J^{-1}\,E \qquad (11.50)$$

Then, combining (11.48), (11.49), (11.50), and the definitions of P^i, M^{ij} and E given in the balance laws for elastic materials of grade 2 in the material description, and also recalling that $dv = J\,dV$ we obtain .

$$p^i dv = P^i\,dV \qquad (11.51)$$

$$m^{ij}\,dv = M^{ij}\,dV \qquad (11.52)$$

$$e\,dv = E\,dV \qquad (11.53)$$

where

$$m^{ij} = p^{[i}\psi^{j]} + \pi^{[ij]} \qquad (11.54)$$

and

$$e = p^i\dot{\psi}_i + \pi^{ij}(\dot{\psi}_i)_{,j} - J^{-1}\,L \qquad (11.55)$$

Thus, from (11.52) and (11.53) we conclude that m^{ij} given by (11.54) and e given by (11.55) are the density of angular momentum and the density of energy expressed in spatial description.

In substituting (11.46), (11.47), (11.51), (11.52) and (11.53) into (11.39), (11.40) and (11.41) we obtain

$$\int_{\mathcal{D}_t} p^i\,dv\,\Big|_{t_1}^{t_2} = \int_{\mathcal{T}}\int_{\partial\mathcal{D}_t} t^{ij}n_j\,da\,dt + \int\int_{\mathcal{T}\,\mathcal{C}_t} [\![\bar{k}^i]\!]d\ell\;\;dt \qquad (11.56)$$

$$\int_{D_t} m^{ij}\, dv \Big|_{t_1}^{t_2} = \int_{\mathcal{T}} \int_{\partial D_t} (t^{[ik} n_k \psi^{j]} + h^{[ji]})\, da\, dt$$

$$+ \int_{\mathcal{T}} \int_{c_t} [\![\bar{k}^{[i}]\!] \psi^{j]}\, d\ell\, dt \qquad (11.57)$$

$$\int_{D_t} e\, dv \Big|_{t_1}^{t_2} = \int_{\mathcal{T}} \int_{\partial D_t} (t^{ij} n_j \dot{\psi}_i + h^{ji} (\dot{\psi}_i)_{,j})\, da\, dt$$

$$+ \int_{\mathcal{T}} \int_{c_t} [\![\bar{k}^i \dot{\psi}_i]\!]\, d\ell\, dt \qquad (11.58)$$

where $c_t = \partial(\sigma_t \cap D_t)$, $D_t = \psi_t(D)$ and

$$\bar{k}^i\, d\ell = K^i_\Sigma\, dL + \pi^{iA} \check{M}_A U_{(N)}\, dL$$

$$= k^i_\sigma\, d\ell + \pi^i U\, d\ell \qquad (11.59)$$

where k^i_σ is the line force on c_t and $\pi^i U$ is the "generalized" force on c_t which will be discussed later (U is the local speed of propagation defined in (10.20)). The identities (11.56), (11.57) and (11.58) are balance laws for the linear momentum, angular momentum and energy, respectively, in the spatial description.

The equivalent form of the balance laws in the spatial description can be obtained in the following way.
First, let us define the generalized traction t^i as follows

$$t^i = t^{ij} n_j - d_j n^{ji} - h^i b \qquad (11.60)$$

where

$$h^i = h^{ji} n_j \qquad (11.61)$$

and

$$d_j h^{ji} = (\delta^k_j - n_j n^k)\, h^{ji}_{,k} \qquad (11.62)$$

is the surface divergence of h^{ji} and $b = tr(b_{k\ell})$ in which $b_{k\ell} = - d_k n_\ell = - d_\ell n_k$ are the components of the second fundamental form of the surface.

On introducing (11.60) into (11.56), (11.57) and (11.58), after integration by parts by means of the formula (6.25) (in spatial representation) we derive

$$\int_{\mathcal{D}_t} p^i \, dv \Big|_{t_1}^{t_2} = \int_{\mathcal{T}} \int_{\partial\mathcal{D}_t} t^i \, da \, dt + \int_{\mathcal{T}} \int_{c_t} [\![k^i]\!] \, d\ell \, dt \tag{11.63}$$

$$\int_{\mathcal{D}_t} m^{ij} \, dv \Big|_{t_1}^{t_2} = \int_{\mathcal{T}} \int_{\partial\mathcal{D}_t} (t^{[i}\psi^{j]} + h^{[ji]}) \, da \, dt \; .$$

$$+ \int_{\mathcal{T}} \int_{c_t} [\![k^{[i}]\!]\psi^{j]} \, d\ell \, dt \tag{11.64}$$

$$\int_{\mathcal{D}_t} e \, dv \Big|_{t_1}^{t_2} = \int_{\mathcal{T}} \int_{\partial\mathcal{D}_t} (t^i\psi_i + h^{ji}(\psi_i)_{;j}) \, da \, dt$$

$$+ \int_{\mathcal{T}} \int_{c_t} [\![k^i\psi_i]\!] \, d\ell \, dt \tag{11.65}$$

where

$$k^i = h^{ji}\hat{m}_j + \bar{k}^i \tag{11.66}$$

is the line force on c_t (on unit arc length of this curve) and $\hat{m}_i$ are the components of unit tangent vector on $\partial\mathcal{D}_t$ at a point of c_t which is also normal to c_t (compare the corresponding vector $\hat{M}$ on Fig. 3).

On introducting (10.18), (11.53), (11.46) (11.47), (11.59) and (11.53) into (11.42) and (11.43) we have

$$\int_{\sigma_t^*} [\![p^i U]\!] \, da = - \int_{\sigma_t^*} [\![t^{ij} n_j]\!] \, da + \int_{c_t^*} [\![\bar{k}^i]\!] \, d\ell \tag{11.67}$$

$$\int_{\sigma_t^*} [\![eU]\!] \, da = - \int_{\sigma_t^*} [\![t^{ij} n_j \dot{\psi}_i + h^{ji} (\dot{\psi}_i)_{,j}]\!] \, da$$

$$+ \int_{c_t^*} [\![\bar{k}^i \dot{\psi}_i]\!] \, d\ell \tag{11.68}$$

where $\bar{k}^i$ was defined in (11.59), as the complementary balance laws for the singular surface σ_t. Of course, these identities can be derived from the balance laws (11.56) and (11.58) by applying to them the Kotchin theorem.

