ABSTRACT

WIGNER SUPERMULTIPELT BASES AND COUPLING COEFFICIENTS

Khursheed Ahmed. Department of Physics
Ph.D. Thesis McGill University

We construct two sets of bases for the Wigner Supermultiplet Scheme — the non-canonical chain
SU(4) ⊃ SU(2) × SU(2). The first ones are analogous to the SU(3) ⊃ O(3) basis states of Bargmann and Moshinsky; they are defined by identifying them with products of powers of certain elementary multiplets. The second are the analogs of the SU(3) ⊃ O(3) states of Elliott; they are obtained by projecting good SU(2) × SU(2) states out of certain simple intrinsic states. Clebsch-Gordan and Wigner coefficients are derived for coupling of states for certain classes of representations.
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DE WIGNER

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AND COUPLING COEFFICIENTS

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states for certain classes of representations.
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INTRODUCTION

§A Wigner Supermultiplet Scheme

In 1937 Wigner\textsuperscript{1}) pointed out that nuclear interactions are approximately invariant under the transformations of SU(4), the special unitary group in four dimensions; the implied identification of nuclear levels with bases of irreducible representations of SU(4) is known as the Wigner supermultiplet scheme. The SU(4) transformations act on the spin and isospin part of the nuclear state; the SU(2) transformations of spin and isospin form an SU(2) x SU(2) subgroup of SU(4). The states which provide a basis for an irreducible representation of SU(4) should be degenerate and form a "supermultiplet"; they all have the same spatial wave function. A supermultiplet contains all values of spin and isospin which are allowed in the particular SU(4) representation and generally contains states of various nuclei.

The irreducible representations of the group SU(4) can be specified by the permutation symmetries of the n-nucleon spin-isospin functions. These symmetries are represented by a Young diagram of at most four rows because there are only four different single nucleon states. The Young diagram is characterized by the partition

\[ n_1 \geq n_2 \geq n_3 \geq n_4 \geq 0, \quad n = n_1 + n_2 + n_3 + n_4 \]  \hfill (I.1)

These are the numbers of boxes in the four rows of the Young diagram. We can put \( n_1 \) nucleons in the \( m_s = \frac{1}{2} \), \( m_t = \frac{1}{2} \)
state, \( n_2 \) nucleons in the \( m_s = \frac{1}{2}, m_t = \frac{1}{2} \) state, \( n_3 \) nucleons in the \( m_s = -\frac{1}{2}, m_t = \frac{1}{2} \) state, and \( n_4 \) nucleons in the \( m_s = -\frac{1}{2}, m_t = -\frac{1}{2} \) state. The resulting state has total \( M_s \) and \( M_t \)

\[
M_s = \frac{1}{2} (n_1 + n_2 - n_3 - n_4) = P
\]

(I.2a)

\[
M_t = \frac{1}{2} (n_1 - n_2 + n_3 - n_4) = P'
\]

(I.2b)

and if we define an additional operator

\[
\omega_{oo} = 2 \sum_{i=1}^{n} S_0^{(i)} T_0^{(i)}
\]

(I.3)

which commutes with the total spin \( S_0 \) and total isospin \( T_0 \), and gives the difference of total proton and total neutron spin. The eigenvalues of \( \omega_{oo} \) for the above state is given by

\[
\omega_{oo} = \frac{1}{2} (n_1 - n_2 - n_3 + n_4) = P''
\]

(I.4)

This state defines a unique irreducible representation of \( SU(4) \), that whose maximum \( M_s \) value is \( P \), whose maximum \( M_t \) value for a state with \( M_s = P \) is \( P' \), and whose maximum \( \omega_{oo} \) value for a state with \( M_s = P, M_t = P' \) is \( P'' \). The labels \((P, P', P'')\) for \( SU(4) \) irreducible representations, or supermultiplets, were introduced by Wigner\(^1\).

The ground state of a nucleus is expected to be characterized by maximum symmetry of its spatial wave function under exchange of the nucleons, and hence, because of Pauli
principle with maximum asymmetry of its spin-isospin state. It is clear from (I.2 and I.4) that \( P, P', P'' \) remain unchanged if we add to every \( n_i \) the same number. Therefore, if \( n \) is increased by 4 the same supermultiplet recurs.

The maximum internal symmetry will occur when the \( n_i \) are as nearly equal to each other as possible for a given value of \( n \). The highest symmetry can thus be obtained for \( n = 4n_1 \) and \( n_1 = n_2 = n_3 = n_4 \). In this case \( P = P' = P'' = 0 \) and the supermultiplet is denoted by \((0,0,0)\); there is only one spin value \( S = 0 \) and one isospin value \( T = 0 \). In the case when \( n \) is even but not a multiple of 4, the highest symmetry is obtained in the case \( n_1 = n_2 \) and \( n_3 = n_4 = n_1 - 1 \) and the corresponding supermultiplet is characterized by \( P = 1, P' = P'' = 0 \) and denoted by \((1,0,0)\). The possible values of spin and isospin are accordingly \( S = 1, T = 0 \) and \( S = 0, T = 1 \). The highest spatial symmetry for odd \( n \) nuclei is different for \( n = 4m + 1 \) and \( n = 4m + 3 \).

In the first case, the highest symmetry is obtained with \( n_1 = m_1 + 1 \) and \( n_2 = n_3 = n_4 = m_1 \), i.e. \( P = P' = P'' = \frac{1}{2} \). The corresponding supermultiplet is denoted by \((\frac{1}{2},\frac{1}{2},\frac{1}{2})\). In the second case, with \( n = 4m + 3 \), the highest symmetry is obtained with \( n_1 = n_2 = n_3 = m + 1 \) and \( n_4 = m \). Thus the lowest supermultiplet is given by \( P = P' = \frac{1}{2}, P'' = \frac{1}{2} \) or \((\frac{1}{2},\frac{1}{2},\frac{1}{2})\). In both cases, the possible values of spin and isospin are \( S = \frac{1}{2}, T = \frac{1}{2} \).

We now look at the nucleon-nucleon interaction and examine to what extent it is invariant under SU(4) transformations.
The spin-isospin part of the two nucleon potential must have the form

\[ V = V_c + V_s (\vec{\sigma}_1 \cdot \vec{\sigma}_2) + V_t (\vec{\tau}_1 \cdot \vec{\tau}_2) + V_{st} (\vec{\sigma}_1 \cdot \vec{\sigma}_2)(\vec{\tau}_1 \cdot \vec{\tau}_2) \]  \hspace{1cm} (1.5)

if it is invariant under spin and isospin transformations separately. The V's are functions of \( r_{12} \). We have omitted tensor and spin-orbit forces, which obviously violate SU(4) invariance. It can be shown that the requirement of SU(4) invariance then implies \( V_s = V_t = V_{st} \). Then (1.5) can be written as

\[ V = V_c + V_s \left\{ \frac{\vec{\sigma}_1 \cdot \vec{\sigma}_2}{2} + \frac{\vec{\tau}_1 \cdot \vec{\tau}_2}{2} + (\vec{\sigma}_1 \cdot \vec{\sigma}_2)(\vec{\tau}_1 \cdot \vec{\tau}_2) \right\} \]  \hspace{1cm} (1.6)

which states that \( V \) depends only on the symmetry of the internal state, i.e., has one value for \( S=1, T=0 \) and \( S=0, T=1 \) and a second, independent, value for \( S=0, T=0 \) and \( S=1, T=1 \). This criterion for SU(4) invariance is met quite well by the phenomenological two-nucleon potential\(^2\) if the tensor force is represented by an "equivalent" central force. The \( S=1, T=0 \) state (deuteron) and \( S=0, T=1 \) (almost bound) see nearly the same potential; the \( S=0, T=0 \) and \( S=1, T=1 \) potentials are very small outside the hard core radius and nearly equal.

This argument for SU(4) invariance breaks down for heavier nuclei. The tensor force becomes less effective
than its "equivalent" central force; spin-orbit forces become important; Coulomb forces violate isospin, and hence SU(4) invariance.

§B Group-theoretical problem of the Supermultiplet Scheme

Wigner observed that a single nucleon has the following four spin-isospin states

\[ |1\rangle = \left| \frac{1}{2}, \frac{1}{2} \right>, \quad |2\rangle = \left| \frac{1}{2}, -\frac{1}{2} \right> \]
\[ |3\rangle = \left| -\frac{1}{2}, \frac{1}{2} \right>, \quad |4\rangle = \left| -\frac{1}{2}, -\frac{1}{2} \right> \]

in the notation \( |m_s, m_t\rangle \).

The total spin \( S \) and the total isospin \( T \) operators, for a nucleus of mass number \( A \), can be written as

\[ S_i = \sum_{j=1}^{\frac{A}{2}} S_i^j, \quad i = 1, 2, 3 \]
\[ T_k = \sum_{j=1}^{\frac{A}{2}} T_k^j, \quad k = 1, 2, 3 \]

respectively. The SU(2) transformations of spin and of isospin through their direct product form the group SU(2)xSU(2) which is locally isomorphic with the orthogonal group in four real-dimensions, O(4).

SU(4) contains all physical unitary transformations of the state (I.7).
The other nine generators of the Supermultiplet scheme are

\[ U_{ik} = 2 \sum_{j=1}^{n} S_i^j T_k^j, \quad i, k = 1, 2, 3 \] (I.9)

The fifteen generators \( S_i, T_i \) and \( U_{ik} \) form a closed set and together they generate the \( SU(4) \) group.

In the language of group-theory the Wigner Supermultiplet scheme corresponds to the study of the irreducible representations of the group chain \( SU(4) \supset SU(2) \times SU(2) \). This unfortunately is not a canonical chain such as \( U(4) \supset U(3) \supset U(2) \supset U(1) \).

The Racah algebra for a canonical chain of unitary groups has been extensively studied and very elegant and powerful techniques have been developed, chiefly by Biedenharn and co-workers\(^3\),\(^4\) However, these methods cannot be used for the \( SU(4) \supset SU(2) \times SU(2) \) group chain.

The problem is complicated by lack of sufficient labels to specify the bases of irreducible representations. In particular, one needs six internal labels to specify the states of irreducible representations of \( SU(4) \), while the subgroup \( SU(2) \times SU(2) \) provides only four labels \( S, T, M_s, M_t \) two labels short of the required number. This circumstance is known as the internal degeneracy problem or the internal labelling problem and is often encountered when group theory is used in nuclear physics. (Other examples are Elliott's
SU(3) ⇒ O(3) model\(^5\) of nucleus and the O(5) ⇒ SU(2)
nuclear seniority model\(^6\); in both cases the subgroups
provide one label less than the required number).

A number of recent papers discuss SU(4) bases in the
SU(2) × SU(2) decomposition. Hecht and Pang\(^7\) give a detailed
account of some special irreducible representations for which
the internal degeneracy problem does not occur. They also give
a large class of Wigner and Racah coefficients for these
particular cases. Brunet and Resnikoff\(^8\) and Draayer\(^9\)
give SU(2) × SU(2) bases for the general irreducible representa-
tions of SU(4). Brunet and Resnikoff use powers of certain
elementary factors to generate SU(2) × SU(2) multiplets in a
manner similar to Bargmann and Moshinsky\(^10\) in the treatment
of SU(3) ⇒ O(3) chain. Draayer follows a method parallel to
the one introduced by Elliott\(^5\) for the SU(3) ⇒ O(3) group
chain. He projects SU(2) × SU(2) states out of certain intrinsic
states, whose \(m_s\) and \(m_t\) quantum numbers provide the missing
labels solving the degeneracy problem.

In the present work we solve the degeneracy problem
following techniques developed by Sharp and von Baeyer\(^11\)
and Sharp and Lam\(^12\), which involve the use of certain
elementary factors as well as projection techniques. In
this respect our work, even though carried out independently,
has some common features with the works of Brunet and Resnikoff,
and Draayer.

The plan of this thesis is as follows. In the next
section we discuss the validity of the Supermultiplet scheme. Chapter I is devoted to the description of SU(4) group with SU(2) x SU(2) decomposition, the internal degeneracy problem and the elementary factors in detail. Polynomial bases for the simple representations [(λ, 0, 0), (0, μ, 0) and (0, 0, ν) types] are developed in chapter II. These simple representations do not involve any labelling problem. Their reduced matrix elements are also worked out. In chapter III we discuss the general irreducible representation (λ, μ, ν) and develop two independent set of bases. The first, using elementary factors, are called Bargmann-Moshinsky states and the second, using projection techniques are referred to as Elliott states. Lastly, in chapter IV we calculate some simple Wigner coefficients for the Supermultiplet scheme.

§C Validity of the Supermultiplet Scheme

Beginning about 1963 there has been a revival of interest in the Supermultiplet Scheme and its predictions have been tested against empirical masses and beta transition rates for a large number of nuclei.\(^{13}\)

Franzini and Radicati\(^{13}\) used the simple mass formula given by Wigner and Feenberg\(^1\). According to this formula the excitation energy of a supermultiplet of mass number A depends linearly on the second order Casimir operator \(C_2\) of SU(4) (the expression I.6, summed over pairs of nucleons, is proportional to \(C_2\), apart from an additive constant). In terms
of the $P$'s, $C_2$ is given by

$$C_2 = P^2 + P'p + P'' + 2P + 2P'$$  \hfill (I.10)

The data analyzed by Franzini and Radicati showed that the simple mass formula is quite accurate even for fairly large values of $A$. Burdet, Maguin and Partensky\textsuperscript{13)} used a more general mass formula involving the higher Casimir operators $C_3$ and $C_4$ which contain three and four nucleon operators; they found agreement with experimental data even for very heavy nuclei. Recently French and Parikh\textsuperscript{13)} have discussed the validity of SU(4) symmetry by the distribution method, in which one considers the energy distribution of the intensities of the different representations. The situation with regard to nuclear masses is summed up by Radicati in his recent review\textsuperscript{13)}:

"We can conclude that the SU(4) invariant forces must play a significant role in determining the average stability of nuclear systems. This of course does not mean that each nuclear state appears at exactly the energy predicted by the mass formula. The SU(4)-violating forces which remove the degeneracy of each supermultiplet necessarily complicate the picture. On the average, however, the position and order of the supermultiplets appear to be those predicted by Wigner".

The hadron operators, whose coupling to leptons is responsible for nuclear beta processes, are generators of SU(4). Thus the isospin operators $T_\pm$, arising from the vector inter-
action, are involved in Fermi transitions while the operators \( \gamma^0 \gamma^\pm \) summed over nucleons, arising from the axial coupling, are responsible for Gamow-Teller transitions. Since only SU(4) generators are involved, beta transitions should be possible only between states of the same supermultiplet according to the supermultiplet model.

Radicati\textsuperscript{13} has summarized the empirical evidence on nuclei up to \( A=26 \) for Gamow-Teller transitions (Fermi transitions test isospin invariance only and not specifically SU(4)). Roughly speaking, transitions inside the same supermultiplet have log ft values around 3.5 while for those between different supermultiplets log ft values exceed 4.5. All transitions are rather slow beyond \( A=26 \) where mixing is expected to be large. Thus it appears that the squares of matrix elements of transitions forbidden by the supermultiplet model are reduced by about 1/100 compared to allowed ones. The evidence favours SU(4) as an approximate symmetry for eight nuclei.

Recently Krüger and vanLeuven\textsuperscript{13} and also Cannata, Leonardi and Rosa-Clot\textsuperscript{13} have used \( \mu \)-capture in an interesting test of SU(4). It turns out that the first forbidden contributions of vector and axial currents to the process can be expressed in terms of operators which are components of an irreducible SU(4) tensor transforming according to the same representation as the generators. They can all be expressed in terms of a single reduced matrix element which can in turn be related to El photoabsorption. Experimental results on nuclei with \( A \) up to
about 40 are in rough agreement with SU(4) symmetry; SU(4) breaking must be taken into account for accuracy better than about 15%.

To sum up, we quote Radicati's opinion that SU(4) today is more in accord with known facts than when it was proposed 34 years ago.
Consider a finite transformation $U$ in a four-dimensional complex space, which transforms a four-component vector $\psi$ into another vector $\psi'$,

\[ \psi \rightarrow \psi' = U\psi. \]

In order that the transformation be unitary, we must have

\[ U^*U = 1, \]

and it is said to be special-unitary if we impose the further condition that

\[ \det U = 1. \]

The group associated with continuous unitary unimodular transformations $U$, in four-dimensional space, is called $SU(4)^*$. We can write

\[ U = \exp(i(\vec{\alpha} \cdot \vec{\lambda})) \quad (1.1) \]

*A good treatment of rank three groups including $SU(4)$ is given by Konuma, Shima and Wada [14]. Sen [15] discusses commutation relations and matrix elements of $SU(4)$ generators.*
where \( \lambda \) is a set of 4x4 hermitian matrices, called the generators of the group, and \( \vec{a} \) is a real vector. For an infinitesimal transformation, Eq. (1.1) can be approximated by

\[
U = 1 + \frac{i}{2} \delta \vec{a} \cdot \lambda
\]

(1.2)

The condition that

\[
\det U = 1 + \frac{1}{2} i \delta \vec{a} \cdot \text{Tr} \lambda + O(\delta \vec{a})^2 \equiv 1
\]

(1.3)

requires that \( \lambda_j \) be traceless matrices. There are \( 4^2 - 1 = 15 \) independent 4x4 matrices. They can be chosen as follows:

\[
(A_{ij})_{mn} = \delta_{im} \delta_{jn} - \frac{1}{4} \delta_{ij} \delta_{mn}, \quad i, j, m, n = 1, 2, 3, 4
\]

(1.4)

The \( A_{ij} \) obey the following commutation relations:

\[
[A_{ij}, A_{kl}] = A_{ih} \delta_{kj} - A_{kj} \delta_{ih}
\]

(1.5)

The four diagonal matrices are not all independent but are subject to the condition:

\[
A_{11} + A_{22} + A_{33} + A_{44} = 0.
\]

(1.6)

We choose the three independent diagonal matrices (the diagonal generators of SU(4), a rank three group) as follows:

\[
\sigma_\circ = \frac{1}{2} (A_{11} - A_{22} + A_{33} + A_{44}), \quad t_\circ = \frac{1}{2} (A_{11} - A_{22} - A_{33} + A_{44})
\]

\[
U_{\circ \circ} = \frac{1}{2} (A_{11} + A_{22} - A_{33} - A_{44})
\]

(1.7)
The Wigner Supermultiplet Scheme dictates that the remaining generators be chosen such that they form two commuting SU(2) groups, namely the s-spin and the t-spin multiplets, satisfying the commutation relations

\[
\begin{align*}
[S_+ , S_-] &= \pm S_+ , \\
[S_+, S_-] &= 2 S_+ , \\
[t_+, t_-] &= \pm t_+ , \\
[t_+, t_-] &= 2 t_+ , \\
[S_i, t_j] &= 0 , \quad i, j = \pm, \text{or } 0.
\end{align*}
\]  

(1.8)

It is easily verified that the following choice of \( s_\pm, t_\pm \), with \( s_0, t_0 \) defined above satisfy these commutation relations:

\[
\begin{align*}
S_+ &= A_{14} + A_{32} , \\
S_- &= A_{41} + A_{23} , \\
t_+ &= A_{42} + A_{13} , \\
t_- &= A_{24} + A_{31} .
\end{align*}
\]  

(1.9)

The remaining nine generators transform under SU(2)xSU(2) as a \((1,1)\) tensor, with the components \( u_{m,n}^{1,1} \) given by

\[
\begin{align*}
U_{1,1} &= A_{12} , \\
U_{1,0} &= \frac{1}{\sqrt{2}} (A_{32} - A_{14}) , \\
U_{1,-1} &= -A_{34} , \\
U_{0,1} &= \frac{1}{\sqrt{2}} (A_{42} - A_{13}) , \\
U_{0,0} &= \frac{1}{\sqrt{2}} (A_{11} + A_{22} - A_{33} - A_{44}) , \\
U_{0,-1} &= \frac{1}{\sqrt{2}} (A_{31} - A_{24}) , \\
U_{-1,1} &= -A_{43} , \\
U_{-1,0} &= \frac{1}{\sqrt{2}} (A_{41} - A_{23}) , \\
U_{-1,-1} &= A_{21} .
\end{align*}
\]  

(1.10)

They are related to the generators of reference 16) in Appendix A.

The non-vanishing commutation relations of these generators can be derived with the help of Eq.(1.5) and are given by:
In tensor notation, Eq. (1.12) can be written as:

\[
[s_{\pm}, u_{mn}] = \sqrt{(1\pm 1)(2\pm 2)} \ u_{m\pm 1,n}, \quad [s_0, u_{mn}] = m \ u_{mn},
\]
\[
[t_{\pm}, u_{mn}] = \sqrt{(1\pm 1)(2\pm 2)} \ u_{m,n\pm 1}, \quad [t_0, u_{mn}] = n \ u_{mn}.
\]  

(1.11)

\[
[u_{l1}, u_{-10}] = -\frac{i}{\sqrt{2}} t_+, \quad [u_{-11}, u_{01}] = -\frac{i}{\sqrt{2}} s_+,
\]
\[
[u_{l1}, u_{00}] = -\frac{i}{\sqrt{2}} s_+, \quad [u_{-11}, u_{00}] = -\frac{i}{\sqrt{2}} t_+.
\]  

(1.12)

where \( s_{\pm} = \frac{1}{\sqrt{2}} s_{\pm}, s_0 \) and \( t_{\pm} = \frac{1}{\sqrt{2}} t_{\pm}, t_0 \) are components of the two spherical tensors, which transform under SU(2) x SU(2) as (1,0) and (0,1) respectively. The symbols in conical brackets in Eq. (1.13) are ordinary Clebsch-Gordan coefficients in the notation \( \langle j, j \mid m, m, m \rangle \).

