PERFECT GRAPHS

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ABSTRACT

This thesis is concerned with perfect graphs. Our main results can be summarized as follows.

- We characterize (by minimal forbidden induced subgraphs) two families of graphs such that for every graph G in the first (or the second) family, the Welsh-Powell (or the Matula) colouring heuristic delivers a perfect order on G. This result has been obtained jointly with V. Chvátal.

- We prove that a graph G is brittle (in the sense of Chvátal) if G does not contain an induced subgraph isomorphic to the graph C_k with $k \geq 5$, or the graph $\overline{P}_5$, or the graph with vertices a,b,c,d,e,f and edges ab,bc,cd,da, de,ef,fc. This result has been obtained jointly with N. Khouzam.

- We prove that in a Meyniel graph, each vertex belongs to a stable set that meets all maximal cliques. We also design a polynomial-time algorithm which, given a Meyniel graph G and a vertex x of G, finds a stable set which contains x and meets all maximal cliques of G.

- We find two new classes of perfect graphs: the class of alternately orientable graphs and the class of alternately colourable graphs. They contain several well-known classes of perfect graphs.

- We prove, jointly with V. Chvátal, the following theorem. If the vertices of a graph G are coloured by two colours so that each P_4 has an even number of vertices of each colour, then G is perfect if and only if each of the two subgraphs of G induced by all the vertices of the same colour is perfect.

- We prove that, as conjectured by Chvátal, a graph is perfect whenever its vertices can be coloured by two colours so that each P_4 has an odd number of vertices of each colour. We shall also present a generalization of this theorem.

RÉSUMÉ

Cette thèse traite des graphes parfaits. Nos résultats principaux peuvent être résumés comme suit.

- Nous caractérisons (par sousgraphes induits interdits) deux familles de graphes telles que, pour chaque graphe G dans la première (ou deuxième) famille, la coloration heuristique de Welsh-Powell (ou de Matula) donne un ordre parfait sur G. Ce résultat est obtenu conjointement avec V. Chvátal.

- Nous prouvons qu'un graphe G est friable, dans le sens de Chvátal, si G ne contient pas un sous graphe induit isomorphique au graphe C_k, avec $k \geq 5$, ou au graphe $\overline{P}_5$, ou au graphe avec les sommets a,b,c,d,e,f, et les arêtes ab,bc,cd,da,de,ef,fc.

- Nous prouvons que dans un graphe de Meyniel, chaque sommet appartient à un ensemble stable qui rencontre toutes les cliques maximales. Nous décrivons aussi un algorithme polynomial lequel, étant donné un graphe G de Meyniel et un sommet x de G, trouve un ensemble stable qui contient x, et rencontre toutes les cliques maximales.

- Nous trouvons deux nouvelles classes de graphes parfaits, qui contiennent quelques autres classes (de graphes parfaits) bien connues.

- Nous prouvons, conjointement avec V. Chvátal, le

théorème suivant. Si les sommets d'un graphe G peuvent être colorés, en deux couleurs, d'une manière telle que chaque P_4 a un nombre pair de sommets de chaque couleur, alors G est parfait si et seulement si chacun des deux sousgraphes, de G, induits par tous les sommets de chaque couleur est parfait.

- Nous résolvons un conjecture de Chvátal: un graphe est parfait si ses sommets peuvent être colorés, par deux couleurs, d'une manière telle que chaque P_4 a un nombre impair de sommets de chaque couleur. Nous présenterons aussi une généralisation de ce théorème.

TABLE OF CONTENTS

1. INTRODUCTION

The subject of this thesis belongs to the theory of graphs. We shall use the standard graph-theoretic terminology throughout the text; for the reader's convenience, all the terms (and their definitions) are listed alphabetically in the Appendix.

Hajnal and Surányi (1958) proved that if G is the complement of a triangulated graph (a graph which contains no chordless cycle with more than three vertices), then the stability number of G equals its clique-cover number (the smallest number of cliques that cover all vertices of G). Berge (1960) proved that if G is a triangulated graph, then its chromatic number equals its clique number. These two results inspired Berge to the notion of a perfect graph: this is a graph in which each induced subgraph has its chromatic number equal to its clique number.

Berge (1962) made two conjectures. First, the Strong Perfect Graph Conjecture states that a graph is perfect if and only if it does not contain an induced subgraph isomorphic to the odd chordless cycle with at least five vertices or to the complement of such a cycle. Second, the Weak Perfect Graph Conjecture states that a graph is perfect if and only if its complement is. Lovász (1972a) proved the Weak Perfect

Graph Conjecture. The Strong Perfect Graph Conjecture remains unsolved. Moreover, nobody has been able to design a polynomial-time algorithm to recognize perfect graphs.

Since Berge publicized his two conjectures, many classes of perfect graphs, along with polynomial-time algorithms for their recognitions, have been identified. It was pointed out that every comparability graph is perfect. König (1916) proved that in a bipartite graph, the number of edges in a largest matching equals the number of vertices in a smallest cover. This theorem implies that line-graphs of bipartite graphs are perfect. Triangulated graphs, comparability graphs and line-graphs of bipartite graphs are sometimes referred to as "classical" perfect graphs.

Chvátal (1981) introduced the notion of a "perfect order". If a graph G admits a perfect order, then a certain colouring heuristic shall always deliver an optimal colouring of G; the graph G is called a "perfectly orderable" graph. All triangulated graphs, all complements of triangulated graphs, and all comparability graphs are perfectly orderable. Chvátal (1981) showed that every perfectly orderable graph G is "strongly perfect," in the sense of Berge and Duchet : each induced subgraph H of G contains a stable set which meets all maximal cliques of H. (Throughout this text, "maximal", and "minimal", are always meant with respect to set-inclusion, not size.)

The Strong Perfect Graph Conjecture can be restated by saying that the only minimal imperfect graphs are the odd (chordless) cycles, except for triangles, and the complements of these cycles. A theorem of Lovász (1972b) states that a minimal imperfect graph G has precisely $\alpha(G).\omega(G) + 1$ vertices ($\omega(G)$ is the clique number of G, and $\alpha(G) = \omega(\bar{G})$). In section 2, we shall reproduce a proof of this result.

In section 3, we reproduce proofs of the following two results. First, Chvátal (1984) proved that no minimal graph G can contain a star-cutset (this is a cutset S containing a vertex which is adjacent to all remaining vertices of S). Second, Henry Meyniel (1984) proved that in every minimal imperfect graph, every two nonadjacent vertices must be endpoints of a chordless path with an odd number of edges. We shall also discuss a few problems related to these results. A conjecture of Chvátal states that no minimal imperfect graph can contain a "skew partition". We shall make a few observations on it. Meyniel defined a graph G to be a "quasi-parity" graph if each induced subgraph H of G is a clique, or contains two vertices which are not endpoints of any chordless path with an odd number of edges. We shall show that the three well-known perfection-preserving operations clique identification, substitution, and amalgam preserve also the property of " being a quasi-parity graph".

In section 4, we present previously known results on triangulated graphs, comparability graphs, line-graphs of bipartite graphs, P_4-free graphs and P_4-sparse graphs. P_4-sparse graphs are graphs in which no two P_4's can share three common vertices. We shall also obtain a new result on P_4-sparse graphs.

In section 5, we present Chvátal's results on perfectly orderable graphs.

The results in section 6 were obtained jointly with V. Chvátal. We characterize (by minimal forbidden induced subgraphs) two families of graphs such that, for every graph G in the first (or the second) family, the Welsh-Powell (or the Matula) colouring heuristic delivers a perfect order on G.

In section 7, we study "brittle" graphs. A graph G is brittle if each induced subgraph H of G contains a vertex which is neither an endpoint nor a midpoint of any P_4 in H. It is easy to see that every brittle graph is perfectly orderable. We shall prove, jointly with N. Khouzam, that a graph is brittle if it does not contain an induced subgraph isomorphic to the chordless cycle with at least five vertices, or the complement of the chordless path with five vertices, or the graph with vertices a,b,c,d,e,f, and edges ab,bc,cd,da,de,ef,fc.

In section 8, we study Meyniel graphs. A graph G is Meyniel of each of its odd cycles (with at least five vertices) contains two chords; G is called a Meyniel graph because it was Meyniel (1976) who established perfection of G. Ravindra (1982) proved that Meyniel graphs are strongly perfect. We shall show that each Meyniel graph G has a stronger property: each vertex of G belongs to a stable set that meets all maximal cliques of G. Furthermore, if a graph is not Meyniel, then it contains an induced subgraph which fails to have this property.

In section 9, we introduce "alternately orientable" graphs and "alternately colourable" graphs. A graph is alternately orientable if it admits an orientation of its edges such that no chordless cycle with at least four vertices contains an induced subgraph with vertices a,b,c, and directed edges ab,bc. A graph is alternately colourable if it admits a colouration of its edges by two colours in such a way that no chordless cycle C with at least four vertices contains a chordless path with three vertices, whose two edges are of the same colour. We shall establish perfection for alternately orientable graphs and for alternately colourable graphs. In addition, we shall prove that a graph G is alternately orientable if each odd cycle (with at least five vertices) contains two non-crossing chords, or if G is a comparability graph, or a

P_4-sparse graph, or a union of two threshold graphs. We shall also prove that a graph G is alternately colourable if G is triangulated or a line-graph of a bipartite graph. Finally, we shall present a polynomial-time algorithm to recognize alternately colourable graphs and alternately orientable graphs.

In section 10, we prove, jointly with Chvátal, the following theorem. If the vertices of a graph G are coloured by two colours in such a way that each P_4 has an even number of vertices of each colour, then G is perfect if and only if each of the subgraphs of G induced by all the vertices of the same colour is perfect. Our theorem implies that a graph is perfect whenever its vertices can be coloured by two colours such that each P_4 has two vertices of each colour.

In section 11, we prove the following theorem. Let the vertices of a graph G be coloured by two colours in such a way that (i) each P_4 is monochromatic (the vertices are of the same colour), or (ii) each P_4 has an odd number of vertices of each colour, and among the three vertices of the same colour of this P_4, at least one vertex does not belong to a monochromatic P_4. Then G is perfect if and only if each of the two subgraphs of G induced by all the vertices of the same colour is perfect. Our theorem implies, as conjectured by Chvátal, that a graph is perfect whenever its vertices can be coloured by two colours such that each P_4 has an odd number of vertices of each colour.

2. PERFECT GRAPHS

The colouring (of vertices) of a graph is an assignment of "colours" to its vertices such that every two adjacent vertices always have different colours. The chromatic number of a graph is the smallest number of colours that suffice to colour it. A graph is called a clique if its vertices are pairwise adjacent. The clique number of a graph is the size of the largest clique in this graph. We denote the chromatic number and the clique number of a graph G by $\chi(G)$ and $\omega(G)$, respectively.

The chromatic number of a graph is at least its clique number, since every two adjacent vertices must receive different colours. Berge (1962) defined a perfect graph as a graph in which every induced subgraph H has $\chi(H) = \omega(H)$. At present, no polynominal-time algorithm to recognize perfect graphs is known, although several large classes of perfect graphs, with polynomial-time recognition algorithms, have been found (see Golumbic (1980) and Berge and Chvátal (1984)).

We define a cycle as a sequence of distinct vertices $v_1, v_2, \ldots, v_k$ with the following properties: $v_i v_{i+1}$ is an edge for $i=1,\ldots,k-1$, and $v_1 v_k$ is an edge. A chord in a cycle $v_1, v_2, \ldots v_k$ is an edge $v_i v_j$ other than $v_i v_{i+1}$ $(1 \le i \le k-1)$ or $v_1 v_k$. A chordless cycle is said to have length k if it consists of k vertices (and k edges). We denote such a cycle by C_k. The complement $\overline{G}$ of a graph $G=(V,E)$ is the graph (V,E^*) such that $uv \in E^*$ if and only if $uv \notin E$ for any vertices u, v in V. We denote the largest number of pairwise nonadjacent vertices in G by $\alpha(G)$. Note that $\alpha(G) = \omega(\overline{G})$ and $\chi(G) \geq \frac{|V|}{\alpha(G)}$ for any graph $G = (V,E)$.

Consider a graph C_{2k+1}, $k \geq 2$. We have $\omega(C_{2k+1}) = 2$, and it is easy to see that $\chi(C_{2k+1}) = 3$. Let S be the largest set of pairwise nonadjacent vertices in C_{2k+1} so that $|S| = \alpha(C_{2k+1})$. We note that $|S| < k+1$, because each vertex x in S must be followed (in cyclic order) by a vertex x' not in S; thus, $\omega(\overline{C}_{2k+1}) = k$.

$$\text{But, } \chi(\overline{C}_{2k+1}) \geq \frac{|V|}{\alpha(\overline{C}_{2k+1})} = \frac{2k+1}{2} > k.$$

We have $\omega(\overline{C}_{2k+1}) = k$, and $(\overline{C}_{2k+1}) = k+1$. Both C_{2k+1} and $\overline{C}_{2k+1}$ are imperfect. A graph is <u>minimal imperfect</u> if it is not perfect, but each of its induced subgraphs is perfect. It is easy to see that both C_{2k+1} and $\overline{C}_{2k+1}$ are minimal imperfect.

1. The Strong Perfect Graph Conjecture (Berge (1962))
   The only minimal imperfect graphs are C_{2k+1} and $\overline{C}_{2k+1}$, $k \geq 2$.

2. The Weak Perfect Graph Conjecture (Berge (1962))
   If a graph G is perfect, then its complement $\overline{G}$ is perfect.

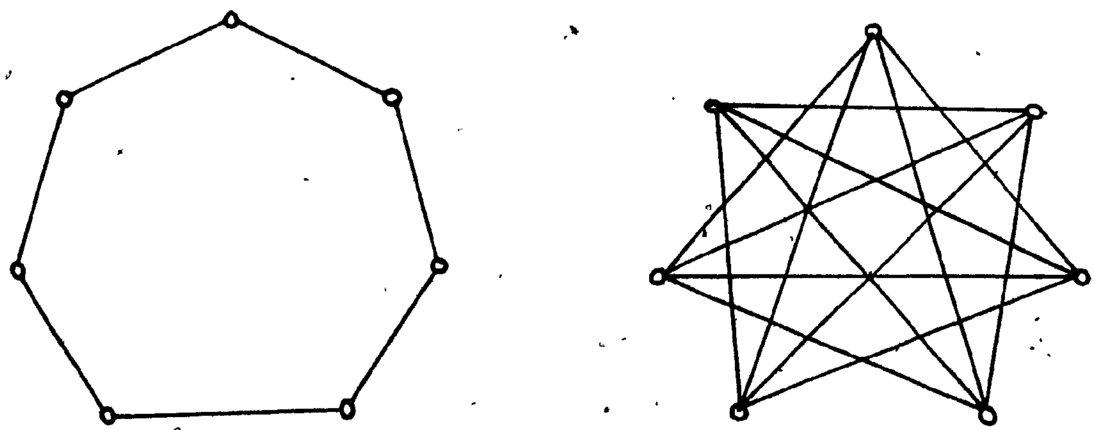


Figure 2.1: the graphs C_7 and $\overline{C}_7$.

The second conjecture was proved by Lovász (1972a). Nowadays, it is called the Perfect Graph Theorem. To see that the Strong Perfect Graph Conjecture implies the Perfect Graph Theorem, consider a perfect graph G. Trivially, G has no induced C_{2k+1} or $\bar{C}_{2k+1}$. Thus, $\bar{G}$ also has no C_{2k+1} or $\bar{C}_{2k+1}$. Now, the Strong Perfect Graph Conjecture implies that $\bar{G}$ is perfect.

We define a path as a sequence of distinct vertices $v_1, v_2, \ldots v_k$ such that $v_i v_{i+1}$ is an edge. A chord of a path $v_1, v_2, \ldots v_k$ is an edge $v_i v_j$ other than $v_i v_{i+1}$. By P_k we denote the chordless path with k vertices. Thus, P_4 is the chordless path with four vertices. It is easy to see that the complement of a P_4 is (isomorphic to) a P_4.

Figure 2.2: a P_4 and its complement.

A graph $G_1 = (V_1, E_1)$ is said to have the <u>P_4-structure</u> of a graph $G_2 = (V_2, E_2)$ if there is a bijection $f: V_1 \rightarrow V_2$ such that a subset S of V_1 induced a P_4 in G_1 if and only if f(S) induces a P_4 in G_2.

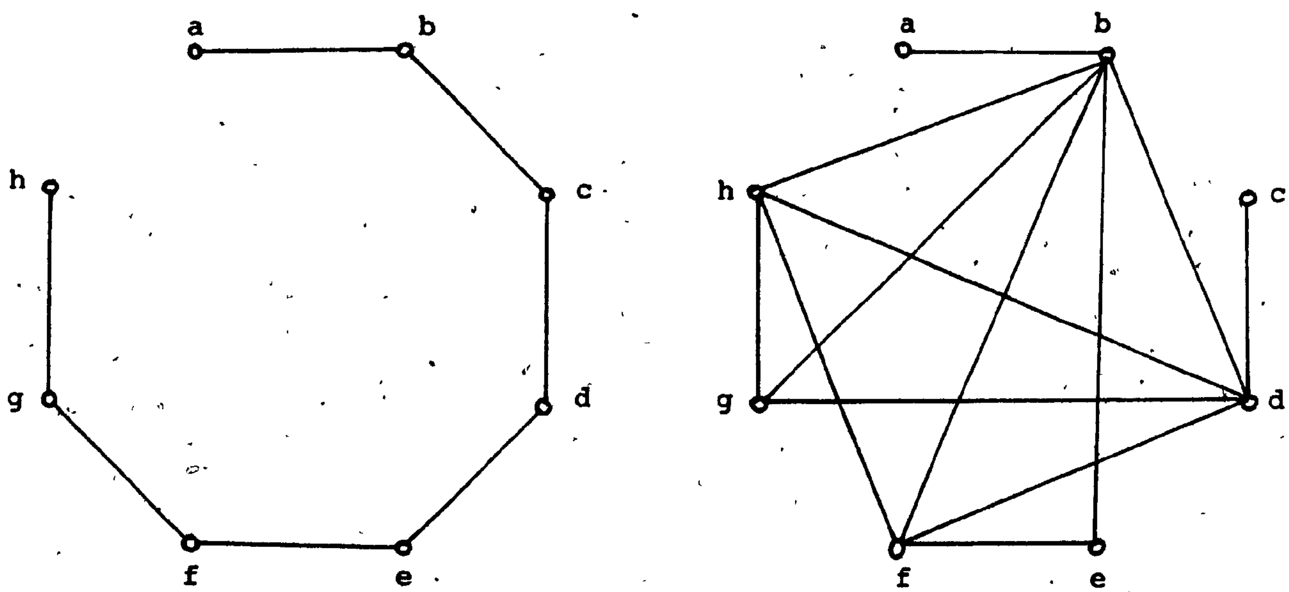


Figure 2.3: two graphs with the same P_4-structure (taken from Chvátal (1982))

Chvátal (1982) introduced the notion of P_4-structure and noted that, since a P_4 is self-complementary,

(i) every graph has the P_4-structure of its complement.

In addition, he proved that

(ii) the only graphs having the P_4-structure of a C_{2k+1} with $k \geq 2$ are C_{2k+1} itself and its complement.

3. The Semi-Strong Perfect Graph Conjecture (Chvátal (1982))

If a graph G has the P_4-structure of a perfect graph, then G is perfect. Note that, by (i) the Semi-Strong Perfect Graph Conjecture implies the Perfect Graph Theorem, and by (ii), the Semi-Strong Perfect Graph Conjecture is implied by the Strong Perfect Graph Conjecture.

Chvátal (1984) proved that no minimal imperfect graph can contain a <u>star-cutset</u> (that is a set S of vertices of G such that G - S is disconnected and some vertex x in S is adjacent to all other vertices of S).

Ryan Hayward (1984) proved that if a graph G does not contain an induced subgraph isomorphic to a chordless cycle with at least five vertices, or to its complement, then G or $\bar{G}$ contains a star-cutset.

Recently, Bruce Reed (1985) used Chvátal's result, Hayward's result and Lovász's Perfect Graph Theorem to prove the Semi-Strong Perfect Graph Conjecture. Actually, he proved the following. (A proper endomorphism of a graph G = (V,E) is a mapping f of the set V into itself such that f(u) and f(v) are adjacent whenever u and v are, and such that the image of V is a proper subset of V.)

The Semi-Strong Perfect Graph Theorem (Reed (1985))

Let G and H be two graphs with the same P_4-structure. Then at least one of the following conditions holds:

(i) H is the complement of G;

(ii) H or $\bar{H}$ contains a star-cutset;

(iii) H or $\bar{H}$ has a proper endomorphism;

(iv) H contains a proper induced subgraph isomorphic to C_5. □

Together with the Semi-Strong Perfect Graph Conjecture, Chvátal made the following two conjectures.

Conjecture A. If the vertices of a graph G are coloured by two colours so that each colour appears at least once, and each P_4 has an even number of vertices of each colour, then G is perfect if and only if each of the two subgraphs induced by all the vertices of each colour is perfect.

Conjecture B. If the vertices of a graph G are coloured by two colours so that each P_4 contains an odd number of vertices of each colour, then G is perfect.

In section 10, we shall present a proof, obtained jointly with Chvátal, of Conjecture A. In section 11, we shall present a proof of Conjecture B. Since both proofs rely on the Perfect Graph Theorem, in the remainder of this section we shall reproduce a proof of this fundamental and important theorem. First, we need introduce a few definitions.

Let $G = (V,E)$ be a graph. A set S of vertices of G is a <u>stable</u> set if no two vertices of S are adjacent. The <u>stability number</u> $\alpha(G)$ of G is the largest number of vertices in a stable set of G. The <u>clique-cover number</u> $\theta(G)$ is the smallest number of cliques needed to cover the vertices of G. If A is a subset of V, then $[A]_G$ will denote the subgraph of G induced by A. When there can be no confusion, we shall write

$$\omega(A) = \omega([A]_G),\ \chi(A) = \chi([A]_G),\quad \alpha(A) = \alpha([A]_G) \text{ and}$$

$$\theta(A) = \theta([A]_G).$$

Let $h_1, h_2, \ldots, h_n$ be a vector of non-negative integers. By $G \circ h$, we denote the graph obtained from G by substituting for each x_i a stable set of h_i vertices $x_i^1, \ldots, x_i^{h_i}$ and joining x_i^s with x_j^t if and only if x_i and x_j are adjacent in G. We say that $G \circ h$ is obtained from G by multiplication of vertices.

Equivalence of (P_1) and (P_2) in the following theorem was established by Lovász (1972a); equivalence of (P_1) and (P_3) was established by Lovász (1972b).

Theorem 2.1 (Lovász (1972b))

For each graph $G = (V,E)$, the following statements are equivalent:

(P_1) $\omega(A) = \chi(A)$ (for all $A \subseteq V$),

(P_2) $\alpha(A) = \theta(A)$ (for all $A \subseteq V$),

(P_3) $\omega(A)\alpha(A) \geq |A|$ (for all $A \subseteq V$).

We shall use the following two lemmas.

Lemma 2.1 (Lovász (1972b))

Let H be obtained from a graph G by multiplication of vertices and let G satisfy (P_2). Then H satisfies (P_2).

Proof

By induction on the number of vertices, let G be a graph and let H be the graph obtained from G by multiplication of vertices. By the induction hypothesis, we may assume that each vertex of G is multiplied at least once, and some vertex of G is multiplied at least twice (for otherwise we are done). Since H can be built up from a sequence of smaller multiplications, it suffices to prove the statement for $H = G \circ x$. Let x' be the copy of x in G.

Assume that G satisfies (P_2). We want to show that $\alpha(G \circ x) = \theta(G \circ x)$. Let $\underline{K}$ be the clique cover of G with $|\underline{K}| = \theta(G) = \alpha(G)$, and let K_x be the clique of $\underline{K}$ containing x.

<u>Case 1:</u> $\alpha(G \circ x) = \alpha(G) + 1$.

The collection of cliques $\underline{K} \cup \{x'\}$ covers all vertices of Gox. Thus $\theta(G \circ x) \leq |\underline{K} \cup \{x'\}| = \alpha(G) + 1 = \alpha(G \circ x)$. Since for any graph F, we have $\theta(F) \geq \alpha(F)$, it follows that

$$\theta(G \circ x) = \alpha(G \circ x).$$

<u>Case 2:</u> $\alpha(G \circ x) = \alpha(G)$.

In this case, no largest stable set of G contains x. Thus the clique $D = K_x - x$ intersects each maximum clique exactly once, so

$$\alpha(G - D) = \alpha(G) - 1.$$

The vertices of the graph G–D can be covered by a collection $\underline{K}'$ of $\alpha(G) - 1$ cliques. Now, $\underline{K}'$ together with the clique $D \cup \{x'\}$ covers Gox, that is

$$\theta(G \circ x) = \alpha(G) = \alpha(G \circ x). \quad \Box$$

<u>Lemma 2.2 (Lovász (1972b)</u>

Let G be a graph such that each proper induced subgraph of G satisfies (P_2). Let H be obtained from G by multiplication of vertices. If G satisfies (P_3), then H satisfies (P_3).

Proof

By induction on the number of vertices, we can assume that H fails to satisfy (P_3) but each proper induced subgraph of H satisfies (P_3). Thus, with X denoting the set of vertices of H, we have

$$\omega(H)\alpha(H) < |X|. \tag{2.1}$$

We may assume that each vertex of G was multiplied at least once, and some vertex u was multipled at least twice. Let $U = \{u^1, u^2, \ldots u^h\}$ be the vertices of H corresponding to u. By the induction hypothesis the graph $H - u^1$ satisfies (P_3), thus we have

$$\begin{aligned} |X| - 1 = |X-u^1| &\leq \omega(X-u^1)\alpha(X-u^1) \\ &\leq \omega(H)\alpha(H) \\ &\leq |X| - 1 \qquad \text{[by (2.1)]} \end{aligned}$$

Thus, equality holds throughout, write

$$\begin{aligned} p &= \omega(X-u^1) = \omega(H) \\ q &= \alpha(X-u^1) = \alpha(H) \\ pq &= |X| - 1 \end{aligned} \tag{2.2}$$

Since G−u satisfies (P_2), H−U satisfies (P_2) by Lemma 2,1. Thus, H−U can be covered by a set of q cliques of H, say $K_1, K_2, \ldots, K_q$. We can choose the clique cover of H-U so that the K_i are pairwise disjoint and $K_1 \geq K_2 \geq \ldots \geq K_q$. We have

$$\sum_{i=1}^{q} K_i = X\text{-}U = X - h = pq - (h-1).$$

Since $|K_i| \leq p$, at most h-1 of the K_i fail to contribute to the sum. Hence,

$$|K_1| = |K_2| = \ldots = |K_{q-h+1}| = p.$$

Let H' be the subgraph of H induced by X' = $K_1 \cup \ldots \cup K_{q-h+1} \cup \{u^1\}$. Thus

$$|X'| = p(q-h+1) + 1 < pq + 1 = |X| \tag{2.3}$$

so, by the induction hypothesis, H' satisfies (P_3), thus

$$\omega(H')\alpha(H') \geq |X'|. \tag{2.4}$$

Since $p = \omega(H) \geq \omega(H')$, we have

$$\alpha(H') \geq \frac{|X'|}{p}$$

$$> q - h + 1.$$

Let S' be a stable set of H' of cardinality q-h+2.

Since $|S' \cap K_i| = 1$ (for $i = 1, q - h + 1$); we have $u^1 \varepsilon S'$. But then $S = S' \cup U$ is a stable set of H with $q + 1$ vertices, contradicting our choice of q. □

Proof of Theorem 2.1

We can assume that the statement is true for each proper induced subgraph of G.

$(P_1) \Rightarrow (P_3)$. Suppose that we can colour each $[A]_G$ by $\omega(A)$ colours. Since at most $\alpha(A)$ vertices can receive the same colour, it follows that $\omega(A)\alpha(A) \geq |A|$.

$(P_3) \Rightarrow (P_1)$. Suppose that $G = (V,E)$ satisfies (P_3). We only need show that $\omega(G) = \chi(G)$.

Suppose that there is a stable set S such that $\omega(G-S) < \omega(G)$. Thus we can colour G-S with $\omega(G) - 1$ colours, and assign a new colour to the vertices of S. This gives $\omega(G) = \chi(G)$.

Now, we can assume that for each stable set S, G-S contains a clique K(S) with $|K(S)| = \omega(G)$. Let $\underline{S}$ be the collection of all stable sets of G. Now, for each $x_i \varepsilon V$, let h_i denote the number of cliques K(S) which contains x_i. Let $H = (X,F)$ be obtained from G by multiplying each x_i by h_i. By Lemma 2.2, we have

$$\omega(H)\alpha(H) \geq |X|$$

But by our choice of H, we have

$$|X| = \omega(G)\,|\underline{S}|$$

$$\omega(H) \leq \omega(G)$$

$$\begin{aligned}\alpha(H) &= \max_{T\in\underline{S}} \sum_{x_i\in T} h_i \\ &= \max_{T\in\underline{S}} \Big[\sum_{S\in\underline{S}} |T \cap K(S)|\Big] \\ &\leq |\underline{S}| - 1 .\end{aligned}$$

which together imply that

$$\omega(H)\alpha(H) \leq \omega(G)(|\underline{S}| - 1) < |X|$$

a contradiction.

$(P_2) \Longrightarrow (P_3)$. Note that G satisfies (P_3) if and only if $\bar{G}$ satisfies (P_3). Therefore;

$$\begin{aligned}G \text{ satisfies } (P_2) &\Longleftrightarrow \bar{G} \text{ satisfies } (P_1) \\ &\Longleftrightarrow \bar{G} \text{ satisfies } (P_3) \\ &\Longleftrightarrow G \text{ satisfies } (P_3). \;\square\end{aligned}$$

To see that Theorem 2.1 implies the Weak Perfect Graph Conjecture, consider a perfect graph G. (That is G satisfies (P_1). By the Theorem 2.1, G satisfies (P_3), so $\bar{G}$ satisfies

(P_3), and so $\overline{G}$ satisfies (P_1), that is $\overline{G}$ is perfect.

