
Higher Spin Fields in Three-Dimensional Gravity

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Abstract

In this thesis, we study the effects of massless higher spin fields in three-dimensional gravity with a negative cosmological constant. First, we introduce gravity in Anti-de Sitter (AdS) space without the higher spin gauge symmetry. We recapitulate the semi-classical analysis that outlines the duality between quantum gravity in three dimensions with a negative cosmological constant and a conformal field theory on the asymptotic boundary of AdS_3 . We review the statistical interpretation of the black hole entropy via the AdS/CFT correspondence and the modular invariance of the partition function of a CFT on a torus. For the case of higher spin theories in AdS_3 we use those modular properties to bound the amount of gauge symmetry present. We then discuss briefly cases that can evade this bound.

Abrégé

Cette thèse s'intéresse à la gravitation en trois dimensions avec une constante cosmologique négative, et en particulier aux effets des champs de masse nulle dont le spin est plus grand que deux. Les interactions gravitationnelles dans l'espace Anti-de Sitter (AdS) sont d'abord décrites sans la symétrie de jauge additionnelle. Nous présentons l'analyse semi-classique qui met en lumière la dualité holographique entre la gravitation quantique en trois dimensions avec une constante cosmologique négative et une théorie conforme des champs (CFT) sur la frontière asymptotique de l'espace AdS_3 . Grâce à cette correspondance AdS/CFT et à l'invariance sous les transformations modulaires de la fonction de partition d'une théorie conforme décrite sur un tore, nous réexaminons l'interprétation statistique de l'entropie des trous noirs. Nous appliquons ensuite ces propriétés modulaires aux théories gravitationnelles avec l'invariance de jauge des spins entiers plus grand que deux, et démontrons ainsi une limite sur le spin maximal présent dans l'espace AdS_3 . Quelques remarques s'ensuivent sur certains cas particuliers pouvant se soustraire à cette limite.

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Lastly, I am sorry but I cannot help myself thanking Alexandra Tchong at little more than she would have liked me to do.

Contribution of Authors

Chapter 3 of the thesis has been published as [1] in January 2011. We modified it only to agree with McGill thesis format. This work has been done under the supervision of Alexander Maloney, in collaboration with Alejandra Castro. In theoretical physics, it is customary to list authors in alphabetical order.

Introduction

One major problem of modern theoretical physics is the quantization of gravity. The need for a quantum theory of gravity is to address long-standing puzzles, such as the place of gravity in a unified description of all interactions, and the characterization of space-time down to scales our universe explored in its infancy. Another phenomenon for which one needs a quantum treatment of gravity is the thermodynamic nature of black holes. On the one hand, classical General Relativity (GR) leads to Birkhoff's theorems for black hole metrics [2]: for a given set of charges (such as mass, angular momentum) there exists a unique black hole metric with those conserved quantities. On the other hand, by studying dynamics of a probe scalar field in the vicinity of a black hole, one can derive the thermodynamical properties for that system. In particular the Bekenstein-Hawking entropy of the black hole scales as the area of the horizon [3, 4]. In this thesis we will review the statistical interpretation for that entropy for the particular case of gravity with a negative cosmological constant in three dimensions [5]. Using the same line of reasoning, we will also generalize the discussion to include higher spin fields and find non-perturbative restrictions on the coupling of these fields to gravity.

The quantization of gravity is notably difficult. At face value, the Einstein-Hilbert action in $d > 2$ dimensions leads to non-renormalizable interactions, hence the usual tools of quantum field theory cannot be put to use. From a bottom-up perspective, the black hole entropy serves as a guide and the 'holographic' approach

to quantum gravity proposes that a gravitational theory in d dimensions can be formulated as a quantum field theory only in $d - 1$ dimensions. As a top-down approach, string theory is an ultraviolet (UV) complete theory which comprises gravity. By smoothing out interactions over two-dimensional surfaces – the world-sheet of a fundamental string – the theory is cured from UV divergences. However the theory is not perfectly understood, in particular there are numerous other fields in the spectrum of the theory. Hence it cannot answer whether the quantization of GR by itself is sensible.

The approach that we use in this thesis lies in between the two points of view just described. We will analyze GR semi-classically as we will define it later in the core of the thesis. We will use the insight gained by the study of string theory over the years, in particular the Anti-de Sitter/Conformal Field Theory correspondence (AdS/CFT) [6, 7]. The AdS/CFT correspondence is a holographic duality which relates quantum gravity in a space with a negative cosmological constant to a conformal theory on its asymptotic boundary. In chapter 2, we will give a heuristic argument for the validity of such duality, and then use it to address the black hole entropy.

This thesis explores three-dimensional gravity with a negative cosmological constant. As we will describe in chapter 2 this theory is locally trivial. Despite this, it has a rich quantum spectrum [8, 9], which we will be able to characterize rather precisely. Understanding these results has broader implications than what will be shown in the thesis. For example, certain extremal black holes in higher dimensions are known to have a scaling limit in which their near-horizon geometry is AdS_3 , the geometry we are studying here. Hence, understanding the quantum effect for GR in AdS_3 background will deepen our knowledge of the dynamics of a large class of black holes. Moreover, some condensed matter theory systems have been shown to be dual to black holes in various dimensions, through the AdS/CFT duality. We refer the interested reader to [10, 11] and references therein.

The thesis is divided as follows. Chapter 2 is a short review of the literature on the subject of three-dimensional quantum gravity and its holographic description as a two-dimensional CFT. In particular, we will define conformal field theories in two dimensions, and study some of their generic properties. In section 2.2, we present semi-classical evidence for the AdS/CFT correspondence following the early work of [12]. In section 2.3 we use this to count the degeneracies of black holes in the large mass regime, shedding light on the black hole entropy. Chapter 3 we generalize the three-dimensional gravity to encompass massless higher spin gauge symmetry and discuss non-perturbative effects of these theories. We present the equation of motions for the higher spin fields in section 3.2 and study the partition function of these theories in section 3.3. Section 3.4 addresses the question of linearization instability that could avoid the bound presented in the present chapter. Chapter 4 is then devoted to some recent results as well as open questions we wish to tackle in future research.

2

Three-dimensional Gravity with a Negative Cosmological Constant

In this chapter we will deal with a semi-classical study of three-dimensional gravity with a negative cosmological constant. The classical action is

$$I = \frac{1}{16\pi G} \int d^3x \sqrt{g} \left(R + \frac{2}{\ell^2} \right), \quad (2.1)$$

where G is Newton's constant and $\Lambda = -\frac{1}{\ell^2}$ is the cosmological constant. A vacuum solution is Anti de-Sitter space (AdS_3), whose metric is

$$ds^2 = -\left(\frac{r^2}{\ell^2} + 1\right)dt^2 + \left(\frac{r^2}{\ell^2} + 1\right)^{-1}dr^2 + r^2d\phi^2, \quad (2.2)$$

where the ϕ coordinate is identified with period 2π . Spatial infinity ($r \rightarrow \infty$) is conformally a cylinder.

General relativity is locally trivial in three dimensions. To understand this statement, we look at the equation of motion of (2.1)

$$\begin{aligned} R_{\mu\nu} &= \left(\frac{R}{2} + \frac{1}{\ell^2} \right) g_{\mu\nu} \\ &= -\frac{2}{\ell^2} g_{\mu\nu}, \end{aligned} \quad (2.3)$$

where the second line is obtained by tracing over the first. The local degrees of freedom are contained in the Riemann curvature tensor, which has $\frac{1}{12}d^2(d^2 - 1)$ independent components in d -dimensions [13]. By using (2.3) and the definition of the Ricci tensor we get the six equations (from the six metric components)

$$g_{\mu\nu} = g^{\alpha\beta} R_{\alpha\mu\beta\nu}, \quad (2.4)$$

which completely fixes the Riemann curvature tensor. Thus any solution of (2.1) will be locally of the form (2.2), possibly with different global identifications.

In this chapter we will give a semi-classical argument for the AdS/CFT correspondence, which we will then use to compute the microscopic entropy of black holes.

2.1 Conformal Field Theory in two dimensions

We review in this section some basic results from quantum field theories in two dimensions invariant under conformal transformations. We denote the coordinates $z, \bar{z}$. A conformal transformation of the plane can be written as $z \mapsto f(z)$ where $f(z)$ is holomorphic. The stress-energy tensor is constrained by conformal invariance

$$T_{z\bar{z}} = 0, \quad T_{zz} = T(z), \quad T_{\bar{z}\bar{z}} = \tilde{T}(\bar{z}). \quad (2.5)$$

One important tool to study Conformal Field Theories (CFT) is the Operator Product Expansion (OPE). In any QFT one can write a product of two local operators as a sum of local operators:

$$\mathcal{O}_i(z_1)\mathcal{O}_j(z_2) = \sum_k c_{ij}^k(z_1 - z_2)\mathcal{O}_k(z_2), \quad (2.6)$$

in the limit $z_1 \rightarrow z_2$. The coefficient functions c_{ij}^k can be singular in that limit, and we will write OPEs up to non-singular terms with the $\simeq$ symbol. Conformal invariance constrains the coefficient functions and makes the OPE convergent around z_2 . From the singular terms of the OPEs of operators in the CFT, one can reconstruct the entire theory [14].

