UNIVARIATE DISTRIBUTION FUNCTIONS:

AN INTERDISCIPLINARY STUDY

by

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ABSTRACT

This dissertation provides a comprehensive coverage of univariate statistical distributions and their uses. 108 simple and compound univariate distributions have been tabulated and studied as to their uses in the different fields of statistical activity.

The fields of application and type of problems to which each of these distributions can be applied are indicated in Chapter 2. A selected reference list is also included in this chapter. These references are intended to explain more precisely the use of the distributions in particular problems and to indicate further references for the reader interested in the more general theory of statistical distributions.

A table of the 108 distribution functions and their parametric values is presented in Chapter 3.

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CHAPTER 1

INTRODUCTION

This dissertation is an expository work covering most of the univariate distribution functions, discrete and continuous, which are known and used at the present time. In the course of finding and tabulating these distributions, much use was made of a group of statistically-oriented periodicals. The major journals covered in this research were the Annals of Mathematical Statistics, the American Statistical Association Journal, the Journal of the Royal Statistical Society, the Calcutta Statistical Association Bulletin, Biometrika, Biometrics, Sankhya, Technometrics, Metron, Econometrika, Psychometrika, Applied Statistics, and the Annals of Eugenics. These periodicals cover the subject of statistical theory and application from a variety of points of view and thus provide a wide scope on the uses of statistical distribution functions in different academic disciplines.

In Chapter 3 of this paper, a table of 108 simple and compound univariate distributions is given. Each distribution function and its parametric range of values are listed here. This group of 108 distributions, the author believes, comprises almost all of the univariate distributions presently applied in any major field of statistical activity.

Chapter 2 provides the basis of this thesis. The author has endeavoured in this chapter to indicate the fields of application and give an insight into the type of problems in which each of these distributions is used. This chapter is intended to provide a comprehensive coverage of the areas of applicability of each included distribution. The impossibility of listing and explaining the individual problems to which these distributions apply is compensated for by the inclusion of a selected set of references after each article. These references are intended to explain more precisely the use of the distribution in particular problems, and to indicate further references for the reader interested in more general theory of statistical distributions. The references are included immediately after each article in order to facilitate the tedious procedure of relating the reference contents to the statements made in each article.

To the author's knowledge, this exposition is the only work of its kind at the present time. G.P. Patil in his book "Classical and Contagious Discrete Distributions" has given a complete bibliography of discrete distributions, but has not considered the continuous cases. Much use was made of Patil's book during the course of this research in the problems of discrete distributions.

Great difficulty was encountered by the author in the search for relevant information on many of the distributions included here. Although almost any statistical article or book will contain information on distribution theory and application, it seems that a comprehensive coverage of the applications of individual distributions has been neglected thus far. It is hoped that this thesis is a step towards remedying this situation, and that further work in this direction will be stimulated.

The importance of a work of this kind is readily apparent. A researcher studying a particular univariate distribution in his discipline, for example in a biological set-up, will see that the same density may be available as a distribution describing some phenomenon in another field, say the physical sciences. A ready comparison of the systems which generate the same distribution in the various fields is available through this dissertation and its reference list. With the broader outlook afforded by this insight, a new approach and possibly new applications in each field can be found. Also, for other research in his particular field of interest, the references cited should be of use to the reader.

The tabulation of distributions in Chapter 3 is an asset to people working on the theoretical aspects of statistical distributions. The table provides an easy method of comparing various frequency forms and of analysing the properties which different distributions have in common.

Due to the large amount of statistical literature on distribution theory, a complete bibliography is not given here. Only those articles dealing mainly with the applications of distribution functions are cited in Chapter 2.

CHAPTER 2

UNIVARIATE STATISTICAL DISTRIBUTIONS AND THEIR APPLICATIONS

1. ARC-SINE DISTRIBUTION

The Arc-Sine distribution is the same as the Beta distribution (4) with parameters $\alpha = \frac{1}{2}$, $\beta = \frac{1}{2}$. It occurs repeatedly in fluctuation theory, and is used to describe the unexpected behaviour of sojourn times in random walk problems.

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2. BERNOULLI DISTRIBUTION

The Bernoulli distribution is a minor variant of the Binomial distribution (16). The sum of n independent random variables having Bernoulli distributions with common parameter p has the Binomial distribution with parameters (n,p).

Tacklind [3] has studied the Bernoulli distribution in the theory of collective risk. It is also used as a distribution for the number of exceedances in a sample taken from an unknown continuous distribution.

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3. TACKLIND, S. "Sur le Risque de Ruine dans les Jeux Inéquitables". Skand. Aktuarietids (1942) pp.1-42.

3. BESSEL DISTRIBUTION

The Bessel family of distributions consists of those distributions which contain a Bessel function in their functional forms. The Bessel distribution considered here is the simplest member of the Bessel family. It is used in the theory of ordinary symmetric random walks to describe the density of the first passage epoch through the point $t > 0$.

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4-12. BETA FAMILY OF DISTRIBUTIONS

The Beta family of distributions consists of variations of Pearson's Type I and Type VI densities (82). The major use of these distributions is in the field of sampling inspection where a Beta-type prior distribution is often assumed. In Bayes procedures the Beta distributions also appear as prior distributions.

In nonparametric statistics the simple Beta distribution (4) arises as the distribution of the α^{th}-order statistic in a sample of size $\alpha + \beta - 1$ from a U(0,1) distribution. Beta distributions also occur in the distribution of certain functions of Gamma and Dirichlet variables. For example, if X and Y are independent Gamma variates, then X/(X + Y) is distributed in the Beta distribution.

The Incomplete Beta distribution (6) is used in tabulating the probabilities for the Beta and Binomial distributions, while the Type I Beta (11) has been shown by Johnson [3] to provide a good fit to cloudiness data.

Beta distributions are used as failure-rate distributions and they also appear frequently in the field of renewal theory.

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9. 13. 14. NONCENTRAL BETA DISTRIBUTIONS

The Noncentral Type I Beta distribution (13) is a mixture of a sequence of Beta distributions and a Poisson density while the Noncentral Type II (14) is a mixture of a Beta sequence and a Negative Binomial distribution. These two distributions along with the simple Noncentral Beta (9) find their main application in calculations of the power functions for the various Beta distributions. A more general distribution covering almost all central, non-central, product and ratio distributions is available in Mathai and Saxena [3].

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15. BILATERAL EXPONENTIAL DISTRIBUTION

The density of the Bilateral Exponential distribution is a convolution of the Exponential density $\alpha \exp(-\alpha x)$ for $x > 0$ with the mirrored density $\alpha \exp(\alpha x)$ for $x < 0$. It is sometimes referred to as the Laplace distribution or the 'first law of Laplace', the second being the Normal distribution.

The Bilateral Exponential distribution is a member of the Double Exponential family of distributions (31). Thus the major use of this distribution is in the statistics of extreme values. In most references, mention of the Bilateral Exponential is limited to those sections discussing convolution theory.

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16, 17, 18. BINOMIAL DISTRIBUTIONS

The Binomial distribution is one of the basic statistical distributions and is very important in theoretical statistics. Its applications extend into every field as it describes the number of occurrences of a specified event in an independent sequence of n trials where each trial results in either a success or a failure.

The Binomial, Compound Binomial, and Mixed Binomial distributions are widely used in the design of sampling surveys and in group testing where the experimenter determines either that all x units are non-defective or that at least one defective is present. They also have applications as failure distributions.

The Binomial distributions also have wide application in the field of quality control. Among their more uncommon uses are the Mendelian theory example in Neyman [7], the effect-of-stress example in Siegel [8], the power-supply and vaccine examples in Feller [3], and the random walk example in Munroe [6].

In biological and medical fields, Binomial distributions are used to describe observational data while the simple Binomial distribution arises in psychological studies as the distribution of the number of isolates in a social group. This distribution also has applications to surgical mortality data and as a stochastic model for the electron multiplier tube.

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19. TRUNCATED BINOMIAL DISTRIBUTION

The Zero-Truncated Binomial distribution is the conditional distribution of the number of successes when it is known that there are no failures. Fisher [2] and Haldane [3] have applied it to problems of human genetics in estimating the proportion of albino children produced by couples capable of producing albinos.

This distribution has many uses in common with the Binomial distribution, in the case of truncated sampling. Finney [1] gives several other particular applications of the Truncated Binomial distribution.

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20. BOREL DISTRIBUTION

The Borel distribution is used as a basic distribution in queuing theory. It describes the number of units served during a busy period in a queue with random arrivals at a rate q per unit time and constant service time β.

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21. BOREL-TANNER DISTRIBUTION

The Borel-Tanner distribution was derived for a particular problem in queuing theory. It represents the probability that exactly x members of a queue will be served before the queue first vanishes, beginning with r members, and with α equal to the traffic intensity. Poisson arrivals and constant service times are underlying assumptions of this distribution.

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22. CAUCHY DISTRIBUTION

The Cauchy distribution is a special case of the Student - t distribution (94) with $v = 1$. The mean and variance of this distribution do not exist, but it is symmetric about its median μ. It plays a role in the Central Limit Theorem for infinite variances.

The distribution of the quotient of two independent standard Normal variates is described by the Cauchy distribution. It can be fitted to growth processes in bacterial colonies and human populations, and is used to describe the distribution of hitting points in two-dimensional Brownian motion.

The Cauchy distribution was proposed as an alternative to the Normal in the statistical analysis of quantal responses in psychometry. The distribution of α - particles from a radioactive source is approximated by this distribution. Cauchy distributions also appear in the theory of random walk and in extreme-value theory.

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23. CHARLIER TYPE B DISTRIBUTION

The Charlier Type B distribution is a series expansion proposed by Charlier in 1905 [2] to represent a discontinuous function in terms of differences of a Poisson variate. The Type B series representation was recommended by Charlier in cases of skewness where the Gram-Charlier Type A distribution (59) was inapplicable.

For practical purposes, the Charlier Type B distribution is useful only if three or four terms in the expansion suffice for fairly good reproduction of the actual values of f(x). Due to difficulties in the use of the series, it is rarely employed at the present time.

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24. CHI DISTRIBUTION

The Chi distribution, as the name suggests, is the distribution of the square root of a Chi-Square variate. The Half-Normal distribution (61) is a special case of the Chi distribution with one degree of freedom.

Cohen [1] gives an example of the use of the Chi distribution to describe the distribution of the radial error in a target analysis problem. Fraser [2] has used this distribution in connection with locally most powerful rank tests for order statistics.

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25. CHI-SQUARE DISTRIBUTION

The Chi-Square distribution is a special case of the Gamma distribution (49) with $\alpha = \frac{n}{2}$ and $\beta = 2$. This distribution, along with the F-distribution (45) and Student's t-distribution (94), plays an important part in the theory of statistics as it describes the sampling distributions of certain statistics in samples from the univariate Normal distribution (77). If a sample of size n is drawn from a Normal distribution with variance σ^2, then the quantity $\frac{(n-1)s^2}{\sigma^2}$, where s^2 is the sample variance, is distributed in the Chi-Square distribution with $n - 1$ degrees of freedom. Thus one use of the Chi-Square distribution is to test hypotheses concerning the variance in samples from Normal populations, and to obtain confidence limits for σ^2.

