Short title of the thesis :

Dynamics of a cluster of pipes in bounded axial flow

DYNAMICS OF A CLUSTER OF PIPES
CONVEYING FLUID IN BOUNDED AXIAL FLOW

by

PAUL BESANCON

A Thesis submitted to the Faculty of Graduate Studies
and Research in partial fulfillment of the
requirements for the degree of
Master of Engineering

Department of Mechanical Engineering
McGill University
Montreal, Canada
July, 1978

ABSTRACT

Flow-induced vibration in tube banks is of great concern, particularly in high performance heat exchangers and nuclear reactor fuel bundles. This thesis deals with the dynamics of a cluster of parallel flexible pipes conveying fluid and simultaneously immersed in bounded axial flow. The equations of small motions are derived taking into account the contribution of the two flows and the hydrodynamic forces which result from the coupling between motions of the cylinders. Solutions are obtained by means of Galerkin's technique and yield the eigenfrequencies of the system. It is shown that for sufficiently high flow velocities, either internal or external, the system is subject to divergence and flutter and that the domain of stability is severely reduced by close spacing in the cluster.

The role of fluid coupling in the propagation of transient perturbations or excitations is also investigated and reveals to be of great importance since it may give rise to appreciable nodes of vibrations, or even to resonances.

RESUME

Les vibrations induites par des fluides s'écoulant dans des faisceaux de tubes sont particulièrement préoccupantes dans les échangeurs thermiques de haute performance et le coeur des réacteurs nucléaires. Cette thèse traite des vibrations d'un faisceau de tubes contenu dans un conduit rigide, et soumis simultanément à un écoulement interne et à un courant externe axial. Dans les équations de mouvement sont pris en compte la contribution des deux fluides ainsi que les forces hydrodynamiques qui résultent du couplage entre les mouvements des tubes. Les solutions sont obtenues à l'aide de la méthode de Galerkin et fournissent les fréquences propres du système. On montre que pour des vitesses de fluide suffisamment élevées, internes ou externes, le système se déstabilise par flambage puis flottement et que le domaine de stabilité est sévèrement réduit par un resserrement étroit du faisceau.

Le rôle de couplage tenu par le fluide externe dans la propagation de perturbations transitoires ou d'excitations est ensuite examiné; il se révèle être assez préoccupant car il peut engendrer des noeuds appréciables de vibrations ou même des résonances.

ACKNOWLEDGEMENTS

The author is deeply grateful to his research director, Professor M.P. Paidoussis, for his valuable guidance and suggestions, and the continuous assistance which he has offered throughout the course of this study.

Sincere thanks are due to Jean-Pierre Chaubernard and Claude Güernier for the fruitful discussions they instigated, and to Miss P. Fong for the excellent job of typing this thesis.

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NOMENCLATURE

1. Latin Alphabet

A_i, A_e	internal and external cross-sectional areas
A_{CH}	channel flow area
a_{rs}, b_{rs}, d_{rs}	coefficients defined in Eq. (3.13)
C_f, C_D	friction drag coefficients defined in Eq. (2.9)
C_{fe}, C_{fi}, C_{fx}	friction coefficients defined in Eq. (2.19)
$C_j, \bar{C}_j$	constants of integration defined in Eq. (4.3)
c, c_f	parameters defined in Eq. (3.5)
c_j	complex constants defined in Eq. (4.1)
$\underset{\sim}{c}, \bar{\underset{\sim}{c}}$	matrices defined in Eq. (4.6)
$\underset{\sim}{C}_v$	matrix of viscous coupling
D_e	external diameter of cylinders
D_h	hydraulic diameter equal to $4A_{CH}/[2\pi(R_o + kR)]$
D_b	drag force at the end of the beam
D/Dt	convective derivative for external flow equal to $\partial/\partial t + U_e\, \partial/\partial x$
EI	flexural rigidity of a cylinder
$(F^j_A)_z, (F^j_A)_y$	inviscid hydrodynamic forces on cylinder j per unit length, in the z- and y-directions
$F^j_{en}, F^j_{et}, F^j_{in}, F^j_{it}$	external and internal forces normal and tangential to the surface of cylinder j
F^j_{px}, F^j_{ph}	forces due to steady-state pressure on cylinder j, per unit length, acting on x- and h^j-directions
g	gravitational constant
G_c	(smallest inter-cylinder gap)/(cylinder radius)

G_w	(smallest cylinder-to-channel gap)/(cylinder radius)
h^j	lateral displacement of cylinder j
h	equal to D_e/D_h
$\vec{i}$	unit vector on x-axis
k	number of cylinders in the cluster
L	length of cylinders
M^j	moment on cross-section of cylinder j
$\underset{\sim}{M}_v$	matrix of added mass coefficients
$\underset{\sim}{M}, \underset{\sim}{C}, \underset{\sim}{K}$	matrices defined in Eq. (3.16)
m	mass of the beam per unit length
N	number of comparison functions used in solving the equation of motion
p_i, p_e	internal and external steady-state pressures
$\bar{p}_i, \bar{p}_e$	mean internal and external pressures at x = L/2
$p_i^j(\tau)$	generalized coordinates defined in Eq. (3.9)
$\underset{\sim}{p}_i$	vector of generalized coordinates defined in Eq. (3.12)
$\underset{\sim}{p}$	vector defined in Eq. (3.14)
Q^j	shear force at a cross-section of cylinder j
q_{et}^j, q_{en}^j	tangential and normal frictional force per unit length due to external flow on cylinder j
r_j	radial coordinates in the polar coordinate systems defined in Appendix A
R_j	external radius of cylinder j
R_o	radius of the external channel
R	external radius common to all cylinders
R_{ij}	distance between the centres of cylinder i and j

$\underset{\sim}{R}_j, \underset{\sim}{I}_j$	vectors defined in Eq. (4.3)
t	time
T^j	tension in the beam j
$\bar{T}^j$	tension at the middle of the beam j
$\underset{\sim}{T}$	matrix defined in Eq. (4.7)
U_e, U_i	external and internal fluid flow velocities
u_e, u_i	dimensionless external and internal fluid flow velocities
v^j, w^j	lateral displacements of cylinder j in the y- and z-directions, respectively
$\vec{V}_i$	absolute velocity of a particle of internal fluid
x, y, z	cartesian coordinates defined in Figure 1
$\underset{\sim}{Y}$	matrix defined in Eq. (3.17)
$\underset{\sim}{z}$	state vector defined in Eq. (3.17)

2. <u>Greek Alphabet</u>

α	dimensionless beam damping coefficient
β_e, β_i	dimensionless external and internal fluid densities
$\gamma_e, \gamma_i, \gamma$	dimensionless parameters defined in Eq. (3.5)
Γ	dimensionless uniform tension
δ	dimensionless parameter defined in Eq. (2.23)
δ_{ij}	Kronecker's delta
ϵ	length/diameter of cylinders
$\varepsilon_{j\ell}, e_{j\ell}, \kappa_{j\ell}, k_{j\ell}$	dimensionless added mass coefficients
$\zeta_{j\ell}, g_{j\ell}, \sigma_{j\ell}, s_{j\ell}$	dimensionless viscous coupling coefficients

$\underset{\sim}{\eta}$	displacement vector defined in Eq. (3.7)
θ_j, θ_o	angular coordinates defined in Figure 2
Θ	dimensionless coefficient defined in Eq. (3.5)
λ_i	dimensionless eigenvalues of the beam equation
$\underset{\sim}{\Lambda}$	eigenvector defined in Eq. (3.19)
μ	internal damping coefficient of beams
ν	Poisson ratio of beam material
ξ	dimensionless axial coordinate
Π_e, Π_i	dimensionless external and internal fluid pressures
ρ_e, ρ_i	external and internal fluid densities
τ	dimensionless time
$\phi_j(r_j, \theta_j)$	fluid potential due to presence of cylinder j
ϕ_j^i	equal to ϕ_j written in terms of the (r_i, θ_i)
ϕ, ϕ^i	total potential and total potential written in terms of the (r_i, θ_i) system
$\phi_i(\xi)$	beam eigenfunctions used as comparison functions
ω	dimensionless eigenfrequency
Ω	angle of incidence to the oncoming flow
$\Omega_j, \bar{\Omega}_j$	real and imaginary part of the eigenfrequency, respectively

Other symbols used are defined in the text.

CHAPTER I

INTRODUCTION

Dynamics of pipes conveying fluid and, to a lesser extent of structures immersed in axial flow, have received notable attention in the past thirty years. The remarkable development of this subject was originally due to concerns experienced with vibrations of the Trans-Arabian pipeline, and more recently with nuclear reactor fuel-element bundles and some heat exchangers where the flow is mainly axial.

The problem of self-excited oscillations of pipes conveying fluid has been studied quite extensively since this phenomenon was first recognized by M. Brillouin in 1885. One of his students, Bourrières, published in 1939 an outstanding paper [1] on the oscillatory instabilities of cantilevered pipes, but which remained unknown until it was rediscovered by Paidoussis in 1972.

In the early fifties several authors derived general mathematical expressions of the equations of motion, and among them Feodos'ev and Housner [2] who found that for sufficiently high flow velocities tubes simply supported at both ends buckled like beams under axial loading. But it was not until 1966 that Gregory and Paidoussis [3] revealed the existence of oscillatory instabilities (flutter) of cantilevered pipes conveying fluid. In 1973 Paidoussis and Issid [4] found that conservative systems, such as simply supported beams, could also be subject to flutter if the flow velocity was increased beyond the point of divergence (buckling), at least according to linear theory.

The dynamics of a flexible cylinder immersed in axially flowing fluid remained uninvestigated until recent years when Paidoussis [5] undertook both theoretical and experimental study of the free motions of a cylinder in axial flow. It was shown that moderate flow velocities conduced to damp free oscillations and to reduce the natural frequencies of vibration, but that sufficiently high flow velocities give rise to fluidelastic instabilities, namely buckling and flutter. However, these instabilities were shown to be of little practical concern since the critical flow velocities required for their inception are far beyond those encountered in engineering applications.

This work was extended by Chen and Wambsganss [6] who showed that the presence of motionless boundaries, such as an enclosing channel, increased the inviscid hydrodynamic forces acting on the cylinder. These forces result from the pressure field generated whenever the fluid is displaced by the moving cylinder and are proportional to the lateral accelerations of the cylinder and to so-called "added mass" coefficients. The effect of confinement was later investigated by Paidoussis [7] for a cluster of cylinders in bounded flow; it was found that proximity to the channel or to other cylinders increased the hydrodynamic virtual mass of a cylinder and that a cluster was severely destabilized by close spacing.

More recently Chen [8] dealt with the inviscid hydrodynamic coupling of a cluster of cylinders in unbounded still fluid. The coupling was expressed mathematically in the form of a non-diagonal mass matrix, thus expressing the force acting on a cylinder as a linear combination of the accelerations of all the cylinders in the bundle. Paidoussis and Suss [9] extended the work by considering both the fluid coupling in the motions of the cylinders and the confining presence of a surrounding channel. Two distinct methods have been developed by Paidoussis, Suss and Pustejovsky [10] for the calculation of the virtual masses of such clusters of parallel cylinders in liquid-filled channels. The first method is based on potential flow theory and the second, less geometrically restrictive but computer-time consuming, makes use of fluid finite elements. From the results thus obtained, it transpires that for tight configurations the critical flow velocities are much lower than those necessary for a solitary cylinder in unbounded flow, and reach values which might be of practical concern.

Despite its practical interest in heat exchangers, the case of a tube subjected to both internal and external axial flow received very little attention until 1978 when the complete study of instabilities of tubular beams induced by internal and external axial flow was first published by Hannoyer and Paidoussis [11]. It was shown that for sufficiently high flow velocities, either internal or external, the pipe is subject to buckling or flutter, and that the effect of the two flows is generally additive for a tubular beam with both ends supported, so that if either flow is just below the corresponding critical value for instability, an increase in the other flow precipitates instability; on the other hand, for cantilevered beams, the effects are not additive, and increase in one flow may actually stabilize a nearly unstable system.

The response of structures to force fields was studied mainly by Chen and Wambsganss [12], Lakis and Paidoussis [13] for random pressure fields, and Paidoussis [7] who produced a general analytical method for the response to an arbitrary force field. In a recent publication, Chen [14] deals with the steady-state response of a row of tubes where one of them is subject to a steady excitation, and shows that the response of the others may be very large if the frequency of excitation is within a certain band of frequencies. In the same paper is also examined the transient response of a tube bank in stationary liquid where a tube rupture is modelled, and it appears that the tubes not directly excited may have a significant amplitude.

The present study is concerned with the dynamics of a cluster of parallel flexible pipes in a cylindrical channel, simultaneously subjected to internal and to external axial flow. The organization of the thesis is as follows:

In Chapter II are detailed and formulated all the forces acting on the system and the general equation of motion is derived.

In Chapter III the equation of motion is rendered dimensionless and solved by means of Galerkin's method, thus leading to the eigenfrequencies of vibration.

Chapter IV deals with the transient response of the structure after initial displacements, or when some cylinders are subjected to prescribed motions.

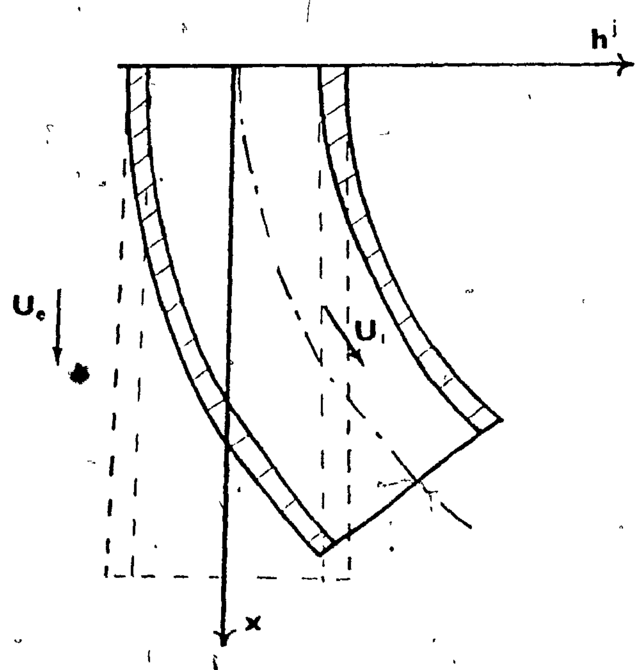
Finally, Chapter V presents the theoretical results and their interpretation. The respective and combined effects on stability of the internal and external flows are discussed, and stability maps are produced to lay stress on the influence of several parameters. Finally, attention will also be focused on problems involved with the response of the tube bank under certain excitations.

CHAPTER II

DERIVATION OF THE EQUATIONS OF MOTION

The system under consideration consists of a cluster of uniform and identical cylinders contained in a rigid cylindrical channel. The axes of the cylinders at rest are all parallel to the axis of the channel which is vertical. This direction will be referred to as that of the x-axis, pointing downwards. The cylinders are slender and we assume that the lateral motions are sufficiently small so that no separation of the flow occurs across the cylinder. An example of such a system is illustrated in Figure 1.

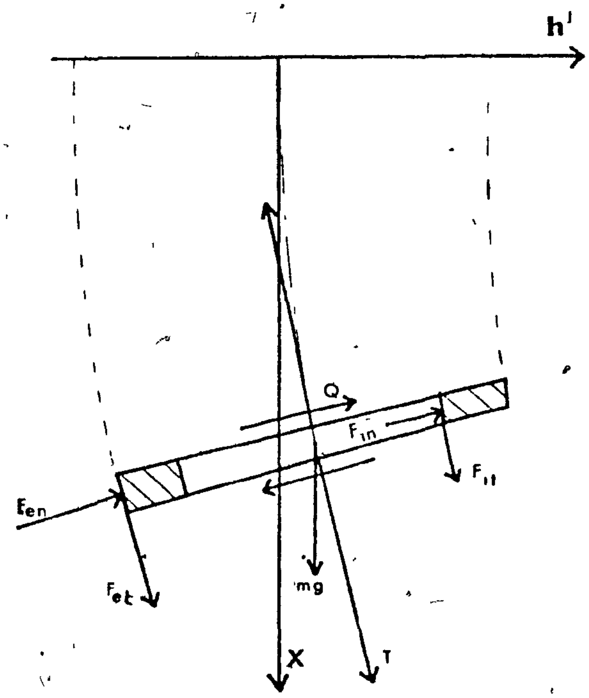
Two incompressible fluids flow in the direction of the positive x-axis, one into the surrounding channel and the other inside the cylinders. Both unperturbed flows are assumed to be perfectly uniform and steady. The displacement of one of the cylinders, say cylinder j, about its position of rest will be denoted by $h^j(x,t)$.



We assume there is no torsion nor rotation around the x-axis. The cylinders are supported at both of their extremities, and it is supposed that the supports produce negligible disturbances to the idealized flow conditions.

2.1 Dynamic Equation of a Cylinder

The equations of motion for lateral displacement of a tube submitted to both internal and external flow have been derived by Paidoussis and Hannoyer and can be found in detail in their papers [4] and [11]. We restrict the analysis to small displacements and small slopes so that the fluid forces on each element of the cylinder may be assumed to be the same as those acting on a corresponding element of a long straight cylinder of the same cross-sectional area and inclination.



The first order balance of forces in the x- and h^j-directions for cylinder j yields the two equations:

$$0 = \frac{\partial T^j}{\partial x} + F^j_{it} + F^j_{et} - (F^j_{in} + F^j_{en}) \frac{\partial h^j}{\partial x} + \frac{\partial}{\partial x}(Q^j \frac{\partial h^j}{\partial x}) + mg \qquad (2.1)$$

$$m \frac{\partial^2 h^j}{\partial t^2} = \frac{\partial Q^j}{\partial x} + F^j_{in} + F^j_{en} + (F^j_{it} + F^j_{et}) \frac{\partial h^j}{\partial x} + \frac{\partial}{\partial x}(T^j \frac{\partial h^j}{\partial x}) \qquad (2.2)$$

The cylinder is considered as a Bernoulli-Euler beam, of mass per unit length m and flexural rigidity EI. It is subjected to gravity force, tension T^j, shear force Q^j, and fluid forces F^j_{in}, F^j_{it}, F^j_{en}, F^j_{et} accounting for the normal and tangential forces per unit length exerted on the beam by the internal and external fluid, respectively.

A third equation is derived from the moments; upon neglecting moments induced by external and internal fluids, we obtain:

$$Q^j = - \frac{\partial M^j}{\partial x} \; . \qquad (2.3)$$

Using a viscoelastic Kelvin-Voigt model to represent the internal damping of the material, we set:

$$M^j = EI \frac{\partial^2 h^j}{\partial x^2} + \mu I \frac{\partial^3 h^j}{\partial t \partial x^2} \; , \qquad (2.4)$$

where μ is the internal damping coefficient.

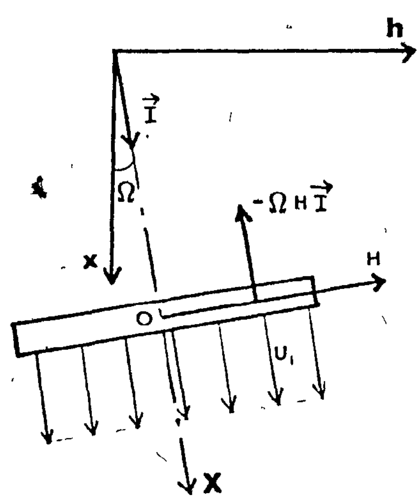
Clearly, Q^j may be eliminated by (2.3) and (2.4), and T^j may be calculated by integration of Eq. (2.1); hence, Eq. (2.2) becomes the equation of motion. The derivation will be completed once we obtain expressions for the fluid forces F^j_{it}, F^j_{in}, F^j_{et}, F^j_{en} and boundary conditions for T^j.

2.2 Internal Fluid Forces

We assume that in any cross-section the axial velocity profile is uniform and that there is no significant secondary cross-flow. Such approximations are valid for a turbulent and fully developed boundary layer and when the curvatures of the flow trajectories are small. Hence, the value of U_i is constant in a cross-section.

In order to take into account the effect of rotation of a cross-section around the z-axis, we define two axes OX, OH attached to the cross-section of the beam and an angle of rotation Ω

$$\Omega = \frac{\partial h}{\partial x} .$$



If we denote $\dot{\Omega} = \frac{d\Omega}{dt}$, the flow velocity relative to the beam has an additional velocity component due to rotation:

$$- \dot{\Omega}H \quad .$$

The rate of change of the flow momentum, inside the control volume Δw of an element of the tube of length dx, may be expressed in terms of the convective derivative of the absolute velocity V_i, as follows:

$$\frac{d}{dt} \int_{\Delta w} \rho_i \vec{V}_i d(\Delta w) \quad .$$

where ρ_i is the internal fluid density,

$\vec{V}_i = \vec{U}_i - \dot{\Omega}H\vec{I} + \frac{d\vec{h}}{dt}$,

$\vec{I}$ is the unit vector on X-axis,

$\frac{d\vec{h}}{dt}$ is the velocity of point 0 on the centerline.

As the size of the control volume remains constant, we may write

$$\rho_i dx \frac{d}{dt} \int_{A_i} (\vec{U}_i - \dot{\Omega}H\vec{I} + \frac{d\vec{h}}{dt}) \, dA_i \quad ,$$

where A_i is the internal cross-sectional area.

This integral simply reduces to

$$\rho_i A_i dx \left(\frac{d\vec{U}_i}{dt} + \frac{d^2\vec{h}}{dt^2}\right) \quad .$$

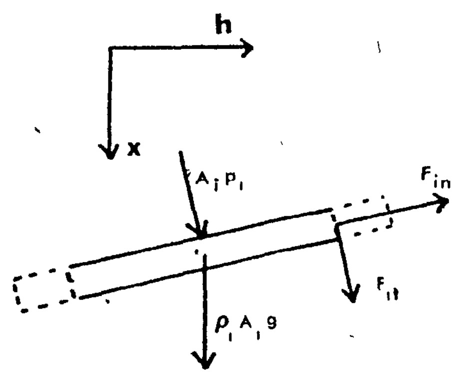
Hence, assuming that the internal velocity is uniform and constant, the rate of change of fluid momentum may be written:

(a) in the x-direction: $\rho_i A_i \dfrac{dU_i}{dt} = \rho_i A_i U_i \dfrac{\partial U_i}{\partial x} = 0$;

(b) in the h^j-direction: $\rho_i A_i \dfrac{d^2h^j}{dt^2} = \rho_i A_i \dfrac{d}{dt} \left(\dfrac{\partial h^j}{\partial x} U_i + \dfrac{\partial h^j}{\partial t}\right)$

$$= \rho_i A_i \left(\frac{\partial^2 h^j}{\partial t^2} + 2U_i \frac{\partial^2 h^j}{\partial x \partial t} + U_i^2 \frac{\partial^2 h^j}{\partial x^2}\right) .$$

The external forces acting on the fluid element balance this change of momentum leading to



$$\rho_i A_i g - \frac{\partial}{\partial x} (A_i p_i) - F^j_{it} + F^j_{in} \frac{\partial h^j}{\partial x} = 0 ,$$

$$- \frac{\partial}{\partial x} \left(A_i p_i \frac{\partial h^j}{\partial x}\right) - F^j_{it} \frac{\partial h^j}{\partial x} - F^j_{in} = \rho_i A_i \left(\frac{\partial}{\partial t} + U_i \frac{\partial}{\partial x}\right)^2 h^j ,$$

where $p_i(x)$ is the average pressure along the median line. To give them the form required for Eqs. (2.1) and (2.2) these two equations may be written as follows:

$$F^j_{it} - F^j_{in} \frac{\partial h^j}{\partial x} = - \frac{\partial}{\partial x} (A_i p_i) + \rho_i A_i g \quad ,$$
$$\tag{2.5}$$
$$F_{in} + F^j_{it} \frac{\partial h^j}{\partial x} = - \frac{\partial}{\partial x} (A_i p_i \frac{\partial h^j}{\partial x}) - \rho_i A_i (\frac{\partial}{\partial t} + U_i)^2 h^j \quad .$$

2.3 Normal and Tangential External Forces

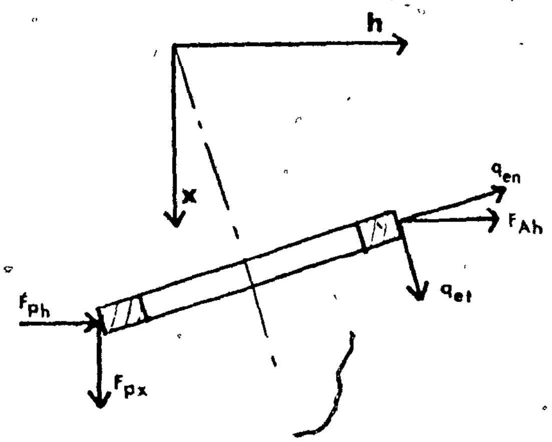
2.3.A. General Formulation

As the secondary (cross) flow cannot be neglected, since the axis of the beam no longer coincides with the axial velocity, it is impracticable to derive dynamic equations from a control volume. Therefore, the three types of forces acting on the external surface which will be taken into account, will be formulated separately.

(i) The lateral force due to the motion of the cylinder in the inviscid fluid: F^j_{Ah} .

(ii) The tangential and normal frictional forces due to the fluid viscous flow: q^j_{et} and q^j_{en} .

(iii) The steady-pressure forces in the x- and h^j-directions: F^j_{px}, F^j_{ph} .



A first order balance of forces in the normal and tangential directions yields:

$$F^j_{en} = F^j_{Ah} + q^j_{en} + F^j_{ph} - F^j_{px} \frac{\partial h^j}{\partial x} ,$$

$$F^j_{et} = F^j_{Ah} + q^j_{et} + F^j_{px} + F^j_{ph} \frac{\partial h^j}{\partial x} .$$

Writing these equations in the form required for (2.1) and (2.2), we obtain:

$$F^j_{et} - F^j_{en} \frac{\partial h^j}{\partial x} = q^j_{et} - q^j_{en} \frac{\partial h^j}{\partial x} + F^j_{px} ,$$

$$F^j_{en} + F^j_{et} \frac{\partial h^j}{\partial x} = F^j_{Ah} + q^j_{en} + q^j_{et} \frac{\partial h^j}{\partial x} + F^j_{ph} . \tag{2.6}$$

We shall now proceed with the formulation of F^j_{Ah}, q^j_{et} and q^j_{en}, F^j_{px} and F^j_{ph}.

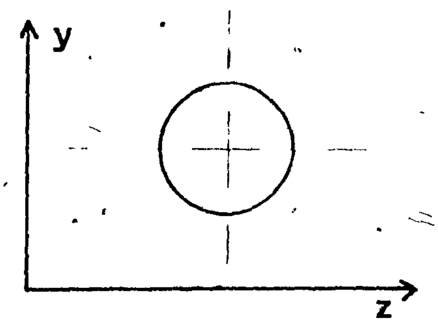
2.3.B. Inviscid Hydrodynamic Forces

According to Lighthill [15], the lateral flow generated by a slender body could be represented by a two-dimensional potential flow. It was shown that the force per unit length due to the inviscid flow around such a body of uniform cross-section could be related to the fluid potential by the contour integral:

$$F_{Ah} = - \int_{S_x} \rho_e \left[\frac{\partial}{\partial t} + U_e \frac{\partial}{\partial x}\right] \phi \ \hat{h} \, ds = - \int_{S_x} \rho_e \frac{D\phi}{Dt} \hat{h} \, ds \tag{2.7}$$

where $\hat{h}$ is the unit vector in the direction of the displacement h, and ϕ the fluid potential.

The integral is taken around the perimeter of the body at position x. In order to allow for the possibilities of coupling between mutually perpendicular motions of neighbouring cylinders, the arbitrary lateral displacement $h^j(x,t)$ is resolved in two orthogonal directions y and z.



By definition, the fluid potential ϕ must satisfy

$$\vec{\nabla}\phi = \text{fluid velocity vector (in the y-z plane)} \quad ;$$

$$\nabla^2\phi = 0 \quad .$$

This equation must be solved with the proper boundary conditions, namely

(i) the fluid velocity normal to the inner surface of the enclosing channel is zero,

(ii) the fluid velocity normal to the surface of each of the k cylinders is equal to the velocity of the cylinder in that direction.

S. Suss [10] has solved this problem (vide Appendix A) and derived the two components on y- and z-axis of the inviscid hydrodynamic force exerted on cylinder j,

$$(F_A^j)_z = - \rho_e A_e \sum_{\ell=1}^{k} \left[\varepsilon_{j\ell} \frac{D^2 w^\ell}{Dt^2} + e_{j\ell} \frac{D^2 v^\ell}{Dt^2} \right] ,$$

$$(F_A^j)_y = - \rho_e A_e \sum_{\ell=1}^{k} \left[\kappa_{j\ell} \frac{D^2 w^\ell}{Dt^2} + k_{j\ell} \frac{D^2 v^\ell}{Dt^2} \right] , \tag{2.8}$$

where A_e and ρ_e are the external cross-section area and fluid density, v^ℓ and w^ℓ the velocity components of cylinder ℓ in y- and z-directions. The dimensionless constants $\varepsilon_{j\ell}$, $e_{j\ell}$, $\kappa_{j\ell}$, $k_{j\ell}$ are the coefficients of the added mass matrix $\underset{\sim}{M}_v$

$$\underset{\sim}{M}_v = - \begin{bmatrix} [\underset{\sim}{\varepsilon}_{j\ell}] & [\underset{\sim}{e}_{j\ell}] \\ [\kappa_{j\ell}] & [\underset{\sim}{k}_{j\ell}] \end{bmatrix} \text{ 2kx2k matrix} .$$

2.3.C. Viscous Frictional Forces

The nature of the force acting on a long inclined cylinder in viscous flow has been studied by Taylor [16] and the relations proposed there have been utilized by Paidoussis [5, 7] for slender, flexible cylinders. The linearized version of the normal and longitudinal components of the force, per unit length, for a displacement h, are given by

$$q_{en} = -\rho_e R U_e^2 C_f \Omega - \rho_e R C_D \frac{\partial h}{\partial t} \,, \qquad (2.9)$$

$$q_{et} = \rho_e R U_e^2 C_f \,,$$

where Ω is the angle of incidence to the oncoming flow and is given by

$$\Omega = \tan^{-1}\left(\frac{\partial h}{\partial x}\right) + \tan^{-1}\left(\frac{1}{U_e}\frac{\partial h}{\partial t}\right) \,.$$

But, in our system, the fluid velocity approaching a particular cylinder j is not purely axial, but has small lateral components $(V_j)_y$ and $(V_j)_z$ and makes an angle with the positive x-axis of $\tan^{-1}[(V_j)_y/U_e]$ in the xy-plane, and an angle of $\tan^{-1}|(V_j)_z/U_e|$ in the xz-plane. Hence, assuming that $(V_j)_y$ and $(V_j)_z$ are of the same order of magnitude as $(1/U)(\partial h^j/\partial t)$, the angles of incidence for the j^{th} cylinder are

$$(\Omega_j)_y = \tan^{-1}\left(\frac{\partial v^j}{\partial x}\right) + \tan^{-1}\left\{\left[\frac{\partial v^j}{\partial t} - (V_j)_y\right]/U_e\right\} \,,$$

$$(\Omega_j)_z = \tan^{-1}\left(\frac{\partial w^j}{\partial x}\right) + \tan^{-1}\left\{\left[\frac{\partial w^j}{\partial t} - (V_j)_z\right]/U_e\right\} \,.$$

Replacing in (2.9) this yields the lateral forces in both directions

$$(q_{en}^j)_y = -\rho_e RU_e C_f\left[\frac{\partial v^j}{\partial t} + U_e\frac{\partial v^j}{\partial x} - (V_j)_y\right] - \rho_e C_D\left[\frac{\partial v^j}{\partial t} - (V_j^o)_y\right] ,$$

$$(q_{en}^j)_z = -\rho_e RU_e C_f\left[\frac{\partial w^j}{\partial t} + U_e\frac{\partial w^j}{\partial x} - (V_j)_z\right] - \rho_e RC_D\left[\frac{\partial w^j}{\partial t} - (V_j^o)_z\right] \quad (2.10)$$

$$q_{et}^j = \rho_e RU_e^2 C_f ,$$

$(V_j^o)_y$ and $(V_j^o)_z$ being the values of $(V_j)_y$ and $(V_j)_z$ when $U_e = 0$. The velocities $(V_j)_y$ and $(V_j)_z$ may be obtained from the fluid potential, taking into account the coupling between the cylinders, and Suss [9] has derived the following expressions for the viscous forces (vide Appendix A)

$$(q_{en}^j)_y = -\rho_e RU_e C_f \sum_{\ell=1}^{k}\left[\sigma_{j\ell}\frac{Dw^\ell}{Dt} + s_{j\ell}\frac{Dv^\ell}{Dt}\right]$$

$$- \rho_e RC_D \sum_{\ell=1}^{k}\left[\sigma_{j\ell}\frac{\partial w^\ell}{\partial t} + s_{j\ell}\frac{\partial v^\ell}{\partial t}\right] ,$$

$$(q_{en}^j)_z = -\rho_e RU_e C_f \sum_{\ell=1}^{k}\left[\zeta_{j\ell}\frac{Dw^\ell}{Dt} + g_{j\ell}\frac{Dv^\ell}{Dt}\right] \quad (2.11)$$

$$- \rho_e RC_D \sum_{\ell=1}^{k}\left[\zeta_{j\ell}\frac{\partial w^\ell}{\partial t} + g_{j\ell}\frac{\partial v^\ell}{\partial t}\right] ,$$

$$q_{et}^j = \rho_e RU_e^2 C_f ,$$

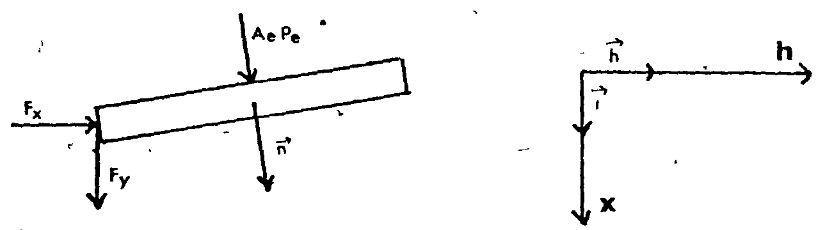
where $\zeta_{j\ell}$, $g_{j\ell}$, $\sigma_{j\ell}$ and $s_{j\ell}$ are the coefficients of the viscous coupling matrix $\underset{\sim}{C}_v$

$$\underset{\sim}{C}_V = \begin{bmatrix} [\underset{\sim}{\varsigma}_{j\ell}] & [\underset{\sim}{g}_{j\ell}] \\ [\underset{\sim}{\sigma}_{j\ell}] & [\underset{\sim}{s}_{j\ell}] \end{bmatrix} \text{ 2kx2k matrix} \quad .$$

2.3.D. <u>Pressure Forces</u>

The easiest way to derive these forces is to consider an element dx of the beam supposed to be immersed in fluid on all sides; the resultant of the hydrostatic forces is the buoyancy force:

$$\vec{F}_B = - \frac{\partial p_e}{\partial x} A_e \vec{i} \quad .$$



Now, if we subtract from the buoyancy force $\vec{F}_B$ the pressure forces acting on the top and bottom flat faces, we shall obtain the resultant of the pressure forces acting on the outer surface, i.e.

$$\vec{F}_p = - \frac{\partial p_e}{\partial x} A_e \vec{i} - [(-A_e p_e \vec{n})_{x+dx} + (A_e p_e \vec{n})_x]$$

$$= - \frac{\partial p_e}{\partial x} A_e \vec{i} + \frac{\partial}{\partial x} (A_e p_e \vec{n}) \quad .$$

Hence, the two components of $\vec{F}_p$ for cylinder j are

$$F^j_{px} = \frac{\partial}{\partial x} (A_e p_e) - A_e \frac{\partial p_e}{\partial x} \quad ,$$

$$F^j_{ph} = \frac{\partial}{\partial x} (A_e p_e \frac{\partial h^j}{\partial x}) \quad . \tag{2.12}$$

2.4 Derivation of the Equations of Small Motions

2.4.A. Equation of Small Motions

Combining equations (2.1) with (2.5) and (2.6) yields

$$0 = \frac{\partial T^j}{\partial x} - \frac{\partial A_i p_i}{\partial x} + \rho_i A_i g + q^j_{et} - q^j_{en} \frac{\partial h^j}{\partial x} + F^j_{px} + \frac{\partial}{\partial x} (Q^j \frac{\partial h^j}{\partial x}) + mg \quad ,$$

$$m \frac{\partial^2 h^j}{\partial t^2} = - EI \frac{\partial^5 h^j}{\partial x^4} - \mu I \frac{\partial^5 h^j}{\partial t \partial x^4} - \frac{\partial}{\partial x} (A_i p_i \frac{\partial h^j}{\partial x}) - \rho_i A_i (\frac{\partial}{\partial t} + U_i \frac{\partial}{\partial x})^2 h^j$$

$$+ F^j_{Ah} + q^j_{en} + q^j_{et} \frac{\partial h^j}{\partial x} + \frac{\partial}{\partial x} (T^j \frac{\partial h^j}{\partial x}) \quad . \tag{2.13}$$

As $\frac{\partial}{\partial x} (Q^j \frac{\partial h^j}{\partial x})$ and $q^j_{en} \frac{\partial h^j}{\partial x}$ are of second order of magnitude with respect to y, they may be neglected. Now, using (2.11), (2.12) and (2.8) in the x-, y- and z-coordinate system, and recalling that v_j and w_j are the velocity components of cylinder j on y- and z-directions, the following expressions are obtained:

$$0 = \frac{\partial}{\partial x} (T^j + p_e A_e - p_i A_i) + (m + \rho_i A_i) g - A_e \frac{\partial p_e}{\partial x} + \rho_e R U_e^2 C_f \quad , \qquad (2.14)$$

$$m \frac{\partial^2 v^j}{\partial t^2} = - EI \frac{\partial^4 v^j}{\partial x^4} - \mu I \frac{\partial^5 v^j}{\partial t \partial x^4} - \rho_i A_i \left(\frac{\partial}{\partial t} + U_i \frac{\partial}{\partial x}\right)^2 v^j$$

$$- \rho_e A_e \sum_{\ell=1}^{k} \left[\kappa_{j\ell} \frac{D^2 w^\ell}{Dt^2} + k_{j\ell} \frac{D^2 v^\ell}{Dt^2} \right]$$

$$- \rho_e R U_e C_f \sum_{\ell=1}^{k} \left[\sigma_{j\ell} \frac{Dw^\ell}{Dt} + s_{j\ell} \frac{Dv^\ell}{Dt} \right] - \rho_e R C_D \sum_{\ell=1}^{k} \left[\sigma_{j\ell} \frac{\partial w^\ell}{\partial t} \right.$$

$$\left. + s_{j\ell} \frac{dv^\ell}{dt} \right] + \rho_e R U_e^2 C_f \frac{\partial v^j}{\partial x}$$

$$+ \frac{\partial}{\partial x} \left[(T^j + A_e p_e - A_i p_i) \frac{\partial v^j}{\partial x} \right] \quad , \qquad (2.15)$$

$$m \frac{\partial^2 w^j}{\partial t^2} = - EI \frac{\partial^4 w^j}{\partial x^4} - \mu I \frac{\partial^5 w^j}{\partial t \partial x^4} - \rho_i A_i \left(\frac{\partial}{\partial t} + U_i \frac{\partial}{\partial x}\right)^2 w^j$$

$$- \rho_e A_e \sum_{\ell=1}^{k} \left[\varepsilon_{j\ell} \frac{D^2 w^\ell}{Dt^2} + e_{j\ell} \frac{D^2 v^\ell}{Dt^2} \right] - \rho_e R U_e C_f \sum_{\ell=1}^{k} \left[\zeta_{j\ell} \frac{Dw^\ell}{Dt} \right.$$

$$\left. + g_{j\ell} \frac{Dv^\ell}{Dt} \right] - \rho_e R C_D \sum_{\ell=1}^{k} \left[\zeta_{j\ell} \frac{\partial w^\ell}{\partial t} + g_{j\ell} \frac{\partial v^\ell}{\partial t} \right]$$

$$+ \rho_e R U_e^2 C_f \frac{\partial w^j}{\partial x} + \frac{\partial}{\partial x} \left[(T^j + A_e p_e - A_i p_i) \frac{\partial w^j}{\partial x} \right] \quad . \qquad (2.16)$$

In Eqs. (2.15) and (2.16), the only unknown quantity is $T^j + A_e p_e - A_i p_i$ which can be obtained by integration of (2.14)

$$T^j + A_e p_e - A_i p_i = (T^j + A_e p_e - A_i p_i)_L + \int_x^L \Big[(m + \rho_i A_i) g - A_e \frac{\partial p_e}{\partial x} + \rho_e R U_e^2 C_f \Big] dx \quad , \tag{2.17}$$

where L is the length of the cylinders. We now need to find the boundary condition $(T^j + A_e p_e - A_i p_i)_L$.