Under infinitesimal conformal transformations $z \mapsto z + \epsilon v(z)$, the stress-energy tensor transforms as [14]

$$\epsilon^{-1}\delta T(z) = -\frac{c}{12}\partial_z^3 v(z) - 2\partial_z v(z)T(z) - v(z)\partial_z T(z), \quad (2.7)$$

and similarly for $\tilde{T}$. This defines the central charges $c, \tilde{c}$ of the CFT. Using this transformation, and the definition of the stress-energy tensor as the generator of coordinate transformations, we arrive at the OPE [14]

$$T(z)T(0) \simeq \frac{c}{2z^4} + \frac{2}{z^2}T(0) + \frac{1}{z}\partial_z T(0). \quad (2.8)$$

We define the Virasoro generators by the Laurent expansion of the stress-energy tensor:

$$L_m = \oint \frac{dz}{2\pi i} z^{m+2} T(z), \quad \tilde{L}_m = \oint \frac{d\bar{z}}{2\pi i \bar{z}} \bar{z}^{m+2} \tilde{T}(z), \quad (2.9)$$

where the integral is performed around the origin. For a general charge associated to a current

$$Q_k = \oint \frac{dz}{2\pi i} j_k(z), \quad (2.10)$$

knowing the OPE of the currents will give the algebra of the charges:

$$[Q_k, Q_l] = \oint \frac{dz_2}{2\pi i} \text{Res}_{z_1 \rightarrow z_2} j_k(z_1) j_l(z_2). \quad (2.11)$$

In the case of the Virasoro generators, we get the Virasoro algebra,

$$\begin{aligned} [L_m, L_n] &= (m-n)L_{m+n} + \frac{c}{12}(m^3 - m)\delta_{m+n,0} \\ [\tilde{L}_m, \tilde{L}_n] &= (m-n)\tilde{L}_{m+n} + \frac{\tilde{c}}{12}(m^3 - m)\delta_{m+n,0} \end{aligned} \quad (2.12)$$

$$[L_m, \tilde{L}_n] = 0.$$

In a CFT, there is an isomorphism between local operators and states of the theory [14], which we won't describe here but we will use it in an abstract way. We can study the spectrum of simultaneous eigenstates of $L_0, \tilde{L}_0$ rather generically. We label states by their weights $(\Delta, \tilde{\Delta})$:

$$L_0|\Omega\rangle = \Delta|\Omega\rangle, \quad \tilde{L}_0|\Omega\rangle = \tilde{\Delta}|\Omega\rangle, \quad (2.13)$$

where the action of the Virasoro generators is understood as the OPE of the stress-energy tensor with the operator isomorphic to the state $|\Omega\rangle$.

We can now state Cardy's Formula [15]. For a CFT on a torus, with a unique vacuum and an energy gap between the vacuum and the first excited state, the asymptotic growth of the density of states as a function of their weights is given by

$$N(\Delta, \tilde{\Delta}) \sim \exp\left(2\pi\sqrt{\frac{c\Delta}{6}}\right) \exp\left(2\pi\sqrt{\frac{\tilde{c}\tilde{\Delta}}{6}}\right). \quad (2.14)$$

We will use this formula in section 2.3 and chapter 3.

2.2 Brown-Henneaux and the AdS/CFT correspondence

In this section we do a thorough classical analysis of Anti-de Sitter gravity, following the pioneering work of Brown and Henneaux [12]. We motivate the importance of boundary conditions at spatial infinity and compute the algebra of the canonical generators. We find that it is the same as that of the conformal group in two dimensions with a non-trivial central charge.

A solution of (2.1) has a metric [12]

$$ds^2 = -\left(\frac{r^2}{\ell^2} + \alpha^2\right) dt^2 + 2A\alpha dt d\phi + \left(\frac{r^2 - A^2}{\ell^2} + \alpha^2\right)^{-1} dr^2 + (r^2 - A^2) d\phi^2, \quad (2.15)$$

for a choice of coordinates with the same global identifications as AdS_3 in the form (2.2). Notice that the dominant terms in the $r \rightarrow \infty$ limit of (2.15) match with those of (2.2).

We can now define the boundary conditions for our solution to be ‘asymptotically Anti-de Sitter’. We act on (2.15) with the isometry group of AdS_3 to generate the boundary conditions

$$g_{\mu\nu} = \begin{pmatrix} -\frac{r^2}{\ell^2} + \mathcal{O}(1) & \mathcal{O}(1/r^3) & \mathcal{O}(1) \\ & \frac{\ell^2}{r^2} + \mathcal{O}(1/r^4) & \mathcal{O}(1/r^3) \\ & & r^2 + \mathcal{O}(1) \end{pmatrix}, \quad (2.16)$$

where $x^\mu = (t, r, \phi)$. We then consider the asymptotic symmetries that leave a metric in the form (2.16). We describe those with vector fields whose Lie derivative of the metric effects the symmetry. These vectors satisfy [5]

$$\begin{aligned}\xi^t &= \ell(T^+ + T^-) + \frac{\ell^3}{2r^2}(\partial_+^2 T^+ + \partial_-^2 T^-) + \mathcal{O}(1/r^4), \\ \xi^r &= -r(\partial_+ T^+ + \partial_- T^-) + \mathcal{O}(1/r), \\ \xi^\phi &= T^+ - T^- - \frac{\ell^2}{2r^2}(\partial_+^2 T^+ - \partial_-^2 T^-) + \mathcal{O}(1/r^4),\end{aligned}\tag{2.17}$$

with

$$2\partial_\pm = \ell \frac{\partial}{\partial t} \pm \frac{\partial}{\partial \phi},\tag{2.18}$$

$$\partial_\pm T^\mp = 0.$$

The gauge freedom is contained in the $\mathcal{O}(1/r)$ and $\mathcal{O}(1/r^4)$ of the r and t, ϕ components respectively. Deformation vectors with only those terms will be referred to as ‘pure gauge’. The Asymptotic Symmetry Group (ASG) is then defined by the vectors (2.17) modulo pure gauge transformations. The ASG is isomorphic to the pseudo-conformal group in two dimensions [12], since (2.18) is the conformal Killing equation in two dimensions for $T^\pm$.

Our goal is to get a canonical realization of the asymptotic symmetries with a non-trivial central charge. We work in the ADM formalism [16], and we write the Hamiltonian generators [12]

$$H[\xi] = \int d^2x \xi^\mu \mathcal{H}_\mu(x) + J[\xi],\tag{2.19}$$

where $\mathcal{H}_\mu$ are the constraint of general relativity, and the $J[\xi]$ are the charges defined so that the generators have well defined variational derivatives. Using the fact that the Poisson bracket of generators is itself a generator up to a constant that can depend only on the asymptotic form of the deformations [12], we can write

$$\{H[\xi], H[\eta]\} = H[\zeta] + K[\xi, \eta].\tag{2.20}$$

We need to check that ζ is an element of the ASG. To do so, we work with pure gauge ξ and η , for which $K[\xi, \eta] = 0$. We arrive at

$$\begin{aligned} \{H[\xi], H[\eta]\} &= \int d^2x d^2y \xi^\mu(x) \eta^\nu(y) \{\mathcal{H}_\mu(x), \mathcal{H}_\nu(y)\} + \{\xi^\mu(x), \eta^\nu(y)\} \mathcal{H}_\mu(x) \mathcal{H}_\nu(y) \\ &\quad + \int d^2x \{\xi^\mu(x), H[\eta]\} \mathcal{H}_\mu(x) - \{\eta^\mu(x), H[\xi]\} \mathcal{H}_\mu(x). \end{aligned} \quad (2.21)$$

With our boundary conditions (2.16), we get that $\zeta \rightarrow [\xi, \eta]$, the Lie bracket, to leading order in $1/r$. Hence

$$\{H[\xi], H[\eta]\} = H[[\xi, \eta]] + K[\xi, \eta]. \quad (2.22)$$

We then compute the central charge $K[\xi, \eta]$, by calculating the Dirac Bracket of the charges $J[\xi]$. We decompose our asymptotic symmetries (2.17) in Fourier series with $T_n^\pm = e^{in(\frac{t}{\ell} \pm \phi)}$. To make a link with the previous section we use the notation $L_n \equiv J[\xi(T_n^+)]$, $\tilde{L}_n \equiv J[\xi(T_n^-)]$. The Dirac brackets are then [12] [5]

$$\begin{aligned} [L_m, L_n]_{D.B.} &= (m - n)L_{m+n} + \frac{c}{12}(m^3 - m)\delta_{m+n,0} \\ [\tilde{L}_m, \tilde{L}_n]_{D.B.} &= (m - n)\tilde{L}_{m+n} + \frac{c}{12}(m^3 - m)\delta_{m+n,0} \end{aligned} \quad (2.23)$$

$$[L_m, \tilde{L}_n]_{D.B.} = 0,$$

with the central charge

$$c = \tilde{c} = \frac{3\ell}{2G}. \quad (2.24)$$

While this derivation is semi-classical in nature, the conclusion is that quantum gravity on AdS_3 can be represented by a CFT on the (t, ϕ) cylinder at asymptotic infinity.

The ADM mass and angular momentum of the bulk solution are given by

$$M = \frac{1}{\ell}(L_0 + \tilde{L}_0), \quad J = L_0 - \tilde{L}_0. \quad (2.25)$$

Note that the AdS_3 background has $M = -\frac{1}{8G} = -\frac{c}{12}$.

2.3 Microscopic Degeneracies of Black Holes

In this section, we want to illustrate one of the main achievements in the study of three-dimensional gravity with a negative cosmological constant: the counting of microscopic states for black holes. The BTZ [8, 9] black hole of mass M and angular momentum J metric can be written as [5]

$$\begin{aligned}
 ds^2 &= -N^2 dt^2 + \rho^2 (N^\phi dt + d\phi)^2 + \frac{r^2}{N^2 \rho^2} dr^2, \\
 N^2 &= \frac{r^2(r^2 - r_+^2)}{\ell^2 \rho^2}, \\
 N^\phi &= -\frac{4GJ}{\rho^2}, \\
 \rho^2 &= r^2 + 4GM\ell^2 - \frac{1}{2}r_+^2, \\
 r_+^2 &= 8G\ell\sqrt{M^2\ell^2 - J^2},
 \end{aligned} \tag{2.26}$$

where ϕ has period 2π . The Bekenstein-Hawking entropy is [5]

$$S = \frac{Area}{4G} = \frac{2\pi\rho(r_+)}{4G} = \pi\sqrt{\frac{\ell}{2G}} \left(\sqrt{M\ell + J} + \sqrt{M\ell - J} \right), \tag{2.27}$$

and the goal of this section is to give a statistical interpretation of this entropy.

To calculate the partition function, we need to analytically continue our metric to Euclidean signature. The asymptotic boundary is now a torus, and we can write the periodic identification with the complex coordinate

$$\begin{aligned}
 z &= \phi + it_E, \\
 z &\sim z + 2\pi \sim z + 2\pi\tau, \\
 \tau &= \frac{1}{2\pi} (\theta + i\beta),
 \end{aligned} \tag{2.28}$$

where t_E is the Euclidean time, θ is the angular potential and β is the inverse temperature. The conformal structure of the asymptotic torus τ labels classical

solutions to (2.1). The BTZ solution is locally AdS_3 and differs from the AdS_3 state through global identifications. For the thermal AdS_3 geometry, we know that there is one contractible cycle in the bulk since the ϕ coordinate shrinks smoothly at the origin. By smoothly, we mean that the $g_{\phi\phi}$ component has a double zero at the origin. The BTZ black hole has its time coordinate shrink to zero size at its horizon. The constraint of having a double zero for g_{tt} fixes the temperature and angular potential in terms of mass and angular momentum. The explicit modular transformation that turns thermal AdS_3 into BTZ is $\tau \mapsto -\frac{1}{\tau}$. We can calculate the partition function for thermal AdS_3 , and use the modular transformation to relate it to the partition function for the BTZ black hole.

