# Group actions on median spaces

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## Contents

**Abstract**

We investigate a particular type of geometry, namely median metric spaces. This encompasses median graphs, as well as simple combinatorial structures known as spaces with walls.

The group-theoretic applications are towards the Kazhdan property and the Haagerup property.

**Résumé**

On s'intéresse à un type particulier de géometrie, les espaces métriques medians. Ceci comprend les graphes medians, ainsi que des structures combinatoires connues sous le nom d'espaces à murs.

Du point de vue de la théorie des groupes, les principales applications sont envers la propriété de Kazhdan et la propriété de Haagerup.

**Acknowledgements**

I thank Dani Wise for being an inspiring supervisor.

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**Some notations**

| | |
|---|---|
| $A \triangle B$ | symmetric difference $(A \setminus B) \cup (B \setminus A)$ of two sets $A$, $B$ |
| $A^c$ | complement of a set $A$, when there is no confusion about the ambient space |
| $\chi_A$ | characteristic function of a set $A$ |
| $\ell_2(I)$ | Hilbert space of real-valued, square-summable functions on $I$ |

# 1 Kazhdan vs. Haagerup

The two properties of groups we focus on are Kazhdan's property T and its nemesis, alternatively known as a-T-menability by group theorists and Haagerup's property by analysts. The former term, due to M. Gromov, has a mnemonic quality to it, as it suggests the negation of property T and the relation to amenability. We prefer the latter term.

We define these properties in terms of isometric actions on (real) Hilbert spaces. We often use "Kazhdan group" as a shorthand for "group with the Kazhdan property", and "Haagerup group" as a shorthand for "group with the Haagerup property". Two related group properties, amenability and property FA, are also briefly reviewed. In the later sections, we will revisit and explain certain examples from this section.

All our groups are discrete and countable. For all practical purposes, the reader may assume that the groups are finitely-generated.

## 1.1 Amenable groups

Among the myriad ways of defining amenability, two are especially significant.

**Definition 1.1.** A group $G$ is *amenable* if there is a $G$-invariant, finitely-additive measure $\mu$ on $\mathcal{P}(G)$ with $\mu(G) = 1$.

**Theorem 1.2 (Følner's Criterion).** *A group $G$ is amenable if and only if there is a sequence $(F_n)_{n\geq 1}$ of finite subsets of $G$ such that for every $g \in G$ we have*

$$\frac{|gF_n \triangle F_n|}{|F_n|} \longrightarrow 0$$

Finite groups and abelian groups are primary examples of amenable groups. On the other hand, free nonabelian groups are not amenable.

**Proposition 1.3.** *The family of amenable groups is closed under taking subgroups, quotients, extensions and ascending countable unions.*

For the purposes of this section only, by a *class* we mean a family of groups having the closure properties listed in the previous proposition. Amenable groups form a class, but there are two more classes we would like to mention.

The smallest class containing the finite groups and the abelian groups is termed the class of *elementary groups*. Elementary groups are amenable, but not every amenable group is elementary.

Amenable groups do not have $\mathrm{F}_2$ as a subgroup. The family of groups without $\mathrm{F}_2$ as a subgroup forms a class. There are groups without $\mathrm{F}_2$ as a subgroup that are not amenable, e.g., Burnside groups of large exponent. We may summarize these ideas in the following chain of strict inclusions:

$$\mathrm{EG} \subset \mathrm{AG} \subset \mathrm{NF}$$

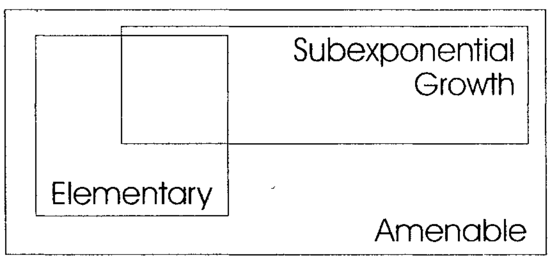
**Example 1.4 (Subexponential growth).** The growth of a finitely-generated group is the rate at which new elements appear as one takes larger and larger balls around the origin in a Cayley graph. There are three ranges of growth:

$$\underbrace{1 \prec n \prec n^2 \prec n^3 \prec \ldots}_{\text{polynomial}} \quad \underbrace{\ldots \prec e^{\sqrt{n}} \prec \ldots}_{\text{intermediate}} \quad \underbrace{\prec e^n}_{\text{exponential}}$$

The following important result is a nice application of Følner's criterion:

**Theorem 1.5.** *Groups of subexponential growth are amenable.*

An outstanding theorem of M. Gromov equates polynomial growth with virtual nilpotency. A result of C. Chou says that elementary groups have either polynomial or exponential growth. Thus elementary groups that are not virtually nilpotent are examples of amenable groups with exponential growth. As free nonabelian groups have exponential growth, we see that both amenable and non-amenable groups can have exponential growth.




Groups of intermediate growth are amenable but not elementary. Currently known groups of intermediate growth, from R. Grigorchuk's first examples to more recent variations, are groups acting on regular trees $\mathcal{T}_n$. Grigorchuk's groups have another interesting feature: they are infinite torsion groups.

**Notes.** Amenable groups were introduced by von Neumann in connection to the Banach-Tarski paradox; see S. Wagon's beautiful book *The Banach-Tarski paradox*.

## 1.2 The Kazhdan property

The Kazhdan property, often called property T, is a form of rigidity.

An isometric action of a group $G$ on a metric space $(X, d)$ is said to be *bounded* if the orbit of some (every) point in $X$ is bounded. For metric spaces in which bounded sets have unique circumcenters, the boundedness of an isometric action is equivalent to the existence of a fixed point.

**Definition 1.6 (Kazhdan group).** A group $G$ has the *Kazhdan property* if every isometric action of $G$ on a Hilbert space has a fixed point.

**Example 1.7 (Amenable groups).** Infinite amenable groups are not Kazhdan. This is immediate if one uses alternate definitions of Kazhdan's property and amenability. Instead, we explicitly describe an action on a Hilbert space that is not bounded.

Let $G = \{g_1, g_2, \ldots\}$ be an infinite amenable group. We start by gaining asymptotic control over the Følner sequence $(F_n)_{n\geq 1}$: by passing to a subsequence if necessary, we may assume that for all $1 \leq i \leq n$ we have:

$$\frac{|g_i F_n \triangle F_n|}{|F_n|} < \frac{1}{n^3}$$

For each $n$, modify the usual linear isometric action of $G$ on $\ell_2(G)$ into an affine one

$$g *_n \phi = g\phi + \sqrt{\frac{n}{|F_n|}}(\chi_{gF_n} - \chi_{F_n})$$

and wrap all these actions into a single action on $\bigoplus_n \ell_2(G)$ by defining:

$$g * (\phi_n)_{n\geq 1} = (g *_n \phi_n)_{n\geq 1}$$

The action is well-defined since for each $g = g_i$ we have:

$$\begin{aligned}\sum_{n\geq 1} \left\| \sqrt{\frac{n}{|F_n|}}(\chi_{g_i F_n} - \chi_{F_n}) \right\|^2 &= \sum_{n\geq 1} \frac{n}{|F_n|} \|\chi_{g_i F_n} - \chi_{F_n}\|^2 = \sum_{n\geq 1} \frac{n}{|F_n|} |g_i F_n \triangle F_n| \\ &= \sum_{n<i} n \frac{|g_i F_n \triangle F_n|}{|F_n|} + \sum_{n\geq i} n \frac{|g_i F_n \triangle F_n|}{|F_n|} \\ &< \sum_{n<i} n \frac{|g_i F_n \triangle F_n|}{|F_n|} + \sum_{n\geq i} \frac{1}{n^2} < \infty\end{aligned}$$

For each $N$, the translate $gF_N$ meets $F_N$ for finitely many $g \in G$. Thus for all other $g$ we have:

$$\|g * 0\|^2 = \sum_{n\geq 1} \left\| \sqrt{\frac{n}{|F_n|}}(\chi_{gF_n} - \chi_{F_n}) \right\|^2 = \sum_{n\geq 1} \frac{n}{|F_n|} |gF_n \triangle F_n| \geq \frac{N}{|F_N|} |gF_N \triangle F_N| = 2N$$

We conclude that, for every $N$, the set $\{g \in G : \|g * 0\| < \sqrt{2N}\}$ is finite. In particular, the action is not bounded.