To complete our discussion in this chapter we have to consider the quantity π^i defined in (11.59) by

$$\pi^i U \, d\ell = \pi^{iA} M_A^{\vee} U_{(N)} \, dL \tag{11.69}$$

First let us note that from the definition of the Jacobian we can obtain

$$\epsilon_{ABC} = J^{-1} \epsilon_{ijk} \psi_{i,A} \psi_{j,B} \psi_{k,C} \tag{11.70}$$

where ϵ_{ABC} and ϵ_{ijk} are the permutation symbols in the material and spatial description, respectively. Then, recalling that the vector $\mathbf{M}^{\vee}$ is defined by

$$M_A^{\vee} = (\boldsymbol{\Sigma} \times \mathbf{N}^{\Sigma})_A = \epsilon_{ABC} \Sigma_B N_C^{\Sigma} \tag{11.71}$$

where $\boldsymbol{\Sigma}$ is tangent unit vector to $C_t^* = \partial(\Sigma_t \cap D)$ we can write

$$\pi^{iA} M_A^{\vee} U_{(N)} \, dL = \pi^{iA} \epsilon_{ABC} \Sigma_B N_C^{\Sigma} \, U_{(N)} \, dL$$

$$= J^{-1} \pi^{iA} \epsilon_{jk\ell} \psi_{j,A} \psi_{k,B} \, \psi_{\ell,C} \Sigma_B N_C^{\Sigma} \, U_{(N)} \, dL$$

$$= \pi^{ij} \epsilon_{jk\ell} \psi_{k,B} \psi_{\ell,C} \Sigma_B N_C^{\Sigma} \, U_{(N)} \, dL \tag{11.72}$$

where in the last equality we have used (11.49).

Now, let us note that (cf. Eringen and Suhubi[62], for example)

$$U_{(N)} = J^{-1} \frac{da}{dA} U = C_{(n)}^{-1} U \tag{11.73}$$

where

$$C_{(n)}^{2} \equiv C_{k\ell}^{-1}\, n^k n^\ell$$

in which

$$C_{k\ell}^{-1} = \psi_{k,A}\psi_{\ell,A}$$

is the Finger deformation tensor, and that

$$\lambda\sigma_k = \psi_{k,A}\, \Sigma_A \tag{11.74}$$

where σ_k is tangent unit vector to $c_t^* = \psi_t(C_t^*)$ and λ defined by $\lambda dL = d\ell$ is the stretch.

From the above we can write

$$\psi_{\ell,C} N_C^{\Sigma} U_{(N)} = J^{-1}\, n_p \psi_{p,C} \frac{da}{dA} \psi_{\ell,C}\, U_{(N)}$$

$$= n_p C_{p\ell}^{-1}\, C_{(n)}^{-2}\, U \tag{11.75}$$

Finally, on substituting (11.75) and (11.74) into (11.72) we obtain

$$\pi^{iA} M_A U_{(N)}\ dL = \pi^{ij} \varepsilon_{jk\ell} \sigma_k e_\ell\, U\, d\ell \tag{11.76}$$

where $e_\ell = n_k C_{k\ell}^{-1}\, C_{(n)}^{-2}$. By comparing this with (11.69) we have that

$$\pi^{i} = \pi^{ij} \varepsilon_{jk\ell} \sigma_k e_\ell \tag{11.77}$$

This completes our discussion in this chapter.

CHAPTER 12

WAVE PROPAGATION IN ELASTIC PLATES

A plate is a 3-dimensional body with one dimension, the thickness, being much smaller than the other two. This fact is used to derive various approximate 2-dimensional theories of plates. It is sufficient for our purpose to mention two main approaches for constructing 2-dimensional equations of motion of thin elastic plates. The first one is the so-called direct approach for Cosserat plates. For this approach, we refer to the monograph by Naghdi [63] where a complete list of references for this subject is given. In this approach the dynamical properties of a plate are represented by assuming the form of a 2-dimensional Lagrangian . Its Lagrangian density function depends on the first order partial derivatives of the position vector of points on the middle surface of a plate (the displacement vector in the linear theory) and on the director field defined over this surface and its partial derivatives of the first order. Thus, this is a first order variational problem. For a concise exposition of this subject we refer to Ericksen [64], in which the problem of wave propagation in elastic shells is also treated.

In the second approach 2-dimensional equations of motion of thin elastic plates are derived from 3-dimensional linear elasticity using the methods of asymptotic expansion with the thickness of the plate as a small parameter. For this approach we refer to Ciarlet and

and Destuynder [65], Gusein-Zade [57] and Raoult [58], among others. The analysis of asymptotic expansions not only justifies various plate models, usually derived by employing a number of approximations or special assumptions (sometimes in an ad-hoc manner) but also gives the limit of applicability of 2-dimensional equations and the corresponding boundary and initial conditions together with sharp convergence estimates (c.f. [65] and [58]). A particular feature of this approach is that the resulting two-dimensional equations of motion are fourth or higher order differential equations. The inverse problem of the calculus of variations associates with these equations Lagrangian density functions which depend also on the second or higher order partial derivatives of motion (i.e. leads to a higher order variational problem).

Finally, let us mention also a (direct) nonlinear theory of elastic shells in which the strain measures depend on the first and second order deformation gradients. The static case of this theory, which is a two-dimensional analog of nonsimple elastic materials and in particular elastic materials of grade 2, has been developed in Cohen and DeSilva [66].

Originally, the theory of wave propagation on surfaces in which a wave curve corresponds to a moving wave front and is a carrier of discontinuities has been elaborated by Cohen and Suh [67]. This theory has been applied to the problems of wave propagation in membranes and shells by Cohen and Barkal in [68,69] and by Pop and Wang in [70]. The problems of wave propagation within the framework of a (direct) linear theory of elastic Cosserat plates was treated by Cohen [71], and in a nonlinear theory of Cosserat shells by Ericksen [64].

