On non-nuclear C*-algebras with the Local Lifting Property

Sacha L'Heveder,

Department of Mathematics and Statistics,

McGill University, Montreal.

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Contents

Abstract

A C*-algebra has the *lifting property (LP)* if any ucp map to a quotient C*-algebra admits a ucp lift. It has the *local lifting property (LLP)* if every ucp map from a finite dimensional operator system admits a ucp lift. The latter, introduced by Kirchberg in 1993 [23], is dual to the weak expectation property introduced by Lance in 1973 [24]. The Choi-Effros lifting theorem [10] implies that nuclear C*-algebras have the LLP, however non-nuclear examples are few and far between. Following a construction of Courtney [13], we present an exposition of results from various fields of C*-algebra theory which we will use to give another such example.

Resumé

Une C*-algèbre a la *propriété de relèvement (local)*, si toute fonction unitale et complètement positive (ucp) à valeur dans une C*-algèbre quotient admet (localement) un relèvement ucp. Cette propriété de Kirchberg (1993) [23] est en dualité avec la propriété d'espérance faible de Lance (1973) [24]. Grâce à un theorème de Choi et Effros [10], nous savons que les C*-algèbres séparables et nucléaires ont la propriété de relèvement local. Pourtant, les exemples non-nucléaires ne sont pas nombreux. Suivant une construction de Courtney [13], nous donnons un exposé de divers résultats venant de la théorie des C*-algèbres et construisons un tel exemple en se basant sur ces résultats.

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Introduction

In his groundbreaking 1993 paper [23], Kirchberg introduced the local lifting property: A unital C*-algebra A has the local lifting property (LLP) if for every unital completely positive (ucp) map $\varphi : E \to B/I$ from a finite-dimensional operator system $E \subset A$ to a quotient C*-algebra, there exists a ucp lifting $\psi : E \to B$. This property can be represented in the following commutative diagram:

$$\begin{tikzcd} & B \arrow[d, "\pi"] \\ A \supset E \arrow[ur, dashed, "\psi"] \arrow[r, "\varphi"'] & B/I \end{tikzcd}$$

Kirchberg showed that this property is dual to Lance's weak expectation property (WEP), leading to many deep results, particularly an equivalence of many important conjectures in operator algebras and various subfields of quantum physics.

Thanks to the completely positive lifting theorem of Choi and Effros (5.1.14), separable nuclear C*-algebras have the LLP, however examples outside of this class were not plentiful. Courtney [13] shows us a way to construct such an example, drawing from various branches of C*-algebra theory. The objective of this thesis is to use the construction of Courtney as a backdrop, allowing us to present proofs of some major results from across the field of C*-algebra theory.

A chronological breakdown of the thesis is as follows: In the first section, we begin at the most very basic level of operator algebras. We then, primarily following [7], formalize the theory of C*-algebras and completely positive maps, including tensor products and characterizations of nuclearity and exactness. In section 3 we introduce multiplier algebras, giving proofs of some classical results, including Brown's stable isomorphism theorem. We also touch on some basic operator K-theory, a useful tool in the study of C*-algebras, particularly pertaining to classification. George A. Elliot famously used the K_0 functor to classify approximately finite dimensional (AF) C*-algebras [16]. We follow this by discussing C*-algebras with real rank zero. The real rank of a C*-algebra, introduced by Brown and Pedersen [5], is the non-commutative analogue of the covering dimension for a topological space. We give the proof of an important result (theorem 4.1.8) showing that this property is equivalent to several others.

In section 5 we discuss the LLP, Lance's WEP, and the theory that follows from Kirchberg's result that they are dual to each other in the sense of proposition 5.1.8. The major consequence of this result is that the (up until recently open) question of Connes' embedding problem was equivalent to a number of conjectures about the WEP and LLP, including the QWEP conjecture (see 5.1.7). We give a proof of a remarkable characterization of the LLP (theorem 5.3.3) from a paper of Ozawa [31], which states that ucp maps to the Calkin algebra $B(\ell^2)/K(\ell^2)$ being liftable is sufficient for the LLP.

Another question of Kirchberg was whether there existed a non-nuclear C*-algebra which had both the LLP and WEP (5.1.15). This was open until a recent paper of Gilles Pisier [36], where he provides an example which satisfies the stronger requirement that the C*-algebra is not even exact. Finally, in section 6 we address the slightly less restrictive task of coming up with examples of non-nuclear C*-algebras with just the LLP. This is done following results of Courtney [13] which build on the work of Hadwin [17], and Loring and Shulman [25] regarding universal C*-algebras - a construction of Blackadar [4].

1 Basics of Operator Algebras

We will first give basic definitions, examples, and results from the basic theory of operator algebras. Following this, we will state the important functional analysis theorems which allow us to say things like "by spectral theory" with impunity. It is more than likely that the reader is already familiar with the content of this section. Proofs are therefore omitted, but included in an appendix for the sake of completeness.

Definition 1.0.1. A complex Banach algebra A is called a *C*-algebra* when equipped with an involution operation $x \mapsto x^*$ such that for all $\alpha, \beta \in \mathbb{C}$, $x, y \in A$

(1) $(\alpha x + \beta y)^* = \bar{\alpha} x^* + \bar{\beta} y^*$

(2) $(xy)^* = y^* x^*$

(3) $(x^*)^* = x$

(4) $\|x^* x\| = \|x\|^2$.

Conditions (1), (2), and (3) define an involution on A, making it a *Banach *-algebra*. The addition of condition (4), the "(strong) C*-condition," defines a *C*-algebra*. Condition 4 implies that the involution is isometric (and thus continuous):

$$(4) \Rightarrow \|x\|^2 \leq \|x^*\|\|x\| \Rightarrow \|x\| \leq \|x^*\|. \text{ Replace } x \text{ with } x^*, \|x^*\| \leq \|x^{**}\| = \|x\|.$$

One can show that 4 is equivalent to the condition $\|x^*x\| = \|x\|\|x^*\|$, and do so without assuming the involution is isometric. This equivalence is nontrivial to prove, however, and we will only make use of condition 4 as stated above.

Definition 1.0.2. A morphism in this category of C*-algebras is called a **-homomorphism*. It is a homomorphism in the usual sense of Banach Algebras, with the added property that it preserves the involution: i.e: $f(x^*) = f(x)^*$. An isomorphism in this category is a *-isomorphism. It follows from the C*-condition that a *-homomorphism is norm non-increasing, and thus a *-isomorphism is isometric.

The canonical example of a C*-algebra is the space of bounded operators on a Hilbert space, denoted $B(\mathcal{H})$. In this case, the involution is given by the adjoint. The conditions of definition 1.0.1 were conceived as an abstract characterization of the structure on $B(\mathcal{H})$ (see A.1.6).

Another example of a C*-algebra is the algebra of complex valued continuous functions vanishing at infinity, $C_0(X)$, on a Hausdorff topological space X. Here the involution is given by complex conjugacy of the function, i.e: where $\bar{f}(x) = \overline{f(x)}\,\forall x \in X$. The C*-condition is immediately obvious in this case. This is an example of a commutative C*-algebra and, in fact, due to the Gelfand representation we may view any commutative C*-algebra as such a space where the topological space X consists of the characters of of the C*-algebra equipped with the weak*-topology (more on this in the appendix).

Definition 1.0.3. We say an element x of a C*-algebra A is:

(1) *Self-adjoint* (or *Hermitian*) if $x^* = x$,
(2) *Normal* if $x^*x = xx^*$,
(3) *Unitary* if $x^*x = xx^* = 1$.

If we view matrices $M \in M_n(\mathbb{C})$ as linear operators on $\mathbb{C}^n$ with the operator norm, we see that the conjugate transpose operation acts as an involution, thus endowing $M_n(\mathbb{C})$ with a C*-algebra structure. In fact, any finite dimensional (by this we mean the dimension of the underlying vector space) C*-algebra is isomorphic to a finite direct sum of complex matrix algebras $M_n(\mathbb{C})$. This is a corollary of the Artin-Wedderburn theorem for semisimple algebras (the self-adjointness of a C*-algebra implies that it is semisimple as an algebra over $\mathbb{C}$).

Definition 1.0.4. An element of A is said to be *positive* if there exists some $y \in A$ such that $x = y^*y$. We'll denote by A_{sa} the set of self adjoint elements and by A_+ the set of all positive elements in A.

The set of self adjoint elements in a C*-algebra forms a locally convex real vector space with a partial ordering, which we'll denote $\leq$. For a positive element we write $x \geq 0$, and we declare $x \geq y$ if $x - y \geq 0$. A particular type of positive element of a C*-algebra is a *projection*. p is a projection if $p = p^2 = p^*p$.

We will mostly concern ourselves with *unital* C*-algebras, that is, those with a multiplicative unit 1 (sometimes denoted $1_A \in A$ if there is ambiguity). There will, however be times when we must prove results in the non-unital case. Luckily we are always able to approximate a unit:

Definition 1.0.5. Let I be an ideal in a C*-algebra A. An *approximate unit* (or *approximate identity*) is a net $\{e_\alpha\} \subset A_{sa}$ such that

$$\lim_\alpha xe_\alpha = x = \lim_\alpha e_\alpha x$$

for every x in A. We say this approximate unit is bounded if there is a uniform bound on elements of the net ($\|e_\alpha\| < M$, $\forall\alpha$), and increasing if $\alpha \leq \beta \Rightarrow e_\alpha \leq e_\beta$. We call an approximate unit *quasicentral* if $\|xe_\alpha - e_\alpha x\| \to 0$.

Every C*-algebra admits an approximate unit, which we can take to be increasing and bounded (by 1). A separable C*-algebra admits a countable (or sequential) approximate unit. A C*-algebra admitting a countable approximate unit is called *σ-unital.* An equivalent condition to admitting a countable approximate unit it that of admitting a *strictly positive element* $a \in A_+$ such that aAa is dense in A (and hence aA and Aa are dense too).

Proposition 1.0.6. Let A be a C*-algebra without identity and denote by $\tilde{A}$ the algebra obtained by adjoining an identity 1 to A. $\tilde{A}$ is a C*-algebra with the following norm:

$$\|\lambda 1 + x\| = \sup_{y \neq 0} \frac{\|\lambda y + xy\|}{\|y\|} \quad \text{for } x \in A,\ \lambda \in \mathbb{C}$$

Proof. First we see that $\tilde{A}$ is a Banach *-algebra with this norm (this just consists of checking conditions 1-3 of the first definition). Now let $0 < \mu < 1$. From the definition of the norm on $\tilde{A}$, there exists some y with $\|y\| = 1$ such that $\mu\|\lambda 1 + x\| < \|\lambda y + xy\|$. Then,

$$\mu^2\|\lambda 1 + x\|^2 < \|\lambda y + xy\|^2 = \|(\lambda y + xy)^*(\lambda y + xy)\| \text{ by condition (5)}$$
$$= \|y^*(\lambda 1 + x)^*(\lambda 1 + x)y\| \leq \|(\lambda 1 + x)^*(\lambda 1 + x)\| \text{ since } \|y\| = 1.$$
$$\text{So, in particular } \|\lambda 1 + x\|^2 \leq \|(\lambda 1 + x)^*(\lambda 1 + x)\| \leq \|(\lambda 1 + x)^*\|\|(\lambda 1 + x)\|$$
$$\Rightarrow \|\lambda 1 + x\| \leq \|(\lambda 1 + x)^*\|$$
$$\text{This means that } \|(\lambda 1 + x)^*\| \leq \|(\lambda 1 + x)^{**}\| = \|\lambda 1 + x\|$$
$$\text{and therefore } \|\lambda 1 + x\|^2 = \|(\lambda 1 + x)^*\|\|\lambda 1 + x\|$$

hence, giving us the strong C*-condition. This shows that every C*-algebra can be isometrically embedded into C*-algebra with unity, called the *unitization*. □

Definition 1.0.7. Let A be a C*-algebra and $E \subset A$ be a self adjoint subspace containing 1_A. We say that a linear functional $\rho : E \to \mathbb{C}$ is a *state* of E if it is a positive linear functional (i.e. that $\rho(x) \geq 0$ if $x \in A_+$) and $\|\rho\| = 1$. The set of states of A is denoted $\mathcal{S}(A)$ and the extreme points of this space are called the *pure states*.

Definition 1.0.8. Let $\mathcal{H}$ be a Hilbert space. A **-representation* $\pi : A \curvearrowright \mathcal{H}$ is a *-homomorphism $\pi : A \to B(\mathcal{H})$. In the language of representation theory, we call such a *-representation *faithful* if it is injective.

Of course if π is a faithful *-representation of a C*-algebra A, A is *-isomorphic to its image, and thus any time we can construct a *-representation of A on a Hilbert space, we may consider A as a *-subalgebra of $B(\mathcal{H})$. We will also sometimes drop the "*-" and just refer to representations of C*-algebras without any ambiguity. Following the Gelfand-Naimark-Segal construction A.1.6, we can build, out of each state, a representation on a specific Hilbert space which has a corresponding so-called cyclic vector.

Definition 1.0.9. An element $\xi \in \mathcal{H}$ is called a *cyclic vector* of the representation $\pi : A \curvearrowright \mathcal{H}$ if the set $\{\pi(x)\xi : x \in A\}$ spans $\mathcal{H}$.

The representation π is called *non-degenerate* if the set

$$\pi(A)\mathcal{H} = \{\pi(x)\xi : \ x \in A, \ \xi \in \mathcal{H}\}$$

is dense in $\mathcal{H}$. In the case where A is unital this is equivalent to having that $\pi(1) = 1$.

Theorem 1.0.10 (Gelfand, Naimark)**.** Let A be a C*-algebra. There exists a faithful *-representation $\pi : A \curvearrowright \mathcal{H}$ on a Hilbert space $\mathcal{H}$ with cyclic vector ξ.

Definition 1.0.11 (Excision)**.** We say that a state ρ can be *excised* if there exists a net of positive elements $0 \leq e_i \leq 1$ such that

$$\|e_i a e_i - \varphi(x) e_i^2\| \to 0 \ \text{ and } \ \rho(e_i) = 1, \ \text{for every } x \in A.$$

An important result of Akemann, Anderson, and Pedersen says that a states lies in the weak*-closure of the pure states if and only if it observes an excision property. The following lemma, which is simply a part of this result, will be useful in writing a short proof of Takesaki's theorem 2.2.12.

Lemma 1.0.12 (Akemann, Anderson, Pedersen)**.** If $\rho \in \mathcal{S}(\mathcal{A})$ is a pure state, it can be excised.

Proof. First we prove the unital case. For any state ρ of a C*-algebra A, the set

$$\mathcal{L}_\rho = \{x \in A : \rho(x^* x) = 0\}$$

is called the *left-kernel* of ρ. Suppose a_i a right approximate unit for $\mathcal{L}_\rho$. That is, for every $x \in \mathcal{L}_\rho$, $\|x - x a_i\| \to 0$. Let $e_i = 1 - a_i$. Note that $x - \rho(x) \in \ker(\rho) = \mathcal{L}_\rho + \mathcal{L}_\rho^*$,

$$\|e_i x e_i - \rho(x) e_i^2\| = \|e_i(x - \rho(x)) e_i\| \to 0.$$

For the non-unital case, let b_j be a quasicentral approximate unit with the property that $\rho(b_j) = 1$ (existence of such an approximate unit follows from [20, theorem 5.4.3]). Then $b_j e_i b_j$ will work, where e_i was found as above for the unitization $\tilde{A}$. $\square$

1.1 Von Neumann Algebras

Definition 1.1.1. A C*-algebra M which is the dual space of a Banach algebra M_* (the "predual") is called a *W*-algebra* (that is, $M = (M_*)^*$). Much like how a norm-closed subalgebra of a C*-algebra is a C*-subalgebra, a $\sigma(M, M_*)$-closed (ultraweakly closed) subalgebra of M is a W*-subalgebra.

When discussing W*-algebras, we will refer to the *ultraweak* or $\sigma(M, M_*)$ topology on a von Neumann algebra, which we will sometimes simply call the σ-topology. For clarity it is perhaps worth mentioning that the predual of a Banach space embed in the dual space (the predual's double dual, if you will), hence why taking the weak*-topology of a the pair (M, M_*) makes sense.

A *-homomorphism between W*-algebras is called a *W*-homomorphism* if it is continuous in the respective ultraweak topologies on the W*-algebras. Predictably, we call a W*-homomorphism $\pi : M \to B(\mathcal{H})$ a W*-representation of M.

Theorem 1.1.2 (Sakai)**.** Every W*-algebra has a faithful W*-representation on some Hilbert space $\mathcal{H}$. Thus every W*-algebra is *-isomorphic to a weakly closed self adjoint subalgebra of $B(\mathcal{H})$.

Definition 1.1.3. A weak operator closed subalgebra of operators $M \subset B(\mathcal{H})$ is called a *von Neumann algebra*. We shall see, thanks to von Neumann's bicommutant theorem, that we could have taken the SOT instead of the WOT.

Let B be any subset of bounded operators on $\mathcal{H}$, a Hilbert space. The *commutant* of B is

$$B' = \{x \in B(\mathcal{H}) : xy = yx, \forall y \in B\}.$$

The double commutant, or *bicommutant*, is simply $B'' = (B')'$. Note that any $x \in B$ certainly commutes with all of B', by definition, and so $B \subset B''$. It is also easy to see that if $A \subset B$, $B' \subset A'$. It follows that $B''' \subset B'$, but we also note that $B' \subset (B')'' = B'''$. Thus,

$$B \subset B'' = B^{(4)} = ...$$

$$B' = B''' = B^{(5)} = ...$$

Theorem 1.1.4 (von Neumann)**.** Let M be a unital (with identity 1) *-algebra. Then the following conditions are equivalent:

(1) M is weak operator closed.

(2) M is strong operator closed.

(3) $M = M''$

Definition 1.1.5. Let A be a C*-algebra and $\pi_u : A \to B(\mathcal{H})$ be the universal representation. We call $\pi_u(A)''$ the *enveloping von Neumann Algebra* of A.

The following theorem allows us to identify the enveloping von Neumann algebra with the double dual A^{**}, which we will do without mentioning, following well established precedent. It was first proposed by Sherman in 1950, and later proved completely by Takeda in 1954.

Theorem 1.1.6 (Sherman, Takeda)**.** Let A be a C*-algebra. Then the double dual, A^{**}, is a von Neumann algebra. In particular, $A^{**} = \pi_u(A)''$.

Remark 1.1.7 (Normal Extension). An important direct consequence of this identification is the following universal property: If $\pi : A \to B(\mathcal{H})$ is a non-degenerate representation, then there exists a unique *normal extension*, a normal representation $\tilde{\pi} : A^{**} \to B(\mathcal{H})$ extending π and such that $\tilde{\pi}(A^{**}) = \pi(A)''$.

Definition 1.1.8. A *trace* τ on a von Neumann Algebra M is a linear map $M_+ \to [0, \infty]$ such that for any $x \in M_+$, $\tau(x^*x) = \tau(xx^*)$. We say a trace is *faithful* if $\tau(x) = 0 \Rightarrow x = 0$, and we call it a *tracial weight* if it is unital.

The standard trace Tr on $B(\mathcal{H})$ is an example of a normal, faithful, tracial state.

1.2 Spectral Theory

Spectral theory takes many different forms. For us, the essential will be to compile results relating to the Gelfand representation, continuous functional calculus, and a spectral theorem. These results are ubiquitous in the basic literature on operator algebras, so we will refer the reader elsewhere for the proof. A good exposition can be found in [39], for example.

Definition 1.2.1. Let A be a C*-algebra and $x \in A$. The *spectrum* of x, denoted $\sigma_A(x)$ is the set of all $\lambda \in \mathbb{C}$ such that $(x - \lambda I)$ is not invertible in A.

Proposition 1.2.2. Let A be a C*-algebra. There is a unique norm on A preserving the C*-algebra structure.

Proof. To see this we use that if $\|\cdot\|$ is a so-called C*-norm on A,

$$\|x\|^2 = \|x^*x\| = \sup\{|\lambda| : \lambda \in \sigma_A(x^*x)\},$$

(the second equality holds because x^*x is normal) and note that the set on the right is independent of the norm on the left. $\square$

The Gelfand Representation is constructed for a general Banach algebra, however we will state the C*-algebra case. Let A be an abelian C*-algebra. We denote by $\sigma(A)$ the set of *characters* of A, continuous *-homomorphisms $\varphi : A \to \mathbb{C}$ with $\|\varphi\| = 1$, which we equip with the weak*-topology as they are a subset of A^*. Sometimes this is called the *spectrum* of A, whence the notation. First we may note that in the unital case, no element of $\ker(\varphi)$ can have a two sided inverse, and $x - \varphi(x) \in \ker(\varphi)$, hence $\varphi(x) \in \sigma(x)$. Then, since $|\varphi(x)| \leq r(x) \leq \|x\|$ and $\varphi(1) = 1$, the assumption that $\|\varphi\| = 1$ is unnecessary in the unital case. One also notes that the space of characters is weak*-closed and bounded when A is unital, so by the Banach-Alaoglu theorem it is compact Hausdorff.

It can be shown (see [35, prop. 1.2.1], for example) that there is a bijection between $\sigma(A)$ and set of maximal ideals in A given by $\varphi \mapsto \ker(\varphi)$. If A is unital and generated by a single x, and $\lambda \in \sigma_A(x)$, $x - \lambda$ generates a maximal ideal (since x generates A). This induces a homeomorphism $\sigma_A(x) \to \sigma(A)$.

The *Gelfand Transform* is the map

$$\Phi : A \to C(\sigma(A)), \quad \Phi(x)(\varphi) = \varphi(x).$$

Theorem 1.2.3 (Gelfand)**.** Let A be a unital abelian C*-algebra. The Gelfand Transform Φ is an isometric *-isomorphism $A \to C(\sigma(A))$.

Corollary 1.2.4. The positive elements A_+ form a convex cone in A_{sa}.

Proof (sketch). It follows from the Gelfand representation that x is positive if and only if $\|1-x\| \leq 1$. With this in mind, showing that A_+ is stable under positive scalar multiplication and addition follows (see [39, p. 1.4.2] for details). $\square$

We noted above that when a single element x generates a unital, abelian C*-algebra A, there is a homeomorphism $\sigma_A(x) \to \sigma(A)$. Taking the inverse of the Gelfand transform we construct an isomorphism $C(\sigma(x)) \to A$ by $f \mapsto f(x)$. This functional calculus is called the *continuous functional calculus*, and satisfies the the following properties

Theorem 1.2.5 (Continuous Functional Calculus)**.** Let A, B be unital C*-algebras, and $x \in A$ be a normal element. Then, for $f \in C(\sigma(A))$,

(1) the map $f \mapsto f(x)$ is a *-homomorphism and if f is a polynomial $f(z) = \sum a_n z^n$, then $f(x) = \sum a_n x^n$,

(2) $\sigma(f(x)) = f(\sigma(x))$,

(3) ff $\varphi : A \to B$ is a *-homomorphism, $\varphi(f(x)) = f(\varphi(x))$, and

(4) if $x_n \to x$ is a sequence of normal elements converging in norm, Ω is a compact neighbourhood if $\sigma(x)$, and $f \in C(\Omega)$, then there exists an N such that $\sigma(x_n) \subset \Omega$ for $n \geq N$ and $f(x_n) \to f(x)$ in norm.

The proofs of (1),(2), and (3) are all straight forward, or follow from 1.2.3. (4) follows from a Stone-Weierstrass argument.

With our new tools, we arrive at equivalent definitions for some C*-algebra properties:

Proposition 1.2.6. $x \in A$ is self adjoint (resp. positive) if and only if $\sigma_A(x) \subset \mathbb{R}$ (resp. $\mathbb{R}^+$).

Theorem 1.2.7 (Multiplication Operator Spectral Theorem)**.** Let $\mathcal{H}$ be a Hilbert space, and $T \in B(\mathcal{H})$. Then there exists a measure space $(X, \mathcal{A}, \mu)$, a continuous, essentially bounded, real valued function f on X, and a unitary operator $U : \mathcal{H} \to L^2(X, \mu)$ such that $\|T\| = \|f\|_\infty$, and $T = U^* \Psi_f U$, where $\Psi_f \in B(L^2(X, \mu))$ is the multiplication operator

$$[\Psi_f(\varphi)](x) := f(x)\varphi(x), \quad \forall \varphi \in L^2(X, \mu).$$

Moreover, if $T \in B(\mathcal{H})$ is a normal operator, f can be taken to be complex valued.

2 C*-algebra Theory

Results in this section are from [7] unless otherwise noted. We are interested in maps between C*-algebras that offer a more general framework than the usual morphisms, the *-homomorphisms. These are completely positive (cp) maps.

It will also sometimes be helpful to work in the context of *operator spaces* (for example in the proof of 5.3.3). For our purposes, these can be viewed as vector subspaces of a C*-algebra. Of course thanks to the Gelfand-Naimark theorem we can forget the C*-algebra altogether and view an operator space as a normed vector space isometrically embedded into $B(\mathcal{H})$. We will also speak of *operator systems*, which are *-closed subspaces of a C*-algebra A containing the unit 1_A. Of course one sees immediately that any C*-algebra is an operator system, and any operator system is an operator space. When studying operator spaces we work with the completely bounded (cb) maps.

Notation: We will denote the algebra of $n \times n$ matrices with entries in A by $M_n(X)$ for any operator space X. If X is a C*-algebra, then so is $M_n(X)$. In particular, it acts on the Hilbert X-module X^n (we will see Hilbert C*-modules a little bit later in 6.2). The involution on a the matrix algebra over an operator system or a C*-algebra is the transpose entry-wise adjoint $[x_{ij}]^* = [x_{ji}^*]$.

2.1 Completely Positive Maps

As we mentioned earlier, a subspace $E \subset A$ of a unital C*-algebra is called an *operator system* if $1_A \in E$ and $E^* = E$. We let A denote a C*-algebra and $\mathcal{H}$ a Hilbert space.

Definition 2.1.1. Let $E \subset A$ be an operator system. An element in $M_n(E)$ is called positive if it is positive in $M_n(A)$, which is well defined since $M_n(A)$ is a C*-algebra. Let B be a C*-algebra (not necessarily unital). A map $\varphi : E \to B$ is called *completely positive (cp)* if each of the maps

$$\varphi_n : M_n(E) \to M_n(B), \quad \varphi_n([x_{ij}]) = [\varphi(x_{ij})]$$

is positive, that is, it sends positive elements to positive elements. The space of completely positive maps from A to B is denoted $CP(A, B)$.

If B is a unital C*-algebra, and φ is unital, we call it a *unital completely positive (ucp)* map. If φ is contractive then we call it *contractive completely positive (ccp)* (you may also see *cpc* in the literature).

Example 2.1.2. Some first examples of completely positive maps are *-homomorphisms. Indeed, if $\varphi : A \to B$ is a *-homomorphism of C*-algebras, then the φ_n are also *-homomorphisms (so they are positive). More generally, if V is an operator on $\mathcal{H}$ (where $B \subset B(\mathcal{H})$) and $\pi : A \to B$ is a *-homomorphism, then the map $p(x) = V^*\pi(x)V$ is completely positive. To see this, recall that x is positive if and only if it can be written as $x = y^*y$. We can write

$$\varphi_n([x_{ij}]) = [\varphi(x_{ij})] = [V^*\pi(x_{ij})V] = (V^n)^*\pi_n(x)(V^n)^*,$$

and since π_n is a *-homomorphism, it follows that

$$\varphi_n([x_{ij}]) = (V^n)^*\pi_n(y)^*\pi_n(y)V^n.$$

We will see in Stinespring's theorem that all completely positive maps can be written as a dilation of a *-homomorphism by a Hilbert space operator in this way.

Example 2.1.3. Positive linear functionals on operator systems are cp. Let E be an operator space and f a positive linear functional. Consider a vector $\xi = (\xi_1, ...\xi_n) \in \ell_n^2$, the n-dimensional Hilbert space, and a positive element $x \in M_n(E)_+$. Then,

$$\langle f_n(x)\xi, \xi\rangle = f(\xi^*[x_{ij}]\xi) \geq 0,$$

because we can once again write $x = y^*y$.

The standard example of a map which is not completely positive is the transpose map $\varphi(\cdot) = (\cdot)^T$ on the algebra $M_2(\mathbb{C})$. Consider the matrix

$$x = \begin{bmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \end{bmatrix} \in M_2(M_2(\mathbb{C})).$$

x is clearly positive, however $\varphi_2(x)$ has determinant -1, so cannot be positive.

The following propositions alow us to characterize the cp maps to and from matrix algebras.

Proposition 2.1.4. Let A be a C*-algebra. The set of completely positive maps $M_n(\mathbb{C}) \to A$ is in bijection with the positive elements of the matrix algebra $M_n(A)_+$ under the identification

$$CP(M_n(\mathbb{C}), A) \ni \varphi \mapsto [\varphi(e_{ij})] \in M_n(A)_+,$$

where e_{ij} are a set of matrix units for $M_n(\mathbb{C})$.