2.4.B. Axial Boundary Conditions

(1) Sliding Support

This case implies that the internal and external fluids come into contact at x = L (otherwise, there is an external tension at x = L, due to the rest of the cylinders).

The only force acting at the end of the tube is a drag force D_b:

$$(T + A_e p_e - A_i p_i)_L = D_b \quad . \tag{2.18}$$

Hannoyer [11] has established a quadratic fit to represent the base drag behind cylindrical tubes:

$$D_b = \frac{1}{2} c_{fe} \rho_e A_e U_e^2 + \frac{1}{2} c_{fi} \rho_i A_i U_i^2 + \frac{1}{2} c_{fx} (\rho_e \rho_i A_e A_i U_e^2 U_i^2)^{\frac{1}{2}} \quad , \tag{2.19}$$

where semi-empirical expressions were found for c_{fe}, c_{fi}, c_{fx}:

$$\left. \begin{array}{l} c_{fe} \sim 0.15 \\ c_{fi} \sim 0.016 \\ c_{fx} \sim 0.109 \end{array} \right\} \text{ for a blunt end piece} \quad .$$

(2) Support Not Free to Move Axially

Let us set

$$T = T^* + T' \;,$$

$$p_e = p_e^* + p_e' \;,$$

$$p_i = p_i^* + p_i' \;,$$

where the asterisk characterizes the zero flow conditions. We have

$$(T^j + A_e p_e - A_i p_i)_L = (T^j + A_e p_e - A_i p_i)_L^* + (T^j + A_e p_e - A_i p_i)_L' \;,$$

and from Eq. (2.17)

$$(T^j + A_e p_e - A_i p_i)_{L/2}^* = (T^j + A_e p_e - A_i p_i)_L^*$$

$$+ \int_{L/2}^{L} \left[(m + \rho_i A_i) g - A_e \frac{\partial p_e}{\partial x} \right] dx \;.$$

Setting $\bar{T}^j = (T^j + A_e p_e - A_i p_i)_{L/2}^*$ (effective tension in the middle of the beam), these two equations give

$$(T^j + A_e p_e - A_i p_i)_L = \bar{T}^j - \int_{L/2}^{L} \left[(m + \rho_i A_i) g - A_e \frac{\partial p_e}{\partial x} \right] dx$$

$$+ (T^j + A_e p_e - A_i p_i)_L' \;. \qquad (2.20)$$

We shall now approximate the tension distribution with an axial stress distribution given by

$$\sigma_{xx} = \frac{T'(x)}{A_e - A_i} ,$$

and the pressure distribution with,

$$\sigma_{rr} + \sigma_{\theta\theta} = 2 \cdot \frac{p_i' A_i - p_e' A_e}{A_e - A_i} .$$

Superposing these two distributions, we obtain the axial strain distribution:

$$\varepsilon_x = \frac{\sigma_{xx} - \nu(\sigma_{rr} + \sigma_{\theta\theta})}{E} = \frac{(T^j)'(x) - 2\nu(p_i' A_i - p_e' A_e)}{E(A_e - A_i)} .$$

From (2.17) we may write

$$\varepsilon_x = \frac{(T^j)_L' + \int_x^L \rho_e R U_e^2 C_f dx - 2\nu(p_i' A_i - p_e' A_e)}{E(A_e - A_i)} .$$

The condition of a non-sliding and support requires

$\int_0^L \varepsilon_x dx = 0$, hence we have:

$$(T^j + A_e p_e - A_i p_i)'_L L = (1 - 2\nu) \int_0^L (A_e p'_e - A_i p'_i)\, dx$$

$$- \frac{1}{2} \rho_e R U_e^2 C_f L^2 \quad . \qquad (2.21)$$

Now, suppose that

$$p'_e = \bar{p}_e + (x - \frac{L}{2}) (\frac{\partial p_e}{\partial x}) \quad ,$$

$$p'_i = \bar{p}_i + (x - \frac{L}{2}) (\frac{\partial p_i}{\partial x}) \quad ,$$

where $\bar{p}_e$ and $\bar{p}_i$ are the mean internal and external pressures at $x = L/2$ and $\frac{\partial p_e}{\partial x}$, $\frac{\partial p_i}{\partial x}$ constants. We can assume that, in the channel, the pressure drop is given by

$$\frac{\partial p_e}{\partial x} = - \frac{F_f}{A_{CH}} + \rho g \quad ,$$

where F_f is the total frictional force per unit length and A_{CH} is the cross-sectional area that is available to flow,

$$F_f = \rho_e R_o U_e^2 C_f + k \rho_e R U_e^2 C_f \quad ,$$

$$A_{CH} = \pi \left[R_o^2 - kR^2 \right] \quad .$$

Hence, replacing in Eqs. (2.21) and (2.20) we obtain:

$$(T^j + A_e p_e - A_i p_i)_L = \bar{T}^j + (1 - 2\nu)(A_e \bar{p}_e - A_i \bar{p}_i)$$

$$- \frac{1}{2} \rho_e R U_e^2 C_f L (1 + \frac{D_e}{D_h}) - (m - \rho_e A_e + \rho_i A_i) g \frac{L}{2} , \qquad (2.22)$$

where $D_h = 4 A_{CH}/2\pi(R_o + kR)$,

$D_e = 2R$.

Now, if we use: $\delta = 0$ if the downstream end is free to move axially ,

$\delta = 1$ if the length is fixed ,

we can combine Eqs. (2.22), (2.18) and (2.17) to obtain only one expression for the two cases:

$$T^j + A_e p_e - A_i p_i = \delta \left\{ \bar{T}^j + (1 - 2\nu)(\bar{p}_e A_e - \bar{p}_i A_i) \right\}$$

$$+ \left\{ (1 - \frac{\delta}{2}) L - x \right\} \left[(m - \rho_e A_e + \rho_i A_i) g \right.$$

$$\left. + \rho_e R U_e^2 C_f (1 + D_e/D_h) \right] + (1 - \delta) D_b . \qquad (2.23)$$

CHAPTER III

THE EQUATIONS OF MOTION AND THEIR SOLUTION

3.1 Formulation of the Equations of Small Motions

Substituting the axial boundary conditions of Eq. (2.23) in the equations (2.15) and (2.16) and re-grouping, one obtains the two equations for small arbitrary motions of cylinder j in y- and z-directions.

$$\mu I \frac{\partial^5 v^j}{\partial x^4 \partial t} + EI \frac{\partial^4 v^j}{\partial x^4} + \rho_e R U_e C_f \sum_{\ell=1}^{k} \left[\sigma_{j\ell} \frac{Dw^\ell}{Dt} + s_{j\ell} \frac{Dv^\ell}{Dt} \right]$$

$$+ \rho_e R C_D \sum_{\ell=1}^{k} \left[\sigma_{j\ell} \frac{\partial w^\ell}{\partial t} + s_{j\ell} \frac{\partial v^\ell}{\partial t} \right] + \rho_e A_e \sum_{\ell=1}^{k} \left[\kappa_{j\ell} \frac{D^2 w^\ell}{Dt^2} + k_{j\ell} \frac{D^2 v^\ell}{Dt^2} \right]$$

$$+ \left[(m - \rho_e A_e + \rho_i A_i)\, g + \rho_e R U_e^2 C_f \left(1 + \frac{D_e}{D_h}\right) \right] x \frac{\partial^2 v^j}{\partial x^2}$$

$$- \left[\delta \left\{ \bar{T} + (1 - 2\nu)(\bar{p}_e A_e - \bar{p}_i A_i) \right\} + \left(1 - \frac{\delta}{2}\right) L \left\{ (m - \rho_e A_e + \rho_i A_i)\, g \right.\right.$$

$$\left.\left. + \rho_e R U_e^2 C_f \left(1 + \frac{D_e}{D_h}\right) \right\} + (1 - \delta)\, D_b - \rho_i A_i U_i^2 \right] \frac{\partial^2 v^j}{\partial x^2}$$

$$+ 2 \rho_i A_i U_i \frac{\partial^2 v^j}{\partial t \partial x} + \left[(m - \rho_e A_e + \rho_i A_i)\, g + \rho_e R U_e^2 C_f \frac{D_e}{D_h} \right] \frac{\partial v^j}{\partial x}$$

$$+ (m + \rho_i A_i) \frac{\partial^2 v^j}{\partial t^2} = 0 \;, \qquad (3.1)$$

$$\mu I \frac{\partial^5 w^j}{\partial x^4 \partial t} + EI \frac{\partial^4 w^j}{\partial x^4} + \rho_e R U_e C_f \sum_{\ell=1}^{k} \left[\zeta_{j\ell} \frac{Dw^\ell}{Dt} + g_{j\ell} \frac{Dv^\ell}{Dt} \right]$$

$$+ \rho R C_D \sum_{\ell=1}^{k} \left[\zeta_{j\ell} \frac{\partial w^\ell}{\partial t} + g_{j\ell} \frac{\partial v^\ell}{\partial t} \right] + \rho_e A_e \sum_{\ell=1}^{k} \left[\varepsilon_{j\ell} \frac{D^2 w^\ell}{Dt^2} + e_{j\ell} \frac{D^2 v^\ell}{Dt^2} \right]$$

$$+ \left[(m - \rho_e A_e + \rho_i A_i)\, g + \rho_e R U_e^2 C_f \left(1 + \frac{D_e}{D_h}\right) \right] x \frac{\partial^2 w^j}{\partial x^2}$$

$$- \left[\delta \left\{ \bar{T} + (1 - 2\nu)(\bar{p}_e A_e - \bar{p}_i A_i) \right\} + \left(1 - \frac{\delta}{2}\right) L \left\{ (m - \rho_e A_e + \rho_i A_i)\, g \right.\right.$$

$$\left.\left. + \rho_e R U_e^2 C_f \left(1 + \frac{D_e}{D_h}\right) \right\} + (1 - \delta)\, D_b - \rho_i A_i U_i^2 \right] \frac{\partial^2 w^j}{\partial x^2}$$

$$+ 2 \rho_i A_i U_i \frac{\partial^2 w^j}{\partial t \partial x} + \left[(m - \rho_e A_e + \rho_i A_i)\, g + \rho_e R U_e^2 C_f \frac{D_e}{D_h} \right] \frac{\partial w^j}{\partial x}$$

$$+ (m + \rho_i A_i) \frac{\partial^2 w^j}{\partial t^2} = 0 \; , \qquad (3.2)$$

where

$$D_b = \frac{1}{2} \left[c_{fe} \rho_e A_e U_e^2 + c_{fi} \rho_i A_i U_i^2 + c_{fx} (\rho_e \rho_i A_e A_i U_e^2 U_i^2)^{\frac{1}{2}} \right] .$$

This general equation of motion applies for different boundary conditions, depending on the case under consideration. Therefore, equations (3.1) and (3.2) are subjected to one of the following conditions:

$$v^j = \frac{\partial^2 v^j}{\partial x^2} = 0 \qquad \text{at } x = 0 \text{ and } x = L$$

for a pinned-pinned cylinder, and

$$v^j = \frac{\partial v^j}{\partial x} = 0 \qquad \text{at } x = 0 \text{ and } x = L \tag{3.3}$$

for a clamped-clamped cylinder. The case of a free end will not be considered here since the proper formulation of the equations of motion would depend on the shape of the free end.

Rearranging the terms of Eqs. (3.1) and (3.2) and expanding the differential operators, we finally get

$$\mu I \frac{\partial^5 v^j}{\partial x^4 \partial t} + EI \frac{\partial^4 v^j}{\partial x^4} + \left\{ (m - \rho_e A_e + \rho_i A_i)\, g + \rho_e R U_e^2 C_f \left(1 + \frac{D_e}{D_h}\right) \right\} x \frac{\partial^2 v^j}{\partial x^2}$$

$$- \left\{ \delta \left[\bar{T} + (1 - 2\nu)(\bar{p}_e A_e - \bar{p}_i A_i) \right] + \left(1 - \frac{\delta}{2}\right) L \left[(m - \rho_e A_e + \rho_i A_i)\, g \right.\right.$$

$$\left.\left. + \rho_e R U_e^2 C_f \left(1 + \frac{D_e}{D_h}\right) \right] + (1 - \delta)\, D_b - \rho_i A_i U_i^2 \right\} \frac{\partial^2 v^j}{\partial x^2}$$

$$+ 2\rho_e A_e U_e \sum_{\ell=1}^{k} \left[\kappa_{j\ell} \frac{\partial^2 w^\ell}{\partial t \partial x} + k_{j\ell} \frac{\partial^2 v^\ell}{\partial t \partial x} \right] + \rho_e A_e U_e^2 \sum_{\ell=1}^{k} \left[\kappa_{j\ell} \frac{\partial^2 w^\ell}{\partial x^2} + k_{j\ell} \frac{\partial^2 v^\ell}{\partial x^2} \right]$$

$$+ \left\{ (m - \rho_e A_e + \rho_i A_i)\, g + \rho_e R U_e^2 C_f \frac{D}{D_h} \right\} \frac{\partial v^j}{\partial x} + \rho_e R U_e^2 C_f \sum_{\ell=1}^{k} \left[\sigma_{j\ell} \frac{\partial w^\ell}{\partial x} \right.$$

$$\left. + s_{j\ell} \frac{\partial v^\ell}{\partial x} \right] + \rho_e R C_D \sum_{\ell=1}^{k} \left[\sigma_{j\ell} \frac{\partial w^\ell}{\partial t} + s_{j\ell} \frac{\partial v^\ell}{\partial t} \right]$$

$$+ (m + \rho_i A_i) \frac{\partial^2 v^j}{\partial t^2} + \rho_e A_e \sum_{\ell=1}^{k} \left[\kappa_{j\ell} \frac{\partial^2 w^\ell}{\partial t^2} + k_{j\ell} \frac{\partial^2 v^\ell}{\partial t^2} \right] = 0 \quad . \qquad (3.4)$$

A similar expression in terms of w^j may be obtained in the y-direction.

3.2 Dimensionless Equations

To render the equation dimensionless before attempting a solution to the problem, the following dimensionless quantities are defined:

$$\xi = \frac{x}{L} \qquad \eta^j = \frac{w^j}{L} \qquad j = 1,2,\ldots k \qquad \eta^{j+k} = \frac{v^j}{L} \qquad j = 1,2,\ldots k$$

$$\beta_e = \frac{\rho_e A_e}{m + \rho_e A_e + \rho_i A_i} \qquad \beta_i = \frac{\rho_i A_i}{m + \rho_e A_e + \rho_i A_i}$$

$$\Pi_e = \frac{\bar{p}_e A_e L^2}{EI} \qquad \Pi_i = \frac{\bar{p}_i A_i L^2}{EI}$$

$$\gamma_e = \frac{\rho_e A_e g L^3}{EI} \qquad \gamma_i = \frac{\rho_i A_i g L^3}{EI} \qquad \gamma = \frac{mgL^3}{EI}$$

$$u_e = \left[\frac{\rho_e A_e}{EI}\right]^{\frac{1}{2}} L U_e \qquad u_i = \left[\frac{\rho_i A_i}{EI}\right]^{\frac{1}{2}} L U_i$$

$$\Gamma = \frac{\bar{T} L^2}{EI} \qquad \varepsilon = \frac{L}{D_e}$$

$$h = \frac{D_e}{D_h} \qquad \alpha = \left[\frac{I}{E(m + \rho_e A_e + \rho_i A_i)}\right]^{\frac{1}{2}} \frac{\mu}{L^2}$$

$$c_f = \frac{4C_f}{\pi} \qquad c = \frac{4}{\pi} L \left(\frac{\rho_e A_e}{EI}\right)^{\frac{1}{2}} C_D$$

$$\theta = \frac{1}{2}\left[c_{fe}u_e^2 + c_{fi}u_i^2 + c_{fx}u_e u_i\right]$$

$$\tau = \left(\frac{EI}{m + \rho_e A_e + \rho_i A_i}\right)^{\frac{1}{2}} \frac{t}{L^2} \qquad (3.5)$$

With these quantities, the dimensionless equation of motion, written in a compact matrix form, takes the form

$$\underset{\sim}{A} \frac{\partial^5 \underset{\sim}{\eta}}{\partial \xi^4 \partial \tau} + \underset{\sim}{I} \frac{\partial^4 \underset{\sim}{\eta}}{\partial \xi^4} + \underset{\sim}{C} \frac{\partial^2 \underset{\sim}{\eta}}{\partial \xi \partial \tau} + \underset{\sim}{E} \frac{\partial^2 \underset{\sim}{\eta}}{\partial \xi^2} + \underset{\sim}{F} \xi \frac{\partial^2 \underset{\sim}{\eta}}{\partial \xi^2}$$

$$+ \underset{\sim}{G} \frac{\partial \underset{\sim}{\eta}}{\partial \xi} + \underset{\sim}{H} \frac{\partial \underset{\sim}{\eta}}{\partial \tau} + \underset{\sim}{M} \frac{\partial^2 \underset{\sim}{\eta}}{\partial \tau^2} = \underset{\sim}{0} \quad . \qquad (3.6)$$

The various matrices are defined as follows:

$$\underset{\sim}{A} = \alpha \underset{\sim}{I} \qquad \underset{\sim}{I} = \text{unit matrix } 2k\text{x}2k$$

$$\underset{\sim}{C} = 2(\beta_i^{\frac{1}{2}} u_i \underset{\sim}{I} + \beta_e^{\frac{1}{2}} u_e \underset{\sim}{M}_v) \quad , \text{ where } \underset{\sim}{M}_v = - \begin{bmatrix} [\varepsilon_{j\ell}] & [\underset{\sim}{e}_{j\ell}] \\ [\underset{\sim}{\kappa}_{j\ell}] & [\underset{\sim}{k}_{j\ell}] \end{bmatrix} ,$$

$$\underset{\sim}{E} = u_e^2 \underset{\sim}{M}_v - \left[\delta \left\{ \Gamma + (1 - 2\nu)(\Pi_e - \Pi_i) \right\} + (1 - \frac{\delta}{2}) \left\{ \gamma - \gamma_e + \gamma_i + \frac{\varepsilon}{2} c_f u_e^2 (1 + h) \right\} + (1 - \delta)\Theta - u_i^2 \right] \underset{\sim}{I} ,$$

$$\underset{\sim}{G} = \frac{1}{2} \varepsilon c_f u_e^2 \underset{\sim}{C}_v + \left[\frac{1}{2} \varepsilon c_f u_e^2 h + \gamma - \gamma_e + \gamma_i \right] \underset{\sim}{I} ,$$

$$\text{where } \underset{\sim}{C}_v = \begin{bmatrix} [\underset{\sim}{\xi}_{j\ell}] & [\underset{\sim}{g}_{j\ell}] \\ [\underset{\sim}{g}_{j\ell}] & [\underset{\sim}{s}_{j\ell}] \end{bmatrix} ,$$

$$\underset{\sim}{F} = \left[\frac{1}{2} \varepsilon c_f u_e^2 (1 + h) + \gamma - \gamma_e + \gamma_i \right] \underset{\sim}{I} ,$$

$$\underset{\sim}{H} = \left[\frac{1}{2} \varepsilon c_f \beta_e^{\frac{1}{2}} u_e + \frac{1}{2} \varepsilon c \beta_e^{\frac{1}{2}} \right] \underset{\sim}{C}_v ,$$

$$\underset{\sim}{M} = \beta_e \underset{\sim}{M}_v + (1 - \beta_e) \underset{\sim}{I} ,$$

$$\underset{\sim}{\eta} = \{\eta^1, \eta^2, \ldots \eta^{2k}\}^T . \tag{3.7}$$

As to the boundary conditions of equation (3.3), they now may be written,

$$\underset{\sim}{\eta} = \underset{\sim}{0} \quad \text{and} \quad \frac{\partial^2 \underset{\sim}{\eta}}{\partial \xi^2} = \underset{\sim}{0} \qquad \text{at } \xi = 0 \text{ and } \xi = 1$$

for pinned ends, and (3.8)

$$\underset{\sim}{\eta} = \underset{\sim}{0} \quad \text{and} \quad \frac{\partial \underset{\sim}{\eta}}{\partial \xi} = \underset{\sim}{0} \qquad \text{at } \xi = 0 \text{ and } \xi = 1$$

for clamped ends.

3.3 Solution of the Dimensionless Equation

An exact solution of equation (3.6) subject to one set of boundary conditions (3.8) is complicated by the presence of the mixed terms. Therefore, Galerkin's approximate method is used to remove the spatial dependence and transform the problem to a more tractable form. To do so, consider the approximate solution

$$\eta^j(\xi,\tau) = \sum_{i=1}^{\infty} \phi_i(\xi) p_i^j(\tau) \quad , \tag{3.9}$$

where $p_i^j(\tau)$ are the generalized co-ordinates and $\phi_i(\xi)$ are the eigenfunctions of the dimensionless beam equation,

$$\phi_i(\xi) = a_i \cos\lambda_i\xi + b_i \sin\lambda_i\xi + c_i \cosh\lambda_i\xi + d_i \sinh\lambda_i\xi \quad . \tag{3.10}$$

The constants a_i, b_i, c_i and d_i may be found by applying the same boundary conditions to (3.10) as those applying to the problem and by normalizing $\phi_i(\xi)$ such that

$$\int_0^1 [\phi_i(\xi)]^2 \, d\xi = 1 \quad . \tag{3.11}$$

If we replace the formulation (3.9) into the equation of motion (3.6) we get an error $\underset{\sim}{\varepsilon}$ instead of $\underset{\sim}{0}$ at the right-hand side. According to Galerkin's method (see Meirovitch [17]) this error, weighted by any comparison function $\phi_r(\xi)$ and integrated over the domain, must be equal to zero. Hence we obtain:

$$\underset{\sim}{A} \sum_{i=1}^{\infty} \dot{\underset{\sim}{p}}_i \int_0^1 \frac{d^4\phi_i}{d\xi^4} \phi_r \, d\,\xi + \underset{\sim}{I} \sum_i \underset{\sim}{p}_i \int_0^1 \frac{d^4\phi_i}{d\xi^4} \phi_r \, d\,\xi$$

$$+ \underset{\sim}{C} \sum_i \dot{\underset{\sim}{p}}_i \int_0^1 \frac{d\phi_i}{d\xi} \phi_r \, d\,\xi + \underset{\sim}{E} \sum_i \underset{\sim}{p}_i \int_0^1 \frac{d^2\phi_i}{d\xi^2} \phi_r \, d\,\xi$$

$$+ \underset{\sim}{F} \sum_i \underset{\sim}{p}_i \int_0^1 \xi \frac{d^2\phi_i}{d\xi^2} \phi_r \, d\,\xi + \underset{\sim}{G} \sum_i \underset{\sim}{p}_i \int_0^1 \frac{d\phi_i}{d\xi} \phi_r \, d\,\xi$$

$$+ \underset{\sim}{H} \sum_i \dot{\underset{\sim}{p}}_i \int_0^1 \phi_i \, \phi_r \, d\,\xi + \underset{\sim}{M} \sum_i \ddot{\underset{\sim}{p}}_i \int_0^1 \phi_i \, \phi_r \, d\,\xi = \underset{\sim}{0} \quad , \qquad (3.12)$$

for $r = 1, 2, 3, \ldots$, where $\underset{\sim}{p}_i = \{p_i^1, p_i^2, p_i^3, \ldots, p_i^{2k}\}^T$, and the dot denotes differentiation with respect to τ. The beam equation, with the boundary conditions of Eq. (3.8) is a self-adjoint problem, which implies, in combination with equation (3.11), that the eigenfunctions $\phi_i(\xi)$ are orthonormal. Let us now define

$$\int_0^1 \frac{d^4\phi_i}{d\xi^4} \phi_r \, d\,\xi = \lambda_r^4 \, \delta_{ri} \quad ,$$

$$a_{ri} = \int_0^1 \frac{d\phi_i}{d\xi} \phi_r \, d\,\xi \quad ,$$

$$b_{ri} = \int_0^1 \frac{d^2\phi_i}{d\xi^2} \phi_r \, d\,\xi \quad ,$$

$$d_{ri} = \int_0^1 \xi \frac{\partial^2 \phi_i}{\partial \xi^2} \phi_r \, d\xi \quad , \tag{3.13}$$

which are given in Table 1 for various boundary conditions. With these coefficients, equation (3.12) becomes

$$\sum_{i=1}^{N} \left[\underset{\sim}{M} \delta_{ri} \ddot{\underset{\sim}{p}}_i + \left[\underset{\sim}{C} a_{ri} + (\underset{\sim}{H} + \underset{\sim}{A} \lambda_i^4) \delta_{ri} \right] \dot{\underset{\sim}{p}}_i + \left[\lambda_i^4 \underset{\sim}{I} \delta_{ri} + \underset{\sim}{E} b_{ri} \right.\right.$$

$$\left.\left. + \underset{\sim}{F} d_{ri} + \underset{\sim}{G} a_{ri} \right] \underset{\sim}{p}_i \right] = \underset{\sim}{0} \quad , \tag{3.14}$$

for $r = 1,2,3,\ldots N$,

where the infinite series has been truncated at $i = N$, the number of comparison functions used. If we define

$$\underset{\sim}{p} = \{\underset{\sim}{p}_1, \underset{\sim}{p}_2, \underset{\sim}{p}_3, \ldots, \underset{\sim}{p}_N\}^T \quad ,$$

Equation (3.14) is now of the form

$$\underset{\sim}{\mathcal{M}} \ddot{\underset{\sim}{p}} + \underset{\sim}{\mathcal{C}} \dot{\underset{\sim}{p}} + \underset{\sim}{\mathcal{K}} \underset{\sim}{p} = \underset{\sim}{0} \quad , \tag{3.15}$$

where

$$\underset{\sim}{\mathcal{M}} = \begin{bmatrix} \underset{\sim}{M} & \underset{\sim}{O} & \ldots & \underset{\sim}{O} \\ \underset{\sim}{O} & \underset{\sim}{M} & \ldots & \underset{\sim}{O} \\ \vdots & \vdots & & \vdots \\ \underset{\sim}{O} & \underset{\sim}{O} & \ldots & \underset{\sim}{M} \end{bmatrix} \quad ,$$

$$\underset{\sim}{C} = \begin{bmatrix} \underset{\sim}{A}\lambda_1^4 & \underset{\sim}{O} & \cdots & \underset{\sim}{O} \\ \underset{\sim}{O} & \underset{\sim}{A}\lambda_2^4 & \cdots & \underset{\sim}{O} \\ \vdots & \vdots & & \vdots \\ \underset{\sim}{O} & \underset{\sim}{O} & \cdots & \underset{\sim}{A}\lambda_N^4 \end{bmatrix} + \begin{bmatrix} \underset{\sim}{C}a_{11} & \underset{\sim}{C}a_{12} & \cdots & \underset{\sim}{C}a_{1N} \\ \underset{\sim}{C}a_{21} & \underset{\sim}{C}a_{22} & \cdots & \underset{\sim}{C}a_{2N} \\ \vdots & \vdots & & \vdots \\ \underset{\sim}{C}a_{N1} & \underset{\sim}{C}a_{N2} & \cdots & \underset{\sim}{C}a_{NN} \end{bmatrix}$$

$$+ \begin{bmatrix} \underset{\sim}{H} & \underset{\sim}{O} & \cdots & \underset{\sim}{O} \\ \underset{\sim}{O} & \underset{\sim}{H} & \cdots & \underset{\sim}{O} \\ \vdots & \vdots & & \vdots \\ \underset{\sim}{O} & \underset{\sim}{O} & \cdots & \underset{\sim}{H} \end{bmatrix}$$

$$\underset{\sim}{K} = \begin{bmatrix} \underset{\sim}{I}\lambda_1^4 & \underset{\sim}{O} & \cdots & \underset{\sim}{O} \\ \underset{\sim}{O} & \underset{\sim}{I}\lambda_2^4 & \cdots & \underset{\sim}{O} \\ \vdots & \vdots & & \vdots \\ \underset{\sim}{O} & \underset{\sim}{O} & \cdots & \underset{\sim}{I}\lambda_N^4 \end{bmatrix} + \begin{bmatrix} \underset{\sim}{E}b_{11} & \underset{\sim}{E}b_{12} & \cdots & \underset{\sim}{E}b_{1N} \\ \underset{\sim}{E}b_{21} & \underset{\sim}{E}b_{22} & \cdots & \underset{\sim}{E}b_{2N} \\ \vdots & \vdots & & \vdots \\ \underset{\sim}{E}b_{N1} & \underset{\sim}{E}b_{N2} & \cdots & \underset{\sim}{E}b_{NN} \end{bmatrix}$$

$$+ \begin{bmatrix} \underset{\sim}{F}d_{11} & \underset{\sim}{F}d_{12} & \cdots & \underset{\sim}{F}d_{1N} \\ \underset{\sim}{F}d_{21} & \underset{\sim}{F}d_{22} & \cdots & \underset{\sim}{F}d_{2N} \\ \vdots & \vdots & & \vdots \\ \underset{\sim}{F}d_{N1} & \underset{\sim}{F}d_{N2} & \cdots & \underset{\sim}{F}d_{NN} \end{bmatrix} + \begin{bmatrix} \underset{\sim}{G}a_{11} & \underset{\sim}{G}a_{12} & \cdots & \underset{\sim}{G}a_{1N} \\ \underset{\sim}{G}a_{21} & \underset{\sim}{G}a_{22} & \cdots & \underset{\sim}{G}a_{2N} \\ \vdots & \vdots & & \vdots \\ \underset{\sim}{G}a_{N1} & \underset{\sim}{G}a_{N2} & \cdots & \underset{\sim}{G}a_{NN} \end{bmatrix} , \qquad (3.16)$$

where $\underset{\sim}{M}$, $\underset{\sim}{C}$ and $\underset{\sim}{K}$ are 2kNx2kN matrices. Finally, introducing the state vector $\underset{\sim}{z} = \{\dot{\underset{\sim}{p}}, \underset{\sim}{p}\}^T$, equation (3.15) may be reduced to a first order differential equation, as follows:

$$\dot{\underset{\sim}{z}} - \underset{\sim}{Y}\,\underset{\sim}{z} = \underset{\sim}{0} \qquad (3.17)$$

where

$$\underset{\sim}{Y} = \begin{bmatrix} -\underset{\sim}{M}^{-1}\underset{\sim}{C} & -\underset{\sim}{M}^{-1}\underset{\sim}{K} \\ \underset{\sim}{I} & \underset{\sim}{O} \end{bmatrix} . \qquad (3.18)$$

A solution of Eq. (3.17) may be written under the classical form

$$\underset{\sim}{z} = \underset{\sim}{\Lambda}\, e^{i\omega\tau} \tag{3.19}$$

which substituted in (3.17) yields the eigenvalue problem

$$(i\omega\, \underset{\sim}{I} - \underset{\sim}{Y})\, \underset{\sim}{\Lambda} = \underset{\sim}{0} \quad . \tag{3.20}$$

A computer program has been written to carry out the computations leading to the eigenvalues $i\omega$ of this equation, and is described in Appendix C.

CHAPTER IV

THE TRANSIENT RESPONSE

In the previous chapter the general equation of motion of the system has been solved with the intent to obtain the different modes of vibrations, and the limits of stability when the flow velocities are increased. But it is also of interest to know the effect of an instantaneous or steady excitation of one or more tubes and its propagation to the neighbouring tubes.

4.1 Response to an Instantaneous Excitation

In this case the initial displacement and velocity state of each cylinder is fixed, and then the system is abandoned to itself. Hence, the problem consists in finding the constants of integration of the general differential equation of the system.

Once equation (3.20)

$$(i\omega \underset{\sim}{I} - \underset{\sim}{Y}) \underset{\sim}{\Lambda} = \underset{\sim}{0} \quad ,$$

has been solved for the eigenvalues $i\omega_j$, and the corresponding eigenvectors $\underset{\sim}{\Lambda}_j$, the solution $\underset{\sim}{z}$ of the differential equation may be written as follows

$$\underset{\sim}{z} = \sum_{j=1}^{4kN} c_j \underset{\sim}{\Lambda}_j e^{i\omega_j \tau} \quad , \tag{4.1}$$

where c_j are arbitrary complex constants. As $\underset{\sim}{Y}$ is a real matrix, the complex values of $i\omega$ occur as complex conjugate pairs, as well as the corresponding eigenvectors. Hence, if we denote by a star the complex conjugate, and order ω_j and $\underset{\sim}{\Lambda}_j$ so that

$$i\omega_i = (i\omega_{i+2kN})^* \quad , \quad \underset{\sim}{\Lambda} = \underset{\sim}{\Lambda}^*_{i+2kN} \quad ,$$

equation (4.1) becomes

$$\underset{\sim}{z} = \sum_{j=1}^{2kN} \left[c_j \, \underset{\sim}{\Lambda}_j \, e^{i\omega_j \tau} + c_{j+2kN} \, \underset{\sim}{\Lambda}^*_j \, e^{-i\omega^*_j \tau} \right] \quad . \tag{4.2}$$

We note that the physical condition for $\underset{\sim}{z}$ being real implies that

$$c_{j+2kN} = c^*_j \quad ;$$

then $\underset{\sim}{z}$ becomes the sum of two complex conjugate expressions which reduces to twice the real part of one of these. Hence, the number of complex constants has been reduced to 2kN, and if we denote

$$\begin{aligned} c_j &= C_j + i\bar{C}_j \quad , \\ \omega_j &= \Omega_j + i\bar{\Omega}_j \quad , \\ \underset{\sim}{\Lambda}_j &= \underset{\sim}{R}_j + i\underset{\sim}{I}_j \quad , \qquad j = 1, 2, \ldots, 2kN, \end{aligned} \tag{4.3}$$

equation (4.2) becomes

$$\underset{\sim}{z} = 2 \sum_{j=1}^{2kN} \Big[C_j (\underset{\sim}{R}_j \cos \Omega_j \tau - \underset{\sim}{I}_j \sin \Omega_j \tau)$$

$$- \bar{C}_j (\underset{\sim}{I}_j \cos \Omega_j \tau + \underset{\sim}{R}_j \sin \Omega_j \tau) \Big] e^{-\bar{\Omega}_j \tau} , \tag{4.4}$$

where $\underset{\sim}{z} = \begin{pmatrix} \dot{\underset{\sim}{p}} \\ \underset{\sim}{p} \end{pmatrix}$, a 4kN vector.