We formally define the canonical ensemble partition function

$$Z_{\text{AdS}}(\tau) = \int \mathcal{D}g e^{-S(g)}, \quad (2.29)$$

for metrics g that are small fluctuations of AdS_3 in the sense of (2.16). More precisely, the integral is over metrics whose asymptotic boundary is a torus with conformal structure τ . To compute this partition function, we expand (2.29) in perturbation theory. The classical contribution and the one-loop part can be computed directly from the gravity theory [17], but we won't reproduce this here¹. Instead, we use what we have learned in section 2.2, and write the partition function as $Z(\tau) = \text{Tr}_{\mathcal{H}} \exp(-\beta M - i\theta J)$, where $\mathcal{H}$ is the Hilbert space of the Virasoro descendants of the vacuum. A generic state in this Hilbert space can be represented by [18, 17]

$$\prod_{n=2}^{\infty} L_{-n}^{u_n} \prod_{n=2}^{\infty} \tilde{L}_{-n}^{u_n} |\Omega\rangle, \quad (2.30)$$

for the CFT ground state $|\Omega\rangle$. Those states are called boundary gravitons, since there are no local propagating degrees of freedom in our theory. Using the Virasoro

¹This computation shows that the partition function is one-loop exact (since the CFT and gravity computation agree), and is a non-trivial check of the derivation of Brown and Henneaux [12]

Algebra (2.12) and the relationship between (M, J) and $(L_0, \tilde{L}_0)$ we arrive at the partition function for AdS_3 [18]

$$Z_{\text{AdS}}(\tau) = \frac{1}{|\eta(\tau)|^2} |q\bar{q}|^{-(c-1)/24} |1-q|^2, \quad (2.31)$$

with the Dedekind η function $\eta(\tau) = q^{1/24} \prod_{n=1}^{\infty} (1 - q^n)$, where $q = e^{2i\pi\tau}$.

We can now evaluate the microscopic degeneracy of a BTZ black hole at fixed energy M and angular momentum J . We first notice that in the large mass regime, we can apply Cardy's formula to count the number of states at large weight $(\Delta, \tilde{\Delta})$, and get the entropy [5]

$$S(\Delta, \tilde{\Delta}) = 2\pi\sqrt{\frac{\Delta c}{6}} + 2\pi\sqrt{\frac{\tilde{\Delta} c}{6}} = \pi\sqrt{\frac{\ell}{2G}} \left(\sqrt{M\ell + J} + \sqrt{M\ell - J} \right). \quad (2.32)$$

We thus reproduce the Bekenstein-Hawking entropy.

We present the derivation of the number of states of [18], which addresses the regime of validity of approximations of the microscopic entropy and its matching to the black hole macroscopic entropy. We apply the transformation $\tau \mapsto -1/\tau$ to the AdS_3 partition function to get

$$\begin{aligned} Z_{\text{BTZ}}(\tau) &= \mathcal{Z}(\tau) \bar{\mathcal{Z}}(\bar{\tau}), \\ \mathcal{Z}(\tau) &= \frac{q_-^{-(c-1)/24} (1 - q_-)}{\eta(-1/\tau)}, \end{aligned} \quad (2.33)$$

for $q_- = e^{-2\pi i/\tau}$. We can use the relationship between the Dedekind η function and the partition of integers $p(N)$ [19] to rewrite the holomorphic piece of the partition function

$$\mathcal{Z}(\tau) = \sum_{\Delta=-c/24}^{\infty} C_{\Delta} q_-^{\Delta}, \quad (2.34)$$

$$C_{\Delta} = p(\Delta + c/24) - p(\Delta + c/24 - 1).$$

We then compute the microcanonical entropy, to count the density of states at fixed weights $N(\Delta, \tilde{\Delta})$. The partition function is holomorphically factorized, and hence

$N(\Delta, \tilde{\Delta}) = \mathcal{N}(\Delta)\mathcal{N}(\tilde{\Delta})$. We perform a Laplace transform to get [18]

$$\begin{aligned}
\mathcal{N}(\Delta) &= \int_{i\epsilon-\infty}^{i\epsilon+\infty} d\tau q^{-\Delta} \mathcal{Z}(\tau) \\
&= \sum_{\Delta'=-c/24}^{\infty} C_{\Delta'} \int_{i\epsilon-\infty}^{i\epsilon+\infty} d\tau q^{-\Delta} q_-^{\Delta'} \\
&= 2\pi \sum_{\Delta'=-c/24}^{\infty} C_{\Delta'} \sqrt{\frac{-\Delta'}{\Delta}} I_1(4\pi\sqrt{-\Delta\Delta'}),
\end{aligned} \tag{2.35}$$

where I_1 is the modified Bessel function of the first kind. We can compare the first two terms in the sum

$$\frac{I_1(4\pi\sqrt{(c-24)\Delta/24})}{I_1(4\pi\sqrt{c\Delta/24})} \sim e^{-2\pi\sqrt{24\Delta/c}} \tag{2.36}$$

for $\Delta c \rightarrow \infty$. This limit is the semi-classical approximation $\ell \gg G$, the same limit one uses to derive the Bekenstein-Hawking entropy. To simplify (2.35) we take the limit $\Delta \gg c$, in which only the first term of the sum contributes. This limit is that of a black hole large in AdS units. We use the asymptotic formula for the Bessel function and the truncated sum of 2.35 [18]

$$S(\Delta, \tilde{\Delta}) = \log \mathcal{N}(\Delta)\mathcal{N}(\tilde{\Delta}) \sim 2\pi\sqrt{\frac{\Delta c}{6}} + 2\pi\sqrt{\frac{\tilde{\Delta} c}{6}}. \tag{2.37}$$

Hence we can reproduce the Bekenstein-Hawking entropy in the large black hole limit.

Higher Spin Theories in AdS_3 and a Gravitational Exclusion Principle

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Abstract

We consider theories of three-dimensional quantum gravity in Anti-de Sitter space which possess massless higher spin gauge symmetry. The perturbative spectrum of the theory includes higher spin excitations which can be organized into vacuum representations of the $\mathcal{W}_N$ algebra; these are higher spin versions of the boundary gravitons. We describe a fundamental bound which relates the value of the cosmological constant to the amount of gauge symmetry present. In the dual CFT language, this is the statement that modular invariance implies that the theory can not be quantized unless the central charge is sufficiently large, i.e. if $c \geq N - 1$. This bound relies on the assumption that all of the perturbative excitations exist as full states in the quantum theory, and can be circumvented if the theory possesses a linearization instability. The $\mathcal{W}_N$ minimal models – recently conjectured to be dual to certain higher spin AdS theories by Gaberdiel and Gopakumar – provide an example of this phenomenon. This result can be regarded as an example of a “gravitational exclusion principle” in Anti-de Sitter space, where a non-perturbative quantum gravity mechanism involving black holes places a limit on the number of light degrees of freedom present.

3.1 *Introduction*

Three-dimensional quantum gravity has proven a useful testing ground for many of our ideas and conjectures concerning the microscopic nature of gravity. One of the most interesting and important conjectures is the proposal that quantum gravity places a fundamental limit on the number of light degrees of freedom present. This conjecture is most commonly discussed in the context of black hole entropy, where it was observed that a large number of light species of identical particles would violate holographic entropy bounds [20, 21]. However, this notion has surfaced in a variety of different guises over the last several decades (for example in [22, 23, 24]). The goal of this paper is to describe a specific three-dimensional scenario where this idea can be put to the test using the precision techniques of AdS/CFT.

We will focus on the case of three-dimensional gravity in asymptotically Anti-de Sitter (AdS) space, and consider theories with massless higher spin gauge fields. These theories possess a large symmetry group which can be regarded as an enhanced version of the conformal symmetry present in every asymptotically AdS theory of gravity. The states organize into representations of this enhanced symmetry group, hence these theories contain a large number of light degrees of freedom. In the context of AdS/CFT, the inclusion of higher spin fields is interesting in its own right. In string theory realizations of AdS/CFT, an infinite tower of massless higher spin fields is expected to emerge when the AdS radius becomes small [25, 26, 27, 28, 29, 30]. The study of such higher spin fields should therefore be regarded as a first step in the study of quantum gravity in AdS beyond the supergravity regime.

We note that the construction of theories of massless higher spin fields is a notoriously delicate procedure. In four space-time dimensions, a consistent theory with an infinite tower of interacting higher spin fields was constructed by Vasiliev [31, 32, 33, 34] (see also the recent progress of [35, 36, 37, 38]). In three space-time dimensions the story is somewhat more straightforward. A simple class of massless higher spin theories in AdS_3 can be formulated using Chern-Simons theory [39, 40].

Unlike Vasiliev’s four dimensional theory, which can be formulated only when there are an infinite number of higher spin fields, this theory exists for both a finite and an infinite number of higher spin fields [41, 42]. These theories describe massless higher spin gauge fields which possess no local degrees of freedom and can be regarded as higher spin versions of the graviton, which itself has no local degrees of freedom in three dimensions.

Despite the fact that these theories have no local degrees of freedom, they have interesting quantum properties which can be understood rather precisely. There are “non-local” degrees of freedom which are associated with boundary excitations of the fields, generalizing the classical results of Brown and Henneaux [12]. In particular, the algebra of the asymptotic symmetry group is enlarged from two copies of the Virasoro algebra to two copies of the $\mathcal{W}_N$ algebra, where N is the highest allowed spin [43, 42]. The central charge of the dual CFT can be computed, and remarkably remains unaffected by the presence of the higher spin fields. A non-trivial check of this story was provided by [44], who computed the one-loop determinant of the gravitational theory and showed that it is precisely the vacuum character of $\mathcal{W}_N$.

Here we will investigate the effect of these higher spin fields on the spectrum of the theory. Classically and at the linearized level the theories seem to be well defined and free of pathologies. We would like to ask what happens once quantum effects are taken into account. Our primary tool will be the AdS/CFT correspondence, which states that to every theory of gravity in asymptotically AdS space there is a dual CFT. Thus the structure of the theory is constrained by conformal invariance. In particular, modular invariance – invariance under large conformal transformations in Euclidean signature – allows us to determine the spectrum of the theory at high energies. This gives Cardy’s formula, which determines the rate of growth for the density of states at high energies.

