PLÜCKER'S NUMBERS IN THE THEORY OF ALGEBRAIC PLANE CURVES

PLÜCKER'S NUMBERS IN THE THEORY OF ALGEBRAIC PLANE CURVES

A Thesis presented in partial fulfilment
for the degree of Master of Arts,
Department of Mathematics.

April 28, 1928.

by

ALICE WILLARD TURNER.

Acknowledgment

Any commendation which this thesis may merit, is in large measure due to Dr. C.T. Sullivan, Peter Redpath Professor of Pure Mathematics; and I take this opportunity to express my indebtedness to him.

April 25, 1928. A.W. Turner.

I N D E X

PLÜCKER'S NUMBERS IN THE THEORY OF ALGEBRAIC PLANE CURVES.

INTRODUCTION

With the advent of the nineteenth century, a new era dawned in the progress of analytic geometry. The appearance of Poncelet's, "Traite des proprietes projectives des figures", in 1822, really initiated modern geometry. Möbius, five years later, in his "Barycentrische Calcul", introduced homogeneous co-ordinates, which greatly facilitated the discussion of descriptive geometrical properties. The outstanding contributor to analytic plane geometry in this significant period, however, was Johann Plücker (1801-68), and we may regard him as the true founder of the modern theory of algebraic curves. He it was who formulated analytically the Principle of Duality, and investigated the geometrical results. Plücker's "Analytisch-Geometrische Entwickelungen" was published in '28-'31, and Steiner - who was really a synthetic geometer - contributed much in 1832. "In the ten years which embrace the publication of the immortal works of Poncelet, Plücker, and Steiner, geometry has made more real progress than in the two thousand years which had elapsed since the time of Apollonius. The ideas which had slowly been taking shape since the time of Descartes suddenly crystallized and almost overwhelmed geometry with an abundance of new ideas and principles." *In Plücker's "Theorie des Algebraischen Kurven" (1835)

* J. Pierpont, Bulletin Amer. Math. Soc. vol. XI. no. 3.

there appeared the analytic relations between the singularities of a curve which are known as Plücker's Equations. Ludwig Otto Hesse subsequently gave a complete theory regarding the inflexions on a curve, and other contributions to the Theory of Higher Plane Curves were made by Chasles, Gergonne, Cayley, Halphen, and Zeuthen. Although German mathematicians, in particular, advanced the subject in the nineteenth century, in recent years, it has especially attracted the attention of Italian geometers.

In the following pages I have endeavoured to give a comprehensive account of Plücker's Numbers and Equations in the Theory of Algebraic Plane Curves. We shall see that these relations enable us to determine the number and species of the simple singularities of a curve. They therefore assume a rôle of fundamental importance in the classification of plane curves.

Moreover, I have confined the scope of this thesis to curves possessing ordinary singularities, k-ple linear branch points with distinct tangents, and ordinary superlinear branch points; that is to say, to simple and ordinary singularities. This limitation enables me to avoid the extensive and difficult considerations involved in the Theory of Functions and Cremona transformations, which are essential to a complete analysis of higher or compound singularities. Nevertheless, Plücker's Equations, properly understood, are applicable to a curve with any singularities whatsoever, for these latter may be regarded as a combination of ordinary simple singularities. In example 47, however, a higher singularity on a unicursal curve is analysed in detail. The zero deficiency in this case, renders the resolution of this singularity possible, without recourse to the Theory of Functions.*

I lay no claim to originality in results obtained, for I have merely attempted to assemble existing knowledge of the subject into a consistent whole. The examples - some of which are logical deductions from independent memoirs on the subject - have been selected chiefly from Hilton's Plane Algebraic Curves, and the solutions probably present certain phases of novelty. Although Section IV deals especially with Plücker's relations, I have thought the preceding material necessary to an adequate explanation of them.

A course of lectures on The Modern Theory of Algebraic Plane Curves (Dr. C.T. Sullivan) has been most helpful in the preparation of this thesis; but in addition I have consulted extensively various works enumerated in the Bibliography.

It may be added that I have tentatively assumed a knowledge of homogeneous co-ordinates, the concepts of calculus employed in analytical geometry, and numerous other comparatively simple results not generally included in elementary courses.

---

*Cf. Cayley: Quarterly Journal of Mathematics vol.VII (1866)
G.A. Bliss: Transactions of American Math.Soc.vol.XXIV(1922)

SECTION I.

SINGULARITIES ON A PLANE ALGEBRAIC CURVE

The curve obtained by equating to zero any non-homogeneous polynomial of degree n in x and y (or the corresponding homogeneous polynomial in x,y,z) is called a PLANE ALGEBRAIC CURVE of degree n. In this thesis we are concerned solely with such curves, and consequently we shall simply use the word 'curve' to denote 'an algebraic plane curve'. It is customary to call a curve of degree n an n-ic; although for n=1,2, .. ,7 we generally retain the familiar terminology line, conic, ... ,septic.

Let us eliminate z between an equation of degree n, $f(x,y,z)=0$, and the equation of a line $\lambda x+\mu y+\nu z=0$. We obtain $f(x,y,-\frac{\lambda x+\mu y}{\nu})=0$, which is a homogeneous equation of degree n in x and y; it therefore represents n straight lines joining (0,0,1) to the intersections of $\lambda x+\mu y+\nu z=0$ with $f(x,y,z)=0$. Thus, A STRAIGHT LINE MEETS AN n-IC IN n POINTS, REAL OR IMAGINARY.

If an n-ic breaks up into one or more rational factors of lower dimensions, the curve is IMPROPER or DEGENERATE; if on the other hand, the n-ic is irreducible, the curve is PROPER or NON-DEGENERATE. Thus, a quintic may consist of a cubic and two straight lines. The sum of the degrees of the factors must, of course, be equal to n. (in the illustration: 3+1+1 =5).

THE INTERSECTIONS OF TWO CURVES.

Before proceeding to the general case, let us investigate the intersections of two specific curves, a quintic and a cubic.

(1). $U_5 \equiv a_0z^5+a_1z^4+a_2z^3+a_3z^2+a_4z+a_5=0$)
)where $a_r b_r$ are
(2). $U_3 \equiv b_0z^3+b_1z^2+b_2z+b_3=0$)homogeneous

polynomials of degree r in x and y.

Multiplying (1) by $1,z,z^2$, and (2) by $1,z,z^2,z^3,z^4$, we obtain eight linear equations in $1,z,z^2,\ldots\ldots,z^7$. By Sylvester's method we can eliminate z, finding as the eliminant:

$$\begin{vmatrix} a_0 & a_1 & a_2 & a_3 & a_4 & a_5 & 0 & 0 \\ 0 & a_0 & a_1 & a_2 & a_3 & a_4 & a_5 & 0 \\ 0 & 0 & a_0 & a_1 & a_2 & a_3 & a_4 & a_5 \\ 0 & 0 & 0 & 0 & b_0 & b_1 & b_2 & b_3 \\ 0 & 0 & 0 & b_0 & b_1 & b_2 & b_3 & 0 \\ 0 & 0 & b_0 & b_1 & b_2 & b_3 & 0 & 0 \\ 0 & b_0 & b_1 & b_2 & b_3 & 0 & 0 & 0 \\ b_0 & b_1 & b_2 & b_3 & 0 & 0 & 0 & 0 \end{vmatrix} = 0.$$

This eliminant is obviously homogeneous in x and y, and therefore represents lines joining (0,0,1) to the intersections of U_5 and U_3. A typical term of this determinant is $\pm a_0^3 b_3^5$, which is of degree 15. Consequently a quintic and a cubic intersect in fifteen points.

If now $a_0=a_1=a_2=b_0=b_1=0$, let us multiply (1) by 1, and (2) by 1, z,
We then obtain the three equations:

$$\left.\begin{array}{l} a_3z^2+a_4z+a_5=0 \\ b_2z+b_3=0 \\ b_2z^2+b_3z=0 \end{array}\right\} \text{eliminating } z:$$

$$\begin{vmatrix} a_3 & a_4 & a_5 \\ 0 & b_2 & b_3 \\ b_2 & b_3 & 0 \end{vmatrix}=0$$

This determinant is evidently of the ninth degree in x and y and therefore represents the nine straight lines joining (0,0,1) to those intersections of the curve which do not coincide with (0,0,1). We saw that a cubic and quintic intersect in fifteen points, and in order to conserve this convention, we say that six of the intersections coincide with (0,0,1). Again, in the above equations, a_3 and b_2 may have a common linear factor. The determinant, after the removal of this factor, (which is a common factor of the terms in the first column), will be reduced to eight; so that we consider seven intersections of the two curves coincident with (0,0,1). If a_3 and b_2 have a common quadratic factor, the degree of the determinant is reduced to seven, and eight intersections coincide with (0,0,1).

The preceding illustration renders a discussion of the general case much simpler.

Let curves be:

$$\left.\begin{array}{l} \text{(i) } U_n \equiv a_0z^n+a_1z^{n-1}+ \ldots\ldots +a_n=0 \\ \text{(ii) } U_N \equiv b_0z^N+b_1z^{N-1}+ \ldots\ldots +b_N=0 \end{array}\right\} \text{where } a_r b_r \text{ are}$$

homogeneous polynomials of degree r in x and y. To eliminate z we multiply (i) by $1, z, \ldots\ldots, z^{N-1}$, and (ii) by $1, z, \ldots\ldots, z^{n-1}$;

thereby obtaining n+N linear equations in $1, z, z^2, \ldots\ldots, z^{n+N-1}$. Their eliminant is homogeneous in x and y and consequently represents lines joining (0,0,1) to the intersections of the two curves. A typical term of the determinant is $\pm a_0^N b_N^n$, that is to say, it is of degree nN. Thus, TWO CURVES OF DEGREE n AND N RESPECTIVELY, INTERSECT IN nN POINTS. Since imaginary roots occur in pairs, the number of real intersections is nN-2I, where I is zero or a positive integer.

Suppose $a_0 = a_1 = \ldots\ldots = a_{k-1} = b_0 = b_1 = \ldots\ldots = b_{K-1} = 0$

The equations (i) and (ii) become

$$\left.\begin{array}{ll}(i)^1. & a_k z^k + a_{k+1} z^{k-1} + \cdots + a_n = 0 \\ (ii)^1. & b_K z^K + b_{K+1} z^{K-1} + \cdots + b_N = 0\end{array}\right\}$$

Multiply $(i)^1$ by $1, z, \ldots, z^{N-K-1}$ and $(ii)^1$ by $1, z, \ldots, z^{n-k-1}$

We thereby obtain n+N-(k+K) linear equations. A typical term of the resultant eliminant is $\pm a_k^{N-K} b_N^{n-k}$ which is of degree nN-kK. This equation represents lines joining (0,0,1) to those intersections of U_n and U_N not coincident with (0,0,1). In order to assign unconditional validity to the statement that two curves of degrees n and N intersect in nN points, we say that kK intersections coincide with (0,0,1).

If a_k and b_K have r linear factors, in common, we may regard the determinant as an (nN-kK-r)-ic, and consequently, kK+r intersections are coincident with (0,0,1).

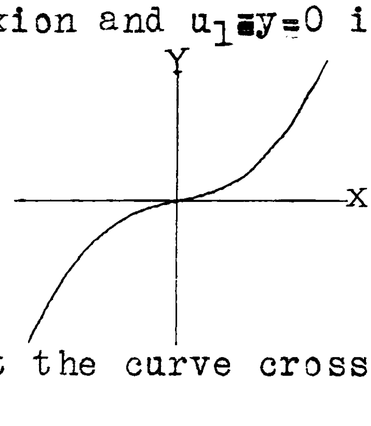
A later section will amplify the preceding paragraphs. We now digress to consider singularities.

SINGULAR POINTS ON CURVES

The general equation of an n-ic in Cartesian co-ordinates is: $a+(b_0x+b_1y)+(c_0x^2+2c_1xy+c_2y^2)+\cdots\cdots+(p_0x^n+p_1x^{n-1}y+\cdots+p_ny^n)=0$ or, more simply, $u_0+u_1+u_2+\cdots\cdots+u_n=0$, where u_r is homogeneous of degree r in x and y. If the origin lies on the curve, evidently $a\equiv u_0=0$, and the equation of the curve becomes: $u_1+u_2+\cdots+u_n=0$. The equation of any line through the origin is $y=mx$, which evidently meets the curve at the origin, since it passes through (0,0); this line, however, meets the curve in two coincident points at the origin, if $b_0+b_1m=0$, and coefficient of $x^2=0$. Thus, $u_1\equiv b_0x+b_1y=0$ is the equation of the TANGENT AT THE ORIGIN. We assume in our discussion that $u_0=0$

INFLEXION AT THE ORIGIN

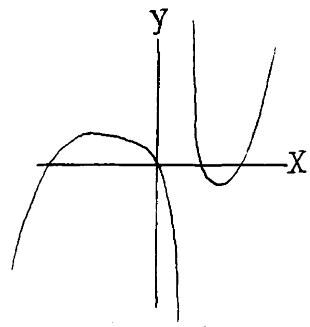
$u_1\neq 0$; equation of form $u_1+u_1v_1+u_3+\qquad+u_n=0$. The origin is an inflexion. $u_1=0$ meets the curve in three points coincident with the origin and is called an inflexional or stationary tangent, or a tangent or 3-point contact. e.g. In cubical parabola $x^3=a^2y$; origin is an inflexion and $u_1\equiv y=0$ is the inflexional tangent.



We notice that the curve crosses tangent at (0,0) and that no curve where $n<3$ can possess this singularity.

UNDULATION AT THE ORIGIN.

$u_1\neq 0$; equation of form: $u_1+u_1v_1+u_1v_2+u_4+\cdots\cdot+u_n=0$. The origin is an undulation. $u_1=0$ meets the curve in four points coincident with the origin, e.g. $a^3(y+x)-2a^2x(y+x)+x^4=0$.



We notice that the curve does not cross the tangent, and that the curve must be of at least the fourth degree to possess this singularity.

TANGENT OF r-POINT CONTACT.

$u_1 \neq 0$; equation of form: $u_1+u_1v_1+u_1v_2+\cdots+u_1v_{r-2}+u_r+\cdots+u_n=0$. $u_1=0$ meets the curve in r points coincident with the origin, and is called a tangent of r-point contact.

DOUBLE POINTS - ACNODES, CRUNODES, CUSPS

$u_2 \neq 0$; equation of form: $u_2+u_3+\cdots+u_n=0$. The origin is a double point on the curve. Every line through the origin meets the curve in two coincident points there, except the two straight lines given by the quadratic $u_2=0$ which meet the curve in three coincident points at the origin. These two lines are the tangents (to the two branches) at the origin. Since a double point is one where the curve cuts itself once, curves possessing such points are frequently called autotomic (self-cutting).

If $u_2=0$ gives tangents real and distinct - (o,0) is a crunode
" " " " imaginary - " " an acnode
" " " " coincident - " " a cusp.

Three familiar curves illustrate these three kinds of double points.

1. Folium of Descartes: $x^3+y^3=3axy$.

$u_2=0$ gives x=o, y=o, for the two tangents. Since they are real and distinct, the origin is a crunode. (fig.1 below).

2. Conchoid of Nicomedes: $(x^2+y^2)(y-a)^2=b^2y^2$; $c^2=\frac{a^2-b^2}{a^2}$, a>b

$u_2=o$ gives $x=\pm icy$ for the two tangents. Since they are imaginary, the origin is an acnode. Although the co-ordinates (o,o) satisfy the equation of the curve, the point is isolated from all the other points as the diagram shows (below). For this reason acnodes are frequently termed isolated points, and some writers call them conjugate points. (fig.2 below)

3. Cissoid of Diocles: $y^2(2a-x)=x^3$

$u_2=o$ gives $y^2=o$, i.e. y=o, y=o, for the two tangents. Since they coincide, the origin is a cusp. (fig.3 below)

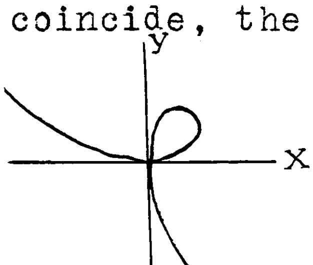


fig.1

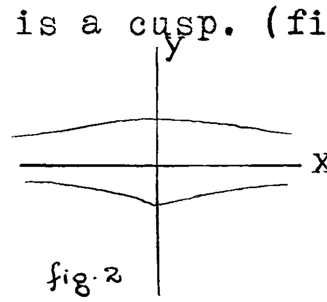


fig.2

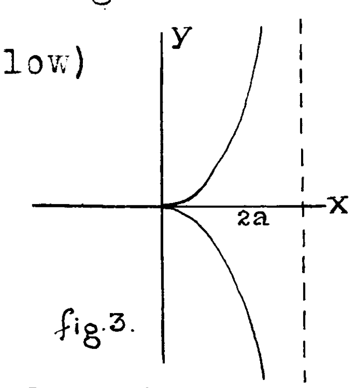
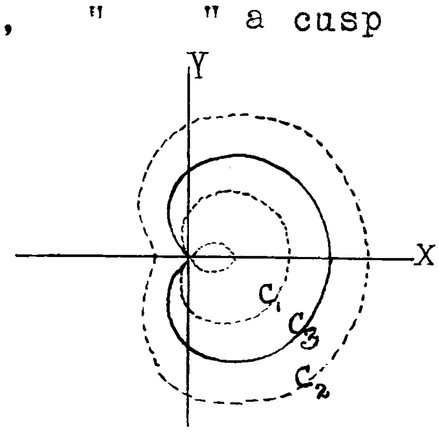


fig.3.