Now, if we recall the definition of $\underset{\sim}{p}$

$$\underset{\sim}{p} = \{\underset{\sim}{p}_1, \underset{\sim}{p}_2, \ldots, \underset{\sim}{p}_\ell, \ldots, \underset{\sim}{p}_N\}^T ,$$

where

$$\underset{\sim}{p}_\ell = \{p_\ell^1, p_\ell^2, p_\ell^3, \ldots, p_\ell^{2k}\}^T$$

and replace in the general approximate solution of Eq. (3.9)

$$\eta^j = \sum_{\ell=1}^{N} \phi_\ell(\xi) \; p_\ell^j(\tau) ,$$

we obtain

$$\underset{\sim}{\eta}(\xi,\tau) = \sum_{\ell=1}^{N} \underset{\sim}{p}_\ell(\tau) \; \phi_\ell(\xi) ,$$

$$\dot{\underset{\sim}{\eta}}(\xi,\tau) = \sum_{\ell=1}^{N} \dot{\underset{\sim}{p}}_\ell(\tau) \; \phi_\ell(\xi) .$$

In terms of the eigenvectors, this expression becomes

$$\begin{pmatrix} \dot{\underset{\sim}{\eta}} \\ \underset{\sim}{\eta} \end{pmatrix} = 2 \sum_{\ell=1}^{N} \sum_{j=1}^{2kN} \phi_\ell(\xi) \Big[C_j (\underset{\sim}{R}_{j,\ell} \cos \Omega_j \tau - \underset{\sim}{I}_{j,\ell} \sin \Omega_j \tau)$$

$$- \bar{C}_j (\underset{\sim}{I}_{j,\ell} \cos \Omega_j \tau + \underset{\sim}{R}_{j,\ell} \sin \Omega_j \tau) \Big] e^{-\bar{\Omega}_j \tau} , \qquad (4.5)$$

where

$$\underset{\sim}{\eta} = \{ \underbrace{\eta_1, \eta_2, \ldots, \eta_k}_{\text{z-direction}}, \underbrace{\eta_{k+1}, \ldots, \eta_{2k}}_{\text{y-direction}} \}^T ,$$

$$\underset{\sim}{R}_j = \{ r^j_{1,1}, r^j_{1,2}, \ldots, r^j_{1,2k}, \ldots, r^j_{N,1}, r^j_{N,2}, \ldots, r^j_{N,2k},$$

$$r^j_{1,2k+1}, r^j_{1,2k+2}, \ldots, r^j_{1,4k}, \ldots, r^j_{N,2k+1},$$

$$r^j_{N,2k+2}, \ldots, r^j_{N,4k} \}^T ,$$

$$\underset{\sim}{R}_{j,\ell} = \{ r^j_{\ell,1}, r^j_{\ell,2}, \ldots, r^j_{\ell,2k}, r^j_{\ell,2k+1}, \ldots, r^j_{\ell,4k} \}^T$$

and similar expressions for $\underset{\sim}{I}_j$ and $\underset{\sim}{I}_{j,\ell}$.

In order to solve for the initialization constants C_j and $\bar{C}_j$, it is more convenient to put Eq. (4.5) into matrix form

$$\begin{pmatrix} \dot{\underset{\sim}{\eta}} \\ \underset{\sim}{\eta} \end{pmatrix} = \underset{\sim}{T} \begin{pmatrix} \underset{\sim}{C} \\ \bar{\underset{\sim}{C}} \end{pmatrix} , \qquad (4.6)$$

where

$$\underset{\sim}{C} = \{C_1, C_2, \ldots, C_{2kN}\}^T ,$$

$$\bar{\underset{\sim}{C}} = \{\bar{C}_1, \bar{C}_2, \ldots, \bar{C}_{2kN}\}^T ,$$

and

$$\begin{aligned}\underset{\sim}{T}(\xi,\tau) = 2\Bigg[& \sum_{\ell=1}^{N} \phi_\ell(\xi)(\underset{\sim}{R}_{1,\ell} \cos \Omega_1\tau - \underset{\sim}{I}_{1,\ell} \sin \Omega_1\tau)\, e^{-\bar{\Omega}_1\tau}, \ldots \\ & \ldots, \sum_{\ell=1}^{N} \phi_\ell(\xi)(\underset{\sim}{R}_{2kN,\ell} \cos \Omega_{2kN}\tau - \underset{\sim}{I}_{2kN,\ell} \sin \Omega_{2kN}\tau)\, e^{-\bar{\Omega}_{2kN}\tau}, \\ & \sum_{\ell=1}^{N} \phi_\ell(\xi)(-\underset{\sim}{I}_{1,\ell} \cos \Omega_1\tau - \underset{\sim}{R}_{1,\ell} \sin \Omega_1\tau)\, e^{-\bar{\Omega}_1\tau}, \ldots \\ & \ldots, \sum_{\ell=1}^{N} \phi_\ell(\xi)(-\underset{\sim}{I}_{2kN,\ell} \cos \Omega_{2kN}\tau - \underset{\sim}{R}_{2kN,\ell} \sin \Omega_{2kN}\tau)\, e^{-\bar{\Omega}_{2kN}\tau}\Bigg] .\end{aligned} \tag{4.7}$$

As the number of degrees of freedom of the system is 4kN, we may set, at time $\tau = 0$, for N values of ξ, the displacement and velocity of each of the k cylinders in both y- and z-directions. This gives rise to N systems of the form of Eq. (4.6), thus allowing for the determination of the 4kN constants C_j and $\bar{C}_j$. Once these initialization constants are determined, the state vector $\{\dot{\underset{\sim}{\eta}}, \underset{\sim}{\eta}\}^T$ is readily obtained, at any time τ and abscissa ξ, by means of Eq. (4.6).

The computer program "TRANSIT" described in Appendix E enables the determination of the initialization constants, and plots the response of the system over a certain period of time.

4.2 Response to a Steady Excitation

The steady excitation of a cylinder may be interpreted either as the result of an applied force field, or as a constrained movement. In the first approach, the force field $f(\xi,\tau)$ is introduced in the equations of motion, and an applied force matrix

$$\underset{\sim}{Q}(\xi,\tau) = \{f_x^1,\ f_x^2,\ \ldots,\ f_x^k,\ f_y^1,\ f_y^2,\ \ldots,\ f_y^k\}^T \quad ,$$

takes the place of the $\underset{\sim}{0}$ matrix at the right-hand side of equation (3.6). After performing the different manipulations, the equation of motion (3.15) becomes

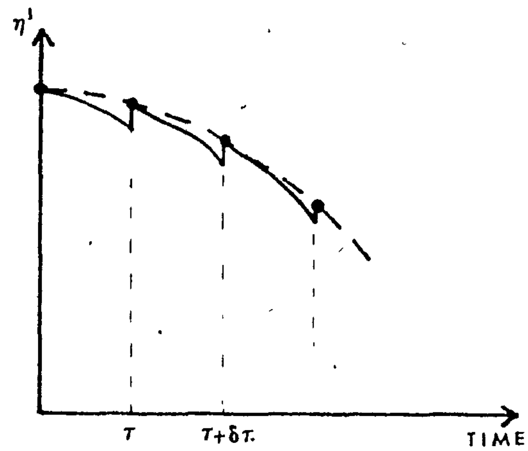
$$\underset{\sim}{\mathcal{M}}\,\ddot{\underset{\sim}{p}} + \underset{\sim}{\mathcal{C}}\,\dot{\underset{\sim}{p}} + \underset{\sim}{\mathcal{K}}\,\underset{\sim}{p} = \underset{\sim}{\mathcal{Q}} \quad , \tag{4.8}$$

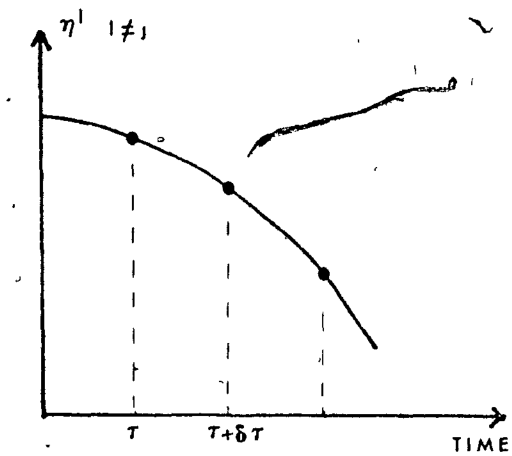
where

$$\underset{\sim}{\mathcal{Q}}(\tau) = \left\{ \int_0^1 \underset{\sim}{Q}\,\phi_1(\xi)\,d\xi,\ \int_0^1 \underset{\sim}{Q}\,\phi_2(\xi)\,d\xi,\ \ldots,\ \int_0^1 \underset{\sim}{Q}\,\phi_N(\xi)\,d\xi \right\}^T .$$

Equation (4.8) may then be decoupled and solved, using for instance an analytical method developed by Paidoussis [7] which only requires that the eigenvalues of the system for free vibrations be all distinct.

In the second approach, the movement of a cylinder, say cylinder j, is constrained, i.e. $\eta^j(\xi,\tau)$ and $\dot{\eta}^j(\xi,\tau)$ are fixed. As the coupling between the cylinders was introduced for arbitrary displacements of each of them, the equations of motions do not have to be modified. The response of the system may then be considered as a succession of infinitely close initial states where the velocity and the displacement of cylinder j are constantly reinitialized to their assigned value at this time, while the other cylinders are left in their actual state.





It should be pointed out that the response sketched above is theoretically exact, the only approximation being the fitting of the actual movement of cylinder j to the assigned one. As for any numerical method, its validity is highly dependent on the interval of time between two successive initializations, but if the frequencies of free and forced vibrations are sufficiently close, the modelization is quite satisfactory.

It is by the use of this numerical method that the computer program "TRANSIT" calculates and plots the response of the system where one (or several) cylinders are given an assigned sinusoidal movement.

CHAPTER V

RESULTS

In the previous chapters, solutions of the general equation of motion of the form

$$\eta(\xi,\tau) = Y(\xi)\ e^{i\omega\tau}$$

have been considered, where ω is a complex dimensionless frequency. The results produced by the computer programs "SOLINTER", "BUCKLING" and "TRANSIT" allow now for the determination of the dynamical behaviour with increasing flow, the stability maps and the response to applied excitations.

Computations have been carried out for systems ranging from one to five cylinders arranged in symmetrical patterns as represented in Figure 3. The confinement of the cylinders is characterized by the two parameters G_c and G_w where

$$G_c = \frac{\text{smallest cylinder-to-cylinder gap}}{\text{radius of cylinders}} ,$$

$$G_w = \frac{\text{smallest cylinder-to-channel gap}}{\text{radius of cylinders}} .$$

As to the dimensionless parameters which appear in the solution equation, they will be generally given values corresponding to a so-called "standard case" where

$$\beta_e = 0.32 \ , \quad \beta_i = 0.26 \ , \quad \varepsilon c_f = 0.25 \ , \quad Pr = 0.3$$

$$\alpha = \gamma_e = \gamma_i = \gamma = \Gamma = \delta = \Pi_e = \Pi_i = c = 0 .$$

We may now proceed with the discussion of the results.

5.1 The Eigenfrequencies of the System

5.1.A General Considerations

First of all some remarks should be made about the ordering of the natural frequencies. It is observed that they occur in "groups", or "frequency bands", each group corresponding to the infinite number of natural frequencies of a solitary tube, and are associated with the axial mode shapes. Of course, in our approximate solution, this infinite number has been truncated to N, the number of beam eigenfunctions $\phi_\ell(\xi)$ used. Within one of these frequency bands there are 2k pairs of complex conjugate eigenvalues, each of these pairs corresponding to a certain cross-sectional (or "lateral") mode, which pattern recurs in each of the N groups. This ordering arises because there are k degrees of freedom in both y- and z-directions, and N degrees in the axial direction. The frequency bands result from the coupling between the cylinders and, as it was emphasized in a previous study [12], the spread of frequencies within each band increases as the relative interstitial gaps G_c and G_w diminish.

5.1.B Behaviour with Increasing External Flow

The dimensionless internal flow velocity u_i is kept at a constant value while the external flow velocity u_e increases from zero. Two Argand diagrams have been plotted for $u_i = 0$

and $u_i = 6$ in the case of a three cylinder clamped-clamped system, and they may be found in Figures 4 and 5, respectively.

For $u_e = 0$, and in the absence of internal dissipation, the eigenfrequencies ω are wholly real which is a characteristic of a conservative system. Thus, the locus of each mode (axial/lateral) starts on the real axis, $Re(\omega)$. For small u_e all the eigenfrequencies become complex with a positive imaginary component, $Im(\omega)$, thus indicating that the flow damps the free motions of the system. With increasing u_e, the frequencies of oscillations $Re(\omega)$ diminish, and eventually that of the first mode vanishes, and at this point (A) its locus reaches the imaginary axis. Continuing from this point the locus bifurcates because the eigenvalues $i\omega$ being real, they are no longer conjugate; thus one branch moves up and the other down the $Im(\omega)$ axis. At slightly higher flow (point B), the imaginary component $Im(\omega)$ vanishes as well for one branch, indicating the onset of *divergence* (buckling). At a yet higher value of u_e the two branches coalesce and leave the imaginary axis at point C to re-enter the stable region. Finally, the locus crosses the real axis at point D which is the threshold of *flutter*, and the system loses stability again. The lowest mode of the group of second axial modes is also plotted and it is seen that it remains in the stable half-plane for the range of flow velocities investigated.

Comparing the two diagrams for $u_i = 0$ and $u_i = 6$ very little difference appears, the behaviour being quite similar.

However, in the case of $u_i = 6$ the instabilities occur for much lower velocities and the system is less damped for small values of u_e.

5.1.C Behaviour with Increasing Internal Flow

Two Argand diagrams have been plotted versus u_i for $u_e = 0$ and $u_e = 2$ in Figures 6 and 7, respectively. As the behaviour is rather different, we shall discuss each of them separately.

In the first case ($u_e = 0$) it is observed that until the first buckling, the system is undamped and the loci remain on the real axis. The first mode buckles at $u_i = 6.315$ and then the second mode at $u_i = 9.036$; at a slightly higher velocity, $u_i = 9.038$, the loci of the modes coalesce on the Im(ω)-axis and leave this axis at symmetrical points, indicating the onset of *coupled-mode flutter*.

In the second case ($u_e = 2$) the loci no longer start on the real axis. The first mode buckles at $u_i = 4.85$ and at $u_i = 8.11$ the two branches coalesce and leave the Im(ω)-axis almost at its intersection with the Re(ω)-axis. Finally, beyond $u_i = 8.5$ the loci of the two modes are almost symmetric, the first mode being unstable and the second stable.

5.2 The Stability Domain

As it may be seen in the Argand diagram of Figures 4 and 5, for each internal flow velocity corresponds a critical external flow velocity $(u_e)_{cr.}$ beyond which the system loses stability for the first time by buckling (point B). The computer program "BUCKLING" allows to plot the curve $(u_e)_{cr.}$ versus u_i, thus delimiting a stable domain between the two coordinate axes, u_i and u_e. We shall now proceed to discuss the influence of the various parameters and conditions on these stability maps.

5.2.A Influence of Confinement

The effect on stability of contiguity between cylinders and proximity to the channel is illustrated in Figure 8. It is seen that at low internal flow velocities, the confinement of the cylinders substantially reduces $(u_e)_{cr.}$ which is more than halved when G_c and G_w decrease from ∞ to 1/4. For high values of u_i, which imply lower $(u_e)_{cr.}$, the effect of confinement is less important and even becomes insignificant at the upper limit of u_i for stability ($u_i \sim 2\pi$). This peculiarity is explained by the static method used to solve the equations: the cylinders are supposed to be still until buckling and hence no hydrodynamic coupling is applied to them. Mathematically, the only term that involves coupling in the absence of external flow is the inertial term, which is multiplied by the instantaneous acceleration; divergence, however, is a phenomenon where time as such does not enter, and this term has no influence on the system. In practical cases, displacements of the cylinders might reduce the stability domain.

Furthermore, it is observed that the tighter the configuration is spaced, the less small values of u_i affect the stability of the system. For $G_c = G_w = \infty$, u_e and u_i have a rather symmetrical influence, but for $G_c = G_w = 1/4$ the threshold of buckling seems independent of the internal velocity until $u_i = 4$, beyond which u_i becomes in its turn the predominant agent. Therefore, for closely spaced configurations, the internal flow will not be prejudicial to stability for $u_1 < 3$ approximately.

5.2.B Influence of the End Conditions

In Figure 9 are shown the domains of stability corresponding to a clamped-clamped and a pinned-pinned three cylinder system. As might be expected, the pinned-pinned system is considerably less stable, and the two curves are almost homothetic in a ratio of around 1/2. This similarity comes from the fact that the boundary conditions exert no influence on the forces, but contribute to stability by altering the modes and frequencies of vibrations. This is why this behaviour is very similar to that observed for a single cylinder in unbounded flow (see Ref. [11] for instance).

5.2.C Influence of the Number of Cylinders

Stability maps have been plotted for systems of one to five cylinders and for various spacings between them. It appears that the number of cylinders does not change significantly the stability regions, especially for close-spaced clusters. Moreover, there is no obvious correlation between the number of cylinders and the destabilizing effect it could induce. For instance, the critical external flow velocity for a typical system ($G_c = G_w = 1/4$, standard parameters, clamped ends) of four cylinders is 3.12 compared to 3.07 for a two cylinder system. This difference is presumably connected to the relative space available to external flow in the cross-section. Shown in Figure 10 are the smallest and the largest stability domains out of the five configurations tested, in clusters where $G_c = G_w = 1$ and $G_c = G_w = 1/4$.

5.2.D <u>Influence of the System Parameters</u>

The effect of changing the various parameters in the equation of motion (α, β_e, β_i, γ_e, γ_i, γ , Γ, δ, εc_f, Π_e, Π_i, c, Pr) were investigated. Each of these parameters has been changed in turn and given extreme, but still physically realistic, values. For all of them, except the pressurization (Π_e, Π_i), the beam tension (Γ) and the viscous frictional coefficient c_f, the modification they create on the critical velocities is quite negligible, always less than 5%. As to the fluid pressure and compression on the beam, they produce similar effects since they intervene in the same term of the general equation of motion. Increasing $\Pi_i - \Pi_e$ or the compression Γ with ends not free to move ($\delta = 1$) gives rise to a significant destabilization which may be viewed in Figure 11 for a three cylinder pinned-pinned system. Thus a compression of $\Gamma = -5.$ corresponds to a curve which is homothetic to the "standard" one in a ratio of 2/3.

On the contrary, the viscous frictional forces, which depend on the group of parameters εc_f, have a tendency to stabilize the system. Thus, increasing εc_f from 0.25 to 1 slightly enlarges the stability domain, as may be observed in Figure 11.

5.3 The Transient Response

In this section we shall be concerned with the transient behaviour of pinned-pinned systems subjected to various excitations. For the sake of simplicity - and incidentally in accordance with observation - all cases looked upon in this study are initialized according to the first axial mode shape, i.e. the initial state of any cylinder j satisfies

$$\eta^j(\xi) = \eta^j(1/2) \sin \pi \xi \quad .$$

5.3.A Response After Initial Displacements

In Chapter IV it is shown that once the initial state of all the cylinders is given, the position and velocity of each of them for all subsequent time are computable. We shall discuss first some peculiar cases which lay emphasis on the multiplicity of the eigenfrequencies for each axial mode.

For better understanding, a system of two symmetrical cylinders (as depicted in Figure 3) will be examined in preference, for in this case the motions in the y- and z-directions are independent, thus leading to simpler interpretation. As mentioned previously, there are in each direction two transverse modes of vibrations associated with two frequencies within each "group" of frequencies. In the first group of frequencies, i.e. those frequencies with the same axial mode, the cross-sectional mode shapes and frequencies are as follows:

i) in the y-direction $\begin{cases} \text{1st mode} & v_1(\tau) = v_2(\tau) & \omega = \omega_1 \ , \\ \text{2nd mode} & v_1(\tau) = -v_2(\tau) & \omega = \omega_2 \ . \end{cases}$

ii) in the z-direction $\begin{cases} \text{1st mode} & w_1(\tau) = w_2(\tau) & \omega = \omega_1' \ , \\ \text{2nd mode} & w_1(\tau) = -w_2(\tau) & \omega = \omega_2' \ . \end{cases}$

The numerical values obtained with $G_c = G_w = 1/4$ and zero flow velocities are the following:

$$\omega_1 = 6.83 \qquad \omega_2 = 9.71 \ ,$$

$$\omega_1' = 8.33 \qquad \omega_2' = 8.92 \ .$$

If the system is given an initial state meeting one of these conditions it will vibrate at the frequency corresponding to the excited mode. In Figures 12 and 13 these initial states are assigned in turn, thus describing respectively the first and second mode in each direction; the cylinders are seen to vibrate with no phase shift, nor damping since they are immersed in still fluid (supposed here to be inviscid).

Now, if we give the system arbitrary initial conditions, the resulting response will be a linear combination of these two modes. For instance, if the first cylinder is displaced in the z-direction while the second is left at rest, it may be deduced from the constants of initialization that the movements of the two cylinders are governed by the relations

$$w_1(\tau) = w_1(0) \cos \frac{\omega_1' + \omega_2'}{2} \tau \cdot \cos \frac{\omega_2' - \omega_1'}{2} \tau ,$$
$$w_2(\tau) = - w_1(0) \sin \frac{\omega_1' + \omega_2'}{2} \tau \cdot \sin \frac{\omega_2' - \omega_1'}{2} \tau . \tag{5.1}$$

In Figure 14 is viewed the beat phenomenon which results from this combination of modes: the "fundamental" frequency $(\omega_1' + \omega_2')/2$ is modulated by the "beat" frequency $(\omega_2' - \omega_1')/2$. The behaviour is similar in the y-direction, but the beat is less evident because ω_1 and ω_2 are more remote from each other. Another important aspect which proceeds from this beating is that the initial amplitude of the first cylinder is fully transmitted to the second, thus expressing a continuous exchange of energy between them.

The case of a three-cylinder configuration has also been plotted in Figure 15 (a and b) to illustrate the coupling between mutually perpendicular displacements. Thus, although the only initial displacement takes place in the z-direction (for the first cylinder), the maximum amplitudes reached by the two others actually occur in the y-direction. Of course, the displacements of cylinders 2 and 3 are symmetric, but their equation of motion may no longer be expressed in a simple form since more than two modes are now involved. Clearly orbital motion of cylinders 2 and 3 takes place.

The effect of a non-zero external flow velocity appears in Figure 16, where $u_e = 1.4$ for the two cylinder system. As expected, the four transverse mode frequencies are lowered and become

$$\omega_1 = 2.65 \quad , \quad \omega_2 = 8.75 \quad ,$$

$$\omega_1' = 6.27 \quad , \quad \omega_2' = 7.38 \quad .$$

In consequence, the "fundamental" frequency is lowered while the "beat" frequency is increased, due to the larger spread $\omega_2 - \omega_1$ of the frequencies. Furthermore, it must be pointed out that in this case equation (5.1) is only an approximation of the equation of motion, for a more careful study of the initialization constants indicates that the other axial modes (mainly the second mode) are slightly excited. It means that the axial mode shape is no longer pure, but is a mixture of several modes which render the modal shape time-dependent. Moreover, there is an exponential damping factor, the effect of which appears clearly on the plot, which halves the amplitude of oscillation by the time $\tau = 20$.

5.3.B Response Under Steady Excitation

In this paragraph we shall consider the response of the system when one of the cylinders (cylinder 1) is constrained to have a sinusoidal motion of constant amplitude and frequency, of the form

$$\eta^1(\tau) = \eta^1(0) \cos \Omega \tau \quad .$$

For a two-cylinder system in still fluid, the differential equations of motions may be integrated by hand; the solutions are derived in Appendix F. If we denote by m_{ij} the coefficients of the mass matrix $\underset{\sim}{M}$, a "resonance" of the second cylinder is shown to occur in the following cases:

i) in the y-direction with $v_1(\tau) = v_1(0) \cos \Omega \tau$:

$$\frac{v_2(\tau)}{v_1(0)} = -\frac{1}{2} \frac{m_{43}}{m_{44}} \Omega \tau \sin \Omega \tau, \text{ if } \Omega = \Omega_{res} = \sqrt{\frac{\pi^4}{m_{44}}} \quad ; \tag{5.2}$$

ii) in the z-direction with $w_1(\tau) = w_1(0) \cos \Omega' \tau$:

$$\frac{w_2(\tau)}{w_1(0)} = \frac{1}{2} \frac{m_{21}}{m_{22}} \Omega' \tau \sin \Omega' \tau, \text{ if } \Omega' = \Omega'_{res} = \sqrt{\frac{\pi^4}{m_{22}}} \quad . \tag{5.3}$$

It seems worthwhile to enumerate some remarks these results arouse. First of all, the "resonance" frequency Ω_{res} is not one of the eigenfrequencies ω described in the previous section, but is located at an intermediate value between them

$$\omega_1 < \Omega_{res} < \omega_2 \quad , \quad \omega'_1 < \Omega'_{res} < \omega_2 \quad .$$

Accordingly, this type of resonance must not be confused with the resonances obtained when the system is excited with a steady sinusoidal force, for in this latter case the resonances are likely to occur at $\Omega = \omega_1$ and $\Omega = \omega_2$. In the case at hand the displacement of one of the cylinders is constrained to remain constant, thus the system is neither free nor forced in the normal way.

Examining equations (5.2) and (5.3) it is noticed that the amplitude of the second cylinder is linearly increasing with time, and that the rate of increase per period is independent of Ω_{res}. The slope is proportional to the off-diagonal terms of the mass matrix, thus underlining the dramatic importance of confinement, on which these terms are highly dependent. In the case where $G_c = G_w = 1/4$, the matrix coefficients are

$$m_{11} = m_{22} = 1.31 \quad , \quad m_{12} = m_{21} = -0.0899 \quad ,$$

$$m_{33} = m_{44} = 1.56 \quad , \quad m_{34} = m_{43} = 0.526 \quad .$$

which yield the dimensionless resonant frequencies

$$\Omega_{res} = 7.90 \quad , \quad \Omega'_{res} = 8.62 \quad ,$$

and a rate of increase of the amplitude per period of 1.06 in the y-direction and of 0.215 in the z-direction. Hence, the resonance is much more pronounced in the y-direction, where in less than one period the amplitude of the second cylinder exceeds the level of excitation of the first one. In Figure 17 is plotted the response of such a system excited in both directions at the corresponding resonant frequency, and the considerable amplitude reached by the second cylinder appears clearly.

The response of systems comprising more than two cylinders is much harder to derive by hand. However, in the case of 3 cylinders, at the middle of a row of rigid tubes centered on the z-axis, an approximate formulation of the response of the two non-excited cylinders is obtained in Appendix F; the equations of motion for displacements in the z-direction are found to be as follows:

$$\frac{w_1(\tau)}{w_1(0)} = \cos \Omega \tau \quad ,$$

$$\frac{w_2(\tau)}{w_1(0)} \sim \frac{1}{2} \frac{m}{M} \Omega \tau \sin \Omega \tau \quad ,$$

$$\frac{w_3(\tau)}{w_1(0)} \sim - \frac{1}{8} \left(\frac{m}{M}\right)^2 \Omega (3 \tau \sin \Omega \tau + \Omega \tau^2 \cos \Omega \tau) \quad ,$$

$$\text{if} \quad \Omega = \Omega_{res} = \sqrt{\frac{\pi^4}{M}} \quad , \tag{5.4}$$

where M is the diagonal term of the mass matrix in the z-direction, and m the off-diagonal term corresponding to the coupling effect between two adjacent cylinders in the z-direction. The validity of equation (5.4) is, however, not reliable after a few periods, for terms of higher order with respect to time have been neglected.

It is observed that the response of the second cylinder is linearly increasing with time, and identical to that obtained in the previous case of two cylinders. As to the third cylinder, its equation of motion contains a quadratic term which exceeds the linear term as soon as $\Omega\ \tau > 3$, i.e. in less than half a period. Hence, even with a relatively large spacing between the cylinders, the displacements of the third cylinder may become quickly important, due to the parabolic evaluation of its amplitude. With $G_c = 1$, $G_w = \infty$ the coefficients of the mass matrix are $M = 1.02$ and $m = -0.0712$ which yield the resonance frequency $\Omega_{res} = 9.77$ and the following equations of motion:

$$w_1(\tau) = w_1(0) \cos \Omega_{res}\ \tau \quad ,$$

$$w_2(\tau) \sim -0.34\ \tau\ w_1(0) \sin \Omega_{res}\ \tau \quad ,$$

$$w_3(\tau) \sim -0.018\ \tau\ w_1(0) \sin \Omega_{res}\ \tau - 0.058\ \tau^2\ w_1(0) \cos \Omega_{res}\ \tau \ .$$

In Figure 18 is given the digital plot corresponding to such a case and, indeed, the parabolic evolution of the third cylinder amplitude may be viewed, as well as the limits of validity of the above formulae. Finally, in Figure 19 (a and b) is plotted the response of a system of three cylinders in classical equilateral configuration, with $G_c = G_w = 1/4$. Cylinder 1 is excited in the z-direction and here again the resonance is more dramatic in the y-direction where in 13 periods the amplitude reaches five times the level of excitation.

5.3.C Precision of the Computer Program "TRANSIT"

It should be recalled that to constrain the movement of the first cylinder the computer program proceeds by continuous reinitializations of its position and velocity in order to fit the prescribed motion. But, as the other cylinders are always left in their actual state, there is no way of recovering the systematic error which is generated whenever the prescribed frequency is different from the frequency of free oscillations of the system when abandoned from the reinitialization state. There is a slight, but generally cumulative, displacement of phase of the non-excited cylinders from the one they should actually have, and this sliding effect is particularly sensitive in cases of resonances where the phase plays an important part. Moreover, little is gained by reducing the time interval between two reinitializations because the number of cumulating errors is increased in the same proportion as the value of each of the errors is reduced. This difficulty was, however, overcome, in the cases of resonance, by giving to the "driving" frequency a value slightly above the actual resonance frequency in order to compensate for this phase shifting.

But besides the necessity of introducing this correction, the computed response proved to corroborate the predicted behaviour, in all cases tested; this agreement is quite satisfactory since the response is in fact more simulated than really computed by the program.

CHAPTER VI

CONCLUSION

The theory developed in this thesis enables a prediction of the dynamical behaviour of a cluster of flexible pipes conveying fluid, immersed in axial flow bounded by a cylindrical channel. Structures of this kind are commonly encountered in practical engineering systems such as heat exchangers and nuclear reactor fuel bundles.

Among the fluid forces acting on the external surface of the tubes, the viscous and inviscid hydrodynamic coupling between motions of the cylinders and the effect of proximity to the channel have been taken into account, according to a method based on potential flow theory presented in previous studies [9, 10]. Expressions derived for these forces, which depend on the presence and motions of all cylinders in the bundle, were incorporated in the general equation of motion of a slender pipe subjected simultaneously to an external axial flow and independent internal flow. The equation of motion takes then a matrix form and was solved by means of Galerkin's approximate technique to yield the eigenfrequencies of the system; these frequencies occur in bands centered around the eigenfrequencies of a single cylinder under the same conditions, and are associated with modes of the same or similar axial shape. It was shown that pipes with both ends supported are subject to divergence (buckling) and then to flutter when increasing either the internal or external flow velocity, or both. This behaviour was expected and may be interpreted as a combination of two previously studied limit cases, namely a solitary pipe in presence of parallel internal and external flows [4, 11], and a cluster of cylinders surrounded by bounded axial fluid flow [10].

As the designer is mostly interested in the critical conditions associated with the first instability encountered with increasing internal or external flow velocities, a simplified analysis referring exclusively to buckling was conducted. It showed that, whereas the two flows play almost interchangeable roles for widely spaced configurations, that of the external flow becomes prominent for compact clusters. For a gap between cylinders or between the cylinders and the channel of 1/4 of the cylinder radius, the divergence due to external flow is independent of the internal fluid velocities, as long as these remain moderate, and occurs for half the velocity necessary in the case of a single unbounded cylinder. However, unless there is appreciable compressive load on the beams, coupled with highly confined geometry, the hydroelastic instabilities are not likely to occur for the velocities usually encountered in practice.

The most original results of this thesis relate to the transient response of such a system under instantaneous perturbations or steady excitations. When the system is released after some of the cylinders are given initial displacements, a beating phenomenon takes place, hence expressing mutual transfer of energy between the cylinders via the surrounding fluid. It may be concluded that if many cylinders are slightly disturbed, the energy fed into the system may eventually concentrate on one cylinder for awhile, thus magnifying its displacements up to abnormal values. Moreover, it was shown that to each cylinder is associated a resonant frequency which is distinct from the eigenfrequencies of the solution equation, the latter being relative to the whole system. Accordingly, in highly symmetric patterns where the resonant frequency is identical for all cylinders, the amplitudes of their oscillations may become considerable if the system is excited at this resonant frequency; indeed computations carried out for rows of three cylinders, indicate that the excitation propagates to the neighbouring cylinders with cumulative effect. This behaviour conduces to many practical implications and may explain the fret marks noticed in some existing cases which result from repeated collisions between adjacent cylinders. Indeed, in closely spaced bundles, even small amplitude vibration results in impact within the structure which, in time, might cause the rupture of the elements with serious consequences.

A large field of investigations is therefore open, pertaining to the origins of the perturbations or excitations that could generate abnormal displacements or resonances. Probable sources of disturbances are a simple shock on some of the cylinders, an arbitrary force field due to external vibration sources, an unsteady fluid flow, or else a random pressure field which may arise from pressure fluctuations in the turbulent boundary layer of a subsonic flow. These types of excitation may feed energy into the system by means of a large spectrum of frequencies; among them one might coincide for awhile with one of the resonant frequencies of some elements and yield worrisome displacements.

Hence, a great deal of work remains to be done on the theoretical side, with the aim to shed light on the transient phenomena of interaction between elements of a structure, and the mechanisms from which resonances could originate.

Finally, it would be worthwhile and presumably workable to conduct some experiments in order to view, at least qualitatively, the beats and resonances which have been brought to light by this thesis.

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	Pinned-Pinned Cylinders	Clamped-Clamped Cylinders
$a_{rs}\ (r \neq s)$	$\frac{2rs}{s^2-r^2}\{(-1)^{r+s}-1\}$	$\frac{4\lambda_r^2\lambda_s^2}{\lambda_s^4-\lambda_r^4}\{(-1)^{r+s}-1\}$
a_{rr}	0	0
$b_{rs}\ (r \neq s)$	0	$\frac{4\lambda_r^2\lambda_s^2}{\lambda_s^4-\lambda_r^4}(\lambda_s\sigma_s-\lambda_r\sigma_r)\{(-1)^{r+s}+1\}$
b_{rr}	$-r^2\pi^2$	$\lambda_r\sigma_r(2-\lambda_r\sigma_r)$
$d_{rs}\ (r \neq s)$	$\frac{4s^3r}{(s^2-r^2)^2}\{1-(-1)^{r+s}\}$	$\frac{(-1)^{r+s}\,b_{rs}}{\{(-1)^{r+s}+1\}} - \frac{3\lambda_s^4+\lambda_r^4}{\lambda_s^4-\lambda_r^4}\,a_{rs}$
d_{rr}	$\frac{1}{2}b_{rr}$	$\frac{1}{2}b_{rr}$

where λ_r and σ_r satisfy

$$\cos\lambda_r\cosh\lambda_r = 1\ ,$$

$$\sigma_r = \frac{\cosh\lambda_r - \cos\lambda_r}{\sinh\lambda_r - \sin\lambda_r}\ .$$

Table 1 The constants a_{rs}, b_{rs} and d_{rs} as defined in Chapter III for pinned-pinned and clamped-clamped beams. The values of λ_r and σ_r for a clamped-clamped beam are tabulated in Bishop and Johnson [18].