**Proposition 1.8.** *The collection of Kazhdan groups is closed under taking finite-index subgroups, quotients, and extensions.*

**Definition 1.9 (Relative Kazhdan).** A group $G$ has the *Kazhdan property relative to a subgroup $H$* if every isometric action of $G$ on a Hilbert space has a point fixed by $H$.

Obviously, we are only interested in the Kazhdan property relative to an infinite subgroup. For example, $\mathbb{Z}^2 \rtimes \mathrm{SL}_2(\mathbb{Z})$ is Kazhdan relative to $\mathbb{Z}^2$.

### 1.2.1 Property FA

There is an important connection between Kazhdan groups and the Bass-Serre theory of groups acting on trees.

**Definition 1.10.** A group $G$ has *property* FA if every action of $G$ on a tree fixes a vertex or an edge.

**Theorem 1.11.** *A group $G$ has property* FA *if and only if the following are satisfied:*
*1) $G$ is finitely generated with finite abelianization*
*2) $G$ is not a nontrivial free product with amalgamation*

For a finitely-generated group, having finite abelianization is equivalent to having no infinite cyclic quotient. The amalgamations $H *_H G$ and $G *_G H$ are considered trivial.

The following important result says in particular that Kazhdan groups are finitely-generated.

**Proposition 1.12.** *Kazhdan groups have property* FA.

**Example 1.13 (Torsion groups).** Finitely-generated torsion groups have property FA, as they satisfy the algebraic criterion described in Theorem 1.11. This observation is valuable for infinite torsion groups only, whose construction is rather delicate. Grigorchuk's groups are examples of infinite finitely-generated torsion groups. They therefore have property FA without being Kazhdan.

Proposition 1.12 is often used for proving that a given finitely-generated group is not Kazhdan. Algebraically, we may prove that the group has infinite abelianization; this method is surprisingly useful. Geometrically, we may actually point out an action of the group on a tree that fixes no vertex or edge; this is certainly more pleasant but it happens less frequently.

**Example 1.14 (Deficient groups).** A group that has a finite presentation with more generators than relators is not Kazhdan, since it has infinite abelianization. Examples include:

- nontrivial free groups: note that they obviously act freely on trees
- infinite 1-relator groups: these include the Baumslag-Solitar groups and the surface groups, with the exception of $\pi_1(\mathbb{S}^2) = 1$ in the orientable case, and $\pi_1(\mathbb{R}\mathrm{P}^2) = \mathbb{Z}_2$ in the nonorientable case.

Likewise for braid groups:

$$B_n = \langle x_1, \ldots, x_{n-1} |\ x_i x_{i+1} x_i = x_{i+1} x_i x_{i+1} \text{ for } 1 \leq i \leq n-2,\ \ [x_i, x_j] = 1 \text{ for } |i-j| \geq 2 \rangle$$

The presentation is not deficient, but the abelianization of $B_n$ is the same as that of the following deficient presentation:

$$\langle x_1, \ldots, x_{n-1} |\ x_i x_{i+1} x_i = x_{i+1} x_i x_{i+1} \text{ for } 1 \leq i \leq n-2 \rangle$$

**Example 1.15** ($\mathrm{SL}_n(\mathbb{Z})$)**.** The group $\mathrm{SL}_2(\mathbb{Z})$ is not Kazhdan, as it is virtually $\mathrm{F}_2$. Moreover, $\mathrm{SL}_2(\mathbb{Z})$ doesn't have property FA. Algebraically, $\mathrm{SL}_2(\mathbb{Z})$ is finitely-generated with finite abelianization $\mathbb{Z}_{12}$, but $\mathrm{SL}_2(\mathbb{Z}) = \mathbb{Z}_4 *_{\mathbb{Z}_2} \mathbb{Z}_6$. Geometrically, $\mathrm{SL}_2(\mathbb{Z})$ acts on $\mathcal{T}_3$.

On the other hand, $\mathrm{SL}_n(\mathbb{Z})$ is Kazhdan for $n \geq 3$.

**Notes.** For property T, the Old Testament is [dlHV89]. The New Testament is [BdlHV03].

What we defined as Kazhdan's property is actually Serre's property FH; for countable discrete groups they are equivalent. The relative Kazhdan property was introduced by G. Margulis.

The reference for property FA and the theory of groups acting on trees is [Ser80].

## 1.3 The Haagerup property

The Haagerup property can be described as a weak form of amenability, or as a strong negation of Kazhdan's property T, hence the alternate name "a-T-menability".

An isometric action of a group $G$ on a metric space $(X, d)$ is said to be *proper* if one of the following equivalent conditions holds:

*a*) for some $x \in X$, the set $\{g \in G : d(x, gx) \leq R\}$ is finite for each $R \geq 0$,

*b*) for all $x \in X$, the set $\{g \in G : d(x, gx) \leq R\}$ is finite for each $R \geq 0$,

*c*) the set $\{g \in G : gB \cap B \neq \emptyset\}$ is finite for each bounded $B \subseteq X$.

**Definition 1.16 (Haagerup group).** A group has the *Haagerup property* if it admits a proper isometric action on a Hilbert space.

Amenable groups have the Haagerup property, as we showed in Example 1.7. An important observation is the following:

**Proposition 1.17 (Kazhdan vs. Haagerup).** *A group that is both Kazhdan and Haagerup, is finite. More generally, a group that is Kazhdan relative to an infinite subgroup cannot be Haagerup.*

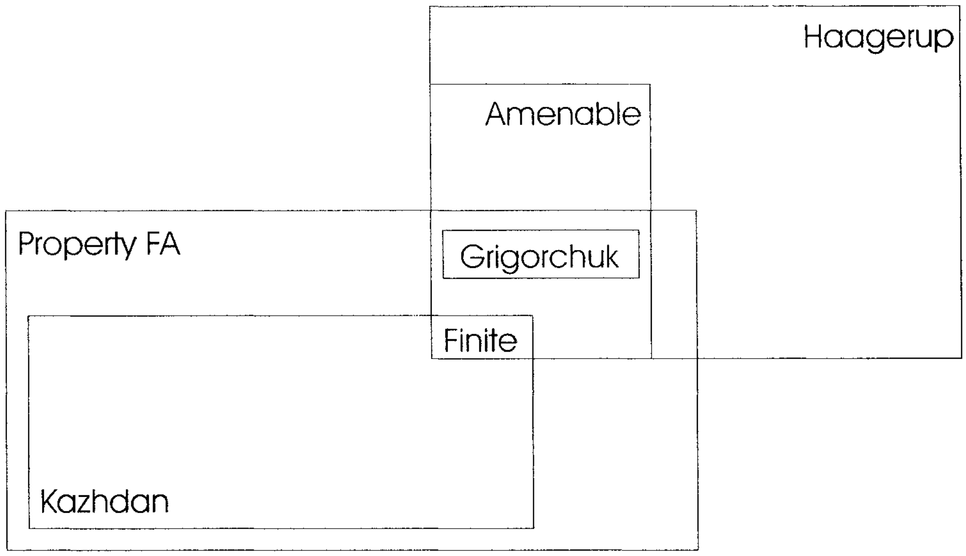
For example, $\mathbb{Z}^2 \rtimes \mathrm{SL}_2(\mathbb{Z})$ is not a Haagerup group. However, both $\mathbb{Z}^2$ and $\mathrm{SL}_2(\mathbb{Z})$ are Haagerup groups, so the family of Haagerup groups is not closed under extensions. Note that $\mathbb{Z}^2 \rtimes \mathrm{SL}_2(\mathbb{Z})$ is not Kazhdan either; in fact, it doesn't even have property (FA). So there are groups that are neither Kazhdan nor Haagerup.

**Problem 1.18.** Suppose that $G$ is not Kazhdan relative to any infinite subgroup. Does it follow that $G$ has the Haagerup property?