This notation is followed throughout this thesis.
§1.2 Root Diagram

The root diagram gives the effect of generators on the eigenvalues of the diagonal generators. The properties of roots, root diagrams, etc., and ways of finding them for semi-simple groups are described in the classical work of Racah. The paper of Behrends et al. is also very useful. We shall summarize the properties of roots and outline the procedure followed to determine the roots of SU(4).

If \( H_i \) (i=1,2,\ldots,\lambda) are the mutually commuting generators and \( E_\alpha \) the remaining generators of a semi-simple group, of rank \( \lambda \), then we have the commutation relations:

\[
[ H_i, H_j ] = 0, \quad i,j = 1,2,\ldots,\lambda
\]

\[
[ H_i, E_\alpha ] = \tau_i(\alpha) \ E_\alpha
\]

where \( r_i(\alpha) \) is the \( i \)th component of the root \( \vec{r}(\alpha) \), that is, the \( r_i(\alpha) \) form a "vector" in an \( \lambda \)-dimensional root space. If \( \vec{r}(\alpha) \) is a root, then \( -\vec{r}(\alpha) = \vec{r}(\alpha) \) is also a root. It is possible to normalize the \( H_i \) such that

\[
\sum_\alpha \tau_i(\alpha) \tau_j(\alpha) = \delta_{ij}
\]

The graphical representation of the root vectors gives the root diagram, (also called vector diagram). For SU(\( n \)) of rank \( (n-1) \) there are \( n(n-1) \) non-zero roots and of these \( n-1 \) are
linearly independent. The multiplicity of the zero root is (n-1).

The following theorem plays a central role in the construction of the root diagram.

If \( \mathbf{r}(\alpha) \) and \( \mathbf{r}(\beta) \) are two root vectors, then

\[
2 \frac{\mathbf{r}(\alpha) \cdot \mathbf{r}(\beta)}{\mathbf{r}(\alpha) \cdot \mathbf{r}(\alpha)} \mathbf{r}(\beta)
\]

is an integer and \( \mathbf{r}(\beta) - 2 \frac{\mathbf{r}(\alpha) \cdot \mathbf{r}(\beta)}{\mathbf{r}(\alpha) \cdot \mathbf{r}(\alpha)} \mathbf{r}(\alpha) \) is also a root.

Graphically this means that a new root can be obtained from \( \mathbf{r}(\beta) \) by reflection with respect to a hyperplane perpendicular to \( \mathbf{r}(\alpha) \). If \( \phi \) be the angle between \( \mathbf{r}(\alpha) \) and \( \mathbf{r}(\beta) \) then it follows from the theorem that

\[
\mathbf{r}(\alpha) \cdot \mathbf{r}(\beta) = \frac{1}{2} m |\mathbf{r}(\alpha)|^2 = \frac{1}{2} n |\mathbf{r}(\beta)|^2,
\]

where \( m \) and \( n \) are integers, and that

\[
\cos^2 \phi = \frac{1}{4} m n
\]

Eq. (1.16) also determines the ratio of lengths of \( \mathbf{r}(\alpha) \) and \( \mathbf{r}(\beta) \).

It turns out that for SU(4) there are twelve non-zero roots, all of equal lengths. The angle between any two adjacent roots is 60° and the three-dimensional root diagram is shown in Fig. (1.1). The components of roots, in the notation \([r_1(H_4), r_2(H_4), r_3(H_4)]\) are

\[
\mathbf{r}(1) = \left( \frac{1}{2}, 0, 0 \right), \quad \mathbf{r}(4) = \left( \frac{1}{4}, \frac{1}{4 \sqrt{3}}, \frac{1}{12} \right),
\]

\[
\mathbf{r}(2) = \left( \frac{1}{4}, \frac{\sqrt{3}}{4}, 0 \right), \quad \mathbf{r}(5) = \left( -\frac{1}{4}, \frac{1}{4 \sqrt{3}}, \frac{1}{12} \right),
\]

\[
\mathbf{r}(3) = \left( -\frac{1}{4}, \frac{\sqrt{3}}{4}, 0 \right), \quad \mathbf{r}(6) = \left( 0, -\frac{1}{2 \sqrt{3}}, \frac{1}{\sqrt{6}} \right),
\]
Fig. (1.1) The Root Diagram for SU(4) ≃ SU(3) chain.
(The r(H₃) axis points upwards, perpendicular to the plane of the paper).

and six others obtained from these using the relation
\[ \hat{r}(-\alpha) = -\hat{r}(\alpha). \]

The root diagram, Fig. (1.1), gives the relative orientation of the generators \( E_{±1}, E_{±2}, \ldots, E_{±6} \), plotted against the axes of the eigenvalues of the diagonal generators \( H_1, H_2, H_3 \). It corresponds to the canonical decomposition \( SU(4) ≃ SU(3) \).

In order to determine the root diagram for \( SU(4) ≃ SU(2) \times SU(2) \) decomposition corresponding to Wigner Supermultiplet Scheme, we find that our generators, Eqs. (1.7, 1.9, 1.10), are related to the generators \( H_i, E_\alpha \) as follows:
It is now easy to obtain the supermultiplet "root diagram" from the $SU(4) \supset SU(3)$ root diagram using vector additions of the roots with the help of Eq.(1.19); it is a projection on the $m,n$ plane of the canonical root diagram of Fig.(1.1). It is shown in Fig.(1.2).

\[\begin{align*}
S_+ &= (E_4 + E_3), \\
S_- &= (E_4 - E_3), \\
S_0 &= \frac{1}{\sqrt{2}} H_1 - \frac{1}{\sqrt{6}} H_2 + \frac{1}{\sqrt{6}} H_3, \\
T_+ &= (E_5 + E_2), \\
T_- &= (E_5 - E_2), \\
T_0 &= \frac{1}{\sqrt{2}} H_1 + \frac{1}{\sqrt{6}} H_2 - \frac{1}{\sqrt{6}} H_3, \\
U_{11} &= \sqrt{2} E_1, \\
U_{10} &= (E_3 - E_4), \\
U_{1-1} &= -\sqrt{2} E_6, \\
U_{01} &= (E_5 - E_2), \\
U_{oo} &= \frac{1}{\sqrt{3}} (\sqrt{2} H_2 + H_3), \\
U_{o-1} &= (E_2 - E_5) , \\
U_{-11} &= -\sqrt{2} E_6, \\
U_{-10} &= (E_4 - E_3), \\
U_{-1-1} &= \sqrt{2} E_{-1}.
\end{align*}\]  

(1.19)

Fig.(1.2) Root diagram for the supermultiplet scheme
(m,n are the eigenvalues of $S_0$, $T_0$ respectively)
§1.3 Fundamental Representations, Weights and Variables

An irreducible representation (hereafter abbreviated as IR) of SU(4) is characterized by a set of three non-negative integers \((\lambda, \mu, \nu)\), which are the numbers of columns containing, respectively, one, two and three rows of boxes in the Young's diagram for the IR, as shown in Fig. (1.3).

![Young's Diagram](image)

Fig. (1.3)

The set \((\lambda, \mu, \nu)\) is more convenient for our present use since we propose to work in Cartan bases; it is related to the characteristic numbers \((P, P', P'')\), introduced by Wigner\(^1\) by:

\[
\begin{align*}
\lambda &= P' + P'' \\
\mu &= P - P' \\
\nu &= P' - P''
\end{align*}
\]  

(1.20)

The \((P, P', P'')\) are associated with the maximum values of the operators \(s_0\), \(t_0\), and \(u_\infty\) for a given supermultiplet. \(P\) is the maximum value of \(s_0\) in the supermultiplet (or the IR), \(P'\) is the maximum value of \(t_0\) for a state with \(s_0 = P\), and \(P''\) is the maximum value of \(u_\infty\) for a state with \(s_0 = P\), \(t_0 = P'\).
The dimensionality of the \((\lambda, \mu, \nu)\) representation is given by:

\[
D_{\lambda\mu\nu} = \frac{1}{12} (\lambda+1)(\mu+1)(\nu+1)(\lambda+\mu+2)(\mu+\nu+2)(\lambda+\mu+\nu+3)
\]  
(1.21)

Having specified the IR we can go on to discuss the weights and the weight diagrams.

By definition \(^{18,19}\), the generators \(H_i (i=1,...,\ell)\) commute among themselves, so that it is possible to diagonalize the \(\ell\) matrices simultaneously. If \(\psi\) is an \(N\)-component basis vector, then we have

\[
H_i \psi = m_i \psi.
\]

The \(\ell\)-component vector \(\vec{m} = (m_1, m_2, \ldots, m_\ell)\) is called the weight and the \(\ell\)-dimensional space is called the weight-space.

According to a theorem by Cartan\(^{20}\) (see also reference \(^{19}, p.7\)), for every simple group of rank \(\ell\) there are \(\ell\) fundamental weights which are called dominant weights, \(\vec{\lambda}(1), \ldots, \vec{\lambda}(\ell)\), such that any other dominant weight \(\vec{M}\) is a linear combination

\[
\vec{M} = \sum_{i=1}^{\ell} \lambda_i \vec{M}(i) = \vec{M} (\lambda_1, \lambda_2, \ldots, \lambda_\ell)
\]

(where \(\lambda_i\) are non-negative integral coefficients), and that there exist \(\ell\) fundamental IR's, each characterized by a fundamental dominant weight \(\vec{\lambda}(i)\).
SU(4), a group of rank three, has three fundamental IR's. They are the two conjugate (unequal) IR's (1,0,0) and (0,0,1), both of dimension four, called the quartet and the antiquartet respectively; and the self-conjugate IR (0,1,0), of dimension six, called the sextet. All the other IR's \((\lambda, \mu, \nu)\) can be generated by taking direct products of suitable powers of these fundamental IR's.

The weight diagrams of the fundamental IR's are shown in Fig.(1.4).

Fig.(1.4) Fundamental Irreducible Representations

The \((1,0,0)\) and \((0,0,1)\) quartets each consist of a single SU(2)xSU(2) multiplet \((\frac{1}{2}, \frac{1}{2})\), in the notation \((s,t)\). Physical realizations of these are the nucleon and the antinucleon quartets, respectively (cf.Eq.(1.1)). The \((0,1,0)\) sextet consists of two SU(2)xSU(2) multiplets \((1,0)\) and
(0,1). Physically they may be associated with the deuteron state \((s=1, t=0)\), together with the (almost bound) singlet-spin states \((s=0, t=1)\), of low energy nucleon-nucleon scattering.

We represent the general state \(^+\) by

\[
| \lambda \mu \nu; \alpha_1 \alpha_2; s, t, m_n \rangle
\]

The \((1,0,0)\) and the \((0,0,1)\) variables are defined as

\[
\eta = |100; \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \rangle, \quad \theta = |100; \frac{1}{2}, -\frac{1}{2}, \frac{1}{2} \rangle, \quad (1.22)
\]

\[
\zeta = |100; \frac{1}{2}, \frac{1}{2}, -\frac{1}{2} \rangle, \quad \xi = |100; \frac{1}{2}, \frac{1}{2}, -\frac{1}{2} \rangle,
\]

and,

\[
\zeta^* = |001; \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \rangle, \quad -\zeta^* = |001; \frac{1}{2}, \frac{1}{2}, -\frac{1}{2} \rangle, \quad (1.23)
\]

\[
-\theta^* = |001; \frac{1}{2}, \frac{1}{2}, -\frac{1}{2} \rangle, \quad \eta^* = |000; \frac{1}{2}, \frac{1}{2}, -\frac{1}{2} \rangle.
\]

They may be identified with the same variables used by Sharp and Jakimow \(^{16}\), with the exception of \(\xi\) and \(\xi^*\) whose signs are reversed. The negative sign in front of \(\zeta^*\) and \(\theta^*\) insures that the \((0,0,1)\) variables transform as complex conjugate to the \((1,0,0)\) variables and at the same time satisfy the Condon-Shortley phase convention.

\(^+\)The nature of the labels \(\alpha_1, \alpha_2\), will be discussed later, in Chapter III. They are not required for simple IR's \((\lambda,0,0)\), \((0,\mu,0)\) and \((0,0,\nu)\) and may be suppressed in these cases. They are, therefore, omitted in Eq.(1.22 and 1.23)
The $(0,1,0)$ variables are

\[
\begin{align*}
\mathcal{P} &= |010; 10, 10\rangle, & \sigma &= |010; 10, 00\rangle, & \tau &= |010; 10, -10\rangle, \\
\mathcal{P}' &= |010; 01, 01\rangle, & \sigma' &= |010; 01, 0-1\rangle, & \tau' &= |010; 01, 0-1\rangle.
\end{align*}
\]

They are related to the corresponding variables of reference 16, by

\[
\begin{align*}
\mathcal{P} &= \beta^*, & \sigma &= \frac{i}{\sqrt{2}} (\gamma^* - \gamma), & \tau &= \beta, \\
\mathcal{P}' &= -\alpha^*, & \sigma' &= \frac{i}{\sqrt{2}} (\gamma^* + \gamma), & \tau^* &= \alpha^*.
\end{align*}
\]

§1.4 Differential Generators

Instead of working with the matrices, we shall express the generators of $SU(4)$ as differential operators using the variables introduced above. It is easy to verify that the commutation relations of the generators are satisfied by these differential operators. They are given in Eq. (1.26);

\[
(\partial_\theta \text{ means } \frac{\partial}{\partial \theta}, \text{ etc. . . . .})
\]

The inner product, in the representation space, is chosen so as to make $\eta$ and $\partial_\eta$, etc. . . . , adjoint (Hermitian conjugate) to each other with respect to this inner product. The inner product is defined as:

\[
\langle \eta^a \zeta^b \ldots \ldots \ | \eta'^a \zeta'^b \ldots \ldots \rangle = \delta_{a'a} \delta_{b'b} \ldots \ldots (1.27)
\]
(25.1)

\[ \rho e_2 + \phi e_2 + \phi e_3 + \rho e_4 = 0 \]

\[ \rho e_2 - \phi e_2 + (\phi e_3 + \phi e_3 - \phi e_3 - \phi e_3 + \phi e_3) = 0 \]

\[ \rho e_3 + \phi e_3 + \phi e_3 - \phi e_3 = 0 \]

\[ \rho e_4 - \phi e_3 + (\phi e_3 + \phi e_3 - \phi e_3 - \phi e_3 + \phi e_3) = 0 \]

\[ \rho e_4 + \phi e_3 + \phi e_3 = 0 \]

\[ \rho e_5 + \phi e_5 + \phi e_5 - \phi e_5 = 0 \]

\[ \rho e_6 + \phi e_6 + \phi e_6 = 0 \]

\[ \rho e_7 + \phi e_7 + \phi e_7 = 0 \]

\[ \rho e_8 + \phi e_8 + \phi e_8 = 0 \]

\[ \rho e_9 + \phi e_9 + \phi e_9 = 0 \]

\[ \rho e_{10} + \phi e_{10} + \phi e_{10} = 0 \]
§1.5 The Internal Labelling Problem

As was mentioned in the Introduction, in the most general IR \((\lambda, \mu, \nu)\) the SU(2)xSU(2) subgroup does not provide enough labels to specify SU(4) bases completely. This is often the case when group theory is applied to nuclear physics and the problem is called Internal Labelling Problem or Internal Degeneracy Problem\(^{12}\).

Racah (ref.\textsuperscript{18}, Section 5) has shown that the number of internal labels necessary to specify the basis states of a general IR of a compact group is \(\frac{1}{2}(r-2)\), where \(r\) is the order of the group (number of generators) and \(\ell\) its rank (number of mutually commuting generators).

For SU(4), \(r=15\) and \(\ell=3\), so that the number of internal labels needed for complete specification of basis states is \(\frac{1}{2}(15-3)=6\), in addition to three Cartan (or Representation) labels \((\lambda, \mu, \nu)\). The SU(2)xSU(2) subgroup provides only four labels, namely, the spin \(s\), isospin \(t\), and their third components \(m\) and \(n\), respectively. We therefore need two extra labels.

Moshinsky and Nagel\textsuperscript{21}) proposed an approach of solving the labelling problem by constructing two operators

\[
\Omega = S_i \omega_{ij} t_j
\]

\[
\Phi = S_i S_j \omega_{ik} \omega_{jk} + \omega_{ki} \omega_{kj} t_i t_j - \epsilon_{ijk} \epsilon_{lmn} S_i \omega_{jm} \omega_{kn} t_l
\]  \(\text{(1.28)}\)

whose eigenvalues might be used to complete the labelling. However, it turns out that the labels obtained in this manner
do not exhibit any obvious symmetry, nor do they correspond to known quantities of physical interest. Besides the algebraic difficulties of following this suggestion, the labels are not necessarily rational numbers. For these reasons this approach has not been developed any further.

A similar labelling problem, in the SU(3) ⊗ O(3) model of the nucleus has been solved by Elliott\(^5\). The subgroup provides one label too few. Elliott projected out SU(3) ⊗ O(3) states from certain "intrinsic states" in the SU(3) ⊗ SU(2) weight diagram (lying in the top or the bottom row) and which are non-degenerate. Each intrinsic state corresponds to a set of projected SU(3) ⊗ O(3) states and the third component of the orbital angular momentum \(K\) of the intrinsic state serves as extra label for the SU(3) ⊗ O(3) states projected out. This approach has also been found useful in solving the labelling problem for the Nuclear Seniority Scheme\(^6\).

We propose to solve the internal labelling problem for the Wigner Supermultiplet Scheme using two different approaches. First, using a suggestion by Sharp and Lam\(^12\) of using certain "elementary factors". These elementary factors will be introduced in the next Section and their use in solving the labelling problem will be discussed in Chapter III. The second manner in which we propose to solve the labelling problem uses Elliott's idea of projection from intrinsic states. We pick certain states lying on a particular rectangular boundary face
of the SU(4) \rightarrow SU(3) weight diagram and use these as intrinsic states to project out supermultiplet states. It turns out that the m and n values of the intrinsic states serve as the extra labels for the supermultiplet states and thus complete the classification. The details of this procedure will also be described in Chapter III.

§1.6 Elementary Factors

We can consider the labelling problem from a different point of view; it is essentially the problem of specifying all the SU(2)xSU(2) multiplets of a given IR\(^{\dagger}\) of SU(4). We need to consider only the heaviest member (with m=s, n=t) of each multiplet because all its lower members can be obtained from it by cranking with the s\(_{\pm}\) and t\(_{\pm}\) operators.

Sharp and Lamb observed that the product of the heaviest states of two or more multiplets from the same or different IR's defines, in general, a multiplet of a higher IR, in particular when unwanted states\(^{*}\) have been projected out of this

\(^{\dagger}\) To avoid confusion, we use different terms, irreducible representation, IR, (referring to the parent group SU(4) and multiplets (referring to IR's of the subgroup SU(2)xSU(2)).

\(^{*}\) The states of IR (\(\lambda, \mu, \nu\)) consist of polynomials of degree \(\lambda, \mu\) and \(\nu\) in the variables of the fundamental IR's (1,0,0), (0,1,0) and (0,0,1) respectively. States of degree \(\lambda, \mu, \nu\) but belonging to IR's lower than (\(\lambda, \mu, \nu\)) are called "unwanted states".
product, the remainder will then be the heaviest state of the multiplet so defined.

As an example, consider \( \eta = |100; \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\rangle \), the heaviest state of the \((\frac{1}{2}, \frac{1}{2})\) multiplet of the IR \((1,0,0)\). Its square

\[ \eta^2 = |200; 11, 11\rangle \]

gives the heaviest state of the \((1,1)\) multiplet of the IR \((2,0,0)\); (no unwanted states involved). Similarly, the product of \( \eta \) and \( \xi = |001; \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\rangle \) gives

\[ \eta \xi^* = |101; 11, 11\rangle \]

the heaviest state of \((1,1)\) multiplet of IR \((1,0,1)\)

We can, therefore, call \( \eta \) and \( \xi^* \) "elementary factors" because the heaviest states of multiplets of some higher IR's could be expressed as products of \( \eta \) and \( \xi^* \).

A systematic study, however, reveals that not all multiplets can be expressed as products of powers of simple weights, for example,

\[ |200; 00, 00\rangle = (\eta \xi - \theta \xi) \]

(ignoring normalization) is such a multiplet, which involves a composite expression of simple weights \( \eta, \xi, \zeta \) and \( \theta \), and not a product of elementary factors. In this respect, \((\eta \xi - \theta \xi)\) is a new elementary factor. Similarly,

\[ |101; 10, 10\rangle = (\eta \xi^* + \zeta \xi^*) \]
is another composite elementary factor.

We can, therefore, define an "elementary factor" as the heaviest state of a multiplet of an IR, such that some multiplet(s) of a higher IR can be expressed as a power of it or as a product involving powers of several elementary factors. It is elementary if it cannot be represented as a product of simpler elementary factors. A set of elementary factors is complete if the heaviest state of each multiplet of every IR is expressed as a product of powers of elementary factors of this set (after projecting out the unwanted states).

The correspondence between multiplets and products of powers of elementary factors is one to one when certain relations are taken into account. These relations arise when a linear superposition of products of powers of elementary factors vanishes identically or is equal to an unwanted expression. In all such cases, to avoid duplication of multiplets, one term in the superposition must be singled out and regarded as redundant.

Sharp and Lam made a systematic search for elementary factors for various group-subgroup decomposition and found that for the \(SU(4) \supset SU(2) \times SU(2)\) scheme, one needs thirteen elementary factors. They also found the redundant combinations for these factors.

