A Class of Residually Finite Groups Isomorphic to Fundamental Groups of $\mathcal{VH}$ Complexes

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May, 2011

A thesis submitted to McGill University in partial fulfillment of the requirements of the degree of Master of Science.

Abstract

In a preprint, Ian Leary inquires whether two hyperbolic finitely presented groups are residually finite. We answer in the affirmative by showing that these groups belong to a class of groups, which we call the *polygonal* $\mathcal{VH}$ or $\mathcal{PVH}$ groups. To prove that a group is $\mathcal{PVH}$ we introduce a systematic tiling method for the standard 2-complex of the group, and deduce from the work of Daniel Wise that hyperbolic $\mathcal{PVH}$ groups are residually finite.

Résumé

Dans un prépublication, Ian Leary se demande si deux groupes finitement hyperboliques et de présentation finie sont résiduellement finis. Nous donnons une réponse positive en montrant que ces groupes appartiennent à une classe de groupes que nous appelons les groupes *polygonale* $\mathcal{VH}$ ou groupes $\mathcal{PVH}$. Pour démontrer qu'un groupe est $\mathcal{PVH}$, nous introduisons une méthode systématique pour couvrir d'un pavage le 2-complexe standard du groupe, et déduisons des travaux de Daniel Wise les groupes $\mathcal{PVH}$ hyperboliques sont résiduellement finis.

Acknowledgments

I thank my supervisor Dr. Daniel T. Wise for giving me a very good question for my thesis and the many hours he spent talking with me and explaining the many intricate geometric concepts in geometric group theory. I also wish to thank him for believing in my mathematical abilities. Without his encouragement to take his topology course when I was still unsure of my life's direction, I never would have applied to the Department of Mathematics at McGill and I never would have realized that my place is with pure mathematics.

I thank Maya Kaczorowski for providing many helpful suggestions and for help with the French abstract. I am grateful to Mark Hagen for explaining how quasiconvexity works by using the Švarc-Milnor lemma. I also thank Jan Feys for additional help with the French abstract, and Dr. Henri Darmon for his review.

I also wish to thank Greg LeBaron, the systems administrator at McGill, for providing me with one of the department's obsolete computers after my laptop ceased to function. I typed my entire thesis on this archaic but usable device. Also thanks goes to the Department of Mathematics at McGill for its warm and intellectually encouraging atmosphere.

I am indebted to my grandmother, Dr. Alenka Paquet. Without her, I probably would not have discovered mathematics and science as a child.

Contents

1 Notation

If X is a cellular complex then X^n denotes its *n-skeleton*. We refer to the cells of X as n-cells, and the cells of the link of a vertex as *edges* and *vertices*. A *polygon* P is a single 2-cell whose boundary is a subdivided circle. The *sides* of P are the 1-cells of this circle. Sides will always refer to these 1-cells, whereas an *edge* may be any 1-cell in some subdivision of the sides. If a side E is subdivided into n 1-cells, then we use write $|E| = n$. The *length* of a word w is denoted by $|w|$.

If G is a group then $H \leq G$ means H is a subgroup of G. Finally, the notation $A - B$ for sets A and B means set difference.

2 Introduction

Ian Leary in [Lea10] introduces two finitely-presented groups, and inquires about their residual finiteness, which is subtly hinted at by their construction. We prove that these groups are residually finite via splitting them using geometric techniques.

We start in this section by introducing presented groups and the word problem. The residual finiteness property for groups and its relation to the word problem are introduced in Section 2.3. At the end of the introduction we give a short account of our results.

Section 3 is technical and gives most of the details needed to understand the main result. In particular, we explain how graphs of spaces represent splittings of a group, and how we can obtain such splittings using so-called $\mathcal{VH}$ complexes. We say how residual finiteness is implied by certain special splittings with a few other conditions.

The main result is contained Section 4. Here we give an algorithm inducing a $\mathcal{VH}$ structure on the standard 2-complex of any finitely presented group satisfying certain conditions. This structure gives a splitting that may be used with hyperbolicity to prove residual finiteness via the observations in Section 3.

2.1 Presentations and the Word Problem

A *presentation* is denoted by $\langle S \mid R\rangle$ where S is a nonempty set of symbols called *generators*, and R is a set of words called *relators*, formed from letters $x \in S$ and x^{-1} where $x \in S$. Thus if $S = \{a, b, c\}$ then $ab^{-1}cc$ is an example of a valid relator. For brevity we write x^n for the string of n copies of x and x^{-n} for n copies of x^{-1}, so that our example would be written $ab^{-1}c^2$.

The *group given by such a presentation* is the free group on S modulo the normal closure of the subgroup generated by the relators. A *presented group* is one given by a presentation. If the presentation requires only a finite number of generators and relators, then we call G *finitely presented*.

One can define such a group directly without reference to the free group construction on a set, and such details are available in the classic text [MKS76], in which the authors develop extensively the purely combinatorial theory of groups.

Every group G has *some* presentation, the most obvious but unwieldy one being the multiplication table of G; that is, G is the set of generators and the relations are every relation in the multiplication table. Two simple examples are the finite cyclic groups $\mathbb{Z}/n\mathbb{Z}$ for some natural number n given by the presentation $\langle x \mid x^n\rangle$ and the free product $\mathbb{Z}/2 * \mathbb{Z}/2$ given by $\langle x, y \mid x^2, y^2\rangle$.

Simply-stated questions about presented groups often turn out to have fiendish solutions, or even no solution at all if one does not have some other purely algebraic description of the corresponding group. One such question that has come to the minds of combinatorial group theorists is the *word problem*: Given a word w in the generators S, does w represent the identity element? We say that a group G has *solvable word problem* if there is some algorithm that will decide in finite time whether any given word represents the identity. Otherwise we say G has *unsolvable word problem*.

Free groups, fundamental groups of closed, orientable surfaces, and $C'(\frac{1}{6})$ groups all have solvable word problems [MKS76]. On the other hand, there are some groups with unsolvable word problems, and some of these are even finitely presented. Collins gives a relatively small example of such a group with ten generators and twenty-seven relators in [Col86] using methods developed by Borisov [Bor69].

2.2 Topological Interpretation of Presentations

Although many interesting and nontrivial results on presented groups have been derived using purely combinatorial means, there is a natural topological interpretation called the *standard* 2*-complex* which is often worthwhile to consider, and it is through topological means which we achieve our results.

Given a presented group G, the standard 2-complex is a CW complex X with $\pi_1(X) \cong G$. It is given by a single 0-cell, a 1-cell for each generator, and a 2-cell for each relator whose attaching map to X^1 is given by the relator. More details are available in Section 3.3.

Example 2.2.1. Consider the group presented by $\langle s, t \mid sts^{-1}t^{-1}\rangle$. The single 2-cell corresponding to $sts^{-1}t^{-1}$ is a square that has its opposite sides identified in an orientation-preserving manner, which results in the familiar genus 1 orientable surface, the torus. The fundamental group of the torus is $\mathbb{Z} \times \mathbb{Z}$ which can be deduced from the presentation.

It is important to note that although groups given by presentations seem algebraic at first, they are more naturally interpreted as geometric objects, and thus conditions on the presentation naturally correspond to conditions on the corresponding complex.