Maybe a historical note should be made here. Fulkerson (1971) independently from Lovász, proved that if the statement of Lemma 2.1 holds (as we know by now, it does), then the Weak Perfect Graph Conjecture holds. Fulkerson's approach, different from Lovász's, relies on the techniques of linear programming.

The Perfect Graph Theorem has enabled Grötschel, Lovász and Schrijver (1982) to design a polynomial-time algorithm which determines the four parameters $\omega(G)$, $\chi(G)$, $\alpha(G)$, $\theta(G)$ of a given perfect graph G. (This algorithm uses the ellipsoid method (see Khachian (1979), Gács and Lovász (1981)), and it does not provide insight to the combinatorial structure of perfect graphs.) It is widely believed that no polynomial-time algorithms exist for determining these four parameters of an arbitrary graph (see Garey and Johnson (1979)). In fact, Cook (1971) proved that the problem of determining whether a prescribed graph has a clique of a prescribed size is NP-complete. At present, no polynomial-time algorithm to solve an NP-complete problem is known.

The Strong Perfect Graph Theorem implies that

there is a good characterization (in the sense of Edmonds) of imperfect graphs. (2.5)

Theorem 2.1 implies that

every minimal imperfect graph G has precisely $\alpha(G).\omega(G) + 1$ vertices. (2.6)

To see that (2.6) implies (2.5), let us call a graph G <u>partitionable</u> if there are integers r, s greater than one such that

(i) G has precisely rs + 1 vertices,

(ii) for each vertex v of G, the vertices of G-v can be partitioned into r disjoint cliques of size s, and into s disjoint stable sets of size r.

Bland, Huang and Trotter (1979) observed that a graph is imperfect if and only if it contains an induced partitionable subgraph. (The "only if" part follows from (2.6). To see the "if" part, first note that a perfect graph H has at most $\alpha(H).\omega(H)$ vertices; now (ii) along with $r,s \geq 2$ implies that $\alpha(G) = r$, $\omega(G) = s$; and so G must be imperfect.) J. Edmonds and K. Cameron (see Cameron (1982)) pointed out that this observation implies (2.5), since conditions (i), (ii) can be verified in polynomial time.

Padberg (1974) proved that if a graph G, with $\alpha = \alpha(G)$ and $\omega = \omega(G)$, is minimal imperfect, then G satisfies the following two properties.

(iii) Each vertex of G is contained in precisely α stable sets of size α, and ω cliques of size ω.

(iv) Each stable set of size α is disjoint from precisely one clique of size ω, and each clique of size ω is disjoint from precisely one stable set of size α.

We shall call a graph <u>(α,ω)-graph</u> if it satisfies (i), (ii), (iii), and (iv). An $(\alpha-\omega)$-graph G is <u>normalized</u> if each edge of G has two endpoints in the same clique of size $\omega(G)$. It is easy to see that every (α,ω)-graph G contains a unique normalized $(\alpha-\omega)$-subgraph H. Examples of normalized (α,ω)-graphs are the graphs $C^{\omega-1}_{\alpha\omega+1}$ with vertices $v_1, v_2, \ldots v_{\alpha\omega+1}$ such that v_i, v_j are adjacent if and only if $|i-j| < \omega$ (as usual, the subscripts are taken modulo $\alpha\omega+1$). Bland, Huang and Trotter (1979), and Chvátal, Graham, Perold and Whitesides (1979) independently found a normalized (3,3)-graph different from C^2_{10}, and a normalized (4,3)-graph different from C^2_{13}. Chvátal, Graham, Perold and Whitesides also presented two methods for constructing infinite families of normalized (α,ω)-graphs. Whitesides (1982) constructed a list of all normalized (4,3)-graphs.

Chvátal (1984) noted that every minimal imperfect graph G must contain

(v) no sets of $\alpha(G) + \omega(G) - 1$ vertices which meets all largest stable sets of G and all largest cliques of G.

(Otherwise, G would contain an induced subgraph H with $(\alpha(G)-1)\cdot(\omega(G)-1)+1$ vertices and $\alpha(H) = \alpha(G) - 1$, $\omega(H) = \omega(G) - 1$.) Chvátal (1976) showed that no $C^{\omega-1}_{\alpha\omega+1}$ with $\alpha > 2$, $\omega > 2$ satisfies (v). However, the following (α,ω)-graph G, constructed by Chvátal, Graham, Perold, and Whitesides, does satisfy (v): G is a (4,4)-graph with vertices $v_1, v_2, \ldots, v_{17}$ such that v_i and v_j are adjacent if and only if $i - j =$ 2,6,7,8,9,10,11 or 15 mod 17.

3. MINIMAL IMPERFECT GRAPHS

In the previous section, we stated that no minimal imperfect graph can contain a star-cutset. In this section, we shall present a proof, due to Chvátal, of this statement. A related conjecture, also due to Chvátal, states that no minimal imperfect graph can contain a "skew partition" (this definition will be given later). Presently, this conjecture is still unresolved. We shall make a few observations on it. Finally, we shall discuss a new result, established by Henry Meyniel, that in a minimal imperfect graph, every two non-adjacent vertices are endpoints of a chordless path with an odd number of edges.

3.1 Star-Cutsets

Recall that a star-cutset of a graph G is a set C of vertices such that G−C is disconnected, and in C there is a vertex adjacent to all other vertices of C.

Theorem 3.1.1 (Chvátal (1984))

No minimal imperfect graph can contain a star-cutset.

Proof

Consider a graph G = (V,E) with a star-cutset C and assume that all proper induced subgraphs of G are perfect;

we only need colour G by k colours with k standing for the clique number of G. Since C is a cutset, we can partition the vertices of V into nonempty disjoint subsets V_1, V_2 such that

$$\text{no vertex in } V_1 \text{ is adjacent to a vertex in } V_2. \quad (3.1)$$

Let G_i $(i = 1,2)$ be the subgraph of G induced by $V_i \cup C$; there is a colouring f_i of G_i by k colours. Since C is a star-cutset, some vertex w in C is adjacent to all other vertices of C; write $v \in S_i$ if $v \in G_i$ and $f_i(v) = f_i(w)$.

Trivially, no two vertices in S_i are adjacent, and $S_i \cap C = \{w\}$, now (3.1) implies that no two vertices in $S = S_1 \cup S_2$ are adjacent. Since $G_1 - S_1$ and $G_2 - V_2$ are coloured by k - 1 colours, neither of these two graphs contains k pairwise adjacent vertices; now (3.1) implies that G - S does not contain a clique of size k. Thus, G - S can be coloured by k - 1 colours; an additional k-th colour may be assigned to all the vertices in S. □

Chvátal has noted that Theorem 3.1.1 implies several well known results on perfection-preserving operations. First, let G_1 and G_2 be two disjoint graphs and let C_i be a clique of G_i with $|C_1| = |C_2| \geq 1$. The graph G, obtained from G_1 and G_2 by choosing a bijection $f: C_1 \to C_2$ and identifying each x in C_1 with f(x) in C_2, is said to arise from G_1 and G_2 by <u>clique identification</u>.

Second, let G_1 and G_2 be two disjoint graphs, and let v be a vertex of G_1. The graph G obtained from $G_1 - v$ and G_2 by joining each vertex in G_2 by an edge to each neighbour of v in $G_1 - v$ is said to arise from G_1 and G_2 by <u>substitution</u>.

Third, let G_1 and G_2 be two disjoint graphs. Let v_i be a vertex of each G_i, let N_i be set of all neighbours of v_i, and let C_i be a subset of N_i such that each vertex in C_i is adjacent to all vertices in N_i and such that $|C_1| = |C_2|$ (note that we can choose C_i to be empty). By an <u>amalgam</u> of G_1 and G_2, we denote the graph G obtained from $G_1 - v_1$ and $G_2 - v_2$ by choosing a bijection $f: C_1 \rightarrow C_2$, identifying each x in C_1 with $f(x)$ in C_2, and joining each vertex in $N_1 - C_1$ by an edge to each vertex in $N_2 - C_2$.

<u>Corollary 3.1.1</u>

If a graph G is obtained from two perfect graphs G_1 and G_2 by clique identification, then G is perfect.

<u>Corollary 3.1.2 (Lovász (1972b))</u>

If a graph G is obtained from two perfect graphs G_1 and G_2 by substitution, then G is perfect.

Corollary 3.1.3 (Burlet and Fonlupt (1982))

If a graph G is the amalgam of two perfect graphs G_1 and G_2, then G is perfect.

Chvátal (1984) showed that the three above corollaries are implied by Theorem 3.1.1 in the following manner. In each of the three cases, assume that G contains a minimal imperfect graph F. Now it is easy to see that F must contain a star-cutset or $\overline{F}$ is disconnected or F has at most two vertices (no minimal imperfect graph can be disconnected). To elaborate on this, we need introduce a few definitions.

Let G = (V,E) be a graph. A cutset of G is a set C of vertices such that G - C is disconnected. A clique cutset of G is a cutset which induces a clique in G. If X is a set of vertices of G, then $N_G(X)$ denotes the set of all vertices y outside X, such that y is adjacent to some vertex of X. (X can consist of a single vertex x in which case we shall write $N_G(x)$ to denote the set of neighbours of x). $\overline{N}_G(X)$ denotes the set of all vertices in $V - (X \cup N_G(X))$. When there can be no confusion, we shall drop the subscript G, and write N(X), $\overline{N}(X)$. A set H of vertices of G is a homogeneous set if $2 \leq |H| < |V|$, and for each x not in H we have either $H \subseteq N(x)$ or $H \cap N(x) = \emptyset$.

If a graph G contains a clique cutset, then G contains a star-cutset. Thus, Theorem 3.1.1 implies that

no minimal imperfect graph can contain a
clique cutset. (3.2)

If a graph G is the amalgam of two graphs G_1 and G_2 such that G has more vertices than both G_1 and G_2, then we say that G has a <u>proper amalgam decomposition</u>. It is easy to see that if a graph G has a homogeneous set or a proper amalgam decomposition, then either G contains a star-cutset, or else $\overline{G}$ is disconnected. Thus, Theorem 3.1.1 implies that

no minimal imperfect graph can contain a
homogeneous set, (3.3)

and that

no minimal imperfect graph can contain a
proper amalgam decomposition. (3.4)

Now if a graph G satisfies the hypothesis of Corollary 3.1.1, then either G is isomorphic to G_1 or G_2, or else G contains a clique cutset. By (3.2), G must be perfect.

If a graph G is obtained by substituting, for a vertex x of G_1, a graph $G_2 = (V_2,E_2)$, then V_2 is a homogeneous set of G. Now, (3.3) implies Corollary 3.1.2.

Finally, it is easy to see that (3.4) implies Corollary 3.1.3.

Incidentally, note that

no minimal imperfect graph G can contain two
vertices x,y with $N(x) \supseteq N(y) \cup \{y\}$. (3.5)

(It suffices to show that either G contains a star-cutset or else $\overline{G}$ is disconnected.)

In latter sections, we shall use Theorem 3.1.1 and properties (3.2), (3.3), (3.5) to generate new classes of perfect graphs.

3.2 A Lemma

Lemma 3.2.1

Let G be a minimal imperfect graph with two disjoint nonempty sets W_1, W_2 of vertices such that no vertex in W_1 is adjacent to a vertex in W_2. Then $\omega(G-W_1) = \omega(G-W_2) = \omega(G)$.

Proof

Assume the contrary: without loss of generality, $\omega(G-W_2) < \omega(G)$. Since G is minimal imperfect, $G-W_2$ is $\omega(G)$-colourable; hence $G-W_2$ contains a stable set S such that $\omega((G-W_2) - S) < \omega(G)$. Since each clique G is fully contained in $G-W_1$ or $G-W_2$, it follows that $\omega(G-S) < \omega(G)$. Next, since G is minimal imperfect, G-S is colourable by $\omega(G)-1$ colours. But then G is $\omega(G)$-colourable, a contradiction. □

3.3 The Skew Partition Conjecture

A graph $G = (V,E)$ is said to have a skew partition if V can be partitioned into four disjoint and nonempty sets V_1, V_2, V_3, V_4 such that

(i) $xy \in E$ whenever $x \in V_1$, $y \in V_2$, and

(ii) $xy \notin E$ whenever $x \in V_3$, $y \in V_4$.

If a graph, with at least five vertices and at least one edge, contains a star-cutset, then it contains a skew partition. (Consider the graph H with at least three vertices and no edge; and consider the graph F with four vertices a,b,c,d and two edges ab,cd. Both H and F contain a star-cutset but not a skew partition.) Thus, the following conjecture of Chvátal implies Theorem 3.1.1.

The Skew Partition Conjecture (Chvátal (1984))

No minimal imperfect graph contains a skew partition.

At present, the Skew Partition Conjecture is unsolved. Furthermore, no one has been able to design a polynomial-time algorithm to recognize the presence of a skew partition in a graph. In this section, we make a few observations concerning this conjecture.

Let G = (V,E) be a graph. Let C be a colouring of G, and let S be a subset of V. By C(S) we shall denote the set of colours of C that appear in S.

Theorem 3.3.1

Let G be a graph with a skew partition V_1,V_2,V_3,V_4, let C_1 be an optimal colouring of $G-V_4$, and let C_2 be an optimal colouring of $G-V_3$. If $|C_1(V_1)| \geq |C_2(V_1)|$ and $|C_1(V_2)| \geq |C_2(V_2)|$ then G is not minimal imperfect.

Proof

Let A consist of all the vertices x in $G-V_3$ such that $C_2(x) \neq C_2(y)$ whenever $y \in V_1 \cup V_2$. Since both C_1 and C_2 use $\omega(G)$ colours (by Lemma 3.2.1), we have $|C_2(A)| \geq |C_1(V_1)| - |C_2(V_2)|$. Choose a subset C^* of $C_2(A)$ that has cardinality $|C_1(V_1)| - |C_2(V_1)|$ and write $x \in A^*$ if and only if $x \in A$, $C_2(x) \in C^*$. Let H be the subgraph of G induced by all the vertices z of $G-V_4$ such that $C_1(z) \in C_1(V_1)$ and by all the vertices y of G_2 such that $C_2(y) \in C_2(V_1) \cup C^*$. Let F be the subgraph of G induced by all the vertices not belonging to H.

Now we have $\omega(H) \leq |C_1(V_1)|$ and $\omega(F) \leq \omega(G) - |C_1(V_1)|$. Since F and H are proper induced subgraphs of G, we have

$$\chi(F) + \chi(H) = \omega(F) + \omega(H) \leq \omega(G) < \chi(G),$$

a contradiction. □

Corollary 3.3.1

Let $G = (V,E)$ be a minimal imperfect graph with a skew partition V_1, V_2, V_3, V_4. Then the set $V_1 \cup V_2$ can not contain a clique of size $\omega(G)$.

Proof

Note that G_1 and G_2 are perfect since they are proper induced subgraphs of G. Let C_1 and C_2 be the optimal colourings of G_1 and G_2 respectively. If $V_1 \cup V_2$ contains a clique of size $\omega(G)$, then (by Lemma 3.2.1) we have $\omega(G_1) = \chi(G_1) = \omega(G_2) = \chi(G_2) = \omega(G)$. Furthermore, we have $\omega(V_1) + \omega(V_2) = \omega(G)$. This implies that $C_1(V_1) = C_2(V_1)$ and $C_1(V_2) = C_2(V_2)$, contradicting Theorem 3.3.1. □

From Corollary 3.3.1, we obtain the following result which was first obtained by Olaru (see Berge and Chvátal (1984)). (This result follows from Corollary 3.3.1 by the Perfect Graph Theorem.)

<u>Corollary 3.3.2</u>

Let $G = (V,E)$ be a minimal imperfect graph with a skew partition V_1, V_2, V_3, V_4. Then the set $V_3 \cup V_4$ can not contain a stable set of size $\alpha(G)$. □

We conclude this section with the following two theorems.

<u>Theorem 3.3.2</u>

Let $G = (V,E)$ be a minimal imperfect graph with a skew partition V_1, V_2, V_3, V_4. Then, there cannot be two vertices x in V_3, y in V_4 such that $N(x) \supseteq V_1$, and $N(y) \supseteq V_1$.

By the Perfect Graph Theorem, Theorem 3.3.2 implies the following theorem.

Theorem 3.3.3

Let $G = (V,E)$ be a minimal imperfect graph with a skew partition V_1, V_2, V_3, V_4. If in V_1, there is a vertex x with $N(x) \cap V_3 = \emptyset$, then for each vertex y in V_2, we have $N(y) \cap V_3 \neq \emptyset$. □

Proof of Theorem 3.3.2

Let C_1 and C_2 be the optimal colourings of G_1 and G_2, respectively. Suppose that the vertices x,y exist. By Theorem 3.3.1, we only need establish $|C_1(V_1)| = |C_2(V_1)|$. In fact, we shall show that $|C_1(V_1)| = |C_2(V_1)| = \omega(V_1)$.

By symmetry, we only need show that $|C_2(V_1)| = \omega(V_1)$. We may assume that $|C_2(V_1)| \leq |C(V_1)|$ for any optimal colouring C of G_2. We may also assume that $|C_2(V_1)| > \omega(V_1)$, for otherwise we are done. Let S be the set of all vertices z of G_2 such that the colour of z appears in V_1. Let G' be the subgraph of G induced by $S \cup \{x\}$. Since G' is a proper induced subgraph of G, G' is a perfect. Hence $\chi(G') = \omega(G') \leq \max(|C_2(V_1)|, \omega(V_1) + 1) \leq |C_2(V_1)|$. Since each vertex in V_1 must receive a colour different from the colour of x, the vertices of S can be coloured by (at most) $|C_2(V_1)|$ colours so that fewer than $|C_2(V_1)|$ colours appear in V_1. This defines a new colouring C of G_2 such that $|C(V_1)| < |C_2(V_1)|$, contradicting our choice of C_2. □

3.4 A Theorem of Meyniel

Two vertices are said to be <u>friends</u> if they are not endpoints of a chordless path with an odd number of edges. (In particular, friends are always nonadjacent.) Henry Meyniel (1985) recently established the following property of minimal imperfect graphs.

<u>Theorem 3.4.1 (Meyniel (1985))</u>

In a minimal imperfect graph, no two vertices can be friends.

To prove Theorem 3.4.1, we shall need the following two lemmas.

<u>Lemma 3.4.1 (Meyniel (1985))</u>

Let G be a graph. If two non-adjacent vertices x,y are not endpoints of the same P_4, then the graph G', obtained from G by identifying x and y, satisfies $\omega(G') = \omega(G)$.

<u>Proof</u>

Clearly $\omega(G') \geq \omega(G)$. We can assume that we have $\omega(G') > \omega(G)$. Then, in G there is a clique K of size $\omega(G)$ with $N(x) \cup N(y) \supseteq K$. Since $N(x) \not\supseteq K$ and $N(y) \not\supseteq K$, it follows that there is a P_4, in $\{x \cup y \cup K\}$, with x,y being two endpoints. □

Lemma 3.4.2 (Meyniel (1985))

Let G be a perfect graph. If G contains two friends x and y, then the graph G', obtained from G by identifying x and y, is perfect.

Proof

By induction on the number of vertices. We shall prove that $\chi(G') = \omega(G')$. Furthermore, by Lemma 3.4.1, we only need prove that $\chi(G) = \chi(G')$.

Consider a colouring of G by $\omega(G)$ colours. If x and y receive the same colour, then this colouring defines obviously the required colouring of G'. We can assume that x and y receive different colours, let us say 1 and 2. Consider the induced subgraph H of G such that H consists of all vertices of colour 1 or 2. If a component C of H contains x, then it cannot contain y; for otherwise x and y are not two friends. Now, interchanging colours 1 and 2 on this component C, we find a colouring of G in $\omega(G)$ colours such that x and y have the same colour. □

Proof of Theorem 3.4.1

Let $G = (V,E)$ be a minimal imperfect graph. Assume that the statement of the theorem is false. Then G has two friendly vertices x,y. Let $G' = (V',E')$ be the graph obtained from G by identifying x and y. We have $\chi(G') \geq \chi(G) > \omega(G)$, and by Lemma 3.4.1, $\omega(G) = \omega(G')$.

By Lemma 3.4.2, each proper induced subgraph of G' is perfect. So, G' is minimal imperfect. By Theorem 2.1, $\omega(G).\alpha(G) = |V| - 1$, and $\omega(G').\alpha(G') = |V'| - 1 = |V| - 2$. Hence, by Lemma 3.4.1, we have $\omega(G).(\alpha(G) - \alpha(G')) = 1$. Since $\omega(G) \geq 2$, this is a contradiction. □

A graph G is a quasi-parity graph if each induced subgraph H of G either contains two friends, or else is a clique. It follows from Theorem 3.4.1 that quasi-parity graphs are perfect. Meyniel (1985) has shown that this class of perfect graph contains all "perfectly orderable" graphs and all "Meyniel" graphs. (These perfect graphs will be investigated in latter sections.)

In subsection 3.2, we have seen that clique identification, substitution and amalgam preserve perfection. In the remainder of this subsection, we shall show that these operations also preserve the property of "being a quasi-parity graph".

Fact 3.4.1

If a graph G is obtained from two quasi-parity graphs G_1 and G_2 by clique identification, then G is a quasi-parity graph.

Proof

We only need show that G is a clique, or G contains two friends.

We can assume that G is not isomorphic to G_1 or G_2. Thus G contains a clique cutset C such that $G = G_1 \cup G_2$ and $G_1 \cap G_2 = C$. Now, it is easy to see that if two vertices x,y are friends in G_i, then x and y are friends in G: the point is that each chordless path that has x and y as two endpoints must be entirely in G_i. Thus, we can assume that each G_i is a clique. Now, x and y are two friends of G whenever $x \in G_1 - C$, $y \in G_2 - C$. □

Fact 3.4.2

If a graph G is obtained from two quasi-parity graphs G_1 and G_2 by substitution, then G is a quasi-parity graph.

Proof

We only need show that G is a clique, or G contains two friends..

By an earlier observation, G contains a homogeneous set Y. Assume that G is obtained by substituting a vertex x of G_1 for G_2. That is $[Y]_G$ is isomorphic to G_2. If there is a chordless path P with two endpoints in G -Y and some interior vertex in Y, then P contains exactly one interior vertex in Y. Thus if two vertices are friends in G_1, then they are friends in G. Now we can assume that G_1 is a clique.

If there is a chordless path P with two endpoints in Y and some interior vertex outside Y, then P contains exactly one interior vertex in N(Y). It follows that P contains exactly three vertices and two edges. Thus if two vertices are friends in G_2, then they are friends in G. We can now assume that G_2 is a clique.

Since G_1 and G_2 are cliques, G is a clique.□

Fact 3.4.3

If a graph G = (V,E) is an amalgam of two quasi-parity graphs G_1 and G_2, then G is a quasi-parity graph.

Proof

We only need prove that G contains two friends, or else G is a clique. We may assume that G is not isomorphic to G_1 or to G_2. Thus we can partition the vertices of G into disjoint sets K, A_1, A_2, B_1, B_2 such that (K is the identified clique)

- $K \cup A_1 \cup A_2 \neq \emptyset$,
- $[K]_G$ is a clique
- We have $xy \in E$ whenever $x \in A_i$, $y \in A_j$ $(i \neq j)$ or $x \in A_i$, $y \in K$,
- We have $xy \notin E$ whenever $x \in B_i$, $y \in (A_j \cup B_j)$, $i \neq j$,
- $|A_i \cup B_i| \geq 2$,
- $A_1 = \emptyset \Longleftrightarrow A_2 = \emptyset$,
- If $A_1 = A_2 = \emptyset$, then in each B_i, there is a vertex x_i with $N(x_i) \supseteq K$

and such that for each $a_i \in A_i$, the graph $G(a_i) = [K \cup A_j \cup B_j \cup \{a_i\}]_G$ is isomorphic to G_j. (We may assume that $A_i = \emptyset$, for otherwise K is a clique cutset of G, and the desired conclusion follows from Fact 3.4.1.)

If the graph $G(a_i)$ contains two friends x,y then x and y are also friends in G: the point is that there can be no chordless path with two endpoints in $G(a_i)$ and more than one interior vertex in $(B_i \cup A_i) - \{a_i\}$. Thus, we may assume that G_1 and G_2 are cliques. It follows that $B_1 = B_2 = \emptyset$. But then G is also a clique. □

Presently, there is no polynomial-time algorithm to recognize quasi-parity graphs. However, the above three facts suggest a natural approach to design such an algorithm. Let G be a graph. Suppose that G satisfies one of the following conditions.

(i) G contains a clique cutset.

(ii) G contains a homogeneous set.

(iii) G contains a proper amalgam decomposition.

Then G can be decomposed into two smaller graphs G_1 and G_2 such that G can be constructed from G_1 and G_2 by one of the three operations, and that G is quasi-parity and only if G_1 and G_2 are both quasi-parity graphs. (This approach is not new; Burlet and Fonlupt (1980) showed that each Meyniel graph G either admits a proper amalgam decomposition or else G is a "basic" Meyniel graph. We shall describe their work in more detail in section 8.) Our challenge is to find some "basic" quasi-parity graphs such that if a quasi-parity graph is not basic, then it satisfies at least one of the three conditions (i), (ii), and (iii).

Sue Whitesides (1981) designed a polynomial-time algorithm to recognize the presence of clique cutset in an arbitrary graph. Polynomial-time algorithms to recognize the presence of a homogeneous set in a graph have been obtained by Maurer (1976), Habib and Maurer (1979) and Cunningham (1982). Cornuéjols and Cunningham (1985) has obtained a polynomial-time algorithm to determine whether a graph admits a proper amalgam decomposition.

Incidentally, note that if a connected graph G contains a homogeneous set, then G admits a proper amalgam decomposition, unless G has a vertex x which is adjacent to all remaining vertices of G (in this case, G is a quasi-parity graph if and only if G - x is a quasi-parity graph).

4. SOME CLASSES OF PERFECT GRAPHS

4.1 Introduction

In this section, we discuss triangulated graphs, comparability graphs, line-graphs of bipartite graphs, P_4-free graphs, and P_4-sparse graphs. The first three classes of perfect graphs are sometimes referred to as "classical" perfect graphs for the reason that they were among the first known classes of perfect graphs.

4.2 Triangulated Graphs

A graph G is <u>triangulated</u> if every cycle with at least four vertices contains a chord. Hajnal and Surányi (1958) proved that complements of triangulated graphs are perfect. Berge (1960) proved that triangulated graphs are perfect. Dirac (1961) showed that every triangulated graph contains a <u>simplicial</u> vertex, that is a vertex whose neighbours form a clique. In sections 5 and 6, we shall see that this special structure suggests a certain "greedy" algorithm to optimally colour triangulated graphs.

Theorem 4.2.1 (Hajnal and Surányi (1958))

If a triangulated graph G is not a clique, then G contains a clique cutset.

Proof

Let $G = (V,E)$ be a triangulated graph. We may assume that G is not a clique; for otherwise we are done. Let C be a minimal cutset of G. Enumerate the connected components of $G-C$ as $C_1, C_2, \ldots, C_k$, $k \geq 2$. If C is a clique, then we are done. Otherwise, there are two nonadjacent vertices a,b in C. Now, since C is a minimal cutset of G, each vertex of C has some neighbour in each C_i. Thus, for each connected component C_i, there is a path P_i with a,b being two endpoints, with all interior vertices belonging to C_i. Note that each P_i contains at least three vertices. Consider two such paths P_i, P_j. They form a chordless cycle with at least four vertices, contradicting our assumption that G is triangulated.□

As triangulated graphs become well known, many interesting properties of them were discovered. In particular, Dirac (1961) showed that every triangulated graph contains a simplicial vertex. (Recall that a vertex is "simplicial" if its neighbours form a clique.)

Theorem 4.2.2 (Dirac (1961))

If a triangulated graph G is not a clique, then G contains two nonadjacent simplicial vertices.

Proof

By induction on the number of vertices. Let G be a triangulated graph. If G is a clique, then we are done. Otherwise, by Theorem 4.2.1, G contains a clique cutset C. Let G_1 and G_2 be two induced subgraphs of G such that $G_1 \cap G_2 = C$, and $G_1 \cup G_2 = G$. Consider the graph G_1. We claim that in $G_1 - C$, there is a simplicial vertex x of G_1.

If G_1 is a clique, then any vertex in $G_1 - C$ can play the role of x. Otherwise, by the induction hypothesis, G_1 contains two nonadjacent simplicial vertices v_1, v_2. Since C is a clique, at least one v_i must lie in $G_1 - C$. Write $x = v_i$ and we have justified our claim.

Similarly, we can find a simplicial vertex y in $G_2 - C$. But then, x and y are two nonadjacent simplicial vertices of G.□

It is easy to see that triangulated graphs can be recognized in polynomial time. Let us elaborate on this point. Suppose we are given a graph $G = (V,E)$ with $|V| = n$. (As usual, we shall assume that G is represented by its adjacency lists.) Now, G is a triangulated graph if and only if

(i) no P_3 extends into a chordless cycle.