**Proposition 1.19.** *The collection of Haagerup groups is closed under taking subgroups, extensions with amenable quotients, and ascending countable unions.*

A simple technique for establishing the Haagerup property is to use group actions on discrete spaces with walls. Some groups that lend themselves to this viewpoint are: Coxeter groups, groups acting on trees, groups acting on cubings.

**Problem 1.20.** The following families are known not to contain any infinite Kazhdan group: braid groups, 3-manifold groups, 1-relator groups. Are these Haagerup families?




**Notes.** The definition of the Haagerup property is essentially the one from M. Gromov's epic essay *Asymptotic invariants of infinite groups.* The chief reference on the Haagerup property is [CCJ+01].

See K. Fujiwara's paper *3-manifold groups and property T of Kazhdan* for a proof of the fact that infinite 3-manifold groups are not Kazhdan, provided that the 3-manifold satisfies Thurston's Geometrization Conjecture.

## 1.4 The framework

One way of distinguishing two groups is to show that they have different isometric actions on a suitably chosen family of metric spaces. Conversely, one may take a well-understood family of metric spaces and use this family as a test-ground for isometric actions. Our formulation of the archetype reads as follows.

Let $\mathbb{X}$ be a family of metric spaces, understood as a type of geometry. The following definitions express the rigidity, respectively the flexibility, of a group with respect to the geometry in question. One may argue that these are properties of actions rather than of groups themselves.

**Definition 1.21 (Property** FX**).** A group has *property* FX if each isometric action on a member of $\mathbb{X}$ is bounded.

**Definition 1.22 (Property** PX**).** A group has *property* PX if it admits a proper isometric action on a member of $\mathbb{X}$.

Notice the vestigial "F", which is only justified for certain metric spaces, e.g., CAT(0) metric spaces. The better known avatars of property FX are: *property* FA, when we consider the family of simplicial trees; *property* F$\mathbb{R}$A, when we consider the family of $\mathbb{R}$-trees; *property* FH, when we consider the family of Hilbert spaces. As far as we know, property PX has only been investigated for Hilbert spaces. We will consider both properties with respect to the family of median metric spaces. Other families worthy of investigation in this light are $L_p$-spaces for $p \neq 2$, ultrametric spaces, etc.

Property FX and property PX are opposite in the sense that a group enjoying both properties is finite. Moreover, every homomorphism from an FX group to a PX group has finite image. This is because property FX is inherited by quotients, while property PX is inherited by subgroups. On the other hand, note that finite groups have both properties.

Typically, there exist groups that are neither FX nor PX. A simple trick for creating such groups is to consider extensions $1 \to F \to G \to P \to 1$ where $F$ is an infinite FX group and $P$ is an infinite PX group.

One expects a certain duality between property FX and property PX, in the sense that each FX statement has a corresponding PX statement, and vice versa.

Two problems arise, the structure problem and the relationship problem. We only formulate them for property FX, but they apply to property PX as well.

The *structure problem* asks for an algebraic characterization of property FX. We have seen such a characterization for property FA.

The *relationship problem* is the following: given two families $\mathbb{X}$ and $\mathbb{Y}$, does property FX imply property FY ? Is property FX equivalent to property FY ? Such relationships, typically established at the metric level between members of $\mathbb{X}$ and members of $\mathbb{Y}$, allow for group-theoretic insights. For example, we have exploited the fact that property FH implies property FA.

When considering isometric actions of countable groups on the family of Hilbert spaces, we are actually looking at one space: $\ell_2$, or $\mathbb{R}^\infty$. More precisely, property FH is equivalent to requiring that every isometric action on $\ell_2$ is bounded, whereas property PH is equivalent to the existence of a proper isometric action on $\ell_2$.

This suggests another possible definition for property FX and property PX, when a single space is taken into account. Our point of view is that property FX and property PX should capture the incompatibility, respectively the compatibility, with a geometry rather than with a particular space. If an intrinsic rigidity of the geometry in question elects a single representative, let it be so.

# 2 Median spaces

We briefly describe median algebras, which are interval structures that enjoy a tripod-like condition. Then we investigate median spaces, which are metric spaces whose geodesic intervals turn the spaces into median algebras.

## 2.1 Median algebras

**Definition 2.1 (Median algebra).** A *median algebra* is a set $X$ with an assignment $(x, y) \mapsto [x, y]$, mapping pairs of points in $X$ to subsets of $X$, so that for any $x, y, z \in X$ the following are satisfied:

- $[x, x] = \{x\}$
- if $z \in [x, y]$ then $[x, z] \subseteq [x, y]$
- $[x, y]$, $[y, z]$, $[z, x]$ have a unique common point, called the *median* of $x, y, z$ and denoted by $m(x, y, z)$

A *morphism of median algebras* is a map $f : X \longrightarrow X'$ between median algebras that is "betweenness preserving", in the sense that $f\big([x, y]\big) \subseteq \big[f(x), f(y)\big]$ for all $x, y \in X$.

**Definition 2.2 (Halfspace).** A subset $A \subseteq X$ is *convex* if $[x, y] \subseteq A$ for all $x, y \in A$. A subset $A \subseteq X$ is a *halfspace* if both $A$ and $A^c$ are convex.

A crucial feature of halfspaces in a median algebra is the following separation property:

**Theorem 2.3.** *Let $X$ be a median algebra and $C_1, C_2$ be disjoint convex sets. Then there is a halfspace $A$ separating $C_1$ and $C_2$, i.e., $C_1 \subseteq A$ and $C_2 \subseteq A^c$.*

**Example 2.4 (Boolean median algebra).** Any power set $\mathcal{P}(X)$ is a median algebra under the interval assignment

$$(A, B) \mapsto [A, B] = \{C : A \cap B \subseteq C \subseteq A \cup B\}$$

The *boolean median* of $A$, $B$, $C$ is $(A \cap B) \cup (B \cap C) \cup (C \cap A) = (A \cup B) \cap (B \cup C) \cap (C \cup A)$. The nonempty halfspaces not containing the empty set are precisely the ultrafilters on $X$. Recall that $\mu$ is an *ultrafilter* on $X$ if :

1) $\emptyset \notin \mu$,
2) $A, B \in \mu$ implies $A \cap B \in \mu$,
3) for all $A \subseteq X$, either $A \in \mu$ or $A^c \in \mu$.

The significance of the previous example is that any median algebra is isomorphic to a subalgebra of a boolean median algebra. Indeed, let $X$ be a median algebra, let $\mathcal{H}$ be the collection of halfspaces of $X$, and denote by $\sigma_x$ the collection of halfspaces containing $x \in X$. We obtain a map $\sigma : X \longrightarrow \mathcal{P}(\mathcal{H})$ that is easily checked to be a median embedding, i.e., an injective median morphism.

We thus have a "boolean method" for proving (non-existential) statements about median algebras. The following result is an illustration of this method.

**Lemma 2.5.** *In a median algebra, the median closure of a finite set is finite.*

Given a median algebra $X$ and a subset $A \subseteq X$, the median closure of $A$ is the smallest subset of $X$ containing $A$ that is stable under taking medians.

*Proof.* Enough to prove for boolean median algebras. For a finite $\mathcal{A} \subseteq \mathcal{P}(X)$ we define:

- $\bigcup \mathcal{A}$ is the collection of arbitrary unions of sets from $\mathcal{A}$
- $\bigcap \mathcal{A}$ is the collection of arbitrary intersections of sets from $\mathcal{A}$

The collection $\langle \mathcal{A} \rangle := \bigcup \bigcap \mathcal{A} = \bigcap \bigcup \mathcal{A}$ is the sublattice of $(\mathcal{P}(X), \cup, \cap)$ generated by $\mathcal{A}$. In particular, $\langle \mathcal{A} \rangle$ is median stable. Clearly $\langle \mathcal{A} \rangle$ is finite, of size at most $2^{2^{|\mathcal{A}|}}$. $\square$

The following proposition gives alternate definitions of median morphisms. Particularly important for later sections is the last characterization, via halfspaces.