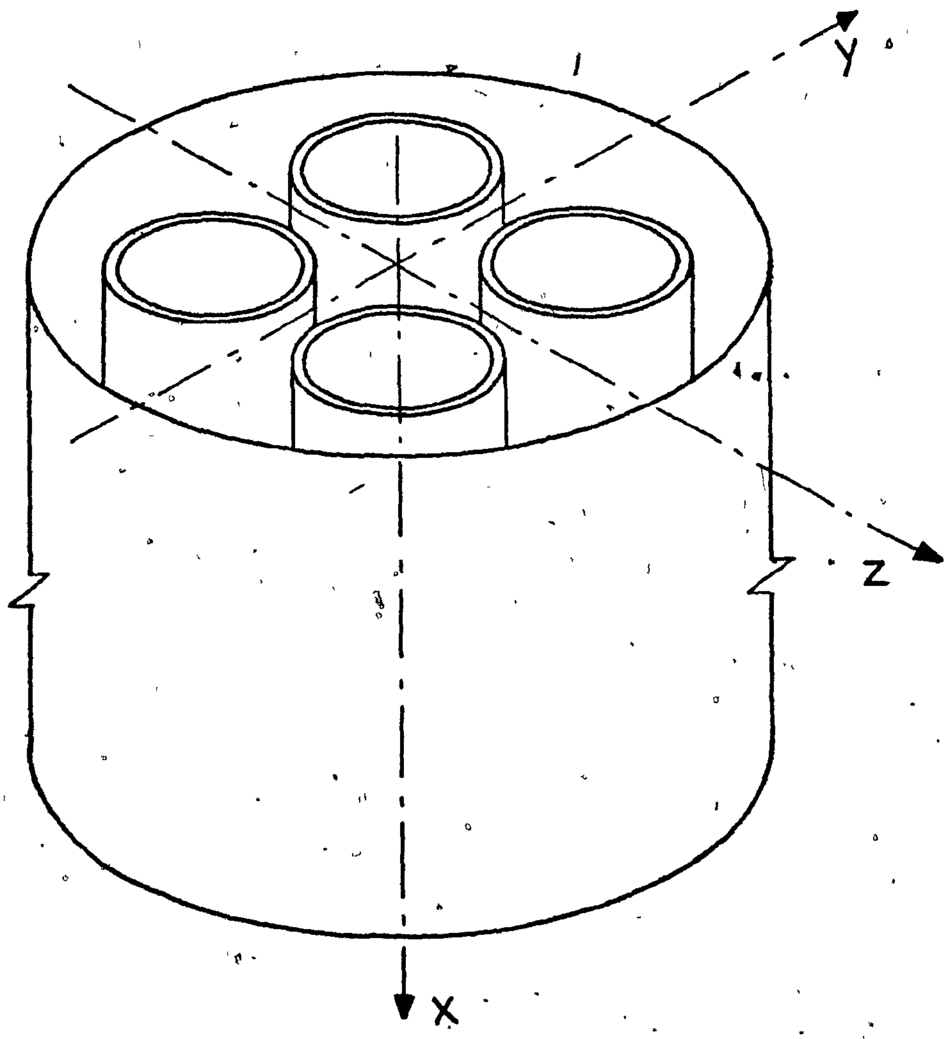


Figure 1 Schematic diagram of the system under consideration

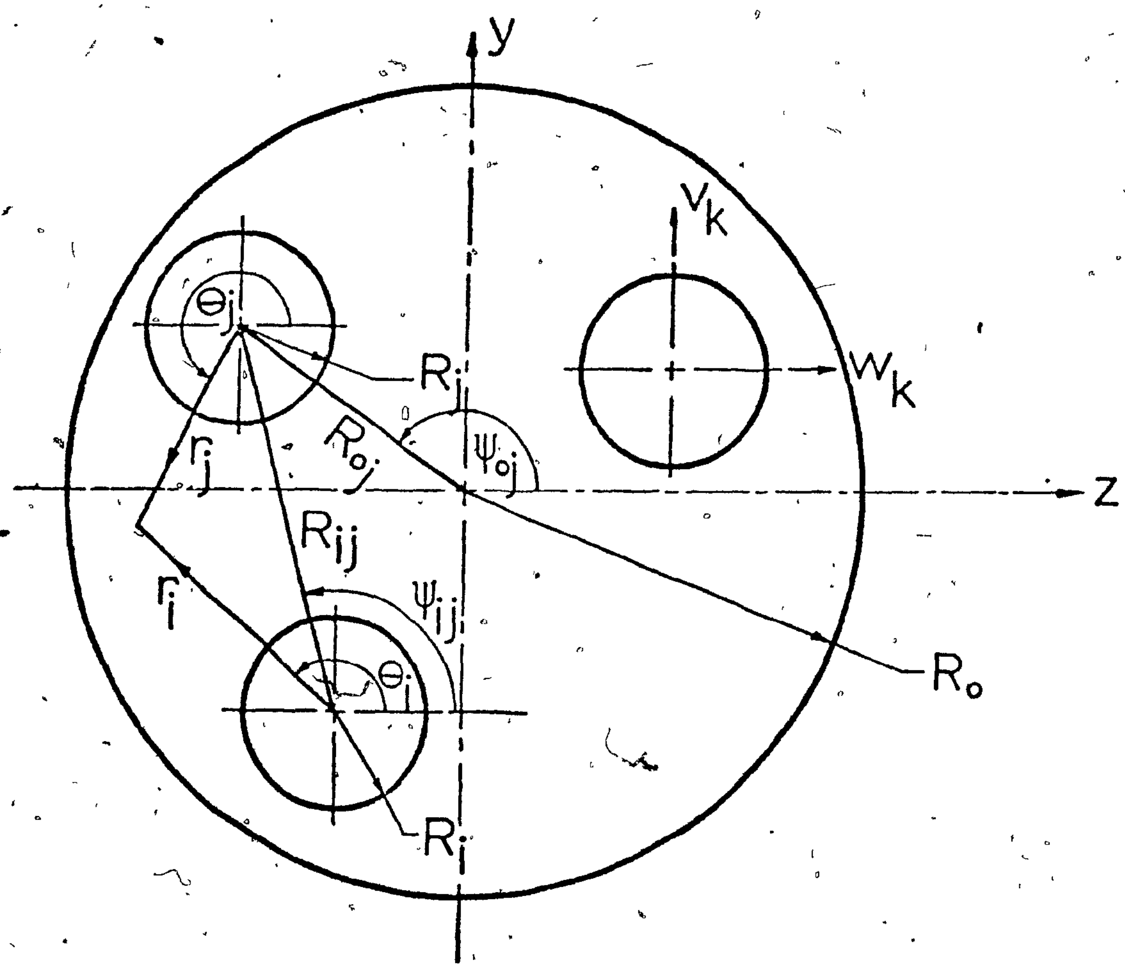


Figure 2 Definition of the coordinate systems and symbols for the determination of the velocity potential

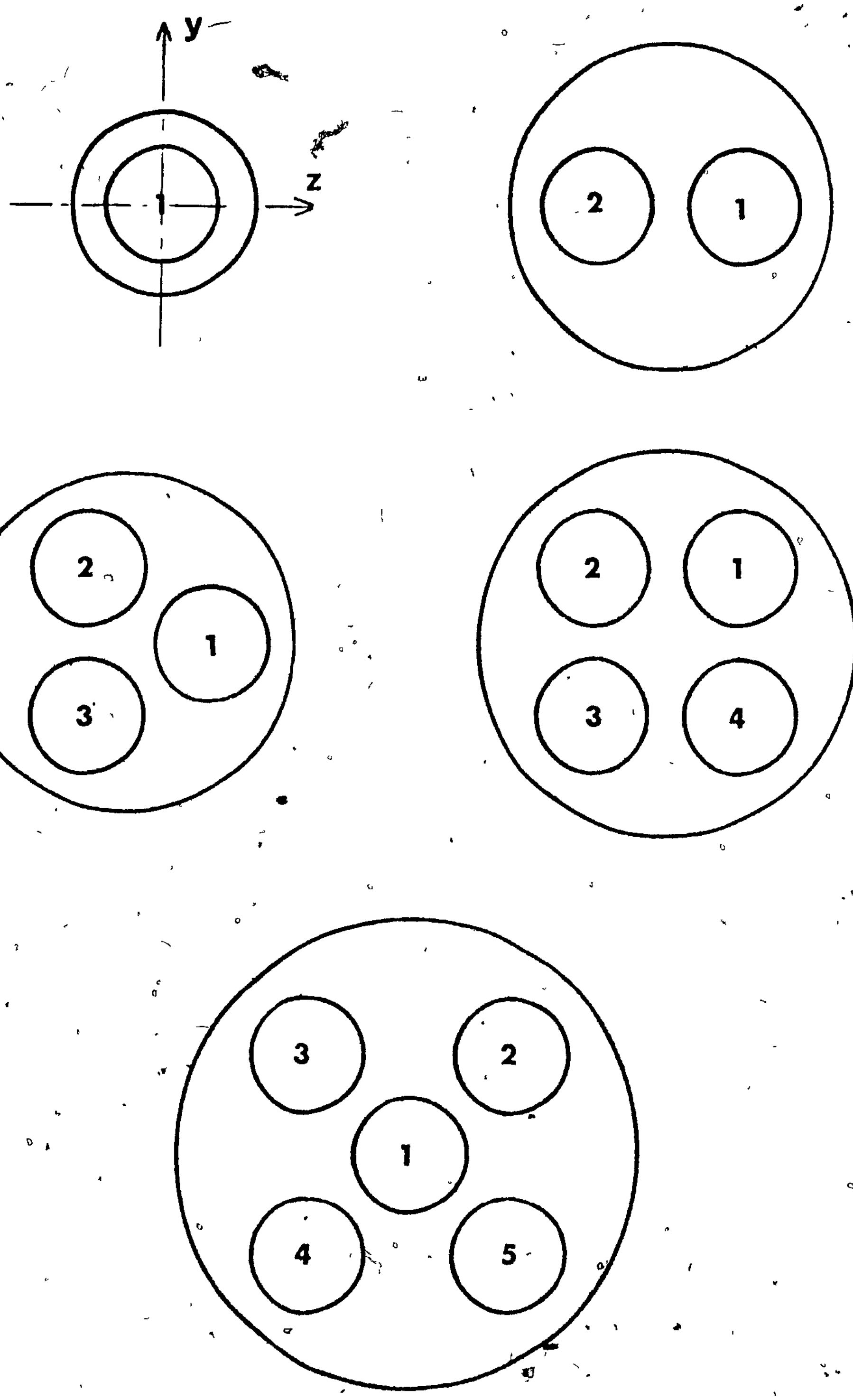


Figure 3 Configurations of the clusters of cylinders considered

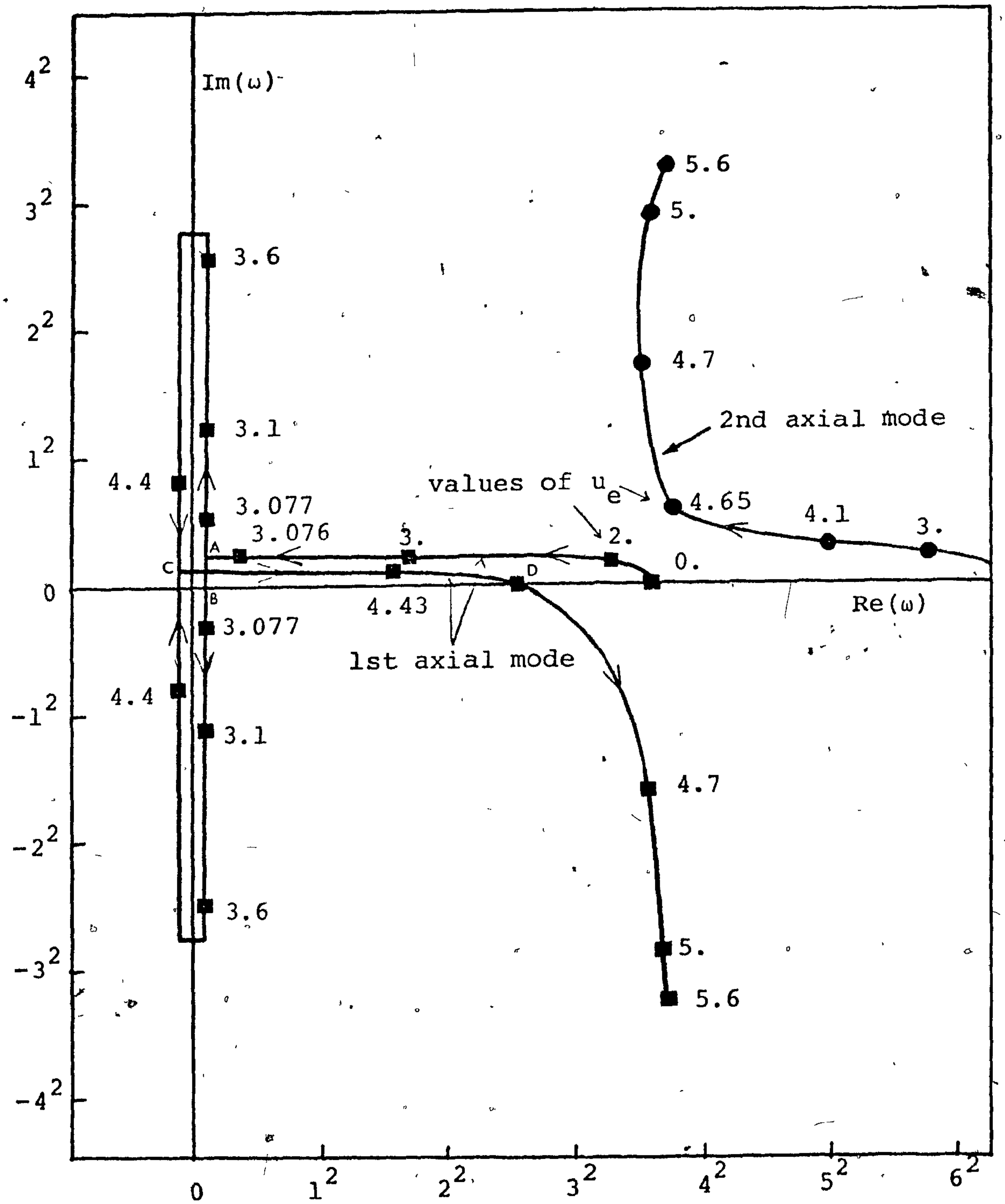


Figure 4 Argand diagram of the complex frequencies ω of the lowest member in each of the groups of 1st and 2nd axial modes, with the external flow u_e as parameter and $u_i = 0$.
Case of a three cylinder clamped-clamped system with standard parameters and $G_c = G_w = 1/4$.
The loci on the Im(ω)-axis have actually been drawn slightly off the axis and parallel to it, for the sake of clarity.

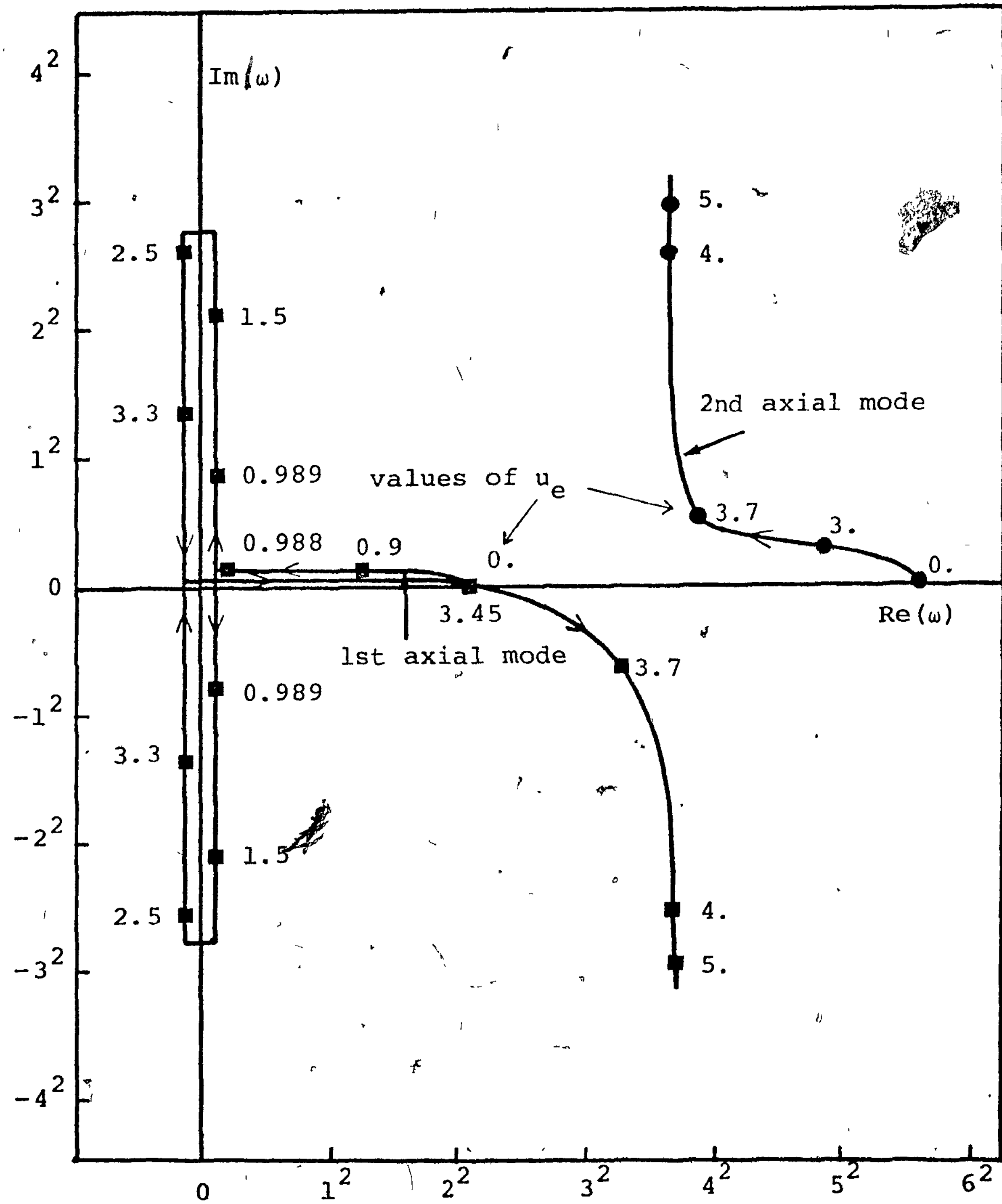


Figure 5 Argand diagram of the complex frequencies ω of the lowest member in each of the groups of 1st and 2nd axial modes, with the external flow as parameter and $u_i = 6$.
Case of a three cylinder clamped-clamped system with standard parameters and $G_c = G_w = 1/4$.

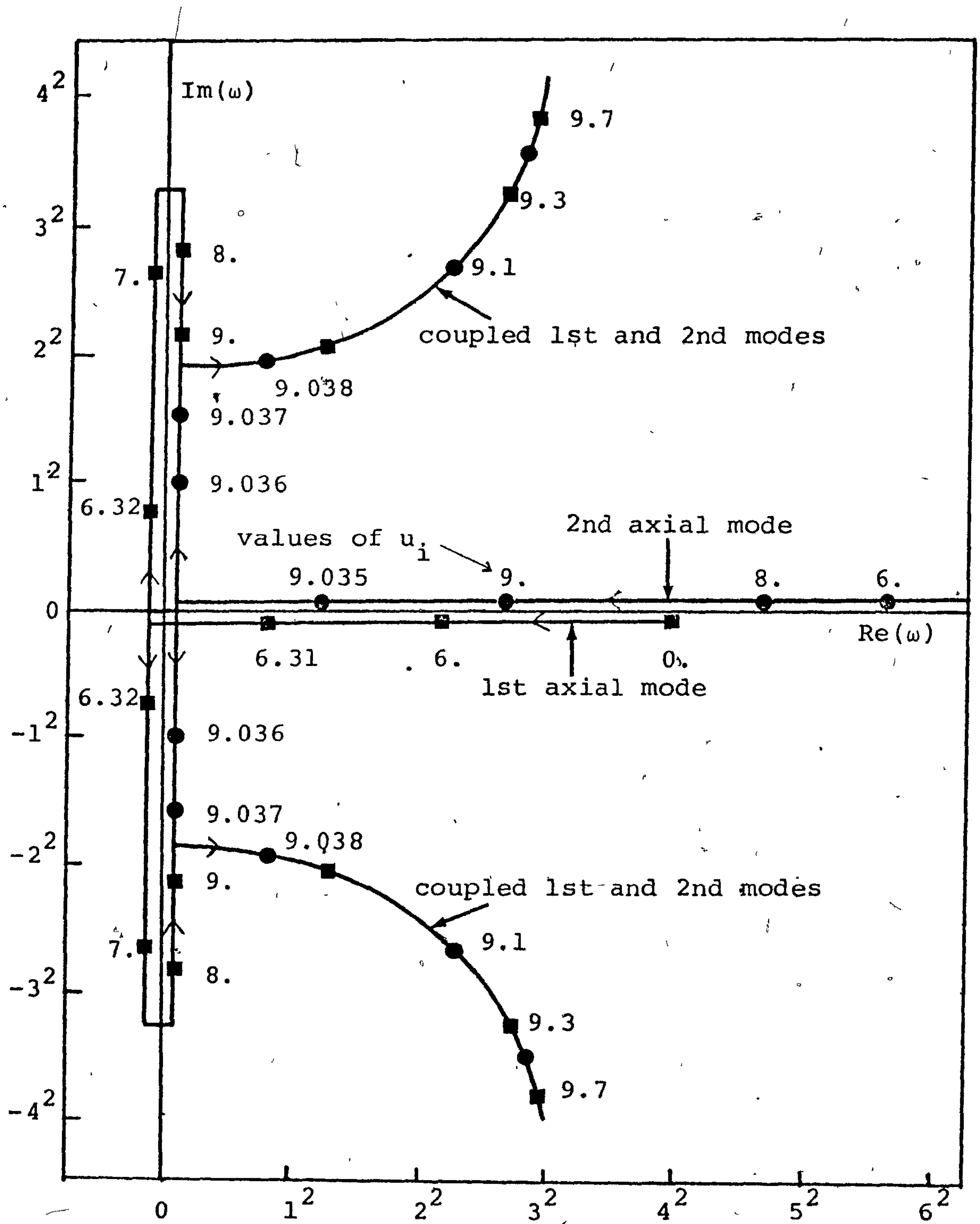


Figure 6 Argand diagram with the internal flow as parameter and $u_e = 0$.
Case of a three cylinder clamped-clamped system with standard parameters and $G_C = G_W = 1/4$.
The loci that actually lie on the axes have been drawn slightly off the axes, but parallel to them for the sake of clarity.

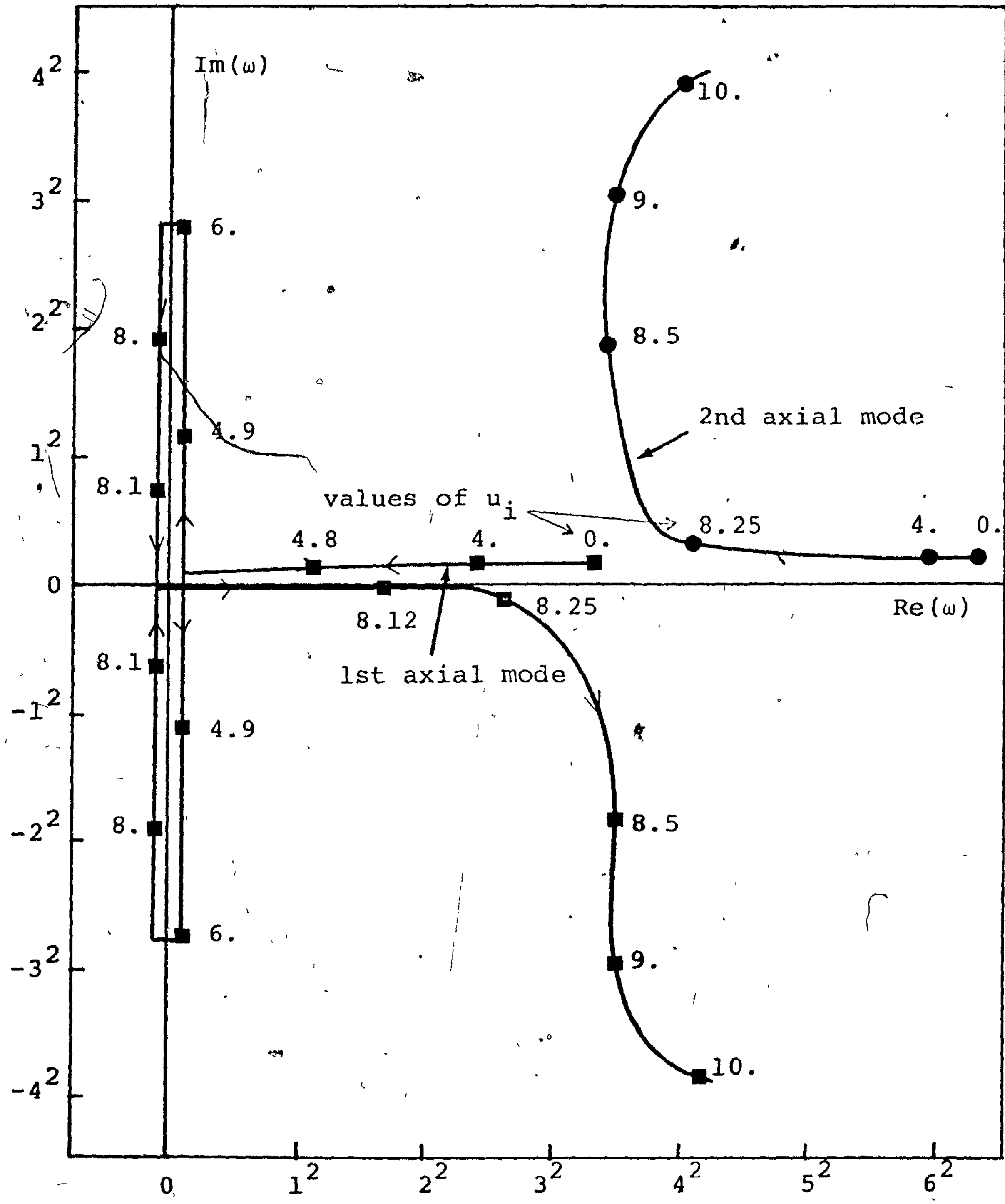


Figure 7 Argand diagram with the internal flow u_i as parameter and $u_e = 2$.
Case of a three cylinder clamped-clamped system with standard parameters and $G_c = G_w = 1/4$.

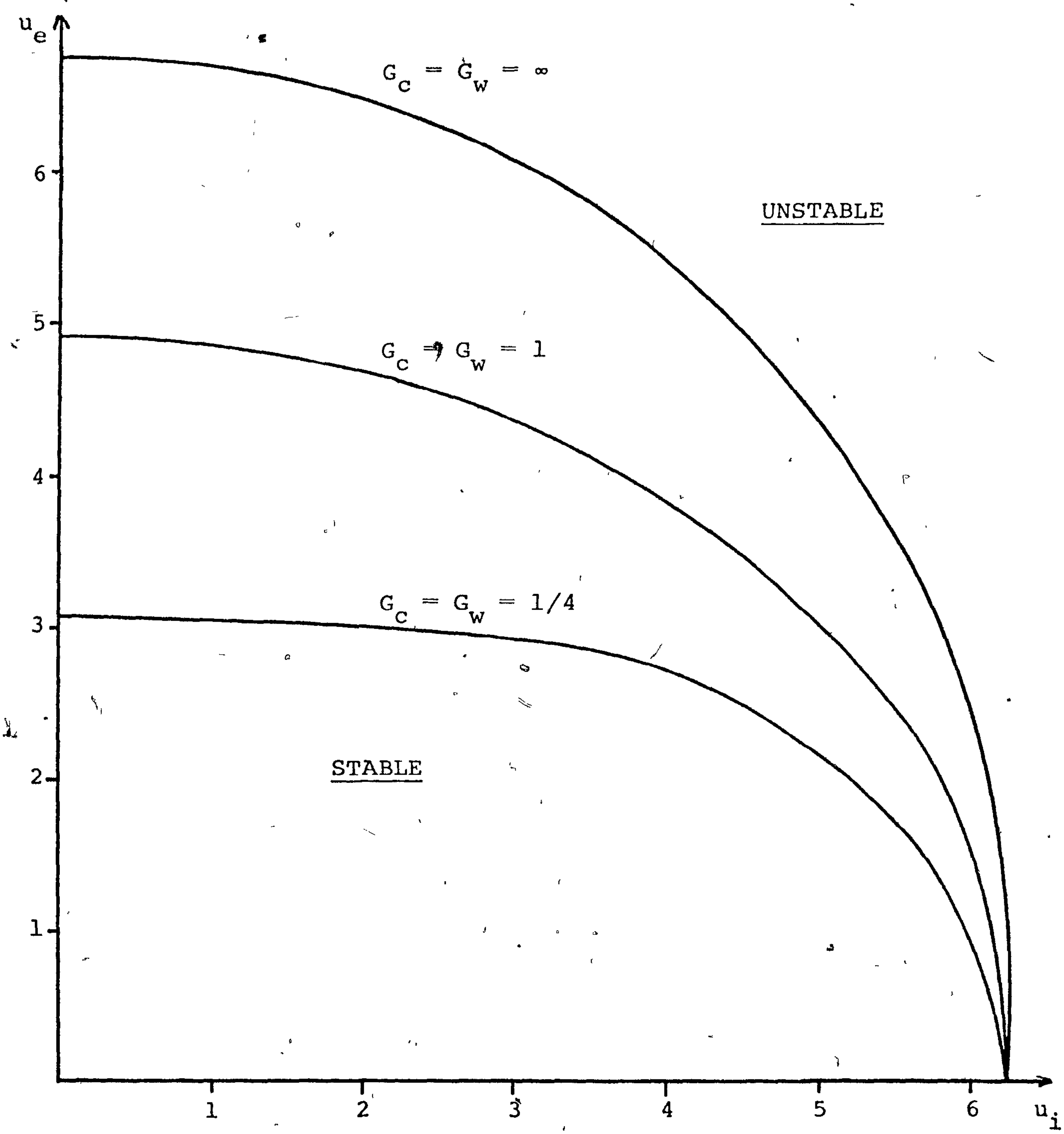


Figure 8 Stability map showing the effect of confinement for a three cylinder clamped-clamped system (standard case)

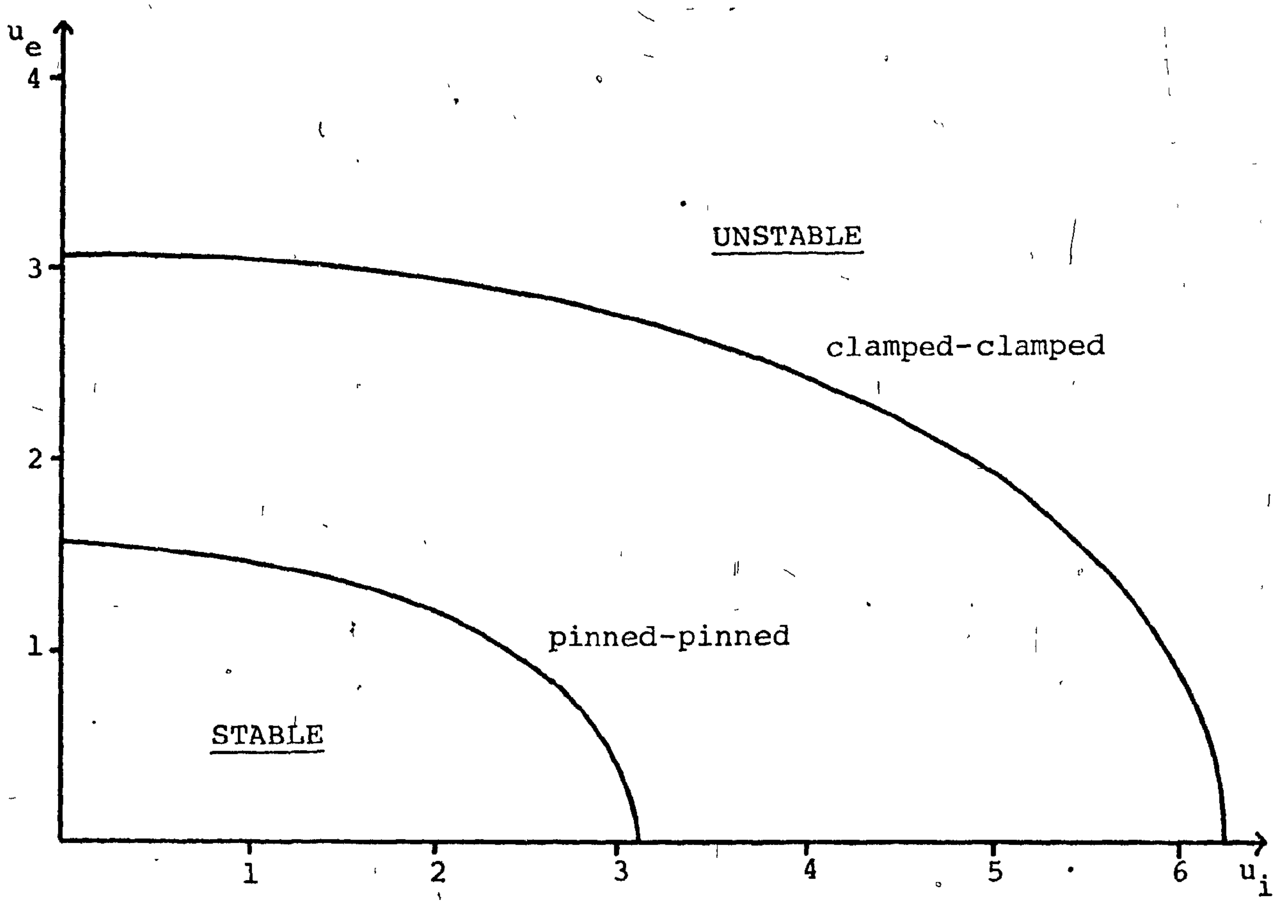


Figure 9 Stability map showing the effect of end conditions for a three cylinder system (standard case).

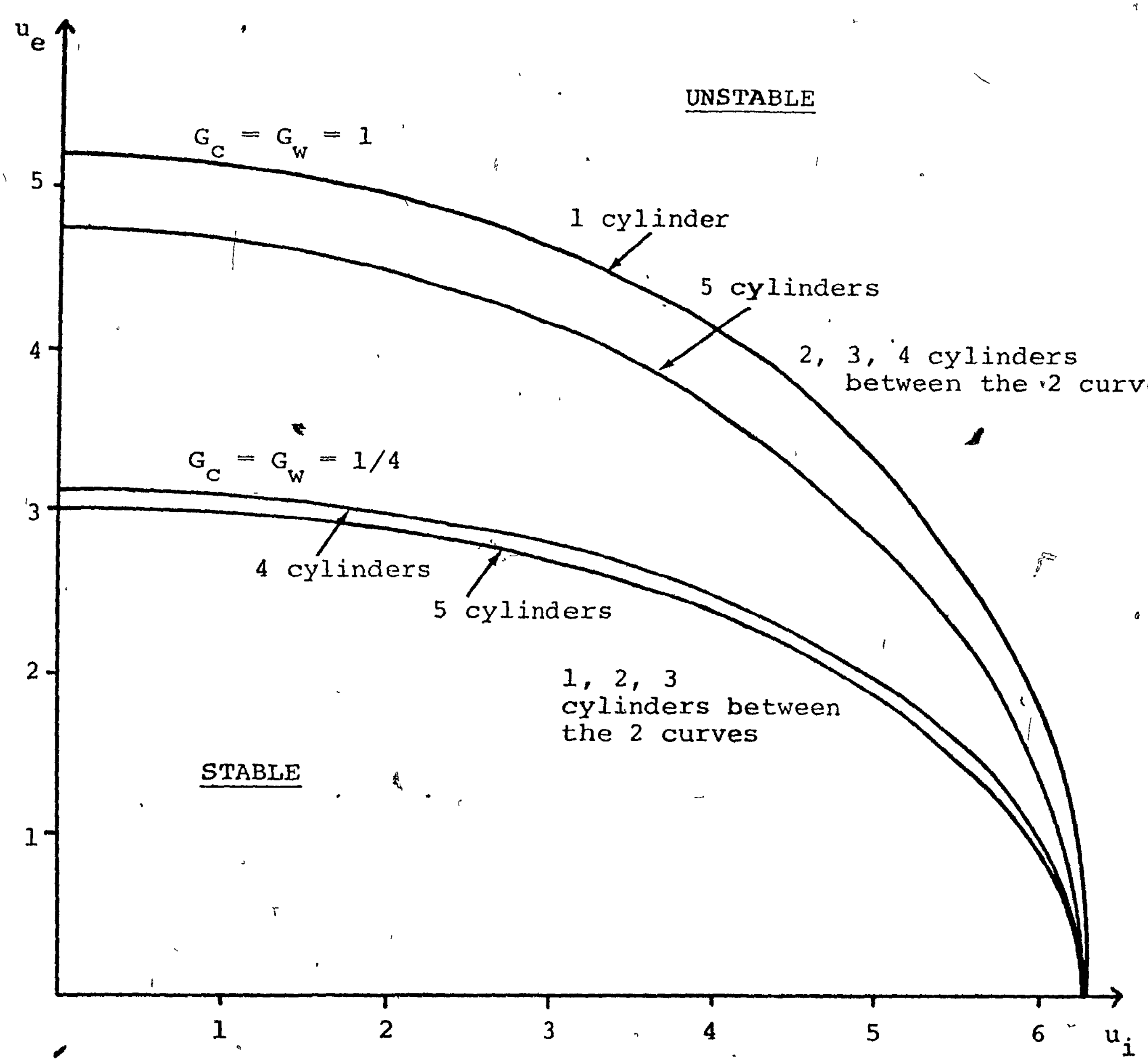


Figure 10 Stability map showing the influence of the number of cylinders for clamped-clamped systems with $G_c = G_w = 1$ and $G_c = G_w = 1/4$ (standard case).

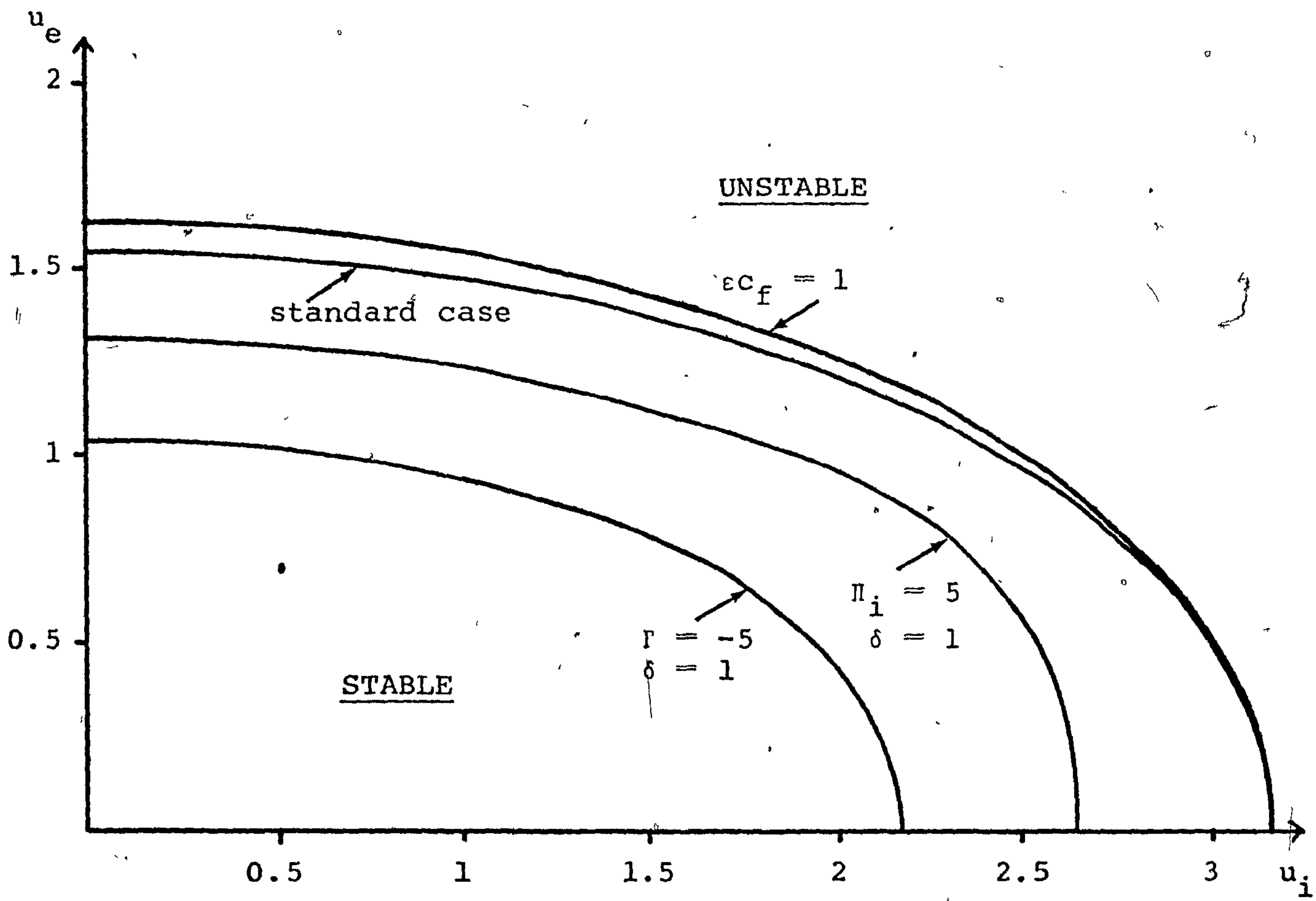


Figure 11 Stability map showing the influence of beam axial tension, Γ, the parameter εc_f, and internal fluid pressure, Π_i, for a three cylinder pinned-pinned system, with $G_c = G_w = 1/4$ (standard case except mentioned parameters).

