Optimization and packings of T-joins and T-cuts

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Abstract

Let G be a graph and T an even cardinality subset of its vertices. We call (G, T) a graft. A T-join is a subgraph of G whose odd-degree vertices are precisely those in T, and a T-cut is a cut $\delta(S)$ where S contains an odd number of vertices of T. An interesting question from a combinatorial optimization perspective is that of finding optimal T-joins and T-cuts. These have applications in various places. We give an overview of several such optimization problems, as well as several algorithms for finding optimal T-joins and T-cuts from the literature.

We then consider a packing problem in grafts. It is a simple observation that the number of edge-disjoint T-joins is at most the number of edges in any T-cut. However it is not known exactly when these quantities are equal. It has been conjectured by Guenin that if G is planar and all T-cuts of G have the same parity and the size of every T-cut is at least k, then G contains k edge-disjoint T-joins. The case $k = 3$ is equivalent to the Four Colour Theorem, and the cases $k = 4$, which was conjectured by Seymour, and $k = 5$ were proved by Guenin. Recently, the case $k = 6$ was settled by Dvořak, Kawarabayashi and Král'. In this thesis, we give a proof of the case $k = 7$.

Abrégé

Soit G un graphe, et T un sous-ensemble de ses sommets de cardinalite pair. Nous appelons (G, T) une greffe. Définissons par T-jointure tout sous-graphe de G dans lequel les sommets de degré impair sont précisement ceux de l'ensemble T, et par T-coupure tout coupure $\delta(S)$ où $|S \cap T|$ est impair. Une question intéressante en optimisation combinatoire est celle de trouver les T-jointures et T-coupures optimales. Nous donnons un aperçu de divers problèmes d'optimisation auxquels ceux-ci s'appliquent, ainsi que plusieurs algorithmes pour trouver les T-jointures et T-coupures optimales.

Nous considérons ensuite un problème d'empaquetage dans les greffes. C'est une observation facile que le nombre de T-jointures arête-disjointes dans le graphe G est au maximum le nombre d'arêtes dans quelconque T-coupure. Cependant on ne sait pas exactement quand ces quantités sont égales. Il a été conjecturé par Guenin que si G est planaire, que tous les T-coupures de G ont la même parité et que et le nombre d'arêtes dans chaque T-coupure est au moins k, alors G contient k T-jointures arête-disjointes . Quand $k = 3$ la question est équivalente au théorème des quatre couleurs, et le cas $k = 4$, ce qui a été conjecturé par Seymour, et $k = 5$ ont été prouvés par Guenin. Récemment, le cas $k = 6$ a été réglé par Dvořak, Kawarabayashi et Král'. Dans cette thèse, nous donnons une preuve pour le cas $k = 7$.

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Table of Contents

List of Tables

List of Figures

Chapter 1

Introduction to T-joins and T-cuts

This thesis focuses on the theory of T-joins and T-cuts. We begin with a short review of definitions and simple lemmas. The reader is assumed to have knowledge of basic graph theory and linear programming. A glossary of terms is provided as an appendix.

Let $G = (V, E)$ be an undirected graph and $S \subseteq V$. The *cut* $\delta(S) \subseteq E$ consists of those edges in E which have exactly one endpoint in S. When it is not clear from the context, we write $\delta_G(S)$ to specify that G is the graph in question. If S or its complement S^c consists of a single vertex v, we call the cut *trivial* and often write $\delta(v)$ instead of $\delta(S)$. Otherwise, the cut is *non-trivial*. The *size* of a cut is $|\delta(S)|$. In the case where S is a single vertex v, we write $d(v)$ (the *degree* of v) for the number of edges incident to v. A cut $\delta(S)$ is *odd* if S or S^c contains an odd number of vertices. Accordingly, in a graph with an odd number of vertices, every cut is odd. Let $\gamma(S)$ denote those edges in E which have both endpoints in S.

Suppose $T \subseteq V$ with $|T|$ even.

Definition 1.0.1. *A set $J \subseteq E$ is called a T-join if for each $v \in V$, $v \in T$ if and only if $|J \cap \delta(v)|$ is odd.*

Definition 1.0.2. *A cut $\delta(S)$ is called a T-cut if $|T \cap S|$ is odd.*

We call a pair (G,T) a *graft*. In this document, unless stated otherwise, G is assumed to have vertex set denoted by V (or $V(G)$) and edge set denoted by E (or $E(G)$). A *packing* of T-joins is a collection of edge-disjoint T-joins; packings of T-cuts are defined similarly.

Suppose that we are given some cost function $c \in \mathbb{R}^E$ on the edges of G. If $F \subseteq E$, we denote by $c(F) = \sum_{e \in F} c_e$ the *c-cost* of F. Following [Gue03] and [Sey81] we use the following notation for the costs of minimum T-joins and T-cuts and the sizes of maximum packings of T-joins and T-cuts.

Definition 1.0.3. *Let (G,T) be a graft. We write*

- *$\tau_c(G,T)$ to denote the cost of a minimum c-cost T-cut in G.*
- *$\nu_c(G,T)$ to denote to denote the size of a maximum collection $\mathcal{C}$ of T-joins in G where each edge e is contained in at most c_e T-joins in $\mathcal{C}$.*

Definition 1.0.4. *We write*

- *$\rho_c(G,T)$ to denote the cost of a minimum c-cost T-join in G*
- *$\mu_c(G,T)$ to denote the size of a maximal collection $\mathcal{C}$ of T-cuts in G where each edge e is contained in at most c_e T-cuts in $\mathcal{C}$.*

When $c \equiv 1$ it is often convenient to drop the subscript c on τ, ν, ρ and μ. In this case, $\nu(G,T)$ and $\mu(G,T)$ refer to the sizes of the largest packings of T-joins and T-cuts, respectively. Sometimes don't specify the graft when it is obvious from the context.

Much of this document is concerned with comparing the quantities τ_c and ν_c, and ρ_c and μ_c for a particular graft and cost function.

The following observation is easy and appears in [CCPS98].

Lemma 1.0.5. *Let (G,T) be a graft. Let J be a T-join and let $S \subseteq V$. Then $|J \cap \delta(S)|$ is odd if and only if $\delta(S)$ is a T-cut.*

Proof. Every edge in J is in one of $\gamma(S)$, $\gamma(S^c)$ or $\delta(S)$. The graph induced by $\gamma(S)$ has an even number of vertices of odd degree. There are an odd number of vertices in S which have odd degree in the graph (V, J) if and only if $S \cap T$ is odd. It follows that there must be an odd number of edges in $J \cap \delta(S)$ if and only if $S \cap T$ is odd. □

In particular, Lemma 1.0.5 implies that any T-join J and any T-cut $\delta(S)$ have nonempty intersection. Hence the following two corollaries.

Corollary 1.0.6. *For every graft* (G, T)*,* $\nu(G, T) \leq \tau(G, T)$

Corollary 1.0.7. *For every graft* (G, T)*,* $\mu(G, T) \leq \rho(G, T)$

We investigate these inequalities in more detail in later chapters. Indeed the motivation for this thesis is a conjecture due to Guenin [Gue03], which proposes that grafts (G, T) where G is planar and where all T-cuts have the same parity satisfy the equality $\nu(G, T) = \tau(G, T)$. The original contribution of the thesis is a proof of the special case of Guenin's conjecture for grafts with $\tau(G, T) = 7$.

In Chapter 2, we study optimizations of T-joins and T-cuts from polyhedral and algorithmic perspectives and give some applications in combinatorial optimization. In Chapter 3 we discuss some known results about packings of T-joins. We then present Guenin's conjecture and introduce a direction for proofs of special cases. Finally, Chapter 4 contains our proof of Guenin's conjecture when $\tau = 7$.

Chapter 2

Optimal T-joins and T-cuts

In this chapter, we study optimal T-joins and T-cuts. The first optimization problem that we look at is that of finding minimum-cost T-joins. Given a graft (G, T) and a cost vector $c \in \mathbb{R}^E$ we wish to find the T-join J with minimal cost $c(J)$. In other words, we seek the solution to the following integer program.

$$\begin{array}{llll} \text{Minimize} & c^T x & & \\ \text{subject to} & x(\delta(v)) & \equiv 1 (\text{mod } 2) & \text{for all } v \in T \\ & x(\delta(v)) & \equiv 0 (\text{mod } 2) & \text{for all } v \notin T \\ & x_e & \in \{0, 1\} & \text{for all } e \in E \end{array} \tag{2.0.1}$$

Observe that when each edge $e \in E$ has unit cost $c_e = 1$, the problem amounts to finding a T-join J with total cost equal to $\rho(G, T)$. This problem can be solved in polynomial time for any cost function c, and we give two different algorithms in this chapter, both presented by Edmonds and Johnson in [EJ73]. The algorithm presented in Section 2.3 relies on a reduction from T-joins to perfect matchings, and in Section 2.2 we study the polytope of T-joins and discuss Edmonds and Johnson's primal-dual algorithm for non-negative cost instances. We begin in Section 2.1 by outlining some related problems and show how optimal T-joins can be used to solve them. The material covered in Sections 2.1 to 2.3 is covered in much greater detail in Chapter 5 of [CCPS98].

We briefly discuss algorithms for computing minimum-cost T-cuts in Section 2.4.

2.1 Applications

We now motivate the study of optimal T-joins with several connections to other combinatorial optimization problems.

2.1.1 Chinese Postman Problem

Perhaps the most well-known application of T-joins is to the Chinese Postman Problem (CPP). Suppose a postman needs to deliver mail to houses along each street of a certain neighbourhood and return to his starting point, while minimizing the amount of walking that he must do. Restated in graph theoretic terms, the problem is to find the minimum length closed walk which visits each edge at least once in a graph. Such a walk is called a postman tour. Consider the following integer program whose solution is the incidence vector of an optimal postman tour.

$$\begin{array}{llll} \text{Minimize} & 1^T x & & \\ \text{subject to} & x(\delta(v)) & \equiv 0 (\text{mod } 2) & \text{for all } v \in V \\ & x_e & \in \mathbb{N}^+ & \text{for all } e \in E \end{array} \tag{2.1.1}$$

If a graph is Eulerian, then by definition, there exists a circuit in G which visits each edge. This is clearly an optimal postman tour, and $x_e = 1 \ \forall \ e$ is a feasible and therefore optimal solution to 2.1.1. Such graphs are easily characterized as follows.

Theorem 2.1.1. *Let $G = (V, E)$ be a connected graph. Then G is Eulerian if and only if every vertex in V has even degree.*

See, for example, [LPV03] for a proof of this fact and a linear-time algorithm for finding an Eulerian circuit when one exists. The problem is slightly trickier in graphs which are not Eulerian. The postman may need to walk along some streets more than once.

Observe that any optimal solution x^* to 2.1.1 satisfies $x^*_e \in \{1, 2\}$ $\forall$ e, for otherwise the solution obtained by subtracting 2 from each component with value larger than 2 is feasible and has lower cost. Consider an optimal solution x^* to 2.1.1 and the graph G' obtained from G by adding $x^*_e - 1$ copies of each edge to G. G' has an Eulerian circuit which corresponds to a postman tour of G. Therefore in order to solve the CPP, we can simply find a smallest set of edges whose duplication in the graph renders it Eulerian. Such a set of edges is called a *postman set*. Let $T = \{v \in V; d(v) \text{ is odd}\}$. Then a T-join is a set of edges whose addition to the graph renders the degree of every vertex even. It follows that a solution to the CPP can be achieved by solving a minimum-cost T-join problem with unit costs.

2.1.2 Shortest paths

T-joins can also be used to solve shortest path problems in undirected graphs with some negative edge costs but no negative cost cycles. Given a graph $G = (V, E)$, a cost vector $c \in \mathbb{R}^E$ and two vertices $r, s \in V$, this is the problem of finding a minimum c-cost path in G from r to s. Assuming that all cycles in G have positive c-cost, the shortest (r, s)-path is the solution to the following integer program.

$$\begin{array}{llll} \text{Minimize} & c^T x & & \\ \text{subject to} & x(\delta(v)) & \equiv 0 (\text{mod } 2) & \text{for all } v \in V \backslash \{r, s\} \\ & x(\delta(v)) & \equiv 1 (\text{mod } 2) & \text{for } v \in \{r, s\} \\ & x_e & \in \mathbb{Z}_+ & \text{for all } e \in E \end{array} \tag{2.1.2}$$

When $c \geq 0$, the shortest path can be found using Djikstra's Algorithm (see for example Chapter 5 of [CCPS98] for a complete discussion of shortest path algorithms). However, classical shortest path algorithms for undirected graphs fail when the cost vector is allowed to take negative values. We now explain how T-joins solve this problem. As an aside, we note that algorithms for finding shortest paths and detecting negative cycles in directed graphs with some negative costs have been known since the 1950's and are due to Bellman-Ford (see [Bel58]) or Floyd-Warshall-Roy ([Flo62]).

If G contains a cycle C with $c(C) < 0$, then 2.1.2 is unbounded from below, i.e. there exist (non-simple) (r, s)-paths of arbitrarily small cost. To see this, let x be any feasible solution to 2.1.2, and let x^- be obtained from x by adding 1 to the variable corresponding to each edge in C. The vector x^- is a feasible solution with cost $c^T x^- = c^T x + c(C) < c^T x$. On the other hand, when there are no negative cost cycles in G, we can find shortest paths.

Observe that any feasible solution to 2.1.2 is an $\{r, s\}$-join. Conversely, let J be a minimum-cost $\{r, s\}$-join. The edges of J clearly contain an (r, s)-path. Under the assumption that there are no negative-cost cycles in G, the edges of J in fact form the union of a simple (r, s)-path and possibly some cycles with cost 0. Otherwise, if J contained a positive-cost cycle, we could remove the edges of the cycle to obtain an (r, s)-path of smaller cost. Therefore we can find the shortest simple (r, s)-path by finding the least cost edge-minimal $\{r, s\}$-join. Indeed, the algorithm discussed in Section 2.3 finds the minimum-cost edge-minimal T-join in instances where G contains some edges with negative costs, but no negative cycles.

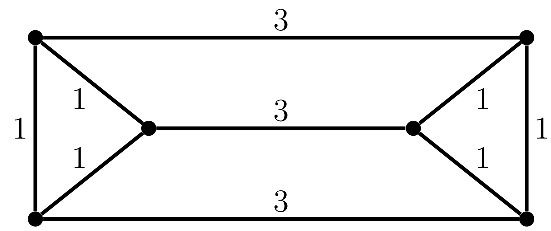
2.2 Linear programs for optimal T-joins

In [EJ73], Edmonds and Johnson study the connection between the Chinese Postman Problem and the previously well-studied field of matching theory. They present two algorithms which solve the special case of minimizing T-joins in grafts (G, T) where the set T consists of the odd-degree nodes of G. This can easily be generalized to all grafts. We discuss their results in the next two sections.

2.2.1 T-join polytope

We have seen that the integer program 2.0.1 describes the optimal T-join. Unfortunately, the corresponding linear program relaxation only provides a lower bound on the optimum which is not exact. To illustrate this, consider the graft (G, T) where G is the graph in Figure 2.2.1 with edge labels corresponding to edge costs and $T = V(G)$. The optimal

Figure 2.1: Example: a graft for which the LP relaxation of 2.0.1 does not give an integral optimal solution



T-join has cost 5, however setting $x_e = \frac{c_e}{5}$, $\forall e \in E$ we obtain a fractional feasible solution to 2.0.1 with cost 3.

Edmonds's Matching Polytope Theorem [Edm65] provides a sharper linear program for the problem. Given $b \in \mathbb{Z}_+^V(G)$, a *b-factor* in G is a set $M \subseteq E$ satisfying $|M \cap \delta(v)| = b_v$ for each vertex v. Edmonds and Johnson observed in [EJ73] that letting $b_v = 1$ for $v \in T$ and $b_v = 0$ for $v \notin T$, we can rewrite the problem of finding a minimum-cost T-join as one of finding a minimum-cost b-factor, then study the following integer program for b-factors.

$$\begin{array}{llll} \text{Minimize} & c^T x & & \\ \text{subject to} & x(\delta(v)) - 2w_v & = b_v & \text{for all } v \in V \\ & w_v & \in \mathbb{Z} & \text{for all } v \in V \\ & x_e & \in \{0,1\} & \text{for all } e \in E \end{array} \tag{2.2.1}$$

This is useful because the problem of finding minimum cost b-factors has been well studied, and there is a known polyhedral description of the solutions and an efficient algorithm for finding solutions to problems of this type. (See [CCPS98] for a detailed discussion of matching problems) Edmonds [Edm65] showed that the incidence vector of a minimum cost b-factor, if it exists, is an optimal solution to the following linear program.

$$
\begin{array}{llll}
\text{Minimize} & c^T x & & \\
\text{subject to} & x(\delta(v)) & = b_v & \text{for all } v \in V \\
& x(\delta(S)\backslash F) + |F| - x(F) & \geq 1 & \text{for all } S \subseteq V \text{ and } F \subseteq \delta(S) \\
& & & \text{such that } |F| + b(S) \text{ is odd.} \\
& 0 \leq \quad x_e & \leq 1 & \text{for all } e \in E
\end{array}
\tag{2.2.2}
$$

Let us take a closer look at the second set of inequalities (known as the *blossom inequalities*) in 2.2.2 from the perspective of our problem. By Lemma 1.0.5, the incidence vector x of any T-join J satisfies $x(\delta(S)) \equiv |S \cap T| \pmod 2$. Thus if $F \subseteq \delta(S)$ and $|F| \not\equiv |S \cap T|$, then there must be some edge in $\delta(S)$ which is either in $J \backslash F$ or in $F \backslash J$. Therefore x satisfies

$$
\begin{array}{r}
x(\delta(S)\backslash F) + |F| - x(F) \geq 1 \text{ for each } S \subseteq V \text{ and } F \subseteq \delta(S) \\
\text{where } |F| \not\equiv |S \cap T| \pmod 2.
\end{array}
\tag{2.2.3}
$$

Thus the polytope of T-joins is the convex hull in $\mathbb{R}^E$ of the integral points which satisfy the first two sets of constraints of 2.2.1 and which satisfy the inequalities 2.2.3. In fact, the inequalities 2.2.3 imply the first set of inequalities in 2.2.1, so it follows that given any graph $G = (V, E)$ and $T \subseteq V$ with $|T|$ even, and $c \in \mathbb{R}^E$, the cost of the minimum-cost T-join in G is given by the optimal solution to the following linear program 2.2.4.

$$
\begin{array}{llll}
\text{Minimize} & c^T x & & \\
\text{subject to} & x(\delta(S)\backslash F) + |F| - x(F) & \geq 1 & \text{for each } S \subseteq V \text{ and } F \subseteq \delta(S) \\
& & & \text{where } |F| \not\equiv |S \cap T| \pmod 2 \\
& 0 \leq \quad x_e & \leq 1 & \text{for all } e \in E
\end{array}
\tag{2.2.4}
$$

Further, Edmonds and Johnson remark in [EJ73] that when $c \geq 0$ the inequalities 2.2.3

can be simplified quite a bit and the cost of the minimum-cost T-join in G is given by the optimal solution to the following linear program 2.2.5.

$$\begin{array}{llll} \text{Minimize} & c^T x & & \\ \text{subject to} & x(\delta(S)) \geq 1 & \text{for each } S \subseteq V \text{ where } |S \cap T| \text{ is odd} & \\ & 0 \leq x_e \leq 1 & \text{for all } e \in E & \end{array} \tag{2.2.5}$$

Their proof of this fact is algorithmic. Indeed a consequence of the fact that minimizing T-joins is a case of b-factors is that the primal-dual Blossom Algorithm for finding optimal b-factors can be used to find optimal T-joins in polynomial time. This algorithm constructs an integral feasible solution to 2.2.5 and one to its dual which have equal objective values. The result follows from strong duality of linear programs.

2.2.2 Reduction to non-negative edge costs

It is worth noting that when studying linear programs and algorithms for minimum-cost T-joins it is actually sufficient to restrict our attention to graphs with non-negative edge costs as we now explain. For any two sets A and B their *symmetric difference* $A\Delta B = (A \cup B)\backslash(A \cap B)$. The following fact is useful.

Lemma 2.2.1. *Let $G = (V, E)$ be a graph, and (G, T) and (G, T') be grafts. Suppose that J is a T-join. Then $J' \subseteq E$ is a T'-join if and only if $J\Delta J'$ is a $T\Delta T'$-join.*

Proof. First, observe that since

$$J\Delta(J\Delta J') = (J \cup (J\Delta J'))\backslash(J \cap (J\Delta J')) = (J \cup J')\backslash(J\backslash J') = J',$$

we just need to show that a vertex v is in $T\Delta T'$ if and only if v is incident to an odd number of edges in $J\Delta J'$.

Let $v \in T\Delta T'$. Then v is in exactly one of T or T'. If $v \in T$, then $|\delta(v) \cap J|$ is odd and $|\delta(v) \cap J'|$ is even. Therefore regardless of the parity of $|\delta(v) \cap (J \cap J')|$, $|\delta(v) \cap (J\Delta J')|$ is even. The case when $v \in T'$ is similar.

Conversely, suppose that $v \in V$ is incident to an odd number of edges in $J\Delta J'$. Then $|\delta(v) \cap J|$ and $|\delta(v) \cap J'|$ must have different parity. Hence $v \in T$ or $v \in T'$ but $v \notin T \cap T'$. $\square$

Now, let (G, T) be a graft and $c \in \mathbb{R}^E$. Let $N = \{e \in E; c_e < 0\}$, and $T' = \{v \in V;$ v has odd degree in N is clearly a T'-join. By Lemma 2.2.1, for any T-join J, $(J\Delta N)$ is a $(T\Delta T')$-join.

Further,

$$\begin{aligned} c(J) &= c(J \setminus N) + c(J \cap N) \\ &= c(J \setminus N) - c(N \setminus J) + c(N \setminus J) + c(J \cap N) \\ &= |c|(J\Delta N) + c(N) \end{aligned}$$

where $|c|_e = |c_e|$. Therefore we can find the optimal T-join by first finding the minimum $|c|$-cost $(T\Delta T')$-join J'. The minimum c-cost T-join is then $J'\Delta N$. For the remainder of this chapter we will focus on grafts with $c \geq 0$.

2.2.3 Dual LP

The dual to 2.2.5 is the linear program below.

$$\begin{array}{llll} \text{Maximize} & 1^T Z & & \\ \text{subject to} & & & \\ & \sum(Z_{\delta(S)} : e \in \delta(S), \delta(S) \text{a } T\text{-cut}) & \leq c_e & \text{for all } e \in E \\ & Z_{\delta(S)} & \geq 0 & \text{for all } T\text{-cuts } \delta(S) \end{array} \tag{2.2.6}$$

As previously mentioned, Edmonds and Johnson's proof of correctness of the Blossom Algorithm for minimizing T-joins constructs a feasible solution to 2.2.6 with optimal value equal to the optimal objective value of 2.2.5. In fact, their proof shows the following theorem, also proved by Lovász.

