BASIC PROPERTIES OF BANACH ALGEBRAS

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Philip Joseph Laufer

Department of Mathematics

McGill University

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PREFACE . . . .

A Banach algebra is at once a Banach space and an algebra with norm satisfying the multiplicative inequality $\|xy\| \leqq \|x\|\|y\|$. Many of the Banach spaces which occur in analysis are at the same time Banach algebras under a definition of multiplication which itself plays a role in analysis. A good example is the Banach space of absolutely integrable functions on the real line with multiplication defined as convolution. Since applications to analysis have always provided the main impetus to the study of Banach spaces, it is rather surprising that there was not a comparable interest shown in Banach algebras. Some of the reason lies, no doubt, in the general unfamiliarity of non-specialists with modern algebra plus the almost universal assumption of finiteness conditions by algebraists. Finiteness conditions rule out the most interesting cases from analysis and make it difficult to ferret out and develop algebraic tools applicable to the infinite case. There were, of course, a number of important attempts to exploit more fully the additional algebraic structure given in a Banach space by an operation of multiplication, some of the pioneers being Nagumo[17)] and Yosida[26)], Murray-von Neumann[14,15,16)] and Stone[24)]. Wiener[25)], although not explicitly drawing attention to the algebraic nature of the tools he was using, made systematic use of the algebraic properties of convolution for establishing certain deep theorems in analysis. However, it remained for Gelfand[4)] in 1941, with his now classical paper on normed rings, to lay the foundation for a theory of Banach algebras. Gelfand's innovation was a systematic use of elementary ideal theory coupled with the Gelfand-Mazur theorem that a normed division algebra (over the complex numbers) must be the complex numbers. His fundamental result was that a semi-simple commutative Banach algebra (with an identity) is isomorphic to an algebra of continuous functions on a compact Hausdorff space. At the same time, Gelfand[5) used his theory of normed rings to give an elegant proof of the well-known theorem of N. Wiener[25) that the reciprocal of a non-vanishing absolutely convergent Fourier series is also an absolutely convergent Fourier series. This proof attracted a great deal of attention to Banach algebras.

Since the appearence of Gelfand's 1941 papers, there has been a rapid growth of interest in Banach algebras. This interest has in turn played an important role in the recent trend among algebraists to dispense with finiteness conditions. The theory of Banach algebras has, generally speaking, developed in two main directions, representing respectively the algebraic and analytic tendencies. The algebraic emphasis has been on structure theory, while the analytic emphasis has been on extending properties of very special Banach algebras to more general cases and on extending analytic function theory to the more general situations provided by Banach algebras. In the following thesis, we shall attempt to bring out the main points in the algebraic line of development.

CHAPTER I

PRELIMINARY NOTIONS

For the sake of completeness we shall touch upon a few preliminary notions required for the exposition of the main theorems in this thesis. Some of these notions are classical and we present them briefly, leaving out some of the longer proofs which are to be found in the works quoted in the bibliography.

1. Topological Concepts

Partial Ordering: A binary relation "$<$" between elements of a class A is called a partial ordering of A (in the weak sense) if it is transitive ($a<b$ and $b<c \Longrightarrow a<c$), reflexive ($a<a$ for every $a \in A$) and if $a<b$ and $b<a \Longrightarrow a = b$. A is called a directed set under "$<$" if it is partially ordered by "$<$" and if for every a and $b \in A$ there exists $c \in A$ such that $c<a$ and $c<b$. A partially ordered set is linearly ordered if either $a<b$ or $b<a$ for every distinct pair a and $b \in A$. A is partially ordered in the strong sense by "$<$" if "$<$" is transitive and irreflexive ($a \nless a$).

Zorn's Lemma: Every partially ordered set A includes a maximal linearly ordered subset. If every linearly ordered subset of A has an upper bound in A, then A contains a maximum element[12)*].

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* The number in the bracket refers to the bibliography.

Theorem 1.1 (Axiom of Choice): If F is a function with domain D such that F(x) is a non-empty set for every $x \in D$, then there exists a function f with domain D such that $f(x) \in F(x)$ for every $x \in D$.

Topological Space: A family F of subsets of a space (set) S is called a topology for S if and only if:

(i) $\emptyset$ and S are in F; where $\emptyset$ denotes the null set;

(ii) If $F_1 \subset F_2$, then $\bigcup \{A : A \in F_1\} \in F$; that is, the union of the sets of any subfamily of F is a member of F;

(iii) The intersection of any finite number of sets of F is a set of F.

If F_1 and F_2 are two topologies for S, then F_1 is said to be weaker than F_2 if and only if $F_1 \subset F_2$.

If S_1 is any family of subsets of S, then the topology generated by S_1, $F(S_1)$, is the smallest topology for S which includes S_1; if $F = F(S_1)$, then S_1 is called a sub-basis for F. It follows readily that $A \in F(S_1)$, if and only if A is $\emptyset$ or S, or if A is a union (perhaps uncountable) of finite intersections of sets in S_1. If every set in $F = F(S_1)$ is a union of sets in S_1, then S_1 is called a basis for F. If a topology F is given for S, then S is called a topological space and the sets of F are the open subsets of S. If A is any subset of S, then the union of all the open subsets of A is called the interior of A and is denoted by int(A); evidently int(A) is the largest open subset of A, and A is open if and only if A = int(A). If $p \in \text{int}(A)$, then A is said to be a neighborhood of p. Neighborhoods are generally, but not always, taken to be open sets. A set of neighborhoods of p is called a neighborhood basis for p if every set which contains p includes a neighborhood of the set. A subset of S is closed (with respect to the topology F) if its complement is open. It follows that $\emptyset$ and S are closed, that that the intersection of any number of closed sets is closed, and that the union of any finite number of closed sets is closed. If A is any subset of S, the intersection of all the closed sets which include A is called the <u>closure</u> of A and is denoted by $\bar{A}$; evidently $\bar{A}$ is the smallest closed set including A, and A is closed if and only if $A = \bar{A}$.

A subset A of a topological space S has the <u>Heine-Borel property</u> if every family of open sets which covers A includes a finite subfamily which covers A. A subset A which has the above property is said to be <u>compact</u>. A set A is <u>dense</u> in S if $\bar{A} = S$, dense in Y if $Y \subset \overline{A \cap Y}$. A set A is <u>separable</u> if there is a countable set which is dense in A. In particular, S is a <u>separable space</u> if there is a countable set which is dense in S.

<u>Hausdorff Spaces</u>: Given a space S and a collection of subsets $\{U_a\}$, called neighborhoods, the space will be called a <u>Hausdorff space</u> or a T_2-space if:

(i) To every point p there is at least one neighborhood U(p) containing it.

(ii) If $U_1(p)$ and $U_2(p)$ are neighborhoods of p, there is at least one neighborhood $U_3(p)$ of p such that $U_3(p) \subset U_1(p) \cap U_2(p)$.

(iii) If U(p) is a neighborhood of p and $q \in U(p)$, then there is a neighborhood U(q) of q with $U(q) \subset U(p)$.

(iv) If $p \neq q$, there exist neighborhoods U(p) of p and U(q) of q such that $U(p) \cap U(q) = \emptyset$.

First Countability Axiom: There is a countable basis at each point of the space.

Second Countability Axiom: The space has a countable basis.

A Hausdorff space satisfying the second countability axiom is separable.

A topological space is locally compact if every point has a closed compact neighborhood. A locally compact space can be made compact by the addition of a single point.

Cartesian Products: The Cartesian product $S_1 \times S_2$ of two sets S_1 and S_2 is defined as the set of all ordered pairs (p, q) such that $p \in S_1$ and $q \in S_2$,

$$S_1 \times S_2 = \{(p, q): p \in S_1 \text{ and } q \in S_2\}.$$

Thus the Cartesian plane of analytical geometry is the Cartesian product of the real line by itself. The topology of $S_1 \times S_2$ can be given by means of the definition of a topological space based on neighborhoods. If now $\sum$ is a basis for S_1 and $\sum'$ is a basis for S_2, we define $\sum''$ of $S_1 \times S_2$ as the totality of all sets of the form $W = U \times V$, where $U \in \sum$, and $V \in \sum'$. The above definition of the Cartesian product obviously extends to any finite number of factors: $S_1 \times S_2 \, \cdots\cdots \times S_n$ is the set of ordered n-ads $(p_1, \cdots\cdots, p_n)$ such that $p_i \in S_i$ for $i = 1, \cdots\cdots, n$.

Theorem 1.2 (Tychonoff): The Cartesian product of a family of compact spaces is compact.

Lemma 1.1. The Cartesian product of a family of Hausdorff spaces is a Hausdorff space[8].

2. Normed Linear Spaces

A <u>normed linear space</u> is a vector space over the real numbers or over the complex numbers on which is defined a non-negative real-valued function called the norm (the norm of x being designated $\|x\|$) such that

(i) $\|x\| = 0 \Leftrightarrow x = 0$

(ii) $\|x + y\| \leqq \|x\| + \|y\|$ (triangle inequality)

(iii) $\|\lambda x\| = |\lambda| \cdot \|x\|$ (homogeneity).

A normed linear space is generally understood to be over the complex number field, the real case being explicitly labeled as a real normed linear space. A normed linear space becomes a metric space if the distance d(x, y) is defined as $\|x - y\|$, and it is called a <u>Banach space</u> if it is complete in this metric, i.e., if whenever $\|x_n - x_m\| \to 0$ as $n, m \to \infty$, then there exists an element x such that $\|x_n - x\| \to 0$ as $n \to \infty$.