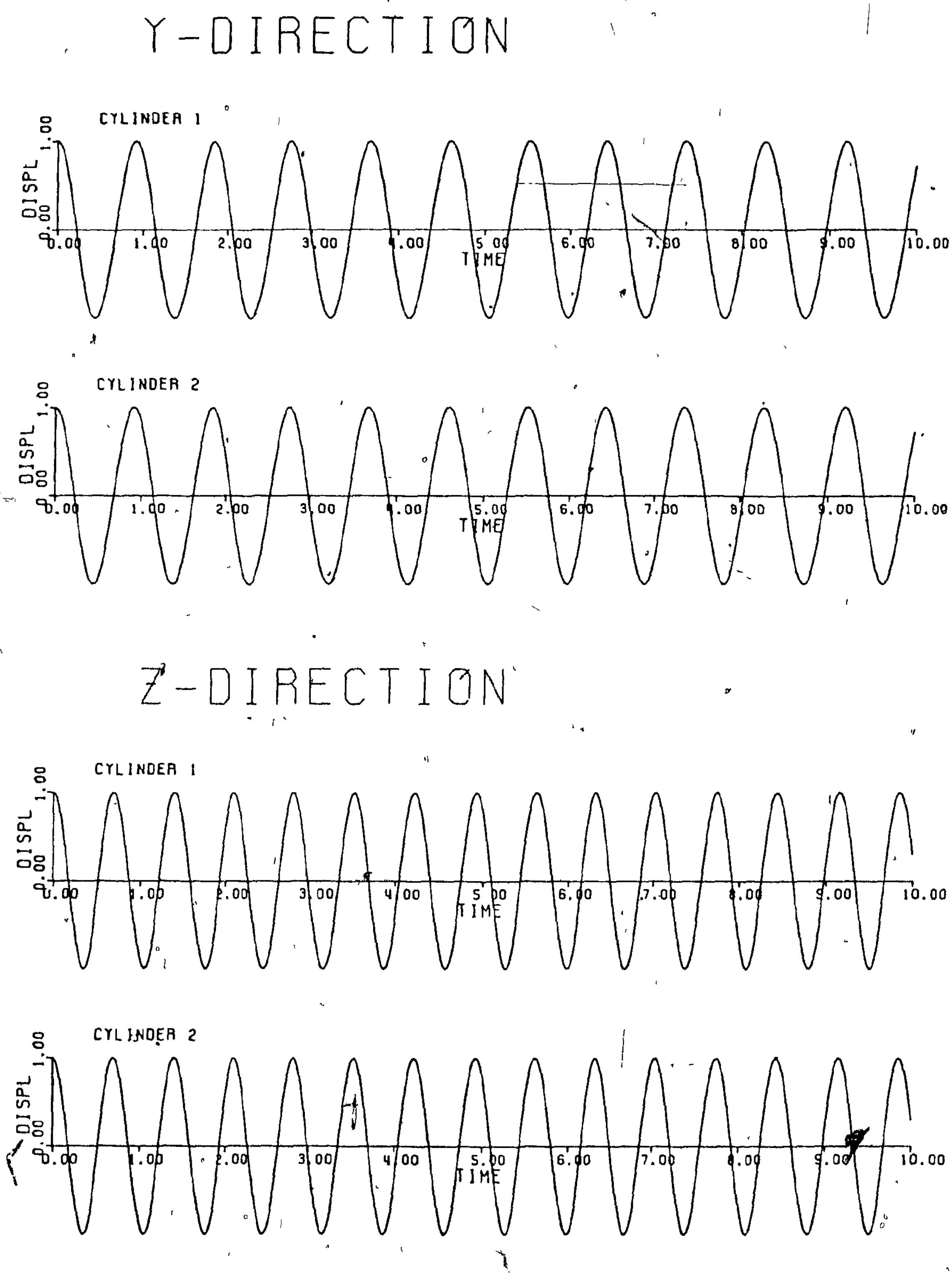
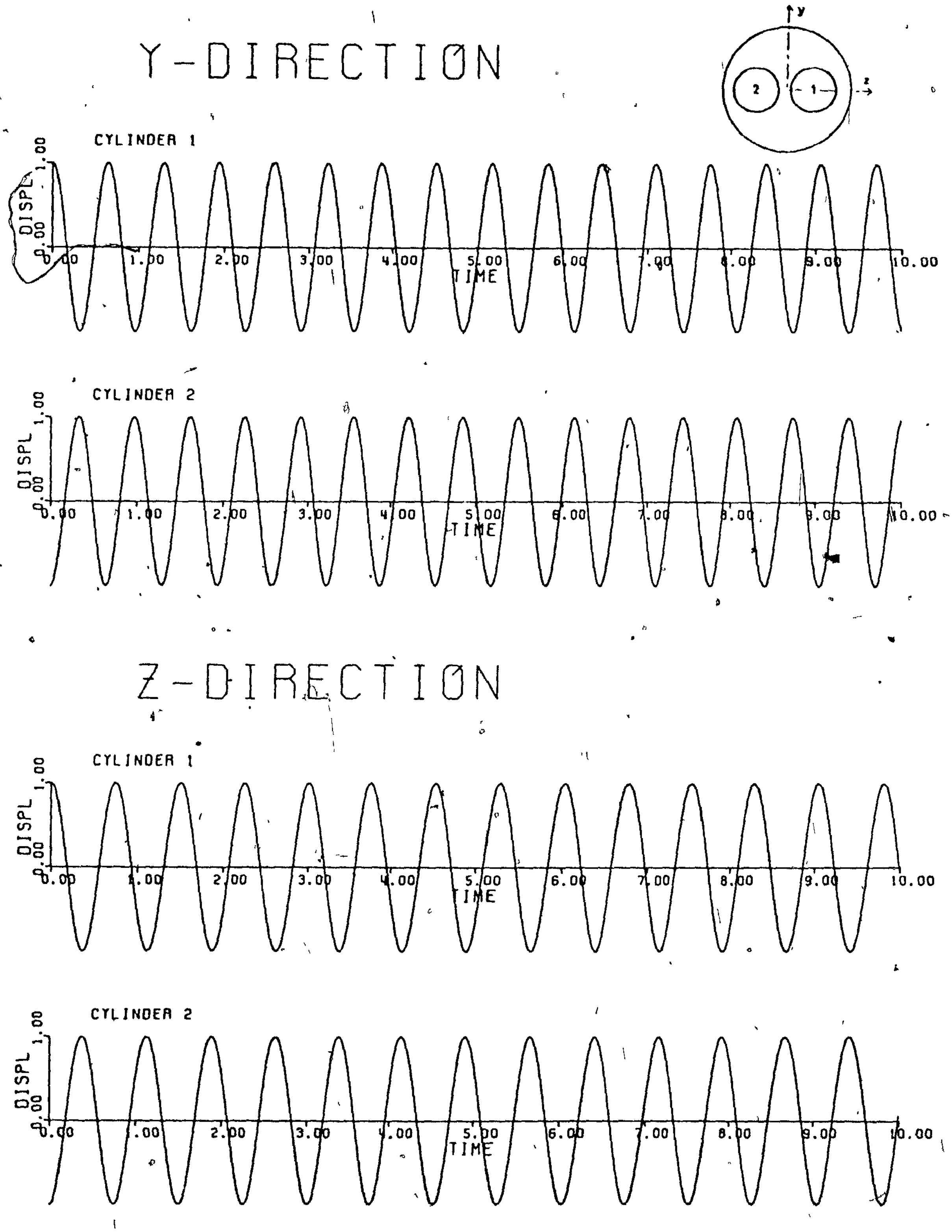
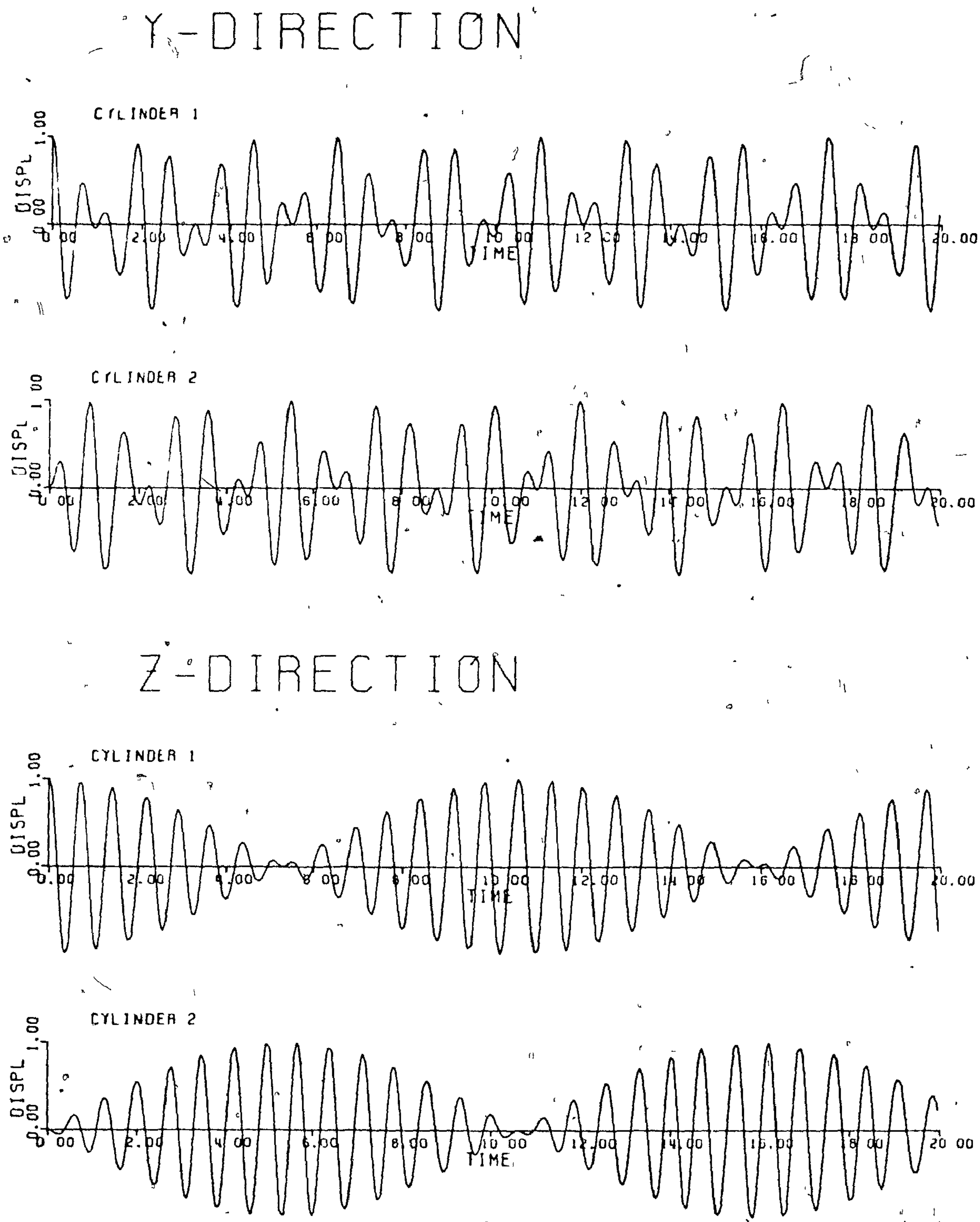


Figure 12 First transverse mode of free vibration in y- and z-directions for a two-cylinder system in still fluid
Standard case with $G_c = G_w = 1/4$.



<u>Figure 13</u> Second transverse mode of free vibration in y- and z-directions for a two-cylinder system in still fluid
Standard case with $G_c = G_w = 1/4$.



<u>Figure 14</u> Response of a two-cylinder system in still fluid when cylinder 1 is given an initial displacement in the y- or z-direction
Standard case with $G_c = G_w = 1/4$.

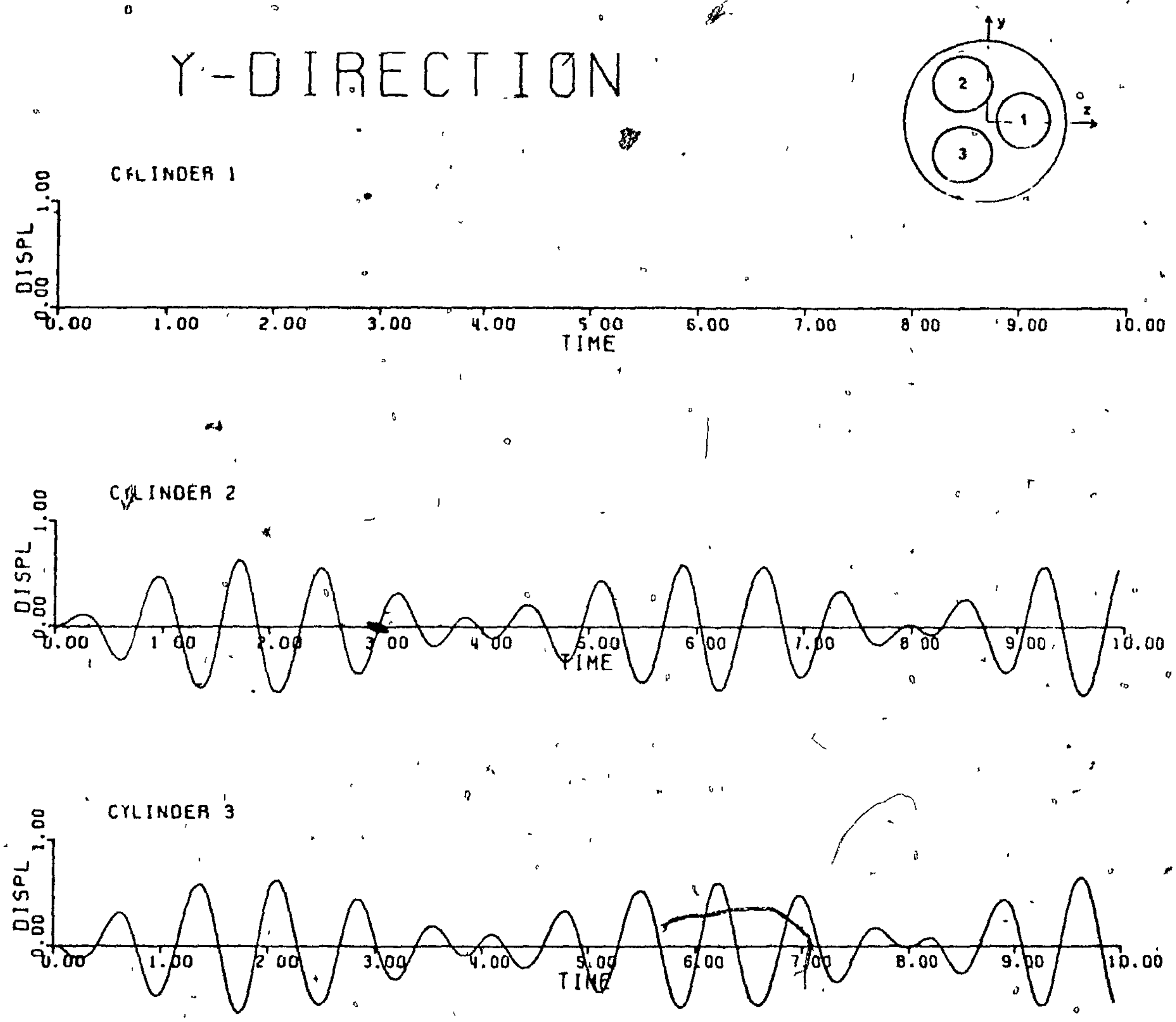


Figure 15(a) Response of a three-cylinder system in still fluid when cylinder 1 is given an initial displacement in the z-direction
Standard case with $G_c = G_w = 1/4$.
The response in the z-direction is plotted in Figure 15(b).

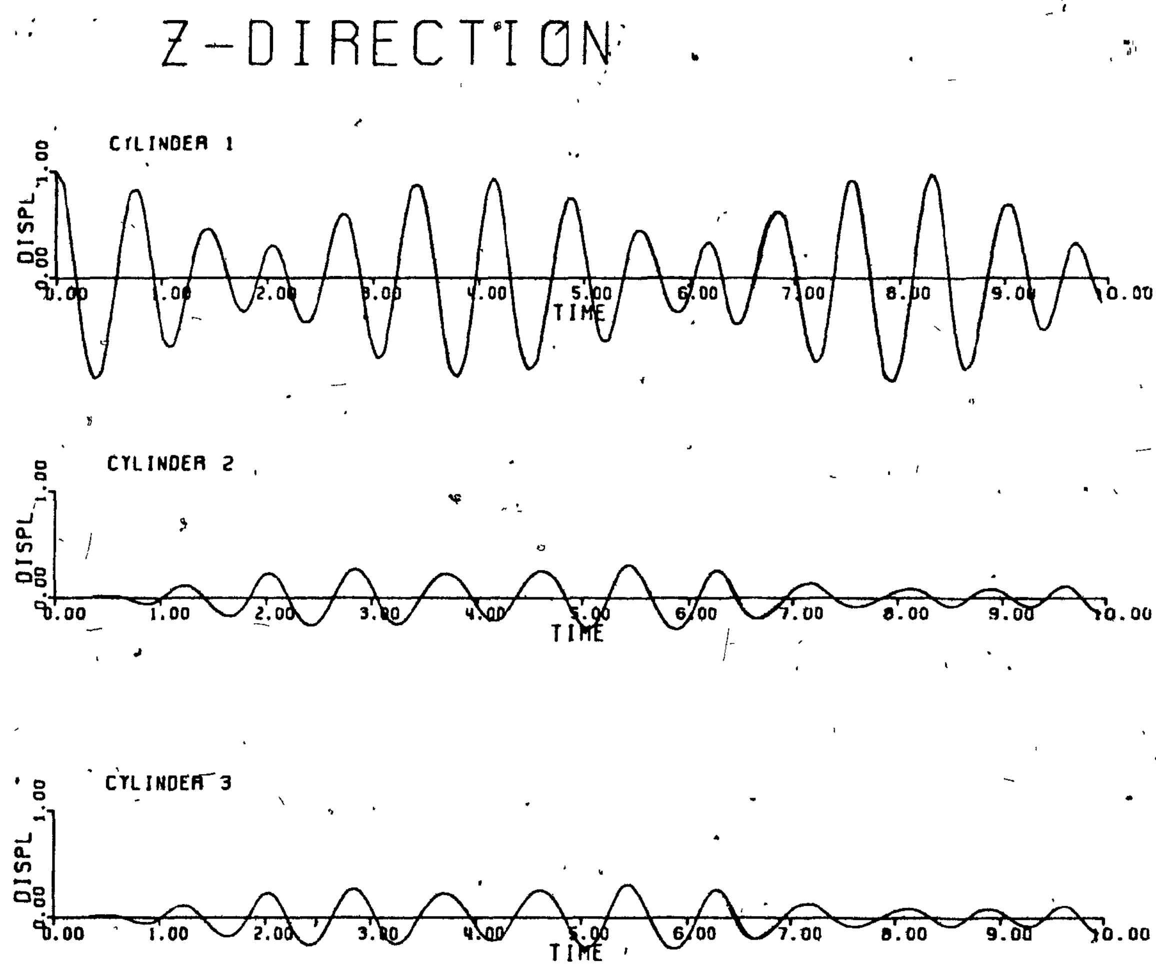

Z-DIRECTION
CYLINDER 1
DISPL
0.00
1.00
0.00
1.00
2.00
3.00
4.00
5.00
6.00
7.00
8.00
9.00
10.00
TIME
CYLINDER 2
DISPL
0.00
1.00
0.00
1.00
2.00
3.00
4.00
5.00
6.00
7.00
8.00
9.00
10.00
TIME
CYLINDER 3
DISPL
0.00
1.00
0.00
1.00
2.00
3.00
4.00
5.00
6.00
7.00
8.00
9.00
10.00
TIME

Figure 15(b)

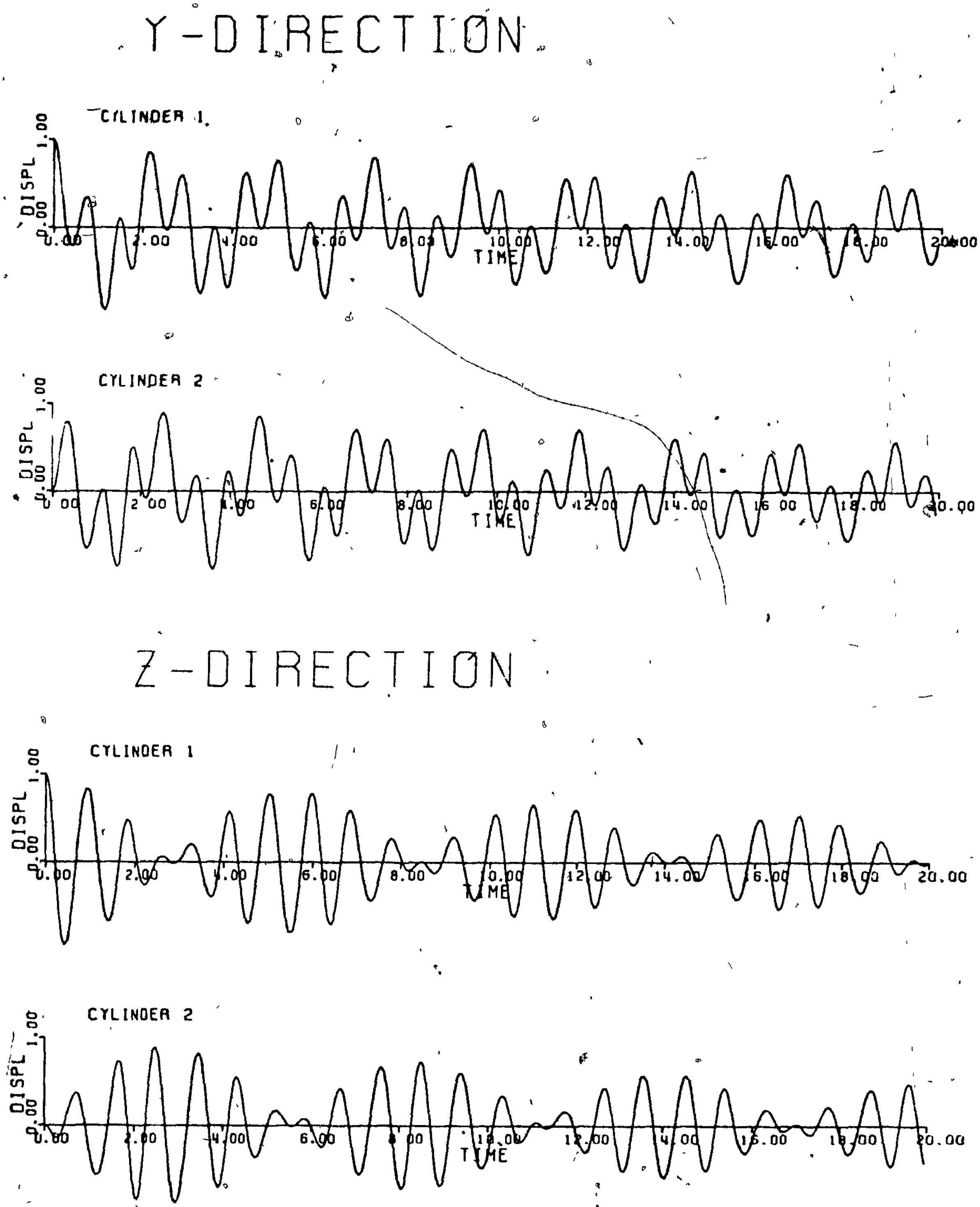


Figure 16 Response of a two-cylinder system in flowing fluid when cylinder 1 is given an initial displacement in the y- or z-direction
Standard case with $G_c = G_w = 1/4$, $u_e = 1.4$, $u_1 = 0$.

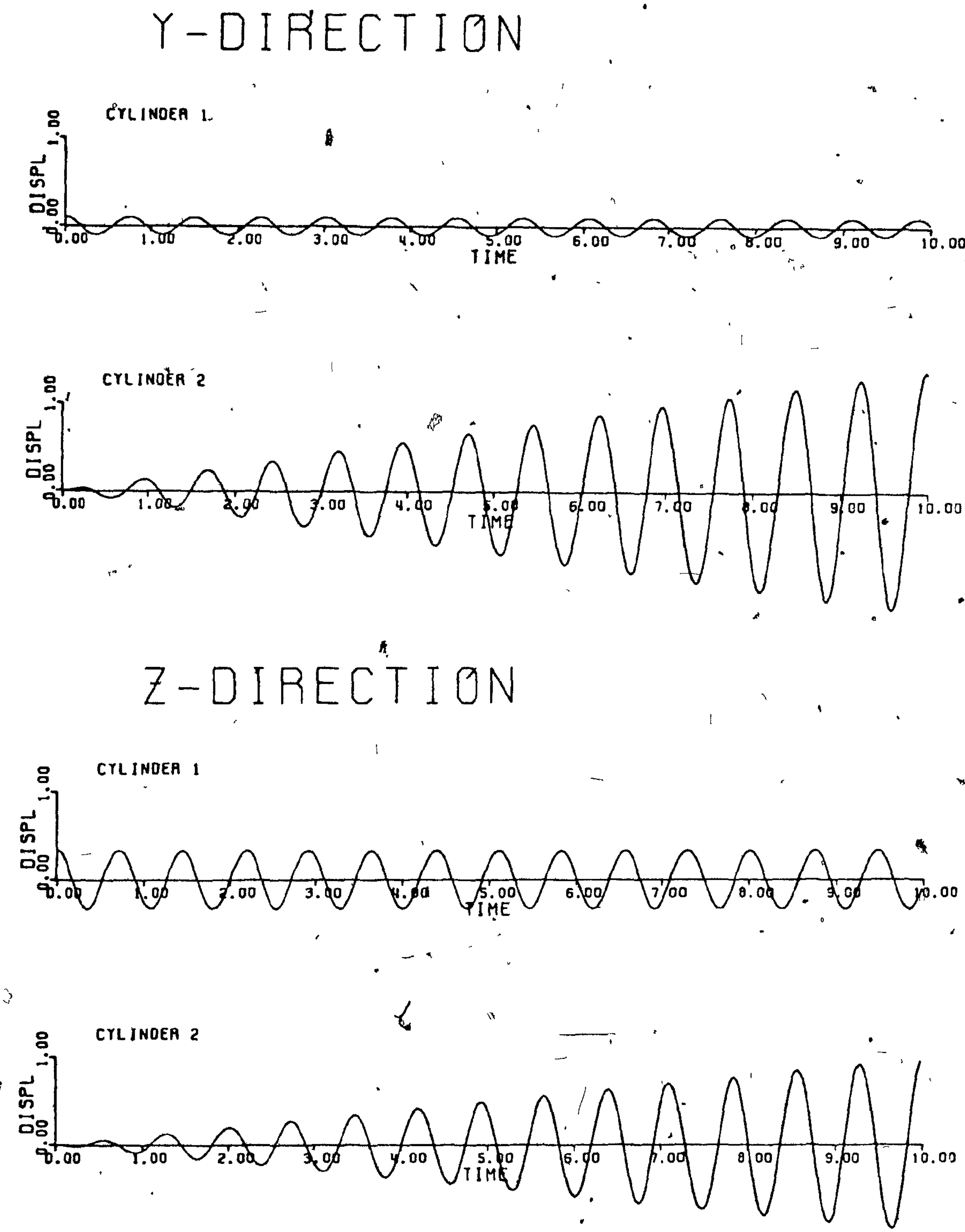


Figure 17 Resonances of a two-cylinder system in still fluid. The first cylinder is excited at $\Omega = 8.3$ in the y-direction, and $\Omega' = 8.62$ in the z-direction with constant imposed amplitudes. The second cylinder is free.
Standard case with $G_c = G_w = 1/4$.

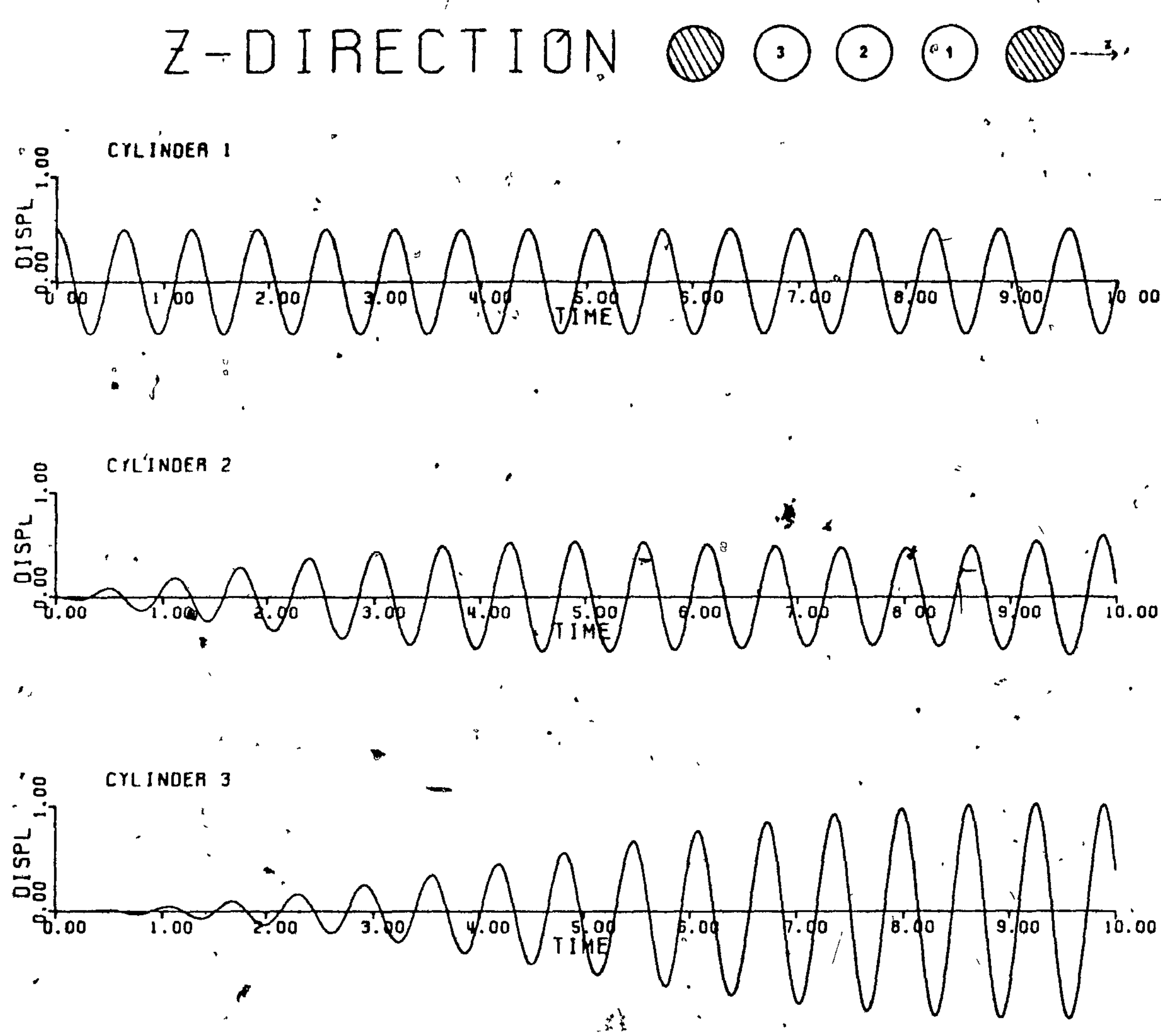


Figure 18 Resonance of a row of three cylinders in still fluid when cylinder 1 is excited at $\Omega = 9.90$ in the z-direction
Standard case with $G_c = 1$, $G_w = \infty$.

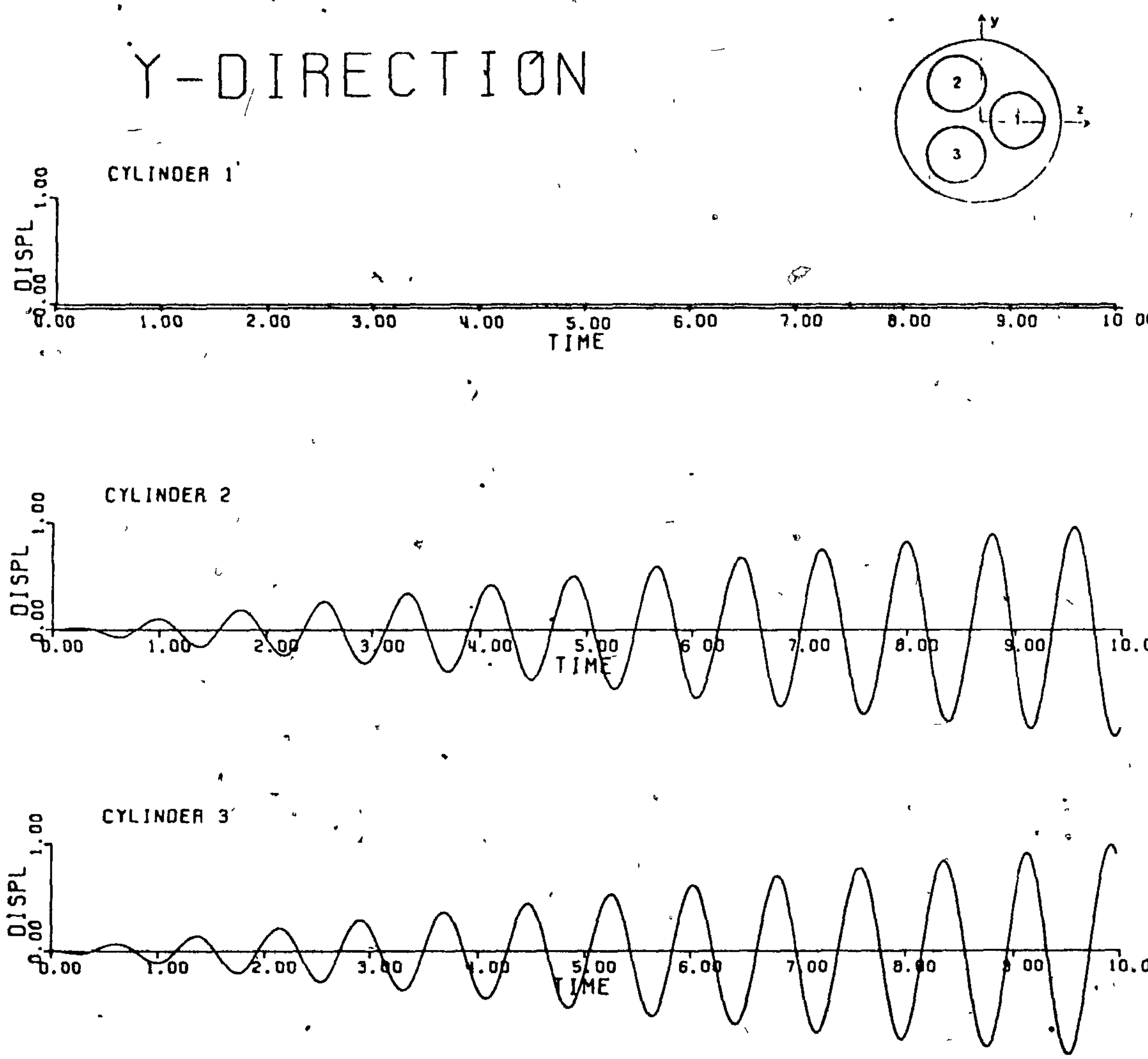


Figure 19(a) Resonance of a three-cylinder system when cylinder 1 is excited at $\Omega = 8.0$ in the z-direction Standard case with $G_C = G_W = 1/4$. The response in the z-direction is plotted in Figure 19(b).

