Ekpyrotic cosmology from classical fields to M-theory

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April 2020

A thesis submitted to McGill University in partial fulfillment of the requirements of the degree of

Master of Science

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Abstract

We provide a comprehensive review of ekpyrotic cosmology, review the basic construction of inflation and compare the mechanisms in which each universe addresses well known problems of standard big bang cosmology, as well as others. We calculate the resultant Gaussian cosmological scalar curvature ζ, generalized Newtonian potential Φ as well as tensor perturbations generated in various ekpyrotic phases of contraction. Regardless of the number of matter fields, we show that the prediction on the tensor spectrum is strongly blue, making the prediction distinct from inflation. We show that single matter field ekpyrotic models generically predict an approximately scale invariant spectrum for Φ and strongly blue spectrum for ζ. Once generalized to two canonically normalized ekpyrotic fields (negative approximately exponential potentials), we show that the generic prediction is the generation of approximately scale invariant entropy perturbations. We describe generally how entropy perturbations in multi-scalar field models may be converted to curvature perturbations and study a specific mechanism. Further we provide a complete description of the covariant approach to cosmological perturbations to second order in gauge invariant entropy and curvature perturbation variables, which we utilize in future work to study non-Gaussianities. We motivate the ekpyrotic model in the context of the strongly coupled limit of $E_8 \times E_8$ heterotic string theory, namely eleven-dimensional supergravity on a manifold with boundary and derive the five dimensional heterotic M-theory action for studying the evolution of the moduli fields acting as ekpyrotic fields. We describe evidence for ekpyrotic-type potentials for moduli arising from the compactification by considering non-perturbative effects from open supermembranes extending between wrapped M-branes. Lastly, we provide an alternative embedding for an ekpyrotic phase of contraction in F-theory compactified on a Calabi-Yau fourfold, more precisely described as a warped IIB supergravity compactification on a Calabi-Yau threefold in the presence of nontrivial R-R and NS-NS three-form fluxes, with all moduli fields stabilized.

Résumé

Nous fournissons une revue complète de la cosmologie ekpyrotique, passons en revue la construction de base de l'inflation et comparons les mécanismes dans lesquels chaque univers aborde les problèmes bien connus de la cosmologie standard du big bang, ainsi que d'autres. Nous calculons la courbure scalaire cosmologique gaussienne résultante ζ, le potentiel Newtonian généralisé Φ ainsi que les perturbations tensorielles générées dans diverses phases de contraction ekpyrotic. Quel que soit le nombre de champs de matière, nous montrons que la prédiction sur le spectre du tenseur est fortement bleue, ce qui rend la prédiction distincte de l'inflation. Nous montrons que les modèles ekpyrotiques de champ de matière unique prédisent de manière générique un spectre invariant à l'échelle approximative pour Φ et un spectre fortement bleu pour ζ. Une fois généralisés à deux champs ekpyrotiques canoniquement normalisés (potentiels approximativement exponentiels négatifs), nous montrons que la prédiction générique est la génération de perturbations d'entropie invariantes approximativement à l'échelle. Nous décrivons généralement comment les perturbations d'entropie dans les modèles de champ multi-scalaires peuvent être converties en perturbations de courbure et étudions un mécanisme spécifique. En outre, nous fournissons une description complète de l'approche covariante des perturbations cosmologiques aux variables de perturbation d'entropie et de courbure invariantes de jauge, que nous utiliserons dans les travaux futurs pour étudier les non-gaussianités. Nous motivons le modèle ekpyrotique dans le contexte de la limite fortement couplée de $E_8 \times E_8$ théorie des cordes hétérotiques, à savoir la supergravité à onze dimensions sur une variété avec frontière et dérivons l'action de la théorie M hétérotique à cinq dimensions pour étudier l'évolution de la champs de modules agissant comme des champs ekpyrotiques. Nous décrivons des preuves de potentiels de type ekpyrotic pour les modules résultant de la compactification en considérant les effets non perturbatifs des supermembranes ouvertes s'étendant entre les M-branes enveloppés. Enfin, nous proposons une intégration alternative pour une phase de contraction ekpyrotique en théorie F compactifiée sur un Calabi-Yau quadruple, plus précisément décrite comme une compactification de supergravité IIB déformée sur un Calabi-Yau triple en présence de RR et NS-NS non triviaux flux à trois formes, avec tous les champs de modules stabilisés.

Contents

List of Figures

Introduction

Inflation is a particularly compelling theory of the early universe, but we would be remiss by inflating our confidence at this point in time, that it is the true history of our universe. Other models are worth studying, because we may show that their predictions are also consistent with observations. As observational signatures begin to lay more stringent boundaries on the landscape of previously acceptable four dimensional field theory models, perhaps at some point we will be able to rule out so many that we are left with only one. Personally, I find this approach rather uninteresting.

Thankfully for me though, there is another approach. Continued observations performed at the quantum level will improve our understanding of a more fundamental theory of particles and perhaps with luck a consistent and predictive theory of quantum gravity will emerge. Up until this point, the theory of supersymmetric strings is a self consistent candidate for such a theory predicting various collections of supersymmetric fermions and bosons; but up to the energy scales probed with modern particle accelerators or high energy particles incoming from the cosmos we have yet to observe any of them. Regardless of whether or not a theory of quantized superstrings is the 'correct' theory, we can learn a great deal from studying it.

Encouragingly, with an understanding of topological structures, differential geometry and quantum mechanics in higher dimensional spaces we can actually perform calculations with closed form solutions. Scattering amplitudes may be calculated in a controlled perturbative expansion in the string coupling and interactions between dynamical stringy objects known as Dp-branes may also be studied in various limits of critical superstring theories. As we will see, these limits of superstring theories provide a playground where we may hope to 'derive' or at the very least motivate a particular cosmological scenario from a top-down perspective. This has been plenty explored for inflation, with a major challenge being the generation of stable or at least long lived De Sitter vacua in superstring theory. This work attempts to rectify that not only is there another cosmologically viable candidate theory in terms of perturbations of classical gravity minimally coupled to a quantized field, but also as a realization within the theory of superstrings.

This candidate theory is known as ekpyrotic cosmology, and serves as an alternative to the inflationary universe paradigm. It is fundamentally different from that of inflation in so far as the universe undergoes a phase of very slow contraction, followed by a 'bounce' and subsequent radiation dominated phase of Friedmann expansion. Section 1 begins with a review of standard big bang cosmology and briefly describes some of the standard problems which theories of the very early universe attempt to address. Section 2 provides a review of generic inflationary cosmology, outlining how the theory addresses the problems of standard big bang cosmology, as well as describes some of the new sets of problems inflation is faced with. Section 3 provides a review of ekpyrotic cosmology in a four dimensional field theoretic model minimally coupled to Einstein gravity, for both single and multi-field models. We also describe how it addresses the problems of standard big bang cosmology as we have done for inflation. Section 4 introduces the coordinate based approach to cosmological perturbations to linear order. We describe the generation of curvature, entropy and tensor perturbations in single and two-field descriptions, discuss the predictions upon each mode and discuss how entropy perturbations may be converted to curvature perturbations in two-field models. We also explore a specific mechanism for the transfer of fluctuations in entropy to curvature. Section 5 introduces the covariant description of cosmological perturbations, which will prove useful in future numerical calculations to explore the generation of non-Gaussianities. Lastly, section 6 motivates the application of string theory to cosmology, where we describe the original ekpyrotic model within the context of the strongly coupled limit of $E_8 \times E_8$ heterotic superstring theory compactified on a warped Calabi-Yau threefold. We also provide a new realization for the model within the context of F-theory compactified on a warped Calabi-Yau fourfold.

This thesis expresses equations in Planck units ($G = c = \hbar = k_B \equiv 1$) unless stated otherwise, the Einstein summation convention is always implied, and within the four dimensional field theory models utilizes the 'mostly minus' signature of the metric $(+, -, -, -)$. Greek characters $\mu, \nu, \rho \cdots \in \{0, 1, 2, 3\}$ generally refer to the four dimensional spacetime coordinates, with 0 denoting the timelike coordinate and $1, 2, 3$ denoting the spacelike coordinates; lowercase Latin characters run solely over the spatial components. The conventions used in string models have been defined case by case in section 6.

1 Standard big bang cosmology

1.1 Friedmann-Lemaître-Robertson-Walker cosmology

Standard big bang cosmology as presented in modern literature is constructed using Einstein's general theory of relativity. General relativity is a vastly improved approximation over Newtonian gravity particularly when describing relativistic, massive dynamical bodies, demonstrating remarkable accuracy and predictive power regarding the evolution of cosmic phenomena such as: stellar mass orbits and the precession of their perihelia [1], galactic evolution, gravitational waves, black holes, gravitational lensing, dark matter, topological defects such as cosmic strings, domain walls and monopoles, and most relevant to this thesis the topological structure of spacetime on cosmological distance scales ($\sim$ 100 Mpc[1]). On cosmological distance scales gravity dominates all other forces: strong, weak and electromagnetic, and general relativity is the effective theory. Furthermore when concepts are generalized and become abstract particularly in the study of quantum gravity, differential geometry is robust both in its calculability and in the study of physical symmetries. Imposition of a physical symmetry or the interpretation of manifested symmetries as physical laws is tractable and through Noether's theorem leads to conserved quantities and charged objects, to name a few: gauge invariance, Poincaré invariance, diffeomorphism invariance, Weyl/conformal invariance, CPT symmetry, BRST invariance and supersymmetry. In a deep respect, physics is the study of symmetry.

Both historical and modern cosmological models describe matter on cosmological scales as a perfect fluid whose properties are described by an equation of state, relating its mass density to its isotropic pressure. This description of matter and energy is motivated in large part by the *cosmological principle*: when considering cosmological distance scales the universe is *spatially* homogeneous and isotropic. Before the end of the twentieth century, the cosmological principle was a postulate allowing one to make simplifications to a complex dynamical framework. Near the dawn of the twenty first century, the cosmological principle accumulated favourable scientific evidence via improved astrophysical experiments and is

[1] $1\ Mpc \approx 3.2616\ ly \approx\ 3.0857 \times 10^{16}\ m$

observed most convincingly in the cosmic microwave background (CMB) radiation displayed in figure 1.

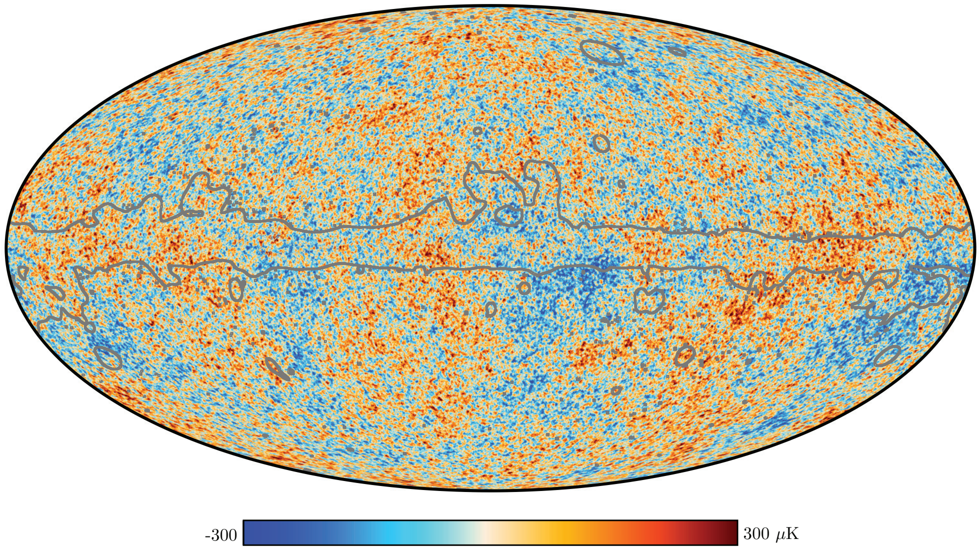

Figure 1: The intensity (temperature) anisotropy map of the cosmic microwave background radiation observed about the isotropic 3 Kelvin background obtained by the Planck collaboration [2]. Color expresses temperature in micro-Kelvin, making evident the high degree of homogeneity, yet distinct presence of small scale inhomogeneities, in the background radiation emitted when the universe cooled enough to become transparent to photons. The regions outlined by the gray lines indicate the regions where foreground emissions are expected to be substantial, mostly in the region of our galactic plane.

Isotropy implies that at all points in the manifold the geometry does not depend on the direction, that is the space looks the same in all directions. Homogeneity implies that the metric is equivalent at all points on the manifold. Spatial homogeneity and isotropy of the universe allows one to express its evolution as a time ordered foliation of three-dimensional space-like hypersurfaces each of which are homogeneous and isotropic. The homogeneity and isotropy of space implies that the space is maximally symmetric, in particular, the space possesses constant Ricci curvature, and possesses the maximum possible number of Killing vectors, for $n = 3$ we have $\frac{1}{2}n(n+1) = 6$ killing vectors: three independent translations and three rotations. Thus we consider our spacetime to be of topology $\mathcal{R} \times \Sigma$, where $\mathcal{R}$ represents the time axis, and Σ is a maximally symmetric three-manifold. The spacetime metric then takes the following form in $(+,-,-,-)$ signature

$$ds^2 = dt^2 - a^2(t)d\sigma^2, \tag{1.1}$$

where $a(t)$ is the scale factor describing the physical size of the manifold, t is the time-like coordinate and $d\sigma^2$ the spatial metric. Therefore the time evolution of the universe is completely described by the scale factor. With suitably scaled radial coordinates the metric on the three-surface Σ may be expressed as

$$d\sigma^2 = \frac{dr^2}{1-kr^2} + r^2 d\Omega^2, \tag{1.2}$$

$$d\Omega^2 \equiv d\theta^2 + \sin^2\theta d\varphi^2, \tag{1.3}$$

with $k \in [-1, 0, 1]$ parameterizing locally (to differentiate global differences such as between the torus and the plane) the three unique homogeneous and isotropic simply-connected three dimensional topological spaces: flat space with null curvature ($k = 0$), sphere of positive curvature ($k = +1$), and hyperbolic space of negative curvature ($k = -1$). In this normalization of the parameter k, the scale factor possesses units of distance, while the radial coordinate r is dimensionless. Therefore, the spacetime metric describing the evolution in size through time of one of the homogeneous and isotropic hypersurfaces may be written as

$$ds^2 = dt^2 - a^2(t)\left[\frac{dr^2}{1-kr^2} + r^2(d\theta^2 + \sin^2\theta d\varphi^2)\right]. \tag{1.4}$$

This is known as the Friedmann-Lemaître-Robertson-Walker metric. The spatial coordinates (r, θ, φ) introduced in equation (1.4) are comoving: every object with constant coordinates has zero peculiar velocity with respect to the expansion or contraction of the universe.

The dynamical variables describing the evolving universe are the components of the metric $g_{\mu\nu}(x^\gamma)$, and they are determined by the Einstein field equations

$$G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = \Lambda g_{\mu\nu} + 8\pi T_{\mu\nu}. \tag{1.5}$$

It is important to note that the cosmological term Λ may always be interpreted as the contribution of vacuum energy to the Einstein equations. Thus it is equally valid to set $\Lambda = 0$ above and include it instead in the energy momentum tensor $T_{\mu\nu}$. $R_{\mu\nu}$ is the Ricci tensor defined in terms of the Riemann curvature tensor $R^{\rho}{}_{\sigma\mu\nu}$ (where the Einstein Summation convention is utilized for repeated indices)

$$R_{\mu\nu} \equiv R^{\lambda}{}_{\mu\lambda\nu} = \partial_{\lambda}\Gamma^{\lambda}_{\nu\mu} - \partial_{\nu}\Gamma^{\lambda}_{\lambda\mu} + \Gamma^{\lambda}_{\lambda\sigma}\Gamma^{\sigma}_{\nu\mu} - \Gamma^{\lambda}_{\nu\sigma}\Gamma^{\sigma}_{\lambda\mu}, \tag{1.6}$$

which in turn is defined via the Christoffel symbols

$$\Gamma^{\sigma}_{\mu\nu} \equiv \frac{1}{2}g^{\sigma\rho}(\partial_{\mu}g_{\nu\rho} + \partial_{\nu}g_{\rho\mu} - \partial_{\rho}g_{\mu\nu}), \tag{1.7}$$

where the partials ∂_{μ} are derivatives with respect to the coordinates x^{μ}

$$\partial_{\mu}\phi \equiv \frac{\partial\phi}{\partial x^{\mu}}. \tag{1.8}$$

The Einstein field equations relate the scale factor $a(t)$ and its derivatives to the matter content via the symmetric rank two energy momentum tensor $T_{\mu\nu}$. The stress energy tensor is the conserved Noether current associated with spacetime translations, thus it obeys the following conservation law

$$\nabla_{\mu}T^{\mu\nu} \equiv \partial_{\mu}T^{\mu\nu} + \Gamma^{\mu}_{\mu\rho}T^{\rho\nu} + \Gamma^{\nu}_{\rho\sigma}T^{\rho\sigma} = 0, \tag{1.9}$$

where the covariant derivative is replaced by a partial derivative for Minkowski spacetime. As previously stated, on large scales we approximate matter and energy in the universe as a perfect fluid characterized by its energy density ε, pressure p and four-velocity U^{μ}. The four-velocity of the perfect fluid may always be expressed in a frame such that it is at rest in comoving coordinates, since the frame that describes the fluid as isotropic that leads to an isotropic metric must be the same as that which describes the metric as isotropic

$$U^{\mu} = (1, 0, 0, 0), \tag{1.10}$$

then the energy-momentum tensor may be expressed as

$$T^{\mu\nu} = (\varepsilon + p)U^\mu U^\nu - pg^{\mu\nu} = \mathrm{diag}(\varepsilon, p, p, p), \tag{1.11}$$

with its trace given by

$$T^\mu_{\ \mu} = \varepsilon - 3p. \tag{1.12}$$

The zero component of equation (1.9) along with the equation of state of matter expressed as

$$p = \omega\varepsilon, \tag{1.13}$$

with ω independent of time yields the conservation of energy equation (continuity equation)

$$\frac{\dot{\varepsilon}}{\varepsilon} = -3H(1+\omega), \tag{1.14}$$

where an overdot indicates derivatives with respect to the coordinate time t and where $H = H(t)$ is the Hubble parameter,

$$H = \frac{\dot{a}}{a}, \tag{1.15}$$

characterizing the rate of expansion of the universe. Equation (1.14) may be integrated to provide an expression for the energy density of matter in terms of the scale factor (for $\omega =$ constant)

$$\varepsilon \propto a^{-3(1+\omega)}. \tag{1.16}$$

The most common cosmological fluids are that of cold matter, radiation and vacuum energy. Each matter type are governed by their respective equation of state and equation (1.16) provides their energy density as they vary with the scale factor

$$p_m = 0, \quad \omega = 0 \Longrightarrow \varepsilon_m \propto a^{-3}, \tag{1.17}$$

$$p_r = \frac{1}{3}\varepsilon_r, \quad \omega = \frac{1}{3} \Longrightarrow \varepsilon_r \propto a^{-4}, \tag{1.18}$$

$$p_\Lambda = -\varepsilon_\Lambda, \quad \omega = -1 \Longrightarrow \varepsilon_\Lambda \propto a^0. \tag{1.19}$$

Returning to the Einstein field equations, we may take the trace of equation (1.5) with $\Lambda = 0$

to find that

$$R = -8\pi T^{\mu}_{\ \mu}. \tag{1.20}$$

Thus the Einstein equations may be rewritten as

$$R_{\mu\nu} = 8\pi(T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T^{\rho}_{\ \rho}). \tag{1.21}$$

In this form one may produce the first and second Friedmann equations respectively

$$H^2 = \frac{8\pi}{3}\varepsilon - \frac{k}{a^2}, \tag{1.22}$$

$$\frac{\ddot{a}}{a} = -\frac{4\pi}{3}(\varepsilon + 3p). \tag{1.23}$$

Equations (1.22) and (1.23) govern the dynamics of the universe on large scales [3–8]. The following useful relation may also be derived using equations (1.14) and (1.22)

$$\dot{H} = -4\pi(\varepsilon + p) + \frac{k}{a^2}. \tag{1.24}$$

1.2 Problems of standard big bang cosmology

1.2.1 Horizon/homogeneity problem

The horizon problem, also known as the homogeneity problem is not a direct conflict of observation and theory, rather it is a statement concerning the statistical unlikeliness that causally disconnected regions of space in the standard big bang cosmological model possess a small variation in their energy densities (temperature). The age of the universe (according to current derivations within the context of ΛCDM) is 13.787 ± 0.020 Gyr [9], this is the amount of time since the cosmological big bang singularity. Since light travels at a constant speed in all frames, photons have therefore travelled a finite distance since the big bang singularity. Thus the boundary of the volume of space centered at an observer indicating the distance from which that observer may receive information is known as the particle horizon.

The CMB, also known as the last scattering surface, are the photons released during the moment of recombination: the moment (referred to as such since it occurs quickly relative to the time at which it begins) when electrons and protons combine to form neutral hydrogen releasing photons in the process. These photons Thomson scatter for a brief period with remaining free electrons, and in particular this is the time, which we will refer to as t_{rec}, the universe becomes transparent to radiation and these are the photons observed in the CMB. Since t_{rec}, photons have traveled mostly without scattering to the present time t_0. Since the universe was opaque to radiation before t_{rec} our observations are limited to the maximum distance light can travel since t_{rec} this is known as the optical horizon.

The universe is observed to be homogeneous via the CMB, and thus the comoving distance over which matter should have been in thermal contact if this is the reason for the homogeneity is the optical horizon. Consider a null geodesic travelling radially in a flat universe, via equation (1.4) we have

$$0 = ds^2 = dt^2 - a^2(t)dr^2. \tag{1.25}$$

The comoving distance d_{opt} traveled by a photon between t_{rec} and t_0, during which the universe is approximately dominated by matter ($a(t) = (t/t_0)^{2/3}$, where we have normalized $a(t)$ such that $a(t_0) = 1$) is then

$$d_{opt} = \int_{t_{rec}}^{t_0} \frac{dt}{a(t)} \approx 3t_0 \left[1 - \left(\frac{t_{rec}}{t_0}\right)^{1/3}\right]. \tag{1.26}$$

The comoving distance of any one point on the CMB to the big bang singularity, during which the universe is approximately dominated by radiation ($a(t) = (t/t_0)^{1/2}$) between $0 < t < t_{eq}$ and approximately by matter between $t_{eq} < t < t_{rec}$ is

$$d_{cmb} = \int_0^{t_{eq}} \frac{dt}{a(t)} + \int_{t_{eq}}^{t_{rec}} \frac{dt}{a(t)} \approx 2t_0^{1/2} t_{eq}^{1/2} + 3t_0^{2/3}\left[t_{rec}^{1/3} - t_{eq}^{1/3}\right]. \tag{1.27}$$

If the isotropy of the CMB is to be explained from causality arguments in the framework of standard big bang cosmology, then d_{cmb} and d_{opt} should be approximately equal. Taking $t_{eq} \approx 10^{11}$ s, $t_{rec} \approx 10^{12}$ s, and as stated previously $t_0 \approx 10^{17}$ s we may calculate

$$d_{opt} \approx 1 \times 10^{28}\ cm \approx 1 \times 10^{10}\ ly, \tag{1.28}$$

$$d_{cmb} \approx 1 \times 10^{26}\ cm \approx 1 \times 10^{8}\ ly. \tag{1.29}$$

Thus $d_{opt} > d_{cmb}$ by approximately two orders of magnitude, that is, the distance scale we observe the CMB to be homogeneous is much larger than the distance scale the CMB had causal contact with at the time the universe became transparent as portrayed in figure 2 [3, 4, 6, 8].

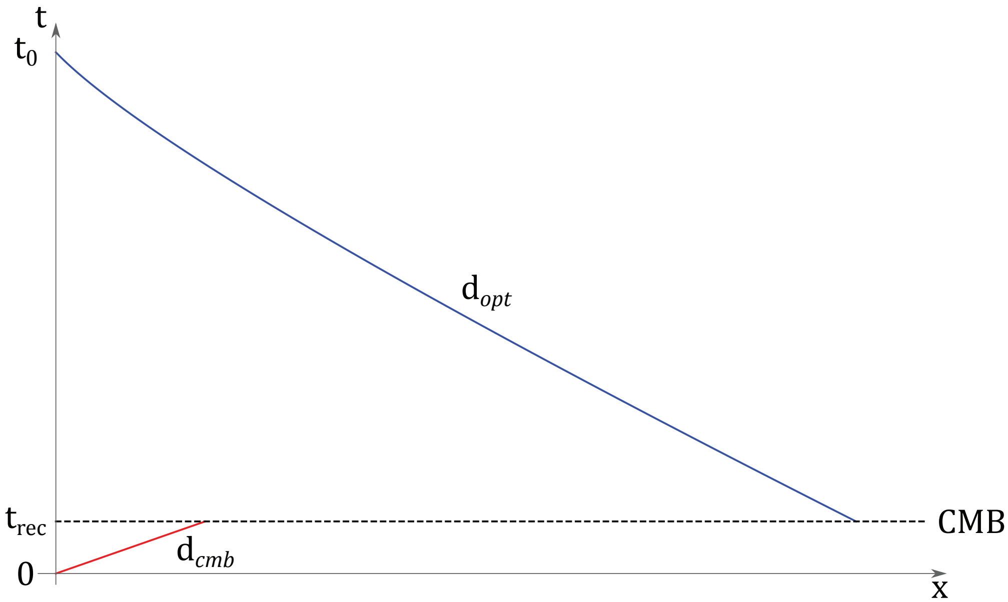

Figure 2: A sketch depicting the horizon problem. The horizontal axis represents the physical distance, the vertical axis represents the coordinate time. The physical distance that a free photon could have travelled from the time of the initial spacetime singularity $t = 0$ to the last scattering surface t_{rec} represented by d_{cmb} is much smaller than the physical distance we observe the CMB to be homogeneous today represented by d_{opt}. Note that this sketch is not to scale, the time $\Delta t_1 = t_{rec} - 0 \ll t_0 - t_{rec} = \Delta t_2$.

1.2.2 Flatness problem

The flatness problem is based upon the current observation that the universe is very close to spatial flatness at the present time [9]. It is a problem of fine tuning: consider the Friedmann equation (1.22) for a flat universe whose energy density we define to be the critical density

ε_c, as well as that of a non flat universe

$$H^2 = \frac{8\pi\varepsilon_c}{3}, \tag{1.30}$$

$$H^2 = \frac{8\pi\varepsilon}{3} - \frac{k}{a^2}, \tag{1.31}$$

by combining equations (1.30) and (1.31) and substituting the definition of the density parameter $\Omega \equiv \varepsilon/\varepsilon_c$, we may recast the above as

$$\Omega(t) = 1 + \frac{k}{a^2H^2} = \frac{1}{1-\gamma(t)}, \qquad \gamma(t) = \frac{3k}{\varepsilon 8\pi a^2} \propto \begin{cases} a, & t \gtrsim t_{eq} \\ a^2, & t \lesssim t_{eq}. \end{cases} \tag{1.32}$$

$\Omega = 1$ is an unstable fixed point, in particular if the universe begins in a configuration where Ω is exactly equal to one, the universe will remain flat indefinitely. However, for any small perturbation from $\Omega = 1$ the universe evolves further from spatial flatness. This may be seen from equation (1.32), since $\Omega(t_0)$ is very close to one, and $\gamma(t)$ decreases as we approach the big bang, the universe must have been even closer to spatially flat in the past. In fact, near the Planck scale

$$|\Omega(10^{-43}\text{ s}) - 1| \sim 10^{-60}. \tag{1.33}$$

That is, the initial spatial curvature of the universe must have been very finely tuned, otherwise our universe would not be as we observe today: it would have collapsed very early on for $k = -1$ or cooled very quickly for $k = +1$ [4, 6, 8, 10].

1.2.3 Singularity problem

The singularity problem is two-fold: not only does the classical description of matter as in standard big bang breakdown at high energies and temperatures and must be replaced by a quantum field theoretic description, but within this classical framework the universe necessarily possesses an initial spacetime singularity; more specifically the singularity is not a coordinate dependent feature. In particular, since the solutions to the scale factor of any classical matter phase are power laws in time, the scale factor $a(t_i = 0) \to 0$ indicating infinite matter and radiation density, infinite spacetime curvature, a finite age for the universe, and the spacetime is null geodesically incomplete [3, 8, 11].

1.2.4 Formation of structure problem

Observations of anisotropies in the CMB have provided evidence that the large scale structure of the universe, galactic clusters and superclusters, originate from primordial cosmological perturbations in energy density that are nearly scale invariant, adiabatic and Gaussian in nature [12]. An immediate question then becomes, what is the origin of these primordial inhomogeneities, and what model successfully predicts their spectrum of fluctuations [4]? In addition, galaxy redshift surveys [13] provide evidence that galaxies and clusters are non-randomly correlated, whose comoving separation are comparable to the comoving horizon at the time of equal matter and radiation. Thus if the density perturbations that generated large scale structure today were generated much before the time of equal matter and radiation, standard big bang cosmology provides no causal explanation for the correlation. In addition, the angular power spectrum of the CMB [2] indicates structure in the CMB on comoving scales larger than the comoving particle horizon at the time of recombination, particularly when using standard big bang cosmology as a description of null geodesic evolution.

1.2.5 Relic problem

If the universe at very early times and thus very high energies is to be described by a grand unified theory of a higher symmetry group, then the absence of stable heavy particle species and topological defects in current observations presents a problem. Many of these unified theories predict copious production of these massive particle species, the most notable being the magnetic monopole. The standard big bang cosmological model provides no mechanism for the removal of these exotic particle species that may have been produced in the very early universe [10].

1.2.6 Cosmological constant problem

While the cosmological constant problem is a deep problem of the theory of quantum fields [14, 15], it may also be posed as a problem of fine tuning in cosmology. Firstly any form of matter that contributes to the energy density of the vacuum acts identically to a cosmological constant term in Einstein's equations. Take for instance scalar field matter (see section 2.1 for an introduction) described by a field ϕ governed by potential $V(\phi)$. If the potential possesses a local minimum at some point ϕ_0, then $\phi(t) = \phi_0$ is a solution to the scalar field equations of motion (the Klein-Gordon equation in an expanding background) and we have an equation of state of the matter field obeying

$$p = -\varepsilon = -V(\phi_0). \tag{1.34}$$

In terms of the Einstein equations, the energy-momentum tensor

$$T_{\mu\nu} = V(\phi_0) g_{\mu\nu}, \tag{1.35}$$

behaves just as a cosmological constant term set to be

$$\Lambda_V = 8\pi V(\phi_0). \tag{1.36}$$

Thus the effective cosmological constant of Einstein's equations Λ_{eff} possesses two contributions, one being the constant that may simply be implemented by hand Λ as done by Einstein himself for the purpose of achieving a static universe [16], as well as the contribution from the vacuum energy density of the matter field $V(\phi_0)$

$$\Lambda_{eff} = \Lambda + 8\pi V(\phi_0). \tag{1.37}$$

Therefore, Λ just as well contributes to the total effective vacuum energy

$$\varepsilon_{eff} = V(\phi_0) + \frac{\Lambda}{8\pi} = \frac{\Lambda_{eff}}{8\pi}. \tag{1.38}$$

Measurements of the cosmological expansion rate today, H_0, forbid a large cosmological constant; an approximate upper limit being (where fundamental constants have been reintroduced) [7, 17]

$$\left|\frac{\Lambda_{eff}}{8\pi G}\right| < 10^{-29}\ \mathrm{g/cm^3} \approx (10^{-11}\ \mathrm{GeV})^4. \tag{1.39}$$

The problem is introduced when one considers that the vacuum energy density predicted by quantum field theory is enormously larger than the bound given in equation (1.39). By summing the zero-point energies of all modes of a Klein-Gordon field of mass m up to high energy cutoff scale $\lambda \gg m$ yields a vacuum energy density (in spherical coordinates) [7, 18]

$$V(\phi_0) = \int_0^{\lambda} \frac{4\pi k^2}{(2\pi)^3}\frac{1}{2}\sqrt{k^2+m^2}\ dk \sim \frac{\lambda^4}{16\pi^2}. \tag{1.40}$$

Taking a cutoff provided by the applicability of general relativity up the the Planck scale $\lambda = (8\pi G)^{-1/2}$, we obtain a vacuum energy density of

$$V(\phi_0) \approx 2 \times 10^{71}\ \mathrm{GeV}^4. \tag{1.41}$$

Thus via the upper bound of equation (1.39) the cosmological term $\Lambda/8\pi G$ must cancel out $V(\phi_0)$ to roughly 115 decimal places, stressing that this is a problem of fine tuning [4,7,14,17].

2 The inflationary universe

Inflation is a model of the very early universe proposed by Alan Guth [19], and has proven robust by solving many of the problems described in section 1. How inflation addresses the standard cosmological problems as well as a brief description of some of the new challenges the theory faces is briefly discussed in section 2.2.

The concept of inflation is to introduce a duration of time in which the universe undergoes a phase of accelerated expansion. The concept is based on a prescription to solve the homogeneity and flatness problems as we will see, and is related to the simple idea of introducing a period in which the comoving Hubble radius is *decreasing* [20, 21]. Most often, the phase of accelerated expansion is approximately exponential, although exactly exponential expansion is not necessary for successful inflation a priori. In fact, any amount of accelerated expansion with $\ddot{a} > 0$, will evolve the spatial curvature to zero.

This phase is implemented before the time of nucleosynthesis (~ 1 s after the big bang singularity, $T \sim 10^{-1}$ MeV), where standard big bang cosmology is then reinstated as the most plausible theory of the early universe, light particle species production and subsequent large scale evolution. The period of inflation may be implemented immediately following the initial singularity before which presumably a theory of quantum gravity governs the dynamics, or it may be implemented after a period of radiation dominated Friedmann expansion.

2.1 Inflation as a dynamical scalar field

In the very early universe, at very high energies, matter described as an ideal gas with the simple equation of state as in equation (1.13) with ω a constant may no longer be reliably applied due to quantum effects. At the present time, quantum field theory is the most successful description of matter and particle interactions at very high energies up to the Planck scale. There are three types of fields currently understood to describe matter: spin 1/2 fermions, spin 1 gauge bosons and spin 0 bosons expressed in this work as a scalar field ϕ. The simplest models of inflation describe the time evolution of the inflationary energy density by a single scalar field known as the *inflaton*. Within the standard model of particle physics the only spin zero boson, the Higgs boson was found to not be a suitable candidate for the inflaton [3, 22], and since the inflaton mass is much larger than the mass scale of the standard model, candidates should come from supersymmetric theories or superstring theories such as Kähler moduli resulting from compactification on Calabi-Yau manifolds [20].

Thus the scalar field (inflaton) may be treated classically or as a quantized field in the appropriate energy regimes: for instance in the study of cosmological perturbations at scales much larger than the Hubble radius fields may be treated classically obeying evolution equations provided by general relativity, and may be promoted straightforwardly to quantum fields by the means of standard canonical quantization methods allowing for the prescription of quantum Bunch-Davies vacuum initial conditions for perturbations.

Formally, we begin by describing matter as a scalar field with Lagrangian $\mathcal{L}_\phi$ whose kinetic term is canonical and is governed by a potential $V(\phi)$ describing self interactions

$$\mathcal{L}_\phi = \sqrt{-g}\left(\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - V(\phi)\right). \tag{2.1}$$

In order to describe the dynamics in a gravitational background this field is minimally coupled (there is no direct coupling between ϕ and the metric $g_{\mu\nu}$) to classical Einstein gravity. In some cases, even seemingly non-minimally coupled theories may be expressed as a minimally coupled theory via a redefinition of the fields. The action describing the dynamics of a scalar matter field in a cosmological background described by classical Einstein gravity is then

$$S = S_{EH} + S_\phi = \int d^4x\sqrt{-g}\left(\frac{1}{16\pi}R + \frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - V(\phi)\right), \tag{2.2}$$

where S_{EH} is the Einstein-Hilbert action and S_ϕ is the scalar matter action, the integration over spacetime coordinates of the Lagrangian density provided in equation (2.1). More generally, the Lagrangian for the scalar field may also possess non-canonical kinetic terms, many of which are natural results of string early universe models, and may be expressed as

$$\mathcal{L}_\phi = G(\phi, g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi) - V(\phi). \tag{2.3}$$

Variation of the Einstein-Hilbert action with respect to the metric provides the vacuum

Einstein equations. Varying the matter sector with respect to the metric, we may derive the stress-energy tensor describing the scalar field

$$T_{\mu\nu} \equiv -\frac{2}{\sqrt{-g}}\frac{\delta S_\phi}{\delta g^{\mu\nu}} = \partial_\mu\phi\partial_\nu\phi - g_{\mu\nu}\left(\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - V(\phi)\right). \tag{2.4}$$

The equation of motion of the scalar field may be found directly by applying the Euler-Lagrange equations to equation (2.1) or varying the matter action in equation (2.2) with respect to the field ϕ. We obtain the Klein-Gordon equation for a scalar matter field in an expanding cosmological background

$$\ddot{\phi} + 3H\dot{\phi} - \nabla^2\phi + \frac{dV}{d\phi} = 0. \tag{2.5}$$

The energy density $\varepsilon(\phi)$ and pressure $p(\phi)$ of the scalar field matter are

$$\varepsilon(\phi) = \frac{1}{2}\dot{\phi}^2 + \frac{1}{2}\frac{(\nabla\phi)^2}{a^2} + V(\phi), \tag{2.6}$$

$$p(\phi) = \frac{1}{2}\dot{\phi}^2 - \frac{1}{6}\frac{(\nabla\phi)^2}{a^2} - V(\phi). \tag{2.7}$$

If we assume the field ϕ to be approximately initially spatially homogeneous ($\partial\phi/\partial x^i = 0$) then the spatial gradient terms appearing in equations (2.5) to (2.7) all become negligible. In fact, this is not a necessary assumption since inflation drives the field towards spatial homogeneity soon after the beginning of inflation. In addition, we may express the equation of state parameter ω_ϕ of the matter field as

$$\omega_\phi = \frac{p_\phi}{\varepsilon_\phi} = \frac{\frac{1}{2}\dot{\phi}^2 - V(\phi)}{\frac{1}{2}\dot{\phi}^2 + V(\phi)}. \tag{2.8}$$

In general ω_ϕ is time-dependent, but bounded from below satisfying the weak energy dominance condition $\varepsilon + p \geq 0$ for any potential satisfying $V(\phi) \geq 0$. The second Friedmann equation (1.23) in a flat universe may be expressed with this new description of matter as

$$H^2 = \frac{8\pi}{3}\left(\frac{1}{2}\dot{\phi}^2 + V(\phi)\right), \tag{2.9}$$

More generally, the second Friedmann equation (1.23) may be expressed to include various matter types by substituting in their energy densities: non-relativistic and cold dark matter ε_m, radiation ε_r, anisotropies ε_σ, the curvature of the universe, and m types of scalar field matter ϕ_i each characterized by their energy density ε_{ϕ_i} and its *constant* equation of state parameter ω_{ϕ_i} (tildes indicate a dimensionful constant)

$$H^2 = -\frac{k}{a^2} + \frac{8\pi}{3}\left(\frac{\tilde{\varepsilon}_m}{a^3} + \frac{\tilde{\varepsilon}_r}{a^4} + \frac{\tilde{\varepsilon}_\sigma}{a^6} + \sum_{i=1}^{m}\frac{\tilde{\varepsilon}_{\phi_i}}{a^{3(1+\omega_{\phi_i})}}\right). \tag{2.10}$$

Within this improved framework, it may be seen from equation (1.23) that an era of accelerated expansion ($\ddot{a} > 0$) takes place if $-\frac{1}{3}\varepsilon > p$ and from equation (2.8) that a scalar field may lead to a type of matter exhibiting negative pressure ($\omega_\phi < 0$).

A successful period of inflation however takes place if the equation of state of the fluid drives accelerated expansion for a sufficient period of time. In particular, the evolution of the scalar field in time must be sufficiently gradual: the potential energy $V(\phi)$ dominates over the kinetic energy $\dot{\phi}^2$ of the field. For large values of the potential the field experiences slow evolution via the Hubble friction term $3H\dot{\phi}$ in equation (2.5). In addition the second derivative of ϕ should be small enough for this state to be maintained for a sufficient amount of time. These conditions are known as the slow-roll conditions [23, 24]

$$\dot{\phi}^2 \ll V(\phi), \qquad |\ddot{\phi}| \ll |3H\dot{\phi}|,\ |V'(\phi)|. \tag{2.11}$$

These two conditions lead to the following two dimensionless quantities respectively, known as the slow-roll parameters which most basically characterize a period of inflation

$$\epsilon_V = \frac{M_P}{2}\left(\frac{V_{,\phi}}{V}\right)^2, \qquad \eta_V = M_P^2\left(\frac{V_{,\phi\phi}}{V}\right). \tag{2.12}$$

Note that η_V may take on either sign, while $\epsilon_V \geq 0$. Related parameters may also be expressed in terms of the Hubble parameter

$$\epsilon_H = -\frac{\dot{H}}{H^2} = -\frac{d\ln H}{dN}, \qquad \eta_H = -\frac{\ddot{\phi}}{H\dot{\phi}} = \epsilon_H - \frac{1}{2\epsilon_H}\frac{d\epsilon_H}{dN}, \tag{2.13}$$

with $dN = Hdt$ the differential number of e-foldings. In the regime of slow-roll governed by the conditions in equation (2.11), these parameters are approximately related as

$$\epsilon_H \approx \epsilon_V, \qquad \eta_H \approx \eta_V - \epsilon_V. \tag{2.14}$$

These parameters should be small, in particular $\epsilon_{H,V}, |\eta_{H,V}| \ll 1$ in the slow-roll regime. It is important to note however that satisfying these conditions does not entirely guarantee a prolonged period of inflation depending on the choice of initial conditions, we may always choose $\dot{\phi}$ large enough such that slow-roll is never satisfied. However it is important to note that the inflationary solution is a dynamical attractor: the inflationary trajectory in the phase space of the inflaton $(\phi,\, \dot{\phi})$ is found from a large domain of initial conditions. This is hinted at by equation (2.10), the energy density of the inflaton field scales approximately as $(\varepsilon_\phi \sim a^0)$ and thus comes to dominate over every other form of matter very quickly; said differently, the energy density of the inflaton field redshifts much slower [3, 8, 21, 25].

There are a plethora of potentials that successfully characterize a period of inflation described by a scalar field, and satisfy the slow-roll conditions. In closing we simply name a few and point the interested reader to exhaustive reviews on slow-roll inflationary models: old inflation (suffers from graceful exit problem [3]), 'new' inflation [23,24], chaotic inflation [26], hybrid inflation [27], natural inflation [28], hilltop inflation [29], inflection point inflation arising from the physics of D-branes (dynamical objects unique to string theory) [30, 31], $f(R)$ Starobinsky type inflation [32, 33] and models non-minimally coupled to gravity [34], as well as generalized models possessing non-canonical kinetic terms known as k-inflation [35] or multiple [36] scalar fields [3, 8, 20, 21, 37, 38].

2.2 Addressing cosmological problems

As foreshadowed, we mentioned that inflation was born from the intention of rectifying the homogeneity and flatness problems. Very simply, if we take the comoving Hubble radius $(aH)^{-1}$ to decrease during a phase of cosmological evolution, this leads to two immediate consequences. The first is that the comoving distance of any one point on the CMB to the big bang singularity can be made to be much larger than the comoving distance travelled by a photon on the CMB and now, so long as inflation occurs for a long enough period of time, ~ 70 e-folds [38], and thus in contrast with equations (1.28) and (1.29)

$$d_{opt} = \int_{t_{rec}}^{t_0} \frac{dt}{a(t)} \ll d_{cmb} = \int_0^{t_{rec}} = \frac{dt}{a(t)}. \tag{2.15}$$

This ensures that all points on the CMB observed today were in causal contact at some time during the inflationary period, addressing and providing an elegant solution to the horizon problem. The second is that the density parameter Ω characterising the spatial geometry of the universe is driven towards flatness, $\Omega \longrightarrow 1$, during inflation. This is seen immediately from equation (1.32) since the comoving Hubble radius is decreasing during inflation.