Proposition 2.1.5. Let A be a C*-algebra. The set of completely positive maps $A \to M_n(\mathbb{C})$ is in bijection with the set of positive linear functionals $M_n(A)_+^*$ under the identification

$$CP(A, M_n(\mathbb{C})) \ni \varphi \mapsto \hat{\varphi} \in M_n(A)_+^*,$$

where we define

$$\hat{\varphi}([a_{ij}]) = \sum \varphi(a_{ij})_{ij}.$$

Theorem 2.1.6 (Stinespring)**.** Let A be a unital C*-algebra and $\varphi : A \to B(\mathcal{H})$ be a completely positive map. Then there exists a Hilbert space $\widehat{\mathcal{H}}$, a linear operator $V : \mathcal{H} \to \widehat{\mathcal{H}}$ and a *-representation $\pi : A \to B(\widehat{\mathcal{H}})$ such that

$$\varphi(x) = V^* \pi(x) V, \ \forall x \in A.$$

Proof. We define a positive semi-definite sesquilinear form on the algebraic tensor $A \odot \mathcal{H}$ by

$$\left\langle \sum b_j \otimes \eta_j, \sum a_i \otimes \xi_i \right\rangle := \sum_{ij} \langle \varphi(a_i^* b_j)\eta, \xi \rangle,$$

this making it a pre-Hilbert space. We then may take $\widehat{\mathcal{H}}$ the completion of the quotient

$$A \odot \mathcal{H} / \{x \in A \odot \mathcal{H} : \langle x, x \rangle = 0\}.$$

Here $A \odot \mathcal{H}$ denotes the *algebraic tensor product*, which we'll define shortly (see 2.2.1). For an element $z \in A \odot \mathcal{H}$, we denote by $z^\wedge$ the image in $\widehat{\mathcal{H}}$. Let $V : \mathcal{H} \to \widehat{\mathcal{H}}$ be defined by $\xi \mapsto (1_A \otimes \xi)^\wedge$, and note that this is a contraction. For $x \in A$ allow $\pi(x)$ to act on the dense subspace $(A \odot \mathcal{H})^\wedge \subset \widehat{\mathcal{H}}$ by

$$\pi(x) \left[\left(\sum_i b_i \otimes \eta_i \right)^\wedge \right] = \left(\sum_i x b_i \otimes \eta_i \right)^\wedge.$$

Indeed, this is a *-representation. Finally, noting that V^* is defined on $(A \odot \mathcal{H})^\wedge$ by $(a \otimes \eta)^\wedge \mapsto \varphi(a^*)^* \eta$, we see that $\varphi(x) = V^* \pi(x) V$ for all $x \in A$. $\square$

Remark 2.1.7. In particular, we see that since a *-representation is isometric, $\|\varphi\| = \|V^*V\| = \|\varphi(1)\|$. We call the triple $(\pi, V, \widehat{\mathcal{H}})$ a *Stinespring Dilation*. If A is non-unital one can still prove a version of Stinespring's theorem, as it can be shown that a ccp map on a non-unital C*-algebra can extend to a ucp map on the unitization. If $\varphi : A \to B$ is a ccp map from a non-unital C*-algebra to a unital one, define $\tilde{\varphi} : \tilde{A} \to B$ by $\tilde{\varphi}(x + \lambda 1_{\tilde{A}}) = \tilde{\varphi}(x) + 1_B$ (see [7, proposition 2.2.1]).

A useful class of completely positive maps are conditional expectations. Here we give the definition, as well as state Tomiyama's theorem and one more elementary, but useful lemma (see [7, p. 12] for the proofs).

Definition 2.1.8. Let $A \subset B$ be C*-algebas. A *conditional expectation* from B onto A is a ccp projection $\Phi : B \twoheadrightarrow A$ that is bi-module (meaning $\Phi(bxb') = b\Phi(x)b'$ for $x \in A$ and $b, b' \in B$).

Theorem 2.1.9 (Tomiyama)**.** Let $\Phi : B \twoheadrightarrow A$ be a projection. Then TFAE:

(1) Φ is contractive,

(2) Φ is ccp,

(3) Φ is a conditional expectation.

Lemma 2.1.10. If $1_M \subset N \subset M$ are von Neumann algebras and τ is normal, faithful, tracial state on M, then there exists a unique, normal, trace preserving conditional expectation $M \twoheadrightarrow N$. In particular, Φ is defined by the relation

$$\tau(a\Phi(x)) = \tau(ax), \quad \forall x \in M, a \in N.$$

2.1.1 Arveson's Extension Theorem

Lemma 2.1.11. Let $E \subset A$ be an operator system, 1_A denote the identity, and $\psi : E \to \mathbb{C}$ a positive linear functional. Then $\|\psi\| = \psi(1_A)$, and any isometric (norm-preserving) extension of ψ to A is also positive.

Proof. Fix some $x \in E$ with $\|x\| \leq 1$ such that $|\psi(x)|$ is close to $\|\psi\|$, say $|\psi(x)| = \|\psi\| - \varepsilon$ for $\varepsilon > 0$. We can rotate $\psi(x)$ (by multiplying it by some scalar $e^{i\theta}$ on the unit circle) and so we'll assume $0 < \psi(x) \in \mathbb{R}$. A positive map is self adjoint, so $2\psi(x) = \psi(x) + \psi(x^*) = \psi(x + x^*)$, and therefore x is also self adjoint.

For a self adjoint $x \in E$, we can use the operator inequality $x \leq \|x\| \cdot 1_A$ and so $\psi(x) \leq \|x\|\psi(1_A)$. Thus $\|\psi\| \leq \psi(1_A)\|x\| + \varepsilon$. We then easily obtain $\|\psi\| = \psi(1_A)$. It follows that any norm preserving extension ψ' of ψ will need to have $\|\psi'\| = \psi'(1_A)$, and will therefore be positive. □

Corollary 2.1.12. Let $E \subset A$ be an operator system and $\varphi : E \to M_n(\mathbb{C})$ a completely positive map. Then φ extends to a completely positive map $A \to M_n(\mathbb{C})$.

Proof. We must make use of the very useful correspondence $CP(E, M_n(\mathbb{C})) \longleftrightarrow M_n(E)^*_+$. For a completely positive map φ we define the corresponding positive linear functional $\hat{\varphi}$ on $M_n(E)$ by

$$\hat{\varphi}([x_{ij}]) = \sum_{i,j} [\varphi(x_{ij})]_{ij}.$$

With this in mind the proof is easy. Given a cp map φ on E we use the lemma above to extend its corresponding positive functional $\hat{\varphi}$ to all of $M_n(A)$. This corresponds to a cp map on A which extends φ. □

Theorem 2.1.13. (Arveson Extension) Let E be an operator system in a C*-algebra A, and $\varphi : E \to B(H)$ a ccp map. Then there exists a ccp map $\psi : A \to B(H)$ extending φ. If A is unital and φ is a ucp map, then we can find a ucp ψ.

Proof. We'll take an increasing net of finite rank projections $P_i \in B(\mathcal{H})$ such that $P_i \to 1$, the identity operator, in the strong operator topology. Denote by φ_i the cp maps which take $x \mapsto P_i\varphi(x)P_i$ for $x \in E$. We can view each of these as maps to matrix algebras $M_{n_i}(\mathbb{C})$, and so by the corollary above, we can define them on all of A. Now we return to viewing the maps φ_i as maps into $B(\mathcal{H})$, passing to a subsequence if necessary, we let ψ be a cluster point of the net φ_i. Clearly ψ is completely positive and extends φ.

Moreover if we had ψ contractive as well, then the maps φ_i would be ccp maps. We can make use of the well known fact that for any Banach space X and von Neumann algebra M, the unit ball of bounded operators $B(X, M)$ is compact in the point-ultraweak topology (we realize this space as the dual of the Banach space $B(X, M_*)$ and apply the Banach-Alaoglu theorem). Then there must be a cluster point ψ in this unit ball, which is once again easily seen to be a cp extension of φ. $\square$

2.2 Tensor Products

We will often need (or want) to discuss tensor products of C*-algebras. Because C*-algebras are at once, both algebraic and analytic objects, defining the tensor product of C*-algebras requires some subtlety. Indeed, we will give the definition of a few types of tensor products, characterized by different analytic properties. In this section we are going to mostly establish notation and quickly go over important properties and results that will be used throughout the thesis. There is, of course, much more depth to to this topic and for a positively comprehensive treatment we refer to the excellent book by Nate Brown and Narutaka Ozawa [7].

The first order of business is the *algebraic tensor product*. That is, given C*-algebras A and B (we could just as well take vector spaces), the tensor product of underlying vector spaces.

Definition 2.2.1. Let $C_c(A \times B)$ denote the space of functions on $A \times B$ with compact support. We denote $\chi_{(x,y)}$ the characteristic function of a given point $(x, y) \in A \times B$. It is not hard to see that indeed, the set $\{\chi_{(x,y)} : x \in A,\ y \in B\}$ forms a basis of $C_c(A \times B)$. Consider the following four classes of elements in $C_c(A \times B)$:

(i) $\chi_{(x_1+x_2,y)} - \chi_{(x_1,y)} - \chi_{(x_2,y)}$,

(ii) $\chi_{(x,y_1+y_2)} - \chi_{(x,y_1)} = \chi_{(x,y_2)}$,

(iii) $\lambda\chi_{(x,y)} - \chi_{(\lambda x,y)}$,

(iv) $\lambda\chi_{(x,y)} - \chi_{(x,\lambda y)}$.

Let K denote the vector space spanned by these four classes of elements. Then we define the *algebraic tensor product* (denoted $A \odot B$) by

$$A \odot B = C_c(A \times B)/K.$$

We will denote the image under this quotient of $\chi_{(x,y)}$ by the familiar $x \otimes y$. These *elementary tensors* span $A \odot B$.

Of course, this algebraic tensor product observes the familiar universal property that if there is a vector space V and a bilinear map $\sigma : A \times B \to V$, there exists a map $\hat{\sigma} : A \odot B \to V$ such that $\hat{\sigma}(x \otimes y) = \sigma(x, y)$. We may also refer to the tensor product of maps, which behave in the usual sense: Given $\phi : A \to C, \psi : B \to D$, there is a map

$$\phi \otimes \psi : A \odot B \to C \odot D : \quad x \otimes y \mapsto \phi(x) \otimes \psi(y).$$

If ϕ, ψ are linear functionals (i.e. C, D in the above are $\mathbb{C}$), since $\mathbb{C} \odot \mathbb{C} \cong \mathbb{C}$ by the isomorphism $\xi \otimes \zeta \mapsto \xi\zeta$, the tensor product map is also a linear functional. We may define multiplication on $A \odot B$ by

$$\left(\sum_i x_i \otimes y_i\right)\left(\sum_j a_j \otimes b_j\right) = \sum_{i,j} x_i a_j \otimes y_i b_j.$$

We may also define an involution on $A \odot B$ in the obvious way:

$$\left(\sum_i x_i \otimes y_i\right)^* = \sum_i x_i^* \otimes y_i^*$$

With these definitions, it is not hard to show that the tensor product of two *-homomorphisms is also a *-homomorphism.

This construction is very straightforward, but this object will become interesting once we choose a norm to equip $A \odot B$ with. Of, course if we want to make a C*-algebra we will need to inherit some analytic properties from an embedding into a Hilbert space. If $A \subset B(\mathcal{H})$ an $B \subset B(\mathcal{K})$ for two Hilbert spaces $\mathcal{H}, \mathcal{K}$, we will want to represent $A \odot B$ in some tensor product of these. We already know about $\mathcal{H} \odot \mathcal{K}$, however it remains to define a Hilbert space structure on this space. It is not a difficult exercise to show that

$$\left\langle \sum_i \xi_i \otimes \eta_i, \sum_j \xi_j' \otimes \eta_j' \right\rangle = \sum_{i,j} \left\langle \xi_i, \xi_j' \right\rangle_{\mathcal{H}} \left\langle \eta_i, \eta_j' \right\rangle_{\mathcal{K}}$$

defines a positive definite sesquilinear form on $\mathcal{H} \odot \mathcal{K}$, making it a pre-Hilbert space. The completion with respect to this form is the Hilbert space, denoted $\mathcal{H} \bar{\otimes} \mathcal{K}$, and is called the *Hilbert space tensor*. If $\mathcal{H}$ and $\mathcal{K}$ have the orthonormal bases $\{\xi_i\}$, $\{\eta_j\}$ respectively, then the elementary tensors $\{\xi_i \otimes \eta_j\}$ form an orthonormal basis of $\mathcal{H} \bar{\otimes} \mathcal{K}$.

The norm induced by this inner product is an instance of a *cross norm*, a norm such that on elementary tensors: $\|\xi \otimes \eta\| = \|\xi\| \|\eta\|$.

If $x \in B(\mathcal{H})$ and $y \in B(\mathcal{K})$ are bounded operators, we define the algebraic tensor product map in the same way we did earlier, except denoting it $x \odot y : \mathcal{H} \odot \mathcal{K} \to \mathcal{H} \odot \mathcal{K}$. The following proposition describes how we impose a norm on these maps to create $B(\mathcal{H} \bar{\otimes} \mathcal{K})$.

Proposition 2.2.2. Given operators $x \in B(\mathcal{H}), y \in B(\mathcal{K})$,

$$[x \otimes y](\xi \otimes \eta) = x(\xi) \otimes y(\eta) \in \mathcal{H} \bar{\otimes} \mathcal{K}.$$

defines a unique operator, and moreover, $\|x \otimes y\| = \|x\|\|y\|$.

Proof. Letting $a \in \mathcal{H} \odot \mathcal{K}$ and using the fact that for any such ξ there is a set $\{k_i\} \subset \mathcal{K}$ such that $a = \sum_i \xi_i \otimes k_i$, where $\{\xi_i\}$ is an orthonormal set in $\mathcal{H}$, [1] we can show that

$$\|\mathrm{id}_{\mathcal{H}} \odot y(a)\|^2 = \left\| \sum_i \xi_i \otimes y(k_i) \right\|^2 \leq \sum_i \|y(k_i)\|^2 \leq \|T\|^2 \|x\|^2.$$

By continuity, one extends $\mathrm{id}_{\mathcal{H}} \odot y$ to a map $\mathrm{id}_{\mathcal{H}} \otimes y$ on the Hilbert space tensor $\mathcal{H} \bar{\otimes} \mathcal{K}$ with norm $\leq \|T\|$ as well. We may do the same for $x \otimes \mathrm{id}_{\mathcal{K}}$, and so if we define

$$x \otimes y = (x \otimes \mathrm{id}_{\mathcal{K}})(\mathrm{id}_{\mathcal{H}} \otimes y),$$

we note that $x \otimes y$ agrees with $x \odot y$ on the algebraic tensor, and obtain the inequality

$$\|x \otimes y\| \leq \|x\|\|y\|.$$

For the other direction we take sequences of unit vectors ξ_n, η_n in $\mathcal{H}, \mathcal{K}$ respectively such that $\|x(\xi_n)\| \to \|x\|$ and $\|y(\eta_n)\| \to \|y\|$ as $n \to \infty$. Since the norm on $\mathcal{H} \bar{\otimes} \mathcal{K}$ is a cross norm, $\|\xi_n \otimes \eta_n\| = 1$ for each n, and moreover we may invoke this again to show that

$$\|(x \otimes y)(\xi_n \otimes \eta_n)\| = \|x(\xi_n) \otimes y(\eta_n)\| = \|x(\xi)\|\|x(\eta)\| \to \|x\|\|y\|.$$

$\square$

As one may have expected, the tensor product operators we just defined satisfy all the properties that we laid out in definition 2.2.1 for the algebraic tensor. Additionally, these

[1] What we have said here is not strictly kosher because we took an $a \in \mathcal{H} \bar{\otimes} \mathcal{K}$, a space which is not complete, however there are ways (see [7] 3.1.10) of making this precise.

operators agree with the multiplication and involution operations we defined. From the canonical isomorphisms $B(\mathcal{H}) \cong B(\mathcal{H}) \otimes \mathbb{C}\mathrm{id}_{\mathcal{K}}$ and $B(\mathcal{K}) \cong \mathbb{C}\mathrm{id}_{\mathcal{H}} \otimes B(\mathcal{K})$, and by [7, prop. 3.1.17], we get a *-homomorphism

$$B(\mathcal{H}) \odot B(\mathcal{K}) \to B(\mathcal{H}\bar{\otimes}\mathcal{K}), \quad \sum_i x_i \otimes y_i \mapsto \sum_i x_i \otimes y_i.$$

We can therefore, extend out idea of tensor product operators to our desired setting of C*-algebras:

Corollary 2.2.3. Let A, B be C*-algebras with representations $\pi_A(A) \subset B(\mathcal{H})$ and $\pi_B(B) \subset B(\mathcal{K})$ on Hilbert spaces $\mathcal{H}, \mathcal{K}$, respectively. Then $\pi_A \odot \pi_B : A \odot B \to B(\mathcal{H}\bar{\otimes}\mathcal{K})$ given by

$$\pi_A \odot \pi_B : a \otimes b \mapsto \pi_A(a) \otimes \pi_B(B)$$

is a *-representation.

Here we use $\otimes$ to mean the tensor product of elements in a C*-algebra, as well as the tensor product of vectors in Hilbert spaces. This abuse of notation is standard and shouldn't cause any confusion.

2.2.1 C*-norms on $A \odot B$

As we have previously alluded to, the act of completing the algebraic tensor product of two C*-algebras so that the end product is also a C*-algebra requires delicate care. Let A, B be C*-algebras and $A \odot B$ be their algebraic tensor.

Definition 2.2.4. A *C*-norm* on $A \odot B$ is a norm $\|\cdot\|_\alpha$ such that for all $x, y \in A \odot B$,

(1) $\|xy\|_\alpha = \|x\|_\alpha \|y\|_\alpha$,
(2) $\|x\|_\alpha = \|x^*\|_\alpha$, and
(3) $\|x^*x\| = \|x\|_\alpha^2$.

Usually, we denote the completion of $A \odot B$ with respect to this norm by $A \otimes_\alpha B$.

In general we will be able to define a few C*-norms on $A \odot B$, however in some cases, particularly the following important example, there is only one.

Proposition 2.2.5. Let A be a C*-algebra. There is a unique norm on $M_n(\mathbb{C}) \odot A$.

Proof. We use the well known isomorphism of algebras

$$M_n(\mathbb{C}) \odot A \cong M_n(A).$$

The fact that $M_n(A)$ is known to be a C*-algebra gives the existence of this norm, and since there is always a unique C*-norm, the uniqueness follows. $\square$

The two tensor products which are commonly used are the following:

Definition 2.2.6 (Max Norm)**.** Let $x \in A \odot B$. We define the maximal norm (commonly referred to as the 'max norm') by

$$\|x\|_{\max} = \sup\{\|\pi(x)\| : \pi : A \odot B \to B(\mathcal{H}) \text{ cyclic *-representation}\}.$$

The completion of $A \odot B$ with respect to $\| \cdot \|_{\max}$ is called the *maximal (max) tensor* and is denoted $A \otimes_{\max} B$.

That this norm is finite follows from the fact that given any such $\pi : A \odot B \to B(\mathcal{H})$ there exist restrictions π_A and π_B, to A and B respectively, with commuting ranges (see [7, theorem 3.2.6]). Then for an elementary tensor $x \otimes y$,

$$\pi(x \otimes y) \leq \|\pi_A(x)\| \|\pi_B(y)\| \leq \|a\| \|b\|.$$

Definition 2.2.7 (Min Norm)**.** Suppose $\pi(A) \subset B(\mathcal{H})$ and $\sigma(B) \subset B(\mathcal{K})$ are faithful representations. We define the *minimal (min)*, sometimes called *spatial norm* on $\sum x_i \otimes y_i \in A \odot B$ by

$$\left\| \sum x_i \otimes y_i \right\|_{\min} = \left\| \sum \pi(x_i) \otimes \sigma(y_i) \right\|_{B(\mathcal{H} \bar{\otimes} \mathcal{K})}.$$

We call the completion of $A \odot B$ with respect to this norm the *min tensor* and denote it either simply by $A \otimes B$ or by $A \otimes_{\min} B$ if there is ambiguity.

That these are norms to begin with (as oppose to just seminorms) is surprisingly delicate to prove. It turns out to be the case, but we will refer to reader to [7] for the proof.

A natural question to ask is whether the min norm is defined independently of the choice of faithful representation. It turns out that this is indeed the case.

Proposition 2.2.8. The minimal tensor product norm is independent of our choice of representations π, σ.

Proof. We will show that the norm is independent of σ, and the argument for π is identical. Let us also assume that the C*-algebras in question are separable (this makes the proof simpler, and one can relatively painlessly deduce the general case). Denote by $\|\cdot\|_{\min}$ the norm induced by the representations π, σ, and by $\|\cdot\|'_{\min}$ the norm induced by π and another faithful representation $\sigma' : B \to B(\mathcal{K}')$.

Take an increasing sequence of finite rank projections $p_1 \leq p_2 \leq \ldots$ in $B(\mathcal{H})$ such that p_n has rank n and $p_n(\xi) \to \xi$ in norm for all $\xi \in \mathcal{H}$. Then, for any $z \in B(\mathcal{H}\bar{\otimes}\mathcal{K})$,

$$\|z\| = \sup_n \{\|(p_n \otimes \mathrm{id}_{\mathcal{K}})z(p_n \otimes \mathrm{id}_{\mathcal{K}})\|\}.$$

For any $\sum x_i \otimes y_i \in A \odot B$,

$$\left\|\sum x_i \otimes y_i\right\|_{\min} = \sup_n \left\|\sum p_n\pi(x_i)p_n \otimes \sigma(y_i)\right\|, \text{ and}$$
$$\left\|\sum x_i \otimes y_i\right\|'_{\min} = \sup_n \left\|\sum p_n\pi(x_i)p_n \otimes \sigma'(y_i)\right\|$$

We took p_n to have rank n, so, in particular, we have that $p_nB(\mathcal{H})p_n \cong M_n(\mathbb{C})$. By 2.2.5 we know that $M_n(\mathbb{C}) \odot B$ has a unique C*-norm, hence,for each n,

$$\left\|\sum p_n\pi(x_i)p_n \otimes \sigma(y_i)\right\| = \left\|\sum p_n\pi(x_i)p_n \otimes \sigma'(y_i)\right\|.$$

The suprema must therefore coincide, and we have shown that $\|\cdot\|_{\min} = \|\cdot\|'_{\min}$. $\square$

Corollary 2.2.9. If A is non-unital, any C*-norm on $A \odot B$ can be extended to a C*-norm on $\tilde{A} \odot B$, where $\tilde{A}$ denotes the unitization, as usual.

See [7, p. 76] for a straight-forward proof. This obviously implies that for two non-unital C*-algebras A, B we can extend any tensor product norm to the algebraic tensor of their unitizations.

Proposition 2.2.10 (Universal Property of $\otimes_{\max}$)**.** Given any *-homomorphism $\pi : A \otimes B \to C$, there exists a unique *-homomorphism $\tilde{\pi} : A \otimes_{\max} B \to C$ which extends π. In particular $\|\cdot\|_{\max}$ is the largest C*-norm on $A \odot B$.

The proof of this is immediate. Just pick a *-representation $\rho : C \to B(\mathcal{H})$, and it follows that $\rho \circ \pi : A \odot B \to B(\mathcal{H})$ is a contractive (w.r.t. $\|\cdot\|_{\max}$) *-homomorphism, which naturally extends to $A \otimes_{\max} B$. Applying universality to $A \odot B \to A \otimes_\alpha B$ gives that $\|\cdot\|_{\max} \geq \|\cdot\|_\alpha$, for any C*-norm $\|\cdot\|_\alpha$.

The next proposition gives an argument based on the excision property which will allow us to skip most of the work in proving Takesaki's theorem (not that there was anything wrong with Takesaki's proof). The details of the proofs we will include, as well as Takesaki's proof are found in [7, section 3.4].

Proposition 2.2.11. Let A, B be unital C*-algebras, $\|\cdot\|_\alpha$ a C*-norm on $A \odot B$, and φ, ψ states on A, B respectively. Then $\varphi \odot \psi$ extends to a state $\varphi \otimes \psi$ of $A \otimes_\alpha B$.

Proof. We need to show that $\varphi \otimes \psi$ is continuous with respect to $\|\cdot\|_\alpha$. Assume without loss of generality (thanks to the Krein-Milman theorem) that φ, ψ are pure states. By 1.0.12, there exist nets e_i, f_i which excise φ, ψ respectively. Then for $x \in A \odot B$,

$$\|x\|_\alpha \geq \lim_i \|(e_i \otimes f_i)x(e_i \otimes f_i)\| = \lim_i \|(\varphi \otimes \psi)(x)(e_i \otimes f_i)^2\| = |(\varphi \otimes \psi)(x)|.$$

□

Theorem 2.2.12 (Takesaki)**.** Let A, B be C*-algebras. $\|\cdot\|_{\min}$ is the smallest C*-norm on $A \odot B$.

Proof. We work first in the unital case, and assume that A, B are separable. In this setting, we are guaranteed the existence of faithful states $p \in \mathcal{S}(A)$ and $\psi \in \mathcal{S}(B)$. If $\|\cdot\|_\alpha$ is any C*-norm on $A \odot B$, we may by proposition 2.2.11 extend $\varphi \odot \psi$ continuously to a state $\varphi \otimes_\alpha \psi \in A \otimes_\alpha B$. By the uniqueness of GNS representations, the induced representation $\pi_{\varphi \otimes_\alpha \psi}|_{A \odot B}$ is unitarily equivalent to $\pi_\varphi \odot \pi_\psi$. Moreover, these representations are faithful because we chose faithful states. Because the min norm is defined independently of the faithful representation, the norm closure of $\pi_\varphi \odot \pi_\psi(A \odot B)$ is isomorphic to the min tensor product $A \otimes B$. This implies that $pi_{\varphi \otimes_\alpha \psi}(A \otimes_\alpha B)$ is also isomorphic to $A \otimes B$. However $A \otimes B$ must then be a quotient of $A \otimes_\alpha B$, which implies that $\|\cdot\|_{\min} \leq \|\cdot\|_\alpha$.

The non-unital case follows from 2.2.9. The non-separable case is a consequence of the inclusion property of $\otimes_{\min}$, which we'll outline below. □

We now give a few more important and useful results regarding the min and max tensor products. We won't include the proofs, however all can be found in chapter 3 of Brown and Ozawa's textbook. The properties in question may seem intuitive, however the proofs are delicate and non-trivial. That being said, the first corollary is an immediate consequence of Takesaki's theorem and the universal property of $\otimes_{\max}$.

Corollary 2.2.13. For any C*-norm $\|\cdot\|_\alpha$ on $A \odot B$, there are surjective *-homomorphisms

$$A \otimes_{\max} B \to A \otimes_\alpha B \to A \otimes_{\min} B.$$

Theorem 2.2.14 (Continuity)**.** Let $\varphi : A \to C$ and $\psi : B \to D$ be completely positive (cp) maps. Then

$$\varphi \odot \psi : A \odot B \to C \odot D$$

extends to cp maps

$$\varphi \otimes_{\max} \psi : A \otimes_{\max} B \to C \otimes_{\max} D, \text{ and}$$
$$\varphi \otimes \psi : A \otimes B \to C \otimes D,$$

with the property that

$$\|\varphi \otimes_{\max} \psi\| = \|\varphi \otimes \psi\| = \|\varphi\|\|\psi\|.$$

Theorem 2.2.15 (Inclusions)**.** Let $A \subset B$ and C be C*-algebras. Then,

(1) $A \otimes C \subset B \otimes C$,
(2) If, for every non-degenerate *-representation $\pi : A \to B(\mathcal{H})$, there exists a ccp map $\varphi : B \to \pi(A)''$ such that $\varphi(x) = \pi(x), \forall x \in A$, then $A \otimes_{\max} C \subset B \otimes_{\max} C$.

Finally, we mention the following result about exact sequences. One will note that we only have included the case of $\otimes_{\max}$. While in the case of inclusions, the max norm required an extra condition, here the min norm is the one requiring an extra condition.

Proposition 2.2.16 (Exact Sequences)**.** Let A, B be C*-algebras, and $I \triangleleft A$ and ideal. If $I \hookrightarrow A \twoheadrightarrow A/I$ is an exact sequence, then the sequence

$$I \otimes_{\max} B \hookrightarrow A \otimes_{\max} B \twoheadrightarrow A/I \otimes_{\max} B$$

is exact.

This is not true for the min tensor product, however there is one class of C*-algebras for which it is true: the exact C*-algebras (see 2.3.22).

2.3 Nuclear and Exact C*-algebras

Now that we have a basic understanding of C*-tensor products, we are prepared to talk about nuclear and exact C*-algebras.

Definition 2.3.1. Let A, B be C*-algebras. A map $\theta : A \to B$ is called *nuclear* if there exist sequences of ccp maps $\varphi_n : A \to M_{k_n}(\mathbb{C})$, and $\psi_n : M_{k_n}(\mathbb{C}) \to B$ such that

$$\|\psi_n \circ \varphi_n(x) - \theta(x)\| \to 0, \quad \forall x \in A.$$

We may call this topology of point-wise convergence in norm the *point-norm* topology. If A and B are unital, we may take ucp maps as oppose to ccp maps. We note immediately that a nuclear map defined in this way is ccp (or ucp in the unital context).

Remark 2.3.2. Instead of matrix algebras $M_{k_n}(\mathbb{C})$, we could have taken our ccp φ_n, ψ_n maps to be to and from a sequence of finite dimensional C*-algebras A_n. Then all we would need to do is represent each A_n in some $M_{k_n}(\mathbb{C})$ and see that there are conditional expectations $\Psi_n : M_{k_n}(\mathbb{C}) \to A_n$. Then if $\Phi_n : A_n \to M_{k_n}(\mathbb{C})$ are the unital embeddings, $\Phi_n \circ \varphi_n$ and $\psi_n \circ \Psi_n$ are the ccp maps that we desire.

We have an associated notion for von Neumann algebras:

Definition 2.3.3. If A is a C*-algebra and M a con Neumann algebra, we call a map $\theta : A \to M$ *weakly nuclear* if there exist ccp maps $\varphi_n : A \to M_{k_n}(\mathbb{C})$, and $\psi_n : M_{k_n}(\mathbb{C})$ such that

$$\eta(\psi_n \circ \varphi_n(x)) \to \eta(\theta(x)), \quad \forall x \in A, \eta \text{ normal functionals in } M_*.$$

This topology of pointwise ultraweak convergence is called the point-ultraweak topology. Once again, if θ is unital we can take ucp maps φ_n, ψ_n.

Proposition 2.3.4 (Compositions)**.** Let $\theta : A \to B$ and $\sigma : B \to C$ be ccp maps. If either of θ or σ is nuclear, then so is $\sigma \circ \theta$.

Proof. If we have for θ, say, ccp maps φ_n, ψ_n to and from matrix algebras $M_{k_n}(\mathbb{C})$ respectively, such that $\psi_n \circ \varphi_n \to \theta$, then we may take $\psi'_n = \sigma \circ \psi_n$. These will be ccp maps with $\psi'_n \circ \varphi_n \to \sigma \circ \theta$. The other case works in the same way. $\square$

Definition 2.3.5. A C*-algebra A is *nuclear* if the identity map $\mathrm{id} : A \to A$ is a nuclear map. One finds other terms for nuclear C*-algebras in the literature, including *amenable* C*-algebras, and C*-algebras with the *completely positive approximation property (CPAP).*

Once again we have a von Neumann version:

Definition 2.3.6. A von Neumann algebra M is called *semidiscrete* if the identity map $\mathrm{id}_M : M \to M$ is weakly nuclear.

When we state Connes' theorem, we will see a definitive link between the C*-algebra and von Neumann cases. In particular, it will follow that A is nuclear if and only if A^{**} is semidiscrete.