We can test (i) as follows. Let a,b,c be the vertices of a P_3, with b being the interior vertex. This P_3 extends into a chordless cycle if and only if there is a connected component C of $\bar{N}(b)$ with $N(a) \cap C \neq \emptyset$ and $N(c) \cap C \neq \emptyset$. Thus, (i) can be tested in $O(n^2)$ steps. Since there are only $O(n^3)$ distinct P_3's in G, this algorithm terminates in $O(n^5)$ steps. Of course, our algorithm is very crude, and there are faster algorithms to recognize triangulated graphs. Leuker (1974), Rose and Tarjan (1975) designed a linear-time algorithm to recognize triangulated graphs. Rose, Tarjan and Leuker (1976) showed that an algorithm of Gavril (1972) can be implemented to find the four parameters $\omega(G)$, $\chi(G)$, $\alpha(G)$ and $\theta(G)$ of a given triangulated graph G in linear time.

3.3 Comparability Graphs

Let X be a set, and let < be a binary and antisymmetric relation on X. The set (X,<) is a <u>partially ordered set</u> (or <u>poset</u> for short) if for each choice of a,b,c in X, we have $a < c$ whenever $a < b$, $b < c$. If $x < y$, then we say that x is <u>comparable</u> to y, and y is comparable to x.

A graph $G = (V,E)$ is a <u>comparability</u> graph if V admits a partial order such that two vertices of V are comparable if and only if they are adjacent. A famous theorem of Dilworth (1950) can be restated by saying that in a comparability graph $G = (V,E)$, the number of vertices in the largest stable set equals the smallest number of cliques that cover V; that is $\alpha(G) = \theta(G)$. This equality also holds for every induced subgraph H of G, since H is a comparability graph. Now, it follows from the Perfect Graph Theorem that the comparability graph G is perfect. (We shall present an easier and more direct proof of this fact in a moment.)

Let $G = (V,E)$ be a graph. By an orientation $\vec{G}$ of G, we denote any of the directed graphs obtained from G by assigning one, and only one, direction to each edge of G. We shall refer to a directed edge as an arc. We say that $\vec{G}$ is acyclic if it does not contain a directed cycle. By a bad P_3 we mean the graph with vertices a,b,c and arcs ab, bc (and no other arc). An orientation $\vec{G}$ of a graph G is transitive if $\vec{G}$ does not contain an induced bad P_3.

By the above definitions, a graph G is a comparability graph if G admits an orientation $\vec{G}$ such that $\vec{G}$ is both acyclic and transitive. (To obtain $\vec{G}$ we only need direct a to b if ab is an edge of G, and $a < b$ in the poset.) The following elegant argument of Berge (1973) shows that comparability graphs are perfect, without relying on Dilworth's theorem. (Fulkerson (1972) also proved that comparability graphs are perfect.)

Theorem 3.3.1 (Berge (1973))

Every comparability graph is perfect.

Proof

By induction on the number of vertices. Let G be a comparability graph. We only need prove that $\omega(G) = \chi(G)$.

By a directed path from v_1 to v_k, we mean a sequence $v_1, v_2, \ldots, v_k$ such that $v_i v_{i+1}$ is an arc. Let $\vec{G}$ be the transitive orientation of G. Let t(x) denote the length of the longest directed path from x plus one. Since $\vec{G}$ is acyclic, t(x) is finite.

By the transitivity of $\vec{G}$, each directed path induces a clique. Let k be the largest number among all numbers t(x). We have $\omega(G) = k$.

Consider a colouring of G by the colours 1,2,...,k such that each vertex x receives the colour t(x). No two adjacent vertices x,y can receive the same colour, because if xy is an arc, then $t(x) > t(y)$. Thus, we have $\chi(G) \leq k$. Since $\chi(G) \geq \omega(G) = k$, it follows that $\chi(G) = \omega(G) = k$. □

Now, we can describe a theorem of Ghouila-Houri (1962).

Theorem 3.3.2 (Ghouila-Houri (1962))

If a graph G admits a transitive orientation, then G admits an orientation which is both acyclic and transitive.

Proof

By induction on the number of vertcies. Let $G = (V,E)$ be a graph that admits a transitive orientation $\vec{G}$.

First, note that if three vertices a,b,c, induce a directed triangle in $\vec{G}$, then no vertex outside $X = \{a,b,c\}$ can be adjacent to exactly one vertex in X.

Now if $\vec{G}$ is acyclic, then we are done. Otherwise, $\vec{G}$ contains a directed cycle. By its transitivity, $\vec{G}$ can not contain a chordless directed cycle with more than three vertices. Thus, we can assume that there are vertices a,b,c with arcs ab, bc, ca. We only need consider two cases.

Case 1. G contains a homogeneous set Y.

Let G_1 be the subgraph of G induced by $(V-Y) \cup \{x\}$, where x is an arbitrary vertex of Y. Let G_2 be the subgraph of G induced by Y. By the induction hypothesis, we can direct the edges of each G_i so that $\vec{G}_i$ is both acyclic and transitive. To obtain a transitive and acyclic orientation of G, we only need add the arcs yz if (i) $y \in Y$, $z \in N(Y)$, $xz \in \vec{G}_1$ or if (ii) $y \in N(Y)$, $z \in Y$, $xz \in \vec{G}_1$.

Case 2. G does not contain a homogeneous set.

We can assume that there is a vertex x outside $X = \{a,b,c\}$ with $|N(x) \cap X| = 2$, for otherwise X is a homogeneous set. Without loss of generality, assume that x is adjacent to b, c, and nonadjacent to a. We must have $cx, xb \in \vec{G}$; for otherwise $\vec{G}$ is not transitive.

Let A be the set of all vertices y such that $cy, yb \in \vec{G}$. We have $|A| \geq 2$, since $x \in A$, and $a \in A$. If A is a homogeneous set, then we are done. Thus, there is a vertex $y \notin A$ with $ya_1 \in E$, $ya_2 \notin E$ for some $a_1, a_2 \in A$. Consider the triangle $\{a_1, b, c\}$, we see that y is adjacent to either b, or c, or both. Consider the triangle $\{a_2, b, c\}$, we see that y must be adjacent to both b and c. Since $ya_2 \notin E$, it follows that $cy, yb \in \vec{G}$. But then y must be in A, a contradiction.□

By an <u>odd walk</u>, we shall mean a sequence of (not necessarily distinct) vertices $v_0, v_1, \ldots, v_{2k}$ such that $v_i v_{i+1}$ is an edge, and $v_i v_{i+2}$ is a nonedge. (As usual, the subscripts are taken modulo 2k+1.) The following result (announced in 1962) of Gilmore and Hoffman provides another characterization of comparability graphs.

<u>Theorem 4.3.3 (Gilmore and Hoffman (1964))</u>

A graph is a comparability graph if and only if it does not contain an odd walk.□

It is easy to see that, by Theorem 4.3.2 and Theorem 4.3.3, comparability graphs can be recognized in polynomial time. (This fact was mentioned in Gilmore and Hoffman (1964).)

4.4 Line-Graphs of Bipartite Graphs

A graph is bipartite if its vertices can be partitioned into two disjoint stable sets. It is easy to see that every bipartite graph is perfect. A line-graph of a bipartite graph G is a graph H whose vertices correspond to the edges of G, two vertices of H being adjacent if and only if their corresponding edges share an endpoint in G. To show that line-graphs of bipartite graphs are perfect, we shall use a well-known result on "matchings" in graphs. We shall need introduce a few definitions.

Let G = (V,E) be a graph. A subset M of E is called a matching of G if no two edges of M share an endpoint. By m (G), we shall denote the number of edges in a largest matching of G. A cover of G is a set C of vertices such that each edge of G has at least one endpoint in C. By c(G) we denote the number of vertices in a smallest cover of G. Clearly, for any graph G, we have $c(G) \geq m(G)$.

Theorem 4.4.2 (Konig (1916)

Every bipartite graph G has m(G) = c(G).

The following elegant proof is due to Lovász (1975).

Proof of Theorem 4.4.2

Let G' be a smallest subgraph of G = (V,E) with

$c(G') = c(G)$. We claim that the edge-set of G' is a matching. This will establish the theorem since we will have $m(G) \geq m(G') = c(G') = c(G)$.

Assume the contrary, and so G' has a vertex x adjacent to two vertices y_1 and y_2. Write $c(G) = c$. By the minimality of G', we have $c(G' - y_1y_2) < c$ and so there is a set S_i, with $S_i \subseteq V$, $|S_i| = c - 1$, such that S_i is a cover of $G - xy_i$. Since the edge xy_i cannot have an endpoint in S_i, we have $x, y_i \notin S_i$.

Write $S = S_1 \cap S_2$, $|S| = t$, $R = (S_1 - S) \cup (S_2 - S) \cup \{x\}$. We have $|R| = 2(c-1-t) + 1 = 2(c - t) - 1$. Note that the vertices of the subgraph $H = [R]_G$, can be partitioned into two disjoint stable sets (H is bipartite). Let T be the smaller of these two stable sets. We have $T \leq c - t - 1$. Now, we claim that $T \cup S$ is a cover of G': if an edge is induced by R, then T covers it; if an edge is not induced by R, then it can meet both S_1 and S_2 only if it has an endpoint in $S_1 \cap S_2 = S$. But then we have $|T \cup S| = c-t-1+t = c - 1 < c(G') = c$, a contradiction.□

Let $L(G)$ be the line-graph of a bipartite graph $G = (V,E)$. To see that $L(G)$ is perfect we only need notice that $\alpha(L(G)) = m(G) = c(G) = \theta(L(G))$; and so $L(G)$ is perfect by the Perfect Graph Theorem.

Edmonds (1965) designed a polynomial-time algorithm to find a largest matching in a graph. The problem of determining the parameter c(G) of a graph G is NP-complete (Cook (1971)).

A claw is the graph with vertices a,b,c,d and edges ab,ac,ad (and no other edge). A diamond is the graph with vertices a,b,c,d and edges ab,bc,cd,bd,ad (and no other edge). We shall call a graph Berge if it does not contain an induced subgraph isomorphic to an odd chordless cycle with at least five vertices, or to the complement of such a cycle.

It is easy to see that if G is a line-graph of bipartite graph, then G is Berge, claw-free, and diamond-free. Parthasarathy and Ravindra proved that claw-free Berge graphs (1976), and diamond-free Berge graphs (1979) are perfect. (Actually, there is a flaw in their proof of the latter result; a correct proof, based in part on the Parthasarathy-Ravindra technique, has been obtained by Tucker (1984).)

4.5 P_4-Free Graphs

A graph is P_4-free if it has no induced P_4. P_4-free graphs have been studied by many people; terms synonymous with "P_4-free graphs" include cographs (Corneil, Lerchs, Stewart-Burlingham (1981)), D*-graphs (Jung (1978)), and HD or Hereditary Dacey graphs (Summer (1979)). Recently Corneil, Perl and STewart-Burlingham (1984) designed a linear-time algorithm to recognize P_4-free graphs.

Lerchs (1971, 1972) and Seinche (1974) independently proved that P_4-free graphs are perfect. In sections 10 and 11, we shall refer to the following theorem of Seinche many times.

<u>Theorem 4.5.1 (Seinche (1974))</u>

If a graph G is P_4-free, then either G or $\bar{G}$ is disconnected.

<u>Proof</u>

Let $G = (V,E)$ be a P_4-free graph.

Suppose both G and $\bar{G}$ are connected. Let A be the smallest induced subgraph of G such that A has at least two vertices and such that A and $\bar{A}$ are both connected. Let x be a vertex whose removal would disconnect A (we can always interchange G and $\bar{G}$, so that this is the case). Since $\bar{A}$ is connected, there is a vertex y in $A - x$ such that $xy \notin E$. Let A' be the connected component of $A - x$ that includes y. Let us partition the set of vertices in A' into disjoint sets R and W such that

(i) $u \in R$ if $ux \notin E$,

(ii) $u \in W$ if $ux \in E$.

Since A is connected, there is a vertex v outside

$A' \cup \{x\}$ such that $vx \in E$; note that $uv \notin E$ for any vertex u in A'. Since A' is connected, there is a path P from x to y; but the only edges leaving A' are edges from W to x, this path must include vertices w in R, z in W such that $zw \in E$. But the vertices v,x,z,w and edges vx,xz,zw form a P_4. □

<u>Corollary 4.5.1 (Lerchs (1972), Seinche (1974))</u>

P_4-free graphs are perfect.

Corollary 4.5.1 follows from Perfect Graph Theorem and Theorem 4.5.1.

<u>Corollary 4.5.2</u>

If G is a P_4-free graph with at least three vertices, then G contains a homogeneous set.

4.6. P_4-Sparse Graphs

A graph $G = (V,E)$ is <u>P_4-sparse</u> if no subset of V, with five vertices, contains two distinct P_4's. By definition, every P_4-free graph is P_4-sparse.

A graph $G = (V,E)$ will be called a <u>spider</u> if its vertices can be labeled $a_1,a_2,\ldots,a_k,b_1,b_2,\ldots,b_k$ or $t,a_1,a_2,\ldots,a_k,b_1,b_2,\ldots b_k$ such that:

(i) $a_i a_j \notin E$ for all i and j

(ii) $b_i b_j \in E$ for all i and j

(iii) $a_i b_j \in E$ if and only if $i = j$

(iv) If t is present, then we have $ta_i \notin E$,
$tb_j \in E$ for all i and j.

The complement of a spider will be called a <u>cospider</u>.

<u>Theorem 4.6.1 (Hoang (1983))</u>

If G is a P_4-sparse graph, then G contains a homogeneous set, or else G, or $\overline{G}$, is a spider.

<u>Proof</u>

Let G = (V,E) be a graph. We can assume that G has more than two vertices, or else G is a spider. Let H be the subgraph of G such that H contains five vertices and at least two P_4's. Our proof is presented in guise of an algorithm. Given, as input the graph G, the algorithm returns as output one of the following:

(i) A subgraph H.

(ii) A homogeneous set Y.

(iii) A spider

(iv) A cospider

If G is a P_4-sparse graph, then (i) can not be returned by the algorithm. Thus, the theorem holds. The algorithm is as follows:

1. If G is P_4-free, then by Corollary 4.5.2, G contains a homogeneous set Y, return Y and stop.

2. Set

   $u \in P$ if $ua_1, ua_2 \in E$ and $ub_1, ub_2 \in E$

   $u \in Q$ if $ua_1, ua_2 \notin E$ and $ub_1, ub_2 \notin E$

   $u \in R$ if $ua_1, ua_2 \notin E$ and $ub_1, ub_2 \in E$

   If some vertex w* other than a_1, b_1, b_2, a_2 lies outside P,Q,R, then return the subgraph H induced by a_1, b_1, b_2, a_2 and w* and stop.

3. As long as there are adjacent vertices $a \in Q$, and $b \in R$, repeat the following operations:

   3.1 If some $w^* \in P$ has $w^*a \notin E$ or $w^*b \notin \varepsilon$ (or both) then return the subgraph H induced by a_1, b_1, b, a and w*, and stop.

   3.2 If some $w^* \in Q$ has $w^*a \in E$ or $w^*b \in E$ (or both) then return the subgraph H induced by a_1, b_1, b, a and w* and stop.

   3.3 If some $w^* \in R$ has $w^*a \in E$ or $w^*b \notin E$ (or both) then return the subgraph H induced by a_1, b_1, b, a and w*, and stop.

   3.4 Delete a from Q, delete b from R, set $a_{k+1} = a$, $b_{k+1} = b$, and replace k by k+1.

4. If k = 2 and some $u \in P$ is nonadjacent to some $v \in R$

then set

$x \leftarrow a_1$, $y \leftarrow b_1$, $z \leftarrow b_2$, $t \leftarrow a_2$,

$a_1 \leftarrow y$, $b_1 \leftarrow t$, $b_2 \leftarrow x$, $a_2 \leftarrow z$,

Replace G by $\overline{G}$, interchange P and Q, and return to step 3.

(Note that a=u, and b=v have just become available.)

5. If $k \geq 3$ and some $u \in P$ is nonadjacent to some $v \in R$, then return the subgraph H induced by a_1,u,b_2,v, and b_3, and stop.
6. If $P \cup Q \neq \emptyset$, then set $Y = \{a_1,a_2,\ldots,a_k, b_1,b_2,\ldots,b_k\} \cup R$, return the homogeneous set Y and stop.
7. If $|R| \geq 2$, then set Y = R. Return the homogeneous set Y and stop.
8. G or $\overline{G}$ is a spider. Return this spider and stop.□

Lemma 4.6.1

Let G be a graph with a homogeneous set Y. If there is a P_4 with at least one vertex in Y and at least one vertex not in Y, then this P_4 has precisely one vertex in Y and three vertices not in Y. Furthermore, if such a P_4 is present, then G is not P_4-sparse.

Proof

Since Y is homogeneous, the set of vertices outside Y can be partitioned into disjoint sets A, B such that, for each vertex u, we have

(i) $u \in A$ if $ux \notin E$ whenever $u \notin Y$, $x \in Y$

(ii) $u \in B$ if $ux \in E$ whenever $u \notin Y$, $x \in Y$

If there is one P_4 with at least one vertex in Y and at least one vertex not in Y, then this P_4 has at least one vertex in B. Thus, such a P_4 can have only one vertex in Y. So, its vertices can be enumerated as a,b,c,d such that we have either $a \in A$, $b,d \in B$, $c \in Y$, or $a,b \in A$, $c \in B$, $d \in Y$. Since $|Y| \geq 2$, there is a vertex e in Y such that a,b,c,d,e induces two distinct P_4's.□

Theorem 4.6.2

If G is a P_4-sparse graph, then G satisfies at least one of the following three conditions.

(i) G is a spider or cospider.

(ii) $\bar{G}$ is disconnected.

(iii) G contains a clique cutset.

Proof

Let G = (V,E) be a P_4-sparse graph. We can assume that G is not a spider or cospider. Now, Theorem 4.6.1 implies that G contains a homogeneous set. Choose Y to be the homogeneous set that maximizes $|Y \cup N(Y)|$. (Recall that N(Y) stands for the set of all vertices x such that $x \notin y$ and $xy \in E$ for some $y \in Y$.)

If $V = Y \cup N(Y)$, then we are done: $\bar{G}$ is disconnected. Thus, the set $\bar{N}(Y) = V - (Y \cup N(Y))$ is nonempty. Let $Z_1, Z_2, \ldots$ be the connected components of $G - (Y \cup N(Y))$. Let $N_1, N_2, \ldots$ be the connected components of the subgraph of $\bar{G}$ induced by $N(Y)$. Let N^* be the union of all components N_i such that N_i consists of a single vertex.

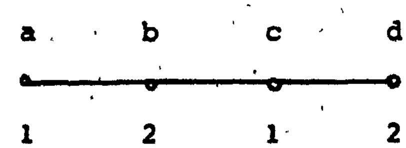
Now, if there is no edge zv with $z \in \bar{N}(Y)$ and $v \in N(Y) - N^*$, then we are done: N^* is a clique cutset in G. Thus, there is an edge zv such that $z \in \bar{N}(Y)$ and v is in some N_i that has at least two vertices. Now, N_i must be a homogeneous set of G. (If there is a vertex x with $N(x) \cap N_i \neq \emptyset$, and $N(x) \not\supseteq N_i$, then we have $x \in \bar{N}(Y)$; since N_i is connected in $\bar{G}$, there are vertices n_1, n_2 in N_i with $xn_1 \in E$, $xn_2 \notin E$, $n_1n_2 \notin E$; thus each vertex in Y forms a P_4 with x, n_1, n_2. By Lemma 4.6.1, this is impossible.) But then we have $|N_i \cup N(N_i)| > |Y \cup N(Y)|$, contradicting our choice of Y. □

5. PERFECTLY ORDERABLE GRAPHS

A natural way of colouring the vertices in a graph is to order them in a sequence $v_1, v_2, \ldots, v_n$. Then, scan the sequence from v_1 to v_n and assign to each v_j the smallest positive integer $f(v_j)$ assigned to none of its neighbours v_i with $i < j$. We shall refer to the graph with the linear order on the set of its vertices as an ordered graph, and to the procedure of assigning colours to the vertices of an ordered graph as the greedy procedure.

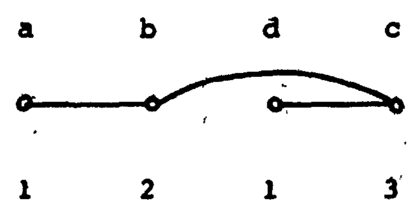
The greedy procedure may not necessarily give the best colouring. Consider the graph P_4 with vertices a,b,c,d and edges ab, bc, and cd, and the following four distinct orderings:

Figure 5.1



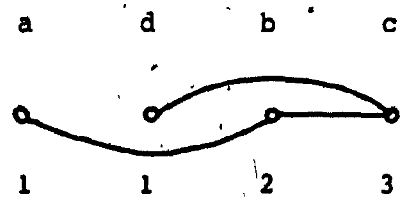
$a < b < c < d$

Figure 5.2



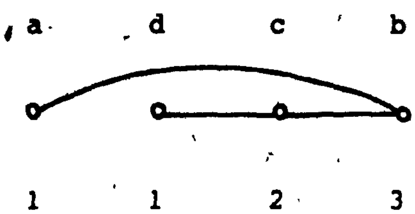
$a < b < d < c$

Figure 5.3



a < d < b < c

Figure 5.4



a < d < c < b

The greedy procedure produces an optimal colouring of the ordered graph in Figure 5.1, but it does not do so for the ordered graphs in Figures 5.2, 5.3, and 5.4. In particular, the graph in Figure 5.1 has f(a) = f(c) = 1, and f(b) = f(d) = 2. The graphs in Figures 5.2 and 5.3 have f(c) = 3, and the graph in Figure 5.4 has f(b) = 3.

<u>Fact 5.1</u>

For every graph, there is always an ordering on which the greedy procedure produces the optimal colouring.

<u>Proof</u>

Let G be an unordered graph. Find the optimal colouring of G by "colours" 1,2,...,k for some k. For each vertex v in G, let g(v) be the colour number assigned to v. Order the vertices of G in a sequence $v_1 < v_2 < \ldots < v_n$ such that $i < j$ whenever $g(v_i) < g(v_j)$. We claim that the colouring f produced by the greedy procedure has $f(v) \leq g(v)$ for any vertex v. Obviously $f(v_1) = g(v_1) = 1$. Consider a vertex v_j, $j > 1$, in the sequence. By the induction hypothesis, each vertex v_i with $i < j$ has $f(v_i) \leq g(v_i)$. Consider all neighbours v_i of v_j such that $i < j$. We know that $g(v_i) < g(v_j)$, because if $g(v_i) = g(v_j)$, then v_i is not a neighbour of v_j. Thus, we have $f(v_i) \leq g(v_i) < g(v_j)$ for all neighbours v_i of v_j. Since $f(v_j) \leq 1 + \max f(v_i)$, it follows that $f(v_j) \leq g(v_j)$. The proof is completed. □

An ordered P_4 with vertices a,b,c,d, edges ab, bc, cd such that $a < b$, $d < c$ is called an obstruction. To put it differently, an obstruction is any one of the three ordered graphs in Figures 5.2, 5.3 and 5.4. As in Chvátal (1981), let the Grundy number be the largest integer $f(v_i)$ used by the greedy procedure. A linear order on the set of vertices of a graph will be called:

(i) admissible if it creates no obstruction.

(ii) perfect if, for each induced subgraph H, the Grundy number of H equals $\chi(H)$.

It is easy to see that every perfect order is admissible. A proof of the converse relies on the following fact.

Lemma 5.1 (Chvátal (1981))

Let G be a graph and let Q be a set of pairwise adjacent vertices in G such that each $w \in Q$ has a neighbour $p(w) \notin Q$; let the vertices $p(w)$ be pairwise nonadjacent. If there is an admissible order $<$ such that $p(w) < w$ for all $w \in Q$, then some $p(w)$ is adjacent to all the vertices in Q.

Proof

By induction on the number of vertices in Q. For each $w \in Q$, the induction hypothesis guarantees the existence of a vertex $w^* \in Q$ such that $p(w^*)$ is adjacent to all the vertices in Q except possibly w. In fact we may assume that $p(w^*)$ is not adjacent to w, for otherwise we are done. Now, it follows that the mapping which assigns w^* to w is one-to-one, and therefore it is onto. In particular, with v standing for that vertex in Q which comes first in the admissible order, there are vertices $b,d \in Q$ such that $b^* = v$ and $c^* = b$. But then there is a contradiction: the vertices a,b,c,d with $a = p(b)$ and $d = p(v)$ constitute an obstruction. □

Theorem 5.1 (Chvátal (1981))

A linear order of the set of vertices of a graph is perfect if and only if it is admissible.

Proof

The "only if" part is trivial; the "if" part will be proved by induction on the number of vertices. Let G be a graph with an admissible order < of the set of its vertices, and let k stand for the Grundy number of this ordered graph. By virtue of the induction hypothesis, it will suffice to show that the chromatic number of G is at least k. Thus, it will suffice to find k pairwise adjacent vertices in G. For this purpose, consider the smallest i such that there are pairwise adjacent vertices $w_{i+1}, w_{i+2}, \ldots, w_k$ with $f(w_j) = j$ for all j. (Note that i is at most k-1, for $k \geq 2$.) If i = 0, then we have found k pairwise adjacent vertices; otherwise each w_j has a neighbour $p(w_j)$ such that $p(w_j) < w_j$ and $f(p(w_j)) = i$. (To see this, suppose there is a vertex w_j with $f(p(w_j)) \neq i$, then we have $j \leq i$, this is a contradiction.) But Lemma 5.1 implies the existence of a vertex v with $f(v) = i$, adjacent to all the vertices w_j, which contradicts the minimality of i. □

A graph is called perfectly orderable if it admits an admissible order. Recognizing perfectly orderable graphs in a polynomial time is an open problem. However, Theorem

5.1 tells us that we can recognize perfectly ordered graphs in a polynomial time. (It is sufficient to look for an obstruction in the ordered graph; if this graph has n vertices then it has at most $\binom{n}{4}$ P_4's.)

A property related to perfection has been studied by Berge and Duchet (1982). A graph is called strongly perfect if each of its induced subgraphs H contains a stable set meeting all the maximal cliques in H. (Here, as usual, "maximal" is meant with respect to set-inclusion, not size. In particular, a maximal clique is not necessarily largest.)

Theorem 5.2 (Berge and Duchet (1982))

Strongly perfect graphs are perfect.

Proof

Let G = (V,E) be a strongly perfect graph.

Using induction on the number of vertices, we only need prove $\chi(G) = \omega(G)$. Let S be a stable set meeting all the maximal cliques in G, H be the subgraph of G induced by V - S. Clearly $\omega(H) = \omega(G) - 1$. By the induction hypothesis, H is perfect, and so $\chi(H) = \omega(H)$. We can colour the pairwise nonadjacent vertices in S by an extra colour and have $\chi(G) = \omega(G)$. □

Theorem 5.3 (Chvátal (1981))

Every perfectly orderable graph is strongly perfect.

Proof

It will suffice to find, in an arbitrary graph G with a perfect order <, a stable set meeting all the maximal cliques in G. We claim that S can be found by the following algorithm: scan the perfect ordering $v_1, v_2, \ldots v_n$ from v_1 to v_n and place each v_j in S if and only if none of its neighbours v_i $(i < j)$ has been placed in S. Indeed, if the resulting stable set is disjoint from some clique Q, then each $w \in Q$ has a neighbour $p(w)$ in S with $p(w) < w$. But then the Lemma 5.1 implies the existence of a vertex $v \in S$ adjacent to all the vertices in Q. Thus, Q is not maximal. □

It may be worth mentioning that

(i) there are strongly perfect graphs which are not perfectly orderable, and

(ii) there are perfect graphs which are not strongly perfect.

An example of (i), taken from Chvátal (1984), is the graph in Figure 5.5; an example of (ii) is any graph $\overline{C}_{2k}$ with $k \geq 3$.

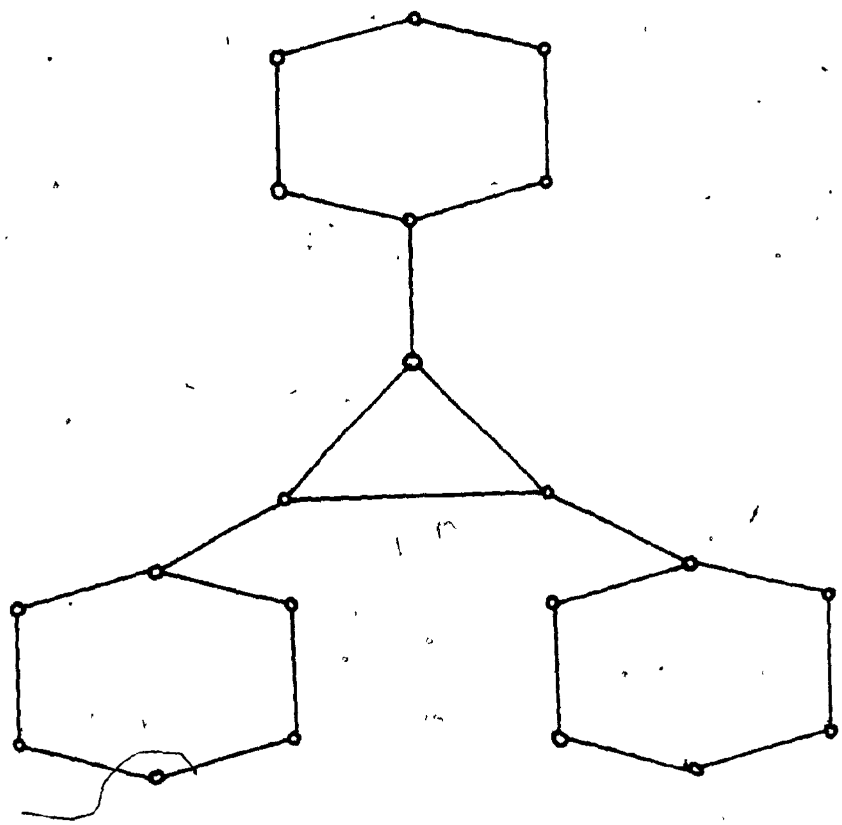


Figure 5.5

6. TWO CLASSES OF PERFECTLY ORDERABLE GRAPHS

6.1 Introduction

In this section, we shall characterize (by minimal forbidden induced subgraphs) two families of graphs such that, for every graph G in the first (or the second) family, the Welsh-Powell (or the Matula) colouring heuristic delivers a perfect order on G. All results presented in this section are obtained jointly with V. Chvátal.