In addition, inflation also addresses the relic problem [23]. A period of inflation simply dilutes the number density of any unobserved products of grand unified theories such as topological defects that may be produced before or during the period of inflation. Ofcourse this is only a natural solution if the reheating temperature after inflation does not exceed the energy scale of the grand unified theory, which would allow for unobserved phenomena to be produced after inflation.

The formation of structure problem is also addressed in part (see discussion below), as seen in figure 3 perturbation modes representing galactic scales as seen today are well within the particle horizon during a period of inflation, allowing for causal contact.

The singularity problem as described in section 1.2.3 is a seemingly unavoidable issue with inflation, in particular it has been shown [39] that by extending the Hawking-Penrose singularity theorems [40] that an eternally inflating background spacetime cannot be past null geodesically complete. This unavoidable singularity raises the important question: how did the universe spawn from the singularity, what are the initial conditions, and how can

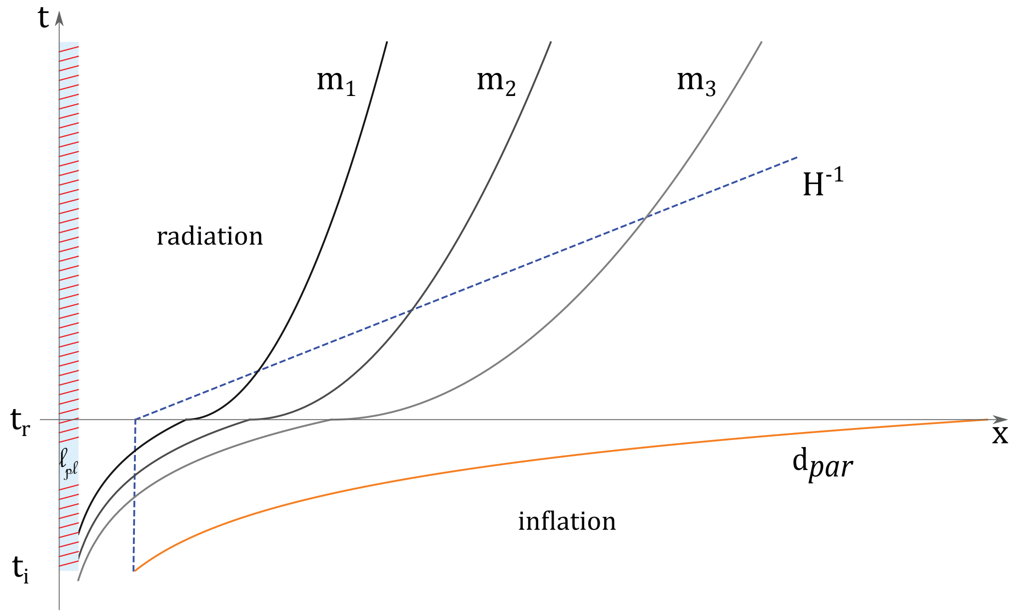

Figure 3: A spacetime diagram whose vertical axis is the cosmological time t and horizontal is physical distance x. The diagram portrays the generic features of the Hubble radius indicated by the blue dashed line, relative to the proper wavelengths of cosmological perturbation modes indicated by the gray-scale solid curves and comoving wavenumber $m_1 > m_2 > m_3$. In a generic phase of inflationary expansion ($t_i \leq t \leq t_r$) the proper wavelengths of perturbation modes emerge length scales smaller than the Planck length l_{pl} and increase very quickly relative to the Hubble radius which is approximately constant. Thus there are two notable periods of evolution with respect to the study of cosmological perturbations: modes evolving on sub-Hubble scales and traversing the curvature scale, and then propagating on super-Hubble scales. The phase of reheating required in all inflationary models converting the inflaton potential to radiation occurs at time t_r and also signals the end of the inflationary phase. A phase of radiation dominated Friedmann expansion then dominates, at which time perturbation modes once again become sub-Hubble. The orange line labels the particle horizon during the inflationary epoch, notice that the physical wavelength of modes are at all times smaller than the particle horizon addressing the formation of structure problem.

we motivate our assumptions of primordial inhomogeneities if we are unable to describe the physics at energies above the Planck scale which necessarily must have been traversed?

This latter point leads to a new problem introduced by inflation [3] known as the trans-Planckian problem for fluctuations. This states that length scales observed today to be on the order of galactic separation, were sub-Planckian during inflation, or in particular, at the time that other relevant density perturbations grew larger than the Hubble radius during inflation as seen in figure 3 [37]. This makes predictions of inflation with respect to the spectrum of density fluctuations particularly sensitive to new physics introduced by a correct theory of quantum gravity. Therefore the formation of structure problem is partially solved, namely that with the assumption that the theory of linear perturbations remains valid throughout the period of inflation, and Bunch-Davies initial conditions are predicted by a theory of quantum gravity, the spectrum of density fluctuations may be predicted accurately and a mechanism for the formation of structure is subsequently provided [3, 37]. Lastly, inflation does not address the cosmological constant problem since it remains to be a finely tuned parameter of the theory.

3 The ekpyrotic universe

Ekpyrosis is presented primarily as a model of the very early universe, but may be further argued to be successfully integrated into cyclical cosmological scenarios [41–45] being preceded particularly by a phase of dark energy domination [46] which current observations indicate we are experiencing at the present day. Ekpyrosis is thus at times presented as a model of both the early and late universe, but standalone is an alternative to the inflationary universe paradigm [19, 44]. The model is introduced as an alternative insofar as it may be argued to address the problems of standard big bang cosmology described in section 1. The original ekpyrotic model [47], is embedded in heterotic M-theory and this realization's primary ingredients for describing the physical observations we observe today are extra dimensions and branes. However, the phase of ekpyrosis may also be discussed entirely in the context of a four dimensional effective field theory. Within this lower dimensional framework, we may discuss the dynamics on large scales as well as the implications of ekpyrosis driven by scalar field matter. What makes a phase of ekpyrosis interesting are the distinct predictions it makes relative to inflationary theories allowing it to be ruled out by observations.

Qualitatively speaking, ekpyrosis is a phase of very slow contraction predating the beginning of time postulated as a spacetime singularity of inflationary and standard big bang cosmological models. The phase of slow contraction is modulated in its simplest form by a single scalar field rolling down a steeply negative exponential potential in an FLRW background, and thus with a nearly constant scale factor and quickly decreasing Hubble radius as demonstrated in figure 5. The phase of slow contraction is followed by a kinetic energy dominated phase and a subsequential cosmological bounce phase which may be either singular or non-singular in nature [12, 44]. As stated previously, the ekpyrotic universe makes distinct predictions from that of inflation to currently unobserved features of the universe. A key generic feature of ekpyrotic phases is the primordial gravitational wave spectrum possessing a blue tilt, in contrast with an approximately scale invariant spectrum as predicted by generic inflation [12, 20, 44, 47, 48].

A topic of contention with the ekpyrotic scenario is the bounce phase, as it shifts from a contracting to an expanding FLRW universe. The recent approach to the problem is to describe the bounce in an entirely non-singular manner which necessarily violates the null energy condition [12,44,49–53]. However problems regarding the stability of the bounce phase in particular with the evolution of curvature perturbations, the evolution of anisotropy, ghosts and tachyons have been raised [54–56]. From the higher dimensional perspective, mechanisms of the particular string theory are used to construct a smooth bounce phase, for instance in the embedding of heterotic M-theory by a shrinking of the fifth dimension simultaneously driving the string coupling to null, while the three large spatial dimensions in which we observe remain large throughout the bounce [12,44,57–62]. A very recent approach to a non-singular bounce has been presented with the use of spacelike branes [52,53,63].

Scenarios of the bounce phase via a ghost condensate possessing higher order kinetic terms [64,65] preceded by a phase of ekpyrotic contraction [12,50] have also been presented. In particular, the authors of [12] argue that the ghost condensate and ekpyrotic scalar field are the same field, where higher derivative kinetic terms become relevant as the universe approaches the bounce and provide consistency relations between the kinetic and potential terms for this to be adequately realized.

3.1 Ekpyrotic cosmology

3.1.1 Single scalar field

To begin the description in a four dimensional effective field theory, we begin with a scalar matter field ϕ with a canonical kinetic term governed by a potential $V(\phi)$ as in equation (2.1); therefore the field is governed again by the Klein-Gordon equation in an expanding cosmological background given by equation (2.5). Approximating the field on large scales to be spatially homogeneous ($\partial\phi/\partial x^i = 0$), we may express the field's energy density ε_ϕ and pressure p_ϕ as in equations (2.6) and (2.7) with the equation of state parameter ω of the field expressed as in equation (2.8).

For this single scalar field model, the general conditions for ekpyrosis are defined in a similar way to slow-roll inflation: in order for large scale density fluctuations generated during the ekpyrotic phase to be approximately scale invariant, the potential $V(\phi)$ must satisfy analogous fast-roll conditions [45] defined in terms of two dimensionless parameters

$$\epsilon_2 \equiv M_P^{-2} \left(\frac{V}{V_{,\phi}}\right)^2, \qquad \eta_2 \equiv 1 - \frac{V_{,\phi\phi}V}{V_{,\phi}^2} \tag{3.1}$$

where M_P is the reduced Planck mass defined as

$$M_P \equiv \sqrt{\frac{\hbar c}{8\pi G}} = \sqrt{\frac{1}{8\pi}}, \tag{3.2}$$

with the second equality following from the conventions used in this thesis. The fast roll condition is met if

$$\epsilon_2 \ll 1, \qquad |\eta_2| \ll 1, \tag{3.3}$$

and satisfying these conditions ensures the potential is steep and approximately exponential

$$V(\phi) \approx -V_0 \exp\left(-\sqrt{\frac{2}{p}}\frac{\phi}{M_P}\right), \tag{3.4}$$

where V_0 is a constant, and $p \ll 1$. Note that the specific value of V_0 is non-physical since a shift in the domain of ϕ effectively alters the set value of V_0. Figure 4 depicts the general features of $V(\phi)$ given in equation (3.4), with $\phi(t_i)$ and $\phi(t_f)$ indicating that the ekpyrotic phase beginning at $t = t_i$ starts at large values of the field and rolls down towards exponentially smaller values. The potential is chosen to be negative in order to allow for the Hubble parameter to pass through zero as is necessary: the positive kinetic energy in equation (2.9) may be cancelled by the negative potential energy.

Equation (2.8) then implies that equation of state parameter $\omega_\phi > 1$. A brief observation of equation (2.10) shows that in a contracting universe without the presence of an ekpyrotic field with $\omega_\phi > 1$, the cosmological evolution quickly becomes dominated by the anisotropy term ($\varepsilon_\sigma \sim a^{-6}$). Introducing the ekpyrotic field with constant (see equation (3.8)) equation of state parameter $\omega_\phi > 1$, then implies that the ekpyrotic field would come to dominate the contracting phase since it scales as an even larger inverse power of the scale factor; the energy density of the ekpyrotic field blueshifts much quicker. This is practically the opposite to what occurs during an inflationary phase: the inflaton $\omega_\phi \approx -1$ comes to dominate the expanding phase as its energy density scales inversely to a very small power of the scale factor; the energy density redshifts much slower.

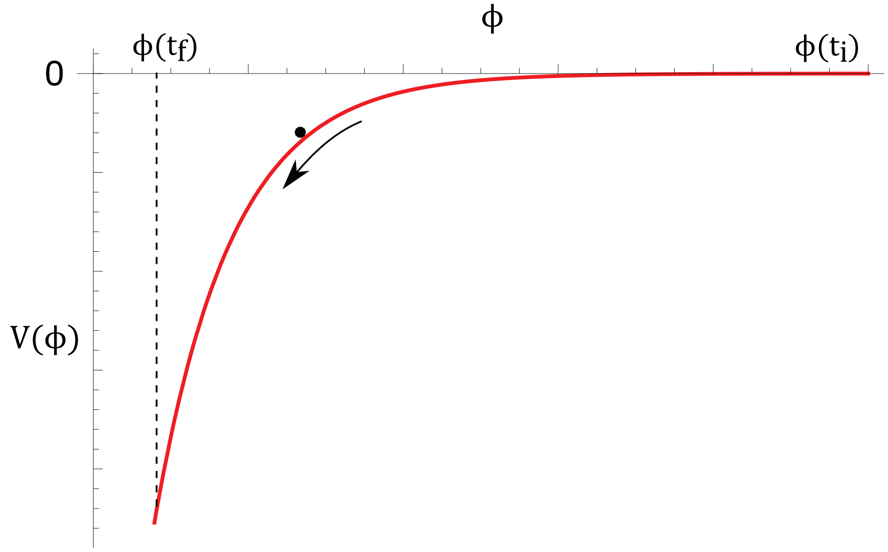

Figure 4: A steep negative exponential potential describing the phase of ekpyrosis.

With the potential for the ekpyrotic field given by equation (3.4) and governed by the Klein-Gordon equation in an FLRW background, one may check that we obtain the following exact scaling solution [58, 66, 67]

$$a(t) \sim (-t)^p \Longrightarrow H(t) = \frac{p}{t}, \qquad \phi(t) = \frac{2}{c} M_P \ln\left(-\sqrt{V_0}\ \alpha\ \frac{t}{M_P}\right), \tag{3.5}$$

$$c \equiv \sqrt{\frac{2}{p}}, \qquad \alpha \equiv \sqrt{\frac{1}{p(1-3p)}}. \tag{3.6}$$

Thus the ekpyrotic solution describes a slowly contracting universe with the rate of contraction modulated by p. One may check that the kinetic and potential energy terms both possess terms that are first order in p, while the first and only contribution to the H^2 term is second order in p, thus we have the following conditions

$$\frac{\dot{\phi}^2}{2} \gg H^2, \qquad |V| \gg H^2. \tag{3.7}$$

With the exact solution of the ekpyrotic field in hand we may calculate its equation of state parameter ω_ϕ and observe its constancy

$$\omega_\phi = \frac{2}{3p} - 1 \gg 1. \tag{3.8}$$

3.1.2 Two scalar fields

The scaling solution described in section 3.1.1 for a single Klein-Gordon field in a cosmological background driven by a negative exponential potential generalizes to two scalar fields straightforwardly [12,68,69] [2]. Consider the generalized potential for two scalar matter fields $\phi^{\mathbf{1}}, \phi^{\mathbf{2}}$ indexed by the bold integer superscript

$$V(\phi^{\mathbf{1}}, \phi^{\mathbf{2}}) = -V_0^{\mathbf{1}} \exp\left(-\sqrt{\frac{2}{q^{\mathbf{1}}}}\frac{\phi^{\mathbf{1}}}{M_P}\right) - V_0^{\mathbf{2}} \exp\left(-\sqrt{\frac{2}{q^{\mathbf{2}}}}\frac{\phi^{\mathbf{2}}}{M_P}\right) \tag{3.9}$$

with $q^{\boldsymbol{i}} \ll 1$ and $V_0^{\boldsymbol{i}}$ are constants for $\boldsymbol{i} = 1, 2$. Each field is a Klein-Gordon scalar field with canonical kinetic terms such that they each obey

$$\ddot{\phi^{\mathbf{1}}} + 3H\dot{\phi^{\mathbf{1}}} + \frac{\partial V(\phi^{\mathbf{1}}, \phi^{\mathbf{2}})}{\partial \phi^{\mathbf{1}}} = 0, \tag{3.10}$$

$$\ddot{\phi^{\mathbf{2}}} + 3H\dot{\phi^{\mathbf{2}}} + \frac{\partial V(\phi^{\mathbf{1}}, \phi^{\mathbf{2}})}{\partial \phi^{\mathbf{2}}} = 0. \tag{3.11}$$

There exists an exact scaling solution to the second Friedmann equation describing two scalar fields

$$H^2 = \frac{8\pi}{3}\left(\frac{1}{2}(\dot{\phi^{\mathbf{1}}})^2 + \frac{1}{2}(\dot{\phi^{\mathbf{2}}})^2 + V(\phi^{\mathbf{1}}, \phi^{\mathbf{2}})\right), \tag{3.12}$$

when accompanied by the combined Klein-Gordon equations (3.10) and (3.11) above

$$a(t) \sim (-t)^{(q^{\mathbf{1}} + q^{\mathbf{2}})}, \qquad H = \frac{q^{\mathbf{1}} + q^{\mathbf{2}}}{t}, \tag{3.13}$$

$$\phi^{\mathbf{1}}(t) = \sqrt{2q^{\mathbf{1}}} M_P \ln\left(-\frac{t}{M_P}\sqrt{\frac{V_0^{\mathbf{1}}}{q^{\mathbf{1}}(1 - 3(q^{\mathbf{1}} + q^{\mathbf{2}}))}}\right), \tag{3.14}$$

$$\phi^{\mathbf{2}}(t) = \sqrt{2q^{\mathbf{2}}} M_P \ln\left(-\frac{t}{M_P}\sqrt{\frac{V_0^{\mathbf{2}}}{q^{\mathbf{2}}(1 - 3(q^{\mathbf{1}} + q^{\mathbf{2}}))}}\right). \tag{3.15}$$

[2] In fact, the scaling solution may be generalized to n scalar fields in the same way [44].

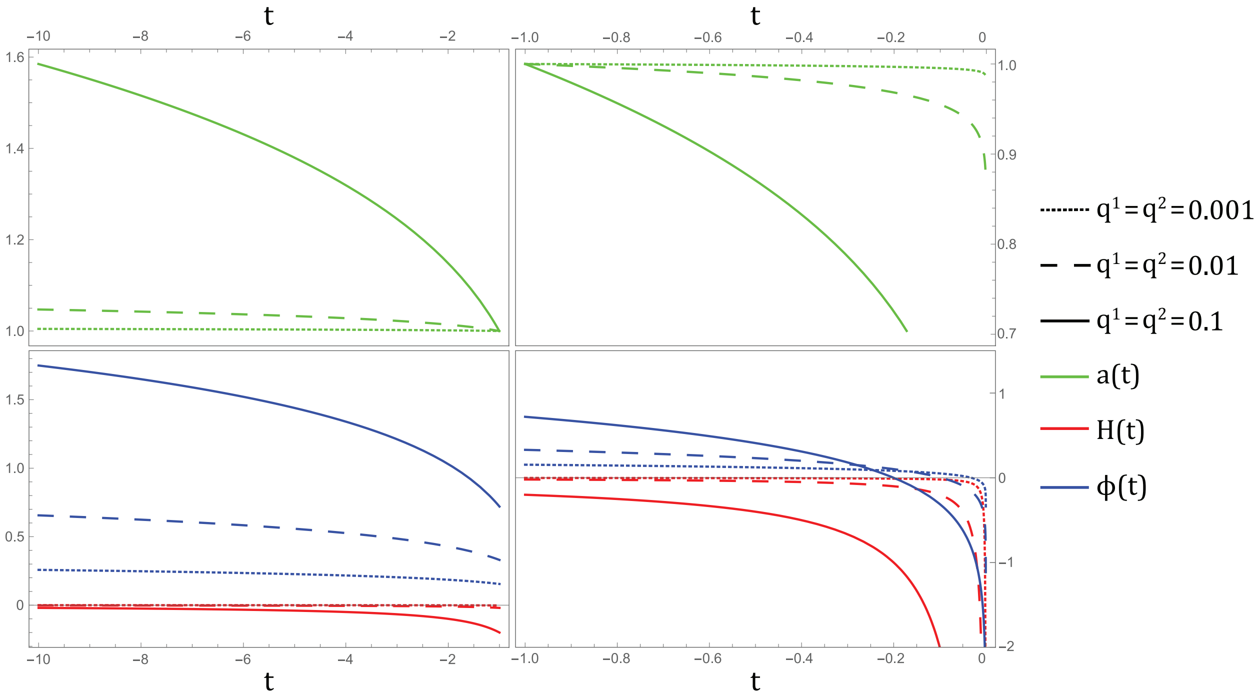

Figure 5: Plots of the ekpyrotic solution as provided in equations (3.13) and (3.14) of two canonical kinetic scalar fields subject to identical ($q^{\boldsymbol{1}} = q^{\boldsymbol{2}}$) exponential potentials as described by equation (3.9). The plots are split into two columns, the left describing the behaviour of the solutions for $|t| > 1$, while the right column describes the behaviour of the solutions for $|t| < 1$. The dotted, dashed and solid curves describe $q^{\boldsymbol{1}} = q^{\boldsymbol{2}} = 0.001, 0.01, 0.1$ respectively. The green, red and blue curves describe the evolution of the scale factor $a(t)$, the Hubble parameter $H(t)$ and the evolution of both scalar fields $\phi^{\boldsymbol{1}}(t)$ and $\phi^{\boldsymbol{2}}(t)$ respectively.

For reference, we calculate the following partial derivatives of the two field ekpyrotic potential equation (3.9) for $\boldsymbol{i}, \boldsymbol{j} = 1, 2$

$$V_{,\phi^{\boldsymbol{i}}\phi^{\boldsymbol{j}}} = -\frac{2V_0^{\boldsymbol{i}}}{q^{\boldsymbol{i}} M_P^2} \exp\left(-\sqrt{\frac{2}{q^{\boldsymbol{i}}}} \frac{\phi^{\boldsymbol{i}}}{M_P}\right) \delta_{\boldsymbol{i}}^{\boldsymbol{j}}. \tag{3.16}$$

It will be useful to define the additional dimensionless fast roll parameter ϵ_1 used in the literature valid for any number of scalar fields [44]

$$\epsilon_1 \equiv \frac{3}{2}(1+\omega) = 1 - \frac{\mathcal{H}'}{\mathcal{H}^2} = -\frac{\dot{H}}{H^2} = -\frac{d\ln H}{dN}, \tag{3.17}$$

with $N \equiv \ln a$ defined as the number of e-folds, and with the final equality following from

$$d\ln H = \frac{1}{H} dH, \qquad dN = d\ln a = H dt = \mathcal{H} d\eta. \tag{3.18}$$

Note that ϵ_1 is actually equivalent to the slow-roll parameter ϵ_H defined in equation (2.13) but we redefine it here and make use of it exclusively in the analysis for ekpyrosis to avoid confusion. A generic phase of ekpyrosis exhibits $\epsilon_1 \gg 1$.

The evolution of the Hubble radius H^{-1} as well as the proper wavelength of cosmological perturbations $\sim a/m$ (with m the comoving wavenumber) in canonical single or two field ekpyrotic backgrounds exhibit universal features. As observed in figure 6 the initial conditions of a period of ekpyrosis entails a Hubble radius far exceeding that of the proper length of relevant perturbation modes, although decreases relatively quickly. Thus quite generically phases of ekpyrotic contraction begin with oscillatory vacuum perturbations at which time the spacetime is approximately locally Minkowskian, and evolves to super-Hubble scales before the end of ekpyrotic contraction where gravitational interactions become increasingly relevant.

3.2 Addressing cosmological problems

The ekpyrotic scenario would not really be worth considering at all if it did not offer solutions to the problems of standard big bang cosmology (see section 1), as does inflation. Here we argue that the ekpyrotic scenario does in fact address many of the problems of standard big bang, promoting it to a competitive theory to that of inflation while some problems remain to be addressed.

The scenario addresses the horizon problem quite simply, the universe may exist for a sufficient time before the bounce such that the causal particle horizon grows larger than the optical horizon calculated in equation (1.28) of section 1.2.1. Allowing the particle horizon to grow sufficiently during ekpyrosis also addresses the structure formation problem: primordial perturbation modes whose wavelengths are relevant to galactic separation scales today are well within the particle horizon and causally connected as seen in figure 6. Once we motivate ekpyrosis in string theory we will also identify perturbation modes with quantum fluctuations on D3-branes providing an explanation as to the origin of the perturbations as well.

The flatness problem is also addressed: consider equation (1.32), since the ekpyrotic field comes to dominate quickly during contraction, the second term $k/(aH)^2 \sim t^2$ during ekpyrosis, while during phases of matter or radiation domination $k/(aH)^2 \sim t^{2/3}$ or $k/(aH)^2 \sim t$

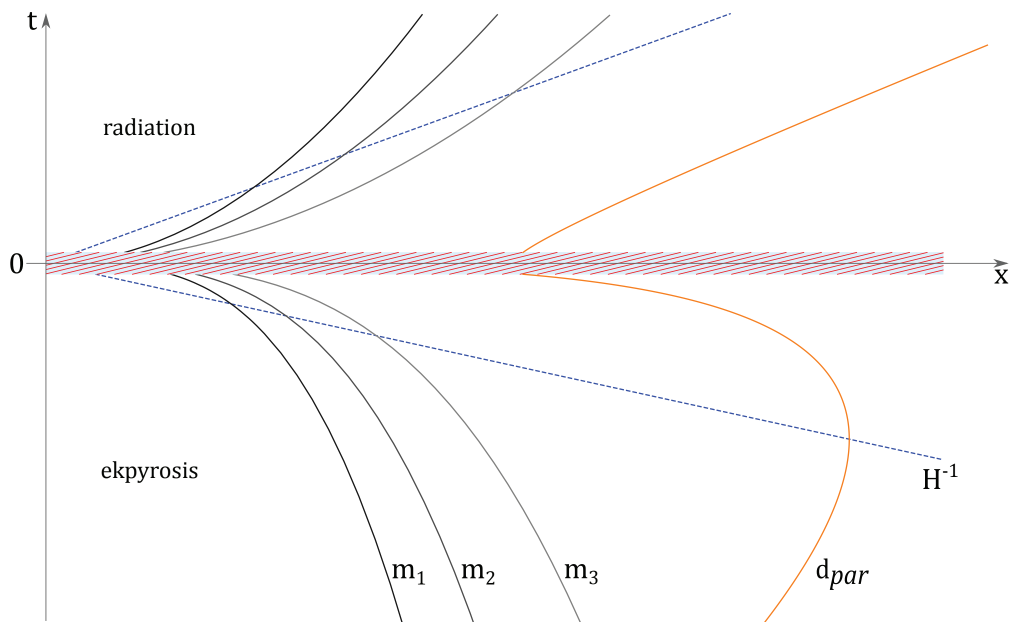

Figure 6: A spacetime diagram whose vertical axis is the cosmological time t and horizontal is physical distance x. The diagram portrays the generic features of the Hubble radius indicated by the blue dashed line, relative to the proper wavelengths of cosmological perturbation modes indicated by the gray-scale solid curves and comoving wavenumber $m_1 > m_2 > m_3$. In a generic phase of ekpyrotic contraction ($t \lesssim 0$) perturbation modes possess proper wavelengths much smaller than the Hubble radius but decrease less quickly, thus there is a period of ekpyrosis where modes evolve on super-Hubble scales. The bounce phase is indicated by the red shaded region and precedes a standard big bang period of radiation domination ($t \gtrsim 0$) in which perturbation modes may re-enter the curvature scale. The orange line labels the particle horizon during the ekpyrotic phase as well as the radiation domination phase after the bounce period. Notice that the physical wavelength of modes are at all times smaller than the particle horizon addressing the formation of structure problem.

respectively. Thus, not only does $\Omega \longrightarrow 1$ during a phase of ekpyrotic contraction (recall t is decreasing) but it is driven to spatial flatness faster than it is driven away from it during radiation and matter domination. Thus, so long as ekpyrotic contraction occurs for a sufficient period of time it will be driven close enough to spatial flatness before radiation domination in order to be consistent with the amount of spatial curvature observed today.

The singularity problem may be approached in one of two ways at this point in time. Either from the string theory perspective (consult section 6) in the sense that as the orbifold dimension collapses the string coupling goes to zero since $R_{11} = g_s^{2/3}$ [70], and so the details of the bounce may be understood in the context of string perturbation theory. Or, we simply never approach the Planck scale during the bounce phase and it is entirely non-singular as in [12].

In general, relics impose strict constraints on bouncing cosmologies since, contrary to inflation, the constructions do not immediately provide a mechanism in which the number density of unobserved relics may be reduced if in fact they are produced. Thus in bouncing models: supersymmetric particles, topological defects, primordial black holes, and other exotic particles [10] should not be produced at all if these phenomena continue to be unobserved. Their non-production is only ensured if the maximal temperature of the thermal history, in particular here referring to the collision energy of higher dimensional D-branes [47], is below that at which a symmetry breaking phase transition occurs, or in which stable or unstable exotic particles may be produced [71]. Of particular relevance here are the gauge symmetries that manifest via compactification of the heterotic string each leading to a corresponding monopole solution: the 't Hooft-Polyakov, Kaluza-Klein and H-monopoles [72].

The foreseeable issue with negative exponential potentials is that at some point, from the four dimensional field theory perspective, the potential must rise above zero in order to imprint upon the universe a positive cosmological constant as is expected with current observations, and so from this perspective the cosmological constant problem remains an issue [9, 73].

4 Linear cosmological perturbations

4.1 Theoretical background

We now introduce a formal description of the theory of cosmological perturbations in four spacetime dimensions within the framework of general relativity, with matter described by a scalar field(s). This allows for the proper treatment of perturbations on scales larger than the Hubble radius, and for relativistic fluids provided the analysis is carried out in a gauge-invariant way [4]. In particular, the gauge freedom can lead to fictitious perturbation modes that do not describe physical inhomogeneities, and instead result due to the properties of the chosen coordinate system; there are four gauge modes representing the invariance under linearized spacetime coordinate transformations.

We begin by perturbing about a spatially flat background FLRW metric ${}^{(0)}g_{\mu\nu}$

$$ds^2 = \left[{}^{(0)}g_{\mu\nu} + \delta g_{\mu\nu}(x^\gamma)\right] dx^\mu dx^\nu, \tag{4.1}$$

such that $|\delta g_{\mu\nu}| \ll |{}^{(0)}g_{\mu\nu}|$. In conformal time η defined as

$$\eta \equiv \int \frac{dt}{a(t)}, \tag{4.2}$$

with derivatives with respect to η represented by a prime $\frac{d\chi}{d\eta} \equiv \chi'$, the background FLRW metric takes on the following form

$${}^{(0)}g_{\mu\nu}dx^\mu dx^\nu = a^2(\eta)(d\eta^2 - \delta_{ij}dx^i dx^j). \tag{4.3}$$

The metric perturbations may be classified into three distinct classifications: scalar modes, vector modes and tensor modes (gravitational waves) [4, 71]. The classification is based on the symmetry properties of the isotropic background which on each spatial hypersurface is translationally and rotationally invariant.

The timelike δg_{00} component behaves as a scalar under rotations and thus

$$\delta g_{00} = 2a^2\varphi, \tag{4.4}$$

where φ is a scalar. The perturbative components δg_{0i} may be decomposed into a sum of components

$$\delta g_{0i} = a^2(B_{,i} + S_i), \tag{4.5}$$

where a comma followed by a spatial index indicates differentiation with respect to the spatial coordinate i

$$B_{,i} \equiv \frac{\partial B}{\partial x^i}, \tag{4.6}$$

and spatial indices may be raised and lowered by the unit metric δ_{ij}. The perturbative components δg_{ij} behave as a tensor under the generators of $SO(3)$, and may also be written as a sum of components

$$\delta g_{ij} = a^2(2\psi\delta_{ij} + 2E_{,ij} + F_{i,j} + F_{j,i} + h_{ij}), \tag{4.7}$$

such that ψ and E are scalars. Note that the vectors S_i and F_i are divergenceless

$$S^i_{,i} = F^i_{,i} = 0, \tag{4.8}$$

(repeated spatial indices are also summed over) and therefore each possess only two independent components. The symmetric rank two tensor h_{ij} is transverse and traceless

$$h^i_{j,i} = 0, \qquad h^i_i = 0. \tag{4.9}$$

Thus scalar perturbations are characterized by the four scalar functions φ, ψ, B and E and the perturbations are induced by inhomogeneities in the energy density of matter. These perturbations exhibit gravitational instability and thus may be the seeds for the formation of large scale structure in the universe. Vector perturbations are described by the two vectors S_i and F_i obeying two constraints. They are related to the rotational motion of the fluid, these modes typically decay relatively quickly. Tensor perturbations are described by the symmetric rank two tensor of dimension three, obeying four constraints, and thus possesses two independent components. These correspond to the two polarization modes of gravitational waves (such as the $+$ and $\times$ polarizations). Tensor perturbations possess no analogue in Newtonian gravity and describe gravitational waves, they are degrees of freedom of the gravitational field. To linear approximation as is done here, gravitational waves do not induce perturbations in the matter sector. Thus in total there are ten independent functions describing the three types of perturbations, this coincides with the number of independent components of $\delta g_{\mu\nu}$. However, this has not yet taken into consideration the specification of gauge. Also importantly, scalar, vector and tensor perturbations (at linear order in fluctuations) are decoupled and may therefore be studied independently [4, 6, 71, 74–77].

For scalar perturbations the metric takes the following form

$$ds^2 = a^2\left[(1+2\varphi)d\eta^2 + 2B_{,i}dx^i d\eta - \left((1-2\psi)\,\delta_{ij} - 2E_{,ij}\right)dx^i dx^j\right], \tag{4.10}$$

where we then define the gauge-invariant quantities Φ and Ψ which completely characterize the two-dimensional space of *physical* scalar perturbations

$$\Phi \equiv \varphi - \frac{1}{a}\left[a(B-E')\right]', \qquad \Psi \equiv \psi + \frac{a'}{a}(B-E'), \tag{4.11}$$

where one may check that these quantities remain invariant under an infinitesimal coordinate transformation to linear order in the transformation law.

To derive the equations for the perturbations, we linearize Einstein's equations (1.5) about a Friedmann universe with small inhomogeneities. We then construct the gauge-invariant perturbations (indicated by an overbar) of the Einstein tensor [4, 78]

$$\overline{\delta G}^0_0 = \delta G^0_0 + ({}^{(0)}G^0_0)'(B-E'), \tag{4.12}$$

$$\overline{\delta G}^0_i = \delta G^0_i - \left({}^{(0)}G^0_0 - \frac{1}{3}\,{}^{(0)}G^k_k\right)(B-E')_{,i}\,, \tag{4.13}$$

$$\overline{\delta G}^i_j = \delta G^i_j - ({}^{(0)}G^i_j)'(B-E'). \tag{4.14}$$

The gauge-invariant stress energy perturbations are defined similarly

$$\overline{\delta T}{}^0_0 = \delta T^0_0 - ({}^{(0)}T^0_0)'(B - E'), \tag{4.15}$$

$$\overline{\delta T}{}^0_i = \delta T^0_i - \left({}^{(0)}T^0_0 - \frac{1}{3}\,{}^{(0)}T^k_k\right)(B - E')_{,i}\ , \tag{4.16}$$

$$\overline{\delta T}{}^i_j = \delta T^i_j - ({}^{(0)}T^i_j)'(B - E'). \tag{4.17}$$

Note that the energy momentum tensor of the background obeys the following constraints

$${}^{(0)}T^0_i = 0, \qquad {}^{(0)}T^i_j \propto \delta_{ij}. \tag{4.18}$$

The gauge-invariant perturbations may then be expressed in terms of the gauge-invariant scalar metric perturbations Φ and Ψ

$$\overline{\delta G}{}^0_0 = \frac{2}{a^2}\left[-3\mathcal{H}(\mathcal{H}\Phi + \Psi') + \nabla^2\Psi + 3k\Psi\right], \tag{4.19}$$

$$\overline{\delta G}{}^0_i = \frac{2}{a^2}\left[\mathcal{H}\Phi + \Psi'\right]_{,i}\ , \tag{4.20}$$

$$\overline{\delta G}{}^i_j = -\frac{2}{a^2}\Bigg[\Bigg(\left[2\mathcal{H}' + \mathcal{H}^2\right]\Phi + \mathcal{H}\Phi' + \Psi'' + 2\mathcal{H}\Psi' \\ -k\Psi + \frac{1}{2}\nabla^2[\Phi - \Psi]\Bigg)\delta^i_j - \frac{1}{2}g^{ij}(\Phi - \Psi)_{,kj}\Bigg], \tag{4.21}$$

with $\mathcal{H}$ defined as

$$\mathcal{H} \equiv \frac{a'(\eta)}{a(\eta)} = aH. \tag{4.22}$$

4.1.1 Scalar mode metric perturbations: single scalar matter field

If we consider scalar field matter governed by the action as in equation (2.1), and represent small inhomogeneities $\delta\phi(\vec{x}, \eta)$ perturbing a homogeneous background $\phi_0(\eta)$

$$\phi = \phi_0 + \delta\phi, \tag{4.23}$$

we may derive the Klein-Gordon equation for the homogeneous component using the perturbed metric (4.10)

$$\phi_0'' + 2\mathcal{H}\phi_0' + a^2 V_{,\phi}(\phi_0) = 0. \tag{4.24}$$

We may also construct the gauge-invariant scalar field perturbation $\overline{\delta\phi}$

$$\overline{\delta\phi} \equiv \delta\phi - \phi_0'(B - E'). \tag{4.25}$$

The gauge-invariant stress energy perturbation describing scalar field matter may then be expressed as

$$\overline{\delta T}{}^0_{\ 0} = \frac{1}{a^2}[-\phi_0'^2\Phi + \phi_0'(\overline{\delta\phi})' + a^2 V_{,\phi}(\overline{\delta\phi})], \tag{4.26}$$

$$\overline{\delta T}{}^0_{\ i} = \frac{1}{a^2}[\phi_0'(\overline{\delta\phi})]_{,i}\ , \tag{4.27}$$

$$\overline{\delta T}{}^i_{\ j} = \frac{1}{a^2}[\phi_0'^2\Phi - \phi_0'(\overline{\delta\phi})' + a^2 V_{,\phi}(\overline{\delta\phi})]\delta^i_j. \tag{4.28}$$

Thus with the metric given by equation (4.10) the gauge-invariant linearized Einstein equations governing and relating matter and metric perturbations

$$\overline{\delta G^\mu_\nu} = 8\pi\overline{\delta T}{}^\mu_{\ \nu}, \tag{4.29}$$

may be expressed using equations (4.19) to (4.21) and (4.26) to (4.28) and the fact that the spatial sector of the stress-energy tensor is diagonal (implying $\Phi = \Psi$) [78]

$$3(k - \mathcal{H}^2)\Phi - 3\mathcal{H}\Phi' + \nabla^2\Phi = 4\pi[-\phi_0'^2\Phi + \phi_0'(\overline{\delta\phi})' + a^2 V_{,\phi}(\overline{\delta\phi})], \tag{4.30}$$

$$\mathcal{H}\Phi + \Phi' = 4\pi[\phi_0'(\overline{\delta\phi})], \tag{4.31}$$

$$\Phi'' + 3\mathcal{H}\Phi' + (2\mathcal{H}' + \mathcal{H}^2 - k)\Phi = 4\pi[-\phi_0'^2\Phi + \phi_0'(\overline{\delta\phi})' - a^2 V_{,\phi}(\overline{\delta\phi})]. \tag{4.32}$$

Equations (4.30) to (4.32) are valid independent of the choice of gauge, and are true for open, flat and closed geometries. Note that the spatial curvature affects the eigenvalues of the Laplacian [78]. For $k = 0$ the eigenvalues of the Laplacian $-\nabla^2$ are m^2 where $m \in (0, \infty)$; for $k = -1$ the eigenvalues of $-\nabla^2$ are $m^2 + 1$ with $m \in (0, \infty)$. In the longitudinal (conformal-Newtonian) gauge ($B = E = 0$), Φ and Ψ have a simple physical interpretation as the amplitude of the metric perturbations in the coordinate system such that φ is a generalization of the Newtonian potential. In particular, in a non-expanding universe, $\mathcal{H} = 0$, the 00 equation of motion governing Φ (4.30) reduces to a Poisson equation where Φ may be interpreted as the classical Newtonian gravitational potential. Thus in an expanding universe, equation (4.30) may be interpreted as a generalization of the Poisson equation and Φ may now be interpreted as the relativistic generalization of the Newtonian potential.

From the background Einstein equations

$$\frac{3}{a^2}[\mathcal{H}^2 + k] = 8\pi\left[\frac{1}{2a^2}\phi_0'^2 + V(\phi_0)\right], \tag{4.33}$$

$$\frac{1}{a^2}[2\mathcal{H}' + \mathcal{H}^2 + k] = 8\pi\left[-\frac{1}{2a^2}\phi_0'^2 + V(\phi_0)\right], \tag{4.34}$$

we may deduce the following relationship

$$\mathcal{H}^2 - \mathcal{H}' + k = 4\pi\phi_0'^2, \tag{4.35}$$

and thus equations (4.30) to (4.32) may be rewritten in a simplified form

$$(4k - 2\mathcal{H}^2 - \mathcal{H}')\Phi - 3\mathcal{H}\Phi' + \nabla^2\Phi = 4\pi[\phi_0'(\overline{\delta\phi})' + a^2 V_{,\phi}(\overline{\delta\phi})], \tag{4.36}$$

$$\mathcal{H}\Phi + \Phi' = 4\pi[\phi_0'(\overline{\delta\phi})], \tag{4.37}$$

$$\Phi'' + 3\mathcal{H}\Phi' + (\mathcal{H}' + 2\mathcal{H}^2)\Phi = 4\pi[\phi_0'(\overline{\delta\phi})' - a^2 V_{,\phi}(\overline{\delta\phi})]. \tag{4.38}$$

Using all three equations (4.36) to (4.38) as well as the background equation of motion for the scalar matter field (4.24), we may obtain the equation of motion for the scalar metric perturbations Φ [12, 44, 58, 74, 76–79]

$$\Phi'' + 2\left(\mathcal{H} - \frac{\phi_0''}{\phi_0}\right)\Phi' - \nabla^2\Phi + \left(2\mathcal{H}' - 2\mathcal{H}\frac{\phi_0''}{\phi_0'} - 4k\right)\Phi = 0. \tag{4.39}$$

We may rearrange equations (4.36) to (4.39) in order to express the gauge-invariant scalar field perturbations entirely in terms of the gauge-invariant scalar metric perturbations Φ and the homogeneous component of the scalar field ϕ_0. Equation (4.40) follows immediately from equation (4.37), and we may add equations (4.36) and (4.38) and substitute in equation (4.39) in order to obtain equation (4.41) [78]

$$(\overline{\delta\phi}) = \frac{1}{4\pi}\frac{1}{\phi_0'}\left[\Phi' + \mathcal{H}\Phi\right], \tag{4.40}$$

$$(\overline{\delta\phi})' = -\frac{1}{4\pi}\frac{1}{\phi_0'}\left[\left(\mathcal{H} - \frac{\phi_0''}{\phi_0'}\right)\Phi' + \left(-\nabla^2 - 4k + \mathcal{H}' - \mathcal{H}\frac{\phi_0''}{\phi_0'}\right)\Phi\right]. \tag{4.41}$$

Alternatively, we may derive a second order differential equation of motion governing the gauge-invariant scalar field perturbations by linearizing the Klein-Gordon equation (derived from Lagrangian (2.1)) about ϕ_0, we express it in flat space ($k = 0$) [74]

$$\overline{\delta\phi}'' + 2\mathcal{H}\overline{\delta\phi}' - \nabla^2\overline{\delta\phi} + V_{,\phi\phi}a^2\overline{\delta\phi} - 4\phi_0'\Phi' + 2V_{,\phi}a^2\Phi = 0. \tag{4.42}$$

4.1.2 Scalar mode metric perturbations: N-scalar matter fields

The above perturbation equations generalize straightforwardly when multiple scalar matter fields are introduced, as we must now consider isocurvature perturbations ('entropy perturbations') in addition to the curvature perturbation ('adiabatic perturbation') [80]. We follow the formalism introduced in [80] in order to study the curvature and isocurvature perturbations by decomposing the scalar field perturbations into those along the background trajectory (adiabatic perturbation) in field space and those orthogonal to it (entropy perturbation).