Definition 2.3.7. A C*-algebra is *exact* if it admits a nuclear faithful representation.

Proposition 2.3.8. Exactness, as defined above, is independent of the faithful representation.

Proof. Suppose $\pi : A \to B(\mathcal{H})$ is a nuclear faithful representation of A, and $\sigma : A \to B(\mathcal{K})$ is a second faithful, non-degenerate, representation. It suffices to show that σ is also nuclear. Let $\varphi_n : A \to M_{k_n}(\mathbb{C})$ and $\psi_n : M_{k_n}(\mathbb{C}) \to B(\mathcal{H})$ be ccp maps so that $\psi_n \circ \varphi_n \to \pi$ in the point-norm topology. We can define a *-homomorphism on the range of π by:

$$\Phi : \pi(A) \to B(\mathcal{K}), \quad \Phi(\pi(x)) = \sigma(x).$$

This is certainly well defined since π is faithful. Since *-homomorphisms are ccp maps, it is also ccp, and hence admits a ccp extension $\bar{\Phi}$ to all of $B(\mathcal{H})$, by Arveson's extension theorem (see 2.1.13). Defining a new sequence of ccp maps $\psi'_n = \bar{\Phi} \circ \psi_n$, we get that

$$\psi'_n \circ \varphi_n = \bar{\Phi} \circ \psi_n \circ \varphi_n \to \bar{\Phi} \circ \pi = \sigma$$

in the point-norm topology, and hence that σ is also nuclear. $\square$

Remark 2.3.9 (Dependence on range). If $\pi : A \to B(\mathcal{H})$ is a faithful representation, we may regard nuclearity of A as the requirement that the map $\mathrm{id}_A = \pi : A \to \pi(A)$ is nuclear. In contrast, for A to be exact we only require that the map π, viewed as a map to all of $B(\mathcal{H})$, is nuclear. The subtlety here is to do wit the range of the map π.

Remark 2.3.10 (Nuclearity Implies Exactness). Of course, looking back to where we showed the composition property of nuclear maps, we see that if A is nuclear, and we take $\pi : A \to B(\mathcal{H})$, indeed A must be exact.

Remark 2.3.11 (Subalgebras). Clearly every subalgebra of an exact C*-algebra is also exact (in fact this is a characterization of exactness), however the same cannot be said for nuclear C*-algebras, particularly because of the dependence on the range.

Here are some more facts about nuclearity. Some are easy to prove, and some (in particular (4)) are highly nontrivial.

Proposition 2.3.12.

(1) Any ideal in a nuclear C*-algebra is nuclear. In fact any hereditary subalgebra is nuclear, see 3.2.1 for the definition of a hereditary subalgebra.
(2) A non-unital C*-algebra A is nuclear if and only if the unitization $\tilde{A}$ is nuclear.
(3) A finite direct sum of nuclear (resp. exact) C*-algebras is nuclear (resp. exact).
(4) Direct limits of nuclear (resp. exact) C*-algebras are nuclear (resp.exact).

Remark 2.3.13. Take the ℓ^∞ direct sum of more than finitely many algebras in (3) above and the statement is no longer true. Consider

$$\bigoplus_{n\in\mathbb{N}} M_n(\mathbb{C}) = \{(x_n) \text{ sequences } : x_n \in M_n(\mathbb{C}),\ \sup_n \|x_n\| < \infty\}.$$

We will see that there are RFD C*-algebras, (i.e. some which embed into the above algebra) which are not exact, for example the full group C*-algebra of a countably generated free group $C^*(\mathbb{F})$ (more on this in the next section). Of course we know that an exact C*-algebra cannot have a non-exact subalgebra, so the above direct sum cannot be exact and hence it is not nuclear.

Example 2.3.14 (AFD C*-algebras). First one mentions that clearly any finite dimensional C*-algebra is nuclear (see remark 2.3.2). We call a C*-algebra *approximately finite dimen-*

sional (AFD) if it is the inductive limit of finite dimensional C*-algebras. With proposition 2.3.12 in mind, the nuclearity of these becomes obvious.

A less trivial example is that the class of abelian C*-algebras are nuclear:

Proposition 2.3.15. Every abelian C*-algebra is nuclear.

Proof. By 2.3.12 we shall only prove the unital case. Thanks to the Gelfand representation, let $A = C(X)$ for some compact Hausdorff space X. Let $F \subset X$ and $\varepsilon > 0$. Then we can find a finite open cover $\{U_1, ..., U_n\}$ of X such that for each $f \in F$ and each $i = 1, ..., n$,

$$|f(x) - f(y)| < \varepsilon, \qquad \text{whenever } x, y \in U_i.$$

Here we are taking x, y to be points in X and not elements of the C*-algebra A as we have done everywhere else (apologies for any confusion). We now take a partition of unity $\{\sigma_1, ..., \sigma_n\}$ subordinate to the open cover $\{U_1, ..., U_n\}$. For those of us whose topology may be rusty, that is a set of functions $X \to [0, 1]$ where the support of each σ_i is contained in U_i and where at each $x \in X$ the sum

$$\sum_{i=1}^{n} \sigma_i(x) = 1.$$

Pick a set of $y_i \in U_i$ and define a unital *-homomorphism (hence, ucp map)

$$\varphi : A \to \mathbb{C}^n, \quad \varphi(f) = (f(y_i), ..., f(y - n)).$$

Next we define

$$\psi : \mathbb{C}^n \to A, \quad \psi(z_1, ..., z_n) = \sum_{i=1}^{n} z_i\sigma_i$$

To show that ψ is completely positive we identify $M_k(\mathbb{C}^n)$ with an n-time direct sum of $M_k(\mathbb{C})$. Let $T \otimes \sigma \in M_k(C(X)) \cong C(X, M_k(\mathbb{C}))$ correspond to the function in $C(X, M_k(\mathbb{C}))$ taking $x \mapsto \sigma(x)T$ and

$$\psi_k : M_k(\mathbb{C}^n) \to M_k(A), \quad \psi_n(T_1, ..., T_k) = \sum_{j=1}^{k} T_j \otimes \sigma_j,$$

for matrices $T_j \in M_j(\mathbb{C})$. It is then not hard to see that ψ_n are all positive.

Thanks to remark 2.3.2 we just need to show that $\psi \circ \varphi(f) \to f$ in norm:

$$\|\psi \circ \varphi(f) - f\| = \left\| f - \sum_{i=1}^{n} f(y_i)\sigma_i \right\| = \left\| f \sum_{i=1}^{n} \sigma_i - \sum_{i=1}^{n} f(y_i)\sigma_i \right\| = \left\| \sum_{i=1}^{n} (f - f(y_i)1)\sigma_i \right\| \leq \varepsilon.$$

□

Any time we can consider a C*-algebra result as a non-commutative analogue of a topological result it makes people happy. From this result one sees that nuclearity presents itself as the non-commutative analogue to having a partition of unity. In particular nuclearity is, in some sense, akin to compactness in the non-commutative setting.

Definition 2.3.16. A von Neumann algebra M is called *injective* if for very operator system $E \subset A$, a unital C*-algebra, and ucp map $\varphi : E \to M$, there is a ucp extension $A \to M$.

We will now state Connes' characterization theorem [11]. This result is very important in the study of von Neumann algebras, but we won't give the proof.

Theorem 2.3.17 (Connes, 1976)**.** Let M be a von Neumann algebra. TFAE:

(1) M is injective,

(2) M is semidiscrete,

(3) M has Schwarz's property P: for any $T \in B(\mathcal{H})$, the weak operator closed convex hull of elements of the form xTx^* contains an element in M',

(4) M has property E (Hakeda-Tomiyama): there exists a norm-one projection

$$B(\mathcal{H}) \twoheadrightarrow M',$$

(5) M is approximately finite dimensional (AFD).

2.3.1 The Original Characterizations

The original definition of nuclearity, as given by Takesaki in 1964 [40], was that a C*-algebra A is nuclear if and only if for any C*-algebra B there is a unique C*-norm on $A \odot B$. In particular, this is equivalent to the statement that $A \otimes B = A \otimes_{\max} B$. We now show the equivalence of this definition with the one given at the start of this section:

Theorem 2.3.18 (Choi, Effros, Kirchberg)**.** A C*-algebra A is nuclear if and only if for all C*-algebras B,

$$A \otimes B = A \otimes_{\max} B.$$

In particular, this means that there is a unique C*-norm on $A \odot B$.

Proof. We will show the easy direction ($\Rightarrow$). The converse is highly nontrivial and we refer to [7, thm 3.8.7] (really, most of the work is done in 3.8.5).

Let $\mathrm{id} : A \to A$ be nuclear. We want to show that for any C*-algebra B the identity map $\mathrm{id}_{A\otimes_{\max}B}\mathrm{id}_A \otimes_{\max} \mathrm{id}_B : A \otimes_{\max} B \to A \otimes_{\max} B$ factors through $A \otimes B$. If this wording seems strangely general, it is because this argument actually works for any nuclear map $\theta : A \to C$ between C*-algebras. We want to show that there exists a ccp map $\Psi : A \otimes B \to A \otimes_{\max} B$ such that

$$\begin{tikzcd}
A \otimes_{\max} B \arrow[rr, "\mathrm{id}"] \arrow[dr, "\pi"'] & & A \otimes_{\max} B \\
& A \otimes B \arrow[ur, "\Psi"'] &
\end{tikzcd}$$

commutes, where π is the canonical quotient map.

Let $\varphi_n : A \to M_{k_n}(\mathbb{C})$ and $\psi_n : M_{k_n}(\mathbb{C}) \to A$ be the ccp maps converging to id_A in the point-norm topology. Recall from proposition 2.2.5 that any matrix algebra has a unique C*-norm, so the following family of diagrams are approximately commutative:

$$\begin{tikzcd}
A \otimes_{\max} B \arrow[r, "\mathrm{id}_A\otimes_{\max}\mathrm{id}_B"] \arrow[d, "\pi"'] \arrow[dr, "\varphi_n\otimes_{\max}\mathrm{id}_B"] & A \otimes_{\max} B \\
A \otimes B \arrow[r, "\varphi_n\otimes\mathrm{id}_B"'] & M_{n_k}(\mathbb{C}) \otimes B. \arrow[u, "\psi_n\otimes_{\max}\mathrm{id}_B"']
\end{tikzcd}$$

Now we define a sequence of ccp maps $\Psi_n : A \otimes B \to A \otimes_{max} B$ by

$$\Psi_n = (\psi_n \otimes_{\max} \mathrm{id}_B) \circ (\phi_n \otimes \mathrm{id}_B).$$

Then, since it is the pointwise limit of these contractive maps, the algebraic tensor map $\mathrm{id}_A \odot \mathrm{id}_B : A \odot B \to A \odot B$ is contractive if viewed as going from $A \odot B$ with the spatial norm to $A \odot B$ with the max norm. Hence it extends to a contractive linear map $\Psi : A \otimes B \to A \otimes_{\max} B$. Moreover Ψ is the point-norm limit of the Ψ_n's so it it completely positive.

This contraction, in combination, with the natural fact that there is a contractive map from the max tensor to the min tensor (by the natural ordering of the norms) gives the equality $A \otimes B = A \otimes_{\max} B$. □

We are also able to now show the following corollary to theorem 2.2.15.

Corollary 2.3.19. If $A \subset B$ and C are C*-algebras, and A is nuclear, then we have a natural inclusion

$$A \otimes_{\max} C \subset B \otimes_{\max} C.$$

Proof. Suppose $\pi : A \to B(\mathcal{H})$ is a faithful representation - in particular, let's view π as a nuclear map onto $\pi(A)$. Then we have ccp maps $\varphi_n : A \to M_{k_n}(\mathbb{C}), \psi_n : M_{k_n}(\mathbb{C}) \to \pi(A)$ so that $\psi_n \circ \varphi_n \to \pi$ in the point-norm topology. By Arveson's extension theorem 2.1.13, we may extend the φ_n's to all of B. Moreover we may view the ψ_n's as maps into the enveloping von Neumann algebra so that $\psi_n \circ \phi_n : B \to A \subset \pi(A)''$. Let $\Phi : B \to \pi(A)''$ any point-ultraweak cluster point of these maps. Φ certainly is ccp and extends π on A, so we may invoke 2.2.15. □

Another interesting tensor product related result is the following:

Proposition 2.3.20. Two C*-algebras A, B are nuclear if and only if $A \otimes B$ is nuclear (there is only one tensor product in this case, by theorem 2.3.18).

Proof. We shall prove the equivalence A, B nuclear if and only of $A \otimes_{\max} B$ is nuclear, but there seems to be no reason this doesn't apply to $\otimes$. The forward direction proceeds very simply. Let $\varphi_n^A : A \to M_{k_n}(\mathbb{C})$, $\psi_n^A : M_{k_n}(\mathbb{C}) \to A$, $\varphi_n^B : B \to M_{l_n}(\mathbb{C})$, $\psi_n^B : M_{l_n}(\mathbb{C}) \to B$ be the ccp maps such that $\psi_n^A \circ \varphi_n^A \to \mathrm{id}_A$ and $\psi_n^B \circ \varphi_n^B \to \mathrm{id}_B$ point-wise in norm. Then it is easy to see that

$$\varphi_n = \varphi_n^A \otimes_{\max} \varphi_n^B : A \otimes_{\max} B \to M_{k_n}(\mathbb{C}) \otimes M_{l_n}(\mathbb{C}) = M_{k_n l_n}(\mathbb{C}), \text{ and}$$
$$\psi_n = \psi_n^A \otimes_{\max} \psi_n^B : M_{k_n l_n}(\mathbb{C}) \to A \otimes_{\max} B$$

are ccp maps such that $\psi_n \circ \varphi_n \to \mathrm{id}_A \otimes_{\max} \mathrm{id}_B = \mathrm{id}_{A \otimes_{max} B}$. For the converse, fix a $b \in B$ with norm 1 and $\varphi \in \mathcal{S}(B)$, a state such that $\varphi(b) = 1$. Define a ccp embedding

$$\iota_b : A \to A \otimes_{\max} B, \quad \iota_b(x) = x \otimes b$$

and a so-called *slice map*[2]

$$\mathrm{id}_A \otimes \varphi : A \otimes B \to A \otimes \mathbb{C} \cong B.$$

Note that $(\mathrm{id}_A \otimes \varphi) \circ \iota_b = \mathrm{id}_A$, so the nuclearity of A follows from the nuclearity of $A \otimes_{\max} B$. One shows B is nuclear in the same way. □

Corollary 2.3.21. A is nuclear if and only if $M_n(A)$ is nuclear.

This is immediate once one remembers that $M_n(A) = M_n(\mathbb{C}) \otimes A$. Finally we give the main characterization of exact C*-algebras:

Theorem 2.3.22 (Kirchberg)**.** Let A be a C*-algebra. TFAE:

(1) A is exact.

(2) Given an exact sequence of C*-algebras $I \hookrightarrow B \twoheadrightarrow B/I$, the sequence

$$I \otimes A \hookrightarrow B \otimes A \twoheadrightarrow B/I \otimes A$$

is exact as well.

In parallel with what we showed above, we have:

Proposition 2.3.23. Two C*-algebras A, B are exact if and only if $A \otimes B$ is exact (this here is just the min tensor).

2.4 Group C*-algebras

Thus far we have mentioned some examples of nuclear and exact C*-algebras, however have failed to give may counterexamples. In this section we give a brief crash course on how one constructs a C*-algebra (or even a von Neumann algebra) out of a discrete group and we shall hopefully be able to gain some interesting information about potential counterexamples.

In what follows let G be a discrete group, and $\{\delta_s : s \in G\}$ the canonical orthonormal basis of $\ell^2(G)$.

[2]this is just the name for a map of the form $\rho \otimes \mathrm{id}_B : A \otimes B \to \mathbb{C} \otimes B$ for some functional $\rho \in A^*$.

Definition 2.4.1. The *left regular representation* of G is

$$\lambda : G \to B(\ell^2), \quad \lambda_s(\delta_t) = \delta_{st}, \, \forall s, t \in G.$$

We can also define the *right regular representation* ρ which takes $\delta_t \mapsto \delta_{ts^{-1}}$.

By the intertwiner $\delta_t \mapsto \delta_{t^{-1}}$ it is easy to see that these two representations are unitarily equivalent.

The *group ring* of G, denoted $\mathbb{C}[G]$ consists of all formal sums

$$\sum_{s \in G} a_s s, \quad a_s \in \mathbb{C}$$

where all but finitely many of the constants a_s are nonzero. The multiplication in $\mathbb{C}[G]$ is defined by $(\sum a_s s)(\sum a_t t) = \sum_{s,t \in G} a_s a_t st$, and there is an involution given by

$$\left(\sum_{s \in G} a_s s \right)^* = \sum_{s \in G} \bar{a}_s s^{-1}.$$

We then extend the left regular representation of G to a *-homomorphism $\lambda : \mathbb{C}[G] \to B(\ell^2(G))$.

Given $f \in \ell^\infty(G)$ and $s \in G$, let $s.f : t \mapsto f(s^{-1}t), t \in G$, define an action (left translation) $G \curvearrowright \ell^\infty(G)$. Regarding $\ell^\infty(G) \subset B(\ell^2(G))$ as multiplication operators (i.e: $f(\delta_t) = f(t)\delta_t$, we see that for every $f \in \ell^\infty(G), s \in G$,

$$\lambda_s f \lambda_s^* = s.f \ .$$

What this means is that the action $G \curvearrowright \ell^\infty(G)$ is *spatially implemented by the left regular representation.*

Definition 2.4.2. The reduced C*-algebra of G, denoted $C^*_\lambda(G)$ (sometimes $C^*_r(G)$, r for "reduced") is the completion of $\mathbb{C}[G]$ with respect to the norm

$$\|x\| = \|\lambda(x)\|, x \in \mathbb{C}[G].$$

If we took the closure with respect to the norm induced by the right representation, we would get a group which is isomorphic, denoted $C^*_\rho(G)$.

Definition 2.4.3. The *full group C*-algebra* associated to G is the completion of $\mathbb{C}[G]$ with respect to the norm

$$\|x\| = \sup\{\|\pi(x)\|,\ \pi : \mathbb{C}[G] \to B(\mathcal{H}) \text{ a cyclic *-representation}\}.$$

The full group C*-algebra is also sometimes called the *universal* group C*-algebra, owing to the following universal property:

Proposition 2.4.4. If $u : G \to B(\mathcal{H})$ is a unitary representation of G, then there is a unique *-homomorphism

$$\pi_u : \mathbb{C}[G] \to B(\mathcal{H})$$

such that $\pi_u(s) = u_s$ for all $s \in G$.

Proposition 2.4.5. If $H \subset G$ is an inclusion of groups, then $C^*_\lambda(H) \subset C^*_\lambda(G)$, and $C^*(H) \subset C^*(G)$ as C*-algebras.

Definition 2.4.6. The *group von Neumann algebra* of G is given by

$$L(G) = (C^*_\lambda(G))'' \subset B(\ell^2(G)).$$

2.4.1 Amenable Groups

An important class of groups that may be mentioned are amenable groups. They lead to some examples of amenable C*-algebras (which are often found under a different name).

Definition 2.4.7. An *invariant mean* on a countable discrete group G is a state $\mu : \ell^\infty(G) \to \mathbb{C}$ which is translation invariant. That is,

$$\mu(s.f) = \mu(f), \quad \forall f \in \ell^\infty(G),\ s \in G.$$

G is called *amenable* if there exists an invariant mean μ.

We say that G has *approximate invariant mean* if for every finite subset $E \subset G$ and $\varepsilon > 0$ there exists a probability measure $\mu \in \ell^1(G)$ such that $\max_{s \in E} \|s.\mu - \mu\| < \varepsilon$.

Example 2.4.8. 1. Finite groups are amenable. Consider the invariant mean which associates to the generating $\chi_s \in \ell^\infty(G)$ a value of $1/|G|$.

2. Abelian groups are amenable
3. Amenability is closed under subgroups, quotients, extensions and direct limits, so any group constructed out of finite oe abelian groups using those operations is amenable. This class is called the *elementary amenable groups*.

Example 2.4.9. The prototypical example of a non-amenable group is a non-abelian free group. The free group on two generators, $\mathbb{F}_2 = \langle a, b \rangle$, admits a paradoxical decomposition: If A^+ is the set of all reduced words starting with a, A^- are those starting with a^{-1}, B^+, B^- are defined analogously, and C is the set $\{1, b, b^2, ...\}$. Then

$$\mathbb{F}_2 = A^+ \sqcup A^- \sqcup (B^+ \setminus C) \sqcup (B^- \cup C)$$

but also $\mathbb{F}_2 = A^+ \sqcup aA^- = b^{-1}(B^+ \setminus C) \sqcup (B^- \cup C)$. Then if μ were an invariant mean on $\mathbb{F}_2$,

$$1 = \mu(1) = \mu(\chi_{A^+}) + \mu(\chi_{A^-}) + \mu(\chi_{B^+ \setminus C}) + \mu(\chi_{B^- \cup C})$$

from the first decomposition. However if μ is invariant

$$= \mu(\chi_{A^+} + a\chi_{A^-}) + \mu(b^{-1}\chi_{B^+ \setminus C} + \chi_{B^- \cup C}) = \mu(1) + \mu(1) = 2.$$

We now give a number of equivalent characterizations of amenability. This list is by no means exhaustive.

Theorem 2.4.10. Let G be a discrete group. TFAE:

(1) G is amenable;

(2) G has approximate invariant mean;

(3) G satisfies the *Følner Condition*: For every finite subset $E \subset G$ and $\varepsilon > 0$, there exists another finite subset $F \subset G$ such that $\max_{s \in E} |sF \triangle F|/|F| < \varepsilon$;

(4) The trivial representation is weakly contained in λ;

(5) $C^*_\lambda(G) = C^*(G)$;

(6) $C^*_\lambda(G)$ has a character;

(7) For every finite subset $E \subset G$, $\| \sum_{s \in E} \lambda_s \| = 1$;

(8) $C^*_\lambda(G)$ is nuclear;

(9) $L(G)$ is semidiscrete.

Remark 2.4.11. Together with previous results we also know that this is equivalent to the von Neumann algebra $L(G)$ being injective.

Remark 2.4.12. Since a non-abelian free group $\mathbb{F}$ is not abelian, it follows that $C^*(\mathbb{F})$ is an example of a non-nuclear C*-algebra. In fact, using the tensorial (original) characterization of exactness, applied to the exact sequence

$$0 \to C^*(\mathbb{F}_2) \otimes I \to C^*(\mathbb{F}_2) \otimes C^*(\mathbb{F}_2) \to C^*(\mathbb{F}_2) \otimes C^*_\lambda(\mathbb{F}_2) \to 0,$$

where I is the kernel of the canonical *-homomorphism $C^*(\mathbb{F}_2) \twoheadrightarrow C^*_\lambda(\mathbb{F}_2)$, S. Wasserman [42] showed that the full group C*-algebra $C^*(\mathbb{F})$ isn't exact (see also [7, prop. 3.7.11]).

Group C*-algebras also provide examples for other types C*-algebras we've defined. For example:

Theorem 2.4.13 (Choi)**.** $C^*(\mathbb{F})$ is RFD for $\mathbb{F}$ a countably generated free group.

2.5 Projective, RFD, and QD C*-algebras

Definition 2.5.1. A C*-algebra A is *projective* if for any C*-algebra B, and two-sided, closed ideal $J \lhd B$, any *-homomorphism $\phi : A \to B/J$ lifts to a *-homomorphism $\psi : A \to B$ so that $\phi = \pi\psi$. This is a standard categorical notion.

Definition 2.5.2. A C*-algebra A is called *residually finite dimensional (RFD)* if its points are separated by finite dimensional representations. That is, for each $0 \neq x \in A$ there is a finite dimensional representation $\pi : A \to B(\mathcal{H})$ such that $\pi(x) = 0$.

Another way of saying this is that A is RFD if there exist finite dimensional *-homomorphisms $\pi_i : A \to M_{k_i}(\mathbb{C})$ such that

$$\bigoplus_{i\in I} \pi_i : A \to \bigoplus_{i\in I} M_{k_i}(\mathbb{C})$$

is faithful. If A is separable we can take the index set I to be countable.

We have the following result which is useful. The proof we'll present is from Terry Loring's book [27] which is an elegant modification of a proof of Goodearl and Menal.

Theorem 2.5.3. A projective, separable C*-algebra is RFD.

Proof. Let A be separable and projective. We may view A as a subalgebra of $B(\ell^2(\mathbb{N}))$, and identify the finite dimensional C*-algebra $M_n(\mathbb{C})$ with the subalgebra $B(\ell^2(1, ..., n)) \subset B(\ell^2(\mathbb{N}))$.

Now denote by E the set of all sequences $(x_n) \subset \prod M_n(\mathbb{C})$ whose strong* limit exists. Since multiplication and involution are strong*-continuous on bounded subsets by definition of the strong*-topology, E is a C*-algebra which admits an obvious surjection $E \twoheadrightarrow B(\ell^2(\mathbb{N}))$, namely taking $(x_n) \mapsto \lim(x_n)$. Then, there is an embedding $A \hookrightarrow E \hookrightarrow \prod M_n(\mathbb{C})$. The coordinate projections of $\prod M_n(\mathbb{C})$ restricted to A are finite dimensional representations of A, and they separate points. $\square$

Definition 2.5.4. Let A be a unital C*-algebra. A representation $\pi : A \subset B(\mathcal{H})$ is called *essential* if $\pi(A)$ contains no nonzero compact operators.

The following results are due to work done in 1980 by M.D. Choi [9].

Theorem 2.5.5 (Choi)**.** Let $\mathbb{F}$ be a finitely or countably generated free group. $C^*(\mathbb{F})$ has no nontrivial projections.

Proof. It will be sufficient to prove for the case of $\mathbb{F} = \mathbb{F}_2$. Let U, V be two generating unitaries for a representation of $C^*(\mathbb{F}_2)$ on some Hilbert space $\mathcal{H}$. Define the C*-algebra

$$\mathcal{U} = \{\text{norm cts } \phi : [0, 1] \to \mathcal{H} : \phi(0) \text{ is a scalar operator}\}.$$

If $\phi \in \mathcal{U}$ is a projection, then $\phi(0) = 0$ or I. By continuity of ϕ, each of the projections $\{\phi(t) : t \in [0, 1]\}$ must be 0 or I respectively, so $\mathcal{U}$ has no nontrivial projection.

By the spectral theorem, we may choose self-adjoint operators $A, B \in B(\mathcal{H})$ such that $e^{iA} = U$ and $e^{iB} = V$. We define two new unitaries in $\mathcal{U}$ by

$$\phi_U(t) = e^{itA}, \text{ and } \phi_V(t) = e^{itB}.$$

The evaluation map ev:$\phi \mapsto \phi(1)$ is a surjective *-homomorphism $\mathcal{U} \twoheadrightarrow C^*(\mathbb{F}_2)$. On the other hand by universality of $C^*(\mathbb{F}_2)$, the assignment $U, V \mapsto \phi_U, \phi_V$ gives a surjective *-homomorphism in the other direction. These two are inverse to each other, so we see that $C^*(\mathbb{F}_2)$ is *-isomorphic to a subalgebra of $\mathcal{U}$, and hence cannot have any nontrivial projection. $\square$

Corollary 2.5.6. If π is a faithful representation of a unital, separable, projective C*-algebra then π is essential.

Proof. Let A be a unital, separable, projective C*-algebra. Since A is separable it can be viewed as a quotient of $C^*(\mathbb{F})$ for a countably generated free group $\mathbb{F}$. Since it is projective A also embeds in $C^*(\mathbb{F})$. Supposing that $\pi(A)$ were to contain a nonzero compact operator K, then it must also contain K^*K. But then the finite rank spectral projections of K^*K lie in $\pi(A) \subset \pi(C^*(\mathbb{F}))$. However $C^*(\mathbb{F})$ (and hence any representation of $C^*(\mathbb{F})$) has no nontrivial projections. □

Definition 2.5.7. A (separable) C*-algebra A is called *quasidiagonal (QD)* if there is a net (resp. sequence) of ccp maps $\phi_n : A \to M_{k_n}(\mathbb{C})$ that are asymptotically multiplicative (i.e: $\|\phi_n(ab) - \phi_n(x)\phi_n(y)\| \to 0$ for all $x, y \in A$), and asymptotically isometric ($\|\phi_n(x)\| \to \|x\|$ for all $x \in A$).

Remark 2.5.8.

1. Because we may view a RFD C*-algebra as faithfully represented in $\prod M_n(\mathbb{C})$ (where the product may be uncountable in the non separable case), such an algebra will be QD.
2. If A is unital and QD then it has a net of ucp maps with the asymptotic properties of the definition above. See [7, lemma 7.1.4] for a quick proof.

We treat the following statement as a definition, but for the traditional definition and a proof of this characterization we direct the reader to [7, proposition 7.2.3].

Definition 2.5.9. A set of vectors $\Omega \subset B(\mathcal{H})$ is *quasidiagonal* if there is an increasing net of finite-rank projections converging strongly to the identity and which commute asymptotically with every $T \in \Omega$. If π is a representation of A on $\mathcal{H}$, then we call it *quasidiagonal* if $\pi(A)$ is a quasidiagonal set in $B(\mathcal{H})$.

Remark 2.5.10. A representation of A being quasidiagonal is not equivalent to A being quasidiagonal as a C*-algebra. This arises from a Fredholm index obstruction (see [7, remark 7.5.3]) due to the fact that we can approximate a separable C*-algebra which is a QD set of operators by a RFD algebra. Voiculescu [41] showed that this issue is avoidable if we only consider essential representations.

Definition 2.5.11. An operator $T \in B(\mathcal{H})$ is *quasitriangular* if there is an increasing net of finite-rank projections (P_n) converging strongly to the identity such that $\|TP_n - P_nTP_n\| \to 0$. If T and T^* are both quasitriangular then T is called *bi-quasitriangular*.

Because $\|TP_n - P_nTP_n\| = \|(TP_n - P_nT)P_n\|$, any QD operator is automatically bi-quasitriangular. This means if π is a QD representation of A, then each $\pi(a), a \in A$ is bi-quasitriangular.

Remark 2.5.12. In particular, by remark 2.5.8 a faithful representation of a separable, projective C*-algebra must have this property.