Theorem 2.2.2 (Edmonds and Johnson [EJ73], Lovász [Lov75])**.** *Let (G, T) be a graft and let $c \in \mathbb{Z}_+^E$. If the total c-cost of every cycle in G is an even integer, then 2.2.6 has an integral optimal solution.*

From this it follows that if the edge costs are integers, then 2.2.6 has a half-integral optimal solution. To see this, consider doubling the cost of each edge.

Seymour proved the strengthening of Theorem 2.2.2 below. Sebő gave an alternate proof in [Seb87].

Theorem 2.2.3 (Seymour [Sey81])**.** *Let (G, T) be a graft, and suppose that G is a bipartite graph. Then the size of the largest collection of edge-disjoint T-cuts is equal to the number of edges in the smallest T-join. In other words, $\rho(G, T) = \mu(G, T)$.*

To see that Theorem 2.2.3 implies Theorem 2.2.2, it suffices to replace each edge e of G with a path of length $2c_e$ to obtain a bipartite graph.

When T contains only two vertices, say $T = \{r, s\}$, it is easy to see that $\rho_c(G, T) = \mu_c(G, T)$, regardless of whether the graph is bipartite. Seymour also gave in [Sey81] a necessary and sufficient condition for $\rho_c(G, T) = \mu_c(G, T)$ to hold when $|T| = 4$. He also observes that in such cases, $\rho_c(G, T) - \mu_c(G, T) \leq 1$. Korach and Penn proved a generalization of these observations.

Theorem 2.2.4 (Korach and Penn [KP92])**.** *Let (G, T) be a graft and let $c \in \mathbb{Z}^E$. Suppose that there exists a minimum c-cost T-join $J \subseteq E$ made up of n_J connected components. Then $\rho_c(G, T) - \mu_c(G, T) \leq n_J - 1$.*

To see the implications of Theorem 2.2.4, observe that when $|T| = 2$, there exists a connected optimal T-join in G (a shortest path), and that when $|T| = 4$ there exists an optimal T-join with at most two connected components.

2.3 Reduction to perfect matchings

We now present another algorithm for finding minimum-cost T-joins, also due to Edmonds and Johnson [EJ73]. This algorithm works by reducing the problem to two other well-studied combinatorial problems, namely those of finding perfect matchings and finding shortest paths. First, observe the following property of the optimal T-join.

Lemma 2.3.1. *Every edge-minimal T-join is the union of $\frac{|T|}{2}$ edge-disjoint paths joining distinct pairs of nodes in T.*

Proof. Consider any set of $\frac{|T|}{2}$ edge-disjoint paths joining distinct pairs of nodes in T. Because the paths are edge-disjoint, the only vertices that will have odd degree in the subgraph formed by the union of the paths are the endpoints of the paths. These are precisely the vertices in T. The union of the paths thus forms a T-join. It remains to show that any T-join $J \subseteq E$ must contain a union of $\frac{|T|}{2}$ edge-disjoint paths joining distinct pairs of nodes in T.

Given J, pick any node $u \in T$ and consider the component H of $G' = (V, J)$ that contains u. H must have an even number of nodes with odd degree, so it must also contain some $v \in T \setminus \{u\}$. Let P be a simple $u - v$ path in H. Then P is a $\{u, v\}$-join. Hence by Lemma 2.2.1, $J \setminus E(P)$ is a T'-join, where $T' = T \setminus \{u, v\}$. (Because $J \triangle E(P) = J \setminus E(P)$ and $T \triangle \{u, v\} = T \setminus \{u, v\}$.) Repeat the process inductively on $J \setminus E(P)$ to obtain the result. $\square$

In fact, when J is a minimum-cost T-join, each of these paths is the shortest (minimum-cost) path between the given pair.

Lemma 2.3.2. *Let $G = (V, E)$, $T \subseteq V$ with $|T|$ even, $c \in \mathbb{R}_+^E$ and J a minimum c-cost T-join. Let $J = \{P_1, \ldots, P_{\frac{|T|}{2}}\}$ be expressed as a union of edge-disjoint paths where u_i and v_i are the endpoints of the path P_i. Then for each $i \in \{1, \ldots, \frac{|T|}{2}\}$, P_i is the shortest (u_i, v_i)-path in G (with respect to the cost function c).*

The idea of Edmonds and Johnson's algorithm is to exploit this fact and find the partition of T into pairs which minimizes the sum of the costs of paths between them. In order to achieve this, we begin by computing the lengths of the shortest paths between each pair of vertices in T. Then we find a minimum-weight perfect matching in the complete graph on T, where edge costs w_{uv} are the lengths of the shortest (u, v)-path for each pair $\{u, v\} \in V$. The set of paths in G corresponding to the perfect matching is edge-disjoint, except possibly for some edges with cost 0, so their symmetric difference is an optimal T-join.

2.3.1 Perfect matchings and all-pairs shortest paths

The key to the simplicity of this algorithm is that it relies on the solutions to two other combinatorial problems. Firstly, the question of how to efficiently find the shortest path between all $\binom{|V|}{2}$ pairs of vertices of a graph $G = (V, E)$ with non-negative edge costs was resolved by Floyd, Warshall and Roy independently. This simple dynamic-programming style algorithm is described in [Flo62]. A perfect matching is a subset $M \subseteq E$ where $|M \cap \delta(v)| = 1$ for each $v \in V$. Hence a perfect matching is simply a 1-factor. As mentioned above, Edmonds's Blossom Algorithm efficiently finds a minimum-weight perfect matching in the graph. Edmonds' study of matchings is fundamental and arises ubiquitously in the field of combinatorial optimization. For a thorough introduction to matching theory, the reader is referred to [CCPS98] or [Sch03].

2.4 Algorithms for minimum T-cuts

The problem of finding a minimum cost T-cut for a graft (G, T) is not quite as easy as that of finding a minimum cost T-join. For one, the problem of finding a minimum T-cut when some edge costs are negative is NP-complete. In other words, no known polynomial time algorithm can find such a cut. Indeed, when $T = \{s, t\}$ and $c \in \mathbb{R}^E$, finding the minimum $-c$-cost T-cut gives a solution to the problem of finding the maximum c-cost $\{s, t\}$-cut (that is, a cut $\delta(S)$ where $|S \cap \{s, t\}| = 1$) and this problem is well-known to be

NP-complete. See [Kar72] for a proof of this fact and [GJ79] for a discussion of problem tractability.

When the edge costs c_e are non-negative, there do exist polynomial time algorithms to find the minimum T-cut. We now discuss two algorithms.

A *Gomory-Hu tree* is a powerful combinatorial structure introduced by Gomory and Hu [GH61] which given a graph G with edge costs $c \in \mathbb{R}_+^E$, encodes the minimum $\{s,t\}$-cut for each pair of vertices $\{s,t\}$ and can be computed in polynomial time. More precisely, the Gomory-Hu tree is defined as follows.

Let $G = (V, E)$ be a graph and $R = (V, F)$ be a tree with the same vertex set. For any edge $f \in F$, we denote by $\delta_R(S_f)$ the cut in R containing only the edge f. Accordingly, S_f and $V \setminus S_f$ correspond to the vertex sets of the two trees obtained from R by deleting f. $\delta_G(S_f)$ is called the *fundamental cut* in G associated with f.

Definition 2.4.1. *Let $G = (V, E)$ be a graph with edge costs $c \in \mathbb{R}_+^E$. A tree $R = (V, F)$ with edge costs $w \in \mathbb{R}_+^F$ is a Gomory-Hu tree if the following holds. Let $\{s,t\} \subseteq V$, and suppose that $uv \in F$ is the minimum w-cost edge on the unique (s,t)-path in R. Then the fundamental cut associated with uv is a minimum c-cost $\{s,t\}$-cut in G.*

For any pair of vertices $\{s,t\}$, the minimum $\{s,t\}$-cut can be found in polynomial time via Ford and Fulkerson's maximum-flow algorithm [FF56] implemented by Edmonds and Karp [EK72]. Gomory and Hu gave an algorithm to construct a Gomory-Hu tree that invokes a minimum $\{s,t\}$-cut algorithm $|V| - 1$ times, as opposed to the $\binom{|V|}{2}$ times that would be necessary to compute all minimum $\{s,t\}$-cuts directly.

Given a graft (G, T) and Gomory-Hu tree for G, it is easy to compute the minimum T-cut in G, as the next lemma shows.

Lemma 2.4.2 (Padberg and Rao [PR82])**.** *Let (G, T) be a graft with edge costs $c \in \mathbb{R}_+^E$, and let $R = (V, F)$ be a Gomory-Hu tree for G with edge costs $w \in \mathbb{R}_+^E$. Then if $\delta_R(S)$ is the minimum w-cost T-cut in R, $\delta_G(S)$ is a minimum c-cost T-cut in G.*

Proof. Let $\delta_G(S)$ be a minimum T-cut in G. It is sufficient to show that there exists an edge $f \in F$ such that $\delta_R(S_f)$ is a T-cut in R and $w(\delta_R(S_f)) = c(\delta_G(S))$. Consider the forest

Z obtained from R by removing the edges in the cut $\delta_R(S)$. Since $S \cap T$ is odd, there must be some tree Q in $S \cap Z$ containing an odd number vertices from T. Let $g = xy \in \delta_R(S)$ with $x \in Q$. Since $w(\delta_R(S_g)) = w(xy)$ is the cost of the minimum $\{x, y\}$-cut in G, and $\delta_G(S)$ is an $\{x, y\}$-cut, it follows that $w(\delta_R(S_g)) \leq c(\delta_G(S))$. But $\delta_R(S_g)$ is a T-cut in R so the result follows. $\square$

More recently, Rizzi [Riz03] gave a simple algorithm for finding a minimum T-cut that requires between $\frac{|V|}{2}$ and $|V| - 1$ minimum $\{s, t\}$-cut computations, and thus is faster than the Gomory-Hu algorithm on average.

Chapter 3

Packing T-joins

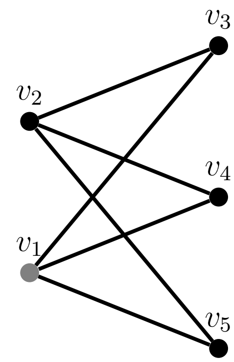
In this chapter, we turn our attention to a packing problem in grafts. Recall that given a graft (G, T), a *packing* of T-joins is a collection of edge-disjoint T-joins and that we use $\tau(G, T)$ to denote the number of edges in a minimum T-cut, and $\nu(G, T)$ the number of T-joins in a maximum packing. Recall also Corollary 1.0.6, which states that $\nu \leq \tau$ for every graft. When equality holds in this equation, we say that the graft *packs*. An interesting open question in combinatorial optimization is to find a precise characterization of grafts which pack. We now overview some of what is known about this question. In Section 3.1 we give a structural description of packing grafts, and in Section 3.2 we discuss some conjectures related to packing grafts.

3.1 Characterization of packing grafts

3.1.1 Graft minors and a structure theorem

We begin with an important example of a graft which does not pack. Consider the graft $(K_{2,3}, T)$, where $K_{2,3}$ is the complete bipartite graph with independent sets of size 2 and 3, and T contains all vertices except one vertex of degree 3. This graft is called the *odd* $K_{2,3}$ and is pictured in Figure 3.1. It is simple to check that the smallest T-cut contains 2 edges

Figure 3.1: An odd $K_{2,3}$



(for example, $\delta(\{v_3\})$ is minimal), while it is impossible to find 2 edge-disjoint T-joins in the odd $K_{2,3}$ (there exist four T-joins which all pairwise intersect).

We now introduce the concept of a *graft minor* before explaining the significance of the odd $K_{2,3}$. Given a graph $G = (V, E)$, and $T \subseteq V$ with $|T|$ even, we define the following three operations.

- To *delete* an edge uv is to replace E with $E \setminus \{uv\}$. We write $G \setminus e$ to denote the resulting graph.
- To *contract* an edge uv is to replace V with $V \setminus \{u, v\} \cup \{w\}$, where w is a new vertex and to replace every edge xu or xv in E with a new edge xw. We write G/e to denote the resulting graph.
- To *T-contract* the edge uv is to replace V with $V \setminus \{u, v\} \cup \{w\}$, where w is a new vertex, to replace every edge xu or xv in E with a new edge xw, and to replace T with $T \setminus \{u, v\}$ if $|T \cap \{u, v\}|$ is even, or with $T \setminus \{u, v\} \cup \{w\}$ if $|T \cap \{u, v\}|$ is odd. We write $G_{(T)}/e$ to denote the resulting graph.

Definition 3.1.1. *We say that a graph G' is a minor of a graph G if G' can be obtained from G by performing a series of edge deletions and contractions.*

Definition 3.1.2. *We say that a graft (G', T') is a T-minor of another graft (G, T) if (G', T') can be obtained from (G, T) by performing a series of edge deletions and T-contractions.*

Observe that if (G', T') is a T-minor of (G, T), then G' is a minor of G, but the converse is not necessarily true.

The following theorem is a consequence of Seymour's max-flow min-cut theorem for binary matroids (see [Sey77]). Codato, Conforti, and Serafini also gave a short combinatorial proof in [CCS96]. We give their proof below.

Theorem 3.1.3. *Let (G, T) be a graft. Then $\nu(G, T) < \tau(G, T)$ if and only if the odd $K_{2,3}$ is a T-minor of (G, T).*

Proof. Let (G, T) be T-minor-minimal such that $\nu = \nu(G, T) < \tau(G, T) = \tau$. That is, every T-minor of (G, T) packs. In particular, for every edge $e \in E(G)$, we have

$$\tau - 1 \leq \tau(G \setminus e, T) = \nu(G \setminus e, T) \leq \nu \leq \tau - 1.$$

Thus, we may assume that $\nu = \tau - 1$ and further that every edge is contained in some minimal T-cut. Let $\Theta = \{v \in T; d(v) = \tau\}$. We claim that $|V \setminus \Theta| \geq 2$. Indeed, let $F_1, \ldots, F_\nu$ be a packing of ν T-joins. Then $E' = E(G) \setminus \{F_1 \cup \cdots \cup F_\nu\}$ is not a T-join. However $|\delta(v) \cap E'| = 1$ for each $v \in \Theta$. Since $T \Delta \{v \in V; |\delta(v) \cap E'| \text{ is odd}\}$ has (nonzero) even cardinality and is contained in $V \setminus \Theta$, we have $|V \setminus \Theta| \geq 2$. Now, let $x, y \in \Theta$ such that their distance in G is minimized amongst all pairs of vertices in Θ.

Claim 3.1.4. *(a) If $\delta(U)$ is a minimum T-cut, then $\delta(U) = \delta(v)$ for some $v \in \Theta$.*

(b) If $\{u, v\}$ is a 2-vertex cutset in G, then $u, v \notin \Theta$.

(c) There exist three internally vertex-disjoint (x, y)-paths P_1, P_2, P_3.

Before proving the claim, we show how it implies the lemma. For $i = 1, 2, 3$ let x_i denote the neighbour of x on the path P_i. By the claim and because every edge is contained in some minimum T-cut, $x_1, x_2, x_3 \in \Theta$. We now consider two cases.

Case 1. $G \setminus \{x_1, x_2, x_3\}$ is connected. Let $F' = (P_1 \cup P_2 \cup P_3) \setminus \{x, x_1, x_2, x_3\}$ and extend F' into a spanning tree F of $G \setminus \{x_1, x_2, x_3\}$. Let e be an edge incident to x in F and let F_1, F_2 be the two components of $F \setminus e$. T-contracting the edges of F_1 into $\{x\}$ and the edges of F_2 into $\{y\}$, then deleting edges between vertices in $\{x_1, x_2, x_3\}$ yields an odd $K_{2,3}$.

Case 2. $G \setminus \{x_1, x_2, x_3\}$ is disconnected. Let $C_1, \ldots, C_k$ be the components of $G \setminus \{x_1, x_2, x_3\}$, numbered so that $|C_j \cap T|$ is odd for $1 \leq j \leq l$ and even for $l < j \leq k$, for some l. Since $x_1, x_2, x_3 \in T$, l is odd. Indeed, since not every $\delta(C_j)$ is a minimum T-cut, we have

$$l\tau < \sum_{j=1}^{k} |\delta(C_j)| \leq \sum_{i=1}^{2} |\delta x_i| \leq 3\tau$$

so $l = 1$. T-contract the edges in C_1 and C_2. For $3 \leq j \leq k$, T-contract the edges of C_j and the edges in $\delta(x_3) \cap \delta(C_j)$ (such edges exist by the claim). Finally, delete edges between the vertices in $\{x_1, x_2, x_3\}$. The resulting minor is an odd $K_{2,3}$.

Proof of Claim: Suppose that either (a) or (b) is false. Then there exists a minimum T-cut $\delta(U)$ and two subgraphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ of G such that E_1 and E_2 are both nonempty and partition $E \setminus \delta(U)$ and such that:

(i) $V_1 \Delta V_2 = V$ (if claim (a) is false)

(ii) $V_1 \cap V_2 = \{v\}$ and $V \setminus (V_1 \cup V_2) = \{u\} = U$ (if claim (b) is false)

Let (G'_1, T_1) be obtained by T-contracting E_2 in G and (G'_2, T_2) be obtained by T-contracting E_1. We have $\tau(G'_1, T_1) = (G'_2, T_2) = |\delta(U)| = \tau$. Hence, for $i = 1$ and $i = 2$, there exist τ edge-disjoint T-joins $F^i_1, \ldots, F^i_\tau$ in G'_i, with $|\delta(U) \cap F^i_j| = 1$ for $j = 1, \ldots, \tau$. We may assume that $\delta(U) \cap F^1_j = \delta(U) \cap F^2_j$ for each j. For each $j = 1, \ldots, \tau$, let $F_j = F^1_j \cup F^2_j$. Since $\nu < \tau$, we may assume that F_1 is not a T-join, so $W(F_1) = T \Delta \{v \in V; |\delta(v) \cap F_1| \text{ is odd}\}$ has nonzero, even cardinality. On the other hand, clearly $(V_1 \Delta V_2) \cap W(F_1) = \emptyset$, so (i) cannot hold. Hence (ii) holds and $W(F_1) = \{u, v\}$. However, $u \in T$ and $|F_1 \cap \delta(u)| = 1$, so $u \notin W(F_1)$ which is a contradiction.

Now, suppose that (c) is false. Then there is a 2-vertex cutset U separating x and y. Because we chose x and y to minimize their distance in G, at least one element, say u of U is in Θ. Hence by (b), $|U| = 1$. But then $\delta(u)$ properly contains a T-cut, which is a contradiction if $u \in T$.

□

Theorem 3.1.3 does give a characterization of those grafts which pack, albeit not an elucidating one, since an efficient way to recognize grafts without an odd $K_{2,3}$-minor is not known.

3.1.2 Lower bounds on τ

As there exist grafts which do not pack, a natural next step is to look for a lower bound on $\nu(G,T)$ in terms of $\tau(G,T)$. DeVos and Seymour proved that such a lower bound does exist for every graft.

Theorem 3.1.5 (DeVos and Seymour, [DS])**.** *Let (G,T) be a graft. Then $\nu(G,T) \geq \frac{1}{6}\lceil \tau(G,T) \rceil$.*

Adding an assumption on (G,T) allows the lower bound to be increased. As we will see in Section 3.2, the assumption that G is Eulerian, or that $E(G)$ is a T-join is a useful one.

Theorem 3.1.6 (DeVos and Seymour, [DS])**.** *Let (G,T) be a graft. Suppose that G is Eulerian, or that $E(G)$ is a T-join. Then $\nu(G,T) \geq \lceil \frac{1}{2}\tau(G,T) \rceil$.*

On the other hand, Devos and Seymour also gave in [DS] a construction of a family of grafts F_r with $\tau(F_r) = 3r$ and $\nu(F_r) = 2r$. The grafts F_r do not satisfy the assumption of Theorem 3.1.6. Rizzi [Riz99] also constructed a family of grafts for which $\nu \leq \lfloor \frac{2}{3}\tau \rfloor - 1$.

3.2 Guenin's conjecture

In this section our primary focus is a conjecture due to Guenin, posed in [Gue03] for which we give a proof of a base case in Chapter 4.

3.2.1 The conjecture and background

The *parity* of a T-cut $\delta(S)$ is the parity of $|\delta(S)|$. Guenin studied grafts in which all T-cuts are of the same parity. The following facts appear in [Gue03, Proposition 1.2].

Lemma 3.2.1. *Let (G,T) be a graft, with $T \neq \emptyset$. Then all T-cuts in G are odd if and only if $E(G)$ is a T-join.*

Proof. The reverse implication follows from Lemma 1.0.5, since for every T-cut $\delta(S)$, we have $E(G) \cap \delta(S) = \delta(S)$. Suppose now that all T-cuts in G are odd. Since $T \neq \emptyset$, there exists some T-cut $\delta(S)$. Suppose there exists $v \notin T$. Then $\delta(S\Delta\{v\})$ is a T-cut as well and thus has the same parity as $\delta(S)$. Observe that

$$\delta(v) = \delta(S\Delta S\Delta\{v\}) = \delta(v)\Delta\delta(S\Delta\{v\}),$$

so $d(v)$ is even. We may assume that there exists a vertex $w \in V(G)$ of odd degree, for otherwise G is Eulerian and has no odd cuts. Since $w \in T$, to finish the proof it remains to be shown that for every vertex $u \in T$, $d(u)$ is odd. Since $\delta(S)$ is a T-cut, so is $\delta(S\Delta u\Delta w)$. Now observe that

$$\delta(u)\Delta\delta(w) = \delta(S\Delta S\Delta\delta(u)\Delta\delta(w)) = \delta(S)\Delta\delta(S\Delta\{u\}\Delta\{w\}).$$

Since $\delta(S)$, $\delta(S\Delta\{u\}\Delta\{w\})$ and $d(w)$ are all odd, $d(v)$ must also be odd. □

Lemma 3.2.2. *Let (G,T) be a graft, with $T \neq \emptyset$. Then all T-cuts in G are even if and only if G is Eulerian.*

Proof. The proof follows easily from the proof of Lemma 3.2.1. □

A *planar graph* is a graph that can be drawn in the plane in such a way that edges may only intersect at their common endpoints. Wagner's Theorem [Wag37] characterizes planar graphs as those which have neither the complete bipartite graph $K_{3,3}$ nor the complete graph K_5 as a minor.