We have worked out the algebraic form of these elementary factors in terms of our variables and they are given, in the notation \((\lambda \nu; \omega; \sigma)\), ignoring normalization, by
\[ f_1 = (100; \frac{1}{2}) = \eta, \]
\[ f_2 = (200; 00) = (\eta \xi' - \theta \varsigma), \]
\[ f_3 = (010; 10) = \rho, \]
\[ f_4 = (010; 01) = \rho', \]
\[ f_5 = (020; 00) = (\sigma^2 - 2\rho \tau), \]
\[ f_6 = (001; \frac{1}{2}) = \xi^*, \]
\[ f_7 = (002; 00) = (\eta \xi^* - \theta \xi^*), \]
\[ f_8 = (101; 10) = (\eta \theta^* + \zeta \xi^*), \]
\[ f_9 = (101; 01) = (\eta \xi^* + \theta \xi^*), \]
\[ f_{10} = (110; \frac{1}{2}) = \eta (\sigma + \sigma') - \sqrt{2} (\rho \theta + \rho' \xi), \]
\[ f_{11} = (011; \frac{1}{2}) = \xi^* (\sigma + \sigma') + \sqrt{2} (\rho \xi^* + \rho' \theta^*), \]
\[ f_{12} = (111; 10) = \eta \xi^* \tau' - \xi \theta^* \rho' - \sigma' (\xi \xi^* - \eta \theta^*) / \sqrt{2} + 3 \rho (\eta \eta^* + \theta \theta^* + \xi^* \xi + \xi^* \xi^*), \]
\[ f_{13} = (111; 01) = \eta \xi^* \tau - \xi \theta^* \rho - \sigma (\theta \xi^* - \eta \xi^*) / \sqrt{2} + 3 \rho' (\eta \eta^* + \theta \theta^* + \xi^* \xi + \xi^* \xi^*). \]

Their redundant combinations are:

\[ f_1 f_5 f_6, f_1 f_{11}, f_6 f_{10}, f_1^2, f_{10}^2, f_9 f_9, f_9 f_{12}, f_8 f_{13}; \]

and \( f_{12} \) or \( f_{13} \) with any of \( f_1, f_e, f_{10}, f_{11}, f_{12}, f_{13} \).
Recently, Brunet and Resnikoff\(^9\) have also given elementary factors and redundant combinations, in agreement with Eqs (1.29) and (1.30).

§1.7 Casimir Operators

A Casimir operator, by definition, is an operator which commutes with all the generators of the group. They are of particular interest in physics since they can characterize IR's and also, they can be related to quantities of physical interest (e.g., mass formulae (cf. Introduction) symmetry breaking interactions, etc.). For a semi-simple group, they are polynomials in the generators\(^18\). The Casimir operators of an arbitrary degree \(k\) can be defined\(^20,21\), for \(\text{SU}(n)\), as follows:

\[
C^{(n)}_R = \sum_{i_1, \ldots, i_k} A_{i_1 i_2} A_{i_2 i_3} \cdots \cdots \cdots A_{i_k i_1} \tag{1.31}
\]

They commute with any generator \(A\) (Eq. 1.4). There are \((n-1)\) independent operators \((C_1=0)\), and their eigenvalues characterize uniquely the IR's.

Jakimow\(^17\) has worked out the eigenvalues of the three \(\text{SU}(4)\) Casimir operators in terms of the representation labels \((\lambda, \mu, \nu)\). We include these expressions here for the sake of completeness. They agree with the expressions used by Burdet et al, (ref. 13.), after the transformation, (Eq. 1.20), from
(\lambda, \mu, \nu) to (P, P', P'') is made.

\[ C_2 = \frac{1}{4} (3\lambda^2 + 4 \mu^2 + 3 \nu^2 + 4 \lambda \mu + 4 \mu \nu + 2 \lambda \nu + 12 \lambda + 16 \mu + 12 \nu) . \]

\[ C_3 = \frac{3}{8} (\lambda - \mu) \left\{ (\lambda+\mu)(\lambda+2\mu+\nu) + 6 \lambda + 4 \mu + 6 \nu + 8 \right\} \]

\[ C_4 = \frac{1}{192} (63 \lambda^4 + 48 \mu^4 + 63 \nu^4 + 96 \lambda \mu^3 + 96 \mu^3 \nu + 168 \lambda^3 \mu + 168 \mu \nu^3 + 84 \lambda^3 \nu + 84 \lambda \nu^3 + \]

\[ = \frac{1}{192} (216 \lambda^2 \mu^2 + 216 \lambda^2 \nu^2 + 216 \mu^2 \nu^2 + 216 \lambda \mu \nu^2 + 216 \lambda \mu \nu^2 + 216 \lambda^2 \mu \nu + 216 \lambda^2 \mu \nu + 150 \lambda^2 \nu^2 + 144 \lambda \mu^2 \nu + 504 \lambda^3 + 504 \nu^3 + 384 \mu^3 + 1152 \lambda^2 \mu + 1152 \mu \nu^2 + 1152 \lambda \mu \nu + 648 \lambda \nu^2 + 648 \lambda^2 \nu + 864 \lambda \mu^2 + 864 \mu^2 \nu + 960 \lambda^2 + 960 \nu^2 + 512 \mu^2 + 1664 \lambda \mu + 1664 \mu \nu + 1408 \lambda \mu - 192 \mu - 192 \nu - 1024 \mu ) . \]
CHAPTER II

SIMPLE REPRESENTATIONS

The simple IR's \((\lambda,0,0), (0,\mu,0)\) and \((0,0,\nu)\) do not involve any internal degeneracy problem\(^\dagger\), their subgroup multiplet labels \(s,t,m,n\) are sufficient to specify the states completely. We shall develop basis states for these IR's in the following Sections.

§2.1 \((\lambda,0,0)\) Representation

The heaviest state of the SU(2)xSU(2) multiplets of the IR \((\lambda,0,0)\) consists of a product of powers of the elementary factors

\[ f_1 = (100; \frac{1}{2}, \frac{1}{2}) = \eta, \quad \text{and} \quad f_2 = (200; 00) = (\eta \xi - \theta \xi), \]

\[ |\lambda 00; st, st\rangle = N_{st}^\lambda \eta^a (\eta \xi - \theta \xi)^b \quad (2.1) \]

\(^\dagger\)The decomposition of IR's of SU(4) into SU(2)xSU(2) multiplets has been discussed in general algebraic form by Racah\(^{24}\)(see also ref.25). These techniques give the sets of \(s,t\) values in a given IR together with their multiplicities. In particular, it turns out that these multiplicities are never greater than one in the following classes of SU(4) representations:

\((\lambda,0,0),(0,\mu,0),(0,0,\nu),(\lambda,1,0),(\lambda,0,1),(1,\mu,0),(0,\mu,1),(1,0,\nu),(0,1,\nu)\).
where $\eta^\lambda_{st}$ is the normalization constant. The degree of this state is $\lambda$, therefore,

$$a + 2b = \lambda \quad (2.2a)$$

All the contribution to $s$ and $t$ comes from $\eta$, each power of which contributes $s=\frac{1}{2}$, $t=\frac{1}{2}$, so that the $s,t$ values of the state (2.1) are

$$s = t = \frac{1}{2} a \quad (2.2b)$$

Solving (2.2a) and (2.2b) gives

$$a = 2s, \quad b = \frac{1}{2} \lambda - s, \quad s = t,$$

therefore,

$$|\lambda 00; ss, ss\rangle = N^\lambda_{st} \eta^{2s} \left( \eta^{\xi-\theta \xi} \right)^{\frac{1}{2} \lambda - s}. \quad (2.3)$$

The constant $N^\lambda_{st}$ is determined by the normalization

$$(N^\lambda_{st})^2 \left\langle \eta^{2s} \left( \eta^{\xi-\theta \xi} \right)^{\frac{1}{2} \lambda - s} | \eta^{2s} \left( \eta^{\xi-\theta \xi} \right)^{\frac{1}{2} \lambda - s} \right\rangle = 1. \quad (2.4)$$

The above scalar product is readily evaluated using the definition (1.27), and we get

$$N^\lambda_{st} = \frac{(2s+1)}{\sqrt{\left(\frac{1}{2} \lambda - s\right)! \left(\frac{1}{2} \lambda + s + 1\right)!}} \quad (2.5)$$

The requirement that the indices in Eq. (2.3) be non-negative integers restricts $s(=t)$ to the values $s=\frac{1}{2} \lambda, \frac{1}{2} \lambda - 1, \ldots, \frac{1}{2}$ or 0 (according as $\lambda$ is an odd or even integer).
The state in Eq. (2.3) gives the heaviest \((m=s, n=s)\) state of the multiplet \((s,s)\). These multiplets for \((\lambda, 0, 0)\) are shown in Fig. (2.1). The state with arbitrary values of \(m, n (|m|, |n| \leq s)\) can be obtained by operating repeatedly with the operators \(s_-\) and \(t_-\) on Eq. (2.3). The general \((\lambda, 0, 0)\) state is then given by:

\[
|\lambda 00; ss, mn\rangle = \frac{(2s+1)(s+m)! (s-m)! (s+n)! (s-n)!}{(\frac{1}{2} \lambda - s)! (\frac{1}{2} \lambda + s+1)!} \frac{1}{2} \\
\times (\eta \xi - \theta \delta)^{\frac{1}{2} \lambda - s} \sum \eta^{\frac{1}{2} (s+m)+r} \xi^{\frac{1}{2} (s+m)-r} \theta^{\frac{1}{2} (s-m)-n+r} \delta^{\frac{1}{2} (s-m)+n-r} \\
\{\eta^{\frac{1}{2} (s+m)+r}\} \{\xi^{\frac{1}{2} (s+m)-r}\} \{\theta^{\frac{1}{2} (s-m)-n+r}\} \{\delta^{\frac{1}{2} (s-m)+n-r}\}
\]

(2.6)

Fig. (2.1) The heaviest states of the multiplets of \(\text{IR} (\lambda, 0, 0)\)
We can check that the states have the correct multiplicity. For \( \lambda \) even, the multiplicity is given by

\[
D_\lambda = \sum_{s=0}^{\lambda/2} (2s+1)^2 = \frac{1}{6} (\lambda+1)(\lambda+2)(\lambda+3),
\]

and for \( \lambda \) odd,

\[
D_\lambda = \sum_{s=\lambda/2}^{\lambda-1} (2s+1)^2 = \sum_{x=0}^{\lambda/2} (2x+2)^2 = \frac{1}{6} (\lambda+1)(\lambda+2)(\lambda+3),
\]

in agreement with the dimensionality formula of IR \((\lambda,0,0)\) (Eq. (1.21) with \( \mu=\nu=0 \)).

An equivalent but a slightly different form of the \((\lambda,0,0)\) states can be written down with the following considerations.

The \((1,0,0)\) quartet, Fig(1.4), can be thought of consisting of two \(s\)-doublets \((\eta \theta, \zeta \xi)\) or two \(t\)-doublets \((\eta \zeta, \xi \eta)\). The \(|\lambda00;ss,mm\rangle\) state can be obtained by forming two Wigner monomials either from the \(s\)-doublets or the \(t\)-doublets, and coupling them together with an \(SU(2)\) Clebsch-Gordan coefficient, ensuring at the same time that the states have correct degree \(\lambda\).

We form two normalized Wigner monomials with the two \(t\)-doublets \((\eta \zeta, \text{having } m=\frac{1}{2}, \text{and } \xi \eta, \text{having } m=-\frac{1}{2})\), in the notation \(|t^m\rangle\)

\[
\begin{align*}
|a\rangle &= \eta^{a+x} \zeta^{a-x} \frac{1}{\sqrt{(a+x)! (a-x)!}}, \\
|b\rangle &= \eta^{b+y} \xi^{b-y} \frac{1}{\sqrt{(b+y)! (b-y)!}}.
\end{align*}
\]
To ensure the correct degree $\lambda$, and the $m$ value of the coupled state, we have the conditions

$$\lambda = 2a + 2b$$

$$m = \frac{1}{2} \left[(a+x) - (a-x) + (b+y) - (b-y)\right] = a - b$$

Solving these, we have

$$a = \frac{1}{4} \lambda + \frac{1}{2} m, \quad b = \frac{1}{4} \lambda - \frac{1}{2} m$$

We now couple the two monomials using a C.G. coefficient to get the right $(t,n)$ values,

$$\langle \lambda_0; s, s, m | n \rangle = \sum_{x} \left| \begin{array}{c} \lambda+x \cr x \end{array} \right| \left| \begin{array}{c} \frac{1}{4} \lambda + \frac{1}{2} m \cr n-x \end{array} \right| \left| \begin{array}{c} \frac{1}{4} \lambda - \frac{1}{2} m \cr n-x \end{array} \right|^{s}$$

$$= \sum_{x} \frac{\eta_{\lambda+x+\frac{1}{2} m} \zeta_{\frac{1}{4} \lambda + \frac{1}{2} m - x} \theta_{\frac{1}{4} \lambda - \frac{1}{2} m + n - x} \xi_{\frac{1}{4} \lambda - \frac{1}{2} m - n + x}}{\left\{ \left(\frac{1}{4} \lambda + \frac{1}{2} m + x\right)! \left(\frac{1}{4} \lambda + \frac{1}{2} m - x\right)! \left(\frac{1}{4} \lambda - \frac{1}{2} m + n - x\right)! \left(\frac{1}{4} \lambda - \frac{1}{2} m - n + x\right)! \right\}^{1/2}}$$

$$\times \left| \begin{array}{c} \frac{1}{4} \lambda + \frac{1}{2} m \\ x \end{array} \right| \left| \begin{array}{c} \frac{1}{4} \lambda - \frac{1}{2} m \\ n-x \end{array} \right|^{s}$$

$$\left| \begin{array}{c} s \\ n \end{array} \right|$$

(2.7)

This form of the $(\lambda,0,0)$ state is sometimes more convenient than the other form Eq(2.6).

The Regge form\textsuperscript{26} of the C.G. coefficient in Eq(2.7) is
Since the $s$ and $t$ play a symmetric role in the supermultiplet scheme, their interchange in Eq.(2.7) should not change the expression. This in effect comes down, apart from a change in the dummy $x$, to the interchange of rows and columns of the Regge form, which is known to have this symmetry. We have thus exhibited a physical representation of the symmetry of the Regge form.

§2:2 (0,0,ν) Representation

The IR (1,0,0) is conjugate to the IR (0,0,1), therefore the states of the simple IR (0,0,ν) are obtained from those of (λ,0,0) by the replacements:

$$\lambda \rightarrow \nu, \; \eta \rightarrow \xi^*, \; \xi \rightarrow \eta^*, \; \xi^* \rightarrow \theta^*, \; \theta \rightarrow -\xi^*$$

The multiplets here correspond to the products of powers of the elementary factors $f_6 = \xi^*, \; f_7 = (\eta^*\xi^* - \theta^*\xi^*)$. The multiplicity of states agrees with the dimensionality of the IR (0,0,ν); Eq.(1.21) with $\lambda = \nu = 0$. 
§2.3 \((0,\mu,0)\) Representations

The heaviest state of the multiplet \((s,t)\) of the simple IR \((0,\mu,0)\) is

\[
\left| 0, \mu; st; st \right> = (-)^{\frac{1}{2}}(\mu-s-t) N_{st}^\mu \rho^s \rho'^t \\
\times \sum_{\gamma} A_\gamma (\sigma^2 - 2\rho\tau')^\gamma (\sigma^2 - 2\rho\tau')^{\frac{1}{2}}(\mu-s-t) - \gamma
\]

(2.10)

where \(N_{st}^\mu\) is the normalization constant and \(A_\gamma\) is a coefficient to be determined. The state (2.10) is characterized by its leading \((\gamma=0)\) term, which is a product of powers of the elementary factors \(f_3 = \rho, f_4 = \rho', f_5 = (\sigma^2 - 2\rho\tau)\).

The requirement that the indices be non-negative integers restricts \(s, t\) to non-negative integer values such that \(\mu-s-t\) is a non-negative even integer. The location of heaviest states of multiplets for IR \((0,\mu,0)\) is shown in Fig.(2.2)

\[\text{Fig.(2.2) The heaviest states of the multiplets of IR } (0,\mu,0)\]
The coefficient $A_x$ is determined by imposing

$$\left( \partial^2 - 2\partial_x \partial_t - \partial^2_x, + 2\partial_x \partial_t \right) |\mu\mu; st; st\rangle = 0$$

This is the condition that the state (2.10) be orthogonal to states containing the SU(4) scalar ($\sigma^2 - 2\sigma_t - \sigma'^2 + 2\sigma'\tau'$) as a factor (such states are called 'unwanted states' and belong to IR's lower than $(0,\mu,0)$). The algebraic calculations are quite straightforward and lead to the result

$$A_x = \left[ x! (2t+2x+1)! (\mu+s-t+2x+1)! \left\{ \frac{1}{2}(\mu-s-t)-x \right\}! \right]^{-1}$$

(2.11)

The normalization constant $N^\mu_{st}$ in Eq.(2.10) is determined by equating the matrix elements

$$\langle o\mu_0; s+1t+1, st | u_{11} | o\mu_0; st, st \rangle = \langle o\mu_0; st, st | u_{-1-1} | o\mu_0; s+1t+1, s+1t+1 \rangle$$

which is based on the observation that $u_{11}$ and $u_{-1-1}$ are Hermitian conjugate to each other. The calculations are carried out in Appendix B, and the result is

$$N^\mu_{st} = \left[ \frac{(\mu+s-t+1)! (\mu-s+t+1)! (\mu+s+t+2)! (\mu-s-t)! (2s+1)! (2t+1)!}{2^{\mu-s-t} (2\mu+2)! s! t!} \right]^{1/2}$$

(2.12)

A state with arbitrary $m,n$ ($|m| \leq s$, $|n| \leq t$) is obtained by operating repeatedly with $s_-$ and $t_-$ operators.

The general, normalized, $(0,\mu,0)$ state is then given by:
\[ |0_{\mu s t}; s t, m n \rangle = N_{\mu s t}^{\mu} s t! t! (-\frac{1}{2}(\mu-s-t)) \]

\[
\times \left[ \frac{2^{s+t+n} (s+m)! (s-m)! (t+n)! (t-n)!}{(2s)! (2t)!} \right]^{1/2}
\]

\[
\times \sum_{x} \frac{(\sigma^2 - 2 \rho \tau')^x (\sigma^2 - 2 \rho \tau)^{1/2}(\mu-s-t)-x}{x! (x+2t+1)! (\mu+s-t-2x+1)! \left\{ \frac{1}{2}(\mu-s-t)-x \right\}!}
\]

\[
\times \sum_{y} \frac{\rho^y \sigma^{s+m-2y} \tau^y-m}{2^y y! (s+m-2y)! (y-m)!} \sum_{\xi} \frac{\rho^x \sigma^{t+n-2\xi} \tau^x-\xi}{2^x \xi! (t+n-2\xi)! (\xi-n)!}
\]

\((\mu, s, t \text{ and } \frac{1}{2}(\mu-s-t) \text{ are non-negative integers).} \quad (2.13)\)

The dimensions of these states can be calculated using only the heaviest states, shown in Fig.(2.2), as follows:

(i) \(\mu\)-even

Since \(s+t\) has the same parity as \(\mu\), i.e. even, we can write \(s+t = 2r\), where \(r\) is a positive integer. The multiplicity of the states is

\[
D_{\mu} = \sum_{r=0}^{\mu/2} \sum_{s=0}^{2r} (2s+1)(2t+1)
\]

\[
= \sum_{r} \sum_{s} (2s+1)(4r-2s+1)
\]

\[
= \frac{1}{12} (\mu+1)(\mu+2)(\mu+3)
\]
(ii) \( \mu \)-odd

In this case, \( s+t = 2r+1 \), and

\[
D_\mu = \sum_{r=0}^{2r+1} \sum_{s=0}^{2r+1} (2s+1)(4r-2s+3)
\]

\[
= \frac{1}{12} (\mu+1)(\mu+2)^2(\mu+3)
\]

in agreement with dimensionality formula for IR \((0,\mu,0)\) Eq.(1.21)(with \(\lambda=\nu=0\)).

\section{2.4 Reduced Matrix Elements}

We shall calculate the reduced matrix elements of the irreducible tensor \( u \), Eq.(1.10), between the SU(2)\( \times \)SU(2) multiplets of simple IR's \((\lambda,0,0)\), \((0,\mu,0)\) and \((0,0,\nu)\).

The Wigner-Eckart theorem can be extended to SU(2)\( \times \)SU(2) \(\cong O(4)\) to define the reduced matrix elements as

\[
\langle s_2 t_2 \| U_{st} \| s_1 t_1 \rangle = \frac{\langle s_2 t_2 ; m_2 n_2 | U_{st} | s_1 t_1 ; m_1 n_1 \rangle \sqrt{(2s_2+1)(2t_2+1)}}{\langle s_1 s \| s_2 \rangle \langle t_1 t \| t_2 \rangle \langle m_1 m 1 \| m_2 m_2 \rangle \langle n_1 n 1 \| n_2 n_2 \rangle}
\]

(2.14)

where the factors in the denominator are ordinary SU(2) Clebsch-Gordan coefficients.

Thus the problem is reduced to calculating certain matrix elements of \( u_{\text{mn}}^{11} \) between the relevant supermultiplet
The action of $u_{11}$ on a state $|s,t\rangle$, in general, gives a linear combination of states $|s',t'\rangle$, where $s' = s+1,s,s-1$ and $t' = t+1,t,t-1$; because $u_{11}$ transforms as a $(1,1)$ tensor under SU$(2) \times$ SU$(2)$ and so it can change $s$ and $t$ only by ±1, or 0.

We shall only give the expressions for non-zero reduced matrix elements. The details of calculations are to be found in Appendix B.

\begin{align}
\langle \lambda 0 0; ss \Vert u \Vert \lambda 0 0; ss \rangle &= \frac{1}{2} (\lambda + 2)(2s+1) \\
\langle \lambda 0 0; s+1,s+1 \Vert u \Vert \lambda 0 0; s+1,s+1 \rangle &= \sqrt{\frac{1}{2} \lambda + s + 2}\left(\frac{1}{2} \lambda - s\right)(2s+1)(2s+3) \\
\langle 0 \mu 0; s+1,t+1 \Vert u \Vert 0 \mu 0; s+1,t+1 \rangle &= \sqrt{(\mu + s + t + 4)(\mu - s - t)(s+1)(t+1)} \\
\langle 0 \mu 0; s+1,t-1 \Vert u \Vert 0 \mu 0; s+1,t-1 \rangle &= \sqrt{(\mu + s - t + 3)(\mu - s + t + 1)(s+1)t}
\end{align}

The $(0,0,v)$ reduced matrix elements are obtained by replacing $\lambda$ by $v$ in Eqs. (2.15a and b).
CHAPTER III

THE GENERAL IR \((\lambda,\mu,\nu)\)

The general representation \((\lambda,\mu,\nu)\) involves internal degeneracy problem, i.e., in addition to the four subgroup labels \(s,t,m,n\) two extra labels are required to specify the basis states uniquely, unlike the simple IR's discussed in the last chapter, for which no labelling problem was encountered.