The basic observation of this note is a simple one. When the value of N is sufficiently large, the number of linearized states in the bulk theory – the number

of higher spin versions of boundary gravitons – exceeds this upper bound set by Cardy’s formula. In order to prevent this we must require that

$$N - 1 \leq c = \frac{3\ell}{2G} . \quad (3.1)$$

Here we have used the Brown-Henneaux expression for the central charge of the theory in terms of the AdS radius ℓ and Newton constant G . Thus the existence of a dual CFT, along with the existence of these boundary excitations, provides a bound on the amount of higher spin gauge symmetry present. An important feature of this result is that when N is large it applies to theories in the semiclassical ($\ell \gg G$) regime. This can be regarded as a “gravitational exclusion principle,” where quantum gravitational effects place an upper bound on the number of light states in the theory.

We note that this bound appears only when non-perturbative effects are included, and that the classical theories discussed above appear to be free of pathology for every value of c and N . It is interesting then to ask exactly what happens when we try to quantize a theory with values of N and c which violate the bound (3.1). One of two things must occur. The first possibility is that the value of ℓ (or G) will be renormalized by quantum effects so that (3.1) is satisfied. In effect, quantum corrections will drive the value of the cosmological constant towards zero to accommodate the large number of degrees of freedom. The second possibility is that some of the dangerous perturbative states are removed from the spectrum upon quantization. This would mean that the theory has a linearization instability; apparently innocuous perturbative states are not in fact linearizations of true states in the Hilbert space. Roughly speaking, these perturbative states are removed to accommodate the finite size of Anti-de Sitter space. It appears that both of these possibilities can be realized in theories of AdS quantum gravity. To see this, we will consider a simple set of CFTs with $\mathcal{W}_N$ symmetry, namely the $\mathcal{W}_N$ minimal models, whose bulk duals were recently discussed in [45].

Finally, we wish to emphasize the intimate connection between the bound (3.1) and the physics of asymptotically AdS black holes. Every classical theory of AdS_3 gravity possesses black hole solutions, the BTZ black holes. The Bekenstein-Hawking entropy of these black holes is precisely given by Cardy's formula for the asymptotic density of states. Thus the bound (3.1) reflects the fact that black holes dominate the spectrum of the theory at high energy. Indeed, we will see that there is a precise sense in which those CFTs which violate the bound (3.1) – such as the $\mathcal{W}_N$ minimal models – are dual to theories of gravity without macroscopic black holes.

In the next section we will review a few salient features of $\mathcal{W}_N$ symmetry and higher spin theories in AdS_3 . In section 3.3 we discuss the bound (3.1) and its application in both the finite N and $N \rightarrow \infty$ case. In section 3.4 we comment on the specific realizations of these conjectures in the $\mathcal{W}_N$ minimal models. In section 3.5 we describe the asymptotic properties of $\mathcal{W}_N$ and $\mathcal{W}_\infty$ vacuum characters.

3.2 Higher spin fields in AdS_3

In this section we summarize the main results of [43, 42, 44] concerning higher spin theories in AdS_3 .

Classical three-dimensional general relativity with a negative cosmological constant can be rewritten as a Chern-Simons gauge theory with gauge group $SO(2, 2) \simeq SL(2, \mathbb{R}) \times SL(2, \mathbb{R})$ [46, 47, 48]. It is easy to generalize this to include a theory with fields of up to spin N . We simply replace the $SL(2, \mathbb{R})$ gauge group by $SL(N, \mathbb{R})$ [42]. In this case the higher spin fields are massless and have no local propagating degrees of freedom; the theory is topological, just as with the spin 2 graviton case. Further, one can take the infinite dimensional extension of $SL(2, \mathbb{R})$ – denoted $hs(1, 1)$ – which will describe a infinite tower of spins in a similar spirit as the Fradkin-Vasiliev theory [39, 40].

To formulate this theory more precisely, we introduce a pair of tensor-valued one

forms

$$e_\mu^{a_1 \cdots a_{s-1}} , \quad \omega_\mu^{a_1 \cdots a_{s-1}} , \quad (3.2)$$

where a_i are Lorentz indices. If the gauge group is $SL(2, \mathbb{R})$ we identify e_μ^a with the dreibein and ω_μ^a with the spin connection. The Chern-Simons gauge fields are the linear combinations

$$A_{(2)}^\pm = J_a \left(\omega_\mu^a \pm \frac{1}{\ell} e_\mu^a \right) dx^\mu , \quad (3.3)$$

where J_a are the generators of $sl(2, \mathbb{R})$. The equations of motion are found by extremizing the Chern-Simons action

$$I_{CS}[A] = \frac{k}{4\pi} \int \text{tr}(A \wedge dA + \frac{2}{3} A \wedge A \wedge A) , \quad (3.4)$$

where tr is the symmetric bilinear form on $SL(2, \mathbb{R})$. The Einstein-Hilbert action is given by

$$I_{EH} = I_{CS}[A_{(2)}^+] - I_{CS}[A_{(2)}^-] , \quad k = \frac{\ell}{4G} . \quad (3.5)$$

where ℓ is the AdS_3 radius and G Newton's constant.

To include the dynamics of the spin s field, we define

$$\begin{aligned} A^+ &= A_{(2)}^+ + T_{a_1 \cdots a_{s-1}} \left(\omega_\mu^{a_1 \cdots a_{s-1}} + \frac{1}{\ell} e_\mu^{a_1 \cdots a_{s-1}} \right) dx^\mu , \\ A^- &= A_{(2)}^- + T_{a_1 \cdots a_{s-1}} \left(\omega_\mu^{a_1 \cdots a_{s-1}} - \frac{1}{\ell} e_\mu^{a_1 \cdots a_{s-1}} \right) dx^\mu , \end{aligned} \quad (3.6)$$

with $s > 2$ and $T_{a_1 \cdots a_{s-1}}$ are generators of the extended gauge group. We can then identify the gauge fields (3.2) with higher spin fields as defined by Fronsdal [49] provided the generators $T_{a_1 \cdots a_{s-1}}$ obey the correct algebra. First, the generators $T_{a_1 \cdots a_{s-1}}$ must be taken to be symmetric and traceless. Second, the J_a and $T_{a_1 \cdots a_{s-1}}$ must form the Lie algebra

$$[J_a, J_b] = \epsilon_{abc} J^c , \quad [J_a, T_{a_1 \cdots a_{s-1}}] = \epsilon^m_{a(a_1} T_{a_2 \cdots a_{s-1})m} . \quad (3.7)$$

One can then consider the Chern-Simons action

$$I_N = I_{CS}[A^+] - I_{CS}[A^-] . \quad (3.8)$$

For $N > 2$, one can check that the linearized fluctuations of the gauge fields around a fixed metric background should satisfy the equations of motion of higher spin fields. More precisely, we have

$$e_\mu^a = e_\mu^{(0)a} + e_\mu^{(1)a} , \quad \omega_\mu^a = \omega_\mu^{(0)a} + \omega_\mu^{(1)a} , \quad (3.9)$$

where the upper script (0) denotes the background and (1) are fluctuations. Treating all other higher spin fields as fluctuations, the linearized Chern-Simons equations are reduced to

$$\nabla^2 \varphi_{\mu_1 \dots \mu_s} - \nabla_{(\mu_1} \nabla^\lambda \varphi_{|\mu_2 \dots \mu_s) \lambda} + \nabla_{(\mu_1} \nabla_{\mu_2} \varphi_{\mu_3 \dots \mu_s) \lambda}^\lambda = 0 , \quad (3.10)$$

where

$$\varphi_{\mu_1 \dots \mu_s} = \frac{1}{s} e_{(\mu_1}^{(0)a_1} \dots e_{\mu_{s-1}}^{(0)a_{s-1}} e_{\mu_s) a_1 \dots a_{s-1}} , \quad (3.11)$$

for $s \geq 2$. At this level the connection $\omega_\mu^{a_1 \dots a_{s-1}}$ becomes an auxiliary field. Equation (3.10) is exactly the equation of motion for a free spin s field propagating on a curved space-time.

There are two very interesting results for these higher spin theories in AdS_3 . First, in [43, 42] the authors computed the asymptotic symmetries of the $SL(N, \mathbb{R}) \times SL(N, \mathbb{R})$ Chern-Simons theories for a given set of boundary conditions. Taking the connection $A_{(2)}^\pm$ on empty AdS_3 as the definition of “asymptotically AdS configurations”, they found all gauge transformations that left the connection invariant up to a constant term with respect to AdS_3 near the boundary. The remarkable result is that the algebra of the asymptotic symmetries is given by two copies of the $\mathcal{W}_N$ algebra. Further, the algebra allows for a central extension and its central charge is

$$c = \frac{3\ell}{2G} . \quad (3.12)$$

It is surprising that the addition of higher spin fields does not affect the central charge. The value in (3.12) is the same as computed by Brown-Henneaux [12] for Einstein gravity with a negative cosmological constant. This results also holds in

the infinite N limit, where the algebra is $\mathcal{W}_\infty$ and the central charge is still (3.12) [43].

The appearance of the centrally extended algebra as studied in [43, 42] is purely classical. The analysis presented in [44] goes one step further and tests whether the $\mathcal{W}_N$ persists at the quantum level. These authors computed the 1-loop determinant associated to the linearized fluctuations (3.11). They found that the full 1-loop contribution of a single spin s field is simply

$$Z^{(s)} = \prod_{n=s}^{\infty} |1 - q^n|^{-2} , \quad (3.13)$$

where $q = \exp(2\pi i\tau)$ and τ is the complex structure of the torus at the boundary of thermal AdS_3 . Therefore for a $SL(N) \times SL(N)$ Chern-Simons theory, which contains a family of spin fields from $s = 2$ up to $s = N$, the 1-loop determinant is given by

$$Z_N^{1\text{-loop}} = \prod_{s=2}^N \prod_{n=s}^{\infty} |1 - q^n|^{-2} = \chi_N \times \bar{\chi}_N , \quad (3.14)$$

with

$$\chi_N = \prod_{s=2}^N \prod_{n=s}^{\infty} (1 - q^n)^{-1} . \quad (3.15)$$

χ_N is precisely the vacuum character of the $\mathcal{W}_N$ algebra. For infinite N the resulting 1-loop determinant is

$$Z_\infty^{1\text{-loop}} = \prod_{s=2}^{\infty} \prod_{n=s}^{\infty} |1 - q^n|^{-2} = \chi_\infty \times \bar{\chi}_\infty , \quad (3.16)$$

where

$$\chi_\infty = M(q) \prod_{n=1}^{\infty} (1 - q^n) , \quad (3.17)$$

and the MacMahon function is defined as

$$M(q) = \prod_{n=1}^{\infty} (1 - q^n)^{-n} . \quad (3.18)$$

The function χ_∞ is the character of the $\mathcal{W}_\infty$ algebra. One nice and unexpected feature is that equations (3.14) and (3.16) can be written as the square of a holomorphic function of q .