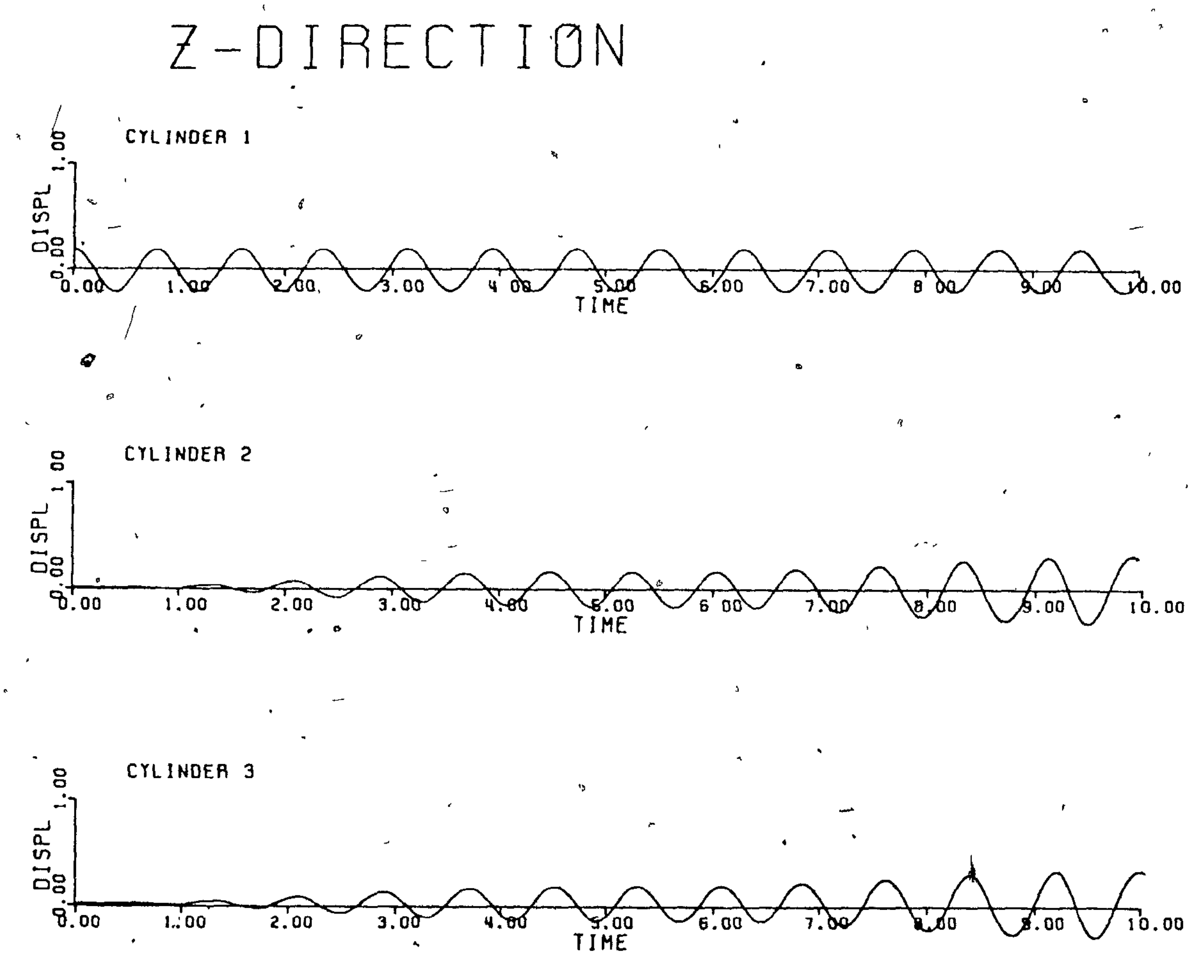

Z-DIRECTION
CYLINDER 1
DISPL
0.00
1.00
0.00
1.00
2.00
3.00
4.00
5.00
6.00
7.00
8.00
9.00
10.00
TIME
CYLINDER 2
DISPL
CYLINDER 3
DISPL

Figure 19(b)

APPENDIX A : THE ADDED MASS AND VISCOUS COUPLING COEFFICIENTS

This appendix is a summary of the work done by S. Suss, M. Pustejovsky and M.P. Paidoussis which can be found in detail in references [9] and [10].

A.1 ~~The~~ added mass coefficients

As shown in chapter 2 the general expression of the inviscid hydrodynamic force given by equation (2.7) may be written in the y- and z- directions for cylinder j as follows:

$$(F_A^j)_y = -\int_0^{2\pi} \rho_e \left[\frac{\partial}{\partial t} + U_e \frac{\partial}{\partial x}\right] \phi_j(R_j, \theta_j) \sin \theta_j \, d\theta_j \,,$$

$$(F_A^j)_z = -\int_0^{2\pi} \rho_e \left[\frac{\partial}{\partial t} + U_e \frac{\partial}{\partial x}\right] \phi_j(R_j, \theta_j) \cos \theta_j \, d\theta_j \,, \qquad \text{(A.1)}$$

where $\phi_j(r_j, \theta_j)$ is the fluid velocity potential at the point $P(r_j, \theta_j)$ due to the motion of cylinder j alone, and (r_j, θ_j) the polar coordinates of P centered on the axis of cylinder j. Let us also denote by R_{ij} the length of the line joining the centre of cylinders i and j and ψ_{ij} the angle this line makes with the positive z-axis, i being the vertex (see figure 2). By convention a zero value of an index refers to a coordinate system at the centre of the enclosing channel, the radius of which is R_0.

The total potential ϕ is the sum of all the potentials ϕ_j; moreover by virtue of linearity, we have

$$\nabla^2 \phi_j = 0 \quad . \tag{A.2}$$

The classical solution of this equation in polar co-ordinates has the form

$$\phi_j(r_j, \theta_j) = \sum_{n=1}^{\infty} \left[A_{nj}\, r_j^n \cos n\theta_j + B_{nj}\, r_j^n \sin n\theta_j \right.$$

$$\left. + C_{nj}\, r_j^{-n} \cos n\theta_j + D_{nj}\, r_j^{-n} \sin n\theta_j \right] \quad . \tag{A.3}$$

To apply the boundary conditions, it is necessary to express each ϕ_j in terms of coordinates centered on each of the other cylinders. This can be done by the use of the complex coordinate transformation

$$r_j e^{i\theta_j} = r_i e^{i\theta_i} - R_{ij} e^{i\psi_{ij}} \qquad \begin{array}{l} i = 0,1,\ldots\, k \quad , \\ j = 0,1,\ldots\, k \quad . \end{array}$$

After lengthy manipulation, the potential due to cylinder j in terms of coordinates centered on cylinder i may be written as

$$\phi_j^i(r_i, \theta_i) = \sum_{n=1}^{\infty} \left\{ \sum_{m=0}^{n} \frac{(-1)^{n-m} R_{ij}^{n-m} r_i^m n!}{m! \; (n-m)!} \right.$$

$$\times \left\{ A_{nj} \cos[m \theta_i + (n-m) \psi_{ij}] + B_{nj} \sin[m \theta_i + (n-m) \psi_{ij}] \right\}$$

$$+ \sum_{m=0}^{\infty} \frac{(-1)^n (n+m-1)! \; r_i^m}{m! \; (n-1)! \; R_{ij}^{n+m}} \times \left\{ C_{nj} \cos[m \theta_i - (n+m) \psi_{ij}] \right.$$

$$\left.\left. - D_{nj} \sin[m \theta_i - (n+m) \psi_{ij}] \right\} \right\} , \qquad \text{(A.4)}$$

which converges for $r_i < R_{ij}$; or alternatively

$$\phi_j^i(r_i, \theta_i) = \sum_{n=1}^{\infty} \left\{ \sum_{m=0}^{n} \text{(same as first internal sum in (A.4))} \right.$$

$$+ \sum_{m=n}^{\infty} \frac{(m-1)! \; R_{ij}^{m-n}}{(n-1)! \; (m-n)! \; r_i^m} \times \left\{ C_{nj} \cos[m \theta_i - (m-n) \psi_{ij}] \right.$$

$$\left.\left. + D_{nj} \sin[m \theta_i - (m-n) \psi_{ij}] \right\} \right\} , \qquad \text{(A.5)}$$

which converges for $r_i > R_{ij}$.

The two boundary conditions require that

(i) the velocity of the fluid normal to the enclosing channel be zero on its surface,

$$\left. \frac{\partial \phi_j^o}{\partial r_o} \right|_{r_o = R_o} = 0 \qquad j = 1, 2, \ldots, k \; ; \qquad \text{(A.6)}$$

(ii) the velocity of the fluid normal to the surface of each cylinder be equal to the velocity of the cylinder in that direction,

$$\left.\frac{\partial \phi^i}{\partial r_i}\right|_{r_i=R_i} = \frac{Dw_i}{Dt} \cos \theta_i + \frac{Dv_i}{Dt} \sin \theta_i \quad , \qquad \text{(A.7)}$$

$$j = 1,2,\ldots, k \quad ,$$

where ϕ^i is the total potential written in terms of co-ordinates centered on cylinder i,

$$\phi^i = \phi_i + \sum_{j=1}^{k}{}^{*} \phi_j^i \quad ,$$

the starred summation excluding j=i.

Now, applying the boundary conditions and isolating the coefficients of cos m θ_i and sin m θ_i, it appears that the constants A_{nj}, B_{nj}, C_{nj} and D_{nj} must be of the following form:

$$\begin{Bmatrix} A_{nj} \\ B_{nj} \end{Bmatrix} = R_j^{1-n} \sum_{\ell=1}^{k} \left[\begin{Bmatrix} \alpha_{nj\ell} \\ \beta_{nj\ell} \end{Bmatrix} \frac{Dw_\ell}{Dt} + \begin{Bmatrix} a_{nj\ell} \\ b_{nj\ell} \end{Bmatrix} \frac{Dv_\ell}{Dt} \right]$$

$$\begin{Bmatrix} C_{nj} \\ D_{nj} \end{Bmatrix} = R_j^{1+n} \sum_{\ell=1}^{k} \left[\begin{Bmatrix} \gamma_{nj\ell} \\ \delta_{nj\ell} \end{Bmatrix} \frac{Dw_\ell}{Dt} + \begin{Bmatrix} c_{nj\ell} \\ d_{nj\ell} \end{Bmatrix} \frac{Dv_\ell}{Dt} \right] \quad . \qquad \text{(A.8)}$$

Substituting these relations into the previous equations, the following eight sets of linear equations with the constants $\alpha_{nj\ell}$ to $d_{nj\ell}$ as unknowns are obtained,

$$\sum_{j=1}^{k}{}^{*}\sum_{n=m}^{\infty}\frac{(-1)^{n-m}\ n!\ R_{ij}^{n-m}\ R_i^{m-1}}{(m-1)!\ (n-m)!\ R_j^{n-1}}\left[\begin{Bmatrix}\beta_{nj\ell}\\ b_{nj\ell}\\ -\alpha_{nj\ell}\\ -a_{nj\ell}\end{Bmatrix}\sin(n-m)\psi_{ij}\right.$$

$$\left.+\begin{Bmatrix}\alpha_{nj\ell}\\ a_{nj\ell}\\ \beta_{nj\ell}\\ b_{nj\ell}\end{Bmatrix}\cos(n-m)\psi_{ij}\right]$$

$$+\sum_{j=1}^{k}{}^{*}\sum_{n=1}^{\infty}\frac{(-1)^{n}\ (n+m-1)!\ R_i^{m-1}\ R_j^{n+1}}{(n-1)!\ (m-1)!\ R_{ij}^{n+m}}\left[\begin{Bmatrix}\delta_{nj\ell}\\ d_{nj\ell}\\ \gamma_{nj\ell}\\ c_{nj\ell}\end{Bmatrix}\sin(n+m)\psi_{ij}\right.$$

$$\left.+\begin{Bmatrix}\gamma_{nj\ell}\\ c_{nj\ell}\\ -\delta_{nj\ell}\\ -d_{nj\ell}\end{Bmatrix}\cos(n+m)\psi_{ij}\right]+m\begin{Bmatrix}\alpha_{mi\ell}\\ a_{mi\ell}\\ \beta_{mi\ell}\\ b_{mi\ell}\end{Bmatrix}-m\begin{Bmatrix}\gamma_{mi\ell}\\ c_{mi\ell}\\ \delta_{mi\ell}\\ d_{mi\ell}\end{Bmatrix}=\begin{Bmatrix}\delta_{1m}\delta_{i\ell}\\ 0\\ 0\\ \delta_{1m}\delta_{i\ell}\end{Bmatrix}, \qquad (A.9)$$

where $m = 1,2,3,\ldots;\ \ell = 1,2,3,\ldots,\ k;\ i = 1,2,\ldots,\ k;$

$$\sum_{n=m}^{\infty}\frac{(-1)^{n-m}\ n!\ R_{oj}^{n-m}\ R_o^{m-1}}{(m-1)!\ (n-m)!\ R_j^{n-1}}\left[\begin{Bmatrix}\alpha_{nj\ell}\\ a_{nj\ell}\\ -\beta_{nj\ell}\\ -b_{nj\ell}\end{Bmatrix}\cos(n-m)\psi_{oj}\right.$$

$$\left.+\begin{Bmatrix}\beta_{nj\ell}\\ b_{nj\ell}\\ \alpha_{nj\ell}\\ a_{nj\ell}\end{Bmatrix}\sin(n-m)\psi_{oj}\right]$$

$$- \sum_{n=1}^{m} \frac{m!\, R_{oj}^{m-n}\, R_j^{n+1}}{(n-1)!\,(m-n)!\, R_o^{m+1}} \left[\begin{Bmatrix} \gamma_{nj\ell} \\ c_{nj\ell} \\ -\delta_{nj\ell} \\ -d_{nj\ell} \end{Bmatrix} \cos(m-n)\psi_{oj} \right.$$

$$\left. - \begin{Bmatrix} \delta_{nj\ell} \\ d_{nj\ell} \\ \gamma_{nj\ell} \\ c_{nj\ell} \end{Bmatrix} \sin(m-n)\psi_{oj} \right] = \{0\} \quad , \tag{A.10}$$

where $m = 1,2,3,\ldots;\ \ell = 1,2,3,\ldots,\ k;\ j = 1,2,3,\ldots,\ k.$

These sets of infinite equations can be truncated to allow for the determination of a finite number of the constants $a_{nj\ell}$ to $d_{nj\ell}$ and $\alpha_{nj\ell}$ to $\delta_{nj\ell}$. Then, performing the operations required by (A.1), the inviscid hydrodynamic forces may be written as

$$\begin{aligned} (F_A^j)_z &= -\rho_e \pi R^2 \sum_{\ell=1}^{k} \left[\varepsilon_{j\ell} \frac{D^2 w^\ell}{Dt^2} + e_{j\ell} \frac{D^2 v^\ell}{Dt^2} \right] , \\ (F_A^j)_y &= -\rho_e \pi R^2 \sum_{\ell=1}^{k} \left[\kappa_{j\ell} \frac{D^2 w^\ell}{Dt^2} + k_{j\ell} \frac{D^2 v^\ell}{Dt^2} \right] , \end{aligned} \tag{A.11}$$

for $j = 1,2,\ldots,\ k$, where $\varepsilon_{j\ell}$, $e_{j\ell}$, $\kappa_{j\ell}$, $k_{j\ell}$ are the non-dimensional coefficients of the added mass matrix which have been found to satisfy

$$\begin{aligned} \varepsilon_{j\ell} &= \delta_{j\ell} + 2\gamma_{1j\ell} \quad , \quad e_{j\ell} = 2c_{1j\ell} \quad , \\ \kappa_{j\ell} &= 2\delta_{1j\ell} \quad , \quad k_{j\ell} = \delta_{j\ell} + 2d_{1j\ell} \end{aligned} \tag{A.12}$$

$\delta_{j\ell}$ being Kronecker's delta.

A.2 The viscous coupling coefficients

This section deals with the resolution of equation (2.10). The only unknowns are the velocities $(v_j)_y$ and $(v_j)_z$ of the oncoming flow in the vicinity of cylinder j; they can be derived from the fluid velocity potential as in the previous section, but computed as if this particular cylinder, the jth, were missing. Hence, the procedure is exactly the same, since the derivation was for an arbitrary number of cylinders; the fluid velocity at any point in the yz-plane, with the jth cylinder left out, may be expressed as

$$\begin{aligned} \nu_y \,(r_j,\theta_j) &= \frac{\partial \phi^j}{\partial r_j} \sin \theta_j + \frac{1}{r_j} \frac{\partial \phi^j}{\partial \theta_j} \cos \theta_j \;, \\ \nu_z \,(r_j,\theta_j) &= \frac{\partial \phi^j}{\partial r_j} \cos \theta_j - \frac{1}{r_j} \frac{\partial \phi^j}{\partial \theta_j} \sin \theta_j \;, \end{aligned} \tag{A.13}$$

where

$$\phi^j = \sum_{i=1}^{k}{}^{*} \phi_i^j \,(r_j,\theta_j) \;, \tag{A.14}$$

the starred summation excluding i=j.

As the distribution of velocities found in this way is not uniform around the cylinder, we shall take a mean value along its circumference, setting

$$\begin{aligned} (V_j)_y &= \frac{1}{2\pi} \int_0^{2\pi} \nu_y \,(R_j,\theta_j)\, d\theta_j \;, \\ (V_j)_z &= \frac{1}{2\pi} \int_0^{2\pi} \nu_z \,(R_j,\theta_j)\, d\theta_j \;. \end{aligned} \tag{A.15}$$

Combining these three last equations in a similar way to the previous section, the velocities are found to be

$$(V_j)_z = \sum_{\ell=1}^{k}{}^{*}\left[\nu_{j\ell}\,\frac{Dw_\ell}{Dt} + n_{j\ell}\,\frac{Dv_\ell}{Dt}\right] ,$$

$$(V_j)_y = \sum_{\ell=1}^{k}{}^{*}\left[\bar{\nu}_{j\ell}\,\frac{Dw_\ell}{Dt} + \bar{n}_{j\ell}\,\frac{Dv_\ell}{Dt}\right] , \tag{A.16}$$

where

$$\begin{Bmatrix}\nu_{j\ell}\\ n_{j\ell}\\ \bar{\nu}_{j\ell}\\ \bar{n}_{j\ell}\end{Bmatrix} = \sum_{s=1}^{k}{}^{*}\sum_{n=1}^{\infty}\left[(-1)^{n-1}\,n\left(\frac{R_{js}}{R_s}\right)^{n-1}\left(\begin{Bmatrix}\alpha_{ns\ell}\\ a_{ns\ell}\\ \beta_{ns\ell}\\ b_{ns\ell}\end{Bmatrix}\cos(n-1)\;\psi_{js}\right.\right.$$

$$\left.+\begin{Bmatrix}\beta_{ns\ell}\\ b_{ns\ell}\\ -\alpha_{ns\ell}\\ -a_{ns\ell}\end{Bmatrix}\sin(n-1)\;\psi_{js}\right) + (-1)^n\,n\left(\frac{R_s}{R_{js}}\right)^{n+1}$$

$$\left.\times\left(\begin{Bmatrix}\gamma_{ns\ell}\\ c_{ns\ell}\\ -\delta_{ns\ell}\\ -d_{ns\ell}\end{Bmatrix}\cos(n+1)\;\psi_{js} + \begin{Bmatrix}\delta_{ns\ell}\\ d_{ns\ell}\\ \gamma_{ns\ell}\\ c_{ns\ell}\end{Bmatrix}\sin(n+1)\;\psi_{js}\right)\right] . \tag{A.17}$$

with $\ell = 1,2,\ldots, k \;,\; j = 1,2,\ldots, k \;;\; \ell \neq j \;,$

the constants $\alpha_{ns\ell}$, $a_{ns\ell}$, $\beta_{ns\ell}$, $b_{ns\ell}$, etc. being evaluated with the jth cylinder missing.

Finally, the normal viscous forces may be put in the form

$$(q_{en}^{j})_y = -\rho_e R U_e C_f \sum_{\ell=1}^{k}\left[\sigma_{j\ell}\frac{Dw^\ell}{Dt} + s_{j\ell}\frac{Dv^\ell}{Dt}\right]$$

$$-\rho_e R C_D \sum_{\ell=1}^{k}\left[\sigma_{j\ell}\frac{\partial w^\ell}{\partial t} + s_{j\ell}\frac{\partial v^\ell}{\partial t}\right] ,$$

$$(q_{en}^{j})_z = -\rho_e R U_e C_f \sum_{\ell=1}^{k}\left[\zeta_{j\ell}\frac{Dw^\ell}{Dt} + g_{j\ell}\frac{Dv^\ell}{Dt}\right]$$

$$-\rho_e R C_D \sum_{\ell=1}^{k}\left[\zeta_{j\ell}\frac{\partial w^\ell}{\partial t} + g_{j\ell}\frac{\partial v^\ell}{\partial t}\right] , \tag{A.18}$$

where $\zeta_{j\ell}$, $g_{j\ell}$, $\sigma_{j\ell}$, $s_{j\ell}$, are called the viscous coupling coefficients and equal, respectively, $-\nu_{j\ell}$, $-n_{j\ell}$, $-\overline{\nu}_{j\ell}$, $-\overline{n}_{j\ell}$, for $j\neq\ell$, and $\zeta_{j\ell} = s_{j\ell} = 1$, $g_{j\ell} = \sigma_{j\ell} = 0$ for $j=\ell$.

The added mass and viscous coupling coefficients are produced and punched out by the computer program 'COUPLAGE' listed in Appendix B. This program is a modified version of Suss' original program 'COUPLING' where the added mass matrix has the opposite sign.

APPENDIX B: THE COMPUTER PROGRAM 'COUPLAGE'

DATA CARDS

Card Number	Symbol	Fortran Name	Designation	Format
1		K,MN	Number of cylinders, number of cylinders at centre of the channel	2I3
2		MM,MP	Number of terms taken in the series solution (see note 1)	2I3
3		IDEA,IPUNCH	See note 2	2I3
4	ψ_{11}, R_{11}	CH(1,1),R(1,1)	parameters of position of each cylinder with respect to the others (see Figure 1)	2F10.5
5	ψ_{12}, R_{12}	CH(1,2),R(1,2)		"
↓				
K+4	ψ_{21}, R_{21}	CH(2,1),R(2,1)		"
↓				
K^2+3	ψ_{kk}, R_{kk}	CH(k,k),R(k,k)		"
K^2+4	R_1, ψ_{01}, R_{01}	RI(1),CI(1), RA(1)	radius of each cylinder parameters of position with respect to the centre of the channel	3F10.5
↓				
$K+K^2+3$	R_k, ψ_{0k}, R_{0k}	RI(k),CI(k), RA(k),		"
$K+K^2+4$	R_o	RO	Radius of the outer channel	F10.5

Note 1: MM > MP. A good convergence is obtained in all cases with MM=15 and MP=11.

Note 2: IDEA=0 if only the added mass matrix is required
=1 otherwise
IPUNCH=0 if the matrices should not be punched
=1 otherwise

Note 3: The angles are expressed in degrees and the lengths in millimeters.

```
C                           ************
C                           *          *
C                           * CCUPLAGE *
C                           *          *
C                           ************
C
C
C*************************************************************************************
C*
C*  K IS THE NUMEER OF CYLINDERS
C*  MN=NUMBER OF CYLINDER AT CENTER OF ARRAY
C*  RC=RADIUS OF ENCLOSING CYLINDER
C*  C1(I)=CH(0,I) ,  RA(I)=R(0,I),  RI(I)=RADIUS OF CYLINDER I
C*  DIMENSION A(MM,MM,K),B(MM,MM,K),W1(MM,MM),W2(MM,MM),W3(MM,MM),W4(MM,MM)
C*  DIMENSION S(2MM+1),CC(2MM+1),F(2MM),R1(2MM+2),R2(2MM+2),R3(2MM+1)        *
C*  DIMENSION G(2MPK,2MPK),W5(2MPK,2K),W6(2MPK),VC(2K,2K)
C*  DIMENSION CH(K,K),R(K,K),RI(K),RA(K),C1(K)
C*  DIMENSION A1(MM),A2(MM),A3(2,2MP(K-1)),A4(2,2(K-1))
C*  DIMENSION VC(2K,25),AM(2*(K+1),2*(K+1))
C*
C*************************************************************************************
C
      IMPLICIT REAL*8(A-H,O-Z)
      DIMENSION A(15,15,3),B(15,15,3)
      DIMENSION W1(15,15),W2(15,15),W3(15,15),W4(15,15)
      DIMENSION G(66,66),W5(66,6),W6(66)
      DIMENSION CH(3,3),R(3,3),RI(3),RA(3),C1(3)
      DIMENSION VC(6,6)
      DIMENSION A1(25),A2(25),A3(2,44),A4(2,4)
      DIMENSION AM(8,8)
      DIMENSION S(50),CC(50),F(50),R1(50),R2(50),R3(50)
      INTEGER DIGIT(10) /'1','2','3','4','5','6','7','8','9','10'/
      INTEGER EC(4)/'(1H0','    ','F12.','5)'/
      READ 99,K,MN
      READ 99,MM,MP
      READ 99,IDEA,IPUNCH
C         READ CH(I,J) AND C1(I) IN DEGREES
      READ200, ((CH(I,J),R(I,J),J=1,K),I=1,K)
C         READ R(I,J),RI(I),RA(I),RC IN THE SAME UNITS OF LENGTH
      READ 201,(RI(I),C1(I),RA(I),I=1,K)
      READ 202,RC
      FF=3.14159265358980D0/180.D0
      PRINT 109,K,MM,MN,MP
      PRINT 101,RC
      PRINT 110
      KKK=K
      IF(K.GT.10) KKK=10
      EC(2)=DIGIT(KKK)
      WRITE(6,EC)((CH(I,J),J=1,KKK),I=1,KKK)
      PRINT 111
      WRITE(6,EC)((R(I,J),J=1,KKK),I=1,KKK)
      PRINT 107
      PRINT 108,(I,RI(I),RA(I),C1(I),I=1,K)
C         DEGREES TO RADIANS
      DO 301 I=1,K
      C1(I)=C1(I)*FF
      DO 301 J=1,K
  301 CH(I,J)=CH(I,J)*FF
      CALL FACT(MM,MP,F)
      M1=2*MP*K
      CALL FIXUP(MM,K,A,B,W1,W2,W3,W4,RC,MN,R1,R2,R3,RI,RA,C1,S,CC,F)
      IF(IDEA.EQ.0)GOTO 477
      DO 1 IDEA=1,K
      KK=2*(K-1)
      CALL FIXUP2(MM,MP,K,KK,M1,A,B,W1,W2,W3,W4,CH,R,RI,R1,R2,R3,S,CC,F,
     *G,W5,W6,IDEA)
      CALL VISCOU(MP,MM,K,IDEA,M1,S,CO,F,CH,R1,RI,A1,A2,A3,A4,W5,A,B)
      IX=IDEA+K
      JX=0
      KK=2*K
      DO 2 J=1,KK
      IF(J.EQ.IDEA.OR.J.EQ.IX)GOTO 2
      JX=JX+1
      VC(IDEA,J)=-A4(1,JX)
      VC(IX,J)=-A4(2,JX)
    2 CONTINUE
```

```
      VC(IDEA,IDEA)=1.D0
      VC(IX,IX)=1.D0
      VC(IDEA,IX)=0.D0
      VC(IX,IDEA)=0.D0
    1 CONTINUE
      KKK=KK
      IF(KK.GT.10)KKK=KK/2
      EC(2)=DIGIT(KKK)
      PRINT 105
      WRITE(6,EC)((VC(I,J),J=1,KKK),I=1,KK)
      IF(KK.LE.10) GO TO 600
      PRINT 105
      KKK=KKK+1
      WRITE(6,EC)((VC(I,J),J=KKK,KK),I=1,KK)
  600 CONTINUE
      IDEA=0
  477 KK=2*K
      CALL FIXUP2(MM,MF,K,KK,M1,A,B,W1,W2,W3,W4,CH,R,RI,R1,R2,R3,S,CC,F,
     *G,W5,W6,IDEA)
      IX=1
      K2=K+2
      DO 81 I=1,M1,MF
      IX=IX+1
      DO 82 J=1,KK
      JX=J+1
      IF(J.GT.K)JX=J+2
      IF(IX.EQ.K2)IX=IX+1
   82 W5(I,J)=2.0D0*W5(I,J)
      J=(I-1+MF)/MF
      W5(I,J)=1.0D0+W5(I,J)
   81 CONTINUE
      IX=0
      DO 85 I=1,M1,MF
      IX=IX+1
      DO 85 J=1,KK
   85 AM(IX,J)=-W5(I,J)
      KKK=KK
      IF(KK.GT.10)KKK=KK/2

      EC(2)=DIGIT(KKK)
      PRINT 103
      WRITE(6,EC)((AM(I,J),J=1,KKK),I=1,KK)
      IF(KK.LE.10) GO TO 601
      PRINT 103
      KKK=KK+1
      WRITE(6,EC)((AM(I,J),J=KKK,KK),I=1,KK)
  601 CONTINUE
      IF(IPUNCH.EQ.0) GO TO 530
      DO 529 I=1,KK
  529 PUNCH 200,(AM(I,J),VC(I,J),J=1,KK)
  530 STOP
   99 FORMAT(2I3)
  101 FORMAT('0'//5X'RADIUS OF THE ENCLOSING CHANNEL RC=',F10.5)
  103 FORMAT('1'//10X,'THE ADDED MASS MATRIX:')
  104 FORMAT('1')
  105 FORMAT('1'//10X'THE VISCOUS COUPLING MATRIX')
  106 FORMAT('0',5D12.5)
  107 FORMAT(///,'0',12X,'I',7X,'R(I)',10X,'R(0,I)',9X,'CH(0,I) IN DEG')
  108 FORMAT('0',10X,I3,3F14.7)
  109 FORMAT('1',5X,'K=',I2,'  MM=',I2,'  CYLINDER AT CENTRE OF ARRAY',I
     *2,'  MF=',I2,//)
  110 FORMAT('0  THE MATRIX DEFINING CH(I,J) IN DEGREES')
  111 FORMAT(///,'0  THE MATRIX DEFINING R(I,J)')
  200 FORMAT(2F10.5)
  201 FORMAT(3F10.5)
  202 FORMAT(F10.5)
      END
/*
```

```
      SUBROUTINE FACT(MM,MP,F)
C
C        GENERATE FACTORIALS SUCH THAT F(J+1)=J FACTORIAL
      DOUBLE PRECISION F(1)
      MTT=2*MP
      MT=MM+1
      IF(MTT.GT.MT)MT=MTT
      F(1)=1.D0
      DO 37 I=2,MT
      IP=I-1
   37 F(I)=IP*F(IP)
      RETURN
      END
      SUBROUTINE GEN(CH,M,S,C)
C       GENERATE C(I) AND S(I) SUCH THAT C(I+1)=DCOS(I*CH)
      IMPLICIT REAL*8(A-H,C-Z)
      DIMENSION C(1),S(1)
      C(1)=1.D0
      S(1)=0.D0
      DO 1 I=2,M
      H=(I-1)*CH
      C(I)=DCOS(H)
      IF(DABS(C(I)).LT.1D-10)C(I)=0.D0
      S(I)=DSIN(H)
      IF(DABS(S(I)).LT.1D-10)S(I)=0.D0
    1 CONTINUE
      RETURN
      END
      SUBROUTINE CCNT(R1,M,A)
C       GENERATE A(I) SUCH THAT A(I+1)=R1**I
      IMPLICIT REAL*8(A-H,C-Z)
      DIMENSION A(1)
      A(1)=1.D0
      DO 1 I=2,M
      IP=I-1
    1 A(I)=R1*A(IP)
      RETURN
      END
      SUBROUTINE TMULT(M,A,B,C)
C        MULTIPLIES TWO UPPER TRIANGULAR MATRICES, A*B=C
      IMPLICIT REAL*8(A-H,O-Z)
      DIMENSION A(M,1),B(M,1),C(M,1)
      DO 1 I=1,M
      DO 1 J=1,M
    1 C(I,J)=0.D0
      DO 2 I=1,M
      DO 2 J=I,M
      DO 4 L=I,J
    4 C(I,J)=C(I,J)+A(I,L)*B(L,J)
    2 CONTINUE
      RETURN
      END
      SUBROUTINE TINV(M,A,B)
C        INVERTS AN UPPER DIAGONAL MATRIX
      IMPLICIT REAL*8(A-H,C-Z)
      DIMENSION A(M,1),B(M,1)
      DO 1 I=1,M
      DO 2 J=1,M
    2 B(I,J)=0.D0
    1 B(I,I)=1.D0/A(I,I)
      N=M-1
      DO 3 I=1,N
      IP=M-I
      DO 4 J=1,I
      JJ=M+1-J
      IG=IP+1
      DO 5 L=IG,JJ
    5 B(IP,JJ)=B(IP,JJ)-A(IP,L)*B(L,JJ)
    4 B(IP,JJ)=B(IP,JJ)/A(IP,IP)
    3 CONTINUE
      RETURN
      END
```

```
      SUBROUTINE DMULT(M,A,B,C)
C        MULTIPLIES MATRICES, A*B=C
      IMPLICIT REAL*8(A-H,O-Z)
      DIMENSION A(M,1),B(M,1),C(M,1)
      DO 1 I=1,M
      DO 1 J=1,M
      C(I,J)=0.D0
      DO 2 L=1,M
    2 C(I,J)=C(I,J)+A(I,L)*B(L,J)
    1 CONTINUE
      RETURN
      END
      SUBROUTINE ADD(M,A,B,C)
C        ADDS TWO MATRICES, A+B=C
      IMPLICIT REAL*8(A-H,O-Z)
      DIMENSION A(M,1),B(M,1),C(M,1)
      DO 1 I=1,M
      DO 1 J=1,M
    1 C(I,J)=A(I,J)+B(I,J)
      RETURN
      END
      SUBROUTINE CMULT(M,X,A)
C        MULTIPLIES A MATRIX BY A CONSTANT
      IMPLICIT REAL*8(A-H,O-Z)
      DIMENSION A(M,1)
      DO 1 I=1,M
      DO 1 J=1,M
    1 A(I,J)=X*A(I,J)
      RETURN
      END
      SUBROUTINE RIPL(M,X,Y,W1,W2,A,B)
C       COMPUTES THE SUMS X*A-Y*B AND X*B+Y*A AND STORES THEM IN W1 AND W2
      IMPLICIT REAL*8(A-H,O-Z)
      DIMENSION X(M,1),Y(M,1),W1(M,1),W2(M,1),A(M,1),B(M,1)
      U=-1.D0
      CALL DMULT(M,X,A,W1)
      CALL DMULT(M,Y,B,W2)
      CALL CMULT(M,U,W2)
      CALL ADD(M,W1,W2,W1)
      CALL DMULT(M,X,B,W2)
      CALL DMULT(M,Y,A,X)
      CALL ADD(M,W2,X,W2)
      RETURN
      END

      SUBROUTINE REPL(M,A,B,C,D,W1,W2)
C      COMPUTES (B*AINV*B+A)INV*(B*AINV*C-D) AND STORES IT IN B
C      COMPUTES -(B*AINV*B+A)INV*(B*AINV*D+C) AND STORES IT IN A
      IMPLICIT REAL*8(A-H,O-Z)
      DIMENSION A(M,1),B(M,1),C(M,1),D(M,1),W1(M,1),W2(M,1)
      U=-1.D0
      CALL TINV(M,A,W1)
      CALL TMULT(M,B,W1,W2)
      CALL TMULT(M,W2,B,W1)
      CALL ADD(M,W1,A,A)
      CALL TINV(M,A,W1)
      CALL DMULT(M,W2,C,B)
      CALL DMULT(M,W2,D,A)
      CALL CMULT(M,U,D)
      CALL ADD(M,B,D,D)
      CALL DMULT(M,W1,D,B)
      CALL ADD(M,A,C,C)
      CALL DMULT(M,W1,C,A)
      CALL CMULT(M,U,A)
      RETURN
      END
      SUBROUTINE DIAG(L,A,B,W1,W2)
      IMPLICIT REAL*8(A-H,O-Z)
      DIMENSION A(L,1),B(L,1),W1(L,1),W2(L,1)
      DO 1 M=1,L
      DO 2 N=1,L
      W1(M,N)=M*A(M,N)
    2 W2(M,N)=M*B(M,N)
    1 W1(M,M)=W1(M,M)-DFLOAT(M)
      RETURN
      END
```

```
      SUBROUTINE VISCOU(M,MM,K,IDEA,M1,S,C,R,CH,R1,RI,W1,W2,W3,W4,W5,A,B
     *)
C        THIS SUBROUTINE CALCULATES THE VISCOUS COUPLING COEFFICIENTS
C        ON CYLINDER IDEA
      IMPLICIT REAL*8(A-H,O-Z)
      DIMENSION W3(2,1),W4(2,1),W5(M1,1),A(MM,MM,K),B(MM,MM,K)
      DIMENSION R(K,1),CH(K,1)
      DIMENSION S(1),C(1),R1(1),RI(1),W1(1),W2(1)
      M2=2*M*(K-1)
      IZ=M*(K-1)
      M3=M+2
      IF(MM.GT.M3)M3=MM
      JX=0
      DO 1 J=1,K
      IF(J.EQ.IDEA)GOTO 1
      JX=JX+1
      R4=R(IDEA,J)/RI(J)
      R6=1.D0/R4
      CALL CONT(R4,MM,R1)
      CALL GEN(CH(IDEA,J),M3,S,C)
      DO 2 N=1,MM
      N1=N-1
      R5=(-1.D0)**N1*R1(N)*DFLOAT(N)
      W1(N)=R5*C(N)
    2 W2(N)=R5*S(N)
      CALL CONT(R6,M3,R1)
      DO 3 L=1,M
      II=(JX-1)*M+L
      N1=L+2
      IP=II+IZ
      R4=0.D0
      R5=0.D0
      R6=0.D0
      R7=0.D0
      DO 4 N=1,MM
      R2=A(N,L,J)
      R3=B(N,L,J)
      R4=R4+W1(N)*R2
      R5=R5+W1(N)*R3
      R6=R6+W2(N)*R2
    4 R7=R7+W2(N)*R3
      R8=(-1.D0)**L*R1(N1)*DFLOAT(L)
      R9=R8*C(N1)
      R10=R8*S(N1)
      W3(1,II)=R9+R4-R7
      W3(1,IP)=R10+R5+R6
      W3(2,II)=R10-R5-R6
    3 W3(2,IP)=R4-R7-R9
    1 CONTINUE
      K1=2*(K-1)
      DO 5 I=1,2
      DO 5 J=1,K1
      W4(I,J)=0.D0
      DO 5 L=1,M2
    5 W4(I,J)=W4(I,J)+W3(I,L)*W5(L,J)
      RETURN
      END
      SUBROUTINE FIXUP(MM,K,A,B,W1,W2,W3,W4,RO,MN,R1,R2,R3,RI,RA,C1,S,CC
     *,F)
      IMPLICIT REAL*8(A-H,O-Z)
      DIMENSION A(MM,MM,K),B(MM,MM,K),W1(MM,1),W2(MM,1),W3(MM,1),RI(1)
      DIMENSION RA(1),C1(1),S(1),CC(1),F(1),R1(1),R2(1),R3(1),W4(MM,1)
      U=-1.D0
      MV=MM+2
      DO 31 L=1,K
      DO 31 I=1,MM
      DO 31 J=1,MM
      A(J,I,L)=0.D0
   31 B(J,I,L)=0.D0
      CALL CONT(RO,MV,R1)
      DO 1 J=1,K
      IF(J.EQ.MN)GOTO 5
      DO 34 N=1,MM
      DO 34 M=1,MM
      W3(M,N)=0.D0
   34 W4(M,N)=0.D0
```

```
      CALL GEN(C1(J),MM,S,CO)
      CALL CONT(RI(J),MV,R2)
      CALL CONT(RA(J),MM,R3)
      DO 2 M=1,MM
      R4=R1(M)/F(M)
      N4=M+1
      N5=M+2
      R6=F(N4)/R1(N5)
      DO 3 N=M,MM
      N1=N-M
      N2=N-M+1
      N3=N+1
      R5=U**N1*F(N3)*R3(N2)*R4/R2(N)/F(N2)
      A(M,N,J)=R5*CO(N2)
    3 B(M,N,J)=R5*S(N2)
      DO 4 N=1,M
      N1=M-N+1
      N3=N+2
      R5=R6*R3(N1)*R2(N3)/F(N)/F(N1)
      W3(M,N)=U*R5*CO(N1)
    4 W4(M,N)=R5*S(N1)
    2 CONTINUE
      CALL REPL(MM,A(1,1,J),B(1,1,J),W3,W4,W1,W2)
      GOTO 1
    5 R6=(RI(MN)/RC)**2
      R5=R6
      DO 7 M=1,MM
      A(M,M,MN)=R6
    7 R6=R6*R5
    1 CONTINUE
      RETURN
      END
      SUBROUTINE FIXUP2(MM,MP,K,KK,M1,A,B,W1,W2,W3,W4,CH,R,RI,R1,R2,R3,S
     *,CO,F,G,W5,W6,IDEA)
      IMPLICIT REAL*8(A-H,O-Z)
      DIMENSION A(MM,MM,K),B(MM,MM,K),W1(MM,1),W2(MM,1),W3(MM,1)
      DIMENSION W4(MM,1),CH(K,1),R(K,1),G(M1,1),W5(M1,1),W6(1)
      DIMENSION RI(1),S(1),CO(1),F(1),R1(1),R2(1),R3(1)
      U=-1.D0
      MS=2*MP+1
      IF(MS.LT.MM)MS=MM
      MW=MM+2
      MQ=MP*K
      IF(IDEA.NE.0)MQ=MP*(K-1)
      IX=0
      DO 9 I=1,K
      IF(I.EQ.IDEA)GOTO 9
      IX=IX+1
      CALL CONT(RI(I),MM,R1)
      JX=0
      DO 10 J=1,K
      IF(J.EQ.IDEA)GOTO 10
      JX=JX+1
      IF(I.EQ.J)GOTO 11
      CALL CONT(RI(J),MW,R2)
      CALL CONT(R(I,J),MS,R3)
      DO 306 N=1,MM
      DO 306 M=1,MM
      W3(M,N)=0.D0
  306 W4(M,N)=0.D0
      CALL GEN(CH(I,J),MS,S,CO)
      DO 12 M=1,MM
      R4=R1(M)/F(M)
      DO 13 N=M,MM
      N1=N-M
      N2=N-M+1
      N3=N+1
```

```
      R5=U**N1*F(N3)*R3(N2)*R4/R2(N)/F(N2)
      W3(M,N)=R5*CC(N2)
 13   W4(M,N)=R5*S(N2)
 12   CONTINUE
      CALL RIPL(MM,W3,W4,W1,W2,A(1,1,J),B(1,1,J))
      DO 14 M=1,MP
      II=(IX-1)*MP+M
      IP=II+MQ
      R4=P1(M)/F(M)
      DO 15 N=1,MP
      JJ=(JX-1)*MP+N
      JP=JJ+MQ
      N1=N+M+1
      N2=N+2
      N3=N+M
      R5=U**N*F(N3)*R4*R2(N2)/F(N)/R3(N1)
      R6=R5*CO(N1)
      FF=R5*S(N1)
      R5=W1(M,N)
      R7=W2(M,N)
      G(II,JJ)=R5+R6
      G(IP,JP)=R5-R6
      G(II,JP)=R7+FF
 15   G(IP,JJ)=FF-R7
 14   CONTINUE
      GOTO 1
 11   CALL DIAG(MM,A(1,1,I),B(1,1,I),W1,W2)
      DO 767 M=1,MP
      II=(JX-1)*MP+M
      IP=II+MQ
      DO 768 N=1,MP
      JJ=(JX-1)*MP+N
      JP=JJ+MQ
      R6=W1(M,N)
      FF=W2(M,N)
      G(II,JJ)=R6
      G(II,JP)=FF
      G(IP,JJ)=U*FF
      G(IP,JP)=R6
 768  CONTINUE
 767  CONTINUE
 1    CONTINUE
 9    CONTINUE
C          TO SOLVE THE EQUATIONS
C
      MT=2*MP*K
      IF(IDEA.NE.0)MT=2*MP*(K-1)
      ID=0
      DO 65 J=1,KK
      DO 65 I=1,MT
      W5(I,J)=0.D0
      II=(J-1)*MP+1
 65   W5(II,J)=1.D0
      CALL LEQTIF(G,KK,MT,M1,W5,ID,W6,IER)
      RETURN
      END
```

APPENDIX C: COMPUTATION OF THE EIGENFREQUENCIES

The frequencies ω of the system are obtained by solving the eigenvalue problem defined in equation (3.20):

$$(i\omega \underset{\sim}{I} - \underset{\sim}{Y}) \underset{\sim}{\Lambda} = \underset{\sim}{0} \quad .$$

The eigenvalues $i\omega$ satisfy the equation

$$\det(i\omega \underset{\sim}{I} - \underset{\sim}{Y}) = 0 \quad .$$

A computer program 'SOLINTER' has been written to generate the matrix $\underset{\sim}{Y}$ for cases of pinned-pinned and clamped-clamped cylinders. A library subroutine (EISPACK) was used to obtain the eigenvalues $i\omega$ of $\underset{\sim}{Y}$. The computer listings and input data cards are listed on the following pages. The program requires as input data the added mass and viscous coupling matrices as punched out by the program 'COUPLAGE'.