The Chi-Square distribution is used extensively in tests of goodness of fit to test whether a postulated probability law for a population is the true one. In the analysis of proportions, Cochran and Cox [2] show how the Chi-Square distribution is used for the statistical analysis of data in $2 \times t$ contingency tables.

The Chi-Square distribution is also used as an income distribution and as a service-time distribution in queuing theory. Parzen [6] shows that the energy of an ideal gas is Chi-Square distributed.

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26. NON-CENTRAL CHI-SQUARE DISTRIBUTION

The Non-Central Chi-Square is the distribution of $y_1^2 + \ldots + y_n^2$, where $y_1, \ldots, y_n$ are independent random variables with $y_i : N(\mu_i, 1)$ and the non-centrality parameter $\Delta^2 = \sum_1^n \mu_i^2$. The probability distribution of the Non-Central Chi-Square was obtained by Fisher [2] in 1928 as a particular case of the distribution of the multiple correlation coefficient.

The major use of this distribution is in computing the power of the Chi-Square test. It is also essentially the distribution of Mahalanobis' D^2 statistic.

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27. CIRCULAR NORMAL DISTRIBUTION

The Circular Normal distribution has widespread application in the analysis of geophysical and meteorological phenomena. The amounts of rainfall per month, the number of occurrences of rainfall of one inch or more per hour, the monthly evaporations from a reservoir, the monthly runoff in inches for the watershed of rivers, and the mean monthly temperatures at a given place can all be described by Circular Normal distributions.

Events or processes which follow a particular cycle are often described by Circular Normal distributions. In vital statistics this distribution is used in infant-mortality studies.

Circular Normal distributions also occur in the monthly egg production per laying hen, in economic time series, and in automobile and pig-iron production series.

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28. CONFLUENT HYPERGEOMETRIC SERIES DISTRIBUTION

The Confluent Hypergeometric Series distribution is so-named because of the use of the confluent hypergeometric function in its density. This distribution contains as a special case the Hyper-Poisson family of distributions (67). Hall [2] has found the Confluent Hypergeometric Series distribution arising in theoretical birth-and-death processes. For a generalization of this density and for a completely general family of distributions the reader is referred to Mathai and Saxena [3,4].

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29. DISTRIBUTION OF THE NUMBER OF EXCEEDANCES

This distribution is the distribution of the number of elements exceeding a given order statistic of a sample. It is a special case of the Polya distribution (87).

It is used in forecasting floods, where we are interested in the frequency and not size of the flood. The same procedure is applied to other meteorological phenomena such as droughts, extreme temperatures (killing frosts), largest precipitations, etc., and permits the forecast of the number of cases surpassing a given severity in the next N years.

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30. DODGE-ROMIG DISTRIBUTION

The Dodge-Romig distribution was proposed as a prior distribution in the theory of sampling inspection. The distribution function consists of a Binomial part and an unspecified part which is usually taken to be Binomial. The $f_N(x)$ is the probability that a lot of N items contains x defectives, where P_1 is the producer's risk point.

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31. DOUBLE EXPONENTIAL DISTRIBUTION

The Double Exponential distribution is an ordinary Exponential distribution (37) together with its reflection about a point c. It is used mainly in extreme value theory where it is a distribution of maximum values.

The distribution of the midrange from a rectangular population is Double Exponential. It is also frequently used in robustness studies. Dalrymple [1] has used the Double Exponential distribution for the design of bridge waterways, and Epstein and Sobel [3] mention it as a possible life-testing distribution.

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32. DOUBLE PARETO DISTRIBUTION

The Double Pareto distribution is a continuous frequency function whose ordinate is the sum of two functions of the Pareto type (79). It is used along with the Pareto distribution as an income distribution in economic statistics.

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33. ENGSET DISTRIBUTION

The Engset distribution is a Truncated Binomial distribution (19). It is used in traffic problems where N is the number of independent traffic sources each at rate α , c is the number of fully accessible servers, and $p = \alpha/b$ where b is the service rate. Riordan [2] illustrates the use of the Engset distribution in the theory of stochastic service systems to determine the equilibrium probability p_j.

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34. ERLANG DISTRIBUTION

The Erlang distribution is a special form of the Generalized Gamma distribution (54) with integral values of p. It often appears in the literature of queuing theory, and it is also used in renewal theory. Fry [4] illustrates its use as a distribution for telephone traffic.

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35. ERROR FUNCTION DISTRIBUTION

The Error Function distribution is the physical interpretation of the Normal distribution (77). In physics, engineering, and chemistry, this distribution is employed in the theory of observational errors. It describes the distribution of accidental errors occurring during the measurement of physical quantities.

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36. EULERIAN DISTRIBUTION

The Eulerian distribution was first used by Bennett [1] and by Fisher [3] to describe the length of germ plasm still heterogenic at an advanced stage of inbreeding. However, as this distribution function assumes the value zero with finite probability e^{-m}, it has become very useful in describing the rainfall in arid regions. The advantage of the Eulerian distribution for the purposes of a water engineer is that there is a probability of e^{-m} of no rain, which cannot be accounted for by a continuous distribution. The Eulerian distribution is continuous over the range of positive values, but has a finite condensation at zero.

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2. CORNISH, E.A. and FISHER, R.A. "The Percentile Points of Distributions Having Known Cumulants". TECHNOMETRICS (May, 1960) pp.209-225.

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37-40. EXPONENTIAL DISTRIBUTIONS

The Exponential distribution is one of the major statistical distributions and its applications are to be found in almost every academic discipline.

The family of Exponential distributions is the best known and most thoroughly explored life-length distribution due to the work of B. Epstein and his associates [7,8,9,10]. However there are many failure-time problems to which the Exponential distribution cannot apply, as its use carries the implication that at any time future life-length is independent of past history. This property of the Exponential distribution limits its use to problems involving the life-length of electron tubes, electric fuses, watch bearings and other materials which, under normal use, do not undergo changes affecting their future life.

Berkson and Gage [2] have proposed the Exponential distribution as a model for the survival rate of cancer patients, and it has found extensive use in other such medical investigations under the assumption of a constant death-rate in the uncured group. Examples of the application of an Exponential distribution to skin sensitivity studies, serial dilution experiments and genetics are given respectively by Manos [13], Peto [17] and Edington [6].

The Exponential distribution also plays a prominent role in physics where it governs radioactive decay, and it holds for distances in time, especially between the happenings of rare events. It is used to describe the distribution of lengths of telephone conversations, and is used as the starting point for the theory of extreme values (Gumbel) [11].

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16. MENDENHALL, W. and HADER, R.J. "Estimation of Parameters of Mixed Exponentially Distributed Failure Time Distributions from Censored Life Test Data". BIOMETRIKA (1958) pp.504-520.

17. PETO, S. "A Dose-Response Equation for the Invasion of Micro-Organisms". BIOMETRICS (1953) pp.320-335.

41. FIRST ASYMPTOTIC DISTRIBUTION OF EXTREME VALUES

This distribution, which we shall refer to as the First Asymptote finds its main use in engineering, breaking strength and meterological problems.

In aeronautics, the First Asymptote is used to determine the life expectancy of aircraftunder specified stresses. Jasper [7] has used it in the estimation of the endurance strength of a ship's structure to determine the design capacity of shipboard stabilization equipment.

In meteorology, the First Asymptote applies to extreme values in pressure, temperature, precipitation, snowfalls and rainfalls. Thom's results [8] on the minimum annual temperatures in certain U.S. cities have led to the discovery of startling inconsistencies in the design temperatures previously used for winter heating systems. Court [1] applied the First Asymptote to wind gusts in cities in the U.S., and his findings have led to the design of new wind resistant structures. The First Asymptote is used also to forecast flood heights, and leads to the knowledge of the return period for chosen design floods appropriate for the construction of bridges, dams, and hydro-electric plants.

Epstein [3] has particularly studied applications of the First Asymptote to breaking strength theory. It yields equations for predicting the breakdown voltage for electrodes in dielectric strength theory.

The First Asymptote is also used to describe bacterial extinction times in the presence of bactericide, and in geological studies of stream deposits to determine the origin of large boulders.

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42. SECOND ASYMPTOTIC DISTRIBUTION OF EXTREME VALUES

The applications of the Second Asymptote lie mainly in meteorological studies.

Thom [5] has used the Second Asymptote for the analysis of wind-speed, while Jenkinson [4] has used it to analyse the floods of the Little River, Westfield, Mass. 1910-1928, and the annual maximum hourly (daily and 4-daily) rainfalls in Naples (1888-1933), Marseille (1882-1946), and Tripoli (1919-1948).

Bernier [1] applied the Second Asymptote to the floods of the Rhine at Rheinfelden, the Colorado River at Black Canyon and the Durance at Archidiacre.

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5. THOM, H.C.S. "Frequency of Maximum Wind Speeds". Proc. Am. Soc. Civ. Eng. (1954).

43. THIRD ASYMPTOTIC DISTRIBUTION OF EXTREME VALUES

The Third Asymptote is used for the distribution of the smallest values taken from the Gamma distribution (49), for normal extremes, and for the analysis of droughts and fatigue failures.

The analysis of droughts by the Third Asymptote may be useful for solving storage and irrigation problems.

Also, it was applied by Weibull to the analysis of breaking strength. The reader is referred to the Weibull distributions (100-104) for further examples.

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44. MODIFIED EXTREME VALUE DISTRIBUTION

The Modified Extreme Value distribution is used in extreme value problems in physics and engineering. The major use of this distribution is as a failure rate distribution, where it has an increasing failure rate. It is also used in breaking strength problems.

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45. F - DISTRIBUTION

The F - distribution, which is also referred to as the Variance-Ratio distribution, is a simple transform of the Beta distribution of the second kind (12). This distribution was first studied by R.A. Fisher, and is used extensively in statistical methodology.

If S_1^2 and S_2^2 are independent estimates (based on samples of sizes n_1 and n_2 respectively) of the variance σ^2 from two Normal populations with equal variances, then their ratio S_1^2 / S_2^2 is distributed in the F - distribution with $n_1 - 1$ and $n_2 - 1$ degrees of freedom. More generally, the ratio of two independent Chi-Square variates over their degrees of freedom will follow the F - distribution.

The F - distribution is frequently used in inference problems regarding the population variances in Normal populations. It is also widely employed in the analysis of variance to test the homogeneity of a set of means, and it has applications to tests of the sample correlation coefficient.

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4b. NON-CENTRAL F DISTRIBUTION

The Non-Central F distribution is based on the ratio of a Non-Central Chi-Square variate (26) to a Central Chi-Square variate (25). The major use of this distribution is to compute the power of the variance-ratio test (F-test), as is illustrated by Hartley and Pearson [3].