**Proposition 2.6.** *Let $f : X \longrightarrow X'$ be a map, where $X$ and $X'$ are median algebras. The following are equivalent:*

*1) $f([x, y]) \subseteq [f(x), f(y)]$ for all $x, y \in X$, i.e., $f$ is a morphism of median algebras*

*2) $f(m(x, y, z)) = m(f(x), f(y), f(z))$ for all $x, y, z \in X$, i.e., $f$ preserves medians*

*3) $f^{-1}(A')$ is a halfspace in $X$ whenever $A'$ is a halfspace in $X'$*

*Proof.* 3) $\Rightarrow$ 2): If $f(m(x, y, z)) \neq m(f(x), f(y), f(z))$ for some $x, y, z \in X$, then there is a halfspace $A'$ in $X'$ so that $f(m(x, y, z)) \in A'$ and $m(f(x), f(y), f(z)) \in X' \setminus A'$. The latter implies, by the convexity of $A'$, that at least two of $\{f(x), f(y), f(z)\}$, say $f(x)$ and $f(y)$, are in $X' \setminus A'$, i.e., $x, y \in f^{-1}(X' \setminus A')$. Then $m(x, y, z) \in f^{-1}(X' \setminus A')$ as well, which is a contradiction.
2) $\Rightarrow$ 1): Let $z \in [x, y]$, i.e., $m(x, y, z) = z$. Then $m(f(x), f(y), f(z)) = f(m(x, y, z)) = f(z)$ which means that $f(z) \in [f(x), f(y)]$.
1) $\Rightarrow$ 3): Note that $f^{-1}(C')$ is convex in $X$ whenever $C'$ is convex in $X'$. Apply this observation to both $A'$ and $X' \setminus A'$. $\square$

**Notes.** We learned about median algebras from [Rol98], which contains the clearest proof of Theorem 2.3 that we know of.

## 2.2 Median spaces

Let $(X, d)$ be a metric space. The *geodesic segment* determined by $x, y \in X$ is defined as $[x, y]_d = \{t \in X : d(x, t) + d(t, y) = d(x, y)\}$.

**Definition 2.7 (Median space).** A metric space $(X, d)$ is *median* if, for each triple $x, y, z \in X$, the geodesic segments $[x, y]_d$, $[y, z]_d$, $[z, x]_d$ have a unique common point.

If $(X, d_X)$ and $(Y, d_Y)$ are median spaces, then $X \times Y$ is a median space under the metric $d((x_1, y_1), (x_2, y_2)) = d_X(x_1, x_2) + d_Y(y_1, y_2)$. The completion of a median space is median.

Trees and $\mathbb{R}$-trees are median spaces. For any measure $\mu$, the space $L_1(\mu)$ of real-valued integrable functions is median.

## 2.3 Constructing a Hilbert space. I

In this section, we show how to obtain a group action on a Hilbert space out of an action on a median space.

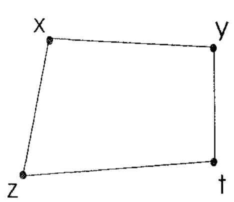
Let $X$ be a median space. Recall that $\sigma_x$ stands for the collection of halfspaces containing a point $x$. Thus $\sigma_x \triangle \sigma_y$ consists of all the halfspaces separating $x$ from $y$. Recall also that $\chi_A$ denotes the characteristic function of a set $A$.

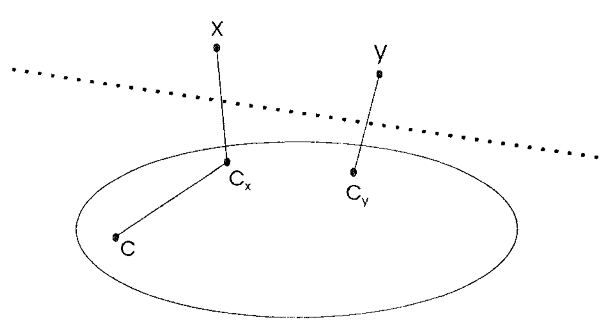
Define $\chi(x,y) := \chi_{\sigma_x \triangle \sigma_y} = |\chi_{\sigma_x} - \chi_{\sigma_y}|$, so $\chi(x,y)$ is a map from the collection of all halfspaces of $X$ to $\{0,1\}$. The following relations are easily checked by halfspace reasoning:

- $\chi(x,y) = \chi(y,x)$
- $z \in [x,y]_d$ if and only if $\chi(x,y) = \chi(x,z) + \chi(z,y)$
- $\chi(x,y) \cdot \chi(x,z) = \chi(x, m(x,y,z))$
- $1-\chi(x,y) = \big(1-\chi(z,x)\big)\big(1-\chi(z,y)\big)$, i.e., $\chi(x,y) = \chi(z,x)+\chi(z,y)-2\chi(z,x)\cdot\chi(z,y)$

**Lemma 2.8.** *If* $\alpha_1\chi(v,x_1) + \ldots + \alpha_n\chi(v,x_n) \geq 0$ *then* $\alpha_1 d(v,x_1) + \ldots + \alpha_n d(v,x_n) \geq 0$.

*Proof.* Say that $xyzt$ is a *rectangle* if $x,t \in [y,z]_d$ and $y,z \in [x,t]_d$. In a rectangle, opposite sides have equal length.




The median closure of a finite set being finite, we may assume that $X_0 = \{v, x_1, \ldots, x_n\}$ is median stable. We proceed by induction. Let $C \subseteq X_0$ be a maximal proper convex subset containing $v$. In particular, $C$ is median stable.

For $x \in X_0 \setminus C$ let $c_x$ be a point in $C$ closest to $x$. For $c \in C$, the median $m(c, c_x, x)$ is in $C$ and closer to $x$ unless $m(c, c_x, x) = c_x$. Thus $c_x \in [x, c]_d$ for all $c \in C$, in particular $c_x$ is unique. One thinks of $c_x$ as the gate of $x$ to $C$.

For all $x, y \in X_0 \setminus C$, $xc_xyc_y$ is a rectangle. Clearly, $c_x \in [x, c_y]_d$ and $c_y \in [y, c_x]_d$. If $y \notin [x, c_y]_d$, let $H$ be a halfspace with $x, c_y \in H$ and $y \notin H$. Then $C \subseteq H$ and hence $X_0 = co(C \cup \{x\}) \subseteq H$, contradicting $y \notin H$. Therefore $y \in [x, c_y]_d$, and $x \in [y, c_x]_d$ by symmetry.

It follows that a halfspace separating some $x \in X_0 \setminus C$ from $C$ actually separates every $x \in X_0 \setminus C$ from $C$. Let us call such halfspaces *significant*. We have:

$$\sum \alpha_i \chi(v, x_i) = \underbrace{\sum_{x_i \in C} \alpha_i \chi(v, x_i) + \sum_{x_i \notin C} \alpha_i \chi(v, c_{x_i})}_{P} \; + \; \underbrace{\sum_{x_i \notin C} \alpha_i \chi(c_{x_i}, x_i)}_{Q} \; \geq 0$$

On significant halfspaces, $P$ vanishes and hence $Q = \sum_{x_i \notin C} \alpha_i \geq 0$. On insignificant halfspaces, $Q$ vanishes hence $P \geq 0$. Thus $P \geq 0$ throughout, and the induction hypothesis applied to $C$ gives

$$\sum_{x_i \in C} \alpha_i d(v, x_i) + \sum_{x_i \notin C} \alpha_i d(v, c_{x_i}) \geq 0$$

and using the fact that $d(c_x, x)$ is independent of $x \notin C$, we obtain:

$$\sum \alpha_i d(v, x_i) = \sum_{x_i \in C} \alpha_i d(v, x_i) + \sum_{x_i \notin C} \alpha_i d(v, c_{x_i}) + \sum_{x_i \notin C} \alpha_i d(c_{x_i}, x_i) \geq 0$$

□

Let $V$ be the vector space generated by $\{\chi(x, y) : x, y \in X\}$. Note that $V$ is an algebra. For $v \in X$, $\Lambda_v = \{\chi(v, x) : x \in X\}$ spans $V$ and Lemma 2.8 allows us to define a positive linear functional $I_v : V \longrightarrow \mathbb{R}$ such that:

$$I_v(\chi(v, x)) = d(v, x)$$

Observe:

$$\begin{aligned} I_v(\chi(x, y)) &= I_v\big(\chi(v, x) + \chi(v, y) - 2\chi(v, m(v, x, y))\big) \\ &= d(v, x) + d(v, y) - 2d(v, m(v, x, y)) = d(x, y) \end{aligned}$$

In particular, $I_v$ doesn't depend on $v$, and we henceforth denote it simply by $I$.