The metric space topology in question has for a basis the family of all open spheres, where the sphere about x_o of radius r, $S(x_o, r)$ is the set $\{x: \|x - x_o\| < r\}$*. The open spheres about x_o form a neighborhood basis for x_o.

It follows directly from the triangle inequality that $\big| \|x\| - \|y\| \big| \leqq \|x - y\|$, so that $\|x\|$ is a continuous function of x.

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* $\{x: \|x - x_o\| < r\}$ is the set of all x such that $\|x - x_o\| < r$.

3. Linear Functionals

Consider a vector space E over the field of complex numbers or more briefly a complex vector space.

Definition. A linear transformation (mapping) from a complex vector space E to a complex vector space E' is a mapping M from E into E' such that

$$M(\alpha x + \beta y) = \alpha Mx + \beta My$$

identically for all complex numbers α and β and all vectors x and y in E.

Linear transformations whose range space E' coincides with the complex vector space C of all complex numbers are called linear functionals. Explicitly: a linear functional on a complex vector space E is a complex-valued function ξ on E such that

(i) ξ is additive (i.e., $\xi(x + y) = \xi(x) + \xi(y)$ for every pair of vectors x and y in E.

(ii) ξ is homogeneous (i.e., $\xi(\alpha x) = \alpha\xi(x)$ for every complex number α and for every vector x in E).

The definition of a conjugate linear functional differs from the one just given in that the equation $\xi(\alpha x) = \alpha\xi(x)$ is replaced by $\xi(\alpha x) = \bar{\alpha}\xi(x)$, where $\bar{\alpha}$ denotes the complex conjugate of α.

If C is the complex number field (with $\|\alpha\| = |\alpha|$), the space of continuous linear mappings (functionals) of a normed linear space X into C is called the conjugate space of X, denoted X*. Since the complex number field is complete, it follows that X* is always a Banach space.

Definition. A bilinear functional on a complex vector space E is a complex-valued function φ on the Cartesian product of E with itself such that if

$$\xi_y(x) = \eta_x(y) = \varphi(x, y),$$

then for every x and y in E, ξ_y is a linear functional and η_x is a conjugate linear functional.

A bilinear functional φ is symmetric if $\varphi(x, y) = \bar{\varphi}(y, x)$ for every pair of vectors x and y.

A bilinear functional φ is positive if $\varphi(x, x) \geqq 0$ for every vector x; we shall say that φ is strictly positive if $\varphi(x, x) > 0$, whenever $x \neq 0$.

Inner Product: An inner product in a complex vector space E, is a strictly positive, symmetric, bilinear functional on E. An inner product space is a complex vector space E with an inner product defined in it.

Hilbert Space: A Hilbert space is an inner product space which, as a metric space, is complete[7)].

Theorem 1.3 (Hahn-Banach): If Y is a linear subspace of the normed linear space X, if F is a bounded linear functional on Y, and if x_o is a point of X not in Y, then F can be extended to Y + (x_o)* without changing its norm[12)].

It follows from Zorn's lemma that a functional F such as above can be extended to the whole space X without increasing

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* (x_o) denotes the linear subspace generated by a subset x_o of the vector space X.

its norm. For the extensions of F are partially ordered by inclusion, and the union of any linearly ordered subfamily is clearly an extension which includes all the members of the subfamily and hence is an upper bound of the subfamily. Therefore, there exists a maximal extension by Zorn's lemma. Its domain must be X, since otherwise it could be extended further by the Hahn-Banach theorem.

4. Rings and Ideals

In order to discuss inverses in a ring without assuming a unit element we introduce the "circle" operation

$$x \circ y = x + y - xy,$$

and call y the _right quasi-inverse_ of x, and x the _left quasi-inverse_ of y. We say that y is a _quasi-inverse_ of x if $x \circ y = y \circ x = 0$, and that x is _quasi-regular_ if it has a quasi-inverse. Under the operation "o" the quasi-regular elements form a group which has zero as an identity.

To treat maximal ideals in rings without unit it is convenient, following Segal[21)] to introduce regular ideals.

Definition. A (right, two-sided) ideal in an algebra is called _regular_ if there exists an element of the algebra that is a (left, two-sided) identity* modulo the ideal.

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* An element e is a left identity modulo an ideal if ex - x is in the ideal for every x in the algebra.

<u>Theorem 1.4</u>. In any ring R an element x has a right quasi-inverse if and only if there exists no regular maximal right ideal modulo which x is a left identity.

<u>Proof</u>. If x has no right quasi-inverse, then the set $\{xy - y: y \in A\}$ is a right ideal not containing x modulo which x is a left identity*, and this ideal can be extended to a regular maximal ideal with the same property by Zorn's lemma. Conversely, if x has a right quasi-inverse y and if x is a left identity for a right ideal I, then $x = xy - y \in I$. Then $y = xy - (xy - y) \in I$ for every $y \in R$ and $I = R$. Thus x cannot be a left identity modulo any proper right ideal.

<u>Theorem 1.5</u>. If M is a regular ideal in a commutative ring R, then M is maximal if and only if R/M is a field.

<u>Proof</u>. If R/M has a proper ideal J, then the union of the cosets in J is a proper ideal of R properly including M. Thus M is maximal in R if and only if R/M has no proper ideals. Therefore, given $X \in R/M$ and not zero, the ideal $\{XY: Y \in R/M\}$ is the whole of R/M. In particular, XY = E for some Y, where E is the identity of R/M. Thus every element X has an inverse and R/M is a field. The converse follows from the fact that a field has no non-trivial ideals. If R is an algebra over the complex numbers, then the above field is a field over the complex numbers.

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* $xy = y \bmod I$ since $xy - y \in I$.

We shall find throughout that the presence of an identity in an algebra makes the theory simpler and more intuitive than is possible in its absence. It is therefore important to observe, as we do in the theorem below, that we can always enlarge an algebra deficient in this respect to one having an identity.

Theorem 1.6. If A is an algebra without an identity, then A can be imbedded as a maximal ideal of deficiency one in an algebra A_e having an identity in such a way that the mapping $I_e \to I = A \cap I_e$ is a one-to-one correspondence between the family of all (right) ideals I_e in A_e which are not included in A and the family of all regular (right) ideals I of A.

Proof. The elements of A_e are the ordered pairs (x, λ), where $x \in A$ and λ is a complex number. Considering (x, λ) as $x + \lambda e$, the definition of multiplication is obviously

$$(x, \lambda)(y, \mu) = (xy + \lambda y + \mu x, \lambda\mu).$$

It is clear that (0, 1) is an identity for A_e and that the correspondence $x \to (x, 0)$ imbeds A in A_e as a maximal ideal with deficiency 1.

Now let I_e be any (right) ideal of A_e not included in A and let $I = I_e \cap A$. I_e must contain an element v of the form (x, -1). Then the element $u = v + e = (x, 0) \in A$ is a left identity for I_e in A_e* and hence automatically for I in A. Thus I is regular in A. Moreover, since $uy - y \in I_e$ and $uy \in A$ for all y, we see that $y \in I_e$ if and only if $uy \in I$.

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* $uy - y = (u - e)y = vy \in I_e$ for all $y \in A_e$.

Conversely, if I is a regular (right) ideal in A and $u = (x, 0)$ is a left identity for I in A, we define I_e as $\{y: uy \in I\}$. Direct consideration of the definition of multiplication in A_e shows that I is a (right) ideal in A_e; hence I_e is a (right) ideal in A_e. It is not included in A since $u(x, -1) = u(u - e) = u^2 - u \in I$ and therefore $u - e = (x, -1) \in I_e$. Moreover, the fact that $uy - y \in I$ for every $y \in A$ shows that $y \in I$ if and only if $uy \in I$ and $y \in A$, i.e., $I = I_e \cap A$.

We have thus established a one-to-one inclusion preserving correspondence between the family of all regular (right) ideals in A and the family of all (right) ideals of A_e not included in A. In particular the regular maximal (right) ideals of A are the intersections with A of the maximal (right) ideals of A_e different from A[12].

5. Radical and Spectrum

The radical is defined to be the union of all quasi-regular right ideals*. Jacobson[9] shows that it is a two-sided ideal, and is in fact equal to the intersection of the regular maximal right ideals or the regular maximal left ideals. A ring is semi-simple if its radical is zero.

Next let A be an algebra over an algebraically closed field F. We say that a non-zero scalar λ is in the spectrum of $x \in A$ if $\lambda^{-1}x$ is not quasi-regular; and 0 is defined to belong to the spectrum of x unless A has a unit element, and x a(two-sided) inverse with respect to it.

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* A right ideal is quasi-regular if its elements are quasi-regular.

CHAPTER II

BANACH ALGEBRAS

1. Elementary Theory

Definition. A *Banach algebra* A is an algebra over the complex numbers, together with a norm under which it is a Banach space and which is related to multiplication by the inequality:

$$\|xy\| \leq \|x\| \|y\|.$$

Theorem 2.1. If $x \in A$ and $\|x\| < 1$, then x is quasi-regular, the quasi-inverse being given by $y = -\sum_1^\infty x^i$, which is also a continuous function of x.