Projectivity is a rare property for C*-algebras, however a useful property that is more common is that of there being a conditionally projective map to A.

Definition 2.5.13. Suppose we have the commutative diagram below in the category of C*-algebras with *-homomorphisms. The *-homomorphism $\alpha : C \to A$ is *conditionally projective* if there exists a *-homomorphism ψ, as in the diagram below, so that the diagram commutes.

$$\begin{tikzcd} C \arrow[r, "\rho"] \arrow[d, "\alpha"'] & B \arrow[d, two heads, "\pi"] \\ A \arrow[r, "\phi"'] \arrow[ur, dashed, "\psi"] & B/I \end{tikzcd}$$

So α being conditionally projective means is that if you can lift $\phi \circ \alpha$ to a *-homomorphism, then you can lift ϕ to a *-homomorphism. It should also be obvious that any *-homomorphism α in where A is a projective C*-algebra is conditionally projective.

Remark 2.5.14 (The case of $C = \mathbb{C}^N$). If $C = \mathbb{C}^N$ and all the C*-algebras and *-homomorphisms in the diagram are unital, then α is conditionally projective whenever a set of orthogonal projections in B/I summing to 1 lift to projections in B summing to 1.

3 Multiplier Algebras

For a C*-algebra A we denote by $M(A)$ the *multiplier algebra* of A, which is the largest C*-algebra which contains A as an ideal. To understand this concretely we embed A in its enveloping von Neumann algebra $\pi_u(A)'' = A^{**}$ and define $M(A)$ as the idealizer of A in A^{**}. For example, the multiplier algebra of $\mathcal{K} = K(\ell^2)$, the compact operators on the separable

Hilbert space, is precisely the bounded operators $B(\ell^2)$. Finally we note that for any unital A, $M(A) = A$, as A cannot be a proper ideal of any C*-algebra.

$M(A)$ may also occasionally be endowed with a topology. One such topology which we will use is the *strict topology*, which is the coarsest topology in which the maps $a \mapsto xa$ and $a \mapsto ax$ are continuous for all $a \in A, x \in M(A)$. We view A in the norm topology for this. The following proposition is easily proved with basic analytic techniques:

Proposition 3.0.1. If x_i is a sequence of bounded self adjoint elements in $M(A)$, and $S \subset A$ is a total subset (meaning the span of S is norm-dense in A), then x_i converges strictly in $M(A)$ if and only if, for all $s \in S$, $x_i s$ is Cauchy in A with the norm topology.

One simply notes that the strict topology is generated by the seminorms $\|x\|_a := \|xa\|+\|ax\|$. The proof follows.

The *corona algebra* is the quotient $Q(A) = M(A)/A$. We denote by e_{11} the projection in $B(\ell^2)$ of a vector onto its first coordinate. Given a unital C*-algebra A, we'll often go between $M(\mathcal{K} \otimes A)$ and A with the map

$$x \mapsto x_{11} = (e_{11} \otimes 1_A)x(e_{11} \otimes 1_A)$$

and the identification $e_{11} \otimes A \sim A$.

Theorem 3.0.2 (Non-commutative Tietze)**.** Let A, B be C*-algebras and $\pi : A \twoheadrightarrow B$ a surjective *-homomorphism. Then π extends uniquely to a *-homomorphism $\hat{\pi} : M(A) \to M(B)$ such that $\hat{\pi}(x)\pi(a) = \pi(x \cdot a)$ for each $x \in M(A), a \in A$. Moreover, if A is σ-unital, the extension is also surjective.

Proof. The normal extension of π to A^{**} clearly provides a suitable extension such that $M(A) \to M(B)$. It remains to check surjectivity in the case of B having a countable approximate unit. For this we require lemma 3.0.3, which can be found in [33], and lemma 3.0.4 which is from [32].

Let A be a C*-algebra. Denote by A^m (respectively A_m) the set of self adjoint elements in $\tilde{A}$ which can be obtained as the limit of a monotone increasing (resp. decreasing) net in $(\tilde{A})_{sa}$.

Lemma 3.0.3 (Pedersen)**.** Let A be a C*-algebra. Then, $A^m \cap A_m = M(A)_{sa}$.

Proof. Define $Q = \{\varphi \in A^* : \varphi \geq 0 \wedge \|\varphi\| \leq 1\}$. Let η denote the evaluation map. This is an isometric map from A^{**}_{sa} onto the set of bounded affine functions on Q which vanish at 0, so that $\eta(A_{sa})$ consists of the continuous functions on Q which vanish at 0.

Let $x \in A^m \cap A_m$, and let $\{y_n\}, \{z_n\} \in \tilde{A}_{sa}$ be the nets such that $y_n \nearrow x$ and $z_n \searrow x$. Recall that by Dini's theorem, a monotone sequence of real valued continuous functions which converges pointwise converges uniformly, and so for any $y \in A_{sa}$,

$$\|\eta(yz_iy) - \eta(yy_iy)\| \to 0.$$

Because η is an isometry, it follows that

$$\|yx - yy_i\|^2 \leq \|y(x - y_i)y\|\|x - y_i\| \leq \|y(z_i - y_i)y\|\|x - y_i\| \to 0$$

Since $yy_i \in A$ we must have that $yx \in A$ for any $y \in A$, and thus $x \in M(A)$. Conversely, suppose $x \in M(A)_{sa}$ and assume $0 \leq x \leq 1$. Let u_λ be an approximate unit in A. Then

$$x^{1/2}u_\lambda x^{1/2} \text{ and } (1-x)^{1/2}u_\lambda(1-x)^{1/2}$$

are increasing nets in A which converge to x and $1 - x$ respectively. Thus $x \in A^m$ and $1 - x \in A^m$ ($\iff x \in A_m$), hence $x \in A^m \cap A_m$. $\square$

Lemma 3.0.4 (Pedersen)**.** If $\phi : A \twoheadrightarrow B$ is a *-homomorphism, $a \in A^+$, and $\phi(a) = b$ then,

$$\phi(\{x \in A^+ : x \leq a\}) = \{y \in B^+ : y \leq b\}.$$

Proof. Let $y \in B^+$ such that $y \leq b$. Because ϕ is a *-homomorphism, $\phi(A^+) = B^+$, and there must exist a self adjoint $z \in A, z \leq a$ such that $\phi(z) = y$. Now let's denote z_+, z_- the positive elements in A such that $z = z_+ - z_-$. We have $\phi(z_+) = y$, $\phi(z_-) = 0$, and $z_+ \leq a + z_-$, therefore the sequence

$$u_n = z_+^{1/2}((1/n) + z_- + a)^{-1}(z_- + a)^{1/2}a^{1/2}$$

converges (since for any operator f, $\{(f + (1/n))^{-1}f\}$ is an approximate identity for $L^* \cap L$, where L is the closed left ideal generated by f, see the proof of theorem 1 of 3.0.4 for details).

We denote by $[f]$ the projection onto the range of an operator f. Set $x := \lim u_n^* u_n$ and note that

$$\phi(x) = \lim b((1/n) + b)^{-1} y((1/n) + b)^{-1} b = [b]y[b] = y.$$

$\square$

Continuation of the proof of Thm 3.0.2. Take a self adjoint element $b \in M(B)$. By lemma 3.0.3 there exist sequences $\{x_n\}$ and $\{y_n\}$ of self-adjoint elements such that $x_n \nearrow b$ and $y_n \searrow b$. Let $\{u_n\}$ be a countable approximate unit for $\ker \pi \subset \tilde{A}$. Choose $s_1, t_1 \in \tilde{A}$ such that $\pi(s_1) = x_1$, $\pi(t_1) = y_1$, and $s_1 \leq t_1$. Now set

$$s_1' := s_1 + (t_1 - s_1)^{1/2} u_1 (t_1 - s_1)^{1/2}.$$

So $\pi(s_1') = x_1$, because $u_1 \in \ker \pi$, and $s_1 \leq s_1' \leq t_1$. Suppose now that we have for $1 \leq k \leq n-1$ elements $\{s_k\}, \{s_k'\}, \{t_k\}$ in $\tilde{A}$ satisfying:

(i) $s_k \leq s_k' \leq s_{k+1} \leq t_{k+1} \leq t_k$ for $k < n-1$

(ii) $\pi(s_k) = \pi(s_k') = x_k$ and $\pi(t_k) = y_k$

(iii) $t_k - s_k' = (t_k - s_k)^{1/2}(1 - u_k)(t_k - s_k)^{1/2}$

By lemma 3.0.4 we can then pick a self-adjoint $t_n \in \tilde{A}$ such that $\pi(t_n) = y_n$ and $s_{n-1} \leq t_n \leq t_{n-1}$. With the same reasoning we can pick a self-adjoint s_n such that $\pi(s_n) = x_n$ and $s_{n-1}' \leq s_n \leq t_n$. Finally set

$$s_n' := s_n + (t_n - s_n)^{1/2} u_n (t_n - s_n)^{1/2}$$

and we have s_n, s_n', t_n satisfying (i), (ii), (iii). Thus we may construct sequences $\{s_n\}, \{s_n'\}$ and $\{t_n\}$ by induction which satisfy the three conditions. Therefore there must be $s \in A^m$ and $t \in A_m$ such that $s_n \nearrow s$, $s_n' \nearrow s$ and $t_n \searrow t$. Since π is normal, $\pi(s) = \pi(t) = b$. Finally,

$$0 \leq t - s \leq t_n - s_n' = (t_n - s_n)^{1/2}(1 - u_n)(t_n - s_n)^{1/2}.$$

Since $\ker \pi$ is an ideal, there exists an open central projection $x \in A^{**}$ such that $\ker \pi = xA^{**} \cap A$ (a projection p is *open* if there is an increasing net of positive operators converging to p). So $B = \pi(A) = (1-x)A$ and we can identify B^{**} with $(1-x)A^{**}$. This means that for each $a \in A^{**}$ we can write $\pi(a) = (1-x)a$. Since $(t_n - s_n)^{1/2} \to (t-s)^{1/2}$ and $1-u_n \to 1-x$ in the SOT, $t - s = (1-x)(t-s)$ (because x is central). This implies that

$$\|t-s\| = \|(1-x)(t-s)\| = \|\pi(t-s\| = \|\pi(t) - \pi(s)\| = 0$$

which means that $t = s \in A^m \cap A_m = M(A)$. So there is a $t \in M(A)$ such that $\pi(t) = b$ for each $b \in M(B)$. □

The above argument is due to Akemann, Pedersen and Tomiyama [1]. The reason for our proving the more involved non-unital case of Tietze's theorem is so that we may apply it to ideals. If we let $J \lhd B$ be an ideal with quotient map $\pi : B \to B/J$. Applying non-commutative Tietze to the map $\mathrm{id}_{\mathcal{K}} \otimes \pi : \mathcal{K} \otimes B \to \mathcal{K} \otimes B/J$, we'll get a *-homomorphism

$$\hat{\pi} : M(\mathcal{K} \otimes B) \to M(\mathcal{K} \otimes B/J).$$

3.1 K-Theory of Stable Multiplier Algebras

Two projections $p, q \in A$ are called *Murray-von Neumann equivalent* if there exists a $v \in A$ such that $v^*v = p$, and $vv^* = q$. We denote this equivalence by $p \sim q$. A projection p is called *infinite* if it is Murray-von Neumann equivalent to a proper sub-projection of itself. That is there is a $q \leq p$ with $q \neq p$ such that $p \sim q$.

A C*-algebra A is called *finite* if $u^*u = 1_A$ implies that $uu^* = 1_A$, in other words the unit is a finite projection. A is called *stably finite* if every matrix algebra $M_n(A)$ is finite.

Proposition 3.1.1. Let $A \neq \mathbb{C}$ be a unital, simple C*-algebra. TFAE:

(1) For every nonzero, positive $x \in A$, there is a $y \in A$ such that $y^*xy = 1_A$,

(2) Every hereditary subalgebra of A contains a (necessarily infinite) projection equivalent to 1_A.

If either of the above conditions is true, we call A *purely infinite*. If A is purely infinite and simple, then for every $n \in \mathbb{N}$, $M_n(A)$ is purely infinite and simple. We now state a result of Rørdam [38], which classifies the A for which the corona algebra $Q(A \otimes \mathcal{K}) = M(A \otimes \mathcal{K})/A \otimes \mathcal{K}$ is simple.

Theorem 3.1.2 (Rørdam)**.** Let A be a unital C*-algebra. Then, the stable multiplier algebra $M(A \otimes \mathcal{K})/(A \otimes \mathcal{K})$ is simple if and only if A is finite dimensional (i.e. $\exists n : \; A \cong M_n(\mathbb{C})$), or A is purely infinite and simple.

We call a multiplier algebra of the form $M(A \otimes \mathcal{K})$ *stable* for any C*-algebra A. That these algebras have trivial K-theory is important in the proof of lemma 4.2.4. The references used for the operator K-theory are the well known book by Bruce Blackadar [3], as well as the introductory book by Rørdam, Larsen, and Laustsen [37]. Obviously operator K-theory is an important field unto itself which requires much study to understand fully, however in the narrow context of Multiplier Algebras we hope to get away with just understanding the basics. First we outline the construction of the first K-theory group K_0.

Given an abelian monoid $(M, +')$, the Grothendieck completion of M is a group with the universal property that any other group containing the homomorphic image of M also contains this group. We define an equivalence class $\sim$ on $M \times M$ by

$$(a_1, a_2) \sim (b_1, b_2) \text{ if there exists a } c \in M : \; a_1 +' b_2 +' c = a_2 +' b_1 +'' c.$$

The *Grothendieck Group* of M is

$$G(M) := M/\sim$$

with operation $+$ such that

$$[(a_1, a_2)] + [(b_1, b_2)] = [(a_1 +' b_1, a_2 +' b_2)].$$

In other words, these equivalence classes $[(a, b)]$ are formal differences $a - b$ of elements in M, and artificially create inverses for elements in M:

$[(a, b)] + [(b, a)] = [(a +' b, b +' a)] \sim [(0, 0)]$ because $a +' b +' 0 +' c = b +' a +' 0 +' c$ with $c = 0$.

The Grothendieck group admits a homomorphism

$$\gamma : M \to G(M) \quad a \mapsto [(a, 0)]$$

called the Grothendieck map.

Given a C*-algebra A, we obtain the group $K_0(A)$ as follows: Let $\mathcal{P}(A)$ denote the set of projections in A, and $\mathcal{P}_n(A) = P(M_n(A)) = P(M_n(\mathbb{C}) \otimes A))$. Then we define

$$\mathcal{P}_\infty(A) = \bigcup_{n=1}^{\infty} P_n(A).$$

Define an equivalence relation $\sim_0$ on $\mathcal{P}_\infty(A)$ by

$$p \sim_0 q \text{ if } \exists v \in M_{n\times m}(A) : v^*v = p,\ vv^* = q.$$

Of course, if p, q are in the same $\mathcal{P}_n(A)$, then this is just the classical Murray-von Neumann equivalence. We may also define an operation $\oplus$ on $\mathcal{P}_\infty(A)$: If $p, q \in P_n(A), P_m(A)$ respectively,

$$p \oplus q = \begin{pmatrix} p & 0 \\ 0 & q \end{pmatrix} \in M_{n+m}(A).$$

We now define a monoid $\mathcal{D}(A) := \mathcal{P}_\infty(A)/\sim_0$ with operation $+$ defined by:

$$[p] + [q] = [p \oplus q].$$

Definition 3.1.3. We define $K_0(A)$ to be the Grothendieck completion of $\mathcal{D}(A)$:

$$K_0(A) = G(\mathcal{D}(A)). \text{ There is a map } [\cdot]_0 : \mathcal{P}_\infty(A) \to K_0(A),\ p \mapsto [p]_0 := \gamma([p]).$$

Now we construct the higher K-group K_1. We once again start with a C*-algebra A, and assume it is unital. We denote by $\mathcal{U}(A)$ the group of unitary elements in A, which forms a topological group with the norm topology. We will denote $u \sim_h v$ when two unitaries $u, v \in \mathcal{U}(A)$ are homotopy equivalent. Of course, any two $x, y \in A$ are homotopic in A via the path $t \mapsto tx + (1-t)y$, however it is not a given that any two elements of $\mathcal{U}(A)$ are homotopic in $\mathcal{U}(A)$.

Similarly to how we started our construction of K_0, we denote $\mathcal{U}_n(A) = \mathcal{U}(M_n(A))$ for $n \in \mathbb{N}$ and $\mathcal{U}_\infty(A)$ the union of the $\mathcal{U}_n(A)$'s. We define the binary operation $\oplus$ on $\mathcal{U}_\infty(A)$ by

$$u \oplus v = \begin{pmatrix} u & 0 \\ 0 & v \end{pmatrix} \in \mathcal{U}_{m+n}(A), \quad \text{for } u \in \mathcal{U}_m(A),\ v \in \mathcal{U}_n(A).$$

We define an equivalence relation $\sim_1$ on $\mathcal{U}_\infty(A)$ by $u \sim_1 v$ if there exists a $k \geq \max m, n$ such that $u \oplus 1_{k-m} \sim_h 1_{k-n} \oplus v$, where 1_k denotes the identity element in $M_k(A)$ ($w \oplus 1_0 = w$ by convention), for each $u \in \mathcal{U}_m(A)$, $v \in \mathcal{U}_n(A)$.

Definition 3.1.4.

$$K_1(A) = \mathcal{U}_\infty(A)/\sim_1$$

The operation $+$ on $K_1(A)$ is defined by $[u]_1 + [v]_1 = [u \oplus v]_1$, where $[\cdot]_1$ denotes the equivalence class with respect to $\sim_1$.

Proposition 3.1.5. Let A be a C*-algebra. Then $K_0(M(A \otimes \mathcal{K})) = K_1(M(A \otimes \mathcal{K})) = 0$.

Proof. Take a sequence v_i of isometries (so since multiplier algebras are always unital, $v_i^* v_i = 1$) in $1 \otimes B(\ell^2) \subset M(A) \otimes_{max} M(\mathcal{K}) \subset M(A \otimes \mathcal{K})$ with pairwise orthogonal ranges (these exist because $M(A \otimes \mathcal{K}) \cong M_n(M(A \otimes \mathcal{K}))$ for all n, and hence is properly infinite, see [3, p. 12.1]). Let p be a projection in $M(A \otimes \mathcal{K})$. With our characterization of strict convergence in 3.0.1, we see that the sums $q = \sigma v_i p v_i^*$ and $\sum v_{i+1} v_i^*$ converge strictly in $M(A \otimes \mathcal{K})$. Then we define

$$w = \begin{bmatrix} 0 & 0 \\ v_1 & \sum v_{i+1} v_i^* \end{bmatrix} \begin{bmatrix} p & 0 \\ 0 & q \end{bmatrix}.$$

$$\text{Then, } w^* w = \begin{bmatrix} p & 0 \\ 0 & q \end{bmatrix}, \quad \text{and } ww^* = \begin{bmatrix} 0 & 0 \\ 0 & q \end{bmatrix}.$$

So, in $K_0(M(A \otimes \mathcal{K}))$, $[q] = ww^* \sim_0 w^* w = [p \oplus q] = [p] + [q]$. This argument of course works for any matrix $p \in \mathcal{P}_\infty(A)$, so we must have that $K_0(M(A \otimes \mathcal{K}))$ is trivial. The same argument works for unitaries so likewise, $K_1(M(A \otimes \mathcal{K})) = 0$. $\square$

The fundamental result of operator K-theory is a Bott periodicity theorem which shows that $K_0(A)$ and $K_1(A)$ are the only two K-theory groups for a given C*-algebra A (see [3], section V.1.2). A consequence of this is that we may construct the following long exact sequence, which becomes cyclic due to the Bott periodicity result:

Theorem 3.1.6 (Six-term exact sequence)**.** Suppose

$$I \xhookrightarrow{\iota} A \xrightarrow{\pi} A/I$$

is a short exact sequence of C*-algebras. Then there is a cyclic exact sequence of groups:

$$\begin{array}{ccccc} K_0(I) & \xrightarrow{\iota_*} & K_0(A) & \xrightarrow{\pi_*} & K_0(A/I) \\ \partial \uparrow & & & & \downarrow \partial \\ K_1(A/I) & \xleftarrow[\pi_*]{} & K_1(A) & \xleftarrow[\iota_*]{} & K_1(I) \end{array}$$

3.2 Hereditary C*-algebras and Stable Isomorphism

Definition 3.2.1. Given a C*-algebra A, a *hereditary* C*-subalgebra $B \subset A$ is one such that whenever $a \in A, b \in B$ are such that $0 \leq a \leq b$, then $a \in B$. If $p \in M(A)$ is a projection, then pAp is a hereditary C*-subalgebra of A called a *corner*. A subalgebra $B \subset A$ is called *full* if it is not contained in any proper ideal, or equivalently if it generates the whole algebra as an ideal, that is, $\overline{\operatorname{span}}(BAB) = A$.

Remark 3.2.2. If $a \in A$ is a positive element (not necessarily a projection), then $\overline{aAa}$ is a hereditary subalgebra (and the smallest one containing a).

Next, we include the following facts from [13], which are surely well known. Suppose $I \triangleleft A$ is a closed two-sided ideal. We may define $M(A, I)$ as the multipliers of A which do not leave I. That is,

$$M(A, I) = \{x \in M(A) : (x \cdot A) \cup (A \cdot x) \subset I\}.$$

Proposition 3.2.3. Let A be a C*-algebra and $I \triangleleft A$ a closed two-sided ideal. Then:

(1) The canonical *-homomorphism $\sigma : M(A, I) \to M(I)$ is surjective.

(2) The image of σ is a hereditary C*-subalgebra of $M(I)$.

(3) The kernel of the Tietze extension $\hat{\pi} : M(A) \to M(A/I)$ of $\pi : A \to A/I$ is $M(A, I)$. If $A, A/I$ are σ-unital, there is a short exact sequence

$$M(A, I) \hookrightarrow M(A) \twoheadrightarrow M(A/I).$$

Proof. (1) By definition of $M(A, I)$, it contains I as an ideal, and there is a unique map $\sigma : M(A, I) \to M(I)$ defined by $\sigma(x)b = x \cdot b$ for all $b \in I, x \in M(A, I)$. If we can show that I is *essential* in $M(A, I)$ (i.e: whenever $xb = 0$ for all $b \in I$, $x = 0$. Equivalently, I intersects trivially with all other nontrivial ideals of $M(A, I)$.) then $\ker(\sigma) = 0$ and we've shown injectivity. So let $xb = bx = 0$ for some nonzero $x \in M(A, I)$ and every $b \in I$ (we will require our ideals to always be two-sided). Then there is a nonzero $a \in A$ such that $xa \neq 0$ and hence $xa \in I$. But then there must be a $b \in I$ such that $0 \neq (xa)b = x(ab)$. But $ab \in I$ so we have a contradiction.

(2) It will suffice to show that $\sigma(M(A,I))M(I)\sigma(M(A,I)) \subset \sigma(M(A,I))$. Let $x_1, x_2 \in M(A,I), y \in M(I)$ and let $z \in M(A)$ act on A such that

$$z \cdot a = x_1 \cdot (y \cdot (x_2 \cdot a)) \text{ and } a \cdot z = ((a \cdot x_1) \cdot y) \cdot x_2$$

for all $a \in A$. Then $z \in M(A,I)$, and

$$\sigma(z)b = z \cdot b = x_1 \cdot (y \cdot (x_2 \cdot b)) = \sigma(x_1) \cdot (y \cdot (\sigma(x_2) \cdot b)) = (\sigma(x_1)y\sigma(x_2)) \cdot b,$$

for all $b \in I$. Likewise $b \cdot \sigma(z) = b \cdot \sigma(x_1)y\sigma(x_2)$. So, $\sigma(x_1)y\sigma(x_2) = \sigma(z) \in \sigma(M(A,I))$.

(3) Let $x \in M(A)$ be in the kernel of $\hat{\pi}$, i.e: $\hat{\pi}(x)\pi(a) = \pi(a)\hat{\pi}(x) = 0, \forall a \in A$. Then, $\pi(x \cdot a) = \pi(a \cdot x) = 0, \forall a \in A$, so $x \cdot a = a \cdot x \in I, \forall a \in A$, and thus $x \in M(A,I)$. □

Lemma 3.2.4 (Pedersen)**.** If B is a C*-algebra containing another C*-algebra A as an essential ideal, then there exists an embedding $B \hookrightarrow M(A)$ taking A to itself.

Proof. Let p be a central projection in B^m (the set of elements of B'' obtained as monotone increasing limits of elements in B) such that $A'' = pB''$. Let

$$\{x \in B : px = 0\} =: J,$$

which is an ideal orthogonal to A. Then the map

$$\tau : B \to A'' \quad x \mapsto px$$

is injective. It is clearly the identity on A, and moreover for each $y \in A$, $x \in B$, $pxy = xy$. This gives the embedding. □

Remark 3.2.5. Applying this to a C*-tensor product, where $A \otimes_\alpha B \subset M(A) \otimes_\alpha M(B)$ as an essential ideal gives us a natural embedding $M(A) \otimes_\alpha M(B) \subset M(A \otimes_\alpha B)$. This embedding is strict in most cases (see [1, Thm. 3.8]).

We make use of the following deep result [2, prop. 2.6] which Terry Loring adapted in [26] so that we may include the norm condition that contractions lift to contractions.

Proposition 3.2.6 (Akemann, Pedersen)**.** If $x_1, ...x_n \in A$ are positive, and $x_i x_j \in I$ for $i \neq j$, then there exist $a_i, ...a_n \in I$ such that $(x_1 - a_1), ...(x_n - a_n)$ are positive and pairwise orthogonal in A.

We will require a few lemmas to prove this, also from [2].

Lemma 3.2.7. If $x = u|x|$ is the $B(\mathcal{H})$ polar decomposition of an element $a \in A \subset B(\mathcal{H})$, a C*-algebra, then for every complex valued f on $\sigma(|x|)$ vanishing at 0, $uf(|x|) \in A$. In particular we can replace $B(\mathcal{H})$ with any von Neumann algebra, so this lemma is allowing us to use polar decomposition in general C*-algebras.

Lemma 3.2.8. Let $x \in A$ be self-adjoint, f be a continuous function on $\sigma(|x|)$, and π denote the quotient map $A \twoheadrightarrow A/J$ for some closed, two-sided ideal J. Then $f(\pi(x)) = \pi(f(x))$.

To prove both of these lemmas we use the Stone-Weierstrass theorem to approximate a continuous function by polynomials (with 0 constant term) because $\sigma(|x|) \subset \mathbb{R}^+$ is compact. This helps because it's easy to see that for such a polynomial,

$$u\left(\sum_i = 1^n \lambda_i |x|^i\right) = \sum_i = 1^n \lambda_i x|x|^{i-1} \in A.$$

A similar fact can be formulated for the second statement. Finally, the Gelfand representation gives the desired result in both cases.

Proposition 3.2.9. Let A be a C*-algebra, $I \triangleleft A$, $x, y \in A$ such that $xy \in I$. Then, there exist $a, b \in I$ such that $(x-a)(y-b) = 0$.

Proof. In the case of x, y positive elements, set $x_1 = (x-y)_+, y_1 = (x-y)_-$ where $(x-y) = (x-y)_+ - (x_y)_-$ is the unique decomposition of the self adjoint element $(x-y)$ into the difference of two orthogonal positive elements. Orthogonality gives that $x_1 y_1 = 0$. Since $x, y \in I$, $\pi(x)\pi(y) = \pi(xy) = 0$, and so we could consider that $\pi(x) - \pi(y)$ is already written as a difference of orthogonal positive elements. So,

$$\pi(x) = (\pi(x) - \pi(y))_+ = (\pi(x-y))_+ = \pi((x-y)_+) = \pi(x_1),$$

where here we applied lemma 3.2.8 to the function $(\cdot)_+$. The same argument gives that $\pi(y) = \pi(y_1)$. Set $a = x - x_1, b = y - y_1$, and note that $(x-a)(y-b) = x_1 y_1 = 0$.

In the case where x, y are non positive, we will use the polar decomposition in A^{**}, thanks to 3.2.7, and write $x = u|x|, y^* = v|y^*|$. Note that by Gelfand theory,

$$|x|^{1/2}|y^*|^{1/2} = \lim_{\varepsilon \to 0}(\varepsilon + |x|)^{-3/2}(x^*x)(yy^*)(\varepsilon + |y^*|)^{-3/2}$$

and since I is a closed ideal, $|x|^{1/2}|y^*|^{1/2}$ lies in I as well. Since these are positive elements, the first part of the proof gives us the existence of $a_1, b_1 \in I$ such that

$$(|x|^{1/2} - a_1)(|y^*|^{1/2} - b_1) = 0.$$

Set $a = u|x|^{1/2}a_1$ and $b = b_1|y^*|^{1/2}v^*$, and note that by an application of 3.2.7 to $(\cdot)^{1/2}$, $u|x|^{1/2}, v|y^*|^{1/2} \in A$ and so $a, b \in I$. Finally, since $y = (y^*)^* = (v|y^*|)^* = |y^*|v^*$,

$$(x - a)(y - b) = (x - a)(|y^*|v^* - b) = u|x|^{1/2}(|x|^{1/2} - a_1)(|y^*|^{1/2} - b_1)|y^*|^{1/2}v^* = 0.$$

□

Remark 3.2.10. In the above proposition, if x, y are positive (resp. self-adjoint), then $(x - a), (y - b)$ can be taken to be positive (resp. self-adjoint). The positive case is baked into the last proof. For the self adjoint case, we let

$$x_1 = [(|x| - |y|)_+]^{1/3} u|x|^{1/3}[(|x| - |y|)_+]^{1/3}$$
$$y_1 = [(|x| - |y|)_-]^{1/3} v|y|^{1/3}[(|x| - |y|)_-]^{1/3}$$

and note that $x_1y_1 = 0$ and clearly since x, y are self adjoint, $u|x|^{1/3} = |x|^{1/3}u$ and $v|y|^{1/3} = |y|^{1/3}v$. If we can show that $\pi(x_1) = \pi(x)$ and $\pi(y_1) = \pi(y)$, it suffices to let $a = x - x_1$ and $b = y - y_1$. To show this we use lemma 3.2.8 and note that $|x|^{1/2}|y|^{1/2} \in I \Rightarrow |x||y| \in I$, so

$$\begin{aligned}\pi(x_1) &= [(\pi(|x|) - \pi(|y|))_+]^{1/3}\pi(u|x|^{1/3})[(\pi(|x|) - \pi(|y|))_+]^{1/3} \\ &= \pi(|x|)^{1/3}\pi(u|x|^{1/3})\pi(|x|)^{1/3} = \pi(x).\end{aligned}$$

Proof of Proposition 3.2.6. The case of $n = 2$ is what we showed in the proof of 3.2.9 (see the last remark). We assume it true for $(n-1)$ and proceed by induction. Let $y = \sum_{i=1}^{n-1} x_i$, and x_n be a positive element such that $x_nx_i, x_ix_n \in I \forall i = 1, ...n-1$. By remark 3.2.10, there exist $a_n, b \in I$ such that $(x_n - a_n)(y - b) = 0$ (since $x_ny \in I$) and $(x_n - a_n), (y - b) \in A^+$. Since $\pi(y) \geq \pi(x_i)$ for each $i < n$, we may apply lemma 3.0.4 $(n-1)$ times to get elements $b_1, ...b_{n-1} \in I$ such that $(x_i - b_i)$ are all positive and satisfy $(x_i - b_i) \leq (y - b)$.