Wishart [6] has considered the Non-Central F distribution in the form of the distribution of the correlation ratio. It is used in testing hypotheses relating to the correlation ratio, the multiple correlation coefficient, and the Studentised D^2 - statistic. It is also useful in analysis of variance problems when there is known to be a real difference between treatments.

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47. FOLDED NORMAL DISTRIBUTION

The Folded Normal distribution arises in studies of empirical data where a Normal distribution is appropriate but the observations are recorded without being given a sign (plus or minus). The measurements are regarded as all having the same sign, conventionally plus, and the Folded Normal distribution is indicated. At the present time, the two main uses of the Folded Normal distribution are in the measurement of the straightness of miniature radio tube leads, and in the determination of the centrality of the sprocket holes in motion picture film.

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4. LEONE, NELSON, and NOTTINGHAM "The Folded Normal Distribution". TECHNOMETRICS (1961).

48. LIMITED GALTON's DISTRIBUTION

The Limited Galton's distribution is a modified form of the Lognormal distribution (72). It is used as a primary distribution in extreme value theory. It is also used in the field of hydrology to represent the distribution of daily discharge observations which are skew.

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49-53. GAMMA FAMILY OF DISTRIBUTIONS

A Gamma distribution can arise in at least two ways which are capable of physical interpretation. In statistical mechanics, a constant force combined with one inversely proportional to the origin (i.e. to age) results in a Gamma-type distribution. Secondly, if a mechanism fails when it has experienced $K \geq 1$ shocks occurring at a Poisson rate α, then the distribution of lifetime is one of the Gamma family.

The Gamma distribution was derived as a life-length distribution by Birnbaum and Saunders [1], and is also frequently seen in the topic of accident theory. It is taken as the underlying distribution of the parameter λ of the Poisson distribution (83) when we assume that all persons are equally susceptible to accidents. This is a basic assumption of the Bates-Neyman model of accident repeatedness.

Drenick [2] and Herd [4] have discussed the use of the Gamma distributions as models for reliability problems. They have also been suggested as income distributions by Klein [6], and as distributions of landholdings by size by Patil [10].

The Gamma distributions are useful for representing the distribution of quantities (such as weight, length, etc.) which cannot be negative or which have a lower limit for their values. Due to the availability of tables of these distributions, they are also used in connection with the fitting and graduation of skew data.

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54. GENERALIZED GAMMA DISTRIBUTION

This distribution includes the Exponential (37), Gamma (49), and Weibull (104) distributions as special cases. It is widely applicable in the area of life-testing and is used to describe time-to-failure data in reliability problems.

The Generalized Gamma Distribution is Pearson's Type III distribution (82), and is only recently being investigated more closely for possible applications. It will prove more useful as statistical techniques are developed for it.

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55. INCOMPLETE GAMMA DISTRIBUTION

The main use of the Incomplete Gamma distribution is in tabulating probabilities for tables of the Gamma (49) and Poisson distributions (83). It also arises in renewal theory as a mortality distribution. Brown and Flood [2] give an example of its use as a distribution for tumbler mortality.

Recently the Incomplete Gamma has been used to fit rainfall data, and Chapman, Kenyon, and Scheffer [4] have used this distribution to fit the migration pattern in an animal population study.

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56. GENERALIZED POWER SERIES DISTRIBUTION

The Generalized Power Series distribution was introduced by Patil [2] in 1961. It is a generalization of the Power Series distribution originally defined by Noack [1] in 1950. Standard discrete distributions such as the Binomial (16), Poisson (83), Negative Binomial (74), and Logarithmic Series distributions (69) can be obtained as special cases of the Generalized Power Series distribution by the proper choice of the range T and the generating function a(x).

This distribution is of theoretical interest in that any properties derived in the statistical analysis of the G.P.S.D. can be immediately applied to the above mentioned special cases.

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57-58. GEOMETRIC DISTRIBUTION

The Geometric distribution is a special case of the Negative Binomial distribution (74) with $r = 1$. It is the distribution of the number of trials required to obtain the first success in a sequence of independent Bernoulli trials in which the probability of success at each trial is θ .

This distribution and its compound form are widely used as discrete failure distributions with constant failure rate. This means that if the life span of a piece of equipment has the Geometric distribution, then no aging takes place with use. Radioactive atoms have this property and are described in the discrete case by the Geometric distribution and in the continuous case by the Exponential (37).

The Geometric distribution is also a waiting-time distribution, and arises in single-server queue theory. In biology, for two species of unsegregated plants, the run lengths of individuals of each species are Geometrically distributed. Lotka [2] shows that the Geometric distribution describes the survival of family names in America. It also has applications to family sizes, and is used as a mortality distribution.

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59. GRAM-CHARLIER TYPE A DISTRIBUTION

The Gram-Charlier Type A distribution is a series expression for a frequency function in terms of derivatives of the Normal curve. The series is used to approximate frequency functions and in problems of practical curve-fitting.

Any function f(x) which together with its first two derivatives is continuous and all the derivatives of which vanish at infinity is capable of representation by a Gram-Charlier Type A Series distribution. However, as faras practical graduation is concerned, it would appear that the Type A series is successful only in cases of moderate skewness. A possibly better form of the series is postulated by Edgeworth [3] through the theory of elementary errors.

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4. KENDALL, M.G. and STUART, A. "The Advanced Theory of Statistics, Volume I". Charles Griffin and Company, 1963.

60. HALF-CAUCHY DISTRIBUTION

The Half-Cauchy distribution is a folded form of the Cauchy density (22). It is a special case of the Generalized Gamma distribution (54), and as such it arises in life-testing problems. This distribution also has possible applications in reliability theory.

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61. HALF-NORMAL DISTRIBUTION

The Half-Normal distribution is an extreme case (with the original expected value equal to zero) of the Folded Normal distribution (47). It is the distribution of $x = |z|$ when z is distributed $N(0,\sigma)$.

Daniel [1] shows how this distribution is used in comparing the sample empirical cumulative distribution with a standard tabled distribution in the interpretation of factorial two-level experiments. In industrial practice the Half-Normal distribution is applied to the same problems as the Folded Normal distribution as well as to feeler gauge measurements of the flatness of objects.

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62. HELMERT DISTRIBUTION

The Helmert distribution, discovered by Helmert in 1875, is a special case of the General Pearson Type III distribution (82). It is the distribution of the sample standard deviation, or equivalently, of the sample variance in samples from a normal population. In the distribution function, σ^2 is the parent variance and s^2 is the sample variance.

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63. HERMITE DISTRIBUTION

This distribution is a special case of the Poisson Binomial distribution (84) for $n = 2$, and may be regarded as the distribution of the sum of two correlated Poisson variables or the distribution of the sum of an ordinary Poisson variable and an independent Poisson doublet variable.

In addition, Kemp and Kemp [2] have shown that it provides a satisfactory approximation to many of the generalized Poisson distributions.

Under the assumption that the distribution of the number of accidents to single cars is Poisson and the distribution of two-car accidents is another independent Poisson variate, the Hermite distribution describes individual car accident statistics.

The overall distribution of mistakes in copying groups of digits also follows the Hermite distribution, if we regard the inversion of neighbouring digits as a Poisson variable.

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64-65. HYPERBOLIC COSINE DISTRIBUTION

The form of the Hyperbolic Cosine distribution shows it to be a variant of the Logistic distribution (71). This distribution is of curiosity value in that it exhibits a "self-reciprocal pair" - the density and its characteristic function differ only by obvious parameters. No practical applications of the Hyperbolic Cosine or its truncated form have been found as of yet. For further studies of the properties exhibited by this distribution see Mathai and Saxena [3].

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66. HYPERGEOMETRIC DISTRIBUTION

The Hypergeometric distribution arises in sampling theory when we consider sampling without replacement. Thus its principal application is in the field of sampling inspection of lots containing a finite number of items. It has also been applied to problems arising in zoological studies in the estimation of the size of an animal population from recapture data.

In sequential procedures, the Hypergeometic distribution is used to find how many items must be sampled from a lot of N items containing K defectives, to produce n nondefectives. It is used to test the equality of two proportions, and has also appeared as a distribution for the number of exceedances in a continuous distribution.

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67. HYPER-POISSON DISTRIBUTION

The Hyper-Poisson distribution is a two parameter generalization of the Poisson distribution (83), and a special case of the three parameter Confluent Hypergeometric Series distribution (28). It is a relatively unexplored distribution, but Bardwell and Crow [1] state that the Hyper-Poisson distribution may have application to accident and contagion phenomena. They also fit this distribution to Student's data [5] on haemacytometer yeast cell counts.

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68. J-SHAPED DISTRIBUTION

The main application of the J-Shaped Family of distributions has been to the theory of failures where they are used to describe the frequency of power band tool failures, the frequency of automatic calculating machine failures, and the frequency of failure at time x of radar components.

J-Shaped distributions also occur in economic statistics where they describe the distribution of wealth in the population at large. Distributions of deaths of centenarians and of infants are often J-Shaped, and it has been shown that the J-Shaped distribution is characteristic of the number of words used once, twice, etc. in a given literary work.

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69-70. LOGARITHMIC SERIES DISTRIBUTION

The Logarithmic Series distribution of R.A. Fisher is obtained by a limiting process from the Negative Binomial distribution (74). Its major use lies in biological studies where it is used in describing the number of individuals per species, the number of species per genus, and the number of genera per sub-family. Williams [7] shows that the number of research papers per biologist in a certain year can be represented by a Logarithmic Series distribution.

This distribution was originally proposed by Fisher to describe the relation between the number of moths of different species caught in a light-trap over a period of time. Harrison [4] has since applied it and the truncated form to the number of insect species infesting different stored products.

In bacterial counts, the number of bacteria per colony follows the Logarithmic Series distribution, while Kendall [5] has described a Markov process which also leads to this distribution. It has recently been found relevant in Operational Research, particularly in the study of inventory control problems.

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71. LOGISTIC DISTRIBUTION

In early studies of the Logistic distribution it was wrongly thought that all growth processes could be represented by a function of its form, with x representing time. More recently however, the Logistic distribution has become increasingly important in biometric research.

It has been developed theoretically for the representation of such phenomena as haemolysis and population growth, and is widely used in studies of dosage. Berkson [1] has established the Logistic distribution as a model for analysing bioassay and other experiments involving quantal response.

Gumbel [5] has shown that the asymptotic distribution of the midrange for initial symmetrical distributions of the Exponential type is Logistic and has indicated extreme value uses. The Logistic distribution is occasionally used to represent data with a pattern similar to the Normal distribution, and Plackett [6] has considered its use in problems involving censored data.

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72. LOGNORMAL DISTRIBUTION

The Lognormal distribution may be defined as the distribution of a variate whose logarithm obeys the Normal probability law(77). Many authors rate it with the Normal distribution for importance as a statistical distribution.