We turn $V$ into an inner product space by defining

$$\langle \phi_1, \phi_2 \rangle = I(\phi_1 \cdot \phi_2)$$

and hence $\|\phi\| = \sqrt{I(\phi^2)}$, e.g. $\|\chi(x, y)\| = \sqrt{d(x, y)}$.

An isometric action of $G$ on $X$ induces an isometric linear action $\phi \mapsto g\phi$ on $V$ defined by $g\phi(H) = \phi(g^{-1}H)$. For the action to be isometric, it suffices to check that it preserves the inner product on a generating set $\Lambda_v$:

$$\begin{aligned} \langle g\chi(v, x), g\chi(v, y) \rangle &= \langle \chi(gv, gx), \chi(gv, gy) \rangle = I\big(\chi(gv, gx) \cdot \chi(gv, gy)\big) \\ &= I\big(\chi(gv, m(gv, gx, gy))\big) = d(gv, m(gv, gx, gy)) \\ &= d(v, m(v, x, y)) = \langle \chi(v, x), \chi(v, y) \rangle \end{aligned}$$

For a fixed $v \in X$, we define an affine isometric action on $V$:

$$(1 - 2g * \phi) = (1 - 2\chi(v, gv))(1 - 2g\phi), \text{ i.e., } g * \phi = \big(1 - 2\chi(v, gv)\big)g\phi + \chi(v, gv)$$

Indeed, observing that $1 - 2\chi(v, gv) = \pm 1$, we have:

$$\|g * \phi_1 - g * \phi_2\| = \|\big(1 - 2\chi(v, gv)\big)(g\phi_1 - g\phi_2)\| = \|g\phi_1 - g\phi_2\| = \|\phi_1 - \phi_2\|$$

Compare the two actions. In the affine action, $g * \chi(v, x) = \chi(v, gx)$, and hence $\Lambda_v$ is invariant under the affine action. The affine action based at $v$ realizes the $G$-action on

$X$ *within* the frame $\Lambda_v$. In the linear action, $g\chi(v,x) = \chi(gv,gx)$, therefore the linear action translates $\Lambda_v$ to $\Lambda_{gv}$. The linear action realizes the $G$-action on $X$ *among* the frames $(\Lambda_v)_{v\in X}$.

By completing $V$ and extending the action if necessary, we may assume that $V$ is a Hilbert space. Since $\|g * 0\| = \|\chi(v,gv)\| = \sqrt{d(v,gv)}$, we obtain:

**Proposition 2.9.** *Every action of a Kazhdan group on a median space is bounded.*

**Proposition 2.10.** *A group that admits a proper action on a median space has the Haagerup property.*

For example, groups acting properly on $\mathbb{R}$-trees are Haagerup groups.

**Conjecture 2.11.** *The converses of Proposition 2.9 and Proposition 2.10 are true.*

## 2.4 Constructing a Hilbert space. II

The relation $d(x,y) = \|\chi(v,x) - \chi(v,y)\|^2$ suggests a second construction, perhaps simpler than the previous one.

**Definition 2.12.** A metric space $(X,d)$ is *negative definite* if $\sum \alpha_i\alpha_j d(x_i,x_j) \leq 0$ for all $x_1,\dots,x_n \in X$ and $\alpha_1,\dots,\alpha_n \in \mathbb{R}$ with $\sum \alpha_i = 0$.

The following observation is sometimes referred to as the GNS construction.

If a metric space $(X,d)$ has the property that there is an inner product space $V$ and a map $\gamma : X \longrightarrow V$ with $d(x,y) = \|\gamma(x) - \gamma(y)\|^2$ for all $x,y \in X$, then $X$ is negative definite since $\sum \alpha_i\alpha_j d(x_i,x_j) = -2\|\sum \alpha_i\gamma(x_i)\|^2 \leq 0$ for $\sum \alpha_i = 0$.

Conversely, let $(X,d)$ be negative definite. Let $V(X)$ be the vector space on $X$, and $V_0(X)$ consist of the vectors in $V(X)$ with zero coefficient sum. On $V_0(X)$ define the inner product $\langle \sum \alpha_i x_i, \sum \beta_j y_j\rangle = -\frac{1}{2}\sum \alpha_i\beta_j d(x_i,y_j)$. Then $d(x,y) = \|x-y\|^2$. Curiously, $\langle x-z, y-z\rangle = < x,y >_z$, where the right-hand side is the inner product in $X$.

**Lemma 2.13.** *Median spaces are negative definite.*

*Proof.* We proceed along the same line as in the proof of Lemma 2.8. Let $x_1,\dots,x_n \in X$, which we may assume to form a median stable set $X_0$. Let $C$ be a maximal proper convex subset of $X_0$. There is a retraction $c : X_0 \longrightarrow C$, associating to each $x_i$ the point in $c_{x_i} \in C$ that is closest to $x_i$. For $x,y \notin C$, $xc_xyc_y$ is a rectangle. Since opposite sides are equal, we conclude that $d(x,c_x) \equiv \delta$ for $x \notin C$. Expand $\sum \alpha_i\alpha_j d(x_i,x_j)$ as follows:

$$\sum_{x_i,x_j\in C} \alpha_i\alpha_j d(x_i,x_j) + \sum_{x_i,x_j\notin C} \alpha_i\alpha_j d(x_i,x_j) + \sum_{x_i\notin C, x_j\in C} \alpha_i\alpha_j d(x_i,x_j) + \sum_{x_i\in C, x_j\notin C} \alpha_i\alpha_j d(x_i,x_j)$$

In the first and second sum, replace $d(x_i,x_j)$ by $d(c_{x_i},c_{x_j})$. In the third and the fourth sum, replace $d(x_i,x_j)$ by $d(c_{x_i},c_{x_j}) + \delta$. We obtain:

$$\sum \alpha_i\alpha_j d(x_i,x_j) = \sum \alpha_i\alpha_j d(c_{x_i},c_{x_j}) + \delta \sum_{x_i\notin C, x_j\in C} \alpha_i\alpha_j + \delta \sum_{x_i\in C, x_j\notin C} \alpha_i\alpha_j \ .$$

Thus $\sum \alpha_i\alpha_j d(x_i, x_j) \leq 0$ since the first sum on the right is non-positive by induction, and the last two sums give $-2\delta(\sum_{x_i \in C'} \alpha_i)^2$. $\square$

**Remark 2.14.** A metric space $(X, d)$ is *hypermetric* if $\sum t_i t_j d(x_i, x_j) \leq 0$ for all $x_1, \ldots, x_n \in X$ and integers $t_1, \ldots, t_n$ with $\sum t_i = 1$. One easily shows that hypermetric spaces are negative definite. An obvious adaptation of the previous proof shows that median spaces are in fact hypermetric.

We now interpret Propositions 2.9 and 2.10 from this point of view. Let $G$ act by isometries on a median space $X$ and consider the inner product space $V_0(X)$. The obvious linear action $\phi \mapsto g\phi$ of $G$ on $V_0(X)$ is isometric, and for a fixed $v \in X$ we "affinize":

$$g * \phi = g\phi + (gv - v)$$

By completing $V_0(X)$ and extending the action if necessary, we may assume that $V_0(X)$ is a Hilbert space. Finally, since $\|g*0\| = \|gv - v\| = \sqrt{d(gv, v)}$, we obtain Propositions 2.9 and 2.10.

# 3 Median graphs

Graphs, which we assume connected and without loops or multiple edges, come equipped with a path metric. One may thus consider median graphs.

Trees are elementary examples of median graphs. The 1-skeleton of the square tiling of the plane is median whereas the 1-skeletons of the hexagonal and triangular tilings are not. The 1-skeleton of an $n$-dimensional cube is median. Note that median graphs are bipartite and do not have $K_{2,3}$ as a subgraph.