6.2 Colouring Heuristics and Perfect Orders

Recall that the greedy procedure (which is a graph-colouring heuristic), given a graph G, proceeds in the following two stages:

(i) impose a linear order $<$ on the set of vertices of G,

(ii) scanning the vertices in this order, assign to each vertex y the smallest positive integer assigned to no neighbour x of y $(x < y)$.

Welsh and Powell (1967) proposed choosing $<$ in such a way that, with $d_G(x)$ standing for the degree of x in G,

$$d_G(x) \geq d_G(y) \text{ whenever } x < y; \qquad (6.2.1)$$

Matula (1968) proposed choosing $<$ in such a way that

$d_H(x) \geq d_H(y)$ whenever $x < y$ and H is the subgraph of G induced by all z with $z \leq y$. (6.2.2)

We shall call a graph G Welsh-Powell perfect if every order $<$ satisfying (6.2.1) is perfect; we shall call G Matula perfect if every order $<$ satisfying (6.2.2) is perfect.

Theorem 6.2.1

The following two conditions are equivalent for every graph G:

(a) All induced subgraphs of G (including G itself) are Welsh-Powell perfect.

(b) G has no induced subgraph isomorphic to one of the graphs $F_1, F_2, \ldots, F_{17}$ in Figure 6.1

Proof

Checking that (a) implies (b) is a routine matter: we only need verify that none of the seventeen forbidden induced subgraphs is Welsh-Powell perfect. (The non-perfect orders satisfying (6.2.1) are suggested by the labels at the vertices in Figure 6.1.)

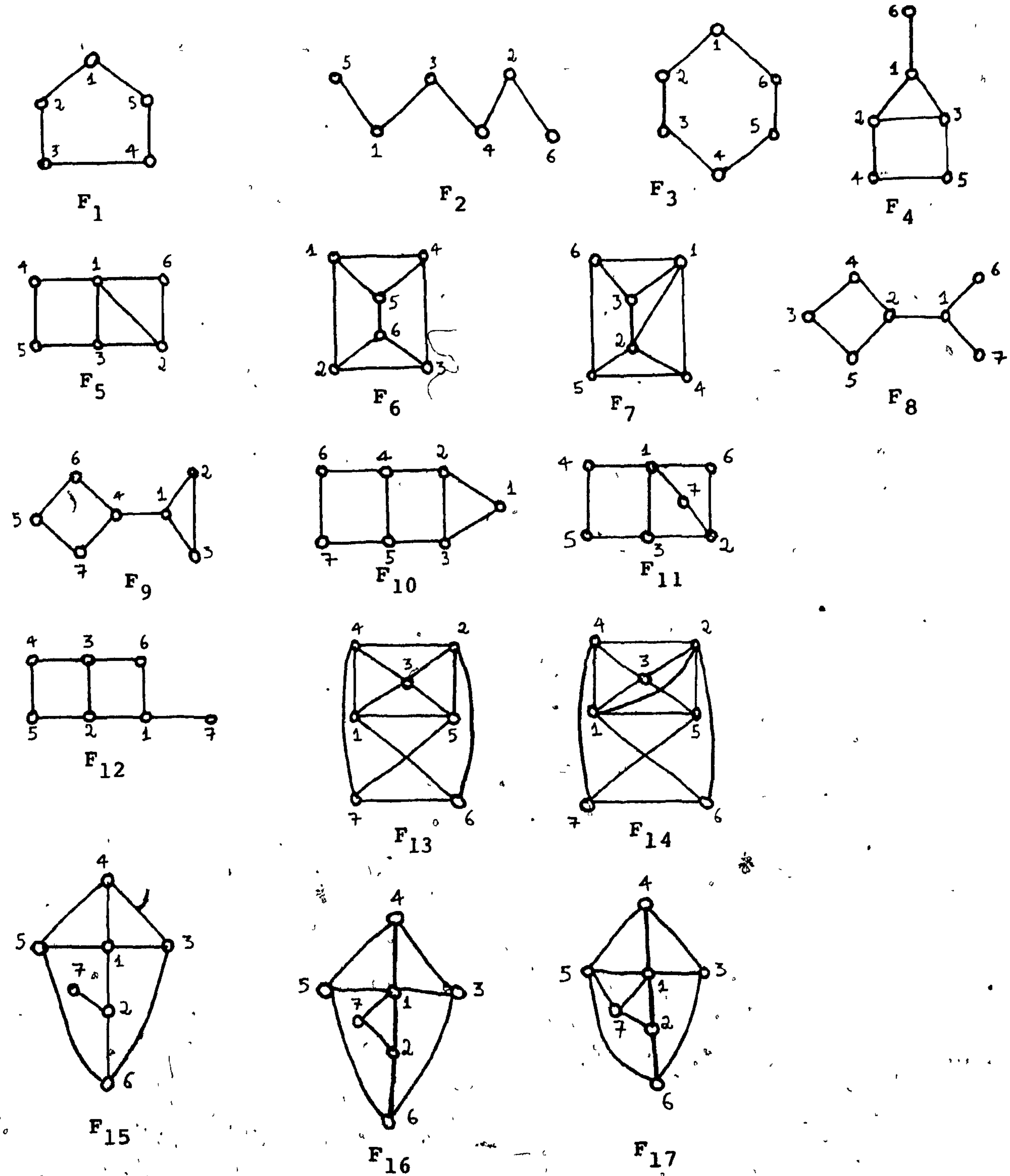


Figure 6.1

To prove that (b) implies (a), consider an arbitrary graph G satisfying (b); we only need prove that G itself is Welsh-Powell perfect. For this purpose, we assume the contrary: now G contains a chordless path with vertices v_0, v_1, v_2, v_3 and edges v_0v_1, v_1v_2, v_2v_3 such that $d_G(v_0) \geq d_G(v_1)$ and $d_G(v_3) \geq d_G(v_2)$. For each vertex w of G other than v_0, v_1, v_2, v_3, write

$$k(w) = a_0(w) + 2a_1(w) + 4a_2(w) + 8a_3(w)$$

with $a_i(w) = 1$ if v_i and w are adjacent, and $a_i(w) = 0$ otherwise. With n_k standing for the number of vertices w such that $k(w) = k$, we have

$$d_G(v_0) = 1 + n_1 + n_3 + n_5 + n_7 + n_9 + n_{11} + n_{13} + n_{15},$$
$$d_G(v_1) = 2 + n_2 + n_3 + n_6 + n_7 + n_{10} + n_{11} + n_{14} + n_{15},$$
$$d_G(v_2) = 2 + n_4 + n_5 + n_6 + n_7 + n_{12} + n_{13} + n_{14} + n_{15},$$
$$d_G(v_3) = 1 + n_8 + n_9 + n_{10} + n_{11} + n_{12} + n_{13} + n_{14} + n_{15}.$$

Hence the inequalities $d_G(v_0) \geq d_G(v_1)$ and $d_G(v_3) \geq d_G(v_2)$ may be written as

$$n_1 + n_5 + n_9 + n_{13} \geq 1 + n_2 + n_6 + n_{10} + n_{14}$$
$$n_8 + n_9 + n_{10} + n_{11} \geq 1 + n_4 + n_5 + n_6 + n_7 \qquad (6.2.3)$$

It is a routine task to verify that every solution of (6.2.3) in nonnegative integers n_k must have at least one of the following properties:

(i) $n_9 \geq 1$

(ii) $n_1 \geq 1$, $n_8 \geq 1$,

(iii) $n_1 \geq 1$, $n_{11} \geq 1$,

(iv) $n_{13} \geq 1$, $n_8 \geq 1$,

(v) $n_{13} \geq 1$, $n_{11} \geq 1$

(vi) $n_{10} \geq 1$, $n_1 \geq 2$,

(vii) $n_{10} \geq 1$, $n_{13} \geq 2$,

(viii) $n_{10} \geq 1$, $n_1 \geq 1$, $n_{13} \geq 1$,

(ix) $n_5 \geq 1$, $n_8 \geq 2$,

(x) $n_5 \geq 1$, $n_{11} \geq 2$,

(xi) $n_5 \geq 1$, $n_8 \geq 1$, $n_{11} \geq 1$.

Another routine task is to verify that

in case (i), F contains F_1,

in case (ii), F contains F_2 or F_3,

in cases (iii) and (iv), F contains F_4 or F_5,

in case (v), F contains F_6 or F_7,

in cases (vi) and (ix), F contains F_2, F_8, F_9, F_{10}, F_{11} or F_{12},

in cases (vii) and (x), F contains F_1, F_{13}, or F_{14},

in cases (viii) and (xi), F contains F_1, F_5, F_{15}, F_{16}, or F_{17}.

Thus, G violates (b); this contradiction completes the proof. □

The analogue of Theorem 6.2.1 that concerns Matula perfect graphs involves a family of graphs that we call bicycles. Take disjoint graphs C, C' and P such that C is a cycle of length three or four, C' is a cycle of length three or four, and P is a path; the path may consist of a single edge or even just a single vertex, except when both C and C' are triangles, in which case we insist on P having at least two edges; a <u>bicycle</u> is a graph obtained from C, C' and P by identifying one endpoint of P with a vertex of C and identifying the other endpoint of P with a vertex of C'. Nine bicycles are shown in Figure 6.2.

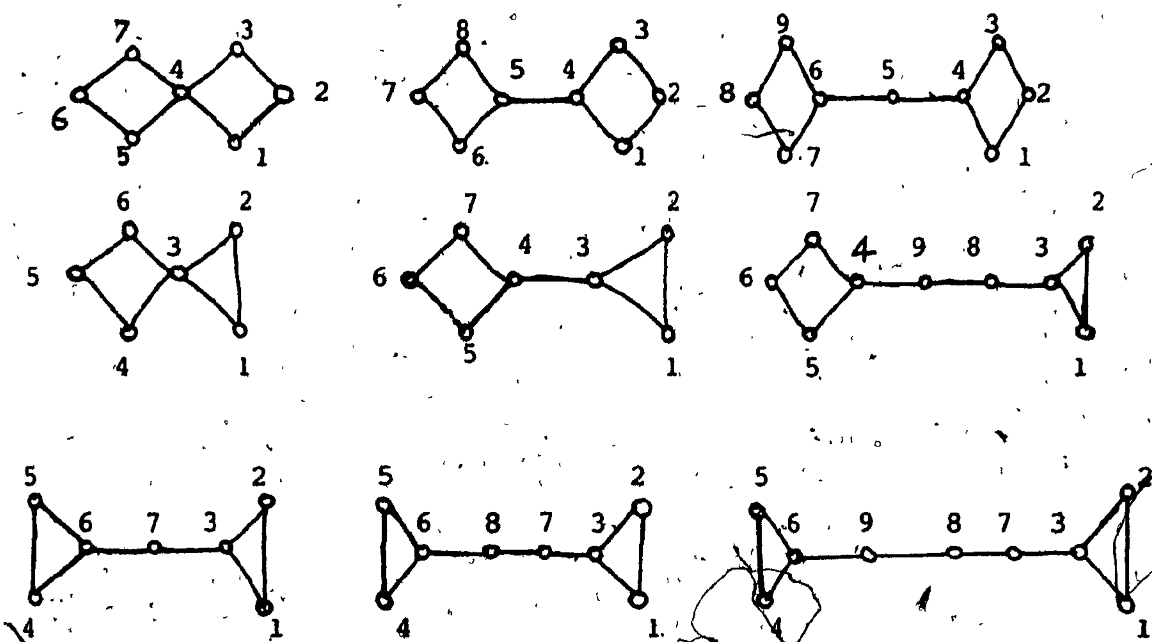


Figure 6.2

Theorem 6.2.2

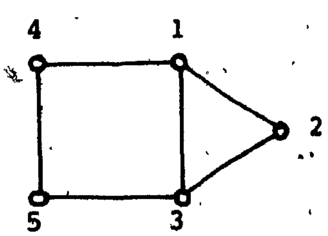
The following four conditions are equivalent for every graph G:

(a) All induced subgraphs of G (including G itself) are Matula perfect.

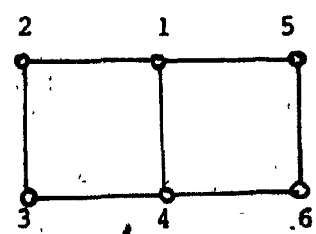
(b) No induced subgraph of G is isomorphic to a chordless cycle of length at least five, or one of the graphs F_{18}, F_{19} in Figure 6.3, or a bicycle.

(c) No induced subgraph F of G contains a chordless path with vertices v_0, v_1, v_2, v_3 and edges v_0v_1, v_1v_2, v_2v_3 such that $d_F(v_1) = 2$ and $d_F(x) \geq 2$ whenever $x \in F$.

(d) No induced subgraph H of G contains a chordless path with vertices v_0, v_1, v_2, v_3 and edges v_0v_1, v_1v_2, v_2v_3 such that $d_H(x) \geq d_H(v_1)$ for all x in H.



F_{18}



F_{19}

Figure 6.3

Proof

Again, checking that (a) implies (b) is a routine matter: we only need verify that none of the forbidden induced subgraphs is Matula perfect. (Non-perfect orders satisfying (6.2.2) are suggested by the labels at the vertices in Figures 6.2 and 6.3.)

To prove that (b) implies (c), consider an arbitrary graph F with the properties specified in (c); we only need show that F contains one of the forbidden induced subgraphs specified in (b). For this purpose, let N stand for the set of common neighbours of v_0 and v_2 in F, and let A_i $(i = 0,2)$ be the component of F−N that contains v_i. We shall distinguish among three cases:

Case 1. $A_0 = A_2$

Case 2. $A_0 \neq A_2$, and some vertex in N has a neighbour in $A_2 - v_2$.

Case 3. $A_0 \neq A_2$, and no vertex in N has a neighbour in $A_2 - v_2$.

In Case 1, the shortest path from v_0 to v_2 in F−N, along with v_1, induces a chordless cycle of length at least five. In Case 2, consider the shortest path P in A_2 such that P has at least one edge, one endpoint of P is v_2, and the other endpoint has a neighbour w in N. Since v_0 and v_2 are the only two neighbours of v_1, we have $w \neq v_1$; now the four vertices v_0, v_1, v_2, w induce a chordless cycle of length four. If P has precisely one edge then P along with v_0, v_1, w induces an F_{18}; if P has precisely two edges then P along with v_0, v_1, w induces an F_{19}; if P has at least three edges then P along with w induces a chordless cycle of length at least five. In Case 3, each vertex in A_2-v_2 has degree at least two in A_2. Since A_2-v_2 is nonempty, it follows that A_2 contains a cycle; let us call this cycle C_2. Now, we shall distinguish between two subcases.

Subcase 3.1 $N - \{v_1\} \neq \emptyset$.

Subcase 3.2 $N = \{v_1\}$.

In Subcase 3.1, F contains a chordless cycle of length four, induced by v_0, v_1, v_2, and a vertex in $N - \{v_1\}$; let us call this cycle C_1. With B standing for the subgraph of F induced by C_1, C_2, and a shortest path joining C_1 and C_2 in A_2, it is easy to verify that B either is a bicycle or else it contains one. In Subcase 3.2, each vertex of A_0 has degree at least two in the subgraph of F induced by $A_0 \cup \{v_1\}$; it follows that A_0 contains a cycle; let us call this cycle C_0. With B standing for the subgraph of F induced by C_0, C_2, and a shortest path joining C_0 to C_2 (necessarily passing through v_1), it is easy to verify that B either is a bicycle or else it contains one.

Thus, we have completed the proof that (b) implies (c).

To prove that (c) implies (d), consider an arbitrary graph G violating (d); we only need show that G violates (c). For this purpose, let F denote the graph obtained from H by deleting all the neighbours of v_1 except v_0 and v_2. Trivially, we have $v_0, v_1, v_2, v_3 \in F$ and $d_F(v_1) = 2$; in addition, each vertex x of F has $d_F(x) \geq d_H(x) - (d_H(v_1) - 2) \geq 2$. Thus, G violates (c).

To prove that (d) implies (a), consider an arbitrary graph G satisfying (d); we only need prove that G is Matula perfect. For this purpose, consider an arbitrary order < satisfying (6.2.2) and an arbitrary chordless path with vertices v_0, v_1, v_2, v_3 and edges v_0v_1, v_1v_2, v_2v_3; without loss of generality, we may assume that $v_1 < v_2$. Now we cannot have both $v_0 < v_1$ and $v_3 < v_2$, for then (d) would be violated by the subgraph H of G induced by all z with $z \leq v_2$. Hence < is a perfect order. □

Condition (c) of Theorem 6.2.2 can be tested in a polynomial time: each possible choice of v_0, v_1, v_2, v_3 may be considered separately (there are only $O(n^4)$ such choices, with n standing for the number of vertices in G) and, as soon as v_0, v_1, v_2, v_3 are fixed, the search for F becomes straightforward. (Letting F stand initially for the graph obtained from G by deleting all the neighbours of v_1 other than v_0 and v_2, we keep replacing F by

F-x as long as F contains a vertex x with $d_F(x) < 2$. If the graph F obtained in the end still includes v_0, v_1, v_2, v_3 then G violates (c); else G satisfies (c) for this particular choice of v_0, v_1, v_2, v_3.)

6.3 Algorithms

As usual, we shall denote the number of edges and the number of vertices of a graph by m and n, respectively; as usual, we shall assume that each graph is specified by its "adjacency lists" enumerating, for each vertex v, all the neighbours of v. In addition, we shall assume that a linear order < on the vertices of G is specified by an ordered list $w_1, w_2, \ldots, w_n$ of vertices such that $w_1 < w_2 < \ldots < w_n$.

We begin by spelling out the details of an algorithm that, given a graph G with a linear order < on the set of its vertices, computes the colouring f defined by (ii). (In the formal presentation of this and the following algorithms, we shall adopt the useful covention of initializing automatically all the numbers as zeros and all the sets as empty.) Here, the variable w_j stands for the vertex that is about to be coloured; the auxiliary variables a_i count the neighbors x of w_j with $f(x) = i$; the variables S_t and k are not needed now, but will be referred to later.

Algorithm A.

For $j = 1, 2, \ldots, n$ do

  for all neighbours x of w_j do $i \leftarrow f(x)$, $a_i \leftarrow a_i + 1$

  endfor

  $t \leftarrow 1$

  while $a_t > 0$ do $t \leftarrow t + 1$ endwhile

  $f(w_j) \leftarrow t$, $S_t \leftarrow S_t \cup \{w_j\}$, $k \leftarrow \max\ (k,t)$

  for all neighbours x of w_j do $i \leftarrow f(x)$, $a_i \leftarrow a_i - 1$

  endfor

endfor

Note that the a_i's are reset to zero at the end of each execution of the main loop, and so the number of positive a_i's never exceeds $d_G(w_j)$. Now it follows that each execution of the main loop takes time at most proportional to $1 + d_G(w_j)$, and the total running time comes to $O(m + n)$.

In particular, given a graph G along with a perfect order on G, Algorithm A finds a minimum colouring of G in $O(m + n)$ steps. Given a graph G along with a perfect order on the complement $\overline{G}$ of G, one may use the same principle to find a minimum colouring of $\overline{G}$. Here, however, care must be taken to keep the running time confined to $O(m + n)$: if G is sparse then even just enumerating all the edges of $\overline{G}$ would require a time far exceeding $O(m + n)$. We get around this difficulty by letting variables b_i count the vertices x with $f(x) = i$. To determine $f(w_j)$, we adjust the b_i's to account only for those vertices x that are adjacent to w_j in the complement of G (keeping track of the smallest candidate for $f(w_j)$ found so far); as soon as $f(w_j)$ is set, the b_i's are reinstated at their original values, and properly adjusted to reflect the appearance of $f(w_j)$. The details can be spelled out as follows.

Algorithm B

```
For j = 1,2,...,n do
    t ← k + 1
    for all neighbours x of w_j do
        i ← f(x), b_i ← b_i - 1
        if b_i = 0 then t ← min (t,i) endif
    endfor
    f(w_j) ← t, S_t ← S_t ∪ { w_j }, k ← max (k,t)
    for all neighbours x of w_j do i ← f(x), b_i ← b_i + 1
    endfor
   b_t ← b_t + 1
endfor
```

Trivially, each execution of the main loop takes time at most proportional to $1 + d_G(w_j)$, and so the total running times comes to $O(m + n)$.

A fast way of finding a largest clique in a graph with a perfect order $<$ has been developed in section 5: if f is the minimum colouring defined by (ii), then any clique consisting of vertices $u_{i+1}, u_{i+2}, \ldots, u_k$ with $f(u_j) = j$ for all j can be enlarged by adjoining a suitably chosen u_i with $f(u_i) = i$ (unless, of course, $i = 0$, in which case the clique clearly has the largest possible number of vertices). A straightforward procedure based on this fact will now be presented as an appendix to our Algorithm A; it relies on the colour classes $S_1, S_2, \ldots, S_k$ produced by the algorithm (and conveniently represented by linked lists); its output is the characteristic function c of the desired clique.

Appendix to Algorithm A

```
For i = k, k - 1, ..., 1 do
    for all v in S_i do
        count ← 0
        for all neighbours w of v do count ← count + c(w)
        endfor
        if count = k - i then new ← v
        endif
    endfor
    c(new) ← 1
endfor
```

Obviously, the running time of this Appendix is $O(m + n)$.

The same principle applies in the context of a graph G with a perfect order on its complement: an appendix to Algorithm B will find a largest clique in $\overline{G}$ (a largest stable set in G). Again, in order to keep the running time to O(m + n), it is crucial to avoid explicit references to the edges of $\overline{G}$. This is now easy to accomplish: we only need replace the test "count = k-i" by "count = 0". Our findings can be summarized as follows.

Theorem 6.3.1

Given any graph G along with a perfect order on G, one can find in time O(m + n) a minimum colouring and a largest clique in G. Given any graph G along with a perfect order on its complement $\overline{G}$, one can find in time O(m + n) a minimum clique cover and a largest stable set in G.□

M. Syslo pointed out to us that O(m + n) steps suffice to compute a linear order < satisfying (6.2.1) and a linear order < satisfying (6.2.2); later on, we discovered that the same fact has been also pointed out by D. Matula and L. L. Beck in 1983. To make our exposition self-contained, we shall now explain the details.

Arranging the vertices into a sequence $w_1, w_2, \ldots, w_n$ such that

$$d(w_1) \geq d(w_2) \geq \ldots \geq d(w_n)$$

is straightforward: having computed first the degrees $d(v)$ of all the vertices v and then the number n_j of vertices of degree j for each j, we only need ensure that $d(w_k) = i$ if and only if

$$\sum_{j=i+1}^{n-1} n_j < k \leq \sum_{j=1}^{n-1} n_j \ . \qquad (6.3.1)$$

For this purpose, we may use pointer variable p_i whose values, initialized as the left-hand side of (6.3.1), are gradually incremented until they reach the right-hand side of (6.3.1). The array r is not needed now but will be referred to later. It keeps track of the rank of each vertex in the linear order: we have $r(v) = k$ if and only if $w_k = v$.

<u>Algorithm C</u>

    <u>For</u> all vertices v <u>do</u>

        <u>for</u> all neighbours of v <u>do</u> $d(v) \leftarrow d(v) + 1$ <u>endfor</u>

    <u>endfor</u>

    <u>for</u> all vertices v <u>do</u> $j \leftarrow d(v)$, $n_j \leftarrow n_j + 1$ <u>endfor</u>

    <u>for</u> $i = n-2, n-3, \ldots, 0$ <u>do</u> $p_i \leftarrow p_{i+1} + n_{i+1}$ <u>endfor</u>

    <u>for</u> all vertices v <u>do</u>

        $i \leftarrow d(v)$, $k \leftarrow 1 + p_i$, $w_k \leftarrow v$, $r(v) \leftarrow k$, $p_i \leftarrow k$

    <u>endfor</u>

An appendix to Algorithm C will permute the sequence $w_1, w_2, \ldots, w_n$ to make it satisfy the condition that, with G_j standing for the subgraph of G induced by $\{w_1, w_2, \ldots, w_j\}$,

each w_j has the smallest degree in G_j.

This can be done iteratively: when $w_j, w_{j+1}, \ldots, w_n$ have been fixed, the values of $d(w_r)$ with $r < j$ are adjusted to the degrees of w_r in G_{j-1}, and the sequence $w_1, w_2, \ldots, w_{j-1}$ permuted to satisfy

$$d(w_1) \geq d(w_2) \geq \ldots \geq d(w_{j-1}).$$

During this process, the pointer variables p_i keep getting adjusted in such a way that the condition

$$d(w_r) \geq i \text{ if and only if } r \leq p_i$$

is maintained for all r with $r < j$.

<u>Appendix to Algorithm C</u>

```
For j = n, n - 1, ..., 4 do
   for all neighbours x of w_j do
      r ← r(x)
      if r < j then do
         i ← d(x), k ← min (p_i, j-1)
         w_r ← w_k, r(w_k) ← r, w_k ← x, r(x) ← k
         d(x) ← i - 1, p_i ← k - 1
```

endif

endfor

endfor

Finally, let $w_1, w_2, \ldots, w_n$ be once again the sequence produced by Algorithm C. (Note that the linear order < defined by $w_n < w_{n-1} < \ldots < w_1$ satisfies (6.2.1) with G replaced by its complement.) An alternative appendix to Algorithm C will permute this sequence to make it satisfy the condition that, with H_j standing for the subgraph of G induced by $\{w_j, w_{j+1}, \ldots, w_n\}$,

each w_j has the largest degree in H_j.

(Note that the linear order < defined by $w_n < w_{n-1} < \ldots < w_1$ satisfies (6.2.2) with G replaced by its complement.) This appendix is an easy variation on the appendix just described: the outer loop now runs for $j = 1, 2, \ldots, n-3$, the test "$r < j$" is replaced by "$r > j$", and the assignment "$k \leftarrow \min(p_i, j-1)$" is replaced by "$k \leftarrow p_i$".

Combining these observations with Theorem 6.3.1, we obtain the following result.

Theorem 6.3.2

Given any graph G that is Welsh-Powell perfect or Matula perfect, one can find in time $O(m + n)$ a minimum colouring and a largest clique in G. Given any graph G whose complement is Welsh-Powell perfect or Matula perfect, one can find in time O(m + n) a minimum clique cover and a largest stable set in G. □

6.4 Additional Remarks

1. The two classes of perfectly orderable graphs presented here are mutually incomparable: the graph F_{19} (see Figure 6.3) is Welsh-Powell perfect but not Matula perfect, the graph F_2 (see Figure 6.1) is Matula perfect but not Welsh-Powell perfect.

2. Recall that a graph G is strongly perfect if, for each induced subgraph F of G, some stable set of F meets all maximal cliques in F. In section 5, we have seen that all perfectly orderable graphs are strongly perfect; it follows that our Theorems 6.2.1 and 6.2.2 delineate two classes of strongly perfect graphs.

An important class of strongly perfect graphs consists of Meyniel graphs (which we shall encounter in section 8) defined as graphs in which every odd cycle has at least two chords: strong perfection of these graphs was established by Ravindra (1982). It is easy to see that every graph satisfying the hypothesis of our Theorem 6.2.2 is a Meyniel graph; however, graph F_{18} in Figure 6.3 satisfies the hypotheses of Theorem 6.2.1 and yet it is not a Meyniel graph.

Incidentally, it is easy to see that the following two conditions are equivalent for every graph G:

(i) both G and its complement $\bar{G}$ are Meyniel graphs,

(ii) G contains no induced subgraph isomorphic to C_5, P_5, or $\bar{P}_5$.

By our Theorem 6.2.1, condition (ii) implies that both G and $\bar{G}$ are Welsh-Powell perfect; in turn, Theorem 6.3.2 guarantees that O(m + n) steps suffice to find a minimum colouring, a largest clique, a minimum clique cover, and a largest stable set of any of these graphs.

7. BRITTLE GRAPHS

Consider a P_4 with vertices a,b,c,d and edges ab,bc,cd. The vertices a,d are called endpoints of this P_4; and the vertices b,c are called midpoints of this P_4. A vertex x of a graph G is said to be sensible if x is not an endpoint of any P_4, or x is not a midpoint of any P_4 in G. Chvátal defined a graph G to be brittle if each induced subgraph H of G has a sensible vertex. In this section, we shall present a sufficient condition for a graph to be brittle. This result was obtained jointly with Nelly Khouzam.

Fact 7.1

Let G be a graph and suppose that G contains a sensible vertex x. Then G is perfectly orderable if and only if G - x is perfectly orderable.

Proof

We only need prove the "if" part. Let $v_1 < v_2 < \ldots < v_n$ be a perfect order of G - x. If x is not an endpoint of any P_4 of G, then $x < v_1 < v_2 < \ldots < v_n$ is a perfect order of G; if x is not a midpoint of any P_4 of G, then $v_1 < v_2 < \ldots < v_n < x$ is a perfect order of G. □

Corollary 7.1

Every brittle graph is perfectly orderable. □

Note that a vertex x is sensible in a graph G if and only if it is sensible in $\bar{G}$. Thus the complement of a brittle graph is a brittle graph.

If a vertex x of a graph G is simplicial, then x is not a midpoint of any P_4 of G. Thus triangulated graphs, and their complements, are brittle.

Fact 7.2

If G is obtained from two perfectly orderable graphs G_1 and G_2 by substitution, then G is perfectly orderable.