In this chapter we shall investigate the problems of wave propagation within the framework of a linear theory of elastic plates based upon the second approach, using results established in Part I of this thesis. From what we have said, it follows that this approach involves higher order variational problems and as such the variational theorems for discontinuous fields from Part I of this thesis can be applied to the investigation of wave curves in the considered plate model.

The flat plate is assumed to be a cylindrical body in R^3, denoted by $B \times [-h,h]$ (see Fig. 7). The thickness 2h of the plate is small compared with its two other dimensions. Following the results derived in Raoult [58] and Gusein-Zade [57] we consider the plate model given by the following 2-dimensional evolution equation for vertical vibrations

$$2\rho h \frac{\partial^2 w}{\partial t^2} + \frac{2}{3} \frac{E}{1-\nu^2} h^3 \Delta^2 w - \frac{34-14\nu}{15(1-\nu)} \rho h^3 \Delta \frac{\partial^2 w}{\partial t^2} = 0 \qquad (12.1)$$

where $w(X_1,X_2,t)$ is the vertical displacement of the middle surface of the plate, ρ is the density of the material ($2\rho h$ is the surface mass density), E is the Young's modulus and ν is the Poisson ratio. Also in (12.1) Δ is the 2-dimensional Laplace operator

$$\Delta = \frac{\partial^2}{\partial X_1^2} + \frac{\partial^2}{\partial X_2^2}$$

and Δ^2 is the 2-dimensional bi-harmonic operator

$$\Delta^2 = \frac{\partial^4}{\partial X_1^4} + 2 \frac{\partial^4}{\partial X_1^2 \partial X_2^2} + \frac{\partial^4}{\partial X_2^4}$$

The equation (12.1) describes linearized transverse vibrations of a plate for which the boundary conditions and the gravitational force have been neglected. If the last term in (12.1) is dropped then the

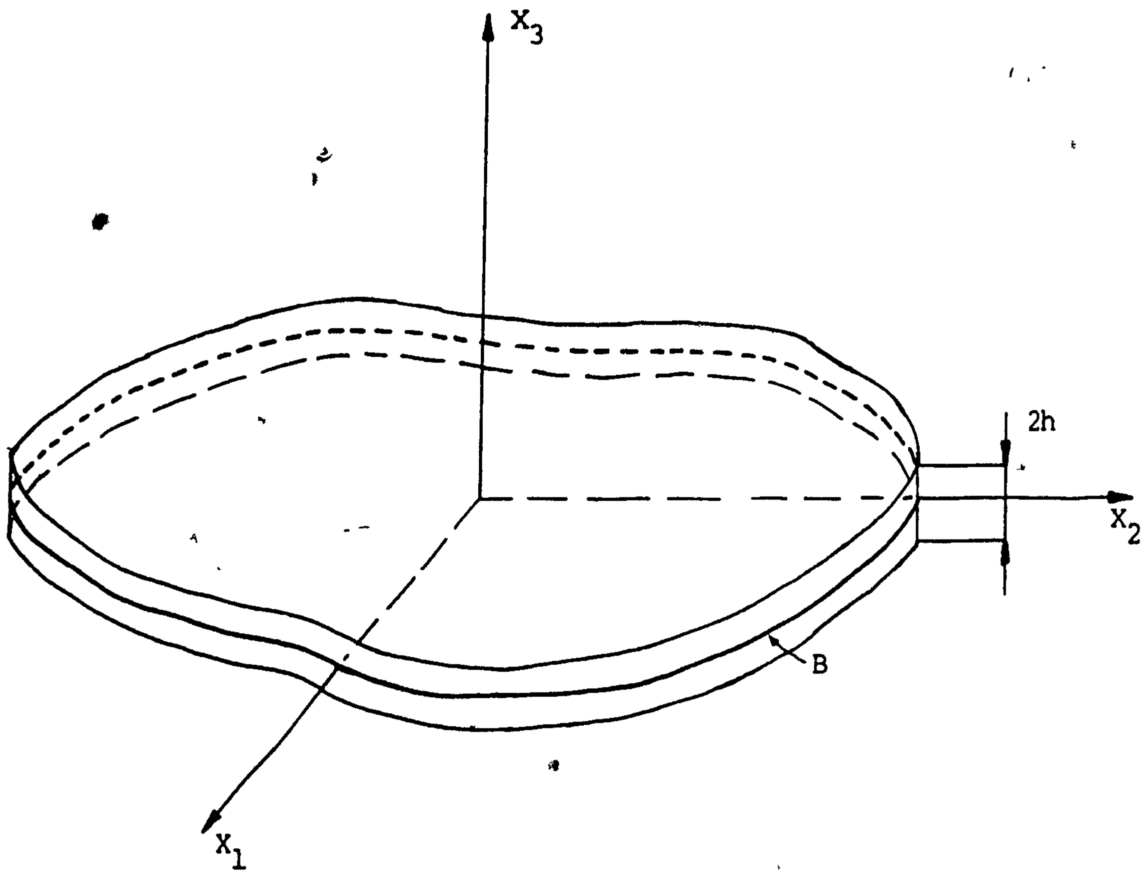


Figure 7

resulting equation is the classical bi-harmonic evolution equation. In Raoult [58] this equation has been obtained as a "0-order" asymptotic expansion and in Gusein-Zade [57] as an equation to an accuracy $0(\varepsilon^{2-\omega})$ where $\varepsilon = \frac{2h}{\ell}$ is the relative thickness with ℓ being the characteristic dimension of the middle surface of a plate and ω is the quantity which characterizes the variability of the state of deformation in time. The equation (12.1) is referred to as a "2-order" asymptotic expansion or as an equation to an accuracy $0(\varepsilon^{4-2\omega})$ in the references mentioned above.

Following [57], if the ω increases in $0(\varepsilon^{2-\omega})$ and $0(\varepsilon^{4-2\omega})$ the accuracy diminishes and for ω=2 the characteristic dimension of the deformation pattern for the plate becomes equal to h. This indicates the essentially 3-dimensional nature of the process.