The computer program prints out the input information, the dimensionless flow velocities and the corresponding 4*K*N eigenvalues for each case. These are printed in columns of thirty, and if there are more than ninety eigenvalues to be printed, they will be printed on two pages.

As underlined previously, the printed eigenvalues are the frequencies ω of the system multiplied by i. The eigenvalues are printed in groups of two columns side by side, the left column being the negative of the imaginary part of ω, and the column on the right the real part of ω.

THE COMPUTER PROGRAM 'SOLINTER'

DATA CARDS

Card Number	Symbol	Fortran Name	Designation	Format
1	k	k	number of cylinders	I2
2	N	N	number of comparison functions	"
3		NV	number of velocity sets	"
4		IBOUND	0 for pinned-pinned, 1 for clamped-clamped	"
5	α	ALPHA	cylinder damping coefficient	F15.6
6	β_e	BETA	external fluid density	"
7	β_i	BETAI	internal fluid density	"
8	γ_e	GAMMAE	defined in equation (3.5)	"
9	γ_i	GAMMAI	defined in equation (3.5)	"
10	γ	GAMMAM	defined in equation (3.5)	"
11	Γ	GAMMAT	uniform tension	"
12	δ	DEL	0. if end free to move, 1. otherwise	"
13	ε	EPS	length/diameter of a cylinder	"
14	πe	PIE	external pressure	"
15	πi	PII	internal pressure	"
16	c	CL	pressure-drag coefficient	"
17	ν	PR	Poisson ratio	"
18	C_f	CF	friction coefficient	"
19	C_{fe}	CFE	defined in equation (2.19)	"
20	C_{fi}	CFI	defined in equation (2.19)	"
21	C_{fx}	CFX	defined in equation (2.19)	"

DATA CARDS (continued)

Card Number	Symbol	Fortran Name	Designation	Format
22	R	R	radius of a cylinder	F15.6
23	R_0	RO	radius of the enclosing channel	"
24	$\underset{\sim}{M}_v$	AM(I,J)	added mass and viscous coupling matrices as punched out by 'COUPLAGE' (one element of each on each card, by columns).	2F10.5
↓	$\underset{\sim}{C}_v$	VC(I,J)		
$4k^2+23$				
$4k^2+24$	u_e, u_i	UL,ULI	1st set of external and internal flow velocities	2F10.5
↓	"	"		"
$4k^2+23+NV$	"	"	last set of external and internal flow velocities	"

All these quantities are dimensionless, except the radii expressed in millimetres.

```
C                             ************
C                             *          *
C                             * SCLINTER *
C                             *          *
C                             ************
C
C      ***********************************************************************
C      THIS PROGRAM 'SCLINTER' COMPUTES THE EIGENFREQUENCIES FOR SETS OF
C      INTERNAL AND EXTERNAL FLOW VELOCITIES.
C      K IS THE NUMBER OF CYLINDERS.
C      N IS THE NUMBER OF COMPARISON FUNCTIONS USED.
C      UL AND ULI ARE THE EXTERNAL AND INTERNAL FLOW VELOCITIES.
C      NV IS THE NUMBER OF VELOCITY SETS.
C      THE MATRICES MUST BE DIMMENSIONNED AS FOLLOWS:
C
C       DIMENSION AM(KK,KK),VC(KK,KK),AM1(KK,KK),C(KK,KK),E1(KK,KK),G(KK,KK)
C      DIMENSION H(KK,KK),WK(KK),WR(2*KK*N),WI(2*KK*N),Y(2*KK*N,2*KK*N)
C       DIMENSION SIS(KK,KK),SIT(KK,KK)
C
C      ***********************************************************************
C
         IMPLICIT REAL*8(A-H,O-Z)
         DIMENSION AM(6,6),VC(6,6),AM1(6,6),C(6,6),E1(6,6),G(6,6),H(6,6)
         DIMENSION WK(6),SIS(6,6),SIT(6,6),Y(72,72),WR(72),WI(72)
         COMMON IBOUND
         INTEGER DIGIT(10) /'1','2','3','4','5','6','7','8','9','10'/
         INTEGER EC(4)/'(1H0','    ','F12.','5)'/
C      READ IN NUMBER OF CYLINDERS AND NUMBER OF COMPARISON FUNCTIONS USED
         READ 100,K,N
         KK=2*K
         KS=2*KK*N
C      READ IN NUMBER OF VELOCITY SETS
         READ 100,NV
C      READ IN BOUNDARY END CONDITIONS (0 FOR P-P,1 FOR C-C)
         READ 100,IBOUND
C      READ IN SYSTEM PARAMETERS
         READ 101,ALPHA,BETA,BETAI,GAMMAE,GAMMAI,GAMMAM,GAMMAT,DEL,EPS,
        *PIE,PII,CL,PR,CF,CFE,CFI,CFX,R,RO
C      READ ADDED MASS MATRIX AND MATRIX OF VISCOUS COUPLING
         READ 102,((AM(I,J),VC(I,J),J=1,KK),I=1,KK)
         PRINT 200,K,N
         PRINT 201,R,RO
         PRINT 202
         PRINT 203,BETA,GAMMAE,PIE
         PRINT 204
         PRINT 205,BETAI,GAMMAI,PII
         PRINT 206,ALPHA,GAMMAT,PR
         PRINT 207,DEL,EPS,CL,CF,CFE,CFI,CFX
         HL=2.D0*R/((RO**2-K*R**2)*2.D0/(RO+K*R))
         PRINT 208,HL
         IF (IBOUND.EQ.0) PRINT 98
         IF (IBOUND.EQ.1) PRINT 99
         GAMMA=GAMMAM-GAMMAE
         PRINT 299
         PRINT 224
         EC(2)=DIGIT(KK)
         WRITE(6,EC)((AM(I,J),J=1,KK),I=1,KK)
         PRINT 225
         WRITE(6,EC)((VC(I,J),J=1,KK),I=1,KK)
         DO 10  I=1,KK
         DO 10  J=1,KK
         SIS(I,J)=AM(I,J)
   10    SIT(I,J)=VC(I,J)
```

```
      DO 20 INR=1,NV
C     READ EXTERNAL AND INTERNAL VELCCITIES
      READ 102,UL,ULI
      PRINT 229,ULI,UL
      CALL SCLN(KK,KS,AM,VC,AM1,C,E1,G,H,WK,WR,WI,Y,ALFHA,BETA,EETAI,
     1GAMMA,GAMMAI,GAMMAT,FIE,PII,EPS,HL,CL,UL,ULI,CF,CFE,CFI,CFX,DEL,
     *PR)
      DC 9 LIP=1,KK
      DC 9 LID=1,KK
      AM(LIP,LID)=SIS(LIP,LIC)
    9 VC(LIP,LID)=SIT(LIP,LIC)
      K1=KS/2
      KR=30
      IF(KS.LE.90)K1=KS
      DO 17 J=1,KR
   17 PRINT 221,(WR(I),WI(I),I=J,K1,KR)
      IF(KS.LE.90)GCTC 16
      PRINT 299
      K1=KS/2+1
      K2=K1+29
      DO 18 J=K1,K2
   18 PRINT 221,(WR(I),WI(I),I=J,KS,30)
   16 CONTINUE
   20 CONTINUE
      STCP
   98 FCRMAT(////,' PINNEC-PINNEC CYLINCERS'/24('*'))
   99 FCRMAT(////,' CLAMPEC-CLAMPEC CYLINDERS'/26('*'))
  100 FCRMAT(I2)
  101 FCRMAT(F15.6)
  102 FORMAT(2F10.5)
  200 FCRMAT('1THERE ARE',I3,' CYLINDERS, AND THE NUMBER CF COMPARISCN F
     *UNCTIONS USEC WAS',I3)
  201 FORMAT('0 THE RACIUS CF EACH CYLINCER IS',F6.1,' MILLIMETRES'
     1//'  THE RACIUS CF THE SURROUNDING CHANNEL IS',F6.1,
     *' MILLIMETRES')
  202 FORMAT(//'0','DIMENSICNLESS EXTERNAL FLOW CUANTITIES:'/41('*'))
  203 FCRMAT(/'0','FLUIC CENSITY EETAE',F10.5/' GAMMAE=',F10.5/
     *' EXTERNAL PRESSURE=',F10.5)
  204 FCRMAT(//'0','DIMENSICNLESS INTERNAL FLOW CUANTITIES:'/41('*'))
  205 FORMAT(/'0','FLUIC CENSITY BETAI=',F10.5/
     *' GAMMAI=',F10.5/
     *' INTERNAL PRESSURE=',F10.5)
  206 FCRMAT(/////' INTERNAL DAMPING ALFHA=',F10.5/
     *' DIMENSIONLESS UNIFCRM TENSICN=',F10.5/
     *' PCISSON RATIC=',F10.5)
  207 FORMAT(//'0','DELTA=',F10.5,5X,'L/C=',F10.5,5X,'CE GRCUPING=',
     *F10.5,/' CF=',F10.5,8X,'CFE=',F10.5,6X,'CFI=',F10.5,5X,'CFX=',F10.
     *5)
  208 FCRMAT('0','D/DH=',F10.5)
  221 FCRMAT('0',5X,2F16.9,5X,2F16.9,5X,2F16.9)
  224 FORMAT('0  THE ACDEC MASS MATRIX')
  225 FORMAT(///,'0  THE MATRIX OF THE VISCCUS COUPLING')
  229 FORMAT('1',/,T10,'U.INT=',F9.5,T30,'U.EXT=',F9.5/,T10,55('*')/)
  299 FORMAT('1')
      END
```

```
      SUBROUTINE SCLN(KK,KS,AM,VC,AM1,C,E,G,H,WK,WR,WI,Y,ALPHA,BETA,BET
     1AI,GAMMA,GAMMAI,GAMMAT,PIE,PII,EPS,HL,CL,UL,ULI,CF,CFE,CFI,CFX,
     *DEL,PR)
      IMPLICIT REAL*8(A-H,O-Z)
      DIMENSION AM(KK,1),VC(KK,1),AM1(KK,1),C(KK,1),E(KK,1),G(KK,1)
      DIMENSION H(KK,1),WK(1),WR(1),WI(1),Y(KS,1)
      COMMON IBOUND
      UN=-1.D0
      PI=3.14159265358980D0
      N=KS/2/KK
      WR(1)=4.73004D0
      WR(2)=7.8532D0
      WR(3)=10.9956D0
      WR(4)=14.1372D0
      WR(5)=17.2788D0
      WI(1)=0.9825022D0
      WI(2)=1.0007773D0
      WI(3)=0.9999665D0
      WI(4)=1.0000015D0
      WI(5)=0.9999999D0
      DO 8 I=6,N
      WR(I)=(2.D0*I+1.D0)*PI/2.D0
    8 WI(I)=1.D0
C     GENERATE MATRICES C,E,G,H,AND M
      TETA=(CFE*UL**2+CFI*ULI**2+CFX*UL*ULI)/2
      X1= 2.D0*UL*DSQRT(BETA)
      X2=UL**2
      X3=0.5D0*EPS*CF*UL**2
      X4=0.5D0*EPS*DSQRT(BETA)*(CF*UL+CL)
      X5=BETA
      X6=UN*(DEL*(GAMMAT+(PIE-PII)*(1.D0-2.D0*PR))+TETA*(1.D0-DEL)+(0.5
     *D0*EPS*CF*UL**2*(1.D0+HL)+(AMMA+GAMMAI)*(1.D0-DEL/2.D0)-ULI**2)
      X7=GAMMA+GAMMAI+0.5D0*EPS*CF*HL*UL**2
      X8=2.D0*ULI*DSQRT(BETAI)
      DO 1 I=1,KK
      DO 2 J=1,KK
      C(I,J)=X1*AM(I,J)
      E(I,J)=X2*AM(I,J)
      G(I,J)=X3*VC(I,J)
      H(I,J)=X4*VC(I,J)
    2 AM1(I,J)=X5*AM(I,J)
      E(I,I)=E(I,I)+X6
      G(I,I)=G(I,I)+X7
      C(I,I)=C(I,I)+X8
    1 AM1(I,I)=AM1(I,I)+1.D0-BETA
C     INVERT M
      CALL LINV1F(AM1,KK,KK,AM,4,WK,IER)
C     FORM PRODUCTS M INVERSE*C, M INVERSE*E, M INVERSE*H, M INVERSE*G
      CALL DMULT(KK,AM,C,AM1)
      CALL DMULT(KK,AM,E,VC)
      CALL DMULT(KK,AM,H,C)
      CALL DMULT(KK,AM,G,E)
C     FIND DIAGONAL ELEMENT OF F
      F=0.5D0*EPS*CF*UL**2*(1.D0+HL)+GAMMA+GAMMAI
C     INITIALIZE Y
      DO 3 I=1,KS
      DC 3 J=L,KS
    3 Y(I,J)=0.0D0
      KN=KK*N
```

```
C    GENERATE Y .
        DO 4 I=1,N
        DO 5 J=1,N
        IF(IBCUNC.EQ.0) GO TO 26
C    PARAMETERS FOR CLAMPED-CLAMPED CYLINDERS
        X3=WR(I)**4
        X5=ALPHA*X3
        IF (I.EQ.J) GO TO 24
        AB=4.D0*(WR(J)*WR(I))**2/(WR(J)**4-WR(I)**4)
        AC=WR(J)*WI(J)-WR(I)*WI(I)
        AA=AB*((-1.D0)**(I+J)-1.D0)
        BB=AB*((-1.D0)**(I+J)+1.D0)*AC
        DD=AB*AC*(-1.D0)**(I+J)-(3.D0*WR(J)**4+WR(I)**4)*AA/(WR(J)**4-WR(I
      *)**4)
        GO TO 25
24      AA=0.0D0
        BB=WR(I)*WI(I)*(2.D0-WR(I)*WI(I))
        DD=BB/2.D0
        GO TO 28
26      CONTINUE
C    PARAMETERS FOR PINNED-PINNED CYLINDERS
        X3=(PI*I)**4
        X5=ALPHA*X3
        IF(I.EQ.J) GO TO 27
        AA=2.D0*I*J*((-1.D0)**(I+J)-1.D0)/(J**2-I**2)
        BB=0.D0
        DD=4.D0*I*J**3*(1.D0-(-1.D0)**(I+J))/((J**2-I**2)**2)
        GOTO 25
27      AA=0.0D0
        BB=UN*(PI*I)**2
        DD=BB/2.D0
28      CONTINUE
        X4=F*DD+X3
        DO 9 II=1,KK
        IP=(I-1)*KK+II
        IX=IP+KN
        DO 10 JJ=1,KK
        JP=(I-1)*KK+JJ
        JX=JP+KN
        Y(IP,JP)=UN*(AA*AM1(II,JJ)+X5*AM(II,JJ)+C(II,JJ))
        Y(IP,JX)=UN*(X4*AM(II,JJ)+VC(II,JJ)*BB+E(II,JJ)*AA)
  10    CONTINUE
        Y(IX,IP)=1.D0
  9     CONTINUE
        GOTO 5
  25    X4=F*DD
        DO 6 II=1,KK
        IP=(I-1)*KK+II
        IX=IP+KN
        DO 7 JJ=1,KK
        JP=(J-1)*KK+JJ
        JX=JP+KN
        Y(IP,JP)=UN*AM1(II,JJ)*AA
        Y(IP,JX)=UN*(BB*VC(II,JJ)+X4*AM(II,JJ)+AA*E(II,JJ))
  7     CONTINUE
  6     CONTINUE
  5     CONTINUE
  4     CONTINUE
C    FIND EIGENVALUES OF Y
        CALL EISPAC(KS,KS,MATRIX('REAL',Y),VALUES(WR,WI))
        RETURN
        END
        SUBROUTINE DMULT(M,A,B,C)
C          MULTIPLIES MATRICES, A*B=C
        IMPLICIT REAL*8(A-H,O-Z)
        DIMENSION A(M,1),B(M,1),C(M,1)
        DO 1 I=1,M
        DO 1 J=1,M
        C(I,J)=0.D0
        DO 2 L=1,M
  2     C(I,J)=C(I,J)+A(I,L)*B(L,J)
  1     CONTINUE
        RETURN
        END
//GO.EISPACLB DD DSN=CFLB.SUBLIB,DISP=SHR
//GO.SYSIN   DD  *
```

APPENDIX D: THE STABILITY DOMAIN

Once the eigenfrequencies $i\omega$ of the system are available, it is theoretically possible to delimit the zones of stability, i.e. where $Re(i\omega) < 0$, with respect to the internal and external flow velocities. But, as it has been observed for pinned-pinned or clamped-clamped end conditions that the first instability to occur is always buckling, it is sufficient to determine the onset of this first buckling. It corresponds to the critical set of internal and external flow velocities $(u_i, u_e)_{crit.}$ at which the eigenfrequency $i\omega$ is annulled for the first time, when one of the two flow velocities is increased.

Then, replacing the condition $i\omega = 0$ in eqn. (3.15) of Chapter 3,

$$\underset{\sim}{M}\,\ddot{\underset{\sim}{p}} + \underset{\sim}{C}\,\dot{\underset{\sim}{p}} + \underset{\sim}{K}\,\underset{\sim}{p} = \underset{\sim}{0} \quad ,$$

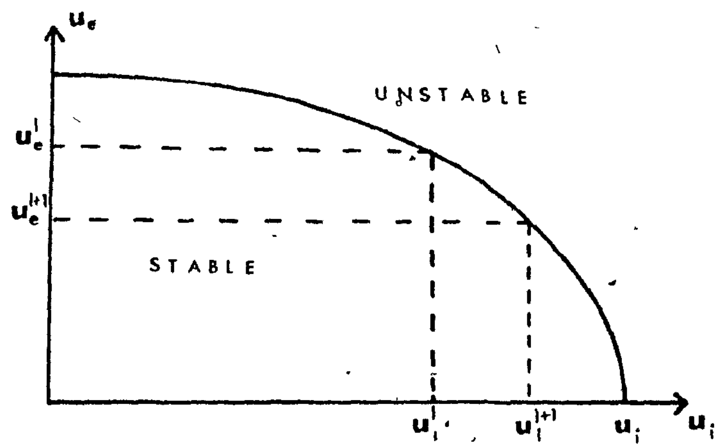
it leads, for a non trivial solution $\underset{\sim}{p}$, to

$$\det \underset{\sim}{K} = \underset{\sim}{0} \quad , \tag{D.1}$$

where $\underset{\sim}{K}$ is defined in eqn. [3.16] and depends on the internal and external flow velocities. Then the problem simply reduces, for a given internal velocity, to finding the lowest external velocity which makes the determinant of $\underset{\sim}{K}$ to vanish. The computer program 'BUCKLING' described in the following pages carries out these operations and allows to delimit the domain of stability.

THE COMPUTER PROGRAM 'BUCKLING'

For an assigned initial internal velocity u_i^ℓ, the program computes the critical external velocity u_e^ℓ at which the first buckling occurs, i.e. when $\det[\mathcal{K}(u_e)]$ vanishes. This routine is reiterated with a higher internal velocity $u_i^{\ell+1}$, till the velocity u_i^n where the system buckles without external flow (u_e^n=0). With the help of these n plots, the curve of the buckling onset may then be drawn.



The numerical method used to solve eqn. (D.1) is the classical secant method applied to the value of the determinant or to its first derivative with respect to u_e. The reason of this latter alternative is that the order of magnitude of the determinant may be very high ($>10^{100}$) even for values of u_e extremely close to the root, which may therefore be missed in the step-by-step method used here. Hence, if such a difficulty is encountered, the program searches for the minimum of the determinant which almost coincides with its root. The program itself selects the appropriate method and, in our trials never failed in its computations.

The program starts at a given value of the internal velocity u_i^1 and settles each time the correct increment $u_i^{\ell+1} - u_i^{\ell}$ in order to satisfy

$$\left| u_e^{\ell+1} - u_e^{\ell} \right| \leqslant D_{max} ,$$

where D_{max} is a given constant. In this way, the curve $u_e(u_i)$ may be drawn with a selected accuracy.

DATA CARDS FOR 'BUCKLING'

Card Number	Symbol	Fortran Name	Designation	Format
1	k	k	number of cylinders	I2
2	N	N	number of comparison functions	"
3		IBOUND	0 for pinned-pinned, 1 for clamped-clamped	"
4	u_i^1	ULI	initial internal flow velocity	F15.6
5		UL1	lower estimated bound for u_e^1	"
6		UL2	upper estimated bound for u_e^2	"
7		XINI	upper bound of the internal velocity step size	"
8	D_{max}	DMAX	maximum allowed value for $\lvert u_e^{\ell}-u_e^{\ell+1}\rvert$	"
9		INDY	maximum number of computed roots (INDY > n)	I2
10	α	ALPHA	system parameters (same as in 'SOLINTER')	F10.5
↓				
28	R_o	RO		
29	M_v	AM(I,J)	added mass and viscous coupling matrices as punched out by 'COUPLAGE'.	2F10.5
↓	C_v	VC(I,J)		
$4k^2+28$				

```
C                  ************
C                  *          *
C                  * BUCKLING *
C                  *          *
C                  ************
C
C*****************************************************************************
C
C    THIS PROGRAM CCMPUTES THE EXTERNAL VELCCITIES AT WHITCH THE FIRST
C    BUCKLING OCCURS,FOR INCREASING INTERNAL VELOCITIES
C    K IS THE NUMBER CF CYLINCERS
C    N IS THE NUMBER CF CCMPARISON FUNCTIONS USED
C    ULI IS THE FIRST INTERNAL VELCCITY
C    UL1 AND UL2 ARE THE 2 EXTERNAL VELOCITIES BETWEEN WITCH THE FIRST
C    RCOT IS SEARCHED (UL1 < UL2)
C    XINI IS THE FIRST INTERNAL VELCCITY STEP SIZE
C    DMAX IS THE MAXIMUM INTERVAL EETWEEN 2 FOLLOWING ROCTS
C    INDYI IS THE MAXIMUM NUMBER OF CCMPUTED ROOTS
C    THE SYSTEM PARAMETERS ARE THE SAME AS IN 'SOLINTER'
C
C    DIMENSION AM(2K,2K),VC(2K,2K),E(2K,2K),G(2K,2K),WR(N),WI(N)
C    DIMENSION SK(2KN,2KN),SK1(2KN,2KN),SK2(2KN,2KN),WK(2KN)
C
C*****************************************************************************
      IMPLICIT REAL*8(A-H,O-Z)
      DIMENSION AM(6,6),VC(6,6),E(6,6),G(6,6),WR(10),WI(10)
      DIMENSION SK(36,36),SK1(36,36),SK2(36,36),WK(36)
      COMMON ALPHA,BETA,BETAI,GAMMA,GAMMAI,GAMMAT,DEL,EPS,HL,
     *PIE,PII,RO,CL,PR,CF,CFE,CFI,CFX,R,IBOUND
      INTEGER DIGIT(10) /'1','2','3','4','5','6','7','8','9','10'/
      INTEGER EC(4)/'(1H0','    ','F12.',',5)'/
      READ 100,K,N
C    READ IN BOUNDARY END CONDITIONS (0 FOR P-P,1 FOR C-C)
      READ 100,IBOUND
      READ 101,ULI,UL1,UL2
      READ 101,XINI,DMAX
      READ 100,INDYI
C    READ IN SYSTEM PARAMETERS
      READ 101,ALPHA,BETA,BETAI,GAMMAE,GAMMAI,GAMMAM,GAMMAT,DEL,EPS,
     *PIE,PII,CL,PR,CF,CFE,CFI,CFX,R,RO
      HL=2.D0*R/((RO**2-K*R**2)*2.D0/(RO+K*R))
      PRINT 200,K,N
      PRINT 201,R,RO
      PRINT 202
      PRINT 203,BETA,GAMMAE,PIE
      PRINT 204
      PRINT 205,BETAI,GAMMAI,PII
      PRINT 206,ALPHA,GAMMAT,PR
      PRINT 207,DEL,EPS,CL,CF,CFE,CFI,CFX
      PRINT 208,HL
      PRINT 209,ULI,UL1,UL2
      PRINT 210,XINI,DMAX,INDYI
      IF (IBOUND.EQ.0) PRINT 98
      IF(IBOUND.EQ.1) PRINT 99
      GAMMA=GAMMAM-GAMMAE
      KK=2*K
      KKN=KK*N
      PRINT 299
C    READ ADDED MASS MATRIX AND MATRIX OF VISCOUS COUPLING
      READ 102,((AM(I,J),VC(I,J),J=1,KK),I=1,KK)
      EC(2)=DIGIT(KK)
      PRINT 224
      WRITE(6,EC)((AM(I,J),J=1,KK),I=1,KK)
      PRINT 225
      WRITE(6,EC)((VC(I,J),J=1,KK),I=1,KK)
      PRINT 300
      DO 8 II=1,INDYI
C    ADJUST STEP SIZE FOR U.EXT
      STEP=1.D0/5/K
11    STEP=STEP/2
      GLI=(UL2-UL1)/STEP
      IF(GLI.LT.10.D0)GO TO 11
      IF(GLI.GT.100) GO TO 7
14    UL=UL1
      INDEX=0
```

```
C     COMPUTE DETERMINANT OF STIFFNESS MATRIX
      A=1.D0
      CALL BUCK(UL,ULI,SK,AM,VC,E,G,KK,KKN,N,WR,WI)
      CALL LINV3F(SK,WR,4,KKN,KKN,A,B,WK,IER)
      IF(A.GT.0.D0) GO TO 1
15    IF(II.GT.1) GO TO 16
      UL2=UL1
      UL1=UL1-1.D0
      IF(UL1.LT.0.D0) GO TO 4
      GO TO 14
16    XINI=XINI/2
      IF(XINI.LE.0.05D0) GO TO 33
      ULI=ULI-XINI
      GO TO 14
1     ULL=UL+STEP
      AA=1.D0
C     COMPUTE DETERMINANT AT NEXT STEP
      INDEX=INDEX+1
      CALL BUCK(ULL,ULI,SK,AM,VC,E,G,KK,KKN,N,WR,WI)
      CALL LINV3F(SK,WR,4,KKN,KKN,AA,BB,WK,IER)
      IF(AA.LT.0.D0) GO TO 2
      RATIO=A/AA*2.D0**(B-BB)
      IF(INDEX.GT.1)GO TO 17
      IF(RATIO.LT.1.D0) GO TO 15
      PRINT 501,ULI,UL1,UL2
17    IF(RATIO.LT.1.D0) GO TO 3
      PRINT 400,ULL,AA,BB
      A=AA
      B=BB
      UL=ULL
      IF(UL.GE.UL2) GO TO 5
      GO TO 1
C     SECANT METHOD TO FIND ROOT OF DETERMINANT(CASE WHERE IT BECOMES<0)
2     GLU=AA*2.D0**(BB-B)
      UL3=(UL*GLU-ULL*A)/(GLU-A)
      IF(INDEX.EQ.1) PRINT 501,ULI,UL1,UL2
      PRINT 400,ULL,AA,BB
      GO TO 32
C     SECANT METHOD ON DERIVATIVE OF DETERMINANT TO FIND ITS MINIMUM
C     CASE WHERE THE DETERMINANT REMAINS ALWAYS NEGATIVE)
3     UL1=UL
      UL2=ULL
      PRINT 298
      A1=1.D0
      AA1=1.D0
      J=0
      STEP=(UL2-UL1)/100
      CALL BUCK(UL1,ULI,SK1,AM,VC,E,G,KK,KKN,N,WR,WI)
      CALL LINV3F(SK1,WR,4,KKN,KKN,A1,B1,WK,IER)
      ULL1=UL1+STEP
      CALL BUCK(ULL1,ULI,SK1,AM,VC,E,G,KK,KKN,N,WR,WI)
      CALL LINV3F(SK1,WR,4,KKN,KKN,AA1,BB1,WK,IER)
      DER1=(AA1-A1+A1*DLOG(2.D0)*(BB1-B1))/STEP
12    CALL BUCK(UL2,ULI,SK2,AM,VC,E,G,KK,KKN,N,WR,WI)
      A2=1.D0
      AA2=1.D0
      CALL LINV3F(SK2,WR,4,KKN,KKN,A2,B2,WK,IER)
      ULL2=UL2+STEP
      CALL BUCK(ULL2,ULI,SK2,AM,VC,E,G,KK,KKN,N,WR,WI)
      CALL LINV3F(SK2,WR,4,KKN,KKN,AA2,BB2,WK,IER)
      DER2=(AA2-A2+A2*DLOG(2.D0)*(BB2-B2))/STEP
      GLO=DER2*2.D0**(B2-B1)
      UL3=(UL1*GLO-UL2*DER1)/(GLO-DER1)
      J=J+1
      JJ=5
      PRINT 500,J,UL1,UL2,UL3
```

```
      IF(DABS(UL3-UL2).LT.1.D-02) GO TO 32
      IF(J.EQ.JJ) GC TO 32
      RATIC=DER2/DER1*2.CO**(B2-B1)
      IF(DABS(RATIO).GT.1.DO) GO TO 31
      UL1=UL2
      DER1=DER2
      A1=A2
      B1=B2
31    UL2=UL3
      STEP=STEP/10
      GC TO 12
32    CONTINUE
      PRINT 401,UL3
      IF(UL1.EC.0.DO) PRINT 503,ULI
      IF(UL3.LT.0.2) GO TO 33
      UL1=UL3-DMAX
      UL2=UL3
      IF(UL1.GT.0.DO) GO TO 13
      UL1=0.DO
      XINI=XINI/2
13    ULI=ULI+XINI
8     CONTINUE
      GO TO 33
4     PRINT 502
      GC TO 33
5     PRINT 403
      GC TO 33
7     PRINT 404
33    CONTINUE
      STCP
98    FORMAT(////,' PINNED-PINNED CYLINCERS'/24('*'))
99    FORMAT(////,' CLAMPED-CLAMPEC CYLINDERS'/26('*'))
  100 FCRMAT(I2)
  101 FORMAT(F15.6)
  102 FCRMAT(2F10.5)
  200 FORMAT('1THERE ARE',I3,' CYLINDERS, AND THE NUMBER CF CCMPARISON F
     *UNCTIONS USED WAS',I3)
201   FORMAT('0 THE RADIUS CF EACH CYLINCER IS',F6.1,' MILLIMETRES'
     1//'  THE RADIUS OF THE SURRCUNDING CHANNEL IS',F6.1,
     *' MILLIMETRES')
202   FORMAT(//'0','DIMENSICNLESS EXTERNAL FLOW QUANTITIES:'/41('*'))
203   FORMAT(/'0','FLUID DENSITY EETAE',F10.5/' GAMMAE=',F10.5/
     *' EXTERNAL PRESSURE=',F10.5)
204   FORMAT(//'0','DIMENSICNLESS INTERNAL FLOW QUANTITIES:'/41('*'))
205   FCRMAT(/'0','FLUID DENSITY EETAI=',F10.5/
     *' GAMMAI=',F10.5/
     *' INTERNAL PRESSURE=',F10.5)
206   FORMAT(/////' INTERNAL DAMPING ALPHA=',F10.5/
     *' DIMENSIONLESS UNIFCRM TENSION=',F10.5/
     *' POISSON RATIC=',F10.5)
207   FORMAT(//'0','DELTA=',F10.5,5X,'L/D=',F10.5,5X,'CC GRCUPING=',
     *F10.5,/' CF=',F10.5,8X,'CFE=',F10.5,6X,'CFI=',F10.5,5X,'CFX=',F10.
     *5)
208   FORMAT('0','D/DH=',F10.5)
209   FORMAT(///'0','ULI=',F8.4,T30,'UL1=',F8.4,T60,'UL2=',F8.4)
210   FORMAT('0','XINI=',F8.4,T30,'DMAX=',F8.4,T60,'INDYI=',I5)
  224 FORMAT('0  THE ADDED MASS MATRIX')
  225 FORMAT(///,'0  THE MATRIX OF THE VISCCUS COUPLING')
298   FORMAT('0')
299   FCRMAT('1')
300   FORMAT('1',///,T10,'INTERNAL AND EXTERNAL FLOW VELCCITIES FCR BLCK
     1LING',/,T10,50('*')///)
400   FORMAT('0',T10,'UL=',F9.5,T30,'A=',D14.7,T55,'B=',D14.7)
401   FCRMAT('0',T10,'LOWEST ROOT FCR BLCKLING U.EXT=',F9.5,/,
     1T10,41('*')////)
403   FORMAT('0',T10,'ROOT FCR BUCKLING AFTER UL2')
404   FORMAT('0',T10,'* WARNING:THERE ARE MCRE THAN 100 STEPS
     1 BETWEEN UL1 AND UL2.REDUCE THE INTERVAL.')
500   FORMAT('0',T10,'J=',I2,T20,'UL1=',F9.5,T45,'UL2=',F9.5,T70,'UL3=',
     1F9.5)
501   FORMAT('0',T8,'* U.INT=',F9.5,T30,'UL1=',F9.5,T55,'UL2=',F9.5,/)
502   FORMAT('0',//,T10,'NC RCOT WAS FOUND BETWEEN U.EXT=0. ANC U.EXT=UL
     *2')
503   FORMAT('0',T10,'OR BUCKLING BEFORE U.INT=',F9.5)
      END
```

```
      SUBROUTINE BUCK(UL,ULI,SK,AM,VC,E,G,KK,KKN,N,WR,WI)
C     THIS SUBROUTINE COMPUTES THE STIFFNESS MATRIX SK
      IMPLICIT REAL*8(A-H,O-Z)
      DIMENSION AM(KK,1),VC(KK,1),E(KK,1),G(KK,1),WR(1),WI(1),SK(KKN,1)
      COMMON ALPHA,BETA,BETAI,GAMMA,GAMMAI,GAMMAT,DEL,EPS,HL,
     *PIE,PII,RO,CL,PR,CF,CFE,CFI,CFX,R,IBOUND
      UN=-1.D0
      PI=3.14159265358980D0
      TETA=(CFE*UL**2+CFI*ULI**2+CFX*UL*ULI)/2
      X2=UL**2
      X3=0.5D0*EPS*CF*UL**2
      X6=UN*(DEL*(GAMMAT+(PIE-PII)*(1.D0-2.D0*PR))+TETA*(1.D0-DEL)+(0.5
     *D0*EPS*CF*UL**2*(1.D0+HL)+GAMMA+GAMMAI)*(1.D0-DEL/2.D0)-ULI**2)
      X7=GAMMA+GAMMAI+0.5D0*EPS*CF*HL*UL**2
      DO 1 I=1,KK
      DO 2 J=1,KK
      E(I,J)=X2*AM(I,J)
2     G(I,J)=X3*VC(I,J)
      E(I,I)=E(I,I)+X6
1     G(I,I)=G(I,I)+X7
      F=0.5D0*EPS*CF*UL**2*(1.D0+HL)+GAMMA+GAMMAI
      DO 4 I=1,N
      DO 5 J=1,N
      IF(IBOUND.EQ.0) GO TO 26
C     PARAMETERS FOR CLAMPED-CLAMPED CASE
      WR(1)=4.73004D0
      WR(2)=7.85320D0
      WR(3)=10.9956D0
      WR(4)=14.1372D0
      WR(5)=17.2788D0
      WI(1)=0.9825022D0
      WI(2)=1.0007773D0
      WI(3)=0.9999665D0
      WI(4)=1.0000015D0
      WI(5)=0.9999999D0
      DO 8 I=6,N
      WR(I)=(2.D0*I+1.D0)*PI/2.D0
8     WI(I)=1.D0
      X3=WR(I)**4
      X5=ALPHA*X3
      IF (I.EQ.J) GO TO 24
      AB=4.D0*(WR(J)*WR(I))**2/(WR(J)**4-WR(I)**4)
      AC=WR(J)*WI(J)-WR(I)*WI(I)
      AA=AB*((-1.D0)**(I+J)-1.D0)
      BB=AB*((-1.D0)**(I+J)+1.D0)*AC
      DD=AB*AC*(-1.D0)**(I+J)-(3.D0*WR(J)**4+WR(I)**4)*AA/(WR(J)**4-WR(I
     *)**4)
      X4=F*DD
      GO TO 25
24    AA=0.D0
      BB=WR(I)*WI(I)*(2.D0-WR(I)*WI(I))
      DD=BB/2.D0
      GO TO 28
26    CONTINUE
C     PARAMETERS FOR PINNED-PINNED CYLINDERS
      X3=(PI*I)**4
      X5=ALPHA*X3
      IF(I.EQ.J) GO TO 27
      AA=2.D0*I*J*((-1.D0)**(I+J)-1.D0)/(J**2-I**2)
      BB=0.D0
      DD=4.D0*I*J**3*(1.D0-(-1.D0)**(I+J))/((J**2-I**2)**2)
      X4=F*DD
      GOTO 25
27    AA=0.0D0
      BB=UN*(PI*I)**2
      DD=BB/2.D0
28    CONTINUE
      X4=F*DD+X3
25    DO 9 II=1,KK
      IP=(I-1)*KK+II
      DO 10 JJ=1,KK
      JP=(J-1)*KK+JJ
      SK(IP,JP)=BB*E(II,JJ)+AA*G(II,JJ)
      IF(II.EQ.JJ) SK(IP,JP)=SK(IP,JP)+X4
10    CONTINUE
9     CONTINUE
5     CONTINUE
4     CONTINUE
      RETURN
      END
```

APPENDIX E - THE COMPUTER PROGRAM "TRANSIT"

This program allows for the determination of the transient response of a pinned-pinned system in two cases:

i) some cylinders are given initial displacements and then the system is abandoned to itself,

ii) some cylinders are constrained to have a sinusoidal displacement of constant amplitude and frequency, and the response of the others is computed.