Firstly, we generalize the scalar matter action defined by the Lagrangian $\mathcal{L}_\phi$ to that involving $\boldsymbol{N}$ scalar fields indexed by a bold Latin superscript

$$\mathcal{L}_\phi = \sqrt{-g}\left(\frac{1}{2}\sum_{\boldsymbol{i}=1}^{\boldsymbol{N}}\left(g^{\mu\nu}\partial_\mu\phi^{\boldsymbol{i}}\partial_\nu\phi^{\boldsymbol{i}}\right) - V(\phi^{\boldsymbol{1}}, \cdots, \phi^{\boldsymbol{N}})\right). \tag{4.43}$$

giving rise to a Klein-Gordon equation in curved spacetime for each scalar field

$$\frac{1}{\sqrt{-g}}\partial_\mu\left[\sqrt{-g}\; g^{\mu\nu}\partial_\nu\phi^{\boldsymbol{i}}\right] + \frac{\partial V(\phi^{\boldsymbol{1}}, \cdots, \phi^{\boldsymbol{N}})}{\partial\phi^{\boldsymbol{i}}} = 0. \tag{4.44}$$

The stress-energy tensor may be generalized from the single field case given in equation (2.4)

$$T_{\mu\nu} \equiv -\frac{2}{\sqrt{-g}}\frac{\delta S_\phi}{\delta g^{\mu\nu}} = \partial_\mu \phi_{\boldsymbol{i}} \partial_\nu \phi^{\boldsymbol{i}} - g_{\mu\nu}\left(\frac{1}{2}g^{\mu\nu}\partial_\mu\phi_{\boldsymbol{i}}\partial_\nu\phi^{\boldsymbol{i}} - V(\phi^{\boldsymbol{1}},\cdots,\phi^{\boldsymbol{N}})\right). \tag{4.45}$$

With each field perturbed about a spatially homogeneous sector $\phi_0^{\boldsymbol{i}}(\eta)$

$$\phi^{\boldsymbol{i}} = \phi_0^{\boldsymbol{i}}(\eta) + \delta\phi^{\boldsymbol{i}}(\eta, \vec{x}), \tag{4.46}$$

the homogeneous sectors of each field satisfy the familiar simplified Klein-Gordon equation separately in an expanding flat FLRW background

$$\phi_0^{\boldsymbol{i}\prime\prime} + 2\mathcal{H}\phi_0^{\boldsymbol{i}\prime} + a^2 V(\phi_0^{\boldsymbol{i}})_{,\phi^{\boldsymbol{i}}} = 0, \tag{4.47}$$

and obey the following second Friedmann equation

$$\mathcal{H}^2 = \frac{8\pi}{3}\left[a^2 V(\phi_0^{\boldsymbol{1}},\cdots,\phi_0^{\boldsymbol{N}}) + \frac{1}{2}\sum_{\boldsymbol{i}=1}^{\boldsymbol{N}}(\phi_0^{\boldsymbol{i}\prime})^2\right], \tag{4.48}$$

as well as the following useful relations

$$\dot{H} = -4\pi\sum_{\boldsymbol{i}=1}^{\boldsymbol{N}}(\dot{\phi}_0^{\boldsymbol{i}})^2, \qquad \mathcal{H}^2 - \mathcal{H}' = 4\pi\sum_{\boldsymbol{i}=1}^{\boldsymbol{N}}(\phi_0^{\boldsymbol{i}\prime})^2. \tag{4.49}$$

Analogous to that of equation (4.42), second order differential equations may be derived for each scalar matter field by linearizing equation (4.44) in the presence of the perturbed metric (4.10), about $\phi_0^{\boldsymbol{i}\star}$ for each $\boldsymbol{i}\star$ in the domain of field indices. In terms of the gauge-invariant scalar field perturbations defined for each scalar matter field as

$$\overline{\delta\phi^{\boldsymbol{i}}} \equiv \delta\phi^{\boldsymbol{i}} - \phi_0^{\boldsymbol{i}\prime}(B - E'), \tag{4.50}$$

and in terms of the gauge-invariant scalar metric perturbation Φ we obtain the following second order differential equations for each field (in the absence of anisotropic stresses $\Longrightarrow$

$\Phi = \Psi$)

$$\overline{\delta\phi}^{\boldsymbol{i}\prime\prime} + 2\mathcal{H}\overline{\delta\phi}^{\boldsymbol{i}\prime} - \nabla^2\overline{\delta\phi}^{\boldsymbol{i}} + a^2\sum_{\boldsymbol{j}=1}^{\boldsymbol{N}}\left(V_{,\phi^{\boldsymbol{i}}\phi^{\boldsymbol{j}}}\overline{\delta\phi}^{\boldsymbol{j}}\right) - 4\phi_0^{\boldsymbol{i}\prime}\Phi' + 2V_{,\phi^{\boldsymbol{i}}}a^2\Phi = 0. \tag{4.51}$$

This differential equation may also be written in gauge dependent form as [80]

$$\delta\phi^{\boldsymbol{i}\prime\prime} + 2\mathcal{H}\delta\phi^{\boldsymbol{i}\prime} - \nabla^2\delta\phi^{\boldsymbol{i}} + a^2\sum_{\boldsymbol{j}=1}^{\boldsymbol{N}}\left(V_{,\phi^{\boldsymbol{i}}\phi^{\boldsymbol{j}}}\delta\phi^{\boldsymbol{j}}\right) + 2a^2V_{,\phi^{\boldsymbol{i}}}\varphi - \phi_0^{\boldsymbol{i}\prime}\left[\varphi' + 3\psi' - \nabla^2(B - E')\right] = 0. \tag{4.52}$$

Finally, for the study of multifield perturbations the Mukhanov-Sasaki gauge (spatially flat gauge) [81, 82] is a convenient choice where the scalar metric perturbation is defined to satisfy $\psi = 0$ [80, 83] and thus via the generalization of the linearized Einstein equations (to incorporate multiple scalar fields) the perturbations may be shown [80] to satisfy

$$\delta\phi^{\boldsymbol{i}\prime\prime} + 2\mathcal{H}\delta\phi^{\boldsymbol{i}\prime} - \nabla^2\delta\phi^{\boldsymbol{i}} + \sum_{\boldsymbol{j}=1}^{\boldsymbol{N}}\left[a^2V_{\phi^{\boldsymbol{i}}\phi^{\boldsymbol{j}}} - \frac{8\pi}{a^2}\left(\frac{a^2}{\mathcal{H}}\phi_0^{\boldsymbol{i}\prime}\phi_0^{\boldsymbol{j}\prime}\right)'\right]\delta\phi^{\boldsymbol{i}} = 0. \tag{4.53}$$

In this gauge, the scalar field perturbations $\delta\phi^{\boldsymbol{i}}$ are sometimes referred to via the Mukhanov-Sasaki variables $Q^{\boldsymbol{i}}$, which have the following definition [75, 80]

$$Q^{\boldsymbol{i}} \equiv \delta\phi^{\boldsymbol{i}} + \frac{\phi_0^{\boldsymbol{i}\prime}}{\mathcal{H}}\psi. \tag{4.54}$$

It is at this point that we define the adiabatic field and entropy fields as a rotation of basis in matter field space as displayed in figure 7. The adiabatic field ξ is defined as the path length of the homogeneous trajectory ('background' trajectory in [80]) in scalar field space and $\boldsymbol{N-1}$ entropy fields $s^{\boldsymbol{i}}$ are defined such that perturbations in these directions are orthogonal to the adiabatic field direction [80].

We work with the adiabatic and entropy field differentials first to ensure that $\vartheta \approx$ constant, we define the perturbations as a rotation of basis [12, 44, 69, 80]

$$d\xi = \cos(\vartheta)\ d\phi_0^{\boldsymbol{1}} + \sin(\vartheta)\ d\phi_0^{\boldsymbol{2}}, \tag{4.55}$$

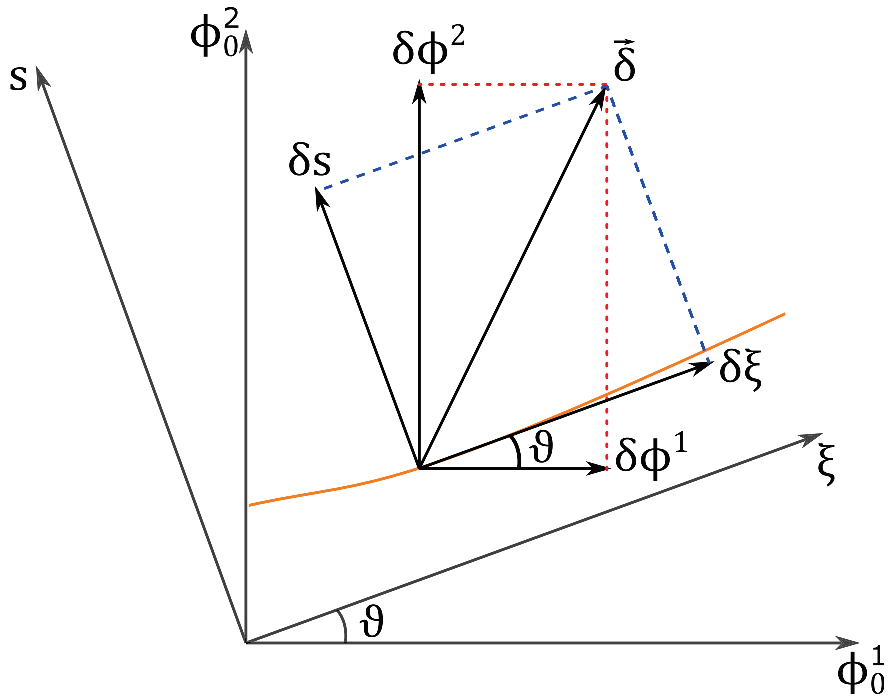

Figure 7: The adiabatic and entropy basis describing perturbations of two scalar matter fields are defined via a rotation in the scalar field spaces by an angle ϑ for each point about the homogeneous trajectory indicated by the orange solid line. An arbitrary perturbation about the homogeneous trajectory is represented by vector $\vec{\delta}$; in field space it may be described by a component along the homogeneous trajectory (adiabatic perturbation) and, for the case of two fields, by a single component orthogonal to the homogeneous trajectory (entropy perturbation). For $\boldsymbol{N}$ scalar fields we would instead include $\boldsymbol{N-1}$ orthogonal entropy fields. The perturbation components in field space $(\phi^{\mathbf{1}}, \phi^{\mathbf{2}})$ are given by $(\delta\phi^{\mathbf{1}}, \delta\phi^{\mathbf{2}})$ and are indicated by the red dotted lines, whereas the perturbation components in the adiabatic and entropy field space (ξ, s) are given by $(\delta\xi, \delta s)$ and are indicated by the blue dashed lines.

$$ds = -\sin(\vartheta)\ d\phi_0^{\mathbf{1}} + \cos(\vartheta)\ d\phi_0^{\mathbf{2}}, \tag{4.56}$$

$$\delta\xi = \cos(\vartheta)\ \delta\phi^{\mathbf{1}} + \sin(\vartheta)\ \delta\phi^{\mathbf{2}}, \tag{4.57}$$

$$\delta s = -\sin(\vartheta)\ \delta\phi^{\mathbf{1}} + \cos(\vartheta)\ \delta\phi^{\mathbf{2}}. \tag{4.58}$$

For the differentials we have the following constraints

$$d\xi\cos(\vartheta) = d\phi_0^{\mathbf{1}}, \qquad d\xi\sin(\vartheta) = d\phi_0^{\mathbf{2}}, \tag{4.59}$$

and thus dividing by the differential proper time we obtain

$$\cos(\vartheta) = \frac{\phi_0^{\mathbf{1}\prime}}{\xi'}, \qquad \sin(\vartheta) = \frac{\phi_0^{\mathbf{2}\prime}}{\xi'}, \qquad \tan(\vartheta) = \frac{\phi_0^{\mathbf{2}\prime}}{\phi_0^{\mathbf{1}\prime}}, \tag{4.60}$$

ensuring that the adiabatic field is the path length of the homogeneous trajectory, and that the modulus of the perturbation vector in field space $\vec{\delta}$ is conserved between coordinate systems

$$\xi' \equiv \sqrt{(\phi_0^{\mathbf{1}\prime})^2 + (\phi_0^{\mathbf{2}\prime})^2}, \qquad |\vec{\delta}| \equiv \sqrt{(\delta\phi^{\mathbf{1}})^2 + (\delta\phi^{\mathbf{2}})^2} = \sqrt{\delta\xi^2 + \delta s^2}. \tag{4.61}$$

Using equation (4.47) for the homogeneous sectors of the two matter fields, we find that the the adiabatic field ξ satisfies its own Klein-Gordon equation

$$\xi'' + 2\mathcal{H}\xi' + a^2 V_\xi = 0, \qquad V_\xi \equiv \cos(\vartheta) V_{,\phi^{\mathbf{1}}}(\phi_0^{\mathbf{1}}, \phi_0^{\mathbf{2}}) + \sin(\vartheta) V_{,\phi^{\mathbf{2}}}(\phi_0^{\mathbf{1}}, \phi_0^{\mathbf{2}}). \tag{4.62}$$

The definition of the entropy field given in equation (4.56) immediately implies that $s' = 0$, thus the entropy field is constant along the homogeneous trajectory and the entropy perturbations are manifestly gauge-invariant [80, 84]. Consequently via the perturbation equation (4.52) we may express the evolution of the entropy perturbations in an entirely gauge-invariant differential equation (in the absence of anisotropic stresses $\Longrightarrow \Phi = \Psi$) [12, 44, 69, 80]

$$\delta s'' + 2\mathcal{H}\delta s' + \left(m^2 + V_{ss}a^2 + 3(\vartheta')^2\right)\delta s = \frac{\vartheta'}{\xi'}\frac{m^2}{2\pi}\Phi. \tag{4.63}$$

Derivatives of the potential with respect to the entropy field s are provided as follows [44]

$$V_s = \frac{1}{\xi'}\left(\phi_0^{\mathbf{1}\prime} V_{,\phi^{\mathbf{2}}} - \phi_0^{\mathbf{2}\prime} V_{,\phi^{\mathbf{1}}}\right), \tag{4.64}$$

$$V_{ss} = \frac{1}{(\xi')^2}\left((\phi_0^{\mathbf{1}\prime})^2 V_{,\phi^{\mathbf{2}}\phi^{\mathbf{2}}} - 2\phi_0^{\mathbf{1}\prime}\phi_0^{\mathbf{2}\prime} V_{,\phi^{\mathbf{1}}\phi^{\mathbf{2}}} + (\phi_0^{\mathbf{2}\prime})^2 V_{,\phi^{\mathbf{1}}\phi^{\mathbf{1}}}\right), \tag{4.65}$$

$$V_{sss} = \frac{1}{(\xi')^3}\Big((\phi_0^{\mathbf{1}\prime})^3 V_{,\phi^{\mathbf{2}}\phi^{\mathbf{2}}\phi^{\mathbf{2}}} - 3(\phi_0^{\mathbf{1}\prime})^2\phi_0^{\mathbf{2}\prime} V_{,\phi^{\mathbf{1}}\phi^{\mathbf{2}}\phi^{\mathbf{2}}} + 3(\phi_0^{\mathbf{2}\prime})^2\phi_0^{\mathbf{1}\prime} V_{,\phi^{\mathbf{1}}\phi^{\mathbf{1}}\phi^{\mathbf{2}}} - (\phi_0^{\mathbf{2}\prime})^3 V_{,\phi^{\mathbf{1}}\phi^{\mathbf{1}}\phi^{\mathbf{1}}}\Big). \tag{4.66}$$

Evident via equation (4.63), the entropy perturbations decouple from the adiabatic perturbations in the case of $\vartheta' = 0$, qualitatively speaking this would imply that the homogeneous trajectory is a straight line in field space. In addition, on large scales the source term on the right hand side becomes negligible and thus either condition leads to a second-order homogeneous differential equation for entropy perturbations decoupled from adiabatic and scalar metric perturbations [80]. Finally it will be useful in further calculations to define generalized fast roll parameters ϵ_3, η_3 in terms of the adiabatic field ξ

$$\epsilon_3 \equiv M_P^{-2} \left(\frac{V}{V_\xi} \right)^2, \qquad \eta_3 \equiv 1 - \frac{V_{\xi\xi} V}{V_\xi^2}. \tag{4.67}$$

The first parameter ϵ_3 is a measure of the steepness of the potential along the homogeneous trajectory in scalar field space. In the case of ekpyrosis we have $\epsilon_3 \ll 1$. One may calculate that for pure exponential potentials as provided in equation (3.9), that $\eta_3 = 0$, thus in one sense η_3 may be interpreted as a measure of departure from exact exponential potentials.

4.1.3 Tensor mode metric perturbations

We now discuss the equations of motion describing the evolution of tensor perturbations in a classical cosmological background and in the presence of scalar field matter. In contrast to scalar metric perturbations described above, tensor modes may also be present in vacuum Einstein gravity, in other words, when there is no coupling to matter [75]. For tensor perturbations the perturbed metric takes the form

$$ds^2 = a^2 \left[d\eta^2 - (\delta_{ij} - h_{ij}) dx^i dx^j \right]. \tag{4.68}$$

The rank two tensor $h_{ij}(\eta, \vec{x})$ is already invariant under infinitesimal coordinate transformations, and thus is a gauge-invariant quantity. We may derive the equation of motion for the tensor perturbations by beginning with the Einstein-Hilbert action coupled to a matter Lagrangian (2.2) and inserting the perturbed metric (4.68). We then truncate terms to second order in tensor perturbations and in the absence of anisotropic stresses we obtain the following equation of motion by varying the action with respect to h_{ij}

$$(h''_{ij} + 2\mathcal{H}h'_{ij} - \nabla^2 h_{ij}) = 16\pi a^2 \overline{\delta T}_{ij}. \tag{4.69}$$

In the presence of scalar field matter, and the absence of anisotropic stresses the above simplifies since $\overline{\delta T}_{ij} = 0$

$$(h''_{ij} + 2\mathcal{H}h'_{ij} - \nabla^2 h_{ij}) = 0. \tag{4.70}$$

As previously stated, the tensor h_{ij} may be decomposed into the two polarization modes (we use the 'plus' +, and 'cross' $\times$ polarization modes)

$$h_{ij}(\eta, \vec{x}) = h_+(\eta, \vec{x})e^+_{ij} + h_\times(\eta, \vec{x})e^\times_{ij}, \tag{4.71}$$

where h_+ and $h_\times$ are scalar functions, and e^+_{ij}, $e^\times_{ij}$ are unit linear polarization tensors defined in terms of the tensor product ($\otimes$) of two dimensional orthogonal basis vectors (e^1_i, e^2_j) [6, 75, 85–87]

$$e^+_{ij} \equiv e^1_i \otimes e^1_j - e^2_i \otimes e^2_j, \qquad e^\times_{ij} \equiv e^1_i \otimes e^2_j + e^2_i \otimes e^1_j. \tag{4.72}$$

Substituting equations (4.71) and (4.72) into the equation of motion for the tensor perturbations (4.70) and converting our equation to Fourier space, we obtain identical equations of motion for both polarizations and each Fourier mode $\vec{m}$ (with mode subscripts suppressed) [44, 74, 75]

$$h''_{(+,\times)} + 2\mathcal{H}h'_{(+,\times)} + m^2 h = 0. \tag{4.73}$$

In many cases it is useful to introduce a rescaled field f defined in terms of h as

$$f \equiv a(\eta)h, \tag{4.74}$$

such that we now have a slightly altered differential equation in terms of f

$$f'' + (m^2 - \frac{a''}{a})f = 0. \tag{4.75}$$

4.1.4 Perturbations as random fields

Finally, we provide the background statistical treatment in which one may characterize cosmological perturbations and describe how this enables one to compare observational data with the predictions of different cosmological models. We have derived sets of equations that, given a set of initial conditions and provided we are able to solve the equations either analytically or numerically, should completely determine the evolution of cosmological perturbations unambiguously through time. However, we recognize that not only do we not have full access to the primordial perturbation spectrum due to us being limited to the information within our past light cone, but the equations we have derived only become accurate at energies below the Planck scale, where general relativity may be trusted. In other words, we have no sound theory (a quantum theory of gravity) capable of accurately providing a set of initial conditions to be fed to our equations in the low energy regime in which the physics is deterministic. Thus, the initial conditions we use are characterized by probability distributions calculated in a quantized field theory coupled to Einstein gravity.

Furthermore, cosmological quantities at late times are well characterized by averaged quantities due to the vastness of our universe, such as the average distance between galaxies or the average temperature of the CMB. For these reasons, it is useful then to treat the perturbation variables (such as Φ) as random fields, and if the field is described purely by a Gaussian distribution (which implies that each Fourier mode of the transformed perturbation variable are uncorrelated) then all odd $\mathcal{N}$-point correlation functions vanish and all even $\mathcal{N}$-point correlation functions may be described in terms of the distribution's variance; in frequency space this is known as the power spectrum. A non vanishing 3-point function (or any odd $\mathcal{N}$-point function) would provide evidence for, or be a requirement of a theory which produces a spectrum deviating from that produced by a Gaussian random field, stated differently the distribution of the perturbation variable is inexactly Gaussian or exhibits non-Gaussianities, which are induced by non-linearities in the equations of motion [21, 86, 88].

Let us define an N dimensional random field $G(\vec{x})$ [89], such that for any fixed $\vec{x}_i \in \mathbb{R}^N$, $G(\vec{x}_i)$ is a random variable, which we may refer to as a realization, described by a cumulative distribution function

$$F_{1,\cdots,n}(g_1,\cdots,g_n) = F[G(\vec{x}_1) \leq g_1, \cdots, G(\vec{x}_n) \leq g_n], \tag{4.76}$$

for any number of points n which we will define as the set $\mathcal{A}$, and where the lowercase $g_i \equiv g(\vec{x}_i)$ denotes a particular value the random variable may take evaluated at a fixed $\vec{x}_i$ for $i \in \mathcal{A}$. All possible values $G(\vec{x}_i)$ may attain for a fixed $\vec{x}_i$ form a statistical ensemble Ω. The marginal probability density function $p_i(g_i)$ for a fixed $\vec{x}_i$ is defined in terms of the marginal cumulative distribution function

$$F_i(g_i) = \lim_{\{g_m | m \in \mathcal{A} \setminus i\} \to \infty} F_{1,\cdots,n}(g_1,\cdots,g_n), \tag{4.77}$$

and the joint probability density function between the n points is defined in terms of the cumulative distribution function $F_{1,\cdots,n}(g_1,\cdots,g_n)$

$$p_i(g_i) \equiv \frac{dF_i(g_i)}{dg_i}, \tag{4.78}$$

$$p_{1,\cdots,n}(g_1,\cdots,g_n) \equiv \frac{\partial^n F_{1,\cdots,n}(g_1,\cdots,g_n)}{\partial g_1,\cdots,\partial g_n}. \tag{4.79}$$

In general it may not be that the probability density function p_i is equal to the probability density function $p_{i'}$ for $i \neq i'$. If however we have equality

$$p_i(g_i) = p_{i'}(g_{i'}), \tag{4.80}$$

then the probability distribution is translationally invariant in $\vec{x}$ and is said to be stationary or exhibit statistical homogeneity. Similarly, the probability density function is rotationally invariant in $\vec{x}$ and is said to exhibit statistical isotropy if

$$p_i(g_i) = p_{R1}(g_{R1}(R\vec{x}_1)), \tag{4.81}$$

for some rotation R applicable to the coordinate space, viewed another way as a rotation of the coordinate system. Finally, invariance under transformations of parity (reversing the handedness of the coordinate system) is a useful property that the random field may possess as it relates realizations of the random field, thereby reducing the number of independent correlation functions that may need to be considered in a given calculation.

We may define the expectation value (ensemble average) of the random field for a given $\vec{x}_i$ and the $\mathcal{N}$-point spatial correlation function respectively

$$\langle G(\vec{x}_i)\rangle \equiv \int_\Omega g_i p_i(g_i) dg_i, \tag{4.82}$$

$$\Xi^{(m)}(\vec{x}_1, \cdots, \vec{x}_\mathcal{N}) \equiv \langle G(\vec{x}_1)\cdots G(\vec{x}_\mathcal{N})\rangle \equiv \int_\Omega g_1 \cdots g_\mathcal{N} p_{1,\cdots,\mathcal{N}}(g_1, \cdots, g_\mathcal{N}) dg_1 \cdots dg_\mathcal{N}. \tag{4.83}$$

Importantly, if a random field is statistically homogeneous then the 2-point correlation function has the following property

$$\Xi^{(2)}(\vec{x}_1, \vec{x}_2) = \Xi^{(2)}(\vec{x}_1 - \vec{x}_2), \tag{4.84}$$

and if the random field is also statistically isotropic then the constraint is strengthened further

$$\Xi^{(2)}(\vec{x}_1, \vec{x}_2) = \Xi^{(2)}(|\vec{x}_1 - \vec{x}_2|). \tag{4.85}$$

Similarly, we define the ensemble variance of the random field as

$$\sigma^2(\vec{x}_1, \vec{x}_2) \equiv \langle G(\vec{x}_1)G(\vec{x}_2)\rangle - \langle G(\vec{x}_1)\rangle\langle G(\vec{x}_2)\rangle, \tag{4.86}$$

noting that the variance may be subject to greater constraints as in equations (4.84) and (4.85) if the random fields are statistically homogeneous and isotropic.

Observations in cosmology are limited to computing spatial averages, thus the ergodic properties of the fields in question is imperative. The ergodic property is the statement that the 2-point correlation function $\langle G(\vec{x}_1)G(\vec{x}_2)\rangle$ (and similarly the m-point correlation function) may be expressed approximately by the spatial average at a fixed $\vec{x}_1 - \vec{x}_2$ for a single realization of the ensemble representing all realizations of the random field. The property follows from homogeneity of the random field in addition to some other weak assumptions, the proof may be found in Weinberg's text as well as Lyth and Liddle [89, 90].

In analyzing perturbations, we will find it necessary to utilize Fourier techniques. Physically, we first define the stochastic properties of the perturbations when treated as random fields at fixed time. Thus the argument of the random field corresponds to the spatial degrees of freedom and we choose to carryout the Fourier analysis in a volume whose side lengths are L. The boundary conditions then immediately imply that the wave vectors $\vec{m}_s$ with index s form a cubic lattice with spacing $2\pi/L$. We then justify that the side lengths of the finite volume are much larger than the observable universe allowing the lattice spacing to go to zero and the wave vectors constitute a justifiably approximate continuous spectrum. The discrete Fourier analysis may then be replaced by the continuous version which allows for more succinct calculations.

The Fourier series and its inverse as well as the infinite space limit ($L \longrightarrow \infty$) corresponding to the Fourier transform and its inverse are defined in this thesis as

$$G(\vec{x}) = \frac{1}{L^3}\sum_s G_s e^{i\vec{m}_s\cdot\vec{x}}, \qquad G_s = \int G(\vec{x}) e^{-i\vec{m}_s\cdot\vec{x}} d^3x, \tag{4.87}$$

$$G(\vec{x}) = \frac{1}{(2\pi)^3}\int G(\vec{m}) e^{i\vec{m}\cdot\vec{x}} d^3m, \qquad G(\vec{m}) = \int G(\vec{x}) e^{-i\vec{m}\cdot x} d^3x. \tag{4.88}$$

We impose that the Fourier coefficients obey a reality condition, $G(-\vec{m}) = G^*(\vec{m})$ which ensures that the complex component of the m^{th} mode cancels out with that of the $-m^{\text{th}}$ mode. More precisely, if $G(\vec{m}) = a(\vec{m}) + ib(\vec{m})$ then $a(\vec{m}) = a(-\vec{m})$ and $b(-\vec{m}) = -b(\vec{m})$. Both the discrete and continuous case satisfy orthogonality relations respectively

$$\int e^{i(\vec{m}_s-\vec{m}_t)\cdot\vec{x}} d^3x = L^3\delta_{st}, \qquad \int e^{i(\vec{m}-\vec{m}')\cdot\vec{x}} d^3x = (2\pi)^3\delta^3(\vec{m}-\vec{m}'). \tag{4.89}$$

First working in a finite volume (discrete case) we define a Gaussian random field by the property that its Fourier coefficients are uncorrelated, by the reality condition we have

$$\langle G_s G_t \rangle = \delta_{s,-t}\langle |G_s|^2\rangle \equiv \delta_{s,-t}\mathcal{P}_{G_s}. \tag{4.90}$$

The factor $\mathcal{P}_{G_s}$ is known as the power spectrum, and in the continuum limit it may be defined in terms of the Fourier transform. A characteristic of Gaussian random fields is that all odd

$\mathcal{N}$-point correlation functions vanish, the 1-point function may also be made to vanish by absorbing the only potentially non zero coefficient of the zero mode into the unperturbed background

$$\langle G_s \rangle = \langle G_{s_1} G_{s_2} G_{s_3} \rangle = \cdots = 0. \tag{4.91}$$

All higher order even $\mathcal{N}$-point functions may be expressed in terms of the 2-point function and therefore in terms of the power spectrum, for example the 4-point function

$$\langle G_{s_1} G_{s_2} G_{s_3} G_{s_4} \rangle = \langle G_{s_1} G_{s_2} \rangle \langle G_{s_3} G_{s_4} \rangle + \langle G_{s_1} G_{s_3} \rangle \langle G_{s_2} G_{s_4} \rangle + \langle G_{s_1} G_{s_4} \rangle \langle G_{s_2} G_{s_3} \rangle. \tag{4.92}$$

In the continuum limit, the 2-point correlation function of two Fourier transformed statistically homogeneous random fields may be expressed as

$$\langle G(\vec{m}) G(\vec{m}') \rangle = \int e^{-i(\vec{m}-\vec{m}')\cdot\vec{x}} d^3x \int \Xi^{(2)}(\vec{z}) e^{i\vec{m}'\cdot\vec{z}} d^3z. \tag{4.93}$$

Using the representation of the Dirac delta and the orthogonality relations given in equation (4.89), we re-express the above as

$$\langle G(\vec{m}) G(\vec{m}') \rangle = (2\pi)^3 \delta^{(3)}(\vec{m} - \vec{m}') \mathcal{P}_G(\vec{m}), \tag{4.94}$$

with the power spectrum $\mathcal{P}_G(\vec{m})$ defined as the Fourier transform of the 2-point spatial correlation function

$$\mathcal{P}_G(\vec{m}) \equiv \int \Xi^{(2)}(\vec{x}) e^{-i\vec{m}\cdot\vec{x}} d^3x, \qquad \Xi^{(2)}(\vec{x}) = \frac{1}{(2\pi)^3} \int \mathcal{P}_G(\vec{m}) e^{i\vec{m}\cdot\vec{x}} d^3m. \tag{4.95}$$

If one also assumes statistical isotropy, then the power spectrum depends only on the modulus of $\vec{m}$ and the 2-point spatial correlation function depends on the modulus of $\vec{x}$, and we may simplify their forms to single integrals

$$\mathcal{P}_G(m) = 4\pi \int_0^\infty r^2 \Xi^{(2)}(r) \frac{\sin(mr)}{mr} dr, \qquad \Xi^{(2)}(r) = \int_0^\infty \frac{m^2 \mathcal{P}_G(m)}{2\pi^2} \frac{\sin(mr)}{mr} dm. \tag{4.96}$$

Similarly, the 4-point function in Fourier space equation (4.92) may be expressed in the continuum limit as

$$\langle G(\vec{m}_1)G(\vec{m}_2)G(\vec{m}_3)G(\vec{m}_4)\rangle = (2\pi)^6\delta^{(3)}(\vec{m}_1+\vec{m}_2)\delta^{(3)}(\vec{m}_3+\vec{m}_4)\mathcal{P}_G(m_1)\mathcal{P}_G(m_3) + 2 \text{ permutations.} \quad (4.97)$$

The random field $G(\vec{x})$ (in position space) is a superposition of Fourier modes, and as we have already stated: Gaussian random fields are defined by the property of possessing uncorrelated (independent) Fourier modes. Therefore, regardless of the probability distribution of the random variable describing the amplitudes of each mode, by the central limit theorem, the sum of the amplitudes of the uncorrelated modes is described by a Gaussian (normal) probability distribution. Therefore the probability distribution of $G(\vec{x})$ for every $\vec{x}$ is described by a Gaussian probability density

$$p(g) = \frac{1}{\sqrt{2\pi}\sigma_g}\exp\left(-\frac{g^2}{2\sigma_g^2}\right), \quad (4.98)$$

where we have assumed the expectation value (first moment) is vanishing. Note that the probability density is independent of $\vec{x}$, thus Gaussianity (uncorrelated Fourier modes) implies statistical homogeneity. If we further assume that the power spectrum is invariant under rotations at every $\vec{m}$, then the use of the expressions in equation (4.96) is justified. The ensemble variance defined in equation (4.86) may be expressed for a statistically homogeneous and isotropic field in terms of the power spectrum $\mathcal{P}_G$ using equations (4.84) and (4.95) (assuming vanishing first moment)

$$\sigma_g^2(\vec{x}) = \langle G^2(\vec{x})\rangle = \Xi^{(2)}(\vec{0}) = \frac{1}{(2\pi)^3}\int_0^\infty \mathcal{P}_G(m)d^3m = \int_0^\infty P_G(m)\frac{dm}{m}. \quad (4.99)$$

In equation (4.99) we have introduced the quantity $P_G(m)$, and define it as the dimensionless power spectrum

$$P_G(m) \equiv \frac{m^3\mathcal{P}_G(m)}{2\pi^2}. \quad (4.100)$$

Thus for a given linear, Gaussian, small perturbation variable in Fourier space such as $\Phi(m)$, we may calculate the field's dimensionless power spectrum if we treat it as a Gaussian random field, since the mean square amplitude of the perturbation variable $|\Phi(m)|^2$ is directly proportional to $\mathcal{P}_\Phi(m)$ by equation (4.94) for the continuous case and equation (4.90) for the discrete case.

To recapitulate: the Fourier coefficients of a Gaussian perturbation have minimal correlation. In particular, all odd $\mathcal{N}$-point correlators of the Fourier coefficients vanish and all even $\mathcal{N}$-point correlators may be expressed in terms of the power spectrum $\mathcal{P}_G$. The Fourier coefficients of a non-Gaussian perturbation possess correlation not only specified by the power spectrum and not necessarily vanishing odd $\mathcal{N}$-point correlations functions in the Fourier coefficients.

The amplitude of metric perturbations may depend on length scale. The spectral index $n_G(m)$ may be defined to characterize the scale dependence of the spectrum

$$n_G - 1 \equiv \frac{d \ln [P_G(m)]}{d \ln m}. \tag{4.101}$$

If $n_G = 1$ the spectrum is said to be scale independent (or flat), otherwise it exhibits either a red ($n_G < 1$) or blue ($n_G > 1$) tilt. If the spectral index is independent of m or we are only considering a small range of scales then the power spectrum $P_G \propto m^{n_G - 1}$. If instead n_G depends on the wave number m the spectral index is said to be running. In this case one possibility is to assume that n_G may be approximated as a linear function of $\ln m$ thereby defining the behaviour of the index through understanding of the rate of change $dn_G/d\ln m$ [4, 89].

We now wish to determine the possible forms of $\mathcal{N}$-point correlators when considering non-Gaussianities obeying translational and rotational invariance. We previously argued that the 1-point function (mean) may be absorbed into the unperturbed background quantity, similarly all correlators between the zero modes may be absorbed into the unperturbed background quantity. The generalized logic of the previous statement is as follows: translational invariance demands that each $\mathcal{N}$-point correlator in Fourier space be proportional to a delta function or a product of up to $\mathcal{N}$ deltas whose arguments are the Fourier modes of the correlator. However a product of delta functions consisting of at least one delta function possessing in its argument the Fourier variable of only one field such as $\delta^{(3)}(\vec{m}_i)$ for $i \in [1, \mathcal{N}]$, immediately implies that the quantity vanishes unless the argument of every delta function of that type present in the term is zero; but the zero modes may be reabsorbed into the unperturbed quantities. Therefore we need not consider contributions to $\mathcal{N}$-point correlation functions containing delta functions of the type $\delta^{(3)}(\vec{m}_i)$. Consequently, the 2-point correlator has the same form as for the Gaussian case. By the above argument, the form of the 3-point correlator for the non-Gaussian case is uniquely

$$\langle G(\vec{m}_1)G(\vec{m}_2)G(\vec{m}_3)\rangle = (2\pi)^3\delta^{(3)}(\vec{m}_1+\vec{m}_2+\vec{m}_3)B_G(m_1,m_2,m_3). \tag{4.102}$$

The factor $B_G(m_1,m_2,m_3)$ is known as the bispectrum, and may be defined in terms of the reduced bispectrum $\mathcal{B}_G(m_1,m_2,m_3)$ and the power spectrum $\mathcal{P}_G$ of each pair of Fourier modes

$$\begin{aligned} B_G(m_1,m_2,m_3) \equiv \mathcal{B}_G(m_1,m_2,m_3)[&\mathcal{P}_G(m_1)\mathcal{P}_G(m_2) \\ &+\mathcal{P}_G(m_2)\mathcal{P}_G(m_3)+\mathcal{P}_G(m_1)\mathcal{P}_G(m_3)]. \end{aligned} \tag{4.103}$$

In position space, we may calculate the 3-point correlators

$$\langle G(\vec{x}_1+\vec{x}_3)G(\vec{x}_2+\vec{x}_3)G(\vec{x}_3)\rangle = \frac{1}{(2\pi)^3}\int B_G(m_1,m_2,m_3)e^{i(\vec{m}_1\cdot\vec{x}_1+\vec{m}_2\cdot\vec{x}_2)}d^3m_1d^3m_2, \tag{4.104}$$

$$\langle G^3(\vec{x}_3)\rangle = \frac{1}{(2\pi)^3}\int B_G(m_1,m_2,m_3)d^3m_1d^3m_2, \tag{4.105}$$

with equation (4.105) providing a correction to the Gaussian spectrum known as the skewness S_G, defined as the third standardized moment in terms of the second and third order cumulants (equal to the moments to the first three orders for a distribution of mean zero)

$$S_G \equiv \frac{\langle G^3(\vec{x})\rangle}{\langle G^2(\vec{x})\rangle^{3/2}}. \tag{4.106}$$

The skewness may be regarded as a measure of asymmetry of the probability distribution of the random variable about the mean. If P_G and $\mathcal{B}_G$ are both independent of scale then $S_G \sim \mathcal{B}_G P_G^{1/2}$, and an approximately Gaussian perturbation will have $\mathcal{B}_G \ll P_G^{-1/2}$.

For a Gaussian random variable, the 4-point correlator of the Fourier coefficients is already non-vanishing. However in the non-Gaussian case, there may be an additional contribution of the form

$$\langle G(\vec{m}_1)G(\vec{m}_2)G(\vec{m}_3)G(\vec{m}_4)\rangle = (2\pi)^3\delta^{(3)}(\vec{m}_1+\vec{m}_2+\vec{m}_3+\vec{m}_4)T_G, \tag{4.107}$$

with T_G known as the trispectrum. Once again, we may define the reduced trispectra $\mathcal{T}_{G_i}$ [91]

$$\begin{aligned} T_G(\vec{m}_1,\vec{m}_2,\vec{m}_3,\vec{m}_4) \equiv \mathcal{T}_{G_1}[\mathcal{P}_G(m_1)\mathcal{P}_G(m_2)\mathcal{P}_G(|\vec{m}_1-\vec{m}_4|) + 23 \text{ permutations}] \\ + \mathcal{T}_{G_2}[\mathcal{P}_G(m_2)\mathcal{P}_G(m_3)\mathcal{P}_G(m_4) + 3 \text{ permutations}], \end{aligned} \tag{4.108}$$

with the permutations running over the arguments of the power spectra. The 4-point function present for a Gaussian perturbation as in equation (4.97) has the same form in position space

$$\langle G(\vec{x}_1)G(\vec{x}_2)G(\vec{x}_3)G(\vec{x}_4)\rangle = \langle G(\vec{x}_1)G(\vec{x}_2)\rangle\langle G(\vec{x}_3)G(\vec{x}_4)\rangle + 2 \text{ permutations}. \tag{4.109}$$

This is known as the disconnected contribution. It does not go to zero as the separation between points paired in a 2 point correlator goes to infinity. The contribution that may be present for a non-Gaussian perturbation given by equation (4.107) is known as the connected contribution, and is only non zero if, for the case of the 4-point function all four points are sufficiently close to one another. We may calculate the fourth order cumulant in position space (for a distribution of mean zero)

$$\langle G^4(\vec{x})\rangle - 3\langle G^2(\vec{x})\rangle^2 \equiv \langle G^4(\vec{x})\rangle_c = \frac{1}{(2\pi)^6}\int T_G d^3m_1 d^3m_2 d^3m_3. \tag{4.110}$$

Equation (4.110) allows us to define the kurtosis as the fourth standardized moment

$$K_G \equiv \frac{\langle G^4(\vec{x})\rangle_c}{\langle G^2(\vec{x})\rangle^2}. \tag{4.111}$$

Similarly as for the bispectrum, if the reduced trispectrum is approximately scale invariant $K_G \sim \mathcal{T}_G P_G$ and the kurtosis provides a measure of non-Gaussianity associated with the trispectrum. This process generalizes for $\mathcal{N}$-point correlators, there will be present disconnected contributions present for both the Gaussian and non-Gaussian case proportional to a product of delta functions (note these delta functions possess a sum of Fourier modes in their argument, not just one, these terms do not automatically vanish for non zero Fourier modes). The connected contribution may or may not be present, depending on the level of non-Gaussianity. If the connected contribution is present, the introduction of a new function analogous to the bispectrum and trispectrum must be introduced, and the disconnected contribution would be proportional to a single delta function whose argument is a sum of all $\mathcal{N}$ Fourier modes.

4.2 Scalar metric perturbations in ekpyrosis

In a contracting ekpyrotic model, scalar and tensor perturbations are generated and have contrasting predictions to inflationary models. It is worth noting that generally speaking, although vector modes grow during a contracting phase, they decay quickly during the subsequent expanding phase [92]. In this subsection, we study the scalar mode metric perturbations generated in single and two field ekpyrotic models.

We will see that single field minimally coupled canonical kinetic ekpyrotic models generally lead to a scale invariant spectrum for the generalized Newtonian potential Φ and a strongly blue spectrum for the curvature perturbation ζ in disagreement with observations. When the model is generalized to two scalar fields an approximately scale invariant spectrum of entropy perturbations may be generated either with a small blue tilt for pure exponential potentials, or a small red tilt for potentials that deviate from pure exponentiality subject to some constraints.

Lastly we study a particular mechanism in which the entropy perturbations generated in two field models may be converted to curvature perturbations via a curve in the trajectory traversed in field space. This allows the curvature perturbation to inherit the approximately scale invariant entropy perturbations generated during the ekpyrotic phase allowing for a comparison with observed data.

Note that in this thesis, we simply assume that a non-singular bounce phase permits the curvature perturbation to traverse the bounce and remain approximately constant or traceable as in [12, 93] through for instance the use of a ghost condensate.

4.2.1 Single matter field

The first rigorous approach to cosmological perturbations in the ekpyrotic scenario was done in [58] where they applied the methods of [78] of evolving Bunch-Davies vacuum modes through an inflationary period. In the following analysis we follow closely the work of [44, 58, 79]. First we express the scaling solution (3.5) in terms of the conformal time η (up to an irrelevant rescaling)

$$a(\eta) = (-\eta)^{p/(1-p)}, \qquad \phi(\eta) = \frac{2}{c(1-p)} \ln(-\eta), \tag{4.112}$$

with c defined as in equation (3.6). We will also make use of the parameter ϵ_1 defined in equation (3.17), with fast-roll achieved for $\epsilon_1 \gg 1$.