A common theme that will recur is a *tiling* of such a complex, which is a subdivision of the cells of the standard 2-complex into smaller cells, often satisfying conditions that allow for some homotopy deformation, which in turn yields a *splitting* of the group. Sometimes in the literature a tiling of a complex by squares is also called a *squaring*.

Since we are only concerned with finite-dimensional CW complexes, the topology will just be the quotient topology, although in the infinite-dimensional case there are subtleties, and the interested reader may consult the appendix of [Hat02] for a gentle introduction to CW complexes or [LW69] for thorough education. Presentations in topology also arise naturally in the descriptions of the fundamental groups for topological manifolds, one example being 3-manifolds that are knot-complements.

2.3 Residually Finite Groups

In geometric group theory one is often interested in presented groups with no obvious algebraic interpretation of any kind, and it is with these groups that the word problem is most difficult. It seems reasonable then to consider classes of groups with additional properties that may ameliorate our troubles with the word problem. One such class is the residually finite groups.

Definition 2.3.1. Let G be a group. We say that G is *residually finite* if for each nontrivial $g \in G$, there exists a finite group F and a homomorphism $\varphi : G \to F$ such that $\varphi(g) \neq 1$.

Succinctly put, a group is residually finite if every nontrivial element survives in some finite quotient. The reader may be familiar with another commonly used definition: a group is residually finite if the intersection of all finite-index normal subgroups is trivial. The two definitions are clearly equivalent. A dynamical systems definition can be found in Ceccherini-Silberstein and Coornaert's book [CSC10]. We shall now give a few examples and some classical results.

Example 2.3.2. Aside from finite groups, perhaps the easiest example of a residually finite group is $\mathbb{Z}$, for if $n \in \mathbb{Z}$ is nonzero then the image of n in the finite quotient $\mathbb{Z}/(n{+}1)\mathbb{Z}$ will be nontrivial. Products of residually finite groups are clearly residually finite. Hence $\mathbb{Z}^n$ is residually finite.

A classical result proved by Baumslag is the following:

Theorem 2.3.3 ([Bau63])**.** *The automorphism group of a finitely generated residually finite group is residually finite.*

Example 2.3.4. Since $\mathbb{Z}^n$ is finitely generated we get from Baumslag's result that $\mathrm{GL}_n(\mathbb{Z})$ is residually finite.

Example 2.3.5. A class of residually finite groups given by presentations in given by Baumslag in [Bau67] (communicated by M. Suzuki):

$$\langle a, b \mid (a^{-1}b^l ab^m)^t = 1\rangle$$

where $l, m, t \in \mathbb{Z}$, $l, m \notin \{0, \pm 1\}$, $t > 1$, and t, l, m are pairwise relatively prime.

These last two examples are interesting because of the brevity of their corresponding proofs, which use only combinatorial methods. Despite these interesting curiosities, few methods are available to determine whether an arbitrary presented group is residually finite.

Example 2.3.6. For an example of groups that are *not* residually finite, we turn to the divisible abelian groups. Recall that an abelian group D is divisible if for each $x \in D$ and nonzero integer n there is a $y \in D$ such that $ny = d$; succinctly, we can divide by nonzero integers. Nontrivial divisible abelian groups *are not* residually finite.

Indeed, suppose that $D \xrightarrow{\varphi} F$ is a group homomorphism with D divisible and F finite. If $x \in D$, then for any nonzero integer n we have $\varphi(x) = \varphi(ny) = n\varphi(y)$ for some $y \in D$. But then if we take $n = |F|$, we get that $\varphi(x) = 0$. Thus φ is the trivial homomorphism, so D has no finite quotients. Since D is infinite, it thus cannot be residually finite. Thus the additive groups $\mathbb{Q}$ and $\mathbb{R}$ are not residually finite.

The most interesting property for us is that finitely presented residually finite groups have solvable word problem [Dys64], and thus we come to the main theme of this thesis; that is, *how can we show whether a group is residually finite?*

Remark 2.3.7. One might wonder if *finitely generated* residually finite groups have solvable word problem. This is not true! Meskin in [Mes74] produces a finitely generated center-by-metabelian residually finite group that has unsolvable word problem.

2.4 Motivation and a Brief Outline

We briefly describe the results, contents, and motivation of this thesis. Most formal definitions are postponed until later.

Recall that if G is a finitely presented group, a fixed presentation of G corresponds to a 2-dimensional cellular complex C called the standard 2-complex. It has the property that $\pi_1(C) \cong G$. *The main result of this thesis is a systematic method that gives under certain conditions, a so-called $\mathcal{VH}$-structure on a standard 2-complex C,* *which is a particularly nice subdivision C. This subdivision induces a splitting of $\pi_1(C)$, which implies residual finiteness.*

The genesis of this thesis is a preprint of Leary [Lea10], who inquires about the residual finiteness of some groups arising out of a construction using certain 2-complexes. One group he gives is the presented group

$$\begin{aligned}\langle a, b, c, d, e, f \mid\ & abcdef, ab^{-1}c^2f^{-1}e^2d^{-1}, a^2fc^2bed \\ & ad^{-2}cb^{-2}ef^{-1}, ad^2cf^2eb^2, af^{-2}cd^{-1}eb^{-2}\rangle.\end{aligned} \tag{1}$$

Leary gives a *tiling* of the standard 2-complex of this group by squares, which suggests a method to show residual finiteness for this and many other groups.

We start with a finitely presented group and give a *systematic* tiling method which works for many groups, including the ones given by Leary. The resulting complexes are $\mathcal{VH}$ *complexes* and are briefly introduced in Section 3.5. The main purpose of this tiling method is to give a practical method to show that a given group actually has a $\mathcal{VH}$ structure.

If the standard 2-complex has a $\mathcal{VH}$ structure, is nonpositively curved, and is also hyperbolic, residual finiteness follows from the results in [Wis, HW08]. We shall survey these results in Section 3.7.

In order to fix a class of groups of study, in Definition 4.2.8 we called any group which is the fundamental group a complex built out of $\mathcal{VH}$ polygons a *polygonal* $\mathcal{VH}$ group. Further, if the corresponding $\mathcal{VH}$ structure is nonpositively curved, then we call such a group a *nonpositively curved* $\mathcal{PVH}$ group. Thus a hyperbolic nonpositively curved $\mathcal{PVH}$ group is residually finite.

3 Background and Observations

3.1 A Short Note on Hyperbolicity

We briefly explain an easily-verified condition implying hyperbolicity, since all of our results assume that our group is hyperbolic. But first, we shall briefly recall hyperbolicity for convenience. In [Gro87], M. Gromov gave three equivalent definitions for a finitely presented group to be hyperbolic, and we shall give the second.

For a finitely presented group G, we fix some presentation $\langle S \mid R\rangle$. We use the standard notation $|g|$ to be the shortest representative of the element $g \in G$ with respect to the given presentation. Then we define a map on $G \times G \to \mathbb{R}$ by

$$(g, h) = \frac{1}{2}(|g| + |h| - |g^{-1}h|).$$

Definition 3.1.1. We say that a finitely presented group G together with a fixed presentation for G is hyperbolic if there is some $\delta \geq 0$ such that for every $g, h, k \in G$,

$$(g, h) \geq \min\{(g, k), (h, k)\} - \delta.$$

Although the condition of hyperbolicity is an important hypothesis in our main results, it is not central in the sense that none of the new developments here make direct use of it. In fact, below we shall see an easily-verified condition for hyperbolicity for the groups in which we have an interest.