In biological problems, the Lognormal distribution has been used in the interpretation of entomological data, in quantitative inheritance, to describe organism growth, and in relation to allometry. It has been applied to the distribution of dosage required to produce a given response in biological assay, and Sartwell [11] has shown that the distribution of incubation periods in a point-source epidemic is approximately Lognormal. Also, Boag [3], Osgood [10], and Tivey [12] have shown that the distributions of duration of survival in several diseases are closely approximated by the Lognormal. It has been used to describe the frequency distribution of deaths due to cancer.

In economics the Lognormal is used as an income distribution, and it occurs in distributions of inheritances, expenditures on particular commodities, total wealth possessed by individual persons, bank deposits, and price change theory.

In anthropometry the Lognormal is used as a distribution for body-weights and heights; in astronomy as a distribution of stars; and in philological problems as the distribution of the number of words in sample sentences from individual authors, or the frequency with which certain authors use nouns.

Industrial applications are found in endurance tests of ball-bearing greases, in data on the distributions of throughputs before failure in acid plants, in discussing loss angles for electrical condensers, in the measurable properties of iron tubes cast in mouldings, and in sound measurements in decibels. It is also well established as a distribution of small particles in small particle statistics, in the theory of breakage, and in probit analysis.

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73. MAXWELL-BOLTZMANN DISTRIBUTION

The Maxwell-Boltzmann distribution is the probability density of the velocity x of a molecule of mass m in a gas at absolute temperature T where $\beta = \frac{m}{2KT}$ and K is Boltzmann's constant. Thus the major applications of the Maxwell-Boltzmann distribution are to dynamic problems in physics and engineering. Middleton [3] illustrates the use of this distribution to describe the first-order distribution density in the equilibrium state of the thermal currents in a linear network. It is also used as a failure-rate distribution.

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74-75. NEGATIVE BINOMIAL DISTRIBUTION

The Negative Binomial distribution was discovered in the course of study of the frequency of death following repeated exposure to a disease or to a toxicant, and it still has application in this field.

In 1919 Greenwood and Woods formulated three models to explain the observed frequencies of accidents-pure chance, proneness, and true contagion. Both the proneness model and the true contagion model result in a Negative Binomial distribution for accident frequency.

The Negative Binomial and its generalized form are also used in problems connected with the number of individuals belonging to given species in samples from plant and animal populations, and have been used as discrete failure distributions.

Anscombe [1] cites a number of ways in which the Negative Binomial distribution can arise, including inverse Binomial sampling, heterogeneous Poisson sampling, and immigration birth-death processes. Also, if colonies are distributed randomly with a Poisson distribution over a fixed area and if the numbers of individuals in each colony are distributed in independent Logarithmic distributions (69), then the total count is distributed in the Negative Binomial distribution.

The Negative Binomial distribution is also used in waiting time problems where it is referred to as the Pascal distribution (81).

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76. NEYMAN TYPE A CONTAGIOUS DISTRIBUTION

This distribution was developed by Neyman [9] in 1939 to describe the distribution of larvae in experimental field plots. The word 'contagious' is used because the distribution can be expected to apply where events are clustered together rather than spread out at random. It is of less general use than many of the other distributions, and is designed mainly for special situations.

It is widely used in describing populations under the influence of contagion: Beall[2] and Evans [4] discuss its use in entomology, Greenwood and Yule [5] in the study of accidents, and Bliss [3] in bacteriology. Barnes and Stanbury [1] have applied the Neyman Type A to plant quadrat work, while Beall [2] has applied it to the quadrat sampling of insect larvae. In industrial and driver accidents, the Neyman Type A distribution arises in the study of accident data.

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77. NORMAL DISTRIBUTION

The Normal distribution is the most important distribution of both theoretical and applied statistics. The assumption of the Normal distribution as the basis for analyzing sample data is common in the development of advanced statistical theory.

Even more importance is lent to this distribution by the Central Limit Theorems, one of which states that the standardized sum of a large number of independent and identically distributed random variables (with finite variance) is approximately Normally distributed regardless of the form of the original distribution. Useful characterization properties such as independence of the sample mean and the sample variance also make the Normal distribution very important in statistical theory.

The practical applications of the Normal distribution are so widespread that enumeration here would be virtually impossible. It is generated in physical and biological fields by models such as Herschel's hypothesis, Hogen's hypothesis, and Maxwell's hypothesis. The theory of errors, quality control of mass-produced product, repeated measurements in biology, physics, and astronomy, life-length theory, the theory of breaking strengths, and extreme-value theory are only a few of the fields in which the Normal distribution plays a prominent role.

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78. TRUNCATED NORMAL DISTRIBUTION

The Truncated Normal distribution arises in various sampling situations where the population is Normally distributed but the sampling space has been restricted for some reason. For example, in various life-testing problems it is desirable to terminate the sampling procedure after a known time has elapsed for financial considerations, as a total rupture of the object being tested represents a complete financial loss.

Thus the Truncated Normal distribution appears in life-testing problems, breaking-strength problems, and as a failure distribution. It is also used to describe the frequency distribution of many industrial quality characteristics in screened inspection lots. These are the most common uses of the Truncated Normal distribution, although it can be applied to any situation to which a Normal distribution can be applied.

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79. PARETO DISTRIBUTION

The Pareto distribution is a continuous analog of the Zeta distribution (108). Its main application is in economic investigations where it is used to represent distributions of income and other economic indices. The essence of the Pareto law of incomes is that there are more poor than rich people. Thus it often gives a good fit as an income distribution, but the observed number of persons with small incomes is generally lower than the theoretical number. Gumbel [3] has also used the Pareto distribution in the statistics of extremes.

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80. TRUNCATED PARETO DISTRIBUTION

The Truncated Pareto distribution applies only to values of x greater than θ (for example, to incomes above a fixed quantity). Many individual incomes are assumed to follow the Truncated Pareto law. Income distributions emanating from income-tax statistics are generally of this form. If the entire distribution is of the Pareto form (79), then the truncated distribution is used for inferences about the inequality of the entire distribution.

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81. PASCAL DISTRIBUTION

The Pascal distribution is the Negative Binomial distribution (74) when r is a positive integer. It is the probability distribution for the waiting time to the r^{th} success in waiting time problems. It also occurs frequently as a failure rate distribution.

The uses of the Pascal distribution will correspond to the uses of the Negative Binomial distribution whenever the parameter r assumes only positive integral values.

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82. PEARSON FAMILY OF DISTRIBUTIONS

The Pearson family of distributions is a system of generalized probability curves developed by Karl Pearson between 1895 and 1916. Pearson formulated twelve distributions in his family by assigning various values to the parameters and solving the first order differential equation $\frac{df}{dx} = \frac{(x - a)f}{b_o + b_1 + b_2x^2}$. Pearson's distributions, Type I - Type XII, appear under their more common names throughout this paper.

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83. POISSON DISTRIBUTION

The Poisson distribution is one of the basic statistical distributions at the present time. It has extremely widespread applications in accident theory, engineering problems, failure problems, and as an approximation to the Binomial distribution (16).

If accidents are assumed to occur by rare chance in a homogeneous population, then the resulting distribution of accident frequencies will be Poisson. This is an underlying assumption of the Bates-Neyman accident proneness model.

Systems which are governed by the Exponential theory of failure produce a Poisson distribution of number of failures during equal intervals of time if we consider instant replacement of systems which have failed. The Poisson distribution is also used when the data consists of the number of failures in each of a large number of equal time intervals.

As an approximation to the Binomial distribution, the applications of the Poisson distribution range from the number of articles lost in subways to the frequency of comets. The Poisson distribution is the limit of the Binomial as $n \to \infty$, and acts as a good approximation when $\theta \to 0$, n is large and $n\theta$ remains constant.

The Poisson distribution is used to describe the radioactive disintegration of alpha particles, electron emission times, the number of counts recorded on a Geiger counter in a unit interval, the number of defects in glass plates, the frequency of dust particles in the air, daily insurance claims, calls on a telephone line, haemocytometer counts of blood cells per square, and the distribution of stars in the sky.

In physics, the random emission of electrons from the filament of a vacuum tube or from a photosensitive substance under the influence of light are described by Poisson distributions. In the fields of operations research and management science, demands for service and also the rate at which service is rendered obey Poisson laws.

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84. POISSON BINOMIAL DISTRIBUTION

The Poisson Binomial is a contagious distribution and has been studied with particular reference to various ecological data by Katti and Gurland [2] and Shumway and Gurland [3].

Under the assumption that clusters (e.g., of plants) are Poisson distributed while individuals within a cluster follow a Binomial distribution, then the overall distribution will be Poisson Binomial. This distribution rapidly approaches the Neyman Type A distribution (76) as n increases; thus attempts to fit it for $n > 2$ have been infrequent.

McGuire, Brindley and Bancroft [5] have fitted the Poisson Binomial with considerable success to the data on the spread of European corn-borers in field corn. More generally, the Poisson Binomial has been suggested for fitting to a large class of contagious biological data.

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85. COMPOUND POISSON DISTRIBUTION

The Compound Poisson distribution was first considered by Greenwood and Yule [2] in 1920. It contains Neyman's Contagious distribution of Type A (76) and the Polya-Eggenberger distribution (89) as special cases. The principle application of the Compound Poisson distribution is to describe industrial accidents, although it has been applied by Palm [5] to problems of telephone traffic and by Lundberg [4] to sickness statistics. It also appears in the field of risk theory, and is used to describe fire damage data.

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86. TRUNCATED POISSON DISTRIBUTION

The Truncated Poisson distribution would arise when one wishes to fit a Poisson distribution to data consisting of numbers of individuals in certain groups which possess a given attribute, but in which a group cannot be sampled unless at least a specified number of its members have the attribute.

Thus the Truncated Poisson distribution has applications wherever the Poisson distribution has applications, if a bounded sample space is being considered.

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87. POLYA DISTRIBUTION

The Polya family of distributions contains the Hypergeometric (66), the Poisson (83), the Binomial (16), and the discrete Uniform distributions (99) as special cases. It is a major distribution in accident proneness, where it postulates identity of the individual with respect to accident proneness, possible presence of contagion, and possible effect of experience gained since entering the particular occupation.

It is used as a prior distribution in the theory of sampling inspection where it results from a Binomial process model if the probability of defectives for different lots has a Beta prior distribution (4).

The Polya distribution also occurred in early studies of the variation of development of electron-photon cascades. In biological applications, the Polya is a minor variant of the Negative Binomial distribution (74), although it represents the number of individuals per quadrat in a growing population better than the Negative Binomial if we consider immediate rather than continuous release of the original progenitors.

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88. POLYA-AEPPLI DISTRIBUTION

Gurland [2] has shown that the limiting case ($n \rightarrow \infty$) of Beall and Rescia's family of distributions is a Polya-Aeppli distribution, and has fitted it to Student's data [5] on yeast-cell counts.

This distribution arises from a population of randomly dispersed colonies when the number of individuals per colony is a Geometric variate. Then the frequency function of the total observed count per sample is Polya-Aeppli.