An example of the three types of double points is also furnished by the limaçon: $(x^2+y^2)^2-2bx(x^2+y^2)=(a^2-b^2)x^2+a^2y^2$. Tangents are real and unequal, imaginary, or coincident, according as $b><=a$. If $b>a$, origin is a crunode, and curve has form C_1 in diagram below.

If $b<a$, " " an acnode " " " " C_2 " " below.

If $b=a$, " " a cusp " " " " C_3 " " below.



Although the cusp presents itself as a species of double point, it is really a distinct singularity, as subsequent work will demonstrate. To differentiate it from the acnode and crunode we call these two latter, NODES; the term double point then includes the two types of singularities, nodes and cusps.

<u>FLECNODE, BIFLECNODE, etc.</u>

$u_1=o$; $u_2=v_1w_1 \neq 0$

(i). equation of form: $v_1w_1+v_1v_2+u_4+u_5+\cdots\cdots+u_n=o$; origin is a flecnode.

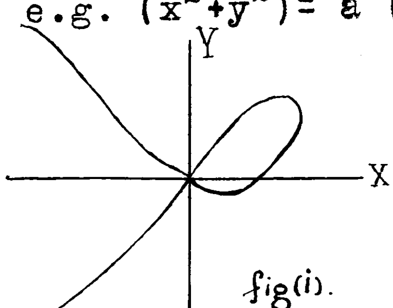
The tangent corresponding to the common linear factor v_1, has 3-point contact with one branch, and thus is an inflexional tangent to this branch.

e.g. $a^2(x+2y)(y-2x)-a(y-2x)x^2+y^4=0$. (fig.i below)

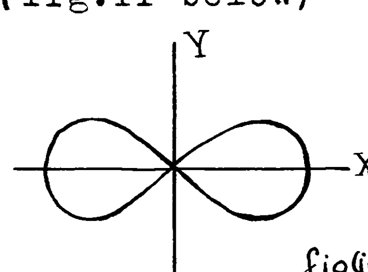
(ii). equation of form: $v_1w_1+v_1w_1v_1' +\cdots\cdots+u_n=o$; origin is a biflecnode.

Both tangents are inflexional.

e.g. $(x^2+y^2)^2= a^2(x^2-y^2)$; (fig.ii below)



fig(i).



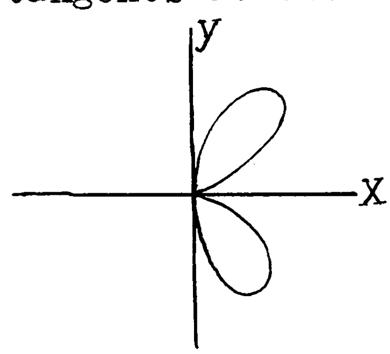
fig(ii).

(iii). Similarly, if v_1(or v_1w_1) is a factor of $u_2,u_3,\cdots\cdots,u_{r-1}$, the corresponding tangent (or tangents) has r-point contact with its branch (or the two branches).

TRIPLE POINT.

$u_1=u_2=o$; $u_3\neq o$; equation of form; $u_3+u_4+\cdots\cdots+u_n=o$; origin is a triple point. Three branches of the curve pass through the origin. Every line through (o,o) meets the curve in three coincident points, except the lines obtained by equating $u_3=o$, which meet the curve in four coincident points there. These lines are the tangents at the origin.

e.g. $axy^2= x^4+y^4$.



The roots of the equation $u_3=o$ admit of four possibilities, for they may be (1) all real and unequal, (2) all real and two equal, (3) all real and all equal. (4) one real and two complex. Consequently, there are four kinds of triple points.

k-PLE POINT

$u_1=u_2=\cdots=u_{k-1}=o$; $u_k\neq 0$; origin is a k-ple point. k distinct branches of the curve pass through the origin, and $U_k=o$ gives the k tangents to these branches there. The greater the value of k, the larger the number of possibilities of the k roots of $u_k=o$. Thus, as k increases, the various species of k-ple points become more and more numerous and complex.

So far the Cartesian system of co-ordinates has been assumed; let us now repeat our investigations where homogeneous co-ordinates form the basic system. The general equation of an n-ic is: $u_o z^n + u_1 z^{n-1} + \cdots + u_n = o$, where u_r is homogeneous of degree r in x and y. If (o,o,1) is on this n-ic, $u_0 = 0$, and equation becomes:

$u_1 z^{n-1} + u_2 z^{n-2} + \cdots + u_n = o$. Consider the intersections of $y = mx$ with this latter curve, and the resultant equation contains x^2 as a factor; hence $u_1 = o$ touches the curve, that is, it is tangent to the curve at (o,o,1); (which is C on our triangle of reference)

INFLEXION.

$u_o = o$; $u_1 \neq o$; equation of form: $u_1 z^{n-1} + u_1 v_1 z^{n-2} + u_3 z^{n-3} + \cdots + u_n = 0$; (o,o,1) is an inflexion.

$u_1 = o$ evidently meets curve in three coincident points at (o,o,1) and is therefore the inflexional tangent.

TANGENT OF r-POINT CONTACT.

$u_o = o$; $u_1 \neq o$; equation of form: $u_1 z^{n-1} + u_1 v_1 z^{n-2} + \cdots + u_1 v_{r-2} z^{n-r-1} + u_r z^{n-r} + \cdots + u_n = o$. $u_1 = o$ has r-point contact at (o,o,1).

k-PLE POINT.

$u_o = u_1 = \cdots = u_{k-1} = o$, $u_k \neq o$; equation of form: $u_k z^{n-k} + u_{k+1} z^{n-k-1} + \cdots + u_n =$; (o,o,1) is a k-ple point on the curve. The tangents at (o,o,1) are given by $u_k = o$. Suppose v_1 is one of these tangents. then it has 2-point contact on its branch. If it is a factor of $u_{k+1}, \ldots, u_{k+r}$, then the order of contact of tangent is (r+2) (Double and triple points are incorporated here, for the sake of brevity).

I have purposely avoided any precise definition of tangency. Certainly it is insufficient to say that a tangent is a line which intersects the curve in two coincident points, for as we have seen in the case of a double point the tangent meets the curve in three coincident points. The only satisfactory definition seems to be that "the tangent is the line of closest possible contact with the curve at that point." (Basset).

Since we can always transfer (o,o) or (o,o,1) to any definite point in the plane, our investigations naturally apply to points other than (o,o) or(o,o,1). However, a curve may possess singularities at infinity, and they can be determined in various ways (Explanations are found in Hilton pp.29-31, and Ganguli, vol.I. pp.71-72).
The following examples afford practical illustrations of the theory.

Ex.1. Discuss the nature of singularities at (o,o),(-1,0), &(1,0) on:

$y^2+x^3(x^2-1)^2(x-2)=o$

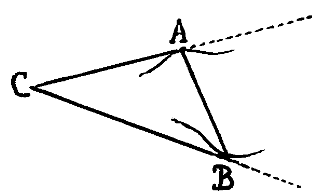
(i) for (o,o)

$u_0=u_1=o$; $u_2=y^2$ gives coincident tangents,$\therefore$(o,o) is a cusp on the curve.

(ii) transfer origin to the point (-1,o); equation assumes form: $(y^2+12x^2)+u_3+\cdots\cdots=o$, the tangents are $y=\pm 2\sqrt{3}\, ix$, that is, imaginary. Thus (-1,o) is an acnode on the curve.

(iii) Transfer origin to (1,o); equation becomes $y^2-4x^2+u_3+\cdots=o$; tangents are $y\pm 2x=o$, and are real and distinct. Thus (1,o) is a crunode on the curve.

Ex.2. An n-ic has the sides CA, CB of the triangle of reference as tangents of r-point contact, A and B being the points of contact. Show that its equation is of the form: $xyu_{n-2}=z^r\, u_{n-r}$, where u_K is homogeneous of degree K in x,y,z,



The equation must contain z^r as factor when =o, and when y=o; and is therefore of form: $xyu_{n-2} = z^r u_{n-r}$.

Ex.3. A line joins two real inflexions on a cubic, show that it passes through a third real inflexion.

Let us take the two real inflexions at A and B, and the inflexional tangents as CA and CB. Then by example 2; n=3, r=3. The equation of the cubic is: $xyu_1= z^3$. AB, i.e. z=o, intersects the cubic at A, B, and I, which is given by $u_1 \equiv ax+$ $+by=o$

Thus I= (-b,a,o) is a real inflexion and $u_1=o$ is the inflexional tangent.

Ex.4. The sides of the triangle of reference have n-point contact with an n-ic. Show that the equation of the n-ic can be put in form:

$xyzu_{n-3}+(ax\pm by)^n+(by+cz)^n+(cz+ax)^n= a^nx^n+b^ny^n+c^nz^n$; the + sign throughout if n is odd, and either sign if n is even.

(i) n odd; for side $x=o$, equation becomes: $(by+cz)^n = o$ which shows that $x=o$ has n-point contact with curve at point $(o,c,-b)$. Similarly for $y=o$, $z=o$.

(ii) n even; reasoning is the same, and we may evidently take either sign.

Ex.5. An n-ic has three tangents having n-point contact. (i) if n is odd the points are collinear (ii) if n is even, either the points are collinear, or the three lines joining each to the intersections of the tangents at the other two are concurrent.

(i). Take the tangents as the sides of the triangle of reference. Then, by example 4, equation has form:

$$xyzu_{n-3}+(ax+by)^n+(by+cz)^n+(cz+ax)^n = a^nx^n+b^ny^n+c^nz^n.$$

The points of contact are:- $(o,-c,b)$; $(-b,a,o)$; $(-c,o,a)$.

Points are collinear if $\Delta \equiv \begin{vmatrix} o & -c & b \\ -c & o & a \\ -b & a & o \end{vmatrix} = o$; which is so.

(ii). Using +sign in $(ax\pm by)^n$ we find, as in (i), that points are collinear.

If, however, we use form: $xyzu_{n-3}+(ax-by)^n+(by+cz)^n+(cz+ax)^n = a^nx^n+b^ny^n+c^nz^n$, the points of contact are:- $(o,-c,b)$, $(-c,o,a)$; $(b.a.o)$

$$\Delta \equiv \begin{vmatrix} o & -c & b \\ -c & o & a \\ b & a & o \end{vmatrix} = -2abc \neq 0;$$ hence points not collienear.

Equations of the three tangents are: $ax-by=0\ldots(1)$
$cz+ax=0\ldots(2)$
$by+cz=0\ldots(3)$

Equation of line joining $(o,-c,b)$ to intersection of (2)and(3)is:

$$\begin{vmatrix} x & y & z \\ c & -c & b \\ b & -a & -\frac{ab}{c} \end{vmatrix} = o$$

i.e. $ax-by=o$. Similarly the other two lines are $by+cz=o$ and $cz+ax=o$. These three lines are concurrent if: $\begin{vmatrix} a & -b & o \\ o & b & c \\ a & o & c \end{vmatrix}=0$, which is so.

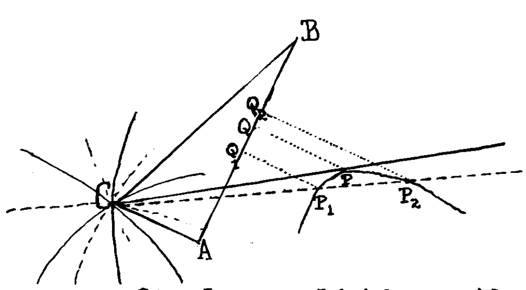
Ex.6. An n-ic has r-point contact with each of $x=o$ and $y=o$ at two distinct points, show that its equation has form:

$xyu_{n-2}=u_2^r\, u_{n-2r}$.

$x=o,y=o$ are bitangents with r-point contact. When $x=o$, u_2^r is a factor of the equation and similarly for $y=o$. Thus equation is: $xyu_{n-2}=u_2^r u_{n-2r}$

Ex.7 In general $n(n-1)-k(k+1)$ tangents to an n-ic can be drawn from the k-ple point $(o,o,1)$. (We exclude the tangents through C to the branches).

equation assumes form: $u_k z^{n-k}+u_{k+1}z^{n-k-1}+\ldots u_n=o$.



CP_1P_2 becomes tangent when $P_1Q_1=P_2Q_2$, i.e. when equation has equal roots in z.

To find condition that $f(x,y,z)=u_k z^{n-k}+\ldots u_n=o$ has equal roots in z, we must find the z - eliminant of $f=o$ and $f^1=o$. The resulting equation will be homogeneous in x and y and therefore represents tangents from $(o,o,1)$ to the n-ic. The degree of the eliminant is the degree of the typical term:

$$\left.\begin{aligned} f &= u_k z^{n-k} + \cdots\cdots + u_n \\ f' &= (n-k)u_k z^{n-k-1} + \cdots\cdots + u_{n-1} \end{aligned}\right\} \text{Typical term is } \bar{K} u_n^{n-k-1} u_k^{n-k}, \text{ i.e. of degree } n(n-k-1)+k(n-k).$$

Thus there are $n(n-1)-k^2$ tangents, including the k tangents to the branches at (o,o,1); therefore, apart from these: $n(n-1)-k^2-k=n(n-1)-k(k+1)$ tangents to the n-ic from (o,o,1).

CONDITIONS FOR MULTIPLE POINTS

1°. $f(x,y)=o$ is the equation of an n-ic in Cartesian form. Let (X,Y) be a point on the n-ic; if we transfer the origin to (X,Y) equation becomes: $f(X+x, Y+y)=o$.

By Taylor's Theorem, $f(X+x, Y+y) = f(X.Y.)+(x\frac{\partial}{\partial X} + y\frac{\partial}{\partial Y})f+\cdots\cdots+\frac{1}{\lfloor n}(x\frac{\partial}{\partial X}+y\frac{\partial}{\partial Y})^n f=o$

Since X,Y on the curve, $f(X,Y)=o$; if it is a double point, $u_1=o$; i.e. $\frac{\partial f}{\partial X} = \frac{\partial f}{\partial Y} = o$.

Then, the three conditions for (X,Y) to be a double point are:

$f(x,y) = \frac{\partial f}{\partial X} = \frac{\partial f}{\partial Y} = o$

The tangents at (X,Y) are given by: $u_2 = x^2\frac{\partial^2 f}{\partial X^2} + 2xy\frac{\partial^2 f}{\partial X \partial Y} + y^2\frac{\partial^2 f}{\partial Y^2} = o$;

They are real and distinct, imaginary, or coincident, according as $\left(\frac{\partial^2 f}{\partial X\partial Y}\right)^2 - \left(\frac{\partial^2 f}{\partial X^2}\right)\left(\frac{\partial^2 f}{\partial Y^2}\right) > < \text{ or } = 0$.

Thus: <u>for a crunode at (X.Y)</u> : $\left(\frac{\partial^2 f}{\partial X\partial Y}\right)^2 > \frac{\partial^2 f}{\partial X^2}\frac{\partial^2 f}{\partial Y^2}$

<u>for an aenode at (X.Y.)</u> . " $<$ "

<u>for a cusp at (X.Y.)</u> : " $=$ "

2° We now repeat the process for homogeneous co-ordinates. Let (X,Y,Z) be a point on the curve, then a point on the line joining this to any other point (x,y,z) is given by, $(\lambda X+\mu x, \lambda Y+\mu y, \lambda Z+\mu z)$, where $\lambda+\mu=1$. This point is on the curve if $f(\lambda X+\mu x, \lambda Y+\mu y, \lambda Z+\mu z)=o$.

By Taylor's theorem: $f(\lambda X+\mu x, \lambda Y+\mu y, \lambda Z+\mu z) \equiv \lambda^n f(X,Y,Z) + \lambda^{n-1}\mu\left(x\frac{\partial}{\partial X}+y\frac{\partial}{\partial Y}+z\frac{\partial}{\partial Z}\right)f + \cdots \frac{\mu^n}{\underline{|n}}\left(x\frac{\partial}{\partial X}+y\frac{\partial}{\partial Y}+z\frac{\partial}{\partial Z}\right)^n f = 0$

For a double point we must have: $\frac{\partial f}{\partial X}=\frac{\partial f}{\partial Y}=\frac{\partial f}{\partial Z}=0.$

The vanishing of these three quantities, in virtue of Euler's relation: $X\frac{\partial f}{\partial X} + Y\frac{\partial f}{\partial Y} + Z\frac{\partial f}{\partial Z} = nf$, implies that $f(X,Y,Z) = 0$. Thus we still have but three independent conditions for a double point.