In addition to solving for the scalar metric perturbations described by the generalized Newtonian potential Φ, we will also track curvature perturbations on spatial hyper surfaces that are comoving with the matter field. The curvature perturbations may be represented in terms of the gauge-invariant perturbation variable ζ in comoving gauge ($E_{,ij} = \delta T^0_i = 0, \zeta = \psi \implies \delta\phi = 0$ [79, 94]) where it represents the curvature perturbation on spatial hypersurfaces, and are not sensitive to the growing mode density perturbation in the contracting phase [58]. The curvature perturbation is defined in terms of Φ [44, 58, 75, 76, 79, 80, 94–97]

$$\zeta \equiv \Phi + \frac{1}{\epsilon_1}\left(\frac{\Phi'}{\mathcal{H}} + \Phi\right) = \Phi - \frac{H}{\dot{H}}\left(\dot{\Phi} + H\Phi\right) = \frac{\mathcal{H}\Phi' + \mathcal{H}^2\Phi}{4\pi\left(\phi_0'\right)^2} + \Phi, \tag{4.113}$$

$$\Phi = -\epsilon_1 \frac{\mathcal{H}}{m^2}\zeta' = \frac{\dot{H}}{H}\frac{a^2}{m^2}\dot{\zeta}, \tag{4.114}$$

which for fluctuations on super-Hubble scales (such that $\Delta\phi$ corrections are negligible) is equal to the curvature perturbation $\mathcal{R}$ [75] introduced in [94]. Working in Fourier space (we suppress subscripts denoting the m^{th} Fourier mode) the generalized Newtonian potential Φ governed by equation (4.39) in flat space $k = 0$ now obeys the following differential equation for each Fourier mode $\vec{m}$

$$\Phi'' + 2\left(\mathcal{H} - \frac{\phi_0''}{\phi_0}\right)\Phi' + 2\left(m^2 + \mathcal{H}' - \mathcal{H}\frac{\phi_0''}{\phi_0'}\right)\Phi = 0, \tag{4.115}$$

where $m = |\vec{m}|$ is the magnitude of the comoving Fourier three-vector. The curvature perturbation ζ obeys the following equation of motion for each mode [74, 79]

$$\zeta'' + 2\frac{z'}{z}\zeta' + m^2\zeta = 0. \tag{4.116}$$

where we have defined the following quantities z, θ in terms of the background

$$z \equiv \frac{a\phi_0'}{\mathcal{H}}, \qquad \theta \equiv \frac{1}{z} = \frac{a'}{a^2\phi_0'}. \tag{4.117}$$

We also define the new variables u and v (the Mukhanov variable [75]) noting that they possess the same m dependence to Φ and ζ respectively and thus have the same spectral properties

$$u \equiv \frac{a}{\phi_0'}\Phi, \qquad v \equiv z\zeta. \tag{4.118}$$

We may simplify the differential equations of motion governing the scalar perturbations (4.115) and (4.116) by working in terms of u, v

$$u'' + m^2 u - \frac{\theta''}{\theta}u = 0, \tag{4.119}$$

$$v'' + m^2 v - \frac{z''}{z}v = 0, \tag{4.120}$$

they are related via

$$v = 2\left[u' + \frac{z'}{z}u\right], \qquad u = -\frac{1}{2m^2}\left[v' + \frac{\theta'}{\theta}v\right]. \tag{4.121}$$

When the equation of state of the scalar matter field is time independent, we may use the scaling solution (4.112) expressed in conformal time to determine the coefficients for equations (4.119) and (4.120)

$$\frac{\theta''}{\theta} = \frac{\epsilon_1}{(\epsilon_1 - 1)^2\eta^2}, \qquad \frac{z''}{z} = \frac{2-\epsilon_1}{(\epsilon_1 - 1)^2\eta^2}. \tag{4.122}$$

With the coefficients expressed in terms of the conformal time η we make the approximation that at early times ($\eta \to -\infty$) the m^2 terms in equations (4.119) and (4.120) dominate over the θ''/θ, z''/z terms ($m^2 \gg z''/z$) mimicking the equation for a simple harmonic oscillator. The scaling solution dictates that $z(\eta) \sim a(\eta)$ since $\mathcal{H}(\eta) \propto \phi_0'(\eta)$, indicating that the physical interpretation of the previous statement is that the (relevant) curvature perturbation modes are well within the Hubble radius (curvature scale), where the solutions are asymptotically constant amplitude oscillations [44, 75]. We thus choose as initial conditions the Minkowski vacuum (Ch. 3 of Birrell & Davies [98]) for a comoving observer in the distant past, since on scales much smaller than the curvature scale the spacetime approaches Minkowski [44, 66, 79] [89, Ch. 24]

$$u \xrightarrow[\eta\to-\infty]{} i(2m)^{-3/2}\exp(-im\eta), \tag{4.123}$$

$$v \xrightarrow[\eta\to-\infty]{} (2m)^{-1/2}\exp(-im\eta). \tag{4.124}$$

As a consistency check to our ekpyrotic initial condition that the space be asymptotically Minkowski in the far past, we may justify this choice of initial condition in another way [58]: in the Newtonian gauge there are two equations relating the scalar field perturbations $\delta\phi$ with the scalar metric perturbations Φ

$$\delta\phi = \frac{2}{\dot{\phi}_0}\left[\dot{\Phi} + H\Phi\right], \tag{4.125}$$

$$\dot{\delta\phi} = -\frac{2}{\dot{\phi}_0}\left[-\frac{\ddot{\phi}_0}{\dot{\phi}_0}\dot{\Phi} + \left[\frac{m^2}{a^2} + \dot{\phi}_0\partial_t\left(\frac{H}{\dot{\phi}_0}\right)\right]\Phi\right]. \tag{4.126}$$

At large $-mt$ for an incoming Minkowski vacuum (as seen by an observer in coordinate time) $\delta\phi \sim e^{-im\eta}/(a\sqrt{2m})$ we find that the scalar metric perturbation

$$\Phi \sim \frac{i\sqrt{p}}{2m^{3/2}t}\exp(-im\eta) \xrightarrow[\eta\to-\infty]{} 0. \tag{4.127}$$

The equations of motion (4.119) and (4.120) are a form of Bessel equation in absence of the term carrying the first order derivative in the dependent variable, and may be solved exactly (see equation (9.1.49) in [99]) by a linear combination of Hankel functions of the first and second kind of order ν ($H^{(1,2)}_{\nu}(x)$) [79, 99]

$$u(x) = x^{1/2}\left[A^{(1)}H^{(1)}_{\alpha_1}(x) + A^{(2)}H^{(2)}_{\alpha_1}(x)\right], \tag{4.128}$$

$$v(x) = x^{1/2}\left[B^{(1)}H^{(1)}_{\alpha_2}(x) + B^{(2)}H^{(2)}_{\alpha_2}(x)\right]. \tag{4.129}$$

Note that $x \equiv m|\eta|$ is a dimensionless time variable, $A^{(1,2)}$, $B^{(1,2)}$ are constant coefficients and the Hankel function order may be expressed in terms of the fast roll parameter ϵ_1

$$\alpha_1 \equiv \sqrt{\frac{\theta''}{\theta}\eta^2 + \frac{1}{4}} = \frac{1}{2}\left|\frac{\epsilon_1 + 1}{\epsilon_1 - 1}\right|, \tag{4.130}$$

$$\alpha_2 \equiv \sqrt{\frac{z''}{z}\eta^2 + \frac{1}{4}} = \frac{1}{2}\left|\frac{\epsilon_1 - 3}{\epsilon_1 - 1}\right|. \tag{4.131}$$

In order to determine the coefficients $A^{(1,2)}$, $B^{(1,2)}$ we use the asymptotic Hankel expression

$$H^{(1,2)}_{\nu}(x) \xrightarrow[|x|\to\infty]{} \sqrt{\frac{2}{\pi x}}\exp\left(\pm i\left[x - \frac{\nu\pi}{2} - \frac{\pi}{4}\right]\right). \tag{4.132}$$

Thus the Minkowski vacuum initial conditions equations (4.123) and (4.124) imply up to an irrelevant overall phase factor [44, 58, 79]

$$u(x) = \frac{\sqrt{\pi x}}{4m^{3/2}}H^{(1)}_{\alpha_1}(x), \tag{4.133}$$

$$v(x) = \frac{\sqrt{\pi x}}{2m^{1/2}}H^{(1)}_{\alpha_2}(x). \tag{4.134}$$

In order to determine the power spectra of ζ and Φ at late times when comoving scales are far outside the curvature scale, one may relate the power spectrum to the growth rate of u, v on super-Hubble scales to the power spectrum at the time of Hubble radius crossing $t_H(k)$; then use the fact that the amplitude of u, v are constant on sub-Hubble scales in order to relate it to the initial vacuum conditions. In this case however, we have the analytic solutions of u, v so we may calculate the power spectra directly. First we express the asymptotic form of the Hankel function $H_\nu^{(1)}(x)$ at late times ($x \to 0$)

$$H_\nu^{(1)}(x) \xrightarrow[x\to 0]{} -\frac{i}{\pi}\Gamma(\nu)\left(\frac{x}{2}\right)^{-\nu}, \tag{4.135}$$

where $\Gamma(\nu)$ is the Gamma function . On large scales the dimensionless power spectrum of Φ is provided by

$$P_\Phi(m) \equiv \frac{m^3}{2\pi^2}\left|\Phi_m\right|^2 = \frac{m^3}{2\pi^2}\left|\frac{u\phi_0'}{a}\right|^2 \propto x^{1-2\alpha_1}. \tag{4.136}$$

Note in the last step we have used equations (4.112) and (4.133) to (4.135). The spectral index n_Φ for the Newtonian potential Φ describing deviation from scale invariance is then

$$n_\Phi - 1 = 1 - 2\alpha_1 = 1 - \left|\frac{\epsilon_1 + 1}{\epsilon_1 - 1}\right| \xrightarrow[\epsilon_1\to\infty]{} 0^-. \tag{4.137}$$

For ekpyrotic models $\epsilon_1 \gg 1$, and thus the scalar metric perturbations in terms of the Newtonian potential approaches *scale invariance* (equal amplitudes at Hubble radius crossing) with a possible *small red tilt*, in particular $n_\Phi \xrightarrow[\epsilon_1\to\infty]{} 1^-$. The spectral index may also be calculated for a *slowly* varying equation of state parameter ω_ϕ, [66]

$$n_\Phi - 1 = -4(\epsilon_2 + \eta_2) \xrightarrow[\epsilon_2,\eta_2\to 0]{} 0^-, \tag{4.138}$$

with ϵ_2, η_2 provided in equation (3.1), and assuming η_2 is also positive we indicate that the limit 0^- is approached from values below zero.

Note that recent works only track the curvature perturbation across a non-singular bounce, as opposed to the generalized Newtonian potential [12, 93]. Using Hwang-Vishniac [100] and Deruelle-Mukhanov [101] matching conditions, it was shown in [102] (see also [103]) that in a particular realization of the ekpyrotic scenario the growing mode Φ does not couple to the non-decaying mode during the Friedmann expansion phase, and should therefore not source the large scale structure today.

We also calculate the dimensionless power spectrum for ζ on large comoving scales given by

$$P_\zeta(m) \equiv \frac{m^3}{2\pi^2}\left|\zeta_m\right|^2 = \frac{m^3}{2\pi^2}\left|\frac{v}{z}\right|^2 \propto x^{3-2\alpha_2}, \tag{4.139}$$

with the spectral index for the scalar curvature perturbations on comoving hypersurfaces n_ζ following

$$n_\zeta - 1 = 3 - 2\alpha_2 = 3 - \left|\frac{\epsilon_1 - 3}{\epsilon_1 - 1}\right| \xrightarrow[\epsilon_1 \to \infty]{} 2^+. \tag{4.140}$$

Thus in contrast to the approximately scale invariant power spectrum P_Φ, the power spectrum for the curvature perturbation P_ζ is *strongly blue.*

Interestingly, the equation of motion for u (4.119) supplemented by the definition of the coefficient (4.122) is invariant under $\epsilon_1 \to \dfrac{1}{\epsilon_1}$. Simultaneously the vacuum initial condition equation (4.123) is independent of ϵ_1, and as a result the spectral index as described in equation (4.137) is invariant under the same transformation. This portrays a type of a duality between a phase of ekpyrosis $\epsilon_1 \gg 1$ and inflation $\epsilon_1 \approx 0$, such that for either model the Φ spectrum is approximately scale invariant [44, 79].

4.2.2 Two matter fields: entropy perturbations I

We now provide the analysis for the generation of a scale-invariant spectrum of entropy perturbations when the scaling solution given by equations (3.13) to (3.15) for two scalar matter fields are realized. We begin with the evolution equation (4.63) for entropy perturbations, and argue via equation (4.60) that the homogeneous trajectory is a straight line in field space

$$\tan(\vartheta) = \frac{\dfrac{M_P\sqrt{2q^{\mathbf{2}}}}{t}}{\dfrac{M_P\sqrt{2q^{\mathbf{1}}}}{t}} = \sqrt{\frac{q^{\mathbf{2}}}{q^{\mathbf{1}}}}. \tag{4.141}$$

Additionally, we calculate explicitly the conformal time derivative of the adiabatic field via equation (4.61)

$$\xi' = \frac{a}{t}\sqrt{2}\sqrt{q^{\mathbf{1}} + q^{\mathbf{2}}}, \tag{4.142}$$

and using equations (3.14) to (3.16) into equation (4.65) we have

$$V_{ss} = -\frac{2}{t^2}(1 - 3(q^{\mathbf{1}} + q^{\mathbf{2}})). \tag{4.143}$$

The evolution equation for entropy perturbations then simplifies to a homogeneous second order equation

$$\delta s'' + 2\mathcal{H}\delta s' + \left[m^2 - a^2\left(\frac{2}{t^2}(1 - 3(q^{\mathbf{1}} + q^{\mathbf{2}}))\right)\right]\delta s = 0. \tag{4.144}$$

Defining the new variable $\delta S \equiv a\delta s$ we obtain

$$\delta S'' - \frac{a''}{a}\delta S + \left[m^2 - a^2\left(\frac{2}{t^2}(1 - 3(q^{\mathbf{1}} + q^{\mathbf{2}}))\right)\right]\delta S = 0. \tag{4.145}$$

We may calculate the prefactor of the second term by taking a derivative of the relationship between $\mathcal{H}$ and H

$$\frac{d}{d\eta}\left(\frac{a'}{a}\right) = a\frac{d}{dt}\left(aH\right), \tag{4.146}$$

as well as using equations (3.17) and (4.49), we obtain

$$\frac{a''}{a} = a^2H^2(2 - \epsilon_1) = -\frac{a^2}{t^2}(q^{\mathbf{1}} + q^{\mathbf{2}}). \tag{4.147}$$

By means of integrating $d\eta = dt/a$ and using appropriate Taylor series expansion in $(q^{\mathbf{1}}+q^{\mathbf{2}})$, we may approximate the time dependent term currently expressed in coordinate time instead in terms of conformal time to linear order

$$\frac{a^2}{t^2} \approx \frac{1}{\eta^2}\left(1 + 2(q^{\mathbf{1}} + q^{\mathbf{2}}) - 4(q^{\mathbf{1}} + q^{\mathbf{2}})\ln(-t)\right). \tag{4.148}$$

However, the final (logarithmic) term in equation (4.148) may be neglected so long as the evolution traverses a region in the domain of cosmological time such that the logarithmic term produces values much less than 1. We will proceed assuming the logarithmic term is negligible, as done in the analysis of [12]. Plugging equations (4.147) and (4.148) into equation (4.145), we obtain the following Bessel equation describing the approximate evolution of entropy perturbations in Fourier space [12]

$$\delta S'' + \left[m^2 - \frac{2}{\eta^2}\left(1 - \frac{3}{2}(q^{\mathbf{1}} + q^{\mathbf{2}})\right)\right]\delta S \approx 0. \tag{4.149}$$

At very early times when the scale of quantum fluctuations is much smaller than the Hubble scale, we may impose standard Bunch-Davies initial conditions [12, 104] that determine the constant coefficients of the general solution of equation (4.149). Up to an irrelevant phase factor we have the following solution for δS with $x \equiv m|\eta|$

$$\delta S = \frac{1}{2}\sqrt{\frac{\pi x}{m}} H^{(1)}_{\alpha_3}(x), \tag{4.150}$$

with the order of the Hankel function provided by

$$\alpha_3 = \sqrt{2\left(1 - \frac{3}{2}(q^{\mathbf{1}} + q^{\mathbf{2}})\right) + \frac{1}{4}} = \frac{3}{2}\left(1 - \frac{2}{3}(q^{\mathbf{1}} + q^{\mathbf{2}}) + \mathcal{O}((q^{\mathbf{1}} + q^{\mathbf{2}})^2)\right). \tag{4.151}$$

Using the asymptotic form of the Hankel function of the first kind provided by equation (4.135) at late times, we may calculate the dimensionless power spectrum for entropy perturbations $P_{\delta s}$

$$P_{\delta s} \equiv \frac{m^3}{2\pi^2}|\delta_s|^2 \approx \frac{m^3}{2\pi^2}\left|\frac{\delta S}{a}\right|^2 \propto m^3 x^{-2\alpha_3}, \tag{4.152}$$

and thus its spectral index [12]

$$n_{\delta s} - 1 \approx 3 - 2\alpha_3, \tag{4.153}$$

$$n_{\delta s} \approx 1 + 2(q^{\mathbf{1}} + q^{\mathbf{2}}) \xrightarrow[q^{\mathbf{1}},q^{\mathbf{2}} \to 0]{} 1^+. \tag{4.154}$$

Thus we have shown that when the scaling solution provided by equations (3.13) to (3.15) for multi-field models is obeyed, or in other words for canonical scalar fields in an expanding FLRW background with purely exponential potentials for both fields of the form as provided by equation (3.9), then an approximately scale invariant spectrum of entropy perturbations with a *small blue tilt* may be generated. The steepness of the tilt is modulated entirely by the steepness of the potential, more specifically by the quantities $q^{\mathbf{1}}, q^{\mathbf{2}}$.

A crucial hinge in this analysis is that if the logarithmic term in equation (4.148) instead dominates the expression, the sign of its coefficient is opposite to that of the preceding term. Thus in this case, it would reverse the results of this analysis: the entropy spectrum would produce a nearly scale invariant spectrum with a *small red tilt*.

4.2.3 Two matter fields: entropy perturbations II

A more general form of the double field potential from that given in equation (3.9) implies that the scaling solution as provided in equations (3.13) to (3.15) are no longer exact solutions. In this section, we follow the analysis of [12] for the case of the two field potential being additively separable for any single field potentials $V^{\mathbf{1}}, V^{\mathbf{2}}$

$$V(\phi^{\mathbf{1}}, \phi^{\mathbf{2}}) = V^{\mathbf{1}}(\phi^{\mathbf{1}}) + V^{\mathbf{2}}(\phi^{\mathbf{2}}), \tag{4.155}$$

subject to the constraint that the potential is symmetric under the interchange of fields

$$\phi^{\mathbf{1}} \longleftrightarrow \phi^{\mathbf{2}}. \tag{4.156}$$

With conditions (4.155) and (4.156) restricting the two-field potential, equation (4.60) immediately implies

$$\vartheta = \frac{\pi}{4}, \tag{4.157}$$

and further that the adiabatic field satisfies via equation (4.55)

$$\xi' = \frac{\sqrt{2}}{2}\left(\phi_0^{\mathbf{1}\prime} + \phi_0^{\mathbf{2}\prime}\right). \tag{4.158}$$

This form of the potential also admits an exact equivalence between second derivatives in the potential

$$V_{ss} = V_{\xi\xi}, \tag{4.159}$$

generically speaking this equivalence is manifest when the following two conditions are satisfied: $V_{,\phi^{\mathbf{1}}\phi^{\mathbf{2}}} = 0$ and $\cos^2(\vartheta) = \sin^2(\vartheta)$. The entropy perturbation equation (4.63) now simplifies

$$\delta s'' + 2\mathcal{H}\delta s' + (m^2 + a^2 V_{\xi\xi})\delta s = 0. \tag{4.160}$$

Since we may no longer readily calculate the term containing $V_{\xi\xi}$, the authors of [12] instead express the potential in terms of the fast roll parameter ϵ_1 by combining equations (3.17), (4.48) and (4.49) obtaining

$$V = \frac{\mathcal{H}^2}{a^2 8\pi}(3-\epsilon_1) = \frac{H^2}{8\pi}(3-\epsilon_1), \tag{4.161}$$

we then derive the crucial relationship between a differential e-folding and that of an infinitesimal change in the adiabatic field using equations (3.17), (3.18), (4.49) and (4.61)

$$dN\sqrt{\frac{\epsilon_1}{4\pi}} = d\xi. \tag{4.162}$$

We may obtain an explicit expression for the derivative of the potential in terms of the fast roll parameter and its derivatives with respect to the e-folding number

$$V_\xi = -\frac{H^2}{\sqrt{4\pi}}\sqrt{\epsilon_1}\left(3-\epsilon_1+\frac{1}{2}\frac{d\ln\epsilon_1}{dN}\right), \tag{4.163}$$

and thus for the second derivative we have

$$\frac{V_{\xi\xi}}{H^2} = -2\epsilon_1^2+6\epsilon_1+\frac{5}{2}\frac{d\epsilon_1}{dN}-\frac{3}{2}\frac{d\ln\epsilon_1}{dN}-\frac{1}{4}\left(\frac{d\ln\epsilon_1}{dN}\right)^2-\frac{1}{2}\frac{d^2\ln\epsilon_1}{dN^2}. \tag{4.164}$$

As done in [105], equation (4.160) is rewritten in terms of the dimensionless time variable x

$$x \equiv \frac{1}{\epsilon_1-1}\frac{m}{aH}, \tag{4.165}$$

as well as rescaling the entropy perturbation variable as $\delta S \equiv a\delta s$ to obtain

$$\begin{aligned}&\left(1-\frac{1}{\epsilon_1-1}\frac{d\ln(\epsilon_1-1)}{dN}\right)^2 x^2\frac{d^2\delta S}{dx^2}+\frac{1}{(\epsilon_1-1)^2}\left[\left(\frac{d\ln(\epsilon_1-1)}{dN}\right)^2-\frac{d^2\ln(\epsilon_1-1)}{dN^2}\right]x\frac{\delta S}{dx}\\&+x^2\delta S+\frac{1}{(\epsilon_1-1)^2}\left[-2-2\epsilon_1^2+7\epsilon_1+\frac{5}{2}\frac{d\epsilon_1}{dN}-\frac{3}{2}\frac{d\ln\epsilon_1}{dN}-\frac{1}{4}\left(\frac{d\ln\epsilon_1}{dN}\right)^2-\frac{1}{2}\frac{d^2\ln\epsilon_1}{dN^2}\right]\delta S=0.\end{aligned} \tag{4.166}$$

This differential equation in the original work is not solved exactly, however a series of approximations are made in order to express it in the form of a Bessel equation: firstly for $\omega \gg 1$ during the phase of ekpyrosis we have $\epsilon_1 \gg 1$, in addition to this it is assumed that the fast roll parameter ϵ_1 is a slowly varying function of the e-folding number N for the observable range of perturbation modes. Via equations (4.161) and (4.163), one may show that to leading order

$$\epsilon_1 \approx \frac{1}{2\epsilon_3}, \tag{4.167}$$

while equations (4.67), (4.161), (4.163) and (4.164) provide the approximate equivalence

$$\frac{d \ln \epsilon_1}{dN} \approx 4\epsilon_1\eta_3. \tag{4.168}$$

The above simplifying assumptions allows one to approximate the differential equation governing entropy perturbations equation (4.166) as

$$x^2\frac{d^2\delta S}{dx^2} + \frac{x^2}{1-8\eta_3}\delta S - 2(1-3(\epsilon_3-\eta_3))\delta S \approx 0. \tag{4.169}$$

Imposing Bunch-Davies initial conditions and using the asymptotic form of Hankel functions as done in previous sections, we may obtain the constant coefficients of the general solution, and in turn obtain up to an irrelevant phase an approximate analytic solution for δS (see 9.1.49 of [99])

$$\delta S = \frac{1}{2}\sqrt{\frac{\pi\lambda x}{m}}H^{(1)}_{\alpha_4}(\lambda x). \tag{4.170}$$

The order of the Hankel function is given as α_4

$$\alpha_4 = \sqrt{2 - 6(\epsilon_3-\eta_3) + \frac{1}{4}} \approx \frac{3}{2} - 2(\epsilon_3-\eta_3) + \mathcal{O}((\epsilon_3-\eta_3)^2), \tag{4.171}$$

and the coefficient λ in the argument is

$$\lambda = \sqrt{\frac{1}{1-8\eta_3}}. \tag{4.172}$$

We once again use the asymptotic form of Hankel functions (at late times $x \to 0$) to calculate the dimensionless power spectrum of entropy perturbations

$$P_{\delta s} \equiv \frac{m^3}{2\pi^2}|\delta_s|^2 \approx \frac{m^3}{2\pi^2}\left|\frac{\delta S}{a}\right|^2 \propto m^3(\lambda x)^{-2\alpha_4}, \tag{4.173}$$

and thus the spectral index

$$n_{\delta s} - 1 \approx 3 - 2\alpha_4 \tag{4.174}$$

$$= 4(\epsilon_3 - \eta_3) \xrightarrow[\epsilon_3,\eta_3 \to 0]{} 0. \tag{4.175}$$

As a consistency check, we may confirm that since for pure exponential potentials as provided in equation (3.9) that $\eta_3 = 0$, the above result is in agreement with equation (4.154).

Since the sign preceding η_3 in equation (4.175) is negative, we may say that the tilt in the entropy perturbation spectrum may be *either slightly blue or slightly red* and is dependent entirely on the sign of η_3 and its relative magnitude with respect to ϵ_3 (note that ϵ_3 is positive definite for real potentials). That being said, we may say that a class of potentials that satisfy the conditions (4.155) and (4.156), in addition to η_3 being positive and dominant over ϵ_3 leads to a entropy perturbation spectrum with a *small red tilt* ($n_{\delta s} < 1$). These two new conditions that ensure a small red tilt may be recast in terms of the potential and its derivatives with respect to the adiabatic field

$$V_{\xi\xi} < -M_P^{-2}V + \frac{3}{4}\frac{V_\xi^2}{V} \qquad \text{(dominance)}, \tag{4.176}$$

$$V_{\xi\xi} < \frac{V_\xi^2}{V} \qquad \text{(positivity)}. \tag{4.177}$$

The authors of [12] briefly state that a potential of the form $V(\phi) \sim \exp(-\phi^n)$ for $n > 2$ fulfills the conditions above, providing an example of a potential which produces a red tilt for large values of the field ϕ.

4.2.4 Two matter fields: converting entropy to curvature

In two field models, the comoving curvature perturbation ζ may be written as (in the absence of anisotropic stresses $\Psi = \Psi$) [80, 106, 107]

$$\zeta = \Phi + H\left(\frac{\dot{\phi}_0^{\mathbf{1}}\delta\phi^{\mathbf{1}} + \dot{\phi}_0^{\mathbf{2}}\delta\phi^{\mathbf{2}}}{(\dot{\phi}_0^{\mathbf{1}})^2 + (\dot{\phi}_0^{\mathbf{2}})^2}\right), \tag{4.178}$$

thus the change in curvature perturbation may be expressed in Fourier space as [80,106,108, 109]

$$\dot{\zeta} = \frac{H}{\dot{H}}\frac{m^2}{a^2}\Phi + \frac{1}{2}H\left(\frac{\delta\phi^{\mathbf{1}}}{\dot{\phi}_0^{\mathbf{1}}} - \frac{\delta\phi^{\mathbf{2}}}{\dot{\phi}_0^{\mathbf{2}}}\right)\frac{d}{dt}\left(\frac{(\dot{\phi}_0^{\mathbf{1}})^2 - (\dot{\phi}_0^{\mathbf{2}})^2}{(\dot{\phi}_0^{\mathbf{1}})^2 + (\dot{\phi}_0^{\mathbf{2}})^2}\right). \tag{4.179}$$

In terms of the adiabatic and entropy fields introduced in section 4.1.2, the change in the comoving curvature perturbation with respect to time may be expressed succinctly as

$$\dot{\zeta} = \frac{H}{\dot{H}}\frac{m^2}{a^2}\Phi + \frac{2H}{\dot{\xi}}\dot{\vartheta}\delta s, \tag{4.180}$$

with

$$\dot{\vartheta} = -\frac{V_s}{\dot{\xi}}. \tag{4.181}$$

An immediate difference is manifest compared to the single field case, evident by the additional source term to the change in curvature perturbation [80]

$$\dot{\zeta}_{(\text{single field})} = \frac{H}{\dot{H}}\frac{m^2}{a^2}\Phi. \tag{4.182}$$

Thus in contrast to the single field case, non-negligible changes in the comoving curvature perturbation may be sourced by entropy perturbations on large scales ($m \to 0$), if the homogeneous trajectory etched in scalar field space is non-linear ($\dot{\vartheta} \neq 0$) [110]. Thus the approximately scale invariant spectrum of entropy perturbations that may be generated by a class of potentials in a two field canonical kinetic model may transfer its spectrum to the comoving curvature perturbation under appropriate conditions.

In this section, we review a specific mechanism [12] on how such a transfer may have occurred, accompanied by a few justifiable assumptions. The mechanism is preceded by a phase of ekpyrosis that generates a nearly scale invariant spectrum of entropy perturbations, such as that governed by a two field potential obeying the conditions provided in the analysis of section 4.2.3. Note that for an exactly exponential phase of ekpyrosis we have $\dot{\vartheta} = 0$, and thus necessarily on large scales entropy perturbations may not source the curvature perturbation and thus may not inherit a scale invariant spectrum during this phase. The conversion of entropy perturbations to curvature perturbations may then follow due to an additional feature in the potential.

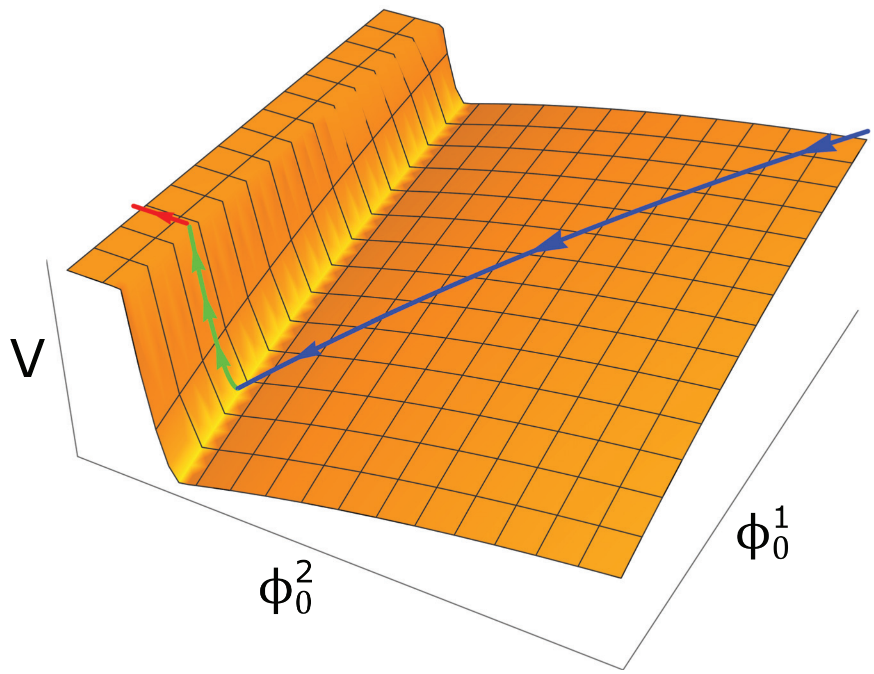

Figure 8: The blue curve traces an example trajectory in scalar field space of a two field ekpyrotic period. The green trajectory indicates the ghost condensate transition phase in which scale invariant entropy perturbations are transferred to the comoving curvature perturbations. During this transition phase, one field ($\phi_0^{\mathbf{2}}$) rapidly loses kinetic energy, while the other field ($\phi_0^{\mathbf{1}}$) approximately maintains its kinetic energy. The red trajectory is the beginning of a phase in which the potential traces positive values in order to satisfy a phase of de Sitter cosmology.

This mechanism allows the potential energy in one of the original field directions, say $\phi_0^{\mathbf{2}}$, to reach a minimum and to then rise rapidly with respect to the ekpyrotic phase, to positive values. During this rapid increase in potential energy in the $\phi_0^{\mathbf{2}}$ direction, the kinetic energy of this field subsequently decreases. An assumption is now made concerning the kinetic energy of the second field $\phi_0^{\mathbf{1}}$, ensuring it remain approximately constant with respect to the large and rapid change in the kinetic energy of $\phi_0^{\mathbf{2}}$. This may be achieved either by $\phi_0^{\mathbf{1}}$ remaining in its ekpyrotic phase, or entering some new phase where the potential energy curve along $\phi_0^{\mathbf{1}}$ is less steep than the aforementioned. The above assumption allows for the geometrical angle ϑ in scalar field space as depicted in figure 7, to rapidly shift from a finite value (for the case of an exact exponential ekpyrotic phase $\vartheta = \arctan\sqrt{q^{\mathbf{2}}/q^{\mathbf{1}}}$ or for the case described in section 4.2.3, $\vartheta = \pi/4$) to

$$\vartheta = \arctan\left(\frac{\dot{\phi}_0^{\mathbf{2}}}{\dot{\phi}_0^{\mathbf{1}}}\right) \approx 0, \tag{4.183}$$

as the rapid increase in potential energy phase of $\phi_0^{\mathbf{2}}$ approaches its end. The accuracy of the approximation made in equation (4.183) is then entirely modulated by the amount of kinetic energy is lost by the field $\phi_0^{\mathbf{2}}$ during the rapid transition phase if the assumed staticity of the kinetic energy of $\phi_0^{\mathbf{1}}$ remains justifiable. This enables a phase in which $\dot{\vartheta} \neq 0$ allowing the comoving curvature perturbation to inherit, on large scales, a scale invariant spectrum from the previously generated entropy perturbations via equation (4.180).

The change in ϑ is argued to occur approximately instantaneously relative to the Hubble time ($1/H$), qualified by a sufficient steepness of the potential in the $\phi_0^{\mathbf{2}}$ direction during the transition phase. To justify this statement, let us indicate variables describing the evolution during the ekpyrotic phase with a subscript *ekp*. We first note the following relationship given by equations (3.13) to (3.15) for the ekpyrotic phase as

$$\frac{1}{H_{ekp}} \gg \frac{1}{(\dot{\phi}_0^{\mathbf{2}})_{ekp}}. \tag{4.184}$$

Via equations (4.64) and (4.181), we approximate the rate of change of ϑ with respect to the cosmological time under the aforementioned assumption that during the rapid transition phase $\dot{\phi}_0^{\mathbf{1}} \approx const.$, equivalently the potential in this field direction does not change appreciably. We denote the variables describing the evolution during the rapid transition phase with the subscript *tra*

$$\dot{\vartheta} \approx \frac{(\dot{\phi}_0^{\mathbf{1}})_{tra}(V_{,\phi^{\mathbf{2}}})_{tra}}{(\dot{\phi}_0^{\mathbf{1}})^2_{tra} + (\dot{\phi}_0^{\mathbf{2}})^2_{tra}}. \tag{4.185}$$

In order to determine the approximate amount of time elapsed during the rapid transition phase, we approximate $(\dot{\phi}_0^{\mathbf{1}})_{tra}$ by $(\dot{\phi}_0^{\mathbf{2}})_{ekp}$ since the time evolution of the two fields were approximately the same during the ekpyrotic phase, and $\phi_0^{\mathbf{1}}$ has approximately maintained its kinetic energy through the rapid transition phase by assumption. Additionally since $(\dot{\phi}_0^{\mathbf{2}})^2_{tra} \leq (\dot{\phi}_0^{\mathbf{2}})^2_{ekp}$, we saturate this bound and replace $(\dot{\phi}_0^{\mathbf{2}})^2_{tra}$ with $(\dot{\phi}_0^{\mathbf{2}})^2_{ekp}$ ensuring a conservative estimate of the elapsed transition time Δt_{tra}, thus we have

$$\Delta t_{tra} \approx \frac{\Delta\vartheta_{tra}}{\dot{\vartheta}_{tra}} \sim \mathcal{O}(1)\left[\frac{(\dot{\phi}_0^{\mathbf{2}})_{ekp}}{(V_{,\phi^{\mathbf{2}}})_{tra}}\right]. \tag{4.186}$$

The deduction of equation (4.184) allows us to derive an approximate condition on the steepness of the potential during the rapid transition phase

$$(V_{,\phi^{\mathbf{2}}})_{tra} \gtrsim (\dot{\phi}_0^{\mathbf{2}})^2_{ekp} \Longrightarrow \Delta t_{tra} \lesssim \frac{1}{(\dot{\phi}_0^{\mathbf{2}})_{ekp}} \ll \frac{1}{H_{ekp}}. \tag{4.187}$$

Equation (4.187) allows us to approximate $\dot{\vartheta}$ as a delta function to be integrated and interpreted as a distribution, we may thus rewrite equation (4.180) to approximate cosmological perturbations on large scales, with t_i denoting the time in which the rapid transition phase begins

$$\zeta \approx \int_{t_i}^{t_i-\Delta t_{tra}} -\frac{2H}{\dot{\xi}}\arctan\left(\sqrt{\frac{q^{\mathbf{2}}}{q^{\mathbf{1}}}}\right)\delta(t-t_i-\frac{\Delta t_{tra}}{2})\delta s\; dt. \tag{4.188}$$

In order to integrate equation (4.188) explicitly we must know the behaviours of each factor of the integrand for $t \in [t_i, t_i + \Delta t_{tra}]$. One may argue that the Hubble parameter is approximately constant during the rapid transition phase, a statement increasing in accuracy by increasing the steepness of the potential during the transition phase in the $\phi_0^{\mathbf{2}}$ direction. The authors of [12] also argue that each of the remaining factors under the integral remain approximately constant (with δs changing by at most a factor of $\mathcal{O}(1)$) during the rapid transition phase with the exception of the delta function, and thus each of their forms are well known: they acquire approximately their ekpyrotic solutions evaluated at $t = t_i$. Therefore, the curvature perturbation inherits the spectrum of entropy perturbations generated throughout the ekpyrotic phase

$$|\zeta| \approx \frac{2H}{\dot{\xi}}\arctan\left(\sqrt{\frac{q^{\mathbf{2}}}{q^{\mathbf{1}}}}\right)\delta s. \tag{4.189}$$

For the specific case of entropy perturbations generated from exact exponential potentials as done in section 4.2.2, we determine the large scale behaviour of entropy perturbations via equations (4.135) and (4.150)

$$|\delta s| = \left|\frac{\delta S}{a}\right| = \frac{1}{2a}\sqrt{\frac{\pi x}{m}}H^{(1)}_{\alpha_3}(x) \xrightarrow[x\to 0]{} \frac{1}{2a}\sqrt{\frac{\pi x}{m}}\left(\frac{1}{\pi}\Gamma(\alpha_3)\left(\frac{x}{2}\right)^{-\alpha_3}\right). \tag{4.190}$$

With the order of the Hankel function α_3 given in equation (4.151) we may Taylor expand the two factors depending on α_3 about $(q^{\mathbf{1}} + q^{\mathbf{2}}) = 0$ to obtain to first order in $(q^{\mathbf{1}} + q^{\mathbf{2}})$

$$|\delta s| \approx \frac{1}{a\sqrt{2}m^{3/2}|\eta|}. \tag{4.191}$$

Using the same approximation as in equation (4.148), the assumption that $q^{\mathbf{1}}, q^{\mathbf{2}} \ll 1$, and the form of the Hubble parameter (3.13) during an exponential ekpyrotic phase we approximate

$$|\delta s| m^{3/2} \approx \frac{|H|}{\sqrt{2}(q^{\mathbf{1}} + q^{\mathbf{2}})}. \tag{4.192}$$

Finally via equation (4.61), we may include $\dot{\xi}$ for the pure exponential ekpyrotic case

$$\dot{\xi} = \frac{\sqrt{2}M_P}{t}\sqrt{q^{\mathbf{1}} + q^{\mathbf{2}}}, \tag{4.193}$$

and we may estimate the inherited comoving curvature perturbation spectrum via equation (4.189)

$$|\zeta| \approx \frac{H}{M_P\sqrt{2\epsilon_3}m^{3/2}}\arctan\left(\frac{q^{\mathbf{2}}}{q^{\mathbf{1}}}\right), \tag{4.194}$$

with ϵ_3 the fast roll parameter defined in terms of the adiabatic field given in equation (4.67).

4.3 Tensor metric perturbations in ekpyrosis

We must first choose a suitable initial condition for the tensor perturbation modes at early times in an ekpyrotic contracting phase. Once again since modes are far inside the curvature scale, it is reasonable to choose the Minkowski vacuum as done for scalar metric perturbation modes in section 4.2.1

$$f \xrightarrow[\eta\to -\infty]{} (2m)^{-1/2}\exp(-im\eta). \tag{4.195}$$

The differential equation in Fourier space describing tensor perturbations for both polarizations of the metric is given in equation (4.75), and is of the same form as the v equation with the same Minkowski vacuum initial condition as in section 4.2.1; thus equation (4.75) is solved exactly by

$$f = a(\eta)h = \frac{\sqrt{\pi x}}{2m^{1/2}} H^{(1)}_{\alpha_2}(x). \tag{4.196}$$

Thus the dimensionless power spectrum and tensor spectral index at late times ($x \to 0$) may be solved for directly using the asymptotic form of the Hankel function equation (4.135) [44, 48, 79]

$$P_h(m) = \frac{m^3}{2\pi^2} \sum_{\lambda=(+,\times)} \left|h^\lambda_m\right|^2 \propto m^3 x^{-2\alpha_2}, \tag{4.197}$$

implying the spectral index for h, n_h (or conventional tensor spectral index n_T)

$$n_T \equiv n_h - 1 = 3 - 2\alpha_2 = 3 - \left|\frac{\epsilon_1 - 3}{\epsilon_1 - 1}\right| \xrightarrow[\epsilon_1 \to \infty]{} 2^+. \tag{4.198}$$

Therefore, ekpyrotic models generically predict a tensor spectrum with a *strongly blue tilt* $n_h \approx 3$ ($n_T \approx 2$). Contrary to the scalar metric perturbations, the tensor spectral index is not invariant under the transformation $\epsilon_1 \to 1/\epsilon_1$. In particular, standard slow-roll inflationary models with $\epsilon_1 \approx 0$ predict a tensor spectrum with a red tilt [79, 87].

5 Non-Gaussianities

Local form non-Gaussianities of the curvature perturbation ζ in ekpyrotic models have been studied in a variety of methods [44, 88, 111–114]. Generally speaking, to explore local form non-Gaussianities one first expands the curvature perturbation variable ζ into increasing powers of its Gaussian component ζ_g (powers of a Gaussian random field is not necessarily Gaussian)

$$\zeta = \zeta_g + \frac{3}{5} f_{NL} \zeta_g^2 + \frac{9}{25} g_{NL} \zeta_g^3 + \cdots . \tag{5.1}$$

These new parameters f_{NL} and g_{NL} now define the reduced bispectrum/trispectra introduced in section 4.1.4. As it turns out for ekpyrotic cosmology, non-Gaussianities are generated not only throughout the phase of ekpyrotic contraction but also through the conversion phase (an example mechanism was presented in section 4.2.4): the phase in which entropy perturbations are converted to curvature perturbations. In [111], the authors derive the equations of motion for entropy and curvature perturbations to third order in perturbation theory using a fully covariant approach, then determine the bispectrum/trispectra by integrating their expression for $\dot{\zeta}$ numerically throughout the ekpyrotic phase of contraction and two distinct types of conversion phases.

In this section, we lay the theoretical groundwork of an entirely covariant approach to cosmological perturbations so that we may pursue the non-Gaussianities of ekpyrotic models in future work. This methodology should be compared with other approaches such as that of Maldacena [115] applied to ekpyrotic models in [113], or that of the δN formalism [116, 117] applied to ekpyrotic models in [112] (and closely related work in [114]) as well as to inflation in [91], and ensure that all methods provide consistent results. The covariant formalism applied to ekpyrotic cosmology has been previously studied in [88, 111, 118], as well as in a non-minimal model [119].

5.1 Covariant formalism of cosmological perturbations

The covariant approach introduced by Ellis and Bruni [120] (with pioneering work done by Hawking [121]) aimed to absolve the gauge dependence of cosmological perturbations analysed in previous works without the need for assuming perturbations to be small as done by Bardeen's coordiante based approach [76]. The gauge dependency makes the perturbations difficult or even impossible to interpret physically particularly at non linear order, and thus defining perturbations in an entirely covariant approach is certainly advantageous. Additionally, the covariant approach proves much more powerful than the coordinate based approach: when working beyond linear order the evolution equations of perturbations may be derived (relatively) succinctly and the covariant formalism is also able to characterize the evolution of perturbations on all scales non-perturbatively. Work by Langlois and Vernizzi [122–125] expanded on the original covariant formalism defining the generalized curvature perturbation on uniform density hypersurfaces and generalized comoving curvature perturbation (whose interpretation is valid on scales large and small). They also determined the evolution equations of the curvature perturbation to second order, and determined the evolution equations for generalized versions of entropy and adiabatic perturbations with respect to those introduced in section 4.1.2 to second order on large scales.