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89. POLYA-EGGENBERGER DISTRIBUTION

The Polya-Eggenberger distribution is identical to the Greenwood-Yule distribution. It is a member of the class of Compound Poisson distributions, (85), and is used to explain probability situations involving contagion (see (76)). The major use of this distribution is to describe accident statistics, but it also arises along with Neyman's distributions in telephone traffic, fire damage, sickness, and life-insurance problems.

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90. RAYLEIGH DISTRIBUTION

The Rayleigh distribution is a special case of the Chi distribution (24) with parameters $n = 2$ and $\sigma = \alpha\sqrt{2}$. It was originally obtained to describe the amplitude of sound derived from independent sources.

The Rayleigh distribution also coincides with a case of the Third Asymptotic distribution of smallest values (43), and is used as a failure rate distribution. It is encountered in dealing with the statistics of the envelope of narrowband normal noise processes, and is used under general conditions to describe wave heights. It is also connected with the problem of random walk.

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91. SECH SQUARE DISTRIBUTION

The Sech Square distribution has been shown by Fisk [4] to be asymptotically equivalent to a form of the Pareto distribution (79) at extreme values of the argument. He has also suggested its use for the graduation of income distributions.

Cox [2] has suggested the Sech Square distribution as an alternative model to the Lognormal (72) in studies of the distribution of lifetimes. Fisk [4] discusses the relative merits of the Sech Square and Lognormal distributions.

Bartlett [1] shows that this distribution occurs as the rate of infection in an epidemic model with zero death and immigration rates.

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92. SHORT DISTRIBUTION

The Short distribution is the convolution of a Poisson distribution (83) and a Neyman Type A distribution (76). It was devised by Cresswell and Froggatt [1] to describe accident data.

Froggatt [2] also considers the possible application of the Short distribution to data on industrial absenteeism. However, estimates of the parameters of the Short distribution are subject to very large sampling errors, which severely limits its practical value.

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93. "STER" DISTRIBUTION

The "Ster" distribution is a theoretical distribution function which can be constructed from any given discrete distribution $p(x)$. After the transformation, each of the probabilities are Sums which are successively Truncated from the original Expectation of the Reciprocal of the random variable x (resulting in the name "Ster" distribution). This distribution is also referred to as Bissinger's System, after its founder Barnard H. Bissinger.

The only information known about the distribution function at the present time is that it is monotone decreasing. No practical applications have been found for the transformation, but it is known that it ties together

Bernoulli numbers, sums of reciprocals of the integers, sums of positive powers of the integers, and Binomial coefficients.

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94. STUDENT - t DISTRIBUTION

This distribution was introduced in 1908 by W.S. Gosset [9] to test the significance of the mean of a single sample from a univariate Normal population. It is one of the basic distributions of theoretical statistics and as such its applications are widespread.

The greatest practical importance of Student's t - distribution is its use in a good number of inference problems regarding mean values in Normal populations with unknown variances. It is also important in the study of robustness of tests.

The t - distribution arises in significance tests of the sample correlation coefficient and Spearman's rank correlation coefficient, and in tests of product-moment correlations. In the analysis of variance, significance tests of the regression coefficient also make use of the Student - t distribution. Pitman [8] and Morgan [7] show that the ratio of population standard deviations in a Bivariate Normal population can be tested with the help of the Student - t distribution.

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95. NON-CENTRAL t DISTRIBUTION

If z is a Standard Normal variate and w is independently distributed as Chi-Square /d where d is the degrees of freedom of the Chi-Square, then $t = \frac{z + \Delta}{\sqrt{w}}$ is distributed in the Non-Central t distribution with noncentrality parameter Δ. This reduces to the familiar Student - t distribution (94) for $\Delta = 0$.

The Non-Central t distribution is used to compare percentage points of Normal populations,to determine confidence intervals for proportions from a Normal population, to compute moments of functions of the Non-Central t statistic, and to determine the power of the Student - t test.

It has applications to testing of hypotheses about the coefficient of variation and two-sided proportion defective problems. It also plays an important role in the computation of operating characteristic curves for sampling inspection by variables procedures, and in designing sampling inspection plans. The WAGR sequential test makes use of this distribution to test hypotheses about proportions from Normal populations.

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96. SUB-POISSON DISTRIBUTION

The Sub-Poisson distribution is a special case of the Hyper-Poisson distribution (67) with $0 < \lambda < 1$. Its applications coincide with those of the Hyper-Poisson family of distributions.

An example of the fit of the Sub-Poisson to the distribution of the numbers of alpha particles emitted by a bar of polonium is given by Bardwell and Crow in Patil [2].

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97. TRIANGULAR DISTRIBUTION

The Triangular distribution is not frequently encountered in practice. Its main application is that it is sometimes used to represent the tail of a bounded smooth density function when there is some reason to be primarily interested in events below a bound which excludes most of the density. It also occurs infrequently in the analysis of life data.

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98. U-SHAPED DISTRIBUTION

The U-shaped distribution arises in random sampling from a sinusoidally oscillating random variable, or in systematic sampling with a sampling interval unequal to a rational fraction of the period of oscillation.

The common practical situation for the U-shaped distribution is to describe the thickness of paper at the cutting points in its manufacture. However recurrent oscillation is common in biological studies. Instantaneous pulse pressures, gas concentrations in the lung, or variables related to menstruation and daily or annual climatic variation could all give rise to U-shaped distributions.

The U-shaped distribution is also used with respect to the degree of cloudiness of the sky, and it occurs as the proportion of trials at which one player is in the lead in a simple coin-tossing game. This could suggest its relevance to economic phenomena.

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99. UNIFORM DISTRIBUTION

The Uniform distribution is encountered in applied statistics in two general types of cases. The first of these is the situation where any value in a range of values is equally likely. The second case is that of approximating a relatively small range of values of another continuous distribution. The importance of the Uniform distribution lies in its utility in statistical theory, as any continuous probability distribution can be readily transformed into the simple Uniform distribution. Due to the rectangular shape of the curve, the Uniform distribution is often called the Rectangular distribution.

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100-104. WEIBULL FAMILY OF DISTRIBUTIONS

The Weibull distribution was introduced on a purely empirical basis by Weibull [4] in his analysis of the distribution of the stress amplitude for constant values of the number of cycles at which a specimen will break. Since then, the Weibull family has received considerable attention in describing the behaviour of life-lengths of materials under normal use.

Pope [3] has used Weibull distributions to represent metal fatigue data and they have also been applied to the lifetime of electron tubes

(Mixed Weibull) and the failure distribution of ball bearings. Weibull distributions occur in describing diffusion-controlled phase transformations in metallic alloys and relaxation in stressed springs.

Specific applications of the Weibull distribution include problems concerning the yield strength of Bofors steel, the fibre strength of Indian Cotton, the size distribution of fly ash, the length of Cyrtoideae, and the fatigue life of St-37 steel.

The Weibull family is a specific case of the Third Asymptotic distribution of extreme values (43), and is perhaps the most popular parametric family of failure distributions at the present time.

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105. WOODBURY DISTRIBUTION

The Woodbury distribution arises when the probability of success at any trial depends linearly upon the number of previous successes. It has applications in both biological and economic fields. It is used as an accident distribution and as the distribution of the number of plants in given areas.

The Woodbury distribution is reasonably easy to handle, and has parameters which are capable of easy physical interpretation. The Negative Binomial (74) and Gram-Charlier distributions (23,59) are close approximations to the Woodbury, but their parameters are not so easily interpreted.

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106. YULE DISTRIBUTION

The Yule distribution is a special case of the Negative Binomial distribution (74) with parameters $p = \exp(-\lambda x)$ and $r = i$. It was used by Yule [4] in his mathematical theory of evolution, and by Furry [2] as a model for cosmic-ray showers. In a pure birth process, the Yule distribution describes the probability that the population numbers exactly n members at time x, where the constant λ determines the rate of increase of the population and i is the population size at time $x = 0$.

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107. Z-DISTRIBUTION

The z-distribution may be regarded as the distribution of the logarithm of the ratio of two independent variates each of which is distributed according to the Chi distribution (24). It was introduced along with the F-distribution (45) by R.A. Fisher [2]. The z-distribution is preferable for practical applications as linear interpolation is more accurate in the tables of this distribution than in the F-distribution.

Fisher's z-distribution is used in the analysis of variance for tests on the ratio of within family variations to between family variations. It is also used in testing the correlation ratio, the multiple correlation coefficient, and for tests of concordance in ranking. Certain confidence intervals for a correlation coefficient and for the difference between two correlation coefficients are constructed with the use of the z-distribution.

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108. ZETA DISTRIBUTION

The Zeta distribution is used in linguistic studies to represent the number of occurrences of the same word in a long text from a given author. It has also been found to give a good representation of the distribution of duplicate holdings (the number of policies held by the same person) of insurance policies.

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CHAPTER 3

In this chapter a table of the distributions discussed in Chapter 2 is presented. The following symbols are used frequently in these distributions:

$$\Gamma(y) = \int_0^\infty x^{y-1} \exp(-x)\, dx \quad .$$ This is the Gamma function.

$$B(\alpha,\beta) = \frac{\Gamma(\alpha)\,\Gamma(\beta)}{\Gamma(\alpha+\beta)} \quad .$$ This is the Beta function.

$$B(x|\alpha,\beta) = [B(\alpha,\beta)]^{-1} x^{\alpha-1} (1-x)^{\beta-1} \quad .$$ This is the density function of the Beta distribution.

$$I_t(x) = \sum_{K=0}^{\infty} \frac{1}{K!\,\Gamma(K+t+1)} \left(\frac{x}{2}\right)^{2K+t} \quad .$$ This is the Bessel function of order $t \geq -1$.

$${}_pF_q(a_1, \ldots, a_p ; b_1, \ldots, b_q : x) = \sum_{r=0}^{\infty} \frac{(a_1)_r \cdots (a_p)_r}{(b_1)_r \cdots (b_q)_r} \frac{x^r}{r!}$$

where $(a)_r = a(a+1) \ldots (a+r-1) = \frac{\Gamma(a+r)}{\Gamma(a)}$.

This is the generalized hypergeometric function with $p + q$ parameters. For further information regarding convergence conditions and for the special cases such as ${}_2F_1$, ${}_1F_1$, Bessel functions of the first kind, etc., the reader is referred to ERDÉLYI, A. et al. "Higher Transcendental Functions", McGraw-Hill Company, 1953. The statistical distributions associated with the function ${}_2F_1$ and other special cases are covered in MATHAI, A.M. and SAXENA, R.K. "On A Generalized Hypergeometric Distribution". METRIKA (1966) pp.127-132.