We shall solve the labelling problem using two different approaches. First, using the elementary factors, in a manner similar to that used by Bargmann and Moshinsky\(^{10}\) in the treatment of \(SU(3)\supseteq O(3)\) and second, following the projection techniques developed by Elliott\(^{5}\), to solve the same \((SU(3)\supseteq O(3))\) degeneracy problem. We shall name the two sets of supermultiplet states after these authors.

§3.1 Bargmann-Moshinsky States

The Cartan type \((\lambda,\mu,\nu)\) states can be constructed out of a product of states of the simple IR's \((\lambda,0,0)\), \((0,\mu,0)\) and \((0,0,\nu)\); i.e., lie in a space spanned by the simple product states, which we define as:
The spin $s_\lambda$ (isospin $s_\lambda$) is first coupled with the spin $s_\nu$ (isospin $s_\nu$) to give $s_{\lambda\nu}$ ($t_{\lambda\nu}$) which is then coupled with $s_\mu$ ($t_\mu$) to get the total spin $s$ (isospin $t$) of the simple-product state*. The last four factors are ordinary Clebsch-Gordan coefficients which bring about the above $O(4)$ couplings.

The simple-product states have definite values of $s,t,m,n$. The space which they span contains not only the states of the IR $(\lambda,\mu,\nu)$ but states belonging to all the IR's in the Clebsch-Gordan series $(\lambda,0,0)\times(0,\mu,0)\times(0,0,\nu)$.

Certain simple product states are of special importance because they correspond to products of powers $s^a_i$ of

* An alternative coupling scheme $s_\lambda^+ s_\mu^+ = s_{\lambda\mu}^+$ and $s^+_{\lambda\mu} s^+_{\nu} = s^+$ is equally acceptable and leads to similar results.
elementary factors (1.29) with the redundant combinations (1.30) eliminated. We call them 'stretched simple-product states' because they are stretched either in s-spin \((s=s_{\lambda}+s_{\mu}+s_{\nu})\) or in t-spin \((t=t_{\lambda}+t_{\mu}+t_{\nu})\). They fall into twelve types, labelled A to L below. Each is characterized by the vanishing of certain (at least five) of the exponents \(a_i\) to make the types mutually exclusive, a product should be assigned to the earliest (alphabetically) type for which it qualifies.

The first six types (A to F) are s-stretched while the last six (G to L) are t-stretched. In each case at most two of the exponents are independent (for fixed \(\lambda,\mu,\nu, s, t\)), in agreement with the fact that two additional labels are needed. We introduce two labels \(k_s, k_t\) defined for the s-stretched states in terms of the independent exponents by:

\[
\begin{align*}
  k_s &= \frac{1}{2}(\lambda+\nu) - a_2 - a_7 \\
  k_t &= \frac{1}{2}(\lambda+\nu) - a_2 + a_7
\end{align*}
\]

(3.2)

For t-stretched states the definitions of \(k_s\) and \(k_t\) are interchanged.

The values of \(a_i\) for the twelve types of stretched states are given in Table I, and the values of \(s_{\lambda} s_{\nu} s_{\mu}, t_{\lambda} t_{\nu}, t_{\mu}\) of these states in terms of \(k_s\) and \(k_t\) are given in Table II. Types F and L have \(k_t=0\) and \(k_s=0\), respectively.

The allowed values of \(k_s\) and \(k_t\) are determined by the
requirement that all the exponents in Table II be non-negative integers. 2s and 2t must have the parity of \(\lambda + \nu\). Then \(k_s, k_t\) take the values in the following ranges:

For \(s\)-stretched states,

\[
|k_t| \leq t
\]  \(3.3a\)

\[
\max \frac{1}{2}(s+t-\mu), s-\mu, |k_t| \leq k_s \leq \min s, \lambda - k_s, \lambda + k_t
\]

and, for \(t\)-stretched states,

\[
|k_s| \leq s
\]  \(3.3b\)

\[
\max \frac{1}{2}(s+t-\mu), t-\mu, |k_s|+1 \leq k_t \leq \min t, \lambda - k_s, \nu + k_s
\]

In both cases,

\[
k_s + k_t + \lambda = \text{even}, \quad k_s - k_t + \nu = \text{even},\]  \(3.3c\)

and no \(k_s = k_t = 0\), unless \(\mu - s - t = \text{even}\).

If the variables that denote the states of the fundamental IR's in (010) and (001) are replaced by the polynomials which represent the same states in the Gelfand basis the stretched simple products states with the allowed values of \(k_s, k_t\) and \(s_\lambda, s_\nu, s_\lambda \nu, t_\lambda \nu, s_\mu, t_\mu\) given by Table II are essentially the states derived independently by Brunet and Resnikoff\(^7\).

Since we work in Cartan basis the stretched simple product state contains components belonging to IR's lower than \((\lambda, \mu, \nu)\); when these unwanted states are projected out the stretched simple-product state is the leading term in a simple product.
state expansion. Each stretched simple-product state appears in the expansion of one and only one state and hence acts as a labelling term or "handle" for the complete state. We choose the phase and normalization of each of these states (which we shall call the Bargmann-Moshinsky states), such that the coefficient of its stretched simple product term is unity and denote the Bargmann-Moshinsky state by

$$|\lambda \mu \nu ; k_s k_t ; s t, m n\rangle_{BM}$$

§3.2 Elliott States

We shall develop a second set of supermultiplet states following Elliott's treatment of SU(3)⊃O(3) nuclear model. We consider certain simple states $|\lambda \mu \nu ; K_s K_t\rangle$ which belong to the IR ($\lambda, \mu, \nu$) and are eigenstates of the $s_0, t_0$ operators with eigenvalues $K_s, K_t$ respectively. They lie in the rectangular face $2(2Z+3Y) = 3(\lambda+2\mu+\nu)$ of the SU(4)⊃SU(3) weight diagram$^{16,17}$, shown in Fig.(3.1) and are non-degenerate with respect to the $s, t, m, n$ labels. We call them the "intrinsic states" (adopting Elliott's terminology) and will project good SU(2)×SU(2) states out of them. The intrinsic states are given by Eq.(16) of ref.(16) and may be written in our present variables as:
Fig. (3.1) The SU(4) \supset SU(3) weight diagram.
(The intrinsic states lie in the rectangular face ABCD)

\[
|\lambda \mu \nu; \kappa_s \kappa_t\rangle = A_{\kappa_s \kappa_t} \xi^{\frac{i}{2}(\lambda - \kappa_s - \kappa_t)} \eta^{\frac{i}{2}(\lambda + \kappa_s + \kappa_t)} x^{\frac{i}{2}(\nu - \kappa_s + \kappa_t)} \zeta^{\frac{i}{2}(\nu + \kappa_s - \kappa_t)} (\sigma - \sigma')^\mu
\]

where the normalization constant

\[
A_{\kappa_s \kappa_t} = \left[ \mu! \left\{ \frac{1}{2}(\lambda - \kappa_s - \kappa_t) \right\}! \left\{ \frac{1}{2}(\lambda + \kappa_s + \kappa_t) \right\}! \left\{ \frac{1}{2}(\nu - \kappa_s + \kappa_t) \right\}! \left\{ \frac{1}{2}(\nu + \kappa_s - \kappa_t) \right\}! \right]^{-1} \left\{ (2\mu + 2)! \right\}^{1/2}
\]

(3.4a)

has been chosen to avoid unimportant factors later. The

\[
\kappa_s, \kappa_t
\]

are restricted to the values:
by the requirement that the exponents in (3.4a) be non-negative integers. The intrinsic states for four cases with \( \lambda, \nu \) both even, even-odd, odd-even and both odd are illustrated in Fig. (3.2).

We expand \( |\lambda \mu \nu ; K_S K_T \rangle \) in simple product state (3.1),

\[
|\lambda \mu \nu ; K_S K_T \rangle = \sum_{s, s', t} |(\lambda_s \nu_s t_{\lambda_s} \nu_{\lambda_s}) \mu_{s'} t_{\mu_{s'}} \rangle_{s t} K_S K_T \rangle B^{\lambda \mu \nu}(S)
\]

where \( S \) stands for the variables \( s, s', t, t' \). The coefficient \( B \) is just the scalar product of \( |\lambda \mu \nu ; K_S K_T \rangle \) with the simple product state on the right hand side of (3.6), which is easily evaluated using the explicit expressions (3.4) and (3.1) together with (2.6, 2.9 and 2.13). Because of the simple form of the intrinsic states (3.4), only a single term, that with \( m_\mu = n_\mu = 0, m_\lambda = n_\lambda = \frac{1}{2}(K_S + K_T), m_\nu = -n_\nu = \frac{1}{2}(K_S - K_T) \) from the quadruple sum in the simple product state (3.1), contributes to \( B \). The sum which arises can be evaluated with the help of the formula

\[
\sum_{x} \left[ x! (2t+2x+1)!! \left( \mu-s-t-2x+1 \right)!! \left( \mu_s-s_t-x \right)!! \right]^{-1}
= 2^{\frac{1}{2}(\mu-s-t)} (2\mu+2)!! \left[ (\mu+s+t+2)!! (\mu-s-t)!! \right]^{-1}
\! \left( \mu-s+t+1)!! (\mu+s-t+1))!! \right]^{-1}
\]

(3.7)
Fig. (3.2) Intrinsic States

(The states to the right of the dotted lines are used to specify Elliott states)
which follows from the normalization of the state (2.10).

The result is

\[ B^{\lambda \mu \nu}_{st \kappa_s \kappa_t} (s_\lambda, s_\nu, s_\lambda \nu, t_\lambda \nu; s_\mu, t_\mu) = (-)^{\frac{1}{2} (\nu + \mu - s_\mu - t_\mu)} s_\nu \]

\[ x \left[ \frac{1}{\sqrt{2} - \frac{1}{2} (s_\lambda + s_\lambda + 2)! (2s_\nu + 1)! (2s_\mu + 1)! (2t_\mu + 1)!} \right] \]

\[ x \left[ \frac{(2s_\lambda + 1)! (2s_\nu + 1)! (2s_\mu + 1)! (2t_\mu + 1)!}{(2s_\lambda - s_\lambda)! (s_\lambda + s_\lambda + 1)! (s_\nu + s_\nu + 1)! (s_\mu + s_\mu + 1)!} \right] \]

To project the intrinsic states onto particular \( s, t \) it is only necessary to drop the \( s, t \) summations in Eq. (3.6).

Thus,

\[ | \lambda \mu \nu ; \kappa_s \kappa_t ; st, mn \rangle \]

\[ = \sum_{S} | (((\lambda, s_\lambda, s_\nu) s_\lambda \nu, t_\lambda \nu) \mu, s_\mu, t_\mu) st, mn \rangle B^{\lambda \mu \nu}_{st \kappa_s \kappa_t} (S) \]

These projected states are not linearly independent in general, for each \( s, t \) it is necessary to retain only a number of independent states equal to the multiplicity of \( s, t \) in the given IR. The labelling of Bargmann-Moshinsky states achieved in the last section with the help of elementary factors is now a valuable guide—\( \kappa_s \kappa_t \) are restricted to the same ranges as \( k_s k_t \), i.e.
For s-stretched states,

$$\left| K_t \right| \leq t$$

$$\max \left\{ \frac{1}{2}(s-t-\mu), s-\mu, \left| K_t \right| \right\} \leq K_s \leq \min \{ s, \lambda-K_t, \nu+K_t \}$$  \hspace{1cm} (3.10a)

and for t-stretched case,

$$\left| K_s \right| \leq s$$

$$\max \left\{ \frac{1}{2}(s+t-\mu), t-\mu, \left| K_s \right| + 1 \right\} \leq K_t \leq \min \{ t, \lambda-K_s, \nu+K_s \}$$  \hspace{1cm} (3.10b)

In both cases,

$$K_s + K_t + \lambda = \text{even}, \hspace{1cm} K_s - K_t + \nu = \text{even}$$  \hspace{1cm} (3.10c)

and no $K_s=K_t=0$, unless $\mu-s-t = \text{even}$

Draayer \cite{8}) found equivalent ranges by a heroic trial and error method. With $K_s, K_t$ thus restricted we call the projected state (3.9) an Elliott state and denote it by a subscript $E$ rather than $P$.

Examination of the expansion (3.9) shows that the Elliott states are independent; for if $K_s, K_t$ lie in the ranges (3.10), the sum (3.9) contains a stretched term, that with $K_s=K_s, K_t=K_t$ which is not included in the expansion of any other state with an equal or greater value of $\left| K_s \right|^2 \left| K_t \right|^2$. That the Elliott states are complete follows from the fact that they are equal in number to the complete set of Bargmann-Moshinsky states. They therefore form a basis.
To expand an Elliott state in Bargmann-Moshinsky states it is necessary to examine only the stretched terms in the expansion (3.9) since each such term characterizes a whole Bargmann-Moshinsky state:

$$|\lambda \mu \nu; K_s K_t; st, mn\rangle = \sum_{k_s k_t} |\lambda \mu \nu; k_s k_t; st, mn\rangle D^{\lambda \mu \nu}_{st} (K_s K_t; k_s k_t) \quad (3.11)$$

Here $D^{\lambda \mu \nu}_{st} (k_s k_t; k_s k_t)$ is just $B^{\lambda \mu \nu}_{st k_s k_t} (S)$, Eq. (3.8), with the variables given their 'stretched' values in terms of $k_s, k_t$ according to Table II. The elements of the transformation matrix are rather simple. If $k_s, k_t$ is of type $E$ or $K$ the matrix element contains a single sum (arising from the fourth or third Clebsch-Gordan coefficient, respectively of Eq. (3.8)); otherwise it contains no sum.

Eq. (3.11) can be solved for Bargmann-Moshinsky states in terms of the Elliott states:

$$|\lambda \mu \nu; k_s k_t; st, mn\rangle = \sum_{k_s k_t} |\lambda \mu \nu; K_s K_t; st, mn\rangle (D^{-1})^{\lambda \mu \nu}_{st} (K_s K_t; k_s k_t) \quad (3.12)$$

for any particular case (we have not succeeded in obtaining an analytic formula for $D^{-1}$ in the general case). Bargmann-Moshinsky states can then be expanded in terms of simple product states by means of Eqs. (3.9 and 3.12); the coefficients are

$$C_{st k_s k_t}^{\lambda \mu \nu} (S) = \sum_{k_s k_t} B^{\lambda \mu \nu}_{st k_s k_t} (S) (D^{-1})^{\lambda \mu \nu}_{st} (K_s K_t; k_s k_t). \quad (3.13)$$
the sum in Eq.(3.13) is over the range (3.10).

§3.3 The Metric Matrices

The Elliott and Bargmann-Moshinsky states are non-orthogonal. The metric matrix (normalizations and overlaps) for Elliott states is written as:

\[
\langle \lambda \mu \nu; k_s k_t; st; mn | \lambda' \mu' \nu'; k_s' k_t'; s' t'; m'n' \rangle \tag{3.14}
\]

\[
= \delta_{\lambda \lambda'} \delta_{\mu \mu'} \delta_{\nu \nu'} \delta_{ss'} \delta_{tt'} \delta_{mm'} \delta_{nn'} \sum_S B^\lambda_S (s) B^{\lambda'}_S (s')
\]

The metric matrix for Bargmann-Moshinsky states is written as:

\[
\langle \lambda \mu \nu; k_s k_t; st; mn | \lambda' \mu' \nu'; k_s' k_t'; s' t'; m'n' \rangle \tag{3.15}
\]

\[
= \delta_{\lambda \lambda'} \delta_{\mu \mu'} \delta_{\nu \nu'} \delta_{ss'} \delta_{tt'} \delta_{mm'} \delta_{nn'} \sum_S C^\lambda_S (s) C^{\lambda'}_S (s')
\]

§3.4 \((K_s, K_t)\) Decomposition for Specific Representations

To illustrate our method we shall determine the \(K_s, K_t\) values for complete classification of \(s, t\) states for a few IR's. If the \(K_s, K_t\) values for a particular \(s, t\) (for a given IR) are desired, Eq.(3.10) is quite helpful, however, for analyzing the entire IR it is more convenient to follow graphical methods to be discussed below.

First we draw the intrinsic states, Fig.(3.2) for the
given IR \((\lambda, \mu, \nu)\), and restrict to the intrinsic states to the right of the dotted line. This is necessary to avoid duplication of states. For each of these intrinsic states we draw the following diagrams, Fig.(3.3), which give the s,t content of each intrinsic state \((K_s, K_t)\) depending upon whether it is s-stretched \((K_s \geq |K_t|)\) or t-stretched \((K_t \geq |K_s|)\). When all the \((K_s, K_t)\) values in the restricted range (to the right of the dotted line) are exhausted we have the required decomposition. Any particular \((s,t)\) value may have several \((K_s, K_t)\) pairs corresponding to it.

The number of these pairs for a particular \((s,t)\) gives the degeneracy of that \((s,t)\) multiplet in the IR \((\lambda, \mu, \nu)\).

The \((K_s, K_t)\) values for the classification of IR's \((2,2,3)\) and \((4,6,2)\) are obtained following the above procedure and are listed in Tables III and IV respectively. Figs.(3.4a) and (3.5a) show the intrinsic states and Figs.(3.4b) and (3.5b) show the degeneracies of the \((s,t)\) multiplets for these IR's. It is found that these degeneracies are in agreement with those obtained by following methods given by Draayer\(^8\) and Perelemow and Popov\(^25\).
\begin{align*}
\text{s-stretched} & \quad (K_s > |K_t|) \\
K_s \leq s & \leq K_s + \mu \\
|K_t| \leq t & \leq \mu - s + 2K_s \\
\text{t-stretched} & \quad (K_t > |K_s|) \\
K_t \leq t & \leq K_t + \mu \\
|K_s| \leq s & \leq \mu - t - 1 - 2K_t
\end{align*}

Fig. (3.3)

\((s,t)\) Contents for a Given \((K_s,K_t)\) Pair
Intrinsic States for the IR (2,2,3)

\[
\begin{array}{cccc}
\frac{1}{2} & 1 & 1 & 1 \\
\frac{3}{2} & 2 & 3 & 2 & 1 \\
\frac{5}{2} & 3 & 4 & 4 & 2 & 1 \\
\frac{7}{2} & 2 & 4 & 4 & 3 & 1 \\
\frac{9}{2} & 1 & 2 & 3 & 2 & 1 \\
\end{array}
\]

Fig. (3.4b)

Degeneracies of the \((s,t)\) multiplets for the IR \((2,2,3)\)
Intrinsic States for IR (4,6,2)

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Fig. (3.5a) Degeneracies of the (s,t) multiplets for the IR (4,6,2)
Chapter IV

WIGNER AND CLEBSCH-GORDAN COEFFICIENTS

The problem of obtaining the coupling coefficients of SU(4) in the non-canonical decomposition SU(4) \rightarrow SU(2) \times SU(2) is a difficult task; as it is indeed even for the simpler SU(3) group in the non-canonical chain SU(3) \rightarrow O(3). Apart from the expected algebraic complications of dealing with a higher group, the internal degeneracy problem makes matters worse— the general \((\lambda, \mu, \nu)\) states which solve the degeneracy problem are non-orthogonal. It is therefore very difficult to obtain the general Wigner coefficients for the supermultiplet bases.

Luckily, in the process of obtaining the general Elliott type state (3.6), we have also obtained certain special Clebsch-Gordan coefficients as a by-product. They couple orthogonal states of any two of the IR's \((\lambda, 0, 0), (0, \mu, 0), (0, 0, \nu)\) to give states which, in general, may involve internal degeneracy and are therefore of special importance. These will be discussed in section 4.2.

Apart from these special cases, we shall confine ourselves only to IR's that do not involve external or internal degeneracies. These IR's fall in following general classes.

\[(\lambda, 0, 0) \quad (0, \mu, 0) \quad (0, 0, \nu) \quad (\lambda, 1, 0) \quad (\lambda, 0, 1)\]

\[(0, \mu, 1) \quad (1, \mu, 0) \quad (1, 0, \nu) \quad (0, 1, \nu)\]  

(4.1)
The corresponding states are orthonormal. We shall develop closed expressions for Wigner coefficients for these cases by extending methods developed by Sharp et al. for SU(3) \(\supseteq O(3)\).