Definition 3.1.2. Let G be a group presented by $\langle S \mid R\rangle$. We say that $r \in R$ is cyclically reduced if for each $a \in S$ there is no string of the form aa^{-1} in any cyclic permutation of r. We say that r is freely reduced just if there is no substring of the form aa^{-1} in r.

Let G be a group presented by $\langle S \mid R\rangle$ whose relators are cyclically reduced. A piece with respect to this presentation is a freely reduced subword that occurs in two distinct relators. Here we make the convention that a subword of a relator r is any substring of any cyclic permutation of r.

Definition 3.1.3. We say that a presentation $\langle S \mid R\rangle$ *satisfies the* $C'(\lambda)$ *(small cancellation) condition* if the length of any piece is less than or equal to $\lambda|r|$ for each $r \in R$ that contains the piece.

For instance, suppose that G is presented by $\langle\ a, b, c \mid aaabc, aba^{8743}c\ \rangle$. The longest piece in this presentation is ab. Since the length of the first word is 5, this group satisfies the $C'(\frac{2}{5})$ condition.

The $C'(\lambda)$ condition is the start of *small cancellation theory*, which analyzes the properties of groups with $C'(\lambda)$ conditions for various values of λ, but we shall only need the following well-known result that allows us to deduce hyperbolicity from a presentation (for some details see [Gro87, LS77]).

Theorem 3.1.4. *If G has a finite $C'(\frac{1}{6})$ presentation then G is hyperbolic.*

We deduce that Leary's group in Equation (1) is hyperbolic from using this condition: each relator has length six, and any word of length two occurs at most once, and so the longest piece must have length one. For longer presentations, the $C'(\frac{1}{6})$ could easily be checked by a computer program.

Remark 3.1.5. Whether every $C'(\frac{1}{6})$ group (hence hyperbolic) is residually finite is an open problem. If this is true, hopefully the proof will only be published after this thesis is submitted!

3.2 Graphs of Groups and of Spaces

Before starting with the main subject matter, we shall describe graphs of groups, and an associated construction, a graph of spaces. A *graph of groups* is given by the following data:

1. A connected graph Γ.
2. A group X_v for each vertex $v \in V(\Gamma)$.
3. A group X_e for each edge $e \in E(\Gamma)$.

4. Corresponding to each edge $e \in E(\Gamma)$, there are two monomorphisms $\imath : X_e \to X_v$ and $\tau : X_e \to X_w$ where $v = \imath(e)$ and $w = \tau(e)$ are the initial and terminal vertices of e respectively.

A *graph of spaces* corresponding to this graph of groups is the same construction, except with each group G being replaced by X_G with $\pi_1(X_G) \cong G$, and the monomorphisms inducing the monomorphisms for groups.

A graph of groups, and correspondingly the graph of spaces X with $\pi_1(X) \cong G$ gives a splitting of G in the obvious way, where the vertex groups are amalgamated along the edge groups. The association between the vertex groups and the edge groups is given by repeated application of the Seifert-van Kampen theorem [Hat02].

3.3 Groups and Complexes

A standard tool to treat groups geometrically will be the correspondence between a presentation and its *standard* 2*-complex*, which was introduced more informally in the introduction.

Definition 3.3.1. Let $G = \langle\ S \mid R\ \rangle$. The *standard* 2*-complex* of G is the cell complex consisting of a single 0-cell, exactly one oriented 1-cell for each $s \in S$, and exactly one 2-cell for each $r \in R$. The attaching maps of the 1-cells are the only possible maps. Let D be a 2-cell and r be the relator associated to D. The attaching map of D is given by dividing the boundary ∂D of D into a graph with $|r|$ edges, and orienting each edge with the corresponding letter of r. The attaching map is then giving by mapping each edge homeomorphically onto the associated 1-cell preserving orientation.

If C is the standard 2-complex associated to G, then $\pi_1(C) \cong G$. The 1-skeleton C^1 has as a fundamental group the free group on S and attaching 2-cells corresponds to taking quotients by successive normal closures of each relator. There is a very nice correspondence between subgroups of G and the covering spaces of C. Subgroups $H \leq G$ correspond to path-connected covering spaces of C whose fundamental group is H, and we can use topological methods to understand the structure of G via these covering spaces.

Definition 3.3.2. A *square tiling* of C is a subdivision of the 2-cells of C into squares.

We shall be interested in exploiting the combinatorial nature of certain tilings to give us a homotopy deformation into a graph of spaces.

3.4 Square Complexes and Nonpositive Curvature

We say a cellular complex X is a *combinatorial square complex* if the attaching map of each two-cell corresponds to a cycle in X^1 of length four, and X is two-dimensional. A square complex is thus a quotient of a disjoint union of copies of $[0,1]^2$, with the attaching maps corresponding to isometries between an edge and one or more edges. Given a square S, a *dual curve to a square* is a smoothly embedded interval in S whose endpoints are 0-cells resulting from a barycentric subdivision of opposite sides. A dual curve looks like a line down the middle of the square.

Notable examples of square complexes are the *nonpositively curved* square complexes, which we shall define in terms of the link.

Definition 3.4.1. Let $v \in X^0$. The *link* of v denoted $\mathrm{link}(v)$ is a one-dimensional cell complex specified as follows. The 0-cells are in bijection with the 1-cells attached to v. For $u, v \in X^0$, we declare that $(u, v) \in X^1$ if and only if the one-cell corresponding to u and the one-cell corresponding to v are in some attaching map of a two-cell.

We may also say more concisely that *two* 0*-cells of the link are connected by a* 1*-cell if and only if the corresponding once-cells in* X *are part of the same square.* Please see Figure 1 for an example.

Definition 3.4.2. A cellular square complex X is *nonpositively curved* if for each $x \in X^0$, the complex $\mathrm{link}(x)$ does not contain any cycles of length strictly less than four. We call any link that does not satisfy this property *bad*.

The tiling of $\mathbb{R}^2$ induced by the $\mathbb{Z}^2$ lattice is an example of a nonpositively curved square complex. The 2-skeleton of a cube is not nonpositively curved. Indeed, the

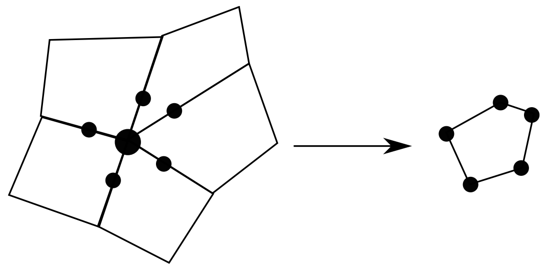

Figure 1: On the left, we wish to calculate the link of the large 0-cell in the middle of the complex. The link consists of one vertex for each 1-cell, and we connect vertices if they correspond to the same square. The result is the graph on the right.

link of any vertex is a three-cycle. Intuitively, positive curvature as in the 2-skeleton of the cube is a local phenomenon that cramps the complex to close in on itself.