NAME OF DISTRIBUTION	DENSITY FUNCTION	PARAMETERS
1. ARC-SINE DISTRIBUTION	$f(x) = \frac{2}{\pi} \arcsin \sqrt{x}$ for $0 \leq x \leq 1$	
2. BERNOULLI DISTRIBUTION	$f(x) = \alpha^{x}(1 - \alpha)^{1-x}$ for $x = 0,1$	$0 < \alpha < 1$
3. BESSEL DISTRIBUTION	$f(x) = \frac{t}{x} I_t(x) \exp(-x)$ for $x > 0$ where $I_t(x)$ is the Bessel function of order t.	$t > 0$
4. BETA DISTRIBUTION	$f(x) = \frac{\Gamma(\alpha + \beta)}{\Gamma(\alpha)\ \Gamma(\beta)} x^{\alpha-1}(1 - x)^{\beta-1}$ for $0 < x < 1$	$\alpha > 0$ $\beta > 0$
5. BETA-BINOMIAL DISTRIBUTION	$f(x) = \binom{n}{x} \frac{\Gamma(\alpha + \beta)\ \Gamma(x + \alpha)\ \Gamma(n + \beta - x)}{\Gamma(\alpha)\ \Gamma(\beta)\ \Gamma(n + \alpha + \beta)}$ for $x = 0,1, \ldots, n$	$\alpha > 0$ $\beta > 0$ n a positive integer

6. BETA, INCOMPLETE DISTRIBUTION	$f(x) = \frac{1}{B(m,n)} \int_0^x x^{m-1}(1-x)^{n-1}\,dx$ for $0 \leq x \leq 1$	$m > 0$ $n > 0$
7. BETA - 1, INVERTED DISTRIBUTION	$f(x) = \frac{1}{B(r,\, n-r)} \; \frac{(x-b)^{n-r-1}\, b^r}{x^n}$ for $0 \leq b \leq x < \infty$	$n > r > 0$ $b \geq 0$
8. BETA - 2, INVERTED DISTRIBUTION	$f(x) = \frac{1}{B(p,q)} \; \frac{x^{p-1}\, b^q}{(x+b)^{p+q}}$ for $0 \leq x < \infty$	$p > 0$ $q > 0$ $b > 0$
9. BETA, NONCENTRAL DISTRIBUTION	$f(x) = B(x \mid \frac{k}{2}, \frac{m}{2}) \; {}_1F_1(\frac{k+m}{2}, \frac{k}{2}, \frac{\lambda^2 x}{2}) \exp(-\lambda^2/2)$ for $0 \leq x \leq 1$ where ${}_1F_1$ is the hypergeometric function of the first kind.	$\lambda > 0$ $k > 0$ $m > 0$
10. BETA - PASCAL DISTRIBUTION	$f(x) = \frac{(r+s-1)!\,(x+n-r-s-1)!\,(x-1)!\,(n-1)!}{(r-1)!\,(s-1)!\,(x-r)!\,(n-s-1)!\,(x+n-1)!}$ for $x = 0,1,2,\ldots$ $x \geq r$	$s > 0$ $n > 0$ $r = 0$ or a positive integer

11. BETA, TYPE I DISTRIBUTION	$f(x) = cx^{s(2m+s+1)/2\ -1}\ (1 - \frac{x}{s})^{s(2n+s+1)/2\ -1}$ for $0 < x < s$ where $c = \frac{1}{s^{s(2m+s+1)/2}}\ B\{s(2m + s + 1)/2,\ s(2n+s+1)/2\}$	$s > 0$ m, n positive integers
12. BETA, TYPE II DISTRIBUTION	$f(x) = \frac{kx^{s(2m+s+1)/2\ -1}}{(1 + \frac{x}{s})^{s(2m+2n+s+1)/2\ +1}}$ for $0 < x < \infty$ where $k = \frac{1}{s^{s(2m+s+1)/2}}\ B\{s(2m + s + 1)/2,\ sn + 1\}$	$s > 0$ m, n positive integers
13. BETA TYPE I, NONCENTRAL DISTRIBUTION	$f(x) = \sum_{j=0}^{\infty} p(j, \frac{1}{2}\Delta^2)\ g(x, m + 2j, n)$ for $0 \leq x \leq 1$, where $g(x, m + 2j, n)$ is Beta with $m + 2j$ and n degrees of freedom and $p(j, \frac{1}{2}\Delta^2) = \frac{(\frac{1}{2}\Delta^2)^j \exp(-\frac{1}{2}\Delta^2)}{j!}$	$\Delta > 0$ m, n positive integers

14. BETA TYPE II, NONCENTRAL DISTRIBUTION	$f(x) = \sum_{j=0}^{\infty} y(\frac{1}{2}p, j, \theta)\ f(x, m + 2j, n)$ for $0 \leq x \leq 1$, where $y(\frac{1}{2}p, j, \theta) = \frac{\Gamma(\frac{p}{2} + j)}{j!\ \Gamma(\frac{p}{2})} \theta^{p/2}(1 - \theta)^j$ and $f(x, m + 2j, n)$ is Beta with $m + 2j$ and n degrees of freedom.	$0 < \theta < 1$ p, m, n positive integers
15. BILATERAL EXPONENTIAL DISTRIBUTION	$f(x) = \frac{\alpha}{2} \exp(-\alpha\lvert x\rvert)$ for $-\infty < x < \infty$	$\alpha > 0$
16. BINOMIAL DISTRIBUTION	$f(x) = \binom{n}{x} \theta^x(1 - \theta)^{n-x}$ for $x = 0,1,2, \ldots, n$	$0 < \theta < 1$ n positive integer
17. BINOMIAL, COMPOUND DISTRIBUTION	$f(x) = \frac{\binom{n}{x} B(\alpha + x, n + \beta - x)}{B(\alpha,\beta)}$ for $x = 0,1,2, \ldots, n$	$\alpha > 0$ $\beta > 0$ n positive integer

18. BINOMIAL, MIXED DISTRIBUTION	$f_N(x) = \sum_{i=1}^{m} W_i \binom{N}{x} p_i^x q_i^{N-x}$ for x = 0,1,2, . . . , N where $\Sigma W_i = 1$	m, N positive integers $0 < p_i < 1$ for all i $q_i = 1 - p_i$
19. BINOMIAL, TRUNCATED DISTRIBUTION	$f(x) = \dfrac{\binom{n}{x} p^x (1-p)^{n-x}}{1-(1-p)^n}$ for x = 1,2, . . . , n (truncated below x = 1)	$0 < p < 1$ n positive integer
20. BOREL DISTRIBUTION	$f(x) = \dfrac{(x\beta q)^{x-1} \exp(x\beta q)}{x!}$ for x = 1, 2, . . .	$\beta > 0$ $q > 0$
21. BOREL - TANNER DISTRIBUTION	$f(x) = \dfrac{r}{(x-r)!} x^{x-r-1} \alpha^{x-r} \exp(-\alpha x)$ for x = r, r + 1, . . .	$\alpha > 0$ r positive integer

22. CAUCHY DISTRIBUTION	$f(x) = \dfrac{\Delta}{\pi[\Delta^2 + (x - \mu)^2]}$ for $-\infty < x < \infty$	$\Delta > 0$ $-\infty < \mu < \infty$
23. CHARLIER TYPE B DISTRIBUTION	$f(x) = \sum_{j=0}^{\infty} a_j D^j p(x\|\theta)$ for $x = 0,1,2,...$ where D^j represents the operation of differentiating j times with respect to θ, and $p(x\|\theta) = \dfrac{\theta^x}{x!} \exp(-\theta)$ and $a_j = \dfrac{\theta^j}{j!} \sum_{x=0}^{\infty} p(x) D^j p(x\|\theta)$, where $p(x)$ is the actual distribution of x.	$\theta > 0$
24. CHI DISTRIBUTION	$f(x) = \dfrac{2(\frac{n}{2})^{\frac{n}{2}} x^{n-1} \exp(-nx^2/2\sigma^2)}{\sigma^n \Gamma(\frac{n}{2})}$ for $x > 0$	$\sigma > 0$ n positive integer

25. CHI-SQUARE DISTRIBUTION	$f(x) = \frac{1}{2^{n/2}\,\Gamma(\frac{n}{2})} \; x^{(n-2)/2} \exp(-x/2)$ for $x > 0$	n positive integer
26. CHI-SQUARE, NONCENTRAL DISTRIBUTION	$f(x) = \frac{1}{2}(\frac{x}{c})^{(n-1)/2} \exp(-\frac{c}{2} - \frac{x}{2})$ $\sum_{j=0}^{\infty} \frac{(\frac{1}{2})^{(n-1)/2+j} \; (\frac{cx}{2})^{(n-1)/2+j}}{j!\,\Gamma(n+j)}$ for $x > 0$	n positive integer $c > 0$
27. CIRCULAR NORMAL DISTRIBUTION	$f(\theta) = \frac{\exp[k\cos(\theta - \theta_o)]}{2\pi\, I_o(k)}$ for $0 \leq \theta \leq 2\pi$ where $I_o(k)$ is the Bessel function of the first kind of pure imaginary argument.	

28. CONFLUENT HYPERGEOMETRIC DISTRIBUTION	$f(x) = \frac{\Gamma(\nu + x)\ \Gamma(\lambda)}{\Gamma(\lambda + x)\ \Gamma(\nu)} \quad \frac{\mu^x}{{}_1F_1(\nu;\ \lambda;\ \mu)}$ for $x = 0,1,2, \ldots$ where ${}_1F_1$ is the hypergeometric function of the first kind.	$\nu > 0$ $\lambda > 0$ $\mu > 0$
29. DISTRIBUTION OF THE NUMBER OF EXCEEDANCES	$f(x) = \frac{\binom{n}{m}\ m\ \binom{N}{x}}{(N + m)\binom{N + n - 1}{x + m - 1}}$ for $x = 0,1, \ldots, N$	m, n, N positive integers
30. DODGE-ROMIG DISTRIBUTION	$f_N(x) = W_1\binom{N}{x}\ p_1^x\ q_1^{N-x} + W_2\ \varphi(x)$ for $x = 0,1,2, \ldots$ where W_1 and $\varphi(x)$ are unspecified.	$W_2 = 1 - W_1$ $0 < p_1 < 1$ $q_1 = 1 - p_1$
31. DOUBLE EXPONENTIAL DISTRIBUTION	$f(x) = ae^{b\lvert x-c \rvert}$ for $-\infty \leq x \leq \infty$	$a,c > 0$ $b < 0$

32. DOUBLE PARETO DISTRIBUTION	$f(x) = \frac{A}{x^{1+\alpha}} + \frac{B}{x^{1+\beta}}$ for $0 \leq x \leq \infty$	$\alpha > 0$ $\beta > 0$ A,B constants to be determined
33. ENGSET DISTRIBUTION	$f(x) = \binom{N}{x} p^x [1 + Np + . . . + \binom{N}{c} p^c]^{-1}$ for $x = 0,1,2, . . .$	N,c positive integers $0 < p < 1$
34. ERLANG DISTRIBUTION	$f(x) = \frac{x^{p-1}}{a^p \Gamma(p)} \exp(-x/a)$ for $x \geq 0$	$a > 0$ p positive integer
35. ERROR FUNCTION DISTRIBUTION	$f(x) = \frac{h}{\sqrt{\pi}} \exp(-h^2 x^2)$ for $-\infty < x < \infty$	$0 < h < \infty$
36. EULERIAN DISTRIBUTION	$f(x) = \frac{1}{p!} x^p \exp(-x)$ for $x > 0$	$p = 0$ or a positive integer