Proof. If $y_n = -\sum_1^n x^i$, then $\|y_m - y_n\| = \|\sum_{m+1}^n x^i\| \leq \sum_{m+1}^n \|x\|^i < \|x\|^{m+1}/(1 - \|x\|) \to 0$ as $n \to \infty$. Thus $\{y_n\}$ is a Cauchy sequence, and its limit y is given by the infinite series $-\sum_1^\infty x^i$ in the usual sense. Then $x + y - xy = \lim(x + y_n - xy_n) = \lim x^n = 0$, and y is a right quasi-inverse of x. Similarly, because y commutes with x, y is a left quasi-inverse and therefore the quasi-inverse of x. Since the series is uniformly convergent in the closed sphere $\|x\| \leq r < 1$, it follows that the quasi-inverse is a continuous function of x in the open sphere $\|x\| < 1$.

Theorem 2.2. In a Banach algebra A with the property that a (right) quasi-inverse of every element in some neighborhood of zero exists, every regular maximal (right) two-sided ideal is closed.

Proof. Let M denote a regular maximal (right) two-sided ideal, then it follows easily that $\overline{M}$* is a (right) two-sided ideal. It is evidently sufficient to show that $\overline{M} = A$ is false. Employing an indirect argument, suppose that $\overline{M}$ is A. Putting e for a (left) two-sided identity modulo M, it is clear that there exists an element of M, m, such that the (right) quasi-inverse of e - m exists. Putting y for this (right) quasi-inverse, then the defining equation of y,

$$(e - m)y = e - m + y,$$

yields a contradiction when reduced modulo M[21].

<u>Definition</u>. The <u>spectral</u> <u>radius</u> of an element $x \in A$, denoted by r(x) is defined as follows:

$$r(x) = \sup |\lambda|, \text{ where } \lambda \in \text{spectrum of } x.$$

In his paper on "normed rings", Gelfand[4] proved the formula for the spectral radius,

$$r(x) = \lim_{n \to \infty} \|x^n\|^{1/n} .$$

Gelfand's proof depends on the application of Taylor's theorem to a certain vector-valued function. However, the proof which we will outline utilizes only numerical functions and this is done by a slight modification of Gelfand's argument.

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* We denote the closure of a set by superposing a bar.

We form the expansion

$$(\lambda x)' = -(\lambda x + \lambda^2 x^2 + \lambda^3 x^3 + \ \ldots\ldots \ + \ldots\ldots + \ )$$ *

which is certainly convergent for complex numbers λ with $|\lambda| < \|x\|^{-1}$. Let f be any fixed continuous functional on the algebra. Then $f[(\lambda x)']$ is a complex-valued function of λ, defined for λ^{-1} not in the spectrum of x, which one verifies by direct computation of the derivative to be analytic. In particular it is analytic for $|\lambda| < [r(x)]^{-1}$. Hence its power series expansion in λ, namely $\sum f(x^n)\lambda^n$ must converge for $|\lambda| < [r(x)]^{-1}$. Let t be any positive real number greater than r(x). Then $\sum |f(t^{-n}x^n)|$ converges. Hence the sequence $t^{-n}x^n$ is bounded for every f. By a well known theorem of Banach, the sequence is bounded in norm. Extracting the n-th root we obtain

(1) $$\limsup \|x^n\|^{1/n} \leq r(x).$$

On the other hand $[r(x)]^n = r(x^n) \leq \|x^n\|$ and hence

(2) $$r(x) \leq \|x^n\|^{1/n} .$$

Combining (1) and (2) we obtain

(3) $$r(x) = \lim_{n\to\infty} \|x^n\|^{1/n} .$$

A particular case of (3) asserts that r(x) = 0 is equivalent to $\|x^n\|^{1/n} \to 0$. Such elements have been called "generalized nilpotent" or "quasi-nilpotent". In the commutative case they constitute precisely the radical, while in the non-

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* $(\lambda x)'$ denotes the quasi-inverse of λx.

commutative case the radical is the union of all ideals which consist entirely of generalized nilpotent elements.

Lemma 2.1. If u is a relative identity modulo a proper regular ideal I, then $d(I, u) \geq 1$.

Proof. If there exists an element $x \in I$ such that $\|u - x\| < 1$, then $u - x$ has a quasi-inverse a, where $(u - x)a - a - (u - x) = 0$. Since x, xa, and $ua - a$ are all in I, it follows that $u \in I$, a contradiction.

Theorem 2.3. If I is a closed ideal in a Banach algebra A, then A/I becomes a Banach algebra if the norm of a coset Y is defined as its distance from the origin: $\|Y\| = \text{glb}\{\|x\|: x \in Y\}$.

Proof. First, $\|Y\| = 0$ if and only if there exists a sequence $x_n \in Y$ such that $\|x_n\| \to 0$. Since Y is closed, this will occur if and only if $0 \in Y$, so that $\|Y\| = 0 \Longleftrightarrow Y = I$.
Next $\|Y_1 + Y_2\| = \text{glb}\{\|x_1 + x_2\|: x_1 \in Y_1,\ x_2 \in Y_2\} \leq \text{glb}\{\|x_1\| + \|x_2\|\}$
$= \text{glb}\{\|x_1\|: x_1 \in Y_1\} + \text{glb}\{\|x_2\|: x_2 \in Y_2\} = \|Y_1\| + \|Y_2\|$.
Similarly $\|\lambda Y\| = |\lambda| \cdot \|Y\|$, and A/I is thus a normed linear space. If $\{Y_n\}$ is a Cauchy sequence in A/I, we can suppose, by passing to a subsequence if necessary, that $\|Y_{n+1} - Y_n\| < 2^{-n}$.
Then we can inductively select elements $x_n \in Y_n$ such that $\|x_{n+1} - x_n\| < 2^{-n}$, for $d(x_n, Y_{n+1}) = d(Y_n, Y_{n+1}) < 2^{-n}$.
Since A is complete, the Cauchy sequence has a limit x_o, and if Y_o is the coset containing x_o, then $\|Y_n - Y_o\| \leq \|x_n - x_o\|$ so that $\{Y_n\}$ has the limit Y_o. That the original sequence converges to Y_o then follows from the general metric space lemma that, if a Cauchy sequence has a convergent subsequence, then it itself

is convergent. Thus A/I is complete.

If X and Y are two of its cosets, then $\|XY\| = \text{glb}\{\|xy\|\}$, where $x \in X$ and $y \in Y$. Hence, $\text{glb}\{\|xy\|\} \leq \text{glb}\{\|x\| \cdot \|y\|\} = \text{glb}\{\|x\|\} \cdot \text{glb}\{\|y\|\} = \|X\| \cdot \|Y\|$. If I is regular and u is a relative identity, then the coset E containing u is the identity of A/I and $\|E\| = \text{glb}\{\|x\|: x \in E\} = \text{glb}\{\|u - y\|: y \in I\} = d(I, u) \geq 1$ by Lemma 2.1. If A does not have an identity and $\|E\| > 1$, then it is possible to renorm A/I with a smaller equivalent norm so that $\|E\| = 1$ [4,8].

<u>Corollary</u>. If I is a regular maximal ideal and if A is commutative, then it follows from Theorem 1.5. and the above theorem that A/I is a normed field.

<u>Theorem 2.4.</u> (<u>Gelfand</u>-<u>Mazur</u>): Every normed field is isometrically isomorphic to the field of complex numbers.

<u>Proof</u>. We have to show that for any element x of the field there is a complex number λ such that $x = \lambda e$. We proceed by contradiction, supposing that $x - \lambda e$ is never zero and therefore that $(x - \lambda e)^{-1}$ exists for every λ. But if f is any continuous linear functional over the field considered as a Banach space, then $f[(x - \lambda e)^{-1}]$, as a function of λ, is seen by direct calculation to have the derivative $f[(x - \lambda e)^{-2}]$, and is consequently analytic over the whole plane. Also $(x - \lambda e)^{-1} \to 0$ as $\lambda \to \infty$, for $(x - \lambda e)^{-1} = \lambda^{-1}(x/\lambda - e)^{-1}$, and $(x/\lambda - e)^{-1} \to -e$ as $\lambda \to \infty$. Thus $f[(x - \lambda e)^{-1}] \to 0$ as $\lambda \to \infty$ and hence $f[(x - \lambda e)^{-1}] \equiv 0$ by Liouville's theorem. It follows from the Hahn-Banach theorem and Zorn's lemma that $(x - \lambda e)^{-1} = 0$, a contradiction[4,12].

Remark. The above proof has not made use of the fact that multiplication is commutative, except for polynomials in a single element x and its inverse. Thus it actually has been shown that the complex number field is the only normed division algebra.

2. Commutative Banach Algebras

In this section we restrict our attention to a commutative Banach algebra A. The results which will be outlined are essentially those obtained by Gelfand[4] in his paper on normed rings, except that we do not assume an identity element.

Theorem 2.5. (Stone-Weierstrass): Let A be a closed* subalgebra of the Banach algebra C(S) of all continuous (real or complex) functions on the locally compact space S which vanish at infinity**. In the case of the complex algebra, suppose that if $x \in A$ then there exists a y in A such that $\bar{x}(p) = y(p)$***. If for every $p_1 \neq p_2$ there is an $x \in A$ such that $x(p_1) \neq 0$, $x(p_2) = 0$, then $A = C(S)$[24].

The Space of Maximal Regular Ideals. Let $\mathfrak{M}$ denote the class of all maximal regular ideals in a commutative Banach algebra A. Unless A is equal to its radical, a case which we exclude, $\mathfrak{M}$ is non-vacuous. If $M \in \mathfrak{M}$, then A/M is a normed field and so, by the Gelfand-Mazur theorem is isomorphic to the complex

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* A is closed in the topology $\|x\| = \sup|x(p)|$.