**§4.1 Definition of Clebsch-Gordan and Wigner Coefficients**

The SU(4) Clebsch-Gordan coefficients are defined as:

\[
\begin{align*}
|\lambda \mu \nu ; k^2 \kappa_t ; st, mn \rangle &= \sum_{s_1, s_2, t_1, t_2, m, n} \left| \lambda_1 \mu_1 \nu_1 ; s_1 t_1, m_1 n_1 \rightangle \left| \lambda_2 \mu_2 \nu_2 ; s_2 t_2, m_2 n_2 \rightangle \\
&= \left\langle \lambda_1 \mu_1 \nu_1, \lambda_2 \mu_2 \nu_2 \middle| \lambda \mu \nu \right\rangle_{s_1, t_1, m_1, n_1, s_2, t_2, m_2, n_2} \\
&= \left\langle \lambda_1 \mu_1 \nu_1, \lambda_2 \mu_2 \nu_2 \middle| \lambda \mu \nu \right\rangle_{s_1, t_1, m_1, n_1, s_2, t_2, m_2, n_2} \\
&= \left\langle \lambda_1 \mu_1 \nu_1, \lambda_2 \mu_2 \nu_2 \middle| \lambda \mu \nu \right\rangle_{s_1, t_1, m_1, n_1, s_2, t_2, m_2, n_2} \\
&= \left\langle \lambda_1 \mu_1 \nu_1, \lambda_2 \mu_2 \nu_2 \middle| \lambda \mu \nu \right\rangle_{s_1, t_1, m_1, n_1, s_2, t_2, m_2, n_2} \left\langle \lambda_1 \mu_1 \nu_1, \lambda_2 \mu_2 \nu_2 \middle| \lambda \mu \nu \right\rangle_{s_1, t_1, m_1, n_1, s_2, t_2, m_2, n_2}
\end{align*}
\]

The symbol in the conical brackets stands for an SU(4) Clebsch-Gordan coefficient (or C.G. coefficient). This definition holds for cases where \((\lambda_1 \mu_1 \nu_1)\) and \((\lambda_2 \mu_2 \nu_2)\) are the type (4.1), i.e. the corresponding states are orthonormal. In the general case, the C.G. coefficients may be defined differently \(11,28\).

The C.G. coefficient in (4.2) factors into a reduced C.G. coefficient and two ordinary C.G. coefficients:

\[
\begin{align*}
\left\langle \lambda_1 \mu_1 \nu_1, \lambda_2 \mu_2 \nu_2 \middle| \lambda \mu \nu \right\rangle_{s_1, t_1, m_1, n_1, s_2, t_2, m_2, n_2} &= \left\langle \lambda_1 \mu_1 \nu_1, \lambda_2 \mu_2 \nu_2 \middle| \lambda \mu \nu \right\rangle_{s_1, t_1, m_1, n_1, s_2, t_2, m_2, n_2} \\
&= \left\langle \lambda_1 \mu_1 \nu_1, \lambda_2 \mu_2 \nu_2 \middle| \lambda \mu \nu \right\rangle_{s_1, t_1, m_1, n_1, s_2, t_2, m_2, n_2} \\
&= \left\langle \lambda_1 \mu_1 \nu_1, \lambda_2 \mu_2 \nu_2 \middle| \lambda \mu \nu \right\rangle_{s_1, t_1, m_1, n_1, s_2, t_2, m_2, n_2} \\
&= \left\langle \lambda_1 \mu_1 \nu_1, \lambda_2 \mu_2 \nu_2 \middle| \lambda \mu \nu \right\rangle_{s_1, t_1, m_1, n_1, s_2, t_2, m_2, n_2} \\
&= \left\langle \lambda_1 \mu_1 \nu_1, \lambda_2 \mu_2 \nu_2 \middle| \lambda \mu \nu \right\rangle_{s_1, t_1, m_1, n_1, s_2, t_2, m_2, n_2} \left\langle \lambda_1 \mu_1 \nu_1, \lambda_2 \mu_2 \nu_2 \middle| \lambda \mu \nu \right\rangle_{s_1, t_1, m_1, n_1, s_2, t_2, m_2, n_2}
\end{align*}
\]
Confining attention to the non-degenerate states, the SU(4) Wigner coefficient is defined with the help of an SU(4) van der Waerden invariant S,

\[ S = \sum_{s_t, s_{t_1}, s_{t_2}, s_{t_3}} \sum_{m, m_1, m_2, m_3, n, n_1, n_2, n_3} |\lambda_1 \mu_1 \nu_1; s_t, m, n_1\rangle |\lambda_2 \mu_2 \nu_2; s_{t_1}, m_1, n_2\rangle |\lambda_3 \mu_3 \nu_3; s_{t_2}, m_2, n_3\rangle \times |\lambda_3 \mu_3 \nu_3; s_{t_3}, m_3, n_3\rangle \]

\[ \times \left( \frac{\lambda_1 \mu_1 \nu_1; \lambda_2 \mu_2 \nu_2; \lambda_3 \mu_3 \nu_3}{s_t, m, n_1; s_{t_1}, m_1, n_2; s_{t_2}, m_2, n_3; s_{t_3}, m_3, n_3} \right) \]

(4.4)

The Wigner coefficient factors into a reduced Wigner coefficient and two 3j-symbols:

\[ \begin{pmatrix} \lambda_1 \mu_1 \nu_1; & \lambda_2 \mu_2 \nu_2; & \lambda_3 \mu_3 \nu_3 \\ s_t, m, n_1; & s_{t_1}, m_1, n_2; & s_{t_2}, m_2, n_3; \end{pmatrix} \]

(4.5)

The Wigner coefficient is related to the C.G. coefficient as follows (details are given in Appendix D).

\[ \begin{pmatrix} \lambda_1 \mu_1 \nu_1; & \lambda_2 \mu_2 \nu_2; & \lambda_3 \mu_3 \nu_3 \\ s_t, m, n_1; & s_{t_1}, m_1, n_2; & s_{t_2}, m_2, n_3; \end{pmatrix} \]

(4.6a)

\[ = \mathcal{E} \left( \begin{pmatrix} \lambda_1 \mu_1 \nu_1; & \lambda_2 \mu_2 \nu_2; & \lambda_3 \mu_3 \nu_3 \\ s_t, m, n_1; & s_{t_1}, m_1, n_2; & s_{t_2}, m_2, n_3; \end{pmatrix} \right) \left( D_3 \right) \]
where \( \varepsilon = (-) \frac{1}{2} (\lambda^2 + \nu^2) - s_3 - m_3 + n_3 \) for \( s \)-stretched states

\( = (-) \frac{1}{2} (\lambda^2 + \nu^2) + \mu_3 + \tau_3 - m_3 + n_3 \) for \( t \)-stretched states \((4.6b)\)

and \( D_3 = \frac{1}{12} (\lambda_3 + 1) (\mu_3 + 1) (\nu_3 + 1) (\lambda_3 + \mu_3 + 2) (\mu_3 + \nu_3 + 2) (\lambda_3 + \mu_3 + \nu_3 + 3) \)

is the multiplicity of the IR \( (\lambda_3, \mu_3, \nu_3) \). Even though the states of our interest are non-degenerate and \( K_s, K_t \) are not needed for labelling yet they are important for the phase considerations. This is related to the fact that we are using Elliott states.

The reduced coefficients are then related by

\[
\begin{pmatrix}
\lambda_1 \mu_1 \nu_1 \\
\lambda_2 \mu_2 \nu_2 \\
\lambda_3 \mu_3 \nu_3
\end{pmatrix}
\begin{pmatrix}
s_1 \\
t_1 \\
s_2 \\
t_2 \\
s_3 \\
t_3
\end{pmatrix}
= \left( \begin{array}{ccc}
\lambda_1 & \mu_1 & \nu_1 \\
\lambda_2 & \mu_2 & \nu_2 \\
\lambda_3 & \mu_3 & \nu_3
\end{array} \right)
\begin{pmatrix}
s_1 \\
t_1 \\
s_2 \\
t_2 \\
s_3 \\
t_3
\end{pmatrix}
\times \left[ \frac{(2S_3 + 1)(2t_3 + 1)}{D_3 \lambda_3 \mu_3 \nu_3} \right]^{1/2} \Delta,
\]

with

\[
\Delta = (-) \frac{1}{2} (\lambda^2 + \nu^2) - s_1 + t_1 + s_2 - t_2 - s_3 \quad \text{for } s \text{-stretched states}
\]

\[
= (-) \frac{1}{2} (\lambda^2 + \nu^2) - s_1 + t_1 + s_2 - t_2 + \mu_3 + t_3 \quad \text{for } t \text{-stretched states}
\]

§4.2 C.G. Coefficients from Elliott States

The Elliott state \((3.6)\) was written as:

\[
|\lambda \mu \nu; K_\lambda K_\mu; st, mn\rangle
\]

\[
= \sum \sum \left| \lambda \mu \nu; s_\lambda s_\mu s_\nu, m_\lambda m_\mu m_\nu \right| |\lambda_\mu \nu; s_\lambda t_\mu t_\nu, m_\lambda m_\mu m_\nu \rangle |000; s_\nu s_\mu m_\nu m_\mu \rangle
\]

\[
x B_{\lambda \mu \nu}^{st} K_{\lambda \mu \nu}^{s_\lambda s_\mu s_\nu t_\lambda t_\mu t_\nu} m_\lambda m_\mu m_\nu m_\mu m_\nu m_\mu m_\nu
\]

\[
x |s_\lambda t_\mu t_\nu; s_\lambda s_\mu s_\nu t_\lambda t_\mu t_\nu\rangle
\]

\[
= \left( \begin{array}{ccc}
s_\lambda & s_\mu & s_\nu \\
m_\lambda & m_\mu & m_\nu
\end{array} \right)
\left( \begin{array}{ccc}
s_\lambda & s_\mu & s_\nu \\
m_\lambda & m_\mu & m_\nu
\end{array} \right)
\left( \begin{array}{ccc}
s_\lambda & s_\mu & s_\nu \\
m_\lambda & m_\mu & m_\nu
\end{array} \right)
\left( \begin{array}{ccc}
s_\lambda & s_\mu & s_\nu \\
m_\lambda & m_\mu & m_\nu
\end{array} \right)
\left( \begin{array}{ccc}
s_\lambda & s_\mu & s_\nu \\
m_\lambda & m_\mu & m_\nu
\end{array} \right)
\left( \begin{array}{ccc}
s_\lambda & s_\mu & s_\nu \\
m_\lambda & m_\mu & m_\nu
\end{array} \right)
\left( \begin{array}{ccc}
s_\lambda & s_\mu & s_\nu \\
m_\lambda & m_\mu & m_\nu
\end{array} \right)
\]
For $\mu=0$, we have

$$\lambda \mu \nu; K_s K_t; st, mn$$

$$= \sum_{s, \lambda, s, \mu, \nu} |\lambda \mu \nu; s, \lambda s, \lambda, m, \nu; s, \mu t, \mu, m, \nu; t, \mu t; s, t; 0, 0\rangle \cdot B_{s, t, K_s K_t}^{\lambda \mu \nu}(s, \lambda s, \lambda; s, \mu t, \mu; s, t; 0, 0)$$

$$\times \langle s, \lambda \mu | \langle s, \lambda \mu | m \rangle \rangle \langle s, \lambda \mu | n \rangle \rangle \langle n, \lambda \mu | n \rangle \rangle \langle n, \lambda \mu | n \rangle \rangle$$

(4.8)

and similarly, for $\nu=0$, (which fixes $K_s = K_t$, see Fig. (3.2))

$$\lambda \mu \nu; K_s K_t; st, mn$$

$$= \sum_{s, \lambda, s, \mu, \nu} |\lambda \mu \nu; s, \lambda s, \lambda, m, \nu; s, \mu t, \mu, m, \nu; t, \mu t; s, t; 0, 0\rangle \cdot B_{s, t, K_s K_t}^{\lambda \mu \nu}(s, \lambda s, \lambda; s, \mu t, \mu; s, t; 0, 0)$$

$$\times \langle s, \lambda \mu | \langle s, \lambda \mu | m \rangle \rangle \langle s, \lambda \mu | n \rangle \rangle \langle n, \lambda \mu | n \rangle \rangle \langle n, \lambda \mu | n \rangle \rangle$$

(4.9)

Thus in these cases, the coefficients $B$ can be identified as reduced C.G. coefficients:

$$\left\langle \lambda \mu \nu; s, \lambda s, \lambda, m, \nu; s, \mu t, \mu, m, \nu; t, \mu t; s, t; 0, 0\right\rangle = B_{s, t, K_s K_t}^{\lambda \mu \nu}(s, \lambda s, \lambda; s, \mu t, \mu; s, t; 0, 0)$$

$$= (-)^{\frac{1}{2}(s + s)} \left[ \frac{(2s + 1)(s + 1)}{(s + 1)! (s + 1)! (s + 1)!} \right]^{\lambda \mu \nu \nu}$$

$$\times \langle s, \lambda \mu | \langle s, \lambda \mu | m \rangle \rangle \langle s, \lambda \mu | n \rangle \rangle \langle n, \lambda \mu | n \rangle \rangle \langle n, \lambda \mu | n \rangle \rangle$$

(4.10)
\[ \left< \lambda \circ \circ \circ ; \circ \mu \circ \circ \circ \left| \lambda \mu \circ \circ \circ \right> = B^{\mu \circ \circ \circ}_{\circ \circ \circ \circ \circ \circ} \left( s_{\lambda} s_{\lambda}; s_{\mu} t_{\mu} \right) \]

\[ = (-)^{1/2} (\mu - s_{\mu} - t_{\mu}) \left( \frac{(2s_{\lambda} + 1) (2s_{\mu} + 1)(2t_{\mu} + 1)}{(\frac{1}{2} \lambda - s_{\lambda})! (\frac{1}{2} \lambda + s_{\lambda} + 1)!} \right)^{1/2} \]

\[ \times \left[ (\mu + s_{\mu} - t_{\mu} + 1)!! (\mu - s_{\mu} + t_{\mu} + 1)!! (\mu + s_{\mu} - t_{\mu} + 2)!! (\mu - s_{\mu} - t_{\mu})!! \right]^{-1/2} \]

\[ \times \left< s_{\lambda}; s_{\mu} \right| s \left< s_{\lambda}; t_{\mu} \right| t \right> \left< k_{S} o \left| k_{S} \right> \left< k_{S} o \right| k_{S} \right> \quad (4.11) \]

Similarly, with \( \lambda = 0 \) (which fixes \( K_{t} = -K_{s} \)), we have:

\[ \left< 0 0 0 0 ; 0 \mu 0 \circ \circ \circ \circ \circ \circ \circ \circ \left| 0 \mu \circ \circ \circ \circ \circ \circ \circ \circ \right> = B^{\circ \circ \circ \circ \circ \circ \circ \circ}_{\circ \circ \circ \circ \circ \circ \circ \circ} \left( o_{\nu} ; s_{\nu} s_{\nu} ; s_{\mu} t_{\mu} \right) \]

\[ = (-)^{1/2} (\mu + \nu - s_{\mu} - t_{\mu} + 1) o_{\nu} \left( \frac{(2s_{\nu} + 1)(2s_{\mu} + 1)(2t_{\mu} + 1)}{(\frac{1}{2} \nu - s_{\nu})! (\frac{1}{2} \nu + s_{\nu} + 1)!} \right)^{1/2} \]

\[ \times \left[ (\mu + s_{\mu} - t_{\mu} + 1)!! (\nu - s_{\mu} + t_{\mu} + 1)!! (\nu + s_{\mu} - t_{\mu} + 2)!! (\nu - s_{\mu} - t_{\mu})!! \right]^{-1/2} \]

\[ \times \left< s_{\nu}; s_{\mu} \right| s \left< s_{\nu}; t_{\mu} \right| t \right> \left< k_{S} o \left| k_{S} \right> \left< -k_{S} o \right| -k_{S} \right> \quad (4.12) \]
§4.3 Gaunt's Formula for (λ00) Type States

The calculations for SU(4) Wigner coefficients are facilitated by the observation that under O(4) the states \(|λ00;ss, mn⟩\) transform as hyperspherical harmonics in four dimensions, and that under O(6) (which is isomorphic to SU(4)) the states \(|0μ0;st, mn⟩\) transform as hyperspherical harmonics in six dimensions. This identification enables us to use the Gaunt formula for coupling of the hyperspherical harmonics to obtain a Gaunt formula for the corresponding supermultiplet states. (The \((0,μ,0)\) Gaunt formula will be discussed later in Section 4.6).

The \(|λ00;ss, mn⟩\) states have been derived under the supermultiplet scheme SU(4) \(\supseteq O(4) \cong SU(2) \times SU(2)\). If we define \(\mathcal{L} = s + t\) and \(\mathcal{K} = s - t\) then it is possible to organize these states with respect to the chain SU(4) \(\supseteq O(4) \supseteq O(3)\) with \(LM\) as the internal labels. The \(LM\) states are then related to the \(st, mn\) states by

\[
|λ00; 2s, LM⟩ = \sum_n |λ00; ss, mn⟩ \langle s \ s | L \rangle
\]

(4.13)

(2s indicates the degree in the \(O(3)\) variables while \(λ\) is the total degree; the difference \(λ - 2s\) is made up by powers of an \(O(3)\) scalar factor). In this notation, the new variables are written as:
\[ |100; 1, 00 \rangle = \alpha = (\Theta - \zeta)/\sqrt{2} \]

\[ |100; 1, 11 \rangle = \eta \]

\[ |100; 1, 10 \rangle = \beta = (\Theta + \zeta)/\sqrt{2} \]

\[ |100; 1, 1-1 \rangle = \xi \]

\( \alpha \) is an \( O(3) \) singlet while \( \eta \) and \( \xi \) form an \( O(3) \) triplet.

Now the hyperspherical harmonics in four dimensions are written as

\[ \Omega_{\ell \mu} = \frac{2 (n-L)! (n+L+1)! (n+1)!}{\pi} \]

\[ \times \sum_{\chi} \frac{(-)^{n-L-2\chi} \left( \cos \chi \right)^{n-L-2\chi} \left( \sin \chi \right)^{2L+2\chi}}{(n-L-2\chi)! (2L+2\chi)!} \]

which is homogeneous in \( \cos \chi \) and \( \sin \chi \), and from these

\[ \frac{\partial \Omega_{\ell \mu}}{\partial \phi} = \frac{\sqrt{\pi}}{\pi} \cos \chi \]

\[ \frac{\partial \Omega_{\ell \mu}}{\partial \mu} = -\frac{i}{\pi} \sin \chi \sin \Theta e^{i\phi} \]

\[ \frac{\partial \Omega_{\ell \mu}}{\partial \phi} = \frac{i\sqrt{\pi}}{\pi} \sin \chi \cos \Theta \]

\[ \frac{\partial \Omega_{\ell \mu}}{\partial \mu} = \frac{i}{\pi} \sin \chi \sin \Theta e^{-i\phi} \]

Thus to within a factor independent of \( LM \), the state \( \lambda_{00}^{2S,L\mu} \) must be the same functions of \( \eta \beta \xi \) as \( \Omega_{\ell \mu} \) is of \( \Omega^{2S}_{00}, \Omega^{2S}_{11}, \Omega^{2S}_{10}, \Omega^{2S}_{1-1} \).

We can identify

\[ \alpha = \sqrt{2} \cos \chi \]

\[ \eta = -i \sin \chi \sin \Theta e^{-i\phi} \]
Thus we can write
\[ |\chi^{00}; 2s, LM \rangle = N \mathcal{Y}_{LM}^{\lambda(2s)} \] (4.18)

the index \( \lambda \) means that the hyperspherical harmonic has been made homogeneous of degree \( \lambda \) in \( \cos \chi \) and \( \sin \chi \) by inserting, where necessary, powers of \( \cos^2 \chi + \sin^2 \chi = \frac{1}{2} (\alpha^2 - \beta^2 + 2\eta \xi) = (\eta \xi - \theta \xi) \) which is a scalar.

The factor \( N \) being independent of \( LM \) can be determined by choosing special values of \( LM \), e.g., \( L=M=s \), and is found to be
\[ N = \left( \frac{2\pi^2}{(\frac{1}{2} \lambda - s)! (\frac{1}{2} \lambda + s + 1)!} \right)^{1/2} \] (4.19)

which completes the identification.

The Gaunt formula for hyperspherical harmonics is given by
\[ \mathcal{G}_{L_1 M_1}^{\lambda_1(2s_1)} (\theta \phi \chi) \mathcal{G}_{L_2 M_2}^{\lambda_2(2s_2)} (\theta \phi \chi) \]
\[ = \sum_{s_3 L_3 M_3} \mathcal{G}_{L_3 M_3}^{\lambda_3(2s_3)} (\theta \phi \chi) \left[ \frac{(2s_1+1)(2s_2+1)}{2\pi^2(2s_3+1)} \right]^{1/2} \left\langle \frac{2s_1}{L_1 M_1}, \frac{2s_2}{L_2 M_2}, \frac{2s_3}{L_3 M_3} \right\rangle \] (4.20)
The last factor in (4.20) is an $O(4) \supset O(3)$ C.G. coefficient which is given in terms of the $O(4) \supset SU(2) \times SU(2)$ C.G. coefficient as

\[
\sum_{L_3} \left\langle \frac{2S_1}{L_1 M_1}, \frac{2S_2}{L_2 M_2}, \frac{2S_3}{L_3 M_3} \right| \sum_{S_3} \left\langle \frac{S_1}{m_1}, \frac{S_2}{m_2}, \frac{S_3}{m_3} \right| \left\langle \frac{n_1}{n_1}, \frac{n_2}{n_2}, \frac{n_3}{n_3} \right\rangle
\]  
(4.21)

Using the identification (4.18), we can express this result as a Gaunt formula for the $|\lambda 00; 2s, LM\rangle$ states, i.e.,

\[
|\lambda 00; 2s_1, L_1 M_1\rangle \left| \lambda 200; 2s_2, L_2 M_2\rangle
\]

\[
= \sum_{S_3} \left| \alpha_1 + \alpha_2 \cdot 0; 0, \frac{2S_3}{L_3 M_3}\right| \frac{N_3}{N_1 N_2} \left[ \frac{(2S_3 + 1)}{(2S_3 + 1)} \right]^{1/2} \left\langle \frac{2S_1}{L_1 M_1}, \frac{2S_2}{L_2 M_2}, \frac{2S_3}{L_3 M_3} \right| \right.
\]

Finally, we obtain the Gaunt formula for the $|\lambda 00; ss, mn\rangle$ states by combining (4.13, 4.19, 4.21 and 4.22) and using some simple properties of ordinary C.G. coefficients. The result is

\[
|\lambda 00; s_1 s_2, m_1 n_1\rangle \left| \lambda 200; s_3 s_2, m_2 n_2\rangle
\]

\[
= \sum_{S_3} \left[ \frac{\left\{ \frac{1}{2}(\lambda_1 + \lambda_2) - S_3 \right\} ! \left\{ \frac{1}{2}(\lambda_1 + \lambda_2) + S_3 + 1 \right\} !}{(2S_3 + 1)(2S_3 + 1)} \right]^{1/2}
\]

\[
x \left\langle \frac{s_1}{m_1}, \frac{s_2}{m_2}, \frac{s_3}{m_3} \right| \left\langle \frac{n_1}{n_1}, \frac{n_2}{n_2}, \frac{n_3}{n_3} \right| \frac{1}{\lambda_1 + \lambda_2} |00; s_3 s_2, m_3 n_3\rangle
\]

(4.23)
This Gaunt formula gives the coupling coefficients of the two states on the left-hand side to the state on the right-hand side, provided that the states are expressed in the same variables η, θ, and ζ. In this respect these coefficients differ from a C.G. coefficient which couples states from different spaces.