We shall henceforth work with *nonpositively curved square complexes*. Actually, the notion link and nonpositive curvature may be generalized to a category of higher dimensional complexes known as *cube complexes* (see [Wis]), and in fact the results of this thesis depend on some results from the theory of cube complexes.

3.5 $\mathcal{VH}$ Complexes

Suppose that X is a cellular complex with a partition of of the one-cells $X^1 = H \cup V$. We shall call cells in H *horizontal*, and cells in V *vertical*. We say that a square complex X is a $\mathcal{VH}$ complex if the attaching map of each 2-cell has either the form $vhv'h'$ or $hvh'v'$ for $h, h' \in H$ and $v, v' \in V$. In this case it follows immediately that a $\mathcal{VH}$ complex is nonpositively curved if there are no cycles of length two in any link, since around any 0-cell the incoming edges alternate between horizontal and vertical.

The first key ingredient in understanding a group whose standard 2-complex has a $\mathcal{VH}$ structure is a result in [Wis06], which for our purposes can be stated as:

Lemma 3.5.1. *If X is a nonpositively curved $\mathcal{VH}$-complex then X splits as a graph of spaces in two ways. Furthermore, each splitting as a graph of spaces corresponds to a splitting of $\pi_1(X)$ as a graph of free groups.*

The sketch of the construction goes as follows. Suppose X has a $\mathcal{VH}$ structure as above. A *vertical leaf* of a vertex is a maximally path-connected subspace of vertical 1-cells. The leaves of the complex then correspond to the vertices of the graph of spaces, with each vertex space being the leaf itself. The edge spaces are more complicated. Two vertex spaces in the graph of spaces are connected by an edge if there is a horizontal edge in the complex X that intersects the two leaves that correspond to these spaces. The edge space itself is taken by choosing a vertex on this horizontal edge, and taking the dual curve D intersecting this vertex. The edge space is then $D \times [0, 1]$, and the attaching map is the obvious one. Please see Figure 2.

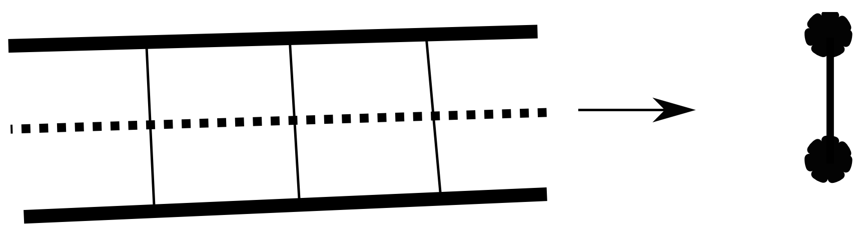

Figure 2: Locally, this is how the graph of spaces is constructed from a $\mathcal{VH}$ complex. The bold horizontal lines are parts of vertical leaves, and they become the vertex spaces represented by large ragged dots. The dotted line in the middle is another leaf that is a hyperplane and becomes the edge space after being crossed by an interval.

3.6 Quasiconvex Hierarchies

The class $\mathcal{QCH}$, or *groups with quasiconvex hierarchies*, is the class of groups that can be constructed from free groups via amalgamations and HNN extensions along quasi-isometrically embedded subgroups.

We shall just need a few basic properties of this class for the rest of the thesis, but first we shall define amalgamations and HNN extensions.

Definition 3.6.1. Let A, B be groups and $i : C \hookrightarrow A$ and $j : C \hookrightarrow B$ be monomorphisms. We define the *amalgamated free product* $A *_C B$ to be the free product $(A * B)/N$ where N is the normal closure of the subgroup generated by the relations $i(c) = j(c)$ for each $c \in C$.

In a graph of spaces, the amalgamated free product corresponds to a graph with two vertices and one edge with the edge corresponding to the subgroup and the vertex groups corresponding to A and to B.

Definition 3.6.2. Let A be a group, and let $B, C \leq A$ be subgroups with $\varphi : B \to C$ an isomorphism. The HNN extension $A*_\varphi$ is the group presented by taking a presentation for A, and adding a new symbol t together with relators $tbt^{-1} = \varphi(b)$ for each $b \in B$.

Before defining groups with a quasiconvex hierarchy, we first define the notion of a quasiconvex subgroup. Quasiconvexity is ubiquitous in geometric theory, and some further information can be found for example in [Gro87, BH99]. We will now turn to the Cayley graph.

Definition 3.6.3 (Cayley Graph)**.** Let G be a finitely generated group and let $A \subseteq G$ be a finite generating set. The Cayley graph $\Gamma = \Gamma(G, A)$ of G with respect to A is a graph Γ with vertex set $V(\Gamma) = G$. We declare that $(g, h) \in E(\Gamma)$ is an edge of Γ if and only if there is an $a \in A$ such that $g = ah$.

The Cayley graph of a group has a graph has a metric d on it, known as the *word metric*. For each $x, y \in G$, $d(x, y)$ is the infimum over all lengths of paths from x to y. This is a well defined nonnegative integer since S is a generating set, so there is some path from x to y. That d is a metric on G is trivial to verify. Furthermore, we can even extend this metric to the entire Cayley graph by considering edges to be copies of $[0, 1]$.

Definition 3.6.4 (Quasiconvex Subgroup)**.** We call a subgroup $H \leq G$ *quasiconvex* if there is an integer $K > 0$ such that for each $x, y \in H$, the geodesic connecting x and y is in the neighbourhood

$$N_K(H) = \{g \in G : d(g, h) < K \text{ for some } h \in H\}.$$

We now state the definition of groups having quasiconvex hierarchy as in [Wis], which is the smallest class of groups satisfying

1. $\{e\} \in \mathcal{QCH}$

2. $\mathcal{QCH}$ is closed under amalgamated free products by quasi-isometrically embedded finitely generated subgroups.

3. $\mathcal{QCH}$ is closed under HNN extensions by quasi-isometrically embedded finitely generated subgroups.

If A is a group, then taking an HNN extension along the trivial group gives the free product $A * \mathbb{Z}$, so that in particular finitely generated free groups belong to $\mathcal{QCH}$.

3.7 From $\mathcal{VH}$-Complex To Residually Finite

Given a finitely presented group G whose 2-complex X has a nonpositively-curved $\mathcal{VH}$-structure, we can apply Lemma 3.5.1 to split G as a graph of free groups. We then split each free group trivially, which shows that G has a length-2 hierarchy.

We require the edge groups to be finitely generated whenever we are almagamating or taking an HNN extension along a quasiconvex subgroup. Recall that the edge groups are just the product of a finite graph with an interval, so they are finitely generated whenever the graph is finite. These edge graphs will always be finite when our complex is compact. Note that this argument also applies to the vertex groups, which are also finitely generated, as they are just the fundamental group of compact graphs.

We still need that the edge groups are quasiconvex, so that the free groups will be amalgamated along finitely generated quasiconvex subgroups. We give the details in the next few paragraphs, assuming some basic covering space theory from algebraic topology, as can be found in [Hat02]. In order to do this, we start with the well-known Švarc-Milnor lemma, a fairly detailed proof of which can be found in [ECH+92].