** A function x vanishes at infinity if for $\varepsilon > 0$ there is a compact set outside of which $|x(p)| < \varepsilon$. *** $\bar{x}(p)$ is conjugate of x(p numbers. There is accordingly associated with M a homomorphism $x \rightarrow x(M)$ of A onto the complex numbers with M as its kernel*. If θ_M is the homomorphism of $\mathfrak{M}$ whose kernel is the regular maximal ideal M, then the number $\theta_M(x)$ is explicitly determined as follows: if e_M is the identity of the field A/M and if X is the coset of A/M which contains x, then $\theta_M(x)$ is that complex number λ such that $X = \lambda e_M$. If x is held fixed and M is varied, then $\theta_M(x)$ defines a complex-valued function x(M) on the set $\mathfrak{M}$ of all regular maximal ideals of A. Next we define a topology in $\mathfrak{M}$ in terms of neighborhoods. Let ε be given, $\varepsilon > 0$, let n be a positive integer, and let $x_1, x_2, \ldots\ldots, x_n$ be any n points of A. Then the set of elements M of $\mathfrak{M}$ with

$$\left|x_i(M) - x_i(M_o)\right| < \varepsilon \ , \quad i = 1, 2, \ldots\ldots, n$$

form a neighborhood of M_o by definition. The set of neighborhoods of M_o is obtained by varying ε, n, and the x's. Then becomes a locally compact Hausdorff space, called the <u>structure</u> space of A, such that each x(M) is a continuous function of M which vanishes at infinity. If A has an identity, then $\mathfrak{M}$ is compact. The range of the function x(M), except possibly for the value 0, is equal to the spectrum of x. Therefore, the max $|x(M)| = r(x) \leq \|x\|$, where r(x) denotes the spectral radius. Hence $x \rightarrow x(\cdot)$ is a continuous homomorphism of A into the Banach algebra $C(\mathfrak{M})$ of all continuous, complex-valued functions defined

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* The kernel of a homomorphism of an algebra is the collection of elements which are mapped into zero by the homomorphism.

on the set of all regular maximal ideals and vanishing at infinity. This homomorphism is an isomorphism if, and only if, A is semi-simple. If for each $x \in A$ there exists a conjugate element $\bar{x}$ such that $\bar{x}(M) = \overline{x(M)}$, then, since the functions $x(M)$ separate elements* of $\mathfrak{M}$, it follows by the Stone-Weierstrass theorem that the functions $x(\cdot)$ form a dense subset of $C(\mathfrak{M})$. If in addition $x_n \to 0$ is equivalent to $x_n(M) \to 0$ uniformly for $M \in \mathfrak{M}$, then the functions $x(\cdot)$ will exhaust $C(\mathfrak{M})$. This will be so, in particular, if the norm in A satisfies the condition that $\|x^2\| = \|x\|^2$. That is, utilizing the above remarks on $x(M)$ and the formula for the spectral radius**, we obtain

$$\max|x(M)| = \lim_{n \to \infty} \|x^{2^n}\|^{1/2^n} = \lim_{n \to \infty} \|x\|^{2^n/2^n} = \lim_{n \to \infty} \|x\| = \|x\|,$$

from which it follows that the uniform convergence induces the convergence in the norm. In this case $r(x) = \|x\|$.

An element $x \in A$ is quasi-regular if, and only if, $x(M) \neq 1$ for all M. If A has an identity, then x will have an inverse in A if, and only if, $x(M) \neq 0$ for all M; i.e., x does not belong to any maximal ideal. The latter observation is the basis for Gelfand's[5)] proof of the Wiener theorem mentioned in the preface. The proof goes briefly as follows:

Take W as the class of all sequences $x = \{\zeta_n\}$ of complex numbers such that $\|x\| = \sum_{-\infty}^{+\infty} |\zeta_n| < \infty$. Let α be a complex

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* The functions $x(M)$ separate elements of $\mathfrak{M}$ if, for $M_1 \neq M_2$ there exists an element $x \in A$ such that $x(M_1) \neq x(M_2)$.

** $r(x) = \lim_{n \to \infty} \|x^n\|^{1/n}$.

number and $x = \{\xi_n\}$, $y = \{\eta_n\}$ elements of W. Then W is a commutative Banach algebra under $\|x\|$ as norm and algebraic operations:

$$\alpha x = \{\alpha\xi_n\}, \quad x + y = \{\xi_n + \eta_n\}, \quad xy = \left\{\sum_{k=-\infty}^{k=+\infty} \xi_k \eta_{n-k}\right\}.$$

The element δ_{0n}, where $\delta_{0n} = 1$ or 0 according as $n = 0$ or $n \neq 0$, is an identity in W. Every homomorphism of W onto the complex numbers is of the form

$$x \longrightarrow \sum_{n=-\infty}^{n=+\infty} \xi_n \, e^{inu} ,$$

where u is a real number. Thus $\mathfrak{M}$ is in one-to-one correspondence (actually homeomorphic)* with the real numbers modulo 2π.

Now if $x(u) = \sum_{n=-\infty}^{n=+\infty} \xi_n e^{inu}$, where $x = \{\xi_n\} \in W$, then the condition $x(u) \neq 0$ for all u implies the existence of $y = \{\eta_n\} \in W$ such that $xy = 1$. Clearly $\sum_{n=-\infty}^{n=+\infty} \eta_n e^{inu} = [x(u)]^{-1}$ so that the Wiener theorem is proved. Gelfand[5] uses the same methods to prove a variety of theorems of the same type. In each case the key problem is the determination of the maximal ideals of the particular Banach algebra under consideration.

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* A one-to-one transformation which is bicontinuous is called a homeomorphism (topological mapping). Two topological spaces are called homeomorphic if one can be homeomorphically mapped on the other.

3. Regular Banach Algebras

In the ideal theory of commutative Banach algebras, the following questions arise:

Question 1. Is every proper closed ideal included in at least one regular maximal ideal?

Question 2. Is every ideal which is included in precisely one maximal regular ideal necessarily equal to that maximal ideal?

Question 3. Is every closed ideal equal to the intersection of maximal ideals containing it?

In studying these and related questions, the following notion of "regularity" of a Banach algebra is of importance.

A commutative Banach algebra A is said to be regular provided, for any closed subset F in $\mathfrak{M}$ and $M_0 \in \mathfrak{M} - F$, there exists $x \in A$ such that $x(M) = 0$ on F and $x(M_0) \neq 0$. We shall outline briefly in this section a few results on regular algebras many of which are due to Silov[23], who has made an exhaustive study of the subject.

We observe first that regularity of A is necessary and sufficient for the given topology in $\mathfrak{M}$ to be equivalent to the following topology which is due to Stone[24]. Let S be a subset of $\mathfrak{M}$ and denote by I(S) the closed ideal obtained as the intersection of all M in S. The closure of S in the Stone topology is defined to be the set of all M in $\mathfrak{M}$ which contain I(S).

An example which shows that the two topologies are not always equivalent is the algebra of all functions f(z) analytic in the circle $|z| < 1$ and continuous in $|z| \leq 1$. If A is regular and F is a closed subset of $\mathfrak{M}$, then the algebra A/I(F) has F as its space of maximal ideals. This result implies that A is normal. In other words, if F_1 and F_2 are disjoint closed sets in $\mathfrak{M}$, and F_2 is compact, then there exists an x such that $x(M) = 0$ on F_1 and $x(M) = 1$ on F_2. A function f(M) defined on $\mathfrak{M}$ is said to belong locally to an ideal I of A if, for every M_o in $\mathfrak{M}$ there exists an x_{M_o} in I and a compact set C_{M_o} disjoint from M_o such that $x_{M_o}(M) = f(M)$ in the complement of C_{M_o}. It is a consequence of normality that, if f(M) belongs locally to A, then f(M) actually belongs to A in the sense that there exists an x in A for which $x(M) = f(M)$ in all of $\mathfrak{M}$.

An ideal in A is said to be primary if it is contained in exactly one maximal regular ideal. If the zero ideal is primary, then A itself is called a primary algebra. In this case A contains exactly one maximal regular ideal. An algebra in which every closed primary ideal is maximal, i.e., an algebra in which Question 2 has an affirmative answer, is called an N*-algebra. If every closed ideal is an intersection of maximal ideals, i.e., the answer to Question 3 is affirmative, then the algebra is called an N-algebra. The terminology here is Silov's[23]. Every N-algebra is an N*-algebra, but the converse is not true as is shown by the following example. Kaplansky[11], has shown that for an arbitrary locally compact abelian group G, all the

closed primary ideals are maximal, hence $L_1(G)$* is an N-algebra. On the other hand, Schwartz[20)] has shown that if G is the three dimensional Euclidean space, then $L_1(G)$ contains a closed ideal which is not an intersection of maximal ideals, that is , $L_1(G)$ is not an N-algebra. However, we have the following criteria for N*- and N-algebras when A is semi-simple and regular.
In order for A to be an N*-algebra it is necessary and sufficient that, corresponding to every $x \in A$ and $M_o \in \mathfrak{M}$ with $x(M_o) = 0$, there exist a sequence $\{x_n\}$ such that $\|x_n\| \to 0$ and $x_n(M) = x(M)$ outside of some compact set C_n which does not contain M_o. The N-algebra criterion is obtained by requiring $\{x_n\}$ to be independent of M_o. Thus, in order for A to be an N-algebra, it is necessary and sufficient that, corresponding to every x, there exist $\{x_n\}$ such that $\|x_n\| \to 0$ and $x_n(M) = x(M)$ outside of some compact set C_n disjoint from the closed set of all M for which $x(M) = 0$.
This result can be used, for example, to show that the algebra V of all continuous functions x(t) of bounded variation on $[0,1]$, with the ordinary algebraic operations and norm

$$\|x\| = \max|x(t)| + \text{var } x(t),$$

is a regular N-algebra. The space of maximal ideals for V is $[0,1]$ with $x(M_t) = x(t)$, where M_t is the maximal ideal associated with t.