As an additional observation, let us note that the coefficient

$$-\frac{34 - 14\nu}{15(1-\nu)}\rho h^3$$

has a negative value. Hence it implies that the fourth order differential terms in (12.1) form a "wave operator" for the quantity Δw, which as a second derivative of w is connected with the curvature of the deformed plate and is also proportional to the bending moment (it will be shown later).

We shall introduce the following notation

$$D = \frac{2}{3}\frac{E}{1-\nu^2}\ h^3$$

$$I = \frac{17 - 7\nu}{15(1-\nu)}\ h^2$$

into (12.1). Then the corresponding Euler operator is given by

$$E(L) = -\mu\ddot{w} + \mu I\Delta\ddot{w} - D\Delta^2 w \tag{12.2}$$

where $\mu = 2\rho h$ is the surface mass density and $\ddot{w} \equiv \frac{\partial^2 w}{\partial t^2}$.

The Lagrangian density function L, which has its Euler operator given by (12.2), has the most general form as follows:

$$L = \frac{1}{2}\mu\dot{w}^2 + \frac{1}{2}\mu I(\nabla\dot{w})^2 - \frac{1}{2}D(\Delta w)^2 - \operatorname{div} Q \tag{12.3}$$

where $Q = (Q_1, Q_2)$ is a two-dimensional vector field on the middle surface B of the plate. Let us note that div Q does not contribute to the Euler operator (12.2), however, it affects the boundary term of the variational problem defined by the Lagrangian density function (12.3).

Now, let us consider the static case of (12.3). In this case

$$-L = \frac{1}{2}D(\Delta w)^2 + \operatorname{div} Q \equiv W \tag{12.4}$$

is the energy of elastic deformation, denoted by W. We require that this W be a quadratic function in $w_{,rs}$ $(r,s = 1,2)$ (W does not involve $w_{,r}$ because no work is done in stretching the plate) which does not depend on the orientation of the coordinate system. Since the matrix

$$\begin{bmatrix} w_{,11} & , & w_{,12} \\ w_{,21} & , & w_{,22} \end{bmatrix}$$

has just two invariants under rotation, i.e. trace and its determinant, it follows that (cf. Gelfand and Fomin [14]) the density of elastic deformation has the form

$$W = \frac{1}{2}D(\Delta w)^2 - (1-\nu)\, D[w_{,11}w_{,22} - (w_{,12})^2] \tag{12.5}$$

It is easy to see that $w_{,11}w_{,22} - (w_{,12})^2$ is the divergence of the vector $(w_{,1}w_{,22}, - w_{,1}w_{,12})$ then from (12.4) and (12.5) we can conclude that the Lagrange density function (12.3) takes the form

$$L = \frac{1}{2}\mu\dot{w}^2 + \frac{1}{2}\mu I(\nabla\dot{w})^2 - \frac{1}{2}D(\Delta w)^2 + (1-\nu)D[w_{,11}w_{,22} - (w_{,12})^2] \quad (12.6)$$

and accordingly, the Lagrangian $\mathcal{L}$ has the following form

$$\mathcal{L} = \int_B L\, dX_1\, dX_2 \quad (12.7)$$

where L is given by (12.6) and B is the middle surface of the plate.

For the Lagrangian density function (12.6) the generalized Cauchy stress tensor t^{3r} and the hyperstress tensor h^{3rs} (compare (11.29) and (11.30)), respectively are given by

$$t^{3r} = \left(\frac{\partial L}{\partial w_{,rs}}\right)_{,s} + \overline{\frac{\partial L}{\partial \dot{w}_{,r}}}^{\bullet}$$

$$= - D(w_{,111} + w_{,122})\delta^{r1} - D(w_{,222} + w_{,121})\delta^{r2}$$

$$+ \mu I\, \ddot{w}_{,r} \quad (12.8)$$

and

$$h^{3rs} = - \frac{\partial L}{\partial w_{,rs}}$$

$$= D(w_{,11} + \nu w_{,22})\delta^{r1}\delta^{s1}$$

$$+ D(w_{,22} + \nu w_{,11})\delta^{r2}\delta^{s2}$$

$$+ (1-\nu)D\, w_{,12}\,(\delta^{r1}\delta^{s2} + \delta^{r2}\delta^{s1}) \quad (12.9)$$

Let us note that the generalized Cauchy stress tensor does not depend on the second invariant of the matrix $[w_{,rs}]$ and that the hyperstress tensor does. The corresponding tractions (cf. (11.34) and (11.35)) on the curve C, which can be the boundary ∂B or an internal curve, take the form

$$t^3 = -D(w_{,111} + w_{,122})n_1 - D(w_{,222} + w_{,121})n_2$$
$$+ \mu I\ddot{w}_{,r}n^r - h^3\Omega - Dd_1 \{(w_{,11} + \nu w_{,22})n_1$$
$$+ (1-\nu)w_{,12}n_2\} - Dd_2 \{(w_{,22} + \nu w_{,11})n_2$$
$$+ (1-\nu)w_{,12}n_1\} \tag{12.10}$$

and

$$h^3 = D(w_{,11} + \nu w_{,22})n_1^2 + D(w_{,22} + \nu w_{,11})n_2^2$$
$$+ 2(1-\nu)D\, w_{,12}n_1n_2 \tag{12.11}$$

where $d_r(\cdot) \equiv (\delta_r^s - n_rn^s)(\cdot)_{,s}$ is the tangential derivative on a curve C, n_r $(r=1,2)$ are the components of the unit normal vector in B to a curve and Ω is the curvature of this curve. The tractions (12.10) and (12.11) can be written in a more familiar form. To this end, first let us note that

$$d_r(\cdot) = m_rm^s(\cdot)_{,s} \equiv m_r\partial_{(m)}(\cdot) \tag{12.12}$$

where m_r $(r=1,2)$ are components of the left-oriented tangent vector to a curve and $\partial_{(m)}(\cdot) \equiv m^s(\cdot)_{,s}$. Then we can write

$$h^3\Omega + d_r(h^{3rs}n_s) = h^3\Omega + m_r\partial_{(m)}(h^{3rs}n_s)$$

$$= h^3\Omega + \partial_{(m)}(h^{3rs}m_rn_s) - h^{3rs}n_s\partial_{(m)}m_r$$

$$= h^3\Omega + \partial_{(m)}(h^{3rs}m_rn_s) - h^3\Omega$$

$$= \partial_{(m)}(h^{3rs}m_rn_s) \qquad (12.13)$$

For the third equality we have used the Frenet formula for curves (cf. [25], for example).