Guenin's conjecture proposes a sufficient condition for a graft (G, T) to pack.

Conjecture 3.2.1 (Guenin)**.** *Let (G, T) be a graft where all T-cuts in G have the same parity. Then (G, T) packs if G is planar.*

Observe that $K_{2,3}$ is planar, and there exist T-cuts of both odd and even parity in the odd $K_{2,3}$. Hence the parity condition in Conjecture 3.2.1 cannot be omitted. The planarity condition may not be removed either. The Petersen graph G_p is not planar, however it can be verified that all $V(G_p)$-cuts in G_p are odd, and that $\tau(G_p, V(G_p)) = 3$. Since G_p is 3-regular, if there exist 3 disjoint $V(G_p)$-joins in G_p, then each one is a perfect matching. However it is well-known that there do not exist 3 disjoint perfect matchings in G_p.

It is a fact that if (G, T) is any graft where each connected component of G contains an even number of vertices from T, there exists a T-join in G. For example, one way to find a T-join is to partition T arbitrarily into $\frac{|T|}{2}$ pairs of vertices, find paths joining the vertices in each pair, and take the symmetric difference of all $\frac{|T|}{2}$ paths. This construction yields a T-join by Lemma 2.2.1. This implies that Conjecture 3.2.1 is true if the assumption is made that $\tau(G, T) = 1$. The case when $\tau(G, T) = 2$ follows from Lemma 3.2.2, for in an Eulerian graph G, $E(G) \setminus J$ is a T-join, for any T-join J. In [Gue03], Guenin proved Conjecture 3.2.1 for the cases when $\tau(G, T) \leq 5$. We discuss his proof in some detail in Section 3.2.2. Recently, Dvořák, Kawarabayashi and Kráľ' [DKK10] settled the case when $\tau(G, T) = 6$, and Edwards and Kawarabayashi [EK11] proved the case $\tau(G, T) = 7$. This proof constitutes the original contribution of this thesis, and is presented in full in Chapter 4.

Let us remark that Conjecture 3.2.1 would be implied by the following conjecture, for it is well-known that no planar graph has a Petersen minor (The Petersen graph has both $K_{3,3}$ and K_5 as minors).

Conjecture 3.2.2 (Guenin [Gue03]). *Let (G,T) be a graft where all T-cuts in G have the same parity. Then (G,T) packs if G does not contain the Petersen graph as a T-minor.*

Given $r \in \mathbb{N}$, an *r-edge-colouring* of a graph G is an assignment of *colours* from the set $\{1,\ldots,r\}$ to the edges of G such that each vertex in $V(G)$ is incident to at most one edge of any given colour. If such an assignment exists we say that G is *r-edge-colourable*. An r-edge-colouring determines a partition of the edges of G into *colour classes* of edges which share the same colour.

In [CJ87], Conforti and Johnson made a weaker version of Conjecture 3.2.2, the parity condition replaced with the requirement that all T-cuts are odd. In particular, as pointed out by Guenin, this case would imply a conjecture of Tutte, recently proved by Robertson, Sanders, Seymour and Thomas (see [RST97], [Tho99]), which states that 3-regular, bridgeless graphs with no Petersen minor have a 3-edge-colouring. To see this, let G be such a graph, $T = V(G)$. Then $E(G)$ is a T-join, so all T-cuts contain at least 3 edges and 3 disjoint T-joins in G would form the colour classes of a 3-edge-colouring.

3.2.2 Outline of Guenin's proof

We now examine Guenin's proof more closely, in preparation for the proof of the special case of Conjecture 3.2.1 when $\tau(G,T) = 7$. Consider a graft (G,T) which is a counterexample to Conjecture 3.2.1. That is, G is planar and all T-cuts in G have the same parity, but (G,T) does not pack. Suppose that (G,T) is minimal in the sense that amongst all such counterexamples, the following are minimized (in order of priority): $|V(G)|$, $\tau(G,T)$ and $|E(G)|$. To simplify notation, let k denote $\tau(G,T)$. Note that we may assume G is a connected graph.

The following two lemmas allows us to study edge-colourings rather than packings of T-joins.

Lemma 3.2.3 (Guenin, [Gue03]). *Let (G,T) be a minimal counterexample to Conjecture 3.2.1. Every non-trivial cut in G has size at least $k+2$.*

Proof. Suppose that there exists a cut $\delta(S)$ in G of size k. Let $S_1 = S$ and $S_2 = V \setminus S$. First, observe that the induced subgraph $G[S_i]$ on each vertex set S_i $(i = 1, 2)$ must be connected. Otherwise, suppose that we can partition S_i into two sets S_i' and S_i'' such that there are no edges between S_i' and S_i'' in G. Since $|S_i \cap T|$ is odd, we may assume that $|S_i' \cap T|$ is also odd. Since G is connected, there must be at least one edge between S_{3-i} and S_i'', so $\delta(S_i')$ is a T-cut of size less than k, a contradiction.

Now, let (G_i, T_i) be the graft obtained by identifying all vertices in S_i into a single node v_i and deleting all loops, and letting $T_i = T \setminus S_i \cup \{v_i\}$. G_i is a minor of G and hence is planar. Further, since the T_i-cuts in G_i correspond to T-cuts in G where all vertices in S_i lie on the same shore it follows that $\tau(G_i, T_i) = k$ and that all T_i-cuts have the same parity.

By the minimality of (G, T) there exist packings $\{B_1^i, \ldots, B_k^i\}$ of T_i-joins in G_i, each containing exactly one edge in $\delta(v_i)$. We may assume that B_j^1 and B_j^2 contain the edge incident to v_1 and v_2 which correspond to the same edge in G, for $j = 1, \ldots, k$. But then $\{B_1^1 \cup B_1^2, \ldots, B_k^1 \cup B_k^2\}$ is a packing of k T-joins in G, a contradiction. □

Lemma 3.2.4 (Guenin, [Gue03])**.** *Let (G, T) be a minimal counterexample to Conjecture 3.2.1. Then G is k-regular and $T = V(G)$.*

Proof. Suppose for a contradiction that there exists $v \in V(G)$ with $v \notin T$ or $d(v) \geq k$. We first prove the following claim.

Claim: Not all edges incident to v are parallel.

Proof of claim: Suppose for a contradiction that all edges in $\delta(v)$ are parallel. We first handle the case where $v \in T$, and thus $d(v) \geq k + 2$. Let $e_1, e_2 \in \delta(v)$ and let $G' = (V, E \setminus \{e_1, e_2\})$ be the graph obtained from G by removing the edges e_1 and e_2. Then $\tau(G', T) = k$, so there exists a packing $\mathcal{J}$ of k T-joins in G' by the minimality of G. But $\mathcal{J}$ is also a packing of k T-joins in G, a contradiction. Now, suppose that $v \notin T$, and let $G' = (V, E \setminus \delta(v))$. Since the edges incident to v are all parallel, it is clear that no minimum T-cut in G contains any edge in $\delta(v)$, so $\tau(G', T) = k$. By minimality of G, there exists a packing $\mathcal{J}$ of T-joins in G' which is also a packing of T-joins in G, a contradiction.

Now, consider a planar embedding of G. Consider the edges of $\delta(v)$ in clockwise order. By the claim, there must exist two consecutive edges u_1v and u_2v where $u_1 \neq u_2$. Let $G' = (V, E \setminus \{u_1v, u_2v\} \cup u_1u_2)$. Observe that the planar embedding of G is easily transformed into a planar embedding of G' and that there are no loops in G'. Let $\delta(S)$ be a T-cut in G. If S is a non-trivial cut or $S = \{v\}$, then $|\delta_G(S)| > k$ by the assumption on v and by Lemma 3.2.3. In fact since all T-cuts have the same parity, $|\delta_G(S)| \geq k + 2$. It follows that $\tau(G', T) = k$. By minimality of G, there exists a packing $\mathcal{J} = \{B_1, \ldots, B_k\}$ of T-joins in G'. If none of these T-joins contains the edge u_1u_2, then $\mathcal{J}$ is also a packing of T-joins in G. Otherwise, say B_1 does contain u_1u_2. Then $\{B_1 \setminus \{u_1u_2\} \cup \{u_1v, u_2v\}, \ldots, B_k\}$ is a packing of T-joins in G. In either case, we have reached a contradiction.

□

A graft (G, T), where $T = V(G)$, G is k-regular, and $\tau(G, T) \geq k$ is sometimes called a *k-graph.*

Suppose that G has a k-edge-colouring. Then Lemma 3.2.4 implies that for any $v \in V(G) = T$, v is incident to exactly one edge of each colour. Thus the edges in each colour class form a T-join, and (G, T) packs.

It follows that in order to prove Conjecture 3.2.1 it is sufficient to prove the following.

Conjecture 3.2.3. *Let G be a k-regular planar graph, and suppose that $|\delta(S)| \geq k$ for each cut $\delta(S)$ where $|S|$ is odd. Then G is k-edge-colourable.*

The case $k = 3$ of Conjecture 3.2.3 is of particular interest. Tait [Tai80] showed that the statement is equivalent to the Four-Colour Theorem, which affirms that the vertices of every planar graph can be coloured using four colours in such a way that the two endpoints of each edge have different colours. This important result in graph theory was proved by Appel and Haken [AH76]. Thus Conjecture 3.2.1 is a strengthening of the Four-Colour Theorem. The proofs of Conjecture 3.2.3 for $k = 4, 5, 6, 7$ all rely on the *discharging* *method*, a proof technique which is at the core of the proof of the Four-Colour Theorem. We defer further discussion of the discharging method to Chapter 4.

Observe that the converse of Conjecture 3.2.3 is always true. That is, if G is a k-regular, k-edge-colourable planar graph, then all cuts $\delta(S)$ where $|S|$ is odd have at least k edges. This is because each colour class in a k-edge-colouring is a $V(G)$-join and must contain an edge from each $V(G)$-cut by Lemma 1.0.5. We close this section with the remark that Seymour has made the following unpublished conjecture which would extend Conjecture 3.2.2 in a fashion analogous to the connection between Conjectures 3.2.1 and 3.2.3.

Conjecture 3.2.4 (Seymour, see [Gue03])**.** *Let G be a k-regular graph with no Petersen minor. Then $|\delta(S)| \geq k$ for each cut $\delta(S)$ where $|S|$ is odd if and only if G is k-edge-colourable.*

Chapter 4

Packing seven T-joins in planar graphs

4.1 Introduction

In this chapter, we prove the following theorem.

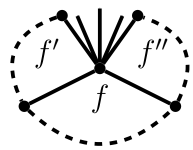
Theorem 4.1.1. *Let G be a 7-regular plane multigraph. If G has no odd cut of size less than 7, then G has a 7-edge-colouring.*

The strategy of the proof is to exclude the existence of a minimal counterexample. We clarify just what we mean by minimal counterexample in Section 4.3, and begin in Section 4.2 with an overview of some important definitions. In the sections that follow, we prove Theorem 4.1.1 by means of a discharging argument.

4.2 Definitions

In the proof of Theorem 4.1.1, we study 7-regular plane graphs with no odd cuts of size less than 7 which are not 7-edge-colourable. We now review some terminology.

Figure 4.1: Illustration of Definition 4.2.2: f' and f'' are f-incident. (dashed lines represent edges in series)



Let $G = (V, E)$ be a 2-edge-connected plane multigraph. A face $f = v_1v_2 \dots v_d$ is an open, connected region in the plane bounded by edges $v_1v_2, \dots, v_{d-1}v_d, v_dv_1$, called the *boundary* of f. We sometimes say that f contains or is adjacent to the edges on its boundary. Two faces of G are *adjacent* if they share a bounding edge in common, and a pair of edges or faces is *incident* if they share one vertex in common.

The *size* of a face f is the number of edges on its boundary. A d-face is a face bounded by exactly d edges. We use an $\leq d$-face and an $\geq d$-face to denote a face of size at most d and at least d, respectively.

Before continuing, we point out the following observation about a minimal counterexample to Theorem 4.1.1, which appears as Lemma 2.10 in [Gue03].

Lemma 4.2.1 (Guenin [Gue03])**.** *Let G be a minimum counterexample to Theorem 4.1.1. Any pair of faces in G share at most one edge. In particular a d-face is adjacent to d distinct faces.*

For the remainder of this chapter, the graphs we study are assumed to be minimum counterexamples to Theorem 4.1.1.

Definition 4.2.2. *Let f, f' and f'' be faces of G. If f' and f'' are both adjacent to f and the edge that f shares with f' is incident to the edge that f shares with f'', then we say that f' and f'' are* f-incident*. (See Figure 4.1)*

We use the term *multigon of order d* to denote the set of 2-faces formed by d parallel edges. Multigons of order 2,3,4,5 are called *bigons, trigons, quadragons* and *quintagons*, respectively. We make use of the notation vw to denote a multigon whose two endpoints are v and w. We say that a face f is *adjacent* to a multigon uv if uv is an edge on the boundary of f. We say that f is *incident* to a multigon if they share one vertex. Two multigons are *incident* if they share a vertex. In order to avoid ambiguity, we always take the order of a multigon to be maximal in the sense that a multigon can only be adjacent to ≥ 3-faces.

In our arguments, we often refer to edge-colourings of graphs. To simplify our arguments, we will use the letters $\alpha, \beta, \delta, \gamma, \epsilon, \phi$ and μ to denote the colours used on edges. We refer to edges by their pairs of endpoints. For example we write $e = vw$ for the edge e with endpoints v and w. Since we are dealing with multigraphs, there may be parallel edges. When we say *an edge* vw, we simply mean one of the edges with endpoints v and w.

4.3 Structure of the proof

By Lemma 2.2 from [Gue03], in order to prove Theorem 4.1.1, it is enough to exclude the existence of a minimal counterexample to Theorem 4.1.1. A minimal counterexample G satisfies the following assumptions:

- G is a 7-regular plane graph,
- every odd cut of G has size at least 7.
- G has no 7-edge-colouring, and

G is minimal in the sense that

- subject to the previous conditions, G has the smallest order,
- subject to the previous conditions, G has as many quintagons as possible,
- subject to the previous conditions, G has as many quadragons as possible,

- subject to the previous conditions, G has as many trigons as possible, and
- subject to the previous conditions, G has as many bigons as possible.

The following lemma is a first observation about such a graph.

Lemma 4.3.1. *The maximum order of a multigon in a minimal counterexample is at most 5 and the sum of the orders of two incident multigons is at most 6.*

Proof. A minimal counterexample G must be connected, for if G is 7-regular and has no odd cuts of size less than 7 then so too must be each connected component, making each component a smaller counterexample. Hence G cannot have any multigons of order 7.

Suppose G has a multigon of order 6. Let v and w be the endpoints of the multigon, and let v be joined by an edge e_1 to another vertex v' and w be joined by en edge e_2 to another vertex w'. Consider the graph G' obtained from G by contracting the edges of the multigon and e_2 into a single vertex. G' is 7-regular and every odd cut in G' has size at least 7. By minimality of G, G' has a 7-edge-colouring. From this colouring, we can obtain a 7-edge colouring of G by assigning the colour of e_1 to e_2, assigning the other six colours to the edges of the multigon and keeping all other colours the same. This is a contradiction.

Now, suppose that G has two adjacent multigons whose orders sum to 7. Then the graph G'' obtained by contracting the two multigons into a single vertex adjacent to all neighbours of the endpoints of the multigons is 7-regular and has no odd cuts of size less than 7. By minimality of G, G'' has a 7-edge colouring which can be easily extended to a 7-edge-colouring of G where the seven edges on the multigons receive seven distinct colours, a contradiction.

□

In the coming sections, we rule out the existence of a counterexample to Theorem 4.1.1 using the discharging method.

4.4 Discharging

4.4.1 Distribution of charges

We now describe the discharging. Let G be a minimal counterexample to Theorem 4.1.1. We distribute charges to the faces, multigons and vertices of G as follows. Every d-face ($d \geq 3$) receives a charge of $d-3$. Every bigon receives a charge of -1, every trigon a charge of -2, every quadragon a charge of -3 and every quintagon a charge of -4. Finally, every vertex receives a charge of 0.5. The total amount of charge distributed is

$$\frac{V}{2} + 2E - 3F = \frac{V}{2} + 7V - 3F = 3\frac{7V}{2} - 3V - 3F = 3(E - V - F)$$

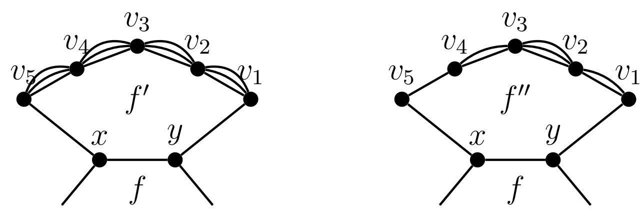
(where V, F, E denote the number of vertices, faces, and edges in G, respectively) since G is 7-regular. Euler's formula for planar graphs shows that the total amount of charge distributed is negative.

We then move the distributed charges according to the rules in Section 4.4.3. We say a face, multigon or vertex x *sends* a quantity q of charge to a face y to mean that x's amount of charge is decreased by q and that of y is increased by q, thus preserving the total amount of charge in the graph. Note that a face or vertex is allowed to send positive quantities of charge even if it does not possess positive charge. Also, fractional quantities of charge may be moved. We proceed through the list of rules in Section 4.4.3, applying each one once, to any vertex, face or multigon where it applies. We then examine the new distribution of charges. In order to prove Theorem 4.1.1, we show that there is a discrepancy between the total amount of charge in the graph before and after applying the discharging rules. More precisely, we show that every vertex and every face has non-negative charge after application of the rules.

4.4.2 Further definitions

Before stating the discharging rules, we need to give a few more definitions, for which we now give some motivation. The ultimate goal of the discharging is to increase the charge of

Figure 4.2: Example: $f' \in T_1(f)$ and $f'' \in T_2(f)$.



multigons to at least 0, without reducing the charge of ≥ 3-faces or vertices, whose initial charge is non-negative, by too much. For our purposes, we prefer for a large face to send some charge to a ≤ 3-face than to a ≥ 4-face. Therefore, it is useful to keep track of the number of ≥ 4-faces adjacent to a given face.

Definition 4.4.1. *A face is d-*big *(respectively $\leq d$-big, $\geq d$-big) if it is adjacent to d (respectively at most d, at least d) ≥ 4-faces.*

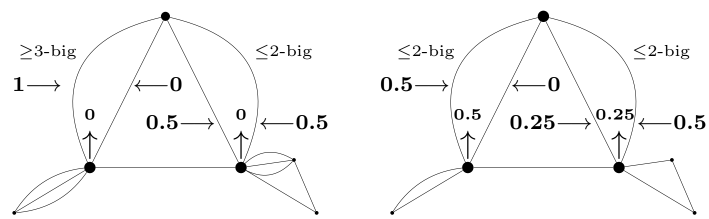
Let G be a minimal counterexample to Theorem 4.1.1 and let f be a ≥ 4-face in G. We classify the ≥ 4-faces adjacent to f into two collections $T_1(f)$ and $T_2(f)$ as follows. Suppose $f' = xyv_1v_2 \dots v_l$ shares the edge xy with f. Let the set $E_f(f') = \{v_1v_2 \dots v_{l-1}v_l\}$ be the edges on the boundary of f'. We define $T_1(f)$ to be the set of faces f' adjacent to f for which each edge in $E_f(f')$ is adjacent to a multigon of order ≥ 3. We also define $T_2(f)$ to be the set of edges f' adjacent to f for which some edge in $E_f(f')$ is adjacent to a bigon or a ≥ 3-face. (See Figure 4.2 for an example.) Clearly each face f' adjacent to f is in exactly one of $T_1(f), T_2(f)$.

Definition 4.4.2. *Let f be a ≥ 4-face in a minimal counterexample G. Suppose $|T_2(f)| = d$. Then we say that f is d-large.*

This definition is extended in the obvious way to characterize f as $\geq d$-large or $\leq d$-large. It follows from the definitions that if f is d-big and d'-large then $d' \leq d$.

Finally, as we closely study the structure of 3-faces in the coming sections, we adopt the following convenient naming convention for 3-faces to reflect the set of multigons to

Figure 4.3: Example of rules for a dangerous 3-face adjacent to a ≤2-big ≥4-face



which they are adjacent. Accordingly, a *1-trigon-3-face* is a 3-face adjacent to a trigon and no other multigons, a *1-trigon-1-bigon-3-face* is adjacent to a trigon, a bigon and no other multigons, and a *0-multigon-3-face* is not adjacent to any multigons. A *1-bigon-3-face* is a 3-face adjacent to one bigon and no other multigons, a *3-bigon-3-face* is adjacent to three bigons and finally, a 3-face is called *dangerous* if it is adjacent to exactly two bigons.

4.4.3 Rules

1 A face f which is ≥6-large sends

1.1 1.5 to adjacent quintagons

1.2 0 to adjacent faces which are f-incident to a quintagon

1.3 1 to adjacent quadragons, trigons and bigons, 3-faces, and faces in $T_1(f)$

1.4 0.5 to all other adjacent ≥4-faces

2 A face f which is 4-large or 5-large sends

2.1 1 to adjacent quadragons, trigons and bigons, 3-faces, and faces in $T_1(f)$

2.2 0.25 to all other adjacent ≥4-faces

3 A face f which is 3-large

sends 1 charge to each adjacent ≤3-face and to each face in $T_1(f)$.

4 A 3-face f which is not dangerous sends charge according to at most one of the following seven cases:

4.1 adjacent to a quadragon: sends 1 to the quadragon

4.2 adjacent to a trigon and a bigon: does not send any charge

4.3 adjacent to one trigon and no bigons: if the trigon is not adjacent to a $\geq$3-big face, sends 0.5 to the trigon, otherwise does not send any charge.

4.4 adjacent to three bigons: does not send any charge

4.5 adjacent to just one bigon: 0.5 charge to the bigon if it is not adjacent to a $\geq$3-big face and 0 if it is.

4.6 adjacent to a $\geq$3-big face: sends 0.5 to each $\leq$2-big $\geq$3-face adjacent to f

4.7 adjacent to two $\geq$3-big faces: sends 1 to each $\leq$2-big $\geq$3-face adjacent to f

5 A dangerous 3-face f sends charge according to at most one of the following three cases:

5.1 adjacent to a $\geq$3-big face: sends 0.5 to each adjacent bigon.

5.2 adjacent to a $\leq$2-big $\geq$4-face f': For each bigon adjacent to f, (see Figure 4.4)

5.2.1 If the bigon is adjacent to a $\geq$3-big face: f sends 0 charge to the bigon.

5.2.2 If the bigon is not adjacent to a $\geq$3-big face, but is incident to a trigon: f sends 0.5 charge to the bigon.

5.2.3 If the bigon is not adjacent to a $\geq$3-big face nor incident to a trigon, but is incident to a 3-face adjacent to f': f sends 0.25 charge to the bigon.