Proof

Write $G = (V,E)$, $G_i = (V_i,E_i)$. Assume that G is obtained by substituting a vertex x_i of G_1 by G_2. That is V_2 is a homogeneous set of G. Let a perfect order of G_1 be $x_1 < x_2 < \ldots < x_{i-1} < x_i < x_{i+1} < \ldots < x_r$. Let a perfect order of G_2 be $v_1 < v_2 < v_3 < \ldots < v_s$. We claim that the order $P = x_1 < x_2 < \ldots < x_{i-1} < v_1 < \ldots < v_s < x_{i+1} < \ldots < x_r$ is a perfect order of G: the point is that if a P_4 has some vertex in V_2 and some vertex in $V-V_2$, then it has precisely one vertex in V_2 and three vertices in $V - V_2$; in other words, we can enumerate its vertices as u_1, u_2, u_3, u_4 with $u_1, u_2, u_3 \in V - V_2$, $u_4 \in V_2$ such that u_1, u_2, u_3, y is a P_4 whenever $y \in V_2$. Thus P contains no obstruction.□

Chvátal suggested the study of the class φ of graphs defined as follows: $G \in \varphi$ if and only if each induced subgraph of G is a comparability graph or else it has a homogeneous set or a sensible vertex. By Facts 7.1 and 7.2, φ is a class of perfectly orderable graphs. Chvátal has constructed the graph shown in Figure 7.1. This graph does not belong to φ and yet it is perfectly orderable.

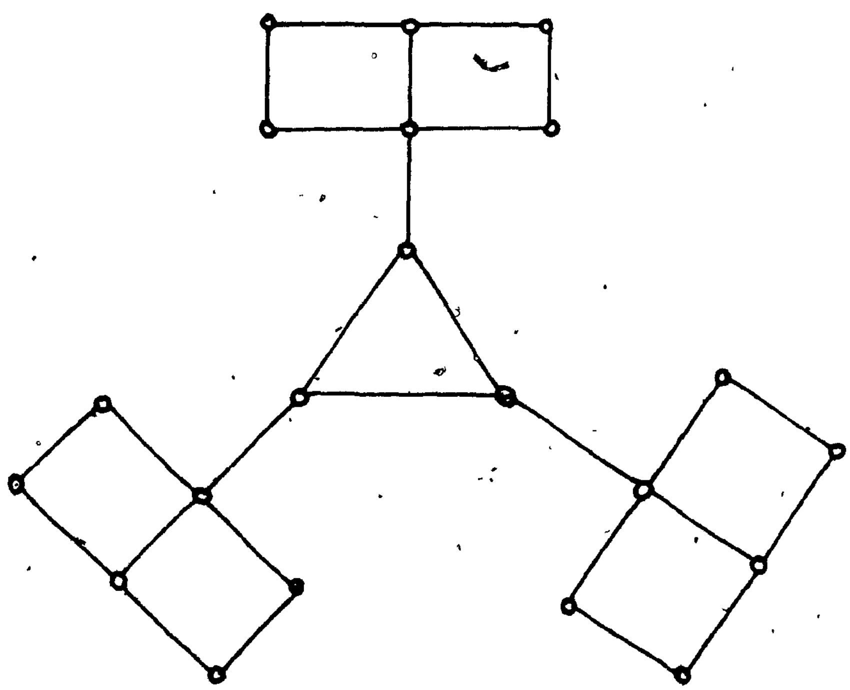


Figure 7.1

We shall investigate a related class φ^* defined as follows: $G \in \varphi^*$ if and only if each induced subgraph of G has a homogeneous set or a sensible vertex. Trivially, φ^* contains all brittle graphs; an example of a graph in φ^* that is not brittle can be obtained by substituting the graph in Figure 7.2(a) for vertices x and y of the graph in Figure 7.2(b).

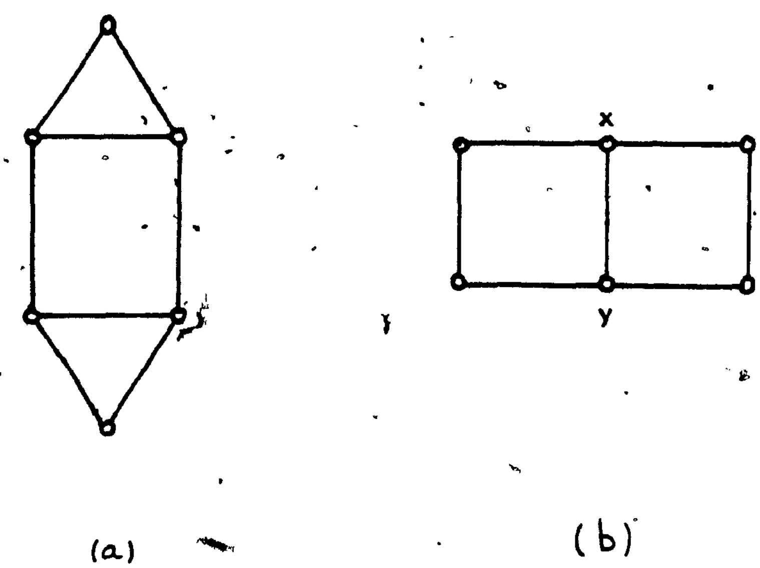


Figure 7.2

Fact 7.3

Let $G = (V,E)$ be a graph which does not contain a homogeneous set. If some vertex x in G is not a midpoint of any P_4, then x is a simplicial vertex of G.

Proof

Consider the set $N(x)$ of all neighbours of x. If $N(x)$ is a clique, then we are home free. Else $N(x)$ induces a component Y with $|Y| \geq 2$ such that Y is connected in $\bar{G}$. We may assume that Y is not a homogeneous set of G. Thus there is a vertex $z \notin (N(x) \cup \{x\})$ with $N(z) \cap Y \neq \emptyset$ and $N(z) \not\supseteq Y$. Since Y is connected in $\bar{G}$, there are vertices $u, t \in Y$ with $ut \notin E$, $zu \in E$, $zt \notin E$. But then x is a midpoint of the P_4 with vertices z, u, x, t, a contradiction.□

Note that a set Y is homogeneous in G if and only if Y is homogeneous in $\bar{G}$, and that a vertex x is not a midpoint of any P_4 in G if and only if x is not an endpoint of any P_4 in $\bar{G}$. Thus the following fact follows from Fact 7.3.

Fact 7.4

Let G be a graph which does not contain a homogeneous set. If G contains a vertex x which is not an endpoint of any P_4 in G, then x is a simplicial vertex of $\bar{G}$. □

Facts 7.3 and 7.4 show that if a graph G in φ* does not contain a homogeneous set, then G or $\bar{G}$ contains a simplicial vertex.

Before presenting the main result of this section, let us establish the following facts.

Fact 7.5

Let G be a graph such that each induced subgraph H of G satisfies at least one of the following conditions.

(i) H contains a homogeneous set.

(ii) H contains a vertex which is not an endpoint of any P_4.

Then each induced subgraph H of G satisfies (ii), and so G is brittle.

Proof

By induction the number of vertices. Let G be a graph satisfying the hypothesis of Fact 7.5. By the induction hypothesis, each proper induced subgraph of G satisfies (ii). If G also satisfies (ii), then we are done. Thus G contains a homogeneous set Y. For each y in Y, the graph $G_y = [\{y\} \cup N(Y) \cup \bar{N}(Y)]_G$ contains a vertex y' that is not an endpoint of any P_4 in G_Y. Similarly, in $[Y]_G$ there is a vertex y" that is not an endpoint of any P_4. If y' = y" then y' is not an endpoint of any P_4 in G: the point is that each P_4, with some vertex in Y and some vertex not in Y, must have one vertex in Y and three vertices not in Y (Lemma 4.6.1).

Now we have y' $\notin$ Y; again by Lemma 4.6.1, y' is not an endpoint of any P_4 in G. □

Fact 7.6

Let G be a graph such that each induced subgraph H of G satisfies at least one of the following conditions:

(i) H contains a homogeneous set.

(ii) H contains a vertex which is not a midpoint of any P_4.

Then each induced subgraph H of G satisfies (ii), and so G is brittle. □

In section 6, we have seen that every Matula perfect graph is brittle. The following theorem describes a class of brittle graphs which contains all Matula perfect graphs and all triangulated graphs.

Theorem 7.1

If a graph G does not contain an induced subgraph isomorphic to a C_k with $k \geq 5$, or a $\overline{P}_5$, or the graph H shown in Figure 7.2b, then G is brittle. □

By Fact 7.6, Theorem 7.1 is implied by the following theorem.

Theorem 7.2

If a graph G does not contain an induced subgraph isomorphic to a C_k with $k \geq 5$, or a $\overline{P}_5$, or the graph H shown in Figure 7.2b, then G satisfies one of the following.

(i) G is a clique.

(ii) G contains a homogeneous set Y such that Y induces a connected subgraph in $\overline{G}$.

(iii) G contains two nonadjacent simplicial vertices.

Note that Theorem 7.2 is best possible in this sense: each of the graphs C_5, C_6, C_7, ... , $\overline{P}_5$ and H fails to satisfy all conditions (i), (ii), (iii) of the theorem. We shall need the following two lemmas.

Lemma 7.1

If a graph G = (V,E) does not contain an induced subgraph isomorphic to a chordless cycle with at least five vertices, or a $\overline{P}_5$, or the graph H shown in Figure 7.2b, then one of the following three conditions holds.

(i) G is a clique.

(ii) G contains a homogeneous set that induces a connected subgraph in $\overline{G}$.

(iii) Every minimal cutset of G is a clique.

Proof

Assume that G is not a clique. Thus G contains a cutset. Consider a minimal cutset C of G. If C is a clique, then we are done. Now the subgraph of $\overline{G}$ induced by C contains at least one connected component Y with at least two vertices. Enumerate the components of G - C as $C_1, C_2, \ldots, C_k$. If Y is homogeneous, then we are done. Else there is a vertex x in some C_i with $N(x) \cap Y \neq \emptyset$ and $N(x) \not\supseteq Y$.

Since Y is connected in $\overline{G}$, there are vertices $y, z \in Y$ with $yz \notin E$, $xy \in E$, $xz \notin E$. Partition the vertices of C_i into disjoint sets A_0, A_1, A_2, A_3 such that

- $t \in A_0$ if $ty, tz \notin E$.
- $t \in A_1$ if $ty \in E$, $tz \notin E$.
- $t \in A_2$ if $ty \notin E$, $tz \in E$.
- $t \in A_3$ if $ty, tz \in E$.

Since $x \in A_1$, A_1 is nonempty. Since $N(z) \cap C_i \neq \emptyset$ (C is a minimal cutset), $A_2 \cup A_3$ is nonempty. Now, note that y and z are two endpoints of a chordless path P_j whose interior vertices lie entirely in C_j. It is easy to see that

$$uv \notin E \text{ whenever } u \in A_1,\ v \in A_2 \tag{7.1}$$

for otherwise any P_j with $j \neq i$ and u,v form a C_k with $k \geq 5$. Next, we claim that

$$uv \notin E \text{ whenever } u \in (A_1 \cup A_2),\ v \in A_3, \tag{7.2}$$

for otherwise P_j and u,v form a $\overline{P}_5$ (if P_j has three vertices), or P_j and v form a C_k with $k \geq 5$ (if P_j has at least four vertices).

Since C_i is connected, there is a path $v_1, v_2, \ldots, v_t$ in C_i such that $v_1 \in A_1$, $v_t \in A_2 \cup A_3$. Taking t as small as possible, we ensure that this path is chordless and (by (7.1), (7.2)) that $v_2 \in A_0$. By the minimality of t, we have $v_3, \ldots, v_{t-1} \in A_0$. If $v_t \in A_2$, then $y, v_1, v_2, \ldots, v_t$, z combined with any P_j $(j \neq i)$ is a chordless cycle of length at least five, a contradiction. Now, we may assume that $v_t \in A_3$. Since $y, v_1, v_2, \ldots, v_t$ is a chordless cycle, we must have $t = 3$. Take any P_j with $j \neq i$. If P_j has precisely three vertices, then these three vertices along with v_1, v_2, v_3 induce the graph H; else P_j along with v_t induces a chordless cycle of length at least five.□

A cutset C of a graph G is simplicial if in each component of G - C, there is a vertex x adjacent to all vertices of C.

Lemma 7.2

Let G = (V,E) be a graph satisfying the hypothesis of Lemma 7.1. Assume that G fails to satisfy either of the conditions (i) and (ii) of this lemma. Then each minimal cutset C of G is a simplicial cutset.

Proof

Enumerate the components of G - C as $C_1, C_2, \ldots, C_k$. We only need show that in each C_i, there is a vertex adjacent to all vertices of C.

Consider a component C_i and a vertex x of C_i such that $|N(x) \cap C_i| \geq |N(y) \cap C_i|$ whenever $y \in C_i$. We may assume that $C \not\subseteq N(x)$, for otherwise we are done. Now, in C, there is a nonemtpy set A of vertices such that $ax \notin E$ whenever $a \in A$.

Let B be the set of all those vertices in C_i that have a neighbour in A. Since C is minimal, $B \neq \emptyset$. Since C_i is connected, there is a path $v_1, v_2, \ldots, v_r$ in C_i such that $v_1 \in B$ and $v_r = x$. Taking r as small as possible, we ensure that the path is chordless and that $v_i \notin B$

whenever $i > 1$. Thus, there is a vertex a in A with

$$av_1 \in E,\ av_i \notin E \text{ for } i > 1. \qquad (7.4)$$

By the maximality of x, there is a vertex y in $N(x) \cap C$ with $yv_1 \notin E$. By Lemma 7.1, C is a clique, and so $ay \in E$. There is a chordless path P_j between a and y with at least one interior vertex, and all interior vertices in C_j for some $j \neq i$. Note that this path has at most four vertices, for otherwise its vertices induce a chordless cycle with at least five vertices.

Now, we have $xv_1 \notin E$ (else P_j and x, v_1 induce a $\overline{P}_5$, or the graph H). This implies that $r \geq 3$. We have $yv_2 \notin E$ (else P_j and v_1, v_2 induce a $\overline{P}_5$, or the graph H). Let m be the smallest subscript such that $yv_m \in E$. We have $m \geq r \geq 3$. By (7.4) the vertices $y, a, v_1, v_2, \ldots, v_m$ induce a chordless cycle with at least five vertices. □

<u>Proof of Theorem 7.2</u>

By induction on the number of vertices.

Let $G = (V,E)$ be a graph satisfying the hypothesis of Theorem 7.2. Assume that G fails to satisfy both (i) and (ii). By Lemma 7.1, G contains a minimal cutset C which is a clique. By Lemma 7.2, C is a simplicial cutset. We only need distinguish among two cases. (A cutset C of a graph G is <u>special</u> if C is simplicial and $G - C$ consists of precisely two components C_1 and C_2 with C_1 having one vertex and C_2 having at least two vertices.)

Case 1: C is not a special cutset.

Enumerate the components of $G - C$ as $C_1, C_2, \ldots, C_k$. If there are two components C_i such that each C_i consists of a single vertex, then condition (iii) is satisfied. Thus we may assume that there are (at least) two components C_1, C_2 such that both components have at least two vertices. Since C is simplicial, in $C_1(C_2)$ there is a vertex $x_1(x_2)$ with $N(x_1) \supseteq C$ $(N(x_2) \supseteq C)$. Let G_1 and G_2 be the two subgraphs of G induced by $(V - C_1) \cup \{x_1\}$ and $(V - C_2) \cup \{x_2\}$, respectively. We only need show that in each G_j,

$$\text{there is a simplicial vertex } y_j \text{ in } G_j - (C \cup \{x_j\}). \quad (7.5)$$

(Since y_j is a simplicial vertex of G, this establishes (iii).)

Consider the graph G_1. By the induction hypothesis, G_1 satisfies at least one of the three properties (i), (ii), (iii). Write $D = C \cup \{x_1\}$. Since x_1 is nonadjacent to each vertex in $G_1 - D$, G_1 is not a clique. If G_1 contains two nonadjacent simplicial vertices, then one of these must be in $G_1 - D$, and so (7.5) is established. Now, we may assume that G_1 contains a homogeneous set Y that induces a connected subgraph of $\bar{G}_1$. We can take Y to be the smallest set of vertices of G_1 with this property.

If $Y \cap D = \emptyset$, then Y is a homogeneous set of G, contradicting our assumption that (ii) fails for G. Note that

$$Y \not\subseteq D, \tag{7.6}$$

because Y induces a connected subgraph of $\bar{G}_1$. Thus, we have

$$Y \cap D \neq \emptyset \text{ and } Y - D \neq \emptyset. \tag{7.7}$$

Now, (7.7) implies that

$$x_1 \in Y. \tag{7.8}$$

(If $x_1 \notin Y$, then $Y \cap C \neq \emptyset$. But then Y is not homogeneous because $ux_1 \in E$ whenever $u \in C$, and $vx_1 \notin E$ whenever $v \in G_1 - D$.)

Consider an arbitrary component C_j of $G_1 - D$. We claim that

$$\text{either } C_j \cap Y = \emptyset \text{ or else } C_j \subset Y. \tag{7.9}$$

If (7.9) fails, then by the connectivity of C_j there are two vertices u,v with $u \in C_j - Y$, $v \in C_j \cap Y$, and $uv \in E$. Since $x_1 \in Y$ (by (7.8)), and $ux_1 \notin E$ (since $u \notin D$), Y is not a homogeneous set, a contradiction. Thus (7.9) holds.

Note that (7.9) implies

$$N_{G_1}(Y) \subsetneq C \qquad (7.10)$$

Let G_Y be the subgraph of G induced by Y. By the induction hypothesis, G_Y must satisfy one of the three conditions (i), (ii), (iii). G_Y is not a clique because $\bar{G}_Y$ is connected. G_Y can not contain a homogeneous set S that induces a connected subgraph of $\bar{G}$: this set S would also be a homogeneous set of G_Y and $|S| < |Y|$, contradicting our choice of Y. Thus, G_Y contains two nonadjacent simplicial vertices y_1, y_2. Since D is a clique, we may assume that $y_2 \in G_1 - D$. From (7.8) it follows that $xy \in E$ whenever $x \in C - Y$ and $y \in Y$. This fact and (7.10) imply that y_2 is also a simplicial vertex of G_1. We have established (7.5) and settled this case.

<u>Case 2</u>: Every minimal cutset C of G is special.

Now, G - C contains precisely two components C_1, C_2, and C_1 has precisely one single vertex x, and C_2 has at least two vertices. Write G' = G - x. By the induction hypothesis, G' satisfies at least one of the three properties (i), (ii), (iii).

If G' is a clique, then each vertex y in C_2 is a simplicial vertex of G. Thus, x and y are two nonadjacent simplicial vertices of G.

If G' contains two nonadjacent simplicial vertices y_1 and y_2, then at least one y_i must lie in C_2 (because $x \cup N(x)$ is a clique). Thus x and y_i are two nonadjacent simplicial vertices.

It remains to show that if G' contains a homogeneous set Y which induces a connected subgraph of $\overline{G}$, then G satisfies at least one of (i), (ii), (iii). In this case, we can take Y to be the smallest homogeneous set in G' that induces a connected subgraph of $\overline{G}$. Let G_Y be the subgraph of G induced by Y. By the induction hypothesis G_Y satisfies at least one of the three properties (i), (ii), (iii).

Since $\overline{G}_Y$ is connected, G_Y is not a clique. G_Y can not contain a homogeneous set Y' which induces a connected subgraph of $\overline{G}$': Y' would be a homogeneous set of G' with $|Y'| < |Y|$, contradicting our choise of Y. Thus, G_Y contains two nonadjacent simplicial vertices y_1, y_2.

Since C is a clique, at least one y_i lies in $Y \cap C_2$. If $N_{G'}(Y)$ induces a clique in G', then we are done: x and y_i are two nonadjacent simplicial vertices of G. Thus we may assume that

$N_{G'}(Y)$ does not induce a clique in G. (7.11)

Write $A = N_{G'}(Y) \cap C$, and $B = N_{G'}(Y) - C$. We claim that

$$ab \in E \text{ whenever } a \in A,\ b \in B. \qquad (7.12)$$

First, note that $Y \cap C \neq \emptyset$ (or else Y is a homogeneous set of G contradicting our assumption on G) and that $Y - C \neq \emptyset$ (or else $Y \subseteq C$ and so $\bar{G}_Y$ is not connected, a contradiction).

Now, we can justify (7.12) as follows. Since $\bar{G}_Y$ is connected, there are nonadjacent vertices b_1, b_2 with $b_1 \in Y \cap C$ and $b_2 \in Y \cap C_2$. If (7.12) fails, then x, a, b, b_1, b_2 induce a $\bar{P}_5$, a contradiction.

Now, (7.11) and (7.12) imply that B contains a set B' such that $|B'| \geq 2$, and B' induces a connected component of the subgraph of $\bar{G}$ induced by B. We may assume that

$$\bar{N}_{G'}(Y) \neq \emptyset, \qquad (7.13)$$

for otherwise (7.12) implies that B' is a homogeneous set in G, inducing a connected subgraph in $\bar{G}$, which contradicts our assumption on G.

By (7.13), $N_{G'}(Y)$ is a cutset in G'. Hence, $N_{G'}(Y)$ is also a cutset in G (since $Y \cap C \neq \emptyset$, we have $C \subseteq (Y \cup N_{G'}(Y))$, and so $N(x) \cap \bar{N}_{G'}(Y) = \emptyset$). Now, G has a minimal cutset C' with $C' \subseteq N_{G'}(Y)$. By assumption of case 2, C' is special. Thus $G - C'$ has precisely two components C_1', C_2' with C_1' consisting of a single vertex c and C_2' containing at least two vertices. Since $Y \cap C \neq \emptyset$ and $Y \cap C' = \emptyset$, x along with all the vertices in $Y \cap C$ belongs to C_2'. Hence x and c are two nonadjacent simplicial vertices of G.□

8. MEYNIEL GRAPHS

8.1 Introduction

This section is concerned with the notion of a "good stable set": we shall say that a stable set S of a graph G is good if S meets all maximal cliques of G. We shall call a graph very strongly perfect if, for each induced subgraph H of G, each vertex of H belongs to a good stable set of H. By this definition, every very strongly perfect graph is strongly perfect.

Henry Meyniel (1976) proved that a graph G is perfect if each odd cycle, with at least five vertices, contains at least two chords. Nowadays, such graphs are called Meyniel graphs. Later, Ravindra (1982) proved that every Meyniel graph is strongly perfect. Meyniel then conjectured that every Meyniel graph is very strongly perfect. In subsection 8.2, we shall prove that a graph is very strongly perfect if and only if it is a Meyniel graph. In subsection 8.3, we design a polynomial-time algorithm which, given a Meyniel graph G and an arbitrary vertex x of G, finds a good stable set of G that contains x. In subsection 8.4, we establish another property, related to perfection, of Meyniel graphs.

8.2 Meyniel Graphs are Very Strongly Perfect

The purpose of this subsection is to prove the following theorem.

Theorem 8.2.1

A graph is very strongly perfect if and only if it is a Meyniel graph.

Our proof relies on the following two lemmas.

Lemma 8.2.1 (Meyniel (1976))

If a graph $G = (V,E)$ is Meyniel, then G contains no odd cycle $v_0v_1 \ldots v_{2k}$ $(k \geq 2)$ such that the path $v_1v_2 \ldots v_{2k}$ is chordless, and v_0 is nonadjacent to some v_i.

Proof of Lemma 8.2.1

If $v_0v_2 \in E$, then consider the largest subscript i such that v_0 is adjacent to $v_1, v_2, \ldots, v_i$, and the smallest subscript j such that $j > i + 1$ and $v_0v_j \in E$. The cycle $v_0v_i \ldots v_j$ is chordless, the cycle $v_0v_{i-1}v_i \ldots v_j$ has exactly one chord and one of these two cycles is odd.

Now, we can assume that $v_0v_2 \notin E$. Consider the smallest even subscript j such that $v_0v_j \in E$ and the largest subscript i such that $i < j - 2$ and $v_0v_i \in E$. The cycle

$v_0v_i \ldots v_j$ is odd (i must be odd) and it has at most one chord. □

Lemma 8.2.2 (Ravindra (1982))

If a graph $G = (V,E)$ contains a cycle $wv_0v_1 \ldots v_k$ such that

(i) v_0 is adjacent to none of the vertices $v_2, v_3, \ldots, v_k$,

(ii) w is not adjacent to v_1, and

(iii) there is a good stable set S, of $G - v_0$, that contains v_1 and v_k,

then G is not a Meyniel graph.

Proof of Lemma 8.2.2

By a starter, we shall mean a cycle that satisfies the conditions (i), (ii) and (iii) of Lemma 8.2.2.

We may assume that

(iv) $v_1v_2 \ldots v_kw$ is the shortest path from v_1 to w which satisfies conditions (i), (ii), (iii).

It follows from (iv) that

(v) the path $v_1v_2 \ldots v_k$ is chordless.

Next, we may assume that

(vi) every v_r adjacent to w has an even subscript r, otherwise the odd cycle $wv_0v_1\ldots v_r$ satisfies the hypothesis of Lemma 8.2.1, and so G is not Meyniel.

Write $y \in S^*$ if $y \in S$ and y is adjacent to some two consecutive vertices v_j, v_{j+1} on the path $v_0v_1\ldots v_k$. We may assume that

(vii) no $v_j \in S^*$ is adjacent to v_0,

otherwise either the cycle $yv_0\ldots v_j$ or the cycle $yv_0\ldots v_{j+1}$ (with yv_j, $yv_{j+1} \in E$) satisfies the hypothesis of Lemma 8.2.1, and so G is not Meyniel. Now, we claim that

(viii) no $y \in S^*$ is adjacent to w.

If (viii) was false, then (iv) would be contradicted by $v_0v_1\ldots v_iyw$ such that i is the smallest subscript with $yv_i \in E$. Next, we may assume that

(ix) each $y \in S^*$ is adjacent to at least three vertices on the path $v_0v_1\ldots v_k$,

otherwise the desired cycle is $wv_r\ldots v_jyv_{j+1}\ldots v_sw$ with r standing for the largest subscript with $r \leq j$, $wv_r \in E$; and s standing for the smallest subscript with $s \geq j + 1$, $wv_s \in E$. It follows that

(x) each $y \in S^*$ is adjacent to precisely three vertices v_{j-1}, v_j, v_{j+1} on the path $v_1, v_2, \ldots, v_j$;

otherwise (iv) would be contradicted by $v_1\ldots v_ryv_s\ldots v_k$ such that r is the smallest subscript with $yv_r \in E$ and s is the largest subscript with $yv_s \in E$.

Now, we can choose our starter so that S^* is minimized. We claim that

(xi) $S^* = \emptyset$;

because for each $y \in S^*$ adjacent to v_{j-1}, v_j, and v_{j+1}, the substitution of y for v_j in the original starter yields a new starter with a smaller S^*, contradicting our choice of the original starter.

Now, note that k is even (by (vi)). Since $S^* = \emptyset$ (by (xi)), each edge of the chordless path $v_1v_2 \ldots v_k$ must have precisely one endpoint in S. Since $v_1 \in S$, we must have $v_3 \in S$, $v_5 \in S$, ..., $v_{k-1} \in S$. But then the edge $v_{k-1}v_k$ has both endpoints in S, a contradiction. □

Proof of Theorem 8.2.1

The "only if" part of the theorem can be settled by observing that if a graph is not Meyniel then it contains an odd cycle C with at least five vertices, and with at most one chord; furthermore, we can assume that the only chord of C (if it is present in C) is a triangulated chord. It suffices to prove that C is not very strongly perfect. We can enumerate the vertices of C as $v_1, v_2, v_3, \ldots, v_t$, (with t being an odd subscript and $t \geq 5$) with edges v_iv_{i+1}, and the edge v_2v_t if C has one chord (otherwise v_2v_t is not present in C). Now, suppose that v_1 belongs to a good stable set S of C. Then we must have $v_3 \in S$,

$v_5 \in S, \ldots, v_{t-2} \in S$; but then v_{t-1} can not be in S, neither can v_t: the maximal clique $v_{t-1}v_t$ is not met by S, a contradiction.

The "if" part is proved by induction on the number of vertices. Let G = (V, E) be a Meyniel graph. By the induction hypothesis, we only need prove that each vertex of G belongs to a good stable set of G. Consider an arbitrary vertex x of G. If x is adjacent to all the vertices of G - x, then $\{x\}$ meet all maximal cliques of G; otherwise choose a vertex y nonadjacent to x such that $|N(y) \cap N(x)| \geq |N(z) \cap N(x)|$ for each vertex z nonadjacent to x.

By the induction hypothesis, G - x is very strongly perfect. Therefore, y belongs to a good stable set S_y of G - x. Let Y be the connected component of the subgraph of G induced by V - N(x) such that Y contains y. By the induction hypothesis, G - Y is very strongly perfect. Thus, x belongs to a good stable set S_x of G - Y. Write $S = S_x \cup (S_y \cap Y)$. Note that there is no edge with one endpoint in S_x and the other endpoint in Y, and so S is a stable set. We only need prove that

S is a good stable set of G.

For this purpose, assume the contrary: some maximal clique C in G is disjoint form S. Note that

$$C \cap Y \neq \emptyset. \qquad (8.2.1)$$

For otherwise, $C \subseteq G - Y$, and so $C \cap S_x \neq \emptyset$, contradicting $C \cap S = \emptyset$. Next, (8.2.1) implies that

$$C \subseteq Y \cup N(x). \tag{8.2.2}$$

Finally, we must have

$$C \cap N(x) \neq \emptyset, \tag{8.2.3}$$

for otherwise $C \subseteq Y$, and so $C \cap (S_y \cap Y) \neq \emptyset$, contradicting $C \cap S = \emptyset$.