Although these one loop determinants were computed directly in the bulk using heat kernel methods, in fact they have a simple physical interpretation. They can be derived using strictly algebraic methods, as traces over the vacuum representations of $\mathcal{W}_N$ and $\mathcal{W}_\infty$. This is the representation where all of the $\mathcal{W}_N$ descendants are linearly independent and have positive norm; i.e. the representation without null vectors. Using this fact, it was further argued in [44] that the partition functions (3.14) and (3.16) are one-loop exact, following [18].

3.3 Partition function and growth of states

We would now like to study the general properties of the partition function of an asymptotically AdS theory of gravity with $\mathcal{W}_N$ symmetry. Our basic observation is that there is a tension between the two essential features described above – the existence of asymptotic conformal symmetry with a finite central charge, and the appearance of the infinite tower of linearly independent, finite norm $\mathcal{W}_N$ descendants. In some cases these features are mutually incompatible.

We start by considering the partition function

$$Z(\tau, \bar{\tau}) = \sum_{\Delta, \bar{\Delta}} d(\Delta, \bar{\Delta}) q^\Delta \bar{q}^{\bar{\Delta}} , \quad (3.19)$$

where $d(\Delta, \bar{\Delta})$ is the number of states with weight $(\Delta, \bar{\Delta})$. We will use the conventional “CFT normalization” for the weights so that the ground state (i.e. empty Anti-de Sitter space) has $\Delta = \bar{\Delta} = -c/24$. This partition function can be regarded as a Euclidean path integral in three dimensions, where we sum over all field configurations such that the metric approaches a torus at asymptotic infinity.¹ With standard Brown-Henneaux boundary conditions this partition function will be a function only of the conformal structure τ of the torus at infinity, and will hence be invariant under the modular transformation $\tau \rightarrow -1/\tau$. In the gravitational

¹The literature on the partition function of AdS₃ gravity is extensive, see e.g. [50, 51, 52, 53, 18] and references therein.

language, this modular transformation is a large diffeomorphism of the bulk which induces a large conformal transformation of the boundary torus.

Modular invariance leads to Cardy's formula [15]

$$\log(d(\Delta, \bar{\Delta})) \sim 2\pi\sqrt{\frac{c\Delta}{6}} + 2\pi\sqrt{\frac{c\bar{\Delta}}{6}} , \quad (3.20)$$

for the number of states at large $\Delta, \bar{\Delta}$. The first assumption involved in the derivation of this formula is that the bulk theory is diffeomorphism invariant in Euclidean signature. The second is that the ground state has finite norm, so that the first excited state has $\Delta, \bar{\Delta} > -c/24$ and is separated by a gap from the ground state. Provided these assumptions are satisfied, equation (3.20) is universal. The details of the bulk theory, such as the specific matter content, will only enter into the subleading corrections to this formula.

This universal behaviour is a consequence of the physics of AdS₃ black holes. Every classical theory of AdS₃ gravity contains black holes [8, 9]. These black holes are quotients of AdS₃, so will necessarily exist as solutions to the equations of motion if AdS₃ itself is a solution to the equations of motion. Their Bekenstein-Hawking entropy is precisely given by equation (3.20) [5]. Thus we expect that in a quantum theory of AdS₃ gravity, there should be states with arbitrarily large weights which describe the BTZ black hole.

Let us now reconsider the higher spin theories in this light. Although we will not be able to compute the partition function exactly, we can compute the tree and one-loop contributions. The vacuum state will just be empty AdS, which contributes to the tree level partition function

$$Z^{(0)} = q^{-c/24} \bar{q}^{-c/24} . \quad (3.21)$$

The one loop piece is also easy to compute. It is given by the trace

$$Z^{(1)} = \text{Tr}_{\mathcal{H}} \left(q^{L_0} \bar{q}^{\bar{L}_0} \right) , \quad (3.22)$$

over the Hilbert space $\mathcal{H}$ of linearized excitations of the theory. This is the space of solutions to the linearized equations of motion (3.10) modulo gauge transformations. Since all local excitations are pure gauge, one might guess that there are no such contributions. However, this is not quite the case as the set of allowed gauge transformations includes only those which vanish sufficiently quickly at infinity. Thus the spectrum includes states obtained by acting on the vacuum state by a linearized gauge transformation at the boundary. Indeed, it was argued that these gauge transformations generate the algebra $\mathcal{W}_N$, which is an extension of the usual Virasoro algebra $\mathcal{W}_2$. Thus the linearized fluctuations of the spin fields are organized into a $\mathcal{W}_N$ character [44],

$$Z^{(1)}(q) = q^{-c/24} \bar{q}^{-c/24} |\chi_N(q)|^2. \quad (3.23)$$

Here $\chi(q)$ is given by the vacuum character (3.15) or (3.17) depending on whether N is finite or infinite.

It is important to emphasize that there is nothing mysterious about the states which contribute to the partition function (3.23). They describe solutions to the equations of motion and can be written out explicitly in the Chern-Simons language. For $N = 2$, of course, they have a simple interpretation; they are the usual boundary gravitons. At the linearized level, these states have finite norm with respect to Klein-Gordon inner product, so appear to represent legitimate states of the free higher spin field theory. The question is whether these states will survive at the non-linear level, and if they do what the implications are for the quantum theory.

The most immediate effect of the higher spin fields is to increase the number of states at each level. In particular, the number of $\mathcal{W}_N$ descendants of a given dimension is larger than the number of Virasoro descendants. But the total number of states is constrained by Cardy's formula (3.20). If the linearized states appearing in (3.23) are to appear as states in the full theory, this is a significant constraint.

To see this let us first consider the case where N is finite. The coefficients p_Δ^N of

the $\mathcal{W}_N$ vacuum character

$$\chi_N = \prod_{s=2}^N \prod_{n=s}^{\infty} (1 - q^n)^{-1} = \sum_{\Delta} p_{\Delta}^N q^{\Delta} , \quad (3.24)$$

can be estimated at large Δ . They grow like

$$\log(p_{\Delta}^N) \sim 2\pi \sqrt{\frac{(N-1)}{6}} \Delta , \quad (3.25)$$

when Δ is large (and in particular if $\Delta \gg N^3$). A derivation of this is given in the appendix, but the origin of this growth can be understood intuitively. When $N = 2$, χ_2 is the vacuum character of the Virasoro algebra. In the absence of null vectors, the number of Virasoro descendants of a given primary state increases like the number of states in a CFT of central charge $c = 1$. That is why the construction CFTs with $c < 1$ (the minimal models) is a highly constrained algebraic problem which requires the existence of null vectors. For $N > 2$, we observe that the character (3.24) is equal to the $(N-1)^{th}$ power of the Virasoro vacuum character times a finite polynomial in q . Thus it is natural to guess that the number of descendants grows like the number of states of a CFT with central charge $(N-1)$. From equation (3.25) we see that this is indeed the case. One just has to verify that this finite polynomial does not lead to cancellations which will spoil this heuristic argument; this computation is described in appendix 3.5.2.

Comparing equations (3.25) and (3.20) it is clear that if $N-1 > c$ then there will be a value of Δ for which p_{Δ}^N will exceed the allowed density of states $d(\Delta, \bar{\Delta})$. Thus some of the linearized states must be removed from the spectrum. Indeed, we will see explicitly that this can happen in certain cases in the next section for the bulk theories dual to the $\mathcal{W}_N$ minimal models.

We note that the situation is even more drastic if N is infinite. The descendants are counted by the $\mathcal{W}_{\infty}$ character

$$\chi_{\infty} = M(q) \prod_{n=1}^{\infty} (1 - q^n)^{-1} = \sum_{\Delta=1}^{\infty} p_{\Delta}^{\infty} q^{\Delta} , \quad (3.26)$$

whose coefficients grow like

$$\log(p_\Delta^\infty) \sim 3 \left(\frac{\zeta(3)\Delta^2}{4} \right)^{1/3}, \quad (3.27)$$

as we show in appendix 3.5.3. The growth of states in (3.27) will always exceed the Cardy growth (3.20) for any finite value of the central charge. Thus in the absence of a linearization instability, the number of perturbative states vastly exceeds the number of black holes states.

Finally, we note that the convergence towards the asymptotic values (3.25) and (3.27) is rather slow. In some cases, this might mean that in order to see that the number of $\mathcal{W}_N$ descendants exceeds the Cardy bound we have to look at states of very high dimension.

As an illustration of this phenomenon, we will consider the following simple example. Let us ask if it is possible to construct a “pure” theory of gravity with $\mathcal{W}_N$ symmetry, in the sense that the only perturbative states are the $\mathcal{W}_N$ descendants described above. Following [51], it is natural to conjecture that this theory is holomorphically factorized. In this case the partition function will be the square of an analytic function $Z(\tau)$ which diverges like $q^{-c/24}$ as $q \rightarrow 0$. $Z(\tau)$ will be a holomorphic, modular invariant function on the upper half τ plane. Using general properties of modular functions (see e.g. [19]) it follows that $Z(\tau)$ is determined uniquely provided we specify the $c/24$ polar terms in the expansion of $Z(\tau)$ around $q = 0$. If the theory is “pure” in the sense defined above, then these polar terms are found by demanding that they match the polar terms in the one loop partition function (3.23). It is then straightforward to compute $Z(\tau)$ for any desired values of N and c and hence determine the number of states of any dimension Δ , using an algorithm similar to that presented in [51].¹ It is then possible to check explicitly that

¹One could also compute $Z(\tau)$ by performing a sum over geometries, following [52]. If we simply sum the holomorphic part of the one-loop determinant over the coset $SL(2, \mathbb{Z})/\mathbb{Z}$, then the resulting $Z(\tau)$ will be the same as that described above. However, if one does not assume holomorphic

for any $N - 1 > c$ there is some value of Δ for which the number of $\mathcal{W}_N$ descendants exceeds to total number of states counted by the partition function $Z(\tau)$.