Here we review the covariant approach to cosmological perturbations [120,125]. We begin by introducing a preferred family of world lines known as fundamental world lines or fluid flow lines, representing the motion of typical observers in the universe. Let the four-velocity vector tangent to said world lines be defined

$$u^\mu = \frac{dx^\mu}{d\tau}, \tag{5.2}$$

with τ the proper time defined along the fundamental world lines. We also define the spatial projection tensor orthogonal to u^μ

$$h_{\mu\nu} \equiv g_{\mu\nu} + u_\mu u_\nu \quad \Longrightarrow \quad h^\mu{}_\nu h^\nu{}_\sigma = h^\mu{}_\sigma, \qquad h_\mu{}^\nu u_\nu = 0. \tag{5.3}$$

Now introduce the covariant definition of the time derivative as the Lie derivative (See Appendix C.2 of Wald [126]) with respect to u^μ for any generic covector X_μ indicated by a hollow dot

$$\mathring{X}_\mu \equiv \mathfrak{L}_u X_\mu \equiv u^\nu \nabla_\nu X_\mu + X_\nu \nabla_\mu u^\nu. \tag{5.4}$$

For scalar quantities, the covariant derivative along u^μ is equivalent to the Lie derivative

$$\mathring{f} = u^\nu \nabla_\nu f. \tag{5.5}$$

In the covariant formalism, it is useful to introduce the covariant derivative projected orthogonally to the four-velocity u^μ denoted as $\mathfrak{D}_\mu$, using our definition for $h_{\mu\nu}$. For a tensor T of arbitrary rank we have

$$\mathfrak{D}_\mu T_{\sigma\cdots}^{\ \nu\cdots} \equiv h_\mu^{\ \gamma} h_\sigma^\omega \cdots h_\delta^{\ \nu} \cdots \nabla_\gamma T_\omega^{\ \delta}. \tag{5.6}$$

Once again for a scalar quantity, the result reduced significantly

$$\mathfrak{D}_\mu f = h_\mu^{\ \nu} \nabla_\nu f = \partial_\mu f + u_\mu \mathring{f}. \tag{5.7}$$

The first covariant derivative of the four-velocity vector is

$$\nabla_\nu u_\mu = \sigma_{\mu\nu} + \omega_{\mu\nu} + \frac{1}{3}\Theta h_{\mu\nu} - a_\mu u_\nu, \tag{5.8}$$

where Θ is the volume expansion (in the case of fluids: fluid expansion), $\sigma_{\mu\nu}$ the symmetric shear tensor ($\sigma_{\mu\nu} u^\nu = 0, \sigma^\mu_{\ \nu} = 0$), $\omega_{\mu\nu}$ the antisymmetric vorticity tensor ($\omega_{\mu\nu} u^\nu = 0$) and $a^\mu \equiv u^\nu \nabla_\nu u^\mu$ the acceleration vector. The volume expansion Θ is defined as

$$\Theta \equiv \nabla_\mu u^\mu. \tag{5.9}$$

Another useful quantity that may be defined is the integrated expansion along u^μ denoted as α

$$\alpha \equiv \frac{1}{3} \int d\tau \ \Theta, \qquad \Theta = 3\mathring{\alpha}. \tag{5.10}$$

As explained in [125], one may identify $\Theta/3$ as the local Hubble parameter when one compares the generalized Klein-Gordon equation derived in the covariant formalism to the usual homogeneous Klein-Gordon equation in FLRW spacetime. Thus the integrated expansion may be interpreted as the number of e-folds measured along the world line of an observer with four-velocity u^μ.

Introducing matter described by N scalar fields minimally coupled to gravity whose matter Lagrangian density is given in equation (4.43), and whose stress-energy tensor is given in equation (4.45), given an arbitrary unit timelike vector field u^μ we may decompose the the stress-energy tensor into components

$$T_{\mu\nu} = (\rho + p)u_\mu u_\nu + q_\mu u_\nu + u_\mu q_\nu + g_{\mu\nu}p + \pi_{\mu\nu}, \tag{5.11}$$

with ρ and p the energy density and pressure respectively, q_μ is the momentum and $\pi_{\mu\nu}$ is the anisotropic stress tensor; all are measured in the frame defined by the four-velocity u^μ. As provided in [125], in the covariant formalism we may define each of the aforementioned quantities in terms of the fields via equation (4.45)

$$\rho \equiv T_{\mu\nu}u^\mu u^\nu = \frac{1}{2}\left(\mathring{\phi}_{\boldsymbol{i}}\mathring{\phi}^{\boldsymbol{i}} + \mathfrak{D}_\mu \phi^{\boldsymbol{i}} \mathfrak{D}^\mu \phi_{\boldsymbol{i}}\right) + V, \tag{5.12}$$

$$p \equiv \frac{1}{3}h^{\mu\sigma}T_{\mu\nu}h^\nu{}_\sigma = \frac{1}{2}\left(\mathring{\phi}_{\boldsymbol{i}}\mathring{\phi}^{\boldsymbol{i}} - \frac{1}{3}\mathfrak{D}_\mu \phi_{\boldsymbol{i}} \mathfrak{D}^\mu \phi^{\boldsymbol{i}}\right) - V, \tag{5.13}$$

$$q_\mu \equiv -u^\nu T_{\nu\sigma}h^\sigma{}_\mu = -\mathring{\phi}_{\boldsymbol{i}}\mathfrak{D}_\mu \phi^{\boldsymbol{i}}, \tag{5.14}$$

$$\pi_{\mu\nu} \equiv h_\mu{}^\sigma T_{\sigma\gamma}h^\gamma{}_\nu - p h_{\mu\nu} = \mathfrak{D}_\mu \phi_{\boldsymbol{i}} \mathfrak{D}_\nu \phi^{\boldsymbol{i}} - \frac{1}{3}h_{\mu\nu}\mathfrak{D}_\sigma \phi_{\boldsymbol{i}} \mathfrak{D}^\sigma \phi^{\boldsymbol{i}}. \tag{5.15}$$

Varying the action with respect to the fields we once again obtain N Klein-Gordon equations as in equation (4.44), provided by

$$\nabla_\mu \nabla^\mu \phi^{\boldsymbol{i}} = \frac{\partial V}{\partial \phi^{\boldsymbol{i}}}. \tag{5.16}$$

However now we consider a decomposition into covariant time-like and space-like gradients defined with respect to the four-velocity of the fluid flow u^μ, one finds that we then obtain

$$\mathring{\mathring{\phi}}^{\boldsymbol{i}} + \Theta\mathring{\phi}^{\boldsymbol{i}} + V_{,\phi^{\boldsymbol{i}}} - \mathfrak{D}_\mu \mathfrak{D}^\mu \phi^{\boldsymbol{i}} - a^\mu \mathfrak{D}_\mu \phi^{\boldsymbol{i}} = 0. \tag{5.17}$$

Restricting the analysis now to two canonically normalized scalar fields, it is once again useful to introduce the adiabatic and entropy covectors establishing a basis in the space of scalar fields. We define a unit vector $\boldsymbol{e}^{\boldsymbol{i}}_{\xi}$ in the direction of the velocity of the two scalar matter fields $\phi^{\mathbf{1}}, \phi^{\mathbf{2}}$ thereby tangent to the background trajectory in field space, and similarly a unit vector $\boldsymbol{e}^{\boldsymbol{i}}_{s}$ orthogonal to the background trajectory. Thus very similarly to the fields introduced in section 4.1.2, we have the following relations where we once again introduce a variable ϑ

$$\boldsymbol{e}^{\boldsymbol{i}}_{\xi} \equiv \frac{1}{\sqrt{(\mathring{\phi}^{\mathbf{1}})^2 + (\mathring{\phi}^{\mathbf{2}})^2}} (\mathring{\phi}^{\mathbf{1}}, \mathring{\phi}^{\mathbf{2}}), \qquad \boldsymbol{e}^{\boldsymbol{i}}_{s} \equiv \frac{1}{\sqrt{(\mathring{\phi}^{\mathbf{1}})^2 + (\mathring{\phi}^{\mathbf{2}})^2}} (-\mathring{\phi}^{\mathbf{2}}, \mathring{\phi}^{\mathbf{1}}), \tag{5.18}$$

$$\cos\vartheta \equiv \frac{\mathring{\phi}^{\mathbf{1}}}{\sqrt{(\mathring{\phi}^{\mathbf{1}})^2 + (\mathring{\phi}^{\mathbf{2}})^2}}, \qquad \sin\vartheta \equiv \frac{\mathring{\phi}^{\mathbf{2}}}{\sqrt{(\mathring{\phi}^{\mathbf{1}})^2 + (\mathring{\phi}^{\mathbf{2}})^2}}, \tag{5.19}$$

such that

$$\delta^{\boldsymbol{j}}_{\boldsymbol{i}} = \boldsymbol{e}^{\boldsymbol{i}}_{\xi} \boldsymbol{e}_{\xi \boldsymbol{j}} + \boldsymbol{e}^{\boldsymbol{i}}_{s} \boldsymbol{e}_{s \boldsymbol{i}}, \tag{5.20}$$

$$\boldsymbol{e}^{\boldsymbol{i}}_{\xi} = (\cos\vartheta, \sin\vartheta), \qquad \boldsymbol{e}^{\boldsymbol{i}}_{s} = (-\sin\vartheta, \cos\vartheta). \tag{5.21}$$

An important variation between the angle ϑ introduced above and that introduced in the coordinate based linear theory (section 4.1.2) ϑ, is that this ϑ is an inhomogeneous quantity which depends on both space and time, as opposed to just time. It will turn out to be the case that once we perturbatively expand fields (with an overbar denoting background quantities) and derive the equations of motion from our covariant approach $\bar{\vartheta} = \vartheta$. The lie derivative of the basis vectors along u^μ then provide

$$\mathring{\boldsymbol{e}}^{\boldsymbol{i}}_{\xi} = \mathring{\vartheta} \boldsymbol{e}^{\boldsymbol{i}}_{s}, \qquad \mathring{\boldsymbol{e}}^{\boldsymbol{i}}_{s} = -\mathring{\vartheta} \boldsymbol{e}^{\boldsymbol{i}}_{\xi}. \tag{5.22}$$

We also abuse notation and define the following 'derivative', although we must recognize that this is not the lie derivative of a scalar field along u^μ

$$\mathring{\xi} \equiv \sqrt{(\mathring{\phi}^{\mathbf{1}})^2 + (\mathring{\phi}^{\mathbf{2}})^2}. \tag{5.23}$$

This definition is a matter of convenience as it allows one to rewrite certain terms or factors succinctly such as

$$\boldsymbol{e}^{\boldsymbol{i}}_{\xi} = \frac{\mathring{\phi}^{\boldsymbol{i}}}{\mathring{\xi}}. \tag{5.24}$$

It is at this point we may define the two covectors ξ_μ and s_μ as linear combinations of covariant derivatives of the two scalar matter fields $\phi^{\mathbf{1}}, \phi^{\mathbf{2}}$ along u^μ

$$\xi_\mu \equiv \boldsymbol{e}^{\boldsymbol{i}}_{\xi} \nabla_\mu \phi_{\boldsymbol{i}} = \cos\vartheta \nabla_\mu \phi^{\mathbf{1}} + \sin\vartheta \nabla_\mu \phi^{\mathbf{2}}, \tag{5.25}$$

$$s_\mu \equiv \boldsymbol{e}^{\boldsymbol{i}}_{s} \nabla_\mu \phi_{\boldsymbol{i}} = -\sin\vartheta \nabla_\mu \phi^{\mathbf{1}} + \cos\vartheta \nabla_\mu \phi^{\mathbf{2}}, \tag{5.26}$$

the inquisitive reader may wish to note the generalization between the above equations and equations (4.55) and (4.56). These are the generalisations of the adiabatic and entropy fields respectively, introduced in the context of linear perturbations in section 4.1.2. The entropy covector s_μ is orthogonal to the four-velocity $u^\mu s_\mu = 0$, whereas the adiabatic covector ξ_μ possesses a component in the direction of the four-velocity $u^\mu \xi_\mu = \mathring{\xi}$.

5.1.1 Generalized background adiabatic and entropic equations of motion

It is instructive to derive the Klein-Gordon equation of motions of the scalar matter fields projected in the adiabatic and entropic directions. We begin with the adiabatic direction, by defining and noting the following quantities

$$\xi^\perp_\mu \equiv \boldsymbol{e}^{\boldsymbol{i}}_{\xi} \mathfrak{D}_\mu \phi^{\boldsymbol{i}} = \xi_\mu + \mathring{\xi} u_\mu, \qquad s^\perp_\mu \equiv \boldsymbol{e}^{\boldsymbol{i}}_{s} \mathfrak{D}_\mu \phi_{\boldsymbol{i}} = s_\mu, \tag{5.27}$$

$$\boldsymbol{e}^{\boldsymbol{i}}_{\xi} \mathring{\phi}_{\boldsymbol{i}} = \mathring{\xi}, \qquad \boldsymbol{e}^{\boldsymbol{i}}_{\xi} \mathring{\mathring{\phi}}_{\boldsymbol{i}} = \mathring{\mathring{\xi}}, \tag{5.28}$$

$$V_\xi \equiv \boldsymbol{e}^{\boldsymbol{i}}_{\xi} V_{,\phi^{\boldsymbol{i}}}. \tag{5.29}$$

Equation (5.27) are the adiabatic and entropic covectors projected onto space, orthogonal to the four-velocity u^μ (we have instead acted on the fields $\phi^{\boldsymbol{i}}$ with $\mathfrak{D}_\mu$ as opposed to ∇_μ). By now contracting $\boldsymbol{e}^{\boldsymbol{i}}_{\xi}$ with equation (5.17) we obtain

$$\mathring{\mathring{\xi}} + \Theta \mathring{\xi} + V_\xi - \boldsymbol{e}^{\boldsymbol{i}}_{\xi} \mathfrak{D}_\mu \mathfrak{D}^\mu \phi_{\boldsymbol{i}} - a^\mu \xi^\perp_\mu = 0. \tag{5.30}$$

Rewriting the fourth term via the generalized Leibniz rule, equation (5.27) and the identity equation (5.20) applied to the entropic portion equation (5.27) to obtain the final term on the right hand side

$$\boldsymbol{e}^{\boldsymbol{i}}_{\xi}\mathfrak{D}_{\mu}\mathfrak{D}^{\mu}\phi_{\boldsymbol{i}} = \mathfrak{D}_{\mu}\left(\boldsymbol{e}^{\boldsymbol{i}}_{\xi}\mathfrak{D}^{\mu}\phi_{\boldsymbol{i}}\right) - \left(\mathfrak{D}_{\mu}\boldsymbol{e}^{\boldsymbol{i}}_{\xi}\right)\mathfrak{D}^{\mu}\phi_{\boldsymbol{i}} = \mathfrak{D}^{\mu}\xi^{\perp}_{\mu} - \left(\boldsymbol{e}_{s\boldsymbol{i}}\mathfrak{D}_{\mu}\boldsymbol{e}^{\boldsymbol{i}}_{\xi}\right)s^{\mu}. \tag{5.31}$$

It may also be found that

$$\boldsymbol{e}_{s\boldsymbol{i}}\nabla_{\mu}\boldsymbol{e}^{\boldsymbol{i}}_{\xi} = -\boldsymbol{e}_{\xi\boldsymbol{i}}\nabla_{\mu}\boldsymbol{e}^{\boldsymbol{i}}_{s} = \frac{1}{\mathring{\xi}}\left(\mathring{s}_{\mu} + \mathring{\vartheta}\xi_{\mu}\right), \tag{5.32}$$

which allows one to find

$$\boldsymbol{e}^{\boldsymbol{i}}_{\xi}\mathfrak{D}_{\mu}\mathfrak{D}^{\mu}\phi_{\boldsymbol{i}} = \mathfrak{D}^{\mu}\xi^{\perp}_{\mu} - \frac{1}{\mathring{\xi}}\left(\mathring{s}_{\mu} + \mathring{\vartheta}\xi^{\perp}_{\mu}\right)s^{\mu}. \tag{5.33}$$

Using the equivalence between equations (5.31) and (5.33), substituting them into equation (5.30), defining the quantity $Y_{(s)}$

$$Y_{(s)} \equiv \frac{1}{\mathring{\xi}}\left(\mathring{s}_{\mu} + \mathring{\vartheta}\xi^{\perp}_{\mu}\right), \tag{5.34}$$

and using the following property suitable for covectors orthogonal to the four-velocity u^{μ}

$$\mathfrak{D}^{\mu}\xi^{\perp}_{\mu} + a^{\mu}\xi^{\perp}_{\mu} = \nabla^{\mu}\xi^{\perp}_{\mu}, \tag{5.35}$$

we arrive at the Klein-Gordon equation describing the field evolution in the adiabatic (longitudinal) direction of N canonically normalized scalar fields minimally coupled to gravity

$$\mathring{\mathring{\xi}} + \Theta\mathring{\xi} + V_{\xi} = \nabla^{\mu}\xi^{\perp}_{\mu} - Y_{(s)}. \tag{5.36}$$

One may carry out a similar procedure for the entropic direction by instead contracting $\boldsymbol{e}^{\boldsymbol{i}}_{s}$ with equation (5.17), we use and define

$$\boldsymbol{e}^{\boldsymbol{i}}_{s}\mathring{\mathring{\phi}}_{\boldsymbol{i}} = \mathring{\theta}\mathring{\xi} \tag{5.37}$$

$$V_s \equiv \boldsymbol{e}_s^{\boldsymbol{i}} V_{,\phi^{\boldsymbol{i}}}, \tag{5.38}$$

to find the similar result of equation (5.30)

$$\mathring{\xi}\mathring{\vartheta} + V_s - \boldsymbol{e}_s^{\boldsymbol{i}} \mathfrak{D}_\mu \mathfrak{D}^\mu \phi_{\boldsymbol{i}} - a^\mu s_\mu = 0. \tag{5.39}$$

Rewriting the last two terms using the identity

$$\boldsymbol{e}_s^{\boldsymbol{i}} \mathfrak{D}_\mu \mathfrak{D}^\mu \phi_{\boldsymbol{i}} = \mathfrak{D}^\mu s_\mu + Y_{(\xi)}, \tag{5.40}$$

$$Y_{(\xi)} \equiv \frac{1}{\mathring{\xi}} \left(\mathring{s}_\mu + \mathring{\vartheta} \xi^\perp_\mu \right) \xi^{\perp\mu}, \tag{5.41}$$

and applying the property to the entropic covector s_μ for orthogonal covectors with respect to the four-velocity equation (5.35), one arrives at the following Klein-Gordon equation in the entropic direction

$$\mathring{\xi}\mathring{\vartheta} + V_s = \nabla_\mu s^\mu + Y_{(\xi)}. \tag{5.42}$$

It is illuminating to notice that the left hand sides of these two equations of motion for the entropic and adiabatic directions of the matter fields equations (5.36) and (5.42) are equivalent to the homogeneous background adiabatic and entropy equations in FLRW, as presented in equations (33) (or equation (4.62) of section 4.1.2) and (46) of the coordiante based approach of [80]. The difference in this case is that the covariantly derived equations mentioned above are exact: they encapsulate the non-linear dynamics of the scalar fields through the source terms on the right hand side.

5.1.2 Evolution equations of the adiabatic and entropy covectors

We now wish to derive the fully non-linear equations governing the evolution of the adiabatic and entropy covectors to second order in the Lie derivative with respect to the four-velocity u^μ. We will find that these equations are very similar (but more general) to those derived in the coordinate approach to liner order in perturbations.

We begin with the evolution equation of ξ_μ, from its definition equation (5.25) we take the Lie derivative with respect to u^μ and utilize the substitution $\mathring{\phi}_{\boldsymbol{i}} = \mathring{\xi} \boldsymbol{e}_{\xi \boldsymbol{i}}$ in the second equality to obtain

$$\mathring{\xi}_\mu = \boldsymbol{e}^{\boldsymbol{i}}_\xi \nabla_\mu \mathring{\phi}_{\boldsymbol{i}} + \mathring{\vartheta} s_\mu = \nabla_\mu \mathring{\xi} + \mathring{\vartheta} s_\mu, \tag{5.43}$$

whereby one applies an additional Lie operator to obtain the second order (in time) equation

$$\mathring{\mathring{\xi}}_\mu = \nabla_\mu \mathring{\mathring{\xi}} + \mathring{\mathring{\vartheta}} s_\mu + \mathring{\vartheta} \mathring{s}_\mu. \tag{5.44}$$

Calling upon the generalized "background" equation for the adiabatic field equation (5.36) derived in section 5.1.1, we eliminate $\mathring{\mathring{\xi}}$ from equation (5.44). However in order to do so entirely, one must first carry out the covariant derivative on V_ξ providing

$$\nabla_\mu V_\xi = V_{\xi\xi} \xi_\mu + V_{\xi s} s_\mu + \frac{V_s}{\mathring{\xi}} (\mathring{s}_\mu + \mathring{\vartheta} \xi_\mu). \tag{5.45}$$

We may then obtain via direct substitution,

$$\begin{aligned} \mathring{\mathring{\xi}}_\mu + \Theta \mathring{\xi}_\mu + \mathring{\xi} \nabla_\mu \Theta + \left(V_{\xi\xi} + \mathring{\vartheta} \frac{V_s}{\mathring{\xi}} \right) \xi_\mu - \nabla_\mu (\nabla^\sigma \xi^\perp_\sigma) \\ = \left(\mathring{\vartheta} - \frac{V_s}{\mathring{\xi}} \right) \mathring{s}_\mu + (\mathring{\mathring{\vartheta}} - V_{\xi s} + \Theta \mathring{\vartheta}) s_\mu - \nabla_\mu Y_{(s)}, \end{aligned} \tag{5.46}$$

with some simplifying notation defined for the second derivatives of the multi-field potential

$$V_{\xi\xi} \equiv \boldsymbol{e}^{\boldsymbol{i}}_\xi \boldsymbol{e}^{\boldsymbol{j}}_\xi V_{,\phi^{\boldsymbol{i}} \phi^{\boldsymbol{j}}}, \qquad V_{ss} \equiv \boldsymbol{e}^{\boldsymbol{i}}_s \boldsymbol{e}^{\boldsymbol{j}}_s V_{,\phi^{\boldsymbol{i}} \phi^{\boldsymbol{j}}}, \qquad V_{s\xi} \equiv \boldsymbol{e}^{\boldsymbol{i}}_s \boldsymbol{e}^{\boldsymbol{j}}_\xi V_{,\phi^{\boldsymbol{i}} \phi^{\boldsymbol{j}}}. \tag{5.47}$$

If we then decompose the above equation (5.46) into longitudinal and orthogonal components by contracting it with u^μ or $h_{\mu\nu}$ respectively, one may perform a consistency check and observe that after utilizing

$$\mathring{V}_\xi = V_{\xi\xi} \mathring{\xi} + \mathring{\vartheta} V_s, \tag{5.48}$$

one obtains the Lie derivative of equation (5.36). On the other hand, the orthogonal component may be expressed as

$$(\mathring{\mathring{\xi}}_\mu)^\perp + \Theta (\mathring{\xi}_\mu)^\perp + \mathring{\xi} \mathfrak{D}_\mu \Theta + \left(V_{\xi\xi} + \mathring{\vartheta} \frac{V_s}{\mathring{\xi}} \right) \xi^\perp_\mu - \mathfrak{D}_\mu (\nabla^\sigma \xi^\perp_\sigma)$$

$$= \left(\mathring{\vartheta} - \frac{V_s}{\mathring{\xi}}\right)\mathring{s}_\mu + (\mathring{\mathring{\vartheta}} - V_{\xi s} + \Theta\mathring{\vartheta})s_\mu - \mathfrak{D}_\mu Y_{(s)}. \quad (5.49)$$

To determine the evolution equation of the entropic covector s_μ, we begin with equation (5.32), take two Lie derivatives with respect to the four-velocity and use equations (5.22) and (5.32) to obtain

$$\mathring{\mathring{s}}_\mu = -\mathring{\mathring{\vartheta}}\xi_\mu - \mathring{\vartheta}\mathring{\xi}_\mu + \frac{\mathring{\mathring{\xi}}}{\mathring{\xi}}(\mathring{s}_\mu + \mathring{\vartheta}\xi_\mu) + \mathring{\xi}\nabla_\mu\mathring{\vartheta}. \quad (5.50)$$

Using the Klein-Gordon equation in the entropic direction equation (5.42) to remove the term proportional to $\nabla_\mu\mathring{\vartheta}$ and using the following relation

$$\nabla_\mu V_s = V_{s\xi}\xi_\mu + V_{ss}s_\mu - \frac{V_\xi}{\mathring{\xi}}(\mathring{s}_\mu + \mathring{\vartheta}\xi_\mu), \quad (5.51)$$

one obtains the evolution equation for the entropic covector to second order in the Lie derivative with respect to u^μ

$$\mathring{\mathring{s}}_\mu - \frac{1}{\mathring{\xi}}(\mathring{\mathring{\xi}} + V_\xi)\mathring{s}_\mu + (V_{ss} - \mathring{\vartheta}^2)s_\mu - \nabla_\mu(\nabla_\sigma s^\sigma)$$
$$= -2\mathring{\vartheta}\mathring{\xi}_\mu + \left[\frac{\mathring{\vartheta}}{\mathring{\xi}}(\mathring{\mathring{\xi}} + V_\xi) - \mathring{\mathring{\vartheta}} - V_{\xi s}\right]\xi_\mu + \nabla_\mu Y_{(\xi)}. \quad (5.52)$$

Once again we may divide this equation into longitudinal and orthogonal components (with respect to u^μ). As a consistency check, after utilization of

$$\mathring{V}_s = V_{\xi s}\mathring{\xi} - \mathring{\vartheta}V_\xi, \quad (5.53)$$

the contraction of equation (5.52) with u^μ (longitudinal component) one may confirm the result is the Lie derivative of equation (5.42)

$$\frac{\mathring{\vartheta}}{\mathring{\xi}}V_\xi - \mathring{\mathring{\vartheta}} - V_{\xi s} = \frac{\mathring{\vartheta}}{\mathring{\xi}}\mathring{\mathring{\xi}} - \frac{1}{\mathring{\xi}}(\mathfrak{D}_\sigma s^\sigma + Y_{(\xi)})^{\circ}. \quad (5.54)$$

The orthogonal component, after using equation (5.27) to simplify the notation since the entropic covectors are entirely spatial: $(\mathring{s}_\mu)^\perp = \mathring{s}_\mu$, $(\mathring{\mathring{s}}_\mu)^\perp = \mathring{\mathring{s}}_\mu$ we obtain

$$\mathring{\mathring{s}}_\mu - \frac{1}{\mathring{\xi}}(\mathring{\mathring{\xi}} + V_\xi)\mathring{s}_\mu + (V_{ss} - \mathring{\vartheta}^2)s_\mu - \mathfrak{D}_\mu(\nabla_\sigma s^\sigma)$$
$$= -2\mathring{\vartheta}(\mathring{\xi}_\mu)^\perp + \left[\frac{\mathring{\vartheta}}{\mathring{\xi}}(\mathring{\mathring{\xi}} + V_\xi) - \mathring{\mathring{\vartheta}} - V_{\xi s}\right]\xi^\perp_\mu + \mathfrak{D}_\mu Y_{(\xi)}. \quad (5.55)$$

Equations (5.49) and (5.55) are among the main results of [125]. Although relatively simple in appearance, they are nevertheless exact as they encapsulate all nonlinearities of the Klein-Gordon equations of multifield models minimally coupled to gravity and should therefore be used as the starting point for deriving perturbed equations to arbitrary order.

5.1.3 Covariant perturbation variables

In this subsection we define the covariant versions of: the comoving energy density perturbation variable, comoving curvature perturbation variable and the curvature perturbation variable on uniform energy density hypersurfaces in the context of double scalar field matter models. Firstly we introduce the covariant comoving energy density perturbation ϵ_μ

$$\epsilon_\mu \equiv \mathfrak{D}_\mu\rho - \frac{\mathring{\rho}}{\mathring{\xi}}\xi^\perp_\mu. \quad (5.56)$$

The goal is to obtain an expression for ξ_μ and s_μ with explicit dependence on the covariant comoving energy density perturbation, to do so we rewrite the components of the stress-energy tensor $T_{\mu\nu}$ in equations (5.12) to (5.15)

$$\rho \equiv \frac{1}{2}\left(\mathring{\xi}^2 + \Pi\right) + V, \quad (5.57)$$

$$p = \frac{1}{2}\left(\mathring{\xi}^2 - \frac{1}{3}\Pi\right) - V, \quad (5.58)$$

$$q_\mu = -\mathring{\xi}\xi^\perp_\mu, \quad (5.59)$$

$$\pi_{\mu\nu} = \Pi_{\mu\nu} - \frac{1}{3}h_{\mu\nu}\Pi, \quad (5.60)$$

$$\Pi_{\mu\nu} \equiv \xi^\perp_\mu\xi^\perp_\nu + s_\mu s_\nu, \qquad \Pi \equiv \xi^\perp_\mu\xi^{\perp\mu} + s_\mu s^\mu. \quad (5.61)$$

By directly substituting ρ described by equation (5.57) into the definition of ϵ_μ described by equation (5.56) and utilization of the following relation (obtained by taking the Lie derivative of equation (5.43))

$$\mathfrak{D}_\mu \mathring{\xi} = \mathring{\xi}_\mu + \mathring{\mathring{\xi}} u_\mu - \mathring{\vartheta} s_\mu, \tag{5.62}$$

we obtain

$$\epsilon_\mu = \mathring{\xi}(\mathring{\xi}_\mu)^\perp - \mathring{\mathring{\xi}} \xi_\mu^\perp + \left(V_s - \mathring{\vartheta}\mathring{\xi}\right) s_\mu + \frac{1}{2}\left(\mathfrak{D}_\mu \Pi - \frac{\mathring{\Pi}}{\mathring{\xi}} \xi_\mu^\perp\right). \tag{5.63}$$

We are now in a position to rewrite equation (5.55) and obtain explicit dependence of the adiabatic and entropic covectors on ϵ_μ. With equations (5.36), (5.54) and (5.63) we rewrite equation (5.55) and obtain

$$\begin{aligned} \mathring{\mathring{s}}_\mu + \left[\Theta - \frac{1}{\mathring{\xi}}\left(\nabla^\sigma \xi_\sigma^\perp - Y_{(s)}\right)\right] \mathring{s}_\mu + \left(V_{ss} + \mathring{\vartheta}^2 - 2\mathring{\vartheta}\frac{V_s}{\mathring{\xi}}\right) s_\mu - \mathfrak{D}_\mu(\nabla_\sigma s^\sigma) \\ = -2\frac{\mathring{\vartheta}}{\mathring{\xi}} \epsilon_\mu + \frac{\mathring{\vartheta}}{\mathring{\xi}}\left(\mathfrak{D}_\mu \Pi - \frac{\mathring{\Pi}}{\mathring{\xi}} \xi_\mu^\perp\right) - \frac{1}{\mathring{\xi}}\left(\mathfrak{D}_\sigma s^\sigma + Y_{(\xi)}\right)^{\circ} \xi_\mu^\perp + \mathfrak{D}_\mu Y_{(\xi)}. \end{aligned} \tag{5.64}$$

One may now note the resemblance the above equation has to the entropy perturbation equation to linear order as seen in equation (48) of [80] when one defines the comoving energy density perturbation variable as in equation (10) of the same paper. This equation will be useful when deriving an approximate form for covariant entropy perturbations in the large scale limit.

We continue to define covariant versions of the perturbation variables commonly used in the coordinate based linearized theory. We define the comoving integrated expansion perturbation variable for the case of N scalar fields

$$\mathcal{R}_\mu \equiv -\mathfrak{D}_\mu \alpha - \frac{\mathring{\alpha}}{(\mathring{\phi}_{\boldsymbol{i}} \mathring{\phi}^{\boldsymbol{i}})} q_\mu. \tag{5.65}$$

The Mukhanov-Sasaki variable may also be extended to a covector for each field

$$Q_\mu^{\boldsymbol{i}} \equiv \mathfrak{D}_\mu \phi^{\boldsymbol{i}} - \frac{\mathring{\phi}^{\boldsymbol{i}}}{\mathring{\alpha}} \mathfrak{D}_\mu \alpha, \tag{5.66}$$

allowing one to express $\mathcal{R}_\mu$ in terms of the Mukhanov-Sasaki variable

$$\mathcal{R}_\mu = \frac{\mathring{\alpha}}{(\mathring{\phi}_{\boldsymbol{i}}\mathring{\phi}^{\boldsymbol{i}})}\mathring{\phi}_{\boldsymbol{j}} Q^{\boldsymbol{j}}_\mu. \tag{5.67}$$

For the case of two scalar matter fields, equation (5.65) reduces using equation (5.59)

$$\mathcal{R}_\mu \equiv -\mathfrak{D}_\mu \alpha + \frac{\mathring{\alpha}}{\mathring{\xi}}\xi^\perp_\mu, \tag{5.68}$$

and the Mukhanov-Sasaki variable in the adiabatic direction may be defined

$$Q_\mu \equiv \boldsymbol{e}_{\xi\boldsymbol{i}} Q^{\boldsymbol{i}}_\mu = \xi^\perp_\mu - \frac{\mathring{\xi}}{\mathring{\alpha}}\mathfrak{D}_\mu \alpha. \tag{5.69}$$

Most relevant for this work, we define the generalized curvature perturbation on uniform energy density hypersurfaces valid for any number of scalar fields [122, 123]

$$\zeta_\mu \equiv \mathfrak{D}_\mu \alpha - \frac{\mathring{\alpha}}{\mathring{\rho}}\mathfrak{D}_\mu \rho, \tag{5.70}$$

as well as the nonlinear nonadiabatic pressure perturbation also valid for any number of scalar fields

$$\Gamma_\mu \equiv \mathfrak{D}_\mu p - \frac{\mathring{p}}{\mathring{\rho}}\mathfrak{D}_\mu \rho. \tag{5.71}$$

However, there are differences between the single and multifield cases, in particular when considering the first Lie derivative of ζ_μ and the relationship between ζ_μ and $\mathcal{R}_\mu$. For a single matter field which may always be described as a perfect fluid by an appropriate choice of u^μ ensuring that q_μ and $\pi_{\mu\nu}$ vanish (see [124] for its extension to a non perfect fluid), ζ_μ satisfies the following exact evolution equation to first order in the Lie derivative [122, 123]

$$\mathring{\zeta}_\mu = \frac{\Theta^2}{3\dot{\rho}}\Gamma_\mu, \qquad \text{(single field)} \tag{5.72}$$

For a fluid whose equation of state may be expressed as $p = p(\rho)$ (barotropic), $\Gamma_\mu = 0$ thus ζ_μ is conserved on all scales to all orders in perturbation theory. However it is important to note that the traditional ζ introduced in the coordinate based approach to perturbation theory at linear order only coincides with the covariant ζ_μ on large scales, where spatial gradient terms may be approximately omitted; on small scales, the two quantities may differ significantly [123]. For a single field, the comoving and uniform density perturbations are related by

$$\zeta_\mu + \mathcal{R}_\mu = -\frac{\mathring{\alpha}}{\mathring{\rho}}\left(\mathfrak{D}_\mu \rho - \frac{\mathring{\rho}}{\mathring{\phi}}\mathfrak{D}_\mu \phi\right). \qquad \text{(single field)} \tag{5.73}$$

The right hand side of equation (5.73) may be interpreted as the result of shifting between hypersurfaces of constant energy density to hypersurfaces of constant ϕ (comoving frame). The term inside the parentheses represents the nonlinear generalization of the comoving energy density for a single scalar field. For a scalar field in perfect fluid form, Γ_μ takes on the reduced form (after making use of $\mathfrak{D}_\mu V = 0$)

$$\Gamma_\mu = \left(1 - \frac{\mathring{p}}{\mathring{\rho}}\right)\mathfrak{D}_\mu \rho = 2\frac{\mathring{\phi}}{\mathring{\rho}}V_{,\phi}\mathfrak{D}_\mu \rho, \qquad \text{(single field)} \tag{5.74}$$

providing a simplified evolution equation for ζ_μ

$$\mathring{\zeta}_\mu = \frac{2}{3}\frac{V_{,\phi}}{\mathring{\phi}^3}\mathfrak{D}_\mu \rho. \qquad \text{(single field)} \tag{5.75}$$

In the large scale limit, the right hand side of equation (5.75) may be neglected enforcing that ζ_μ be conserved.

For two scalar matter fields the story changes: we instead have the generalized relationship between ζ_μ and $\mathcal{R}_\mu$

$$\zeta_\mu + \mathcal{R}_\mu = -\frac{\mathring{\alpha}}{\mathring{\rho}}\epsilon_\mu, \tag{5.76}$$

and in contrast with the single field case, the stress-energy tensor for two or more fields may in general be described by a dissipative fluid as described in [124]. The adiabatic Klein-Gordon equation equation (5.36) may be rewritten as a continuity equation for the total energy density and pressure equations (5.57) and (5.58)

$$\mathring{\rho} + \Theta(\rho + p) = \mathcal{D}, \tag{5.77}$$

with the dissipative term defined as

$$\mathcal{D} = \mathring{\xi} \left(\nabla^{\mu} \xi_{\mu}^{\perp} - Y_{(s)} \right) + \frac{1}{3}\Theta\Pi + \frac{1}{2}\mathring{\Pi}. \tag{5.78}$$

It was shown in [124] that the evolution equation for ζ_μ with matter described by a dissipative fluid is given now by

$$\mathring{\zeta}_{\mu} = \frac{\Theta^2}{3\mathring{\rho}}(\Gamma_\mu + \Sigma_\mu), \tag{5.79}$$

with the additional (relative to the single field case equation (5.75)) source term Σ known as the dissipative nonadiabatic pressure perturbation is defined in terms of properties of the dissipative fluid

$$\Sigma_\mu \equiv -\frac{1}{\Theta}\left(\mathfrak{D}_\mu \mathcal{D} - \frac{\mathring{\mathcal{D}}}{\mathring{\rho}} \mathfrak{D}_\mu \rho \right) + \frac{\mathcal{D}}{\Theta^2}\left(\mathfrak{D}_\mu \Theta - \frac{\mathring{\Theta}}{\mathring{\rho}} \mathfrak{D}_\mu \rho \right). \tag{5.80}$$

From equations (5.57) and (5.58), we may express the pressure and energy density of the two scalar fields in terms of one another

$$p = \rho - 2V - \frac{2}{3}\Pi, \tag{5.81}$$

and substitute this into the definition of the nonadiabatic pressure perturbation equation (5.71) providing,

$$\Gamma_\mu = \left(1 - \frac{\mathring{p}}{\mathring{\rho}} \right) \mathfrak{D}_\mu \rho - 2\mathfrak{D}_\mu V - \frac{2}{3}\mathfrak{D}_\mu \Pi. \tag{5.82}$$

The Lie derivative of equation (5.81) and

$$\mathfrak{D}_\mu V = V_\xi \xi_\mu^{\perp} + V_s s_\mu, \tag{5.83}$$

into equation (5.82) finally provides

$$\Gamma_\mu = 2\frac{\mathring{\xi}}{\mathring{\rho}}V_\xi \epsilon_\mu - 2V_s s_\mu - \frac{2}{3}\left(\mathfrak{D}_\mu \Pi - \frac{\mathring{\Pi}}{\mathring{\rho}}\mathfrak{D}_\mu \rho\right). \tag{5.84}$$

To recapitulate, we now have a complete description of the first order (in the Lie derivative) evolution of ζ_μ provided by equation (5.75) for the single field case and equation (5.79) for the multifield case. For the two field case, the two source terms for the evolution of ζ_μ, Γ_μ and Σ_μ are provided in equations (5.80) and (5.84) respectively. We also have expressions relating ζ_μ and $\mathcal{R}_\mu$ for the single field case (5.73) and multifield cases (5.76). In this paper, we will be interested in tracking perturbations far beyond the Hubble radius and we will see that in this large scale limit (and also the linear perturbation limit [125]) that the source terms for the evolution of ζ_μ simplify significantly.

5.1.4 Large scale limit of covectors and linear perturbations

The primary aim of this subsection is to determine the first order (in the Lie derivative) evolution equation of the covariant curvature perturbation variable on uniform density hypersurfaces ζ_μ, as well as the second order (in the Lie derivative) evolution equation for the entropic covector s_μ on scales much larger than the Hubble radius. In order to accurately approximate the dynamical equations on large scales (colloquially known as the long wavelength approximation), one should linearize the equations with respect to the spatial gradient, as done in works such as [127–129] where they perform spatial gradient expansions of Einstein's equations. We represent equalities valid on large scales by $\stackrel{\sim}{=}$, unless otherwise stated.

One may immediately determine from their definitions (5.27) that the spatially projected adiabatic and entropic covectors $\xi^\perp_\mu$ and $s^\perp_\mu = s_\mu$ are first-order with respect to the spatial gradient. Further the scalar quantities $Y_{(s)}, Y_{(\xi)}$ (5.34) and (5.41) are second order with respect to spatial gradients as they are second order in $\xi^\perp_\mu$ and s_μ. Thus, equations (5.36) and (5.42) immediately simplify in the large scale limit

$$\mathring{\mathring{\xi}} + \Theta\mathring{\xi} + V_\xi \stackrel{\sim}{=} 0, \tag{5.85}$$

$$\mathring{\vartheta} \simeq -\frac{V_s}{\mathring{\xi}}. \tag{5.86}$$

Once again, a reader familiar with the coordinate based linear theory of two-field perturbation models may notice the seemingly equivalent forms of the above equations equations (5.85) and (5.86) with equations (4.62) and (4.181). However, these equations are entirely inhomogeneous in the restricted sense that they capture all non-linearities of the dynamics on large scales only [3].

However, the following term is second order in spatial gradients provided one assumes that the four-velocity may be chosen such that a^μ is at a minimum first order in spatial gradients as is shown explicitly in [125]

$$\nabla^\mu \xi_\mu^\perp = \mathfrak{D}^\mu \xi_\mu^\perp + a^\mu \xi_\mu^\perp. \tag{5.87}$$

Thus with the above realizations in mind in addition to the fact that the terms containing Π are third order in spatial gradients, the evolution equations equations (5.49) and (5.64) of the orthogonally projected covectors $\xi_\mu^\perp$ and $s_\mu^\perp = s_\mu$ to first order in spatial gradients are

$$(\mathring{\mathring{\xi}}_\mu)^\perp + \Theta(\mathring{\xi}_\mu)^\perp + \mathring{\xi}\mathfrak{D}_\mu\Theta + \left(V_{\xi\xi} - \mathring{\vartheta}^2\right)\xi_\mu^\perp \simeq 2\left(\mathring{\vartheta} s_\mu\right)^{\circ} - 2\mathring{\vartheta}\frac{V_\xi}{\mathring{\xi}} s_\mu, \tag{5.88}$$

$$\mathring{\mathring{s}}_\mu + \Theta \mathring{s}_\mu + \left(V_{ss} + 3\mathring{\vartheta}^2\right) s_\mu \simeq -2\frac{\mathring{\vartheta}}{\mathring{\xi}}\epsilon_\mu. \tag{5.89}$$

Finally, we linearize in spatial gradients the evolution equation to first order in the Lie derivative of ζ_μ provided in equation (5.79) by neglecting higher-order spatial gradients in the two source terms Σ_μ and Γ_μ. The nonadiabatic pressure perturbation becomes

$$\Gamma_\mu \simeq 2\frac{\mathring{\xi}}{\mathring{\rho}} V_\xi \epsilon_\mu - 2V_s s_\mu, \tag{5.90}$$

and the dissipative nonadiabatic pressure perturbation Σ_μ may be entirely omitted since it is at a minimum third order in spatial gradients. Equation (5.79) may then be approximated

[3] Evidence for the validity of the separate universe picture: this concept may be used to describe an inhomogeneous universe on large scales as a collection of Friedmann homogeneous universes [116, 117].

on large scales

$$\mathring{\zeta}_\mu \cong \frac{2}{3\mathring{\xi}^3} V_\xi \epsilon_\mu + \frac{2\Theta}{3\mathring{\xi}^2} V_s s_\mu. \tag{5.91}$$

Equations (5.79), (5.88) and (5.89) are two of the three main results of section 5.1.4, and will be used when deriving the perturbative equations up to third order which in turn will allow us to describe the generation of non-Gaussianities, with a focus on the ekpyrotic scenario.