5.2.4 If the bigon is not adjacent to a $\geq$3-big face, but is incident to a bigon adjacent to f': f sends 0 charge to the bigon.

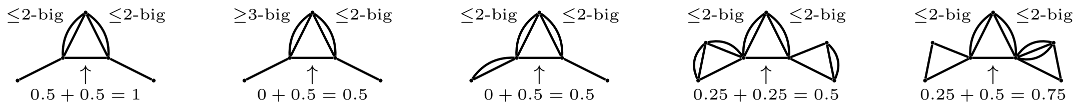
5.2.5 Otherwise: f sends 0.5 to the bigon

5.3 adjacent to a 3-face: sends 0.5 to each bigon of f adjacent to only $\leq$2-big faces

6 A $\leq$2-big $\geq$4-face f sends

6.1 1 to adjacent quadragons

Figure 4.4: Rules for a dangerous 3-face adjacent to a ≤2-big ≥4-face



6.2 to an adjacent trigon t which is also adjacent to a ≥3-big face: 0.25 for each other trigon that is incident to t but not adjacent to f.

6.3 to an adjacent trigon t which is also adjacent to a ≤2-big face: 0.5 plus an additional 0.25 for each trigon that is incident to t but not adjacent to f.

6.4 0.5 charge to those adjacent bigons which are not adjacent to a ≥3-big face and 0 to those bigons which are.

6.5 Further, for each bigon on an adjacent dangerous 3-face f', (see Figure 5)

6.5.1 If the bigon is adjacent to a ≥3-big face: f sends 0 charge to f'.

6.5.2 If the bigon is not adjacent to a ≥3-big face, but is incident to a trigon: f sends 0.5 charge to f'.

6.5.3 If the bigon is not adjacent to a ≥3-big face nor incident to a trigon, but is incident to a 3-face adjacent to f: f sends 0.25 charge to f'.

6.5.4 If the bigon is not adjacent to a ≥3-big face, but is incident to a bigon adjacent to f: f sends 0 charge to f'.

6.5.5 Otherwise: f sends 0.5 to f'

6.6 and sends 0.25 to any adjacent 3-face f'' which is adjacent to one ≤2-big face and one bigon that is only adjacent to ≤3-big faces

7 A vertex sends charge according to at most one of the following seven cases:

7.1 incident to a quintagon: sends 0.5 to the quintagon

7.2 incident to a quadragon: sends 0.5 to the quadragon

7.3 incident to one trigon: sends 0.5 to the trigon

7.4 incident to two trigons: sends 0.25 to each trigon

7.5 incident to three bigons:

7.5.1 If one of the bigons is adjacent to a $\geq$3-big face, sends 0.25 to each of the other bigons which is not.

7.5.2 Otherwise, if the vertex is incident to exactly one of the bigons on a dangerous 3-face sends 0.5 to that bigon.

7.6 incident to two bigons and no trigon: If only one of the bigons lies on a dangerous 3-face, sends 0.5 to that bigon, otherwise sends 0.25 to each bigon

7.7 incident to one bigon and no other multigons: sends 0.5 to the bigon

4.5 Final charge of vertices and $\geq$ 3-big faces

We now begin to analyze the final amount of charge in G. In this section, we establish the non-negative final charge of vertices and $\geq$ 3-big faces. In order to analyze the final charge of multigons and $\leq$ 3-big faces we need to gain further insight into the structure of the minimal counterexample G. The remainder of the proof is organized as follows. In Section 4.6 we introduce some tools which are useful for analyzing the faces of G. In Section 4.7 we analyze the structure of 3-faces and their adjacent faces, then prove that 3-faces have non-negative final charge. Then in Section 4.8 we show that all multigons have non-negative final charge. Finally Sections 4.9 and 4.10 are devoted to analyzing the structure and final charge of $\leq$ 3-big 4 and $\geq$ 5-faces, respectively. This will exclude the existence of G and therefore complete the proof.

4.5.1 Final charge of vertices

The initial charge of every vertex is 0.5, and none of Rules 1 through 6 involve vertices. Therefore it follows directly from Rules 7 and Lemma 4.3.1 that the final amount of charge of every vertex is non-negative.

4.5.2 Final charge of $\geq$ 3-big faces

Lemma 4.5.1. *The final charge of every ≥ 3-big face is non-negative.*

Proof. Let f be an l-face and suppose that f is d-big for some $d \geq 3$. Then if f is d'-large, clearly $d' \leq d$.

First, suppose f is $\geq$6-large. Then Rules 1 dictate how f sends charge. By Rule 1.4 there are at least 6 faces to which f sends at most 0.5 units of charge. Since the initial charge of f is $l-3$, f can distribute at least $(l-3) - 0.5 * (6) = l-6$ charges amongst the $l-6$ remaining faces. If f is adjacent to a quintagon, then by Rules 1.1 and 1.2, f sends 1.5 charge to the quintagon and 0 to the each face f-incident to the quintagon. f sends at most 1 charge to each remaining face, so the final charge of f is non-negative.

Now, suppose f is 4-large or 5-large. Then f sends charge as dictated by Rules 2. There are at least 4 faces to which f sends 0.25 units of charge by Rule 2.2. Since the initial charge of f is $l-3$, f can distribute at least $(l-3) - 0.25 * (4) = l-4$ charges amongst the remaining $l-4$ faces. According to Rule 2.1, f sends at most 1 charge to each of those faces, so the final charge is non-negative.

If f is 3-large, then the initial charge of f is $l-3$. Since Rule 3 is the only rule which has f send any charge, there are at least three faces adjacent to f to which f sends 0 charge. f sends at most 1 charge to each of the remaining $l-3$ faces, so the final charge is non-negative.

Finally, if f is 0-, 1- or 2-large, then there is no rule by which f sends charge, so the final charge is at least $l-3 \geq 0$.

$\square$

4.6 Properties of a minimal counterexample

We now examine the structure of a minimal counterexample G more closely. The ideas developed in this section are frequently used and are crucial to an easy understanding of the proofs in the coming sections. The next lemma is analogous to Lemma 3 in [DKK10].

The next result follows directly from Lemma 3.2.3 and the fact that G is 7-regular.

Lemma 4.6.1. *Every non-trivial odd cut in a minimal counterexample has size at least nine.*

Definition 4.6.2. *Let G be a plane graph. Suppose that for some set of vertices $v_1, v_2, ..., v_m$ (m even), G contains edges $v_2v_3, v_4v_5, ..., v_{m-2}v_{m-1}, v_mv_1$ (G may or may not contain edges $v_1v_2, \dots, v_{m-1}v_m$) and it is possible to draw a closed curve in the plane that intersects G only at $v_1, ..., v_m$. Then we say that the ordered set of vertices $(v_1, v_2, ..., v_m)$ are* eligible. *Given such a set of eligible vertices, the operation of removing an edge v_iv_{i+1} and adding an edge $v_{i-1}v_i$ for every even i (taking indices modulo m) is called a $v_1...v_m$-*swap.

If G is a plane graph then it is clear that the resulting graph is also a plane graph. We will now see that when m is 4 or 6, the key properties of the minimal counterexample are preserved by swapping.

Lemma 4.6.3. *Let G be a minimal counterexample and let m be either 4 or 6. Any graph G' obtained by a $v_1v_2...v_m$-swap for eligible vertices $v_1, v_2, ..., v_m$ is 7-regular and has no odd cut of size less than 7.*

Proof. It is clear from the definition of the swap that G' is 7-regular. Let $\delta_{G'}(S)$ be an odd cut in G'. Observe that any edge which doesn't have endpoints v_i and v_{i+1} for $i \in \{1, ..., m\}$ is in the cut if and only if it is in the corresponding cut $\delta_G(S)$ in G. By symmetry, we may assume that S contains at most 3 vertices in $\{v_1, ..., v_m\}$. For $v_l \in S$, the vertices v_{l-1}, v_{l+1} may either lie on the same side of the cut or on different sides. Further, S contains at most 2 vertices v_l in $\{v_1, ..., v_m\}$ with the property that only one of v_{l-1}, v_{l+1} is in S. If $v_l \in S$ and $|\{v_{l-1}, v_{l+1}\} \cap S| = 0$ or 2 , then $\delta(v_l) \cap \delta_{G'}(S) = \delta(v_l) \cap \delta_G(S)$. If $|\{v_{l-1}, v_{l+1}\} \cap S| = 1$ then $|\delta(v_l) \cap \delta_{G'}(S) - \delta(v_l) \cap \delta_G(S)| \leq 1$. Hence $||\delta_{G'}(S)| - |\delta_G(S)| \leq 2|$. From Lemma 4.6.1, it follows that $|\delta_{G'}(S)| \geq 7$. $\square$

While the graph G is assumed not to be seven edge colourable, Guenin showed that there does exist a colouring of the edges of G which is similar to a 7-edge-colouring.

Definition 4.6.4. *Let G be a 7-regular plane graph and $e = uv$ be an edge of G. An* e-colouring *of G is a colouring of the edges of G with seven colours such that each edge except e is assigned one colour and e is assigned a set of three or more colours. Further both u and v are incident to an odd number of edges of each colour and every other vertex is incident to exactly one edge of each colour. Finally, the number of colours assigned to e is minimized among all such colourings.*

Lemma 4.6.5 (Guenin [Gue03])**.** *Let G be a minimal counterexample to Theorem 4.1.1. Then for every edge e of G, there exists an e-colouring of G.*

Definition 4.6.6. *Let G be a minimal counterexample to Theorem 4.1.1, let e be an edge of G and let c be any colour. An odd cut $M_c \in E(G)$ with $e \in M_c$ is called a* c-mate *if the following property holds. For every colour $c' \neq c$, M_c contains exactly one edge (possibly e) coloured with c'.*

Guenin showed the following strengthenings of Lemma 4.6.5.

Lemma 4.6.7 (Guenin [Gue03])**.** *Let G be a minimal counterexample to Theorem 4.1.1, let e be any edge of G and let c be any colour. There exists a non-trivial c-mate in G for this e-colouring.*

We now show two more properties of e-colourings.

Proposition 4.6.8. *Let G be a minimal counterexample and e an edge of G. In each e-colouring, every non-trivial c-mate contains at least five edges (possibly including e) assigned the colour c.*

Proof. Let M_c be a c-mate. Lemma 4.6.1 implies that M_c contains at least 9 edges. Since at least 3 colours appear on e, and M_c only contains one edge of every colour other than c, at least 5 edges must be coloured with c. □

Lemma 4.6.9. *In any e-colouring of G, no colour is assigned to more than one edge of a given multigon.*

Proof. Given an e-colouring of G, suppose there is some multigon with two edges assigned the colour α. Clearly e must be an edge of the multigon. By relabelling the colours on the multigon we may assume that α is assigned to e and to some other edge e'. Let β be another colour assigned to e. Then by assigning β to e' instead of α and removing α and β from the list of colours assigned to e, we obtain either a proper 7-edge-colouring of the edges of G or a colouring of the edges of G satisfying the conditions of an e-colouring, but where e receives fewer colours. Both cases are a contradiction. □

We now state a fact that will be useful for several proofs in the coming sections.

Proposition 4.6.10. *Let G be a minimal counterexample, and let G' be obtained from G by a $v_1v_2v_3v_4$-swap for some eligible vertices (v_1, v_2, v_3, v_4). If G' has a 7-edge-colouring, then no edge v_1v_2 can have the same colour as any edge v_1v_3, v_2v_4 or v_3v_4.*

Proof. Consider a 7-edge-colouring of G'. Suppose that some edge v_1v_2 is coloured α. It is clear that no other edge incident to v_1 or v_2 can be coloured α. Suppose for a contradiction that some edge v_3v_4 is coloured α. Then consider the following colouring of G: Let an edge v_2v_3 and an edge v_1v_4 have colour α and all other edges keep the same colour as in the 7-edge-colouring of G'. It is straightforward to check that in this colouring every vertex in G is incident to one edge of every colour, so it is a 7-edge-colouring of G, which is impossible. □

A consequence of Proposition 4.6.10 is the following useful fact that we will now state as a proposition.

Proposition 4.6.11. *Let G be a minimal counterexample and suppose that for some eligible set of vertices (v_1, v_2, v_3v_4), the graph G' obtained from G by the $v_1v_2v_3v_4$-swap has a 7-edge-colouring. Then if there exists an edge $e = v_1v_2$ in G there exists an e-colouring of G where e is coloured with three colours, say α, β and ϕ, an edge v_2v_3 as well as an edge v_4v_1 are coloured ϕ and all other edges are coloured with the same colour as in the edge colouring of G'.*

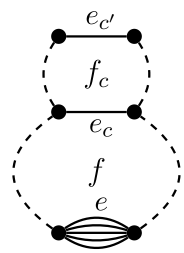
Proof. By the definition of the swap, G' contains at least one edge v_3v_4 and at least two edges v_1v_2. By Proposition 4.6.10, every 7-edge colouring of G' assigns three distinct colours to these three edges, say α, β and ϕ. Then by assigning colours α, β and ϕ to e and the colour ϕ to v_2v_3 and v_4v_1 and assigning each other edge the colour it received in the 7-edge-colouring of G' we obtain the desired e-colouring of G. □

To demonstrate how we can use the the lemmas in this section to show properties of a minimal counterexample, we give an application in the next lemma. The key to solving Lemma 4.6.12 is that given the e-colouring, we find a set of edges (the edges of the quintagon) that must be in every c-mate for every c. We then use the existence of mates for every colour to show that there exist many cuts (one for each colour) which pairwise intersect only in the edges of the quintagon. This allows us to draw conclusions about the sizes of faces that contain edges from these cuts.

Lemma 4.6.12. *In a minimal counterexample G, every face f adjacent to a quintagon is $\geq$6-big.*

Proof. Let e be the edge of the quintagon adjacent to f and consider an e-colouring of G. Since e receives at least three colours, we may assume by Lemma 4.6.9 that all seven colours appear on edges of the quintagon. Let c be any of the seven colours assigned to edges in the e-colouring. Then every c-mate M_c must contain the edges of the quintagon and the remaining edges in M_c are coloured with c by the definition of mates. Further, Proposition 4.6.8 implies that that there are at least five edges coloured c in M_c. Because G is planar, every cut must contain an even number of edges on the boundary every face. Therefore M_c contains an edge $e_c \neq e$ which is adjacent to f and to some other face $f_c \neq f$ (recall that any pair of faces in G have at most one boundary edge in common and that e_c cannot be parallel to e by Lemma 4.6.9). M_c also contains an edge $e'_c \neq e_c$ adjacent to f_c. Because e_c and e'_c are both coloured c, as long as e_c is not incident to the quintagon they cannot share an endpoint in common so the face f_c is a $\geq$4-face. (See Figure 4.5) The faces f_c are distinct for distinct colours c, and there are at most two colours c for which e_c is incident to the quintagon (since there are two edges f-incident to the quintagon) hence

Figure 4.5: Illustration for Lemma 4.6.12.



f is $\geq$5-big. We now argue that we may assume there is actually at most one colour c for which e_c is incident to the quintagon.

In fact, we may assume that the four edges incident to the quintagon have the same colour, from which it follows that f is $\geq$6-big. To see this, let v and w be the endpoints of the quintagon and let v' and w' be neighbours on f of v and w respectively. Since f is $\geq$5-big (and therefore not a triangle), $v' \neq w'$. Consider the graph G' obtained from G by the $vww'v'$-swap. G' has a multigon of order 6, and G doesn not by Lemma 4.3.1 so by minimality G' has a 7-edge-colouring. In this colouring we may assume by Proposition 4.6.10 that the edges between v and w are assigned colours $\alpha, \beta, \delta, \gamma, \phi$ and ϵ and the edge $v'w'$ as well as the other two edges incident to v or w have colour μ. It follows that there exists an e-colouring of G where the edges between v and w receive all seven colours, the other four edges incident to v or w receive the colour μ and all other colours remain the same as in the colouring of G'.

□

The next two lemmas generalize the ideas in the previous lemma. The idea is that we are given a set S of edges which is contained in every mate, upon which appear many different colours. We can then use the existence of mates for every colour to show that there exist many cuts which pairwise intersect in the edges contained in S. This allows us to draw conclusions about the sizes of faces that contain edges from these cuts. We will use them extensively in the remainder of the proof.

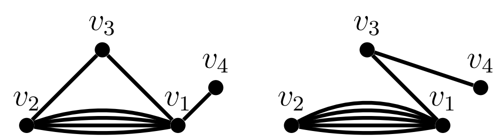
Proposition 4.6.13. *Let $G = (V, E)$ be a minimal counterexample, $e = uv$ an edge of G. Fix an e-colouring and a colour ϕ. Suppose that both u and v are incident to exactly one edge of each colour, except possibly ϕ. Suppose that there exists $S \subseteq E$ containing exactly one edge of each colour (possibly including e). Suppose further that for each $c \neq \phi$, and every c-mate M_c, $S \subseteq M_c$. Let f' and f'' be two distinct nonadjacent faces in G, and suppose that these are the only pair of faces intersecting S in an odd number of edges. Then f',f'' are both 6-big.*

Proof. For $c \neq \phi$, consider any c-mate M_c where $c \neq \phi$. Recall that since M_c is a cut, it must contain an even number of edges adjacent to every face. By Proposition 4.6.8, $M_c \setminus S$ consists of at least four edges, each of which is coloured with c. Therefore f' must be adjacent to an edge $e_c \in M_c \setminus S$ coloured with c. f' shares the edge e_c with some face f_c $(f_c \neq f'')$. f_c contains another edge $e'_c \in M_c \setminus S$ coloured with c, because by assumption f_c has an even number of edges in S on its boundary. Since no vertex has more than one incident edge coloured with c, f_c is $\geq$4-face. Because f_c are distinct for different colours c, f' is adjacent to at least six $\geq$ 4-faces and hence is $\geq$6-big. Symmetrically, f'' is also $\geq$6-big. □

Proposition 4.6.14. *Let $G = (V, E)$ be a minimal counterexample, $e = uv$ an edge of G. Fix an e-colouring of G and colours ϕ and μ. Suppose that both endpoints of e are incident to exactly one edge of each colour (possibly including e) , except possibly ϕ. Suppose that there exists $S \subseteq E$ containing one edge of each colour except μ. Suppose further that for each $c \neq \phi$, and every c-mate M_c, $S \subseteq M_c$. Let f' and f'' be two distinct nonadjacent faces in G, and suppose that these are the only pair of faces intersecting S in an odd number of edges. Then f',f'' are both 4-big.*

Proof. Let $c \neq \phi$ and let M_c be any c-mate where $c \neq \phi$. M_c must contain an even number of edges adjacent to every face. M_c contains the edges in S, so by Proposition 4.6.8 the remaining edges consist of at least four edges, each of which is coloured with c plus one edge coloured with μ. Therefore either f' or f'' must be adjacent to an edge $e_c \in M_c \setminus S$ coloured with c which it shares with some face f_c $(f_c \neq f' or f'')$. f_c contains another edge

Figure 4.6: Swap used in the proof of Lemma 4.7.1



$e'_c \in M_c \setminus S$ coloured with c since by assumption it has an even number of edges in S on its boundary. Since no vertex has more that one incident edge coloured with c, f_c is $\geq$4-face. Because for this is true for every colour $c \neq \phi$, and when $c = \mu$, both f' and f'' are adjacent to such a face, at least one of f' or f'' must be $\geq$4-big. □

4.7 Final charge of $\leq$ 2-big 3-faces

4.7.1 Structure of 3-faces

In this section, we discuss the structure of 3-faces in a minimal counterexample. This will allow us to analyze the final charge of 3-faces and of multigons. In the hypotheses of Lemmas 4.7.1 through 4.7.11, the graph G in question is assumed to be a minimal counterexample to Theorem 4.1.1.

Lemma 4.7.1. *If a 3-face $f = v_1v_2v_3$ is adjacent to a quadragon v_1v_2, then f is adjacent to no other multigon, the endpoints of the quadragon are incident to no other multigon, the other face adjacent to the quadragon v_1v_2 is $\geq$6-big, and the faces adjacent to both f and one of the edges v_2v_3 and v_3v_1 are $\geq$6-big.*

Proof. Let f' be the face adjacent to the edge v_1v_3, f'' the other face adjacent to the quadragon and f''' the face adjacent to the edge v_2v_3. Let $v_4 \neq v_3$ be a neighbour of v_1 on the boundary of f'. Consider the graph G' obtained from G by the $v_1v_2v_3v_4$-swap (see Figure 1). G' contains one more quintagon than G and therefore by the minimality of G has a 7-edge-colouring. From symmetry and Propositions 4.6.10 and 4.6.11 it follows that for any edge $e = v_1v_2$, G has an e-colouring where the edges v_1v_2 receive colours $\alpha, \beta, \delta, \gamma, \epsilon$ and ϕ, the edge v_2v_3 is coloured μ and v_1v_3 is coloured ϕ.

Because v_1v_3 is coloured ϕ, for any colour $c \neq \phi$, every c-mate M_c must contain both the quadragon v_1v_2 and the edge v_2v_3. Proposition 4.6.13 shows that both f' and f'' are $\geq$6-big. Observe that the other two edges incident to v_2 must have the same colour so v_2 cannot be incident to another multigon. Repeating the argument with the roles of v_1 and v_2 reversed shows that f''' is also $\geq$6-big. □

Lemma 4.7.2. *If a 3-face f is adjacent to a trigon, then f is adjacent to at most one other multigon which, if it exists, must be a bigon.*

Proof. Let $f = v_1v_2v_3$ and consider the odd cut $\{v_1, v_2, v_3\}$. Let $A = \{v_iv_j \in E(G) : i, j \in \{1, 2, 3\}\}$. Since G is 7-regular, the number of edges in the cut is

$$21 - 2|A|.$$

Therefore, by Lemma 4.6.1, there can be at most 6 edges in A. The lemma follows.

□

Lemma 4.7.3. *If a 3-face f is adjacent to a trigon and a bigon, then the trigon and the bigon are both adjacent to ≥6-big faces.*

Proof. Let $f = v_1v_2v_3$ and suppose that there is a trigon v_1v_2 and a bigon v_1v_3. Let f' and f'' be the other faces adjacent to the trigon and the bigon respectively and let $v_4 \neq v_3$ be adjacent to v_1 on the boundary of f''. Consider the graph G' obtained from G by the $v_1v_2v_3v_4$-swap. Since G' has one more quadragon and as many multigons of higher order as G, it follows from the minimality of G that G' has a 7-edge-colouring. By Propositions 4.6.10 and 4.6.11, for any $e = v_1v_2$ there exists an e-colouring of G where e is assigned colours α, β, and γ, the edges parallel to it are coloured δ and ϕ, the edges v_1v_3 are coloured with ϵ and μ, the edge v_1v_4 and v_2v_3 are both coloured ϕ and all other colours are the same as in the 7-edge-colouring of G'.