It is amusing to work this out explicitly for the case $c = 24$ where the holomorphic part of the partition function is, up to an additive constant, equal to the Klein's J -invariant $J(\tau)$. The q expansion is

$$Z(\tau) = q^{-1} + (\text{const}) + 196884q + 21493760q^2 + 864299970q^3 + \dots \quad (3.28)$$

One can compare this to the asymptotic growth of the vacuum character

$$Z^{(1)}(\tau) = q^{-1}\chi_N = q^{-1} + q + 2q^2 + 3q^3 \dots \quad (3.29)$$

It is a surprising (but true) fact that when $N > 25$ the coefficients of (3.29) become larger than those of (3.28) for some value of Δ . For N very large this occurs when $\Delta \approx 10^5$ and the coefficients are of order 10^{1000} .¹ The explanation of this curiously large value of Δ is the following. The J -function happens to be well approximated by Cardy's formula for small values of Δ , whereas the corresponding asymptotic formula for χ_N is only a good approximation for relatively large (of order 10^4) values of Δ . The lesson is that while the first few coefficients in expressions like (3.29) may appear small, this does not tell the full story!

3.4 Minimal Models and Black Holes

In this section we comment on the $\mathcal{W}_N$ minimal models, which provide specific and calculable examples of $\mathcal{W}_N$ symmetric CFTs with central charges $c < N - 1$. Thus they lie on the other side of the bound (3.1). This bound was motivated in

factorization and instead sums the full one loop determinant (3.23) over $SL(2, \mathbb{Z})/\mathbb{Z}$ one finds results which are not consistent with a quantum mechanical interpretation, as in [18].

¹In fact, we can improve this argument a bit by noticing that the full partition function must be a $\mathcal{W}_N$ character, so that every time a primary state appears in the theory this leads to additional $\mathcal{W}_N$ descendants at higher order. When $N = \infty$ and $c = 24$, for example, this leads to a negative number of $\mathcal{W}_N$ primaries at $\Delta \approx 60000$ if there are no null vectors.

part by bulk gravity considerations, so one might expect that the bulk duals to these minimal models have several rather unusual properties. Indeed these minimal models are just like their Virasoro cousins in that they contain only a finite number of primary states, so in a sense they are dual to theories of gravity without black holes.

The $\mathcal{W}_N$ minimal model at level k can be described in terms of the coset WZW model

$$\frac{su(N)_k \oplus su(N)_1}{su(N)_{k+1}}, \quad (3.30)$$

where the subscripts give the level of the algebra. The central charge is

$$c = (N - 1) \left(1 - \frac{N(N + 1)}{(N + k)(N + k + 1)} \right), \quad (3.31)$$

and is strictly less than $N - 1$ for finite values of N and k . When $N = 2$ these coincide with the usual (Virasoro) minimal models, and it can be proven that there are no other unitary CFTs with $c < 1$. We do not know of a similar proof for higher values of N .

We note that, from a quantum gravity perspective, these higher N minimal models are much more interesting than their Virasoro ($c < 1$) cousins. That is because c can be taken to be large provided that N is also large, so that the theories are dual to macroscopic theories of three-dimensional gravity with AdS radius large in Planck units. Thus one would expect that all of the familiar features of classical three-dimensional gravity – in particular the BTZ black holes – to arise in this limit.

Unfortunately, the bulk duals of these theories are not known explicitly. However, when N and k are taken to infinity with the ratio k/N fixed, the bulk dual was conjectured to be an infinite tower of higher spin fields along with a pair of complex scalar fields [45]. In this limit the central charge goes to infinity, meaning that the AdS radius is infinite in Planck units. For finite values of N this bulk theory should presumably be augmented by terms involving the curvature of AdS space. These modifications are not known, but based on the above considerations we can describe

some basic features of the bulk dual of the $\mathcal{W}_N$ minimal models for finite N and k .

We first note that, as emphasized in the previous section, the theory must have null vectors, meaning that certain higher spin versions of the boundary gravitons are removed from the spectrum. Indeed, one can check explicitly that the $\mathcal{W}_N$ minimal models have null vectors. For the $\mathcal{W}_N$ descendants of the vacuum, the first null vector appears at dimension $\Delta = k + 1 - \frac{c}{24}$. Indeed, the vast majority of the $\mathcal{W}_N$ descendants will be projected out of the spectrum at high order.

In fact, for large values of Δ the spectrum of the $\mathcal{W}_N$ minimal model consists *entirely* of descendant states, rather than primary states.¹ In particular, these theories have only a finite number of primaries, hence they have a state with largest dimension. The dimension of this highest dimension state can be estimated, and is of order

$$\Delta_{\text{max}} \sim k^2 N , \quad (3.32)$$

when k and N are large.

We note that this is in drastic contrast to our semiclassical expectations. BTZ black holes exist as classical solutions of the equations of motion for any value of the mass and angular momentum². In particular the theory contains black holes whose horizon size (r_+) is large compared to both the Planck length G and the AdS radius ℓ . In the dual CFT language, this means that we expect there to be primary states of arbitrarily high dimension. More precisely, we expect that the CFT should include primary states with dimensions Δ such that

$$c\Delta \sim \left(\frac{r_+^2}{G_3^2} \right) \gg 1 , \quad (3.33)$$

and

$$\left(\frac{\Delta}{c} \right) \sim \left(\frac{r_+^2}{\ell^2} \right) \gg 1 . \quad (3.34)$$

¹We are grateful to M. Gaberdiel for discussions related to this point.

²More precisely, BTZ black holes exist for values of mass and angular momentum such that $M\ell \geq J$.

For states where the second inequality is valid, Cardy's formula can be used to compute the entropy. Although we might expect that the allowed values r_+ (and hence Δ) will be quantized in the full quantum theory, we still expect that there should still be a tower of black hole states with arbitrarily large dimension.

However, this is not what is indicated by equation (3.32) for finite values of k and N . The only states with arbitrarily high dimension are descendant states, i.e. higher spin versions of the boundary gravitons, as well as lower order primaries dressed with descendants. We conclude that the bulk theory dual to these minimal models must be a strange object indeed, in that it does not possess standard BTZ black holes with arbitrarily large mass and angular momentum.

It is also worth investigating how the large N and k limit is approached, and whether in this regime the minimal $\mathcal{W}_N$ models contain large black holes in the sense of (3.33) and (3.34). In the 't Hooft limit, as defined in [45], both N and k are infinity with the ratio

$$\lambda = \frac{N}{k + N} \quad (3.35)$$

fixed and the highest dimension growth is given by (3.32). The central charge in this regime becomes infinitely large, $c \simeq N(1 - \lambda^2)$, and therefore the bulk cosmological constant is effectively zero. Instead we could take the limit $k \rightarrow \infty$ and N large but fixed. This corresponds to the free theory limit where the central charge is simply $c \simeq N - 1$, but $\Delta_{\max}$ can now be arbitrarily large. In this limit we see that the linearization instability disappears and the theory has arbitrarily large dimension states which can be interpreted as black hole microstates. It would be interesting to compute the degeneracies of these states and see if they can indeed be interpreted as black holes (see [54] for recent progress in this direction).

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3.5 Asymptotic Behaviour of χ_N and χ_∞

Here we collect several asymptotic formulas for $\mathcal{W}_N$ characters used in section 3.3 and sketch the derivations of these formulas.

3.5.1 Asymptotics of the Partition Function $F(q)$

As a warmup we first estimate the growth of the coefficients of partition function $F(q)$ defined as

$$F(q) = \sum_{n=0}^{\infty} p(n)q^n = \prod_{n=1}^{\infty} (1 - q^n)^{-1} . \quad (3.36)$$

where $p(n)$ is the number of partitions of the integer n . Our goal is to approximate $p(n)$ for large values of n . This is a classic computation which we review here for the sake of completeness.

We start with the inverse Laplace transform

$$p(n) = \frac{1}{2\pi i} \int_{\mathcal{C}} \frac{F(q)}{q^{n+1}} dq , \quad (3.37)$$

where $\mathcal{C}$ is a simple contour that encloses the origin. Since $F(q)$ has poles for $|q| = 1$ we must keep the contour $\mathcal{C}$ inside the unit circle in the complex q plane. Our strategy is to choose a contour $\mathcal{C}$ which approaches the unit circle $|q| = 1$ where we can approximate $F(q)$ by elementary functions.

Our next step is to write $F(q)$ as an elliptic modular function

$$F(e^{2\pi i\tau}) = e^{i\pi\tau/12} \eta(\tau)^{-1} , \quad (3.38)$$

with $\eta(\tau)$ the Dedekind eta function and $q = e^{2\pi i\tau}$. The eta function transforms simply under modular transformations. In particular,

$$\eta(-1/\tau) = (-i\tau)^{1/2} \eta(\tau) , \quad (3.39)$$

so that

$$F(e^{2\pi i\tau}) = \exp\left(\frac{i\pi}{12}(\tau + \tau^{-1})\right) (-i\tau)^{1/2} F(e^{-2\pi i/\tau}) . \quad (3.40)$$

When the imaginary part of τ is very small, so that we are close to $|q| = 1$, $F(e^{-2\pi i/\tau})$ approaches one and equation (3.40) becomes

$$F(e^{2\pi i\tau}) \sim \exp\left(i\pi(\tau + \tau^{-1})/12\right) (-i\tau)^{1/2} , \quad (3.41)$$

so that

$$p(n) \sim \int_{i\epsilon}^{i\epsilon+1} \exp\left(-2\pi i\tau\left(n - \frac{1}{24}\right) + \frac{\pi i}{12\tau}\right) (-i\tau)^{1/2} d\tau . \quad (3.42)$$

Using the saddle point approximation we obtain

$$p(n) \sim (\text{const}) \frac{1}{n} \exp\left(\pi \sqrt{\frac{2n}{3}}\right) . \quad (3.43)$$

This estimate is valid only in the limit $n \rightarrow \infty$. By refining the above argument we can estimate the size of the error terms in this approximation (see e.g. [19]).

3.5.2 Asymptotics of the $\mathcal{W}_N$ character χ_N

We now turn to the vacuum character for $\mathcal{W}_N$,

$$\chi_N = \left(\prod_{n=1}^{N-1} (1 - q^n)^{N-n} \right) F(q)^{N-1} = \sum_{n=0}^{\infty} p_n^N q^n . \quad (3.44)$$

whose coefficients are again given by the contour integral

$$p_n^N = \int_C \frac{\chi_N}{q^{n+1}} dq . \quad (3.45)$$

Again, it is necessary to keep the contour within the unit circle $|q| = 1$, where χ_N diverges. Our goal is to obtain an approximate expression for p_n^N by estimating χ_N when $|q| \rightarrow 1$.