For convenience in later use we introduce an abbreviation and write the Gaunt formula as

\[
\begin{align*}
\langle \lambda_1 \cdot \cdot \cdot ; s_1 s_1, m_1 m_1 \rangle \langle \lambda_2 \cdot \cdot \cdot ; s_2 s_2, m_2 m_2 \rangle &= \sum_{s_3} \langle \lambda_1 \cdot \cdot \cdot , s_1 s_1, m_1 m_1 \rangle \langle \lambda_2 \cdot \cdot \cdot , s_2 s_2, m_2 m_2 \rangle \\
&\times \left\{ \begin{array}{c}
\lambda_1 \cdot \cdot \cdot , s_1 \\
\lambda_2 \cdot \cdot \cdot , s_2 \\
\lambda_1 + \lambda_2 \cdot \cdot \cdot , s_3
\end{array} \right\}
\end{align*}
\]  

(4.24a)

where,

\[
\left\{ \begin{array}{c}
\lambda_1 \cdot \cdot \cdot , s_1 \\
\lambda_2 \cdot \cdot \cdot , s_2 \\
\lambda_1 + \lambda_2 \cdot \cdot \cdot , s_3
\end{array} \right\} = \left[ \begin{array}{c}
\frac{1}{2}(\lambda_1 + \lambda_2 - s_3)! \\
\frac{1}{2}(\lambda_1 + \lambda_2 + s_3 + 1)! \\
(2s_1 + 1)! (2s_2 + 1)!
\end{array} \right] ^{1/2}
\left[ \begin{array}{c}
\frac{1}{2} \lambda_1 - s_1! \\
\frac{1}{2} \lambda_1 + s_1 + 1! \\
\frac{1}{2} \lambda_2 - s_2! \\
\frac{1}{2} \lambda_2 + s_2 + 1! \\
(2s_3 + 1)!
\end{array} \right]
\]

Because of the similar form of \((\lambda, 0, 0)\) and \((0, 0, \nu)\) states, the Gaunt formula for coupling of \((0, 0, \nu)\) states is obtained simply by replacing \(\lambda\) by \(\nu\) in the above equation. (See also Eq. (2.9)).
§4.4 The SU(4) Invariants

The invariant \( S \) in the definition (4.4) of the SU(4) Wigner Coefficient can be constructed by multiplying suitable powers of the scalars formed by coupling simple representations. These scalars arise in the following Clebsch-Gordan series and are underlined where they occur.

\begin{align*}
(100) \times (001) & = (101) + (000) \\
(010) \times (010) & = (020) + (101) + (000) \\
(100) \times (100) \times (010) & = (210) + (020) + 2(101) + (000) \\
(001) \times (001) \times (010) & = (012) + (020) + 2(101) + (000) \\
(010) \times (010) \times (101) & = (121) + 2(210) + 2(012) + (202) + 2(020) + 4(101) + (000)
\end{align*}

etc.

We adopt a notation such that, for example, \((100; 100; 010)\) refers to the SU(4) scalar formed by coupling \((100)\) in space 1, \((100)\) in space 2, and \((010)\) in space 3.

The algebraic expressions for these scalars can be calculated as follows.

Consider the scalar \( B_{12} = (100, 001, 000) \). It is particularly simple because in the C.G. series

\[(100) \times (001) = (101) + (000)\]

the IR \((101)\) has no \( s=t=0 \) state. Thus the \( O(4) \) coupling is sufficient to write the scalar \( B_{12} \) as
\( B_{12} = (100, 001, 000) \)

\[
= \sum_{mn} |100; \frac{1}{2}, m n_1 \rangle |001; \frac{1}{2}, -m - n_2 \rangle \left( \begin{array}{c}
\frac{1}{2} \\
\frac{1}{2} \\
0
\end{array} \right) \left( \begin{array}{c}
m - m_0 \\
n - n_0
\end{array} \right)
\]

\[
= \sum_{mn} |100; \frac{1}{2}, m n_1 \rangle |001; \frac{1}{2}, -m - n_2 \rangle \frac{(-1)^{m-n}}{2}
\]

\[
= (\eta_1 \eta_1^* + \zeta_1 \zeta_2^* + \Theta_1 \Theta_2^* + \xi_1 \xi_2^*)
\]

(4.26)

ignoring normalization. There are six scalars of this type

\[
B_{ij} = \eta_i \eta_j^* + \zeta_i \zeta_j^* + \Theta_i \Theta_j^* + \xi_i \xi_j^*, \quad i \neq j, \; i, j = 1, 2, 3.
\]

(4.27)

In general, however, in addition to the desired scalar there are other \( s=t=0 \) states belonging to some IR's in the C.G. series (4.25). It is then necessary to ensure the proper SU(4) coupling together with the O(4) coupling. As an example consider \( D=(100;100;010) \) which is, of course, orthogonal to the \( s=t=0 \) states of the IR's in the C.G. series. If we write

\[
D = (100;100;010)
\]

(4.28)

\[
= \sum_{s m_1 m_2} |100; \frac{1}{2}, m, n_1 \rangle |100; \frac{1}{2}, m_2, n_2 \rangle |010; s, 1-s : m_3, n_3 \rangle
\]

\[
x \left( \begin{array}{c}
\frac{1}{2} \\
\frac{1}{2} \\
s
\end{array} \right) \left( \begin{array}{c}
m_1 \\
m_2 \\
m_3
\end{array} \right) \left( \begin{array}{c}
n_1 \\
n_2 \\
n_3
\end{array} \right) A_5
\]
then the coefficient $A_s$, which determines the SU(4) coupling, is determined by demanding that $D$ be orthogonal to the other states at $s=t=0$, e.g., by operating with some of the operators $u_{mn}$ and equating the result to zero. We find $A_s = (-1)^s$. Hence, the expression for $D$ in the 1, 2, and 3 variables is:

$$D = \left( \theta_1 \xi_2 - \xi_1 \theta_2 \right) \rho_3 + \frac{1}{\sqrt{2}} (\xi_1 \eta_2 - \xi_2 \eta_1 + \zeta_1 \theta_2 - \zeta_2 \theta_1) \sigma_3 + (\eta_1 \zeta_2 - \eta_2 \zeta_1) \tau_3$$

$$+ (\xi_1 \xi_2 - \xi_2 \xi_1) \rho'_3 + \frac{1}{\sqrt{2}} (\xi_2 \eta_2 - \xi_1 \eta_1 - \zeta_1 \theta_2 + \zeta_2 \theta_1) \sigma'_3 + (\eta_1 \theta_2 - \eta_2 \theta_1) \tau'_3$$

The other scalars may be determined in a similar manner.

When we have a product of these scalars the result is still a scalar, i.e., the invariant $S$. However, it is the degree of variables from each space which determines the Wigner coefficient that $S$ defines, Eq.(4.4). This will be seen more clearly in the next Section.

§4.5 Wigner Coefficients with arbitrary $\lambda$ or $\nu$.

We shall combine the results from the previous Sections to obtain Wigner coefficients for certain classes of non-degenerate states with arbitrary $\lambda$ or $\nu$.

(i) The Wigner coefficient $(\lambda_1 00; \lambda_2 00 ; 00 \lambda_1 + \lambda_2)$:

Consider the invariant

$$S = N \mathcal{B}^{\lambda_1}_{15} \mathcal{B}^{\lambda_2}_{23}$$

(4.30a)
\[ N = \left[ \frac{6}{\lambda_1! \lambda_2! (\lambda_1 + \lambda_2 + 3)!} \right]^{1/2} \]  

(4.30b)

is the normalization constant. \( B_{13}^{\lambda_1} \) can be written as

\[ B_{13}^{\lambda_1} = \lambda_1! \sum_{s m n} |\lambda_0 0 0; s, s, m, n_1\rangle |0 0 \lambda_1; s, s, -m, -n_2\rangle_{3}^{(\lambda_1 + m_1 - n_1)} \]  

(4.31)

Thus \( S \) can be used to define a reduced Wigner coefficient:

\[ S = \sum_{s_1, s_2, s_3} |\lambda_0 0 0; s_1, s_2, m, n_1\rangle |\lambda_0 0 0; s_2 s_2, m_2, n_2\rangle_{2} |0 0 \lambda_1 + \lambda_2; s_3 s_3, m_3, n_3\rangle_{3} \]

\[ \times \left( \lambda_0 0 0 \lambda_2 0 0 0 0 \lambda_1 + \lambda_2 \right)_{s_1 s_2 s_3} \left( s_1, s_2, s_3 \right)_{m_1, m_2, m_3, n_1, n_2, n_3} \]  

(4.32)

With the help of expansion (4.31) \( S \) is written as

\[ S = N \lambda_1! \lambda_2! \sum_{s_1, s_2, s_3} |\lambda_0 0 0; s_1, s_2, m_1, n_1\rangle |\lambda_2 0 0; s_2 s_2, m_2, n_2\rangle_{2} |0 0 \lambda_1 + \lambda_2; s_3 s_3, m_3, n_3\rangle_{3} \]

\[ \times \left( \lambda_1 0 0 + \lambda_2 0 0 - m_1 - n_1 \right)_{s_1 s_2 s_3} \left( 0 0 \lambda_1 + \lambda_2 s_2 s_2 - m_2 - n_2 \right)_{3} \]

Using the Gaunt formula (4.23), we can couple the states in variable 3 to give

\[ S = \sqrt{\frac{6 \lambda_1! \lambda_2!}{(\lambda_1 + \lambda_2 + 3)!}} \sum_{s_1, s_2, s_3} |\lambda_0 0 0; s_1, s_2, m_1, n_1\rangle |\lambda_2 0 0; s_2 s_2, m_2, n_2\rangle_{2} |0 0 \lambda_1 + \lambda_2; s_3 s_3, m_3, n_3\rangle_{3} \]

\[ \times \left[ \frac{\left( \frac{1}{2} \lambda_1 (\lambda_1 + \lambda_2) - s_3 \right)! \left( \frac{1}{2} (\lambda_1 + \lambda_2 + s_3 + 1)! \times (2s_1)! \left( 2s_2 \right)! \left( 2s_3 + 1 \right)!} \right]^{1/2}} \left( \frac{1}{2} \lambda_2 - s_2 \right)! \left( \frac{1}{2} (s_1 + s_2 + 1) \right)! \left( \frac{1}{2} \lambda_2 - s_2 \right)! \left( \frac{1}{2} (s_1 + s_2 + 1) \right)! \]

\[ \times \left( s_1 s_2 s_3 \right)_{s_1 s_2 s_3} \left( n_1, n_2, n_3 \right) \]  

(4.33)
(the phase factors all cancelled with the extra phases obtained while converting the ordinary C.G. coefficients in the Gaunt formula to 3-j symbols).

On comparison of Eqs. (4.32 and 4.33) we immediately get the reduced Wigner coefficient:

\[
\left( \begin{array}{ccc} \lambda_1 \rho_0 & \lambda_2 \rho_0 & \lambda_1 + \lambda_2 \\ s_1 & s_2 & s_3 \end{array} \right) \quad (4.34)
\]

\[
= \left\{ \frac{6 \lambda_1! \lambda_2! \left[ \frac{1}{2} (\lambda_1 + \lambda_2) - s_3 \right]! \left[ \frac{1}{2} (\lambda_1 + \lambda_2 + s_3 + 1) \right]! (2s_1 + 1)(2s_2 + 1)(2s_3 + 1)!}{(\lambda_1 + \lambda_2 + 3)! (\frac{1}{2} \lambda_1 - s_1)! (\frac{1}{2} \lambda_1 + s_1 + 1)! (\frac{1}{2} \lambda_2 - s_2)! (\frac{1}{2} \lambda_2 + s_2 + 1)!} \right\}^{1/2}
\]

(ii) The Wigner Coefficient \((\lambda_1 00; \lambda_2 00; 0 1 \lambda_1 + \lambda_2 - 2)\):

Consider the invariant

\[
S = N \, B_{13}^{\lambda_1^{-1}} \, B_{23}^{\lambda_2^{-1}} \, D
\]

where

\[
N = \left( \frac{2}{(\lambda_1 - 1)! (\lambda_2 - 1)! (\lambda_1 + \lambda_2 - 2)!} \right)^{1/2}
\]

is the normalization constant and \(D\) is the scalar (4.28).

The invariant \(S\) defines a reduced Wigner coefficient:

\[
S = \sum_{s_1 s_2 s_3 t_3} \left( \begin{array}{c} \lambda_1 \rho_0 \lambda_2 \rho_0 0 1 \lambda_1 + \lambda_2 - 2 \\ s_1 s_2 s_3 t_3 \end{array} \right) \left( \begin{array}{c} \lambda_1 \rho_0 \lambda_2 \rho_0 0 1 \lambda_1 + \lambda_2 - 2 \\ m_1 m_2 m_3 \end{array} \right) \left( \begin{array}{c} t_1 t_2 t_3 \\ n_1 n_2 n_3 \end{array} \right)
\]

\[
\times \left( \begin{array}{ccc} \lambda_1 \rho_0 & \lambda_2 \rho_0 & 0 1 \lambda_1 + \lambda_2 - 2 \\ s_1 s_2 s_3 t_3 \end{array} \right) \left( \begin{array}{ccc} s_1 & s_2 & s_3 \\ t_1 & t_2 & t_3 \\ m_1 & m_2 & m_3 \\ n_1 & n_2 & n_3 \end{array} \right)
\]

\[
(4.36)
\]
In terms of (4.28 and 4.31) the invariant \( S \) is written as:

\[
S = (\lambda_1^{-1})! (\lambda_2^{-1})! N
\]

\[
x \sum_{s_i' s_i'' m_i' n_i'} |100; s_i' s_i'' m_i' n_i'\rangle |00; \lambda_1^{-1}; s_i' s_i'' m_i' n_i'\rangle (-)^{\lambda_1^{-1}-m_i'-n_i'}
\]

\[
x \sum_{s_i' s_i'' m_i' n_i'} |\lambda_2^{-1} 100; s_i' s_i'' m_i' n_i'\rangle |00; \lambda_2^{-1}; s_i' s_i'' m_i' n_i'\rangle (-)^{\lambda_2^{-1}-m_i'-n_i'}
\]

\[
x \sum_{s_i' s_i'' m_i' n_i'} |100; \frac{1}{2} \frac{1}{2}, m_i'' n_i''\rangle |00; \frac{1}{2} \frac{1}{2}, m_i'' n_i''\rangle |010; s_3' 1-s_3', m_3'' n_3''\rangle
\]

\[
=(-)^{s_3'} \left( \begin{array}{ccc} \frac{1}{2} & \frac{1}{2} & s_3' \\ m_i'' & m_3'' & m_i'' \end{array} \right) \left( \begin{array}{ccc} \frac{1}{2} & \frac{1}{2} & 1-s_3' \\ n_i'' & n_3'' & n_3'' \end{array} \right)
\]

(4.37)

\[
= (\lambda_1^{-1})! (\lambda_2^{-1})! N
\]

\[
x \sum_{s_i s_i' s_i'' m_i n_i} |\lambda_1 00; s_i s_i' s_i'' m_i n_i\rangle |\lambda_2 00; s_2 s_2' s_2'' m_2 n_2\rangle |01 \lambda_1^{-1} \lambda_2^{-2}; s_3 s_3', m_3'' n_3''\rangle
\]

\[
x \sum_{s_i' s_i'' m_i' n_i'} \left\{ \begin{array}{ccc} \lambda_1^{-1} & 0 & 0 \\ s_i' & \frac{1}{2} & s_i \\ \lambda_1^{-1} & 0 & 0 \end{array} \right\} \left\{ \begin{array}{ccc} \lambda_2^{-1} & 0 & 0 \\ s_i' & \frac{1}{2} & s_i \\ \lambda_2^{-1} & 0 & 0 \end{array} \right\}
\]

\[
=(-)^{s_3'} \left( \begin{array}{ccc} \frac{1}{2} & \frac{1}{2} & 1-s_3' \\ m_i'' & m_3'' & m_i'' \end{array} \right) \left( \begin{array}{ccc} \frac{1}{2} & \frac{1}{2} & 1-s_3' \\ n_i'' & n_3'' & n_3'' \end{array} \right)
\]

(4.38)
where we have used the Gaunt formula (4.24) to couple the states in space 1, the states in space 2, and the first two states in space 3; the resulting state in space 3 is then coupled with the last state ((010) type) in space 3 using an SU(4) C.G. coefficient (4.12). We have thus obtained Eq.(4.38) in the same form as Eq.(4.36) and the Wigner coefficient in (4.36) is obtained by comparing these equations. However, before we make the comparison, we can further simplify Eq. (4.38) using properties of 3-j symbols \(^{30}\). The ten 3-j symbols in Eq.(4.38), summed over their m (and n) values, contract to give two 9-j symbols and two 3-j symbols; the latter are identical to the ones appearing in Eq.(4.36). We can now compare the two equations and get for the reduced Wigner coefficient:

\[
\begin{pmatrix}
\lambda_1 & \lambda_2 & \lambda_3 & \lambda_4 & \lambda_5 & \lambda_6 & \lambda_7 & \lambda_8 & \lambda_9 \\
\ell_1 & \ell_2 & \ell_3 & \ell_4 & \ell_5 & \ell_6 & \ell_7 & \ell_8 & \ell_9
\end{pmatrix}
\]

(4.39)

\[
\frac{2(\lambda_1-1)! (\lambda_2-2)!}{(\lambda_1+\lambda_2-2)!} \left\{ \begin{array}{ccc}
\lambda_1 & \lambda_2 & \lambda_3 \\
\ell_1 & \ell_2 & \ell_3
\end{array} \right\} \left\{ \begin{array}{ccc}
\lambda_4 & \lambda_5 & \lambda_6 \\
\ell_4 & \ell_5 & \ell_6
\end{array} \right\} \left\{ \begin{array}{ccc}
\lambda_7 & \lambda_8 & \lambda_9 \\
\ell_7 & \ell_8 & \ell_9
\end{array} \right\}
\]

\[
\sum \left\{ \begin{array}{ccc}
S_1' & S_2' & S_3' \\
\ell_1' & \ell_2' & \ell_3'
\end{array} \right\} \left\{ \begin{array}{ccc}
S_1 & S_2 & S_3 \\
\ell_1 & \ell_2 & \ell_3
\end{array} \right\} \left\{ \begin{array}{ccc}
S_4' & S_5' & S_6' \\
\ell_4' & \ell_5' & \ell_6'
\end{array} \right\} \left\{ \begin{array}{ccc}
S_7' & S_8' & S_9' \\
\ell_7' & \ell_8' & \ell_9'
\end{array} \right\}
\]

\[
x \left\{ \begin{array}{ccc}
S_1' & S_2' & S_3' \\
\ell_1' & \ell_2' & \ell_3'
\end{array} \right\} \left\{ \begin{array}{ccc}
S_1 & S_2 & S_3 \\
\ell_1 & \ell_2 & \ell_3
\end{array} \right\} \left\{ \begin{array}{ccc}
S_4' & S_5' & S_6' \\
\ell_4' & \ell_5' & \ell_6'
\end{array} \right\} \left\{ \begin{array}{ccc}
S_7' & S_8' & S_9' \\
\ell_7' & \ell_8' & \ell_9'
\end{array} \right\} \times (-)^{S_3} (2S_3+1)
\]

The summation indices take only the following range of values:

\[
s_1' = s_1 \pm \frac{1}{2}, \quad s_2' = s_2 \pm \frac{1}{2}, \quad s_1' = |s_1' - s_2'|, \quad s_3' = 0, 1.
\]
(iii) The Wigner Coefficient \( \lambda_1^0 1 : \lambda_2^0 0 1 : 1^0 \lambda_1^1 + \lambda_2^2 \)

We shall consider one more Wigner coefficient in the present class (i.e., with arbitrary \( \lambda \)). However, since the details are similar to the cases discussed before, we shall only give the result and the invariant \( S \) which defines this Wigner coefficient.

\[
S = N \cdot B_1^{1 \lambda_1} B_2^{\lambda_2} B_{31}
\]

where,

\[
N = \left( \frac{2}{\lambda_1! \lambda_2! (\lambda_1 + \lambda_2 + 2)! (\lambda_1 + \lambda_2 + 4)} \right)^{\frac{1}{2}}
\]  

(4.40)

(4.41)

the resulting reduced Wigner coefficient is:

\[
\begin{pmatrix}
\lambda_1^0 1 \\
\lambda_2^0 0 1 \\
1^0 \lambda_1^1 + \lambda_2^2
\end{pmatrix}
\]

\[
\begin{pmatrix}
s_1, t_1 \\
s_2, s_2 \\
s_3, t_3
\end{pmatrix}
\]

\[
= \left( \frac{2 \lambda_1! \lambda_2!}{(\lambda_1 + \lambda_2 + 2)! (\lambda_1 + \lambda_2 + 4)} \right)^{\frac{1}{2}} (-)^{\lambda_1 + \lambda_2 + s_2 + t_3 + 1}
\]

\[
\left[ (2s_1 + 1)(2s_2 + 1)(2s_3 + 1)(2t_1 + 1)(2t_2 + 1)(2t_3 + 1) \right]^{\frac{1}{2}}
\]

\[
\sum_{s_1', s_2', s_3'} \begin{pmatrix}
0 0 \lambda_1 \\
0 0 \lambda_2 \\
0 0 \lambda_1 + \lambda_2
\end{pmatrix}
\begin{pmatrix}
s_1' \\
s_2' \\
s_3'
\end{pmatrix}
\begin{pmatrix}
\lambda_1^0 1 \\
\lambda_2^0 1 \\
\lambda_1^1 + \lambda_2^2
\end{pmatrix}
\begin{pmatrix}
s_1, t_1 \\
s_2, s_2 \\
s_3, t_3
\end{pmatrix}
\]

\[
x \left[ \begin{pmatrix}
0 0 \lambda_1 \\
0 0 \lambda_2 \\
0 0 \lambda_1 + \lambda_2
\end{pmatrix}
\begin{pmatrix}
s_1' \\
s_2' \\
s_3'
\end{pmatrix}
\begin{pmatrix}
\lambda_1^0 1 \\
\lambda_2^0 1 \\
\lambda_1^1 + \lambda_2^2
\end{pmatrix}
\begin{pmatrix}
s_1, t_1 \\
s_2, s_2 \\
s_3, t_3
\end{pmatrix}
\right]
\]

\[
x \left\{ \begin{pmatrix}
s_1, s_2, s_3 \\
t_1, s_2, t_3
\end{pmatrix}
\begin{pmatrix}
s_1' \\
\frac{1}{2}
\end{pmatrix}
\begin{pmatrix}
s_2 \\
\frac{1}{2}
\end{pmatrix}
\begin{pmatrix}
s_3 \\
\frac{1}{2}
\end{pmatrix}
\right\}, \text{ with } s_1' = s_1 + \frac{1}{2}, s_2' = s_2 + \frac{1}{2}, s_3' = s_3 + \frac{1}{2}
\]

The last two symbols in curly brackets are 6-j symbols. For the other three symbols, refer to Eqs.(4.10 and 4.24).
§ 4.6. The Wigner Coefficient \((0\mu_1^0; 0\mu_2^0; 0\mu_1+\mu_2^0)\):

We shall consider only one case—the stretched coupling of the IR \((0\mu_1^0)\) and \((0\mu_2^0)\).