The tangents at the double point are given by the homogeneous expression; $\left(x\frac{\partial}{\partial X}+y\frac{\partial}{\partial Y}+z\frac{\partial}{\partial Z}\right)^2 f \equiv x^2\frac{\partial^2 f}{\partial X^2}+y^2\frac{\partial^2 f}{\partial Y^2}+z^2\frac{\partial^2 f}{\partial Z^2}+2yz\frac{\partial^2 f}{\partial Y\partial Z}+2zx\frac{\partial^2 f}{\partial Z\partial X}+2xy\frac{\partial^2 f}{\partial X\partial Y}=0 \ldots\ldots (i).$

Equation (i) will represent two straight lines if D=o (#see footnote)

Here $D \equiv \begin{vmatrix} f_{11} & f_{12} & f_{13} \\ f_{21} & f_{22} & f_{23} \\ f_{31} & f_{32} & f_{33} \end{vmatrix}$ In this multiply the columns by X,Y,Z in turn, and then use

Euler's formula:

$$D = \frac{n}{XYZ}\begin{vmatrix} f_1 & f_{12} & f_{13} \\ f_2 & f_{22} & f_{23} \\ f_3 & f_{32} & f_{33} \end{vmatrix}$$ Since $f_1 = f_2 = f_3 = 0$, evidently D=o

---

#. In $f(x,y,z) = ax^2+by^2+ez^2+2fyz+2gzx+2hxy=0$, $D = \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}$

$f=0$ represents two straight lines, if D=o. If lines are coincident $A=B=C=F=G=H=0$; $D=0$; where $A=\frac{\partial D}{\partial a}=bc-f^2$, etc., $F=\frac{1}{2}\frac{\partial D}{\partial f}=gh-af$, etc.; $BC-F^2=aD$, etc., $GH-AF=fD$, etc.

If $F=G=H=0, D=0$, then at least two of A,B,C must vanish, say B and C. $B=ca-g^2$; $C=ab-h^2$ whence $a^2bc=g^2h^2=a^2f^2$; i.e., A=o. Thus the four conditions $F=G=H=D=0$ suffice to make $f=0$ represent two coincident lines. In the above exercise $f_1=\frac{\partial f}{\partial X}$, etc., $f_{12}=\frac{\partial^2 f}{\partial X\partial Y}$ etc.

Cf. Muir's Theory of Determinants.

If the two lines are coincident, we have, in addition to D=o, $A=B=C=F=G=H=0$

$$\left.\begin{array}{ll} A\equiv f_{22}\, f_{33} - f_{23}^2 = 0; & F \equiv f_{12}\cdot f_{13} - f_{11}\cdot f_{23} = 0 \\ B\equiv f_{33}\, f_{11} - f_{31}^2 = 0; & G \equiv f_{12}\cdot f_{23} - f_{22}\, f_{13} = 0 \\ C\equiv f_{11}\cdot f_{22} - f_{12}^2 = 0; & H \equiv f_{13}\cdot f_{23} - f_{33}\cdot f_{12} = 0 \end{array}\right\}$$

Now, $f_{12}(xf_{11}+yf_{12}+zf_{13}) - f_{11}(xf_{21}+yf_{22}+zf_{23}) = (n-1)f_1f_{12} - (n-1)f_2f_{11}$, by Euler.

i.e. $0 = (n-1)f_1f_{12}-(n-1)f_2f_{11}$ (for L.H .S. = 0 identically)

i.e. $\frac{f_1}{f_{11}} = \frac{f_2}{f_{12}} \left(= \frac{f_3}{f_{13}} \ldots\text{by symmetry}\right)$ Multiply numerators and denominators by X,Y,Z and add;

$$\frac{f_1}{f_{11}} = \frac{f_2}{f_{12}} = \frac{f_3}{f_{13}} = \frac{\sum Xf_1}{\sum Xf_{11}} = \frac{nf}{(n-1)f_1} = 0$$; it therefore follows that

$f_1 = f_2 = f_3 = 0$

Hence, for a cusp, we have four conditions as in Cartesian coordinates.

A summary of our results then is:

conditions that (X,Y,Z) is a node of $f=0$ are $f_1 = f_2 = f_3 = 0$

" " " " " cusp " "are $f = f_{11}\, f_{23} - f_{12}f_{13} = f_{22}f_{31} - f_{23}f_{21} = f_{33}f_{12} - f_{31}f_{32}$.

Although our considerations have been confined to double points, we may in a precisely analogous manner demonstrate that for a k-ple point on the curve, all partial derivations up to and including the order (k-1) vanish.

8. Find the double points on $x^4-2ay^3-3a^2y^2-2a^2x^2+a^4=0$

at a double point (X.Y.); $f(X.Y) = \frac{\partial f}{\partial X} = \frac{\partial f}{\partial Y} = 0$.

$$\left.\begin{array}{l}\frac{\partial f}{\partial X}=4X^3-4a^2X=0;\ X=0;\pm a.\\ \frac{\partial f}{\partial Y}=-6aY^2-6a^2Y=0;\ Y=0,-a.\end{array}\right\}$$ $f(a,o)=f(-a,o)=f(o,-a)$ o; hence the double points are $(a,o),(-a,o),(o,-a)$

For all three points: $\frac{\partial^2 f}{\partial X\partial Y}=o$; $\frac{\partial^2 f}{\partial X^2}\cdot\frac{\partial^2 f}{\partial Y^2}=-^{ve}$.

Hence $\left(\frac{\partial^2 f}{\partial X\partial Y}\right)^2 > \frac{\partial^2 f}{\partial X^2}\frac{\partial^2 f}{\partial Y^2}$,whence the three points are crunodes.

Ex.9. Find double points on: $y(x+3)^2=4(4x-3y)(2x-3y-6)$

$$\left.\begin{array}{l}\frac{\partial f}{\partial X}=64X-78Y-2YX-96\\ \frac{\partial f}{\partial Y}=-X^2-78X+72Y-63\end{array}\right\}$$ $X=-3$; $Y=-4$; $f(-3,-4)=o$,hence is a double point.

$$\frac{\partial^2 f}{\partial X^2}=64-2Y=72;\ \frac{\partial^2 f}{\partial Y^2}=72;\ \frac{\partial^2 f}{\partial X\partial Y}=78+2X=72^2=\frac{\partial^2 f}{\partial X^2}\cdot\frac{\partial^2 f}{\partial Y^2}$$

whence $(-3,-4)$ is a cusp on this cubic.

Ex.10. For what value of k has $x^3+y^3+z^3=k(x+y+z)^3$ a double point?

At a double point: $\frac{\partial f}{\partial X}=\frac{\partial f}{\partial Y}=\frac{\partial f}{\partial Z}=o$ (this implies $f=o$)

$$\left.\begin{array}{l}\frac{\partial f}{\partial X}=3X^2-3k(X+Y+Z)^2=o\\ \frac{\partial f}{\partial Y}=3Y^2-3k(X+Y+Z)^2=o\\ \frac{\partial f}{\partial Z}=3Z\ \ -3k(X+Y+Z)^2=o\end{array}\right\}$$ $(X,Y,Z)\equiv(1,1,1)$; since $x=y=z$, $3x^3=k(3x)^3$, i.e. $k=\frac{1}{9}$.

Putting $k=\frac{1}{9}$, equation is $x^3+y^3+z^3-\frac{1}{9}(x+y+z)^3=o$. The tangents at (X,Y,Z) are represented by:

$$x^2\frac{\partial^2 f}{\partial X^2}+y^2\frac{\partial^2 f}{\partial Y^2}+z^2\frac{\partial^2 f}{\partial Z^2}+2yz\frac{\partial^2 f}{\partial Y\partial Z}+2zx\frac{\partial^2 f}{\partial Z\partial X}+2xy\frac{\partial^2 f}{\partial X\partial Y}=o\ \cdots\cdots\ (i)$$

$\frac{\partial^2 f}{\partial X^2}=\frac{\partial^2 f}{\partial Y^2}=\frac{\partial^2 f}{\partial Z^2}=5$; $\frac{\partial^2 f}{\partial X\partial Y}=\frac{\partial^2 f}{\partial Y\partial Z}=\frac{\partial^2 f}{\partial Z\partial X}=-2$. $z=o$ meets (i) where $5x^2-4xy+5y^2=o$; i.e. $x=\frac{4\pm 2i\sqrt{21}}{10}$, that is, two imaginary lines.

Thus (i) represents two imaginary tangents at $(1,1,1)$ and consequently $(1,1,1)$ is an acnode on this curve.

Ex.11. If $S_1=0, S_2=0, \ldots, S_r=0$ have a k-ple point at P, and $C_1=0, C_2=0, \ldots, C_r=0$ are any other curves, $f\equiv C_1S_1+C_2S_2+\cdots+C_rS_r=0$ has a k-ple point at P.

Clearly $f_{\text{at } P}=0$; $\frac{\partial f}{\partial x}=C_1\frac{\partial S_1}{\partial x}+S_1\frac{\partial C_1}{\partial x}+C_2\frac{\partial S_2}{\partial x}+S_2\frac{\partial C_2}{\partial x}+\cdots\cdots$
$+C_r\frac{\partial S_r}{\partial x}+S_r\frac{\partial C_r}{\partial x}=0$

$$\left(\frac{\partial}{\partial x}\right)^{k-1}f=C_1\left(\frac{\partial}{\partial x}\right)^{k-1}S_1+\cdots\cdots+S_1\left(\frac{\partial}{\partial x}\right)^{k-1}C_1+C_2\left(\frac{\partial}{\partial x}\right)^{k-1}S_2+\cdots\cdots+S_2\left(\frac{\partial}{\partial x}\right)^{k-1}C_2+\cdots$$
$$+C_r\left(\frac{\partial}{\partial x}\right)^{k-1}S_r+\cdots\cdots+S_r\left(\frac{\partial}{\partial x}\right)^{k-1}C_r=0$$

Leibnitz expansion

Similarly for $\frac{\partial f}{\partial y}, \ldots\ldots, \left(\frac{\partial}{\partial y}\right)^{k-1}f$. Thus $f=0$, has a k-ple point at P.

Ex.12. If $u=0, v=0$ are straight lines and $p=0$, $r=0$ are (n-2)-ics, then $pu^2+2quv+rv^2=0$ is an n-ic with a node at P, the intersection of $u=0$ and $v=0$

$f=0$, $\frac{\partial f}{\partial x}=u^2\frac{\partial p}{\partial x}+2up\frac{\partial p}{\partial x}+2uv\frac{\partial q}{\partial x}+\cdots=0$, evidently vanishes at P. Similarly $\frac{\partial f}{\partial y}=0$, at P

Consequently P is a node on the curve.

The most general equation of degree n in Cartesian co-ordinates is:-

$$a+(b_0x+b_1y)+(c_0x^2+2c_1xy+c_2y^2)+\cdots\cdots+(p_0x^n+\cdots\cdots+p_ny^n)=0 \ldots\ldots(1)$$

Evidently there are $1+2+3+\cdots+(n+1)=\frac{1}{2}(n+1)(n+2)$ terms in this equation. For the corresponding homogeneous equation we merely multiply each term by the appropriate power of z, which naturally does not affect the number of terms. The number of independent constants in (1) is equal to one less than the number of terms it contains, since the generality of (1) remains unaltered when each term is divided by a and new constants are substituted for the ratios of the original coefficients to a. Hence the general equation of an n-ic contains $\frac{1}{2}n(n+3)$ independent constants, $(\frac{1}{2}(n+1)(n+2)-1=\frac{1}{2}n(n+3))$, and therefore the curve can be made to satisfy the same number of independent conditions. If these latter are that the curve is to pass through $\frac{1}{2}n(n+3)$ assigned points, we have $\frac{1}{2}n(n+3)$ linear relations between the coefficients. These $\frac{1}{2}n(n+3)$ conditions determine the coefficients uniquely, so that: ONE AND ONLY ONE n-IC CAN BE FOUND PASSING THROUGH $\frac{1}{2}n(n+3)$ GIVEN POINTS. Thus, for the determination of a conic, we require five points, for a cubic nine, for a quartic fourteen, etc. It is necessary to amplify the preceding remarks by the phrase, "in general", in order to exclude the possibility of inconsistent or dependent conditions. For example,suppose in the determination of a conic, four of the five points are collinear, then infinitely many conics (consisting of two straight lines) pass through the five points.

It is quite legitimate for the arbitrary points to determineuniquely a degenerate curve, for we did not reject this possibility in our considerations. e.g. Nine points determine a cubic, but if three of these are collinear, the cubic consists of this line and a conic through the remaining six points. The curve determined will be non-degenerate or proper, if no group of the $\frac{1}{2}n(n+3)$ given points lies on a curve of anorder lower than n.

To be given that a point is a node on a curve is equivalent to three linear relations between the coefficients viz: $f(X,Y)=o$; $\frac{\partial f}{\partial X}=o$; $\frac{\partial f}{\partial Y}=0$. If the given point is a cusp, we also have $\left(\frac{\partial^2 f}{\partial X \cdot \partial Y}\right)^2 = \left(\frac{\partial^2 f}{\partial X^2}\right)\left(\frac{\partial^2 f}{\partial Y^2}\right)$; that is to say, a cusp at a given point is equivalent to four linear relations between the coefficients. For a k-ple point, $1+2+3+\cdots+k=\frac{1}{2}k(k+1)$ terms of the general equation are absent, and the information that the curve has a k-ple point at a given point is therefore equivalent to $\frac{1}{2}k(k+1)$ linear relations between the coefficients.

A node on the n-ic, not at a given point, imposes but one condition upon the coefficients, obtained by eliminating x and y from: $f(x,y)=o$, $\frac{\partial f}{\partial x}=o$; $\frac{\partial f}{\partial y}=o$.

A cusp on the curve, not at a given point, is equivalent to two relations found by eliminating x and y from the last three equations and $\left(\frac{\partial^2 f}{\partial x \partial y}\right)^2 = \frac{\partial^2 f}{\partial x^2} \cdot \frac{\partial^2 f}{\partial y^2}$.

Similarly a k-ple point not at a given point imposes $\frac{1}{2}k(k+1)-2$ relations.

We easily infer that in general, A FINITE NUMBER OF n-ICS CAN BE FOUND HAVING δ NODES AND κ CUSPS AND SATISFYING r OTHER CONDITIONS, where $r = \frac{1}{2}n(n+3)-\delta-2\kappa$.

It is important to realize that these results claim no universality, and we must consequently exercise due caution in applying them.

Ex.13. To be given a k-ple point and the tangents at that k-ple point, is equivalent to $\frac{1}{2}k(k+3)$ linear relations between the coefficients. The given k-ple point $\sim \frac{1}{2}k(k+1)$ relations, and since each tangent at the point imposes one relation between the coefficients, the k tangents impose k.

$\therefore$ number of relations $= \frac{1}{2}k(k+1)+k = \frac{1}{2}k(k+3)$

Ex.14 $S+a_1S_1+a_2S_2+\cdots\cdots+a_rS_r=o$ is the equation of an n-ic with s given nodes and preassigned tangents at these points, where $S=o, S_1=o, \ldots, S_r=o$ are n-ics each possessing the s given nodes. Find r.

r given nodes $\sim$ 3s relations

Nodal tangents (two at each node) $\sim$ 2s relations

$\therefore r=\frac{1}{2}n(n+3)-(3s+2s)=\frac{1}{2}n(n+3)-5s.$

Ex.15. To be given an inflexion is equivalent to two conditions(#)

Let equation of curve be: $(b_0x+b_1y)+(c_0x^2+2c_1xy+c_2y^2)+u_3+\cdots$

$$\cdots+u_n=o$$

$b_0x+b_1y=o$ is the inflexional tangent, and when $b_0x+b_1y=o$, then $c_0x^2+2c_1xy+c_2y^2=o$. Substitute for $m=\frac{y}{x}=-\frac{b_0}{b_1}$ in $u_2=o$, and we have: $b_1^2c_0-2b_0b_1c_1+c_2b_0^2=o$

i.e. $f_1(b_0b_1c_0c_1c_2)=o$; $f_2(b_0b_1c_0c_1c_2)=o$. That is, we have two relations between the coefficients.

(#) See Ganguli, vol.I;p.60.Ex.2. If a point is to be an inflexion on a curve that amounts to three conditions.

Ex.16. Show that to be given three collinear inflexions on a cubic is equivalent to five (not six) conditions. For instance, show that a singly infinite family of cubics can be drawn with three given collinear inflexions and a given node.