Now we define a C*-subalgebra

$$A_0 = \{x \in A : (x_n - a_n)x = x(x_n - a_n) = 0\}.$$

Note that $(y-b)(x_n - a_n)$, and since $0 \leq x_i - b_1 \leq y - b$, $x_i - b_i \in A_0$, for each $i = 1, ...n-1$. Now note that $\{z_i := x_i - b_i : 1 \leq i \leq n-1\}$ satisfy $z_i z_j \in I \cap A_0, \forall i \neq j$, so by assumption there are $c_1, ...c_{n-1} \in I\mathcal{A}$, such that $x_i - b_i - c_i$ are all positive in A_0 and

$$(x_i - b_i - c_i)(x_j - b_j - c_j) = 0, \ \forall i \neq j.$$

Now, set $a_k = b_k + c_k$ for $1 \leq k \leq n-1$, and note that since $x_k - a_k$ are in A_0, $(x_k - a_k)(x_n - a_n) = (x_n - a_n)(x_k - a_k) = 0$ for all $1 \leq k \leq n$. □

Corollary 3.2.11. Let A be a unital C*-algebra and $I \triangleleft A$. If $x_1, ...x_n$ are pairwise orthogonal, positive, contractions in A/I, then these lift to $\bar{x}_1, ...\bar{x}_n \in A$ which are also pairwise orthogonal, positive, and contractive.

Proof. By 3.2.6, there are $y_1, ...y_n \in A$ such that $\pi(y_i) = x_i$. Define $\bar{x}_i = f(y_i)$, where $f(\lambda) = \min(1, \lambda)$. Making use of 3.2.8,

$$\pi(\bar{x}_i) = \pi(f(y_i)) = f(\pi(y_i)) = f(x_i) = x_i,$$

because x_i are contractions. Finally,

$$\bar{x}_i \bar{x}_j = f(y_i) f(y_j) = 0.$$

□

3.2.1 Brown's Stable Isomorphism Theorem

The next theorem we will prove is the stable isomorphism theorem due to Lawrence Brown [6]. We will give the full proof following Brown's initial work. As usual, $\otimes$ denotes the min tensor.

Definition 3.2.12. A pair of C*-algebras A, B are *stably isomorphic* if $A \otimes \mathcal{K} \cong B \otimes \mathcal{K}$, where $\mathcal{K}$ is the algebra of compact operators on the separable Hilbert space $\mathcal{K}(\ell^2)$.

Let A be a C*-algebra with $I \triangleleft A$ a dense two-sided ideal. We know [14, Prop. 1.7.2] that A admits an increasing approximate unit of elements in I. A helpful corollary to this is that if we have a right ideal $R \triangleleft A$ which generates a dense two sided ideal, then we can assume this approximate unit takes the form $\sum_i r_i^* r_i$, where $r_i \in R$. Noting the basic fact that in a C*-algebra,

$$a^*b + b^*a \leq a^*a + b^*b$$

it follows that any self adjoint element x of the two sided dense ideal $I = R^*R$ is abounded above by a sum $\sum a_i^* a_i$ with $a_i \in R$. Then we can take

$$r_j = c_j(\varepsilon + \sum c_i^* c_i)^{1/2}.$$

Lemma 3.2.13. If p is a full projection in $M(A)$ (that is, pAp is a full corner), $e \in A$, and $\varepsilon > 0$, then there are $a_1, ...a_n \in A$ such that $\sum a_i^* p a_i \leq 1$ and $\|(1 - \sum a_i^* p a_i)e\| < \varepsilon$.

Proof. Considering that pA is a right ideal generating a dense two sided ideal if p is a full projection, we apply the preceding argument to get an increasing approximate unit of the form $\sum a_i^* p a_i$. Then we may make an appropriate choice of n based on ε. □

Lemma 3.2.14. Once again let $p \in M(A)$ be a full projection, and suppose A has a strictly positive element e. Then there exist $(a_i) \in A$ such that $\sum a_i^* p a_i = 1$ with convergence in the strict topology.

Proof. Suppose $n_1 < n_2 < ...n_k$. For some k, suppose we have $s_k = \sum_1^{n_k} a_i^* p a_i \leq 1$ and $\|(1 - s_k)e\| < 1/k$. By 3.2.13, there exist $b_1, ...b_m \in A$ such that

$$s' := \sum_1^m b_i^* p b_i \leq 1, \text{ and } \|(1 - s_k)^{1/2}(1 - s')(1 - s_k)^{1/2} e\| < 1/(k + 1).$$

Let $n_{k+1} = n_k + m$ and $a_{n_k + j} = b_j(1 - s_k)^{1/2}$. So the $s_k \to 1$ strictly. Now note that for arbitrary $x \in A$, the sequence $x^*(1 - \sum_1^n a_i^* p a_i)x$ is monotone decreasing and has a subsequence (the s_k's) going to zero, hence goes to zero. Finally, since

$$\left\| \left(\sum_1^n a_i^* p a_i \right) x \right\| \leq \left\| x^* \left(\sum_1^n a_i^* p a_i \right) x \right\|^{1/2},$$

indeed the sum converges strictly to 1. □

Now let e_{ij} be a generating set of matrix units for the C*-algebra $\mathcal{K} = K(\ell^2)$ of compact operators on the separable Hilbert space. We also note that in the subsequent proofs Brown makes liberal use of the fact that $M(A)\otimes M(\mathcal{K}) \subset M(A\otimes\mathcal{K})$ embeds as a subalgebra. This is because $A\otimes\mathcal{K}$ is an essential ideal of $M(A)\otimes M(\mathcal{K})$, and follows from 3.2.4, as was noted by Akemann and Pedersen in [1] (see remarks after corollary 3.6).

Lemma 3.2.15. Let $p \in M(A)$ be a full projection, and $e \in A$ a strictly positive element. Then,

(1) There exists a partial isometry $u \in M(A\otimes\mathcal{K})$ such that

$$u^*u = 1\otimes e_{11}, \quad \text{and} \quad uu^* \leq p\otimes 1$$

(2) There exists a $v \in M(A\otimes\mathcal{K})$ such that

$$v^*v = 1, \quad \text{and} \quad vv^* = p\otimes 1$$

Proof. (1) Take the sequence (a_i) as in 3.2.14 and let $u = \sum pa_i \otimes e_{i1} \in M(A\otimes\mathcal{K})$. That the sum converges strictly is not terribly hard to see. Let's denote by u_n the partial sums $\sum_1^n pa_i \otimes e_{i1}$. Because finite sums of the form $b_{jk}\otimes e_{jk}$ are dense in $A\otimes\mathcal{K}$, and $\|u_n\| \leq 1$, it will suffice to show that the u_n converge in the seminorms $\|\cdot\|_{b_{jk}\otimes e_{jk}}$.

First note that $(b\otimes e_{jk})(pa_i\otimes e_{i1}) = 0$ unless $i = k$, so obviously $(b\otimes e_{jk})u_n$ will converge. For the other side, note that:

$$\|(u_n - u_m)(b\otimes e_{jk})\|^2 = \|(b^*\otimes e_{jk})(u_n-u_m)^*(u_n-u_m)(b\otimes e_{jk})\| \leq \|b^*\sum_{m+1}^{n} a_i^* p a_i)b\| \to 0$$

That the above goes to zero is thanks to 3.2.14. So indeed the u_n are convergent in the seminorms we wanted, and hence we have strict convergence.

(2) Let N_j be disjoint infinite subsets of $\mathbb{N}$ such that $N = \cup_j N_j$, and let $e_j = \sum_{i\in N_j} 1\otimes e_{ii} \in M(A\otimes\mathcal{K})$. We will proceed in a similar way to our previous proofs. Suppose we have for some fixed n, a set of partial isometries $\{v_k : 1 \leq k \leq 2(n-1)\}$ such that:

(i) $v_k^*v_k$ are all mutually orthogonal, as are $v_kv_k^*$,
(ii) $\sum_{i=1}^{2n-3} v_k^*v_k = \sum_{i=1}^{n-1} e_j$,
(iii) $\sum_{i=1}^{2n-2} v_k^*v_k \leq \sum_{i=1}^{n} e_j$,
(iv) $\sum_{i=1}^{2n-3} v_kv_k^* \leq (p\otimes 1)\sum_{i=1}^{n-1} e_j$, and

(v) $\sum_{i=1}^{2n-2} v_k v_k^* = (p \otimes 1) \sum_{i=1}^{n} e_j$.

We build v_{2n-1} and v_{2n} recursively. One may easily adapt the argument in (1) to obtain a partial isometry u such that $u^*u = e_n$ and $uu^* \leq (p \otimes 1)e_n$. It is not hard to see that there exists a partial isometry $w \in M(A \otimes \mathcal{K})$ such that $w^*w \leq e_{n+1}$ and $ww^* = (p \otimes 1)e_n$ (indeed once the second condition is satisfied, adjusting to cover the first poses no issue). Then we define:

$$v_{2n-1} = u\left(\sum_{i=1}^{n} e_j - \sum_{i=1}^{2n-2} v_k^* v_k\right), \text{ and } v_{2n} = (p \otimes 1)\left(\sum_{i=1}^{n} e_j - \sum_{i=1}^{2n-1} v_k v_k^*\right) w$$

By construction, the v_k satisfy conditions (i)-(v). Finally we set $v = \sum v_k$, with strict convergence. $\square$

Corollary 3.2.16. If B is a full corner of A, and A has a strictly positive element, then B is stably isomorphic to A.

Proof. If B is a full corner, then $B = pAp$ for a full projection $p \in M(A)$, so we may make the identification

$$B \otimes \mathcal{K} \cong (p \otimes 1)(A \otimes \mathcal{K})(p \otimes 1).$$

The isomorphism $A \otimes \mathcal{K} \to B \otimes \mathcal{K}$ is induced by the partial isometry v as above. $\square$

We are now ready to prove Brown's main theorem. The case of a full corner was just proven, but to extend the result to any hereditary subalgebra Brown makes use of a clever matrix trick:

Theorem 3.2.17 (Brown)**.** Let B be a hereditary subalgebra of a C*-algebra A, and suppose both have a strictly positive element. Then A is stably isomorphic to B.

Proof. Let $M_2(A) = A \otimes M_2(\mathbb{C})$ be the algebra of 2×2 matrices with entries in A, and $\{e_{ij} : i, j = 1, 2\}$ be matrix units for $M_2(\mathbb{C})$. Define the following subalgebra $C \subset M_2(A)$:

$$C = \left\{\sum a_{ij} \otimes e_{ij} : a_{11} \in B,\ a_{12} \in \overline{BA},\ a_{21} \in \overline{AB},\ \text{and } a_{22} \in A\right\}.$$

Using coordinate projections, we may easily see that B is isomorphic to the full corner $B \otimes e_{11}$ of C, and A to the full corner $A \otimes e_{22}$. If x_1, x_2 are strictly positive elements of B and A

respectively, then $x_1 \otimes e_{11} + x_2 \otimes e_{22}$ is a strictly positive element of C. We now simply invoke corollary 3.2.16 to see that C is stably isomorphic to both A and B, and hence A is stably isomorphic to B. $\square$

4 C*-algebras with Real Rank Zero

The real rank of a C*-algebra was introduced by Brown and Pedersen [5] as a non-commutative analogue of the notion of dimension for a topological space. Results in this section are due to Lawrence Brown and Gert Pederson [5] unless otherwise noted.

4.1 The Real Rank of a C*-algebra

Definition 4.1.1. The *real rank* of a unital C*-algebra A, denoted $\mathrm{RR}(A)$, is the smallest integer such that for any n-tuple of self adjoint elements $(x_1, ..., x_n)$ in A with $n \leq \mathrm{RR}(A)+1$, and every $\varepsilon > 0$, there is another n-tuple of self adjoint elements $(y_1, ..., y_n)$ such that $\sum y_i^2$ is invertible and $\| \sum (x_i - y_i)^2 \| < \varepsilon$.

For non-unital A we define the real rank of A to be the real rank of its unitization. If A has real rank zero, this means that any self adjoint element can be approximated by an invertible self adjoint element.

Proposition 4.1.2. Any von Neumann Algebra has real rank zero.

Proof. Let $A \subset B(\mathcal{H})$ be a von Neumann Algebra and $x \in A_{sa}$. Consider the spectral projection of x corresponding to the Borel subset $[-\varepsilon, \varepsilon]$, and denote this projection p. Then $y = (1-p)x + \varepsilon p$ has self adjoint inverse by spectral theory.

That is, recalling the multiplication operator version of the spectral theorem (1.2.7), $\mathcal{H}$ can be identified with some $L^2(X, \mu)$ by a unitary, and there is a real valued function f and multiplication operator Ψ_f corresponding to the element x, so that $[x\varphi](t) = f(t)\varphi(t)$. Then, p is given by the multiplication operator $f_\varepsilon := \chi_{[-\varepsilon,\varepsilon]} \circ f$ and y is given by $g = (1-f_\varepsilon)f + \varepsilon f_\varepsilon$. This function is real valued because x is self adjoint and it's not hard to see that $1/g$ is bounded and real valued as well, hence showing y has a self adjoint inverse. Finally,

$$\|x - y\| \leq \|px\| + \varepsilon \|p\| = 2\varepsilon.$$

$\square$

Lemma 4.1.3. If $x, y \in A_+$ are such that $\|xy\| \le \varepsilon^2$, then if $z = x - y$,

$$\|z - (x+y)\| \le 2\varepsilon, \quad \|z_+ - x\| \le \varepsilon, \quad \|z_- - y\| \le \varepsilon.$$

Proof. First we recall that $|x|$ denotes $\sqrt{x^*x}$ for $x \in A$, a C*-algebra. It is immediate that

$$\|z^2 - (x+y)^2\| = \|2(yx + xy)\| \le 4\varepsilon^2$$

The square root is operator monotone, that is, if $x \le y$ in a C*-algebra, then $x^{1/2} \le y^{1/2}$. [3] Moreover, on positive elements the square root is subadditive, so

$$x + y \le (z^2 + 4\varepsilon^2)^{1/2} \le |z| + 2\varepsilon.$$

Similarly,

$$|z| \le x + y + 2\varepsilon \Rightarrow \||z| - (x+y)\| \le 2\varepsilon.$$

Now, $2(z_+ - x) = |z| + z - 2x = |z| - (x+y) \le 2\varepsilon$, and by analogous reasoning we obtain the third and final inequality. $\square$

Lemma 4.1.4. If $x, y \in A_{sa}$ are such that $\|x - y\| \le \varepsilon$, then for $\delta^2 = (\|x\| + \|y\|)\varepsilon$,

$$\||x| - |y|\| \le \delta, \quad \|x_+ - y_+\| \le (\delta + \varepsilon)/2, \quad \|x_- - y_-\| \le (\delta + \varepsilon)/2.$$

Proof. First we note that $\|x^2 - y^2\| \le \delta^2$ and so by the same operator monotonicity of the square root argument as above, $\||x| - |y|\| \le \delta$. Finally,

$$2(x_+ - y_+) = (|x| + x) - (|y| + y) = (|x| - |y|) + (x - y) \Rightarrow \|x_+ - y_+\| \le (\delta + \varepsilon)/2.$$

The same argument gives the third and final inequality. $\square$

Lemma 4.1.5. If A is unital, p is a projection in A, and $x \in A$ such that $b = (1-p)x(1-p)$ is invertible in $(1-p)A(1-p)$, then x is invertible in A if and only if $a - cb^{-1}d$ is invertible in pAp, where $a = pxp, c = px(1-p)$, and $d = (1-p)xp$.

[3] On invertible $y \ge 0$ (otherwise take $y + \delta 1$) suppose that $x^2 \le y^2 \Rightarrow \|y^{-1}x^2y^{-1}\| \le 1$. Then $w = y^{-1/2}xy^{-1/2} = y^{-1/2}zy^{1/2}$, where $z = xy^{-1}$ (so w, z are similar, and t is normal). Then, $\|w\| = r(w) = r(z) \le \|z\| = \sqrt{\|z^*z\|} = \sqrt{\|y^{-1}x^2y^{-1}\|} \le 1 \Rightarrow x \le y$.

Proof. We note that we may define a correspondence

$$A \longleftrightarrow \begin{pmatrix} pAp & pA(1-p) \\ (1-p)Ap & (1-p)A(1-p) \end{pmatrix}$$

It's obvious that the map from A is onto, and if x is to be mapped to the zero element then, $pxp = (1-p)x(1-p) = 0$, so it is also one-to-one. In view of this correspondence, $x \in A$ can be written according to the decomposition

$$x = \begin{pmatrix} a & c \\ d & b \end{pmatrix} = \begin{pmatrix} p & cb^{-1} \\ 0 & 1-p \end{pmatrix} \begin{pmatrix} a - cb^{-1}d & 0 \\ 0 & b \end{pmatrix} \begin{pmatrix} p & 0 \\ b^{-1}d & 1-p \end{pmatrix}$$

Both the triangular matrices correspond to invertible elements in A, so provided b is invertible, x is invertible if and only if $a - cb^{-1}d$ is invertible. □

Theorem 4.1.6 (Brown, Pederson)**.** Let A be a unital C*-algebra. If $\mathrm{RR}(A) = 0$, then $\mathrm{RR}(pAp) = 0$, for all projections $p \in M(A)$. Conversely, if $\mathrm{RR}(pAp) = \mathrm{RR}((1-p)A(1-p)) = 0$, then $\mathrm{RR}(A) = 0$.

Proof. Suppose A is unital. Then of course $p \in A = M(A)$. If $\mathrm{RR}(A) = 0$ and $x \in pAp$ is self adjoint, then $x \in A$ (in particular, $x + 1 - p \in A_{sa}$) and we can find self adjoint and invertible $y \in A$ such that $\|x + 1 - p - y\| \leq \varepsilon$. If we let $b = (1-p)y(1-p)$, then

$$\|1 - p - b\| = \|(1-p)[x + 1 - p - y](1-p)\| \leq \varepsilon.$$

Assuming $\varepsilon < 1$, b is invertible in $(1-p)A(1-p)$ and so invoking lemma 4.1.5, because b and y are invertible, this means that

$$z := pyp - py(1-p)b^{-1}(1-p)yp \in pAp$$

is invertible. We obtain the estimate $\|b^{-1}\| \leq (1-\varepsilon)^{-1}$ by approximating the Neumann series of b^{-1}. Then we can bound:

$$\|py(1-p)b^{-1}(1-p)yp\| \leq (1-\varepsilon)^{-1}\|py(1-p)\|^2 \leq (1-\varepsilon)^{-1}\varepsilon^2.$$

Then, since $\|x - pyp\| = \|x - y\| \leq \varepsilon$,

$$\|x - z\| \leq \|x - pyp\| + (1-\varepsilon)^{-1}\varepsilon^2 \leq \varepsilon + (1-\varepsilon)^{-1}\varepsilon^2.$$

Thus, $\mathrm{RR}(pAp) = 0$.

Conversely, suppose (once again for A unital) that $\mathrm{RR}(pAp) = \mathrm{RR}((1-p)A(1-p)) = 0$. Take $x \in A_{sa}$ and write is according to the decomposition from 4.1.5 noting that since $x = x^*$, $d = (1-p)xp = [px(1-p)]^* = c^*$:

$$x = \begin{pmatrix} a & c \\ c^* & b \end{pmatrix}.$$

Also we remark that a, b are self adjoint in this case. Let $\varepsilon > 0$ and find some $b_0 \in (1-p)A(1-p)$ self-adjoint and invertible such that $\|b - b_0\| \leq \varepsilon$. Next we can find $a_0 \in pAp$ self adjoint such that $a_0 - cb^{-1}c^*$ is invertible in pAp and $\|a - a_0\| \leq \varepsilon$. Lemma 4.1.5 shows that

$$x_0 = \begin{pmatrix} a_0 & c \\ c^* & b_0 \end{pmatrix}$$

is invertible in A and of course we see that $\|x - x_0\| \leq \varepsilon$. Thus, $\mathrm{RR}(A) = 0$. □

The case of non-unital A follows if one of the projections p or $(1-p)$ is taken to be in A (we simply follow the same steps noting that the unitzation of $(1-p)A(1-p)$ is $(1-p)A(1-p) + \mathbb{C}(1-p)$). However, if A is non-unital and both p and $(1-p)$ are not in A, the proof becomes more involved. Fortunately, we will not require this case, as the projections in the theorem below are all in A.

Lemma 4.1.7. Let $x_1, ...x_n \in A$. If there is a projection $p \in A$ such that $\|(1-p)x_k\| < \varepsilon$ for each k, then there exists an approximate unit for A consisting of projections.

The above can be found in the proof of theorem 3.1 of [15]. We have the following characterizations of real rank zero algebras. We follow the proofs from [5] and [34].

Theorem 4.1.8 (Brown, Pedersen)**.** Let A be a unital C*-algebra. TFAE:

(1) $\mathrm{RR}(A) = 0$,
(2) (FS) The elements in A_{sa} with finite spectra are dense in A_{sa},
(3) (HP) Every hereditary C*-subalgebra $B \subset A$ has an approximate unit consisting of projections,

(4) For any two orthogonal elements $x, y \in A_+$ and $\varepsilon > 0$ there is a projection p in A such that $\|(1-p)x\| \leq \varepsilon$ and $py = 0$,

(5) For any two orthogonal elements $x, y \in A_+$ and $\varepsilon > 0$ there is a projection $p \in A$ such that $\|(1-p)x\| \leq \varepsilon$ and $\|py\| \leq \varepsilon$,

(6) For any two elements $x, y \in A_+$ and $\varepsilon > 0$ such that $\|xy\| \leq \varepsilon^2$ there is a projection $p \in A$ such that $\|(1-p)x\| < \varepsilon$ and $\|py\| < \varepsilon$.

Proof. First we prove the equivalence (1) $\Rightarrow$ (6) $\Rightarrow$ (5) $\Rightarrow$ (1). The implication (6) $\Rightarrow$ (5) is immediate. Let $\varepsilon > 0$. To show (1) $\Rightarrow$ (6) take an $\varepsilon_1 > 0$ such that

$$\|xy\|^{1/2} + (((2\|x-y\| + \varepsilon_1)\varepsilon_1)^{1/2} + \varepsilon_1)/2 < \varepsilon.$$

Next choose an invertible $z \in A_{sa}$ such that $\|x - y - x\| < \varepsilon_1$, and note that since z is invertible, $0 \notin \sigma(z)$ so by the spectral theorem there is a projection p such that $pz = z_+$ and of course $(1-p)z = z_-$. Then,

$$\|x - z_+\| \leq \|x - (x-y)_+\| + \|(x-y)_+ - z_+\|.$$

By lemma 4.1.3, $\|x - (x-y)_+\| \leq \|xy\|^{1/2}$, and by applying the second inequality of 4.1.4 to $x - y$ and z, $\|(x-y)_+ - z_+\| \leq (((\|x-y\| + \|z\|)\varepsilon_1)^{1/2} + \varepsilon_1)$. Finally noting that $\|z\| \leq \|x - y\| + \varepsilon$ by assumption, we get that

$$\|x - z_+\| \leq \|xy\|^{1/2} + (((2\|x-y\| + \varepsilon_1)\varepsilon_1)^{1/2} + \varepsilon_1)/2 < \varepsilon.$$

A similar argument gives us $\|y - z_-\| < \varepsilon$. Hence,

$$\Rightarrow \|(1-p)x\| = \|(1-p)(x - z_+)\| < \varepsilon, \text{ and } \|py\| = \|p(y - z_-)\| < \varepsilon.$$

Next we want to show the implication (5) $\Rightarrow$ (1). Take an $x \in A_{sa}$ and $\varepsilon > 0$. x_+ and x_- are orthogonal in A_+ so we may, by assumption, choose a projection $p \in A$ such that $\|(1-p)x_+\| \leq \varepsilon$ and $\|px_-\| \leq \varepsilon$. Then,

$$\begin{aligned}\|x - [pxp + (1-p)x(1-p)]\| &= \|(1-p)xp + px(1-p)\| = \|px(1-p)\| \\ &= \|px(1-p)\| = \|p[x_+ - x_-](1-p)\| < 2\varepsilon,\end{aligned}$$

and

$$-\varepsilon p \leq pxp, \quad \text{and} \quad (1-p)x(1-p) \leq \varepsilon(1-p).$$

Therefore,

$$y = pxp + 2\varepsilon p + (1-p)x(1-p) - 2\varepsilon(1-p)$$

is self adjoint, invertible in A_{sa}, and $\|x - y\| \leq 4\varepsilon$. Hence, $\mathrm{RR}(A) = 0$, and we've completed $(1) \Rightarrow (6) \Rightarrow (5) \Rightarrow (1)$.

Next we prove $(5) \Rightarrow (2) \Rightarrow (3) \Rightarrow (4) \Rightarrow (5)$. $(4) \Rightarrow (5)$ is clear. We shall now show $(5) \Rightarrow (2)$. Let $x \in A_{sa}$ and take real numbers r, t such that $r \leq x \leq t$. Let $s = (r+t)/2$ and for $\varepsilon > 0$, let p be a projection such that $\|(1-p)(x-s)_+\| < \varepsilon$ and $\|p(x-s)_-\| < \varepsilon$. Then,

$$\begin{aligned} \|x - [pxp + (1-p)x(1-p)]\| &= \|(1-p)xp + px(1-p)\| = \|px(1-p)\| \\ &= \|px(1-p) + ps(1-p)\| = \|p[(x-s)_+ - (x-s)_-](1-p)\| < 2\varepsilon \end{aligned}$$

Moreover,

$$(s-\varepsilon) \leq sp - p((x-s)_+ - \varepsilon)p \leq sp - p(x-s)p = pxp \leq tp,$$

and similarly,

$$r(1-p) \leq (1-p)x(1-p) = (1-p)(x-s)(1-p) + s(1-p) \leq (s+\varepsilon)(1-p).$$

We already showed that $(5) \Rightarrow (1)$, so by theorem 4.1.6, pAp and $(1-p)A(1-p)$ are both real rank zero, and we may replace x in the above by pxp and $(1-p)x(1-p)$. Starting with $\|x\| \leq 1$ and letting $\varepsilon_n = 2^{-n}\varepsilon$, after n iterations we end up with $\{p_k : 1 \leq k \leq 2^n\}$, orthogonal projections summing to 1, and such that

$$\left\| x - \sum p_k x p_k \right\| \leq (\varepsilon_1 + ... + \varepsilon_n) \leq 2\varepsilon,$$

and

$$(k-1)2^{-n+1} - 1 - \varepsilon \leq p_k x p_k \leq k2^{-n+1} - 1 + \varepsilon, \ \forall k.$$

If we set

$$x_n = \sum_k (k2^{-n+1} - 1)p_k,$$

then x_n has finite spectrum by construction, and

$$\|x - x_n\| \leq \|x - \sum_k p_k x p_k\| + \|\sum_k p_k x p_k - x_n\|$$
$$\leq 2\varepsilon + \left\| \sum_k p_k x p_k - (k2^{-n+1} - 1)p_k \right\| \leq 3\varepsilon + 2^{-n+1}$$

Hence, we have found a dense subset of elements with finite spectrum.

(2) $\Rightarrow$ (3), i.e. (FS)$\Rightarrow$(HP). Let $B \subset A$ be a hereditary subalgebra. By 4.1.7, if $x_1, ...x_n \in B$ and $\varepsilon > 0$, we must find a projection p so that $\|(1-p)x_k\| < \varepsilon$ for each k. However we note that, in fact,

$$\|(1-p)x_k\|^2 \leq \left\|(1-p)\sum_k x_k x_k^*\right\|,$$

so it suffices to show that we can find p for a single positive x. Furthermore, we may assume that $\|x\| = 1$, by scaling. Choose $\delta > 0$ so that $6\delta > \varepsilon - \varepsilon^2$. There will be large enough N such that $\delta + \delta^{2/N} \geq 1$. Since we assume (FS), there exists a y in A_{sa} with $\|x - y\| \leq \delta$, and $0 \leq y \leq 1$ (by spectral theory). The map $t \mapsto t^{1/n}$ is continuous, so we also have $\|x^{1/n} - y^{1/n}\| \leq \delta$.

Let q be the spectral projection of y onto the interval $[\delta, 1]$, which gives $\|(1-q)y\| \leq \delta$. Then, because we chose N sufficiently large, $\|(1 - y^{2/N})q\| \leq \delta$. Thus,

$$\|(1-q)x\| \leq \|(1-q)(x-y\| + \|y - qy\| \leq 2\delta$$

and, because we assumed $\|x\| = 1$,

$$\|x^{1/n} q x^{1/n} - q\| \leq 3\delta.$$

Letting $z = x^{1/n} q x^{1/n} \in B$, we see that

$$\|z - z^2\| \leq 6\delta < \varepsilon - \varepsilon^2 = (1-\varepsilon)\varepsilon.$$

Hence, $\sigma(z) \subset [0, \varepsilon] \cup [1-\varepsilon, 1]$. Letting p denote the spectral projection of z corresponding to the interval $[1/2, 1]$, we note that $p \in B$, because $z \in B$, and $\|z - p\| \leq \varepsilon$. Then,

$$\|(1-p)x\| \leq \varepsilon + \|(1-z)x\| \leq \varepsilon + 3\delta + \|(1-q)x\| < \varepsilon + 5\delta < 2\varepsilon.$$

Hence, p forms our approximate unit.