On substituting (12.13) with h^{3rs} given by (12.9) into (12.10) and after simple manipulations in (12.11) we obtain

$$t^3 = -D\partial_{(m)}(\Delta w) + \mu I\partial_{(m)}\ddot{w} + D(1-\nu)\partial_{(m)}[w_{,11}n_2m_2 - w_{,12}(m_2n_1 + m_2n_2) + w_{,22}n_1m_1] \qquad (12.14)$$

$$h^3 = D[\Delta w - (1-\nu)(w_{,11}n_2^2 - 2w_{,12}n_1n_2 + w_{,22}n_1^2)] \qquad (12.15)$$

which have the same form (except that in (12.14) we have also the term $\mu I\partial_{(m)}\ddot{w}$ related to the shear effect) as those given in Duvaut and Lions [72] (eq. (2.54) on p. 206) and in Gelfand and Fomin [14] (eqs. (61) and (62) on p. 166).

Now, we shall investigate the problems of wave propagation in the plate model we have considered above. Recall that a wave curve Σ_t in the middle surface B of the plate corresponds to a moving wave front and is a carrier of simple discontinuities in the partial derivatives of the vertical displacement $w(X_1, X_2, t)$.

The order of the wave corresponds to the lowest order derivative of w with respect to time which is discontinuous across the wave curve Σ_t. It is very important to point out clearly that the wave front propagating through B represents a two-dimensional process within the framework of a two-dimensional plate model as we have described this model in the first part of this chapter. From the previous discussion of the accuracy and the limits of applicability of this model derived from the asymptotic expansion of three-dimensional plate equations and from the fact that a wave curve is a mathematical idealization of a domain in B of finite area where derivatives of $w(X_r,t)$ $(r=1,2)$ change rapidly, it is obvious that a justification of the considered wave problem has to be given. In other words, the consistency of the order of the wave with the accuracy and limits of applicability of the plate model derived by asymptotic expansion methods has to be studied in order to ensure that this wave represents a meaningul process. Any such analysis must technically be very involved and is outside the scope of this work.

In this thesis, we shall consider third order waves; their meaning within the plate model is given by the following simple observation. If we apply to an infinite plate on elastic support a constant load P concentrated on the line $X_2 = 0$, then by the symmetry of this problem, the resulting deflection w is independent of X_1. Formally, this problem is identical to the problem of a beam on elastic support with a constant load applied at a point. This latter problem has been investigated by v.Kármán and Biot [73]. The explicit solution for deflection of this beam problem (p. 273 in [73]) shows that the resulting deflection curve has the shape of damped waves with a discontinuous third derivative at the point of load application . Also, it can be easily seen that the first and second order derivatives of this deflection curve are continuous everywhere. In our case of a plate this solution implies that the first and second derivative of w are continuous everywhere and that the third order derivative $w_{,222}$ suffers a discontinuity across the line $X_2 = 0$ (see Fig. 8). Let us note that this third derivative of w is proportional to the shear force (cf. (12.10)), i.e. the jump of the shear force across the line $X_2 = 0$ is equal to a constant load P concentrated on this line.

The existence of this jump and its physical meaning justifies the investigation of third order waves in the plate model. The consistency of the acceleration waves (recall that the second derivatives of w with respect to X_1 and X_2 are proportional to the bending moments - cf. (12.9)) remains an open problem.

From Theorem 6.1 and eqs. (12.8) and (12.9), it follows that for the third order wave, the only nontrivial jump is (6.33). This jump is now given by

$$[\![-D(\Delta w)_{,r} n^r + \mu I \, \ddot{w}_{,r} n^r]\!] = 0 \tag{12.16}$$

From kinematical conditions of compatibility (or directly from (10.6)) we have that

$$[\![\Delta w_{,r} n^r]\!] = - U_{(n)}^{-3} \, \tilde{a} \tag{12.17}$$

and

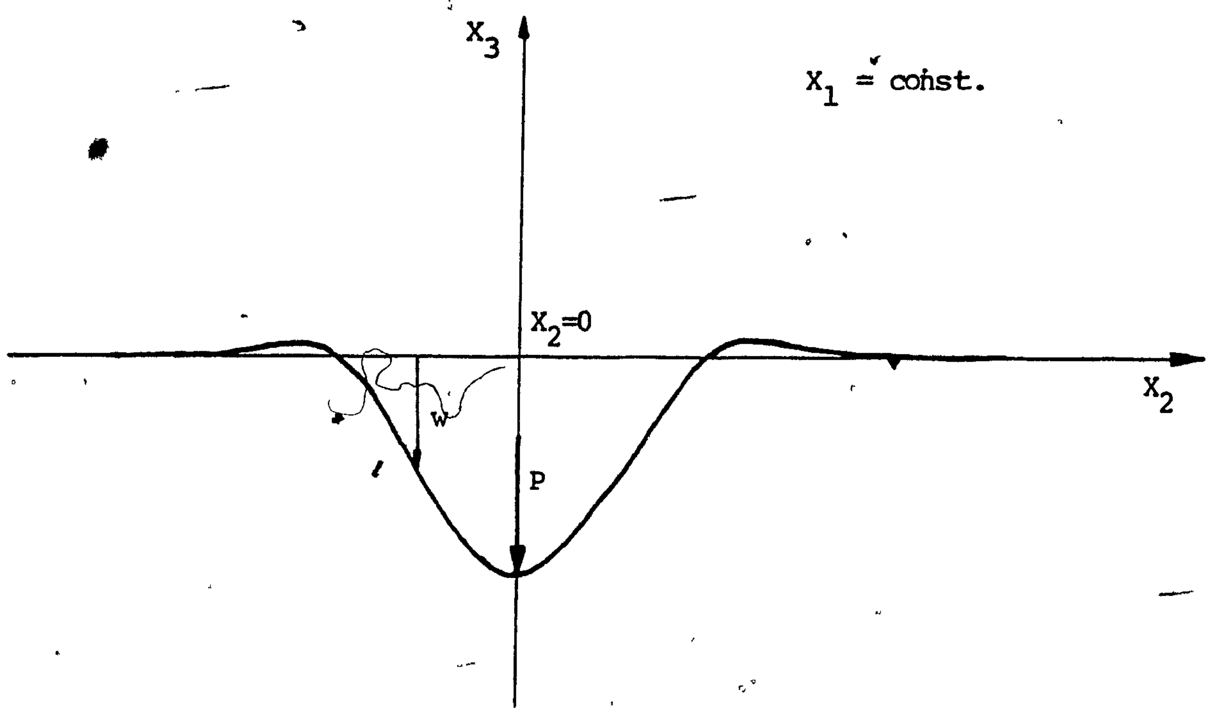
$$[\![\ddot{w}_{,r} n^r]\!] = - U_{(n)}^{-1} \, \tilde{a} \tag{12.18}$$