The third and final important result of section 5.1.4 is determining the behaviour of ϵ_μ on large scales, and understanding the repercussions on the dynamical equations. We make use of Einstein's equations to derive constraints on the energy ρ and momentum q_μ in order to determine the non linear covariant version of a generalized Poisson equation, which in turn will tell us how ϵ_μ behaves on larges scales based on its dependence on spatial gradients.

We begin by projecting Einstein's equations in the direction of the four-velocity u^μ to acquire the covariant energy constraint

$$u^\mu G_{\mu\nu} u^\nu = 8\pi\rho. \tag{5.92}$$

The authors of [125] claim that if one assumes u^μ to be orthogonal to spacelike foliated hypersurfaces one may use the Gauss-Codacci relations [126] in addition to the the spatially projected decomposition of the covariant derivative of the four-velocity (see equation (5.8), note that $\omega_{\mu\nu} = 0$ below since u^μ is assumed to be hypersurface orthogonal)

$$\mathfrak{D}_\nu u_\mu = \sigma_{\mu\nu} + \frac{1}{3}\Theta h_{\mu\nu}, \tag{5.93}$$

in order to rewrite the energy constraint as

$$\frac{1}{2}\left({}^{(3)}R + \frac{2}{3}\Theta^2 - \sigma_{\mu\nu}\sigma^{\mu\nu}\right) = 8\pi\rho. \tag{5.94}$$

${}^{(3)}R$ is the intrinsic Ricci scalar of the spacelike hypersurfaces orthogonal to u^μ. For completeness, we mention the Gauss-Codacci relations

$${}^{(3)}R_{\mu\nu\sigma}{}^{\gamma} = h_\mu{}^\delta h_\nu{}^\omega h_\sigma{}^\alpha h^\gamma{}_\beta R_{\delta\omega\alpha}{}^\beta - K_{\mu\sigma}K_\nu{}^\gamma + K_{\nu\sigma}K_\mu{}^\gamma, \tag{5.95}$$

$$\mathfrak{D}_\mu K^\mu{}_\nu - \mathfrak{D}_\nu K^\mu{}_\mu = R_{\sigma\gamma} n^\gamma h^\sigma{}_\nu. \tag{5.96}$$

These equations relate the three-curvature of spacelike hypersurfaces to the spacetime curvature in which the hypersurfaces are embedded. The rank two tensor field $K_{\mu\nu}$ is defined as

$$K_{\mu\nu} \equiv h_\mu{}^\sigma \nabla_\sigma u_\nu = \frac{1}{2}\mathfrak{L}_u h_{\mu\nu}, \tag{5.97}$$

and acts as a measure of the amount of bending the spacelike hypersurfaces exhibit with respect to the spacetime in which it is embedded.

The mixed projection of Einstein's equations on the other hand yields the momentum constraint in covariant form

$$u^\nu G_{\nu\sigma} h_\mu{}^\sigma = 8\pi q_\mu, \tag{5.98}$$

which once again may be written via the Gauss-Codacci relations and equation (5.93) as

$$\mathfrak{D}_\nu \sigma_\mu{}^\nu - \frac{1}{3}\mathfrak{D}_\mu \Theta = 8\pi q_\mu. \tag{5.99}$$

Combining equations (5.94) and (5.99) (the energy and momentum constraints) we obtain the nonlinear covariant version of a generalized Poission equation

$$\frac{1}{2}\mathfrak{D}_\mu \left({}^{(3)}R - \sigma_{\nu\gamma}\sigma^{\nu\gamma}\right) + \Theta \mathfrak{D}_\nu \sigma_\mu{}^\nu = 8\pi \tilde{\epsilon}_\mu. \tag{5.100}$$

The authors of [125] introduce here an alternative definition of the comoving energy density $\tilde{\epsilon}_\mu$ since in the linear limit it is equivalent to ϵ_μ. They show that this definition of the energy density covector

$$\tilde{\epsilon}_\mu \equiv \mathfrak{D}_\mu \rho - \Theta q_\mu = \mathfrak{D}_\mu \rho + \Theta \mathring{\xi} \xi^\perp_\mu, \tag{5.101}$$

in the covariant description of cosmological perturbations in general differs from ϵ_μ in the case of two fields

$$\tilde{\epsilon}_\mu - \epsilon_\mu = \frac{1}{\mathring{\xi}}\left(\mathcal{D} - \frac{1}{3}\Theta\Pi\right)\xi^\perp_\mu, \tag{5.102}$$

but are equal on large scales

$$\tilde{\epsilon}_\mu \stackrel{\sim}{=} \epsilon_\mu. \tag{5.103}$$

Finally, with a particular interest in equations (5.76), (5.89) and (5.91) we may state that ϵ_μ may be neglected on large scales due to equation (5.100): the terms on the left hand side are all proportional to a projected gradient $\mathfrak{D}_\mu X$. In particular, from the definition of the Ricci scalar in terms of derivatives of the metric [126] it may be shown that the term $\mathfrak{D}_\mu{}^{(3)}R$ is third order in spatial gradients. Additionally, the shear in general rapidly decreases in an expanding perturbed FLRW universe, and is far less blue-shifted than the dominant energy contribution in ekpyrotic contracting models (ie. the ekpyrotic field) [111,125]. Thus in the large scale limit with negligible shear the comoving energy density perturbation covector ϵ_μ may be neglected and as consequences: ζ_μ and $\mathcal{R}_\mu$ coincide up to a sign, and the evolution equations may be simplified to become a closed coupled system

$$\left.\begin{array}{c} \zeta_\mu + \mathcal{R}_\mu \stackrel{\sim}{=} 0 \\ \mathring{\mathring{s}}_\mu + \Theta \mathring{s}_\mu + \left(V_{ss} + 3\mathring{\vartheta}^2\right) s_\mu \stackrel{\sim}{=} 0 \\ \mathring{\zeta}_\mu \stackrel{\sim}{=} -\dfrac{2}{3}\dfrac{\Theta^2}{\mathring{\rho}} V_s s_\mu \end{array}\right\} \qquad \text{(negligible shear).} \tag{5.104}$$

Furthermore, the spatial components of equation (5.100) may be linearized, producing a relativistic Poisson equation

$$\frac{1}{a^2}\nabla^2\left[\psi + H(aB - a^2\dot{E})\right] = 4\pi\delta\epsilon, \tag{5.105}$$

with $\delta\epsilon$ defined in equation (5.138), explicitly displaying that the comoving energy density perturbation $\delta\epsilon$ is second order in spatial gradients and thus may also be neglected on large scales. Thus, once the authors of [125] reproduce the linearized equations of motion via the covariant formalism of the coordinate based approach as in section 4 or [80] (see the following section 5.1.5 for more explicit definitions of the linear perturbation variables)

$$\ddot{\delta\xi} + 3H\dot{\delta\xi} + \left(\bar{V}_{\xi\xi} - \dot{\bar{\vartheta}}^2\right)\delta\xi - \frac{1}{a^2}\nabla^2\delta\xi = 2\left(\dot{\bar{\vartheta}}\delta s\right)^{\cdot} - 2\frac{\bar{V}_\xi}{\dot{\bar{\xi}}}\dot{\bar{\vartheta}}\delta s$$

$$
- 2\bar{V}_\xi\varphi + \dot{\bar{\xi}}\left[\dot{\varphi} + 3\dot{\psi} - \nabla^2\left(\frac{B}{a} - \dot{E}\right)\right], \quad (5.106)
$$

$$
\ddot{\delta s} + 3H\dot{\delta s} + \left(\bar{V}_{ss} + 3\dot{\bar{\vartheta}}^2\right)\delta s - \frac{1}{a^2}\nabla^2\delta s = -2\frac{\dot{\bar{\vartheta}}}{\dot{\bar{\xi}}}\delta\epsilon, \quad (5.107)
$$

$$
\dot{\zeta} = \frac{2}{3}\frac{\bar{V}_\xi}{\dot{\bar{\xi}}^3}\delta\epsilon - 2\frac{H}{\dot{\bar{\xi}}}\dot{\bar{\vartheta}}\delta s + \frac{1}{3}\frac{1}{\dot{\bar{\xi}}a^2}\nabla^2\delta\xi, \quad (5.108)
$$

$$
\zeta + \mathcal{R} = -\frac{\dot{\bar{\alpha}}}{\dot{\bar{\rho}}}\delta\epsilon, \quad (5.109)
$$

they are able to make very similar conclusions to that of equation (5.104) for linear perturbations

$$
\zeta + \mathcal{R} \overset{\sim}{=} 0, \quad (5.110)
$$

$$
\ddot{\delta s} + 3H\dot{\delta s} + \left(\bar{V}_{ss} + 3\dot{\bar{\vartheta}}^2\right)\delta s \overset{\sim}{=} 0, \quad (5.111)
$$

$$
\dot{\zeta} \overset{\sim}{=} -2\frac{H}{\dot{\bar{\xi}}}\dot{\bar{\vartheta}}\delta s. \quad (5.112)
$$

Namely that on large scales: the curvature perturbation variables ζ and $\mathcal{R}$ coincide up to a sign, and the evolution equations for the entropy perturbation and curvature perturbation simplify once again to a closed coupled system.

5.1.5 Second and third perturbative order

As we have stated in closing of the previous subsection, within the covariant formalism the authors of [125] reproduce the dynamical equations governing linear entropy and adiabatic perturbations found in the coordinate based approach for two fields as presented in section 4.1.2; they also reproduce the dynamical equation governing ζ, the curvature perturbation variable on uniform density hypersurfaces. Within the coordinate based approach, the evolution of second order perturbations for the single field case has been studied in [130–135] and multifield case in [136].

Below we follow the works of [123, 125] and derive evolution equations of gauge-invariant perturbative quantities from the ξ_μ, s_μ and ζ_μ equations, at the non-linear level. To derive perturbative equations at higher order in perturbation theory we begin by expanding *all* fields $X(t, x^i)$ as

$$X(t, x^i) \equiv \bar{X}(t) + \delta X^{(1)}(t, x^i) + \delta X^{(2)}(t, x^i) + \delta X^{(3)}(t, x^i), \tag{5.113}$$

where we have altered notation from previous sections denoting background quantities with an overbar as opposed to a subscript naught, and $\delta X^{(1)}, \delta X^{(2)}, \delta X^{(3)}$ are the first, second and third-order contributions respectively. The first and higher order contributions should be understood as the quantities that solve the equations of motion perturbed to the respective order. Additionally, where doing so is unambiguous the superscript (1) of the first order perturbation may be dropped as seen for example in the first order perturbation of the entropy and adiabatic covectors ($\delta s = \delta s^{(1)}$, $\delta\xi = \delta\xi^{(1)}$) and the curvature perturbation on uniform energy density hypersurfaces ($\zeta = \zeta^{(1)}$). We also fix u^μ such that $u_i = 0$, thus determining the remaining component u_0 entirely in terms of metric quantities; we do this to ensure that $\xi_i^{\perp} = \xi_i$.

Firstly, we derive the evolution equations for the adiabatic and entropy fields to second order. We determine the background equations from equations (5.36) and (5.42), and expand equations (5.25) and (5.26) to second order and obtain the following expressions for the spatial components of the adiabatic and entropic fields

$$\ddot{\bar{\xi}} + 3H\dot{\bar{\xi}} + \bar{V}_\xi = 0, \qquad \dot{\bar{\vartheta}} = -\frac{\bar{V}_s}{\dot{\bar{\xi}}} = \dot{\vartheta}, \tag{5.114}$$

$$\delta\xi_i = \frac{\dot{\bar{\phi}}^{\mathbf{1}}}{\dot{\bar{\xi}}}\partial_i\delta\phi^{\mathbf{1}} + \frac{\dot{\bar{\phi}}^{\mathbf{2}}}{\dot{\bar{\xi}}}\partial_i\delta\phi^{\mathbf{2}} \equiv \partial_i\delta\xi, \qquad \delta\xi \equiv \frac{\dot{\bar{\phi}}^{\mathbf{1}}}{\dot{\bar{\xi}}}\delta\phi^{\mathbf{1}} + \frac{\dot{\bar{\phi}}^{\mathbf{2}}}{\dot{\bar{\xi}}}\delta\phi^{\mathbf{2}}, \tag{5.115}$$

$$\delta s_i = \frac{\dot{\bar{\phi}}^{\mathbf{1}}}{\dot{\bar{\xi}}}\partial_i\delta\phi^{\mathbf{2}} - \frac{\dot{\bar{\phi}}^{\mathbf{2}}}{\dot{\bar{\xi}}}\partial_i\delta\phi^{\mathbf{1}} \equiv \partial_i\delta s, \qquad \delta s \equiv \frac{\dot{\bar{\phi}}^{\mathbf{1}}}{\dot{\bar{\xi}}}\delta\phi^{\mathbf{2}} - \frac{\dot{\bar{\phi}}^{\mathbf{2}}}{\dot{\bar{\xi}}}\delta\phi^{\mathbf{1}}, \tag{5.116}$$

$$\delta\xi_i^{(2)} \equiv \partial_i\delta\xi^{(2)} + \frac{\dot{\bar{\vartheta}}}{\dot{\bar{\xi}}}\delta\xi\partial_i\delta s - \frac{1}{\dot{\bar{\xi}}}V_i, \quad \delta\xi^{(2)} \equiv \frac{\dot{\bar{\phi}}^{\mathbf{1}}}{\dot{\bar{\xi}}}\delta\phi^{\mathbf{1}(2)} + \frac{\dot{\bar{\phi}}^{\mathbf{2}}}{\dot{\bar{\xi}}}\delta\phi^{\mathbf{2}(2)} + \frac{1}{2\dot{\bar{\xi}}}\delta s\dot{\delta s}, \tag{5.117}$$

$$\delta s_i^{(2)} \equiv \partial_i\delta s^{(2)} + \frac{\delta\xi}{\dot{\bar{\xi}}}\partial_i\dot{\delta s}, \quad \delta s^{(2)} \equiv -\frac{\dot{\bar{\phi}}^{\mathbf{2}}}{\dot{\bar{\xi}}}\delta\phi^{\mathbf{1}(2)} + \frac{\dot{\bar{\phi}}^{\mathbf{1}}}{\dot{\bar{\xi}}}\delta\phi^{\mathbf{2}(2)} - \frac{\delta\xi}{\dot{\bar{\xi}}}\left(\dot{\delta s} + \frac{\dot{\bar{\vartheta}}}{2}\delta\xi\right), \tag{5.118}$$

$$V_i \equiv \frac{1}{2}(\delta s \partial_i \dot{\delta s} - \dot{\delta s} \partial_i \delta s). \tag{5.119}$$

Note that the fields $\bar{\phi}^{\boldsymbol{i}}$ solve the usual Klein-Gordon equation in a cosmological background, and a correspondence between the angle introduced in the coordinate theory and the angle introduced in the covariant theory has been introduced $\bar{\vartheta} = \vartheta$.

Crucially now, we must ensure gauge invariance of the perturbation variables on large scales at minimum (with the exception of the adiabatic perturbation variable). We first observe the general tensor transformation law generated by a vector Λ^μ which allows us to write down how tensors transform to all orders in a perturbative expansion [137]

$$\tilde{\boldsymbol{T}} \longrightarrow e^{\mathfrak{L}_\Lambda} \boldsymbol{T}. \tag{5.120}$$

If one expands the generator of transformations perturbatively as $\Lambda = \sum_n \frac{1}{n!}\Lambda_{(n)}$ we may then determine the transformation law of tensors expanded perturbatively. In particular, the coordinates of the tensor field transform as (as a particular case)

$$\tilde{x}^\mu \longrightarrow x^\mu = \tilde{x}^\mu - \Lambda^\mu_{(1)} + \frac{1}{2}\Lambda^\nu_{(1)}\Lambda^\mu_{(1),\nu} - \Lambda^\mu_{(2)}, \tag{5.121}$$

and the perturbations of the tensor field transform as

$$\delta \boldsymbol{T}^{(1)} \longrightarrow \delta \boldsymbol{T}^{(1)} + \mathfrak{L}_{\Lambda_{(1)}} \boldsymbol{T}^{(0)}, \tag{5.122}$$

$$\delta \boldsymbol{T}^{(2)} \longrightarrow \delta \boldsymbol{T}^{(2)} + \mathfrak{L}_{\Lambda_{(2)}} \boldsymbol{T}^{(0)} + \frac{1}{2}\mathfrak{L}^2_{\Lambda_{(1)}} \boldsymbol{T}^{(0)} + \mathfrak{L}_{\Lambda_{(1)}} \delta \boldsymbol{T}^{(1)}. \tag{5.123}$$

Let us deal with the entropic covector first. Since s_μ vanishes at zeroth order, one may observe from equation (5.122) that s_μ is automatically gauge-invariant and first order. However, it is not automatically gauge-invariant at second order and the corresponding gauge transformation law by equation (5.123) is

$$\delta s^{(2)}_\mu \longrightarrow \delta s^{(2)}_\mu + \mathfrak{L}_{\Lambda_{(1)}} \delta s_\mu. \tag{5.124}$$

We observe the second term on the right hand side of the above via the definition of the Lie derivative in the large scale limit (neglecting higher orders in the spatial gradient)

$$\mathfrak{L}_{\Lambda_{(1)}}\delta s_i^{(1)} \overset{\sim}{=} \Lambda^0_{(1)}\partial_0\delta s_i. \tag{5.125}$$

Thus due to commuting partials and since $\delta s_i = \partial_i \delta s$, the spatial components of $\delta s_\mu^{(2)}$ transform on large scales as

$$\delta s_i^{(2)} \overset{\sim}{\longrightarrow} \delta s_i^{(2)} + \Lambda^0_{(1)}\partial_0\delta s_i = \delta s_i^{(2)} + \Lambda^0_{(1)}\partial_i\dot{\delta s}. \tag{5.126}$$

In addition, since at linear order

$$\frac{\delta\xi}{\dot{\xi}} \longrightarrow \frac{\delta\xi}{\dot{\xi}} + \Lambda^0_{(1)}, \tag{5.127}$$

the following quantity transforms as

$$\partial_i\delta s^{(2)} = \delta s_i^{(2)} - \frac{\delta\xi}{\dot{\xi}}\partial_i\dot{\delta s} \overset{\sim}{\longrightarrow} \delta s_i^{(2)} + \Lambda^0_{(1)}\partial_i\dot{\delta s} - \left(\frac{\delta\xi}{\dot{\xi}} + \Lambda^0_{(1)}\right)\partial_i\dot{\delta s} = \partial_i\delta s^{(2)}, \tag{5.128}$$

and is thus gauge-invariant. Thus we have proved the gauge invariance of δs generally, and the gauge invariance of $\delta s^{(2)}$ on large scales.

With gauge invariance in mind, the spatial components of the curvature perturbation on uniform energy density hypersurfaces ζ_μ up to second order should be expressed as [123]

$$\delta\zeta_i = \partial_i\zeta, \qquad \zeta \equiv \delta\alpha - \frac{H}{\dot{\rho}}\delta\rho, \tag{5.129}$$

$$\zeta_i^{(2)} = \partial_i\zeta^{(2)} + \frac{\delta\rho}{\dot{\rho}}\partial_i\dot{\zeta}, \tag{5.130}$$

$$\zeta^{(2)} \equiv \delta\alpha^{(2)} - \frac{H}{\dot{\rho}}\delta\rho^{(2)} - \frac{\delta\rho}{\dot{\rho}}\left[\dot{\zeta} + \frac{1}{2}\left(\frac{H}{\dot{\rho}}\right)^{\cdot}\delta\rho\right], \tag{5.131}$$

and one may check using the transformation laws in equations (5.122) and (5.123) that $\zeta, \zeta^{(2)}$ are gauge invariant on large scales.

Importantly, due to the fact that the perturbed integrated expansion term $\delta\alpha$ is related to the scalar metric perturbations E, B, ψ introduced in the coordinate based approach of section 4 [123, 125]

$$\delta\alpha \equiv -\psi + \frac{1}{3}\int \nabla^2 \left(\frac{B}{a} - \dot{E}\right) dt, \tag{5.132}$$

and on large scales

$$\delta\alpha \stackrel{\sim}{=} -\psi. \tag{5.133}$$

Thus the quantity ζ defined in equation (5.129) coincides with the curvature perturbation variable $\zeta_{(\mathrm{Bardeen})}$ defined in the coordinate based approach of section 4 on large scales up to a sign

$$-\zeta_{(\mathrm{Bardeen})} \stackrel{\sim}{=} \zeta. \tag{5.134}$$

The adiabatic covector does not vanish at zeroth order, and thus the first order perturbation is not automatically gauge invariant. The adiabatic perturbations are thus highly dependent on the choice of gauge, and we are free to choose their definitions for convenience. As hinted at in the previous paragraph, it turns out that we may define an approximate comoving gauge on large scales which is done so by exploiting the gauge-variance of the adiabatic quantities. Consider perturbatively expanding the momentum density provided by equation (5.59) and recalling that with our choice of u^μ we have $\xi_i^\perp = \xi_i$

$$q_i = -\dot{\xi}\xi_i, \tag{5.135}$$

$$\delta q_i = -\partial_i \left(\dot{\bar{\xi}}\delta\xi\right), \tag{5.136}$$

$$\delta q_i^{(2)} = -\partial_i \left(\dot{\bar{\xi}}\delta\xi^{(2)} + \frac{1}{2}\frac{\ddot{\bar{\xi}}}{\dot{\bar{\xi}}}\delta\xi^2 + \dot{\bar{\vartheta}}\delta\xi\delta s\right) - \frac{1}{\dot{\bar{\xi}}}\delta\epsilon\partial_i\delta\xi + V_i. \tag{5.137}$$

One may immediately notice that, with the exception of the V_i term in $\delta q_i^{(2)}$, setting $\delta\xi = \delta\xi^{(2)} = 0$ causes all terms in $\delta q_i, \delta q_i^{(2)}$ to vanish. Thankfully, from its definition (5.119) we see that V_i vanishes when $\dot{\delta s} = f(t)\delta s$; this condition is met for super-Hubble modes in both inflationary and ekpyrotic models due to the suppression of spatial gradient terms [111, 125]. Therefore on large scales we may define an approximate comoving gauge at second order ($\delta q = \delta q^{(2)} = 0$) after setting $\delta\xi = \delta\xi^{(2)} = 0$.

Finally we determine up to second order the perturbations in the covariant comoving energy density covector ϵ_μ. From the definition (5.56) we find up to second order

$$\delta\epsilon_i = \partial_i \delta\epsilon, \qquad \delta\epsilon \equiv \delta\rho - \frac{\dot{\bar{\rho}}}{\dot{\bar{\xi}}}\delta\xi, \tag{5.138}$$

$$\delta\epsilon_i^{(2)} = \partial_i \delta\epsilon^{(2)} + \frac{\delta\xi}{\dot{\bar{\xi}}}\partial_i \dot{\delta\epsilon}^{(1)} - 3HV_i, \tag{5.139}$$

$$\delta\epsilon^{(2)} \equiv \delta\rho^{(2)} - \frac{\dot{\bar{\rho}}}{\dot{\bar{\xi}}}\delta\xi^{(2)} - \frac{\delta\xi}{\dot{\bar{\xi}}}\left[\delta\dot{\epsilon} + \frac{1}{2}\left(\frac{\dot{\bar{\rho}}}{\dot{\bar{\xi}}}\right)^{\cdot}\delta\xi + \frac{\dot{\bar{\rho}}}{\dot{\bar{\xi}}}\dot{\bar{\vartheta}}\delta s\right]. \tag{5.140}$$

Only in the approximate comoving gauge defined on large scales via the discussion above (ie. when V_i is negligibly small and we have set $\delta\xi = \delta\xi^{(2)} = 0$) may the quantity $\delta\epsilon^{(2)}$ be interpreted as the comoving energy density at second order [125]. We now wish to understand how the second order perturbation of the comoving energy density $\delta\epsilon^{(2)}$ behaves on large scales. To do so, we make use of the constraint equations derived at the close of section 5.1.4. By expanding the spatial components of equation (5.101) on large scales such that $\tilde{\epsilon}_\mu \overset{\sim}{=} \epsilon_\mu$, we obtain a Poisson equation for $\delta\epsilon_i^{(2)}$

$$\delta\epsilon_i^{(2)} \overset{\sim}{=} \partial_i \delta\rho^{(2)} - 3H\delta q_i^{(2)} - \delta\Theta\delta q_i, \tag{5.141}$$

$$\delta\Theta \equiv -3H\varphi - 3\dot{\psi} + \nabla^2\left(\frac{B}{a} - \dot{E}\right). \tag{5.142}$$

Using equation (5.139) and the fact that $\delta\epsilon^{(1)}$ is negligible on large scales as explained in section 5.1.4, we rewrite the above equation (5.141) and obtain

$$\partial^2 \delta\epsilon^{(2)} \overset{\sim}{=} 3H\partial^i V_i. \tag{5.143}$$

Thus when V_i is negligible as we have explained is the case for super-Hubble perturbation modes in both ekpyrotic and inflationary models, we have

$$\delta\epsilon^{(2)} \overset{\sim}{=} 0. \tag{5.144}$$

We are almost ready to derive the entropic and curvature evolution equations at second order in perturbations on large scales. However, we will require relations translating perturbations of Lie derivatives of covectors to perturbations of cosmological time derivatives of covectors. First let us consider a general covector X_μ and determine the components of its Lie derivative. For convenience we replace the covariant derivatives in the definition of the Lie derivative by partials (which we may do since the Lie derivative may be defined for any derivative operator) and write

$$\mathring{X}_\mu = u^\nu \partial_\nu X_\mu + X_\nu \partial_\mu u^\nu. \tag{5.145}$$

At zeroth order we have

$$\bar{\mathring{X}}_\mu = \left[\dot{\bar{X}}_0, \vec{0}\right], \tag{5.146}$$

with the presupposition that the spatial components $\bar{X}_i$ vanish [125]. Note that the bar is *above* the Lie derivative operator ($\mathring{}$), this is to signify that this is the background quantity *of the Lie derivative of the covector* X_μ. Recalling our choice for the four-velocity vector u^μ such that $u_i = 0$, we may determine its remaining components in terms of scalar metric perturbations. It will be sufficient to calculate solely the time component of u^μ to continue with the calculations; to do so one should make use of the fact that $u_\mu u^\mu = 1$ and $u_i = 0$ to obtain the following two relations

$$u^0 = \frac{1}{u_0}, \qquad u_\mu = g_{\mu 0} u^0. \tag{5.147}$$

From the above we obtain an expression for u^0 in terms of the metric perturbation as expressed in the coordinate theory $g_{00} = 1 + 2\varphi$; we then expand the metric perturbation as in equation (5.113) to third order to obtain

$$u^0 = \frac{1}{\sqrt{g_{00}}} \approx 1 - \varphi - \varphi^{(2)} - \varphi^{(3)} + \frac{3}{2}\varphi^2 + 3\varphi\varphi^{(2)} - \frac{5}{2}\varphi^3. \tag{5.148}$$

Expanding the covector fields to first order as $\mathring{X}_\mu = \bar{\mathring{X}}_\mu + \delta(\mathring{X}_0)$ and $X_\mu = \bar{X}_\mu + \delta X_\mu$, inserting them into equation (5.145) along with equation (5.148) and neglecting all terms second order or higher in perturbations we obtain

$$\delta(\mathring{X}_0) = \delta\dot{X}_0 - \left(\bar{X}_0\varphi\right)^{\cdot}, \qquad \delta(\mathring{X}_i) = \delta\dot{X}_i - \bar{X}_0\partial_i\varphi. \tag{5.149}$$

We may carry out the same procedure expanding covectors to second order and maintaining terms second order in perturbations

$$\delta(\mathring{X}_\mu)^{(2)} = -\bar{\mathring{X}}_\mu - \delta(\mathring{X}_\mu) + (1 - \varphi - \varphi^{(2)} + \frac{3}{2}\varphi^2)\dot{\bar{X}}_\mu + (1-\varphi)\delta\dot{X}_\mu \\ + \delta\dot{X}_\mu^{(2)} + \bar{X}_0\partial_\mu(-\varphi - \varphi^{(2)} + \frac{3}{2}\varphi^2) - \delta X_0\partial_\mu\varphi. \tag{5.150}$$

We care only for the spatial components which reduces the above using equations (5.146), (5.148) and (5.149), and simplifies further when concerned only with large scale fluctuations

$$\delta(\mathring{X}_i)^{(2)} = \delta\dot{X}_i^{(2)} + \bar{X}_0\partial_i(-\varphi^{(2)} + \frac{3}{2}\varphi^2) - \varphi\delta\dot{X}_i - \delta X_0\partial_i\varphi, \tag{5.151}$$

$$\delta(\mathring{X}_i)^{(2)} \cong \delta\dot{X}_i^{(2)} - \varphi\delta\dot{X}_i. \tag{5.152}$$

In exactly the same method, we calculate perturbations of the second order Lie derivative and after plenty of algebra we obtain for each respective perturbative order

$$\bar{\mathring{\mathring{X}}}_\mu = \left[\ddot{\bar{X}}_0, \vec{0}\right], \tag{5.153}$$

$$\delta(\mathring{\mathring{X}}_i) = \delta\ddot{X}_i - (\bar{X}_0\partial_i\varphi)^{\cdot} - \dot{\bar{X}}_0\partial_i\varphi, \tag{5.154}$$

$$\delta(\mathring{\mathring{X}}_i)^{(2)} = \delta\ddot{X}_i^{(2)} - 2\varphi\delta\ddot{X}_i - \dot{\varphi}\delta\dot{X}_i + (2\varphi\dot{\bar{X}}_0 - 2\delta\dot{X}_0 + \dot{\varphi}\bar{X}_0)\partial_i\varphi \\ + (\varphi\bar{X}_0 - \delta X_0)\partial_i\dot{\varphi} - (\bar{X}_0 + \dot{\bar{X}}_0)\partial_i\varphi^{(2)} + \frac{3}{2}\bar{X}_0\partial_i(\varphi^2). \tag{5.155}$$

Since we are interested in evolution equations on large scales, equation (5.155) reduces when considering the lowest order in spatial gradient terms

$$\delta(\mathring{\mathring{X}}_i)^{(2)} \cong \delta\ddot{X}_i^{(2)} - 2\varphi\delta\ddot{X}_i - \dot{\varphi}\delta\dot{X}_i. \tag{5.156}$$

We are finally in the position to derive the evolution equations for second order entropy and curvature perturbations. We begin with equation (5.89) and expand all spatial components to second order in perturbations neglecting gradients of the comoving energy density at first order $\delta\epsilon_i^{(1)}$ (due to the arguments of section 5.1.4), and make use of the first order perturbation equation to eliminate terms arriving at

$$\delta(\mathring{\mathring{s}}_i)^{(2)} + 3H\delta(\mathring{s}_i)^{(2)} + \left(\bar{V}_{ss} + 3\dot{\bar{\vartheta}}^2\right)\delta s_i^{(2)} + \delta\Theta\partial_i\dot{\delta s} + \left[\delta V_{ss} + 6\dot{\bar{\vartheta}}\delta(\mathring{\vartheta})\right]\partial_i\delta s \,\widetilde{=}\, -2\frac{\dot{\bar{\vartheta}}}{\dot{\bar{\xi}}}\delta\epsilon_i^{(2)}, \quad (5.157)$$

with [111]

$$\delta V_{ss} \,\widetilde{=}\, \bar{V}_{sss}\delta s - 2\frac{\bar{V}_{s\xi}}{\dot{\bar{\xi}}}\dot{\delta s}. \quad (5.158)$$

We may now use our equations (5.152) and (5.156) to translate the perturbations of Lie derivatives to perturbations of cosmological time derivatives. In addition to the first equation in equation (5.116) ($\delta s_i = \partial_i\delta s$), we obtain

$$\delta(\mathring{s}_i)^{(2)} \,\widetilde{=}\, \dot{\delta s}_i^{(2)} - \varphi\dot{\delta s}_i = \dot{\delta s}_i^{(2)} - \varphi\partial_i\dot{\delta s}, \quad (5.159)$$

$$\delta(\mathring{\mathring{s}}_i)^{(2)} \,\widetilde{=}\, \ddot{\delta s}_i^{(2)} - 2\varphi\ddot{\delta s}_i - \dot{\varphi}\dot{\delta s}_i = \ddot{\delta s}_i^{(2)} - 2\varphi\partial_i\ddot{\delta s} - \dot{\varphi}\partial_i\dot{\delta s}. \quad (5.160)$$

We may also take advantage of the Lie derivative for scalar quantities equation (5.5), and expanding as in equation (5.113) to find

$$\delta(\mathring{\vartheta}) = -\frac{1}{\dot{\bar{\xi}}}\left[\bar{V}_s\varphi + \delta V_s + \dot{\bar{\vartheta}}\left(\dot{\delta\xi} - \dot{\bar{\vartheta}}\delta s\right)\right], \quad (5.161)$$

$$\delta V_s \equiv \bar{V}_{s\xi}\delta\xi + \bar{V}_{ss}\delta s - \frac{\bar{V}_\xi}{\dot{\bar{\xi}}}\left(\dot{\delta s} + \dot{\bar{\vartheta}}\delta\xi\right). \quad (5.162)$$

Substituting equations (5.118), (5.142) and (5.159) to (5.161) into the evolution equation for entropy perturbations on large scales equation (5.157), we deduce for large scale fluctuations

$$\ddot{\delta s}^{(2)} + 3H\dot{\delta s}^{(2)} + \left(\bar{V}_{ss} + 3\dot{\bar{\vartheta}}^2\right)\delta s^{(2)} \,\widetilde{=}\, -\frac{\dot{\bar{\vartheta}}}{\dot{\bar{\xi}}}\dot{\delta s}^2 - \frac{2}{\dot{\bar{\xi}}}\left(\ddot{\bar{\vartheta}} + \dot{\bar{\vartheta}}\frac{\bar{V}_\xi}{\dot{\bar{\xi}}} - \frac{3}{2}H\dot{\bar{\vartheta}}\right)\delta s\dot{\delta s}$$

$$-\left(\frac{1}{2}\bar{V}_{sss}-5\frac{\dot{\bar{\vartheta}}}{\dot{\bar{\xi}}}\bar{V}_{ss}-9\frac{\dot{\bar{\vartheta}}^3}{\dot{\bar{\xi}}}\right)\delta s^2-2\frac{\dot{\bar{\vartheta}}}{\dot{\bar{\xi}}}\delta\epsilon^{(2)}. \quad (5.163)$$

The evolutionary equation governing the second order curvature perturbation on large scales for the two field case may be deduced using the dissipative fluid description, by expanding equation (5.91) to second order in perturbations. It was found in [123, 125] that on large scales the equation takes the following form

$$\dot{\zeta}^{(2)} \cong -\frac{H}{\dot{\bar{\xi}}^2}\left[2\dot{\bar{\vartheta}}\dot{\bar{\xi}}\delta s^{(2)}-\left(\bar{V}_{ss}+4\dot{\bar{\vartheta}}^2\right)\delta s^2+\frac{\bar{V}_{\xi}}{\dot{\bar{\xi}}}\delta s\dot{\delta s}-\frac{2\bar{V}_{\xi}}{3H\dot{\bar{\xi}}}\delta\epsilon^{(2)}\right]. \quad (5.164)$$

Note that in both equations (5.163) and (5.164) the final terms involving the second order comoving energy density perturbation may be ignored on large scales via the arguments preceding equation (5.144). As in the linear theory, the second order entropy and curvature perturbations evolve independently of adiabatic perturbations on large scales, and perturbations in entropy source perturbations in curvature.

Subsequent work in [111] expanded on the work by Langlois and Vernizzi above and derived equations of motion to third perturbative order allowing one to make predictions on the trispectrum. We do not express the full equations to third order in this thesis in the interest of brevity.

6 String cosmology

The four-dimensional scalar field effective actions of previous sections constitutes a well established framework in which various cosmological models may be explored; the framework is tractable, allowing for physical parameters to be calculated making specific cosmological models quantitative and falsifiable. However, the four dimensional scalar field effective descriptions studied in sections prior is a coarse-graining of a more fundamental theory, of which describes physics at higher energies and smaller length scales.

One may attempt to gain a deeper understanding of very early universe cosmology in two distinct methods. Among the two is the bottom-up effective field theory approach where we describe interactions and physics perturbatively at energies far below a high energy cutoff scale; in this energy regime certain interactions are perturbatively suppressed by powers of their coupling, manifesting themselves as irrelevant, decoupling from the low energy degrees of freedom rendering the low energy physics independently studiable. This description can be very useful as previous sections have shown, but irrelevant interactions introduce ultraviolet divergences, and unless the theory's renormalization group flow possesses a nontrivial ultraviolet fixed point it is non-renormalizable, and thus we cannot fully provide a description of the physics beyond the cutoff. A pedagogical example of such a divergence with particular relevance is the short-distance divergence and non-renormalizability of quantum gravity. By dimensional analysis (in Planck units), we may observe that the theory is ultraviolet divergent due to the coupling (Newton's G) being of negative mass dimension, and is non-renormalizable in the sense that the divergences appearing in the perturbative expansion may not be absorbed by a finite number of counterterms.

However, understanding the behaviour of the gravitational interaction at high energies is essential if we wish to make important progress in our understanding of the origin of our universe. For example, if one wishes to invalidate or provide a description of a particular inflationary model in which perturbation modes traverse scales smaller than the Planck length (trans-Planckian problem [8]), we must be able to describe the evolution of these modes. This leads us to the alternative method, a top-down description in which an ultraviolet complete theory constrains effective theories that appear to provide a consistent description at low energies [18, 20, 72, 138–140].

String theory's general freedom from ultraviolet (short distance) divergence is exemplified by the calculation of the torus (T^2) vacuum amplitude of bosonic string theory, which we provide a lightning review of, a full derivation may be found in [18]. The gauge fixed (via Fadeev-Popov procedure) BRST (Becchi-Rouet-Stora-Tyutin) invariant (guaranteeing that amplitudes for BRST equivalent states are equal) S-matrix of a general bosonic string theory

$$\mathbf{S}_{j_1\cdots j_n}(k_1,\cdots,k_n) = \sum_{\substack{\text{compact}\\ \text{topologies}}} \int_F \frac{d^\mu t}{n_R} \int [d\phi\; db\; dc] \exp\left(-S_m - S_g - \lambda\chi\right) \times \prod_{(a,i)\notin f} \int d\sigma_i^a \prod_{k=1}^{\mu} \frac{1}{4\pi}(b, \partial_k \hat{g}) \prod_{(a,i)\in f} c^a(\hat{\sigma}_i) \prod_{i=1}^{n} \hat{g}(\sigma_i)^{1/2} \mathcal{V}_{j_i}(k_i,\sigma_i), \quad (6.1)$$

may be applied to the torus without vertex operator insertions corresponding to initial and final states: the vacuum amplitude of the string spectrum. After dividing out the volume of the conformal killing group of the torus, including the appropriate anticommuting ghost insertions b, c from the two metric moduli and two conformal killing vectors we obtain

$$Z_{T^2} = \int_{F_0} \frac{d\tau d\bar{\tau}}{4\tau_2} \left\langle b(0)\tilde{b}(0)\tilde{c}(0)c(0) \right\rangle_{T^2}, \quad (6.2)$$

with $\tau \equiv \tau_1 + i\tau_2$ containing the two metric moduli of the unique closed oriented surface with Euler number zero. Performing the ghost path integrals we obtain the following result as a sum over the transverse closed string Hilbert space excluding ghosts, the $\mu = 0, 1$ oscillators[4] and the non-compact momenta valid for a general CFT and $d \geq 2$ noncompact flat spacetime dimensions

$$Z_{T^2} = iV_d \int_{F_0} \frac{d\tau d\bar{\tau}}{4\tau_2} (4\pi^2\alpha'\tau_2)^{-d/2} \sum_{i\in\mathcal{H}^\perp} q^{h_i - 1}\bar{q}^{\tilde{h}_i - 1}, \quad (6.3)$$

where $q \equiv \exp(2\pi i\tau)$, $h_i, \tilde{h}_i$ are the weights of the transverse states related to local operators via the state-operator isomorphism, V_d is a spacetime volume factor and α' is the Regge slope. After integrating over the standard fundamental region for the moduli space of (diff

[4] the ghost contributions cancel two sets (left and right moving) of bosonic oscillators.

$\times$ Weyl)-inequivalent metrics of the torus

$$F_0 = \left\{ \tau \in \mathbb{C} : -\frac{1}{2} \leq \text{Re } \tau \leq \frac{1}{2},\ |\tau| \geq 1 \right\}, \tag{6.4}$$

also indicated by the grey shaded region in figure 9, one finds that the vacuum amplitude Z_{T^2} does not contain the region that is present in the field theory calculation as a sum of point particle paths with the topology of a circle indicated by the union of the pink and grey shaded region of figure 9. Additionally, as long as the theory is free of tachyons the additional divergence present in the $\tau_2 \longrightarrow \infty$ limit does not manifest, since the behaviour of q decays exponentially for all string states with the exception of the tachyon where the sign of the exponent flips and instead grows exponentially without bound.

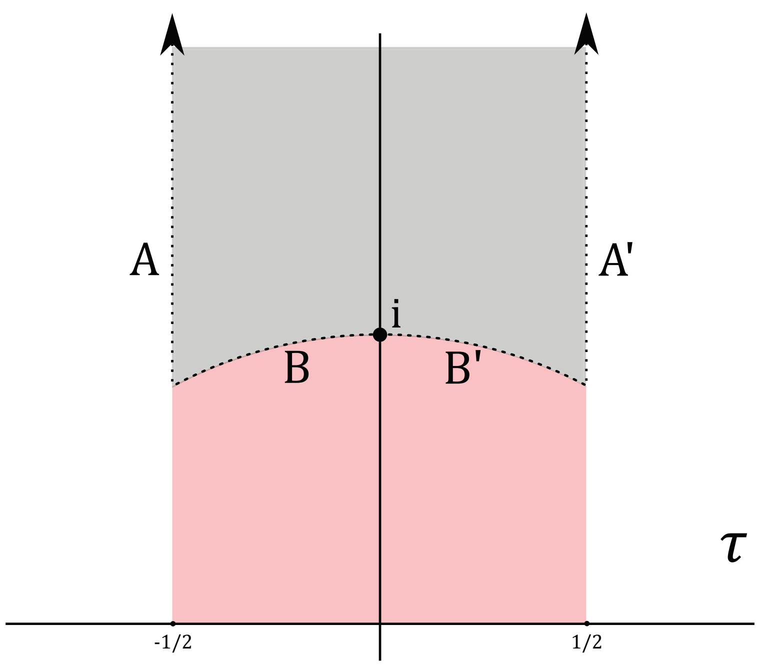

Figure 9: The grey shaded domain is the standard fundamental region F_0 for the moduli space of the torus, indicating the moduli space of (diff $\times$ Weyl)-inequivalent metrics in order to prevent over counting of physically identical configurations. The partial arcs of the unit circle B and B' are identified as well as the semi-infinite straight lines A and A'. The pink shaded region is also included when one attempts to calculate the analogous vacuum amplitude in field theory, the region includes points that makes the integration divergent, in particular the $\tau_2 \longrightarrow 0$ limit.

This is in part the motivation for the application of string theory to very early universe cosmology. Scattering amplitudes of fundamental strings may be handled perturbatively at energies above the Planck scale free of ultraviolet divergences, and at energies below the Planck scale physics is described by an effective quantum field theory coupled to general relativity. Famously, the vanishing of the Weyl anomaly in the Polyakov path integral formulation [141] of pure bosonic string theory, or alternatively the preservation of Lorentz invariance after quantization of one dimensional objects in flat spacetime determines the critical spacetime dimension $D = 26$, thus extra dimensions are commonplace among all self-consistent string theories. Infamously, since the bosonic string possesses vacuum instabilities in the form of tachyons and is obviously void of fermions it is an unrealistic theory of our universe.