One of the motivating facts about median graphs is the close relationship with CAT(0) cube complexes, or *cubings* for short. Cubings are simply-connected complexes of nonpositive curvature made out of standard euclidean cubes. Every gluing of a cube is an isometry on each face of the cube. The nonpositive curvature condition can be expressed as follows: [no bigons] no two 2-cubes share adjacent edges, and [no triangle] if three $(n+2)$-cubes share an $n$-cube and pairwise share $(n+1)$-cubes, then they are faces of an $(n+3)$-cube.

For example, a cubing made out of 1-cubes, i.e. segments, is a tree. A cubing made out of 2-cubes, i.e. squares, can be described as a simply-connected square complex with at least 4 squares around each vertex.

**Theorem 3.1.** *The* 1*-skeleton of a cubing is a median graph. Conversely, every median graph is the* 1*-skeleton of a cubing.*

Roughly speaking, one obtains a cubing from a median graph by "filling in" isometric copies of euclidean cubes, that is by inductively adding an $(n+1)$-dimensional cube whenever its $n$-skeleton is present.

**Notes.** CAT(0) cube complexes were introduced in M. Gromov's landmark paper *Hyperbolic groups.*

Theorem 3.1 is proved in [Rol98], [Che00], [Ger98].

## 3.1 Spaces with walls

**Definition 3.2 (Space with walls).** Let $X$ be a set. A *wall* in $X$ is a partition of $X$ into 2 subsets called *halfspaces.* We say that $X$ is a *space with walls* if $X$ is endowed with a collection of walls, containing the trivial wall $\{\emptyset, X\}$, and so that any two distinct points are separated by a finite, non-zero number of walls. Note that a wall separates two distinct points $x, y \in X$ if $x$ belongs to one of the halfspaces determined by the wall, while $y$ belongs to the other halfspace.

A *morphism of spaces with walls* is a map $f : X \longrightarrow X'$ between spaces with walls with the property that $f^{-1}(A')$ is a halfspace of $X$ for each halfspace $A'$ of $X'$.

For a given $x \in X$, we let $\sigma_x$ denote the collection of halfspaces containing $x$. Note that the number of walls separating $x, y \in X$ is $\frac{1}{2}|\sigma_x \triangle \sigma_y|$.

**Definition 3.3 (Wall metric).** The *wall metric* $d_w$ on a space with walls $X$ is defined by $d_w(x,y) = \frac{1}{2}|\sigma_x \triangle \sigma_y|$.

A group acts on a space with walls $X$ by permuting the walls. Consequently, it acts by isometries on $(X, d_w)$.

Median graphs are the main examples of spaces with walls. As we will show in Section 3.2, they are "universal spaces with walls", in the sense that every space with walls admits a canonical embedding in a median graph in such a way that the wall structure is preserved.

The proper halfspaces in a median graph $X$ have a simple description:

$$H_{xy} = \{z \in X : d(z,x) < d(z,y)\}$$

where $xy$ is an edge. Therefore, the path metric and the wall metric coincide.

**Notes.** The notion of a space with walls is due to F. Haglund and F. Paulin [HP98]. Our definition differs from the original one in that we insist on the presence of the trivial wall. This minor modification is needed for a morphism of spaces with walls to be well-defined. Moreover, the trivial wall is present in median algebras.

### 3.1.1 Application: Kazhdan property and Haagerup property

Suppose $G$ acts on a space with walls $X$. Let $\mathcal{H}$ denote the collection of halfspaces of $X$. Then $G$ acts linearly isometrically on $\ell_2(\mathcal{H})$ via $g\phi(H) = \phi(g^{-1}H)$.

Fix a basepoint $v \in X$ and consider the affine action:

$$g * \phi = g\phi + (\chi_{\sigma_{gv}} - \chi_{\sigma_v})$$

Then $\|g * 0\|^2 = \|\chi_{\sigma_{gv}} - \chi_{\sigma_v}\|^2 = |\sigma_{gv} \triangle \sigma_v| = 2d_w(gv, v)$. We conclude:

**Proposition 3.4.** *Every action of a Kazhdan group on a space with walls is bounded.*

In particular, Kazhdan groups have property FA.

**Proposition 3.5.** *A group that admits a proper action on a space with walls has the Haagerup property.*

In particular, groups that admit proper actions on trees, e.g. free groups, are therefore Haagerup groups.

Finitely-generated Coxeter groups act properly on spaces with walls. There are at least three ways of interpreting their wall structure; see [NR03]. On the other hand, they have property FA provided that their defining matrix $(m_{ij})$ has finite entries only.

The usefulness of Proposition 3.4 and Proposition 3.5 resides in the fact that spaces with walls can be read off in many geometric contexts.

**Problem 3.6.** Give examples of non-Kazhdan groups with the property that every action on a space with walls is bounded.

Note that such groups have property FA.

**Problem 3.7.** Give examples of Haagerup groups that do not admit proper actions on spaces with walls.

## 3.2 From spaces with walls to median graphs

A common way of enriching a space $X$ is by devising a notion of ultrafilter, which is roughly a collection of objects related to $X$, so that to each point in $X$ corresponds a unique ultrafilter. The ultrafilters tagged by points of $X$ are called principal ultrafilters, and so $X$ embeds in a larger set of ultrafilters. Such a procedure provides a completion or a compactification of a space. In our case, it provides a *cubulation.*

**Definition 3.8 (Ultrafilter).** An *ultrafilter* on a space with walls $X$ is a nonempty collection $\omega$ of halfspaces that satisfies:

- $A \in \omega$ and $A \subseteq B$ imply $B \in \omega$
- either $A \in \omega$ or $A^c \in \omega$ but not both.

This notion is different from the notion of ultrafilter mentioned in Example 2.4. Intuitively, an ultrafilter is a coherent orientation of the walls. Note that every ultrafilter contains $X$.

For every $x \in X$, the collection $\sigma_x$ of halfspaces containing $x$ is an ultrafilter, called the *principal* ultrafilter at $x$. In addition, we consider the *almost principal* ultrafilters, that is ultrafilters $\omega$ with $\omega \triangle \sigma_x$ finite for some (every) principal ultrafilter $\sigma_x$.

Let $\mathcal{C}^1(X)$ be the graph whose vertices are the almost principal ultrafilters on $X$, and whose edges are defined by: $\omega_1$ is adjacent to $\omega_2$ if $\frac{1}{2}\left|\omega_1 \triangle \omega_2\right| = 1$.

If $\omega_1$, $\omega_2$ are almost principal ultrafilters, then elements of $\omega_1 \triangle \omega_2$ come in pairs $\{A, A^c\}$, so we may think of them as being walls, more specifically walls on which $\omega_1$ and $\omega_2$ have opposite orientation. Thus two ultrafilters are adjacent in $\mathcal{C}^1(X)$ if there is exactly one wall on which they have different orientation.

**Lemma 3.9.** *In $\mathcal{C}^1(X)$, the distance between $\omega_1$ and $\omega_2$ is $\frac{1}{2}\left|\omega_1 \triangle \omega_2\right|$.*

*Proof.* If $\omega_1 = \theta_1, \ldots, \theta_{m+1} = \omega_2$ is a path connecting $\omega_1$ to $\omega_2$, then

$$\frac{1}{2}\left|\omega_1 \triangle \omega_2\right| = \frac{1}{2}\left|(\theta_1 \triangle \theta_2) \triangle \ldots \triangle (\theta_m \triangle \theta_{m+1})\right| \leq \sum_{1 \leq i \leq m} \frac{1}{2}\left|\theta_i \triangle \theta_{i+1}\right| = m$$

Conversely, let $\omega_1 \triangle \omega_2 = \{A_1, \ldots, A_n, A_1^c, \ldots, A_n^c\}$ with $A_i \in \omega_1 \setminus \omega_2$ and $A_i^c \in \omega_2 \setminus \omega_1$. Assume each $A_i$ minimal in $\{A_i, \ldots, A_n\}$ and define $\theta_1 = \omega_1$, $\theta_{i+1} = \theta_i \triangle \{A_i, A_i^c\}$ for $i \leq n$. Then $\theta_{n+1} = \omega_1 \triangle \{A_1, \ldots, A_n, A_1^c, \ldots, A_n^c\} = \omega_2$.