37. EXPONENTIAL DISTRIBUTION	$f(x) = \frac{1}{\theta} \exp(-x/\theta)$ for $x > 0$	$0 < \theta < \infty$
38. EXPONENTIAL, GENERALISED DISTRIBUTION	$f(x) = \frac{\alpha^a(\alpha + 1)^{p-1}}{\Gamma(p)} \; x^{p-1} \exp[-(a + 1)x] \quad {}_1F_1(a;\ p;\ y)$ for $0 < x < \infty$	$a > 0$ $p > 0$ $\alpha > 0$
39. EXPONENTIAL, TRUNCATED DISTRIBUTION	$f(x) = \frac{\exp(-x)}{\exp(-\theta) - \exp(-b)}$ for $\theta < x < b$	$\theta > 0$ $b > 0$
40. EXPONENTIAL, TWO-PARAMETER DISTRIBUTION	$f(x) = \frac{1}{\beta} \exp[-(\frac{x - \lambda}{\beta})]$ for $\lambda \leq x < \infty$	$0 < \beta < \infty$ $-\infty < \lambda < \infty$
41. EXTREME VALUE: FIRST ASYMPTOTIC DISTRIBUTION	$f(x) = \frac{1}{\beta} \exp[- y - \exp(-y)]$ for $-\infty < x < \infty$ where $y = \frac{x - \lambda}{\beta}$	$0 < \beta < \infty$ $-\infty < \lambda < \infty$

42. EXTREME VALUE: SECOND ASYMPTOTIC DISTRIBUTION	$f(x) = \frac{k}{\nu}\left(\frac{\nu}{x}\right)^{k+1} \exp\left[-\left(\frac{\nu}{x}\right)^{k}\right]$ for $x > 0$	$\nu, k > 0$
43. EXTREME VALUE: THIRD ASYMPTOTIC DISTRIBUTION	$f(x) = \frac{k}{-\nu}\left(\frac{x}{\nu}\right)^{k-1} \exp\left[-\left(\frac{x}{\nu}\right)^{k}\right]$ for $x < 0$	$\nu < 0$ $k > 1$
44. EXTREME VALUE, MODIFIED DISTRIBUTION	$f(x) = \frac{1}{\lambda} \exp\left[-\left(\frac{\exp(x) - 1}{\lambda}\right) + x\right]$ for $x \geq 0$	$\lambda > 0$
45. F - DISTRIBUTION	$f(x) = \frac{\Gamma[(\nu_1 + \nu_2)/2]}{\Gamma(\nu_1/2)\,\Gamma(\nu_2/2)} \; \frac{x^{(\nu_1/2) - 1}}{(1 + x\nu_1/\nu_2)^{(\nu_1+\nu_2)/2}} \; (\nu_1/\nu_2)^{\nu_1/2}$ for $x > 0$	ν_1, ν_2 positive integers

46. F, NONCENTRAL DISTRIBUTION	$f(x) = \exp(-\lambda^2/2) \dfrac{\Gamma(\frac{k+m}{2})}{\Gamma(\frac{k}{2})\,\Gamma(\frac{m}{2})} \dfrac{x^{(k/2)\ -1}}{(1+x)^{(k+m)/2}}\ {}_1F_1(\frac{k+m}{2}, \frac{k}{2}, \frac{\lambda^2}{2}\ \frac{x}{1+x})$ for $x > 0$	$\lambda > 0$ k,m positive integers
47. FOLDED NORMAL DISTRIBUTION	$f(x) = \dfrac{1}{\sqrt{2\pi}\ \sigma} \{\exp[-(x - \mu)^2/2\sigma^2] + \exp[-(x + \mu)^2/2\sigma^2]\}$ for $x \geq 0$	$\sigma \geq 0$ $-\infty < \mu < \infty$
48. GALTON'S LIMITED DISTRIBUTION	$w(x)\ dx = \text{const.}\ \exp[-\frac{1}{2}\{k(b - \log(a-x))\}^2]\ d[k(b - \log(a-x))]$ for $0 < x < \infty$	$-\infty < b < \infty$ $a > 0$
49. GAMMA DISTRIBUTION	$f(x) = \dfrac{1}{\beta^\alpha\ \Gamma(\alpha)}\ x^{\alpha-1} \exp(-x/\beta)$ for $x > 0$	$\alpha > 0$ $\beta > 0$

50. GAMMA - 1 DISTRIBUTION	$f(x) = \frac{y(yx)^{r-1} \exp(-yx)}{(r-1)!}$ for $x \geq 0$	$r > 0$ $y > 0$
51. GAMMA - 2 DISTRIBUTION	$f(x) = \frac{vy(vyx/2)^{v/2-1} \exp(-vyx/2)}{2(v/2-1)!}$ for $x \geq 0$	$v > 0$ $y > 0$
52. GAMMA - 1, INVERTED DISTRIBUTION	$f(x) = \frac{(y/x)^{r+1} \exp(-y/x)}{y(r-1)!}$ for $x \geq 0$	$r > 0$ $y > 0$
53. GAMMA - 2, INVERTED DISTRIBUTION	$f(x) = \frac{2(vs^2/2x^2)^{(v+1)/2} \exp(-vs^2/2x^2)}{(v/2-1)!\,(vs^2/2)^{1/2}}$ for $x \geq 0$	$s > 0$ $v > 0$

54. GAMMA, GENERALIZED DISTRIBUTION	$f(x) = \frac{p}{a^d\,\Gamma(\frac{d}{p})}\, x^{d-1} \exp[-(x/a)^p]$ for $0 \leq x < \infty$	$a > 0$ $d > 0$ $p > 0$
55. GAMMA, INCOMPLETE DISTRIBUTION	$f(x) = \frac{1}{\Gamma(n)} \int_o^x x^{n-1} \exp(-x)\ dx$ for $0 < x < \infty$	$n > 0$
56. GENERALIZED POWER SERIES DISTRIBUTION	$f(x) = \frac{a(x)\theta^x}{\sum_{x\epsilon T} a(x)\theta^x}$ for $x\epsilon T$, where $a(x)$ is a function of x with $a(x) > 0$	$\theta > 0$
57. GEOMETRIC DISTRIBUTION	$f(x) = \theta(1 - \theta)^{x-1}$ for $x = 1,2,3, \ldots$	$0 < \theta < 1$

58. GEOMETRIC, COMPOUND DISTRIBUTION	$f(x) = \frac{1}{B(\alpha,\beta)} \int_0^1 p^{\alpha+x-2} (1-p)^{\beta} dp$ for $x = 0,1,2,\ldots$	$\alpha > 0$ $\beta > 0$ $0 < p < 1$
59. GRAM-CHARLIER TYPE A SERIES DISTRIBUTION	$f(x) = \sum_{j=0}^{\infty} C_j H_j(x) \phi(x)$ for $-\infty < x < \infty$, where $\phi(x)$ is the probability density function of the standardized normal, and $H_j(x)$ is the Tchebycheff-Hermite polynomial of degree j in x defined by $(-1)^j \frac{d^j}{dx^j} \phi(x) = H_j(x) \phi(x)$, and $C_j = \frac{1}{j!} \int_{-\infty}^{\infty} H_j(x) f(x) dx$	
60. HALF - CAUCHY DISTRIBUTION	$f(x) = \frac{2}{\pi} (1 + x^2)^{-1}$ for $0 < x < \infty$	

61. HALF-NORMAL DISTRIBUTION	$f(x) = \frac{1}{\sigma} \frac{\sqrt{2}}{\sqrt{\pi}} \exp(-x^2/2\sigma^2)$ for $0 \leq x \leq \infty$	$0 < \sigma < \infty$
62. HELMERT DISTRIBUTION	$dF = \frac{n^{(n-1)/2}}{2^{(n-3)/2}\ \Gamma[(n-1)/2]} (s/\sigma)^{n-2} \exp(-ns^2/2\sigma^2) \frac{ds}{\sigma}$ for $0 \leq s \leq \infty$	$0 < \sigma < \infty$ n positive integer
63. HERMITE DISTRIBUTION	$p(k) = \exp(-\lambda) \sum_{t=0}^{\infty} \frac{\lambda^t}{t!} \binom{2t}{k} p^k q^{2t-k}$ for $k = 0,1,2, \ldots$	$\lambda > 0$ $0 < p < 1$ $q = 1 - p$
64. HYPERBOLIC COSINE DISTRIBUTION	$f(x) = \frac{1}{\pi \cosh x}$ for $-\infty < x < \infty$	

65. HYPERBOLIC, TRUNCATED DISTRIBUTION	$f(x) = \frac{1}{px}$ for $\exp(-p) \leq x \leq 1$	$p > 0$
66. HYPERGEOMETRIC DISTRIBUTION	$f(x) = \frac{\binom{a}{x}\binom{b}{n-x}}{\binom{a+b}{n}}$ for $x = 0,1,2, \ldots, n$ or a	n,a,b positive integers
67. HYPER-POISSON DISTRIBUTION	$f(x) = \frac{\Gamma(\lambda)\ \theta^{x}}{{}_1F_1[1;\ \lambda;\ \theta]\ \Gamma(\lambda + x)}$ for $x = 0,1,2, \ldots$	$\theta > 0$ $\lambda > 0$
68. J-SHAPED DISTRIBUTION FAMILY	$f(x) = \frac{2ar}{b^{2r}} (b - x)(2bx - x^2)^{r-1} + \frac{1 - a}{b}$ for $0 < x < b$	$0 < r < 1$ $b > 0$ $0 < a \leq 1$

69. LOGARITHMIC SERIES DISTRIBUTION	$f(x) = \frac{\theta^x}{-x \log(1 - \theta)}$ for $x = 1,2, \ldots$	$0 < \theta < 1$
70. LOGARITHMIC SERIES, TRUNCATED DISTRIBUTION	$f(x) = \frac{\beta\theta^x}{x}$ for $x = 1,2, \ldots, d$ where $\beta = \sum_{x=1}^{d} \frac{\theta^x}{x}$	$0 < \theta < 1$ d positive integer
71. LOGISTIC DISTRIBUTION	$f(x) = \frac{\exp[-(x - \alpha)/\beta]}{\beta\{1 + \exp[-(x - \alpha)/\beta]\}^2}$ for $-\infty < x < \infty$	$\beta > 0$ $-\infty < \alpha < \infty$
72. LOGNORMAL DISTRIBUTION	$f(x) = \frac{1}{\sqrt{2\pi}\ \sigma x} \exp[-(\ln x - \mu)^2/2\sigma^2]$ for $x > 0$	$\sigma > 0$ $-\infty < \mu < \infty$