For any colour $c \neq \phi$, every c-mate must contain both the trigon v_1v_2 and the bigon v_1v_3. Proposition 4.6.13 shows that f' and f'' are $\geq$6-big.

□

Lemma 4.7.4. *If a 3-face f is adjacent to a trigon, then at least one of f and the trigon is adjacent to a $\geq$4-big face.*

Proof. Let $f = v_1v_2v_3$ and suppose that there is a trigon v_1v_2, let f' be the face adjacent to the edge v_1v_3 and let f'' be the face adjacent to the trigon. Let $v_4 \neq v_3$ be adjacent to v_1 on the boundary of f' and consider the graph G' obtained from G by the $v_1v_2v_3v_4$-swap. Since G' has one more quadragon than G, it follows from the minimality of G that G' has a 7-edge-colouring. By Propositions 4.6.10 and 4.6.11, for any edge $e = v_1v_2$, there exists an e-colouring of G where e is assigned colours α, β, and γ, the edges parallel to it are coloured δ and ϕ, the edges v_1v_4 and v_2v_3 are both coloured ϕ, and all other edges have the same colours as in the colouring of G'.

For any colour $c \neq \phi$, every c-mate M_c must contain both the trigon v_1v_2 and the edge v_1v_3. Proposition 4.6.14 shows that at least one of f' and f'' is $\geq$4-big.

□

Lemma 4.7.5. *If a 3-face f is adjacent to three bigons, then at least two of those bigons are adjacent to $\geq$4-big faces.*

Proof. Let $f = v_1v_2v_3$, let f' be the face adjacent to the bigon v_1v_3 and let f'' be the face adjacent to the bigon v_1v_2. Let $v_4 \neq v_3$ be adjacent to v_1 on f' and consider the graph G' obtained from G by the $v_1v_2v_3v_4$-swap. Since G' has one more trigon and as many multigons of higher order as G, it follows from the minimality of G that G' has a 7-edge-colouring. By Propositions 4.6.10 and 4.6.11, for $e = v_1v_2$, there exists an e-colouring of G where e is assigned colours α, β, and γ, the edge parallel to e is coloured ϕ, the edge v_1v_4 is coloured ϕ, the edges of the bigon v_2v_3 are assigned ϕ and μ, the edges of the bigon v_1v_3 are coloured δ and ϵ, and all other edges have the same colours as in the colouring of G'.

Since ϕ appears on the bigon v_2v_3, for any colour $c \neq \phi$, every c-mate M_c must contain both the bigon v_1v_2 and the bigon v_1v_3. Proposition 4.6.14 shows that at least one of f' and f'' is $\geq$4-big.

Repeating the argument with the roles of v_1 and v_2 reversed, and again with those of v_2 and v_3 reversed shows that at least one of the other two bigons on f is also adjacent to a $\geq$4-big face.

□

Lemma 4.7.6. *If a 3-face $f = v_1v_2v_3$ is adjacent to three bigons, then neither v_1, v_2 nor v_3 is incident to a trigon.*

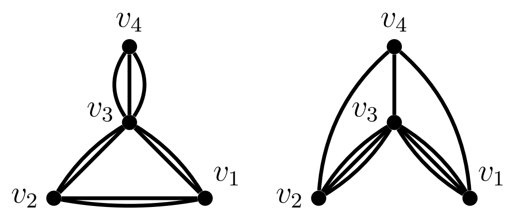
Proof. Let $f = v_1v_2v_3$ and suppose for a contradiction (without loss of generality) that G has a trigon v_3v_4. Consider the graph G' obtained from G by removing two edges v_1v_2 and two edges v_3v_4 and adding edges v_1v_4, v_2v_4, v_1v_3, v_2v_3 as in Figure 2. Observe that G' is a 7-regular planar graph. It can also be verified using Lemma 4.6.1 that every odd cut in G' contains at least seven edges. G' has one more quadragon and as many multigons of higher order as G and therefore has a 7-edge-colouring. We may assume by symmetry that in this colouring, the edges v_1v_3 are coloured α, β, γ, the edges v_2v_3 are coloured δ, ϵ, ϕ, the edge v_3v_4 is coloured μ and the edges v_1v_4 and v_2v_4 are coloured δ and α, respectively. But then we can obtain a 7-edge-colouring of G by colouring two edges v_1v_2 and two edges v_3v_4 with α and δ, colouring the bigon v_1v_3 with α, β, the bigon v_2v_3 with ϵ, ϕ and giving all other edges the same colour as in the colouring of G'. This is a contradiction.

□

Lemma 4.7.7. *If a trigon t is adjacent to a 3-face, then t cannot be incident to another trigon.*

Proof. Suppose that G contains a trigon $t = v_1v_2$ and another trigon $t' = v_2v_3$. Further, suppose for contradiction that for some other vertex v_4, G has a 3-face $v_1v_2v_4$. Consider the graph G' obtained from G by the $v_1v_2v_3v_4$-swap. G' has one more quadragon than G, and therefore has a 7-edge-colouring. We may assume that in this colouring, the edges of

Figure 4.7: Swaps used in the proof of Lemma 4.7.6



the quadragon v_1v_2 are coloured α, β, γ and δ. The other three edges incident to v_2 must then be coloured ϕ, μ and ϵ. Then the edge v_3v_4 must be coloured with one of α, β, γ or δ. But this is a contradiction to Propostion 4.6.10.

□

Lemma 4.7.8. *If a vertex v is incident to two bigons and one trigon, then the trigon cannot be adjacent to a 3-face.*

Proof. Suppose that for vertices v_1, v_2, v_3 there is a trigon vv_1, a bigon vv_2 and another bigon vv_3, and suppose for contradiction that there is also an edge v_1v_2. Then consider the graph G' obtained from G by the $vv_1v_2v_3$-swap. G' has one more quadragon than G, so by the minimality of G it has a 7-edge-colouring. We may assume that the edges of the quadragon v_1v are coloured α, β, γ and δ and the other three edges incident to v are coloured ϕ, μ and ϵ. Then the edge v_2v_3 is coloured either α, β, γ or δ which is a contradiction to Proposition 4.6.10.

□

Lemma 4.7.9. *Every bigon in G is be adjacent to at most one dangerous 3-face.*

Proof. Let $f = v_1v_2v_3$ be a dangerous 3-face, with bigons v_1v_2 and v_2v_3, and let $f' = v_2v_3v_4$ be another dangerous 3-face.

First, suppose that the bigons adjacent to f' are v_2v_3 and v_3v_4. Then consider the graph G' obtained from G by the $v_1v_2v_4v_3$-swap. Since G' has two more trigons and as many multigons of higher order as G, there exists a 7-edge-colouring of G'. Observe that in this colouring one of the edges v_3v_4 must have the same colour as one of the edges v_1v_2, a contradiction to Proposition 4.6.10.

Now, suppose that the bigons adjacent to f' are v_2v_3 and v_2v_4. Then G' obtained from G by the $v_1v_2v_4v_3$-swap has one more trigon than G and consequently, a 7-edge-colouring. In this colouring, one of the edges v_3v_4 has the same colour as an edge v_1v_2, a contradiction to Proposition 4.6.10.

□

Lemma 4.7.10. *If a dangerous 3-face $f = v_1v_2v_3$ is adjacent to a 3-face $f' = v_1v_3v_4$ then f' cannot be adjacent to any multigon. Further, either the bigon v_2v_3 or the edge v_1v_4 is adjacent to a $\geq$4-big face, and either the bigon v_1v_2 or the edge v_3v_4 is adjacent to a $\geq$4-big face.*

Proof. f is adjacent to bigons v_1v_2 and v_2v_3 and shares the edge v_1v_3 with f'. Suppose that there is a bigon v_1v_4. Then the graph G' obtained from G by the $v_2v_3v_4v_1$-swap has two more trigons than G and consequently a 7-edge-colouring. But in this colouring, one of the edges v_1v_4 must have the same colour as one of the edges v_2v_3, a contradiction to Proposition 4.6.10. It follows that f' can not be adjacent to any multigon.

It remains true however that G' has a 7-edge-colouring as G' has one more trigon than G. By Propositions 4.6.10 and 4.6.11, for an edge $e = v_2v_3$, G has an e-colouring where the bigon v_2v_3 is coloured with α, β, γ and ϕ, the edges of the bigon v_1v_2 are coloured δ and ϕ, the edge v_3v_4 is coloured ϕ, and all other colours are as in the colouring of G'.

Then for any colour $c \neq \phi$, any c-mate must contain the bigon v_2v_3, the edge v_1v_3 and the edge v_1v_4. From Proposition 4.6.14 it follows that either the bigon v_2v_3 or the edge v_3v_4 must be adjacent to a $\geq$4-big face. Symmetrically, either the bigon v_1v_2 or the edge v_3v_4 is adjacent to a $\geq$4-big face.

□

Lemma 4.7.11. *If a 3-face $f = v_1v_2v_3$ is adjacent to a bigon $b = v_1v_2$ and a 3-face $f' = v_1v_3v_4$, then at least one of f' and the bigon is adjacent to a $\geq$4-big face.*

Proof. Lemma 4.7.10 implies that one of the following cases must occur.

1. f' is adjacent to a trigon. If the trigon has endpoints v_3 and v_4, then the graph G' obtained from G by the $v_1v_2v_3v_4$-swap has one more quadragon than G and therefore has a 7-edge-colouring. Observe that in this colouring, one of the edges v_3v_4 must have the same colour as one of the edges v_1v_2, a contradiction to Propostion 4.6.10. Similarly, if the trigon has endpoints v_1 and v_4, then the graph G'' obtained from G by the $v_2v_3v_4v_1$-swap is a contradiction to Proposition 4.6.10. Therefore, f' cannot be adjacent to a trigon.

2. f' is adjacent to a bigon v_1v_4. In this case, consider the graph G' obtained by the $v_1v_2v_3v_4$-swap. G' has one more trigon than G and therefore has a 7-edge-colouring. By Propositions 4.6.10 and 4.6.11, for $e = v_1v_2$, there exists an e-colouring of G where the edges v_1v_2 are coloured α, β, γ and ϕ, the edges of the bigon v_1v_4 are coloured μ and ϕ, and the edge v_2v_3 is coloured ϕ. It follows that for $c \neq \phi$ any c-mate must contain the edges v_1v_2, v_1v_3 and v_3v_4. From Proposition 4.6.14 it follows that either the bigon v_1v_2 or the edge v_3v_4 is adjacent to a $\geq$4-big face.

3. f' is adjacent to a bigon v_3v_4. In this case, as in Case 2, the $v_1v_2v_3v_4$-swap shows that G has an e-colouring where for every $c \neq \phi$ every c-mate must contain the edges v_1v_2, v_1v_3 and v_3v_4. Since v_3v_4 is a bigon, all seven colours appear on these edges and it follows from Proposition 4.6.13 that both the bigons v_1v_2 and v_3v_4 are adjacent to $\geq$6-big faces.

4. f' is not adjacent to any multigons. In this case, the graph G' obtained by the $v_1v_2v_3v_4$-swap has one more bigon than G and hence has a 7-edge-colouring. By Propositions 4.6.10 and 4.6.11, for $e = v_1v_2$, there exists an e-colouring of G where the edges v_1v_2 are coloured α, β, γ and ϕ, and the edges v_1v_4 and v_2v_3 are both coloured ϕ. It follows that for $c \neq \phi$ any c-mate must contain the edges v_1v_2, v_1v_3 and v_3v_4. Just as in Case 2, it follows that either the bigon v_1v_2 or the edge v_3v_4 is adjacent to a $\geq$4-big face.

□

4.7.2 Analysis of final charge

Lemma 4.7.12. *The final charge of every 3-face is non-negative.*

Proof. Let f be a 3-face. The initial charge of f is 0. By Lemmas 4.6.12, 4.7.1 and 4.7.2, one of the following cases must be true.

1. f is adjacent to a quadragon. In this case, by Lemma 4.7.1, f is adjacent to two $\geq$6-big faces, and hence receives 2 units of charge. By Rule 4.1, f sends only 1 charge to the quadragon, so the final charge is non-negative.

2. f is adjacent to a trigon and a bigon. Then f sends 0 charge by Rule 4.2, so the final charge is non-negative.

3. f is adjacent to a trigon and no other multigon. If f is adjacent to a $\geq$4-big face, from which it receives 1 charge, then f sends at most 0.5 to the trigon by Rule 4.3 and possibly 0.5 to another adjacent face by Rule 4.6. Otherwise, by Lemma 4.7.4 the trigon is adjacent to a $\geq$4-big face, so f sends 0 charge by Rule 4.3. In both cases the final charge is non-negative.

4. f is adjacent to one bigon. In this case, if f is adjacent to a $\geq$3-big face, then f receives at least 1 charge by Rule 3, and must send at most 0.5 to the bigon, and possibly 0.5 to another face by Rules 4.5 and 4.6, so the final charge is non-negative. If the bigon is adjacent to a $\geq$3-big face but f is not, then f sends 0 charge by Rule 4.5. If neither the bigon nor f is adjacent to a $\geq$3-big face, and f is adjacent to two $\leq$2-big $\geq$4-faces, then f receives 0.25 charge from each one by Rule 6.6 and sends 0.5 to the bigon by Rule 4.5. Finally if f is adjacent to a 3-face f', f' cannot be a dangerous 3-face by Lemma 4.7.10. Further, by Lemma 4.7.11 either f' is adjacent to a $\geq$4-big face, in which case it sends at least 0.5 charge to f by Rule 4.6, or the bigon is adjacent to a $\geq$4-big face and f need not send any charge to it by Rule 4.5. In all cases, the final charge of f is non-negative.

5. f is adjacent to two bigons. Then f is a dangerous 3-face. If f is adjacent to a $\geq$3-big face, then f receives 1 charge by Rule 3 and sends at most 0.5 to each of the bigons by Rules 5 and the final charge is non-negative. If f is adjacent to a $\leq$2-big $\geq$4-face, then by Rules 5.2 and 6.5, for each bigon adjacent to f, f receives a certain amount of charge and sends exactly the same amount to the bigon. The charge sent is at most the charge received in this case. Finally, if f is adjacent to a 3-face, then Lemma 4.7.10 shows that f receives at least 0.5 charge per bigon which is not adjacent to a $\geq$4-big face by Rules 1.3 and 2.1 and sends 0.5 to each such bigon by Rule 5.1. Hence the final charge of f is non-negative.

6. f is adjacent to three bigons. In this case, f sends out 0 charge, so its final charge is non-negative.

7. f is not adjacent to any multigons. In this case f only sends out charge if it is adjacent to one or two $\geq$3-big faces by Rules 4. In this case it follows from Rules 1.3 and 2.1 and Rules 4.6 and 4.7 that f sends out at most the amount of charge that it receives from other faces so the final charge is non-negative.

□

4.8 Final charge of multigons

There are no rules by which multigons ever send charge to other faces, so we just need to show each multigon receives at least enough charge to make their final charge non-negative.

Lemma 4.8.1. *The final charge of every quintagon is 0.*

Proof. The initial charge of a quintagon is -4. By Lemma 4.6.12, every quintagon is adjacent to two $\geq$6-faces and therefore receives a total of 3 charges from adjacent faces by Rule 1.1. Also, each quintagon receives 0.5 units of charge from both of its vertices by Rule 7.1. The total charge received is 4, so the final charge of every quintagon is 0. □

Lemma 4.8.2. *The final charge of every quadragon is 0.*

Proof. The initial charge of a quadragon is -3. Each quadragon receives 0.5 units of charge from both of its vertices by Rule 7.2. By Rules 1.3, 2.1, 3, 4.1 and 6.1, a quadragon receives 1 charge from each adjacent face. Therefore the total charge received is 3, and the final charge is 0. □

Lemma 4.8.3. *The final charge of every trigon is non-negative.*

Proof. Let t be a trigon. Its initial charge is -2.

If t is adjacent to two $\geq$3-big faces, then it receives 1 charge from each one by Rules 1.3, 2.1 and 3 and the final charge is non-negative.

Suppose t is adjacent to a 3-face. Then by Lemma 4.7.7, each of its endpoints sends 0.5 charge to it according to Rule 7.3. If the other face adjacent to t is a $\geq$ 3-big face then that face sends 1 charge to t by rules 1.3, 2.1 and 3 and the total charge received is 2. If t is adjacent to a 3-face and a $\leq$ 2-big $\geq$ 4-face then each of those faces sends at least 0.5 to t by Rules 4.3 and 6.3. Otherwise, if t is adjacent to two 3-faces, then by Lemma 4.7.3, neither of them is adjacent to any other multigons. Each sends 0.5 charge to t by Rule 4.3 and the total charge received is at least 2.

If t is adjacent to one $\geq$3-big face, and the other face adjacent to t is a $\leq$ 2-big $\geq$4-face, it follows from Rules 6.2, 7.3 and 7.4 that the total charge received from the endpoints of t and the $\geq$4-face is at least 1, so the final charge of t is non-negative.

Finally, if t is adjacent to two $\leq$2-big $\geq$4-faces, then it follows from Rule 6.3 that t receives 0.5 from each adjacent face, plus an additional 0.25 for each endpoint which is incident to another trigon. By Rules 7.3 and 7.4, each vertex sends 0.25 if it is incident to another trigon, otherwise it sends 0.5 so the total charge received is 2 and the final charge is non-negative.

□

Lemma 4.8.4. *The final charge of every bigon is non-negative.*

Proof. Let b be a bigon. Its initial charge is -1.

If b is adjacent to a $\geq$3-big face, then it receives 1 charge from that face by Rules 1.3, 2.1 and 3, so its final charge is non-negative.

Suppose that both faces adjacent to b are $\leq$2-big. Every such $\geq$4-face sends 0.5 units of charge to b by Rule 6.4. If b is adjacent to a 3-face f, then by Lemma 4.7.3, f cannot be adjacent to a trigon. If f is adjacent to no other bigons, then b receives 0.5 charge from f by Rule 4.5. If f is adjacent to one other bigon (f is dangerous), then it follows Rules 5.2, 6.5 and 7.5, 7.6, 7.7 that the total amount of charge received from f and the endpoints of b is at least 0.5. If f is adjacent to two other bigons, then the other two bigons are adjacent to $\geq$4-big faces by Lemma 4.7.5. Lemma 4.7.6 and Rules 7.5.1, 7.6 and 7.7 imply that each endpoint of b sends at least 0.25 charge to it. By Lemma 4.7.9, b cannot be adjacent to two 3-faces which are both adjacent to more than one bigon, so the Lemma is proved. □

4.9 Final charge of $\leq$ 2-big 4-faces

4.9.1 Structure of 4-faces

Lemma 4.9.1. *If a 4-face $f = v_1v_2v_3v_4$ is adjacent to a quadragon v_1v_2, then f can not be adjacent to any other multigon.*

Proof. First, suppose there is a bigon v_2v_3. Then the graph G' obtained from G by the $v_1v_2v_3v_4$-swap has one more quintagon than G and hence has a 7-edge-colouring by the minimality of G. By symmetry we may assume that in this colouring the colours assigned to the edges of the quintagon are $\alpha, \beta, \gamma, \delta$ and ϵ, and that the edge v_2v_3 is coloured ϕ. Then one of the two edges between v_3 and v_4 must be coloured with $\alpha, \beta, \gamma, \delta$ or ϵ, a contradiction to Proposition 4.6.10. Similarly, there cannot be a multigon between v_1 and v_4.

Now, suppose there is a bigon v_3v_4. Then the graph G' obtained from G by the $v_1v_2v_3v_4$-swap has a quintagon v_1v_2, a trigon v_3v_4 and consequently a 7-edge-colouring. In any

7-edge-colouring of G' some colour, must appear on both the trigon and the quintagon. This is a contradiction to Proposition 4.6.10.

□

Lemma 4.9.2. *If a 4-face $f = v_1v_2v_3v_4$ is adjacent to a quadragon v_1v_2, then both the quadragon and the edge v_3v_4 are adjacent to $\geq$6-big faces.*

Proof. The graph G' obtained from G by the $v_1v_2v_3v_4$-swap has one more quintagon than G and therefore by the minimality of G has a 7-edge-colouring. In this colouring, we may assume by Proposition 4.6.10 that the edges of the quintagon are coloured with $\alpha, \beta, \gamma, \delta$ and ϵ and that the two edges v_3v_4 are coloured ϕ and μ. Hence for any $e = v_1v_2$ there exists an e-colouring of G where the edges on the quadragon are coloured $\alpha, \beta, \gamma, \delta, \epsilon$ and ϕ, the edges v_2v_3 and v_1v_4 are coloured ϕ and the edge v_3v_4 is coloured μ.

Then for any colour $c \neq \phi$, any c-mate must contain the quadragon v_1v_2, and the edge v_3v_4. It follows from Proposition 4.6.13 that both the quadragon v_1v_2 and the edge v_3v_4 must be adjacent to a $\geq$6-big face f'. More strongly, observe that f' is $\geq$6-large and that $f \in T_1(f')$.

□

Lemma 4.9.3. *If a 4-face $f = v_1v_2v_3v_4$ is adjacent to a trigon v_1v_2, then f is adjacent to at most one other multigon which, if it exists, must be a bigon.*

Proof. First, suppose there is a trigon v_2v_3. Then the graph G' obtained from G by the $v_1v_2v_3v_4$-swap has a quadragon between v_1 and v_2 and hence has a 7-edge-colouring by the minimality of G. By symmetry we may assume that in this colouring the colours assigned to the edges of the quadragon are α, β, γ and δ, and that the two edges between v_2 and v_3 is coloured ϵ and ϕ. Then one of the two edges between v_3 and v_4 must be coloured with α, β, γ or δ, a contradiction to Proposition 4.6.10.

Now, suppose there is a trigon v_3v_4. Then the graph G' obtained from G by the $v_1v_2v_3v_4$-swap has a quadragon v_1v_2, and another quadragon v_3v_4 and consequently has a

7-edge-colouring. In any 7-edge-colouring of G' some colour must appear on both of the quadragons, a contradiction to Proposition 4.6.10.