In both cases the program computes the y- and z-displacements of three (or less) selected cylinders at equal intervals of time and, if desired, plots the response over a certain period of time.

INPUT

Like the other programs it requires as input the system parameters, the added mass and viscous coupling coefficients. At non-dimensional time $\tau = 0$ all the cylinders are supposed to be at their position of rest except KI cylinders which are either simply displaced or submitted to a steady excitation of frequency OMEGA. As the system possesses N degrees of freedom in the x-direction (see Chapter V for further details), the displacements of the KI cylinders must be given at N abscissae ξ_i $(i = 1,\ldots, N)$. The selected cylinders for which the response is computed and the time intervals are also to be defined.

OUTPUT

The program calculates and prints the mass matrix $\underset{\sim}{M}$, the eigenvalues $i\omega$, the initial state vector $\{\dot{\underset{\sim}{\eta}}(0), \underset{\sim}{\eta}(0)\}^T$, the constants of initialization $\{\underset{\sim}{c}, \bar{\underset{\sim}{c}}\}$ and the displacements of the selected cylinders in the y- and z-directions at the first abscissa where the initial displacements were given, i.e. $\xi = \xi_1$. Note that in order to spare paper, every other computed point only is printed. If the digital plot is not desired some cards have to be removed as indicated on the listing.

DATA CARDS FOR "TRANSIT"

Card Number	Symbol	Fortran Name	Designation	Format
1	K	K	number of cylinders	I2
2	N	N	number of comparison functions	I2
3	α	ALPHA		
↓			system parameters as in "SOLINTER"	F10.5
21	R[illegible]	RO		
22	M [illegible]	AM(I,J)		
↓			added mass and viscous coupling matrices as punched out by "COUPLAGE"	2F10.5
$4K^2+21$	$\underset{\sim}{C}_v$	VC(I,J)		
$4K^2+22$	u_e, u_i	UL,ULI	external and internal flow velocities	2F10.5
$4K^2+23$		KI	number of displaced cylinders at time $\tau = 0$	I2
$4K^2+24$	ξ_i	XI(I)	i^{th} abscissa where an initial state is given ($i = 1,\ldots, N$)	F10.5
$4K^2+25$		ICYL	initially displaced cylinder number	I2,8X
$4K^2+25$		ZDISPL	amplitude of displacement in z-direction	F10.5
$4K^2+25$		YDISPL	amplitude of displacement in y-direction	F10.5
$4K^2+25$		OMEGA	frequency of excitation if any (blank otherwise)	F10.5
$4K^2+25$			This card is repeated KI times. If $i < N$ return to ξ_i for another abscissa ξ_{i+1}	
$4K^2$+N*KI+25		NCYL(I)	plotted cylinder numbers (maximum 3)	3I2
$4K^2$+N*KI+26		TIME	period of time over which the response is computed	F10.5
$4K^2$+N*KI+27		MNCP	maximum number of computed points	I3
$4K^2$+N*KI+28		NPPP	number of computed points per period of the first excited cylinder (nothing for free vibrations)	I3

```
C                     ***********
C                     *         *
C                     * TRANSIT *
C                     *         *
C                     ***********
C
C     ************************************************************************
C
C     THIS PROGRAM CCMPUTES AND PLOTS THE RESPONSE OF PINNED-PINNED
C     CYLINDERS SUBJECTED TO AN INSTANTANEOUS OR STEADY EXCITATION.
C
C     K IS THE NUMBER OF CYLINDERS.
C     N IS THE NUMBER OF COMPARISON FUNCTIONS USED.
C     UL AND ULI ARE THE EXTERNAL AND INTERNAL FLOW VELOCITIES.
C
C     THE MATRICES ARE DIMENSIONNED AS FOLLOWS:
C
C     AM(2K,2K),VC(2K,2K),AM1(2K,2K),C(2K,2K),E1(2K,2K),G(2K,2K),H(2K,2K)
C     WK(2K),SIS(2K,2K),SIT(2K,2K),Y(4KN,4KN),WR(4KN),WI(4KN)
C     CT(4KN),AS(4K,4K),BAS(4KN,4KN),WAK(4KN),ZP(4KN,4KN),STC(901,6)
C     NCYL(K),IPL(2K),ZO(2K),YO(2K),CM(2K),XI(K)
C
C     ************************************************************************
C
      IMPLICIT REAL*8(A-H,O-Z)
      DIMENSION AM(4,4),VC(4,4),AM1(4,4),C(4,4),E1(4,4),G(4,4),H(4,4)
      DIMENSION WK(4),SIS(4,4),SIT(4,4),Y(24,24),WR(24),WI(24)
      DIMENSION CT(8),AS(8,8),BAS(8,8),WAK(8),ZP(8,8),STC(601,4)
      DIMENSION NCYL(3),IPL(6),ZO(6),YO(6),CM(6),XI(3)
      INTEGER DIGIT(10) /'1','2','3','4','5','6','7','8','9','10'/
      INTEGER EC(4)/'(1HO',',     ',',F12.','5)'/
C     READ IN NUMBER OF CYLINDERS AND NUMBER OF COMPARISON FUNCTIONS USED
      READ 100,K,N
      KK=2*K
      KS=2*KK*N
      KKN=KK*N
      KKK=4*K
C     READ IN SYSTEM PARAMETERS
      READ 101,ALPHA,BETA,BETAI,GAMMAE,GAMMAI,GAMMAM,GAMMAT,DEL,EPS,
     *PIE,PII,CL,PR,CF,CFE,CFI,CFX,R,RC
      HL=2.D0*R/((RC**2-K*R**2)*2.D0/(RC+K*R))
      PRINT 200,K,N
      PRINT 201,R,RC
      PRINT 202
      PRINT 203,BETA,GAMMAE,PIE
      PRINT 204
      PRINT 205,BETAI,GAMMAI,PII
      PRINT 206,ALPHA,GAMMAT,PR
      PRINT 207,DEL,EPS,CL,CF,CFE,CFI,CFX
      PRINT 208,HL
      GAMMA=GAMMAM-GAMMAE
C     READ ADDED MASS MATRIX AND MATRIX OF VISCOUS COUPLING
      READ 102,((AM(I,J),VC(I,J),J=1,KK),I=1,KK)
      EC(2)=DIGIT(KK)
      PRINT 299
      PRINT 224
      WRITE(6,EC)((AM(I,J),J=1,KK),I=1,KK)
      PRINT 225
      WRITE(6,EC)((VC(I,J),J=1,KK),I=1,KK)
      DO 1 I=1,KK
      DO 2 J=1,KK
    2 AM1(I,J)=BETA*AM(I,J)
    1 AM1(I,I)=AM1(I,I)+1.D0-BETA
      PRINT 226
      WRITE(6,EC)((AM1(I,J),J=1,KK),I=1,KK)
      DO 10 I=1,KK
      DO 10 J=1,KK
      SIS(I,J)=AM(I,J)
   10 SIT(I,J)=VC(I,J)
C     READ EXTERNAL AND INTERNAL VELOCITIES
      READ 102,UL,ULI
      PRINT 229,ULI,UL
      CALL SOLN(KK,KS,AM,VC,AM1,C,E1,G,H,WK,WR,WI,Y,ALPHA,BETA,BETAI,
     1GAMMA,GAMMAI,GAMMAT,PIE,PII,EPS,HL,CL,UL,ULI,CF,CFE,CFI,CFX,DEL,
     *PR,ZP)
```

```
      DC 9 LIP=1,KK
      DC 9 LID=1,KK
      AM(LIP,LID)=SIS(LIP,LID)
    9 VC(LIP,LID)=SIT(LIP,LID)
      K1=KS/2
      KR=30
      IF(KS.LE.90)K1=KS
      DC 17 J=1,KR
17    PRINT 221,(WR(I),WI(I),I=J,K1,KR)
      IF(KS.LE.90)GCTC 16
      PRINT 299
      K1=KS/2+1
      K2=K1+29
      DC 18 J=K1,K2
  18  PRINT 221,(WR(I),WI(I),I=J,KS,30)
16    CONTINUE
C
C    THE TRANSIENT RESPCNSE
C
      PRINT 299
      PI=3.14159265358980O
      DC 50 I=1,KS
50    CT(I)=0.
      KCF=0
      READ 100,KI
      DC 52 I=1,N
      READ 101,XI(I)
      DC 51 J=1,KI
      READ 104,ICYL,ZDISPL,YDISPL,CMEGA
      IP=(I-1)*KKK+ICYL
      CT(IP+KK)=ZDISPL
      CT(IP+3*K)=YDISPL
      IF(CMEGA.EQ.0) GC TO 51
      KCF=KCF+1
      IPL(KCF)=IP
      ZO(KCF)=ZDISPL
      YO(KCF)=YDISPL
      CM(KCF)=CMEGA
      IF(I.EQ.1) PRINT 700,J,CMEGA
51    CONTINUE
      TC=0.
      XP=XI(I)
      CALL ASSEMB(TO,XP,AS,K,N,KK,KS,KKN,KKK,ZP,WR,WI,FI)
      DC 53 II=1,KKK
      DC 53 JJ=1,KS
53    EAS(KKK*(I-1)+II,JJ)=AS(II,JJ)
52    CONTINUE
      PRINT 500
      PRINT 501,(CT(I),I=1,KS)
      CALL LEQTIF(EAS,1,KS,KS,CT,0,WAK,IER)
      PRINT 503
      PRINT 504,(CT(I),I=1,KS)
      KP=3
      IF(K.LT.3) KP=K
      KPP=2*KP
C    READ THE NUMBERS OF THE THREE CYLINDERS TO BE COMPUTED
      READ 103,(NCYL(I),I=1,KP)
      READ 101,TIME
      READ 105,MNCP
      IF(KCF.GT.0) GC TC 69
C
C    FREE VIBRATIONS
C
      TCSTEP=TIME/400.
      IF(TOSTEP.GT.0.05) TCSTEP=0.05
      NCP=TIME/TCSTEP+1
      IF(NCP.GT.MNCP) NCP=MNCP
      XP=XI(1)
      DC 55 IND=1,NCP
      CALL ASSEMB(TC,XP,AS,K,N,KK,KS,KKN,KKK,ZP,WR,WI,FI)
      DO 56 I=1,KKK
      WAK(I)=0.
      DC 57 J=1,KS
57    WAK(I)=WAK(I)+AS(I,J)*CT(J)
56    CONTINUE
```

```
      DC 58 IJ=1,KP
      STO(IND,IJ)=WAK(KK+NCYL(IJ))
      STO(INC,IJ+K)=WAK(KK+K+NCYL(IJ))
58    CONTINUE
      TC=TC+TCSTEP
55    CONTINUE
      GC TO 71
C
C   FORCED VIBRATIONS
C
69    CONTINUE
      READ 105,NPPF
      TOSTEP=2*PI/NPPP/CM(1)
      NCP=TIME/TCSTEP+1
      IF(NCP.GT.MNCP) NCP=MNCP
      DO 70 IND=1,NCP
      DO 72 I=1,N
      XP=XI(I)
      CALL ASSEMB(TC,XP,AS,K,N,KK,KS,KKM,KKK,ZP,WR,WI,FI)
      DC 72 II=1,KKK
      DC 72 JJ=1,KS
72    BAS(KKK*(I-1)+II,JJ)=AS(II,JJ)
      DC 73 I=1,KS
      WAK(I)=0.
      DC 73 J=1,KS
73    WAK(I)=WAK(I)+BAS(I,J)*CT(J)
      DO 74 IJ=1,KP
      MA=KK+NCYL(IJ)
      STO(IND,IJ)=WAK(MA)
74    STO(INC,IJ+K)=WAK(MA+K)
      DC 76 I=1,KS
76    CT(I)=WAK(I)
      DO 75 I=1,KS
      DC 75 J=1,KCF
      IF(IPL(J).NE.I) GC TC 75
      CT(I)=-Z0(J)*CM(J)*DSIN(OM(J)*TO)
      CT(I+K)=-Y0(J)*CM(J)*DSIN(CM(J)*TC)
      CT(I+KK)=Z0(J)*DCCS(CM(J)*TC)
      CT(I+3*K)=Y0(J)*DCCS(CM(J)*TC)
75    CONTINUE
      CALL LEQT1F(BAS,1,KS,KS,CT,0,WAK,IER)
      TC=TC+TOSTEP
70    CONTINUE
C
C   PRINT OUT
C
71    PRINT 505,XI(1)
      DC 62 I=1,NCF,2
      TC=TOSTEP*(I-1)
      PRINT 506,TO,(STC(I,J),J=1,KFP)
62    CONTINUE
C
C   PLCT
C   IF A DIGITAL PLOT IS NCT DESIREC REMCVE THE CARDS FFCM 'CALL PLCTCN'
C   TO 'CALL ENDPLT' PLUS THE 2 CCNTROL CARDS OF THE CALCOMP PLCTTER
C
      CALL PLOTCN
      CALL SYMBCL(1.,10.,0.4,11HZ-DIRECTION,0.0,11)
      CALL SYMBCL(14.,10.,0.4,11HY-DIRECTION,0.0,11)
      TTT=TIME/10.
      CALL PLOT(0.,8.,-3)
      DC 59 J=1,2
      XCR=(J-1)*13.C0
      YOR=(KP-1)*(J-1)*3.
      DC 60 I=1,KP
      CALL PLOT(XOR,YOR,-3)
      CALL AXIS(0.,0.,4HTIME,-4,10.,0.,0.0,TTT)
      CALL AXIS(0.,0.,5HDISPL,5,1.,90.,0.0,1.)
      GR=DFLCAT(NCYL(I))
      CALL SYMBCL(0.5,1.2,0.12,9HCYLINDER .0.,9)
      CALL NUMBER(1.58,1.2,0.12,GR,0.,-1)
      DC 61 II=1,NCP
      XX=(II-1)*TOSTEP*10./TIME
      YY=STO(II,I+(J-1)*KP)
      IF(YY.GT.1.5) NCP=II
      IPEN=-2
      IF(II.EQ.1)IPEN=0
      IF(II.EQ.NCP) IPEN=-24
      CALL SMOOT(XX,YY,IPEN)
```

```
61    CONTINUE
      XCR=0.
      YCR=-3.
60    CONTINUE
59    CONTINUE
      CALL ENDPLT
      STOP
98    FORMAT(////,' PINNED-PINNED CYLINDERS'/24('*'))
  100 FORMAT(I2)
  101 FORMAT(F15.6)
  102 FORMAT(2F10.5)
103   FORMAT(3I2)
104   FORMAT(I2,8X,4F10.5)
105   FORMAT(2I3)
  200 FORMAT('1THERE ARE',I3,' CYLINDERS, AND THE NUMBER OF COMPARISON F
     *UNCTIONS USED WAS',I3)
201   FORMAT('0 THE RADIUS OF EACH CYLINDER IS',F6.1,' MILLIMETRES'
     1//'   THE RADIUS OF THE SURROUNDING CHANNEL IS',F6.1,
     *' MILLIMETRES')
202   FORMAT(//'0','DIMENSIONLESS EXTERNAL FLOW QUANTITIES:'/41('*'))
203   FORMAT(/'0','FLUID DENSITY BETAE',F10.5/' GAMMAE=',F10.5/
     *' EXTERNAL PRESSURE=',F10.5)
204   FORMAT(//'0','DIMENSIONLESS INTERNAL FLOW QUANTITIES:'/41('*'))
205   FORMAT(/'0','FLUID DENSITY BETAI=',F10.5/
     *' GAMMAI=',F10.5/
     *' INTERNAL PRESSURE=',F10.5)
206   FORMAT(/////' INTERNAL DAMPING ALPHA=',F10.5/
     *' DIMENSIONLESS UNIFORM TENSION=',F10.5/
     *' POISSON RATIO=',F10.5)
207   FORMAT(//'0','DELTA=',F10.5,5X,'L/D=',F10.5,5X,'CD GROUPING=',
     *F10.5,/' CF=',F10.5,8X,'CFE=',F10.5,6X,'CFI=',F10.5,5X,'CFX=',F10.
     *5)
208   FORMAT('0','D/DH=',F10.5)
  221 FORMAT('0',5X,2F16.9,5X,2F16.9,5X,2F16.9)
  224 FORMAT('0  THE ADDED MASS MATRIX')
  225 FORMAT(///,'0  THE MATRIX OF THE VISCOUS COUPLING')
226   FORMAT(///,'0 THE MASS MATRIX')
229   FORMAT('1',T10,'U.INT=',F9.5,T30,'U.EXT=',F9.5,T70,
     *'EIGENVALUES I*OMEGA OF THE SYSTEM',/,T10,36('*'),T70,34('*')/)
  299 FORMAT('1')
500   FORMAT('0',10X,'INITIAL STATE VECTOR'/)
501   FORMAT('0',10X,F12.5)
503   FORMAT('1'//10X,'CONSTANTS OF INITIALIZATION',//)
504   FORMAT('0',10X,D15.5)
505   FORMAT('1'/T10,'DISPLACEMENTS OF THE PLOTTED CYLINDERS IN THE Z AN
     *D Y DIRECTIONS AT X=',F8.5/T10,78('*')//)
506   FORMAT('0','TIME=',F7.3,T20,6(F7.3,8X))
700   FORMAT('0',10X,'CYLINDER',I2,' IS EXCITED AT THE FREQUENCY OMEGA=
     *',F8.5//)
      END


      SUBROUTINE SOLN(KK,KS,AM,VC,AM1,C,E,G,H,WK,WR,WI,Y,ALPHA,BETA,BET
     1AI,GAMMA,GAMMAI,GAMMAT,FIE,PII,EPS,HL,CL,UL,ULI,CF,CFE,CFI,CFX,
     *DEL,PR,ZP)
      IMPLICIT REAL*8(A-H,O-Z)
      DIMENSION AM(KK,1),VC(KK,1),AM1(KK,1),C(KK,1),E(KK,1),G(KK,1)
      DIMENSION H(KK,1),WK(1),WR(1),WI(1),Y(KS,1)
      DIMENSION ZP(KS,1)
      UN=-1.D0
      PI=3.141592653589 8D0
      N=KS/2/KK
C     GENERATE MATRICES C,E,G,H,AND M
      TETA=(CFE*UL**2+CFI*ULI**2+CFX*UL*ULI)/2
      X1= 2.D0*UL*DSQRT(BETA)
      X2=UL**2
      X3=0.5D0*EPS*CF*UL**2
      X4=0.5D0*EPS*DSQRT(BETA)*(CF*UL+CL)
      X5=BETA
      X6=UN*(DEL*(GAMMAT+(FIE-PII)*(1.D0-2.D0*PR))+TETA*(1.D0-DEL)+(0.5
     *D0*EPS*CF*UL**2*(1.D0+HL)+GAMMA+GAMMAI)*(1.D0-DEL/2.D0)-ULI**2)
      X7=GAMMA+GAMMAI+0.5D0*EPS*CF*HL*UL**2
      X8=2.D0*ULI*DSQRT(BETAI)
```

```
      DO 1 I=1,KK
      DO 2 J=1,KK
      C(I,J)=X1*AM(I,J)
      E(I,J)=X2*AM(I,J)
      G(I,J)=X3*VC(I,J)
      H(I,J)=X4*VC(I,J)
    2 AM1(I,J)=X5*AM(I,J)
      E(I,I)=E(I,I)+X6
      G(I,I)=G(I,I)+X7
      C(I,I)=C(I,I)+X8
    1 AM1(I,I)=AM1(I,I)+1.D0-BETA
C     INVERT M
      CALL LINV1F(AM1,KK,KK,AM,4,WK,IER)
C     FORM PRODUCTS M INVERSE*C, M INVERSE*E, M INVERSE*H, M INVERSE*G
      CALL DMULT(KK,AM,C,AM1)
      CALL DMULT(KK,AM,E,VC)
      CALL DMULT(KK,AM,H,C)
      CALL DMULT(KK,AM,G,E)
C     FIND DIAGONAL ELEMENT OF F
      F=0.5D0*EPS*CF*UL**2*(1.D0+HL)+GAMMA+GAMMAI
C     INITIALIZE Y
      DO 3 I=1,KS
      DO 3 J=1,KS
    3 Y(I,J)=0.0D0
      KN=KK*N
C     GENERATE Y
      DO 4 I=1,N
      DO 5 J=1,N
C     PARAMETERS FOR PINNED-FINNED CYLINDERS
      X3=(PI*I)**4
      X5=ALPHA*X3
      IF(I.EQ.J) GO TO 27
      AA=2.D0*I*J*((-1.D0)**(I+J)-1.D0)/(J**2-I**2)
      BB=0.D0
      DD=4.D0*I*J**3*(1.D0-(-1.D0)**(I+J))/((J**2-I**2)**2)
      GOTO 25
   27 AA=0.0D0
      BB=UN*(PI*I)**2
      DD=BB/2.D0
   28 CONTINUE
      X4=F*DD+X3
      DO 9 II=1,KK
      IP=(I-1)*KK+II
      IX=IP+KN
      DO 10 JJ=1,KK
     JP=(I-1)*KK+JJ
     JX=JP+KN
     Y(IP,JP)=UN*(AA*AM1(II,JJ)+X5*AM(II,JJ)+C(II,JJ))
     Y(IP,JX)=UN*(X4*AM(II,JJ)+VC(II,JJ)*BB+E(II,JJ)*AA)
   10 CONTINUE
     Y(IX,IP)=1.D0
    9 CONTINUE
     GOTO 5
   25 X4=F*DD
     DO 6 II=1,KK
     IP=(I-1)*KK+II
     IX=IP+KN
     DO 7 JJ=1,KK
     JP=(J-1)*KK+JJ
     JX=JP+KN
     Y(IP,JP)=UN*AM1(II,JJ)*AA
     Y(IP,JX)=UN*(BB*VC(II,JJ)+X4*AM(II,JJ)+AA*E(II,JJ))
    7 CONTINUE
    6 CONTINUE
    5 CONTINUE
    4 CONTINUE
C    FIND EIGENVALUES OF Y
     CALL EISPAC(KS,KS,MATRIX('REAL',Y),VALUES(WR,WI),
    *VECTOR(ZP))
     RETURN
     END
```

```
      SUBROUTINE DMULT(M,A,B,C)
C        MULTIPLIES MATRICES, A*B=C
      IMPLICIT REAL*8(A-H,O-Z)
      DIMENSION A(M,1),B(M,1),C(M,1)
      DO 1 I=1,M
      DO 1 J=1,M
      C(I,J)=0.D0
      DO 2 L=1,M
    2 C(I,J)=C(I,J)+A(I,L)*B(L,J)
    1 CONTINUE
      RETURN
      END
      SUBROUTINE ASSEMB(TC,XI,AS,K,N,KK,KS,KKN,KKK,ZP,WR,WI,PI)
      IMPLICIT REAL*8(A-H,O-Z)
      DIMENSION ZP(KS,1),WR(1),WI(1),AS(KKK,1)
      DO 1 I=1,KKK
      DO 1 J=1,KS
    1 AS(I,J)=0.
      DO 2 J=1,KKN
      DC=DCOS(WI(2*J-1)*TC)
      DS=DSIN(WI(2*J-1)*TC)
      DE=DEXP(WR(2*J)*TC)
      DO 2 L=1,N
      PHI=DSIN(PI*L*XI)
      DO 3 I=1,KK
      M=KK*(L-1)+I
      AS(I,J)=AS(I,J)+2*PHI*(ZP(M,2*J-1)*DC-ZP(M,2*J)*DS)*DE
      AS(I+KK,J)=AS(I+KK,J)+2*PHI*(ZP(M+KKN,2*J-1)*DC-ZP(M+KKN,2*J)*DS)
     **DE
      AS(I,J+KKN)=AS(I,J+KKN)-2*PHI*(ZP(M,2*J)*DC+ZP(M,2*J-1)*DS)*DE
      AS(I+KK,J+KKN)=AS(I+KK,J+KKN)-2*PHI*(ZP(M+KKN,2*J)*DC+ZP(M+KKN,2*J
     *-1)*DS)*DE
    3 CONTINUE
    2 CONTINUE
      RETURN
      END
//GO.PLOTTAPE DD UNIT=TAPE8,VOL=SER=PLOTTP,LABEL=(1,SL),DISP=NEW
//GO.EISPACLB DD DSN=CFLB.SUBLIB,DISP=SHR
//GO.SYSIN  DD  *
```

APPENDIX F - RESONANCES OF ROWS OF TWO AND THREE CYLINDERS

F.1 Resonances of Two Cylinders

Consider the case of two pinned-pinned cylinders arranged as in Figure 3 and not subjected to any external or internal flows ($u_e = u_i = 0$). As the tubes are immersed in still fluid, it is reasonable to consider that they vibrate in their first axial mode only, and under these conditions the general equation of motion (3.15)

$$\underset{\sim}{M}\ddot{\underset{\sim}{p}} + \underset{\sim}{C}\dot{\underset{\sim}{p}} + \underset{\sim}{K}\underset{\sim}{p} = \underset{\sim}{0} ,$$

becomes

$$\underset{\sim}{M}\ddot{\underset{\sim}{\eta}} + \underset{\sim}{K}\underset{\sim}{\eta} = \underset{\sim}{0} , \qquad \text{(F.1)}$$

where $\underset{\sim}{M}$ is the mass matrix and $\underset{\sim}{K}$ the stiffness matrix

$$\underset{\sim}{M} = \beta_e \underset{\sim}{M}_v + (1 - \beta_e)\, \underset{\sim}{I} \quad , \quad \underset{\sim}{K} = \pi^4\, \underset{\sim}{I} \; .$$

Because of the symmetry of the system, $\underset{\sim}{M}$ may be written as

$$\underset{\sim}{M} = \begin{bmatrix} m_{11} & m_{12} & 0 & 0 \\ m_{21} & m_{22} & 0 & 0 \\ 0 & 0 & n_{11} & n_{12} \\ 0 & 0 & n_{21} & n_{22} \end{bmatrix} ,$$

and then the equations of motion (F.1) become decoupled in the two directions; we may solve for instance for motions in the z-direction, the following two equations:

$$m_{11}\,\ddot{w}_1 + m_{12}\,\ddot{w}_2 + \pi^4\,w_1 = 0 \quad , \tag{F.2}$$

$$m_{21}\,\ddot{w}_1 + m_{22}\,\ddot{w}_2 + \pi^4\,w_2 = 0 \quad , \tag{F.3}$$

Now, if we suppose that the movement of cylinder 1 is fixed, Eq. (F.3) allows for the determination of the response of cylinder 2. The two cross-sectional modes of vibration in the z directions are easily obtained as follows:

i) 1st mode $\quad \xrightarrow{\;2\;} \qquad \xrightarrow{\;1\;} \qquad w_1 = w_2$

Equation (F.3) becomes

$$(m_{22} + m_{21})\,\ddot{w}_2 + \pi^4\,w_2 = 0 \quad ,$$

the solution of which is

$$w_2 = w_2^0 \cos \omega_1 \tau \quad , \quad \text{where } \omega_1 = \sqrt{\frac{\pi^4}{m_{21} + m_{22}}} \quad .$$

ii) 2nd mode $\quad \xleftarrow{\;2\;} \qquad \xrightarrow{\;1\;} \qquad w_1 = -\,w_2$

Equation (F.3) becomes

$$(m_{22} - m_{21})\,\ddot{w}_2 + \pi^4\,w_2 = 0 \quad ,$$

which has as solution

$$w_2 = w_2^0 \cos \omega_2 \tau \quad , \quad \text{where } \omega_2 = \sqrt{\frac{\pi^4}{m_{22} - m_{21}}} \quad .$$

Here ω_1 and ω_2 are the two resonant frequencies for the system in the z-direction and they are the eigenvalues in the z-direction given by the computer program "SOLINTER".

If alternatively cylinder 1 is given a sinusoidal movement of constant amplitude -

$$w_1 = w_1^0 \cos \Omega \tau \quad , \tag{F.4}$$

equation (F.3) becomes

$$m_{22} \ddot{w}_2 + \pi^4 w_2 = m_{21} w_1^0 \Omega^2 \cos \Omega \tau \quad , \tag{F.5}$$

with the initial conditions $w_2(0) = \dot{w}_2(0) = 0$.

This differential equation may be very easily solved using the Laplace transform

$$\mathcal{L}(w_2(\tau)) = W_2(p)$$

which reduces Eq. (F.5), with the given initial conditions, to

$$(m_{22} p^2 + \pi^4) W_2 = - m_{21} w_1^0 \Omega^2 \frac{p}{p^2 + \Omega^2} \quad ,$$

or

$$W_2(p) = - \frac{m_{21}}{m_{22}} w_1^0 \Omega^2 \frac{p}{(p^2 + \Omega^2)(p^2 + \frac{\pi^4}{m_{22}})} \quad .$$

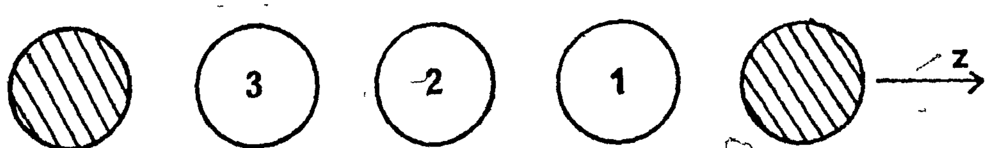
The inverse Laplace transform of this expression may be found in any table of Laplace transforms (reference [19] for instance) and takes a very simple form in the case of resonance, i.e. when $\Omega^2 = \frac{\pi^4}{m_{22}}$

$$w_2(\tau) = \frac{1}{2}\,\frac{m_{21}}{m_{22}}\,w_1^0\,\Omega\,\tau\,\sin\Omega\,\tau \quad , \quad \text{where } \Omega = \sqrt{\frac{\pi^4}{m_{22}}} \quad . \qquad (F.6)$$

Thus, the amplitude of vibrations of the second cylinder is linearly increasing.

F.2 Resonances of Three Cylinders in a Row

Consider a row of cylinders where, except for the middle three cylinders, all others are assumed to be rigid.



Under the same assumptions as in the case of a system of two cylinders, the equations of motion for cylinders 2 and 3 in the z-direction become

$$m_{21}\,\ddot{w}_1 + m_{22}\,\ddot{w}_2 + m_{23}\,\ddot{w}_3 + \pi^4\,w_2 = 0 \quad ,$$
$$m_{31}\,\ddot{w}_1 + m_{32}\,\ddot{w}_2 + m_{33}\,\ddot{w}_3 + \pi^4\,w_3 = 0 \quad . \qquad (F.7)$$

Because of the symmetry of the configuration, we have $m_{22} = m_{33} = M$ and $m_{21} = m_{23} = m_{32} = m$; furthermore, the hydrodynamic coupling between cylinders 1 and 3 can be neglected, yielding $m_{31} = 0$. Equation (F.7) may then be written as follows:

$$M \ddot{w}_2 + \pi^4 w_2 + m \ddot{w}_3 = - m \ddot{w}_1 \quad ,$$

$$m \ddot{w}_2 + M \ddot{w}_3 + \pi^4 w_3 = 0 \quad . \tag{F.8}$$

As in the previous section we consider a steady constrained motion for cylinder 1 of the form

$$w_1 = w_1^0 \cos \Omega \tau \quad ,$$

and null initial conditions for the two other cylinders:

$$w_2(0) = w_3(0) = \dot{w}_2(0) = \dot{w}_3(0) = 0 \quad .$$

Using again the Laplace transforms $W_2(p)$ and $W_3(p)$ the system (F.8) reduces to

$$(M p^2 + \pi^4) W_2 + m p^2 W_3 = m \Omega^2 \frac{p}{p^2 + \Omega^2} w_1^0 \quad ,$$

$$m p^2 W_2 + (M p^2 + \pi^4) W_3 = 0 \quad . \tag{F.9}$$

The Laplace transforms are readily obtained

$$W_2 = \frac{m \Omega^2 (p^2 + \pi^4/M) p}{M\left[(p^2 + \frac{\pi^4}{M})^2 - (\frac{m}{M} p^2)^2\right](p^2 + \Omega^2)} w_1^0 \quad ,$$

$$W_3 = \frac{-\ m^2\ \Omega^2\ p^3}{M^2\left[(p^2 + \frac{\pi^4}{M})^2 - (\frac{m}{M}\ p^2)^2\right](p^2 + \Omega^2)}\ w_1^0 \quad .$$

If we make the assumption that $\frac{m}{M} << 1$ and set

$$\Omega = \sqrt{\frac{\pi^4}{M}} \quad ,$$

first approximations for W_2 and W_3 may be obtained as follows:

$$W_2 \sim \frac{m\ \Omega^2\ p}{M(p^2 + \Omega^2)^2}\ w_1^0 \quad ,$$

$$W_3 \sim \frac{-\ m^2\ \Omega^2\ p^3}{M^2(p^2 + \Omega^2)^3}\ w_1^0 \quad , \qquad \text{(F.10)}$$

the inverse transform of which may be found in Ref. [19], i.e.

$$\frac{w_2}{w_1^0} \sim \frac{1}{2}\,\frac{m}{M}\ \Omega\ \tau\ \sin\ \Omega\ \tau \quad ,$$

$$\frac{w_3}{w_1^0} \sim -\ \frac{1}{8}\ (\frac{m}{M})^2\ \Omega\ (3\ \tau\ \sin\ \Omega\ \tau + \Omega\ \tau^2\ \cos\ \Omega\ \tau) \quad . \qquad \text{(F.11)}$$

Thus, the amplitude of the second cylinder is linearly increasing, with time while that of the third cylinder eventually becomes almost parabolic.