We start by noting that χ_N differs from $F(q)^{N-1}$ only by the prefactor in parenthesis in (3.44). We then define the log of the polynomial prefactor in (3.44)

$$g(z) = \sum_{n=1}^{N-1} (N-n) \log(1 - q^n) = - \sum_{m=1}^{\infty} \sum_{n=1}^{N-1} (N-n) \frac{e^{-2\pi znm}}{m} \quad (3.46)$$

where $q = e^{-2\pi z}$. Our strategy will be to apply the Abel-Plana formula

$$\sum_{n=0}^{\infty} f(n) = \int_0^{\infty} f(x) dx + \frac{1}{2}f(0) + i \int_0^{\infty} \frac{f(ix) - f(-ix)}{e^{2\pi x} - 1} dx , \quad (3.47)$$

which relates an infinite sum to the residues of a complex function. To use this formula we will first take a derivate of (3.46) and add the $m = 0$ contribution

$$g'(z) = 2\pi \sum_{m=0}^{\infty} \sum_{n=1}^{N-1} (N-n) n e^{-2\pi z n m} - \frac{\pi}{3} N(N^2 - 1) . \quad (3.48)$$

so that (3.47) gives

$$g'(z) = \frac{1}{2z} N(N-1) - \frac{\pi}{6} N(N^2 - 1) + \mathcal{O}(z) \quad (3.49)$$

where we neglect terms which vanish in the $z \rightarrow 0$ limit, where $|q| \rightarrow 1$. Thus

$$g(z) = \frac{N(N-1)}{2} \log(z) + g_0 - \frac{\pi}{6} N(N^2 - 1) z + \mathcal{O}(z^2) \quad (3.50)$$

where we have introduced a constant of integration g_0 .

Following the arguments of section 3.5.1, we approximate the contour integral by the value of χ_N close to $|q| = 1$ where

$$\chi_N \sim z^{N(N-1)} \exp \left(-\frac{\pi}{6} N(N^2 - 1) z - \frac{\pi}{12} (N-1)(z - z^{-1}) \right) . \quad (3.51)$$

We have neglected an overall constant prefactor. Thus

$$p_n^N \sim i \int_{\mathcal{C}} z^{N(N-1)} \exp \left(-\frac{\pi}{6} N(N^2 - 1) z - \frac{\pi}{12} (N-1)(z - z^{-1}) + 2\pi n z \right) dz . \quad (3.52)$$

and the saddle point approximation

$$p_n^N \sim (\text{const}) n^{-\frac{1}{2}N(N-1) - \frac{3}{4}} \exp \left(\pi \sqrt{\frac{2(N-1)n}{3}} \right) . \quad (3.53)$$

gives an estimate for p_n^N which is valid in the large n limit. We note that the constant multiplying (3.53) depends on N . In the figure(3.1) we compare the asymptotic formula (3.53) with the actual values (3.44). We note that the p_n^N approach their asymptotic values more slowly as N increases. Indeed one can check that the error terms in this approximation are negligible only when $n \gg N^3$.

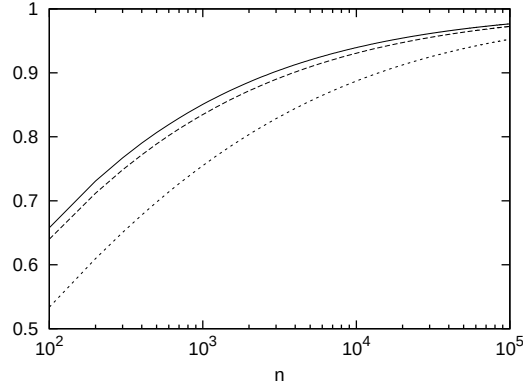


Figure 3.1: For $N = 2$ (straight line), $N = 3$ (long dash) and $N = 6$ (short dash), we plot the ratio of numerical value of $\log(p_n^N)$ over the approximated value given by (3.53). As N increases, we require larger values of n to reach the Cardy regime.

3.5.3 Asymptotics of the $\mathcal{W}_\infty$ character χ_∞ and the MacMahon Function

For the character of $\mathcal{W}_\infty$ we need to be a little more careful; we refer the reader to [55] for a more detailed analysis. The character is given by

$$\chi_\infty = M(q)F(q)^{-1} , \quad (3.54)$$

with $F(q)$ given by (3.36) and $M(q)$ the MacMahon function

$$M(q) = \prod_{n=1}^{\infty} (1 - q^n)^{-n} \quad (3.55)$$

As before we compute the coefficients of χ_∞ using a contour integral

$$p_n^\infty = \frac{1}{2\pi i} \int_{\mathcal{C}} \frac{\chi_\infty(q)}{q^{n+1}} dq . \quad (3.56)$$

where $\mathcal{C}$ encloses the origin and is contained in the unit circle.

We start by approximating the MacMahon function $M(q)$. Defining the logarithm

$$g(z) \equiv \log M(q) = - \sum_{n=1}^{\infty} n \log(1 - e^{-2\pi n z}) . \quad (3.57)$$

with $q = e^{-2\pi z}$ and applying (3.47) we find

$$g(z) = - \int_0^\infty x \log(1 - e^{-2\pi x z}) dx + 2 \int_0^\infty \frac{x}{e^{2\pi x} - 1} \log(2 \sin(\pi x z)) dx$$

$$g(z) = \frac{\zeta(3)}{4\pi^2 z^2} + \frac{1}{12} \log z + \frac{1}{12} (1 - \gamma + 6\zeta'(2)) + \mathcal{O}(z^2) \quad (3.58)$$

where we have used the Taylor expansion of $\log(2\sin(\pi z))$ at $z \rightarrow 0$ and computed the integrals explicitly. From this we can read off the behaviour of $M(q)$ at small z

$$M(e^{-2\pi z}) \sim z^{1/12} \exp\left(\frac{\zeta(3)}{4\pi^2 z^2}\right) . \quad (3.59)$$

where we have neglected an overall constant prefactor.

This leads to an approximate expression for χ_∞

$$\chi_\infty(e^{-2\pi z}) \sim z^{-5/12} \exp\left(\frac{\zeta(3)}{4\pi^2 z^2} + \frac{\pi}{12}(z - z^{-1})\right) . \quad (3.60)$$

so that

$$p_n^\infty = i \int_{\mathcal{C}} z^{-5/12} \exp\left(\frac{\zeta(3)}{4\pi^2 z^2} + \frac{\pi}{12}(z - z^{-1}) + 2\pi n z\right) dz , \quad (3.61)$$

and the saddle point approximation gives

$$p_n^\infty \sim (\text{const}) n^{-19/36} \exp\left(3 \left(\frac{\zeta(3)}{4} n^2\right)^{1/3} \left[1 - \frac{\pi^2}{18} \left(\frac{2}{\zeta(3)^2 n}\right)^{1/3}\right]\right) . \quad (3.62)$$

for large n . We note that this grows like $e^{n^{2/3}}$, which is faster than the $e^{n^{1/2}}$ behaviour obtained for finite N . In figure (3.2) we compare the asymptotic growth (3.62) with the actual coefficients of (3.54).

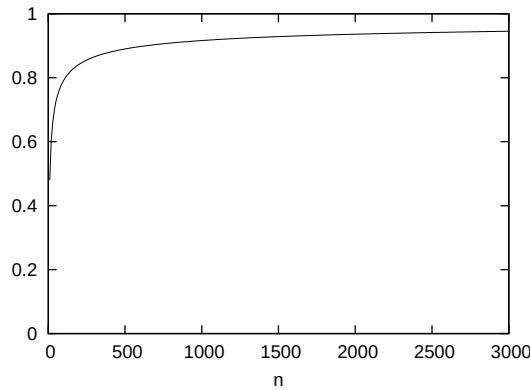


Figure 3.2: Ratio of $\log(p_n^\infty)$ over the saddle point approximation (3.62).

4

Discussion

In this thesis we have reviewed the current status of quantum gravity in three dimensions with a negative cosmological constant. We also considered massless higher spin fields and their interaction with gravity. We have found non-perturbative restrictions on the coupling of higher spin fields to gravity. We discussed $\mathcal{W}_N$ minimal models which provide an example of higher spin theories in AdS_3 that avoid the bound presented in chapter 3. This is understood by the presence of null vectors, which remove certain higher spin versions of the boundary graviton from the quantum spectrum. However, these theories appear to lack black holes with arbitrarily large charges.

To address the issue of black holes in higher spin theories on AdS_3 , we study the recent proposal of [56] for a spin-3 charged black hole. In chapter 3, we defined our boundary conditions by the requirement on the Chern-Simons connection $A - A_{\text{AdS}} \sim \mathcal{O}(1)$ at the asymptotic boundary. We can try to relax our boundary conditions to allow for a finite spin-3 charge. In [56] it is shown to be equivalent to a deformation of the boundary CFT by source terms for weights $(3, 0)$ and $(0, 3)$ primary fields. We rewrite (3.7) for the $SL(3, \mathbb{R})$ case as in [56]

$$\begin{aligned} [L_i, L_j] &= (i - j)L_{i+j} \\ [L_i, W_m] &= (2i - m)W_{i+m} \\ [W_m, W_n] &= \frac{\sigma}{3}(m - n)(2m^2 + 2n^2 - mn - 8)L_{m+n} \end{aligned} \tag{4.1}$$

where σ is a negative parameter appearing as a prefactor in the representation of the W 's. Note that L_0 and W_0 commute. Hence we can generalize our partition function to be taken as the trace over simultaneous eigenstates of L_0 and W_0 [56]

$$Z(\tau, \alpha, \bar{\tau}, \bar{\alpha}) = \text{Tr}_{\mathcal{H}} q^{L_0 - c/24} \bar{q}^{\bar{L}_0 - c/24} u^{W_0} \bar{u}^{\bar{W}_0} \quad (4.2)$$

where $u = e^{2\pi i \alpha}$. Define the weight with respect to W_0 to be

$$W_0 |\Omega\rangle = \delta |\Omega\rangle \quad (4.3)$$

and similarly with $\tilde{W}_0$.