We proved in example 3 that a straight line passing through two real inflexions passes through a third; hence the conditions imposed by three collinear inflexions number but five.

Let a cubic have a node at(0,0,1) and inflexions at (1,0,0),(0,1,0),(1,m,0). Then $r=\frac{1}{2}3(3+3)-3-5=1$; that is, one parameter is at our disposal. Form of cubic is: $f_3=xy(y-mx)+\lambda z\ (mxy-m^2x^2-y^2)=0$

Evidently (0,0,1) is a node on $f_3=0$, and we must now show that (1,0,0), (0,1,0), (1,m,0) are inflexions.

Equation of tangent at (1,0,0) is $U_1=y+\lambda mz=0$

$$\left[\left(x\frac{\partial f}{\partial X}+y\frac{\partial f}{\partial Y}+z\frac{\partial f}{\partial Z}\right)=0\ ;(X,Y,Z)\equiv(0,0,1)\right]$$

$$\left\{x\frac{\partial}{\partial X}+y\frac{\partial}{\partial Y}+z\frac{\partial}{\partial Z}\right\}^2 f_3=2y^2+2\lambda mz-4zx\lambda m^2-4mxy$$

$$=(y+\lambda mz)(2y-4mx)$$

$$=u_1v_1;$$ whence (0,0,1) is an inflexion.

Similarly for (0,1,0) and (1,m,0). Thus it is possible to draw a one parameter (or singly infinite) family of cubics with three given collinear inflexions and a given node.

DOUBLE POINTS ON AN n-IC

There is a limit to the number of double points an algebraic curve can possess. For example, a quartic cannot have more than three double points; for if it has four we can describe a conic through these four double points and any fifth point; the conic would therefore intersect the quartic in nine points, which is absurd - since a conic and a quartic intersect in only eight points ($n=2$, $N=4$, p.4)

We shall now prove the general theorem that: AN n-IC CANNOT HAVE MORE THAN $\frac{1}{2}(n-1)(n-2)$ DOUBLE POINTS.

If the curve has $\frac{1}{2}(n-1)(n-2)+1$ double points, it is possible to describe an $(n-2)$-ic through these double points and also through $(n-3)$ other points, since $\frac{1}{2}(n-1)(n-2)+1+n-3 = \frac{1}{2}(n-2)(n+1)$ $[(=\frac{1}{2}N(N+3); N=n-2)]$

At each double point there are two intersections of the $(n-2)$-ic and the n-ic. Consequently the n-ic and the $(n-2)$-ic intersect in $2(\frac{1}{2}(n-1)(n-2)+1)+n-3=n(n-2)+1$ points; but this is impossible, since they cannot intersect in more than $n(n-2)$ points.

We have thus assigned a definite limit to the number of double points on an n-ic and in so doing, we assume that the curve is non-degenerate.

SECTION II

TANGENTIAL EQUATIONS; POLAR RECIPROCATION; SUPERLINEAR BRANCHES.

Johann Plücker in his Theorie der Algebraischen Kurven wrote: "If a point move continuously along a straight line while the straight line rotates continuously about the point, one and the same curve is enveloped by the line and described by the point." Thus every curve(#note) has two equations, one in point co-ordinates and the other in line co-ordinates, depending upon whether we have considered the curve as traced by a moving point or as enveloped by a moving line. Hence whenever we demonstrate any descriptive theorem whatsoever by point co-ordinates, we simultaneously demonstrate the correlative theorem for line or tangential co-ordinates, and vice-versa. The importance of this dual aspect of a curve will become quite apparent in the sequel.

From previous acquaintance with point and line co-ordinates, (herein assumed) we know that $\lambda x+\mu y+\nu z=0$ represents the equation of a line (λ,μ,ν) in point co-ordinates, or the equations of a point (x,y,z) in line co-ordinates. That is to say, $\lambda x+\mu y+\nu z=0$ represents a pencil of lines on point $P(x,y,z)$ or a pencil of points on line $L(\lambda,\mu,\nu)$.

(#) The point and the line are exceptions to this dual principle. The point, which has the line as envelope, cannot be regarded as a locus, while the line, which is a locus of points, cannot be regarded as an envelope. Consequently a point has only a line equation, and a line has only a point equation.

TANGENTIAL EQUATIONS.

If the line $\lambda x+\mu y+\nu z=0$ is tangent to a curve $f(x,y,z)=0$, a relation $\phi(\lambda,\mu,\nu)=0$ exists between λ,μ,ν. This equation $\phi(\lambda,\mu,\nu)=0$, satisfied by λ,μ,ν, is called the tangential or line equation of $f(x,y,z)=0$.

If the point equation is given in Cartesian co-ordinates, the corresponding line equation expresses the relation existing between λ and μ when $\lambda x+\mu y+1=0$ touches the curve.

The number of line elements (m) common to $\lambda x'+\mu y'+\nu z'=0$ and $\phi(\lambda,\mu,\nu)=0$ is called the CLASS OF THE CURVE, just as the number of point elements (n) common to $\lambda' x+\mu' y+\nu' z=0$, & and $f(x,y,z)=0$, is called the DEGREE or ORDER OF THE CURVE. In section I we proved that the number of point elements common to two curves (i.e. their intersections) of degree n and N is nN. We can show by an exactly similar procedure that the number of line elements (i.e. tangents) common to two curves of class m and M respectively is mM.

If we are given the equation $f(x,y,z)=0$, of degree n, we know that any straight line in the plane meets the curve in n points, real or imaginary; whereas, given an equation $\phi(\lambda,\mu,\nu)=0$, of class m, we know that through any point in the plane m tangents, real or imaginary, can be drawn to the curve. This notion of the order and class of a curve is due to Gergonne.

Obviously, the degree of a curve is not necessarily equal to its class. Subsequent examples will substantiate this remark.

TO FIND TANGENTIAL EQUATION FROM POINT EQUATION.

(a) First Method:

$\lambda x+\mu y+\nu z=0$ intersects $f_n(x,y,z)=0$, where $f(x,y,-\frac{\lambda x+\mu y}{\nu})=0$. This equation is homogeneous in x and y and hence is the product of: $(a_1x+b_1y)(a_2x+b_2y)\ldots\ldots(a_nx+b_ny)=0$. Each of these factors represents a pencil of points on range $CP_{i=1,2,\ldots,n}$, where P_i is the intersection of the curve and the line. The line will be tangent if two ranges coincide. Hence the tangential equation is the condition that the above equation has two equal roots. This is given by $\phi(\lambda,\mu,\nu)=0$, the discriminant of $f(x,y,-\frac{\lambda x+\mu y}{\nu})=0$

(b) Second Method:

Let $\lambda x+\mu y+\nu z=0$ touch the curve at (α,β,γ).

Equation tangent there is: $x\frac{\partial f}{\partial\alpha}+y\frac{\partial f}{\partial\beta}+z\frac{\partial f}{\partial\gamma}=0$.

Comparing the two equations which define the same line, we have:

$$\frac{\frac{\partial f}{\partial\alpha}}{\lambda}=\frac{\frac{\partial f}{\partial\beta}}{\mu}=\frac{\frac{\partial f}{\partial\gamma}}{\nu};\text{ also by Euler: }\alpha\frac{\partial f}{\partial\alpha}+\beta\frac{\partial f}{\partial\beta}+\gamma\frac{\partial f}{\partial\gamma}=nf(\alpha,\beta,\gamma)$$

Upon elimination of α,β,γ we obtain, as in (a), $\phi(\lambda,\mu,\nu)=0$

Ex.18. A conic is of degree 2 and class 2.

Conic has equation: $f(x,y,z)\equiv ax^2+by^2+cz^2+2fyz+2gzx+2hxy=0$

Using method (b); tangent at (α,β,γ) is: $x\frac{\partial f}{\partial\alpha}+y\frac{\partial f}{\partial\beta}+z\frac{\partial f}{\partial\gamma}=0$

This is to be the same as $\lambda x+\mu y+\nu z=0$

$$\frac{\frac{\partial f}{\partial \alpha}}{\lambda} = \frac{\frac{\partial f}{\partial \beta}}{\mu} = \frac{\frac{\partial f}{\partial \gamma}}{\nu} = R(\text{say})$$

i.e.
$$\left.\begin{aligned} \lambda R &= a\alpha + h\beta + g\gamma \\ \mu R &= h\alpha + b\beta + f\gamma \\ \nu R &= g\alpha + f\beta + c\gamma \\ 0 &= \lambda\alpha + \mu\beta + \nu\gamma \end{aligned}\right\} \text{ Eliminate } R, \alpha, \beta, \gamma \text{ and: } \begin{vmatrix} a & h & g & \lambda \\ h & b & f & \mu \\ g & f & c & \nu \\ \lambda & \mu & \nu & 0 \end{vmatrix} = 0$$

i.e. $A\lambda^2 + B\mu^2 + C\nu^2 + 2F\mu\nu + 2G\nu\lambda + 2H\lambda\mu = 0$, the tangential equation, which is of class 2.

Ex.19. Find tangential equation of: $3(x+y) = x^3$

$\lambda x + \mu y + 1 = 0$ intersects curve where: $3\left(x - \frac{\lambda x + 1}{\mu}\right) = x^3$

$$\left.\begin{aligned} f &\equiv \mu x^3 - 3(\mu x - \lambda x - 1) = 0 \ldots\ldots(i) \\ \frac{\partial f}{\partial x} &\equiv 3\mu x^2 - 3\mu + 3\lambda = 0 \ldots\ldots(ii) \end{aligned}\right\} \text{Multiply (i) by 3. (ii) by } x.$$

$$\left.\begin{aligned} 3\mu x^3 - 9\mu x + 9\lambda x + 9 &= 0 \ldots\ldots(i)' \\ 3\mu x^3 - 3\mu x + 3\lambda x &= 0 \ldots\ldots(ii)' \end{aligned}\right\} \text{substract}$$

$$6x(\lambda - \mu) = -9$$

$\therefore x = \frac{-3}{2(\lambda - \mu)}$; substitute this in (i) and:

$9\mu - 4(\lambda - \mu)^3 = 0$.....required tangential equation.

Ex.20. Find tangential equation of: $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}} z^{\frac{2}{3}}$

$f(x, y, -\frac{\lambda x + \mu y}{\nu}) = 0$; $\nu^{\frac{2}{3}}(x^{\frac{2}{3}} + y^{\frac{2}{3}}) = a^{\frac{2}{3}}(\lambda x + \mu y)^{\frac{2}{3}}$

i.e. $\nu^{\frac{2}{3}}(\xi^{\frac{2}{3}} + 1) = a^{\frac{2}{3}}(\lambda\xi + \mu)^{\frac{2}{3}}$.....(i) , $\xi = \frac{x}{y}$

Differentiate with respect to ξ ; $\quad \frac{\nu^{\frac{2}{3}}}{\xi^{\frac{1}{3}}} = \frac{a^{\frac{2}{3}}\lambda}{(\lambda\xi+\mu)^{\frac{1}{3}}}$

Cube and simplify; $\xi = \frac{\mu\nu^2}{a^2\lambda^3 - \nu^2\lambda}$, substitute in (i):

$$\nu^{\frac{2}{3}}(a^2\lambda^3 - \nu^2\lambda)^{\frac{2}{3}} = \mu^{\frac{2}{3}}(a^2\lambda^2 - \nu^2)$$

Cube and: $\nu^2(a^2\lambda^3 - \nu^2\lambda)^2 = \mu^2(a^2\lambda^2 - \nu^2)^3$

Whence $\nu^2(\lambda^2 + \mu^2) = a^2\mu^2\lambda^2$.....tangential equation.

N.B. If we cube f =o, we have $a^2z^2 - x^2 - y^2 = 3x^{\frac{2}{3}}y^{\frac{2}{3}}.(a^{\frac{2}{3}}\ z^{\frac{2}{3}})$

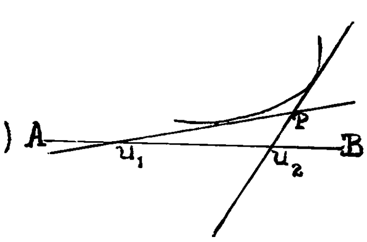
Cubing again, $(a^2z^2 - x^2 - y^2)^3 = 27a^2x^2y^2z^2$......i.e.n=6

TO DERIVE POINT EQUATION FROM THE TANGENTIAL.

(a) First Method:

Pencil of lines $\lambda x + \mu y + \nu z = o$ has in common with curve line elements for which $\phi(\lambda, \mu, -\frac{\lambda x + \mu y}{z}) = o$ represents points on AB.

$$\phi\left(\lambda, \mu, -\frac{\lambda x + \mu y}{z}\right) = u_1 . u_2 \ldots\ldots . u_n = o \ldots . (i)$$



If two points coincide, P is on the curve, and then required equation is f(x,y,z)=o, which is the discriminant of (i).

(b).Second Method:

Let (x,y,z) be a point on the curve, and (λ', μ', ν') be tangent. Then equation of point of contact is: $\lambda\frac{\partial\phi}{\partial\lambda'} + \mu\frac{\partial\phi}{\partial\mu'} + \nu\frac{\partial\phi}{\partial\nu'} = o$

Hence x: y: z = $\frac{\partial\phi}{\partial\lambda'} : \frac{\partial\phi}{\partial\mu'} : \frac{\partial\phi}{\partial\nu'}$; also by Euler:

$(\lambda'\frac{\partial}{\partial\lambda'} + \mu'\frac{\partial}{\partial\mu'} + \nu'\frac{\partial}{\partial\nu'})\phi \equiv n\,\phi(\lambda', \mu', \nu')$. . Eliminating λ', μ', ν', between these three equations we obtain f(x,y,z) =o.

Ex.21. Find point equation of: $(-\lambda)^{p+q} p^p q^q = \mu^p \nu^q (p+q)^{p+q}$

$$\phi\left(\lambda,\mu,-\frac{\lambda x+\mu y}{z}\right)=0 \text{ is: } (-1)^{p+q}\lambda^{p+q}p^p q^q z^q=(-1)^q\mu^p(\lambda x+\mu y)^q(p+q)^{p+q}$$

$$(-1)^{p+q}\xi^{p+q}p^p q^q z^q=(-1)^q(\xi x+y)^q(p+q)^{p+q}\ \dots\dots\dots(i);\quad \xi=\frac{\lambda}{\mu}$$

Differentiate w.r. to ξ: $(-1)^p(p+q)z^q p^p q^q \xi^{p+q-1} = xq(\xi x+y)^{q-1}(p+q)^{p+q}\ \dots(ii)$.

Divide (i) by (ii): $\frac{\xi}{p+q}=\frac{\xi x+y}{xq}$; $\xi=-\frac{y(p+q)}{xp}$

Substitute for ξ in(i); $z^q y^p = x^{p+q}$, required point equation

Here $n = m = p+q$

POLAR RECIPROCATION

The general principles of polar reciprocation were first enunciated by M. Poncelet.

Consider a base-conic C and a curve S. As a point P describes S its polar with respect to C will have an envelope S', which is called the POLAR RECIPROCAL OF S. If P and Q are neighbouring points on S, their polars will evidently meet in R, the pole of P Q. In the limit, $Q \to P$, QP becomes tangent to S at P, and R becomes the point of contact of tangents to S' corresponding to P. Hence if any tangent to S corresponds to a point on S', the point of contact of that tangent to S will correspond to the tangent through the point on S'. Thus the relation between S, S', is reciprocal; that is to say, the curve S may be generated from S' in precisely the same manner as S' is generated from S.

If L is any line and P its pole, L intersects S in points that are poles of tangents to S', through the pole of L. Hence as many tangents to S' can be drawn through a point as there are points on S lying on a straight line. This is equivalent to saying that the class of S' is equal to the degree of S, or reciprocally.

DEGREE OF CURVE = CLASS OF ITS POLAR RECIPROCAL.

Let us select for the base-conic, the imaginary circle $x^2+y^2+z^2=0$, and let the tangential equation of the curve be $\phi(\lambda,\mu,\nu)=0$

The polar (#) of (x',y',z') is: $xx'+yy'+zz'=0$; this will be an element (line element) of curve $\phi(\lambda,\mu,\nu)=0$, if $\phi(x',y',z')=0$ Hence the equation of the polar reciprocal of $\phi(\lambda,\mu,\nu)=0$ is $\phi(x,y,z)=0$. Thus, THE POLAR RECIPROCAL OF $f(x,y,z)=0$ IS $\phi(x,y,z)=0$, where $\phi(\lambda,\mu,\nu)=0$ is the line equation of $f(x,y,z)=0$

Similarly, for the Cartesian equation $f(x,y)=0$, the polar reciprocal will be $\phi(x,y)=0$, (where $\phi(\lambda,\nu)=0$ is condition that $\lambda x+y+\nu=0$ touches the curve), if the base-conic is the parabola $x^2+2y=0$.