where $\tilde{a} = [\![\dddot{w}]\!]$ is the wave amplitude.

On substituting (12.17) and (12.18) into (12.16) we obtain the following propagation condition

$$(-D + \mu I \, U_{(n)}^2) \, \tilde{a} = 0 \tag{12.19}$$

from which it follows that the speed of propagation $U_{(n)}$ is given by



$$w = C\, e^{\pm \alpha X_2} (\cos \alpha X_2 \mp \sin \alpha X_2)$$

where the upper sign holds for $X_2 < 0$ and the lower sign holds for $X_2 > 0$.

$$[\![w_{,2}]\!] = [\![w_{,22}]\!] = 0, \; [\![w_{,222}]\!] = 8C\alpha^3$$

Figure 8

$$U_{(n)} = \left(\frac{D}{\mu I}\right)^{1/2} \tag{12.20}$$

The equation for the wave amplitude $\tilde{a}$ can be obtained from the equation of motion (12.1). By taking the jump of this equation across the singular curve Σ_t we obtain

$$[\![-D\Delta^2 w + \mu I \Delta \ddot{w}]\!] = 0 \tag{12.21}$$

First, let us note that from kinematical conditions of compatibility we have

$$\begin{aligned}
[\![\Delta\ddot{w}]\!] &= \frac{\delta}{\delta t}[\![\Delta\dot{w}]\!] - U_{(N)}\,[\![\Delta\dot{w}_{,r}n^r]\!] \\
&= \frac{\delta}{\delta t}\left\{\frac{\delta}{\delta t}[\![\Delta w]\!] - U_{(n)}\,[\![\Delta w_{,r}n^r]\!]\right\} \\
&\quad - U_{(n)}n^r\left\{\frac{\delta}{\delta t}[\![\Delta w_{,r}]\!] - U_{(n)}\,[\![\Delta w_{,rs}n^s]\!]\right\} \\
&= -\,2U_{(n)}\frac{\delta}{\delta t}[\![\Delta w_{,r}n^r]\!] + U^2_{(n)}\,[\![\Delta w_{,rs}n^r n^s]\!] \\
&= 2U^{-2}_{(n)}\frac{\delta}{\delta t}\tilde{a} + U^2_{(n)}\,[\![\Delta w_{,rs}n^r n^s]\!]
\end{aligned} \tag{12.22}$$

where we have used (12.17) and the fact that $[\![\Delta w]\!] = 0$ and

$$\frac{\delta n_r}{\delta t} = -\,d_r\,U_{(n)} = 0$$

($U_{(n)}$ is a constant).

On substituting (12.22) into (12.21) we obtain

$$[\![\Delta w_{,rs}\,(\delta^{rs} - n^r n^s)]\!] = 2U^{-2}_{(n)}\frac{\delta}{\delta t}\tilde{a} \tag{12.23}$$

Now, from the geometrical and kinematical conditions of compatibility we have

$$[\![\Delta w_{,rs}(\delta^{rs} - n^r n^s)]\!] = -\Omega_{rs} [\![\Delta w_{,p} n^p]\!] (\delta^{rs} - n^r n^s)$$

$$+ n_r n_s [\![\Delta w_{,pq} n^p n^q]\!] (\delta^{rs} - n^r n^s)$$

$$= - \Omega [\![\Delta w_{,p} n^p]\!]$$

$$= \Omega U^{-3}_{(n)} \tilde{a} \tag{12.24}$$

where $\Omega_{rs} = -d_r n_s = - d_s n_r$ and $\Omega = \Omega_{rr}$ is the curvature of a wave curve.

Introducing (12.24) into (12.23) we obtain the following equation for the wave amplitude $\tilde{a}$:

$$\frac{\delta}{\delta t} \tilde{a} = \frac{1}{2} \Omega U_{(n)} \tilde{a} \tag{12.25}$$

To integrate this equation, first, let us introduce the distance parameter σ along the normal trajectory of the wave front defined by

$$\sigma = U_{(n)} (t-t_1) \tag{12.26}$$

for $t > t_1$ and then notice that this wave front consists of a parallel family of curves ($U_{(n)}$ is constant). This last remark implies that the curvature for this family of curves is given by (cf. Cohen and Suh [67])

$$\Omega = \frac{\Omega(o)}{1-\Omega(o)\sigma} \tag{12.27}$$

for all sufficiently small σ, where $\Omega(o)$ is the initial curvature for $t = t_1$ (i.e. $\sigma = 0$).

If we notice that

$$\frac{\delta \tilde{a}}{\delta t} = U_{(n)} \frac{d\tilde{a}}{d\sigma} \tag{12.28}$$

then introducing (12.28) and (12.23) into (12.25) we obtain

$$\frac{d\tilde{a}}{d\sigma} = \frac{\Omega(o)}{2(1-\Omega(o)\sigma)}\,\tilde{a} \tag{12.29}$$

This equation can be easily integrated and the result is

$$\tilde{a}(\sigma) = \tilde{a}(o)\,\left|1-\Omega(o)\sigma\right|^{-1/2} \tag{12.30}$$

which expresses the decay law for the wave amplitude.