We shall now discuss in more detail Question 1 posed at the beginning of this section. If A is an arbitrary commutative

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* $L_1(G)$ are the complex functions on the group G which are summable with respect to left Haar measure, with multiplication defined as convolution.

semi-simple Banach algebra, it has been observed by Mackey[13], that an affirmative answer to the above question can be given under the following two assumptions: (a) A is regular and (b) the set of elements x, such that x(M) vanishes outside some compact subset of $\mathfrak{M}$, is dense in A. The proof goes as follows: Let I be a closed proper ideal and assume that there is no M such that $I \subseteq M$. If C is any compact subset of $\mathfrak{M}$, then regularity of A implies the existence of a $u \in I$ such that $u(M) = 1$ for $M \in C$. Now if $x(M) = 0$ for $M \notin C$, then, by semi-simplicity, we have $xu = x$. Hence $x \in I$. Thus I contains every element x such that x(M) vanishes outside a compact set. It follows from (b) and the closure of I that $I = A$, a contradiction. Therefore, $I \subseteq M$ for some $M \in \mathfrak{M}$. It should be remarked that there is no problem here if A has an identity element, for then a proper ideal can always be embedded in a maximal (automatically regular) ideal.

CHAPTER III

BANACH *-ALGEBRAS

1. Commutative *-algebras

We shall call a Banach algebra A a *-algebra if it admits an involution *: that is, a conjugate linear anti-automorphism of period two. We say that A is symmetric if every x^*x is quasi-regular. The meaning of symmetry is easily understood in the commutative case: it is precisely equivalent to the statement that each regular maximal ideal is self-adjoint ($M^* = M$). Moreover the functions $x(M)$ then have the property that $x^*(M)$ is complex conjugate of $x(M)$. If we add the assumption $\|x^2\| = \|x\|^2$ then as noted previously(page 19), the algebra A can be completely identified with the algebra $C(\mathfrak{M})$ of all continuous complex functions vanishing at infinity on the space $\mathfrak{M}$ of regular maximal ideals.

An important fact is that symmetry of a commutative *-algebra can be proved from a suitable assumption on the norm.

Theorem 3.1. Let A be a commutative *-algebra in which $\|x^*x\| \geq k\|x^*\|\|x\|$ holds for every x, with k independent of x. Then A is symmetric.

If $k = 1$, the hypothesis simplifies to the equality $\|x^*x\| = \|x^*\|\|x\|$. In this case, with the additional assumption of a unit element, the theorem was proved by Gelfand and Neumark[6]. In the generality stated above, the theorem is likewise due to Arens[2]. We shall outline a slightly modified version of Aren's proof. First, if necessary we adjoin a unit element; as norm we may take

(1) $$\|x + \lambda\| = \|x\| + |\lambda|.$$

It is easy to verify that our hypothesis survives, perhaps with a smaller value of k. So, after a change of notation, we may simply assume that A has a unit element. Next we prove:

(2) $$r(x^*x) = r(x^*)r(x) \ ,$$

(3) $$k^2\|x\| \leq r(x) \leq \|x\|.$$

Proof of 2. In any commutative Banach algebra we have $r(xy) \leq r(x)r(y)$; thus only the reverse inequality in (2) needs proof. We have

$$r(x^*x) = \lim_{n \to \infty} \|(x^*x)^n\|^{1/n}$$

while

$$\|(x^*x)^n\| = \|x^{*n}x^n\| \geq k\|x^{*n}\|\|x^n\|.$$

On passage to the limit we obtain $r(x^*x) \geq r(x^*)r(x)$.

Proof of 3. Let y be self-adjoint ($y^* = y$). Then $\|y^2\| \geq k\|y\|^2$ and by induction on n

$$\|y^{2^n}\| \geq k^{2^n-1}\|y\|^{2^n} .$$

On extracting the 2^n-th root and passing to the limit we get $r(y) \geq k\|y\|$. We now apply this with $y = x^*x$:

(4) $$r(x^*x) \geq k\|x^*x\| \geq k^2\|x^*\|\|x\| .$$

On the other hand

$$(5) \qquad r(x^*x) \leq r(x^*)r(x) \leq \|x^*\|r(x) \ .$$

From (4) and (5) we obtain $k^2\|x\| \leq r(x)$. The other half of (3) holds in any Banach algebra.

In summary: the spectral radius $r(x)$ gives a norm equivalent to the original norm; and if we use it as a new norm we achieve the equality $\|x^*x\| = \|x^*\|\|x\|$. To complete the proof that the algebra A is symmetric, we follow the elegant method given by Aren's[1].

Set

$$x(M) = a + bi \ , \quad x^*(M) = c + di \ ,$$

where a, b, c, d are real numbers. It is required to prove that $a - c = 0$ and $b + d = 0$. Assume the contrary; for example, let $b + d \neq 0$. Define

$$y = \frac{1}{b + d}\left[x + x^* - (a + c)e\right] \ ,$$

where e is the identity of A. Evidently y is a Hermitian element and

$$y(M) = \frac{1}{b + d}\left[x(M) + x^*(M) - (a + c)\right] = i \ .$$

This means $y - ie$ has no inverse in A. Therefore,

$$(y - ie)^* = y^* + ie$$

has no inverse, whence it lies in some maximal ideal M'. Thus $(y + ie)(M) = 0$ or $y(M') = -i$. Hence for an arbitrary positive number N

$$(6a) \qquad (y + iNe)(M) = i(1 + N),$$

$$(6b) \qquad (y - iNe)(M') = -i(1 + N) .$$

Applying (1) to (6a) and (6b) we obtain

$$(7a) \qquad 1 + N \leq \|y + iNe\| ,$$

$$(7b) \qquad 1 + N \leq \|y - iNe\| .$$

Hence, by applying the condition $\|x^*x\| = \|x^*\|\|x\|$ to the elements $y + iNe$ and $(y + iNe)^* = y - iNe$, we obtain

$$(1 + N)^2 \leq \|y + iNe\|\|y - iNe\| = \|y^2 + N^2e\| \leq N + N^2 ,$$

i.e., $(1 + N)^2 \leq N + N^2$ for arbitrary $N \geq \|y^2\|$, a contradiction. Hence, $b + d = 0$. Applying the same argument to ix in place of x, we obtain $a - c = 0$, whence that $x^*(M) = \overline{x(M)}$, i.e., A is symmetric.

2. States and Representations

We now turn our attention to non-commutative Banach algebras. Nearly all work in this field has thus far been confined to algebras possessing an involution with suitable properties, which makes it possible to make connections with the highly developed theory of Hilbert space. We proceed to describe the mechanism that achieves this connection.

By a <u>state</u> on a *-algebra we mean a linear functional f such that for every x, $f(x^*)$ is the complex conjugate of f(x) and $f(x^*x) \geq 0$. These functionals are also called <u>positive</u>.

If A is a *-algebra of operators on a Hilbert space H, and ξ is a fixed vector in H, we may construct a state by defining $f(T) = (T\xi, \xi)$ for all T in A.

The first step is to use the state f to define an inner product on A via the definition $(x, y) = f(y^*x)$. This inner product is positive semi-definite, which is enough to yield the Cauchy-Schwartz inequality:

$$(1) \qquad |f(y^*x)|^2 \leq f(x^*x)f(y^*y) \ .$$

The elements with $(x, x) = f(x^*x) = 0$ form, by (1), a left ideal I in A, and the quotient space A/I is in a natural way a pre-Hilbert space.* For each fixed x in A the mapping

$$y + I \rightarrow xy + I$$

is a linear transformation T_x on A/I. In other words, there is a unique way of passing from a state on A to a representation of A by linear transformations on a pre-Hilbert space. The vital thing is to know that the linear transformations T_x are continuous, and so can be extended to the completion of A/I. For this we must have

$$(2) \qquad f(y^*x^*xy) \leq K(x)f(y^*y)$$

for all x and y, where K(x) may depend on x. We shall now

* A Hilbert space, which may not be complete.

prove (2) under either of two hypotheses.

(a) Suppose * is continuous. Let z be a self-adjoint element of norm less than 1. The binomial expansion for $(1 - z)^{1/2}$ converges. We write u (formally if there is no unit element) for the sum, and k = uz; k is an actual ring element. We observe (by the continuity of *) that u* = u. Hence $k^*k = y^*u^2y = y^*(1 - z)y$. From $f(k^*k) \geq 0$ we deduce

$$(3) \qquad f(y^*zy) \leq f(y^*y) .$$

Now put $z = (x^*x)/(\|x^*x\| + \varepsilon)$ and take the limit as $\varepsilon \to 0$; we obtain

$$(4) \qquad f(y^*x^*xy) \leq \|x^*x\| f(y^*y) .$$

This is a strengthened form of (2).