Ex.22. Find polar reciprocal of $\phi(\lambda,\mu,\nu)=0$ with respect to base conic: $ax^2+by^2+cz^2+2fyz+2gzx+2hxy=0$

Polar of (x',y',z') is: $x(ax'+hy'+gz')+y(hx'+by'+fz')+z(gx'+fy'+cz')=0$

Hence polar reciprocal is: $\phi(ax+hy+gz,hx+by+fz,gx+fy+cz)=0$

(#) We assume a knowledge of polar properties with respect to conics: e.g. equ. polar of (x',y',z') is: $\left(x'\frac{\partial}{\partial x}+y'\frac{\partial}{\partial y}+z'\frac{\partial}{\partial z}\right)\phi=0$; $\phi=0$ is conic.

Ex.23. Show that polar reciprocal of $x^p y^q = a^{p+q}$ with respect to a circle whose centre is at the origin is another curve of same kind

Conic is $x^2+y^2=r^2$; polar of (x',y') is: $xx'+yy'=r^2$ same as $\lambda x+\mu y+1=0$ } whence

$$\lambda = -\frac{x'}{r^2} \; ; \mu = -\frac{y'}{r^2}$$

Tangential equation of curve from Ex.21 is:

$$(-1)^{p+q} p^p q^q = \lambda^p \mu^q a^{p+q} (p+q)^{p+q}$$

i.e. $$p^p q^q = \frac{x^p y^q a^{p+q} (p+q)^{p+q}}{r^{2(p+q)}}$$

i.e. $$x^p y^q = \left(\frac{r^2 \, p^{\frac{p}{p+q}} \, q^{\frac{q}{p+q}}}{a} \right)^{p+q}$$

i.e. $x^p y^q = k^{p+q}$; a curve of same kind.

Ex.24 Find polar reciprocal of $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$

From Ex.19, $(z=1)$, tangential equation is: $\lambda^2+\mu^2=a^2\lambda^2\mu^2$

Selecting $x^2+2y=0$ as base-conic, polar reciprocal is:

$x^2+y^2=a^2x^2y^2$, or $\frac{1}{x^2}+\frac{1}{y^2}=\frac{1}{c^2}$; where $c^2=\frac{1}{a^2}$

SINGULARITIES ON A CURVE AND ITS RECIPROCAL

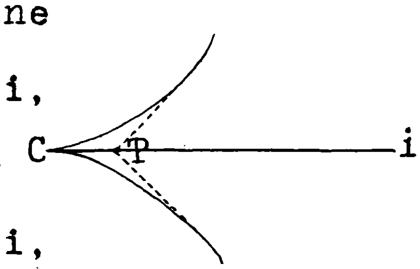
We found for the two polar reciprocal curves S and S' that the line and point properties of one are exactly the same as the point and line properties of the other. We shall now investigate this correspondence for singularities.

A NODE ON A CURVE CORRESPONDS TO A BITANGENT ON THE RECIPROCAL CURVE, and vice-versa.

Two branches of a curve, with a distinct tangent to each, pass through a node. Hence to a node and its tangent corresponds a tangent with two distinct points of contact, (i.e. a bitangent), in the reciprocal curve. Assuming the base-conic to be real, then to a crunode corresponds a real bitangent with real points of contact, while to an acnode or isolated point corresponds a real bitangent with imaginary points of contact.

TO A CUSP C AND CUSPIDAL TANGENT i CORRESPOND IN THE RECIPORCAL CURVE AN INFLEXIONAL TANGENT c AND ITS INFLEXION I.

For the cusp has the property that every line through C meets the curve in two points, except i, which meets it at three points in C. Of the tangents from any point P on i, one coincides with i, excepting that P be at C, when three coincide with i.



Hence from any point on c two tangents can be drawn to the reciprocal curve coinciding with c itself, unless the point be at I, when three tangents coincide with c. Also any line through I meets the reciprocal curve at one point in I, except c, which meets it thrice at I. Consequently I is an inflexion and c the inflexional tangent.

Similarly, to a triple, quadruple,. . . .,k-ple point with distinct tangents corresponds a tangent with three, four,. . . . ,k distinct points of contact (we call these latter 3-fold,4-fold,. .k-fold tangents). To a triple point with coincident tangents corresponds the tangent at a point of undulation, and so for quadruple,....., k-ple points with coincident tangents.

We note in passing, that the polar reciprocal of a flecnode is evidently a double tangent which has a contact of the first order at one point of the reciprocal curve and touches it at a cusp at the other; while to a biflecnode corresponds a pair of cusps with a common cuspidal tangent.

Consider a curve C of degree n and class m, and possessing δ nodes, κ cusps, τ bitangents, ι inflections. The six quantities $n, m, \delta, \kappa, \tau, \iota$ are called the PLÜCKER'S NUMBERS OF THE CURVE.

In virtue of the foregoing considerations we see that by interchanging n and m, δ and τ, κ and ι, we have the Plücker's numbers of the reciprocal curve - which we denote symbolically R(C). A schematic arrangement places the correspondence in evidence:

	C	R(C)
degree	n	m
class	m	n
nodes	δ	τ
cusps	κ	ι
bitangents	τ	δ
inflexions	ι	κ

In section I we found that the number of conditions necessary to determine an n-ic is $\frac{1}{2}n(n+3)$, and that δ nodes and κ cusps on a curve impose $\delta + 2\kappa$ conditions. Thus, if we are given that a curve, having δ nodes and κ cusps, satisfies r other conditions, we know that $\frac{1}{2}n(n+3)=\delta+2\kappa+r$....(i). Obviously, the polar reciporcal of degree m, possessing τ nodes and ι cusps, is also subjected to r conditions, i.e. $\frac{1}{2}m(m+3)=\tau+2\iota+r$....(ii). From (i) and (ii) eliminating r:

$$\tfrac{1}{2}n(n+3)-\delta-2\kappa=\tfrac{1}{2}m(m+3)-\tau-2\iota$$ PLÜCKER'S EQUATION.

SUPERLINEAR BRANCHES.

We conclude this section with a few remarks concerning Superlinear Branches. Since an adequate discussion of the principles involved properly belongs to the subject of Algebraic Functions, we tacitly assume certain results without proof.

Suppose f(x,y)=o is the equation of a curve passing through O (o,o), then y on any branch through O is expressible in form:

$y=ax+b\omega^{\beta}x^{\frac{\beta}{\alpha}}+c\omega^{\gamma}x^{\frac{\gamma}{\alpha}}+\ldots\ldots\ldots\ldots$, where $\omega^{\alpha}=1$; a,b,c,....are constants, and α, β, γ,...... are positive integers in ascending order of magnitude.

The tangent at O is y=ax. The entire portion of the curve near O, obtained by giving to ω each of the α roots of unity is called a SUPERLINEAR BRANCH OF ORDER α. Each $\sqrt[\alpha]{1}=\omega$ gives a PARTIAL SUPERLINEAR BRANCH, while if $\alpha=1$, the branch is LINEAR. It follows that if O is an ordinary point on f=o, only one linear branch (the curve itself) passes through O, while at a k-ple point with distinct tangents there are k linear branches. If, however, at a k-ple point, two or more tangents coincide, we have superlinear branches (obviously, a cusp is the simplest case here).

If we select axes so that y =o is tangent, the curve $y^k = ax^{k+1} + bxy^k + cx^{k-1}y^2 + \ldots\ldots$, has an ordinary (#) superlinear branch of order k at (o,o); the expansion of y near O is:

$$y = A\omega x^{\frac{k+1}{k}} + B\omega^2 x^{\frac{k+2}{k}} + \ldots\ldots\ldots; \quad \omega = \sqrt[k]{1} \ldots\ldots (i)$$

APPLICATION TO THE INTERSECTIONS OF CURVES AT SINGULAR POINTS.

We proved in section I that an n-ic and an N-ic intersect in Nn points.

$$f(x,y) \equiv a_o y^n + a_1 y^{n-1} + \ldots + a_n \equiv (y-u_1)(y-u_2)\ldots\ldots(y-u_n) = o$$

$$F(x,y) \equiv b_o y^N + b_1 y^{N-1} + \ldots + b_N \equiv (y-v_1)(y-v_2)\ldots\ldots(y-v_N) = o$$

In these equations a_r, b_r are polynomials of r-th degree in x, while u_i and v_j are functions of x given by (i) for x sufficiently small. (We assume that the y-axis is not tangent to either curve, and that it meets the curve in finite points only; also, that no other intersection of f and F takes place on x=o, except at the origin) Corresponding to a point not at the origin on the y-axis, u_i or v_i will contain a constant term.

Eliminating y in the two equations, we have an eliminant $\phi(x)=o$ of degree nN. $\phi(x) \equiv (u_1 - v_1)\ldots\ldots(u_1 - v_N)\ldots\ldots(u_n - v_1)\ldots\ldots(u_n - v_N) = \prod_{i \neq j}(u_i - v_j) = o$

(so-called because there are no special relations between the coefficients.

We require the number of intersections at O, which is the number of zero roots of $\phi(x)=0$. This number (ϵ) is the exponent of the lowest powers of x in $\phi(x)$, i.e. it is the product of the lowest powers of x in each $(u_i - v_j)$, through O. To illustrate, we consider six important cases.

Case 1. $f(x,y)=0$ has a double point at the origin.

$F(x,y)=0$ has O an ordinary point, but does not touch $f=0$ at O.

Two linear branches pass through O on f, and one on F.

$$\left.\begin{array}{l} u_1 = ax + bx^2 + \cdots\cdots\cdots \\ u_2 = a'x + b'x^2 + \cdots\cdots\cdots \end{array}\right\}$$

$$v_1 = Ax + Bx^2 + \cdots\cdots\cdots$$

$$\Pi(u_i - v_j) \equiv \{(a-A)x+\cdots\}\{(a'-A)x+\cdots\} = cx^2 + \cdots$$

$$\therefore \epsilon = 2$$

Case 2. $f(x,y) = 0$ has a cusp at the origin.

$F(x,y) = 0$ has O as an ordinary point, but touches $f(x,y)$ o at O.

$$\left.\begin{array}{l} u_1 = ax^{\frac{3}{2}} + bx^2 + \cdots\cdots\cdots \\ u_2 = -ax^{\frac{3}{2}} + bx^2 - \cdots\cdots\cdots \end{array}\right\}$$

$$v_1 = Ax^2 + Bx^3 + \cdots\cdots\cdots \qquad (y=0 \text{ is tangent})$$

$$\Pi(u_i - v_j) \equiv \{ax^{\frac{3}{2}} + \cdots - Ax^2 \cdots\}\{-ax^{\frac{3}{2}} \quad -Ax^2 \quad\} = cx^3 + \cdots\cdots\cdots$$

$$\therefore \epsilon = 3.$$

Case 3

$$\left.\begin{array}{l} f(x,y)=0 \\ F(x,y)=0 \end{array}\right\} \text{ have nodes at 0 with common tangents.}$$

$$\left.\begin{array}{l} u_1 = ax + bx^2 + \cdots\cdots \\ u_2 = a'x + b'x^2 + \cdots \end{array}\right\}$$

$$\left.\begin{array}{l} v_1 = ax + Bx^2 + \\ v_2 = a'x + B'x^2 + \cdots \end{array}\right\} \quad (A=a,\ A'=a' \text{ since tangents common.})$$

$$\Pi(u_i - v_j) \equiv \{(b-B)x^2 + \cdots\}\{(a-a')x + \cdots\}\{(a'-a)x + \cdots\}\{(b'-B)x^2 + \cdots\} = c^2x^6 + \cdots$$

$$\therefore\ \epsilon = 6$$

Case 4 $f(x,y)=0$ has a cusp at 0

$F(x,y)=0$ has a triple point at 0, and two tangents coincide with the cuspidal tangents. (Evidently both curves have superlinear branches)

$$\left.\begin{array}{l} u_1 = +ax^{\frac{3}{2}} + bx^2 + \cdots \\ u_2 = -ax^{\frac{3}{2}} + bx^2 - \cdots \end{array}\right\}$$

$$\left.\begin{array}{l} v_1 = a'x + b'x^2 + \cdots \\ v_2 = +Ax^{\frac{3}{2}} + Bx^2 + \cdots \\ v_3 = -Ax^{\frac{3}{2}} + B'x^2 - \cdots \end{array}\right\}$$

$$\Pi(u_i - v_j) = \{-a'x + \cdots\}\{x^{\frac{3}{2}}(a-A) - \cdots\}\{x^{\frac{3}{2}}(a+A) \cdots\}\{-a'x + \cdots\}\{x^{\frac{3}{2}}(-A-a) \cdots\}$$
$$\{x^{\frac{3}{2}}(A-a) + \cdots\} = cx^8 + \cdots\cdots$$

$$\therefore\ \epsilon = 8$$

Case 5 $f(x,y)=0$ has k linear branches through O } no two of the

$F(x,y)=0$ " K " " " " } k plus K tangents coincide

$u_i = a_i x + b_i x^2 + \cdots\cdots$

$v_j = a'_j x + b'_j x^2 + \cdots\cdots$

$\prod(u_i - v_j) = \{x(a_1 - a'_1) + \cdot\ \} \cdots kK$ factors $= cx^{kK} + \cdots$

$\therefore \epsilon = kK$

Case 6. $f(x,y)=0$ has k linear branches through O

$F(x,y)=0$ " " " " " "; and the

tangents are common.

$u_i = a_i x + b_i x^2 + \cdots$

$v_j = a_j x + B_j x^2 + \cdots$

$\prod(u_i - v_j) = \{x^2(b_i - B_j) \cdots\} \cdots k$ factors $\cdot \{x(a_1 - a_2) \cdots\} \cdots$

$\cdots k(k-1)$ factors $= x^{2k} x^{k(k-1)} = x^{k(k+1)} + \cdots$

$\therefore \epsilon = k(k+1)$

Ex. 25/. Two curves have linear branches touching the same tangent at O, one having p-point contact and the other q-point contact with the tangent, where $p \geqslant q$

How many intersections at O ?

Taking $y = 0$ as tangent :

$u_1 = ax^p + bx^{p+1} + \cdots\cdots$

$v_1 = Ax^q + Bx^{q+1} + \cdots\cdots$

$\prod(u_i - v_j) = \{ax^p + \cdots - Ax^q - \cdots\} = cx^q + \cdots\cdots \quad \therefore \epsilon = q$; i.e. curves meet q times at O.

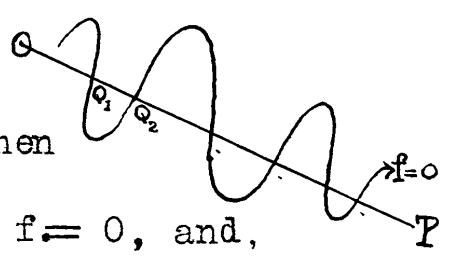
SECTION III

POLAR CURVES.

We saw that the tangents which can be drawn to a curve from an arbitrary point are m in number, where m is the class of the curve. We naturally seek to find the relation which exists between m and n. Waring was the first to prppose a solution, and fixed n^2 as the maximum number of tangents which could be drawn from a point to the n-ic. Poncelet, however, produced substantial evidence to invalidate this contention, and from his own investigations he formulated the theorem: "The number of tangents which can be drawn from a point to a curve of order n is in general, and at most, n(n-1)." He intimated that this limit was subject to reduction when the curve possessed double points, but the satisfactory explanation of such cases was given by his contemporary, Plücker. Poncelet's method of solution inbolves a knowledge of Polar Curves.

POLAR CURVES.