Since S_y is a good stable set of $G - x$, C must include a vertex v_1 of S_y. We must have $v_1 \in N(x)$, for otherwise $v_1 \in Y$, and so $v_1 \in C \cap S$, contradicting $C \cap S = \emptyset$. Write $A = N(v_1) \cap Y$. By (8.2.1), we have $A \neq \emptyset$; note that $y \notin A$ since both y and v_1 belong so S_y. Since Y is connected, there is a path in Y from a vertex in A to y. Consider a shortest such path P. We can enumerate the vertices of P as $v_2, v_3, \ldots, v_k$ with $v_2 \in A$, $v_i \notin A$ for $i \geq 3$, and $v_k = y$. Note that

$$wv_1 \in E \text{ whenever } w \in N(y) \cap N(x). \tag{8.2.4}$$

If (8.2.4) was false then the cycle $x, v_1, v_2, \ldots, v_k, w$ (with $x = v_0$) would satisfy conditions (i), (ii), (iii) of the Lemma 8.2.2 and so G would not be a Meyniel graph, a contradiction.

Now, (8.2.4) holds. Since $v_1 \in N(v_2) \cap N(x)$ but $v_1 \notin N(y) \cap N(x)$, and since $|N(y) \cap N(x)| \geq |N(v_2) \cap N(x)|$ by our choice of y, there must be a vertex w in $(N(y) - N(v_2)) \cap N(x)$. Let i be the smallest subscript such that $wv_i \in E$ and $i \neq 1$; note that $i \geq 3$. If i is even then $wv_1v_2\ldots v_i$ is a chordless cycle with at least five vertices; if i is odd then $wv_1v_2\ldots v_i$ is an odd cycle with at least five vertices and only one chord. In both cases, we arrive at a contradiction.□

8.3 Finding Good Stable Sets of Meyniel Graphs

Burlet and Fonlupt (1984) showed that all connected Meyniel graphs can be constructed from certain "basic Meyniel graphs" by repeated applications of amalgam. In this section, we are going to rely on this result to design a polynomial-time algorithm which, given a Meyniel graph G and any vertex x of G, finds a good stable set of G that contains x. First, we need introduce a few definitions.

A graph G = (V,E) is basic Meyniel if V can be partitioned into disjoint sets K, B, S^* with the following properties.

- $[B]_G$ is a two-connected bipartite graph (possibly $B = \emptyset$).
- $[K]_G$ is a clique.
- We have $xy \in E$ whenever $x \in B$, $y \in K$.

- S^* is a stable set of G, and, for each vertex x in S^*, we have $|N(x) \cap B| \leq 1$.

Recall that a graph $G = (V,E)$ has a proper amalgam decomposition if V can be partitioned into disjoint sets K, A_1, B_1, A_2, B_2 with the following properties.

- $K \cup A_1 \cup A_2 \neq \emptyset$.
- $[K]_G$ is a clique.
- We have $xy \in E$ whenever $x \in A_i$, $y \in A_j$ $(i \neq j)$ or $x \in A_i$, $y \in K$.
- We have $xy \notin E$ whenever $x \in B_i$, $y \in (A_j \cup B_j)$, $i \neq j$.
- $|A_i \cup B_i| \geq 2$.
- $A_1 = \emptyset$ if and only if $A_2 = \emptyset$.
- If $A_1 = A_2 = \emptyset$, then in each B_i there is a vertex x_i with $N(x_i) \supseteq K$.

Note that if a graph G has a proper amalgam decomposition, then G is an amalgam of its induced subgraphs G_1, G_2 defined as follows:

(i) if $A_i \neq \emptyset$, then $G_i = [K \cup A_i \cup B_i \cup \{a_j\}]_G$ where a_j is a vertex of A_j,

(ii) if $A_i = \emptyset$, then $G_i = [K \cup B_i \cup \{x_j\}]_G$.

It is easy to verify that G is a Meyniel graph if and only if G_1, G_2 are both Meyniel graphs. Burlet and

Fonlupt (1984) designed a polynomial-time algorithm to recognize a Meyniel graph. They proved that if G is a connected Meyniel graph, then either G is basic Meyniel or else G has a proper amalgam decomposition. Furthermore, they proved that this decomposition is polynomial: G (and its subgraphs produced by the proper amalgam decomposition) can only be decomposed into a polynomial number of smaller graphs which are basic Meyniel graphs (these graphs can be recognized in polynomial time). We shall assume that we have the following procedure GRENOBLE(G) (a modified version of the algorithm given in Section 5 of Burlet and Fonlupt (1984)) which, given a Meyniel graph G, finds a proper amalgam decomposition of G, or else it shows that G is a basic Meyniel graph.

Procedure GRENOBLE(G)

Input. A Meyniel graph $G = (V,E)$.

Output. 1. G is basic Meyniel: a partition of G into sets K,B,S^*.

2. G has a proper amalgam decomposition: a partition of V into sets K,A_1,B_1,A_2,B_2, or (if $A_1 = A_2 = \emptyset$) sets $K,A_1,A_2,\{x_1\},\{x_2\}$.

The following procedure FIND(G,x,S) performs the following operation: given an input a Meyniel graph G and a vertex x of G, FIND returns as output a good stable set S of G such that S contains x.

Procedure FIND(G,x,V)

Input. A Meyniel graph $G = (V,E)$ and a vertex x of G.

Output. A good stable set S of G such that S contains x.

Begin

1. If G is disconnected, then find the connected components $C_1, C_2, \ldots, C_k$ of G. Find the subscript j such that $x \in C_j$.
   - For i = I to k do call FIND(C_i, x_i, S_i), where x_i is an arbitrary vertex of C_i for $i \neq j$, and $x_i = x$ for $i = j$.
   - Let $S \leftarrow S_1 \cup S_2 \cup \ldots \cup S_k$, return S and stop.
2. Call GRENOBLE(G). If G is basic Meyniel then go to 3, else go to 4.
3. (Now, the sets K,B,S* are returned.) If $x \in K$, then go to 3.1, else partition B into two stable sets B_1, B_2 such that $B = B_1 \cup B_2$. If $x \in B$ then go to 3.2, else go to 3.3.

   3.1 Let $S \leftarrow \{x\} \cup S'$, where $S' = S^* - N(x)$, return S and stop.

3.2 Find the subscript i such that $x \in B_i$, and let $S \leftarrow B_i \cup S'$, where $S' = \{y \mid N(y) \cap B_i = \emptyset\}$, return S and stop.

3.3 (Now, $x \in S^*$.) Execute the following steps.

3.3.1 If there is a vertex x' in K with $xx' \notin E$, then let $S \leftarrow \{x'\} \cup (S^* - N(x'))$, return S and stop.

3.3.2 Note that there is a subscript i with $N(x) \cap B_i = \emptyset$. Let $S \leftarrow B_i \cup S'$ where $S' = \{y \mid y \in S^*, \text{ and } N(y) \cap B_i = \emptyset.\}$, return S and stop.

4. (Now, the sets K, A_1, B_1, A_2, B_2 or the sets $K, B_1, B_2, \{x_1\}, \{x_2\}$ are returned. For the remaining steps, a_j will be an arbitrary vertex of A_j, however if $A_j = \emptyset$, then we let $a_j = x_j$.) Execute the following steps.

4.1 If $x \in K$ then

. for i = 1 to 2 do call FIND(G_i,x,S_i) where $G_i = [K \cup A_i \cup B_i \cup \{a_j\}]_G$ with $i \neq j$.

. let $S \leftarrow S_1 \cup S_2$, and return S and stop.

4.2 If $x \in A_1$ then

. call FIND(G_1,x,S_1), where $G_1 = [K \cup A_1 \cup B_1 \cup \{a_2\}]_G$,

. call FIND(G_2,x,S_2), where $G_2 = [K \cup A_2 \cup B_2 \cup \{x\}]_G$,

. let $S \leftarrow S_1 \cup S_2$, return S and stop.

4.3 If $x \in A_2$, then interchange A_1 and A_2, B_1 and B_2, and go to step 4.2.

4.4 (Now, $x \in (B_1 \cup B_2)$) If $x \in B_2$, then interchange B_1 and B_2, interchange A_1 and A_2. (Now, $x \in B_1$.) Execute the following steps.

- call FIND(G_1,x,S_1), where $G_1 = [K \cup A_1 \cup B_1 \cup \{a_2\}]_G$,
- find a vertex y' in $S_1 \cap (K \cup A_1 \cup \{a_2\})$,
- call FIND(G_2,y',S_2) with $G_2 = [K \cup A_2 \cup B_2 \cup \{a_1\}]_G$ in case $y' \in (K \cup \{a_2\})$, and $G_2 = [K \cup A_2 \cup B_2 \cup \{y'\}]_G$ in case $y' \in A_1$,
- let $S \leftarrow S_1 \cup S_2$, return S and stop.

end (procedure).

Proof of Correctness of Procedure FIND

First, we shall show that the procedure works correctly on all basic Meyniel graphs.

Step 3.1 Trivial: any maximal clique not meeting S' must meet x.

Step 3.2 Suppose that there is a maximal clique C not meeting S. (We shall show that C can not exist.) First, we claim that

$$C \cap S^* = \emptyset. \qquad (8.3.1)$$

Assume that (8.3.1) is false. By our choise of S, for each vertex $u \in S^* - S$, there is a vertex $u' \in B_i$ with $uu' \in E$. Since $|N(u) \cap B| \leq 1$, we have $N(u) \subsetneq N(u')$. Thus, any maximal clique containing u must contain u'. This shows that C is met by S, a contradiction. Hence (8.3.1) holds.

Since $B \neq \emptyset$, K can not be a maximal clique of G. So we have $C \not\subseteq K$. This fact and (8.3.1) imply that there is a vertex y in $C \cap B_j$ with $i \neq j$. Since $[B]_G$ is a two-connected bipartite graph, there is some y' in B_i with $yy' \in E$. Now, note that B_j is a stable set, and $uv \in E$ whenever $u \in K$, $v \in B$. These facts and (8.3.1) imply that any maximal clique containing y must contain y'. Hence C is met by S. This is the desired contradiction.

<u>Step 3.3.1</u> Similar to Step 3.1.

<u>Step 3.3.2</u> Similar to Step 3.2.

For steps 4.1, 4.2, 4.3, note that S is a stable set of G. We may assume that S is not a good stable set of G, for otherwise we are done. So there is a maximal clique C with $C \cap S = \emptyset$, $C \cap G_i \neq \emptyset$, $i = 1,2$. For each of the steps 4.1, 4.2, 4.4, we are going to show that C can not exist. If

$A_1 = A_2 = \emptyset$, then each maximal clique of G lies entirely in G_1 or in G_2. Thus we can assume that $A_1 \neq \emptyset$ and $A_2 \neq \emptyset$. It follows that $C \subseteq (K \cup A_1 \cup A_2)$.

Step 4.1 Since $C \subseteq (K \cup A_1 \cup A_2) \subseteq N(x) \cup \{x\}$, C must meet x, a contradiction.

Step 4.2 Since $C \cap S_2 = \emptyset$ and S_2 is a good stable set of G_2, we have $C' = C \cap (K \cup A_1) \neq \emptyset$. Note that $x \notin C'$ since $C \cap S_1 = \emptyset$. But then in G_1, $C' \cup \{a_2\}$ is a maximal clique, and this maximal clique is not met by S_1 contradicting our assumption that S_1 is a good stable set of G_1.

Step 4.4 Note that y' must exist because in G_1 there must be a maximal clique containing a_2, and this maximal clique must be met by S_1. If $y' = a_2$ or $y' \in A_1$, then we can apply the analysis of Step 4.2; otherwise we can apply the analysis of Step 4.1.□

8.4 Another Characterization of Meyniel Graphs

Recall that two vertices of a graph G are two friends if they are not endpoints of a chordless path with an odd number of edges. As we have seen in section 3, no two vertices can be friends in a minimal imperfect graph. Meyniel (1985) showed that if G is a Meyniel graph, then either G is a clique or else G contains two friends. In this subsection, we are going to establish a stronger property of Meyniel graphs. (A vertex x of a graph $G = (V,E)$ is <u>universal</u> if $\{x\} \cup N(x) = V$.)

<u>Theorem 8.4.1</u>

A graph G is a Meyniel graph if and only if, for each induced subgraph H of G and for each vertex x of H, one of the following two conditions holds.

(i) x is a universal vertex of H.

(ii) x is a friend of some vertex in H.

<u>Proof</u>

The "if" part is easy; to prove the "only if" part, consider an arbitrary vertex x of a Meyniel graph $H = (V,E)$. We can assume that x is not a universal vertex of H. Let x' be a vertex in $A = V - (N(x) \cup \{x\})$ such that for each z in A, we have

$$|N(x') \cap N(x)| \geq |N(z) \cap N(x)|. \tag{8.4.1}$$

We claim that x and x' are friends. Suppose that our claim is false. Then there is a chordless path $v_1 v_2 \ldots v_{2k}$ with $k \geq 2$, $x = v_1$, $x' = v_{2k}$. Note that there must be a vertex v_0 in $N(x) \cap N(x')$ with $v_0v_3 \notin E$; for otherwise $|N(v_3) \cap N(x)| > |N(x') \cap N(x)|$, contradicting (8.4.1). But then the cycle $v_0v_1 \ldots v_{2k}$ satisfies the hypothesis of Lemma 8.2.1, contradicting our assumption that H is a Meyniel graph. □

9. ALTERNATELY ORIENTABLE GRAPHS AND ALTERNATELY COLOURABLE GRAPHS

9.1 Introduction

In this section, we establish a property of minimal imperfect graphs, and use this property to generate two classes of perfect graphs. The first class contains all comparability graphs, all triangulated graphs, and several other classes of perfect graphs. The second class contains all triangulated graphs, and all line-graphs of bipartite graphs.

9.2 Alternating Orientation of Perfect Graphs

By a hole, we mean a chordless cycle with at least four vertices. Recall (from section 3) that a bad P_3 is a graph with vertices a,b,c and arcs (directed edges) ab,bc (and no other arcs). An orientation $\vec{G}$ of a graph G is an alternating orientation if no hole of $\vec{G}$ contains a bad P_3. Such a graph G is called an alternately orientable graph.

Theorem 9.2.1

Every alternately orientable graph is perfect.

To prove Theorem 9.2.1, we shall rely on the following results.

First, Ghouila-Houri (see section 4) proved that G is a comparability graph if and only if G admits an orientation $\vec{G}$ such that $\vec{G}$ does not contain a bad P_3 ($\vec{G}$ could be cyclic).

Second, Chvátal (see section 3) proved that no minimal imperfect graph G can contain a star cutset. We are going to use Chvátal's theorem to establish a certain property of minimal imperfect graphs.

Theorem 9.2.2

Let G be a minimal imperfect graph. Then each P_3 of G extends into a hole.

Proof

We are going to prove a stronger statement. We only need prove that for any graph G with at least two vertices, at least one of the following two properties (i), (ii) holds.

(i) G contains a star-cutset.

(ii) Each induced P_3 in G extends into a hole.

(If G is minimal imperfect, then by Chvátal's theorem, (i) must fail for G.)

If (ii) fails, then some P_3 with vertices a,b,c and edges ab,bc does not extend into a hole. But then b and all its neighbours except a and c form a cutset in G (separating a from c); hence (i) holds.□

Proof of Theorem 9.2.1

Let G be an alternately orientable graph, and let $\vec{G}$ be its alternating orientation. If G is not perfect, then $\vec{G}$ contains a minimal imperfect graph. Thus without loss of generality, we may assume that G is minimal imperfect. Now $\vec{G}$ must contain a bad P_3, for otherwise by Gouila-Houri's theorem G is perfect, a contradiction. But by Theorem 9.2.2 this bad P_3 extends into a hole. This is also a contradiction. □

9.3 Subclasses of Alternately Orientable Graphs

1. By definition, the class of alternately orientable graphs contains all comparability graphs and all triangulated graphs.

2. A graph is i-triangulated if each of its odd cycles with at least five vertices contains at least two non-crossing chords. Gallai (1962) proved that every i-triangulated graph is perfect. Burlet and Fonlupt (1984) proved a decomposition theorem for i-triangulated graphs. They proved that every i-triangulated graph G is either a "basic i-triangulated graph" or else G contains a simplicial clique

cutset (defined in section 7). A graph $G = (V,E)$ is a <u>basic i-triangulated</u> graph if V can be partitioned into disjoint sets A,K,S such that

(i) A can be partitioned into stable sets $A_1, A_2, \ldots, A_k$, $k \geq 3$, each A_i contains at least two vertices, and two vertices of A are adjacent if and only if they belong to different stable sets,

(ii) K induces a clique in G,

(iii) We have $xy \in E$ whenever $x \in A$, $y \in K$,

(iv) S is a stable set and for each $x \in S$ we have $|N(x) \cap A| \leq 1$.

(Note that each vertex in S is simplicial.)

<u>Theorem 9.3.1</u>

Every i-triangualted graph is alternately orientable.

<u>Proof</u>

By induction on the number of vertices. Let $G = (V,E)$ be an i-triangulated graph. If G is basic i-triangulated, then an alternating orientation $\vec{G}$ of G can be obtained as follows. We direct x to y whenever $x \in A_i$, $y \in A_j$ with $i > j$, or $x \in A$, $y \notin A$, or $x \in K$, $y \in S$. Now, we can assume that G contains a simplicial clique cutset C. Let G_1 and

G_2 be two induced subgraphs of G such that $G = G_1 \cup G_2$, and $G_1 \cap G_2 = C$. By the induction hypothesis, G_1 and G_2 are alternately orientable. Let $\vec{G}_1$ and $\vec{G}_2$ be the alternating orientations of G_1 and G_2, respectively. Now, note that for each $\vec{G}_1$, the direction of each edge uv of C is immaterial: no such edge can belong to a hole (if uv belongs to a hole C_k of G_1, then in $G_2 - C$ there is a vertex x, adjacent to both u and v, such that the subgraph induced by x and this C_k contains an odd cycle with at least five vertices with at most one chord). The above remark shows that $\vec{G} = \vec{G}_1 \cup (\vec{G}_2 - C)$ is an alternating orientation of G. □

3. Let u and v be two vertices of a graph G. We say that u dominates v if $N(u) \cup \{u\} \supseteq N(v)$. (It is easy to see that domination is transitive: if x dominates y and y dominates z, then x dominates z.) Chvátal and Hammer (1973) defined a graph to be a threshold graph if for any two vertices u,v, either u dominates v, or v dominates u.

Theorem 9.3.2

If a graph G is union of two threshold graphs then G is alternately orientable.

Proof

Let G be union of two threshold graphs G_1 and G_2. Now, the edges of each G_i can be directed so that

(i) if a,b,c induces a P_3 in G_i, with b being the interior vertices of this path, then b is directed to both a and c.

We can realize (i) by directing x to y if xy is an edge of G_i and x dominates y in G_i. (If x and y dominate each other then xy can be directed either way.)

Next, it is easy to see that each edge of a C_4 of G can belong only to one G_i. That is, it can not belong to the intersection of G_1 and G_2. It follows that

(ii) each C_4 of G can be decomposed into two P_3's, one of these belongs to G_1 and not G_2, the other belongs to G_2 and not G_1.

Since G can not contain a hole with more than four vertices, (i) and (ii) imply that G admits an alternating orientation. □

4. Golumbic, Monma and Trotter (1984) found yet another class of alternately orientable graphs. They proved that every "tolerance graph" admits an alternating orientation. It was this result that motivated our work.

5. We are going to show that every P_4-sparse graph is alternately orientable. Note that there are P_4-sparse

graphs which are not i-triangulated. One such graph is shown in Figure 9.1

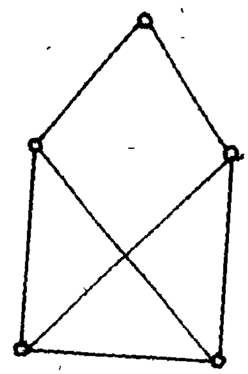


Figure 9.1

<u>Theorem 9.3.3</u>

Every P_4-sparse graph is alternately orientable.

<u>Proof</u>

By induction on the number of vertices.

Let $G = (V,E)$ be a P_4-sparse graph. By Theorem 4.6.2, we only distinguish among three cases.

Case 1: G is a spider or a cospider.

In this case, G is a triangulated graph and therefore an alternately orientable graph.

Case 2: G or $\overline{G}$ is disconnected.

Partition V into two disjoint sets V_1, V_2 such that for every choice of $x \in V_1$, $y \in V_2$,

if G is disconnected, then $xy \notin E$, and

if $\overline{G}$ is disconnected, then $xy \in E$.

By the induction hypothesis, the graph $G_i = [V_i]_G$ admits an alternating orientation $\vec{G}_i$. If G is disconnected, then $\vec{G} = \vec{G}_1 \cup \vec{G}_2$ is an alternating orientation of G. If $\overline{G}$ is disconnected, then G admits an alternating orientation $\vec{G} = \vec{G}_1 \cup \vec{G}_2 \cup \vec{X}$, where $\vec{X}$ is the set of all arcs xy with $x \in V_1$, $y \in V_2$, and $xy \in E$.

Case 3: G contains a clique cutset C.

Without loss of generality, we may assume that C is a minimal cutset of G. Let G_1 and G_2 be two induced subgraphs of G such that $G = G_1 \cup G_2$ and $G_1 \cap G_2 = C$. By the induction hypothesis, each G_i is alternately orientable. So G_i admits an alternating orientation $\vec{G}_i$. Now we claim that

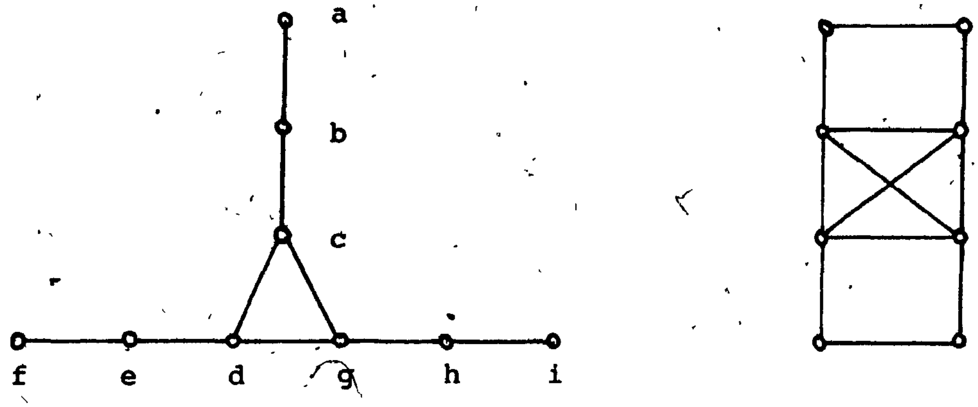
$\vec{G} = \vec{G}_1 \cup (\vec{G}_2 - C)$ is an alternating orientation of G.

To justify our claim, it sufficies to show that no vertices v_1, v_2 of C extend into a hole H in G_2: if H exists, then we can enumerate the vertices of H as $v_1, v_2, v_3, \ldots, v_k$ (with edges $v_i v_{i+1}$ and edge $v_1 v_k$) so that, for each $i > 2$, $v_i \in G_2 - C$ (note that $v_i \notin G_1$). We have $k = 4$ for otherwise G is not P_4-sparse. By the minimality of C, there is a vertex $x \in G_1 - C$ with $xv_1 \in E$. But then the five vertices x, v_1, v_2, v_3, v_4 contain two distinct P_4's, a contradiction. □

9.4 Perfect Graphs Which are Not Alternately Orientable

1. A graph G is a weakly triangulated graph if G does not contain a hole or the complement of a hole. Hayward (1984) proved that every weakly triangulated graph is perfect. He also showed that weakly triangulated graphs are not necessarily alternately orientable. Consider the complement $\bar{H}$ of the graph H shown in Figure 9.2a.

First, note that if xy and uv are two edges of a graph G, with x,y being nonadjacent to u,v, in G, then $\{x,y,u,v\}$ induces a C_4 in $\bar{G}$. We shall write $xy \to uv$ to mean that x and y are directed to u and v in $\bar{G}$. Now without loss of generality, we may assume that in $\bar{H}$ we have $ab \to fe$. This forces the relations $bc \to fe$, $cg \to fe$, $gh \to fe$, $hi \to fe$,



(a) (b)

Figure 9.2

ab → ed, ab → dg, ab → gh, ab → hi, bc → hi, cd → hi, ed → hi, fe → hi. But the relations hi → fe and fe → hi imply that $\overline{H}$ can not admit an alternating orientation.

2. The graph shown in Figure 9.2b is alternately orientable but not strongly perfect.

3. Since complements of triangulated graphs are perfectly orderable, the complement of the graph shown in Figure 9.2a is perfectly orderable but not alternately orientable.

4. We are going to construct a Meyniel graph which is not alternately orientable. Let G be a Meyniel graph and let x be a vertex of G. It is easy to see that the graph G' obtained from G by <u>duplicating</u> x (that is, adding

a vertex x' nonadjacent to x and joining x' to z if and only if xz is an edge of G) remains a Meyniel graph.

Consider the Meyniel graph G shown in Figure 9.3.

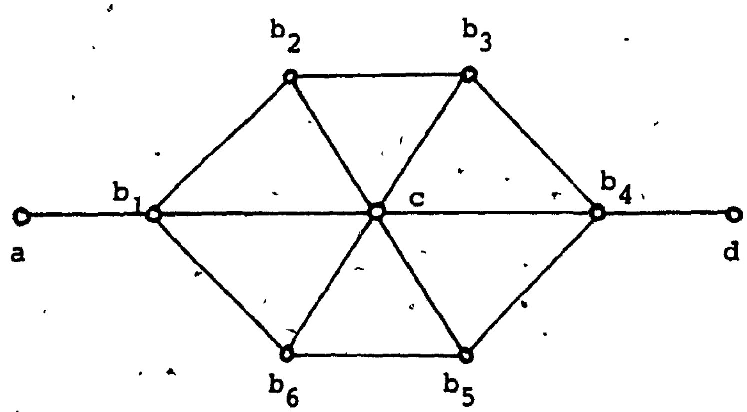


Figure 9.3

The graph H obtained from G, by first duplicating c, and then duplicating b_1 and b_4, is a Meyniel graph. But H can not admit an alternating orientation: without loss of generality, we may assume that b_1 is directed to b_2.

This forces b_1 to be directed to a, and so b_1 is forced to be directed to c. Directing b_1 to b_2 also forces b_3 to be directed to b_2, and b_3 to be directed to b_4. Directing b_3 to b_4 forces d to be directed to b_4, and so c is forced to be directed to b_4. But now, b_1cb_4 is a bad P_3.

9.5 Alternating Colouration of Perfect Graphs

We say that a graph G admits an alternating colouration if the edges of G can be coloured by two colours such that no hole of G contains a monochromatic P_3, that is a P_3 whose two edges are of the same colour. Recall that a line-graph of a graph H is a graph G whose vertices are edges of H, two vertices of G being adjacent if and only if they share an endpoint as edges of H. We shall say that a graph is alternately colourable if it admits an alternating colouration. It is easy to show that a graph G is a line-graph of bipartite graph if and only if the edges of G can be coloured by two colours such that the edges of each colour form vertex-disjoint cliques. Thus every line-graph of bipartite graphs is alternately colourable. Furthermore, by definition, every triangulated graph is alternately colourable.

Theorem 9.5.1

Every alternately colourable graph is perfect.

It is easy to see that if G is a line-graph of bipartite graphs then G must be claw-free and Berge. Chvátal and Sbihi (1985) designed a polynomial-time algorithm to recognize claw-free Berge graphs. In the process of doing so, they found many graphs which are claw-free Berge, and which do not admit an alternating colouration. One such graph is shown in Figure 9.4.

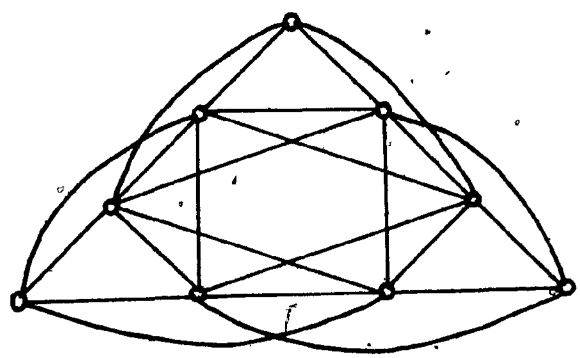


Figure 9.4

The graph shown in Figure 9.5 is alternately orientable but is not alternately colourable. The graph $\overline{C}_6$ is alternately colourable but is not alternately orientable (also, it is not a quasi-parity graph).

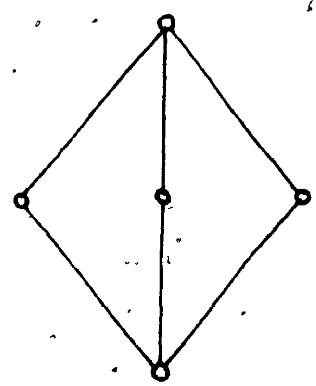


Figure 9.5

Proof of Theorem 9.5.1

Let G be a graph with an alternating colouration. If G is not perfect then G contains a minimal imperfect graph. Thus without loss of generality, we may assume that G is minimal imperfect. Now, G must contain a monochromatic P_3, or else G is a claw-free Berge graph and by Parthasarathy and Ravindra's theorem, G is perfect, a contradiction. But by Theorem 9.2.2, this monochromatic P_3 extends into a hole. This is a contradiction. □

9.6 A Recognition Algorithm

In this section, we show that the problem of determining whether a graph admits an alternating orientation (or colouration) can be solved in polynomial time. Let G = (V,E) be a graph. First, we want to partition the edges of G into "equivalence classes" $E_1, E_2, \ldots$ by the following recursive rule: two edges e_1 and e_2 belong to the same E_i if and only if

(i) e_1 and e_2 belong to the same hole, or

(ii) there are edges e_3, e_4 in E_i such that e_1 and e_3 belong to the same hole, and e_2 and e_4 belong to the same hole.