Lemma 3.7.1 (Švarc-Milnor Lemma)**.** *If G acts properly and cocompactly by isometries on a proper geodesic metric space X, then G is finitely generated and for each $x \in X$, the map $f : G \to X$ given by $f(g) = gx$ is a quasi-isometry.*

Recall that a *proper* metric space is one whose finite closed balls are compact. Given a nonpositively curved square complex, its universal covering space admits a metric, the CAT(0) metric [Wis], in which each cube has the induced metric from Euclidean space.

If we now consider the edge space, we can consider a loop H that generates it, which will be a hyperplane or dual curve. Sageev showed in [Sag95] that this hyperplane will be convex in the universal cover $\hat{X}$ of X. This universal cover is a proper metric space, so if we want to apply Švarc-Milnor to show that G is quasi-isometric to X, we need to prove that it acts properly and cocompactly. That the action is proper follows from standard covering space theory. *Cocompact* means that $\hat{X}/G$ is compact. Since $\hat{X}/G \cong X$ and X is compact, cocompactness also follows.

Thus G is quasi-isometric to $\hat{X}$. It should be fairly clear that here we consider G as a metric space as it is identified with its Cayley graph. Note that we never mentioned an explicit generating set for the Cayley graph, but it turn out that a quasi-isometric embedding is independent of the generating set chosen [BH99].

Furthermore, we have a quasi-isometry $G \to \hat{X}$ for each $x \in X$ given by the map $g \mapsto gx$, so we can choose $x = h \in H$. Now our edge group G_H is just the stabilizer of H in $\hat{X}$, and we can restrict the map $G \to \hat{X}$ to $G_H \to H$ because G_H stabilizes H. We have a commutative square of metric spaces:

$$\begin{array}{ccc} H & \longrightarrow & X \\ \uparrow & & \uparrow \\ G_H & \longrightarrow & G \end{array}$$

We will be done if we can show that the map $G_H \to H$ is a quasi-isometry. By Švarc-Milnor lemma, it suffices to show that G_H acts properly and cocompactly on H. The action is cocompact because the original hyperplane is compact in X and the action is proper, since its just the restriction of the action of G on $\hat{X}$, which is proper. Thus $G_H \to H$ is a quasi-isometric embedding. Thus the edge groups are quasiconvex in the vertex groups.

We now use the following two theorems (see also [Wis09] for a summary).

Theorem 3.7.2 ([Wis])**.** *If G is hyperbolic with a quasiconvex hierarchy, then G is the fundamental group of a compact virtually special nonpositively curved cube complex.*

Theorem 3.7.3 ([HW08])**.** *Let B be a compact connected cube complex. If B is virtually special, then $\pi_1(B)$ is linear.*

Recall that a group H is linear if there is a monomorphism $H \hookrightarrow \mathrm{GL}_n(k)$ for some integer $n \geq 1$ and some field k. By a classical theorem of Mal'cev [Mal40], every finitely generated linear group is residually finite.

Remark 3.7.4. As previously mentioned, *cube complex* is just the natural analogue of the square complex in higher dimensions. We shall very briefly sketch a few of the ideas behind these two theorems which were prove by Haglund and Wise in [HW08]. Special cube complexes were found to have the unexpected property that their fundamental groups embed into *right angled Artin groups* or *right angled Coxeter groups*. Actually Haglund and Wise define two types of special cube complexes, corresponding to each case.

If B is virtually special and compact, we can find a finite cover B' whose fundamental group is a subgroup of a right-angled Coxeter group, which is known to be linear since B is compact. Finally, since B' is a finite cover, $\pi_1(B')$ has finite index in $\pi_1(B)$, so that $\pi_1(B)$ is also linear.

Hence we have the corollary that completes our investigation.

Corollary 3.7.5. *If G is a finitely presented hyperbolic group whose standard* 2-*complex has a nonpositively curved $\mathcal{VH}$-structure then G is residually finite.*

Theorem 3.1.4 tells us that we need only check the $C'(\frac{1}{6})$ condition to check hyperbolicity, which is what we shall use in practice.

4 Results: From Group to $\mathcal{VH}$ Complex to Residually Finite

4.1 Some Discussion

In Section 3, we have observed that if a finitely presented group G has a nonpositively curved $\mathcal{VH}$ structure and G is hyperbolic, then G is residually finite. That is, each nonpositively curved hyperbolic $\mathcal{PVH}$ is residually finite. Given a finitely presented group without any additional data, we still need to find a $\mathcal{VH}$ structure on it. In other words, in order to apply the observations of the previous section, we need to prove that G is actually in $\mathcal{PVH}$. In Section 4.2 we give a constructive procedure to give a $\mathcal{VH}$ structure on a 2-complex satisfying the so-called *triangle inequality*, and finally in Section 4.3 we show with a few figures how these methods work for the groups introduced by Leary in his preprint [Lea10].

4.2 Main Result: Conditions for a Tiling

In an arbitrary square complex S, each square has two associated *dual curves*. These are the 1-cells that connect the two 0-cells resulting from the barycentric subdivision of opposite sides of a square. A *hyperplane* is an immersion $h : [0, n] \to S$ for some n such that consecutive 0-cells are mapped to barycentric vertices on opposite sides of the same square and two consecutive images $(n-1, n)$ and $(n, n+1)$ lie in the interiors different squares.

By abuse of language, we shall refer to the hyperplane in S as the map h, so that h will denote the map and the set $h([0, n])$. Under mild assumptions that we shall describe below, a maximal hyperplane will be such that $S - h$ consists of two connected components. In such cases, it is natural to introduce a wallspace.

Definition 4.2.1. Let S be a set. A *wallspace* on S is a collection of pairs $\{(P_i, Q_i)\}_{i \in I}$ of subsets of S for some index set I such that each pair is a partition of S. We call the pairs *walls*.

A relevant example is a wallspace on $D = [0,1]^2$ with each wall (P_i, Q_i) being defined by a smooth 1-cell h connecting two distinct 0-cells on the boundary ∂D. In this case, $S - h$ consists of two connected components C_1 and C_2, and we set $P_i = C_1$ and $Q_i = C_2 \cup h$. We shall call such wallspaces *smooth*. Usually when drawn, a wallspace is represented by such lines, although the general definition admits all sorts of bizarre examples which we shall happily dismiss. In this case, we shall refer to the wall by abuse of language as h itself and we say two walls *intersect* if their corresponding curves intersect.

In such smooth examples, the curves defining our wallspace naturally give a 2-complex, and for our purposes we will only consider examples in which the wallspace gives us a square complex.

Definition 4.2.2. Let S be a 2-complex. The dual complex to S, denoted by S^*, is the complex defined as follows. S^* has exactly one 0-cell for each 2-cell of S. Two 0-cells of S^* are connected if and only if their corresponding squares in S are adjacent. Finally, any 1-skeleton of a square in S^* bounds a 2-cell.

Remark 4.2.3. Although we shall consider only 2-complexes S such that S^* is a square complex, S itself may not be a square complex. For instance, the large 2-cell in Figure 3 is a hexagon. Our application of the dual complex construction will be to a 2-complex, some of whose cells represent walls. We introduce the concept of a wallspace because it has a natural generalisation to higher dimensional cube complexes, and in this setting the dual complex is known as Sageev's construction [Sag95].