We claim that each $\theta_i$ is an ultrafilter. Since $\theta_{i+1}$ is obtained from $\theta_i$ by exchanging $A_i$ for $A_i^c$, and since exchanging a minimal halfspace in an ultrafilter for its complement results in an ultrafilter, we are left with showing that $A_i$ is minimal in $\theta_i$. Suppose there is $B \in \theta_i$, $B \subsetneq A_i$. Then $B \notin \omega_2$ because $A_i \notin \omega_2$. As

$$\theta_i = \left(\omega_1 \setminus \{A_1, \ldots, A_{i-1}\}\right) \cup \{A_1^c, \ldots, A_{i-1}^c\}$$

we necessarily have $B \in \omega_1 \setminus \{A_1, \ldots, A_{i-1}\}$. We obtain $B \in \{A_i, \ldots, A_n\}$ which contradicts the fact that $A_i$ is minimal in $\{A_i, \ldots, A_n\}$. □

**Lemma 3.10.** *In $\mathcal{C}^1(X)$, $\omega \in [\omega_1, \omega_2] \Leftrightarrow \omega_1 \cap \omega_2 \subseteq \omega \Leftrightarrow \omega \subseteq \omega_1 \cup \omega_2$.*

*Proof.* We have $\omega \in [\omega_1, \omega_2] \Leftrightarrow |\omega_1 \triangle \omega| + |\omega \triangle \omega_2| = |\omega_1 \triangle \omega_2| \Leftrightarrow \omega_1 \cap \omega_2 \subseteq \omega \subseteq \omega_1 \cup \omega_2$. The equivalence $\omega_1 \cap \omega_2 \subseteq \omega \Leftrightarrow \omega \subseteq \omega_1 \cup \omega_2$ holds for arbitrary ultrafilters. $\square$

**Proposition 3.11.** *$\mathcal{C}^1(X)$ is a connected median graph.*

*Proof.* Since geodesic intervals in $\mathcal{C}^1(X)$ are of boolean type, the median in $\mathcal{C}^1(X)$ of a triple of vertices $\omega_1$, $\omega_2$, $\omega_3$, has to be the boolean median

$$m(\omega_1, \omega_2, \omega_3) = (\omega_1 \cap \omega_2) \cup (\omega_2 \cap \omega_3) \cup (\omega_3 \cap \omega_1)$$

One easily checks that $m(\omega_1, \omega_2, \omega_3)$ is indeed a vertex in $\mathcal{C}^1(X)$. $\square$

**Proposition 3.12.** *There is a bijective correspondence between the halfspaces of $X$ and the halfspaces of $\mathcal{C}^1(X)$ given by $A \mapsto H_A = \{\omega \in \mathcal{C}^1(X) : A \in \omega\}$.*

*Proof.* Each $H_A$ is convex: if $\omega_1, \omega_2 \in H_A$ and $\omega \in [\omega_1, \omega_2]$, then $A \in \omega_1 \cap \omega_2 \subseteq \omega$ hence $\omega \in H_A$. The complement of $H_A$ in $\mathcal{C}^1(X)$ is $H_{A^c}$. Therefore $H_A$ is a halfspace in $\mathcal{C}^1(X)$ for every halfspace $A$ in $X$.

The mapping is injective: $\sigma_x \in H_A$ if and only if $x \in A$, i.e., $\sigma^{-1}(H_A) = A$.

The mapping is surjective. Let $H$ be a halfspace in $\mathcal{C}^1(X)$. Assume that $H$ is proper, as $H_\emptyset = \emptyset$ and $H_X = \mathcal{C}^1(X)$. Then $H$ cuts some edge $\theta_1\theta_2$: $\theta_1 \in H$ and $\theta_2 \notin H$. Suppose the edge $\theta_1\theta_2$ is obtained by exchanging $A \in \theta_1$ for $A^c \in \theta_2$. We claim that $H = H_A$. If $\omega \in H_A$, i.e. $A \in \omega$, then $\theta_1 \subseteq \theta_2 \cup \omega$ and the convexity of $\mathcal{C}^1(X) \setminus H$ implies $\omega \in H$. Thus $H_A \subseteq H$. Similarly $H_{A^c} \subseteq \mathcal{C}^1(X) \setminus H$, which by complementation becomes $H \subseteq H_A$. $\square$

As $\mathcal{C}^1(X)$ is median, the wall metric and the path metric coincide. Proposition 3.12 provides another explanation. The path metric counts the walls $\{A, A^c\}$ in $\omega_1 \triangle \omega_2$. The wall metric counts the walls $\{H_A, H_{A^c}\}$ separating $\omega_1$, $\omega_2$ in $\mathcal{C}^1(X)$, and a wall $\{H_A, H_{A^c}\}$ separates $\omega_1$, $\omega_2$ if and only if $\{A, A^c\}$ is in $\omega_1 \triangle \omega_2$.

**Proposition 3.13.** *The map $\sigma : X \longrightarrow \mathcal{C}^1(X)$ given by $x \mapsto \sigma_x$ is an injective morphism of spaces with walls and an isometric embedding when $X$ is equipped with the wall metric.*

*Proof.* $\sigma$ is a morphism of spaces with walls as $\sigma^{-1}(H_A) = A$, and $d_w(x, y) = \frac{1}{2}|\sigma_x \triangle \sigma_y|$, the right-hand side being the distance between $\sigma_x$ and $\sigma_y$ in $\mathcal{C}^1(X)$. $\square$

**Proposition 3.14.** *$\mathcal{C}^1(X)$ is the median closure of $\{\sigma_x : x \in X\}$.*

*Proof.* Let $M \subseteq \mathcal{C}^1(X)$ be the median closure of $\{\sigma_x : x \in X\}$. We proceed by contamination, assuming that $\omega \in M$ and $\omega\omega'$ is an edge in $\mathcal{C}^1(X)$, and proving that $\omega' \in M$. Suppose the edge $\omega\omega'$ is obtained by exchanging $A^c \in \omega$ for $A \in \omega'$. Let $\zeta \in M \cap H_A$ be closest to $\omega$. See figure 1.

We claim that $\zeta = \omega'$. Note that $\zeta \in [\omega, \sigma_x]$ for all $x \in A$ (hint: $m(\omega, \zeta, \sigma_x)$) and $\omega' \in [\omega, \zeta]$ (hint: bipartite), hence $\zeta \in [\omega', \sigma_x]$, i.e., $\zeta \subseteq \omega' \cup \sigma_x$ for all $x \in A$. If there is $B \in \zeta \setminus \omega'$ then $B \in \sigma_x$ for all $x \in A$, so $A \subseteq B$ and hence $B \in \omega'$, which is a contradiction. Thus $\zeta \subseteq \omega'$, so $\zeta = \omega'$. $\square$

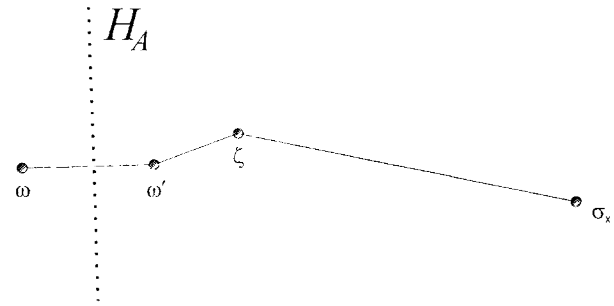



Figure 1: The principal ultrafilters span.

In particular, if $X$ is a median graph then $\sigma : X \longrightarrow \mathcal{C}^1(X)$ is a median isomorphism. One checks that $m(\sigma_x, \sigma_y, \sigma_z) = \sigma_{m(x,y,z)}$, i.e., $\sigma$ is a median morphism. Therefore all vertices in $\mathcal{C}^1(X)$ are principal ultrafilters, in other words $\sigma$ is surjective. Note that a median isomorphism between median graphs is a graph isomorphism as well.