(3) $\Rightarrow$ (4). Let x, y be orthogonal elements in A_+, and $B = \overline{xAx} \subset A$ be the hereditary subalgebra generated by x. By assumption (3) = (HP), there is for any $\varepsilon > 0$ a projection $p \in B$ such that $\|(1-p)x\| < \varepsilon$. Since y is orthogonal to x, it annihilates B, hence $py = 0$. This completes the equivalence (5) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) $\Rightarrow$ (5), and thus completes the proof. $\square$

Immediately from the equivalence (1) $\iff$ (3) we obtain the following corollary:

Corollary 4.1.9. If $B \subset A$ is a hereditary subalgebra and $\mathrm{RR}(A) = 0$, then $\mathrm{RR}(B) = 0$.

The process of determining $\mathrm{RR}(A)$ for an arbitrary C*-algebra is not straight forward, however there are many examples of common C*-algebras which can be shown to be real rank zero thanks to the following proposition (a collection of results from [5]).

Proposition 4.1.10.

(1) If $\mathrm{RR}(A) = 0$, then for every n, $\mathrm{RR}(M_n(A)) = 0$.

(2) If $(A_i)_i$ is a net of real rank zero C*-algebras and

$$A = \varinjlim A_i,$$

then $\mathrm{RR}(A) = 0$.

(3) If $\mathrm{RR}(A) = 0$ and B is approximately finite dimensional (AFD), that is, it is a direct limit of finite dimensional C*-algebras $(B_i)_i$, then $\mathrm{RR}(A \otimes B) = 0$.

Proof. (1) Since $M_n(M(A)) = M(M_n(A))$, there is no harm in assuming A to be unital.[4] Let us assume that for $n \leq N$, $\mathrm{RR}(M_n(A)) = 0$ and we shall proceed by induction. Denote by e_N the projection onto the span of the first N basis vectors. Consider the projection $p = 1 \otimes e_N$. Then,

$$pM_{N+1}(A)p \cong M_N(A) \text{ and } (1-p)M_{N+1}(A)(1-p) \cong A.$$

We assumed $\mathrm{RR}(A) = 0$ and $\mathrm{RR}(M_N(A)) = 0$, so by theorem 4.1.6, $\mathrm{RR}(M_{N+1}(A)) = 0$.

[4] please excuse the ambiguity in the notation for a multiplier algebra $M(\cdot)$ and matrix algebra $M_n(\cdot)$.

(2) Assume A is unital and that $1_A \in A_i, \forall i$. If A_i is not unital take the unitization to be $\tilde{A}_i = A_i \oplus \mathbb{C}1_A$. However, if A_i is unital, but. $1_{A_i} \neq 1_A$, then take $B_i = A_i + \mathbb{C}(1_A - 1_{A_i}$, which has real rank zero and and note that A is the direct limit of the B_i. Pick some $x \in A_{sa}$ and choose $\varepsilon > 0$. For each i, we may find an x_i, self-adjoint in A_i, such that $\|x - x_i\| < \varepsilon/2$. Because $\mathrm{RR}(A_i) = 0$, there exists a $y_i \in A_i$, self-adjoint and invertible, such that $\|x_i - y_i\| < \varepsilon/2$. Of course an invertible element in A_i will be invertible in the direct limit, and $\|x - y_i\| < \varepsilon$. Hence, $\mathrm{RR}(A) = 0$.

(3) Once again we may assume B to be unital such that each of the B_i's contains 1_B. For each i note that we may write

$$A \otimes B_i = \bigoplus_{k=1}^{N} M_{n_k}(A).$$

By (1), each $M_{n_k}(A)$ has real rank zero, and therefore so does $A \otimes B_i$ (being a finite direct sum). Since $A \otimes B$ is the direct limit of the net $A \otimes B_i$, each of which has real rank zero, $\mathrm{RR}(A \otimes B) = 0$ by (2). $\square$

4.2 Lifting Projections

We now note some more results compiled in [5]. Lemma 4.2.1 is due to Zhang, and was used by Brown and Pederson to show one of their main results, that multiplier algebras of matroid C*-algebras have real rank zero. A C*-algebra is called *matroid* if for any finite number of elements $x_1, ...x_n$ there are $y_1, ...y_n$ residing in a subalgebra isomorphic to a matrix algebra, with $\|x_i - y_i\| < \varepsilon$. That proposition 4.2.2, and hence by extension, corollary 4.2.3 follows from 4.2.1 is due to George A. Elliot.

Lemma 4.2.1 (S. Zhang)**.** Let $I \triangleleft A$ is a closed ideal with $\mathrm{RR}(I) = 0$, and $B \subset A$ is a hereditary subalgebra. If $p \in B/(B \cap I)$ is a projection which lifts to a projection in A, then p lifts to a projection B.

Proof. Let $p \in B + I$ be a projection which we may write as $p = b + x$ with self-adjoint $b \in B$ and $x \in I$, and let us find another projection $q \in B$ such that $p - q \in I$. Consider the corner pIp, which is a hereditary subalgebra of I which has RR(0). Hence, by theorem

4.1.8 (3), there is an approximate unit of projections for pIp. In practice, this means that for every $\varepsilon > 0$, there is a projection $r \leq p$ such that $\|px^2p(1-r)\| < \varepsilon^2$. Then,

$$\|xp(1-r)\|^2 = \|(xp(1-r))^*xp(1-r)\| = \|1-r\|\|px^2p(1-r)\| = \|px^2p(1-r)\| < \varepsilon^2,$$

and hence, $\|x(p-r)\| = \|xp(1-r)\| < \varepsilon$. Now note that

$$p - r = p(p-r) = b(p-r) + x(p-r).$$

Letting $p_1 = p - r$, and $b_1 = bp_1b$, we get $p_1 = b_1 + x_1$, for some $x_1 \in I$, and

$$\|x_1\| \leq 2\|b\|\varepsilon + \varepsilon^2.$$

If ε is small enough, then $\|b_1 - b_1^2\| < 1/4$, and there is a neighbourhood V of $1/2$ so that $V \cap \sigma(b_1) =$. Letting $f \in C(\sigma(b_1))$ be such that $f(t) = 0$ for $t < 1/2$ and $f(t) = 1$ for $t > 1$, by continuous functional calculus $q = f(b_1)$ is a projection with $p - q \in I$. □

Proposition 4.2.2. If $I \triangleleft A$ is a closed ideal with $\mathrm{RR}(I) = 0$ and the induced group homomorphism $\pi_* : K_0(A) \to K_0(A/I)$ is surjective, then every projection in A/I lifts to a projection in A. (See 3.1 for the definitions of K_0 and K_1)

Proof. Because π_* is surjective, any projection from A/I lifts to a projection in $M_n(A)$, for some n. However $A \subset M_n(A)$ is hereditary (consider p the projection onto the first coordinate, so that $A = pM_n(A)p$), so by 4.2.1, such a projection projection also lifts to one in A. □

Corollary 4.2.3. If $I \triangleleft A$ is a closed ideal with $\mathrm{RR}(I) = 0$ and $K_1(I) = 0$, then every projection in A/I lifts to a projection in A.

Proof. Consider the following part of the six-term exact sequence in K-theory (see 3.1.6 - also [3] for details):

$$K_0(A) \xrightarrow{\pi_*} K_0(A/I) \xrightarrow{\partial} K_1(I) = 0$$

The exactness of this segment of the sequence implies of course that π_* is surjective, and so we may invoke proposition 4.2.2. □

Lemma 4.2.4 (Courtney).

(1) Any finite collection of pairwise orthogonal non-zero projections in $M(\mathcal{K} \otimes Q(\mathcal{K}))$ (which sum to 1) lifts to pairwise orthogonal projections in $M(\mathcal{K} \otimes M(\mathcal{K}))$ (which sum to 1).

(2) For any $N \geq 1$, any unital *-homomorphism $\mathbb{C}^N \to M(\mathcal{K} \otimes Q(\mathcal{K}))$ lifts to a unital *-homomorphism $\mathbb{C}^N \to M(\mathcal{K} \otimes M(\mathcal{K}))$.

Proof. Assertion (2) follows directly from (1). 3.2.11 states that $p_1, ...p_n$ lift to some $x_1, ...x_n \in M(\mathcal{K} \otimes M(\mathcal{K}))$ which are pairwise orthogonal positive contractions, but not necessarily projections. For each i define

$$I_i = \overline{x_i M(\mathcal{K} \otimes M(\mathcal{K}), \mathcal{K} \otimes \mathcal{K}) x_i},$$

$$A_i = \overline{x_i M(\mathcal{K} \otimes M(\mathcal{K})) x_i} \quad \text{and,} \quad B_i = p_i M(\mathcal{K} \otimes Q(\mathcal{K})) p_i,$$

Giving us for each i, the short exact sequence $I_i \hookrightarrow A_i \twoheadrightarrow B_i$.

Because the x_i are pairwise orthogonal, if we can lift the p_i to projections in the A_i, the lifts will also be pairwise orthogonal. Fix an i and note that I_i is a hereditary subalgebra of $M(\mathcal{K} \otimes M(\mathcal{K}), \mathcal{K} \otimes \mathcal{K})$ (specifically, the one generated by x_i). By 3.2.3(2), I_i embeds in $M(\mathcal{K} \otimes \mathcal{K}) = B(\ell^2)$ as a hereditary subalgebra. $B(\ell^2)$ has real rank 0, a property which will pass to I_i by corollary 4.1.9.

$\mathcal{K} \otimes Q(\mathcal{K})$ is simple, σ-unital, and purely infinite, so by a result of Rørdam (3.1.2, or [38, theorem 3.2], also independently proven by S. Zhang) the corona algebra $Q(K \otimes Q(K))$ is simple. Then, $K \otimes Q(K)$ is the only closed two-sided ideal in $M(K \otimes Q(K)$, so B_i embeds either as a hereditary subalgebra of $M(K \otimes Q(K))$, or as a hereditary subalgebra of $K \otimes Q(K)$. Either way, B_i embeds as a hereditary subalgebra of a σ-unital C*-algebra. Since B_i is also σ-unital (i.e. there is a strictly positive element), in both cases we may invoke Brown's stable isomorphism theorem 3.2.17 to determine that we are in one of the two following situations:

$$\text{either } \mathcal{K} \otimes B_i \cong M(\mathcal{K} \otimes Q(\mathcal{K})), \quad \text{or} \quad \mathcal{K} \otimes B_i \cong \mathcal{K} \otimes Q(\mathcal{K}).$$

By the stability of K_0 for C*-algebras of the form $\mathcal{K} \otimes A$, we have

$$K_0(\mathcal{K} \otimes B_i) = K_0(B_i) = K_0(Q(\mathcal{K})) \quad \text{or} \quad K_0(B_i) = K_0(M(\mathcal{K} \otimes Q(\mathcal{K})).$$

Since $K_0(M(\mathcal{K})/\mathcal{K}) = K_0(M(\mathcal{K})) = 0$ and by 3.1.5, in both cases, $K_0(B_i) = 0$. Recall that $\mathrm{RR}(I_i) = 0$, and note that the induced group homomorphism $\pi_{i*} : K_0(A_i) \to K_0(B_i)$ must be surjective (because $K_0(B_i)$ is trivial). Hence, by proposition 4.2.2, the projections p_i in B_i lift to A_i. □

5 The Local Lifting and Weak Expectation Properties

In this section we introduce the local lifting property (LLP). Of the many characterizations of this property, one which is of particular interest is 5.3.3. Although we only wish to apply the result to C*-algebras, we include much of the operator space theory leading to the original operator space result. We also introduce the LLP's cousin, Lance's weak expectation property (WEP). Unless otherwise noted $\otimes$ denotes the min tensor product, as usual. Most of the work on this topic was at least initiated by Eberhard Kirchberg in the foundational paper [23], and much of it has been compiled by Narutaka Ozawa in [30].

5.1 First definitions and historical context

Let A, B be C*-algebras and $J \lhd B$ a (closed, two-sided) ideal. We call a ccp map $\varphi : A \to B/J$ *ccp liftable* if there exists a ccp map $\psi : A \to B$ such that $\pi\psi = \phi$, where $\pi : B \to B/J$ is the quotient map. φ is called *locally ccp liftable* if for every finite dimensional operator system $E \subset A$, the restriction $\varphi|_E$ is ccp liftable. If we have unital C*-algebras (as we usually will) we can replace ccp with ucp in this definition and be perfectly satisfied (this follows from an argument based on 5.1.13 - see [7, lem. 13.1.2]). We shall henceforth abide by this new definition.

Definition 5.1.1. We say A has the *lifting property (LP)* if for every B, J, and φ as above, φ is ucp liftable. A has the *local lifting property (LLP)* if every φ is locally ucp liftable.

Remark 5.1.2. The LLP, as is implied by the name, is a local version of the LP. It is immediate, then, that the LP implies the LLP.

We say $A \subset B$ is *(weakly) cp complemented* in B if there is a ucp map $\varphi : B \to A$ (resp A^{**}) such that $\varphi|_A = \mathrm{id}|_A$. An example of an inclusion of weakly cp complemented C*-algebras is: for $H \leq G$ discrete groups, then natural inclusion of C*-algebras $C^*(H) \subset C^*(G)$. The conditional expectation defined simply by $s \mapsto 0$, $\forall s \in G \setminus H$ is sufficient.

Definition 5.1.3. A has the *weak expectation property (WEP)* if it is weakly cp complemented in $B(H)$, where one faithfully represents $A \subset B(H)$. A is said to be QWEP if it is the quotient of a C*-algebra with the WEP.

Remark 5.1.4. Arveson's theorem 2.1.13 essentially states that $B(\mathcal{H})$ is an injective von Neumann algebra. Because of this, the definition of WEP is independent of the choice of faithful representation.

Remark 5.1.5. If instead we have von Neumann algebras $M \subset N$, with τ a faithful normal trace on N, then recall that by lemma 2.1.10 there is a unique trace preserving conditional expectation $\Phi : N \twoheadrightarrow M$ with the relation $\tau(a\Phi(x)) = \tau(ax))$ for $a, x \in M$. Thus M is cp complemented in N (also weakly cp complemented by the Bicommutant theorem).

The obvious example of a C*-algebra with the WEP is $B(\mathcal{H})$. In studying the WEP and LLP of separable C*-algebras we note that $B(\mathcal{H})$ is universal because it is injective and contains all separable C*-algebras. The other object which is universal in a sense similar to this is $C^*(\mathbb{F})$, because every separable C*-algebra is a quotient of it. Moreover, $C^*(\mathbb{F})$ has the LP (and the LLP).

Proposition 5.1.6 (Kirchberg)**.** $C^*(\mathbb{F})$ has the LP for $\mathbb{F}$ a countable free group.

Proof. First we shall show that a *-homomorphism $\theta : C^*(\mathbb{F}) \to B/J$ is ucp liftable. Let $U_1, U_2, ...$ be the free generators of $C^*(\mathbb{F})$ and $x_1, x_2, ... \in B$ be contractive lifts of $\theta(U_1), \theta(U_2), ...$ Let

$$\hat{x}_n = \begin{pmatrix} x_n & (1 - x_n x_n^*)^{1/2} \\ (1 - x_n x_n^*)^{1/2} & -x_n^* \end{pmatrix} \in M_2(B).$$

Evidently, $\hat{x}_n$ is a unitary. By universality of $C^*(\mathbb{F})$ there exists a *-homomorphism $\rho : C^*(\mathbb{F}) \to M_2(B)$ with $\rho(U_n) = \hat{x}_n$. Composition with the projection onto the (1,1) entry gives the desired lifting $C^*(\mathbb{F}) \to B$.

Given an arbitrary ucp $\varphi : C^*(\mathbb{F}) \to B/J$ we now want to show there exists a ucp lift. We assume that B/J is separable because $\mathbb{F}$ is countable. By the Kasparov-Stinespring dilation theorem 6.2.5, there exists a *-homomorphism

$$\theta : C^*(\mathbb{F}) \to M(\mathcal{K} \otimes B/J)$$

such that $\varphi(x) = (\theta(x))_{11}$ for $z \in C^*(\mathbb{F})$ (here $\mathcal{K}$ denotes the compact operators on the separable Hilbert space). We can now invoke the non-commutative Tietze theorem 3.0.2 which states that the *-homomorphism $\pi : \mathcal{K} \otimes B \twoheadrightarrow \mathcal{K} \otimes B/J$ extends to a surjective *-homomorphism

$$\tilde{\pi} : M(\mathcal{K} \otimes B) \twoheadrightarrow M(\mathcal{K} \otimes B/J).$$

We can now use the first part of the proof to find a ucp map $\rho : C^*(\mathbb{F}) \to M(\mathcal{K} \otimes B)$ such that $\theta = \tilde{\pi}\rho$. Then defining a new ucp map

$$\psi : C^*(\mathbb{F}) \to B, \quad \psi(x) = (\rho(x))_{11}$$

gives the ucp lift of φ. $\square$

It seems proper to mention now that it was conjectured by Kirchberg that all separable C*-algebras are QWEP. This is known as the *QWEP conjecture* and much work was done on this. In particular it was shown that the following conjectures are all equivalent to this conjecture, including Connes' embedding problem, a big (up until recently open) problem in von Neumann algebra theory.

Theorem 5.1.7. The following are equivalent:

(1) (QWEP conjecture) Every separable C*-algebra is QWEP;

(2) $C^*(\mathbb{F}_\infty)$ has the WEP;

(3) There is a unique C*-norm n $C^*(\mathbb{F}) \odot C^*(\mathbb{F})$ (here $\mathbb{F}$ is the countably generated free group $\mathbb{F}_\infty$). In particular, $C^*(\mathbb{F}) \otimes C^*(\mathbb{F}) = C^*(\mathbb{F}) \otimes_{\max} C^*(\mathbb{F})$;

(4) Every C*-algebra with the LLP has the WEP;

(5) Tsirelson's Problem in quantum information theory, see [19, p. 5];

(6) (Connes' Embedding Problem) Any separable II$_1$-factor embeds into R^ω, the ultrapower of the hyperfinite II$_1$ factor R. [5]

[5] A *factor* $N \subset M$ is a subalgebra of a von Neumann algebra with trivial (up to scalar multiplication)

This study was initiated when Kirchberg proved the remarkable proposition below, establishing the duality between the LLP and WEP that we have been referring to:

Proposition 5.1.8 (Kirchberg)**.** Let A, B be C*-algebras. Then,

(1) $A \otimes B = A \otimes_{\max} B$ if A has the WEP and B has the LLP,

(2) $A \otimes C^*(\mathbb{F}) = A \otimes_{\max} C^*(\mathbb{F})$ if and only if A has the WEP,

(3) $B(\mathcal{H}) \otimes B = B(\mathcal{H}) \otimes_{\max} B$ if and only if B has the LLP.

However, a consequence of the groundbreaking 2020 paper of MIP*=RE in quantum complexity theory [19] was a refutation of Tsirelson's problem, and hence Connes' embedding problem as well. This leads to a cascade of results related to the WEP and LLP. In particular, it implies that all of the statements in the above theorem are false.
The proof of 5.1.6 is due to Kirchberg [22]. The question of properly distinguishing between the LP and LLP in the separable setting was interesting. In the non-separable setting there exist examples of C*-algebras with the LLP but not the LP, such as ℓ^∞/c_0 (see [30, p. 10]). In the separable setting it was shown that the LLP and LP are equivalent if the QWEP conjecture were true. Of course, the aforementioned refutation of the latter means that as of 2020, there is still no answer to the following question:

Question 5.1.9. Does there exist a separable C*-algebra with the LLP but not the LP?

The following is a useful characterization of the LLP which is certainly known to experts. Our proof follows remarks after proposition 6.5 of [12].

Corollary 5.1.10 (Arveson, Effros, Haagerup)**.** Let $\phi : A \to C^*(\mathbb{F})/J$ be a the identity on a unital C*-algebra A (viewed as a quotient of $C^*(\mathbb{F})$ for some free group $\mathbb{F}$). Then A has the LLP if and only if ϕ locally lifts.

Proof. The *only if* direction is trivial. Now if we suppose $E \subset A$ is a finite dimensional operator space, and let ψ denote the lift of the restriction $\phi|_E$. If we denote by ρ the

center. A factor is of type II_1 if every projection can be written as the sum of two Murray-von Neumann equivalent projections, and the identity projection is finite. A factor is *hyperfinite (AFD)* if it contains a dense increasing union of finite dimensional subalgebras. Connes [11] proved there is a unique such factor, denoted R.

surjective *-homomorphism $C^*(\mathbb{F}) \to A$, then given any ucp map to a quotient C*-algebra $\varphi : A \to B/J$, the composition $\varphi \circ \rho : C^*(\mathbb{F}) \to B/J$ is ucp and using the well known fact that $C^*(\mathbb{F})$ has the lifting property, there is a ucp lift $C^*(\mathbb{F}) \to B$. The composition of ψ with this lift gives us the desired lifting of of φ on E, and hence we've shown A has the LLP. $\square$

Remark 5.1.11. With a little work it's not hard to see that in the case of separable A and countably generated free group $\mathbb{F}$, A has the LP if and only if ϕ lifts.

As an immediate consequence of the previous corollary, we get a new class of examples of C*-algebras observing the LLP and LP:

Corollary 5.1.12. Every projective C*-algebra has the LLP. Every separable projective C*-algebra has the LP.

For the second assertion remember that a separable C*-algebra is a quotient of $C^*(\mathbb{F})$, for $\mathbb{F}$ countably generated, which has the lifting property.

Perhaps the most well-known class of examples of C*-algebras admitting the LLP are the separable, nuclear C*-algebras. This follows from the Choi-Effros lifting theorem, below. An elegant proof due to Arveson is included in [7, appendix C]. The main ingredient in this proof is the following useful lemma:

Lemma 5.1.13 (Arveson)**.** Let E be a separable operator system, B a separable C*-algebra and $I \triangleleft B$ a ideal. The set of ccp maps $E \to B/I$ which are liftable is closed in the point-norm topology.

Proof. Let $\varphi : E \to B/I$ be a ccp map and $\pi : B \twoheadrightarrow B/I$ the quotient map. Suppose there is a sequence of ccp maps $\psi'_n : E \to B$ such that $\pi \circ \psi'_n \to \varphi$ in the point-norm topology. Since E is separable we may choose $\{x_k\}$ to be a norm dense subset. Passing to a subsequence if necessary, the point-norm convergence implies that for every $k \leq n$,

$$\|\pi \circ \psi'_n(x_k) - \varphi(x_k)\| < 1/2^n.$$

It can be shown (by a simple induction argument - see [7, appendix C] for details), that there exists another sequence of ccp maps $\psi_n : E \to B$ such that

$$\|\pi \circ \psi_n(x_k) - \varphi(x_k)\| < 1/2^n, \text{ and } \|\psi_{n+1}(x_k) - \psi_n(x_k)\| < 1/2^{n-1}, \quad k \leq n,$$

and $\pi \circ \psi'_m = \pi \circ \psi_n$. Then the ψ_n converge to some ccp ψ on the norm dense subset $\{x_n\}$, they do so on the whole space. Moreover ψ is certainly a ccp lift of φ. $\square$

Theorem 5.1.14 (Choi, Effros)**.** Let A be a separable C*-algebra. Every nuclear ccp map $A \to B/I$ is ccp liftable.

Proof. It suffices to show that any ccp map $\varphi : M_n(\mathbb{C}) \to B/I$ is liftable. Then since any nuclear ccp map factors approximately through matrix algebras, and the point-norm limit of liftable maps is liftable, the theorem follows. By the bijective correspondence in 2.1.4, we take the corresponding positive $a = [\varphi(e_{ij})] \in M_n(B/I)$ for the matrix units $\{e_{ij}\}$. Since $\pi_n : M_n B \twoheadrightarrow M_n B/I$ is a *-homomorphism, a lifts to some $b \in M_n(B)_+$. Applying the correspondence again gives some map $\psi : M_n(\mathbb{C}) \to B$ which is a ccp lifting of φ. $\square$

As an immediate consequence, every ccp map from a nuclear C*-algebra to a quotient is liftable. In particular this implies that every map from a separable nuclear C*-algebra is ccp liftable (just compose with the nuclear map id_A). Hence, separable, nuclear C*-algebras have the LP (and the LLP).

In the same paper [23] where Kirchberg launches the study of the LLP and WEP, proves most of the results found in this section, and poses the QWEP conjecture, the following question is also asked:

Question 5.1.15 (Kirchberg)**.** Does there exist a non-nuclear C*-algebra with the LLP and WEP?

In 2020, Gilles Pisier [36] constructs a first example of a C*-algebra with these properties. In fact, concrete examples of a non-nuclear C*-algebra with just the LLP have been hard to come by. We will show a class of examples satisfying these modified criteria.

5.2 Operator Spaces and Operator Space Duality

Definition 5.2.1. Let $X \subset A$ be an operator space in a C*-algebra A. A vector space homomorphism $\varphi : X \to Y \subset B$ to another operator space is called *completely bounded (cb)* if each of the maps

$$\varphi_n : M_n(X) \to M_n(Y), \quad \varphi_n([x_{ij}]) = [\varphi(x_{ij})]$$

is bounded with respect to the norms on $M_n(X)$ and $M_n(Y)$ inherited from $M_n(A)$ and $M_n(B)$. If φ is completely bounded we assign to it a norm

$$\|\varphi\|_{cb} = \sup_n \|\varphi_n\|,$$

the completely bounded (cb) norm. In the language of operator spaces we say that φ is *completely contractive* if $\|\varphi\|_{cb} < 1$ and *completely isometric* if each φ_n is isometric. The space of all completely bounded maps $X \to Y$ forms a Banach space with the cb norm and is sometimes denoted $CB(X, Y)$.

The next thing we'll need is some basic facts about operator space duality. Let E, X be an operator spaces. Here and for the rest of the section we will simply write $\otimes$ for the min tensor of operator spaces, which is defined as the norm closure of the algebraic tensor $E \odot X$ in $B(\mathcal{H} \otimes K)$, where we represent $E \subset B(\mathcal{H})$ and $X \subset B(K)$.

For an operator space $E \subset B(\mathcal{H})$ we denote by E^* its Banach dual space, and will proceed to define an operator space structure. For $x = [x_{kl}] \in M_m(E)$ we define

$$\theta_x : E^* \to M_m(\mathbb{C}), \quad \theta_x(\varphi) = [\varphi(x_{kl})].$$

Denote by $B_m(E)$ the closed unit ball of $M_m(E)$. The following isometric inclusion gives the operator space structure on E^*:

$$\Theta : E^* \ni \varphi \mapsto (\theta_x(\varphi))_x \in \prod_{m\in\mathbb{N}} \prod_{x\in B_m(E)} M_m(\mathbb{C}) \subset B(\mathcal{H})$$

Theorem 5.2.2 (Operator Space Duality)**.** Let E, X be operator spaces, and X finite dimensional. There is an isometric identification between $E^* \otimes X$ and the space of completely bounded maps $CB(E, X)$. If both operator spaces are infinite dimensional then there is an isometric inclusion $E^* \otimes X \subset CB(E, X)$.

Proof. The construction we will use is as follows.First we'll suppose that X is finite dimensional. For any $\varphi = [\varphi_{ij}] \in M_n(E^*) = E^* \otimes M_n(\mathbb{C})$ define

$$T_\varphi : x \mapsto [\phi_{ij}(x)] \in M_n(\mathbb{C}), \text{ for } x \in E.$$

So T_φ is a cb map $E \to M_n(\mathbb{C}) = X$, and it is readily available from the definitions and from the fact that $M_n(M_m(\mathbb{C})) = M_m(M_n(\mathbb{C}))$ that the bijection $\varphi \leftrightarrow T_\varphi$ is isometric:

$$\begin{aligned}\|\varphi\|_{M_n(E^*)} &= \sup\{\|(\theta_x)_n(\varphi)\|_{M_n(M_m(\mathbb{C}))} : m \in \mathbb{N}, x \in B_m(E)\} \\ &= \sup\{\|(T_\varphi)_m(x)\|_{M_m(M_n(\mathbb{C}))} : m \in \mathbb{N}, x \in B_m(E)\} = \|T_\varphi\|_{cb}\end{aligned}$$

This concludes the finite dimensional case. Now we let X be a general operator space. For $z = \sum_i \varphi_i \otimes x_i \in E^* \odot X$, we'll define our operator $T_z : E \to X$ by

$$T_z(e) = \sum_i \varphi_i(x)x_i, \text{ for } x \in E.$$

Now let $X \subset B(\mathcal{H})$, and for simplicity let's assume $\mathcal{H}$ is separable. Take P_n an increasing sequence of finite rank projections in $B(\mathcal{H})$ converging to 1 in the strong operator topology, and with the property that $\mathrm{rank} P_n = n$. Denote by $\Phi_n : B(\mathcal{H}) \to M_n(\mathbb{C})$ the compression by P_n ($x \mapsto P_n x P_n$), and fix a $z \in E^* \odot X$. We remark that

$$T_{(\mathrm{id}\otimes\Phi_n)(z)}(x) = \sum_i \varphi_i(x)\Phi_n(x_i) = (\Phi_n \circ T_z)(x).$$

So when we pass to the finite dimensional case we just proved, it suffices to first make the identification $T_{(\mathrm{id}\otimes\Phi_n)(z)} \sim (\mathrm{id} \otimes \Phi_n)(z)$ and then use this fact.

$$\|z\|_{min} = \sup_n \|(\mathrm{id} \otimes \Phi_n)(z)\|_{E^* M_n(\mathbb{C})} = \sup_n \|\Phi_n \circ T_z\|_{cb} = \|T\|_{cb}.$$

□

Given a C*-algebra B, a finite dimensional operator space F and an ideal $J \triangleleft B$ there is a contractive isomorphism

$$(F \otimes B)/(F \otimes J) \to F \otimes B/J$$

which will be useful because we can view $F \otimes B/J$ as a quotient space of cb maps $E^* \to B$.

5.3 Calkin Algebra Characterization of the LLP

Let ℓ^2 denote the separable Hilbert space, and $\mathcal{K} = K(\ell^2)$ the ideal of compact operators in $B(\ell^2)$. The quotient algebra $B(\ell^2)/\mathcal{K}$ is known as the *Calkin algebra*.