The first observation is that the decay law (12.30) (and eq. (12.29)) does not involve any material constant. Also, the amplitude equation (12.29) has exactly the same form as that obtained by Cohen and Suh [67] (eq. (4.23) in [67]) for waves propagating through elastic surfaces. In conclusion, we should expect that the amplitude equation (12.29) (and also the decay law (12.30)) is rather universal for waves propagating through thin elastic plates and shells, although future study is needed to justify fully this expectation.

The similar statement has been made by Chen [74]. He concluded that the amplitudes of acceleration waves in 3-dimensional media almost always obey the Bernoulli equation.

Finally, let us mention that our approach to wave propagation in the plate model considered here is complementary to that studied by Ericksen [64] and Cohen [71] where a direct approach to Cosserat plates has been employed.

APPENDIX I

An (infinitesimal) deformation of a (smooth) hypersurface Σ in R^N is a one-parameter family of (smooth) hypersurfaces

$$(-\varepsilon,\varepsilon) \ni s \mapsto \Sigma(s) \subset R^N. \tag{I.1}$$

There exists a differentiable function $f: \Omega \to R^1$ defined in an open set $\Omega \subset R^N$ containing $\Sigma(s)$, $s \in (-\varepsilon,\varepsilon)$, such that each s is a regular value of f and $\Sigma(s) = f^{-1}(s)$, $\Sigma(0) \equiv \Sigma$. For each X, in a coordinate neighbourhood V of X in R^N the deformation (I.1) induces the following mapping

$$(-\varepsilon,\varepsilon) \ni s \mapsto \chi(s)(u) = \chi(u,s) \in V \cap \Sigma(s) \tag{I.2}$$

such that

$$R^{N-1} \supset U \ni u \mapsto \chi(u,s) \in V \cap \Sigma(s) \tag{I.3}$$

is a local parametrization of $\Sigma(s)$ and for s=0 (I.3) is a local parametrization at $X \in \Sigma$

A unit normal vector N at $X' \in \Sigma(s)$ is given by

$$N_A = |\text{Grad } f|^{-1} f_{,A} |_{X' \in f^{-1}(s)} \tag{I.4}$$

By differentiation of the superposition $s \mapsto f \circ \chi(s)$, of the mapping (I.2) and the function f, we obtain

$$f_{,A} \frac{d\chi^A(s)}{ds} \Big|_{s=0} = 1 \tag{I.5}$$

On substituting the expression (I.4) into (I.5) we conclude that the quantity

$$N_A \frac{d\chi^A(s)}{ds} \Big|_{s=0}$$

is independent of the choice of local parameterizations.

APPENDIX II

Consider a one-parameter family

$$(-\varepsilon,\varepsilon) \ni s \mapsto (\Sigma(s),\ \phi(s)) \tag{II.1}$$

where $\varepsilon > 0$ and for each $s, \phi(s)(X) = \phi(X,s)$, $X = (X_A) \in R^m$, is a tensor valued mapping such that $\Sigma(s)$ is its singular nypersurface (cf. definitions in Chapter 2). We assume that for each s the hypersurface $\Sigma(s)$ is a compact manifold (i.e. without boundary). These hypersurfaces can be given by

$$\Sigma(s): \quad \Phi(X,s) = 0 \tag{II.2}$$

where Φ is a smooth function such that Grad $\Phi \neq 0$ and $\frac{\partial\Phi}{\partial s} \neq 0$ on $\Sigma(s)$.

Let $\theta(\Phi)$ be the characteristic function of the region $\Phi > 0$, i.e.

$$\theta(\Phi) = \begin{cases} 0 \text{ for } & \Phi \leq 0 \\ 1 \text{ for } & \Phi > 0 \end{cases}$$

then we have

$$\int_{R^m} \theta(\Phi)\ f(X)\ dV = \int_{\Phi>0} f(X)\ dV \tag{II.3}$$

where f is a smooth function with compact support and $dV = dX_1 \ldots dX_m$.

Following ideas presented in Gelfand and Shilov [75] we have

$$\theta'(\Phi) = \delta(\Phi) \tag{II.4}$$

which is understood in the sense that

$$\text{Grad}\ \theta(\Phi) = \delta(\Phi)\ \text{Grad}\ \Phi \qquad \text{(II.5)}$$

where the generalized function $\delta(\Phi)$ is defined by

$$\int_{R^m} \delta(\Phi)\ f(X)\ dV = \int_{\Phi=0} f(X)\ \omega \qquad \text{(II.6)}$$

in which the (m-1)-differential form ω has been defined by

$$d\Phi \wedge \omega = dV \qquad \text{(II.7)}$$

In [75], it is proved that such form in fact exists in some region containing $\Sigma(s)$ and is unique in the sense that it depends only on $\Phi(X,s)$ by which this $\Sigma(s)$ is given in (II.2). Of course, (II.6) is independent of the choice of $\Phi(X,s)$.

Consider a one-parameter family of action integrals

$$A(s) = \int_{\Omega\setminus\Sigma(s)} L(s)\ dV$$

$$= \int_{\Omega\setminus\Sigma(s)} L(X_A, \phi(s), \phi_{,A}(s), \phi_{,AB}(s))\ dV \qquad \text{(II.8)}$$

where $(\Sigma(s), \phi(s))$ is a pair as it has been defined in (II.1) and such that $\phi(s)$ and its partial derivatives vanish outside of a bounded region Ω in R^m containing $\Sigma(s)$ for each $s \in (-\varepsilon,\varepsilon)$.