73. MAXWELL-BOLTZMANN DISTRIBUTION	$f(x) = \frac{4}{\sqrt{\pi}} \beta^{3/2} x^2 \exp(-\beta x^2)$ for $x > 0$	$\beta > 0$
74. NEGATIVE BINOMIAL DISTRIBUTION	$f(x) = \frac{\Gamma(r + x)}{x!\,\Gamma(r)} p^r (1 - p)^x$ for $x = 0,1,2, \ldots$	$0 < p < 1$ $r > 0$
75. NEGATIVE BINOMIAL, GENERALIZED DISTRIBUTION	$f(x) = \frac{\alpha^a (\alpha + 1)^{p-a}}{\Gamma(p)\; x!} \frac{\Gamma(p + x)}{(\alpha + 2)^{p+x}} {}_2F_1[a, p + x; p; \frac{1}{\alpha + 2}]$ for $x = 0,1,2, \ldots$	$a > 0$ $p > 0$ $\alpha > 0$
76. NEYMAN TYPE A DISTRIBUTION	$P_n = \frac{c^n}{n!} \exp(-\lambda) \sum_{k=0}^{\infty} \frac{k^n [\lambda \exp(-c)]^k}{k!}$ for $n = 0,1,2, \ldots$	λ, k positive constants

77. NORMAL DISTRIBUTION	$f(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp[-\frac{1}{2}(\frac{x-\mu}{\sigma})^2]$ for $-\infty < x < \infty$	$0 < \sigma < \infty$ $-\infty < \mu < \infty$
78. NORMAL, TRUNCATED DISTRIBUTION	$f(x) = \frac{C}{\sqrt{2\pi}\,\sigma} \exp[-\frac{1}{2}(\frac{x-\mu}{\sigma})^2]$ for $\lvert x - \mu \rvert < a\sigma$ where $\frac{1}{C} = \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} \exp(-t^2/2)\, dt$	$0 < \sigma < \infty$ $-\infty < \mu < \infty$ a is the truncation point
79. PARETO DISTRIBUTION	$f(x) = \frac{\alpha}{x_o} (x_o/x)^{\alpha+1}$ for $x > x_o$	$\alpha > 0$ x_o constant
80. PARETO, TRUNCATED DISTRIBUTION	$f_\theta(x) = \frac{\frac{1}{\theta}(\frac{\theta}{x})^2}{1 - \frac{\theta}{b}}$ for $0 < \theta < x < b$	$0 < \theta < b$

81. PASCAL DISTRIBUTION	$f(x) = \binom{x + r - 1}{x} q^x p^r$ for $x = 0,1,2, . . .$	$0 < p < 1$ $q = 1 - p$ r positive integer
82. PEARSON FAMILY OF DISTRIBUTIONS	$\frac{df}{dx} = \frac{(x - a)f}{b_o + b_1x + b_2x^2}$ where f is the frequency function. The explicit solutions are classified into types (I - XII) according to the nature of the roots of the equation $b_o + b_1x + b_2x^2 = 0$.	
83. POISSON DISTRIBUTION	$f(x) = \frac{\lambda^x \exp(-\lambda)}{x!}$ for $x = 0,1,2, . . .$	$\lambda > 0$
84. POISSON BINOMIAL DISTRIBUTION	$f(x) = \exp(-a) \sum_{t=0}^{\infty} \frac{a^t}{t!} \binom{nt}{x} p^x(1 - p)^{nt-x}$ for $x = 0,1,2, . . .$	$a > 0$ $0 < p < 1$ n,t positive integers

85. POISSON, COMPOUND DISTRIBUTION	$f(x) = \frac{\exp(-\lambda)\ c^x}{x!} \sum_{k=0}^{\infty} [\lambda \exp(-c)]^k$ for $x = 0,1,2, \ldots$	$\lambda > 0$ $c > 0$
86. POISSON, TRUNCATED DISTRIBUTION	$f(x) = \frac{\mu^x}{[\exp(\mu) - 1]x!}$ for $x = 1,2, \ldots$ (truncated below $x = 1$)	$\mu > 0$
87. POLYA DISTRIBUTION	$f_N(x) = \binom{N}{x} \frac{\Gamma(\alpha + \beta)\ \Gamma(\alpha + x)\ \Gamma(\beta + N - x)}{\Gamma(\alpha)\ \Gamma(\beta)\ \Gamma(\alpha + \beta + N)}$ for $x = 0,1,2, \ldots$	N,α,β positive integers
88. POLYA - AEPPLI DISTRIBUTION	$P_r = [4m/(a + 2)^2]^r \exp[-2m/(a + 2)] \sum_{s=0}^{r-1} \frac{\binom{r-1}{s}}{(r-s)!} b^s$ for $r \geq 1$ (an integer) where $b = a(a + 2)/4m$	m,a positive constants

89. POLYA - EGGENBERGER DISTRIBUTION	$P_n = \frac{1}{n!} \frac{\Gamma(n + \frac{h}{d})}{\Gamma(\frac{h}{d})} (1 + d)^{\frac{-d}{h}} (\frac{1}{1 + d})^n$ for n = 0,1,2, . . .	$d > 0$ $h > 0$
90. RAYLEIGH DISTRIBUTION	$f(x) = \frac{1}{\alpha^2} x \exp[- \frac{1}{2}(x/\alpha)^2]$ for $x > 0$	$\alpha > 0$
91. SECH SQUARE DISTRIBUTION	$dF(x) = \frac{\beta \exp(\alpha + \beta x)}{[1 + \exp(\alpha + \beta x)]^2} dx$ for $- \infty < x < \infty$	$\beta > 0$ $- \infty < \alpha < \infty$
92. SHORT DISTRIBUTION	$f(x) = \frac{\exp[-(\lambda + \theta)]}{x!} \sum_{k=0}^{\infty} (k\theta + \phi)^x \frac{[\lambda \exp(-\theta)]^k}{k!}$, for x = 0,1,2, . . . where ϕ is the parameter of a Poisson distribution.	$\lambda > 0$ $\theta > 0$

93. STER DISTRIBUTION	$f(t) = \frac{1}{1 - p(0)} \sum_{t+1}^{\infty} \frac{p(x)}{x}$ for $t = 0,1,2, \ldots$, where $p(x)$ is any discrete distribution over the range $x = 0,1,2, \ldots$	
94. STUDENT - t DISTRIBUTION	$f(x) = \frac{\Gamma[(v + 1)/2]}{\sqrt{v\pi}\ \Gamma(\frac{v}{2})} (x^2/v + 1)^{-(v+1)/2}$ for $-\infty < x < \infty$	v positive integer
95. STUDENT - t, NONCENTRAL DISTRIBUTION	$f(x) = \frac{[n/(n + x^2)]^{(n+1)/2}}{2^{(n-1)/2} \sqrt{\pi n}\ \Gamma(\frac{n}{2})} \exp[-n\Delta^2/2(n + x^2)]$ $\int_0^\infty v^n \exp[-\frac{1}{2}(v - \frac{\Delta x}{(n + x^2)^{1/2}})^2]\, dv$ for $-\infty < x < \infty$	$\Delta > 0$ n positive integer

96. SUB-POISSON DISTRIBUTION	$f(x) = \frac{\Gamma(\lambda)}{\Gamma(\lambda + x) \;\; {}_1F_1(1;\ \lambda;\ \mu)} \mu^x$ for $x = 0,1,2, . . .$	$0 < \lambda < 1$ $\mu > 0$
97. TRIANGULAR DISTRIBUTION	$f(x) = \frac{2(b - x)}{b^2}$ for $0 \leq x \leq b$	$0 < b < \infty$
98. U - SHAPED DISTRIBUTION	$f(x)\ dx = y_0(1 - \frac{x^2}{a^2})^m$ for $- a \leq x \leq a$ where the origin of x is at mean = mode	$- 1 < m < 0$
99. UNIFORM DISTRIBUTION	$f(x) = \frac{1}{\beta - \alpha}$ for $\alpha < x < \beta$	α, β such that $\beta - \alpha > 0$

100. WEIBULL DISTRIBUTION	$f(x) = \frac{m}{\alpha} x^{m-1} \exp(-x^m/\alpha)$ for $x \geq 0$	$\alpha > 0$ $m > 0$
101. WEIBULL, COMPOSITE (r - component) DISTRIBUTION	$f(x) = f_j(x)$ for $\Delta_j \leq x \leq \Delta_{j+1}$, $j = 0,1, . . . , r$ where $f_j(x) = \frac{\beta_j(x - \gamma_j)^{\beta_j - 1}}{\alpha_j} \exp[-(x - \gamma_j)\beta_j/\alpha_j]$	Δ_j's are partition parameters with $\Delta_o = \gamma_1$ and $\Delta_{r+1} = \infty$ Each $\alpha_j, \beta_j > 0$
102. WEIBULL, MIXED DISTRIBUTION	$f(x) = \sum_{i=1}^{k} p_i f_i(x)$ where $f_i(x) = \frac{\beta_i(x - \gamma_i)^{\beta_i - 1}}{\alpha_i} \exp\{-(x - \gamma_i)^{\beta_i}/\alpha_i\}$ for $\gamma_i < x < \infty$	$0 < p_i < 1$ $\sum_{i=1}^{k} p_i = 1$ $\alpha_i, \beta_i > 0 \quad \forall_i$

103. WEIBULL, SIMPLE DISTRIBUTION	$f(x) = \frac{\beta(x - \gamma)^{\beta-1}}{\alpha} \exp[-(x - \gamma)^{\beta}/\alpha]$ for $x \geq \gamma$	$\alpha > 0$ $\beta > 0$
104. WEIBULL, THREE PARAMETER DISTRIBUTION	$f(x) = m(x - G)^{m-1}\ \theta^{-1} \exp[-\theta^{-1}(x - G)^{m}]$ for $x > G$	$\theta > 0$ $m > 0$ G any real number
105. WOODBURY DISTRIBUTION	$P(n,x) = \frac{1}{x!}\left[(\frac{p}{c})(\frac{p}{c} + 1)(\frac{p}{c} + 2) \ . \ . \ . \ (\frac{p}{c} + x - 1)\right]$ $\sum_{r=0}^{x} (-1)^{r} \binom{x}{r} (q - cr)^{n}$ for $x = 0,1,2, \ . \ . \ .$	$0 < p < 1$ $-\infty < c < \infty$ $q = 1 - p$ r,n positive integers

106. YULE DISTRIBUTION	$f(x) = \binom{n-1}{n-i} \exp(-i\lambda x)[1 - \exp(-\lambda x)]^{n-i}$ for $x \geq 0$	n,i positive integers $n \geq i \geq 1$ $\lambda \geq 0$
107. Z - DISTRIBUTION	$f(z)dz = 2c \exp(n_1 z)[1 + \frac{n_1}{n_2} \exp(2z)]^{-(n_1+n_2)/2} dz$ for $-\infty < z < \infty$	n_1, n_2 positive integers
108. ZETA DISTRIBUTION	$f(x) = [\zeta(k)\, x^k]^{-1}$ for x = 1,2, . . . where $\zeta(k) = \sum_{x=1}^{\infty} x^{-k}$	$k > 0$