(b) Suppose f is continuous. In (1) replace x by $(x^*x)^{2^i}y$:

$$(5) \qquad f[y^*(x^*x)^{2^i}y]^2 \leq f(y^*y)f[y^*(x^*x)^{2^{i+1}}y] .$$

Extract the 2^{i+1}-th root in (5), and multiply the results for i = 0, 1, ..., n - 1. We obtain

$$(6) \qquad f(y^*x^*xy) \leq f(y^*y)^{1-2^{-n}} f[y^*(x^*x)^{2^n}y]^{2^{-n}} .$$

By the continuity of f,

$$(7) \qquad f|y^*(x^*x)^{2^n}y| \leq \|f\|\|y^*\|\|y\|\|(x^*x)\|^{2^n} .$$

On applying (7) to (6) and passing to the limit we deduce

$$(8) \qquad f(y^*x^*xy) \leq r(x^*x)f(y^*y) .$$

This is an improvement on (4) .

In the event that A has a unit element and * is continuous, f is automatically continuous, for by replacing y by 1 in (1) and (4) we get

$$|f(x)|^2 \leq f(1)^2 \|x^*x\| \quad . \tag{9}$$

Moreover now that we know f to be continuous, we can repeat our argument using (8) instead of (4). The result we obtain is an improvement of (9):

$$|f(x)|^2 \leq f(1)^2 r(x^*x) \quad . \tag{10}$$

The continuity of a state f can be proved for suitably well behaved algebras even if they lack a unit element. To prove this we begin with the observation that it is sufficient to show that f is bounded on the positive elements of norm 1. Suppose on the contrary that $f(x_i) \rightarrow \infty$, where $x_i = 0$, $\|x_i\| = 1$. We can suppose that $f(x_i) > 2^i$. Write $y = \sum 2^{-i} x_i$. The difference between y and any partial sum is non-negative; hence $f(y) > i$ for every i, a contradiction.

By a *-representation of a *-algebra A we shall mean a *-preserving homomorphism of A into the algebra of bounded operators on a Hilbert space. A *-representation is called <u>cyclic</u> if there exists a vector of the Hilbert space whose transforms are dense. If A has a unit element, then the *-representation induced by a state is cyclic: it turns out that the image of 1 is the desired cyclic vector. If A is a C*-algebra*, Segal[22] gets a cyclic vector even without a unit

* A uniformly closed self-adjoint algebra of operators on Hilbert space.

element, by making use of the fact that any C*-algebra has an "approximate unit"* in a rather strong sense. The connection between states and representations is completed by the remark that any *-representation is a direct sum of cyclic representations and a trivial** representation.

The next step in the theory is to consider the set of states of norm 1. These form a convex*** set, and the extreme points are precisely those which give rise to *-representations which are irreducible in the sense of admitting no closed invariant subspaces. The Krein-Millman**** theorem then assures us that any *-algebra, which possesses a faithful *-representation, has a <u>complete set of irreducible</u> *-<u>representations</u>*****.

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* An approximate unit of a C*-algebra A is a directed system V_u of elements of A such that $\|V_u\| \leq 1$ and $\lim_u V_u U = U$ for every U in A.

** A representation is trivial if it sends every element into the zero operator.

*** A convex set is one which, if it contains the points x and y, contains the segment xy.

**** The Krein-Millman theorem asserts that a convex set, in the conjugate space of a Banach space, which is compact in the weak (neighborhood) topology, is the weak compactification of the convex core of its extreme points, an extreme point being one which is not an inner point of a segment contained in the set.

***** A collection of representations of A is called complete if no element of A except zero is mapped into zero by every representation in the collection.

<u>Remark</u>. The results presented in the above section can be given a quantum mechanical interpretation. They grew out of investigations which tried to give a broader and more rigorous treatment of certain parts of quantum mechanics, particularly of the principle that the spectral values of an observable are determined by the behaviour of the observable in the irreducible representations of the operator algebra describing the physical system in question. The original mathematical model for the observables in quantum theory was the class of all self-adjoint elements of the algebra C of all bounded operators on a Hilbert space. However, it seemed desirable to consider the collection of self-adjoint elements in less restricted algebras of operators as a possible model for the observables, the most notable reason being: (a) the lack of a physical reason why every self-adjoint operator should correspond to an observable, (b) the serious mathematical difficulties in quantum electrodynamics.

The concept of a state was introduced by von Neumann and Weyl (independently) in a quite general fashion and in the case of the algebra C just mentioned it was shown that an element of the Hilbert space of norm unity (a wave function) gave rise to a pure state*. In more general systems the question of the existence of pure states was left open, but we should mention that the proof of this existence is now based on the Krein-Millman theorem which was proved only fairly recently.

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* A pure state is one which is not a linear combination with positive coefficients of two other states.

3. C*-algebras

Following the terminology of Segal[22], we define a C*-algebra to be a uniformly closed self-adjoint algebra of operators on Hilbert space*. One may give an equivalent intrinsic set of axioms due to Gelfand and Neumark[6]: a C*-algebra is a *-algebra in which, for every x, $\|x^*x\| = \|x\|^2$. Gelfand and Neumark included the assumption of symmetry in the postulates for a C*-algebra but Fukamiya[3] showed this to be redundant.

We shall now discuss questions of structure theory, confining ourselves to C*-algebras. In the light of Jacobson's structure theory of rings, we set forth the problem as follows. Let A be a C*-algebra, $\{P_i\}$ its primitive ideals**. To know the structure of A, we have to know: (a) the structure of the primitive C*-algebras A/P_i, (b) how these algebras combine to give A.

It is instructive to review the commutative case. Here each A/P is simply the field of complex numbers, and the way they combine is fully described by saying that we get all continuous complex functions vanishing at infinity on the space of maximal ideals.

In attempting to generalize to the non-commutative case, the first step is to select the analogue of the space of maximal

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* The Hilbert space is not necessarily separable.

** A (two-sided) ideal P in a ring A is called primitive if 0 is the only $x \in A$ such that $Ax \subset P$.

ideals. An appropriate selection is the structure space of Jacobson[10], i.e., the space of primitive ideals topologized (after Stone) by making the closure of the set $\{P_i\}$ of primitive ideals the totality of all primitive ideals containing $\bigcap P_i$. Next we consider the possibility of a representation by continuous functions. For an element x in A and a primitive ideal P, we may form the image x(P) of A mod P. Naturally, it is not promising to speak of the continuity of x(P), for we cannot compare elements in the unrelated algebras A/P. Instead of x(P) we consider $\|x(P)\|$; this is a real-valued function on the structure space X whose continuity is perfectly meaningful. It turns out, as we will illustrate in the following that, the functions $\|x(P)\|$ are continuous if and only if X is a Hausdorff space.

This focuses attention on the problem of determining reasonable conditions which will assure us that X is a Hausdorff space. This problem is taken up by Kaplansky[11] in the case where each A/P is known to be the algebra of all completely continuous operators** on a Hilbert space. Such an algebra is called a CCR-algebra ("completely continuous representations"). The main structure theorem asserts that any CCR-algebra possesses a composition series* I_r such that each I_{r+1}/I_r has a Hausdorff structure space.

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* A well ordered ascending chain I_r of closed two-sided ideals.

** An operator on a Hilbert space is completely continuous if it is the limit in the uniform topology of operators with finite dimensional range.

Lemma 3.1. Let X be a topological space at each point of which a C*-algebra A_x is given and let A be a self-adjoint algebra of functions from X to $\{A_x\}$, with $f(x) \in A_x$, satisfying the following two conditions: (a) $\|f(x)\|$ is continuous at 0, that is, if f(x) = 0, then for any $\varepsilon > 0$, there exists a neighborhood U of x such that $\|f(x)\| < \varepsilon$ for x in U.
(b) $\|f(x)\|$ is bounded, and A is complete under the norm $\|f\| = \sup\|f(x)\|$. Then for any self-adjoint element $f \in A$, the spectrum of f(x) is a continuous function of x in the following sense: for any $x \in X$ and $\varepsilon > 0$, there is a neighborhood U of x such that for all $y \in U$, the spectrum of f(y) is contained in an ε-neighborhood of the set consisting of 0 and the spectrum of f(x).

Proof. Write V for the set consisting of 0 and the spectrum of f(x), and W for an ε-neighborhood of it. Let p be a continuous real-valued function which vanishes on V and is equal to 1 on the complement of W. Then g = p(f) vanishes at x, and so by the continuity of the norm, $\|g(y)\| < 1$ for y in a suitable neighborhood U of x. For y in U the spectrum of g(y) must lie in W.

We now quote the following useful result[11)].

Lemma 3.2. Let A be any ring and B either a two-sided in A, a subring of the form eAe, or a subring of the form (1 - e)A(1 - e), e being an idempotent. Then there is a one-one correspondence between the primitive ideals of B and those primitive ideals of A containing B. The mapping is implemented by $P \to P \cap B$, P primitive in A, and it is a homeomorphism in the topologies of the structure spaces of A and B.

In the context of C*-algebras, the use of the Jacobson structure space is perhaps open to suspicion on two grounds. (1) A primitive ideal is the kernel of a purely algebraic irreducible *-representation. For C*-algebras one naturally prefers to use irreducible *-representations. Since in the latter case irreducibility means the absence of closed invariant subspaces, the connection between the two concepts is not clear. However, in one direction we can clear up the ambiguity: any primitive ideal P is also the kernel of an irreducible *-representation. To see this, we note that there is a regular maximal ideal M such that P is the kernel of the natural representation on A/M. Now in the terminology of Segal[22)], there exists a state* vanishing on M. By an appropriate application of the Krein-Millman** theorem we can get further a pure state*** w vanishing on M. The *-representation attached to w is irreducible and has P as kernel. Whether it is true conversely that the kernel of an irreducible *-representation is primitive is an open question; but in any event the structure of Jacobson is the smaller of the two spaces, and for many purposes this more or less justifies its use.