The following theorem from a paper of Ozawa [31] characterizes the local lifting property in terms of ucp maps to the Calkin algebra. We'll write the proof in the C*-algebra case, however it appears that there is no way to avoid passing through the more general case of operator systems with the operator local lifting property (the operator space analogue of the LLP). We are afforded some simplification thanks to the characterization in proposition 5.1.10 however it is essentially the same proof as proposition 2.9 of [31]. We'll use the following technical lemma:

Let $\theta : B \to B(\ell^2)$ be a ucp map with the exactness property that $\theta(J) \subset \mathcal{K}$. Then for a finite dimensional operator system $E \subset A$ (in fact it is true for an operator space), there are natural ucp maps

$$\dot{\theta} : B/J \to B(\ell^2)/\mathcal{K}, \quad x + J \mapsto \theta(x) + \mathcal{K}$$
$$\text{and } \check{\theta} : (E \otimes B)/(E \otimes J) \to (E \otimes B(\ell^2))/(E \otimes \mathcal{K}).$$

The easiest way to view $\check{\theta}$ is through the identification in theorem 5.2.2 and the isomorphism described afterwards. If we view an element $u \in (E \otimes B)/(E \otimes J)$ as a cb map $\varphi_u : E^* \to B$ mod $CB(E^*, J)$, then $\varphi_{\check{\theta}(u)} = \theta \circ \varphi_u$ is the cb map $E^* \to B(\ell^2)/\mathcal{K}$ corresponding to $\check{\theta}(u)$.

Lemma 5.3.1. Let Θ denote the set of all ucp maps $\theta : B \to B(\ell^2)$ as above with the property that $J \subset \mathcal{K}$. For $u \in E \otimes B/J$ we have:

$$\|u\|_{E \otimes B/J} = \sup_{\theta \in \Theta} \|(\mathrm{id}_E \otimes \dot{\theta})(u)\|_{E \otimes B(\ell^2)/\mathcal{K}} \tag{1}$$

$$\|u\|_{(E \otimes B)/(E \otimes J)} = \sup_{\theta \in \Theta} \|\check{\theta}\|_{(E \otimes B(\ell^2))/(E \otimes \mathcal{K})} \tag{2}$$

Proof. Let $v \in E \otimes B$ with $\|v + E \otimes J\| > 1$ in the quotient space. Since B is separable, so is J, and so there exists a positive element $h \in J$ with $0 \leq h \leq 1$ such that $\overline{hJh} = J$ (a so-called strictly positive element). For each n we'll define a projection

$$p_n = \chi_{[1/n,1]}(h) \in J^{**} \text{ s.t. } \lim \|(1 - p_n)h\| = 0$$

Of course, by density this means that for any $x \in J$, $\lim \|(1 - p_n)x\| = 0$. Next we'll define a set of projections h_n such that $h_n \geq p_n$. Let $f_n \in C_0(0, 1]$ be defined

$$f_n(t) = nt \text{ if } 0 \leq t \leq 1/n, \text{ and } f_n(x) = 1 \text{ if } 1/n \leq t$$

and set $h_n = f_n(h)$. Then it is clear to see that $h_n p_n = p_n$ and so in the usual ordering on C*-algebras, $h_n \geq p_n$. Therefore for each n,

$$\|(1 \otimes (1 - h_n))v(1 \otimes (1 - h_n))\|_{E \otimes B^{**}} \geq \|(1 \otimes (1 - p_n))v(1 \otimes (1 - p_n))\|_{E \otimes B}$$
$$\geq \|v + E \otimes J\|_{(E \otimes B)/(E \otimes J)} > 1$$

For each n take a ucp map $\theta_n : (1 - p_n)B(1 - p_n) \to M_{k_n}(\mathbb{C})$, a finite dimensional C*-algebra of corresponding dimension such that

$$\|(\mathrm{id}_E \otimes \theta_n)(1 \otimes (1 - p_n))v(1 \otimes (1 - p_n))\| > 1.$$

Then we can define a $\theta'_n : B \to M_{k_n}(\mathbb{C})$ by $\theta'_n(x) = \theta_n((1 - p_n)b(1 - p_n))$ and finally a $\theta : B \to \prod_n M_{k_n}(\mathbb{C})$ by $\theta(b) := (\theta'_n(b))_n$.

We can canonically view $\prod_n M_{k_n}(\mathbb{C}) \subset B(\oplus_n \mathbb{C}^{k_n})$ and note that θ maps J into the compact operators $\mathcal{K}(\oplus_n \mathbb{C}^{k_n}) = \left(\prod_n M_{k_n}(\mathbb{C})\right)_{c_0}$ because of how we've defined θ in terms of the vanishing projections $(1 - p_n)$. Then, by the inequality we showed earlier,

$$\|\hat{\theta}(v + E \otimes J)\| > 1$$

This gives the second equation. The first equation follows the second in combination with remembering there is this contractive isomorphism

$$(F \otimes B)/(F \otimes J) \to F \otimes B/J$$

for any finite dimensional operator space F and ideal in a C*-algebra $J \triangleleft B$. □

Remark 5.3.2. In words, what equation (1) in the lemma essentially states is that we may approximate (in norm) a ucp map $\varphi : E^* \to B/J$ by a composition $\theta \circ \varphi : E^* \to B(\ell^2)/\mathcal{K}$ for $\theta \in \Theta$. Equation (2) states that if we lift some map $\theta \circ \phi$ as above to $B(\ell^2)$, then that lift is $\check{\theta}$ and in norm, a limit point of these such maps will be a lift of φ to B.

Theorem 5.3.3. Let A be a C*-algebra. A has the LLP if and only if any ucp map $A \to B(\ell^2)/\mathcal{K}$ has a ucp lift.

Proof. Proving the *only if* portion is straight forward. We let $\varphi : A \to B(\ell^2)/\mathcal{K}$ be a ucp map, and $E \subset A$ a finite dimensional operator system. The local lifting property gives a ucp lift $E \to B(\ell^2)$. Since $B(\ell^2)$ is an injective object in the category of operator systems (Arveson's theorem 2.1.13), this map can be extended to a ucp map on all of A.

To produce the *if* direction let us assume that any ucp map into the Calkin algebra has a ucp lift and consider the isomorphism $\varphi : A \to C^*(\mathbb{F})/J$ which arises from the universality of $C^*(\mathbb{F})$. Let $E \subset A$ be a finite dimensional operator system. Under the identification from theorem 5.2.2

$$CB(E^{**}, C^*(\mathbb{F})/J) \cong E^* \otimes C^*(\mathbb{F})/J$$

let $u \in E^* \otimes C^*(\mathbb{F})/J$ be the restriction of the identity map $\varphi|_E$. For any $\theta \in \Theta$, $\mathrm{id} \otimes \dot{\theta}(u)$ corresponds to the composition $\dot{\theta} \circ \varphi|E$ - colored blue in the commutative diagram below.

$$\begin{tikzcd}
 & C^*(\mathbb{F}) \arrow[r, "\theta"'] \arrow[d, two heads] & B(\ell^2) \arrow[d, two heads] \\
A \supset E \arrow[ur, dashed] \arrow[urr, dashed, bend left, "\widehat{\dot{\theta}\circ\varphi|_E}\sim\check{\theta}(u)"] \arrow[r, "\varphi|_E\sim u"'] \arrow[rr, bend right, "\dot{\theta}\circ\varphi|_E\sim\mathrm{id}\otimes\dot{\theta}(u)"'] & C^*(\mathbb{F})/J \arrow[r, "\dot{\theta}"'] & B(\ell^2)/\mathcal{K}
\end{tikzcd}$$

By assumption, this ucp map into the Calkin algebra has a ucp lift $\widehat{\dot{\theta} \circ \varphi|_E}$ which corresponds in $(E \otimes B(\ell^2))/(E \otimes \mathcal{K})$ to $\check{\theta}(u)$. Then by equation (2) in lemma 5.3.1 there is a limit point - precisely a $u \in (E \otimes C^*(\mathbb{F}))/(E \otimes J)$ with cb norm approximated by the cb norm of the $\check{\theta}(u)$ for all $\theta \in \Theta$. But these are of course ucp maps so they have cb norm 1, and hence u corresponds to a ucp map $E \to C^*(\mathbb{F})$ which lifts $\varphi|_E$. This proves the LLP by proposition 5.1.10. $\square$

6 Non-nuclear C*-algebras with the LLP

We are able to give a class of examples of non-nuclear C*-algebras with the LLP. We use techniques of Kichberg and results of Courtney to obtain a characterization of the LLP. We then make note of the improvements of Loring and Shulman on the work of Hadwin which implies that the class of examples we will define satisfies this new characterization.

6.1 Universal C*-algebras

We want to emulate the idea of a presentation for a group, but in the case of C*-algebras. Of course, these are much more complicated structures and we will see that there is a little work to be done to guarantee that what we obtain in the end has a C*-algebra structure. Indeed, unlike groups, rings, and algebras, we don't define C*-algebras as quotients of a free object, but instead the GNS construction says we must realize them as operators on a Hilbert space with certain topological properties. The following construction is due to Blackadar [4].

Let $\mathcal{G} = \{x_i : i \in I\}$ be a set of generators and $\mathcal{R}$ a set of relations which we'll assume all take the form $\|p(x_{i_1}, \ldots, x_{i_n})\| < r$, for n finite, $0 \leq r \leq \infty$, and p a polynomial in the variables $\{x_{i_j}, x_{i_j}^* : 1 \leq j \leq n\}$. Note that when $r = 0$, the relation is in fact an algebraic relation between the generators and scalars. Something else worth mentioning is that this norm $\|\cdot\|$ is just an abstract norm. It means nothing until we identify these generators with operators on a Hilbert space. At that point the norm will be the natural norm on $B(\mathcal{H})$, but until then it is strictly a fictitious norm. Additionally, we don't necessarily assume that the scalars are present in the generated algebra.

For our pairing $(\mathcal{G}, \mathcal{R})$, a *representation* is a set of operators $\{T_i\} \in B(\mathcal{H})$ with an assignment $\mathcal{G} \to \{T_i\}$ so that the T_i satisfy the relations in $\mathcal{R}$, where in $B(\mathcal{H})$ the complex coefficients of p may be regarded as scalar multiples of the the identity operator. Such a representation ρ may be extended to an injective *-homomorphism $\tilde{\rho}$ from $F(\mathcal{G})$, the free algebra with involution on $\mathcal{G}$, into $B(\mathcal{H})$.

The pair $(\mathcal{G}, \mathcal{R})$ is *admissible* (in the sense that it will generate a C*-algebra) if there exists a representation as described above, and that the relations in $\mathcal{R}$ imply a bound on the generators x_i. That is, whenever $\{T_i^\alpha\}$ are representations on Hilbert spaces $\mathcal{H}^\alpha$ for $\alpha \in J$, then $\oplus_\alpha T_i^\alpha \subset B(\oplus_\alpha \mathcal{H}^\alpha)$. For an admissible pair $(\mathcal{G}, \mathcal{R})$, we can define a C*-seminorm (a seminorm with the C*-condition $\|x^* x\| \leq \|x^*\| \|x\|$) on the free algebra $F(\mathcal{G})$ by

$$|||x||| = \sup\{\|\tilde{\rho}(x)\| : \rho \text{ is a representation of } (\mathcal{G}, \mathcal{R})\}$$

Definition 6.1.1. We define the *universal C*-algebra* $C^*\langle \mathcal{G}, \mathcal{R} \rangle$ to be the completion of

$$F(\mathcal{G}) / \{x : |||x||| = 0\}$$

with respect to $|||\cdot|||$. $C^*\langle\mathcal{G},\mathcal{R}\rangle$ has the universal property that any representation of $(\mathcal{G},\mathcal{R})$ extends uniquely to a representation of $C^*\langle\mathcal{G},\mathcal{R}\rangle$. We will denote the unitization of $C^*\langle\mathcal{G},\mathcal{R}\rangle$ by $C^*_u\langle\mathcal{G},\mathcal{R}\rangle$.

Remark 6.1.2. Any C*-algebra A can easily be viewed as a universal C*-algebra with the generators $\mathcal{G} = A$ and the relations being all the C*-relations between elements of A.

Example 6.1.3. In this section we'll discuss several families of universal C*-algebras which were studied in [13]:

1. The *universal unital C*-algebra associated to a single polynomial relation.* We let $p \in \mathbb{C}[x]$ be a polynomial with at least one root with norm smaller than $C > 0$ (this will guarantee the existence of a representation), and present the algebra:

$$C^*_u\langle x : \|x\| \leq C,\ p(x) = 0\rangle.$$

   Notice here that the relation $p(x) = 0$ has been written as an agebraic relation, however we also have bounded the generator, so we've certainly got an admissible set of generators and relations. Tatiana Shulman showed that for $p(x) = x^n$, the above algebra is projective (see [25] corollary 3).
2. The *Pythagorean C*-algebras*,

$$\mathcal{P}_n = C^*\langle x_1, \ldots, x_n : \sum_i x_i^* x_i = 1\rangle.$$

   Note that the relation in these algebras implies that each of the terms of the polynomial $x_i^* x_i$ must have norm at most 1, and so by the C* condition $\|x_i\| \leq 1$. By theorem 4.1 of [13], $\mathcal{P}_n$ has the LP for $n \geq 1$.
3. The *universal unital C*-algebra generated by a contraction* which will be denoted by $\mathcal{A}$ was extensively studied in [12] and has generators and relations

$$\mathcal{A} = C^*_u\langle x : \|x\| \leq 1\rangle.$$

   Concretely, for some Hilbert space $\mathcal{H}$, this is the C*-subalgebra $C^*(T, I) \subset B(\mathcal{H})$ generated by the identity I and a universal contraction $T : \mathcal{H} \to \mathcal{H}$. T is a *universal contraction* if for every other contraction $S \in B(\mathcal{H})$ there is a *-homomorphism

$C^*(T, I) \to C^*(S, I)$ taking T to S. We note that if T and S are two universal contractions, then by definition they admit a *-isomorphism $C^*(T, I) \cong C^*(S, I)$, and so $\mathcal{A}$ is indeed defined independently of the contraction.

The following few results are due to Courtney and Sherman [12]. Proposition 6.1.6 is a key step in obtaining "exactness"-type result.

Lemma 6.1.4. If π is a faithful representation of a projective C*-algebra A on a separable Hilbert space $\mathcal{H}$, then for any $a \in A$, $\pi(a) \in \pi(A)$ is the norm limit of a sequence of nilpotent operators in $B(\mathcal{H})$ if and only if the $\sigma(\pi(a))$ is connected and contains 0.

In particular, if T is a universal contraction in $B(\mathcal{H})$, then there is a sequence of nilpotent operators converging to T in norm.

Proof. We begin by citing a result of Apostol, Foias, and Voiculescu [8, Thm 2.7] which states that an element $T \in B(\mathcal{H})$ is in the norm closure of the set of nilpotents if and only if:

(i) T is bi-quasitriangular,

(ii) $\sigma(T)$ and $\sigma_{ess}(T)$ are connected, and

(iii) $\sigma_{ess}(T)$ contains 0.

Let π be a faithful representation of A on a separable $\mathcal{H}$, which, because A is projective, is essential by 2.5.6 and so $\sigma(\pi(a)) = \sigma_{ess}(\pi(a))$. We remarked in 2.5.12 that $\pi(a)$ will also be a bi-quasitriangular operator for each $a \in A$. Hence, $\sigma(\pi(a))$ is connected and contains 0, if and only if each $\pi(a)$ is the norm limit of a sequence of nilpotents.

Of course the spectrum of a universal contraction T must be in the closure of the unit disc, and the sequence of nilpotents can be modified so that the T_n^n criterion is satisfied as well. □

Lemma 6.1.5. Let $\lambda \in \mathbb{D}$, the unit disc, and T be a universal contraction operator on a separable Hilbert space $\mathcal{H}$. There exists a sequence of contraction operators T_n such that $(T_n - \lambda I)^n = 0$ and $T_n \to T$ pointwise in norm.

Proof. Since the spectrum $\sigma(T - \lambda I) = \overline{\mathbb{D}} - \lambda$ is connected and contains 0, by 6.1.4, there are nilpotent operators $N_n \in B(\mathcal{H})$ such that $N_n \to T - \lambda I$ in norm and $N_n^n = 0$ for each

n. We would like to edit this sequence so that we only have contraction operators. To this end, for each $n \geq 1$ let $T_n := c_n N_n + \lambda I$ where $c_n = 1$ if $N_n + \lambda I$ is a contraction, and

$$c_n = \frac{1 - |\lambda|}{\|N_n + \lambda I\| - |\lambda|}$$

otherwise (i.e. when $\|N_n + \lambda I\| > 1$). Then $(T_n - \lambda I)^n = (c_n N_n)^n = 0$ and since $\|N_n + \lambda I\| \to \|T\| = 1$, $c_n \to 1$. Thus, $T_n \to T$ in norm and the definition of the constants c_n ensure that each T_n is a contraction. $\square$

Proposition 6.1.6. Let $\lambda \in \mathbb{D}$. Any faithful, unital representation $\pi : \mathcal{A} \to B(\mathcal{H})$ factors through $A_{\lambda,n} = C^*_u\langle x_n : \|x_n\| \leq 1, (x_n - \lambda)^n = 0\rangle$. That is, there exist maps $\psi_n : A_{\lambda,n} \to B(\mathcal{H})$ such that $\psi_n \circ \phi_n$ converges to π in the point-norm topology, where $\phi_n : \mathcal{A} \to A_{\lambda,n}$ are the canonical inclusions due to the inclusions of the sets of relations.

Proof. Concretely, we can view the unital C*-algebras $A_{\lambda,n}$ as those generated by the universal contractions T_n from Lemma 6.1.5 in the same way as we did for $\mathcal{A}$. That is, there are surjective homomorphisms

$$\psi_n : A_{\lambda,n} \to C^*(T_n, I) \subset B(\mathcal{H})$$

defined by the canonical identification $x_n \to T_n$. Then, since the T_n are built in Lemma 6.1.5 so that $T_n \to T$ pointwise in norm, we must have that for any $a \in \mathcal{A}$,

$$\|\psi_n \circ \phi_n(a) - \pi(a)\| \to 0.$$

$\square$

6.1.1 Lifting Polynomial Relations

We briefly discuss the business of lifting relations in a C*-algebra. This follows from work of Hadwin [17], and comments by Loring and Shulman [25] and is pertinent to our study of the universal C*-algebra associated to a polynomial relation.

In the case of $p(z) = z^n$, Olsen and Pederson [29] proved that the relation lifts. In particular, they showed that nilpotent elements have nilpotent lifts. Hadwin, then extended this result.

Theorem 6.1.7 (Olsen, Pederson)**.** Let A be a C*-algebra, $I \triangleleft A$ a closed, two-sided ideal, $n \geq 0$ be an integer, and $y \in A/I$ such that $y^n = 0$. Then there exists an element $x \in A$ such that $x^n = 0$ and $\pi(x) = y$, where $\pi : A \twoheadrightarrow A/I$ is the quotient map.

Theorem 6.1.8 (Hadwin)**.** Let $y \in A/I$ satisfy the polynomial relation $f(z) = 0$, where $f \in \mathbb{C}[z]$ is the minimal polynomial of y, with N roots, and $\pi : A \to A/I$ denote the quotient map. Then there exists an orthogonal family $p_1, ..., p_N$ of projections in A/I summing to 1 such that the following statements are equivalent:

1. There exists an orthogonal family of projections $q_1, ..., q_N \in A$, summing to 1, such that $\pi(q_i) = p_i$.
2. There exists $x \in A$ such that $\pi(x) = y$ satisfying the polynomial relation $f(x) = 0$.

The construction of these projections is as follows: We can write

$$p(z) = (z - \lambda_N)^{k_N} \cdots (z - \lambda_1)^{k_1}, \quad \lambda_i \neq \lambda_j, \text{ when } i \neq j.$$

Let M denote the matrix consisting of the direct sum of the N $k_i \times k_i$ Jordan blocks for the eigenvalues λ_i. f being the minimal polynomial, $\mathbb{C}[y] \cong \mathbb{C}[z]/f\mathbb{C}[z]$ which is also isomorphic to $\mathbb{C}[M]$. Then we can find polynomials $f_1, ...f_N$ such that: if $f_i(y) =: e_i \in A/I$, $\sum_i e_i = 1$, $e_i e_j = \delta_{ij} e_j$, and $e_i(y - \lambda_i)^{k_i} e_i = 0$. Let $s = \sum_i e_i^* e_i$, and note that s is invertible and has the property that for each i, $e_i^* s = s e_i$. Then

$$s^{-1/2} e_i^* s^{1/2} = s^{1/2} e_i s^{-1/2},$$

which implies that $s^{1/2} e_i s^{-1/2}$ is self adjoint. Moreover by definition of the e_i's, it is also a projection. Denoting these

$$p_i = s^{1/2} e_i s^{-1/2}, \quad i = 1, ..., N,$$

we have a family of projections in A/I summing to 1. It turns out that these p_i are exactly the projections that satisfy the theorem (see [17, thm. 2] for details).

Remark 6.1.9 (Hadwin). It follows from this that a family of projections in A/I lifts to projections in A if and only if the self adjoint element $p_1 + 2p_2 + ... + Np_N$ lifts to a self adjoint element of A with finite spectrum. By the equivalence (1) $\iff$ (2) in 4.1.8, it follows that this is true whenever $\mathrm{RR}(A) = 0$. We then obtain the following corollaries:

Corollary 6.1.10.

(1) If A has real rank zero, then any $y \in A/I$ satisfying a polynomial relation $f(y) = 0$ lifts to an $x \in A$ such that $f(x) = 0$.

(2) If M is a von Neumann algebra, then any $y \in M/I$ satisfying a polynomial relation $f(y) = 0$ lifts to an $x \in M$ such that $f(x) = 0$.

6.1.2 The Universal C*-algebra of a Polynomial is not always Nuclear

Let $p \in \mathbb{C}[z]$ be a polynomial. We show that the algebra

$$C_u^*\langle x : \|x\| \leq C,\ p(x) = 0\rangle$$

is not exact (therefore not nuclear) in general. We follow the paper of Courtney [13] as usual, as well as the work of Courtney and Sherman [12] on $\mathcal{A}$, the universal algebra of a contraction (see example 6.1.3(3)).

Definition 6.1.11. We say a unital C*-subalgebra $A \subset B$ embeds *relatively weakly injectively* if there exists a weak expectation $\Phi : B \twoheadrightarrow \pi_u(A)^{**}$ (i.e. a ucp map such that $\Phi(a) = \pi_u(a)$, $\forall a \in A$). Here $\pi_u : A \to B(\mathcal{H}_u)$ is the universal *-representation of A from the GNS construction.

Using this new terminology, we could say that A has the WEP if A embeds relatively weakly injectively into $B(\mathcal{H})$ for some Hilbert space $\mathcal{H}$.

Proposition 6.1.12. $C^*(\mathbb{F}_2)$ embeds relatively weakly injectively into $\mathcal{A}$. In fact they embed relatively weakly injectively into each other - see [12, theorem 6.10].

Proof. Let $\pi_u : C^*(\mathbb{F}_2) \to B(\mathcal{H}_u)$ be the universal representation, and U_1, U_2 be the images of the generators. Let A_1, A_2 be self adjoint elements of $B(\mathcal{H}_u)$ such that $U_j = e^{iA_j}$ and

$$C^*(U_1, U_2) \subset C^*(A_1, A_2) = C^*\left(\frac{A_1}{\alpha} + i\frac{A_2}{\alpha}\right) \subset C^*(U_1, U_2)'',$$

where $\alpha = \|A_1 + iA_2\|$. $\mathcal{A}$ is generated by a single contraction, call it x, which admits a decomposition $x = x_1 + ix_2$. The $e^{i\alpha x_j}$ are unitaries in $\mathcal{A}$ and $C^*(e^{i\alpha x_1}, e^{i\alpha x_2}) \subset C^*(x_1, x_2) =$

$\mathcal{A}$. Because $\mathcal{A}$ is the universal unital C*-algebra generated by a contraction and $(A_1+iA_2)/\alpha$ is a contraction, we get a surjective unital *-homomorphism

$$\psi : \mathcal{A} \twoheadrightarrow C^*\left(\frac{A_1}{\alpha} + i\frac{A_2}{\alpha}\right), \quad \psi(x_j) := A_j/\alpha.$$

The universality of $C^*(\mathbb{F}_2)$ means that there is a surjective unital *-homomorphism

$$\phi : C^*(U_1, U_2) \twoheadrightarrow C^*(e^{i\alpha x_1}, e^{i\alpha x_2}), \quad \phi(U_j) = e^{i\alpha x_j}.$$

$\psi \circ \phi = \mathrm{id}_{C^*(U_1,U_2)}$, so $\mathcal{A}$ contains a relatively weakly injectively embedded copy of $C^*(\mathbb{F}_2)$. $\square$

Remark 6.1.13. Of course this means that there is a copy of $C^*(\mathbb{F})$ for a free group $\mathbb{F}$ embedded in $\mathcal{A}$ as a subalgebra. As we mentioned in 2.4.12, $C^*(\mathbb{F})$ is never exact, for a non-abelian free group $\mathbb{F}$, and an exact C*-algebra cannot contain a non-exact subalgebra, hence $\mathcal{A}$ is not exact.

Proposition 6.1.14. Let $\lambda \in \mathbb{C}^\times$ and $C > |\lambda|$. Then there exists an $N > 0$ such that whenever $(x-\lambda)^N | p(x)$ for $p(x) \in \mathbb{C}[x]$, the universal C*-algebra $\mathrm{A} = C^*\langle x : \|x\| \leq 1, p(x) = 0\rangle$ is not nuclear.

Proof. Fix some λ and assume WLOG (by scaling) that $C = 1$. For each $n \geq 1$, let

$$A_{\lambda,n} = C^*_u\langle x : \|x\| \leq C, (x-\lambda)^n = 0\rangle.$$

Clearly if $(x-\lambda)^n | p(x)$, then A surjects onto $A_{\lambda,n}$, or equivalently $A_{\lambda,n}$ embeds in A. If A is to have a nuclear faithful representation, then so must $A_{\lambda,n}$, by composition. Hence it suffices to show that for some N, $A_{\lambda,N}$ is not exact. Recall that we denote by $\mathcal{A}$ the universal C*-algebra of a contraction. By proposition 6.1.6, the representation π factors through the $A_{\lambda,n}$. In other words, there exist homomorphisms $\phi_n : \mathcal{A} \to A_{\lambda,n}$ and $\psi_n : A_{\lambda,n} \to B(\mathcal{H})$ such that for each $a \in \mathcal{A}$, $\|\psi \circ \phi(a) - \pi(a)\| \to 0$.

If the $A_{\lambda,n}$ were exact for more than finitely many n, then it would imply that those ψ_n would be nuclear. However then π would be nuclear and so $\mathcal{A}$ would need to be exact. However this is absurd as we just showed that $\mathcal{A}$ is not exact. $\square$

6.2 Hilbert C*-Modules

A Hilbert C*-module behaves much like a Hilbert space, except that the inner product on the module may take values in a general C*-algebra as oppose to just $\mathbb{C}$.

Let B be a C*-algebra. A *pre-Hilbert B-module* is a complex vector space E equipped with a map $\langle \cdot, \cdot \rangle : E \times E \to B$ satisfying:

(i) $\langle x, y\alpha + z\beta \rangle = \langle x, y \rangle \alpha + \langle x, z \rangle \beta$ for all $x, y, z \in E, \alpha, \beta \in \mathbb{C}$

(ii) $\langle x, y \rangle b = \langle x, ya \rangle$ for $x, y \in E,\ b \in B$

(iii) $\langle x, y \rangle = \langle y, x \rangle^*$

(iv) $\langle x, x \rangle \geq 0$ and $\langle x, y \rangle = 0$ if and only if $x = 0$.

Define a norm on such a module by $\|x\| = \langle x, x \rangle^{1/2}$ (it is easy to check that this indeed defines a norm on E).

Definition 6.2.1. A *Hilbert B-module* is the completion of a pre-Hilbert A-module with respect to the norm $\| \cdot \|$ defined above.

Let B be a C*-algebra. B is itself a Hilbert B-module with the inner product $\langle x, y \rangle = x^* y$. For any Hilbert B-module E, we denote by E^n the direct sum of n copies of E, and it is not hard to see that B^n is also a Hilbert B-module with the inner product $\langle \oplus_i x_i, \oplus_i y_i \rangle. = \sum_i \langle x_i, y_i \rangle$. We may extend this definition naturally to the case of $n = \infty$ without any issue. We denote by $\mathcal{L}(E, F)$ the set of adjointable bounded linear maps $E \to F$ for two Hilbert B-modules E, F. We simplify the notation to $\mathcal{L}(E)$ in the case $E = F$.

If A is a C*-algebra and E a Hilbert B-module for some C*-algebra B, given some ucp map $\phi : A \to \mathcal{L}(E)$ we may define an inner product taking values in B on $A \odot E$ by

$$\left\langle \sum_i a_i \otimes x_i, \sum_j b_j \otimes y_j \right\rangle = \sum_{i,j} x_i \phi(a_i^* b_j) y_j.$$

Let $\{V_i\}$ be a countable collection of finite dimensional vector spaces and define a inner product on each $V_i \otimes B$ by $\langle v \otimes x, w \otimes y \rangle = \langle \xi, \eta \rangle x^* y$. The Hilbert direct sum $\mathcal{H}_B = \bigoplus_{i=1}^{\infty} (V_i \otimes B)$ with inner product $\langle \xi \otimes x, \eta \otimes y \rangle = \langle \xi, \eta \rangle x^* y$ is a Hilbert C*-module, named the *Hilbert space over B*. It's not hard to see that we may view $\mathcal{H}_B$ as the tensor $\mathcal{H} \otimes B$

where $\mathcal{H}$ is the separable Hilbert space ℓ^2. This is a very important example as we will see in the stabilization theorem below. This theorem is due to Kasparov in [21], however we follow a simplified proof found in the literature (see [28], for example) using polar decomposition as oppose to a lengthy Gram-Schmidt process.

For E_1, E_2 Hilbert B-modules and $x \in E_1, y, x \in E_2$, we define the functions

$$\theta_{x,y} \in \mathcal{L}(E_2, E_1), \quad \theta_{x,y}(z) = x\langle y, z\rangle.$$

Note that $\theta_{x,y}^* = \theta_{y,x}$. If $T \in \mathcal{L}(E_2, E_1)$, then $T \cdot \theta_{x,y} = \theta_{T(x),y}$. In Particular,

$$\theta_{x,y}\theta_{u,v} = \theta_{x(y,u),v} = \theta_{x,v(u,y)}.$$

Define the space $\mathscr{K}(E_2, E_1)$ to be the closure of the linear span of $\theta_{x,y}$ in $\mathcal{L}(E_2, E_1)$. If $E_2 = E_1 = E$, then we write $\mathscr{K}(E)$.