Superstring theories on the other hand are vastly more promising, with five known weakly coupled theories subject to varying constraint algebras and all forms possessing some form of spacetime supersymmetry. These superstring theories exhibit the desired results of spacetime fermions, the absence of tachyons and are free of various anomalies (the violation of classical symmetries after quantization). Additionally, string theory produces extended objects that are unique to the theory such as Dp-branes, orbifolds and orientifold planes separating it from pure supergravity. Dp-branes in particular are dynamical objects, may inherit gauge symmetry groups that may spontaneously break down to the standard model gauge group and may possesses a variety of spatial dimensions depending on the superstring theory. It was these realizations that motivated the idea that our observable universe may in fact be described by the worldvolume of a D3-brane [72].

6.1 Ekpyrosis from heterotic M-theory

6.1.1 Ekpyrosis with a bulk M5-brane

The original ekpyrotic scenario [47] was introduced in a particular compactification scheme of the heterotic [142] $E_8 \times E_8$ superstring theory at strong coupling (Hořava-Witten theory) known as *heterotic M-theory*. It was shown [60, 143] that the ten dimensional heterotic $E_8 \times E_8$ superstring theory at strong coupling corresponds to eleven dimensional M-theory compactified on an $S^1/\mathbb{Z}_2$ orbifold, where the dilaton appearing in the heterotic superstring theory at weak coupling is reinterpreted as the orbifold dimension of $S^1/\mathbb{Z}_2$ via the string coupling. Furthermore, this low energy limit of M-theory is effectively described by an eleven dimensional supergravity theory describing bulk interactions and is only free of gauge and gravitational anomalies if the two ten dimensional orbifold fixed hyperplane boundaries possess identical E_8 Yang-Mills gauge theories; this is Hořava-Witten theory. In fact it is now widely understood that each of the five nonanomalous superstring theories and eleven dimensional M-theory are related to one another by various S and T dualities, exchanging the strong and weak coupling and momentum and winding respectively, and are therefore all viewed to be limits of a single theory [20, 72, 144, 145].

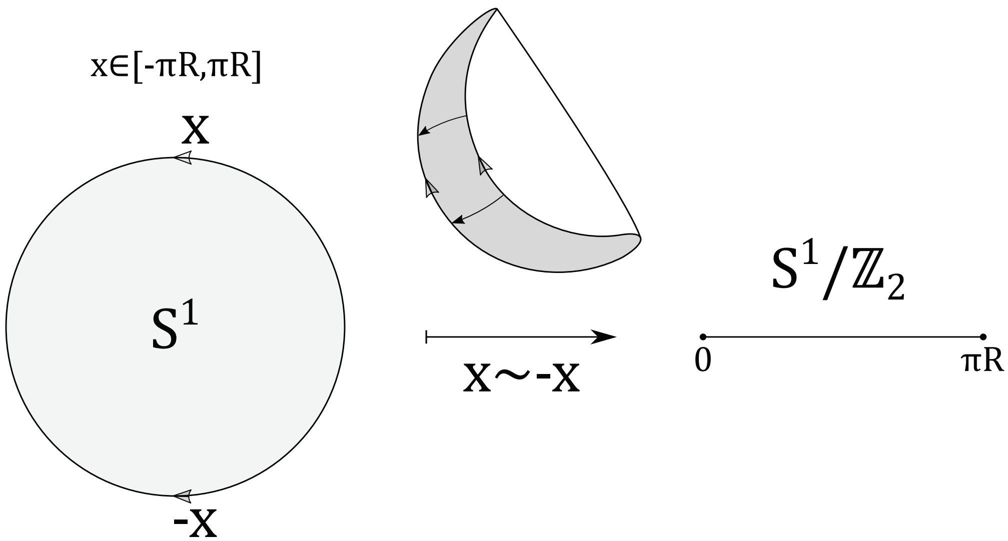

Figure 10: A topological depiction of the quotient space $S^1/\mathbb{Z}_2$, beginning with S^1 we identify all points on the circle to an equivalence class defined by $x \sim -x$, effectively folding over and gluing two arcs of the circle together. Thus every point in the quotient space corresponds to an orbit of points in S^1 consisting of its images under the action of the isometry group $\mathbb{Z}_2$. Generally speaking orbifolds are singular at the fixed points of the discrete symmetry group, but a closed string theory is consistent on spaces with orbifold singularities so long as the twisted states which are also required by modular invariance are included in the spectrum.

The compactification scheme known as heterotic M-theory is more precisely: a compactification of six of the remaining nine spatial dimensions of Hořava-Witten theory on a Calabi-Yau threefold. Witten has shown [61] that there exists a consistent compactification of Hořava-Witten theory on a deformed Calabi-Yau threefold that permits a supersymmetric theory in four dimensions such that the ratio of the number of supercharges to the smallest spinor representation $\mathcal{N} = 1$, and Hořava demonstrated [146] that gaugino condensation provides a possible mechanism for supersymmetry breaking from the eleven dimensional M-theory. Upon compactification on the three-fold, these six dimensions become internal in the sense that they must be much smaller than the orbifold dimension in order to obtain the correct magnitude of Newton's constant from the four dimensional perspective [61, 147]. Newton's constant G_N and the grand unified coupling α_{GUT} may be expressed in terms of the eleven dimensional gravitational coupling κ_{11}, the volume of the Calabi-Yau space V_{CY} and the size of the orbifold dimension πR_{11}

$$G_N = \frac{\kappa_{11}^2}{16\pi^2 V_{CY} R_{11}}, \qquad \alpha_{GUT} = \frac{(4\pi\kappa_{11}^2)^{2/3}}{2V_{CY}}. \tag{6.5}$$

By setting the above equal to their phenomenological values, there emerges a natural hierarchy of energy scales, dictating the sizes of the compactification radii of the eleventh dimension as well as the Calabi-Yau volume. Ultimately the number of effective dimensions increases with energy: the universe is first effectively four, then five dimensional, the orbifold dimension being the fifth, and finally eleven dimensional. Thus, there is an energy regime in which the universe is effectively five dimensional and is the regime chosen to study the ekpyrotic model. It then becomes very relevant to derive said five-dimensional effective action that is the low-energy effective description of Hořava-Witten theory and in the limit $R_{11} \longrightarrow 0$, leads to an $\mathcal{N} = 1$ supergravity theory. As we will demonstrate in section 6.1.2, the compactification of Hořava-Witten theory in the presence of a non-vanishing four-form field strength in the internal Calabi-Yau directions (G-flux, see also [148, 149] for more investigations of compactifications with G-flux that preserve supersymmetry and handle anomalies) allows for the explicit derivation of the effective action describing the five dimensional spacetime, and may also be shown to be a gauged version of $\mathcal{N} = 1$ supergravity action [59, 150–152].

Importantly, it was also shown [59, 151, 152] that the potentials arising from compactification support BPS (Bogolmon'yi-Prasad-Sommerfeld) D3-brane solutions of the equation of motion, with the vacuum consisting of two D3-branes coincident with the orbifold fixed planes bounding the orbifold dimension; cosmological solutions of this vacuum configuration are presented in [153]. It is the worldvolume of one of these boundary branes that is deemed as our observable (3+1) dimensional universe. The model is made realistic by ensuring heterotic M-theory includes the standard model gauge group $SU(3)_C \times SU(2)_L \times U(1)_Y$

and three families of quarks and leptons at least on the visible D3-brane by the inclusion of M5-branes in the bulk spacetime, also effectively eliminating anomalies [154–160]. For the interested reader, one may find literature on the four dimensional limit of both Hořava-Witten theory [151, 161–171] and heterotic M-theory [172–184].

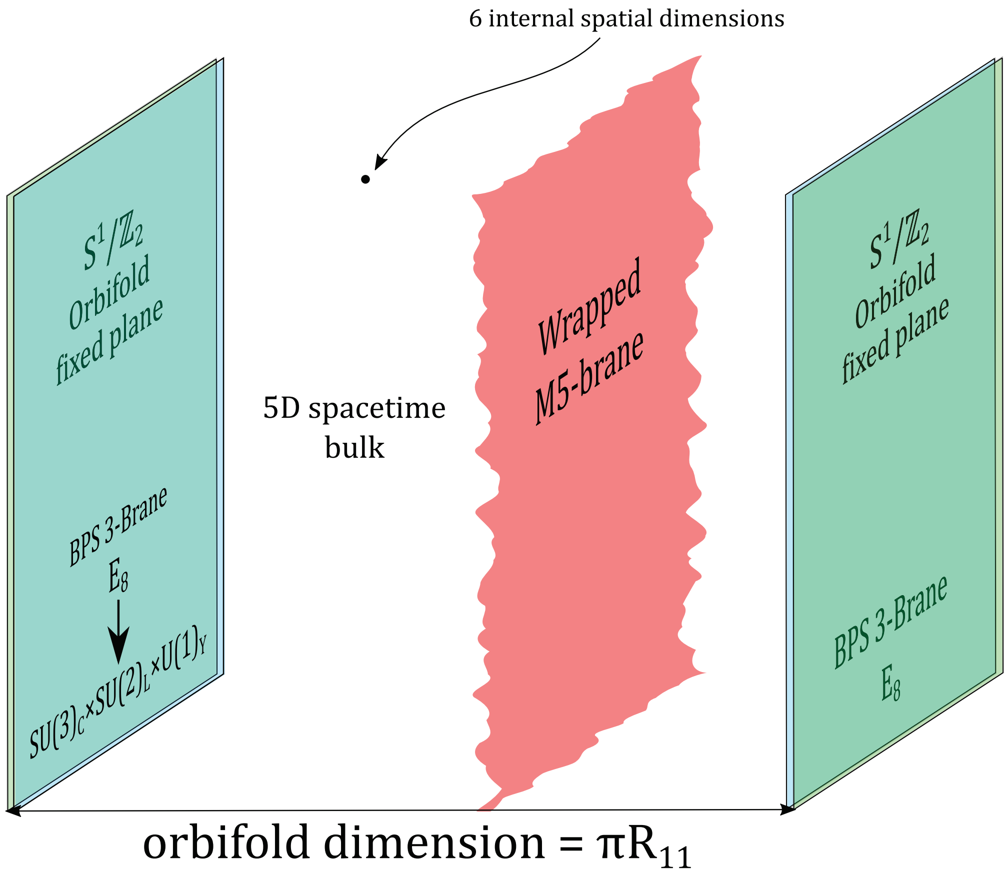

Figure 11: A depiction of the original ekpyrotic model embedded in the compactification of eleven dimensional M-theory on an $S^1/\mathbb{Z}_2$ orbifold, with a further compactification of six spatial dimensions on a Kähler manifold with three complex coordinates and vanishing first Chern class, otherwise known as a Calabi-Yau threefold. The orbifold dimension is determined by the radius R_{11} of S^1. BPS D3-branes may exist and coincide with the fixed orbifold planes bounding the eleventh dimension. Each point in the five dimensional spacetime bulk possesses six internal spatial dimensions that are much smaller than the orbifold dimension. The inclusion of a wrapped M5-brane is forced in order to make the theory nonanomalous in their construction of inducing the standard model gauge group on one of the boundary D3-branes. The M5-brane is wrapped on holomorphic curves in the Calabi-Yau manifold and appear as D3-branes in the effective five dimensional spacetime theory.

The included bulk M5-brane is permitted to move along the orbifold dimension and the potential governing its motion is known to receive contributions from non-perturbative dynamics. As an example, contributions to the superpotential via the exchange of supermembranes stretched between the boundary brane and M5-brane in the effective four-dimensional theory have been calculated explicitly [185–187], and in conjunction with the M5-brane contribution to the Kähler potential as calculated in [188] one may calculate the contribution to the potential from this particular non-perturbative effect. More recently, open membrane instanton interactions between M5-branes are known to give rise to exponential potentials and have been utilized in the construction of an assisted inflation model in heterotic M-theory [189]. As in this inflationary model, these potentials may be utilized to argue the inevitability of collision of the M5-brane with one of the fixed orbifold planes causing the M5-brane to dissolve via a non-perturbative phase transition known as a small instanton transition [190, 191].

The transition alters the M5-brane to a small instanton on the Calabi-Yau space at the boundary brane and is then smoothed into a holomorphic vector bundle, effectively being dissolved and absorbed, altering the properties of the boundary brane. In particular, small instanton phase transitions have the ability to alter the number of quark and lepton families, the gauge group (or both) on the boundary brane via the alteration of the third Chern class of the associated vector bundle by inheriting all or part of the base component, or(and) by inheriting the pure fiber component of the class of holomorphic curves in which the M5-branes may be wrapped, respectively. Thus the details of the phase transition depends on the topological structure of the bulk M5-brane being dissolved and absorbed [190].

The initial conditions of this rendition of the ekpyrotic scenario are motivated by maximizing symmetry by ensuring the boundary branes begin as BPS states: they remain invariant under a nontrivial subalgebra of the full spacetime supersymmetry algebra [72]. The BPS condition not only ensures initial homogeneity but also spatial flatness. The condition also demands that the boundary branes be parallel and the bulk M5-brane be initially nearly stationary. The BPS initial condition configuration in heterotic M-theory is appealing as it has been shown to be an appropriate background for reducing to four dimensional $\mathcal{N} = 1$ supergravity theories [59].

The finding of a dynamical attractor mechanism for driving the universe to the BPS state configuration just described from generalized initial conditions would also make the model more appealing, and perhaps provide a mechanism for a cyclic universe in this particular string background. The authors of [47] also heuristically propose how the bulk M5-brane may appear: as a spontaneous peeling mechanism from one of the boundary BPS D3-branes, a process akin to bubble nucleation as the universe undergoes a first order phase transition and decays from a metastable vacuum with the new phase expanding at the nucleation rate [4, 192]. A more precise mechanism for the nucleation of bulk M5-branes or motivation for their initial existence would also be a welcome addition.

6.1.2 Five dimensional effective action of heterotic M-theory

In this section we review the derivation of the five dimensional effective action of heterotic M-theory as found in [59, 150]. To do so, we first present the methodology as carried out by Witten [61] of preserving $\mathcal{N} = 1$ supersymmetry in four spacetime dimensions to first order in a perturbative expansion of the four form field strength of Hořava-Witten theory after compactification on a deformed Calabi-Yau space[5].

The conventions used throughout are as follows and differ slightly from the original literature: the coordinates describing the eleven dimensional spacetime $\mathcal{M}_{11}$ are x^I whose indices are capitalized Latin characters $I, J, K, L, \cdots \in \{0, 1, \cdots, 9, 11\}$ (notice that 10 is omitted). The two ten-dimensional hyperplanes that bound the eleventh dimension fixed by the $\mathbb{Z}_2$ symmetry are denoted by $\mathcal{M}_{10}^{(i)}$, $i = 1, 2$. Capitalized barred Latin indices $\bar{I}, \bar{J}, \bar{K}, \bar{L}, \cdots \in \{0, 1, \cdots, 9\}$ are utilized for the ten dimensional space orthogonal to the orbifold direction. Upon compactification on a Calabi-Yau threefold, coordinates of the five dimensional effective spacetime $\mathcal{M}_5$ are indicated by lowercase Greek indices at the beginning of the alphabet $\alpha, \beta, \gamma, \cdots \in \{0, \cdots, 3, 11\}$, and Latin characters appearing at the beginning of the alphabet refer to directions of the Calabi-Yau space $A, B, C, \cdots \in \{4, \cdots, 9\}$; their lowercase are utilized when holomorphic and anti-holomorphic indices are required $a, b, c, \cdots \in \{4, 5, 6\}$, $\bar{a}, \bar{b}, \bar{c}, \cdots \in \{4, 5, 6\}$. Occasionally, we will refer to the space tangent to the Calabi-Yau space as well as the orbifold dimension with capitalized Latin characters appearing at the end of the alphabet $\cdots, X, Y, Z \in \{4, \cdots, 9, 11\}$. Once the orbifold planes

[5]See Ch. 17 of [145] or Ch. 10 of [144] for information of flux compactification schemes and Ch. 14 of [145] or Ch.9 of [144] for information of compactifications on Calabi-Yau manifolds.

effectively become four dimensional they are indicated by $\mathcal{M}_4^{(i)}$ with $i = 1, 2$, and Greek indices appearing at the middle of the alphabet are used to describe directions tangent to this space $\mu, \nu, \rho, \cdots \in \{0, \cdots, 3\}$. The eleven dimensional Dirac matrices Γ^I are 32×32 real matrices and obey the Clifford algebra $\left\{\Gamma^I, \Gamma^J\right\} = 2g^{IJ}$, thus the spinors in eleven dimensions are Majorana spinors with 32 real components (notice the spinor indices have been suppressed). Additionally, we define $\Gamma^{I_1\cdots I_n} \equiv \Gamma^{[I_1} \cdots \Gamma^{I_n]} = \frac{1}{n!}(\Gamma^{I_1} \cdots \Gamma^{I_n} \pm \text{ permutations})$[6].

We begin with the effective description of the strongly coupled heterotic $E_8 \times E_8$ superstring theory as maximal eleven dimensional supergravity with E_8 Yang-Mills gauge theories inhabiting the boundary orbifold planes. Neglecting spacetime fermions, the bosonic sector is described by the following action [193]

$$S = S_{SG} + S_{YM}, \tag{6.6}$$

with

$$S_{SG} = -\frac{1}{2\kappa_{11}^2} \int_{\mathcal{M}_{11}} \sqrt{-g} \left[R + \frac{1}{24} G_{IJKL} G^{IJKL} + \frac{\sqrt{2}}{1728} \epsilon^{N_1 \cdots N_{11}} C_{N_1 N_2 N_3} G_{N_4 \cdots N_7} G_{N_8 \cdots N_{11}} \right], \tag{6.7}$$

and the two ten dimensional E_8 Yang-Mills theories on each fixed plane is described by

$$S_{YM} = -\frac{1}{8\pi\kappa_{11}^2} \left(\frac{\kappa_{11}}{4\pi}\right)^{2/3} \left[\int_{\mathcal{M}_{10}^{(1)}} \sqrt{-g} \left\{ \operatorname{tr}(F^{(1)})^2 - \frac{1}{2} \operatorname{tr} R^2 \right\} + \int_{\mathcal{M}_{10}^{(2)}} \sqrt{-g} \left\{ \operatorname{tr}(F^{(2)})^2 - \frac{1}{2} \operatorname{tr} R^2 \right\} \right]. \tag{6.8}$$

In the above: κ_{11} is the eleven dimensional gravitational coupling, $F^{(i)}_{\bar{I}\bar{J}} \equiv \partial_{\bar{I}} A^{(i)}_{\bar{J}} - \partial_{\bar{J}} A^{(i)}_{\bar{I}} + [A^{(i)}_{\bar{I}}, A^{(i)}_{\bar{J}}]$ are the field strengths defined in terms of the two E_8 Yang-Mills gauge fields $A^{(i)}_{\bar{I}}$, the four form $G_{IJKL} = 24\partial_{[I} C_{JKL]}$ is the field strength associated with the three-form potential C_{IJK} of the supergravity multiplet. It also must be noted that $\frac{1}{30}\operatorname{Tr}(F^2) = \operatorname{tr}(F^2) \equiv F^z F^z$ (with the index $z \in \{1, \cdots, 248\}$ denoting the adjoint representation of E_8) relating adjoint representation traces (Tr) to fundamental representation traces (tr),

[6] See App. B of [72] for a review of spinors and supersymmetry in various dimensions, as well as differential forms and generalized gauge fields.

and $\mathrm{tr}(R^k)$ terms refers to the trace in the vector representation of $SO(1,10)$; additionally $\mathrm{tr}(\underbrace{\star \wedge \cdots \wedge \star}_{k \text{ times}}) \equiv \mathrm{tr}(\star^k)$, where $(\star)$ may be replaced for either the field strength two form F or the curvature two form R [60, 144, 194][7]. The Yang-Mills gauge fields $A^{(i)}_{\bar{I}}$ may be expressed as Lie-algebra one-forms defined in terms of the matrices λ^z in this case in the adjoint representation of E_8, and they define the field strength two form

$$A = A^z_{\bar{I}} \lambda^z dx^{\bar{I}}, \qquad F = \frac{1}{2} F_{\bar{I}\bar{J}} dx^{\bar{I}\bar{J}} dx^{\bar{I}} \wedge dx^{\bar{J}} = dA + A \wedge A. \tag{6.9}$$

Similarly the curvature two-form may be defined in terms of the spin connection one-form which is in turn defined in terms of the λ^z matrices in the fundamental representation of the Lorentz algebra $SO(1,10)$

$$\tilde{\omega} = \tilde{\omega}^z_I \lambda^z dx^I, \qquad R = d\tilde{\omega} + \tilde{\omega} \wedge \tilde{\omega}. \tag{6.10}$$

Finally, covariant derivatives acting on a spinor η appearing as $D_I \eta$ are defined in terms of the spin connection [195]

$$D_I \eta = \partial_I \eta + \frac{1}{4} \tilde{\omega}_{IJK} \Gamma^{JK} \eta, \tag{6.11}$$

where the indices J, K above refer to the flat (tangent) space Clifford algebra and may be related to the curved (base) indices (I) via an elfbein field [144].

As explained in [60], in order to ensure the full action (including the fermionic sector) is locally supersymmetric, additional matter interaction terms are included in the action with the purpose of cancelling out variations induced by supersymmetry transformations. However, it is not possible to cancel all of the anomalous variations with the introduction of matter couplings, one must provide a correction to the Bianchi identity $dG = 0$ which induces an additional variation of the eleven dimensional supergravity Lagrangian appearing in S_{SG} (including fermionic sector). In particular, since the gauge and gravitational anomalies are localised on the boundary from the eleven-dimensional perspective one must include source

[7] See Ch. 5 of [144] for discussions of differential forms, characteristic classes and an introduction to the analysis of anomalies.

terms localised on the hyperplanes [60, 152] such that the component of the five form reads

$$(dG)_{11\bar{I}\bar{J}\bar{K}\bar{L}} = -\frac{1}{2\sqrt{2}\pi}\left(\frac{\kappa_{11}}{4\pi}\right)^{2/3}\left\{J^{(1)}\delta(x^{11}) + J^{(2)}\delta(x^{11}-\pi R_{11})\right\}_{\bar{I}\bar{J}\bar{K}\bar{L}}\,, \tag{6.12}$$

with the sources provided by

$$J^{(i)} = \left(\mathrm{tr}\left(F^{(i)}\wedge F^{(i)}\right) - \frac{1}{2}\,\mathrm{tr}\,(R\wedge R)\right), \tag{6.13}$$

$$= 6\left(\mathrm{tr}\left(F^{(i)}_{[\bar{I}\bar{J}}F^{(i)}_{\bar{K}\bar{L}]}\right) - \frac{1}{2}\,\mathrm{tr}\left(R_{[\bar{I}\bar{J}}R_{\bar{K}\bar{L}]}\right)\right). \tag{6.14}$$

As an aside, the analysis [60] of the gauge anomalies localized on the boundary of the eleventh dimension also leads to the following relation between the Yang-Mills gauge coupling λ^2 and the eleven-dimensional gravitational coupling providing reason as to why it does not appear explicitly in equation (6.8)

$$\lambda^2 = 2\pi(4\pi\kappa_{11}^2)^{2/3}. \tag{6.15}$$

It is also exceptionally useful to interpret the existence of source terms in the Bianchi identity as excitations of M5-brane charges as explained in [59] by an analogy with D-branes in type II theories, note that they must be supported by delta function sources due to singularities associated with the presence of branes [61].

The existence of these localised source terms introduce complications when one attempts to perform a Calabi-Yau compactification of Hořava-Witten theory. Namely, the four form field strength G may not be conveniently made to vanish due to the sources located at the fixed points of $S^1/\mathbb{Z}_2$. Imposing a null G along with selecting a metric on $\mathcal{M}_{11}$ that permits the infinitesimal transformation of the supercharges represented by the Majorana spinor η to be covariantly conserved (satisfy the Killing spinor equation) is the simplest method to compel the supersymmetry transformation law of the gravitino Ψ_I to vanish, and preserve supersymmetry (note that the spinor index is suppressed) [59, 61, 152]

$$\delta\Psi_I = D_I\eta + \frac{\sqrt{2}}{288}\left(\Gamma_{IJKLM} - 8g_{IJ}\Gamma_{KLM}\right)G^{JKLM}\eta + \cdots\,. \tag{6.16}$$

In order to obtain unbroken $\mathcal{N} = 1$ supersymmetry in four dimensions, this transformation law must vanish for four linearly independent choices of η forming a four-component Majorana spinor [144, Ch. 9.4].

In particular, it does *not* seem possible [61] to determine a vacuum with the property that $\mathrm{tr}(F \wedge F) = \frac{1}{2}\,\mathrm{tr}(R \wedge R)$ pointwise, and thus equations (6.12) and (6.13) generically imply a nontrivial G-flux for Calabi-Yau compactification. This is in contrast with the weakly coupled heterotic string, where upon embedding the spin connection into one of the E_8 gauge groups (sometimes known as the standard embedding) we may set $\mathrm{tr}(F^{(1)} \wedge F^{(1)}) = \mathrm{tr}(R \wedge R)$ as well as $F^{(2)} = 0$, allowing one to consistently set the antisymmetric tensor gauge field strength H to zero since we have effectively removed sources appearing in the Bianchi identity [59, 61, 144, 152]

$$dH \sim \mathrm{tr}(F^{(1)} \wedge F^{(1)}) + \mathrm{tr}(F^{(2)} \wedge F^{(2)}) - \mathrm{tr}(R \wedge R). \tag{6.17}$$

Encouragingly, one may preserve supersymmetry perturbatively in the presence of G-flux by allowing corrections to the metric about a zeroth order background. By equation (6.12) we may identify G to be of order $\kappa_{11}^{2/3}$ and thus set $G = 0$ at zeroth order. We then choose the zeroth order metric to be of product topology $X \times S^1/\mathbb{Z}_2 \times \mathbb{M}^4$ where X is a Calabi-Yau manifold and $\mathbb{M}^4$ is four dimensional Minkowski spacetime

$$ds_{11}^2 = \eta_{\mu\nu}dx^\mu dx^\nu + R_0^2(dx^{11})^2 + V_0^{1/3}\Omega_{AB}dx^A dx^B, \tag{6.18}$$

with $\eta_{\mu\nu}$ the Minkowski metric tensor, the metric on S^1 is simply $(dx^{11})^2$ and the metric tensor Ω_{AB} describes the Calabi-Yau space with Kähler form $\omega = i\Omega_{a\bar{b}}dx^a \wedge d\bar{x}^{\bar{b}}$. We also introduce scaling parameters R_0 and V_0 for the orbifold radius and Calabi-Yau volume respectively, note that this choice is arbitrary, the zeroth order metric may be rescaled. At order $\kappa_{11}^{2/3}$ the equation of motion for G

$$D_I G^{IJKL} = \frac{\sqrt{2}}{1152}\epsilon^{JKLN_1\cdots N_8}G_{N_1\cdots N_4}G_{N_5\cdots N_8}, \tag{6.19}$$

reduces significantly since $G \wedge G$ is order $\kappa_{11}^{4/3}$ to

$$D_I G^{IJKL} = 0. \tag{6.20}$$

Supplementing this equation of motion with the Bianchi identity (6.12) which introduces localised sources (or boundary conditions in the 'downstairs' interpretation), and ensuring the supersymmetry transformation law of the gravitino (6.16) vanishes to first order in G allows one to derive a set of equations dictating the corrections to the metric (6.18) hence distorting the background Calabi-Yau threefold. Incidentally, a correction to the Majorana spinor η is also induced since a first order perturbation to the metric $g_{IJ} \longrightarrow g_{IJ} + h_{IJ}$ modifies the covariant derivative of a spinor as

$$D_I\eta \longrightarrow D_I\eta - \frac{1}{8}(D_J h_{KI} - D_K h_{JI})\Gamma^{JK}\eta. \tag{6.21}$$

It is in this manner that the supersymmetry transformation law of the gravitino may be made to vanish at order $\kappa_{11}^{2/3}$ in the presence of G sources at the fixed points of $S^1/\mathbb{Z}_2$ (in fact the solution is actually more general, allowing for the inclusion of M5-branes in the bulk since they also act as magnetic sources for G [61, 194]). The form of the corrected metric from that of equation (6.18) that preserves supersymmetry at order $\kappa_{11}^{2/3}$ may be expressed in a block diagonal form [61]

$$ds_{11}^2 = (1+b)\eta_{\mu\nu}dx^\mu dx^\nu + R_0^2(1+\gamma)(dx^{11})^2 + V_0^{1/3}(\Omega_{AB} + h_{AB})dx^A dx^B. \tag{6.22}$$

In order to preserve Lorentz-invariance in four dimensions the perturbations $b, \gamma,\ h_{AB}$ are constrained to be functions of the directions x^Y, tangent to $X \times S^1$. Defining the following three objects whose indices are raised and lowered by the metric on Calabi-Yau space Ω_{AB} in terms of the Kähler form ω_{AB},

$$\beta_A = \omega^{BC} G_{ABC11}, \tag{6.23}$$

$$\theta_{AB} = \omega^{CD} G_{ABCD}, \tag{6.24}$$

$$\alpha = \omega^{AB}\theta_{AB} = \omega^{AB}\omega^{CD} G_{ABCD}, \tag{6.25}$$

we quote the set of identities derived in [61] from the equation of motion (6.20) and the Bianchi identity (6.12) that these objects must obey to order $\kappa_{11}^{2/3}$. They are

$$D_{\bar{a}}\beta_{\bar{b}} - D_{\bar{b}}\beta_{\bar{a}} = 0, \tag{6.26}$$

$$-\frac{i}{2}D^A\theta_{A\bar{b}} + \frac{1}{4}D_{\bar{b}}\alpha - \frac{i}{2}D_{11}\beta_{\bar{b}} = 0, \tag{6.27}$$

$$D_{11}\beta_{\bar{b}} - D^b\theta_{b\bar{b}} = 0, \tag{6.28}$$

$$D_{11}\beta_{\bar{a}} + \frac{i}{4}D_{\bar{a}}\alpha = 0, \tag{6.29}$$

$$D^A\beta_A = 0, \tag{6.30}$$

in conjunction with an additional two which arise via contracting the Bianchi identity (6.12) with $\omega^{AB}\omega^{CD}$ and ω^{CD} respectively

$$D_{11}\alpha + 4i(D^{\bar{a}}\beta_{\bar{a}} - D^a\beta_a) = -\frac{1}{2\sqrt{2}\pi}\left(\frac{\kappa_{11}}{4\pi}\right)^{2/3} J\left\{\delta(x^{11}) - \delta(x^{11} - \pi R_{11})\right\}, \tag{6.31}$$

$$\begin{aligned} D_{11}\theta_{a\bar{a}} + (D_a\beta_{\bar{a}} - D_{\bar{a}}\beta_a) + i\left(D^{\bar{b}}G_{\bar{b}a\bar{a}11} - D^bG_{ba\bar{a}11}\right) \\ = -\frac{1}{\sqrt{2}\pi}\left(\frac{\kappa_{11}}{4\pi}\right)^{2/3} J_{a\bar{a}}\left\{\delta(x^{11}) - \delta(x^{11} - \pi R_{11})\right\}. \end{aligned} \tag{6.32}$$

J_{AB} and hence $J \equiv \omega^{AB}J_{AB}$ in the above are defined in terms of the sources $J^{(i)}$ introduced in equation (6.13), with the spin connection embedded in the $F^{(1)}$ gauge group

$$J^{(1)}_{ABCD} = -J^{(2)}_{ABCD} \equiv J_{ABCD} = \frac{1}{2}\left(\mathrm{tr}(R^{(\Omega)} \wedge R^{(\Omega)})\right)_{ABCD}, \tag{6.33}$$

$$J_{AB} = J_{ABCD}\omega^{CD} = 3\,\mathrm{tr}(R^{(\Omega)}_{[AB}R^{(\Omega)}_{CD]})\omega^{CD}, \tag{6.34}$$

with $R^{(\Omega)}_{AB}$ the curvature of the zeroth-order metric on the Calabi-Yau space Ω_{AB}. We now express the second term appearing in the supersymmetry transformation of the gravitino (6.16) in terms of the quantities defined in equations (6.23) to (6.25)

$$\begin{aligned} &\frac{\sqrt{2}}{288}dx^I\left(\Gamma_{IJKLM} - 8g_{IJ}\Gamma_{KLM}\right)G^{JKLM}\eta = \\ &\qquad \frac{\sqrt{2}}{288}\Big[dx^{11}\left(-3\alpha - 24i\beta_{\bar{b}}\Gamma^{\bar{b}}\right) + dx^{\bar{b}}12i\beta_{\bar{b}} \end{aligned}$$

$$+ dx^a \left(36i\beta_a + (36i\theta_{a\bar{b}} - 3\alpha g_{a\bar{b}})\Gamma^{\bar{b}} + (-36G_{a\bar{a}\bar{b}11} - 6i(g_{a\bar{a}}\beta_{\bar{b}} - g_{a\bar{b}}\beta_{\bar{a}}))\,\Gamma^{\bar{a}\bar{b}}\right)$$
$$+dx^\mu\Gamma_\mu\left(-3\alpha - 12i\beta_{\bar{b}}\Gamma^{\bar{b}}\right)\Big]\,\eta. \quad (6.35)$$

Introducing a corrected spinor $\tilde{\eta} = e^{-\psi}\eta$ where ψ is an order $\kappa_{11}^{2/3}$ correction we calculate from the transformation of the covariant derivative of the spinor under a perturbation in the metric (6.21), the first term in the supersymmetry transformation of the gravitino (6.16) explicitly in terms of the metric perturbations introduced in equation (6.22) assuming a covariantly constant spinor $D_I\eta$ with respect to the unperturbed metric

$$dx^I D_I\tilde{\eta} = \Bigg(-dx^Y\partial_Y\psi + dx^\mu\Gamma_\mu\left(\frac{1}{4}\partial_{11}b + \frac{1}{4}\partial_{\bar{a}}b\Gamma^{\bar{a}}\right)$$
$$- dx^a\left(-\frac{1}{4}\partial_{11}h_{\bar{a}a}\Gamma^{\bar{a}} + \frac{1}{8}(\partial_{\bar{a}}h_{\bar{b}a} - \partial_{\bar{b}}h_{\bar{a}a})\Gamma^{\bar{a}\bar{b}} + \frac{1}{4}g^{b\bar{b}}\partial_b h_{a\bar{b}}\right)$$
$$-dx^{\bar{a}}\left(-\frac{1}{4}g^{a\bar{b}}\partial_{\bar{b}}h_{a\bar{a}}\right) - \frac{1}{4}dx^{11}\partial_{\bar{a}}\gamma\Gamma^{\bar{a}}\Bigg)\,\tilde{\eta}. \quad (6.36)$$

To be specific, the index Y above refers to the the directions tangent to $X \times S^1$. Finally substituting equations (6.35) and (6.36) into the gravitino transformation law (6.16), demanding that it vanishes to first order in G in order to preserve supersymmetry and making use of the identities in equations (6.26) to (6.30), we find that the supersymmetry variation of the gravitino vanishes to first order if the metric variations b, γ, h_{AB} satisfy the following set of equations

$$i\sqrt{2}\beta_{\bar{a}} = 6\partial_{\bar{a}}b = -24\partial_{\bar{a}}\psi = -3\partial_{\bar{a}}\gamma, \quad (6.37)$$

$$\frac{1}{2\sqrt{2}}\alpha = 6\partial_{11}b = -24\partial_{11}\psi, \quad (6.38)$$

$$\partial_{\bar{a}}h_{b\bar{b}} - \partial_{\bar{b}}h_{b\bar{a}} = -\sqrt{2}\left(G_{b\bar{a}\bar{b}11} + \frac{i}{6}(\Omega_{b\bar{a}}\beta_{\bar{b}} - \Omega_{b\bar{b}}\beta_{\bar{a}})\right), \quad (6.39)$$

$$\Omega^{\bar{b}b}D_{\bar{b}}h_{b\bar{a}} = -\frac{i\sqrt{2}}{3}\beta_{\bar{a}}, \quad (6.40)$$

$$\partial_{11}h_{a\bar{b}} = -\frac{1}{\sqrt{2}}\left(i\theta_{a\bar{b}} - \frac{1}{12}\alpha\Omega_{a\bar{b}}\right), \quad (6.41)$$

preserving $\mathcal{N} = 1$ supersymmetry in four spacetime dimensions.

In [152] these relations were solved explicitly by utilizing the Hodge dual of the antisymmetric four form G as well as the sources $J^{(i)}$, transforming the equations of motion involving a newly defined six-form and then solving said equations by performing a harmonic expansion in an orthonormal set of eigenmodes of the Hodge-de Rham Laplacian [196] on the Calabi-Yau space[8]. The metric corrections to first order are then fixed in terms of the modes of the six-form, or in the case of the scalar corrections the inner product of the six-form with the Calabi-Yau Kähler form. The terms in the expansion include massive (eigenmodes with negative eigenvalue) and massless (zero eigenvalue) contributions as well as a pure gauge term that may be chosen freely.

At this point, it is sufficient to quote a simplified result of the mentioned analysis above. In doing so, we omit all massive modes since they turn out to decouple at linear order [150], additionally we include only one massless zero mode that is related to the Calabi-Yau breathing mode (which corresponds to a rescaling of X, $\Omega_{AB} \longmapsto c\ \Omega_{AB}$). The metric corrections may then be expressed as follows [59, 152]

$$b = -\frac{\sqrt{2}}{3} R_0 V_0^{-2/3} \alpha_B (|x^{11}| - \frac{R_{11}}{2}\pi), \tag{6.42}$$

$$\gamma = \frac{2\sqrt{2}}{3} R_0 V_0^{-2/3} \alpha_B (|x^{11}| - \frac{R_{11}}{2}\pi), \tag{6.43}$$

$$h_{AB} = \frac{\sqrt{2}}{3} R_0 V_0^{-2/3} \alpha_B (|x^{11}| - \frac{R_{11}}{2}\pi) \Omega_{AB}, \tag{6.44}$$

$$G_{ABCD} = \frac{1}{6} \alpha_B \epsilon_{ABCD}{}^{EF} \omega_{EF}\ \epsilon(x^{11}), \tag{6.45}$$

$$G_{ABC11} = 0. \tag{6.46}$$

In the full expansion, there exist multiple α_i for each mode i in the expansion, but since we are utilizing a single mode it is sufficient to define a single α_B as

$$\alpha_B \equiv -\frac{1}{8\sqrt{2}\pi v} \left(\frac{\kappa_{11}}{4\pi}\right)^{2/3} \int_X \omega \wedge \mathrm{tr}(R^{(\Omega)} \wedge R^{(\Omega)}), \qquad v \equiv \int_X \sqrt{\Omega}, \tag{6.47}$$

with v the volume of the Calabi-Yau space. $\epsilon(x^{11})$ appearing in the non-vanishing component

[8] See Ch. 12 of [196] for harmonic differential forms and analysis of zero modes.

of G_{ABCD} is a step function defined as

$$\epsilon(x^{11}) = \begin{cases} -1 & \text{if } x^{11} < 0, \\ +1 & \text{if } x^{11} \geq 0. \end{cases} \tag{6.48}$$

It is also important to note that α_B is quantized since $\operatorname{tr} R \wedge R$ is an element of $H^{2,2}(X, \mathbb{Z})$ [9]. In the D-brane perspective, the configuration given in equations (6.42) to (6.46) if we were to also include all other massless modes may be interpreted as a linearized solution for a collection of M5-branes located at the fixed points of the orbifold dimension. The standard embedding determines the amount of M5-brane charge, $-\frac{1}{2} \operatorname{tr} R \wedge R$ and $\frac{1}{2} \operatorname{tr} R \wedge R$, at each fixed point.

The intention now then, is to perform the compactification of six spatial dimensions on a Calabi-Yau threefold such that the five dimensional background metric instead preserves one fourth (eight) of the thirty-two supercharges as opposed to one eighth (four) of them. This may be done in a perturbative expansion in G by ensuring the nullification of the supersymmetry transformations to order $\kappa_{11}^{2/3}$ ('first' perturbative order) of the fermionic sector. One may check the preservation of supersymmetry explicitly by verifying the Killing spinor equations or checking that the theory derived is a gauged version of a known $\mathcal{N} = 1$ supersymmetric theory in five dimensions as done in [150]. In particular, performing the reduction directly on the background metric given in equation (6.22) constrained by equations (6.42) to (6.46) preserves only four supercharges and also leads to an action explicitly dependent on the orbifold coordinate x^{11} [59].

However, we want to perform the reduction to five dimensions in such a way such that the background just mentioned above may still be recovered from the five dimensional theory. This has been shown to be possible by absorbing the metric perturbations into the five-dimensional metric moduli, specifically, the scale factors b, γ of the four dimensional spacetime and of the orbifold may be absorbed into the five-dimensional metric and similarly the variation of the Calabi-Yau volume along the orbifold may be absorbed into a

[9] See [197] for an explanation of how quantization follows.

modulus V. Technically speaking one performs a Kaluza-Klein reduction on the metric

$$ds_{11}^2 = V^{-2/3} g_{\alpha\beta} dx^\alpha dx^\beta + V^{1/3} \Omega_{AB} dx^A dx^B, \tag{6.49}$$

and as shown in [150] doing so on this background preserves eight supercharges from the eleven-dimensional theory. Importantly, this procedure differs from a standard reduction of eleven-dimensional supergravity on a Calabi-Yau space [198, 199], with the reasoning that the non-vanishing component of the four form field G_{ABCD} as seen in equation (6.45) must be accounted for explicitly as it may not be absorbed in a corresponding antisymmetric tensor field moduli of the five-dimensional theory. A non-vanishing antisymmetric tensor field configuration is known as the nonzero mode of the reduction and in this case the field strength is not a harmonic form [196](G does not have vanishing Bianchi identity[10]). In this case, the four-form field strength may be identified with the fourth de Rham cohomology[11] group of X, $H^4(X)$, but in general G_{ABCD} is a linear combination of all harmonic $(2,2)$ forms [59][12].

In the reduction, the total bulk field content of the five dimensional theory is given by the gravity multiplet $(g_{\alpha\beta}, \mathcal{A}_\alpha, \psi^i_\alpha)$ along with the universal hypermultiplet $(V, \sigma, \xi, \bar{\xi}, \zeta^i)$, with 'universal' referring to the fact that these fields are present regardless of the particulars of the Calabi-Yau manifold. ψ^i_α and ζ^i are the gravitini and the hypermultiplet fermions respectively, with $i = 1, 2$. We also have the five-dimensional $h^{(1,1)}$ vector[13] field $\mathcal{A}_\alpha$, an antisymmetric three-form potential $C_{\alpha\beta\gamma}$ which may be dualized to a scalar σ whose charge is α_B and quantized, and a complex scalar ξ; all give rise to their respective five dimensional

[10] See Ch. 14.3.3 of [196] for more discussions on nonzero modes of the reduction.

[11] Cohomology is any vector space with a nilpotent operator $\mathcal{B}$ ($\mathcal{B}^2 = 0$), de Rham cohomology is the cohomology of the exterior derivative d acting on differential forms. Dolbeault cohomology is the cohomology of $\partial, \bar{\partial}$ (the $(1,0)$ and $(0,1)$ parts of d) acting on (p,q) forms. The pth de Rham cohomology of a real manifold K is the quotient space $H^p(K) =$ (closed p-forms on K)/(exact p-forms on K), with its dimension given by the topology dependent Betti number. The (p,q) Dolbeault cohomology of a complex manifold K may be defined as the quotient space $H^{p,q}_{\bar{\partial}}(K) =$ ($\bar{\partial}$-closed (p,q)-forms in K)/($\bar{\partial}$-exact (p,q)-forms in K), with its dimension given by the topology dependent Hodge number $h^{p,q}$. In particular, for complex Kähler manifolds the Dolbeault cohomologies become equivalent $H^{p,q}_{\bar{\partial}}(K) = H^{p,q}_{\partial}(K) \equiv H^{p,q}(K)$ [72, 200].

[12] See Ch. 15.5.3 of [196] for discussions on the Hodge decomposition.