Let C be a 2-cell. We can obtain $\mathcal{VH}$-structure of C by constructing dual curves or a wallspace that corresponds to the $\mathcal{VH}$ complex. If our wallspace satisfies certain axioms which will be listed later, then the dual complex with be a $\mathcal{VH}$-complex. Please see Figure 3 for an example of this, which also illustrates the dual complex construction.

We shall now describe explicitly which conditions we need on a wallspace in order for the dual construction to give us a $\mathcal{VH}$-complex.

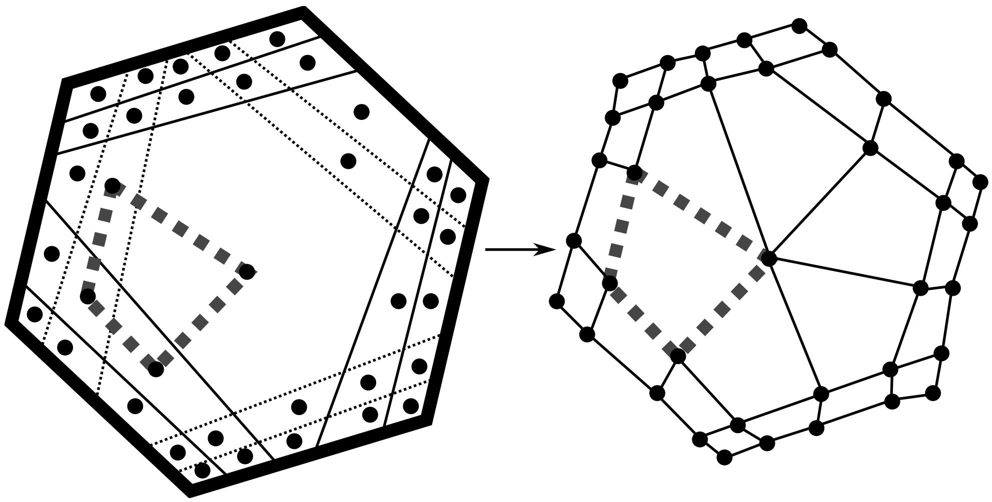

Figure 3: From wallspace to $\mathcal{VH}$-complex. In the left hexagon, a dual square is shown by the larger dotted line. On the right, the dual $\mathcal{VH}$-complex has been constructed.

Proposition 4.2.4. *Suppose that P is a polygon with sides $S_1, \ldots, S_k$ and that each side S_k is subdivided $p(k)$ times. Suppose that the sides are classified as either* horizontal *or* vertical*, and that there is a wallspace whose lines pair either horizontal edges or vertical edges, in which case we call the walls horizontal or vertical respectively. If no two walls of the same type intersect and no side contains the endpoints of the same wall, then the dual complex P^* is a $\mathcal{VH}$-complex.*

Proof. Consider a vertex $x \in P^*$. It corresponds to a 2-cell in P, whose boundary in P consists of alternating horizontal and vertical 1-cells. If these 1-cells were not alternating, then we would have two walls of the same type intersecting, which is excluded by hypothesis.

Furthermore, each vertex in P is the intersection of at most two walls, for otherwise we would again have walls of the same type intersecting. Hence any 1-cell starting at x is the side of a square, so P^* is a square complex. Furthermore, since the walls are already divided into horizontal and vertical types, and the only intersections are between horizontal and vertical walls, P^* is also a $\mathcal{VH}$ complex. ■

Recall that we would like to give a $\mathcal{VH}$-structure for the standard 2-complex of a finitely-presented group. Thus we would like to give each 2-cell corresponding to a relator a $\mathcal{VH}$-structure. Thus we consider each 2-cell a polygon P and we subdivide the sides in order to construct a wallspace on P. Finally we use the dual construction to obtain the dual P^*, which will be a $\mathcal{VH}$ structure. Notice that P^* will still clearly correspond to the original P whose sides have been subdivided.

Here, the sides of P will correspond to subwords of the relator and each side of P will be classified as either horizontal or vertical. Finally, we subdivide the sides of P into edges so that we can match edges of the same class by walls.

For the second step, we are thus led to a matching problem. In the sequel a polygon is a one-dimensional simplicial complex homeomorphic to a circle, and a edge of such a polygon is any 1-simplex.

Definition 4.2.5. Let P be a polygon and let $E_1, \ldots, E_k$ be distinct sides of P. Suppose each edge E_i is subdivided into n_i edges. An admissible pairing of these is an equivalence relation $\sim$ on the subdivided edges such that:

1. If $u \sim v$, $u \in E_i$ and $v \in E_j$ then $i \neq j$.
2. If $u \sim v$ and $w \sim x$, then w and x lie in the same connected component of $P - u \cup v$.
3. Each equivalence class has two members.

Suppose the sides of a polygon are divided into two classes $H_1, \ldots, H_m$ and $V_1, \ldots, V_n$, which we shall call horizontal and vertical respectively, and there is a subdivision of these edges so that there is an admissible pairing of the horizontals and an admissible pairing of the verticals. If we construct the dual complex to both pairings considered together, then by Proposition 4.2.4 we get a $\mathcal{VH}$-structure on this polygon.

Theorem 4.2.6. *Suppose that $E_1, \ldots, E_k$ are sides of a polygon, and $|E_i| = n_i$. If $\sum n_i$ is even, then there exists an admissible pairing of these vertices if and only if for each i, we have the triangle inequality*

$$n_i \leq \sum_{j \neq i} n_j. \tag{2}$$

Proof. Suppose that $E_1, \dots, E_k$ are ordered clockwise around the polygon.

We construct the admissible pairing via steps, each step pairing two edges. Each step will consist of choosing E_i with $|E_i|$ maximal, and take an unpaired $v \in E_i$ such that v is closest to another E_j with $i \neq j$. We then pair v with the nearest edge not in E_i. We continue until all edges can be paired, and we contend that if the triangle inequality holds, then all edges will be paired. Please see Figure 4 for an example of the first three steps of such a procedure.

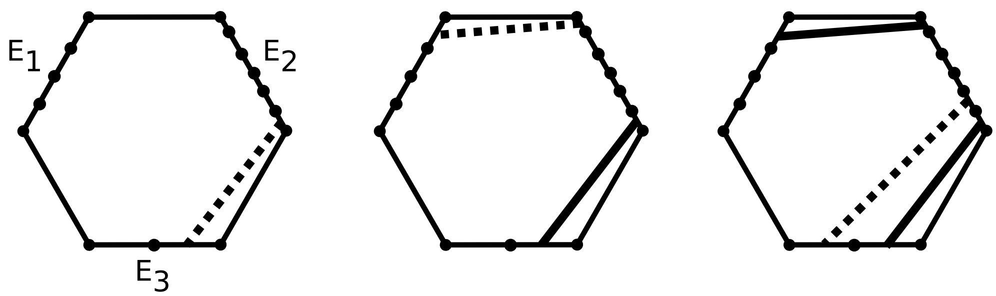

Figure 4: A hexagon, with three sides E_1, E_2 and E_3 chosen for pairing. The sides are subdivided, and the number of edges in each side satisfies the triangle inequality. The procedure for pairing is shown, with the dotted line showing the current pairing.