To find the equivalent classes, we only need construct certain classes $E_1^*, E_2^*, \ldots$ by this rule: two edges e_1, e_2 belong to the same E_i^* if and only if they form a P_3, and

(iii) this P_3 extends into a hole.

We can test (iii) as follows. Let a,b,c be the vertices of a P_3, with b being the interior vertex. This P_3 extends into a hole if and only if there is a connected component C of $\bar{N}(b)$ with $N(a) \cap C \neq \emptyset$, and $N(c) \cap C \neq \emptyset$.

The desired equivalence classes $E_1, E_2 \ldots$ can be found by recursively merging two classes E_i^*, E_j^* if and only if they intersect. Now, once the direction (colour) of an edge in each E_i is fixed, the directions (colours) of all other edges in this E_i are determined. We can assume that each edge is forced to accept only one direction (colour), for otherwise G is not alternately orientable (colourable). Now, the resulting orientation (colouration) is alternating if and only if no bad P_3 (monochromatic P_3) extends into a hole.

10. EVEN DECOMPOSITIONS

10.1 The Main Results

In this section, we give a proof, obtained jointly with Chvátal, of the following theorem.

Theorem 10.1.1

Let the vertices of a graph G be coloured red and white in such a way that each induced P_4 in G has an even number of vertices of each colour. Then G is perfect if and only if each of its two subgraphs induced by all the vertices of the same colour is perfect. □

This theorem reduces the task of testing perfection of G into the task of testing perfection of two nonempty vertex-disjoint induced subgraphs of G as soon as the vertices of G can be coloured red and white in such a way that

(i) each induced P_4 in G has an even number of vertices of each colour,

(ii) each of the two colours appears on at least one vertex of G.

Not every perfect graph can be coloured in this way: for example, see any of the three graphs in Figure 10.1.

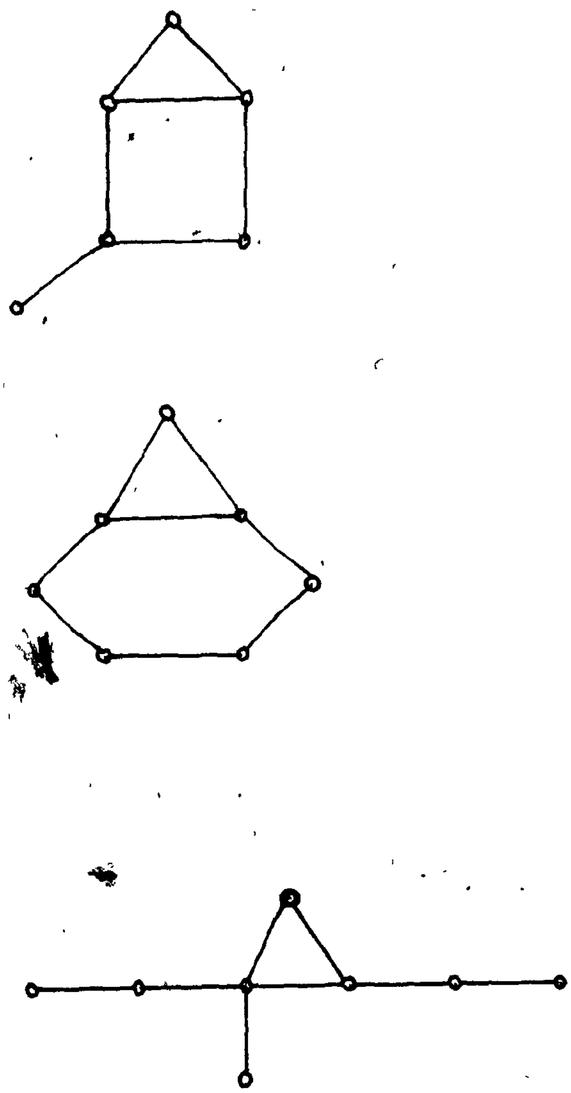


Figure 10.1

Graphs that do admit two-colourings with properties (i) and (ii) are recognizable in a polynomial time: when "v is red" and "v is white" are represented by "$x_v = 1$" and "$x_v = 0$", respectively, condition (i) assumes the form of a (small) system of linear congruences modulo two. Now we only need find out if this system, with x_w set at zero for an arbitrary but fixed vertex w, has a nonzero solution; this can be done routinely by Gaussian elimination.

10.2 Auxillary Results

Our proof of Theorem 10.1.1 relies on the following results concerning perfect graphs. First, the Perfect Graph Theorem states that a graph is perfect if and only if its complement is. Second, as mentioned previously, Parthasarathy and Ravindra proved that every claw-free Berge graph is perfect. Third, as mentioned in section 3, the following three statement are true.

No minimal imperfect graph can contain a homogeneous set.

No minimal imperfect graph can contain a clique set.

No minimal imperfect graph can contain vertices v, w with $N(v) \subseteq \{w\} \cup N(w)$.

By virtue of the above facts, the validity of Theorem 10.1.1 is guaranteed by the following result:

Theorem 10.2.1

Let the vertices of a graph G be coloured by two colours red and white in such a way that each P_4 has an even number of vertices of each colour, and that each colour appears at least once. Then

(i) G or $\overline{G}$ is a claw-free Berge graph, or

(ii) G or $\overline{G}$ contains a homogeneous set, or

(iii) G or $\overline{G}$ contains a clique cutset, or

(iv) G or $\overline{G}$ contains vertices v and w, with $N(v) \subseteq \{w\} \cup N(w)$. □

If G has at least three vertices, then any of properties (ii), (iii), (iv) of Theorem 10.2.1 implies that G or $\overline{G}$ has a star-cutset; hence Theorem 10.2.1 implies the following fact.

Corollary 10.2.1

If G satisfies the hypothesis of Theorem 10.2.1, then G or $\overline{G}$ is a claw-free Berge graph, or else G or $\overline{G}$ has a star-cutset. □

One graph that satisfies the hypothesis of Theorem 10.2.1 is the graph G obtained from four disjoint complete graphs on vertices a_i, b_i, c_i, d_i, e_i, f_i (i = 1,2,3,4) by adding edges

$$a_i d_{i+1}, d_{i+1} b_i, b_i e_{i+1}, e_{i+1} c_i, c_i f_{i+1}, f_{i+1} a_i$$

for all i = 1,2,3,4 (with subscript 5 interpreted as 1); neither G nor $\overline{G}$ has a star-cutset.

We shall prove Theorem 10.2.1 by proving the following two lemmas.

Lemma 10.2.1

Let the vertices of a graph G be coloured red and white in such a way that each induced P_4 in G has two vertices of each colour. Then G has at least one of properties (i), (ii), (iii), (iv) in Theorem 10.2.1.

Lemma 10.2.2

Let the vertices of a graph G be coloured red and white in such a way that the hypothesis of Theorem 10.2.1 is satisfied and that all four vertices of some induced P_4 in G have the same colour. Then G has at least one of properties (ii), (iii), (iv) in Theorem 10.2.1.□

One graph that satisfies the hypothesis of Lemma 10.2.1 is the graph obtained from disjoint copies G_1, G_2, G_3, G_4 of the graph G_i shown in Figure 10.2 by joining, for each $i = 1,2,3$, each of the vertices a_i, b_i, c_i, d_i to each of the vertices $a_{i+1}, b_{i+1}, c_{i+1}, d_{i+1}$. This graph G has none of the properties (i), (ii), (iii) in Theorem 10.2.1

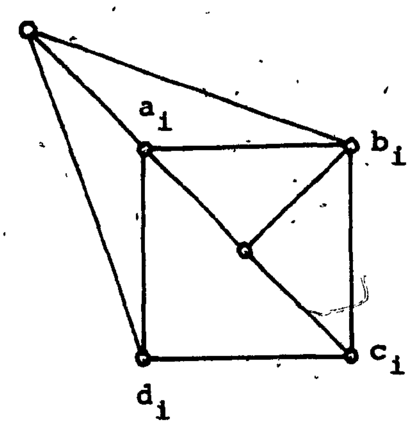


Figure 10.2

One graph that satisfies the hypothesis of Lemma 10.2.2 is the graph obtained from the graph shown in Figure 10.3 by joining each vertex labelled R_1 or W_1 to both vertices labelled R_C, and joining each vertex labelled W_2 to all the vertices labelled R_C or R_S. This graph G has none of the properties (i), (ii), (iii) in Theorem 10.2.1.

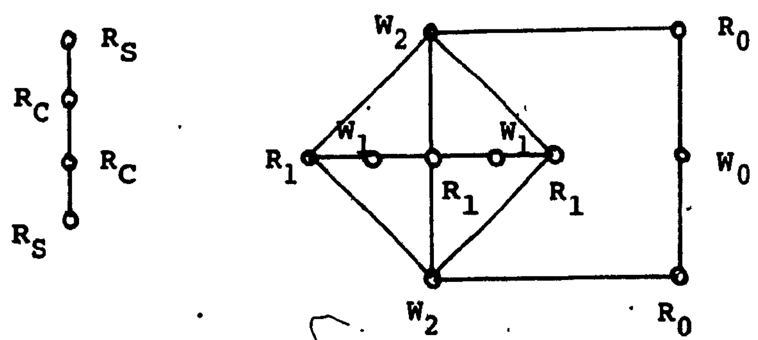


Figure 10.3

We shall derive Lemma 10.2.2 from a statement involving the following notion, suggested to us by Minoru Ishii: an <u>alignment</u> in a graph is a sequence $Q_1, Q_2, \dots, Q_k$ of sets of vertices such that each Q_i induces a P_4, and each Q_i with $i \geq 2$ has precisely one vertex outside $Q_1 \cup Q_2 \cup \dots \cup Q_{i-1}$. The alignment is called <u>full</u> if each vertex of the graph belongs to at least one Q_i.

<u>Lemma 10.2.3</u>

If some alignment in a graph G does not extend to a full alignment then G has at least one of properties (ii), (iii), (iv) in Theorem 10.2.1.

To derive Lemma 10.2.2 from Lemma 10.2.3, denote the set of vertices of the monochromatic P_4 by Q_1 and consider an arbitrary alignment $Q_1, Q_2, \dots, Q_k$ that extends the alignment Q_1. An easy induction on i shows that all four vertices in Q_i must have the colour of Q_1; since each of the two colours appears on at least one vertex of G, the alignment $Q_1, Q_2, \dots, Q_k$ can not be full.

Now, we only need prove Lemma 10.2.1 and 10.2.3.

<u>10.3 The Proofs</u>

Throughout this section, we let E stand for the set of edges of G.

Proof of Lemma 10.2.1

We shall often rely on the following theorem of Seinsche (see section 4):

if G contains no induced P_4, then G or $\bar{G}$ is disconnected.

Let G satisfy the hypothesis of the lemma; let R and W stand for the subgraphs of G induced by all the red vertices and all the white vertices, respectively. Given any two disjoint sets S and T of vertices in G, we shall partition S into three subsets as follows:

$u \in S_0(T)$ if $u \in S$ and $uv \notin E$ whenever $v \in T$,

$u \in S_2(T)$ if $u \in S$ and $uv \in E$ whenever $v \in T$,

$u \in S_1(T)$ if $u \in S$ and $u \notin S_0(T) \cup S_2(T)$.

We shall often rely on the following observation, applying to any component A or R and any component B of W:

$$N(z) \subseteq A \cup B \text{ whenever } z \in A_1(B) \cup B_1(A). \qquad (10.3.1)$$

By symmetry, we only need prove (10.3.1) with $z \in A_1(B)$. Note that B includes adjacent vertices x,y such that $xz \in E$, $yz \notin E$. If z had a neighbour w outside $A \cup B$ then trivially $w \in W-B$; but then wzxy would be a badly coloured P_4.

The remainder of our proof amounts to a case analysis in the guise of an algorithm. During the execution of this algorithm, G may be replaced by its complement; note that both the hypothesis and the conclusion of the lemma are invariant under this transformation.

0. If W is connected then replace G by its complement. (By Seinsche's theorem, the complement of W is disconnected.)

1. Now W is disconnected. If no two vertices of W are adjacent and no two vertices of R are adjacent then stop: G is bipartite, and so $\bar{G}$ is claw-free. If no two vertices of W are adjacent and R is connected, then stop: W is a clique cutset in the complement of G (by Seinsche's theorem, the complement of R is disconnected). If no two vertices of W are adjacent, some two vertices of R are adjacent, and R is disconnected, then switch colours.

2. Now W is disconnected and it has a component B with $|B| \geq 2$. If B is a homogeneous set, then stop; else there are vertices r,s,t such that $r \in R$, $s,t \in B$ and $rs \in E$, $rt \notin E$. Let A be the component of R that contains r. If $|A| = 1$ then stop: in this case, G is disconnected or else $N(r) \subseteq N(w)$ for some w in R. (To see this, let R* stand for the set of all the vertices in R that have at least one neighbour in B. If some w in R is adjacent to all the vertices in B then (10.3.1) guarantees that $N(r) \subseteq N(w)$; else (10.3.1) guarantees that there is no edge xy with $x \in R^* \cup B$,

$y \notin R^* \cup B$.) If $|A| \geq 2$ then

go to 3 in case $A = A_1(B)$,

go to 4 in case $A_0(B) \neq \emptyset$, $A_1(B) \neq \emptyset$, $A_2(B) = \emptyset$,

go to 8 in case $A_1(B) \neq \emptyset$, $A_2(B) \neq \emptyset$.

(Since $r \in A_1(B)$, all the eventualities are covered.)

3. Now there are a component A of R and a component B of W such that $|A| \geq 2$ and $A = A_1(B)$; furthermore, W is disconnected.

If $B_1(A) = \emptyset$ then stop: (10.3.1) implies that W is a homogeneous set. If $B_1(A) \neq \emptyset$ and $B_2(A) \neq \emptyset$ then switch colours and go to 8.

Now we have $B_1(A) \neq \emptyset$ and $B_2(A) = \emptyset$. If $B_0(A) \neq \emptyset$ and R is disconnected then switch colours and go to 4; else stop: (10.3.1) implies that there are no edges xy with $x \in A \cup B$, $y \notin A \cup B$, and so G is disconnected.

4. Now there are a component A of R and a component B of W such that $A_0(B) \neq \emptyset$, $A_1(B) \neq \emptyset$, $A_2(B) = \emptyset$; furthermore, W is disconnected.

If A is not a clique then go to 7; if A is a clique then proceed as follows.

If R is connected or $B = B_1(A)$ then stop: by (10.3.1), A is a clique cutset. If R is disconnected and $B \neq B_1(A)$ then note that $B_2(A) = \emptyset$, $B_1(A) \neq \emptyset$ and $B_0(A) \neq \emptyset$; if B is not a clique then switch colours and go to 7.

5. <u>Now there are a component A of R and a component B of W such that A is a clique, $A_1(B) \neq \emptyset$, $A_2(B) = \emptyset$ and B is a clique, $B_1(A) \neq \emptyset$, $B_2(A) = \emptyset$; furthermore, W is disconnected.</u>

Extend the subgraph of G induced by $A \cup B$ into a maximal connected induced subgraph H of G such that every component A^* of $H \cap R$ and every component B^* of $H \cap W$ have the following properties:

(a) A^* is a clique and a component of R,

(b) B^* is a clique and a component of W,

(c) $A_2^*(B^*) = \emptyset$ and $B_2^*(A^*) = \emptyset$.

If H = G then go to 6; if G is disconnected then stop; else find an edge xy such that $x \notin H$, $y \in H$. We may assume (by switching colours if necessary) that $x \in R-H$ and $y \in W \cap H$.

Let $\tilde{A}$ be the component of R that contains x and let $\tilde{B}$ be the component of W that contains y. By (a), we have $\tilde{A} \cap H = \emptyset$; by (b), we have $\tilde{B} \subset H$. We claim that

$$\tilde{A}_2(B^*) = \emptyset \text{ for every component } B^* \text{ of } H \cap W: \quad (10.3.2)$$

since H is connected, there is a component A^* of $H \cap R$ such that $B_0^*(A^*) \neq B^*$. By (c), we have $B_1^*(A^*) \neq \emptyset$; now (10.3.1) implies that $N(z) \cap \tilde{A} = \emptyset$ whenever $z \in B_1^*(A^*)$; it follows that $\tilde{A}_2(B^*) = \emptyset$.

If $\tilde{A} = \tilde{A}_1(B^*)$ for some component B^* of $H \cap W$ then stop:

(10.3.1) implies $N(z) \subseteq \tilde{A} \cup B^*$ whenever $z \in \tilde{A}$, and so B^* is a clique cutset. If $\tilde{A} = \tilde{A}_1(B^*)$ for no component B^* of $H \cap W$ then go to 7. (In this case, we have $B_2^*(A) = \emptyset$ for each component B^* of $H \cap W$: else we would have $\tilde{A}_0(B^*) = \emptyset$, and so $\tilde{A}_1(B^*) = \tilde{A}$ by virtue of (10.3.2), contradicting the assumption. But then maximality of H implies that $\tilde{A}$ is not a clique; in addition, we have $\tilde{A}_2(\tilde{B}) = \emptyset$, $\tilde{A}_1(\tilde{B}) \neq \emptyset$, and $\tilde{A}_1(\tilde{B}) \neq \tilde{A}$.)

6. <u>Now every component of R is a clique and every component of W is a clique; furthermore, $A_2(B) = \emptyset$ and $B_2(A) = \emptyset$ for every component A of R and every component B of W.</u>

Stop: G is claw-free Berge. (G is claw-free since (10.3.1) guarantees that each N(z) is covered by two cliques; G is Berge simply because it satisfies the hypothesis of the lemma.)

7. <u>Now there are a component A of R and a component B of W such that $A_0(B) \neq \emptyset$, $A_1(B) \neq \emptyset$, and $A_2(B) = \emptyset$; furthermore, W is disconnected and A is not a clique</u>.

We shall distinguish among three cases.

<u>Case 7.1</u>: Some u in $A_0(B)$ is nonadjacent to some v in $A_1(B)$.

Replace G by its complement and go to 8: we claim that u has no neighbours in W. To justify this claim, note first that the shortest path from u to v in A has precisely three vertices, for otherwise A would contain a (badly coloured)

P_4. Next, note that the midpoint x of this path must be in $A_1(B)$: if it were in $A_0(B)$ then there would be a badly coloured P_4 consisting of u,x,v, and a vertex in $N(v) \cap B$. Finally, if u had a neighbour z in W-B then zuxv would be a badly coloured P_4 by virtue of (10.3.1).

<u>Case 7.2</u>: Every vertex in $A_0(B)$ is adjacent to every vertex in $A_1(B)$, but $A_0(B)$ is not a clique.

In this case, consider the subgraph of G induced by $A_0(B)$; the complement of this graph has a component H with at least two vertices. Stop: we claim that H is a homogeneous set. To justify this claim, assume the contrary. Now there are vertices x,y,z with $x,y \in H$, $z \notin H$, $xz \in E$, $yz \notin E$ and $xy \notin E$. Trivially, $z \in W-B$; but then (10.3.1) implies that yvxz is a badly coloured P_4 whenever $v \in A_1(B)$.

<u>Case 7.3</u>: Every vertex in $A_0(B)$ is adjacent to every vertex in $A_1(B)$, but $A_1(B)$ is not a clique.

In this case, consider the subgraph of G induced by $A_1(B)$; the complement of this graph has a component H with at least two vertices. Stop: we claim that H is a homogeneous set. To justify this claim, assume the contrary. Now there are vertices x,y,z with $x,y \in H$, $z \notin H$, $xz \in E$, $yz \notin E$ and $xy \notin E$. By (10.3.1), we have $z \in B$; but then yuxz is a badly coloured P_4 whenever $u \in A_0(B)$.

8. <u>Now there are a component A of R and a component B of W such that $A_1(B) \neq \emptyset$, $A_2(B) \neq \emptyset$.</u>

Again, we shall distinguish among three cases.

Case 8.1: Some vertex w in $A_2(B)$ is not adjacent to all the vertices in $A_1(B)$.

In this case, let us first show that

$$\text{no vertex in } A_1(B)-N(w) \text{ has a neighbour in } A_0(B). \qquad (10.3.3)$$

Assuming the contrary, we find vertices u,v such that $v \in A_1(B)-N(w)$ and $u \in A_0(B) \cap N(v)$. Next, we find adjacent vertices x,y in B such that $x \in N(v)$ and $y \notin N(v)$. Finally, if $uw \notin E$ then uvxw is a badly coloured P_4; if $uw \in E$ then vuwy is a badly coloured P_4.

Next, let us show that

$$N(v) \cap A_2(B) \subseteq N(w) \text{ whenever } v \in A_1(B) - N(w) \qquad (10.3.4)$$

Assuming the contrary, we find a vertex z in $N(v) \cap A_2(B)$ such that $z \notin N(v)$; but then vzyw is a badly coloured P_4 whenever $y \in B - N(v)$.

If no two vertices in $A_1(B) - N(v)$ are adjacent then stop: (10.3.1), (10.3.3), (10.3.4) imply that $N(v) \subseteq N(w)$ whenever $v \in A_1(B) - N(w)$. Otherwise, the subgraph of G induced by $A_1(B) - N(w)$ has a component H with at least two vertices; stop: we claim that H is a homogeneous set. To justify this claim, assume the contrary. Now there are vertices x,y,z such that $x,y \in H$, $z \notin H$, $xz \in E$, $yz \notin E$ and $xy \in E$. Trivially, $z \notin R-A$; by (10.3.1), we have $z \notin W - B$; by (10.3.3) we have $z \notin A_0(B)$. Furthermore, $z \notin A_1(B)$, for otherwise $z \in A_1(B) \cap N(w)$, and so wzxy is a badly coloured P_4. Thus, we may assume $z \in B \cup A_2(B)$; now (10.3.4) with $v = x$ implies $z \in N(w)$; but then again $wzxy$ is a badly coloured P_4.

<u>Case 8.2</u>: Every vertex in $A_2(B)$ is adjacent to every vertex in $A_1(B)$, and $A_2(B)$ is not a clique.

In this case, consider the subgraph of G induced by $A_2(B)$; the complement of this graph has a component H with at least two vertices. Stop: we claim that H is a homogeneous set. To justify this claim, assume the contrary. Now there are vertices x,y,z such that $x,y \in H$, $z \notin H$, $xz \in E$, $yz \notin E$ and $xy \notin E$. Trivially, $z \in A_0(B)$ or $z \in W - B$; if $z \in A_0(B)$ then $zxty$ is a badly coloured P_4 whenever $t \in B$; if $z \in W - B$ then (10.3.1) guarantees that $zxty$ is a badly coloured P_4 whenever $t \in A_1(B)$.

<u>Case 8.3</u>: Every vertex in $A_2(B)$ is adjacent to every vertex in $A_1(B)$, and $A_2(B)$ is a clique.

Stop: we claim that $N(v) \subseteq N(w) \cup \{w\}$ whenever $v \in A_1(B)$ and $w \in A_2(B)$. To justify this claim, assume the contrary. Now there is a vertex u in $N(v)$ such that $u \notin N(w) \cup \{w\}$. By (10.3.1), we must have $u \in A_0(B)$; but then $uvwz$ is a badly coloured P_4 whenever $z \in B - N(v)$. □

<u>Proof of Lemma 10.2.3</u>

Let G be a graph with an alignment $Q_1,Q_2,\ldots,Q_k$ that does not extend into a full alignment. Without loss of generality, we may assume that the alignment $Q_1,Q_2,\ldots,Q_k$

is maximal.

We shall define certain sets $C_1, C_2, \ldots, C_k$ and $S_1, S_2, \ldots, S_k$ such that

$$C_i \cap S_i = \emptyset \text{ and } C_i \cup S_i = Q_1 \cup Q_2 \cup \ldots \cup Q_i$$

for all i. To begin, enumerate the vertices of Q_1 as x_1, x_2, x_3, x_4 in such a way that $x_1x_2, x_2x_3, x_3x_4 \in E$ (and $x_1x_3, x_1x_4, x_2x_4 \notin E$); then set $C_1 = \{x_2, x_3\}$ and $S_1 = \{x_1, x_4\}$. Next, when C_i and S_i have been defined for some i smaller than k, let x be the vertex in Q_{i+1} that does not belong to $C_i \cup S_i$. If $|C_i \cap Q_{i+1}|$ is odd then set $C_{i+1} = C_i \cup \{x\}$, $S_{i+1} = S_i$. If $|C_i \cap Q_{i+1}|$ is even then set $C_{i+1} = C_i$, $S_{i+1} = S_i \cup \{x\}$.

Next, write $C = C_k$, $S = S_k$, $A = C \cup S$ and set

$u \in B_0$ if $u \notin A$ and $uv \notin E$, $uw \notin E$ whenever $v \in C_1$, $w \in S_1$

$u \in B_1$ if $u \notin A$ and $uv \in E$, $uw \notin E$ whenever $v \in C_1$, $w \in S_1$

$u \in B_2$ if $u \notin A$ and $uv \in E$, $uw \in E$ whenever $v \in C_1$, $w \in S_1$.

It is easy to see that each vertex u outside A belongs to one of the sets B_0, B_1, B_2: otherwise u along with some three vertices in Q_1 would induce a P_4, contradicting maximality of the alignment.

We claim that

$uv \notin E$ and $uw \notin E$ whenever $u \in B_0$, $v \in C$, $w \in S$

$uv \in E$ and $uw \notin E$ whenever $u \in B_1$, $v \in C$, $w \in S$

$uv \in E$ and $uw \in E$ whenever $u \in B_2$, $v \in C$, $w \in S$.

This claim is easy to justify: if it failed, then some vertex u outside A would have an odd number of neighbours in some Q_i. But then u along with some three vertices in Q_i would induce a P_4, contradicting maximality of the alignment.

Finally, let us distinguish among four cases.

Case 1: $B_1 = \emptyset$. In this case, A is a homogeneous set.

Case 2: $B_1 \neq \emptyset$ and some two vertices in S are adjacent. In this case, the subgraph of G induced by S has a component H with at least two vertices; we claim that H is a homogeneous set. Assuming the contrary, we find vertices x,y,z such that $x,y \in H$, $z \notin H$ and $xz \in E$, $yz \notin E$; since H is connected, we may assume that $yx \in E$. Trivially, $z \in C$. But then x,y,z and any vertex in B_1 induce a P_4, contradicting maximality of the alignment.

Case 3: $B_1 \neq \emptyset$ and some two vertices in C are nonadjacent. This case reduces to Case 2 when G is replaced by its complement and C interchanged with S.

Case 4: Every two vertices in S are nonadjacent and every two vertices in C are adjacent. In this case, $N(w) \subseteq \{v\} \cup N(v)$ whenever $w \in S$ and $v \in C$. □

11. ODD DECOMPOSITIONS

11.1 The Results

Vašek Chvátal conjectured that a graph is perfect whenever its vertices can be coloured by two colours in such a way that each chordless path with four vertices and three edges has an odd number of vertices of each colour. The main purpose of this section is to prove Chvátal's conjecture.

Theorem 11.1.1

If the vertices of a graph G can be coloured by two colours in such a way that each induced P_4 has an odd number of vertices of each colour, then G is perfect. □

Note that the hypothesis of Theorem 11.1.1 can be tested by solving a small system of linear congruences modulo two: each variable in the system corresponds to a vertex, and each congruence, corresponding to an induced P_4, requires that the sum of the four variables be odd.

Our proof relies on the following results concerning perfect graphs. First, by the Perfect Graph Theorem, a graph is perfect if and only if its complement is. Second, Seinche (section 4) proved that if G is a P_4-free graph, then G or $\overline{G}$ is disconnected. Finally, as mentioned in section 3, the following three statements hold.

No minimal imperfect graph can contain a homogeneous set. (11.1.1)

No minimal imperfect graph can contain a clique cutset. (11.1.2)

No minimal imperfect graph can contain two vertices u,v with $N(u) \subseteq \{v\} \cup N(v)$. (11.1.3)

By virtue of the above facts, the validity of Theorem 11.1.1 is guaranteed by the following result:

Theorem 11.1.2

If the vertices of a graph G are coloured by two colours in such a way that each chordless path with four vertices and three edges have an odd number of vertices of each colour, then

(i) G or $\bar{G}$ is bipartite, or
(ii) G or $\bar{G}$ contains a homogeneous set, or
(iii) G or $\bar{G}$ contains a clique cutset, or
(iv) G or $\bar{G}$ contains two vertices u,v with $N(u) \subseteq \{v\} \cup N(v)$.

One graph that satisfies the hypothesis of Theorem 11.1.2 is shown in Figure 11.1. This graph has none of the properties (i), (ii) and (iii) of Theorem 11.1.2. (To show that the graph satisfies the hypothesis of Theorem 11.1.2, we assign colours to its vertices. R denotes "red" and W denotes "white".)

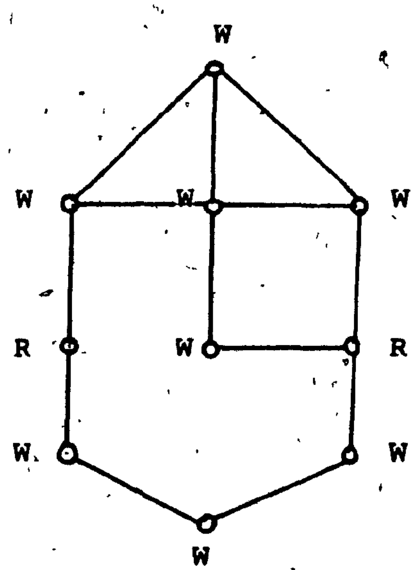


Figure 11.1

Finally, we shall present a generalization of Theorem 11.1.1. First, let us consider a graph whose vertices are coloured by two colours. A P_4 is said to be monochromatic if all of its four vertices receive the same colour. A P_4 is well odd-coloured if it has an odd number of vertices of each colour and if among the three vertices of the same colour of this P_4, at least one vertex does not belong to a monochromatic P_4.