In [56], the authors find a solution for the Chern-Simons connection for which $\frac{\partial \Delta}{\partial \alpha} = \frac{\partial \delta}{\partial \tau}$ and that has the same holonomies around the $z, \bar{z}$ cycles in the Euclidean geometry as the BTZ black hole. The first condition is necessary for their solution to have charges that obey the first law of thermodynamics. The holonomy is a gauge invariant quantity that the authors of [56] use to characterize their solution as a black hole. However, in the gauge they use the g_{tt} component never vanishes.

There are several interesting questions emerging from the analysis in [56]. The most basic question is whether there exists a gauge transformation of their solution for which the metric has an event horizon. The issue of boundary conditions should also be analyzed in terms of the metric and the spin-3 field. The generalized characters [56]

$$\chi(q, u) = \text{Tr}_{\mathcal{H}} q^{L_0 - c/24} u^{W_0} \quad (4.4)$$

and their asymptotic behaviour could also reveal intriguing properties of higher spin theories in three dimensions with a negative cosmological constant.

BIBLIOGRAPHY

- [1] A. Castro, A. Lepage-Jutier, and A. Maloney, “Higher Spin Theories in AdS3 and a Gravitational Exclusion Principle,” *JHEP* **1101** (2011) 142, 1012.0598.
- [2] S. Hawking and G. Ellis, *The Large scale structure of space-time*. Cambridge Monographs on Mathematical Physics. Cambridge University Press, 1973.
- [3] J. D. Bekenstein, “Black holes and the second law,” *Nuovo Cim. Lett.* **4** (1972) 737–740.
- [4] S. W. Hawking, “Particle Creation by Black Holes,” *Commun. Math. Phys.* **43** (1975) 199–220.
- [5] A. Strominger, “Black hole entropy from near-horizon microstates,” *JHEP* **02** (1998) 009, hep-th/9712251.
- [6] J. M. Maldacena, “The large N limit of superconformal field theories and supergravity,” *Adv. Theor. Math. Phys.* **2** (1998) 231–252, hep-th/9711200.
- [7] E. Witten, “Anti-de Sitter space and holography,” *Adv. Theor. Math. Phys.* **2** (1998) 253–291, hep-th/9802150.
- [8] M. Banados, C. Teitelboim, and J. Zanelli, “The Black hole in three-dimensional space-time,” *Phys. Rev. Lett.* **69** (1992) 1849–1851, hep-th/9204099.
- [9] M. Banados, M. Henneaux, C. Teitelboim, and J. Zanelli, “Geometry of the (2+1) black hole,” *Phys. Rev.* **D48** (1993) 1506–1525, gr-qc/9302012.
- [10] S. A. Hartnoll, “Lectures on holographic methods for condensed matter physics,” *Class.Quant.Grav.* **26** (2009) 224002, 0903.3246.
- [11] C. P. Herzog, “Lectures on Holographic Superfluidity and Superconductivity,” *J.Phys.A* **A42** (2009) 343001, 0904.1975.
- [12] J. D. Brown and M. Henneaux, “Central Charges in the Canonical Realization of Asymptotic Symmetries: An Example from Three-Dimensional Gravity,” *Commun. Math. Phys.* **104** (1986) 207–226.
- [13] S. M. Carroll, *Spacetime and geometry: An introduction to general relativity*. Addison-Wesley, 2004.

- [14] J. Polchinski, *String theory. Vol. 1: An introduction to the bosonic string*. Cambridge Monographs on Mathematical Physics. Cambridge University Press, 2001.
- [15] J. L. Cardy, “Operator Content of Two-Dimensional Conformally Invariant Theories,” *Nucl. Phys.* **B270** (1986) 186–204.
- [16] R. Arnowitt, S. Deser, and C. Misner, “Republication of: The dynamics of general relativity,” *General Relativity and Gravitation* **40** (2008) 1997–2027.
- [17] S. Giombi, A. Maloney, and X. Yin, “One-loop Partition Functions of 3D Gravity,” *JHEP* **0808** (2008) 007, 0804.1773.
- [18] A. Maloney and E. Witten, “Quantum Gravity Partition Functions in Three Dimensions,” *JHEP* **02** (2010) 029, 0712.0155.
- [19] T. M. Apostol, *Modular Functions and Dirichlet Series in Number Theory*. Graduate Texts in Mathematics. Springer-Verlag, 2nd ed., 1990.
- [20] J. D. Bekenstein, “Entropy bounds and black hole remnants,” *Phys. Rev.* **D49** (1994) 1912–1921, gr-qc/9307035.
- [21] G. ’t Hooft, “Dimensional reduction in quantum gravity,” gr-qc/9310026.
- [22] L. Susskind, “Trouble for remnants,” hep-th/9501106.
- [23] R. Bousso, “The holographic principle,” *Rev. Mod. Phys.* **74** (2002) 825–874, hep-th/0203101.
- [24] N. Arkani-Hamed, L. Motl, A. Nicolis, and C. Vafa, “The string landscape, black holes and gravity as the weakest force,” *JHEP* **06** (2007) 060, hep-th/0601001.
- [25] P. Haggi-Mani and B. Sundborg, “Free large N supersymmetric Yang-Mills theory as a string theory,” *JHEP* **04** (2000) 031, hep-th/0002189.
- [26] B. Sundborg, “Stringy gravity, interacting tensionless strings and massless higher spins,” *Nucl. Phys. Proc. Suppl.* **102** (2001) 113–119, hep-th/0103247.
- [27] S. E. Konstein, M. A. Vasiliev, and V. N. Zaikin, “Conformal higher spin currents in any dimension and AdS/CFT correspondence,” *JHEP* **12** (2000) 018, hep-th/0010239.
- [28] A. M. Polyakov, “Gauge fields and space-time,” *Int. J. Mod. Phys.* **A17S1** (2002) 119–136, hep-th/0110196.
- [29] A. Mikhailov, “Notes on higher spin symmetries,” hep-th/0201019.

- [30] A. Y. Segal, “Conformal higher spin theory,” *Nucl. Phys.* **B664** (2003) 59–130, [hep-th/0207212](#).
- [31] E. S. Fradkin and M. A. Vasiliev, “Cubic Interaction in Extended Theories of Massless Higher Spin Fields,” *Nucl. Phys.* **B291** (1987) 141.
- [32] E. S. Fradkin and M. A. Vasiliev, “On the Gravitational Interaction of Massless Higher Spin Fields,” *Phys. Lett.* **B189** (1987) 89–95.
- [33] M. A. Vasiliev, “Equations of motion for interacting massless fields of all spins in (3+1)-dimensions,”. In *Moscow 1990, Proceedings, Symmetries and algebraic structures in physics, pt. 1* 15–33.
- [34] M. A. Vasiliev, “More on equations of motion for interacting massless fields of all spins in (3+1)-dimensions,” *Phys. Lett.* **B285** (1992) 225–234.
- [35] E. Sezgin and P. Sundell, “Massless higher spins and holography,” *Nucl. Phys.* **B644** (2002) 303–370, [hep-th/0205131](#).
- [36] I. R. Klebanov and A. M. Polyakov, “AdS dual of the critical $O(N)$ vector model,” *Phys. Lett.* **B550** (2002) 213–219, [hep-th/0210114](#).
- [37] S. Giombi and X. Yin, “Higher Spin Gauge Theory and Holography: The Three-Point Functions,” *JHEP* **09** (2010) 115, [0912.3462](#).
- [38] S. Giombi and X. Yin, “Higher Spins in AdS and Twistorial Holography,” [1004.3736](#).
- [39] M. P. Blencowe, “A CONSISTENT INTERACTING MASSLESS HIGHER SPIN FIELD THEORY IN $D = (2+1)$,” *Class. Quant. Grav.* **6** (1989) 443.
- [40] E. Bergshoeff, M. P. Blencowe, and K. S. Stelle, “AREA PRESERVING DIFFEOMORPHISMS AND HIGHER SPIN ALGEBRA,” *Commun. Math. Phys.* **128** (1990) 213.
- [41] C. Aragone and S. Deser, “HYPERSYMMETRY IN $D = 3$ OF COUPLED GRAVITY MASSLESS SPIN $5/2$ SYSTEM,” *Class. Quant. Grav.* **1** (1984) L9.
- [42] A. Campoleoni, S. Fredenhagen, S. Pfenninger, and S. Theisen, “Asymptotic symmetries of three-dimensional gravity coupled to higher-spin fields,” *JHEP* **11** (2010) 007, [1008.4744](#).
- [43] M. Henneaux and S.-J. Rey, “Nonlinear $W(\text{infinity})$ Algebra as Asymptotic Symmetry of Three-Dimensional Higher Spin Anti-de Sitter Gravity,” [1008.4579](#).

- [44] M. R. Gaberdiel, R. Gopakumar, and A. Saha, “Quantum W-symmetry in AdS_3 ,” 1009.6087.
- [45] M. R. Gaberdiel and R. Gopakumar, “An AdS_3 Dual for Minimal Model CFTs,” 1011.2986.
- [46] A. Achucarro and P. K. Townsend, “A Chern-Simons Action for Three-Dimensional anti-De Sitter Supergravity Theories,” *Phys. Lett.* **B180** (1986) 89.
- [47] E. Witten, “(2+1)-Dimensional Gravity as an Exactly Soluble System,” *Nucl. Phys.* **B311** (1988) 46.
- [48] A. Achucarro and P. K. Townsend, “Extended supergravities in $d = (2+1)$ as Chern-Simons Theories,” *Phys. Lett.* **B229** (1989) 383.
- [49] C. Fronsdal, “Massless Fields with Integer Spin,” *Phys. Rev.* **D18** (1978) 3624.
- [50] J. M. Maldacena and A. Strominger, “ $\text{AdS}(3)$ black holes and a stringy exclusion principle,” *JHEP* **12** (1998) 005, hep-th/9804085.
- [51] E. Witten, “Three-Dimensional Gravity Revisited,” arXiv:0706.3359 [hep-th].
- [52] R. Dijkgraaf, J. M. Maldacena, G. W. Moore, and E. P. Verlinde, “A black hole farey tail,” hep-th/0005003.
- [53] P. Kraus and F. Larsen, “Partition functions and elliptic genera from supergravity,” *JHEP* **01** (2007) 002, hep-th/0607138.
- [54] E. Kiritsis and V. Niarchos, “Large- N limits of 2d CFTs, Quivers and AdS_3 duals,” 1011.5900.
- [55] E. M. Wright, “Asymptotic partition formulae, I: Plane partitions,” *Quart. J. Math.* **2** (1931) 177–189.
- [56] M. Gutperle and P. Kraus, “Higher Spin Black Holes,” *JHEP* **1105** (2011) 022, 1103.4304.