(2) For commutative Banach algebras, it has been found that the structure space has unsatisfactory properties, and that the right topology is the weak topology introduced by Gelfand. However, for commutative C*-algebras, the two are known to

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* A state is a complex-valued linear functional f such that f(x*) is the complex conjugate of f(x).

** For a statement of the Krein-Millman theorem, see page 32.

*** For the definition of a pure state, see the footnote on pg.33.

coincide. The next three results will indicate that for arbitrary C*-algebras, the Jacobson structure space is reasonably well behaved.

Having selected the structure space X of primitive ideals $\{P_i\}$ in A, we may represent an arbitrary element x of A by the set $\{x_i\}$ of its images in the C*-algebras $\{A/P_i\}$. We shall write x(P) for the value of x at the point P of X. Utilizing a theorem of Kaplansky[11']), we can say that this functional representation preserves norm; that is, we have $\|x\| = \sup\|x(P)\|$, taken over P in X. However, in this context where X is the structure space, even more is true: the sup is attained for some P in X. Because of the identity $\|x^*x\| = \|x\|^2$, one needs to prove this only in the case $x \geq 0$, and we may assume $\|x\| = 1$. To say that $\|x(P)\|$ is less than 1 is to say that the spectrum of x(P) does not contain 1; that is, that $-x$ has a quasi-inverse modulo P. If this is true for every P, then $-x$ has a quasi-inverse modulo every primitive ideal. It is known that this implies that $-x$ has a quasi-inverse in A itself, contradicting $\|x\| = 1$.

Lemma 3.3. Let x be a self-adjoint element of a C*-algebra with structure space X. Let E be a closed set of real numbers containing 0. Then the set Z of $P \in X$, such that the spectrum of x(P) is contained in E, is a closed subset of X.

Proof. Suppose that Q is in the closure of Z, and x(Q) has a in its spectrum, $a \notin E$. Let p be a continuous real-valued function vanishing on E but not a. Then p(x) vanishes on Z but not at Q, contradicting the definition of the topology of X. Q.E.D.

The next result indicates that it is quite generally true that the functions on the structure space "vanish at infinity".

Lemma 3.4. Let x be any element of a C*-algebra with structure space X, and ε a positive number. Then the set K of $P \in X$ for which $\|x(P)\| \geq \varepsilon$ is a compact subset of X.

Proof. Because of the identity $\|x^*x\| = \|x\|^2$, we need consider only the case where x is self-adjoint. Let $\{F_j\}$ be a family of (relatively) closed subsets of K having void intersection. We must prove that a finite subset of the F's already have void intersection. Let I_j be the intersection of the primitive ideals comprising F_j, H_o the ideal generated by the I_j's (in the purely algebraic sense), and H the closure of H_o. We observe that H is not contained in any of the primitive ideals comprising K. For if $H \subset R$, $R \in K$, then R contains each I_j. By the definition of the topology of the structure space, we see that R lies in each F_j, a contradiction. Next we remark that A/H is semi-simple (indeed a C*-algebra). Hence, H is the intersection of the primitive ideals containing it. We write L for this set of primitive ideals, and observe (as just shown) that L is disjoint from K. Hence, for every Q in L, we have $\|x(Q)\| < \varepsilon$. Write $r = \sup\|x(Q)\|$, taken over Q in L. If we write x_1 for the image of x mod H, we see that r is precisely the sup of the norms of all the images of x_1 at primitive ideals of A/H. As we remarked above, such a sup is actually attained; in other words we have $r = \|x(Q_o)\|$ for a suitable Q_o in L. Hence r is itself less than ε. Let p(t) be a continuous real-valued function of the real variable t, which vanishes for $|t| \leq r$, equals 2 for $|t| \geq \varepsilon$, and is linear between;

write $z = p(x)$. Then z is in H since it vanishes on L, and $\|z(P)\| = 2$ for P in K. Since H_0 is dense in H, there exists an element y in H_0 with $\|y - z\| < 1$. The element y must already lie in the union of a finite number of I's, say $I_1, \ldots, I_r$. Then $F_1 \cap \cdots \cap F_r$ must be void; for at any primitive ideal in this intersection y would vanish, whereas $\|y(P)\| \geq 1$ throughout K.

The question as to when the functional representation on the structure space gives functions with continuous norm is completely answered in the following theorem.

Theorem 3.2. For any C*-algebra A with structure space X, the following statements are equivalent: (1) X is Hausdorff, (2) for any $x \in A$, the function $\|x(P)\|$ is continuous on X.

Proof. That (2) implies (1) is immediate. If Q, $R \in X$, $Q \neq R$, then there exists $x \in A$ vanishing say at Q, but not at R. The continuous real-valued function $\|x(P)\|$ yields disjoint neighborhoods of Q, R.

To prove that (1) implies (2), we begin by investigating continuity of $\|x(P)\|$ at 0. Take a fixed $Q \in X$, and let I be the closure of the set of all $x \in A$ such that x(P) vanishes in a neighborhood of Q; I is a closed two-sided ideal in A. We claim that I = Q. If not, since I is the intersection of the primitive ideals containing it, I will also be contained in a different primitive ideal Q_0. By the Hausdorff separation property, there exists a neighborhood U of Q whose closure does not contain Q_0. By the definition of the topology of X, A contains an element vanishing on U but not at Q_0, and this is a contradiction. At this point we know that anything vanishing at Q is a limit of elements vanishing in a neighborhood of Q; this proves continuity of $\|x(P)\|$ at 0.

We now pass to the general proof of continuity. Because of the equation $\|x^*x\| = \|x\|^2$, it is enough to do this for self-adjoint x. Let $Q \in X$ and $\varepsilon > 0$ be given, and write $r = \|x(Q)\|$. It follows from Lemma 3.1 that in a suitable neighborhood of Q, $\|x(P)\| < r + \varepsilon$. To complete the proof it will suffice to show that the set of P with $\|x(P)\| > r - \varepsilon$ is open; or alternatively, that the set of P with $\|x(P)\| \leq r - \varepsilon$ is closed. This is a consequence of Lemma 3.3 [11)].

<u>Remark</u>. If we combine Lemma 3.4 and Theorem 3.2, we see that when the structure space is Hausdorff it is also locally compact.

In the following theorem we treat the fundamental case in which we are able to prove that the structure space is Hausdorff.

<u>Theorem 3.3</u>. Let A be a C*-algebra in which for every primitive ideal P, A/P is finite dimensional and of order independent of P. Then the structure space of A is Hausdorff.

<u>Proof</u>. We shall not prove this by considering the structure space directly; instead, following the idea of Kaplansky[11')], we provisionally introduce another space. Let M be the finite-dimensional C*-algebra (a full matrix algebra) to which each A/P is isomorphic. Let Y be the space of all *-homomorphisms of A into M, including the 0 homomorphism. In the weak topology, Y is a compact Hausdorff space, and the elements of A are represented by continuous functions from Y to M. Each primitive

ideal in A gives rise to an orbit of points in Y, the orbit being in fact induced by the group G of *-automorphisms of M. Now G is compact in its natural topology, and the mapping from G onto an orbit is readily seen to be continuous. Hence the orbits are closed, and we may form a well defined quotient space X relative to this decomposition of Y. The points of X are of course in one-one correspondence with the set consisting of the structure space of A and a point at infinity. Being a continuous image of Y, X is again compact. We can no longer speak of elements of A as being represented by continuous functions x(a) on X; but the function $\|x(c)\|$ is constant on orbits, and so gives us a uniquely defined function on X, which is again continuous. Moreover , these functions $\|x(a)\|$ exist in sufficient abundance to separate points of X, for given two distinct points of X, we can find an element $x \in A$ vanishing at one but not at the other. From this it follows that X is Hausdorff. Q.E.D.

In order to see how the above considerations fit into a more general framework, we make a few remarks and consider only the homogeneous case.* Without the aid of a well behaved *-operation, it seems to be difficult to construct a satisfactory theory; in general terms one may trace the trouble to the fact that the group of automorphisms of M is not compact. Hence , let us assume that A has a continuous *-operation which is symmetric. Then the construction of the above space Y, and its reduction to X, go through; this appears to be a satisfactory beginning for the

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* A Banach algebra A such that all A/P are isomorphic to a fixed finite-dimensional matrix algebra M.

theory. Still, one is troubled by the fact that there are three further possible choices for a space of primitive ideals: (1) the structure space, (2) primitive ideals with the weak topology induced by traces of elements, (3) the space of maximal ideals (in Gelfand's sense) of the center Z of A. If A is a C*-algebra, it is known that all four versions coincide, but the general situation is not clear. In particular, the choice (3) is in jeopardy, since perhaps Z = 0. The case of degree two is an exception, since then $(xy - yx)^2$ is always in the center.