By convexity considerations, *if* we manage a pairing that satisfies (1) and (3), it will also be admissible. Suppose we represent the number of unpaired edges in each step of this pairing by an ordered vector so that before any edges are paired, our vector is $(n_1, \dots, n_k)$. Using our pairing strategy, each step will consist of subtracting 1 from two components, one component being maximal. We shorten the vector by dropping zero entries whenever they appear.

It thus suffices to prove that any vector $(n_1, \dots, n_k)$ can be completely reduced to the zero vector if and only if the triangle inequality in Equation (2) holds. Suppose that there is an admissible pairing. If the triangle inequality does not hold, then there exists an i such that $n_i > \sum_{j \neq i} n_j$. But each vertex in V_i must be paired with a vertex outside of V_i, and this is obviously impossible with this violation of the triangle inequality.

Now suppose the triangle inequality holds. If $n_i = 1$ for all i, then there is an admissible pairing since $\sum n_i$ is even.

Otherwise, choose an i such that $n_i \geq n_j$ for each j. Subtract one from n_i and an adjacent component. We claim that the triangle inequality holds on the new vector, and for simplicity we write this new vector as $(n_1, \ldots, n_{i-1} - 1, n_i - 1, \ldots, n_p)$.

Indeed, if $n_i > n_j$ for each $j \neq i$ then the triangle inequality holds. If there is more than one maximum then $n_i = n_j$ for some j. In this case, we cannot have just $n_i - 1$ and n_j left in the new vector, for otherwise before applying the reduction we would have an odd number of edges left unpaired. So the only possibilities are $n_i - 1$ and $n_j - 1$ left in the vector, or $n_i - 1$, n_j and some other nonzero entry, in which case the triangle inequality still holds.

Since the triangle inequality holds after each step, there are always sufficiently many edges to be paired, and so there is an admissible pairing. ■

Remark 4.2.7. An admissible pairing of a polygon with sides $E_1, \ldots, E_k$ correspond naturally to a tree considered as a disc diagram. Given such a subdivision of each edge E_i of P into n_i edges, if the triangle inequality holds then there is a tree T with ∂T corresponding to the polygon. The length of the side of T corresponding to E_i is $|E_i|$.

Suppose now that each 2-cell corresponding to a relator is given such a $\mathcal{VH}$-structure. Of course, the subdivisions for each 2-cell should be consistent so that the quotient map is combinatorial. We still do not necessarily have a nonpositively curved $\mathcal{VH}$-structure on our 2-complex, even though each 2-cell does have a nonpositively curved $\mathcal{VH}$-structure. It may happen that in the gluing process, some vertices will have a small link.

This can only happen with the 0-cells on the boundary of each 2-cell. Consider each 2-cell as a polygon P. If some 0-cell is on the interior of a side of the polygon, then it cannot have a bad link, since it already has two squares sharing the same 1-cell in P. But since the subdivisions were consistent across 2-cells, the same 0-cell in the quotient will have this same property in any other 2-cell, showing that a bad link can not occur at this 0-cell.

Thus the only possibility for a bad link are links of 0-cells on the corner of two meeting sides. In this case, a bad link will occur if such a corner is paired with another corner in the gluing. In other words, our $\mathcal{VH}$-complex will be nonpositively curved if and only if there are no repeated corners. Since our complex only has finitely many corners, we can verify easily whether such a complex is nonpositively curved. This prompts the following definition.

Definition 4.2.8 (Polygonal $\mathcal{VH}$)**.** We call a group G a *polygonal* $\mathcal{VH}$ group or $\mathcal{PVH}$ group if G is the fundamental group of a complex X made up of polygons glued together, each with a $\mathcal{VH}$ structure. Furthermore, if $E_1, \ldots, E_n$ are the sides of any one of these polygons, then $|E_i| \leq \sum_{j\neq i} |E_j|$. If X is nonpositively curved, then we call G a *nonpositively curved* $\mathcal{PVH}$ *group.*

Any $\mathcal{PVH}$ group is finitely presented since there are a finite number of polygons and a finite number of sides on each polygon. In this terminology, Corollary 3.7.5 can be rephrased by saying that any hyperbolic nonpositively curved $\mathcal{PVH}$ group is residually finite. Given a $\mathcal{VH}$ complex constructed out of polygons, the $C'(\frac{1}{6})$ condition can easily be verified by writing down the associated presentation, which can in turn be easily derived by ignoring the $\mathcal{VH}$ structure and considering each polygon as a 2-cell.

4.3 Examples and Explicit Calculations

There are two examples from Leary's paper [Lea10] that we shall consider. The first example is the group

$$\begin{aligned}\langle a, b, c, d, e, f \mid\ & abcdef, ab^{-1}c^2f^{-1}e^2d^{-1}, a^2fc^2bed \\ & ad^{-2}cb^{-2}ef^{-1}, ad^2cf^2eb^2, af^{-2}cd^{-1}eb^{-2}\rangle.\end{aligned}$$

Leary proved that this group is nontrivial, torsion free, and acyclic, and asked whether it is also residually finite.

If we consider each relator to be a polygon whose sides are given by powers of generators, then this presentation satisfies the triangle inequality, so we can apply Lemma 4.2.6 and obtain a $\mathcal{VH}$-structure on this complex. Checking that there are no repeated corners, we get nonpositive curvature. Furthermore, this group is obviously

$C'(\frac{1}{6})$, for any piece in the above relators is at most one generator. Hence this group is hyperbolic, and so we can apply Corollary 3.7.5 and deduce that this group is residually finite.

We shall now illustrate the algorithm as described in the proof of Lemma 4.2.6 on one of the relators of this group. We first need to arbitrarily decide which of the generators will be horizontal and which will be vertical. Here we shall decide that the vertical generators will be $V = \{a, c, e\}$ and the horizontal ones will be $H = \{b, d, f\}$.

Each relator now corresponds to a polygon whose sides are maximal adjacent vertical sets of generators or maximal adjacent horizontal sets. Consider the second relator $ab^{-1}c^2f^{-1}e^2d^{-1}$ for instance. One vertical side of its polygon is c^2. As in the terminology of the Lemma 4.2.6, we would like to pair edges of the sides according the rules we have defined. As it stands, the pairing is not possible, so we need to subdivide the sides of the polygon. If we subdivide each 1-cell corresponding to a generator once, we get a polygon as in Figure 5.