Let O be a fixed point, and $f=0$ an n-ic; also let OP be any line meeting curve in $Q_{i=1,2,\dots n}$. The locus of P such that $\sum \frac{OQ_i}{PQ_i} = 0$ is called the FIRST POLAR CURVE OF O. with respect to $f=0$. Similarly, if $\sum_{i \neq j} \frac{OQ_i\, OQ_j}{PQ_i\, PQ_j} = 0$, then locus of P is the SECOND POLAR CURVE of O w. r. to $f=0$, and, in general, k-th polar curve of O is locus of P such that :



$$\sum \frac{OQ_{i_1} OQ_{i_2} \cdots\cdots OQ_{i_k}}{PQ_{i_1} PQ_{i_2} \cdots\cdots PQ_{i_k}} = 0$$

Let $O(X,Y,Z)$ be any given point, $P(x,y,z)$ a second point, and $f(x,y,z)$ 0 the equation of the n-ic. The co-ordinates of a point Q which divides OP in ratio $\lambda:\mu$ $(\lambda+\mu=1)$ are:

$\lambda x+\mu X, \lambda y+\mu Y, \lambda z+\mu Z$. If Q lies on curve :

$f(\lambda x+\mu X, \lambda y+\mu Y, \lambda z+\mu Z) = 0.$ Expanding by Taylor's Theorem:

(i) $\lambda^n f(x,y,z) + \lambda^{n-1}\mu\left(X\frac{\partial}{\partial x}+Y\frac{\partial}{\partial y}+Z\frac{\partial}{\partial z}\right)f + \cdots \cdots + \frac{\mu^n}{\lfloor n}\left(X\frac{\partial}{\partial x}+Y\frac{\partial}{\partial y}+Z\frac{\partial}{\partial z}\right)^n f = 0$

or

(ii) $\mu^n f(X,Y,Z) + \mu^{n-1}\lambda\left(x\frac{\partial}{\partial X}+y\frac{\partial}{\partial Y}+z\frac{\partial}{\partial Z}\right)f + \cdots + \frac{\lambda^n}{\lfloor n}\left(x\frac{\partial}{\partial X}+y\frac{\partial}{\partial Y}+z\frac{\partial}{\partial Z}\right)^n f = 0$

If Q lies on the k-th polar of O, the sum of the products of the roots taken k at a time of the equation in $\frac{\lambda}{\mu}$ must be zero. Hence the equation of k-th polar curve of O is:

(1) $\left(X\frac{\partial}{\partial x}+Y\frac{\partial}{\partial y}+Z\frac{\partial}{\partial z}\right)^k f = 0$ or (2) $\left(x\frac{\partial}{\partial X}+y\frac{\partial}{\partial Y}+z\frac{\partial}{\partial Z}\right)^{n-k} f = 0.$

THE k-TH POLAR OF AN n-IC IS AN (n-k)- IC. Hence the first, second, ..., n-2, n-1, polars of an n-ic are of degrees n-1, n-2, ..., 2,1, respectively.

Ex.26 The r-th polar curve of O is the s-th polar curve of O with respect to the (r-s)-th polar curve.

O(o,o,1), r-th polar curve is: $\frac{\partial^r f}{\partial z^r} = 0 \ldots\ldots (i)$

(r-s)-th polar curve is: $\frac{\partial^{r-s} f}{\partial z^{r-s}} \equiv F = 0$

s-th polar curve of O w.r. to $F = 0$ is:

$\frac{\partial^s F}{\partial z^s} \equiv \frac{\partial^s}{\partial z^s}\left(\frac{\partial^{r-s} F}{\partial z^{r-s}}\right) = 0$; i.e. $\frac{\partial^r f}{\partial z^r} = 0 \ldots\ldots\ldots \ldots (ii)$

Clearly , (i) and (ii) are identical.

Ex.27. The k-th polar curve of P with respect to an n-ic having an (n-1)-ple point at O is an (n-k)-ic having an (n-k-1)-ple point at O/ $f \equiv z u_{n-1} + u_n = 0$.

k-th polar of P(X,Y,Z) is: $\left(X\frac{\partial}{\partial x} + Y\frac{\partial}{\partial y} + Z\frac{\partial}{\partial z}\right)^k f = 0$

i.e. $z\left(X\frac{\partial}{\partial x} + Y\frac{\partial}{\partial y}\right)^k u_{n-1} + kZ\left(X\frac{\partial}{\partial x} + Y\frac{\partial}{\partial y}\right)^{k-1} u_{n-1} + \cdots\cdots = 0$*

In this (n-k)-ic, highest power of z is unity, hence O(o,o,1) is an (n-k-1)-ple point at O.

Ex.28. On an n-ic there are 2n(n-2) points whose polar conics are parabolas.

Polar conic of P(X,Y) w.r. to f(x ,y)=o is:

$$\left(x^2\frac{\partial^2}{\partial X^2} + 2xy\frac{\partial^2}{\partial X\partial Y} + y^2\frac{\partial^2}{\partial Y^2}\right) f + u_1 + u_0 = 0.$$

For a parabola, u_2 must be a perfect square, i.e.

$\left(\frac{\partial^2 f}{\partial X\partial Y}\right)^2 = \frac{\partial^2 f}{\partial X^2}\frac{\partial^2 f}{\partial Y^2}$; ∴ locus of P is:

$\left(\frac{\partial^2 f}{\partial x\partial y}\right)^2 = \frac{\partial^2 f}{\partial x^2}\cdot\frac{\partial^2 f}{\partial y^2}$ ....a 2(n-2)-ic.

This intersects the n-ic in 2.n(n-2) points.

Let us now suppose that point O(X,Y.Z) is an r-ple point on the curve f = o, and, for simplicity, take (X,Y.Z) ≡ (o,o,1)

$$f \equiv u_r z^{n-r} + u_{r+1} z^{n-r-1} + \cdots\cdots + u_n = 0.$$

k-th polar of O is:

$$\frac{\partial^k f}{\partial z^k} \equiv \frac{(n-r)!}{(n-r-k)!} u_r z^{n-r-k} + \frac{(n-r-1)!}{(n-r-k-1)!} u_{r+1} z^{n-r-k-1} + \cdots\cdots = 0;\ n-r > k.$$

---

* i.e. $\left\{\left(X\frac{\partial}{\partial x} + Y\frac{\partial}{\partial y}\right)^k + k\left(X\frac{\partial}{\partial x} + Y\frac{\partial}{\partial y}\right)^{k-1} Z\frac{\partial}{\partial z} + \cdots\cdots Z\left(\frac{\partial}{\partial z}\right)^k\right\}(z u_{n-1} + u_n) = 0.$

Form of this last equation shows that: ANY POLAR CURVE OF AN r-PLE POINT O ON A GIVEN CURVE HAS AN r-PLE POINT AT O WITH SAME TANGENTS, $u_r = o$

Suppose that the tangent $u_1 = o$ at O has r-point contact with curve: $f \equiv u_1 z^{n-1} + u_1 v_1 z^{n-2} + \dots\dots + u_1 v_{r-2} z^{n-r+1} + u_r z^{n-r} + \dots + u_n = o.$

k-th polar of O is ; $\frac{\partial^k f}{\partial z^k} \equiv \frac{u_1 (n-1)!}{(n-k-1)!} z^{n-k-1} + u_1 v_1 \frac{(n-2)!}{(n-k-2)!} z^{n-k-2} + \dots = 0; n-r > k$

FOR ANY POLAR CURVE OF O, SAME TANGENT $u_1 = o$ HAS r-POINT CONTACT AT O.

INTERSECTION OF AN n-IC AND FIRST POLAR

Let us select C(o,o,1) as an intersection, O as (1,o,o) and any point B (o,1,o).

$f \equiv az^n + (b_o x + b_1 y) z^{n-1} + (c_o x^2 + 2c_1 xy + c_2 y^2) z^{n-2} + \dots\dots + u_n = o$

First polar of O ($\equiv$ A) is: $\frac{\partial f}{\partial x} \equiv b_o z^{n-1} + 2(c_o x + c_1 y) z^{n-2} + \dots\dots + v_{n-1} = o$

(1) If C on $f = o$ and $\frac{\partial f}{\partial x} = o$, then $a = b_o = o$

Tangent to $f = o$ at C is $y = o$, which passes through O($\equiv$A). Hence, in general, intersections of a curve and first polar of a point are the points of contact of tangents drawn from a point to the curve.

$\therefore m = n(n-1)$

(ii) If C is a node on $f = o$, then $a = b_o = b_1 = o$

C is an ordinary point on the first polar, with the tangent at C to the polar not coincident with either nodal tangent. In this instance, from p.37 case (i), the number of intersections of $f = o$ and $\frac{\partial f}{\partial x} = o$ is two.

$\therefore m = m(n-1) - 2$

(iii) If C is a cusp on $f = o$, with $x = o$ the cuspidal tangent, $a = b_o = b_1 = c_1 = c_2 = o$

$\frac{\partial f}{\partial x} = 2c_o x z^{n-2} + \ldots\ldots\ldots = 0$. On the first polar, C is an ordinary point with the tangent coinciding with cuspidal tangent. Consequently, from p. 37 case (ii), the number of intersections of $f = o$ and $\frac{\partial f}{\partial x} = 0$ at C is three.

$\therefore m = n(n-1) - 3$

Since, then, we have a diminution of two for a node on the curve and a diminution of three for a cusp, in the number of intersections of the curve and first polar, for δ nodes and κ cusps we have a reduction of $2\delta + 3\kappa$ in the class of the curve. Hence, for an n-ic possessing δ nodes, and κ cusps

$m = n(n-1) - 2\delta - 3\kappa \}$ PLÜCKER'S EQUATION.

We can obtain the equation uniting n, m, τ, i, in a precisely analagous manner by means of line co-ordinates, but we employ the principles of polar reciprocation.

If the n-ic has τ bitangents, i inflexions, the polar reciprocal curve of degree m and class n, has τ nodes and i cusps.

$\therefore n = m(m-1) - 2\tau - 3i \}$ PLÜCKER'S EQUATION.

An n-ic has not, in general, any double points, for if it had, the point equation would be specialized, since certain functions of the coefficients would vanish (see page 15). It is for this reason that $m = n(n-1)$ IN GENERAL.

Reciprocally, a curve of assigned class, has not in general any double lines (bitangents or inflexional tangents), for the existence of such would cause certain functions of the coefficients in the line equation to vanish.

Ex. 29. Show that tangents to: $y^p z^q = x^{p+q}$, drawn from $(1,\alpha,\beta)$ touch the same at points lying on the hyperbola:

$(p+q)\ yz = (\alpha pz + \beta qy)x$

First polar of $(1,\alpha,\beta)$ is: $(p+q)x^{p+q-1} = y^{p-1} z^{q-1} (\alpha pz + \beta qy)$.

This meets curve where: $(p+q)yz = (\alpha pz + \beta qy)x$. $\left[y^{p-1} z^{q-1} = \frac{x^{p+q}}{yz}\right]$

Ex. 30. A k-ple point with distinct tangents is equivalent to $\frac{1}{2}k(k-1)$ nodes.

Take $(0,0,1)$ as k-ple point.

$f \equiv u_k z^{n-k} + u_{k+1} z^{n-k-1} + \cdots + u_n = 0.$

First polar curve of $(0,1,0)$ is $\frac{\partial f}{\partial y} = 0$

$\frac{\partial f}{\partial y} \equiv v_{k-1} z^{n-k} + v_k z^{n-k-1} + \cdots\cdots + v_{n-1} = 0$. This $(n-1)$-ic has a $(k-1)$-ple point at O $(0,0,1)$ with tangents $v_{k-1} = 0$.

These tangents are distinct from those given by $u_k = 0$, since, by assumption, u_k contains no repeated factor. Hence, from p.39 case (v), there are $k(k-1)$ intersections at O. Since at a node the number of intersections is 2, we may regard a k-ple point $\sim$ $\frac{1}{2}k(k-1)$ nodes.

Ex.31 A sextic cannot have a 3-ple point, 1 node, and six cusps. 3-ple point $\sim \frac{1}{2}3(3-1) = 3$ nodes. $\delta = 3 + 1 = 4$

By Plücker, $m = n(n-1) - 2\delta - 3\kappa$.

i.e. $m = 6(6-1) - 2.4 - 3.6 = 4$

The reciprocal of a 3-ple point is a tritangent, which intersects curve in 6 points; this is impossible since the reciprocal curve is a quartic, for a line meets the quartic in but 4 points. (class of curve = m = degree of R(C).)

HESSIAN.

We shall now discuss a curve which is covariantly* related to the original n-ic. This derived curve, the Hessian, is defined as THE LOCUS OF POINTS WHOSE POLAR CONICS DEGENERATE INTO TWO STRAIGHT LINES, and, it is so-called, because it was first studied by the German mathematician, Hesse.

---

* C.A. Scott p.269. The general algebraic idea of a covariant is formulated in the definition:

Any function of the coefficients and variables that is unchanged by linear transformations, save as to a power of the modulus of transformation is called a Covariant.

The polar conic of P (X,Y,Z) w. r. to $f(x,y,z)=0$ is:

$\left(x\frac{\partial}{\partial X}+y\frac{\partial}{\partial Y}+z\frac{\partial}{\partial Z}\right)^2 f=0$. If this degenerates into two straight lines its discriminant must vanish, i.e.

$$\begin{vmatrix} \frac{\partial^2 f}{\partial X^2} & \frac{\partial^2 f}{\partial X\partial Y} & \frac{\partial^2 f}{\partial X\partial Z} \\ \frac{\partial^2 f}{\partial X\partial Y} & \frac{\partial^2 f}{\partial Y^2} & \frac{\partial^2 f}{\partial Y\partial Z} \\ \frac{\partial^2 f}{\partial X\partial Z} & \frac{\partial^2 f}{\partial Y\partial Z} & \frac{\partial^2 f}{\partial Z^2} \end{vmatrix} = 0$$

Therefore, locus of P is:

$$H \equiv \begin{vmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x\partial y} & \frac{\partial^2 f}{\partial x\partial z} \\ \frac{\partial^2 f}{\partial x\partial y} & \frac{\partial^2 f}{\partial y^2} & \frac{\partial^2 f}{\partial y\partial z} \\ \frac{\partial^2 f}{\partial x\partial z} & \frac{\partial^2 f}{\partial y\partial z} & \frac{\partial^2 f}{\partial z^2} \end{vmatrix} = 0$$

Since each of the elements in the above determinant is of degree $(n-2)$ in x,y,z, the equation is of degree $3(n-2)$, i.e.

HESSIAN OF AN n-IC IS A $3(n-2)$- IC

We may express the above equation in form:

$$z^2 H \equiv (n-1)^2\left\{2\frac{\partial f}{\partial x}\cdot\frac{\partial f}{\partial y}\cdot\frac{\partial^2 f}{\partial x\partial y}-\left(\frac{\partial f}{\partial y}\right)^2\frac{\partial^2 f}{\partial x^2}-\left(\frac{\partial f}{\partial x}\right)^2\frac{\partial^2 f}{\partial y^2}\right\}$$
$$+n(n-1)\left\{\frac{\partial^2 f}{\partial x^2}\frac{\partial^2 f}{\partial y^2}-\left(\frac{\partial^2 f}{\partial x\partial y}\right)^2\right\}f \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (i)$$

To effect this result we multiply the columns by x,y,z in turn, and add the first two columns to the third, and then perform a similar operation upon the rows. The use of Euler's relations then enables us to obtain (i), which is frequently abbreviated in the form:

$$z^2 H \equiv (n-1)^2 K+n(n-1)\left\{\frac{\partial^2 f}{\partial x^2}\cdot\frac{\partial^2 f}{\partial y^2}-\left(\frac{\partial^2 f}{\partial x\partial y}\right)^2\right\}f .$$

Ex. 32. Find the Hessian of: $y^p z^q = x^{p+q}$

$$\frac{\partial f}{\partial x} = (p+q)\, x^{p+q-1}\ ; \qquad \frac{\partial^2 f}{\partial x^2} = (p+q)(p+q-1)\, x^{p+q-2}$$

$$\frac{\partial f}{\partial y} = -p\, z^q y^{p-1}\ ; \qquad \frac{\partial^2 f}{\partial y^2} = -p(p-1)\, z^q y^{p-2}\ ; \qquad \frac{\partial^2 f}{\partial x \partial y} = 0.$$

Substituting these values in (i):

$$z^2 H \equiv (p+q-1)^2 \left\{ -p^2(p+q)(p+q-1)\, y^{2p-2} z^{2q} x^{p+q-2} + (p+q)^2 p(p-1)\, y^{p-2} z^q x^{2p+2q-2} \right\}$$

$$+(p+q)(p+q-1) \left\{ -p(p-1)(p+q)(p+q-1)\, x^{p+q-2} y^{p-2} z^q \right\} \left(x^{p+q} - y^p z^q\right)$$

i.e. $z^2 H \equiv -pq\,(p+q)\,(p+q-1)^2\, x^{p+q-2}\, y^{2p-2}\, z^{2q} = 0$

i.e. $H = x^{p+q-2}\, y^{2(p-1)}\, z^{2(q-1)}$

INTERSECTIONS OF CURVE AND HESSIAN

The intersections of $f=0$ and $H=0$ are the same as those of $f=0$ and $K=0$, excepting that $f=0$ and $K=0$ also meet twice at each intersection of $f=0$ and $z=0$. We select the point $O(0,0,1)$ on $f=0$ and the tangent there $y=0$ (We notice that O is not on $z=0$).

$$f \equiv b_1 y\, z^{n-1} + (c_0 x^2 + 2c_1 xy + c_2 y^2) z^{n-2} + \ldots\ldots + u_n = 0$$

Polar conic of O is: $\frac{\partial^{n-2} f}{\partial z^{n-2}} \equiv c_0 x^2 + 2c_1 xy + c_2 y^2 + (n-1) b_1 yz = 0$

If this conic degenerates into two straight lines, then either $c_0=0$ and the coefficient of z^{n-1} is a factor of the coefficient of z^{n-2}, i.e. O is an inflexion, or $b_1=0$, and O is a multiple point. This proves that: THE HESSIAN MEETS CURVE ONLY AT THE INFLEXIONS AND MULTIPLE POINTS.