The \((0,\mu,0)\) state (Eq. (2.13)) was written as

\[ |0\mu; st, mn\rangle = N_{st}^{\mu} (-)^{\frac{1}{2}(\mu-s-t)} \]

\[ \times \sum_x \frac{c(x+2\rho\tau)^{x}(\sigma^2-2\rho\tau\tau)^{\frac{1}{2}(\mu-s-t)-x}}{x! (2x+2t+1)! \{\frac{1}{2}(\mu-s-t)-x\}! \{\mu-s+t+2x+1\}!} \]

\[ \times s! \left[ \frac{2^{s+m} (s+m)! (s-m)!}{(2s)!} \right]^{\frac{1}{2}} \sum_y \frac{\rho_y \sigma^{s+m-2\nu} \tau \gamma^{y-m}}{2^y y! (s+m-2y)! (y-m)!} \]

\[ \times t! \left[ \frac{2^{t+n} (t+n)! (t-n)!}{(2t)!} \right]^{\frac{1}{2}} \sum_z \frac{\rho_z \sigma^{t+n-2\xi} \tau \zeta^{z-n}}{2^z z! (t+n-2z)! (z-n)!} \]

Now the Eq. (2.3) of reference (27) can be re-written as

\[(\sigma^2-2\rho\tau)^{x} \sum_y \frac{\rho_y \sigma^{L+M-2\nu} \tau \gamma^{y-M}}{2^y y! (L+M-2y)! (y-M)!} \]

\[= \left[ \frac{x! (2L+2x+1)!}{2^{L+M} (L+x)! (L+M)! (L-M)! (2L+1)!} \right]^{\frac{1}{2}} |L+2x\rangle \]

\[ \left. \left| L \begin{array}{c} \frac{1}{2} \end{array} M \right) \right| \quad \quad (4.43) \]

where \( |L+2x\rangle \) is an \(O(3)\) state and \(L+2x\) is the total degree.

In terms of (4.43), the \((0,\mu,0)\) state can be written as:
\[ |0_{\mu o; st, mn} \rangle = N_{st}^{\mu} (-\frac{1}{2})^{(\mu+s-t)} \frac{1}{s! t!} 2^{\frac{1}{2}(\mu+s+t)} \frac{1}{(2s+1)!(2t+1)!} \]

\[ \times \sum_{p} \left[ \frac{\frac{1}{2}((\mu-p+t)!)(s+p)!}{\frac{1}{2}((\mu-p-t)!(\mu-p+t+1)!(s-p)!(s+p+1)!)} \right]^{1/2} \]

\[ \times \left| \mu-p \right\rangle \left| t n \right\rangle \left| s m \right\rangle \] (4.44)

where the new summation dummy \( p = \mu - t - 2x \) takes only the alternate integral values. The \( O(3) \) state \( \left| p \right\rangle_{s m} \) is written in the \( \rho, \sigma, t \) variables while the state \( \left| \mu-p \right\rangle \left| t n \right\rangle \) is written in \( \rho', \sigma', t' \).

Under \( O(3) \) the state \( \left| p \right\rangle_{L M} \) transforms in the same way as \( Y_{LM}(\theta, \phi) \), the spherical harmonics. This fact has been used in reference (27) to derive a Gaunt formula for the \( O(3) \) states, viz.,

\[ \left| p_1 \right\rangle_{L_1 M_1} \left| p_2 \right\rangle_{L_2 M_2} = \sum_{L_3} C_{p_1 p_2}^{P_1 P_2} \left| p + p_2 \right\rangle_{L_1 L_2 L_3} \left\langle L_1 L_2 L_3 \right| \left( L_1 M_1, L_2 M_2 L_3 M_3 \right) \] (4.45a)

where \( C \) stands for

\[ C_{p_1 p_2}^{P_1 P_2} = 2^{\frac{1}{2}(L_1+L_2-L_3)} \frac{\{1/2(P_1+L_1)\}! \{1/2(P_2+L_2)\}! \left(2L_1+1\right)\left(2L_2+1\right)!}{\{1/2(P_1-L_1)\}! \{1/2(P_2-L_2)\}! \left(2P_1+1\right)\left(2P_2+1\right)!} \]

\[ \times \frac{\{1/2(P_1+P_2-L_3)\}! \left(P_1+P_2+L_3+1\right)!}{\{1/2(P_1+P_2+L_3)\}! \left(2L_3+1\right)!} \frac{1}{(2L_3+1)} \left\langle L_1 L_2 L_3 \right| \left( O O O \right) \] (4.45b)

Consider now the scalar \( C_{13} = (010; 000; 010) \),

\[ C_{13} = (\rho_3 \tau_3 - \sigma_3 \sigma_3 + \tau_1 \rho_3 - \rho_1' \tau_3' + \sigma_1' \sigma_3' - \tau_1' \rho_3') \] (4.46)

\[ = (\cos \Theta_{13} - \cos \Theta_{13}) \]
where \( \theta_{13} \) is the angle between the 1 and 3 directions and \( \theta_{1',3'} \) is the angle between the 1' and 3' directions. Similarly, we can write

\[
C_{23} = (000; 010; 010)
\]

\[
= (P_2 T_3 - \sigma_2 \sigma_3' + \sigma_3' T_3 - \sigma_3 T_3')
\]

\[
= (\cos \theta_{3'2'} - \cos \theta_{23})
\]

We form an SU(4) scalar invariant

\[
S = N C_{13}^{\mu_1} C_{23}^{\mu_2}
\]

where

\[
N = \left( \frac{2}{\mu_1! \mu_2! (\mu_1 + \mu_2 + 3)! (\mu_1 + \mu_2 + 2)!} \right)^{1/2}
\]

is the normalization constant.

We can now define a Wigner coefficient:

\[
S = \sum_{s_1 s_2 s_3 \underline{t}_1 \underline{t}_2 \underline{t}_3} \sum_{m_1 m_2 m_3} \left| 0 \mu_1 0; s_1 t_1, m_1 n_1 \right> \left| 0 \mu_2 0; s_2 t_2, m_2 n_2 \right> \left| 0 \mu_1' + \mu_2 0; s_3 t_3, m_3 n_3 \right>
\]

\[
\times \left( \begin{array}{ccc} 0 & \mu_1 & \mu_2' \mu_2 \\ 0 & \mu_2 & \mu_1' + \mu_2 \\ 0 & \mu_1' + \mu_2 & 0 \end{array} \right) \left( \begin{array}{ccc} s_1 & s_2 & s_3 \\ t_1 & t_2 & t_3 \\ m_1 & m_2 & m_3 \end{array} \right) \left( \begin{array}{ccc} n_1 & n_2 & n_3 \end{array} \right)
\]

Expanding (4.48a) using binomial theorem we get:
Using (4.45) we couple the states in $3,3'$ variables to get

\begin{align}
S &= N \sum_{\mu_1, \mu_2} (-)^{\mu_1 + \mu_2} \sum_{s_1, s_2} \sum_{s_1', s_2'} (-)^{s_1 + s_2 + s_1' + s_2'} \sum_{m_1, m_2, m_1', m_2'} (-)^{m_1 + m_2 + m_1' + m_2'} \\
&\times \begin{pmatrix}
\mu_1 - p_1 \\
- t_1, n_1, t_1, n_1', s_1, m_1, s_1, -m_1, 3 \\
\mu_2 - p_2 \\
- t_2, n_2, t_2, -n_2, s_2, m_2, s_2, -m_2, 3
\end{pmatrix} \begin{pmatrix}
p_1 \\
s_1, m_1, -s_1, m_1, 1 \\
p_2 \\
s_2, m_2, -s_2, m_2, 1
\end{pmatrix} \begin{pmatrix}
\mu_1 + p_1 \\
- t_1, n_1, t_1, n_1', s_1, m_1, s_1, -m_1, 3 \\
\mu_2 + p_2 \\
- t_2, n_2, t_2, -n_2, s_2, m_2, s_2, -m_2, 3
\end{pmatrix} \\
&= N \sum_{\mu_1, \mu_2} (-)^{\mu_1 + \mu_2} \sum_{s_1, s_2, s_3} \sum_{s_1', s_2', s_3'} (-)^{s_1 + s_2 + s_1' + s_2' + s_3 + s_3'} \sum_{m_1, m_2, m_1', m_2', m_3, m_3'} (-)^{m_1 + m_2 + m_1' + m_2' + m_3 + m_3'} \\
&\times \begin{pmatrix}
\mu_1 - p_1 \\
- t_1, n_1, t_1, n_1', s_1, m_1, s_1, -m_1, 3 \\
\mu_2 - p_2 \\
- t_2, n_2, t_2, -n_2, s_2, m_2, s_2, -m_2, 3
\end{pmatrix} \begin{pmatrix}
p_1 \\
s_1, m_1, -s_1, m_1, 1 \\
p_2 \\
s_2, m_2, -s_2, m_2, 1
\end{pmatrix} \begin{pmatrix}
\mu_1 + p_1 \\
- t_1, n_1, t_1, n_1', s_1, m_1, s_1, -m_1, 3 \\
\mu_2 + p_2 \\
- t_2, n_2, t_2, -n_2, s_2, m_2, s_2, -m_2, 3
\end{pmatrix} \\
&\times \begin{pmatrix}
- t_3, n_3, -t_3, -n_3, s_3, -m_3, s_3, m_3, 3
\end{pmatrix} \begin{pmatrix}
p_1 + p_2 \\
s_3, m_3, -s_3, m_3, 1
\end{pmatrix} \
\end{align}

(4.51)
To evaluate the reduced Wigner coefficient in Eq. (4.49) we take scalar product of $S$ with the three states

$$
|0 \mu_1 0; s_1 t_1, -s_1 t_1 \rangle \quad |0 \mu_2 0; s_2 t_2, -s_2 t_2 \rangle \quad |0 \mu_3 0; s_3 t_3, -s_3 t_3 \rangle
$$

and divide by the $3$-$j$ symbols

$$
\begin{pmatrix}
  s_1 & s_2 & s_3 \\
  -s_1 & -s_2 & s_3
\end{pmatrix}
\begin{pmatrix}
  t_1 & t_2 & t_3 \\
  -t_1 & -t_2 & t_3
\end{pmatrix}
$$

(these special values of $m$'s and $n$'s have been chosen to simplify the calculations; the reduced coefficient being independent of the $m$ and $n$ values). We get

$$
\begin{align*}
&\left(0 \mu_1 0; 0 \mu_2 0; 0 \mu_3 0\right) \\
&\left(s_1 t_1; s_2 t_2; s_3 t_3\right)
\end{align*}
$$

$$
\begin{align*}
&= \frac{12 \mu_1 ! \mu_2 !}{(\mu_1 + \mu_2 + 2) (\mu_1 + \mu_2 + 3)!} \left(\frac{1}{2}(s_1 + s_2 + s_3 - t_1 - t_2 - t_3)^2\right) \\
&\times N_{s_1 t_1}^{\mu_1} N_{s_2 t_2}^{\mu_2} N_{s_3 t_3}^{\mu_3 + \mu_2} 2^{\mu_1 + \mu_2 + s_1 + s_2 + t_1 + t_2} \\
&\times \frac{s_1! s_2! s_3! t_1! t_2! t_3!}{(2s_1)!(2s_2)!(2s_3)!(2t_1)!(2t_2)!(2t_3)!} \left(\frac{1}{2} \begin{pmatrix}
  s_1 & s_2 & s_3 \\
  t_1 & t_2 & t_3
\end{pmatrix}
\right) \left(\begin{pmatrix}
  0 & 0 & 0 \\
  0 & 0 & 0
\end{pmatrix}\right)
\end{align*}
$$

$$
\begin{align*}
&\times \sum_{p_2} (-1)^{p_2} \frac{\left\{ \frac{1}{2}(p_2 - t_2) \right\}! \left(\frac{1}{2}(s_2 + p_2) \right)!}{\left\{ \frac{1}{2}(p_2 - t_2) \right\}! \left(\frac{1}{2}(s_2 + p_2) \right)!}
\end{align*}
$$

$$
\begin{align*}
&\times \sum_{p_1} (-1)^{p_1} \frac{\left\{ \frac{1}{2}(p_1 - t_1) \right\}! \left(\frac{1}{2}(s_1 + p_1) \right)!}{\left\{ \frac{1}{2}(p_1 - t_1) \right\}! \left(\frac{1}{2}(s_1 + p_1) \right)!}
\end{align*}
$$

$$
\begin{align*}
&= \frac{12 \mu_1 ! \mu_2 !}{(\mu_1 + \mu_2 + 2) (\mu_1 + \mu_2 + 3)!} \left(\frac{1}{2}(s_1 + s_2 + s_3 - t_1 - t_2 - t_3)^2\right) \\
&\times N_{s_1 t_1}^{\mu_1} N_{s_2 t_2}^{\mu_2} N_{s_3 t_3}^{\mu_3 + \mu_2} 2^{\mu_1 + \mu_2 + s_1 + s_2 + t_1 + t_2} \\
&\times \frac{s_1! s_2! s_3! t_1! t_2! t_3!}{(2s_1)!(2s_2)!(2s_3)!(2t_1)!(2t_2)!(2t_3)!} \left(\frac{1}{2} \begin{pmatrix}
  s_1 & s_2 & s_3 \\
  t_1 & t_2 & t_3
\end{pmatrix}
\right) \left(\begin{pmatrix}
  0 & 0 & 0 \\
  0 & 0 & 0
\end{pmatrix}\right)
\end{align*}
$$

$$
\begin{align*}
&\times \sum_{p_2} (-1)^{p_2} \frac{\left\{ \frac{1}{2}(p_2 - t_2) \right\}! \left(\frac{1}{2}(s_2 + p_2) \right)!}{\left\{ \frac{1}{2}(p_2 - t_2) \right\}! \left(\frac{1}{2}(s_2 + p_2) \right)!}
\end{align*}
$$

$$
\begin{align*}
&\times \sum_{p_1} (-1)^{p_1} \frac{\left\{ \frac{1}{2}(p_1 - t_1) \right\}! \left(\frac{1}{2}(s_1 + p_1) \right)!}{\left\{ \frac{1}{2}(p_1 - t_1) \right\}! \left(\frac{1}{2}(s_1 + p_1) \right)!}
\end{align*}
$$
It turns out that the $p_1, p_2$ sums in the last equation can be carried out by identifying these sums with Eq. (3.7) after a change in dummies $p_1 = \mu_1 - t_1 - 2x_1$, and $p_2 = \mu_2 - t_2 - 2x_2$, and the final result is

\[
\begin{pmatrix}
\mu_1 & \mu_2 & \mu_1 + \mu_2 \\
 s_1 t_1 & s_2 t_2 & s_3 t_3
\end{pmatrix}
\]

\[
\times \frac{(s_1 + s_2 + s_3 - t_1 - t_2 + t_3)^{1/2}}{(\mu_1 + \mu_2 + 2) ! (\mu_1 + \mu_2 + 3) !}
\]

\[
\times \frac{s_3 ! t_3 ! (2s_1 + 1) ! (2s_2 + 1) ! (2t_1 + 1) ! (2t_2 + 1) !}{[ (2s_1) ! (2s_2) ! (2t_1) ! (2t_2) ! (2t_3) ! ]^{1/2}}
\]

\[
\times \left( \begin{pmatrix} s_1 & s_2 & s_3 \\ t_1 & t_2 & t_3 \end{pmatrix} \right) s_1 ^ {t_1} + s_2 ^ {t_2} + s_3 ^ {t_3}
\]

\[
\times \frac{\mu_1 + \mu_2}{s_3 t_3}
\]

\[
\times \frac{N_{s_3 t_3}}{N_{s_1 t_1} N_{s_2 t_2}}
\]

(4.55)

The value of $N_{st}^\mu$ is given by Eq. (2.12).
APPENDIX A

Relationship of our Generators with those of Jakimow and Sharp

Jakimow and Sharp\textsuperscript{16}) work in the SU(4)\:\!\!\:\!\!\Rightarrow SU(3) scheme. Since we have referred to their work frequently we give the relationship of our generators with theirs.

\begin{align*}
S_+ &= U_+ + S_- \\
S_- &= U_- + S_+ \\
S_o &= M - \frac{1}{2} Y + \frac{2}{3} Z \\
t_+ &= -V_- R_- \\
t_- &= -V_+ R_+ \\
t_o &= M + \frac{1}{2} Y - \frac{2}{3} Z \\
\omega_{11} &= T_+ \\
\omega_{10} &= \frac{i}{4\sqrt{2}} (S_+ - U_+) \\
\omega_{1-1} &= -W_+ \\
\omega_{01} &= \frac{1}{4\sqrt{2}} (R_- - V_-) \\
\omega_{00} &= Y + \frac{2}{3} Z \\
\omega_{0-1} &= \frac{i}{4\sqrt{2}} (V_+ - R_+) \\
\omega_{-11} &= -W_- \\
\omega_{-10} &= \frac{i}{4\sqrt{2}} (U_- - S_-) \\
\omega_{-1-1} &= T_- \\
\end{align*}

Apart from trivial normalization and phases our generators are identical with those of Hecht and Pang\textsuperscript{7}) and Draayer\textsuperscript{9}).
APPENDIX B.

Calculation of Matrix Elements

(i) $(0, \mu, 0)$ matrix elements and the normalization constant $N_{st}^\mu$

We shall calculate the matrix elements

\[
\langle 0|\mu; s+1 t+1, s+1 t+1 | u_{11} | 0|\mu; s t, st \rangle
\]

and

\[
\langle 0|\mu; s t, st | u_{11} | 0|\mu; s+1 t+1, s+1 t+1 \rangle
\]

and equate them to compute $N_{st}^\mu$.