□

Lemma 4.9.4. *Suppose a 4-face $f = v_1v_2v_3v_4$ is adjacent to a trigon v_1v_2. Let f' be the face adjacent to v_3v_4. Then either the trigon is adjacent to a ≥4-big face or $f \in T_1(f')$.Further, f' is not a dangerous 3-face.*

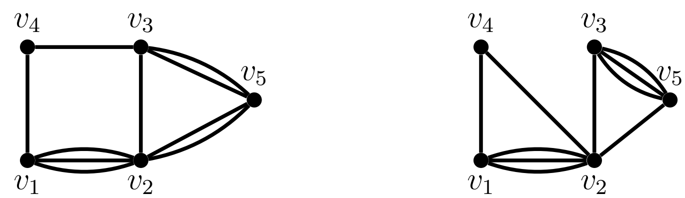
Proof. Let f' be the other face adjacent to v_3v_4 and f'' be the other face adjacent to the trigon. The graph G' obtained from G by the $v_1v_2v_3v_4$-swap has one more quadragon than G and as many multigons of higher order and therefore, by the minimality of G, a 7-edge-colouring. By Proposition 4.6.10, we may assume that in this colouring, the edges of the quadragon receive colours α, β, γ and δ and the two edges v_3v_4 are coloured ϕ and μ. It follows that for an edge e on the trigon v_1v_2, there exists an e-colouring of G where e is coloured with α, β, γ, the edges parallel to it with δ and ϕ, the edges v_2v_3 and v_1v_4 are coloured ϕ and the edge v_3v_4 is coloured μ. Then for every colour $c \neq \phi$, any c-mate must contain both the trigon v_1v_2 and the edge v_3v_4. Since 6 colours appear on those edges it follows from Proposition 4.6.14 that either f' or f'' is ≥4-big and it is also clear that f' cannot be a dangerous 3-face. If f' is ≥4-big, then it is ≥4-large with $f \in T_1(f)$, as the only face adjacent to f that is not f-incident to f' is a trigon.

□

Lemma 4.9.5. *If a 4-face $f = v_1v_2v_3v_4$ is adjacent to a trigon v_1v_2 and a bigon v_3v_4, then the trigon and the bigon are each adjacent to a ≥6-big face.*

Proof. Let f' and f'' be the other faces adjacent to the trigon and the bigon, respectively. The graph G' obtained from G by the $v_1v_2v_3v_4$-swap has one more quadragon and as many multigons of higher order as G and therefore, by the minimality of G, a 7-edge-colouring. In this colouring, suppose that the edges between v_1 and v_2 are coloured α, β, γ and δ. Then by Proposition 4.6.10 the edges between v_3 and v_4 must have colours ϕ, μ and ϵ. It follows that for an edge e on the trigon v_1v_2, there exists an e-colouring of G in which e is

Figure 4.8: Swap used in the proof of Lemma 4.9.6



coloured with α, β, γ, the edges parallel to it with δ and ϕ, the edges v_2v_3 and v_1v_4 with ϕ, and the edges of the bigon v_3v_4 with ϵ and μ.

Then for every colour $c \neq \phi$, any c-mate must contain both the trigon v_1v_2 and the bigon v_3v_4. From Proposition 4.6.13 it follows that f' and f'' are both ≥6-big.

□

Lemma 4.9.6. *If a 4-face $f = v_1v_2v_3v_4$ is adjacent to a trigon v_1v_2, then f is adjacent to at most one dangerous 3-face. If f is adjacent to a dangerous 3-face f' then f' must be f-incident to the trigon. Further, either the other face that is f-incident to the trigon, or the face adjacent to the bigon on f' which is not incident to the trigon must be ≥4-big.*

Proof. By Lemma 4.9.4, the edge v_3v_4 is not adjacent to a dangerous 3-face. Suppose that the edge v_2v_3 is adjacent to a dangerous 3-face $v_2v_3v_5$. Then the graph G' obtained by the $v_2v_4v_3v_5$-swap has one more trigon than G and just as many multigons of higher order and therefore has a 7-edge-colouring. In this colouring, we may assume by symmetry and Proposition 4.6.10 that the edges v_3v_5 are coloured α, β, γ, the edge v_2v_3 is coloured δ, v_2v_5 is coloured ϵ and v_2v_4 is coloured ϕ. It follows that there exists an e-colouring of G where the edges of the bigon v_3v_5 are coloured α, β, γ and ϕ, the edge v_3v_4 and one edge v_2v_5 are coloured ϕ. Therefore, for $c \neq \phi$, every c-mate must contain the bigon v_3v_4 and the edges v_2v_3 and v_1v_4. The result follows from Proposition 4.6.14.

□

Lemma 4.9.7. *Let $f = v_1v_2v_3v_4$ be a 4-face. Suppose f is not adjacent to any multigons of order $\geq$3. Then*

(a) If f is adjacent to four bigons, then either $b = v_1v_2$ or $b' = v_3v_4$ is adjacent to a $\geq$4-big face.

(b) If f is adjacent to a bigon $b = v_1v_2$ and a 1-bigon-3-face f' adjacent to v_3v_4 which is adjacent to one bigon b', then either b, f' or b' is adjacent to a $\geq$4-face.

(c) If f is adjacent to a bigon $b = v_1v_2$ and a dangerous 3-face f' adjacent to v_3v_4, then both b and one of the bigons on f' are adjacent to $\geq$6-big faces.

(d) If f is adjacent to zero or one bigon, then f cannot be adjacent to two dangerous 3-faces which are not f-incident.

(e) If f is adjacent to zero or one bigon, then if f is adjacent to a dangerous 3-face f' and a 1-bigon-3-face f'' which are not f-incident, then both f'' and one of the bigons on f' are adjacent to $\geq$6-big faces.

Proof. (a) Let f' be the other face adjacent to b and f'' be the other face adjacent to b'. Since the graph G' obtained from G by the $v_1v_2v_3v_4$-swap contains two more trigons and as many multigons of higher order as G, it follows from the minimality of G that G' has a 7-edge-colouring. By symmetry and Proposition 4.6.10, we may assume that in this colouring the edges of the trigon v_1v_2 are coloured α, β and γ, the edges of the trigon v_3v_4 are coloured δ, ϵ and ϕ and the edges v_1v_4 and v_2v_3 are both coloured with μ. It follows that for an edge e with endpoints v_1 and v_2, G has an e-colouring where e is coloured with α, β and γ, and the edge parallel to e with ϕ. The edges of the bigon between v_3 and v_4 are coloured δ and ϵ, and the edges of each of the other two bigons are coloured ϕ and μ and all other edges have the same colour as in the colouring of G'. Neither v_1 nor v_2 has more than one incident edge of any other colour than ϕ and for any $c \neq \phi$ any c-mate M_c must contain both the bigon v_1v_2 and the bigon v_3v_4 so Proposition 4.6.14 implies that either f' or f'' is $\geq$4-big.

(b) Let f share the edge v_3v_4 with the 3-face $f' = v_3v_4v_5$ and v_4, v_5 be the endpoints of b'. The graph G' obtained from G by the $v_1v_2v_3v_4$-swap has one more trigon and as many multigons of higher order as G and therefore has a 7-edge-colouring. By symmetry and by Propostion 4.6.10, we may assume that in this colouring, the edges v_1v_2 are coloured α, β, γ and the edges v_3v_4 are coloured ϕ and δ. It follows that for an edge e on b there exists an e-colouring of G where e is coloured α, β, γ, and the edge parallel to e as well as an edge v_1v_4 and an edge v_2v_3 are coloured ϕ. For any $c \neq \phi$, every c-mate must contain b and the edge v_3v_4. If the edges of the b' are coloured ϵ and μ, then Proposition 4.6.13 shows that both b and b' are adjacent to ≥6-faces. If not, then the edge v_3v_5 is coloured either ϵ or μ and Propostion 4.6.14 shows that either b or v_3v_5 is adjacent to a ≥4-big face.

(c) Let f share the edge v_3v_4 with the dangerous 3-face $f' = v_3v_4v_5$. The same argument as in part (b) applies here to show that for $e = v_1v_2$ there exists an e-colouring of G where e is coloured α, β, γ, v_3v_4 is coloured δ, and the edge parallel to e as well as an edge v_1v_4 and an edge v_2v_3 are coloured ϕ, but in this case the edges of either the bigon v_3v_5 or the bigon v_4v_5 must be coloured with ϵ and μ and Proposition 4.6.13 shows that both that bigon and b are adjacent to ≥6-big faces.

(d) Suppose that the edge v_1v_2 is adjacent to a dangerous 3-face $v_1v_2v_5$ and the edge v_3v_4 is adjacent to a dangerous 3-face $v_3v_4v_6$. The graph G' obtained from G by the $v_1v_2v_3v_4$-swap contains at least one more bigon than G and as many multigons of higher order, so G' has a 7-edge-colouring. By Proposition 4.6.10, we may assume that in this colouring the edges v_1v_2 are coloured α, β, the edges v_1v_5 are coloured γ, δ, the edges v_2v_5 are coloured ϵ, ϕ, and one of the edges v_3v_4 is coloured ϕ. Let $e = v_1v_2$ and consider the e-colouring of G where e is coloured α, β, ϕ, an edge v_1v_4 as well as an edge v_2v_3 are coloured ϕ, and all other edges have the same colour as in the e-colouring of G'. Then for any $c \neq \phi$, every c-mate must contain the bigon v_1v_5, the edges v_1v_2 and v_3v_4, and either the bigon v_3v_6 or the bigon v_4v_6. But both v_3v_6 and v_4v_6 must contain an edge coloured with a colour that appears on v_1v_5, v_1v_2 or v_3v_4 so this is a contradiction.

(e) Suppose that the edge v_1v_2 is adjacent to a dangerous 3-face $v_1v_2v_5$ and the edge v_3v_4 is adjacent to a 3-face $v_3v_4v_6$ adjacent to a bigon v_3v_6. As in the proof of part (d), for $e = v_1v_2$, there exists an e-colouring of G where e is coloured α, β, ϕ, the edges v_1v_2 are coloured α, β, the edges v_1v_5 are coloured γ, δ, the edges v_2v_5 are coloured ϵ, ϕ, an edge v_1v_4 as well as an edge v_2v_3 are coloured ϕ, and the edge v_3v_4 is coloured δ. Since the bigon v_3v_6 must contain an edge coloured with some colour that appears on v_1v_5, v_1v_2 or v_3v_4, for $c \neq \phi$, every c-mate must contain the edges v_1v_5, v_1v_2, v_3v_4 and v_4v_6. The desired result follows from Proposition 4.6.13.

□

4.9.2 Analysis of final charge

Lemma 4.9.8. *The final charge of every 4-face is non-negative.*

Proof. Let f be a 4-face. The initial charge of f is 1. Lemmas 4.6.12, 4.9.1 and 4.9.3 imply that one of the following cases must occur.

1. f is adjacent to a quadragon. In this case, f must send one charge to the quadragon by Rule 6.1. By Lemma 4.9.2, f is adjacent to a $\geq$6-large face f' from which it receives 1 charge by Rule 1.3. Observe that if the quadragon is f-incident to a dangerous 3-face f'', then one of the bigons adjacent f'' lies on the same $\geq$6-big face as the quadragon, so f only needs to send at most 0.5 charge to f'' by Rules 6.5. Since by Rule 6.6 f sends at most 0.25 charge to non-dangerous 3-faces and by Lemma 4.9.1 f is adjacent to no other multigons, the total charge sent by f is no more than 2 and the final charge is non-negative.

2. f is adjacent to a trigon. In this case, if the trigon is adjacent to a $\geq$4-big face, then by Lemma 4.7.7 and Rule 6.2, f only sends 0.25 to the trigon if one of the faces f-incident to it is $\geq$4-face (to which it sends 0 charge). f sends 0.5 to the trigon if both faces f-incident to it are $\geq$4-faces by Rule 6.3. It follows from Lemma 4.9.6 that f sends a total of at most 0.75 charge to the trigon and the two faces f-incident to

it. If the remaining face adjacent to f is a 3-face, it cannot be dangerous by Lemma 4.9.4, so it receives at most 0.25 charge from f by Rule 6.6. If the remaining face is a bigon then by Lemma 4.9.5 f sends no charge to it by Rule 6.4.

If the trigon is not adjacent to a $\geq$4-big face, then Lemma 4.9.4 implies that f is adjacent to a $\geq$4-big face from which it receives 1 charge by Rule 2.1. Lemma 4.9.6 shows that f sends at most a total of 2 charges to the trigon and the two faces f-incident to it combined by Rules 6. Therefore the final charge of f is non-negative.

3. f is adjacent to zero, one, two, three, or four bigons and no multigons of order $\geq$2. Lemma 4.9.7 and Rules 6.4, 6.5 and 6.6 imply that f sends at most 0.5 to any pair of its adjacent faces which are not f-incident. In this case it is easy to see that the final charge of f is non-negative.

□

4.10 Final charge of $\leq$ 2-big $\geq$ 5-faces

4.10.1 Structure of $\geq$ 5-faces

Lemma 4.10.1. *If a $\geq$5-face $f = v_1, \ldots, v_l$ is adjacent to a quadragon $q = v_2, v_3$ then f is also adjacent to a $\geq$4-face f' which is not f-incident to q. Further, if f is a 5-face then f' is $\geq$ 6-big.*

Proof. The graph G' obtained from G by the $v_1v_2v_3v_4$-swap has one more quintagon than G and therefore has a 7-edge-colouring. By Propositions 4.6.10 and 4.6.11, for $e = v_2v_3$, there exists an e-colouring of G where the edges of q are coloured with $\alpha, \beta, \gamma, \delta, \epsilon, \phi$ and there are edges v_1v_2 and v_3v_4 coloured ϕ. For $c \neq \phi$, every c-mate M_c must contain the edges in q. The other edges in M_c consist of one edge coloured μ and at least four edges coloured c. M_c must contain an edge e_c which is adjacent to f but not f-incident to q. Observe that e_μ must be coloured μ and be adjacent to a $\geq$4-face f'. If f is a 5-face, we may assume that the edge v_4v_5 is coloured with μ and v_5v_1 is coloured α so all mates of colours $\beta, \gamma, \delta, \epsilon, \mu$ must contain v_4v_5, and there must exist at least five $\geq$4-faces with which the face f' adjacent to v_4v_5 shares an edge. Since f is $\geq$5-face, f' is $\geq$6-big.

$\square$

Lemma 4.10.2. *Let $f = v_1 \dots v_5$ be a 5-face. Suppose that v_1v_2 is adjacent to a trigon t. Denote by f' and f'' the faces adjacent to the edges v_3v_4 and v_4v_5, respectively. Denote by f''' the other face adjacent to t. Then f' and f'' cannot be any combination of only dangerous 3-faces, trigons, and quadragons. Further, if f' is a dangerous 3-face or a multigon of order $\geq$3, then at least one of f'' or f''' must be $\geq$4-big. If f' and f" are some combination of bigons and 1-bigon-3-faces, then both t and one of f' or f'' is adjacent to a $\geq$5-big face.*

Proof. The graph obtained from G by the $v_1v_2v_3v_5$-swap has one more quadragon than G and therefore has a 7-edge-colouring. By Propositions 4.6.10 and 4.6.11, for $e = v_1v_2$ there exists an e-colouring of G where the edges v_1v_2 are coloured $\alpha, \beta, \gamma, \delta$ and ϕ, and one edge each of v_2v_3 and v_5v_1 is coloured ϕ.

Suppose first that each of f' and f'' is a dangerous 3-face or a multigon of order $\geq$3. For any $c \neq \phi$, any c-mate must contain all the edges v_1v_2 (on which appear five colours) plus a set of three edges incident to either v_3, v_4 or v_5. There are only a total of 4 possible such sets and some colour appearing on the trigon v_1v_2 appears on each of them. It follows from the definition and existence of c-mates that this is a contradiction.

Now, if f' is a dangerous 3-face, a trigon or a quadragon then a similar argument shows that all c-mates for $c \neq \phi$ must contain the edge v_4v_5 which must be coloured ϵ or μ. It follows from Proposition 4.6.14 that either f'' or f''' is $\geq$4-big.

If f' and f'' are bigons, then in the e-colouring, two edges of one of them, say without loss of generality f' are coloured ϵ and μ. For $c \neq \phi$, every c-mate must contain both edges of f' as well as the three edges v_1v_2 so Proposition 4.6.13 implies that both t and f' are adjacent to $\geq$6-big faces.

If f' is a 1-bigon-3-face and f'' is a bigon or a 1-bigon-3-face, then two edges of one of f' or f'' are coloured ϵ and μ, and mates of at most one colour can contain edges of the other. An argument similar to the proof of Proposition 4.6.13 shows that both t and one of f' and f'' are adjacent to $\geq$5-big faces.

□

Lemma 4.10.3. *Let $f = v_1 \dots v_5$ be a 5-face. Suppose that v_1v_2 is adjacent to a dangerous 3-face f_1, and denote by f_2 and f_3 the faces adjacent to the edges v_3v_4 and v_4v_5 respectively. Suppose that neither of the faces adjacent to v_2v_3 and v_5v_1 are multigons. Then f_2 and f_3 cannot be any combination of only dangerous 3-faces, trigons and quadragons. Further, if f_2 is a bigon and f_3 is a trigon or a dangerous 3-face, then both f_2 and a bigon adjacent to f_1 must be adjacent to $\geq$3-big faces.*

Proof. The graph G' obtained from G by the $v_1v_2v_3v_5$-swap has one more bigon than G and just as many multigons of higher order so G' has a 7-edge-colouring. By Propositions 4.6.10 and 4.6.11, for $e = v_1v_2$ there exists an e-colouring of G where e is coloured α, β, γ and the edges v_2v_3 and v_5v_1 are both coloured ϕ.

First, suppose f_1 and f_2 are both dangerous 3-faces. Then for $c \neq \phi$, every c-mate must contain the edge v_1v_2, both edges of a bigon adjacent to f_1, as well as either v_3v_4 and a bigon adjacent to f_2 or v_4v_5 and a bigon adjacent to f_3. There must be three colours c_1, c_2, c_3 for which a c_i-mate M_{c_i} contains the same bigon adjacent to f_1. If this bigon is coloured with δ and ϵ, then the other remaining edges in M_{c_i} are two edges coloured with ϕ and μ and some additional edges coloured c_i. But at most one such mate can contain the edge v_3v_4 and a bigon adjacent to f_2 and at most one can contain the edge v_4v_5 and a bigon adjacent to f_3, a contradiction. Similar arguments show that a contradiction is also reached if f_1 and f_2 are both trigons, or if they are a trigon and a dangerous 3-face.

Also, if f_2 is a bigon and f_3 is a dangerous 3-face or a trigon, then as above, there must be 3 colours c_1, c_2, c_3 for which some c_i-mates M_{c_i} all contain the same bigon adjacent to f_1. It follows that they all three must contain the edges of the bigon f_2 as well. We may assume by symmetry that the bigon adjacent to f_1 that they all contain is coloured with

δ and ϵ and that f_2 is coloured ϕ and μ. A similar argument to the proof of Proposition 4.6.13 shows that both f_1 and f_2 are adjacent to $\geq$3-big faces.

$\square$

Lemma 4.10.4. *Let $f = v_1 \dots v_5$ be a 5-face. If f is not adjacent to any multigons of order $\geq$3, and f is adjacent to three bigons which are not adjacent to three consecutive faces around the boundary of f, then at least one of the bigons is also adjacent to a $\geq$3-big face.*

Proof. Suppose there are bigons adjacent to the edges v_1v_2, v_3v_4 and v_4v_5. The graph G' obtained from G by the $v_1v_2v_3v_5$-swap has one more trigon than G and just as many multigons of higher order, and therefore has a 7-edge-colouring by the minimality of G. It follows from Proposition 4.6.10 and Proposition 4.6.11 that for $e = v_1v_2$, there exists an e-colouring of G where the edges v_1v_2 are coloured $\alpha, \beta, \gamma, \phi$ and an edge v_2v_3 as well as an edge v_5v_1 are coloured with ϕ. For $c \neq \phi$, every c-mate must contain the edges of the bigon v_1v_2, as well as both edges of either the bigon v_3v_4 or the bigon v_4v_5. Therefore we may assume without loss of generality that the edges v_3v_4 are coloured δ and ϵ, and the edges v_4v_5 with μ and α. For $c \neq \phi$ or α, every c-mate contains the edges v_3v_4 and a similar argument to the proof of Proposition 4.6.14 shows that one of the bigons v_1v_2 or v_3v_4 is adjacent to a $\geq$3-big face.

$\square$

Lemma 4.10.5. *Let $f = v_1 \dots v_l$ be a 2-big l-face, for $l = 6$ or 7. Suppose that v_1v_2 is adjacent to a quadragon q and v_2v_3 is adjacent to a ≥ 4-face. Then one of the faces in the set $\mathcal{F}$ of faces adjacent to the edges $\{v_3v_4, \dots, v_lv_1\}$, say f', is ≥ 4-big. Further, if $\mathcal{F} \setminus f'$ contains only dangerous 3-faces, quadragons, and trigons then f' is ≥ 6-big.*

Proof. By Lemma 4.10.1, at most one of the ≥ 4-faces is f-incident to the quadragon. Consider the graph obtained from G by the $v_lv_1v_2v_3$-swap. This graph has one more quintagon than G so by Propositions 4.6.10 and 4.6.11, for $e = v_1v_2$ there is an e-colouring of G where the edges of the quadragon are coloured $\alpha, \beta, \gamma, \delta, \epsilon$ and ϕ and the edges v_lv_1 and v_2v_3 are both coloured ϕ.

For any $c \neq \phi$, a c-mate M_c must contain the edges of the quadragon q. The remaining edges in M_c consist of one edge coloured μ and at least four edges coloured c. M_c must contain an edge $e_c \in \mathcal{F}$ which cannot be adjacent to a multigon of order ≥3, a dangerous 3-face or a 1-trigon-1-bigon-3-face. Therefore, if $\mathcal{F} \setminus f'$ contains only dangerous 3-faces, quadragons, and trigons then for each $c \neq \phi$, e_c is adjacent to f' so f' is $\geq$ 6-big by Proposition 4.6.13.

□

Lemma 4.10.6. *Let $f = v_1 \dots v_l$ be a 2-big l-face, for $l = 5, 6$ or 7. Suppose that every $\leq$ 4-face adjacent to f is a trigon or a dangerous 3-face. Suppose that v_1v_2 is adjacent to a $\geq$ 4-face and v_2v_3 is adjacent to a trigon t. Let $e' \neq v_1v_2$ on the boundary of f be adjacent to a $\geq$ 4-face. Let f' be the other face adjacent to t. Then either f' is $\geq$ 3-big, or e' is adjacent to a $\geq$ 4-big face. Finally, e' cannot be incident to t.*

Proof. The graph obtained from G by the $v_2v_3v_4v_1$-swap has one more quadragon than G and hence has a 7-edge-colouring. By Propositions 4.6.10 and 4.6.11, for $e = v_2v_3$ there exists an e-colouring of G where the edges of t are coloured $\alpha, \beta, \gamma, \delta, \phi$ and the edges v_1v_2 and v_3v_4 are coloured ϕ.