(i) If $b_1 \neq 0$, $c_0 = 0$,

$$K_{(3k-4)} \equiv \left\{(8b_1 c_1^2 - 6b_1^2 d_1)y - 6b_1^2 d_0 x\right\} z^{3n-5} + v_2 z^{3n-6} + \dots\dots\dots$$

Hence f = o and K = o (and consequently curve & H=o) meet only once at O.

i.e. CURVE AND HESSIAN MEET ONCE AT EACH INFLEXION.

(ii) If $b_1 = 0$, $c_0 \neq 0$, f = o has a double point at O.

Suppose O is a node.

$$K \equiv 8(c_1^2 - c_0 c_2)(c_0 x^2 + 2c_1 xy + c_2 y^2) z^{3n-6} + v_3 z^{3n-7} + \dots\dots$$

Hence K has a node at O with same nodal tangents as f = o . Hence, from p. 38 case (iii), K = o (and H=o) and curve have six intersections at O. i.e.

CURVE AND HESSIAN HAVE SIX INTERSECTIONS AT A NODE.

(iii) Suppose O is a cusp with y=o the cuspidal tangent.

$$f \equiv y^2 z^{n-2} + (d_0 x^3 + 3d_1 x^2 y + 3d_2 xy^2 + d_3 y^3) z^{n-3} + \dots\dots\dots + u_n = 0$$

$$K \equiv -24(d_0 x + d_1 y) y^2 z^{3n-7} + v_4 z^{3n-8} + \dots\dots\dots$$

Hence O is a triple point on K = o (and on H = o), with two tangents coincident with the cuspidal tangent. From p. 38 case (iV), H = o and f=o meet eight times at O.

i.e. CURVE AND HESSIAN HAVE EIGHT INTERSECTIONS AT A CUSP.

The curve (an n-ic) and its Hessian (a 3(n-2)-ic) intersect in 3n (n-2) points. If, then, an n-ic possesses δ nodes, κ cusps, and i inflexions, from (i), (ii), (iii), we have:

$$3n\ (n-2) = i + 6\delta + 8\kappa, \text{ or :}$$

$\iota = 3n(n-2) - 6\delta - 8\kappa$ } PLÜCKER'S EQUATION.

Since R(C), of degree m, has τ nodes, ι cusps, and κ inflexions:

$\kappa = 3m(m-2) - 6\tau - 8\iota$ } PLÜCKER'S EQUATION.

Ex.33 A k-ple point O of a curve is in general a (3k-4)-ple point of the Hessian, and the tangents to the curve at O, are tangents to the Hessian at O.

Let $y = o$ be one of the tangents to curve at O(o,o,1),

then $f \equiv yu_{k-1}z^{n-k} + u_{k+1}z^{n-k-1} + \dots\dots\dots\dots + u_n = 0$

$K \equiv \left\{ 2\frac{\partial f}{\partial x}\cdot\frac{\partial f}{\partial y}\frac{\partial^2 f}{\partial x\partial y} - \left(\frac{\partial f}{\partial x}\right)^2\frac{\partial^2 f}{\partial y^2} - \left(\frac{\partial f}{\partial y}\right)^2\frac{\partial^2 f}{\partial x^2}\right\}$

$\frac{\partial f}{\partial x} = yv_{k-2}z^{n-k} + \dots\dots$; $\frac{\partial^2 f}{\partial x^2} = yv'_{k-3}z^{n-k} + \dots\dots\dots$;

Retaining only highest power of z in K, since each term contains either $\frac{\partial f}{\partial x}$ or $\frac{\partial^2 f}{\partial x^2}$, y is a factor of the coefficient of this power of z. i.e. $K \equiv yU_{3k-5}z^{3n-3k} + U_{3k-3}z^{3n-3k-1} + \dots\dots$ (i)

i.e. $K \equiv V_{3k-4}z^{3n-4-(3k-4)} + \dots\dots\dots$ (ii)

From (ii) we see that O (o,o,1) is a (3k-4)-ple point on $K = o$, and consequently on $H = o$

From (i) we see that $y=o$ is a tangent to the Hessian, and we can show in a similar way that the remaining (k-1) tangents to $f=o$ at O are also tangents to $H = o$ at O.

Ex. 34 (i) The locus of the points whose polar conic with respect to a given n-ic touches a given line is a 2(n-2)-ic.

(ii) This 2(n-2)-ic separates the plane into two portions in one of which the n-ic has no real inflexion or crunode, while in the other there is no acnode.

(iii) If the given line is tangent at an inflexion O, the 2(n-2)-ic has a node at O with the given line as one tangent there.

(i) Polar conic of P (X,Y,Z) is:

$(x\frac{\partial}{\partial X}+y\frac{\partial}{\partial Y}+z\frac{\partial}{\partial Z})^2 f=0$; if this touches $z=0$:

$x^2\frac{\partial^2 f}{\partial X^2}+2xy\frac{\partial^2 f}{\partial X\partial Y}+y^2\frac{\partial^2 f}{\partial Y^2}$ is a perfect square;

$\therefore$ Locus of P is : $\left(\frac{\partial^2 f}{\partial x^2}\right)\left(\frac{\partial^2 f}{\partial y^2}\right)=\left(\frac{\partial^2 f}{\partial x\,\partial y}\right)^2$ .... a 2(n-2)-ic

(ii) $F\equiv\frac{\partial^2 f}{\partial x^2}\cdot\frac{\partial^2 f}{\partial y^2}-\left(\frac{\partial^2 f}{\partial x\,\partial y}\right)^2=0$

If $F>0$, the degenerate conic of a point on H is imaginary, hence the point is not a real inflexion or crunode.

If $F<0$, the degenerate conic is real, hence the point is not an acnode.

(iii) If z=o is tangent at O (1,o,o), an inflexion, n-ic is:

$z(x^{n-1}+u_1x^{n-2})+u_3x^{n-3}+\dots\dots=0$

$F\equiv zv_1x^{2n-6}+v_3x^{2n-7}+\dots\dots=0$

It follows that (1,o,o) is a double point on F=o with the inflexional tangent to n-ic (z=o) tangent to one branch.

Ex.35. If a curve has r-point contact with its tangent at O, the Hessian has (r-2)- point contact with the same tangent at O; and the curve and Hessian meet (r-2) times at O.

Take $y=0$ as tangent at O.

$f \equiv y\,(a+u_1+u_2+\cdots\cdots+u_{r-2}) + \phi = 0$; degree of lowest term in ϕ is r.

Forming Hessian: ($z=1$, in (i) p.47)

$y\,F(x,y) + \psi = 0$; degree of lowest term in ψ arising from

$\left(\frac{\partial f}{\partial y}\right)^2 \frac{\partial^2 f}{\partial x^2}$, is $(r-2)$. i.e. $y=0$ has $(r-2)$-point contact with

Hessian. By ex.25,p.39 the curve and the Hessian meet $(r-2)$

times at O $(r-2=q<p=r)$.

SECTION IV

PLÜCKER'S NUMBERS

(↑)

As we have previously intimated, Plücker's Numbers (*) are the six quantities $n, m, \delta, \kappa, \tau, \iota$ by which we denote the degree (order), class, nodes, cusps, bitangents, and inflexions of an algebraic plane curve. Replacing these by $m, n, \tau, \iota, \delta, \kappa$, in order, we have the Plucker's numbers of the reciprocal curve. We have also independently established three equations expressing relations between these numbers, viz:

$$\tfrac{1}{2}n\ (n+3) - \delta - 2\kappa = \tfrac{1}{2}m(m+3) - \tau - 2\iota \qquad \text{(p 35)}$$

$$m = n(n-1) - 2\delta - 3\kappa \qquad \text{(p 44)}$$

$$\iota = 3n(n-2) - 6\delta - 8\kappa \qquad \text{(p 50)}$$

By polar reciprocation, we also deduced:

$$n = m(m-1) - 2\tau - 3\iota \qquad \text{(p 44)}$$

$$\kappa = 3m(m-2) - 6\tau - 8\iota \qquad \text{(p 50)}$$

Various other relations between the six numbers may be found, and we include the most useful of these in the list given below. Of these nine equations only three are independent, since given any three, we may deduce the remaining six. Equations (5) - (9) were not included in Plücker's original formulae. These latter consisted of the

---

(*). Some writers call these the "characteristics" of the curve, but this term is frequently reserved for the two quantities (p,l), where, for a system of curves:
p = number of curves which pass through a given point.
l = " " " " touch a given line.

first four equations, together with two rather cumbersome relations expressing 2δ and 2τ in terms of m, τ, i and n, δ, κ, respectively (See Ex.36).

PLÜCKER'S EQUATIONS.

(1) . . . . . $m = n(n-1) - 2\delta - 3\kappa$

(2) . . . . . $n = m(m-1) - 2\tau - 3i$

(3) . . . . . $i = 3n(n-2) - 6\delta - 8\kappa$

(4) . . . . . $\kappa = 3m(m-2) - 6\tau - 8i$

(5) . . . . . $\frac{1}{2}n(n+3) - \delta - 2\kappa = \frac{1}{2}m(m+3) - \tau - 2i$

(6) . . . . . $\frac{1}{2}(n-1)(n-2) - \delta - \kappa = \frac{1}{2}(m-1)(m-2) - \tau - i \quad (= D)$.

(7) . . . . . $i - \kappa = 3(m-n)$

(8) . . . . . $2(\tau - \delta) = (m+n-9)(m-n)$

(9) . . . . . $n^2 - 2\delta - 3\kappa = m^2 - 2\tau - 3i$

Assuming (1), (3), (5), [or any three of the above equations] we may deduce the others by simple Algebra.e.g.:

TO OBTAIN (7): Eliminate δ between (1) and (3)

TO OBTAIN (8): Multiply (5) by 2, and arrange in form: $2(\tau - \delta) = m(m+3) - n(n+3) - 4(i - \kappa)$; in this substitute value for $(i - \kappa)$ from (7)

TO OBTAIN(2): Add (1) and (8) and to the result add $3(i - \kappa) = 9m - 9n$, from (7).

TO OBTAIN (9): Substract (2) from (1).

TO OBTAIN (6): Substract (5) from (9).

TO OBTAIN (4): In (7) substitute for n from (2)

Clearly Plücker's Numbers are unaltered by projection. Curves having the same Plücker's Numbers belong to the same TYPE or class, and as a consequence, these numbers assume a very important rôle in the classification of plane curves — as a subsequent paragraph will show.

DEFICIENCY.

It is customary to denote equation (6) by the symbol D, which is called the deficiency of the curve. Since $\frac{1}{2}(n-1)(n-2)$ is the maximum number of double points an n-ic can possess (p.24), D is the number by which the actual number of double points falls short of this upper limit. From the form of (6) we easily infer that a curve and its polar reciprocal have the same deficiency.

If the co-ordinates of any point on an n-ic can be expressed rationally in terms of a single parameter, the n-ic is UNICURSAL(†). It can be shown that THE DEFICIENCY OF A UNICURSAL CURVE IS ZERO, and conversely, IF D=o, THE n-ic IS UNICURSAL.

For a conic: $n = m = 2$; $\delta = \kappa = \tau = \iota = 0$. From elementary analytic geometry we know that the co-ordinates of any point (x,y) on the conic are expressible in terms of a parameter μ. (Scott p.134)

If a cubic has zero deficiency, it has one double point (a,b). The line $y-b = t(x-a)$, for any t, intersects the cubic in two points at (a,b), and hence the third intersection will be a rational function of t.

(†). so-called because such a curve consists of a single circuit; it is unipartite.

The general theorem, stated above, was proved as early as 1865 by Clebsch, and since that date, numerous other proofs have appeared. We assume this theorem without proof in Ex.47.

CLASSIFICATION OF CUBICS AND QUARTICS.

The classification of cubic and quartic curves has been attempted, with considerable difficulty, by many eminent mathematicians, notably amongst them, Newton, Mobius, Wiener, Zeuthen, and Plücker. Although we are particularly interested in that classification which utilizes Plücker's formulae, we comment upon a second one, for purposes of comparison.

A classification of cubics in relation to the line at infinity ($\mathcal{L}_\infty$) gives rise to 14 genera, containing in all 78 species of cubics.

Basing a classification upon deficiency, however, cubics may be divided into 2 main groups, according as D is 1 or 0. These two groups include 3 types of cubics. All cubics which do not possess a double point belong to the first group (D=1), while the second group (D=0) includes nodal and cuspidal cubics. From Plücker's Equations we readily deduce the following table:

TYPE	n	m	δ	κ	τ	ι	D
I	3	6	0	0	0	9	1
II	3	4	1	0	0	3	0
III	3	3	0	1	0	1	0

Illustrations of types II and III are the curves discussed in Ex. 16, p. 23, and Ex, 19, p.28.

Quartics classified in relation to the [illegible] belong to 9 genera, which Plücker subdivided into 152 species (or more).

Where deficinecy is the basic criterion, however, quartics may be confined to 4 main groups, according as D is 3, 2, 1, or 0. These 4 groups include the 10 types listed in the chart below. (As for n=3, we deduce this scheme in virtue of Plücker's relations).

Type	n	m	δ	κ	τ	i	D
I	4	12	0	0	28	24	3
II	4	10	1	0	16	18	2
III	4	9	0	1	10	16	2
IV	4	8	2	0	8	12	1
V	4	7	1	1	4	10	1
VI	4	6	0	2	1	8	1
VII	4	6	3	0	4	6	0
VIII	4	5	2	1	2	4	0
IX	4	4	1	2	1	2	0
X	4	3	0	3	1	0	0

Ex.36. Prove that: $\tau=\delta+\frac{1}{2}(n^2-2n-2\delta-3\kappa)(n^2-9-2\delta-3\kappa)$

From (8): $\tau=\delta+\frac{1}{2}(m^2-n^2-9m+9n)$

In this substitute from (1) for m:

$$\tau=\delta+\tfrac{1}{2}\{n^4-2n^3-n^2(4\delta+6\kappa-9)+2n(2\delta+3\kappa-9)+12\delta\kappa+4\delta^2+9\kappa^2+18\delta+27\kappa\}$$

$$\tau=\delta+\tfrac{1}{2}\{n^2-2n-2\delta-3\kappa\}\{n^2-9-2\delta-3\kappa\}$$

N.B. This relation is a simplified expression for Plücker's original equation.

Ex.37 Find curves for which $\delta=\tau$

If $\delta=\tau$, from (8) we have: (i) n=m, or (ii) n+m=9

(i) n=m, then $i=k$ from (7). The curve and its reciprocal are of same class.

(ii) n + m = 9.

(a) $n \neq 2$, for, if it were, m = 2 (Ex.18 p.27), and $n + m = 4 \neq 9$.

(b) n = 3, m = 6. From previous chart for cubics, the cubic is of type I.

$\delta = k = \tau = 0$; $i = 9$; $D = 1$

(c) n=4, m=5. From chart for quartics, curve belongs to type VIII.

$\delta = 2, k = 1, \tau = 2, i = 4, D = 0.$

(d) n=5, m=4. Reciprocal curve where $n' = 4$, $m' = 5$ has numbers $\delta' = 2; k' = 1, \tau' = 2, i' = 4$, $D = 0$, and ∴ quintic has:

$\delta = 2, k = 4, \tau = 2, i = 1, D = 0.$

(e) n=6, m=3. Reciprocal curve is a cubic of type I, and ∴ the sextic has Plücker's numbers: $\delta = \tau = 0$, $k = 9$, $i = 0$, $D = 1$

Ex.38. If $n > m$, then $k > i$ and $\delta >, =, < \tau$ as $n + m >, =, < 9$

From (7). $i - k = 3(m-n) = -^{ve}$, since $n > m$

$\therefore \ i < k$

From (8) $2(\tau - \delta) = (m-n)(m+n-9)$

$= (-^{ve} \text{ quantity })(m+n-9)$

If $m+n \leq 9$, $2(\tau - \delta)$ is $+^{ve}$, i.e. $\delta \leq \tau$.