The first matrix element is easy to calculate since the operation of $u_{11}$ on the $|0\mu0; st, st\rangle$ state leads to a unique state as depicted in Fig. (A.1)

\[
U_{11} | 0|\mu; s t, st \rangle = C | 0|\mu; s+1 t+1, s+1 t+1 \rangle
\]

where $C$ is a constant to be determined. Using the differential operator $u_{11} = \rho \partial_\tau + \rho' \theta_\tau$ and the expressions (2.10) and (2.11) we have

\[
U_{11} | 0|\mu; s t, st \rangle = 2 (\mu + s + t + 4) \frac{N_{st}^\mu}{N_{s+1 t+1}^\mu} | 0|\mu; s+1 t+1, s+1 t+1 \rangle
\]

Therefore,

\[
C = \langle 0|\mu; s+1 t+1, s+1 t+1 | u_{11} | 0|\mu; s t, st \rangle
= 2 (\mu + s + t + 4) \frac{N_{st}^\mu}{N_{s+1 t+1}^\mu}
\]  

(A.1)
The second matrix element is more difficult to calculate because the operation of \( u_{-1-1} \) on the state \( |s+1\ t+1\rangle \) leads to a linear combination of states of the multiplets \( |s\ t\rangle, |s+1\ t\rangle \) and \( |s\ t+1\rangle \), with \( m=s \) and \( n=t \) in each case; Fig.(A.2). Thus the state of our interest \( |s\ t\rangle \) has to be singled out. We find that

\[
\begin{align*}
\langle \bar{0} \mu \sigma; st, st | u_{-1-1} | \bar{0} \mu \sigma; s+1 t+1, s+1 t+1 \rangle &= \frac{N_{s+1 t+1}^{\mu}}{N_{st}^{\mu}} \frac{(\mu-s-t)(s+1)(t+1)}{2(2s+3)(2t+3)} |\bar{0} \mu \sigma; st, st \rangle \\
\text{orthogonal states.}
\end{align*}
\]

Therefore,

\[
\begin{align*}
&= \frac{N_{s+1 t+1}^{\mu}}{N_{st}^{\mu}} \frac{(\mu-s-t)(s+1)(t+1)}{2(2s+3)(2t+3)} \\
\text{(A.2)}
\end{align*}
\]
Since $u_{11}$ and $u_{-1-1}$ are Hermitian conjugate to each other, the matrix elements (A.1) and (A.2) must be equal, and we get

$$N_{s+t+1}^{\mu} = \frac{2}{N_{st}^{\mu}} \left[ \frac{(\mu+s+t+4)(2s+3)(2t+3)}{(\mu-s-t)(s+1)(t+1)} \right]^{1/2}$$  \hspace{1cm} (A.3)

and iteration of this equation gives

$$N_{st}^{\mu} = D \left[ \frac{2^{s+t}(\mu+s+t+2)!!(\mu-s-t)!!(2s+1)!!(2t+1)!!}{s!t!} \right]^{1/2}$$  \hspace{1cm} (A.4)

The constant $D$ can be evaluated at a boundary state $|s_b, t_b\rangle$ of the IR $(0, \mu, 0)$, where

$$S_b + t_b = \mu, \quad S_b - t_b = s - t$$

i.e.

$$S_b = \frac{1}{2}(\mu + s - t), \quad t_b = \frac{1}{2}(\mu - s + t)$$

therefore, the state (2.10) at the boundary becomes:

$$|\rho_{0}; S_b t_b, s t_b\rangle = N_{s_b t_b}^{\mu} \frac{\rho_{s_b}^{s_b} \rho_{t_b}^{t_b}}{(2s_b+1)!!(2t_b+1)!!} = \frac{\rho_{s_b}^{s_b} \rho_{t_b}^{t_b}}{\sqrt{s_b! t_b!}}$$

the right-hand side is written using the normalization convention (1.27). We therefore have

$$N_{s_b t_b}^{\mu} = \frac{(2s_b+1)!!(2t_b+1)!!}{\sqrt{s_b! t_b!}}$$

Equating this value of $N_{s_b t_b}^{\mu}$ and that obtained from Eq. (A.4), with $s=s_b$, $t=t_b$, we get

$$D = \left[ \frac{(\mu+s-t+1)!!(\mu-s+t+1)!!}{2^{\mu}(2\mu+2)!!} \right]^{1/2}$$  \hspace{1cm} (A.5)
Substituting this value of $D$ in Eq. (A.4) we get the expression for the normalization constant (2.12)

$$N_{\mu}^{st} = \frac{((\mu+s-t+1)!!)^{(\mu-s-t+1)!!}(\mu+s+t+2)!!(\mu-s-t)!!(2s+1)!!(2t+1)!!}}{2^{\mu-s-t} (2\mu+2)!! s! t!}$$

Combining Eqs. (A.1), (A.2) and (A.3), we have

$$C = \langle o\mu o; s+1, s+1, t+1 | u_{s+1} | o\mu o; s, t, st \rangle$$

$$= \langle o\mu o; st, st | u_{s+1} | o\mu o; s+1, t+1, s+1, t+1 \rangle$$

$$= \left[ \frac{(\mu+s+t+4)(\mu-s-t)(s+1)(t+1)}{(2s+3)(2t+3)} \right]^{1/2}$$

(A.6)

Using the definition (2.14), we obtain the reduced matrix element

$$\langle o\mu o; s+1, t+1 | u_{s+1} | o\mu o; s, t \rangle = \frac{C \left[ (2s+3)(2t+3) \right]^{1/2}}{\langle s+1 | s+1 | t+1 \rangle}$$

$$= \left[ (\mu+s+t+4)(\mu-s-t)(s+1)(t+1) \right]^{1/2}$$

(A.7a)

and similarly,

$$\langle o\mu o; st | u_{t} | o\mu o; s+1, t+1 \rangle = \frac{C \left[ (2s+1)(2t+1) \right]^{1/2}}{\langle s+1 | s+1 | s+1, t+1 \rangle}$$

$$= \left[ (\mu+s+t+4)(\mu-s-t)(s+1)(t+1) \right]^{1/2}$$

(A.7b)
Next, we shall evaluate the matrix elements

\[ A = \langle 0 \mu 0; s-1 t+1, s-1 t+1 | u_{m,1} | 0 \mu 0; st, st \rangle \]

and

\[ B = \langle 0 \mu 0; s+1 t-1, s+1 t-1 | u_{m,1} | 0 \mu 0; st, st \rangle \]

We have,

\[ u_{m,1} | 0 \mu 0; st, st \rangle = A | 0 \mu 0; s-1 t+1, s-1 t+1 \rangle + A' | 0 \mu 0; s+1 t+1, s-1 t+1 \rangle \]

(See Fig. (A.3)).

Using the differential form

\[ u_{m,1} = \rho' \partial_{\rho'} + \tau \partial_{\tau} \]

and the \((0, \mu, 0)\) state Eq. (2.10) and following through the details of algebra, comparing coefficients etc., we get

\[ A = \langle 0 \mu 0; s-1 t+1, s-1 t+1 | u_{m,1} | 0 \mu 0; st, st \rangle = \left[ \frac{(\mu-s+t+3)(\mu+s-t+1)s(t+1)}{(2s+1)(2t+3)} \right]^{1/2} \text{(A.8a)} \]

By making use of the symmetry between \(s\) and \(t\), we can write the second matrix element simply by interchanging \(s\) and \(t\) in the above matrix element, i.e.,
\[ B = \bra{0 \nu_0; s+1 \ t-1, s+1 \ t-1} u_{-1} \ket{0 \nu_0; s \ t, s t} \]
\[ = \left( \frac{(\mu+s-t+3)(\mu-s+t+1)(s+1) t}{(2s+3)(2t+1)} \right)^{1/2} \]

(A.8b)

The reduced matrix elements are then found to be

\[ \bra{0 \nu_0; s-1 \ t+1} u \ket{0 \nu_0; s \ t} = \frac{A \left( \frac{(2s+1)(2t+3)}{t} \right)^{1/2}}{\langle s \ | \ t \rangle \langle t \ | \ s \rangle} \]
\[ = \left[ t (s+1) (\mu+s-t+1)(\mu-s+t+3) \right]^{1/2} \]

(A.9a)

and similarly,

\[ \bra{0 \nu_0; s+1 \ t-1} u \ket{0 \nu_0; s \ t} = \frac{B \left( \frac{(2s+3)(2t+1)}{s} \right)^{1/2}}{\langle s \ | \ t \rangle \langle t \ | \ s \rangle} \]
\[ = \left[ s (t+1) (\mu+s-t+1)(\mu-s+t+3) \right]^{1/2} \]

(A.9b)

(ii) \((\lambda,0,0)\) matrix elements

The action of \( u_{oo} = \frac{1}{2}(\eta_\eta + \xi_\xi - \theta_\theta - \zeta_\zeta) \) on the \((\lambda,0,0)\) states, Eq.(2.3) leads to:

\[ u_{oo} \ket{\lambda \nu_0; ss; ss} = A \ket{\lambda \nu_0; ss, ss} + B \ket{\lambda \nu_0; s+1, ss, s} \]
After carrying out the differentiation and comparing coefficients of powers of $\eta$, we have

$$A = \langle \lambda_00; s0, s0 | U_{oo} | \lambda_00; s0, s0 \rangle = \frac{s(\lambda+2)}{2(s+1)}$$

so that the reduced matrix element

$$\langle \lambda_00; s0 \| u \| \lambda_00; s0 \rangle = A \frac{(2s+1)}{\langle s \| s \rangle} = \frac{1}{2}(\lambda+2)(2s+1)$$

(A.10)

Next,

$$U_{11} | \lambda_00; s0, s0 \rangle = C | \lambda_00; s+1, s+1, s+1 \rangle$$

and we get,

$$C = \langle \lambda_00; s+1, s+1, s+1 | U_{11} | \lambda_00; s0, s0 \rangle$$

$$= \langle \lambda_00; s0, s0 | U_{11} | \lambda_00; s+1, s+1, s+1 \rangle$$

$$= \left[ \frac{(\frac{1}{2} \lambda-s) \left(\frac{1}{2} \lambda+s+1\right) (2s+1)}{(2s+3)} \right]^{1/2}$$

therefore,

$$\langle \lambda_00; s+1, s+1 \| u \| \lambda_00; s0, s0 \rangle = C \frac{(2s+3)}{\langle s \| s \rangle^{1/2}} \langle s \| s \rangle^{1/2}$$

$$= \left[ \frac{(\frac{1}{2} \lambda-s) \left(\frac{1}{2} \lambda+s+1\right) (2s+1)(2s+3)}{(2s+3)} \right]^{1/2}$$

(A.11a)

Similarly,

$$\langle \lambda_00; s0 \| u \| \lambda_00; s+1, s+1 \rangle = \left[ \frac{(\lambda-s) \left(\lambda+s+1\right) (2s+3)}{(2s+3)} \right]^{1/2}$$

(A.11b)
APPENDIX C

Conjugation of States

If we conjugate the variables, i.e., in the basis states replace \( \eta \rightarrow \eta^* \), \( \theta \rightarrow \theta^* \) etc., it is easy to see from the form of Eq.(2.6) that the \((\lambda,0,0)\) state transforms as follows:

\[
|\lambda 0 0; ss, mn\rangle^* = (-)^{m-n} |0 \lambda 0; ss, -m-n\rangle
\]

(A.12)

and similarly on conjugation the \((0,0,v)\) state transforms as:

\[
|0 0 v; ss, mn\rangle^* = (-)^{m-n} |0 0 v; ss, -m-n\rangle
\]

(A.13)

The fundamental IR \((0,1,0)\) is self-conjugate and the starred variables are given by

\[
\rho^* = \tau, \quad \sigma^* = -\sigma, \quad \tau^* = \rho,
\]

\[
\rho^* = -\tau, \quad \sigma^* = \sigma', \quad \tau^* = -\rho'
\]

(A.14)

Thus, on conjugation the \((0,\mu,0)\) state, Eq.(2.13) transforms as:

\[
|0 \mu 0; st, mn\rangle^* = (-)^{s+m-n} |0 \mu 0; st, -m-n\rangle
\]

(A.15)

Now, the general Elliott state for the IR \((\lambda,\mu,v)\) is:

\[
|\lambda \mu v; K_2 K_2; st, mn\rangle
\]

(A.16)

\[
= \sum_{s_a s_v s_{\mu}} \sum_{m_\lambda m_\mu \eta_\lambda \eta_\mu} |\lambda 0 0; s_a s_v, m_\lambda \eta_\lambda \rangle |0 \mu 0; s_{\mu} t_{\mu}, m_\mu \eta_\mu \rangle |0 0 v; s_v t_v, m_v \eta_v \rangle
\]

\[
\times B_{st}^{\lambda \mu v} (s_a s_v; s_{\lambda v} t_{\lambda v}; s_{\mu} t_{\mu})
\]

\[
\times \left\langle s_\lambda s_v \mid s_{\lambda v} s_{\mu} \mid s \right\rangle \left\langle s_\lambda s_v t_{\lambda v} m_\mu \eta_\mu \mid n \right\rangle \left\langle t_{\lambda v} t_{\mu} t \right\rangle
\]

\[
\left\langle m_\lambda m_\mu m_{\lambda v} m_{\mu} \eta_\lambda \eta_\mu \eta_{\lambda v} \eta_{\mu} \right\rangle
\]
On conjugation $\lambda$ and $\nu$ interchange roles which changes the intrinsic states, Fig.(3.2), and hence also changes the $K_s, K_t$ values specifying a given Elliott state. With the help of ranges (3.10) we find that only the smaller (in magnitude) of $K_s, K_t$ changes sign. Thus, for $s$-stretched states ($K_s > |K_t|$), $K_t$ changes sign and the coefficient $B$ changes to

$$B^{\nu \mu \lambda}_{st-k_s-k_t} (s_s v_s s_n t_n s_m t_m) = (-)$$

while for $t$-stretched states ($K_t > |K_s|$), $K_s$ changes sign and

$$B^{\nu \mu \lambda}_{st-k_s-k_t} (s_s v_s s_n t_n s_m t_m) = (-)$$

Combining Eqs.(A.12, A.13, A.15, A.17 and A.18) we find that under conjugation Elliott state (A.16) transforms as follows:

For $s$-stretched states ($K_s > |K_t|$),

$$|\lambda \mu \nu ; k_s k_t, st, m n|^* = (-)^{\frac{1}{2}(\lambda +\nu) - s - m + n} |\nu \mu \lambda ; k_s - k_t, st, - m - n\rangle$$

and for $t$-stretched states ($K_t > |K_s|$)

$$|\lambda \mu \nu ; k_s k_t, st, m n|^* = (-)^{\frac{1}{2}(\lambda +\nu) + \mu + t - m + n} |\nu \mu \lambda ; k_s - k_t, st, - m - n\rangle$$
Relationship between C.G. and Wigner Coefficients

We have defined SU(4) C.G. coefficients (4.2) by

\[ |\lambda_1 \mu_1 \nu_1; s_1, t_1, m_1, n_1 \rangle_{12} = \sum_{s_1, t_1, m_1, n_1} |\lambda_2 \mu_2 \nu_2; s_2, t_2, m_2, n_2 \rangle_{12} \]

and Wigner coefficient (4.4) by

\[ \sum_{s_1, t_1, m_1, n_1} \sum_{s_2, t_2, m_2, n_2} \left( \frac{1}{s_1 s_2 s_3 t_1 t_2 t_3} \right) \]

To find the relationship between these coefficients, we recall that to within a factor depending on relative phase, the invariant \( S \) can be written as

\[ S = \sum_{s_1, t_1, m_1, n_1} \sum_{s_2, t_2, m_2, n_2} \left( \frac{1}{s_1 s_2 s_3 t_1 t_2 t_3} \right) \]

where \( D_{\lambda \mu \nu} \) is the dimensionality of the IR \((\lambda, \mu, \nu)\) given by Eq.(1.21). The conjugate state in the above equation is given
by Eq. (A.19 or A.20) so that

\[ S = \sum_{s, t, m_n} \left| \lambda_3 \mu_1 \nu_1 ; s t_3 ; m_1 n_1 \right> \left< \nu_3 \mu_3 \nu_3 ; s t_3 , -m_3 -n_3 \right| \xi \left( D_{s t_3 \nu_3 \nu_3} \right)^{-\frac{1}{2}} \]  

(A.24)

where the phase factor

\[ \xi = (-)^{\frac{1}{2} \lambda_3 + \lambda_2 + \lambda_1} - s - m_3 + n_3 \]  

(A.25a)

\[ = (-)^{\frac{1}{2} \lambda_3 + \lambda_2 + \lambda_1 + t_3 - m_3 - n_3} \]  

(A.25b)

(Even though there is no internal degeneracy problem for the IR's under consideration, yet \( K_s, K_t \) play a role in determining the phase factor. This is because we are using Elliott states).

Combining Eqs. (A.21 and A.24) we get

\[ S = \sum_{s, t, m_n} \sum_{s, t_3, m_n} \left| \lambda_1 \mu_1 \nu_1 ; s t_1 ; m_1 n_1 \right> \left< \lambda_2 \mu_2 \nu_2 ; s t_2 , m_2 n_2 \right| \nu_3 \mu_3 \nu_3 ; s t_3 , -m_3 -n_3 \right> \xi \left( D_{s t_3 \nu_3 \nu_3} \right)^{-\frac{1}{2}} \]  

(A.26)

\[ \times \left< \lambda_1 \mu_1 \nu_1 ; \lambda_2 \mu_2 \nu_2 \right| \nu_3 \mu_3 \nu_3 \nu_3 ; s t_3 , m_3 n_3 \right> \]

Comparing (A.22) and (A.26), we get the desired relationship

\[ \left( \lambda_1 \mu_1 \nu_1 ; \lambda_2 \mu_2 \nu_2 ; \lambda_3 \mu_3 \nu_3 \right) \]

\[ \left( s t_1 m_1 n_1 ; s t_2 m_2 n_2 ; s t_3 m_3 n_3 \right) \]

(A.27)

\[ \left< \lambda_1 \mu_1 \nu_1 ; \lambda_2 \mu_2 \nu_2 \right| \nu_3 \mu_3 \nu_3 \nu_3 ; s t_3 , m_3 n_3 \right> \xi \left( D_{s t_3 \nu_3 \nu_3} \right)^{-\frac{1}{2}} \]

with \( \xi \) given by Eq. (A.25).
The reduced coefficients are related by

\[
\begin{pmatrix}
\lambda_1 \mu_1 \nu_1 & \lambda_2 \mu_2 \nu_2 & \nu_3 \mu_3 \lambda_3 \\
\mu_1 & \mu_2 & \mu_3 \\
\end{pmatrix}
\begin{pmatrix}
s_1 t_1 \\
s_2 t_2 \\
s_3 t_3 \\
\end{pmatrix}
= \frac{\langle \nu_1 \nu_2 \nu_3 \mu_3 \lambda_3 \rangle}{\lambda_3 \mu_3 \nu_3} \Delta \left[ \frac{(2s_3+1)(2t_3+1)}{D_3 \mu_3 \nu_3} \right]^{1/2}
\]

\[
\Delta = \begin{cases} 
-\frac{1}{2} (\lambda_3 + \nu_3) - s_1 + s_2 + t_1 - t_2 - s_3 & \text{for s-stretched states} \\
-\frac{1}{2} (\lambda_3 + \nu_3) + \mu_3 - s_1 + s_2 + t_1 - t_2 + t_3 & \text{for t-stretched states}
\end{cases}
\]
CONCLUSIONS

We have constructed two sets of bases for the Wigner supermultiplet scheme, solving the internal labelling problem for SU(4) ⊃ SU(2) x SU(2) decomposition. Such bases, apart from inherent interest, will be useful in any serious attempt to review the status of the supermultiplet model.

The first set of bases, which we call the Bargmann-Moshinsky states, are defined by identifying them with products of elementary factors. The missing labels $k_s$, $k_t$ are defined by powers of two of the elementary factors $f_2$ and $f_7$. Each state is characterized by a unique stretched term in its simple product state expansion. This facilitates expansion of an arbitrary state in Bargmann-Moshinsky states and allows evaluation of generator matrix elements, transformation matrix elements and Clebsch-Gordan coefficients directly in the non-canonical chain.

The second set of bases, which we call Elliott states, are obtained by projecting them from non-degenerate intrinsic states lying in a rectangular boundary face of the SU(4) ⊃ SU(3) weight diagram. The third components of spin and isospin of the intrinsic state $K_s$, $K_t$ serve as the missing labels for the projected states and resolve the s, t degeneracy inside a supermultiplet.

Both the bases are non-orthogonal; the metric matrices (overlaps and normalizations) are evaluated explicitly for
the Elliott states and implicitly for the Bargmann-Moshinsky states.

These states can be identified with the spin isospin part of the nuclear wave functions in the supermultiplet approximation; in this connection all the states of a supermultiplet have the same symmetry under the exchange of nucleons and hence correspond to the same spatial wave function.

Clebsch-Gordan and Wigner coefficients have been evaluated for coupling of states of certain classes of representations involving no external multiplicity. They can be directly identified with the many-particle spin-isospin fractional parentage coefficients\(^{31}\). Their usefulness also arises in connection with the SU(4) Wigner-Eckart theorem. A physical operator (e.g. energy, transition operator etc.) is expanded as a sum of irreducible SU(4) tensor components. The matrix element of such an operator between the supermultiplet states can be expressed as a product of a reduced matrix element and an SU(4) Wigner coefficient (or when there is external multiplicity a sum of such products).
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Table II

Parameters of twelve types of stretched simple product states in terms of $k_s, k_t$

$$
\begin{array}{cccccccc}
\text{A} & \frac{1}{2}(k_s+k_t) & \frac{1}{2}(k_s-k_t) & k_s & k_t & s-k_s & t-k_t \\
\text{B} & \frac{1}{2}(k_s+k_t) & \frac{1}{2}(k_s-k_t) & k_s & k_t & s-k_s & t-k_t+1 \\
\text{C} & \frac{1}{2}(k_s+k_t) & \frac{1}{2}(k_s-k_t) & k_s & -k_t & s-k_s & t+k_t \\
\text{D} & \frac{1}{2}(k_s+k_t) & \frac{1}{2}(k_s-k_t) & k_s & -k_t & s-k_s & t+k_t+1 \\
\text{E} & \frac{1}{2}(k_s+k_t) & \frac{1}{2}(k_s-k_t) & k_s & s+t-u-k_s & s-k_s & u-s+k_s \\
\text{F} & \frac{1}{2}k_s & \frac{1}{2}k_s & k_s & l & s-k_s & t+1 \\
\text{G} & \frac{1}{2}(k_s+k_t) & \frac{1}{2}(k_s-k_t) & k_s & k_t & s-k_s & t-k_t \\
\text{H} & \frac{1}{2}(k_s+k_t) & \frac{1}{2}(k_s-k_t) & k_s & k_t & s-k_s+1 & t-k_t \\
\text{I} & \frac{1}{2}(k_s+k_t) & \frac{1}{2}(k_s-k_t) & -k_s & k_t & s+k_s & t-k_t \\
\text{J} & \frac{1}{2}(k_s+k_t) & \frac{1}{2}(k_s-k_t) & -k_s & k_t & s+k_s+1 & t-k_t \\
\text{K} & \frac{1}{2}(k_s+k_t) & \frac{1}{2}(k_s-k_t) & s+t-u-k_t & k_t & u-t+k_t & t-k_t \\
\text{L} & \frac{1}{2}k_t & \frac{1}{2}k_t & l & k_t & s+1 & t-k_t \\
\end{array}
$$
### Table III

**$(K_{s},K_{t})$ Decomposition of IR $(2,2,3)$**

(All the digits are to be divided by 2)

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Table IV

\((K_s,K_t)\) Decomposition of IR \((4,6,2)\)

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### Table IV (...contd.)

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|   | 2 | (3, 1)         |
|   | 3 | (2, 1)         |