For $c \neq \phi$, there is a set of edges coloured with $\alpha, \beta, \gamma, \delta, \phi$ that must be contained in every c-mate M_c. M_c must also contain another edge e_c adjacent to f. If e_c is adjacent to a trigon or a dangerous 3-face, then mates of at most one colour can contain e_c, and the edges in M_c which share a vertex with e_c are coloured ϵ, μ and c. The other edges in the mate consist of at least three more edges coloured c so there exists a ≥4-face which shares an edge coloured c with f'.

Further, since $e' \neq v_1v_2$ is adjacent to a ≥4-face, every μ-mate and ϵ-mate must contain e'. We may assume by symmetry that $e' \neq v_3v_4$ and is coloured μ so f' as well as the face that f shares with e' each share an edge coloured ϵ with a ≥4-face. Consider the mates of the remaining colours α, β, γ, and δ. If there exist mates of at least two of those colours that don't contain e' then f' is a ≥3-big face. Otherwise there exist mates of at least three of those colours which do contain e', so either f' is a g ≥3-big face, or e' is adjacent to a ≥4-big face (to see this, observe that f is also a ≥4-face adjacent to e'.)

□

Lemma 4.10.7. *Let $f = v_1 \dots v_l$ be a 2-big l-face, for $l = 5, 6$ or 7. Suppose that every ≤ 4-face adjacent to f is a trigon or a dangerous 3-face. Suppose that v_1v_2 is adjacent to a ≥ 4-face and v_2v_3 is adjacent to a dangerous 3-face v_2v_3w. Let $e' \neq v_1v_2$ on the boundary of f be adjacent to a ≥ 4-face. Let f' be the other face adjacent to v_2w. Then either f' is ≥ 3-big, or e' is adjacent to a ≥ 4-big face.*

Proof. The graph obtained from G by the $v_2wv_3v_1$-swap has one more trigon and just as many multigons of higher order as G and therefore has a 7-edge colouring. By Propositions 4.6.10 and 4.6.11, there exists a 7-edge-colouring of G where the edges v_2w are coloured $\alpha, \beta, \gamma, \phi$, the edge v_1v_2 and an edge v_3w are coloured ϕ and the edge v_2v_3 is coloured δ. The remainder of the argument is identical to the proof of Lemma 4.10.6 and is left to the reader.

□

Lemma 4.10.8. *Let $f = v_1 \dots v_l$ be a 0-big l-face, for $6 \leq l \leq 10$. Suppose that f is not adjacent to either*

- *four faces, each of which is either a bigon, a 1-bigon-3-face or a 1-trigon-3-face,*
- *two faces, each of which is are either a bigon, a 1-bigon-3-face or a 1-trigon-3-face, plus one 0-multigon-3-face, or*
- *three 0-multigon-3-faces.*

Then if t is a trigon adjacent to f, the other face adjacent to t is ≥ 3-big.

Proof. Suppose that the edge v_1v_2 is adjacent to t. Following previous arguments, for $e = v_1v_2$ on t there exists an e-colouring of G where the edges of t are coloured $\alpha, \beta, \gamma, \delta$ and ϕ and the edges v_lv_1 and v_2v_3 are both coloured ϕ. For $c \neq \phi$, every c-mate must contain the edges of t plus some other edge e_c adjacent to f but not incident to t. If e_c is adjacent to a trigon or to a dangerous 3-face then mates of at most one colour can contain

e_c. In this case, since e_c and the two edges in the mate with which e_c shares a vertex are coloured ϕ, μ and c, the remaining edges in the mate consist of at least three edges coloured with c. Therefore there is a $\geq$4-face which shares an edge coloured c with the face adjacent to t. Similarly, if mates of more than one colour contain the edges of a bigon, a 1-bigon-3-face or a 1-trigon-3-face, then there must exist edges coloured ϵ and μ contained in all mates of all but at most one of those colours. Further, for every such colour the face adjacent to t must share an edge of the colour of the mate with a $\geq$4-face. The same is true if mates of more than two colours contain edges of a 0-multigon-3-face. It follows from $6 \leq l \leq 10$ and that f is not adjacent to any of the combinations of faces in the statement of the lemma that the face adjacent to t is $\geq$3-big. □

4.10.2 Analysis of final charge

The following easy observation follows from the fact that any face sends at most 1 charge to any face which is adjacent to it and will be the principal criterion that we will use for proving that the final charge of a $\geq$5-face is non-negative.

Fact 4.10.9. *If there exists a set of six (resp. five, four, three) edges adjacent to f such that the difference between the total amount of charge sent to those faces and the amount of charge received by f from all other faces is at most 3 (resp. 2,1,0), then f has non-negative final charge.*

Let f be an l-face, for $l \geq 5$. The initial charge of f is $l - 3$.

It is easy to see from Rules 6 that f sends either 0, 0.25, 0.5, 0.75 or 1 charge to each of its adjacent faces. f sends 0 charge to any $\geq$4-face. By Lemma 4.6.12, there are eight other types of faces that may be adjacent to f; they are quadragon, trigon, bigon, dangerous 3-face, 1-bigon-3-face, 1-trigon-3-face, 1-bigon-1-trigon-3-face, 0-multigon-3-face.

Reading Tables 4.1, 4.2 and 4.3

We now analyze the final charge of 2-big, 1-big and 0-big faces separately. In the proofs of Lemmas 4.10.10, 4.10.11 and 4.10.12, the charges that a face f of each kind (2-big, 1-big, 0-big) sends according to the rules in Section 4.4.3 are summarized in Tables 4.1, 4.2 and 4.3. Each table should be read as follows. Let f be a $\geq$ 5-face. Suppose that a face f' is adjacent to f and f-incident to faces f_1 and f_2 and that we know the amount of charge that f sends to f'. Locate the row of the table amongst those where the entry in the second column corresponds to the type of f', for which the corresponding entry in the first column is the smallest amount of charge greater than or equal to the amount sent to f' by f. Then one of the faces f-incident to f', say f_1, must be of one of the types specified in the third column entry and receive at most the amount of charge in the fourth column entry from f. Similarly for the type and amount of charge sent to the other face f_2. Note that the order of the faces around the boundary of f does not matter for the purposes of reading the tables.

For example, according to Table 4.1 if a 2-big $\geq$ 5-face f sends 1 charge to a dangerous 3-face, then each of the faces f-incident to it must be a trigon, 1-trigon-3-face, $\geq$4-face or 1-trigon-1-bigon-3-face. It can be observed from Tables 4.1, 4.2 and 4.3 that a $\leq$ 2-big $\geq$ 5-face f never sends a total of more than 1.75 charge to any two f-incident faces.

The entries in Tables 4.1, 4.2 and 4.3 are all easy to deduce from Rules 6, however we point out the more subtle observations.

First, since f is $\leq$2-big, it follows from Propostion 4.6.14 that any quadragon adjacent to f must also be adjacent to a $\geq$4-big face. Because G is 7-regular, if a dangerous 3-face is f-incident to a quadragon, then one of the bigons adjacent to it is also adjacent to a $\geq$4-face. From this, Rule 6.5.1 and Lemma 4.3.1 it follows that f sends at most 0.5 charge to any face f-incident to a quadragon. Second, according to Rules 6.2 and 6.3, f may send 0.75 charge to a trigon only if at least one endpoint is incident to another trigon, and may send 1 charge to a trigon only if both of its endpoints are incident to other trigons. Because G is 7-regular and from Lemma 4.7.7, it follows that if G sends 0.75 to a trigon then it must be f-incident to a $\geq$ 4-face, and if G sends 1 charge to a trigon then it must be f-incident to two $\geq$ 4-faces.

Lemma 4.10.10. *The final charge of every 2-big $\geq$ 5-face is non-negative.*

Let f be a 2-big l-face. Let $f = v_1 \dots v_l$.

Table 4.1: Summary of charges sent by a 2-big $\geq$5-face to adjacent faces

Charge to f'	**Type of f'**	**f_1 can be...**	**Max charge to f_1**	**f_2 can be...**	**Max charge to f_2**
1	quadragon	any except trigon or quadragon	0.5	any except trigon or quadragon	0.5
1	dangerous 3-face	trigon 1-trigon-3-face[a] $\geq$4-face 1-trigon-1-bigon-3-face[a]	0.75 0 0 0	trigon 1-trigon-3-face[a] $\geq$4-face 1-trigon-1-bigon-3-face[a]	0.75 0 0 0
1	trigon	$\geq$4-face	0	$\geq$4-face	0
0.75	dangerous 3-face	dangerous 3-face 1-bigon-3-face 1-trigon-3-face[b] 1-trigon-1-bigon-3-face[b] 0-multigon-3-face	0.75 0.25 0 0 0	trigon 1-trigon-3-face[a] $\geq$4-face 1-trigon-1-bigon-3-face[a]	0.75 0 0 0
0.75	trigon	$\geq$4-face	0	any face	1
0.5	dangerous 3-face	quadragon any except quadragon	1 0.75	 any except quadragon	 0.75
0.5	trigon	any except quadragon	1	any except quadragon	1
0.5	bigon	quadragon trigon any except trigon or quadragon	1 0.75 0.5	quadragon trigon any except trigon or quadragon	1 0.75 0.5
0.25	1-bigon- 3-face	quadragon any except quadragon	1 0.75	quadragon any except quadragon	1 0.75
0	1-trigon-3-face	any face	1	any face	1
0	1-trigon-1-bigon-3-face	any face	1	any face	1
0	0-multigon-3-face	any face	1	any face	1

Proof. [a] f' incident to the trigon on the 1-trigon-3-face or 1-trigon-1-bigon-3-face

[b] f' not incident to the trigon on the 1-trigon-3-face or 1-trigon-1-bigon-3-face

First, suppose $l \geq 8$. f sends 0 charge to each of the two $\geq$ 4-faces adjacent to it. If there are four other faces to which f sends at most 0.75 charge, then the final charge of f is non-negative by Fact 4.10.9. Otherwise, if there are less than four other faces to which f sends at most 0.75 charge, then there are at least three faces to which f sends 1 charge. Because any face that is f-incident to a face that receives 1 charge from f receives at most 0.75 from f, it follows that f sends charge to adjacent faces as follows (possibly after relabeling vertices). The faces receiving 1 charge from f are adjacent to v_1v_2, v_3v_4 and

v_5v_6 and the $\geq$ 4-faces are adjacent to v_2v_3 and v_4v_5. But then, since the faces adjacent to v_6v_7 and v_lv_1 are not f-incident to $\geq$ 4-faces they must receive at most 0.5 charge each from f. Then f sends a total of at most 3 charge to the faces adjacent to v_lv_1, v_1v_2, v_2v_3, v_4v_5, v_5v_6 and v_6v_7, so the final charge of f is non-negative by Fact 4.10.9.

Suppose now that $l \leq 7$. We consider the following cases.

1. f is adjacent to a quadragon v_1v_2.

1.1 $l = 5$.

Lemma 4.10.1 implies that one of the faces which is not f-incident to the quadragon is $\geq$6-big and sends 0.5 charge to f by Rule 1.3. Since f sends at most 0.5 charge to any face f-incident to a quadragon and 0 charge to the $\geq$6-big face, f sends out at most 2.5 charge, so the final charge of f is non-negative.

1.2 $l = 6$ or 7.

If neither of the two $\geq$4-faces is f-incident to the quadragon, then the quadragon, the two faces f-incident to the quadragon and the two $\geq$4-faces form a set of five faces to which f sends at most 2 charge, so the final charge of f is non-negative by Fact 4.10.9.

Otherwise, let f' be the $\geq$4-face which is not f-incident to the quadragon, and let f_1 and f_2 be the pair of $\leq$3-faces amongst those not f-incident to the quadragon, to which the total amount of charge sent by f is minimized. Then there are two possibilities according to how Rules 6 were applied.

- The total amount of charge sent to f_1 and f_2 is 1.75 or 2. Then both f_1 and f_2 are dangerous 3-faces, quadragons or trigons so f' is $\geq$6-big by Lemma 4.10.5 and sends at least 0.5 charge to f by Rule 1.4.

- The total amount of charge sent to f_1 and f_2 is at most 1.5.

The difference between the total amount of charge sent to f', f_1 and f_2 and the amount of charge received by f is at most 1.5 and the total amount of charge sent to the quadragon and the two faces f-incident to it is at most 1.5, so by Fact 4.10.9 the final charge is non-negative.

2. f is not adjacent to any quadragons.

If f is adjacent to a 0-multigon-3-face, a 1-trigon-3-face or a 1-trigon-1-bigon-3-face, then there are three faces to which f sends 0 charge by Rules 6. Therefore, we may consider the following two subcases.

2.1 f is adjacent to a bigon or a 1-bigon-3-face b.

Since f sends at most 0.5 charge to b and 0 charge to each of the two $\geq$ 4-faces, if there are two other faces which each receive at most 0.75 charge from f, then f has non-negative final charge by Fact 4.10.9. Suppose there do not exist two faces which each receive at most 0.75 charge from f. If $l = 6$ or $l = 7$, then there are at least two faces to which f sends 1 charge. Because no two f-incident faces both receive 1 charge form f, and because no face that is f-incident to a bigon or 1-bigon-3-face receives 1 charge, it follows that f sends charge to adjacent faces as follows (possibly after relabeling vertices). The faces adjacent to v_1v_2 and v_3v_4 receive 1 charge from f. The faces adjacent to v_2v_3 and v_4v_5 are the $\geq$ 4-faces. The face adjacent to v_lv_1 cannot be b, and since the face adjacent to v_lv_1 is not f-incident to a $\geq$ 4-face, it cannot receive more than 0.5 charge from f. Therefore f sends a total of at most 1 charge to b, the faces adjacent to b, v_lv_1, v_2v_3 and v_3v_4 and has non-negative final charge by Fact 4.10.9.

If $l = 5$, there is at least one face that receives 1 charge (call it f') from f. Note that f' cannot be f-incident to b. If the two $\geq$ 4-faces are f-incident, then f' must be f-incident to a $\geq$ 4-face and to a face which receives at most 0.5 charge (because it is not f-incident to a $\geq$ 4-face), so the total charge sent out by f is at most 2. Finally, we consider the case where the two $\geq$ 4-faces are not f-incident. It follows from Lemma 4.10.2 that if there is a trigon f-incident to both $\geq$ 4-faces, then it receives at most 0.5 charge from f by Rule 6.2. Therefore, f' is a dangerous 3-face. Either f' or b is f-incident to the remaining $\leq$ 3-face, which if it receives 0.75 charge must be a trigon. By Lemma 4.10.2, since that trigon is f-incident one $\geq$ 4-face, then either f sends to it at most 0.25 charge by Rule 6.2, or f is adjacent to a $\geq$ 4-big face from which it receives at least 0.25 charge. Therefore the difference between the charge sent to other faces and the charge received by f is at most 2 so the final charge of f is non-negative.

2.2 Except for the two $\geq$4-faces, f is adjacent to only trigons and dangerous 3-faces.

Without loss of generality, suppose that v_1v_2 is adjacent to a $\geq$4-face, and v_2v_3 is not. Let f' be the $\geq$ 4-face adjacent to f but not to v_1v_2.

It follows from Lemma 4.10.6 that no trigon can be f-incident to both $\geq$4-faces, and that if v_2v_3 is adjacent to a trigon t, then either t is adjacent to a $\geq$ 3-face, or f' is $\geq$ 4-big. Therefore t receives at most 0.75 charge and the difference between the amount of charge sent to t and the amount of charge received by f' is at most 0.25.

Similarly, by Lemma 4.10.7 and Rules 6.5, if v_2v_3w is a dangerous 3-face, the difference between the amount of charge sent to v_2v_3w and the amount of charge received by f from f' is at most 0.75 if the other face f-incident to the dangerous 3-face is a trigon, 1-trigon-3-face, 1-trigon-bigon-3-face or $\geq$4-face and at most 0.5 otherwise.

Therefore, for any pair of f-incident faces, one of which is f-incident to a $\geq$4-face, the difference between the total amount of charge sent to those faces and the amount of charge received by the other $\geq$4-face is at most 1.25. Any other pair of f-incident faces receives a total of at most 1.75 charge. It follows that there always exists a set of five edges such that the difference between the total amount of charge sent by f to those faces and the total charge received is at most 2, so the final charge of f is non-negative by Fact 4.10.9.

□

Lemma 4.10.11. *The final charge of every 1-big $\geq$ 5-face is non-negative.*

Proof. Let f be a 1-big l-face. Let $f = v_1 \dots v_l$.

Table 4.2: Summary of charges sent by a 1-big $\geq$5-face to adjacent faces

Charge to f'	**Type of f'**	**f_1 can be...**	**Max charge to f_1**	**f_2 can be...**	**Max charge to f_2**
1	quadragon	any except trigon or quadragon	0.5	any except trigon or quadragon	0.5
1	dangerous 3-face	trigon 1-trigon-3-face[a] $\geq$4-face 1-trigon-1-bigon-3-face[a]	0.75 0 0 0	trigon 1-trigon-3-face[a] $\geq$4-face 1-trigon-1-bigon-3-face[a]	0.75 0 0 0
0.75	dangerous 3-face	dangerous 3-face 1-bigon-3-face 1-trigon-3-face[b] 1-trigon-1-bigon-3-face[b] 0-multigon-3-face	0.75 0.25 0 0 0	trigon 1-trigon-3-face[a] $\geq$4-face 1-trigon-1-bigon-3-face[a]	0.75 0 0 0
0.75	trigon	$\geq$4-face	0	any face	1
0.5	dangerous 3-face	quadragon any except quadragon	1 0.75	 any except quadragon	 0.75
0.5	trigon	any except quadragon	1	any except quadragon	1
0.5	bigon	quadragon trigon any except trigon or quadragon	1 0.75 0.5	quadragon trigon any except trigon or quadragon	1 0.75 0.5
0.25	1-bigon- 3-face	quadragon any except quadragon	1 0.75	quadragon any except quadragon	1 0.75
0	1-trigon-3-face	any face	1	any face	1
0	1-trigon-1-bigon-3-face	any face	1	any face	1
0	0-multigon-3-face	any face	1	any face	1

[a] f' incident to the trigon on the 1-trigon-3-face or 1-trigon-1-bigon-3-face

[b] f' not incident to the trigon on the 1-trigon-3-face or 1-trigon-1-bigon-3-face

f sends 0 charge to the $\geq$4-face adjacent to it. f can only send 0.75 charge to a trigon if the trigon is f-incident to the $\geq$4-face. To any pair of f-incident faces, neither of which is f-incident to the $\geq$4-face, f sends a total of at most 1.5 charge. If f sends 0.75 to a trigon t, then the total charge that f sends to t, the $\leq$3-face f-incident to t and the other face f-incident to that face is at most 2.25. Therefore if x denotes the number of trigons to which f sends 0.75 charge, the total charge sent to other faces by f is at most

$$0 + 2.25x + 1.5(\frac{l - 1 - 3x}{2}) \leq l - 3$$

when $l \geq 9$.

Suppose $l = 5, 6, 7$ or 8. By Lemma 4.6.12 f is not adjacent to a quintagon. We consider the cases when f is and is not adjacent to a quadragon separately.

1. f is adjacent to a quadragon v_1v_2.

1.1 $l = 5$.

In this case, we may assume by Lemma 4.10.1 that the edge v_3v_4 is adjacent to a $\geq$ 6-big face which sends 0.5 charge to f. Further f sends at most 0.5 charge to each of the faces adjacent to the edges v_2v_3 and v_5v_1. Therefore, if f sends at most 0.5 charge to the face f' adjacent to v_4v_5 then the final charge is non-negative. Suppose that f sends more than 0.5 charge to f'. By Lemma 4.10.1, f' cannot be a quadragon and by Lemma 4.10.2 f cannot be a trigon. By Lemma 4.10.3, if f' is a dangerous 3-face then the face f'' adjacent to v_5v_1 cannot be a multigon, and it can easily be verified using Table 4.2 that the total amount of charge that f sends to f' and f'' is at most 1. Thus the final charge of f is non-negative.

1.2 $l \geq 6$.

If there are either

- four faces, each of which is a bigon, 1-bigon-3-face, 1-trigon-3-face, dangerous 3-face f-incident to a bigon or dangerous 3-face f-incident to two other dangerous 3-faces,
- two faces, each of which is a bigon, 1-bigon-3-face or 1-trigon-3-face, plus one 0-multigon-3-face, or
- two 0-multigon-3-faces

adjacent to f. Then the reader may verify with Table 4.2 that using these faces, the two faces f-incident to the quadragon, the $\geq$ 4-face and if necessary some other arbitrary faces, we can find a set of five faces to which f sends a total of at most 2 charge and so the final charge of f is non-negative by Fact 4.10.9.

Otherwise, by an argument similar to Lemma 4.10.10, Case 1.2, the $\geq$ 4-face adjacent to f is $\geq$ 6-big and sends 0.5 charge to f. If there are two faces adjacent to f, other than the faces f-incident to the quadragon and the $\geq$ 4-face, to which f sends a total of at most 1.5 charge, then the final charge of f is non-negative. It is straightforward to verify using Table 4.2 that this must be the case.

2. f is not adjacent to a quadragon

Let the $\geq$4-face be adjacent to the edge v_1v_2. Suppose that f is adjacent to two f-incident faces f_1 and f_2 such that f_1 is a bigon, a 1-bigon-3-face, a 1-trigon-3-face, a 1-trigon-1-bigon-3-face, or a 0-multigon-3-face and f_2 is not f-incident to the $\geq$4-face. Then f sends at most 0.5 charge to f_1 and further the total amount of charge sent to f_1 and f_2 by f is at most 1. Therefore the reader may verify using Table 4.2 that if $l \geq 6$ and there are

- Three consecutive faces each of which is a bigon, 1-bigon-3-face, 1-trigon-3-face, 1-trigon-1-bigon-3-face, or 0-multigon-3-face, or
- Two f-incident faces each of which is a 1-bigon-3-face, 1-trigon-3-face, 1-trigon-1-bigon-3-face, or 0-multigon-3-face, or
- Two non f-incident faces each of which is a bigon, 1-bigon-3-face, 1-trigon-3-face, 1-trigon-1-bigon-3-face, or 0-multigon-3-face, or
- A 0-multigon-3-face and another face which is a bigon, 1-bigon-3-face, 1-trigon-3-face, 1-trigon-1-bigon-3-face, or 0-multigon-3-face

adjacent to f, then using these faces, the faces f-incident to them, the $\geq$4-face and possibly some other arbitrary faces we can find a set of six faces to which f sends a total of at most 3 charge. Similarly, if $l = 5$ and there are

· Four consecutive faces each of which is a bigon, 1-bigon-3-face, 1-trigon-3-face, 1-trigon-1-bigon-3-face, or 0-multigon-3-face, or

· Three non-consecutive faces each of which is a bigon, 1-bigon-3-face, 1-trigon-3-face, 1-trigon-1-bigon-3-face, or 0-multigon-3-face, or

· A 0-multigon-3-face and another face which is a 1-bigon-3-face, 1-trigon-3-face, 1-trigon-1-bigon-3-face, or 0-multigon-3-face, or

· A 0-multigon-3-face and a bigon, neither of which is f-incident to the $\geq$4-face

adjacent to f, then the total charge that f sends to adjacent faces is at most 2 so the final charge is non-negative.