**Proposition 3.15.** *For every morphism of spaces with walls $f : X \longrightarrow X'$, there is a unique median morphism $f_* : \mathcal{C}^1(X) \longrightarrow \mathcal{C}^1(X')$ making the following diagram commute:*

$$\begin{array}{ccc} X & \xrightarrow{f} & X' \\ {\scriptstyle \sigma_X}\downarrow & & \downarrow{\scriptstyle \sigma_{X'}} \\ \mathcal{C}^1(X) & \xrightarrow{f_*} & \mathcal{C}^1(X') \end{array}$$

*Proof.* Define $f_*(\omega) = \{A' \subseteq X' \text{ halfspace} : f^{-1}(A') \in \omega\}$. Then $f_*(\omega)$ is an ultrafilter on $X'$ whenever $\omega$ is an ultrafilter on $X$. Moreover, $f_*(\omega_1)\triangle f_*(\omega_2)$ is finite whenever $\omega_1 \triangle \omega_2$ is finite, since the halfspace equation $f^{-1}(A') = A$, where $A \subsetneq X$, has finitely many solutions (for $x \in A$ and $y \in A^c$, any solution $A'$ will separate $f(x)$ from $f(y)$). Thus $f_*$ is well-defined.

That $f_*$ is a median morphism can be seen either by checking that it is a morphism of spaces with walls, or by checking that $f_*$ is median preserving. Finally, $f_*(\sigma_x) = \sigma_{f(x)}$. In general, $f_*$ need not be a graph morphism. This happens if the halfspace equation $f^{-1}(A') = A$, where $A \subsetneq X$, has more than one solution.

Uniqueness: $f_*$ is determined on $\{\sigma_x : x \in X\}$, which generate $\mathcal{C}^1(X)$. $\square$

It follows from Proposition 3.15 that a group action on a space with walls $X$ extends uniquely to a group action on its 1-cubulation $\mathcal{C}^1(X)$. Since the embedding $X \hookrightarrow \mathcal{C}^1(X)$ is isometric, the action on $X$ is bounded, respectively proper, if and only if the extended action on $\mathcal{C}^1(X)$ is bounded, respectively proper.

**Example 3.16 (3D Hex).** Let us cubulate the 1-skeleton of the hexagonal tiling of the plane. See figure 2.

The choice of halfspaces is independent along the three directions $X$, $Y$, $Z$. But this is also the case for the 1-skeleton of the usual tiling of $\mathbb{R}^3$ by 3-dimensional cubes. Since this is already a median graph, we conclude that it is the 1-cubulation of the hexagonal tiling of the plane.

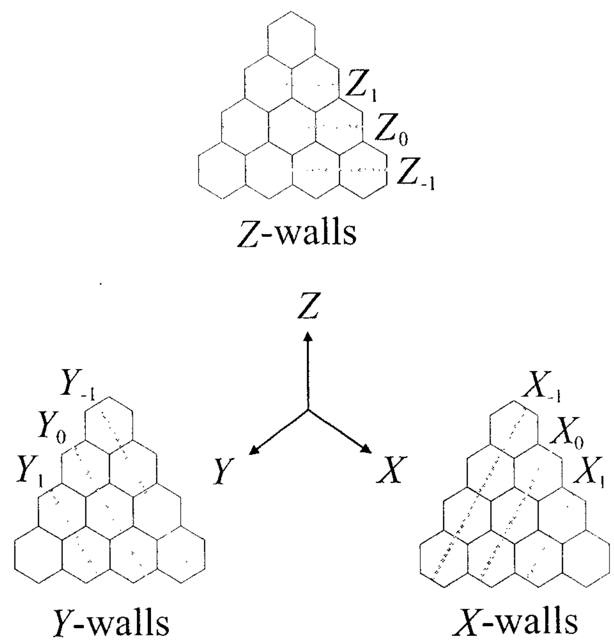



Figure 2: Cubulating the hexagonal tiling

**Notes.** This procedure, implicit in [Sag95], was made explicit in [Nic] and independently in [CN03]. We learned the term "cubulation" from Dani Wise.

### 3.2.1 Application: Ends of pairs of groups

The number of ends, much like the growth function, is an asymptotic invariant of groups defined in terms of their Cayley graphs. In what follows, it is convenient to consider finitely-generated groups.

**Ends of graphs.** Let $X$ be a locally finite connected graph. The *number of ends* of $X$ is the supremum of the number of infinite components one obtains by excising arbitrary finite sets from $X$. By local finiteness, excising arbitrary finite sets is equivalent to excising arbitrary bounded sets, which is equivalent to excising larger and larger balls around some basepoint.

The boundary $\partial A$ of a set $A \subseteq X$ is the set of edges connecting $A$ to $A^c$. Observe that $X$ has more than 1 end iff there is $A \subseteq X$ with $A$, $A^c$ infinite and $\partial A (= \partial A^c)$ finite.

**Ends of groups.** The *number of ends* of a group $G$ is the number of ends of a Cayley graph of $G$. For example, the trivial group has 0 ends, $\mathbb{Z}^2$ has 1 end, $\mathbb{Z}$ has 2 ends and $\mathrm{F}_2$ has infinitely many ends. In fact, these are the only possibilities: the number of ends of any finitely-generated group is one of 0, 1, 2, $\infty$.

A group has 0 ends if and only if it is virtually trivial, i.e. finite, and it has 2 ends if and only if it is virtually $\mathbb{Z}$. A theorem of J. Stallings characterizes the groups with more than 1 end as being those that split over a finite subgroup, which leaves the groups with 1 end as the generic case.

**Ends of pairs of groups.** One is lead to a notion of relative ends by the desire to have an invariant that encodes splittings over arbitrary subgroups. As Stallings' theorem suggests, one does not expect a splitting, but rather a virtual splitting. After all, the number of ends is a coarse notion so it is only natural that we have to settle for coarse splittings. Here is the notion of relative ends that has gained preeminence. We stress the fact that the finite-generation assumption applies to the ambient group only.

Given a group $G$ and a subgroup $K$, the number $e(G, K)$ of ends of $G$ relative to $K$ is the number of ends of a coset graph $\Gamma\backslash K$, the quotient by $K$ of a Cayley graph $\Gamma$ of $G$. For example, a cuspified $n$-gon suitably decorated to provide a covering of the wedge of two circles shows that the number of relative ends may take any value in $\mathbb{N} \cup \infty$.

The number of relative ends partially fulfills its purpose: if $G$ virtually splits over $K$ then $e(G, K) > 1$, but the converse fails in general.

At the core of Bass-Serre theory stands the fact that a group splits if and only if it acts on a tree without bounded orbits. Combining with Stallings' theorem, we deduce that a group with more than 1 end has an action on a tree without bounded orbits. This time, there is a satisfactory relative analogue.

**Theorem 3.17.** *For a finitely generated group $G$, the following are equivalent:*

*1) $G$ has more than 1 end relative to a subgroup*

*2) $G$ acts on a median graph without bounded orbits.*

Here is an outline of the proof. Start by translating the fact that $e(G, K) > 1$ into the existence of certain subsets of $G$ called $K$-sets. Provided that $G$ acts without bounded orbits on a median graph, focus on the wall structure of the graph and choose a suitable halfspace $H$ that will become a $G_H$-set, where $G_H$ is the stabilizer of $H$, and thus $G$ will have more than 1 end relative to $G_H$. Conversely, if $e(G, K) > 1$ then consider a space with walls whose underlying set is $G$ and whose walls are translates of a $K$-set. Now consider a median graph corresponding to this space with walls as described in Section 3.2.

An instantaneous consequence of Theorem 3.17 is that Kazhdan groups have no codimension-1 subgroups. To ask whether the converse holds amounts to Problem 3.6.

We have gathered enough evidence in order to assert that median graphs are a meaningful generalization of trees. Let us summarize.

Groups acting on trees were among the first examples of Kazhdan groups or Haagerup groups, depending on the flavor of the action. We have extended these ideas to median graphs in two ways: via spaces with walls, and as a particular instance of actions on median spaces.

The interplay between finite splittings, multiple ends and actions on trees are key ideas in the Bass-Serre theory. The corresponding triumvirate consists of arbitrary splittings, multiple relative ends and actions on median graphs.

Finally, the nonpositive curvature and the simple-connectivity of trees is present in cubings.

On the downside, a serious casualty is the algebraic criterion for bounded actions provided by Theorem 1.11.

**Notes.** Theorem 3.17 is a version of a theorem due to M. Sageev [Sag95]. Sageev's result is further explored in [Ger98], [Rol98].

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