From the homogeneous case treated in Theorem 3.3, we pass on to the case where A is a C*-algebra such that each A/P is finite-dimensional with a fixed upper bound on the order. This hypothesis can be described more briefly by saying that A satisfies a polynomial identity. In order to clarify our further considerations, we make a few general remarks at this point. Consider the algebra of n by n matrices over a field. In the notation of Kaplansky, this satisfies a polynomial identity

$$\left[x_1, \ldots\ldots, x_{r(n)}\right] = 0 ,$$

where r(n) is a certain function of n, and matrix algebras of higher order do not satisfy this identity. Amitsur and Levitzki have shown that r(n) = 2n, but for our purposes, all that matters is that some identity shall exist that characterizes n by n matrices. Now let A be any Banach algebra; let C_n denote the set of all primitive ideals P such that A/P is a k by k matrix algebra with $k \leq n$, and let I_n be the intersection of these ideals. Then A/I_n satisfies the above polynomial identity, and so does every primitive image of A/I_n. It follows that C_n is a closed subset of the structure space. Moreover, it follows from Lemma 3.2 that I_{r-1}/I_r is homogeneous: each of its primitive images is an r by r matrix algebra.

Let A be any C*-algebra satisfying a polynomial identity. The above defined chain of ideals I_n reaches 0 in a finite number of steps. We have thus constructed a finite composition series for A, with the property that every factor algebra has a Hausdorff structure space.

Composition Series. By a composition series of a C*-algebra A, we mean a well-ordered ascending series of closed two-sided ideals I_s, beginning with 0 and ending with A, and such that for any limit ordinal u, I_u is the closure of the union of the preceding I's. The use of an ascending series here, as opposed to a descending series, is typical of ring theory, and is analogous to the superiority of minimal over maximal ideals.

Theorem 3.4. Let A be a C*-algebra such that every primitive image A/P is finite-dimensional. Then A has a composition series I_s such that each I_{s+1}/I_s satisfies a polynomial identity.

Proof. In the structure space X of A, let C_n be the set of primitive ideals P such that A/P has degree not greater than n. Then X is the union of the countable family of closed sets C_n, and hence, one of them, say C_r, must have a nonvoid interior U. Let I denote the intersection of the primitive ideals comprising the complement of U. Since the latter is closed, these are precisely the primitive ideals containing I; in particular we see that I is nonzero. By Lemma 3.2, the primitive ideals in I itself are in one-one correspondence with the members of U. It follows that the primitive images of I are all of degree not greater than r. Hence I satisfies a polynomial identity: to be precise, the identity for r by r matrices. This is the beginning of our composition series. The algebra A/I again satisfies the hypothesis of our theorem (any primitive image of A/I is a primitive image of A), and we continue by transfinite induction.

As we observed above, a C*-algebra B with a polynomial identity has a (finite) composition series such that all factor algebras possess a Hausdorff structure space. In particular the first nonzero ideal in this series has a Hausdorff structure space, and by Lemma 3.2 the latter is homeomorphic to an open subset of the structure space of B. If we combine this with Theorem 3.4, and another application of Lemma 3.2, we obtain the following corollary.

Corollary. If A is a C*-algebra such that every primitive image A/P is finite-dimensional, then the structure space of A has a nonvoid open Hausdorff subset.

From a certain point of view, the study of C*-algebras may be divided into two parts: the determination of the primitive ones, and the study of how the latter combine to form a general C*-algebra. We now pick out a class of C*-algebras for which the first problem evaporates, and so attention is concentrated on the second.

Definition. A CCR-algebra is a C*-algebra for which every primitive homomorphic image A/P is isomorphic to the algebra of all completely continuous operators* on a Hilbert space.

We now prove our main structure theorem.

Theorem 3.5. A CCR-algebra has a composition series I_s such that each I_{s+1}/I_s has a Hausdorff structure space.

Proof. Let A be the algebra and X its structure space. Select a self-adjoint element x in A whose spectrum lies in (0, 1) and actually contains 1. At any $P \in X$, the spectrum of x(P) is a finite or countable set with at most 0 as a limit point. Let p(t) be a continuous real-valued function, vanishing in a neighborhood of 0, satisfying p(1) = 1, and, say, linear between. We pass to y = p(x), and observe that every y(P) has a finite spectrum lying between 0 and 1. Let B denote the intersection of the primitive ideals containing y, or in other words, the set of P with y(P) = 0; let Y be the structure space of B. Lemma 3.2 shows that B is again CCR, and that Y is in a natural way an open subset of X. We will now show that Y has an open Hausdorff subset.

For this purpose we first note that $y(Q) \neq 0$ for any $Q \in Y$. For n = 2, 3, ..., let C_n denote the set of $Q \in Y$ for which the spectrum of y(Q) lies in the closed set consisting of 0 and the closed interval from 1/n to 1. By Lemma 3.3 C_n is closed. Also, since the spectrum of each y(Q) is finite, $Y = \bigcup C_n$. Hence one of the C's, say C_r, has a nonvoid interior U. Let q(t) be a

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* See the footnote on page 35.

continuous real-valued function satisfying $q(0) = 0$, $q(t) = 1$ for $t \geq 1/r$, and write $z = q(y)$. Then at every point of C_r, z maps into a nonzero self-adjoint idempotent, and this is a fortiori true at all points of the closure V of U. Let J be the intersection of the primitive ideals in B which comprise V. Let $D = B/J$; the structure space of D can be identified with V. Let e denote the homomorphic image of z mod J; then e is a self-adjoint idempotent not vanishing at any point of V. We consider finally the algebra eDe; by Lemma 3.2 its structure space is again homeomorphic to V. Now it follows from Lemma 3.2 again that the primitive ideals in eDe are of the form $R \cap eDe = eRe$, where R is primitive in D. Thus the primitive homomorphic images of eDe are of the form $e_1(D/R)e_1$, e_1 being the image of e mod R. We know that D/R is the algebra of all completely continuous operators on a Hilbert space. It follows that $e_1(D/R)e_1$ is finite-dimensional. In short, all the primitive images of eDe are finite-dimensional. The corollary of Theorem 3.4 is therefore applicable, and tells us that V has a nonvoid open Hausdorff subset, say Z. The intersection T of Z and U is a nonvoid open Hausdorff subset of Y. The same set T is open in X (since Y is open in X). Let I be the intersection of the primitive ideals comprising the complement of T. Then I is a nonzero closed two-sided ideal in A whose structure space is homeomorphic to the Hausdorff space T (Lemma 3.2 is being used again). This is the beginning of our composition series; we continue with a similar treatment of A/I, and so on by transfinite induction,[11)].

Our study of CCR-algebras now stands as follows. We have established the existence of a composition series such that all the factor algebras have a Hausdorff structure space. Next, if A is a CCR-algebra with a Hausdorff structure space X, then Lemma 3.4 and Theorem 3.2 show that X is locally compact, and that the representation of A on X gives us functions with continuous norm vanishing at infinity.

4. W*-algebras

Some tentative investigations indicate that the prospect is not very bright for developing the theory just described beyond CCR-algebras. However, this outlook changes if we are willing to strengthen the assumption of uniform closure to that of weak closure. We define a W*-algebra to be a weakly closed self-adjoint algebra of operators on a Hilbert space.
For W*-algebras there is a highly developed structure theory, due to Murray and von Neumann[14, 15, 16].

Murray and von Neumann considered only the separable Hilbert space, and concerned themselves largely with factors*. However, neither the results nor the methods change very much if one drops both these restrictions. We shall proceed to briefly describe the structure of an arbitrary W*-algebra.

The dominating role is played by the projections**. We call a projection e abelian if eAe is commutative; call e finite if right and left inverses coincide in the algebra eAe. We say that A is of type I if it is generated by abelian projections,

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* Factors are algebras whose center is the complex numbers.

** A projection is a self-adjoint idempotent.

of type II if it contains no abelian projections but is generated by finite projections, of type III if it contains no finite projections. A known result is that, any W*-algebra is a unique direct sum of algebras of types I, II and III.

Algebras of type I are completely known. By the methods of multiplicity theory, one decomposes the algebra into direct summands, one for each cardinal number, which are uniform in a suitable sense. A uniform algebra attached to the cardinal K can be, in a sense, described as an K by K total matrix algebra over a commutative algebra.

About W*-algebras of type III little is known except that they exist, and that in the non-separable case a cardinal-valued dimension theory can be constructed.

There remains the most interesting case of algebras of type II. In studying these, the first step is to single out those which are finite.* In accordance with the terminology of Murray and von Neumann, we say that such an algebra is of type II_1. An arbitrary algebra of type II decomposes uniformly into uniform parts, a typical uniform summand being a K by K matrix algebra over an algebra of type II_1. Thus the study of algebras of type II can be reduced essentially to the case of II_1, and one can furthermore reduce to the case where the commuter is of type II_L.

The most important fact about an algebra of type II_1 is the existence of a trace. For algebras that are not necessarily

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* Those for which the unit element is finite, as defined above.

factors, the trace is a linear function $x \longrightarrow T(x)$ from the algebra to the center, which is the identity on the center, is positive,* and satisfies $T(xy) = T(yx)$. The proof of the existence of the trace, even after many simplifications, is still quite elaborate; it seems that it would be most desirable to have a simpler proof.

As regards the structure theory of algebras of type II_1, the known facts are as follows. Murray and von Neumann[16)] define a factor of type II_1 to be <u>approximately finite</u> if it is the weak closure of the union of an ascending chain of finite-dimensional algebras. They show that all approximately finite factos are isomorphic, and that there exist factors that are not approximately finite. Thus, at the present writing, it is not known whether there exist more than two non-isomorphic factors of type II_1.

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* $x \geq 0$ implies $T(x) \geq 0$.

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