It remains to investigate the case when f is not adjacent to any set of faces described above. Consider the two faces f-incident to the $\geq$ 4-face. If each of v_lv_1 and v_2v_3 is a bigon, 1-bigon-3-face, 1-trigon-3-face, 1-trigon-1-bigon-3-face, or 0-multigon-3-face, then the total amount of charge that f sends to the faces adjacent to $v_{l-1}v_l, v_lv_1, v_1v_2, v_2v_3$, and v_3v_4 is at most 2, so the final charge is non-negative. Thus, for the remainder of the proof we may assume that the $\geq$ 4-face is f-incident to only trigons or dangerous 3-faces.

We first consider the case where one of the faces f-incident to the $\geq$4-face is a trigon t. Without loss of generality, assume t is adjacent to v_2v_3. The graph obtained by performing the $v_1v_2v_3v_l$-swap on G has one more quadragon than G and as many multigons of higher order and therefore has a 7-edge-colouring. From Propositions 4.6.10 and 4.6.11 it follows that for $e = v_2v_3$ there exists an e-colouring of G where the edges of t are coloured $\alpha, \beta, \gamma, \delta, \phi$ and the edges v_1v_2 and v_3v_4 are both coloured ϕ. For $c \neq \phi$, every c-mate M_c must contain the edges of t, plus an edge coloured ϵ, one coloured μ and at least four more coloured c. M_c must also contain an edge e_c adjacent to f but not f-incident to t. Observe that neither e_ϵ nor e_μ can be adjacent to a trigon or dangerous 3-face. Therefore we may assume (since f is not adjacent to any of the combinations of faces listed above) that the $l-3$ faces adjacent to the edges $v_4v_5, \ldots, v_lv_1$ contain two f-incident faces g_1 and g_2 where either

- g_1 is a bigon, and g_2 is either a bigon, a 1-bigon-3-face, 1-trigon-3-face, 1-trigon-1-bigon-3-face, a trigon or a dangerous 3-face, or
- g_1 is either a 0-multigon-3-face, a 1-bigon-3-face, or a 1-trigon-3-face, and g_2 is either a trigon or a dangerous 3-face

and the remaining $l-5$ edges are a collection of trigons and dangerous 3-faces. Further, e_ϵ and e_μ must be the same edge, adjacent to either g_1 or g_2. We will assume that $e_\epsilon = e_\mu$ is adjacent to g_1 and the argument is similar if it is g_2. Both M_ϵ and M_μ contain two edges adjacent to g_1 coloured ϵ and μ so the face adjacent to t as well as the face adjacent to g_1 (not f) must be adjacent to two $\geq$4-faces with which they share edges coloured ϵ and μ respectively.

If an edge $e' = v_iv_{i+1}$ $(i = 4, ..., l)$ is adjacent to a trigon or dangerous 3-face then mates of at most one colour may contain e'. Also, no mates can contain the edge which f shares with g_2. Since $l \leq 8$, at least one more mate M_c, for $c \neq \epsilon, \mu$ must contain the edge that f shares with g_1. If g_1 is not a 0-multigon-3-face, then M_c contains the two edges adjacent to g_1 coloured ϵ and μ so t and g_1 are adjacent to $\geq$3-big faces. If g_1 is a 0-multigon-3-face, then t is still adjacent to a $\geq$3-big face. This is since mates of at most one colour can contain the edge adjacent to g_1 which is not coloured with ϵ nor μ and because if for some c, a mate M_c contains an edge which f shares with a trigon or dangerous 3-face then the face adjacent to t shares an edge coloured c with a $\geq$4-face.

Therefore, f sends at most 0.25 charge to t, and 0 charge to g_1. Since at least one of the faces f-incident to g_1 receives at most 0.75 charge and one of every pair of f-incident faces receives at most 0.75 charge, and $l \geq 5$ there is at least one other face to which f sends at most 0.75 charge. These, along with the $\geq$4-face adjacent to v_1v_2 form a set of four faces to which f sends a total of at most 1 charge, so the final charge of f is non-negative by Fact 4.10.9.

Finally, if both faces f-incident to the $\geq$4-face are dangerous 3-faces, then consider the dangerous 3-face v_2v_3w. By performing the $v_2wv_3v_1$-swap on G and applying Propositions

4.6.10 and 4.6.11, it can be seen that for $e = v_2w$ there exists an e-colouring of G where the edges of the bigon v_2w are coloured α, β, γ and ϕ, an edge v_3w and the edge v_1v_2 are coloured ϕ and the edge v_2v_3 is coloured δ. For $c \neq \phi$, every c-mate M_c must contain the bigon v_2w and the edge v_2v_3, plus another edge adjacent to f. The faces adjacent to the edges $v_4v_5, \ldots, v_lv_1$ consist of two edges g_1 and g_2 as described above plus a collection of trigons and dangerous 3-faces. If follows that the bigon v_2w is adjacent to a $\geq$ 3-big face, so f sends at most 0.5 charge to the dangerous 3-face adjacent to v_2v_3. Similarly, f sends at most 0.5 charge to the dangerous 3-face adjacent to v_lv_1. f sends a total of at most 1 charge to g_1 and one of the faces f-incident to g_1. Therefore the faces adjacent to v_lv_1, v_1v_2, v_2v_3, g_1 and one face f-incident to g_1 form a set of five edges to which f sends at most 2 charges, so the final charge of f is non-negative by Fact 4.10.9.

□

Lemma 4.10.12. *The final charge of every 0-big $\geq$ 5-face is non-negative.*

Proof. Let f be a 0-big l-face. Let $f = v_1 \ldots v_l$.

f never sends more than a total of 1.5 charge to any pair of f-incident faces. Therefore when $l \geq 12$, the total charge sent out by f is at most $l - 3$ and the final charge is non-negative. In fact, there is at least one face adjacent to f to which f sends at most 0.5 charge so when $l = 11$, f sends at most 7.5 to the other 10 faces and the final charge is non-negative.

Consider the case when $l \leq 10$. We will analyze the cases for $l = 5$ and $l \geq 6$ separately. Note that f is not adjacent to a quadragon by Lemma 4.10.1.

1. $6 \leq l \leq 10$.

If f is adjacent to either

- four faces, each of which is either a bigon, a 1-bigon-3-face or a 1-trigon-3-face,

Table 4.3: Summary of charges sent by a 0-big ≥5-face to adjacent faces

Charge to f'	**Type of f'**	**f_1 can be...**	**Max charge to f_1**	**f_2 can be...**	**Max charge to f_2**
1	dangerous 3-face	trigon 1-trigon-3-face[a] ≥4-face 1-trigon-1-bigon-3-face[a]	0.5 0 0 0	trigon 1-trigon-3-face[a] ≥4-face 1-trigon-1-bigon-3-face[a]	0.5 0 0 0
0.75	dangerous 3-face	dangerous 3-face 1-bigon-3-face 1-trigon-3-face[b] 1-trigon-1-bigon-3-face[b] 0-multigon-3-face	0.75 0.25 0 0 0	trigon 1-trigon-3-face[a] ≥4-face 1-trigon-1-bigon-3-face[a]	0.5 0 0 0
0.5	dangerous 3-face	any except quadragon	0.75	any except quadragon	0.75
0.5	trigon	any except quadragon	1	any except quadragon	1
0.5	bigon	any except quadragon	0.5	any except quadragon	0.5
0.25	1-bigon- 3-face	any except quadragon	0.75	any except quadragon	0.75
0	1-trigon-3-face	any face	1	any face	1
0	1-trigon-1-bigon-3-face	any face	1	any face	1
0	0-multigon-3-face	any face	1	any face	1

[a] f' incident to the trigon on the 1-trigon-3-face or 1-trigon-1-bigon-3-face

[b] f' not incident to the trigon on the 1-trigon-3-face or 1-trigon-1-bigon-3-face

- two faces, each of which is are either a bigon, a 1-bigon-3-face or a 1-trigon-3-face, plus one 0-multigon-3-face, or
- three 0-multigon-3-faces,

then by taking these faces, the faces f-incident to them, and possibly some other arbitrary faces we can find a set of six faces to which f sends a total of at most 3 charges so the final charge of f is non-negative by Fact 4.10.9.

Otherwise, by Lemma 4.10.8, every trigon adjacent to f is also adjacent to a $\geq$ 3-big face, and receives no charge from f. Further if f sends 1 charge to a dangerous 3-face d, then each of the faces f-incident to d must be either a trigon, a 1-trigon-3-face or a 1-trigon-1-bigon-3-face to which f sends 0 charge. If f sends 0.75 to a dangerous 3-face f_1, then one face f-incident to it receives 0 charge from f, and the other is a 3-face f_2. If

f_2 is not dangerous, then the total charge sent to f_1 and f_2 is at most 1. If f_2 is dangerous then the total charge sent to f_2 and the other face f-incident to f_2 is at most 0.75.

If f sends 0.75 charge to each of a pair of f-incident faces, then they are both dangerous 3-faces and f sends 0 charge to each of the two faces f-incident to them. When $l \geq 6$, along with any other pair of f-incident faces, these form a set of six edges to which f sends at most 3 charge, so the final charge is non-negative. Otherwise f sends a total of at most 1 charge to each pair of f-incident faces, so when $l \geq 6$ the total charge sent by f is at most $l - 3$.

2. $l = 5$.

It is sufficient to consider the following cases.

2.1. f is adjacent to a trigon t.

Let t be adjacent to the edge v_1v_2 and let f_1, f_2, f_3 and f_4 be the faces adjacent to v_2v_3, v_3v_4, v_4v_5 and v_5v_1, respectively so that f_1 and f_4 are f-incident to t. It follows easily from the proof of Lemma 4.10.2 that at most one of f_2 and f_3 can be a trigon, dangerous 3-face or 1-trigon-1-bigon-3-face.

Suppose that one of them (without loss of generality, f_2) is a trigon, dangerous 3-face or 1-trigon-1-bigon-3-face. Then f_3 must be a bigon, a 1-bigon-3-face, a 1-trigon-3-face or a 0-multigon-3-face and by Lemma 4.10.2, must be adjacent to a $\geq$3-big face and receive no charge from f. If f_2 is a trigon or 1-trigon-1-bigon-3-face then it also receives 0 charge from f, so the final charge of f is non-negative. If f_2 is a dangerous 3-face and f_3 is a bigon, then the charge sent to each of f_1, f_2, f_3 and f_4 is at most 0.5 so the final charge is non-negative. If f_2 is a dangerous 3-face and f_3 is not a bigon, then by Lemma 4.10.3, it cannot be the case that both f_1 and f_4 are dangerous 3-faces, so Table 2 shows that the total amount of charge sent to f_1, f_2 and f_4 is at most 2 and the final charge is non-negative.

If neither of f_2 and f_3 is a trigon, dangerous 3-face or 1-trigon-1-bigon-3-face, then the total amount of charge sent to f_1 and f_2 is at most 1, as is the total amount of charge sent to f_3 and f_4, so the final charge of f is non-negative.

2.2. f is adjacent to no multigons of order ≥ 3.

Recall from Rules 6.5 that if a dangerous 3-face is f-incident to one bigon then it receives at most 0.5 charge from f, and if it is f-incident to 2 bigons it receives no charge from f. Therefore it follows from Lemma 4.10.4 that if f is adjacent to four or five bigons then its final charge is non-negative. If f is adjacent to three bigons, then each of the other faces adjacent to f is f-incident to either two bigons or one bigon and a 3-face so f sends a total of at most 0.5 to each of those faces, and at most 0.5 to each of the bigons so the final charge is non-negative. If f is adjacent to one or two bigons, and the other faces adjacent to f are 3-faces, then it is clear from the Rules 6 for dangerous 3-faces adjacent to f and the rules summarized in Table 2 that f sends out a total of at most 2 charges.

2.3 f is not adjacent to any multigons.

Then f sends charge only to dangerous 3-faces and 1-bigon-3-faces. Lemma 4.10.3 implies that f cannot be adjacent to three non-consecutive dangerous 3-faces, or more than three dangerous 3-faces. When f is adjacent to three consecutive, or fewer than three dangerous 3-faces, it follows from the fact that f sends at most 0.5 charge to a dangerous 3-face which is f-incident only to 1-bigon-3-faces or dangerous 3-faces, that the total amount of charge sent out by f is at most 2.

□

4.11 Finale

Recall that we would like to prove Theorem 4.1.1 by excluding the existence of a minimal counterexample. Accordingly, let G be a minimal counterexample and assign charges to the faces and vertices of G as described in Section 4.4, and apply the rules of Section 4.4.3. Since the initial amount charge assigned to G is negative and the final charge of every face and vertex is non-negative, this is a contradiction to the assumption that the rules conserve charge in G.

Chapter 5

Conclusion

We close with a brief summary. In Chapters 1 and 2 we studied optimizations of T-joins and T-cuts with some applications. In Chapter 3 we introduced the main problem of interest to our endeavour, Conjecture 3.2.1 and in Chapter 4 we gave a proof of the special case when $k = 7$. Along the way we have noted several existing conjectures, and we now restate those open problems which are of highest interest to us.

Does Conjecture 3.2.1 hold for grafts where $\tau \geq 8$?

Conjecture (Guenin)**.** *Let (G, T) be a graft where all T-cuts in G have the same parity. Then (G, T) packs if G is planar.*

A proof of Guenin's more general conjecture, which would imply Conjecture 3.2.1 would also be a beautiful and deep result.

Conjecture (Guenin [Gue03])**.** *Let (G, T) be a graft where all T-cuts in G have the same parity. Then (G, T) packs if G does not contain the Petersen graph as a T-minor.*

Terminology

Graph Basics

Term	Notation	Definition
Graph	$G = (V, E)$	A set V of vertices along with a set E of pairs of vertices (edges)
Edge	$e = uv$	An unordered pair of vertices $u, v \in V$ (called the endpoints of e)
Vertex set of G	$V(G)$	The set of vertices of a graph G
Edge set of G	$E(G)$	The set of edges of a graph G
Simple graph		A graph with at most one edge between any pair of vertices
Multigraph		A graph where multiple edges between any pair of vertices are allowed
e incident to v		There exists an edge $e = uv \in E$
u adjacent to v		There exists an edge $uv \in E$
Degree of v	$d(v)$	The number of edges incident to v
k-regular graph		A graph where all vertices have degree k
k-edge-colouring of G		An assignment of k colours to the edges of G such that no pair of edges that share an endpoint are assigned the same colour
Subgraph of G		A graph whose vertex set is a subset of $V(G)$ and whose edge set is a subset of $E(G)$
Induced subgraph of G	G[S]	For $S \subseteq V(G)$, the subgraph of G with vertex set S and edge set containing all edges in $E(G)$ with both endpoints in S

Graph Connectivity

Term	Notation	Definition
Path	$v_1 v_2 \dots v_l$	A sequence of vertices, where each vertex is adjacent to those preceding and following it in the sequence
(u, v)-path		A path whose first vertex is u and whose last vertex is v
Cycle	$v_1 v_2 \dots v_1$	A path with no repeated vertices, except for the first and last which are the same
Tour	$e_1 e_2 \dots e_1$	A sequence of edges, where each edge shares an endpoint with those preceding and following it in the sequence such that no edge is repeated except the first and the last edge which are the same.
Eulerian graph		A graph in which there exists a tour containing all edges.
Cut	$\delta(S)$	Given $S \subseteq V$, the set of edges which have exactly one endpoint in S
$\{u, v\}$-cut		A cut $\delta(S)$ where $\lvert S \cap \{r, s\} \rvert = 1$
Connected graph		A graph in which there exists a (u, v)-path for every pair of vertices $u, v \in V$
Bridgeless graph		A graph which contains a cut of size 1
k-edge-connected graph		A graph which contains no cut of size $k - 1$

Types of Graphs

Term	Notation	Definition
Complete bipartite graph	$K_{m,n}$	The graph with vertex set $V = V_1 \cup V_2$, $V_1 = [n]$, $V_2 = [m]$ and edge set $E = \{uv;\ u \in V_1, v \in V_2\}$
Complete graph	K_n	The graph with vertex set $V = [n]$ and edge set $E = \{uv;\ u, v \in V\}$
Tree	T	A graph not containing any cycles
Planar graph		A graph which can be drawn in the plane such that pairs of edges may intersect only at their common endpoints
Plane graph		A planar graph, embedded in the plane such that pairs of edges may intersect only at their common endpoints

Miscellaneous

Term	Notation	Definition
Matching		A family of pairwise disjoint edges
Perfect matching		A matching containing an edge adjacent to each vertex $v \in V$

References

[AH76] K. Appel and W. Haken, *Every Planar Map is Four Colourable*, American Mathematical Society **82** (1976), no. 5. 26

[Bel58] R. Bellman, *On a Routing Problem*, Quarterly of applied mathematics **16** (1958), no. 1, 87. 6

[CCPS98] W. Cook, W. Cunningham, W. Pulleyblank, and A. Schrijver, *Combinatorial Optimization*, Wiley Interscience, 1998. 2, 4, 6, 8, 14

[CCS96] P. Codato, M. Conforti, and C. Serafini, *Packing T-joins*, Journal of Graph Theory **22** (1996), no. 4, 293–296. 19

[CJ87] M. Conforti and E.L. Johnson, *Two Min-Max theorems for graphs noncontractible to a four wheel, preprint, IBM Thomas J*, Watson Research Center (1987). 24

[DKK10] Z. Dvořák, K. Kawarabayashi, and D. Král', *Packing six T-joins in plane graphs.* 23, 39

[DS] M. DeVos and P.D. Seymour, *On Packing T-Joins.* 21

[Edm65] J. Edmonds, *Maximum matching and a polyhedron with 0, 1-vertices*, Journal of Research of the National Bureau of Standards **69** (1965), no. 1-2, 125–130. 8

[EJ70] J. Edmonds and E.L. Johnson, *Matching: a well-solved class of integer linear programs*, Combinatorial structures and their applications: proceedings (1970), 89.

[EJ73] ______, *Matching, Euler tours and the Chinese postman*, Mathematical programming **5** (1973), no. 1, 88–124. 4, 7, 8, 9, 12, 13

[EK72] J. Edmonds and R.M. Karp, *Theoretical improvements in algorithmic efficiency for network flow problems*, Journal of the ACM (JACM) **19** (1972), no. 2, 248–264. 15

[EK11] K. Edwards and K. Kawarabayashi, *Seven-edge-colouring in seven regular planar graphs*, Manuscript (2011). 23

[FF56] L.R. Ford and D.R. Fulkerson, *Maximal flow through a network*, Canadian Journal of Mathematics **8** (1956), no. 3, 399–404. 15

[Flo62] R.W. Floyd, *Algorithm 97: shortest path*, Communications of the ACM **5** (1962), no. 6, 345. 6, 14

[GH61] R.E. Gomory and T.C. Hu, *Multi-terminal network flows*, Journal of the Society for Industrial and Applied Mathematics **9** (1961), no. 4, 551–570. 15

[GJ79] M.R. Garey and D.S. Johnson, *Computers and intractability. A guide to the theory of NP-completeness. A Series of Books in the Mathematical Sciences*, WH Freeman and Company, San Francisco, Calif, 1979. 15

[Gue03] B. Guenin, *Packing T-Joins and Edge Colouring in Planar Graphs.* 2, 3, 22, 23, 24, 25, 27, 29, 30, 40, 84

[Kar72] RM Karp, *Reducibility among combinatorial problems, 85–103*, Proc. Sympos., IBM Thomas J. Watson Res. Center, Yorktown Heights, NY, 1972. 15

[KP92] E. Korach and M. Penn, *Tight integral duality gap in the chinese postman problem*, Math. Program. **55** (1992), no. 2, 183–191. 12

[Lov75] L. Lovász, *2-matchings and 2-covers of hypergraphs*, Acta Mathematica Hungarica **26** (1975), no. 3, 433–444. 12

[LPV03] L. Lovász, J. Pelikán, and K. Vesztergombi, *Discrete mathematics: elementary and beyond*, Springer Verlag, 2003. 5

[PR82] M.W. Padberg and M.R. Rao, *Odd minimum cut-sets and b-matchings*, Mathematics of Operations Research (1982), 67–80. 15

[Riz99] R. Rizzi, *Indecomposable r-graphs and some other counterexamples.* 21

[Riz03] ______, *A simple minimum T-cut algorithm*, Discrete applied mathematics **129** (2003), no. 2-3, 539–544. 16

[RST97] N. Robertson, P. Seymour, and R. Thomas, *Tutte's edge-colouring conjecture*, Journal of Combinatorial Theory, Series B **70** (1997), no. 1, 166–183. 24

[Sch03] A. Schrijver, *Combinatorial optimization: polyhedra and efficiency*, Springer Verlag, 2003. 14

[Seb86] A. Sebő, *Finding the T-join structure of graphs*, Mathematical programming **36** (1986), no. 2, 123–134.

[Seb87] ______, *A quick proof of Seymour's theorem on T-joins*, Discrete Mathematics **64** (1987), no. 1, 101–103. 12

[Sey77] P.D. Seymour, *The matroids with the max-flow min-cut property*, Journal of Combinatorial Theory, Series B **23** (1977), no. 2-3, 189–222. 19

[Sey81] ______, *On odd cuts and plane multicommodity flows*, Proceedings of the London Mathematical Society **3** (1981), no. 1, 178. 2, 12

[Tai80] PG Tait, *Remarks on the colouring of maps*, Proc. Roy. Soc. Edinburgh, vol. 10, 1880, pp. 501–503. 26

[Tho99] R. Thomas, *Recent excluded minor theorems for graphs*, London Mathematical Society Lecture Note Series (1999), 201–222. 24

[Wag37] K. Wagner, *"Uber eine Eigenschaft der ebenen Komplexe*, Mathematische Annalen **114** (1937), no. 1, 570–590. 23