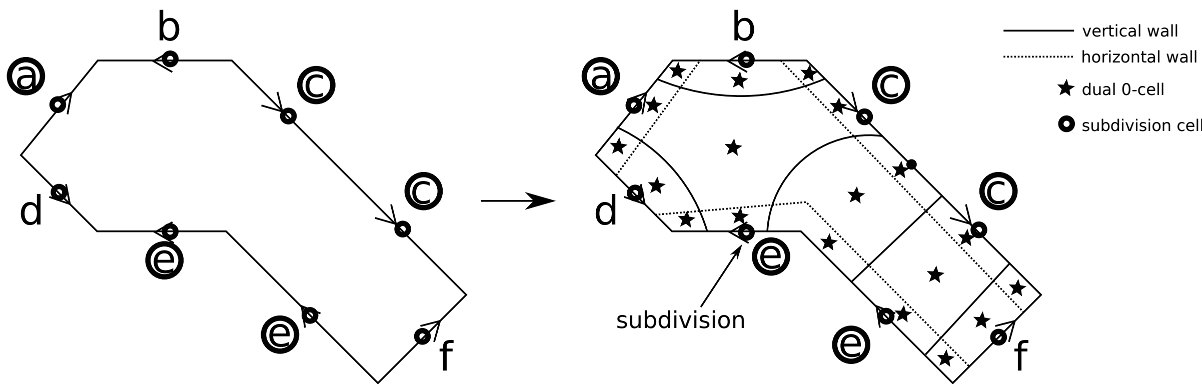

Figure 5: The first few steps to tiling the second relator in Leary's example. Consider the polygon on the left. As in the legend, the annuli represent one subdivision of each 1-cell corresponding to a relator. The polygon on the right is the wallspace constructed from the algorithm in Lemma 4.2.6. The stars represented the dual 0-cells that will be the new 0-cells of the $\mathcal{VH}$-structure on our relator.

Note that the subdivision should be made in reference to the generators once and for all, so that the subdivisions are consistent across relators. We are, after all, attempting to add a $\mathcal{VH}$-structure to the entire 2-complex. We now have side lengths for the vertical sides to be $2, 4$ and 4, and for the horizontal, $2, 2$ and 2. These satisfy the triangle inequality, so the algorithm will work.

The next step is to use the matching algorithm to match horizontal edges with other horizontal edges not on the same side, and the same with vertical edges. This is also shown in the figure. We then apply the dual complex construction as in Figure 6 to get the desired $\mathcal{VH}$-structure.

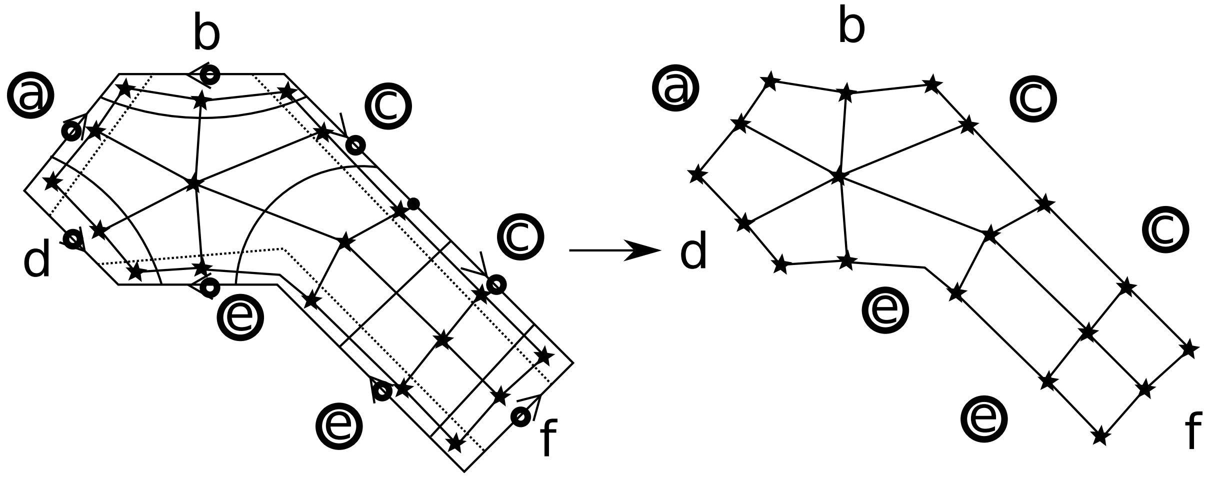

Figure 6: We now connect the dual 0-cells (the stars) by an edge if and only if their corresponding 2-cells in the original polygon are adjacent. We get a new polygon on the right, which happens to be a tiling of the old one!

The second example with which Leary is concerned is a group defined as follows. We let $n = 4$, and for each $i \in \mathbb{Z}/n\mathbb{Z}$ we define the two words $A_i = a_i a_{i+2} a_i^{-2} a_{i+2}^{-1} a_i$ and $B_i = b_i b_{i+2} b_i^{-2} b_{i+2}^{-1} b_i$ and the eight words given by $a_i A_i B_i A_{i+1} B_i A_{i+2} B_i A_{i+3} B_i$ and $b_i B_i A_i^{-1} B_i A_{i+1}^{-1} B_i A_{i+2}^{-1} B_i A_{i+3}$. Again, by inspection this group is $C'(\frac{1}{6})$ and these relators also satisfy the triangle inequality by grouping the relators with a has horizontals for instance. It takes a bit longer, but by looking at this presentation for repeated corners shows that there are none.

Remark 4.3.1. Some care must be used in selecting the horizontal and vertical generators. Since not all possibilities will give the triangle inequality condition, regardless of the number of subdivisions on each side. However, since this technique only applies to finitely presented groups, one only has to check a finite number of cases to find the good cases, if any.

5 Concluding Remarks

We shall briefly indicate possible future directions of this type of work with a minimum of technical detail.

In Section 3.7 we stated Theorem 3.7.2 from [Wis] that allowed us to eventually deduce the residual finiteness of a hyperbolic nonpositively curved $\mathcal{PVH}$ group. Although Theorem 3.7.2 is impressive, it relies on hundreds of pages of heavy machinery. It would be nice to have an alternative proof of this without relying on Theorem 3.7.2.

Another interesting idea is suggested in [Wis03], which gives the following theorem.

Theorem 5.0.2. *Suppose G splits as a finite graph of finitely generated free groups. If each edge group incident at each vertex group is malnormal, then G is residually finite.*

A subgroup $H \leq G$ is called *malnormal* if for each $g \notin H$, we have $gHg^{-1} \cap H = \{1\}$ —sort of an opposite condition to normality. According to [Wis01], one can verify malnormality by examining the attaching map $f : E \to V$ where E is an edge space and V is a vertex space—in this case by abuse of notation we consider E as the graph *before* taking its product with $[0, 1]$. We then take the inverse limit of the diagram $E \to V \leftarrow E$, which in concrete terms is the *graph fiber product.*

This inverse limit can be described as a graph whose vertices are pairs $(e_1, e_2) \in E^2$ such that $f(e_1) = f(e_2)$, and such that (e_1, e_2) and (d_1, d_2) are connected by an edge if and only if $f(e_1)$ and $f(d_1)$ are connected in V. One can verify that this fiber product with the obvious maps has the usual universal mapping property.

The *diagonal component* of the fiber product are pairs (e, e) for all $e \in E$, and all such edges incident to these vertices. It turns out that the subgroup given by the edge space is malnormal with respect to an incident vertex space if the fiber product without the diagonal component is a disjoint union of trees, also known as a *forest.*

If Leary's groups had turned out to have this property once tiled, then this would be a nice lighter alternative to our approach here. I wrote a small computer program to calculate this fiber product, but the result for Leary's first group, at least under one tiling, was not a forest.

It would be interesting to determine conditions on a presentation, or a tiling, which would indicate malnormality.

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