Theorem 11.1.3

If the vertices of a graph G are coloured by two colours in such a way that each colour appears at least once and that each induced P_4 is either monochromatic or well odd-coloured, then G is perfect if and only if each of the two subgraphs of G induced by all the vertices of the same colour is perfect.

One graph that satisfies the hypothesis of Theorem 11.1.3 (but not the hypothesis of Theorem 11.1.1) is shown in Figure 11.2. (As in Figure 11.1, R denotes "red" and W denotes "white".) Neither this graph nor its complement contains a homogeneous set or a clique cutset.

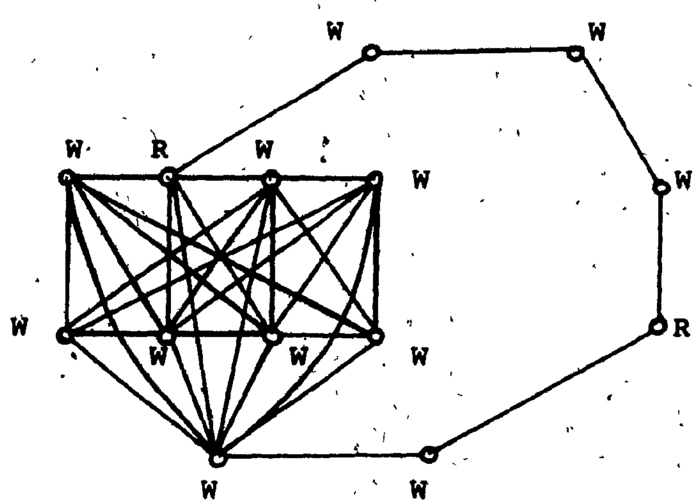


Figure 11.2

Unlike the case of Theorem 11.1.1, we do not know how difficult it is to test the hypothesis of Theorem 11.1.3. However, Chvátal (1985) found a common generalization of Theorem 11.1.3 and Theorem 10.1, whose hypothesis can be tested in a polynomial time. This common generalization involves the following notion: vertices x and y in the same graph are called siblings if there is a set S of three vertices such that both $S \cup \{x\}$ and $S \cup \{y\}$ induce a P_4.

Theorem 11.1.4 (Chvátal (1985))

Let the vertices of a graph G be coloured red and white in such a way that every two siblings have the same colour and that each colour appears at least once. Then G is perfect if and only if each of its two subgraphs induced by all the vertices of the same colour is perfect. □

It is easy to see that Theorem 11.1.4 implies Theorem 11.1.3 and Theorem 10.1. (The proof of Theorem 11.1.4 relies on Theorems 11.1.3 and 10.1.) Chvátal has noted that, given a graph G with n vertices, one can test whether G satisfies the hypothesis of Theorem 11.1.4 in $O(n^5)$ steps. To see this, we construct the sibling graph of G that has the same vertices as G, with any two vertices adjacent if and only if they are siblings in G. Clearly, G satisfies the hypothesis of Theorem 11.1.4 if and only if the sibling graph of G is disconnected.

11.2 The Proofs

Proof of Theorem 11.2.1

Let G be a graph satisfying the hypothesis of Theorem 11.1.2.

We shall write G = (V,E) and refer to the two colours as red and white; the two subgraphs of G induced by all the red vertices and by all the white vertices will be denoted by R and W, respectively. Given any two nonempty disjoint subsets S and T of V, we shall partition S into three subsets as follows:

$u \in S_0(T)$ if $u \in S$ and $uv \notin E$ whenever $v \in T$,

$u \in S_2(T)$ if $u \in S$ and $uv \in E$ whenever $v \in T$,

$u \in S_1(T)$ if $u \in S$ and $u \notin S_0(T) \cup S_2(T)$.

Let us make note of a simple fact.

Fact 11.2.1

Let Y be a subset of R such that the complement of the graph induced by Y is connected; let W be partitioned into disjoint sets P and Q such that $P = P_0(Q)$ and $Q_2(Y) \neq \emptyset$. Then $P_1(Y) = \emptyset$.

Proof of Fact 11.2.1

Assume the contrary, so that some vertex z in P is adjacent to some but not all vertices in Y. Since the complement of the graph induced by Y is connected, there must be vertices x and y in Y with $xy \notin E$, $xz \in E$, $yz \notin E$. But then zxwy is a badly coloured P_4 whenever $w \in Q_2(Y)$, a contradiction.

The following corollary of Fact 11.2.1 will be used over and over again.

Fact 11.2.2

Let R be partitioned into disjoint sets Y and Z such that the complement of the graph induced by Y is connected, $|Y| \geq 2$, and $Z_1(Y) = \emptyset$; let W be partitioned into disjoint sets P and Q such that $P = P_0(Q)$ and $Q_2(Y) \neq \emptyset$. If $P \neq P_0(Y)$ or $Q_1(Y) = \emptyset$, then Y is a homogeneous set.

Proof of Fact 11.2.2

Since $Z_1(Y) = \emptyset$, we only need prove that $W_1(Y) = \emptyset$. In fact, we only need prove that $Q_1(Y) = \emptyset$, as $P_1(Y) = \emptyset$ is guaranteed by Fact 11.2.1. Thus, we may assume that $P \neq P_0(Y)$; now $P_1(Y) = \emptyset$ implies $P_2(Y) \neq \emptyset$, and $Q_1(Y) = \emptyset$ follows from Fact 11.2.1 with P and Q interchanged.

A component of a graph will be called big if it has at least two vertice. The remainder of the proof is presented in the guise of an algorithm.

0. If W is connected then replace G by its complement. (By Seinsche's theorem, W or its complement is disconnected.)

1. <u>Now, W is disconnected.</u>

If W has no big component then go to 9; if R has no big component then switch colours and go to 9. Now both R and W have big components; we shall distinguish between two cases.

<u>Case 1.1</u>: There are no big components A of R, B of of W with $A_1(B) \neq \emptyset$.

If there is no edge xy with x in a big component A of R and y in a big component B of W then go to 8; else consider this edge. If B is homogeneous then stop; else there is a component C of R with $C_1(B) \neq \emptyset$. Since we are in Case 1.1, this component C consists of a single vertex c. Stop: we claim that $N(c) \subseteq N(x)$. (To justify this claim, assume the contrary: $cd \in E$ and $xd \notin E$ for some vertex d. Trivially, $d \in W$; furthermore, $d \notin B$, for otherwise $x \in A_1(B)$, contradicting the assumption that $A_1(B) = \emptyset$. But now $d \in W - B$, and so xbcd is a badly coloured P_4 whenever b is a neighbour of c in B.)

<u>Case 1.2</u>: There are big components A of R, B of W with $A_1(B) \neq \emptyset$.

If $A_0(B) \neq \emptyset$, go to 7; if $A_0(B) = \emptyset$ and $A_2(B) \neq \emptyset$, go to 6; if $A_0(B) = \emptyset$ and $A_2(B) = \emptyset$, go to 2.

2. <u>Now, W is disconnected and there are big components</u>

A of R, B of W with $A = A_1(B)$.

Note that $B_1(A) \cup B_2(A) \neq \emptyset$. If $B = B_1(A)$ then go to 3; if $B_0(A) \neq \emptyset$ and $B_1(A) \neq \emptyset$ then switch colours and go to 7. Now only two cases remain to be considered.

Case 2.1: $B_1(A) = \emptyset$ and $B_2(A) \neq \emptyset$.

If A is not a clique then stop: the complement of the subgraph induced by A has a big component Y and Fact 11.2.2 (with Q = B) guarantees that Y is homogeneous. Now A is a clique. If R is connected then stop: A is a clique cutset. Now R is disconnected. If A is homogeneous then stop; else there is a component C of W with $C_1(A) \neq \emptyset$. If $C = C_1(A)$ then stop: A is a clique cutset. (Otherwise, some vertex c in C is adjacent to some vertex d in R - A. Consider an arbitrary vertex b in $B_2(A)$. If $bd \in E$ then cdba is a badly coloured P_4 whenever $a \in A - N(c)$; if $bd \notin E$ then dcab is a badly coloured P_4 whenever $a \in A \cap N(c)$.) Now $C \neq C_1(A)$ but $C_1(A) \neq \emptyset$, and so C is big. If $C_0(A) \neq \emptyset$ then switch colours and go to 7; if $C_0(A) = \emptyset$ then switch colours and go to 6.

Case 2.2: $B_0(A) = \emptyset$, $B_1(A) \neq \emptyset$, $B_2(A) \neq \emptyset$.

If R is disconnected then switch colours and go to 6. Now, R is connected, and so R = A. Let C consist of all the vertices in R that have neighbours in W-B; note that C is a cutset (every path from B to W-B must pass through C). If C is a clique then stop (C is a clique cutset); else there are nonadjacent vertices u and v in C. Now the complement of R has a (big) component Y containing u and v. Stop: Fact 11.2.2 (with Q = B) guarantees that Y is homogeneous.

3. <u>Now, W is disconnected and there are big components A of R, B of W with $A = A_1(b)$, $B = B_1(A)$.</u>

If G is disconnected then stop; else there is an edge xy with $x \in A \cup B$, $y \notin A \cup B$. If $x \in B$ and $y \in R-A$ then switch colours; now there is an edge vw with $v \in A$ and $w \in W-B$. If $N(w) \supseteq A$ then go to 4; else go to 5.

4. <u>Now, there are big components A of R, B of W such that $A = A_1(B)$, $B = B_1(A)$ and such that some vertex w in W-B has $N(w) \supseteq A$.</u>

If A is not a clique then stop: the complement of A has a big component Y and Fact 11.2.2 (with Q = W-B) guarantees that Y is homogeneous. Now A is a clique. If A is a clique cutset then stop; else some vertex in B has a neighbour r in R-A. We claim that

$$N(r) \not\supseteq B.$$

To justify this claim, find vertices a in A and b,c in B with $ab \in E$, $ac \notin E$. If $wr \in E$ then we must have $rc \notin E$ (else awrc would be a badly coloured P_4); if $wr \notin E$ then we must have $rb \notin E$ (else wabr would be a badly coloured P_4).

Now switch colours, replace w by r, and go to 5.

5. <u>Now, there are big components A of R, B of W</u>

such that $A = A_1(B)$, $B = B_1(A)$ and such that some vertex w in $W-B$ has $N(w) \cap A \neq \emptyset$, $N(w) \not\supseteq A$.

We claim that

there are no vertices a_1, a_2, a_3 in A with
$wa_1, a_1a_2, a_2a_3 \in E$ and $wa_2, wa_3, a_1a_3 \notin E$. (11.2.1)

To justify this claim, assume the contrary and let b_i ($i = 2,3$) be an arbitrary neighbour of a_i in B. We must have $b_2a_1 \in E$ (else $wa_1a_2b_2$ is a badly coloured P_4), $b_2a_3 \notin E$ (else $a_3b_2a_1w$ is a badly coloured P_4), $b_3a_1 \notin E$ (else $wa_1b_3a_3$ is a badly coloured P_4), $b_3a_2 \notin E$ (else $wa_1a_2b_3$ is a badly coloured P_4), and $b_2b_3 \notin E$ (else $a_1b_2b_3a_3$ is a badly coloured P_4). But then $b_2a_2a_3b_3$ is a badly coloured P_4.

Next, writing $C = N(w) \cap A$, we claim that

some vertex x in C has $N(x) \supseteq A-C$. (11.2.2)

To justify this claim, consider any vertex x in C that maximizes the size of $N(x) \cap (A-C)$. If $N(x) \supseteq A-C$ then we are done; else there is a vertex z in $A-C$ with $xz \notin E$. Since A contains no P_4, the shortest path from x to z in A has precisely three vertices; let y be the interior vertex of this path. By (11.2.1) with $a_1 = x$, $a_2 = y$, $a_3 = z$, we must have $y \in C$. By the choice of x, there must be a vertex t in $A-C$ with $xt \in E$, $yt \notin E$. By (11.2.1) with $a_1 = x$, $a_2 = t$, $a_3 = z$, we must have $zt \notin E$. But then $txyz$ is a badly coloured P_4.

Now we shall distinguish between two cases.

Case 5.1: C is a clique.

Let D stand for the set of vertices in B that have neighbours in A−C. We claim that, with x as in (11.2.2),

$$D \subseteq N(x). \qquad (11.2.3)$$

To justify this claim, consider an arbitrary vertex d in D; there is a vertex a in A−C with ad ε E. We must have xd ε E, for otherwise wxad would be a badly coloured P_4.

Next, since $A = A_1(B)$, there is a vertex b in B with xb ∉ E; by (11.2.3), we have b ∉ D. Since $B = B_1(A)$, there is a vertex a in A with ba ε E; since b ∉ D, we have a ε C. We claim that

$$N(a) \supseteq A-C. \qquad (11.2.4)$$

To justify this claim, assume the contrary: some vertex y in A−C has ya ∉ E. Since $A = A_1(B)$, there is a vertex c in B with yc ε E. Note that c ε D, and so xc ε E by (11.2.3). We must have ac ∉ E (else wacy is a badly coloured P_4), bc ε E (else baxc is a badly coloured P_4), and by ∉ E (as b ∉ D). But then ycba is a badly coloured P_4.

Now stop: $N(y) \subseteq \{a\} \cup N(a)$ whenever y ε A−C. (Otherwise there would be a vertex z with yz ε E, az ∉ E. By (11.2.4), we must have z ε W. If z ε B then (11.2.4) guarantees that wayz is a badly coloured P_4; if $z \in W-B$ then (11.2.4) and $b \notin D$ guarantee that zyab is a badly coloured P_4.)

<u>Case 5.2</u>: C is not a clique.

Now the complement of the subgraph induced by C has a big component Y. Stop: We claim that Y is homogeneous. (To justify this claim, write $Z = R-Y$. We only need show that $Z_1(Y) = \emptyset$ for the rest will follow from Fact 11.2.2 with $Q = W-B$. To show that $Z_1(Y) = \emptyset$, assume the contrary: some vertex z in R-Y is adjacent to some but not all the vertices in Y. Trivially, $z \in A-C$; since the complement of the subgraph induced by Y is connected, there are vertices u,v in Y with $uv \notin E$, $uz \in E$, $vz \notin E$. Consider any neighbour b of z in B. We must have $ub \in E$, for otherwise wuzb would be a badly coloured P_4. But $ub \in E$ implies $vb \in E$, as Fact 11.2.1 with $Q = W-B$ guarantees $B_1(Y) = \emptyset$. Now wvbz is a badly coloured P_4).

6. <u>Now, W is disconnected and there are big components A of R, B of W such that $A_0(B) = \emptyset$, $A_1(B) \neq \emptyset$, $A_2(B) \neq \emptyset$.</u>

We shall distinguish between two cases.

<u>Case 6.1</u>: There is no edge with one endpoint in A and the other endpoint in W-B.

Let C stand for the set of the vertices in B that have neighbours in R-A; note that C is a cutset (every path from A to W-B must pass through C). If C is a clique then stop

(C is a clique cutset); else there are nonadjacent vertices u and v in C. Now the complement of the graph induced by B has a (big) compnoent Y containing u and v. Stop: Fact 11.2.2 (with colours switched and Q = A) guarantees that Y is homogeneous.

Case 6.2: There is an edge with one endpoint in A and the other endpoint in W-B.

Write $u \in A_1^*(B)$ if $u \in A_1(B)$ and $N(u) \supseteq A_2(B)$; write $v \in A_2^*(B)$ if $v \in A_2(B)$ and $N(v) \supseteq A_1(B)$. We claim that

$$\text{no vertex in W-B has a neighbour in } (A_1(B)-A_1^*(B)) \cup (A_2(B)-A_2^*(B)). \qquad (11.2.5)$$

To justify this claim, assume the contrary. Now there are nonadjacent vertices u,v such that $u \in A_1(B)$, $v \in A_2(B)$, and such that $uw \in E$ or $vw \in E$ (or both) for some vertex w in W-B. Next, there are vertices b,c in B such that $ub \in E$ and $uc \notin E$; of course $vb \in E$ and $vc \in E$. If $uw \in E$ and $vw \in E$ then uwvc is a badly coloured P_4; if $uw \in E$ and $vw \notin E$ then wubv is a badly coloured P_4; if $uw \notin E$ and $vw \in E$ then wvbu is a badly coloured P_4.

Next, we claim that

$$\text{some vertex w in W-B has a neighbour in } A_2^*(B). \qquad (11.2.6)$$

To justify this claim, recall that there is an edge xw with $x \in A$ and $w \in W-B$; by (11.2.5), we must have $x \in A_1^*(B) \cup$

$A_2^*(B)$. If $x \in A_2^*(B)$ then (11.2.6) holds; thus, we may assume $x \in A_1^*(B)$. Now there is a vertex b in B with $xb \notin E$. Consider an arbitrary vertex v in $A_2(B)$: we must have first $wv \in E$ (else wxvb would be a badly coloured P_4) and then $v \in A_2^*(B)$ by (11.2.5). Hence, (11.2.6) holds again.

With w as in (11.2.6), write $S = N(w) \cap A_2(B)$. We claim that

$$va \in E \text{ whenever } v \in S \text{ and } a \in A-S. \qquad (11.2.7)$$

To justify this claim, assume the contrary: $va \notin E$ for some v in S and for some a in A-S. By (11.2.5), we have $S \subseteq A_2^*(B)$, and so $a \in A_2(B)-S$. But then abvw is a badly coloured P_4 whenever $b \in B$.

The remainder of the argument relies only on (11.2.7). If S is a clique then stop: $N(u) \subseteq \{v\} \cup N(v)$ whenever $u \in A_1(B)$ and $v \in S$. (Otherwise, there would be a vertex z with $uz \in E$, $vz \notin E$. Necessarily, $z \in W-B$; but then zuvb is a badly coloured P_4 whenever $b \in B-N(u)$.) If S is not a clique then the complement of the subgraph induced by S has a big component Y. Stop: Fact 11.2.2 (with Q = B or Q = W-B) guarantees that Y is homogeneous.

7. <u>Now, there are big components A of R, B of W such that $A_0(B) \neq \emptyset$, $A_1(B) \neq \emptyset$.</u>

Note that

$$\text{there is no edge } uv \text{ with } u \in A_0(B),\ v \in A_1(B): \qquad (11.2.8)$$

else, finding vertices x,y in B with $xy \in E$ and $vx \in E$, $vy \notin E$, we would obtain a badly coloured P_4 $(uvxy)$.

Since A contains no P_4, Seinsche's theorem guarantees that A splits into nonempty parts S and T such that $xy \in E$ whenever $x \in S$, $y \in T$. Without loss of generality, we may assume that $A_0(B) \cap T \neq \emptyset$; now (11.2.8) implies $A_0(B) \cup A_1(B) \subseteq T$, and so $S \subseteq A_2(B)$. If S is a clique then stop: $N(u) \subseteq \{w\} \cup N(w)$ whenever $u \in A_0(B)$ and $w \in S$. (Otherwise, there would be a vertex z with $uz \in E$ and $wz \notin E$. Necessarily, $z \in W-B$; but then zuwb is a badly coloured P_4 whenever $b \in B$.) If S is not a clique then the complement of the subgraph induced by S contains a big component Y. Stop: Fact 11.2.2 (with $Q = B$) guarantees that Y is homogeneous.

8. <u>Now, both R and W have big components, but no edge has one endpoint in a big component of R and the other endpoint in a big component of W.</u>

Stop: we claim that G is disconnected. (To justify this claim, assume the contrary: now there is a path $v_1, v_2, \ldots, v_k$ such that v_1 is in a big component of R and v_k is in a big component of W. Choosing k as small as possible, observe that $\{v_2\}$ is a component of W, $\{v_3\}$ is a component of R, and $v_4 \in W$. But then $v_1v_2v_3v_4$ is a badly coloured P_4.)

9. <u>Now, no two vertices in W are adjacent.</u>

The following elegant argument, proposed by Bruce Reed, shows that G is perfectly orderable. Trivially, there is a linear order $<$ on the set of vertices of G such that

$$x < y \text{ whenever } x \in R,\ y \in W$$

and such that

$$c < d \text{ whenever } c,d \in R \text{ and } |N(c) \cap W| > |N(d) \cap W|;$$

it is easy to verify that no P_4 with vertices a,b,c,d and edges ab,bc,cd has $a < b$ and $d < c$.

We shall present a lengthier but self-contained argument, providing more insight into the structure of G. First, if R is connected then stop: by Seinsche's theorem, the complement of R is disconnected, and so W is a clique cutset in the complement of G. Now

R is disconnected;

we shall distinguish among three cases.

Case 9.1: Some vertex in a big compnoent of R has at least two neighbours in W.

Among all the vertices in big components of R, choose a vertex a that has the largest number of neighbours in W. Let A be the big component of R that contains a; write $Y = N(a) \cap W$ and note that $|Y| \geq 2$. If some vertex in R-A has a neighbour in Y then stop: Fact 11.2.2 (with colours switched and $Q = A$) guarantees that Y is homogeneous. Now

there is no edge with one endpoint in R-A and the other endpoint in Y. (11.2.9)

Write

$$x \in A_0 \text{ if } x \in A \text{ and } N(x) \cap W = \emptyset,$$
$$x \in A_1 \text{ if } x \in A-A_0 \text{ and } N(x) \cap W \subseteq Y,$$
$$x \in A_2 \text{ if } x \in A \text{ and } N(x) \cap W \not\subseteq Y. \quad (11.2.9)$$

Note that $(A-A_2) \cup Y$ is a component of $G-A_2$ by virtue of (11.2.9); since $A \neq R$, it follows that A_2 is a cutset of G. If $A_2 = \emptyset$ then stop: G is disconnected. Now

$$A_1 \neq \emptyset \text{ and } A_2 \neq \emptyset. \quad (11.2.10)$$

We claim that

no vertex z in A_2 has a neighbour y in Y. (11.2.11)

To justify this claim, assume the contrary. Since z has a neighbour w in W-Y, we must have $az \in E$ (else ayzw would be a badly coloured P_4). But the choice of a guarantees the existence of a vertex x in W with $ax \in E$, $zx \notin E$; now xazw is a badly coloured P_4.

From (11.2.11), it follows that

there is no edge xz with $x \in A_1$, $z \in A_2$: (11.2.12) else yxzw would be a badly coloured P_4 whenever $y \in N(x) \cap Y$ and $w \in N(z) \cap W$.

By Seinsche's theorem, A splits into nonempty parts S and T such that $xy \in E$ whenever $x \in S$, $y \in T$. Without loss of generality, we may assume that $a \in T$; now (11.2.10) and (11.2.12) imply that $A_1 \cup A_2 \subseteq T$, and so $S \subseteq A_0$. If $|S| \geq 2$ then stop (S is homogeneous); else let s be the unique vertex in S. If $A_0 = \{s\}$ then stop: By (11.2.9) and (11.2.12), $A_1 \cup Y$ is a component of $G-A_0$, and so $\{s\}$ is a clique cutset of G. If $A_0 \neq \{s\}$ then stop: $N(t) \subseteq \{s\} \cup N(s)$ whenever $t \in A_0-\{s\}$.

Case 9.2: No vertex in a big component of R has two or more neighbours in W, but some vertex w in W has at least two neighbours in some big component A of R.

Write

$x \in A_0$ if $x \in A$ and $N(x) \cap W = \emptyset$,

$x \in A_1$ if $x \in A \cap N(w)$,

$x \in A_2$ if $x \in A-(A_0 \cup A_1)$.

Note that $A_0 \cup A_1$ is a component of $G-(A_2 \cup \{w\})$; since $A \neq R$, it follows that $A_2 \cup \{w\}$ is a cutset of G. If $A_2 = \emptyset$ then stop: $\{w\}$ is a clique cutset. Now

$$A_1 \neq \emptyset \text{ and } A_2 \neq \emptyset. \qquad (11.2.13)$$

In addition,

$$\text{there is no edge } xy \text{ with } x \in A_1,\ y \in A_2: \qquad (11.2.14)$$

else wxyz with $z \in N(y) \cap W$ would be a badly coloured P_4.

By Seinsche's theorem, A splits into nonempty parts S and T such that $xy \in E$ whenever $x \in S$, $y \in T$. Without loss of generality, we may assume that $A_1 \cap T \neq 0$; now (11.2.13) and (11.2.14) imply that $A_1 \cup A_2 \subseteq T$, and so $S \subseteq A_0$. If $|S| \geq 2$ then stop (S is homogeneous); else let s be the unique vertex in S. If $A_0 = \{s\}$ then stop: by (11.2.14), A_1 is homogeneous. If $A_0 \neq \{s\}$ then stop: $N(t) \subseteq \{s\} \cup N(s)$ whenever $t \in A_0-\{s\}$.

Case 9.3: No vertex in a big component of R has two or more neighbours in W, and no vertex in W has two or more neighbours in the same big component of R.

Consider an arbitrary big component A of R and write

$$x \in A_0 \text{ if } x \in A \text{ and } N(x) \cap W = \emptyset,$$
$$x \in A_1 \text{ is } x \in A \text{ and } N(x) \cap W \neq \emptyset.$$

Note that

$$\text{there is no edge } yz \text{ with } y \in A_1,\ z \in A_1; \qquad (11.2.15)$$

else xyzw with $x \in N(y) \cap W$, $w \in N(z) \cap W$ would be a badly coloured P_4.

By Seinsche's theorem, A splits into nonempty parts

S and T such that $xy \in E$ whenever $x \in S$, $y \in T$. By (11.2.15), we must have $A_1 \subseteq S$ or $A_1 \subseteq T$. Without loss of generality, we may assume that $A_1 \subseteq T$, and so $S \subseteq A_0$. If $|S| \geq 2$ then stop (S is homogeneous); else let s be the unique vertex in S. If $A_0 \neq \{s\}$ then stop: $N(t) \subseteq \{s\} \cup N(s)$ whenever $t \in A_0-\{s\}$. Now we have

$$|A_0| = 1. \tag{11.2.16}$$

Finally, let Q stand for the union of all the sets A_0 (one for each big component A of R). By (11.2.16), no two vertices in $W \cup Q$ are adjacent; by (11.2.15), no two vertices in R-Q are adjacent. Stop: G is bipartite.

The proof is completed.□

To prove Theorem 11.1.3, we shall need a result established by Chvátal and the author in the previous section. This result can be restated as follows.

Theorem 11.2.1

Let G be a minimal imperfect graph and let S be a set of vertices such that S induces a P_4 in G. Then the vertices of G can be enumerated as $v_1, v_2, v_3, v_4, v_5, \ldots, v_n$ in such a way that $S = \{v_1, v_2, v_3, v_4\}$ and that each v_j with $j > 4$ forms a P_4 with some three vertices v_i such that $i < j$.□

Proof of Theorem 11.1.3

If the statement was false, then the smallest counterexample would be minimal imperfect. Thus, we only need show that no graph G satisfying the hypothesis of Theorem 11.1.3 is minimal imperfect. Assume such a graph to exist. (We want to arrive at a contradiction.) There must be a set S of four vertices such that S induces a monochromatic P_4 in G; for otherwise G would satisfy the hypothesis of Theorem 11.1.1, and so G is perfect, a contradiction. Since G is minimal imperfect, its vertices can be enumerated as $v_1, v_2, \ldots, v_n$ as in Theorem 11.2.1. In particular, v_1, v_2, v_3 and v_4 have the same colour (because they belong to S). Now, let j be the smallest subscript such that v_j has the colour different from that of v_4. Note that we have $j \geq 5$. By Theorem 11.2.1, v_j forms a P_4 with some three vertices v_i with $i < j$. Observing that each v_i with $i < j$ belongs to a monochromatic P_4, we conclude that the P_4 containing v_j and the three vertices v_i $(i < j)$ is neither monochromatic nor well odd-coloured. This is the desired contradiction. □

APPENDIX

Adjacent : two vertices are adjacent if and only if they are joined by an edge.

Bijection : a mapping one-to-one and onto.

Chord : a chord in a cycle $v_1, v_2, \ldots, v_k$ is an edge $v_i v_j$ other than $v_i v_{i+1}$ $(1 \leq i \leq k)$ or $v_1 v_k$.

Chromatic number : the smallest number of colours that suffice to colour a graph.

Clique : a set of pairwise adjacent vertices.

Clique number : the number of vertices of the largest clique in a graph.

Colouring : an assignment of "colours" to vertices such that adjacent vertices always have different colours.

Complement : the complement of a graph $G = (V,E)$ is denoted by $\bar{G} = (V,E')$ with the same set of vertices, and the set E' of edges such that for any two vertices x,y in V, we have $xy \in E'$ if and only if $xy \notin E$.

Connected : a graph is connected if there is at least a path between any two vertices.

Cutset : a set of vertices such that its

Cutset (con't)	:	removal would disconnect a connected graph.
Cycle	:	a cycle is a path from a vertex x to a vertex y with the edge xy.
Edge	:	see Graph.
Graph	:	an ordered pair (V,E) such that V is a set and E is a set of two-point subset of V. The elements of V are called vertices and the elements of E are called edges.
Induced subgraph	:	a graph $H = (V_H, E_H)$ is an induced subgraph of a graph $G = (V,E)$ if $V_H - V$ and for each edge xy in E, we have $xy \in E_H$ if and only if both x and y are in V_H.
Neighbour	:	a vertex x is a neighbour of vertex y if x and y are adjacent.
Path	:	a sequence of distinct vertices $v_1, v_2, \ldots, v_n$ such that $v_i v_{i+1} \in E$ $(i \leq 1 \leq n-1)$.
Stable set	:	a set of pairwise nonadjacent verticex.
Vertex	:	see Graph.

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