Now, let us write the integral (II.8) in the following way

$$A(s) = \int_{\Omega} \{L_+(s)\ \theta(\Phi) + L_-(s)\ \theta(-\Phi)\}\ dV \qquad \text{(II.9)}$$

where

$$L_{\pm}(s) = L(X_A, \tilde{\phi}^{(\pm)}(s), \tilde{\phi}^{(\pm)}_{,A}(s), \tilde{\phi}^{(\pm)}_{,AB}(s))$$

in which $\tilde{\phi}^{(+)}(s)(X) = \tilde{\phi}^{(+)}(X,s)$ and $\tilde{\phi}^{(-)}(s)(X) = \tilde{\phi}^{(-)}(X,s)$ are smooth extensions of $\phi|\Omega^+(s)$ and $\phi|\Omega^-(s)$, respectively, such that $\tilde{\phi}^{(+)}(s)$, $\tilde{\phi}^{(-)}(s)$ and their partial derivatives vanish outside of Ω. In the above $\Omega^+(s)$ and $\Omega^-(s)$ are the sub-domains of Ω divided by $\Sigma(s)$, i.e. $\Omega = \Omega^+(s) \cup \Omega^-(s) \cup \Sigma(s)$.

By differentiation (II.9) we obtain

$$\frac{dA(s)}{ds}\Big|_{s=0} = \int_\Omega \left\{ \frac{dL_+(s)}{ds}\Big|_{s=0} \theta(\Phi) + \frac{dL_-(s)}{ds}\Big|_{s=0} \theta(-\Phi) \right\} dV$$

$$+ \int_\Omega \left\{ L_+(0) \frac{\partial}{\partial s} [\theta(\Phi)]_{s=0} + L_-(0) \frac{\partial}{\partial s} [\theta(-\Phi)]_{s=0} \right\} dV \qquad \text{(II.10)}$$

First, let us note that

$$\frac{\partial}{\partial s} [\theta(\Phi)]_{s=0} = \delta(\Phi) \frac{\partial \Phi}{\partial s}\Big|_{s=0}$$

$$\frac{\partial}{\partial s} [\theta(-\Phi)]_{s=0} = -\delta(\Phi) \frac{\partial \Phi}{\partial s}\Big|_{s=0} \qquad \text{(II.11)}$$

Next, let us assume that $\Phi(X,s)$ is the Euclidean distance of X from the $\Phi = 0$ hypersurface. Then, following [75], we have that the differential form ω coincides with the Euclidean element area $d\Sigma_{m-1}$ on $\Sigma(s)$. In this case, it should be clear (cf. eq. (10.16)) that

$$\frac{\partial \Phi}{\partial s}\Big|_{s=0} = - \delta\Sigma \qquad \text{(II.12)}$$

where $\delta\Sigma$ is the normal variation of the hypersurface Σ defined in Chapter 4.

On substituting (II.11) and (II.12) into (II.10), and on using (II.6) with $\omega = d\Sigma_{m-1}$, we finally derive

$$\frac{dA(s)}{ds}\bigg|_{s=0} = \int_{\Omega} \frac{dL(s)}{ds}\bigg|_{s=0} dV - \int_{\Sigma} [\![L]\!] \,\delta\Sigma \;\; d\Sigma_{m-1} .$$

which does not depend on smooth extensions.

By using ideas developed in Gelfand and Shilov [75], and in particular in Chapter III of this reference, we could find the expression for the second variation of the action integral in the class of discontinuous fields, however, we have not considered the second variation in this thesis, hence this important problem is left for the future.

APPENDIX III

Let us recall that a singular hypersurface propagating through a material has been represented by

$$\sigma_t: \phi(x_r,t) = 0 \quad (r=1,\dots,N) \tag{III.1}$$

or by

$$\Sigma_t: \Phi(X_r,t) = 0 \quad (r=1,\dots,N) \tag{III.2}$$

in the spatial and material description, respectively, where

$$\phi(\psi_r(X_r,t),t) \equiv \Phi(X_r,t)$$

in which $x_r = \psi_r(X_r,t)$ has been defined in Chapter 10 in the case of "small" deformations. Because the hypersurface has no singular points then we have that grad ϕ and Grad Φ on σ_t and Σ_t, respectively, do not vanish.

Now, let us define $\bar{\phi}$ and $\bar{\Phi}$ by

$$\bar{\phi} = \frac{\phi}{|\text{grad } \phi|} \tag{III.3}$$

and

$$\bar{\Phi} = \frac{\Phi}{|\text{Grad } \Phi|} \tag{III.4}$$

respectively, then the induced Euclidean measures on σ_t and Σ_t are defined respectively, by

$$d\bar{\phi} \wedge d\sigma_{N-1} = dv_N \tag{III.5}$$

$$d\bar{\Phi} \wedge d\Sigma_{N-1} = dV_N \tag{III.6}$$

where $d\bar{\phi}$ and $d\bar{\Phi}$ are the differentials of $\bar{\phi}$ and $\bar{\Phi}$, respectively. If we recall that $dv_N = J\, dV_N$ where J is the Jacobian of the transformation $x_r = \psi_r(X_r,t)$, then from (III.5) and (III.6) we have that

$$d\bar{\Phi} \wedge d\Sigma_{N-1} = d\bar{\phi} \wedge J^{-1}\, d\sigma_{N-1} \qquad \text{(III.7)}$$

From (III.3), (III.4) and (10.12) we can obtain on ϕ_t and Σ_t:

$$d\bar{\phi} = n_r dx^r \qquad \text{(III.8)}$$

$$d\bar{\Phi} = N_r dX^r \qquad \text{(III.9)}$$

On substituting (III.8) and (III.9) into (III.7) and on using the fact that $dx_r = \psi_{r,s} dX^s$ we obtain

$$N_r dX^r \wedge d\Sigma_{N-1} = J^{-1}\, n_s \psi_{s,r} dX^r \wedge d\sigma_{N-1} \qquad \text{(III.10)}$$

which by (10.15) is well-defined.

Now, to prove (10.11) it is enough to introduce into (III.10) the relation

$$J^{-1} \frac{|(\text{grad}\ \phi)_s\ \text{Grad}\ \psi_s|}{|\text{grad}\ \phi|}\ N_r = J^{-1}\, n_s \psi_{s,r}$$

which follows from (10.12), and then using the fact that if γ is a (N-1)-differential form then

$$N_r dX^r \wedge \gamma = 0$$

implies that $\gamma = 0$ on the $\Phi = 0$ hypersurface (cf. Gelfand and Shilov [75], p. 221).

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