[13] $h^{(p,q)}$ is the Hodge number specifying the dimension of the Dolbeault cohomology group $H^{p,q}$.

field strengths $\mathcal{F}_{\alpha\beta}$, $G_{\alpha\beta\gamma\delta}$ and X_α in the following way

$$C_{\alpha\beta\gamma} \quad \widehat{=} \quad G_{\alpha\beta\gamma\delta} = 24\partial_{[\alpha}C_{\beta\gamma\delta]}, \tag{6.50}$$

$$C_{\alpha AB} = \frac{1}{6}\mathcal{A}_\alpha \omega_{AB} \quad \widehat{=} \quad G_{\alpha\beta AB} = \mathcal{F}_{\alpha\beta}\omega_{AB}, \tag{6.51}$$

$$C_{ABC} = \frac{1}{6}\xi\omega_{ABC} \quad \widehat{=} \quad G_{\alpha ABC} = \partial_\alpha \xi \omega_{ABC}, \tag{6.52}$$

$$\mathcal{F}_{\alpha\beta} = \partial_\alpha \mathcal{A}_\beta - \partial_\beta \mathcal{A}_\alpha, \tag{6.53}$$

$$X_\alpha = \partial_\alpha \xi, \tag{6.54}$$

and the nonzero mode defined as in equation (6.45). In the above, ω_{ABC} is a harmonic $(3,0)$ form on the Calabi-Yau space, and ξ is the complex scalar zero mode arising due to ω_{ABC} [59, 150].

In [150], the complete five-dimensional effective action is computed including matter fields on the boundary, however here we omit the latter for lack of necessity. Using the field configurations given in equations (6.45) and (6.50) to (6.54) the effective action from equations (6.6) to (6.8) is found to be

$$S_5 = S_{grav} + S_{hyper} + S_{bound}, \tag{6.55}$$

with each action defined as

$$S_{grav} = -\frac{1}{2\kappa_5^2}\int_{\mathcal{M}_5}\left[R + \frac{3}{2}\mathcal{F}_{\alpha\beta}\mathcal{F}^{\alpha\beta} + \frac{1}{\sqrt{2}}\epsilon^{\alpha\beta\gamma\delta\epsilon}\mathcal{A}_\alpha\mathcal{F}_{\beta\gamma}\mathcal{F}_{\delta\epsilon}\right], \tag{6.56}$$

$$\begin{aligned} S_{hyper} = -\frac{1}{2\kappa_5^2}\int_{\mathcal{M}_5}\sqrt{-g}\Big[& V^{-2}\partial_\alpha V \partial^\alpha V + 2V^{-1}\partial_\alpha\xi\partial^\alpha\bar{\xi} + \frac{1}{24}V^2 G_{\alpha\beta\gamma\delta}G^{\alpha\beta\gamma\delta} \\ & + \frac{\sqrt{2}}{24}\epsilon^{\alpha\beta\gamma\delta\epsilon}G_{\alpha\beta\gamma\delta}(i(\xi\partial_\epsilon\bar{\xi} - \bar{\xi}\partial_\epsilon\xi) + 2\alpha_B\mathcal{A}_\epsilon) + \frac{1}{3}V^{-2}\alpha_B^2\Big], \end{aligned} \tag{6.57}$$

$$\begin{aligned} S_{bound} = -\frac{1}{2\kappa_5^2}\left[-2\sqrt{2}\int_{\mathcal{M}_4^{(1)}}\sqrt{-g}V^{-1}\alpha_B + 2\sqrt{2}\int_{\mathcal{M}_4^{(2)}}\sqrt{-g}V^{-1}\alpha_B\right] \\ - \frac{1}{16\pi\alpha_{GUT}}\sum_{i=1}^{2}\int_{\mathcal{M}_4^{(i)}}\sqrt{-g}V \operatorname{tr}\left(F_{\mu\nu}^{(i)2}\right). \end{aligned} \tag{6.58}$$

In the above, higher derivative terms have been truncated, and this action has been checked to be consistent with a reduction of the eleven-dimensional equations of motion. The four-form five dimensional field strength $G_{\alpha\beta\gamma\delta}$ is constrained by the Bianchi identity determined by equation (6.12)

$$(dG)_{11\mu\nu\rho\sigma} = -\frac{\kappa_5^2}{2\sqrt{2}\pi\alpha_{GUT}} \left(J^{(1)}\delta(x^{11}) + J^{(2)}\delta(x^{11} - \pi R_{11})\right)_{\mu\nu\rho\sigma}, \tag{6.59}$$

with the currents defined as in equation (6.13). The five dimensional gravitational coupling κ_5 is related to the eleven dimensional gravitational coupling κ_{11} as

$$\kappa_5^2 = \frac{\kappa_{11}^2}{v}. \tag{6.60}$$

One should note that the 'cosmological' potential terms for V in the bulk and boundary arise due to the nonzero mode of the reduction, and the strength determined by α_B. In [59, 150] it is determined explicitly that the action above is in fact a gauged $\mathcal{N} = 1$ supergravity theory in five dimensions with the presence of bulk and boundary potentials arising from the nonzero modes of the reduction.

The final piece of the puzzle is to recover from the five dimensional theory the deformations of the Calabi-Yau background as in equation (6.22). To generate some intuition, we note that in the compactification scheme done above to four dimensions, we stated that the Bianchi identity provided source terms for M5-branes with equal and opposite charges located at the orbifold fixed points. In the five-dimensional perspective, these M5-branes are effectively D3-branes that span the four-dimensional spacetime along with a two-cycle in the Calabi-Yau space. Furthermore, since we have restricted ourselves to the Calabi-Yau breathing modes, we keep only the M5-brane which is associated with the two-cycle defined by the Kähler form. Thus we expect our vacuum solution to correspond to two parallel D3-branes located at the ends of the orbifold dimension.

With the ansatz

$$ds_5^2 = a(x^{11})^2 dx^\mu dx^\nu \eta_{\mu\nu} + b(x^{11})^2 (dx^{11})^2, \tag{6.61}$$

$$V = V(x^{11}), \tag{6.62}$$

and all other fields vanishing, the general solution that obeys the equations of motion derived from the five dimensional effective action equations (6.55) to (6.58), with a_0, b_0, c_0 constants determined by initial conditions is

$$a = a_0 D^{1/2}, \tag{6.63}$$

$$b = b_0 D^2, \tag{6.64}$$

$$V = b_0 D^3, \tag{6.65}$$

$$D = \frac{\sqrt{2}}{3}\alpha_B |x^{11}| + c_0. \tag{6.66}$$

$D(x^{11})$ is a harmonic function, fixed by the sources localised on the orbifold boundaries. Generically speaking, from the point of view of the five-dimensional bulk theory with no information of the source terms, we may have an arbitrary number of D3-branes between the orbifold fixed planes in the x^{11} direction. Generally, in order to have a $(D-2)$-brane in a D dimensional theory we would require a $(D-1)$ form field to be present in the action, or equivalently the presence of its dualized field strength to a zero form, otherwise known as a cosmological constant term. In our case the cosmological potential terms in the five dimensional bulk and four dimensional boundary embedded in five dimensions dualize to five form fields, and therefore couple to D3-branes located anywhere between the boundaries [150].

However brane solutions have singularities at the location of the branes needing to be supported by source terms [59,61]. Thus in this particular case we have two of them, located at the two fixed points of the orbifold dimension. Thus the form of the harmonic function $D(x^{11})$ is two sections each linearly dependent on the orbifold coordinate in the domains $x^{11} \in [0, \pi R_{11}]$ and $x^{11} \in [-\pi R_{11}, 0]$, identified at $x^{11} = 0$ and $x^{11} = \pm\pi R_{11}$ with slopes $\frac{\sqrt{2}}{3}\alpha_B$ and $-\frac{\sqrt{2}}{3}\alpha_B$ respectively. Thus we have a Poisson type equation for $D(x^{11})$

$$\partial_{11}^2 D = \frac{2\sqrt{2}}{3}\alpha_B(\delta(x^{11}) - \delta(x^{11} - \pi R_{11})), \tag{6.67}$$

indicating a solution representing two parallel three-branes located at the orbifold fixed planes as expected. Note that this is an exact solution to the action, and once linearizing to order $\kappa_{11}^{2/3}$ one may check upon appropriate fixing of the constants a_0, b_0, c_0 that we recover the solution of the background preserving four supercharges given by equations (6.22) and (6.42) to (6.46). Preservation of half of the supersymmetries from the five dimensional theory may be checked explicitly by noting the satisfaction of the Killing spinor equations given by the supersymmetry transformations of the gravitini $\delta\psi_\alpha^i = 0$ and the hypermultiplet fermions $\delta\zeta^i = 0$ as done in [59]. These Killing spinor equations confirm that we have found a BPS state preserving four of the eight bulk supercharges on each boundary D3-brane [59].

In closing, it is argued that the D3-brane embedded in the orbifold fixed plane $\mathcal{M}_4^{(1)}$ in the five dimensional theory may be interpreted as the appropriate vacuum solution providing the background in which a reduction to an $\mathcal{N} = 1$ supergravity theory in four dimensions with preserved Poincaré invariance may be performed. This boundary brane also possesses the low energy gauge and matter fields required [150, 152] for relevant low energy particle phenomenology.

6.1.3 Interbrane interactions

As alluded to in section 6.1.1 a non-vanishing net force between the boundary D3-branes (M9-branes wrapped on the internal Calabi-Yau space [185, 186, 201]) coincident with the orbifold planes or, between the boundary D3-branes and D3-branes existing in the bulk spacetime (wrapped M5-branes on a holomorphic curve on the internal Calabi-Yau space) are argued to provide cosmological potential terms for Kähler moduli or volume modulus of the $S^1/\mathbb{Z}_2$ orbifold, describing either the position of the bulk brane or separation length of the boundary branes, respectively. Non-perturbative effects [202–206] are known to provide contributions to the potential energy of the moduli fields, or contributions to the superpotential [185–187] in the setting of an $\mathcal{N} = 1$ supergravity theory. Here we motivate a basic form for a cosmological potential arising from M-theory supermembrane instantons, and conjecture, subject to the form of the Kähler potential and perturbative corrections to it, that it may drive an ekpyrotic phase of evolution.

Within the context of heterotic M-theory it is argued [185–187] that open (oriented such that the orbifold coordinate is tangent to the membrane) supermembranes extending either between the bulk M5-brane and one of the M9-branes or between the boundary M9-branes, generate similar nonperturbative contributions to the superpotential. The approach to calculating the nonperturbative contribution from open supermembranes follows the fundamental understandings for calculating M-brane instanton effects described in [205, 206]. In the compactification of heterotic M-theory on $S^1/\mathbb{Z}_2$ to an $\mathcal{N}=1$ theory in four dimensions [150], a direct comparison is made between the fermionic two-point function that are superpartnered to their respective moduli and the fermionic bilinear interaction term coupled to the superpotential in the $\mathcal{N}=1$ supergravity theory in four dimensions, derived in [150]. The instanton contributions to the superpotential are extracted by explicitly computing the contributions to the two-point function, from open supermembranes wrapping either a subset of the orbifold $S^1/\mathbb{Z}_2$ or the entire interval in product with a holomorphic curve in the Calabi-Yau space; the specificity of the wrapping is in large part to ensure that the membranes are supersymemtric in the background product topology defined by $\mathcal{M}_{11} = X \times S^1/\mathbb{Z}_2 \times \mathbb{M}^4$. The two wrappings correspond to whether the supermembranes extend between the bulk M5-brane and the M9-brane, or the two boundary M9-branes respectively.

Thus the open supermembrane action is required, and given explicitly in [186, 187]. In the low energy limit ($R_{11} \longrightarrow 0$) the action of the supermembrane with boundary strings is shown to be equivalent to the $E_8 \times E_8$ heterotic superstring wrapped on a holomorphic curve in the internal Calabi-Yau space, once again in order to preserve $\mathcal{N}=1$ supersymmetry. Thus in this limit, the non-perturbative contributions from open supermembrane instantons are equivalent to considering non-perturbative contributions of heterotic superstring instantons.

The non-perturbative contributions to the superpotential from open supermembranes wrapped once around the orbifold and holomorphic curve in the internal Calabi-Yau space at low energy were calculated to be of the following forms

$$W_{M9-M9} \propto e^{-\frac{T}{2}\sum_{k=1}^{h^{1,1}} \omega_k T^k}, \qquad W_{M5-M9} \propto e^{-\frac{T}{2}\mathbf{Y}}. \tag{6.68}$$

The T^k in the first expression are redefinitions of the Kähler moduli arising from the harmonic $(1,1)$-form expansion of the Kähler form of the Calabi-Yau threefold. The number of Kähler moduli are thus determined by the Hodge number $h^{1,1}$. The dimensionless field $\mathbf{Y}$ in the second expression is the bosonic component of the four dimensional $\mathcal{N}=1$ translational supermultiplet of the M5-brane [187,188], itself defined in terms of the $h^{1,1}$ $(1,1)$-form Kähler moduli, the translational mode along the orbifold direction from the tensor supermultiplet of the worldvolume theory of the M5-brane, and the axionic moduli of the M5-brane arising from the field strength two form from the same tensor supermultiplet. The prefactor T is defined identically in both expressions, it is a dimensionless parameter defined in terms of the membrane tension T_M, the circumference of the orbifold πR_{11} and the volume of the holomorphic curve $\mathfrak{v}$

$$T = T_M \pi R_{11} \mathfrak{v}. \tag{6.69}$$

Lastly, the ω_ks appearing in the first expression in equation (6.68) are dimensionless, inversely proportional to the volume of the holomorphic curve $\mathfrak{v}$ and proportional to an integral over the curve with the integrand defined as the pullback onto the curve of the kth harmonic form appearing from the harmonic expansion of the Kähler form.

Via the bosonic sector of the effective four dimensional effective supergravity theory derived in [150], we may derive an effective potential for one or multiple of the $h^{1,1}$ Kähler moduli or the translational $\mathbf{Y}$ modulus. The relevant terms are the usual formula for the scalar potential as given in equations (6.86) to (6.88), with the implied summation over $i, j \dots \in \{1, \cdots, h^{1,1}, h^{1,1}+1, h^{1,1}+2\}$. It is straightforward to see that the superpotentials provided above in equation (6.68) lead generically to steep negative exponential potentials in a four dimensional effective theory for large T. The caveat is that the Kähler potential generally leads to non-canonical kinetic terms appearing in the Lagrangian, and thus specific models must be studied carefully.

6.2 Ekpyrosis from F-theory

Up until this point, we have proposed that the ekpyrotic scenario may be embedded in the five dimensional effective action derived from the strongly coupled limit of $E_8 \times E_8$ heterotic superstring theory. This then raises the question as to whether the model may be embedded in another critical string theory. Here, we propose that the moduli stabilisation[14] procedure of [207] in type IIB supergravity may lead to a natural embedding for the ekpyrotic scenario.

[14] Moduli stabilization is the process of determining vacua in which all moduli from the compactification possess positive mass squared [20].

In the description below, note that capitalized Latin characters appearing in the middle of the alphabet $M, N, P, \cdots \in \{0, 1, \cdots, 9\}$ are used to indicate the coordinates x^M of a ten-dimensional manifold. Upon compactification, lower case Latin characters $m, n, p, \cdots \in \{4, \cdots, 9\}$ will be used to index the coordinates y^m of the compact six manifold X, and Greek characters $\mu, \nu, \rho, \cdots \in \{0, \cdots, 3\}$ to refer to the coordinates of the four-dimensional Minkowski spacetime x^μ.

To leading order in the Regge slope α' and and the string coupling g_s, the low energy effective action of type IIB superstring theory is a ten dimensional supergravity theory, which unlike IIA supergravity may not be derived from eleven-dimensional supergravity [144]. The field content of IIB supergravity is the massless spectrum of the type IIB superstring, for both the bosonic and fermionic sectors. The fermionic sector is comprised of two right-handed Majorana-Weyl dilatini and two left-handed Majorana-Weyl gravitini, but this sector will not be relevant for this discussion. In the string frame (ie. the Ricci scalar is multiplied by the string coupling, a useful form when studying string perturbation theory) the bosonic sector of the IIB supergravity theory may be expressed as

$$S_{IIB} = S_{NS} + S_R + S_{CS}, \tag{6.70}$$

where

$$S_{NS} = \frac{1}{2\kappa^2} \int d^{10}x \sqrt{-g} e^{-2\Phi} \left(R + 4\partial_\mu \Phi \partial^\mu \Phi - \frac{1}{2}|H_3|^2 \right), \tag{6.71}$$

$$S_R = -\frac{1}{4\kappa^2} \int d^{10}x \sqrt{-g} \left(|F_1|^2 + |\tilde{F}_3|^2 + \frac{1}{2}|\tilde{F}_5|^2 \right), \tag{6.72}$$

$$S_{CS} = -\frac{1}{4\kappa^2} \int C_4 \wedge H_3 \wedge F_3. \tag{6.73}$$

The above comprises the closed string NS-NS (Neveu-Schwarz) sector, the closed string R-R (Ramond) sector and a Chern-Simons term respectively. The NS-NS sector consists of the metric, the two form B_2 with field strength $H_3 = dB_2$ and the dilaton Φ. The R-R sector is constituted by the zero, two and four forms C_0, C_2, C_4 with field strengths defined by $F_p = dC_{p-1}$. The field strengths of the forms just mentioned may be redefined into the gauge invariant quantities indicated by the tilde $\tilde{F}_3 = F_3 - C_0 \wedge H_3$ and $\tilde{F}_5 = F_5 - \frac{1}{2} C_2 \wedge H_3 + \frac{1}{2} B_2 \wedge F_3$.

The five form $\tilde{F}_5$ is self dual satisfying $\tilde{F}_5 = *_{10}\tilde{F}_5$, with $*_D$ defining the Hodge dual in D dimensions. Additionally we may redefine the fields

$$G_3 \equiv F_3 - \tau H_3, \qquad \tau \equiv C_0 + ie^{-\Phi}, \tag{6.74}$$

and re-express the action in the Einstein frame, which is more useful for studying gravitational effects, by performing a Weyl rescaling

$$g_{E,MN} \equiv e^{-\Phi/2} g_{MN}, \tag{6.75}$$

arriving at

$$S_{IIB} = \frac{1}{2\kappa^2}\int d^{10}x\sqrt{-g_E}\left[R_E - \frac{|\partial\tau|^2}{2(\mathrm{Im}(\tau))^2} - \frac{|G_3|^2}{2\mathrm{Im}(\tau)} - \frac{|\tilde{F}_5|^2}{4}\right] - \frac{i}{8\kappa^2}\int \frac{C_4 \wedge G_3 \wedge \bar{G}_3}{\mathrm{Im}(\tau)}, \tag{6.76}$$

with the overbar on G_3 denoting the complex conjugate. The ten dimensional gravitational coupling $\tilde{\kappa}_{10}$ is related to the Regge slope, the string coupling $g_s = e^{\Phi}$ and κ as $2\tilde{\kappa}_{10}^2 = 2\kappa^2 g_s^2 = (2\pi)^7(\alpha')^4 g_s^2$ [18, 20, 72, 144]. $|G_3|^2$ is defined as

$$|G_3|^2 = \frac{1}{3!} g_E^{M_1N_1} g_E^{M_2N_2} g_E^{M_3N_3} G_{M_1M_2M_3}\bar{G}_{N_1N_2N_3}. \tag{6.77}$$

The goal is to consider a warped compactification of type IIB supergravity from a ten dimensional manifold on a six dimensional space X with compact topology such that $\mathcal{M} = \mathcal{M}_4 \times X$, with $\mathcal{M}_4$ denoting a four-dimensional Minkowski spacetime. Let us begin by introducing a warped metric ansatz for $\mathcal{M}$ of the following form

$$ds^2 = e^{2A(y)}\eta_{\mu\nu}dx^\mu dx^\nu + e^{-2A(y)}g_{mn}dy^m dy^n. \tag{6.78}$$

By Poincaré invariance of $\mathcal{M}_4$, the complex warp factor $A(y)$ is permitted to depend only on the coordinates of X, the three-form G_3 possesses necessarily vanishing components in

$\mathcal{M}_4$, and that $\tilde{F}_5$ take on the following form

$$\tilde{F}_5 = (1 + *_{10}) d\alpha(y) \wedge dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3, \tag{6.79}$$

with $\alpha(y)$ a complex scalar. There is a well known no-go [208–210] theorem for IIB compactifications, stating that if the internal space is compact and non singular and localised brane or orientifold sources are absent then the warp factor is necessarily trivial in the leading supergravity approximation. This is straightforwardly demonstrated by deriving the equations of motion by variation of the action with respect to the metric in order to produce a governing equation for the warp factor $A(y)$ [20, 144]. With the inclusion of localised sources this equation takes on the following form

$$\Delta e^{4A} = \frac{e^{8A}}{2\mathrm{Im}(\tau)} |G_3|^2 + e^{-4A} \left(|\partial \alpha|^2 + |\partial e^{4A}|^2 \right) + 2\kappa^2 e^{2A} \mathcal{J}_{loc}, \tag{6.80}$$

with Δ the Laplacian on X and $\mathcal{J}_{loc}$ defined in terms of the stress-energy tensor $(T_{MN})_{loc}$ and action S_{loc} describing localised sources (D-branes or orientifold planes, ie. DBI + CS)

$$\mathcal{J}_{loc} \equiv \frac{1}{4} \left(g_E^{mn} T_{mn} - g_E^{\mu\nu} T_{\mu\nu} \right)_{loc}, \qquad (T_{MN})_{loc} = -\frac{2}{\sqrt{g_E}} \frac{\delta S_{loc}}{\delta g_E^{MN}}. \tag{6.81}$$

If we omit localised sources $\mathcal{J}_{loc} = 0$ and integrate equation (6.80) over the internal manifold X, the left hand side vanishes since it is a total derivative, while each term on the right hand side is positive definite, immediately implying A, α are constant and G_3 vanishes.

Indeed, it has been carefully studied that additional string sources invalidate the no-go theorem and permit warped compactifications, in particular the inclusion of these sources leads to $\mathcal{J}_{loc} < 0$ and the triviality of A, α, G_3 no longer immediately follows [208, 211, 212]. Thus a warped geometry is a necessary requirement in order to generate a non-trivial three-form flux.

Taking the brane sources into account, we may determine a set of constraints. The sources contribute to the Bianchi identity for $\tilde{F}_5$

$$d\tilde{F}_5 = H_3 \wedge F_3 + 2\kappa^2 T_3 \rho_3^{loc}, \tag{6.82}$$

where ρ_3^{loc} is the D3 charge density from the sources which are supported by delta functions and T_3 is the D3-brane tension, which for a general Dp-brane is

$$T_p \equiv \frac{1}{g_s(2\pi)^p(\alpha')^{(p+1)/2}}. \tag{6.83}$$

Now inserting equation (6.79) into equation (6.82) and subtracting the result from equation (6.80) one finds [20, 144]

$$\Delta(e^{4A}-\alpha) = \frac{e^{8A}}{24\mathrm{Im}(\tau)}|iG_3 - *_6G_3|^2 + e^{-4A}|\partial(e^{4A}-\alpha)|^2 + 2\kappa^2 e^{2A}(\mathcal{J}_{loc} - T_3\rho_3^{loc}). \tag{6.84}$$

When integrated over the internal space X, the left hand side vanishes and the terms associated with non-local sources are once again positive definite. Considering only sources that satisfy the BPS bound $\mathcal{J} \geq T_3\rho_3^{loc}$, equation (6.84) is only satisfied if all sources considered in fact *saturate* the BPS bound $\mathcal{J} = T_3\rho_3^{loc}$, which are D3-branes, O3-planes and D7-branes wrapped on four-cycles [20, 144]. Furthermore, the three-form flux must be imaginary self dual (ISD) $*_6G_3 = iG_3$, and the warp factor is required to be equivalent to the four-form potential $e^{4A} = \alpha$. These criteria define ISD solutions. Thus in the presence of sources obeying the BPS condition, only ISD solutions are permitted.

Compactification on a Calabi-Yau three-fold with a single Kähler modulus (contained in the superfield ρ), before fluxes are turned on, there exist the massless complex structure moduli ζ_α ($\alpha = 1, \cdots, h^{2,1}$) and axiodilaton τ. The Kähler potential in this compactification scheme of IIB at tree level (leading order in string loop expansions) denoted as $\mathcal{K}_0$ receives contributions from all three types of superfields

$$\mathcal{K}_0 = -3\ln\left(-i(\rho-\bar{\rho})\right) - \ln\left(-i(\tau-\bar{\tau})\right) - \ln\left(-i\int_X \Omega\wedge\bar{\Omega}\right). \tag{6.85}$$

The holomorphic three-form Ω depends on the complex structure moduli ζ_α. In general, the Kähler potential $\mathcal{K}$ defines the Kähler metric $\mathcal{K}_{i\bar{j}}$

$$\mathcal{K}_{i\bar{j}} \equiv \frac{\partial}{\partial\phi^i}\frac{\partial}{\partial\bar{\phi}^j}\mathcal{K}(\phi,\bar{\phi}), \tag{6.86}$$

and this in turn defines the kinetic and potential terms of the complex scalar fields of the effective $\mathcal{N} = 1$ supergravity theory in four dimensions

$$\mathcal{L}_\phi = -\mathcal{K}_{i\bar{j}} \partial^\mu \phi^i \partial_\mu \bar{\phi}^j - V_F. \tag{6.87}$$

The F-term potential V_F is completely determined in terms of $\mathcal{K}$ and the superpotentnial W (in units of $M_{pl} = 1$)

$$V_F(\phi^i, \bar{\phi}^i) = e^{\mathcal{K}} \left[\mathcal{K}^{i\bar{j}} D_i W \overline{D_j W} - 3|W|^2 \right], \qquad D_i W \equiv \partial_i W + (\partial_i \mathcal{K}) W. \tag{6.88}$$

Note that in general, the indices i, j run over all complex scalars of the supergravity theory, and thus in our case run over all moduli (ρ, τ, ζ_α for all α).

In the presence of nonzero three-form flux, two crucial consequences manifest. The first is the generation of the Gukov-Vafa-Witten (GVW) flux superpotential for the Calabi-Yau moduli, which we denote here as W_0 [213]

$$W_0 = \int_X G_3 \wedge \Omega. \tag{6.89}$$

The Kähler potential in our case given in equation (6.85), satisfies

$$\mathcal{K}_0^{\rho\bar{\rho}} \partial_\rho \mathcal{K}_0 \partial_{\bar{\rho}} \mathcal{K}_0 = 3, \tag{6.90}$$

and since the superpotential in equation (6.89) is independent of the Kähler modulus, the resulting scalar potential is of characteristic no-scale type [214, 215]

$$V_F = e^{\mathcal{K}_0} \sum_{i,j \neq \rho} \mathcal{K}_0^{i\bar{j}} D_i W_0 \overline{D_j W_0}. \tag{6.91}$$

V_F is positive semi-definite and since at the minimum of the potential $D_i W_0 = 0 \; \forall \; i \neq \rho$, then by equation (6.91) $V_F = 0$ at the minimum. However, supersymmetry is only preserved if $D_i W = 0 \; \forall \; i$, and since in general we may have nonperturbative corrections to the superpotential that depend on the Kähler modulus ρ supersymmetry is not preserved in general.

The second consequence of nonzero three-form flux is that all complex structure moduli and axiodilaton acquire supersymmetric masses, with the mass scale dictated by the flux quantization condition. The reception of a mass may be understood most directly by observing that the moduli fields with the exception of ρ experience a potential in the third term of the IIB supergravity action given in equation (6.76) [20, 144, 207]. The acquisition of a mass stabilizes these moduli as they may be integrated out, and an effective theory for the Kähler modulus ρ is established [20, 216]. In turn W_0 may be treated as a constant, which may be understood by expanding the flux superpotential W_0 in powers of the non-Kähler moduli fields about the supersymmetric minimum [20].

The final step in our construction then is to consider corrections to the no scale model, by searching for corrections to the superpotential affecting the Kähler modulus, hence stabilizing it by providing it a mass. Let us note quickly that the process of integrating out the complex structure moduli and axiodilaton is justifiable so long as their mass scale is much larger than that of the Kähler modulus [216]. The Kähler potential receives perturbative corrections in the α', g_s expansions; conversely the superpotential receives none to any order in either expansion largely due to the shift symmetry protecting Kähler moduli [217, 218]. Therefore with respect to corrections to the superpotential we consider solely nonperturbative corrections. Additionally, we will omit higher order corrections to the tree level Kähler potential given in equation (6.85) which may technically alter the potential for the Kähler moduli, but ensuring the volume modulus is stabilized at large values relative to the string scale ensures that omitting these corrections is consistent [207].

There are two well-known non-perturbative effects we will remark on that may contribute to the superpotential. These are gluino condensation in the gauge theory inhabiting stacks of N_{D7} D7-branes wrapping internal four-cycles of the Calabi-Yau space [219–221], and Euclidean D3-branes wrapping four-cycles, also known as D3-brane instantons [222, 223]. Respectively, these two mechanisms generate superpotentials that may be expressed as

$$W_{\lambda\lambda} = \mathcal{A}_1(\zeta^\alpha, Y_k)e^{-\frac{2\pi}{N_{D7}}T} \propto e^{-\frac{T_3\mathcal{V}_4}{N_{D7}}}, \qquad W_{ED3inst} = \mathcal{A}_2(\zeta^\alpha, Z_{k'})e^{-2\pi T}, \tag{6.92}$$

where T is a Kähler modulus, $\mathcal{V}_4$ is the warped volume of the four-cycle, and $\mathcal{A}_1(\zeta^\alpha, Y_k)$ and $\mathcal{A}_2(\zeta^\alpha, Z_{k'})$ are independent of the Kähler moduli but depend on the complex structure moduli and on the respective positions of any D-branes $Y_k, Z_{k'}$ indexed by k, k'.

Thus the primary result is that there is evidence for nonperturbative corrections to the superpotential that are exponential in form contributing to the GVW flux superpotential. Promisingly for the ekpyrotic scenario, we may generalize the construction to multiple Kähler moduli, and superpotentials from nonperturbative effects may be generated for each Kähler modulus T_i [224, 225] mimicking an effective multi-field model in four spacetime dimensions. Thus in general, we have for the superpotential including either or a combination of the non-perturbative superpotentials described in equation (6.92)

$$W = W_0 + \sum_i \mathcal{A}_i e^{-a_i T_i} + \cdots \,, \tag{6.93}$$

the ellipsis denoting sub-leading nonperturbative corrections [20]. However, for the time being we restrict the analysis to a single Kähler modulus

$$W = W_0 + \mathcal{A} e^{ia\rho} + \cdots \,, \tag{6.94}$$

with $\mathcal{A}, a$ being determined from the details of either or both of the nonperturbative effects described above. Using the superpotential given in equation (6.94), the Kähler potential given in equation (6.85) (with only ρ remaining), plugging into equation (6.88), setting the axion in the ρ modulus to zero and letting $\rho = i\sigma$ we obtain for the scalar potential of the non-canonical scalar field σ

$$V_F(\sigma) = \frac{a\mathcal{A}e^{-a\sigma}}{2\sigma^2}\left(\frac{1}{3}\sigma a\mathcal{A}e^{-a\sigma} + W_0 + \mathcal{A}e^{-a\sigma}\right). \tag{6.95}$$

A few comments are in order concerning the range of validity of this potential. In particular, $\sigma \gg 1$ to ensure that the supergravity approximation is valid and as stated earlier, that perturbative corrections to the Kähler potential are negligible [207]. We also require $a\sigma \gg 1$ in order for the sub-leading nonperturbative corrections in equation (6.93) to be neglected [20]. Lastly, by equation (6.96) the Kähler modulus is stabilized in a controlled manner for exponentially small values of the GVW superpotentnial $W_0 \ll \mathcal{A}$ achieved through tuning the fluxes H_3, F_3 as in [226]. We discuss these last few points in more detail below.

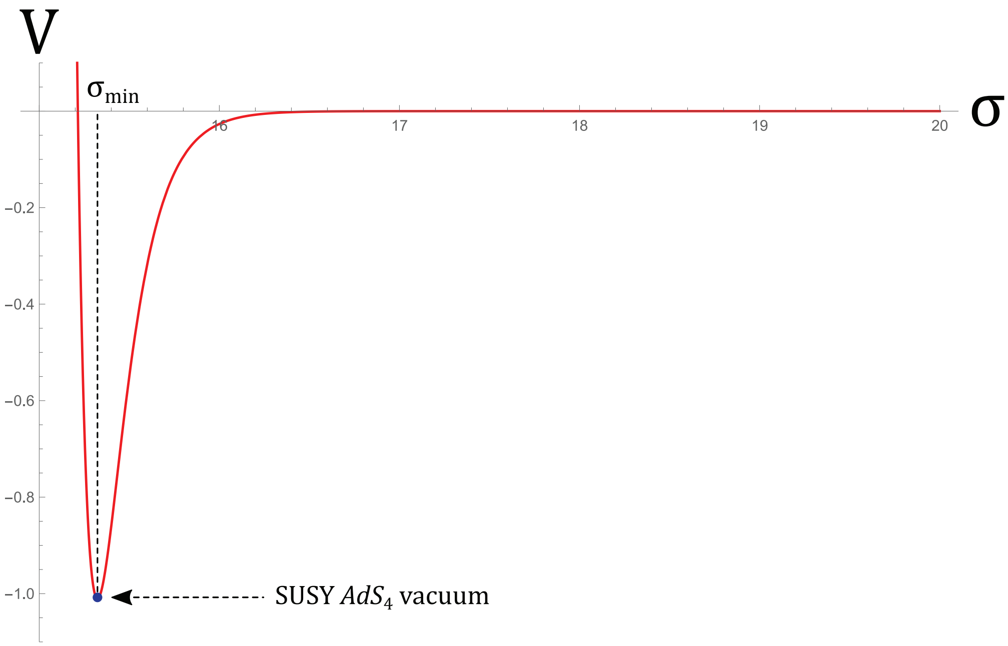

Figure 12: The potential (multiplied by an overall factor of 10^{44}) arising from the compactification of F-theory on a Calabi-Yau four-fold in the presence of flux. By tuning fluxes $W_0 = -10^{-20}$ and the contribution from non-perturbative effects as assumed large $\mathcal{A} = 10^{20}$. The steepness of the potential is well motivated to be $a = 2\pi$, which corresponds to a value of $p \approx 0.05 \ll 1$ defined in the single field scaling solution of section 3. The supersymmetric AdS minimum in this case is located at $\sigma_{min} \approx 15.3 \gg 1$.

The potential in equation (6.95) is plotted in figure 12. The ground state σ_{min} is supersymmetric ($D_i W = 0 \; \forall \; i$) since we have in addition $D_\rho W = 0$, and the ground state may be expressed in terms of W_0

$$W_0 = -\mathcal{A} e^{-a\sigma_{min}} \left(\frac{2}{3} a\sigma_{min} + 1 \right), \tag{6.96}$$

and by equation (6.94)

$$W(\sigma_{\rm min}) = -\frac{2}{3} \mathcal{A} a \sigma_{min} e^{-a\sigma_{min}}. \tag{6.97}$$

Thus the minimum of the potential given in equation (6.95) and hence the vacuum energy is

$$V_{Fmin} = -\frac{a^2 \mathcal{A}^2}{6\sigma_{min}} e^{-2a\sigma_{min}}, \tag{6.98}$$

and since the relevant portion of the potential and ground state are negative the only space-time permitted constrained by maximal symmetry is AdS_4.

We also remark, that this mechanism involves competition between the GVW flux superpotential given in equation (6.89) and the nonperturbative superpotentials given in equation (6.92), where the GVW flux superpotential is made small by fine-tuning fluxes in order to stabilize the volume; this addresses the Dine-Seiberg problem of runaway vacua [227].

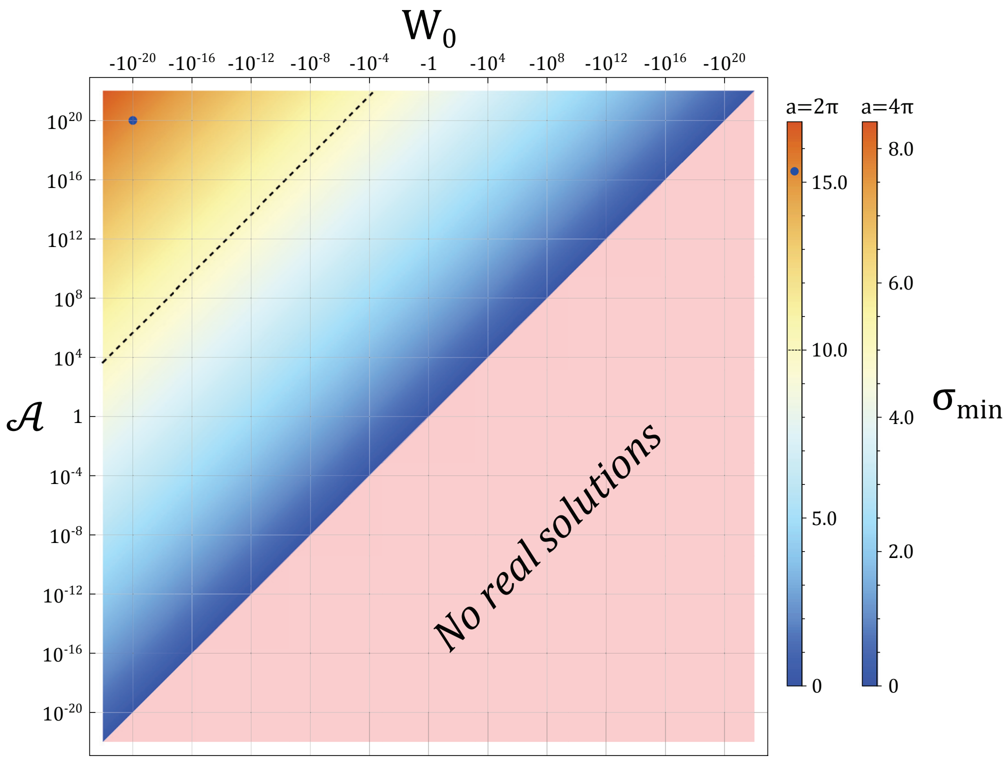

Figure 13: A density plot whose vertical and horizontal axes are logarithmically scaled, characterizing the magnitude of the supersymmetric minimum as a function of the GVW flux superpotential W_0 and the prefactor determining the contribution from nonperturbative effects $\mathcal{A}$, for fixed $a = 2\pi, 4\pi$. The blue point indicates the magnitude of the supersymmetric minimum resulting from the tuned values for W_0 and $\mathcal{A}$ in plotting figure 12. The dashed black line indicates the threshold in which the supersymmetric minima are larger than one order of magnitude above 1. The blue point and black dashed line are only present for the $a = 2\pi$ plot.

The exemplary potential provided in [207] was plotted for $W_0 = -10^{-4}$, $\mathcal{A} = 1$ and a value for the slope of the exponential $a = 0.01 \ll 1$. Here, we note that the nonperturbative contributions in equation (6.92) motivate a value for $a \geq 2\pi$. We argue that this is well suited for the ekpyrotic scenario, where saturating the bound corresponds to a value for

$p \approx 0.05$ in the single field case in four dimensions (see section 3). Taking $\mathcal{A}, a, W_0$ to all be real and W_0 to be negative, also generally ensures that we obtain a scalar potential featuring a negative exponential potential.

However, for perturbative corrections to the Kähler potential to be consistently neglected the field values should obey $\sigma \gg 1$, in other words it should be stabilized at large values [221]. Figure 13 displays the magnitude of σ_{min} as a function of the 'parameter' space of values $(W_0, \mathcal{A})$, obtained by solving equation (6.96) numerically. Taking W_0 and $\mathcal{A}$ to be real-valued, implies the bound $|\mathcal{A}| \geq |W_0|$, such that real-valued solutions exist for σ_{min}. When the aforementioned bound is saturated the location of σ_{min} is null. We inevitably conclude that for non-perturbative corrections to be leading, $|W_0|$ is forced to increasingly smaller negative values and $\mathcal{A}$ is forced to increasingly larger values. This behaviour is accentuated further with increasing a, as displayed in figure 13 by the secondary legend. Thus for larger values of the steepness of the exponential, a higher degree of fine tuning of integral fluxes is required for W_0 [221].

Discussion

We have argued here that ekpyrotic cosmology is at the very least an interesting alternative to the inflationary paradigm by describing how it addresses the fundamental problems that originally plagued standard big bang cosmology.

From the four dimensional quantum field theory perspective minimally coupled to classical Einstein gravity, an ekpyrotic phase of contraction dominated by a single scalar matter field with Minkowski vacuum initial conditions generically predicts a strongly blue tilted curvature perturbation spectrum in contention with current observations. Thus we argue that in order to be consistent with current observations, it seems that the model should be generalized to two scalar matter fields or deviate slightly from a purely exponential potential. We argue this explicitly by generalizing to two scalar matter fields, and show that an approximately scale invariant spectrum of entropy perturbations may be produced. We then argue that this scale invariant spectrum may be transferred to the curvature perturbation spectrum, since the entropy perturbations may act as a source term so long as there is a nonlinear curve traversed in scalar field space. We then study a specific mechanism in which this transference may be achieved, allowing agreement with current observations. It is important to note that the original single field theory thus should be modified in order to be consistent with observations. Many will view this as a contrivance to fit the data, and rightly so, however one should also recall the numerous times this was done for inflation.

At the non-linear level, we introduce the covariant description of cosmological perturbations. In future work, this framework will prove useful to calculate explicitly the non-Gaussianities produced both during the ekpyrotic phase of contraction as well as during the specific phase in which the curvature perturbation spectrum is inherited from the entropy perturbation spectrum. Although we have not done the explicit calculations in this thesis, they have been previously explored and we remark that they have been shown to be markedly more prominent in ekpyrotic models than in general inflationary models [44]. We have also argued that the predictions on the tensor spectrum is independent of the number of scalar matter fields so long as the model admits approximately the same behaviour for the scale factor with respect to the single field case. We show that the generic prediction is a strongly blue tilted spectrum, with the tensor spectral index $n_T \approx 2$. Thus we provide two sources of testable predictions which differ from that of inflationary models, as non-Gaussianities of inflationary models tend to be suppressed and the tensor spectrum is approximately scale invariant.

From a more fundamental perspective the ekpyrotic model is quite naturally embedded in string theory in so far as negative exponential potentials governing moduli fields are commonplace due to nonperturbative effects. We provide the explicit example of open supermembranes extending between wrapped M-branes at low energy in heterotic M-theory. Further, we motivate two instances in which the ekpyrotic model may be introduced, within the context of M-theory compactified on an $S^1/\mathbb{Z}_2$ orbifold or F-theory compactified on a Calabi-Yau fourfold in the presence of nontrivial three-form flux.

We note that the proposition of the ekpyrotic scenario in type IIB supergravity compactified on a warped Calabi-Yau threefold has not yet been put forward, and is a new idea. This familiar construction stabilizes all complex structure moduli and axiodilaton, with the Kähler modulus acting as the ekpyrotic field. In fact this stabilization procedure admits similar solutions for multiple Kähler moduli and thus may be an interesting embedding for multi-field models. We do however conclude that in order for non-perturbative contributions to the superpotential to be leading corrections, the integral three-form fluxes must be finely tuned which worsens as the steepness of the ekpyrotic potential increases.

Acknowledgements

I am eternally grateful to R. Brandenberger, V. Kaspi and L. Trottier for granting me the wonderful opportunity through the McGill Space Institute graduate fellowship, to explore high energy theoretical physics, cosmology and astrophysics. I am also especially indebted to K. Dasgupta, H. Bernardo, A. Maloney, V. Errasti Diez, S. Caron-Huot, O. Hernandez, R. Namba, E. Ferreira, B. Cyr, and G. Franzmann for their guidance and mentorship, and commend each of them for their professionalism and dedication to their respective fields of research. My thanks also extends to fellow graduate students and colleagues particularly within the cosmology and high energy groups, the McGill Space Institute and MGAPS committee. I also thank J.-L. Lehners for useful discussions related to non-gaussianities. On a more personal note, I am the most grateful to my parents P. & T. Muscat; I carry you in the highest regard, respect you the most and am most grateful for you, as you have provided and shown me more than I will ever have the ability to appropriately thank you for. I am also particularly grateful for my partner C. Strutynski, my second family S.M.C.J. Roscanu, countless close friends, and those of the past, for the cherished memories created throughout our journey through life together in search of the hardest laughter and most interesting answers. This work has largely been financially supported by the McGill Space Institute graduate fellowship, as well as NSERC SAP grant.

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