" " $= 9$, $2(\tau - \delta) = 0$, i.e. $\delta = \tau$

" " $\geqslant 9$, $2(\tau - \delta)$ is $-^{ve}$, i.e. $\delta > \tau$

Ex. 39 Show that: (i) $m = 2(n-1) + 2D - \kappa$

(ii) $i = 3(n-2) + 6D - 2\kappa$

(iii) $\delta = \frac{1}{2}(n-1)(n-2) - D - \kappa$

(i) From (1): $m = n(n-1) - (2\delta + 2\kappa) - \kappa$

From (6): $2\delta + 2\kappa = (n-1)(n-2) - 2D$

i.e. $m = n(n-1) - (n-1)(n-2) + 2D - \kappa$

$m = 2(n-1) + 2D - \kappa$

(ii) From (7): $i = 3(m-n) + \kappa$

Substitute for m from (i):

$i = 3\{2n-2 + 2D - \kappa - n\} + \kappa$

$\therefore\ i = 3(n-2) + 6D - 2\kappa$

(iii) Follows at once from (6). $\left[\tau = \frac{1}{2}(m-1)(m-2) - D - i \text{ from (6) also}\right]$

N.B. From the above expressions we easily express Plücker's numbers in terms of 2(D-1), viz:

$$\left.\begin{array}{l} m + \kappa - 2n \\ n + i - 2m \\ n(n-3) - 2(\delta + \kappa) \\ m(m-3) - 2(\tau + i) \end{array}\right\} = 2(D-1) \ldots \text{(IV)}$$

Ex. 40. If D=o, and $n > 4$, not all the double points are cusps.

When $D = 0, \delta = 0$, from (6): $\kappa = \frac{1}{2}(n-1)(n-2)$

from (3): $i = 3n(n-2) - 4(n-1)(n-2)$

i.e. $i = -n^2 + 6n - 8$; for $n > 4$ the R.H.S. is negative and obviously $i \not< 0$.

Ex.41 If $D=0$, $m \leqslant 2(n-1)$; while $m \geqslant \frac{1}{2}(n+2)$ if n is even, and $m \geqslant \frac{1}{2}(n+3)$ if n is odd.

From first two equations in Ex.39(IV), if D o;

$m = 2(n-1) - \kappa$, i.e., $m \leqslant 2(n-1)$

$m = \frac{1}{2}(n+2+i)$. [OMIT From equation (3) we see that if n is even the curve has an even number of inflexions, while if n is odd the number of inflexions is odd. For n even then, $m \geqslant \frac{1}{2}(n+2)$;] From equation (3) we see that if n is even the curve has an even number of inflexions, while if n is odd the number of inflexions is odd.

For n even then, $m \geqslant \frac{1}{2}(n+2)$. For n odd, since i at least one, $m \geqslant \frac{1}{2}(n+3)$.

Ex.42 If $D=1$, then $i+\kappa = n+m$.

Equ.(6) for $D=1$, gives:

$$2\delta + 2\kappa = n^2 - 3n$$

By (1) $\underline{2\delta + 3\kappa} = \underline{n^2 - n - m}$

$$\therefore \kappa = 2n - m \ldots (a)$$

From (7) $\underline{i - \kappa} = \underline{3m - 3n}$

$$\therefore i = 2m - n \ldots . (b)$$

Adding (a) and (b): $i + \kappa = m + n$.

Ex. 43. If the deficiency of a curve is even, the number of double points is even if degree of curve is $4p+2$ or $4p+1$, and is odd if degree is $4p$ or $4p-1$. ($p = +^{ve}$integer)

$S \equiv \delta + \kappa = \frac{1}{2}(n-1)(n-2) - D_{EVEN}$

S is ∴ even if $\frac{1}{2}(n-1)(n-2)$ is even, and odd if $\frac{1}{2}(n-1)(n-2)$ is odd.

For $n = 4p+2$

$\frac{1}{2}(n-1)(n-2) = 2p(4p+1) = \text{even}.$

For $n = 4p+1$

$\frac{1}{2}(n-1)(n-2) = 2p(4p-1) = \text{even}.$

For $n = 4p$

$\frac{1}{2}(n-1)(n-2) = (4p-1)(2p-1) = \text{odd}.$

For $n = 4p-1$

$\frac{1}{2}(n-1)(n-2) = (2p-1)(4p-3) = \text{odd}$

Ex. 44 (i) If $D=o$, $\iota=n$; then $m=n+1$

(ii) If $D=o$, $\iota=m$; then $m=n+2$

(i) From (7): $\kappa = 4n - 3m$
From Ex. 39 (i) $\kappa = 2(n-1) - m$
$\therefore m = n+1$

(ii) $\kappa = 3n - 2m$
$\kappa = 2(n-1) - m$
$\therefore m = n+2$

Ex.45. Find Plücker's Numbers for the Hessian of the general n-ic. The n-ic has in general, no double points, and the Hessian then has no double points (we assume this almost self-evident result). Let the numbers be $n', m', \delta', \kappa', \tau', \iota'$

$n' = 3(n-2)$ .. .. (p.47)

$m' = 3(n-2)(3n-7)$ $\quad (m' = n'(n'-1)$ since $\delta' = \kappa' = 0)$

$\delta' = \kappa' = 0$

$\tau' = \frac{27}{2}(n-1)(n-2)(n-3)(3n-8)$ ...... from (8)

$\iota' = 9(n-2)(3n-8)$ ..... from (7)

MULTIPLE POINTS WITH DISTINCT TANGENTS

In the preceding discussion we have not considered singularities other than double points (and double lines).

We now show that: PLÜCKER'S EQUATIONS HOLD FORMALLY IF WE REGARD A k-PLE POINT WITH DISTINCT TANGENTS, EQUIVALENT TO $\frac{1}{2}k(k-1)$ NODES; (AND RECIPROCALLY, A k-PLE TANGENT WITH DISTINCT POINTS OF CONTACT EQUIVALENT TO $\frac{1}{2}k(k-1)$ BITANGENTS).

Since only three of Plücker's Equations are independent, it is only necessary to show that this equivalence satisfies equations (1) and (3) [(2) and (4) then follow by reciprocation]. We must therefore prove:

(i) that a curve and first polar meet $k(k-1)$ times at O (the k-ple point).

(ii) that a curve and Hessian meet $3k(k-1)$ times at O.

We proved (i) in Ex.30 p.45.

The intersections of curve and Hessian at O depend only on $(o, \epsilon > o)$

Consequently apart from k-ple point we may assume that the curve has δ_2 nodes, κ cusps, τ bitangents, ι inflexions.

From the reciprocal curve: $\kappa - \iota = 3(n-m)$ . . . (7)

From (i) and (2) : $m = n(n-1) - 2\delta_2 - 3\kappa - k(k-1)$

Eliminating m in these equations:

$\iota = 3n(n-2) - 6\delta_2 - 8\kappa - 3k(k-1)$. Hence, the Hessian intersects the curve in $3k(k-1)$ points at k-ple point O.

If in Ex.30 p.45 we take $P \equiv O(o,o,1)$, the first polar has a k-ple point at O with same tangents as the n-ic. The total number of intersections at O will be $k(k+1)$; so that 2k of the tangents from a k-ple point must be considered coincident with the tangents at that point.

Ex.46. If the only singularities of an n-ic are linear branch points with distinct tangents, $D = \frac{1}{2}(m-2n+2)$

By above theorem, $\kappa = o$, and result follows at once from Ex.39 (i) p.59

MULTIPLE POINTS WITH SUPERLINEAR BRANCHES.

At a k-ple point, however, two or more of the k tangents may be coincident, and superlinear branches arise. We now prove that: PLÜCKER'S EQUATIONS STILL HOLD IF A k-PLE POINT O WITH L ORDINARY SUPERLINEAR BRANCHES, IS CONSIDERED EQUIVALENT TO $\frac{1}{2}k(k-3)+L$ NODES AND (k-L) CUSPS.

We require to show that (i) the n-ic and first polar intersect in $2\left\{\frac{1}{2}k(k-3)+L\right\}+3(k-L)$ points at O and (ii) the n-ic and Hessian meet $6\left\{\frac{1}{2}k(k-3)+L\right\}-8(k-L)$ times at O.

(i) (a) A curve has a k-ple point O with L superlinear branches of orders $r_1, r_2, \ldots r_L$, having distinct tangents $(k=\sum r_j)$; and a second curve has a (k-1)-ple point at O such that the tangent to the branch of order r_t of the first curve is a tangent to a branch of order r_t-1 of the second. How many of their intersections coincide at O ?

$$\left.\begin{array}{l} \text{For first curve (U): } y = A\omega_r x^{\frac{r+1}{r}} + \ldots\ldots\ldots \\ \text{" second " (V): } y = a\omega_{r-1} x^{\frac{r}{r-1}} + \ldots\ldots\ldots \end{array}\right\} \text{common tangent } y=0$$

For a partial j-th superlinear branch of V, $\epsilon_j = r.\left(\frac{r+1}{r}\right)$,

hence for total intersections over total superlinear branch of V:

$\epsilon = \sum \epsilon_j = (r-1)(r+1) = r^2 - 1$

Order of O on V is k-1; order of O for superlinear branches other than the one just considered is (k-1)-(r-1) = k-r

Number of intersections of superlinear branches making up this order with the superlinear branches considered is: r(k-r).

Hence total $\epsilon = \sum_{i=1}^{L}\{(r_i^2 - 1) + r_i(k-r_i)\} = k\sum r_i - L = k - L$

(b) At an ordinary superlinear branch of order k:

$$y^k = ax^{k+1} + \dots\dots\dots + u_n = 0 \sim f(x,y)$$

$$y^{k-1} = Ax^k + \dots\dots\dots + u_{n-1} = 0 \sim \frac{\partial f}{\partial y} \text{ (first polar)}$$

i.e. A superlinear branch on first polar curve is of order one lower than on the n-ic , and the tangent is the same. Clearly, the curve and the first polar are U and V of (a), and intersect in k^2-L points. $k^2 - L = 2\left\{\frac{1}{2}k\ (k-3) + L\right\} + k - L$, which proves (i).

For (ii):

(c) If a curve has (k-L) cusps, (some may have coincident tangents), the reciprocal curve has (k-L) inflexions (some may coincide). Hence, by Ex.35 p.51 the Hessian intersects the reciprocal curve in (k-L) points corresponding to these cusps on the original curve.

Suppose that apart from O there are δ_2 nodes, κ_2 cusps, τ_2 bitangents, and ι_2 inflexions on the curve. For reciprocal curve, we have:

$n = m(m-1) - 2\tau_2 - 3\iota_2 \ldots\ldots\ldots (2)$

$m = n(n-1) - 2\delta_2 - 3\kappa_2 - (k^2 - L) \ldots\ldots\ldots (1)$ & (i)

$\kappa_2 = 3m(m-2) - 6\tau_2 - 8\iota_2 - (k-L) \ldots (3)$ & (c)

Eliminating m and τ_2:

$\iota_2 = 3n(n-2) - 6\delta_2 - 8\kappa_2 - 6\left\{\frac{1}{2}k(k-3)+L\right\} - 8(k-L)$; which proves (ii)

Hence, the point singularity of the superlinear branch points is equivalent to $\frac{1}{2}k(k-3)+L$ nodes and $(k-L)$ cusps.

By reciprocation, Plücker's Equations hold if a k-fold tangent having L points of contact, and consequently k+L intersections with the curve, is counted as equivalent to $\frac{1}{2}k\cdot(k-3)+L$ bitangents, and $(k-L)$ inflexions.

HIGHER SINGULARITIES.

Given any singularity at O, how many nodes, cusps, bitangents, and inflexions, must be considered as coinciding with O, in order that Plücker's Equations may hold?

Let the numbers be $\delta_1, \kappa_1, \tau_1, \iota_1$, and suppose that the first polar and Hessian respectively, intersect the given n-ic in α and β points coinciding with O. Then:

(i) ... $2\delta_1 + 3\kappa_1 = \alpha$; $6\delta_1 + 8\kappa_1 + \iota_1 = \beta$

The reciprocal of O is a line united with the reciprocal of bitangents and inflexional tangents at O. Let the first polar and Hessian, respectively, (of R(C)) intersect the reciprocal curve in ρ and σ points on this line which is the reciprocal of O. Then:

(ii) ... $2\tau_1 + 3i_1 = \rho$; $6\tau_1 + 8i_1 + \kappa_1 = \sigma$.

Adding (i) and (ii): $3(\alpha+\rho) = \beta + \sigma$, which the numbers must satisfy.

To prove this consistent, let $\delta_2, \kappa_2, \tau_2, i_2$ be the number of nodes, etc., not coincident with O. We have

$$\left.\begin{aligned} \alpha &= n(n-1) - m - 2\delta_2 - 3\kappa_2 \\ \beta &= 3n(n-2) - 6\delta_2 - 8\kappa_2 - i_2 \\ \rho &= m(m-1) - n - 2\tau_2 - 3i_2 \\ \sigma &= 3m(m-2) - 6\tau_2 - 8i_2 - \kappa_2 \end{aligned}\right\} \text{ i.e. } 3(\alpha+\rho) = \beta + \sigma.$$

Although (i) and (ii) are consistent, they are not independent, but admit of infinitely many solutions. To make the solution definite, we must define deficiency at higher singularities. Such a definition is beyond the scope of this thesis, but we consider a specific illustration in Ex. 47 below.

We notice that (a) k-ple points with distinct tangents and (b) ordinary superlinear branch points, are special cases of this generalized treatment.

e.g. (a) $\delta_1 = \frac{1}{2}k(k-1)$; $\kappa_1 = 0$; $\tau_1 = 0$; $i_1 = 0$.

(b) $\delta_1 = \frac{1}{2}k(k-3) + L$; $\kappa_1 = k - L$; $\tau_1 = 0$; $i_1 = 0$; $[\rho = 0$; $\sigma = k - L]$

$\alpha = k^2 - L$; $\beta = k(3k-1) - 2L$; $\rho = 0$; $\sigma = k - L$.

Ex.47. Discuss the singularity (o,o,1) on: $y^p z^q = x^{p+q}$.

From Ex.21,p.30; R(C) is: $y^p z^q = \alpha x^{p+q}$, i.e. curve and reciprocal have the same singularities. B and C are reciprocal singularities.*

*($\because$ polar at B is tangent at C)

First polar curve of B(o,1,o) is: $y^{p-1} z^q = 0$; hence at (o,o,1): $\alpha = (p-1)(p+q)$

From Ex.32 p.48, Hessian is: $x^{p+q-2} y^{2(p-1)} z^{2(q-1)} = 0$; hence at (o,o,1):

$\beta = 2(p-1)(p+q) + p(p+q-2) = 3p(p+q) - 4p - 2q$

Since singularities at B and C are reciprocal, interchanging p and q: $\rho = (q-1)(q+p)$; $\sigma = 3q(p+q) - 4q - 2p$.

$$3(\alpha + \rho) = \beta + \sigma$$

$$\left.\begin{array}{lll} \text{(i)} & 2\delta_1 + 3\kappa_1 = \alpha; & 6\delta_1 + 8\kappa_1 + i_1 = \beta \\ \text{(ii)} & 2\tau_1 + 3i_1 = \rho; & 6\tau_1 + 8i_1 + \kappa_1 = \sigma \end{array}\right\}$$

These are satisfied by:

$$\left.\begin{array}{l} \delta_1 = \frac{1}{2}p(p+q) - 2p - \frac{1}{2}q + \frac{3}{2}F \\ \kappa_1 = p - F \\ \tau_1 = \frac{1}{2}q(p+q) - 2q - \frac{1}{2}p + \frac{3}{2}F \\ i_1 = q - F \end{array}\right\}$$

F is any symmetric function of p and q and we find δ_1 etc. only by method of trial.

Although the singularity is analysed, it is not definite, for F may be ANY symmetric function of p and q.

Evidently the curve is unicursal, for $x:y:z = t^{p}: t^{p+q}: 1$

Assuming that $D = 0$ (p. 56), we can calculate F.

$$D = \frac{1}{2}(n-1)(n-2) - \delta - \kappa = \frac{1}{2}(p+q-1)(p+q-2) - \sum \text{double points} = 0$$

$\therefore$ Nodes and cusps at B and C $= \delta_1 + \kappa_1 + \tau_1 + \iota_1 = \frac{1}{2}(p+q-1)(p+q-2)$ ......(a)

Also from above, $\delta_1 + \kappa_1 + \tau_1 + \iota_1 = \frac{1}{2}(p+q-1)(p+q-2) + F - 1$ ....(b)

From (a) and (b): $F = 1$.

The singularity at the superlinear branch point (not an ordinary superlinear branch point since coefficients specialized), is now resolved in such a way that Plücker's Equations hold formally.

Theoretically, the process of analysing singularities is possible provided we know the deficiency. It can also be shown that if the co-ordinates of any point on an n-ic can be expressed rationally in terms of elliptic functions, the curve has unit deficiency. All such considerations, however, are without the scope of this thesis.

B I B L I O G R A P H Y

Ball, W.W.R.: Short History of Mathematics.

Basset, A.B.: Cubic and Quartic Curves.

Cayley A. : Collected Papers.

Fink : Short History of Mathematics (translation).

Forsyth, A.R.: Theory of Functions.

Frost, P. : Curve Tracing.

Ganguli, S : Theory of Plane Curves.

Hilton, H. : Plane Algebraic Curves.

Salmon, G. : Higher Plane Curves.

Scott, C.A. : Modern Analytical Geometry.

Smith, D. : History of Modern Mathematics.

Periodicals, etc. referred to in the footnotes.