Lemma 6.2.2. If E is a Hilbert B-module, and T is a positive element of $\mathscr{K}(E)$, then T is strictly positive if and only if T has dense range.

Proof. If T is strictly positive, then by definition, $T\mathscr{K}(E)$ is dense in $\mathscr{K}(E)$. Note that $[\mathscr{K}(E)](E)$ is dense in E, and so

$$\overline{T(E)} = \overline{T[\mathscr{K}(E)](E)} = \overline{[\mathscr{K}(E)](E)} = E.$$

In the other direction, suppose now that T has dense range and for arbitrary $x \in E$ take a sequence $\{x_n\} \subset E$ such that $Tx_n \to x$. Then for any $y \in E$, $\theta_{x,y} = \lim \theta_{x_n,y}$. However note that each $T\theta_{x_n,y} \in T\mathscr{K}(E)$, so $T\mathscr{K}(E)$ is dense and T is strictly positive. □

Theorem 6.2.3 (Kasparov Stabilization)**.** Let B be a unital C*-algebra and E a countably generated Hilbert B-module. Then $E \oplus \mathcal{H}_B \cong \mathcal{H}_B$.

Proof. Let $\{x_i\}$ be a countable system of generators for E where each generator is repeated infinitely often, and $\{e_i\}$ the standard orthonormal basis for $\mathcal{H}_B$. Define:

$$T : \mathcal{H}_B \to E \oplus H_B, \quad T(e_i) = 2^{-i}x_i \oplus 4^{-i}e_i$$

It is not hard to see that $T \in \mathcal{L}(\mathcal{H}_B, E \oplus \mathcal{H}_B)$, since we may decompose T as

$$T = \sum 2^{-i}\theta_{x_i, 2^{-i}e_i}$$

which in fact shows that $T \in \mathscr{K}(\mathcal{H}_B, E \oplus \mathcal{H}_B)$.

Because each x_i is repeated infinitely many times, for each fixed i there are infinitely many k such that $x_i \oplus 2^{-k} e_k \in \operatorname{ran}(T)$. Hence $x_i \oplus 0 \in \overline{\operatorname{ran}(T)}$ and by the same argument on the x_i's, $0 \oplus e_i \in \overline{\operatorname{ran}(T)}$ for each i. Thus, $\operatorname{ran}(T)$ is dense in $E \oplus \mathcal{H}_B$. Now consider the operator

$$T^*T = \begin{pmatrix} 4^{-4} & & & 0 \\ & 4^{-8} & & \\ & & 4^{-12} & \\ 0 & & & \ddots \end{pmatrix} + \begin{pmatrix} 4^{-2}\langle x_1, x_1 \rangle & 4^{-3}\langle x_1, x_2 \rangle & \cdots \\ 4^{-3}\langle x_2, x_1 \rangle & 4^{-4}\langle x_2, x_2 \rangle & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix}$$

and denote these two parts K and K' respectively where we note that $K, K' \geq 0$. $\operatorname{ran}(K)$ is dense and so K is strictly positive by lemma 6.2.2 and T^*T is also strictly positive. Then we use the lemma again to see that T^*T has dense range, and therefore so does $|T|$. Now define on the range of $|T|$ a map

$$V : \mathcal{H}_B \to E \oplus \mathcal{H}_B, \quad V(|T|\xi) = T\xi.$$

Because $\|V(|T|\xi)\| = \||T|\xi\|$, we may continuously extend V to all of $\mathcal{H}_B$, where it extends to a unitary $\mathcal{H}_B \to E \oplus \mathcal{H}_B$. $\square$

Lemma 6.2.4. If E is a countably generated and full B-module, and B has a countable approximate unit, $E^\infty \cong \mathcal{H}_B$.

Proof. We'll begin by proving the *claim* that if E is full and has a completely positive element (or has a countable approximate unit), then we may decompose $E^\infty = B \oplus F$ for some Hilbert B-module F:

Because B has a strictly positive element, we may take a sequence $\{e_i\} \subset E$ such that $\sum \langle e_i, e_i \rangle = 1$ in $M(B)$ with convergence in the strict topology (see [6] Lemma 2.3). Define a $T : B \to E^\infty$ by $T(b) = (e_i b)_i$. Since $\langle (e_i b)_i, (e_i b)_i \rangle = b^* b$, we see indeed that $(e_i b)_i \in E^\infty$. Next define $T^* : E^\infty \to B$ by $T^*(x_i) = \sum \langle e_i, x_i \rangle$. Because we have a Cauchy-Schwarz inequality in a Hilbert C*-module ($\langle x, y \rangle \langle y, x \rangle = \|\langle x, x \rangle\|_A \langle y, y \rangle$), $T^*(x_i)$ will converge in norm to some $a \in A$. Clearly $T^*T = \mathrm{id}_A$ and $T \oplus \mathrm{id} : A \oplus (1 - TT^*)E^\infty \to E^\infty$ is an isomorphism.

Finally,

$$E^\infty = (E^\infty)^\infty = (B \oplus F)^\infty = \mathcal{H}_B \oplus F^\infty = \mathcal{H}_B$$

where the last isomorphism is from theorem 6.2.3 because F must be countably generated (as it is a complemented submodule of the countably generated module E^∞). $\square$

The following is a version of Stinespring's theorem (due to Kasparov), which was adapted by Courtney to consider the module $A \otimes_\phi B$ for a ucp $\phi : A \to B$ instead of $A \otimes_\phi \mathcal{H}_B$ as in the standard version [21, Thm 3].

Theorem 6.2.5 (Kasparov)**.** Let A, B be unital C*-algebras with A separable and $\phi : A \to B$ a ucp map. Then there exists a *-homomorphism $\Phi : A \to M(\mathcal{K} \otimes B)$ such that $\forall a \in A$,

$$(\Phi(a))_{11} = (e_{11} \otimes 1_B)\Phi(a)(e_{11} \otimes 1_B) = e_{11} \otimes \phi(a) = \phi(a).$$

Proof. Let $E = A \otimes_\phi B$. Since ϕ is unital and A is separable, E has a countable generating set, and is full. Then by lemma 6.2.4, $E^\infty \cong \mathcal{H}_B$.

Let $\pi_1 : A \to \mathcal{L}(E), \pi_1(a)(a' \otimes b) = aa' \otimes b$ be the unital *-homomorphism induced by the left action of A on the algebraic tensor $A \odot B$. Likewise, let $\pi_\infty : A \to \mathcal{L}(E^\infty)$ be the homomorphism

$$\pi_\infty(a)[(a_n \otimes b_n)_n] = (aa_n \otimes b_n)_n.$$

Define $W \in \mathcal{L}(B, E)$ by $W(b) = 1_A \otimes b$ and a map W^* on elementary by

$$W^*(\sum a_i \otimes b_i) = \sum \phi(a_i)b_i.$$

$$\left\| W^* \left(\sum_{i=1}^n a_i \otimes x_i \right) \right\|^2 = \left\| \sum_{i,j=1}^n \langle x_i, \phi(a_i^*)\phi(a_j)x_j \rangle \right\| \leq \|\phi(1)\| \cdot \left\| \left\langle \sum_{i=1}^n a_i \otimes x_i, \sum_{i=1}^n a_i \otimes x_i \right\rangle \right\|$$

The inequality follows from the fact that for an arbitrary completely positive map φ, $\varphi(x^*x) \leq \|\varphi(1)\|\varphi(x^*x)$, which is a direct consequence of the classical Stinespring theorem. We apply this fact to the cp map $\varphi = id \otimes \phi : M_n \otimes A \to \mathcal{L}(\mathcal{H}_B^n)$. So now it is clear that for any $a \otimes b$ such that $\langle a \otimes b, a \otimes b \rangle = 0$, $W^*(a \otimes b) = 0$, and hence by continuity we may extend the map W^* to all of E, noting that it is indeed adjoint to W.

Moreover we observe that $[W^*\pi_1(a)W](b) = \phi(a)b$ so indeed $W^*\pi_1(\cdot)W = \phi$. We also note that since $W^*W = 1_{\mathcal{L}(B)}$, $WW^* \in \mathcal{L}(B)$ is a projection and W is a unitary in

$\mathcal{L}(B, WW^*(E))$. By the Kasparov stabilization theorem and the remark at the start of the proof we get an isomorphism

$$[(1 - WW^*)(E)] \oplus E^\infty \cong [(1 - WW^*)(E)] \oplus \mathcal{H}_B \cong \mathcal{H}_B.$$

Suppose that $U \in \mathcal{L}(\mathcal{H}_B, [(1-WW^*)(E)] \oplus E^\infty)$ is a unitary which implements this isomorphism. Then $V := W \oplus U$ is a unitary implementing

$$B \oplus \mathcal{H}_B \cong WW^*(E) \oplus [(1 - WW^*)(E)] \oplus E^\infty \cong E \oplus E^\infty.$$

This implies that $V^*\pi_\infty(\cdot)V = W^*\pi_1(\cdot)W \oplus U^*\pi_\infty(\cdot)U : A \to \mathcal{L}(B + \mathcal{H}_B)$ is a unital *-homomorphism. Let $\Psi : \mathcal{L}(B + \mathcal{H}_B) \tilde{\to} M(\mathcal{K} \otimes B)$ be the *-isomorphism which takes (projection onto B) to $e_{11} \otimes 1_B$, and define $\Phi = \Psi \circ V^*\pi_\infty(\cdot)V$.

$$(\Phi(a))_{11} = (e_{11} \otimes 1_B)\Phi(a)(e_{11} \otimes 1_B) = \Psi(W^*\pi_1(a)W \oplus 0_{\mathcal{H}_B}) = \Psi(\phi(a)) = e_{11} \otimes \phi(a).$$

□

6.3 A class of C*-algebras with the LLP

We conclude by giving a class of examples of C*-algebras with the LLP. This is the work of Kristin Courtney [13], albeit with a strong Kirchberg flavor.

Lemma 6.3.1 (Kirchberg)**.** Let $A \subset B(\mathcal{H})$ be a C*-algebra with strictly positive element. Then A has the WEP if an only if $M(A)$ has the WEP.

The proof relies on the fact that the identity map of $M(A)$ approximately factors through $l^\infty(A)$ by completely positive contractions (see [23, Observation 5.3 (iii), (viii)]). Given this, one uses the corresponding result for $l^\infty(A)$ (see [23, Cor. 3.3 (i)]).

Proposition 6.3.2. Let $\pi : M(\mathcal{K}) \to Q(\mathcal{K})$ be the quotient map and recall that the noncommutative Tietze extension theorem gives us a surjective *-homomorphism extending $\mathrm{id}_\mathcal{K} \otimes \pi$

$$\hat{\pi} : M(\mathcal{K} \otimes M(\mathcal{K})) \to M(\mathcal{K} \otimes Q(\mathcal{K})).$$

A separable unital C*-algebra A has the LLP if and only if any unital *-homomorphism $\rho : A \to M(\mathcal{K} \otimes Q(\mathcal{K}))$ lifts to a ucp map $\theta : A \to M(\mathcal{K} \otimes M(\mathcal{K}))$.

Proof. Let us first assume A is a separable unital C*-algebra with the LLP. The compact operators $\mathcal{K}$ is a nuclear C*-algebra, and $B(\ell^2) = M(\mathcal{K})$ has Lance's weak expectation property (WEP), so $\mathcal{K} \otimes M(\mathcal{K})$ has the WEP. We remark that separable C*-algebras have a strictly positive element by virtue of admitting a countable approximate unit. So by lemma 6.3.1, $M(\mathcal{K} \otimes M(\mathcal{K}))$ has the WEP.

Now let's fix a ucp map $\phi : A \to Q(\mathcal{K})$. Karparov's Stinespring theorem 6.2.5 gives us a unital *-homomorphism $\Phi : A \to M(\mathcal{K} \otimes Q(\mathcal{K})$ satisfying $(\Phi(a))_{11} = \phi(a)$. By assumption there is a map $\theta : A \to M(\mathcal{K} \otimes M(\mathcal{K}))$ extending Φ, i.e: such that $\hat{pi}\theta = \Phi$. Then define a ucp map $\psi : A \to M(\mathcal{K})$ by $\psi(a) = (\theta(a))_{11}$. The commutative diagram below (adapted from [13]) may be a helpful way of visualizing this:

$$\begin{tikzcd}
 & M(\mathcal{K} \otimes M(\mathcal{K})) \arrow[d, "\hat{\pi}"'] \arrow[dr, "(\cdot)_{11}"] & \\
 & M(\mathcal{K} \otimes Q(\mathcal{K})) \arrow[dr, "(\cdot)_{11}"] & M(\mathcal{K}) \arrow[d, "\pi"] \\
A \arrow[uur, dashed, bend left, "\theta"] \arrow[ur, "\Phi"] \arrow[rr, bend right, "\phi"'] & & Q(\mathcal{K})
\end{tikzcd}$$

It is clear to see that ψ extends ϕ to $M(\mathcal{K})$, and so by the characterization in theorem 5.3.3, A has the LLP. $\square$

Theorem 6.3.3. Let A be a separable unital C*-algebra and $\alpha : \mathbb{C}^N \to A$ a conditionally projective unital *-homomorphism for some $N \geq 1$. Then A has the LLP.

Proof. Let $\rho : A \to M(\mathcal{K} \otimes Q(\mathcal{K}))$ be a unital *-homomorphism. Then $\rho \circ \alpha : \mathbb{C}^N \to M(\mathcal{K} \otimes Q(\mathcal{K}))$ is also a unital *-homomorphism. By statement (2) of lemma 4.2.4, any unital *-homomorphism $\mathbb{C}^N \to M(\mathcal{K} \otimes Q(\mathcal{K}))$ lifts to a unital *-homomorphism $\mathbb{C}^N \to M(\mathcal{K} \otimes M(\mathcal{K}))$. This implies that ρ has a lift and by 6.3.2 we've shown that A has the LLP. $\square$

In the hopes of tying many of the prior results together we now are able to show that our example of a universal unital C*-algebra generated by a single polynomial relation abides by this version of the LLP. In particular we arrive at the following corollary:

Corollary 6.3.4. Let $C > 0$ and $p \in \mathbb{C}[z]$ a polynomial whose roots λ are such that $|\lambda| < C$. Then the universal C*-algebra $A = C^*_u\langle x : \|x\| < C,\, p(x) = 0\rangle$ has the LLP.

Proof. Write p as $p(z) = (z - \lambda_N)^{k_N} \cdots (z - \lambda_1)^{k_1}$. In [25, remark 12], we are shown how to construct a conditional projection $\mathbb{C}^{N-1} \to A$. Letting B/I be a C*-algebra where elements satisfy the relations $p(z) = 0$ and $\|z\| < C$, we immediately note that by universality there exists a *-homomorphism $A \to B/I$. Applying the construction of 6.1.8 to any $y \in B/I$, we get a family of orthogonal projections summing to 1 which lift to projections (also summing to 1) in B. Moreover if $\|y\| < C$, then by the proof of [25, thm. 9], $\|x\| < C$. In particular (recalling remark 2.5.14) there is a lift $A \to B$ and we have that there exists a conditional projection

$$\mathbb{C}^{N-1} \to A.$$

$\square$

In 6.1.14 we showed that a large subclass of these C*-algebras are non-nuclear, and hence provide examples of non-nuclear C*-algebras with the local lifting property.

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A Fundamental Results in Operator Theory

A.1 The Gelfand-Naimark-Segal (GNS) Construction

The *Gelfand-Naimark* theorem realizes a C*-algebra, an abstract algebraic object, as a norm-closed algebra of bounded operators on a Hilbert space. The Gelfand-Naimark-Segal (GNS) construction, which provides the machinery behind this theorem, is at the heart of all C*-algebra theory, and can be found all over the literature. Our particular approach follows the book of Kadison and Ringrose [20].

Proposition A.1.1. Let A be a C*-algebra, $E \subset A$ be a self-adjoint subspace containing 1_A, and $x \in E$. If $\lambda \in \sigma_A(x)$, then there is a state ρ of E such that $\rho(x) = \lambda$.

Proof. If $\lambda \in \sigma(x)$ then for complex numbers α and β,

$$\alpha\lambda + \beta \in \sigma(\alpha x + \beta 1) \Rightarrow |\alpha\lambda + \beta| \leq \|\alpha x + \beta 1\|$$

$$\text{Define } \rho_0(\alpha x + \beta 1) = \alpha\lambda + \beta \text{ on } B = \{\alpha x + \beta 1 : \alpha, \beta \in \mathbb{C}\}$$

ρ_0 defines a linear functional on the subspace $B \subset E$ with $\rho_0(x) = \lambda$, $\rho_0(1) = 1$ and $\|\rho\| = 1$. By the Hahn-Banach theorem we can extend ρ_0 to ρ a bounded linear functional on E with $\|\rho\| = 1$. Since ρ is bounded, it is positive and therefore a state. □

Theorem A.1.2. Let A be a C*-algebra, $E \subset A$ a self-adjoint subspace containing 1_A, and $x \in E$. Then:

(1) If $\rho(x) = 0$ for each state ρ of E, then $x = 0$.

(2) If $\rho(x)$ is real for each state ρ of E, then x is self-adjoint

(3) If $\rho(x) \geq 0$ for each state of E, then $x \in E_+ = E \cap A_+$

(4) If x is normal, there exists a state ρ of E such that $|\rho(x)| = \|x\|$

Proof. (1) Suppose x is self adjoint and $\rho(x) = 0$ for each state ρ of E. Then $\sigma(x) = \{0\}$ and so $\|x\|$ which coincides with the spectral radius of x is 0, and therefore $x = 0$. Now suppose x is an arbitrary element in the subspace E. Then x can be written as $y + iz$ for y, z self-adjoint elements. $\rho(x) = 0 \Rightarrow \rho(y) = \rho(z) = 0 \Rightarrow y = z = 0$ and therefore $x = 0$.

(2) Suppose $\rho(x) \in \mathbb{R}$ for each state ρ of E. Then $\rho(x - x^*) = \rho(x) - \overline{\rho(x)} = 0$ so $x = x^*$.

(3) If $\rho(x) \geq 0$ for each state of E, then by part 2 x is self adjoint. By proposition A.1.1, $\sigma(x) \subset \mathbb{R}^+$ and so $x \in E_+$.

(4) If x is normal, it has spectral radius $r(x) = \|x\|$ so there is $\lambda \in \sigma(x)$ such that $|\lambda| = \|x\|$. By the above proposition $\lambda = \rho(x)$ for some ρ, and thus $\|x\| = \rho(x)$.

□

The set of states of $E \subset A$ is called the *state space* of E, which we denote by $\mathcal{S}(E)$. It is a subspace of the unit sphere in the Banach dual space E^* and is closed and convex in the weak* topology $\sigma(E^*, E)$.

The state space $\mathcal{S}(E)$ is convex, and is thus is equal to the convex hull of its extreme points. These points must exist due to the Krein-Milman theorem, which states that for a compact, convex set X in a locally convex space, there exists at least one extreme point and X is in fact the convex hull of these points. We call the extreme points of $\mathcal{S}(E)$ *pure states* and denote the subspace of pure states $\mathcal{P}(E)$. We will use the following corollary to the Krein-Milman theorem later on.

Corollary A.1.3. Let V be a locally-convex topological vector space and X a nonempty compact convex subset. If ρ is a continuous linear functional on V, there is an extreme point $x_0 \in X$ such that $\mathrm{Re}(\rho(x)) \leq \mathrm{Re}(\rho(x_0)$ for all $x \in X$.

For the proof, simply note that if we denote $c = \sup\{\mathrm{Re}(\rho(x)) : x \in X\}$, then the set $\{x \in X : \rho(x) = c\}$ is a compact face (nonempty convex subset Y with the fact that any 'line' $\alpha x + (1 - \alpha)y,\ x, y \in Y :\ \alpha \in [0, 1]$) of X. In particular it is a nonempty compact convex set in V so it has an extreme point. An extreme point of a face of X will be an extreme point of x with $\mathrm{Re}(\rho(x_0)) = c \geq \mathrm{Re}(\rho(x)), \forall x \in X$.

Theorem A.1.4. Let A be a C*-algebra, $E \subset A$ a self-adjoint subspace containing 1_A, and $x \in E$. Then:

(1) If $\rho(x) = 0$ for each pure state ρ of E, then $x = 0$.

(2) If $\rho(x)$ is real for each pure state ρ of E, then x is self-adjoint

(3) If $\rho(x) \geq 0$ for each pure state of E, then $x \in E_+ = E \cap A_+$

(4) If x is normal, there exists a pure state ρ of E such that $|\rho(x)| = \|x\|$

Proof. We just remarked above that any state is the weak*-limit of a convex combination of pure states, so by the corresponding results in A.1.2, the first three facts follow. For the last assertion suppose that x is normal and, by A.1.2(4), let c be a scalar and τ a state such that $\tau(x) = c$ with $|c| = \|x\|$. Denote the evaluation map (at an element $a \in E$) ev_a on E^*, the Banach dual space. Let a be a complex number with $|a| = 1$ and such that $\tau(ax) = |c| = \|x\|$. From corollary A.1.3 applied to $P(E) \subset S(E)$ with linear functional ev_{ax}, there is an element $\rho_0 \in \mathcal{P}(E)$ such that

$$\begin{aligned} \|x\| \geq |\rho_0(x)| &\geq \mathrm{Re}(\mathrm{ev}_{ax}(\rho_0)) \geq \sup\{\mathrm{Re}(\mathrm{ev}_{ax}(\rho)) : \rho \in \mathcal{S}(E)\} \\ &\geq \mathrm{Re}(\mathrm{ev}_{ax}(\tau)) = \mathrm{Re}(\tau(ax)) = \|A\| \end{aligned}$$

□

We note that given a vector in a Hilbert space $\xi \in \mathcal{H}$, the map $\omega_\xi : \eta \mapsto \langle \eta\xi, \xi \rangle$ defines a positive linear functional (when $\|\xi\| = 1$, a state) on $\mathcal{H}$. Such maps ω_ξ are called *vector states* of $\mathcal{H}$. Of course if $\pi : A \curvearrowright \mathcal{H}$ is a representation with cyclic vector ξ, then $x \mapsto \langle \pi(x)\xi, \xi \rangle$ defines a vector state. The GNS construction in lemma A.1.6 will show that any state can be obtained as the cyclic vector of a representation behaving in this manner.

Proposition A.1.5. $\mathcal{L}_\rho$ is a closed left ideal of A, and for any $x \in \mathcal{L}_\rho$ and $y \in A$, $\rho(y^*x) = 0$.

Proof. Since a state is self adjoint (because it is positive), we may define an inner product on A by $\langle x, y \rangle_0 = \rho(y^*x)$. Then the left kernel is simply $\mathcal{L}_\rho = \{x \in A : \langle x, x \rangle_0 = 0\}$, so it is a subspace of A. Then

$$\langle x + \mathcal{L}_\rho, y + \mathcal{L}_\rho \rangle = \langle x, y \rangle_0 = \rho(y^*x) \tag{3}$$

gives a definite inner product on $A/\mathcal{L}_\rho$. If $x \in \mathcal{L}_\rho$ and $y \in A$,

$$|\rho(y^*x)|^2 \leq \rho(y^*y)\rho(x^*x) = 0,$$

so $\rho(y^*x) = 0$. If we replace y with y^*yx,

$$\rho((yx)^*(yx)) = \rho((y^*yx)^*a) = 0,$$

so $yx \in \mathcal{L}_\rho$ and $\mathcal{L}_\rho$ is a left ideal. Since ρ is continuous $\mathcal{L}_\rho$ is closed. □

Lemma A.1.6 (GNS Construction)**.** For each state ρ of A, there is a *-representation $(\pi_\rho, \mathcal{H}_\rho)$ with a unit cyclic vector ξ such that $\rho(x) = \langle \pi_\rho(x)\xi, \xi\rangle$.

Proof. The inner product in equation 3 above gives a pre-Hilbert space structure to the quotient $A/\mathcal{L}_\rho$ for any ρ. Let $\mathcal{H}_\rho$ be the Hilbert space obtained as the completion of this space. The map

$$\pi(x)(y + \mathcal{L}_\rho) = xy + \mathcal{L}_\rho$$

defines, for each $x \in A$, a linear operator acting on the pre-Hilbert space $A/\mathcal{L}_\rho$ (because $\mathcal{L}_\rho$ is an ideal). If we can show that π is bounded on $A/\mathcal{L}_\rho$, it may be extended continuously to a bounded operator on $\mathcal{H}_\rho$. A similar argument to what we used to show the left kernel is a left ideal works and will show $\pi(x) \leq \|x\|$. The continuous extension of π to all of $\mathcal{H}_\rho$ will be denoted π_ρ. It is not hard to see π_ρ is a homomorphism (simply check that is is the case on $A/\mathcal{L}_\rho$) and note that $A/\mathcal{L}_\rho$ is everywhere dense in $\mathcal{H}_\rho$. Likewise, the following shows that π_ρ is self adjoint.

$$\langle \pi_\rho(x)(y + \mathcal{L}_\rho), z + \mathcal{L}_rho\rangle = \langle xy + \mathcal{L}_rho, z + \mathcal{L}_rho\rangle = \rho(z^*xy)$$
$$= \rho((x^*z)^*y) = \langle y, x^*z + \mathcal{L}_\rho\rangle = \langle y, \pi(x^*)(z + \mathcal{L}_\rho)\rangle.$$

Thus, π_ρ defines a *-homomorphism $A \to B(\mathcal{H}_\rho)$. If A us unital, then the cyclic vector can easily be seen to be the image ξ of $1 + \mathcal{L}_\rho$ in $\mathcal{H}_\rho$. If A is non-unital, take an approximate unit e_α and since positive linear functionals are bounded, the image s of e_α in $\mathcal{H}_\rho$ will converge to some cyclic vector ξ for π_ρ.

Finally, clearly $\langle x\xi, \xi\rangle = \langle x + \mathcal{L}_\rho, 1 + \mathcal{L}_\rho = \rho(x)$ for all $x \in A$. □

Corollary A.1.7. If $0 \neq x \in A$, there exists a pure state ρ on A so that $\pi_\rho(x) \neq 0$, where π_ρ is the representation obtained from ρ via the GNS construction

Proof. By A.1.4(1) there exists a pure state ρ such that $\rho(x) \neq 0$. Equivalently, thanks to the GNS construction $\langle \pi_\rho(x)\xi, \xi\rangle \neq 0$, where ξ is the cyclic vector for π_ρ. Thus, $\pi_\rho(x) \neq 0$. □

In particular, this last corollary tells us the the representations obtained in a GNS manner from the pure states of A are numerous enough to separate points in A.

Theorem A.1.8 (Gelfand, Naimark)**.** Let A be a C*-algebra. There exists a faithful *-representation $\pi : A \curvearrowright \mathcal{H}$ on a Hilbert space $\mathcal{H}$ with cyclic vector ξ.

Proof. Let $\mathcal{S}_0$ be any family of states containing all the pure states of A. Let

$$\phi = \bigoplus_{\rho \in \mathcal{S}_0} \pi_\rho : A \to \bigoplus_{\rho \in \mathcal{S}_0} \mathcal{H}_\rho,$$

where π_ρ are associated the GNS representations. For any $x \in \ker(\phi)$, $\pi_\rho(x) = 0$ for each state $\rho \in \mathcal{S}_0$. However by corollary A.1.7 tells us that the pure states separate points, so x must be 0, and thus ϕ is faithful. □

Remark A.1.9. In the case where we choose $\mathcal{S}_0 = \mathcal{S}$, the whole state space, the representation $\phi = \oplus_{\mathcal{S}} \pi_\rho : A \curvearrowright \oplus_{\mathcal{S}} \mathcal{H}_\rho$ is called the *universal *-representation* of A. Due to the Gelfand-Naimark theorem, we will henceforth treat a C*-algebra A and its representation as a norm closed *-subalgebra $A \subset B(\mathcal{H})$ for some Hilbert space $\mathcal{H}$ interchangeably.

With this representation theorem in mind, let us now show that our C*-algebra definitions of certain types of elements coincide with certain properties of Hilbert space operators.

Proposition A.1.10. $x \in B(\mathcal{H})$ is (in the sense of C*-algebras):

(1) self-adjoint if and only if $\langle x\xi, \xi \rangle \in \mathbb{R}$ for all $\xi \in \mathcal{H}$,

(2) normal if and only if $\|x\xi\| = \|x^*\xi\|$ for all $\xi \in \mathcal{H}$,

(3) a projection if and only if p is the orthogonal projection onto a closed subspace of $\mathcal{H}$.

A.2 Locally Convex Topologies on $B(\mathcal{H})$

Now that we know that the study of C*-algebras can be viewed as the study of closed subalgebras of operators in $B(\mathcal{H})$, we should be aware of other topologies on $B(\mathcal{H})$ which are useful.

We can define the *strong operator topology (SOT)* on a subset $M \subset B(\mathcal{H})$ to be the topology with a base of neighbourhoods of the form $V(x_0; \xi_1, ...\xi_n; \varepsilon)$ for $\xi_1, ...\xi_n \in \mathcal{H}$ and $\varepsilon > 0$ where

$$V(x_0; \xi_1, ...\xi_n; \varepsilon) = \{x \in M : \|(x - x_0)\xi_j\| < \varepsilon \, \forall j = 1, ...n\}$$

Accordingly, $x_n \to x$ in the strong operator topology if and only if $\|(x - x_n)\xi\| \to 0 \, \forall \xi \in \mathcal{H}$. The generating family of seminorms for the SOT is $x \mapsto \|x\xi\|$ for each $\xi \in \mathcal{H}$.

We define the *weak operator topology (WOT)* on $B(\mathcal{H})$ to be the weak topology generated by the following family of linear functionals

$$F_w = \{w_{\xi,\eta} : B(\mathcal{H}) \to \mathbb{C} : \xi, \eta \in \mathcal{H}\} \quad \text{where} \quad w_{\xi,\eta}(x) = \langle x\xi, \eta\rangle$$

In other words, it is the weakest (coarsest) topology such that each of the above maps are continuous. A base of neighbourhoods in the weak operator topology is of the form $V(x_0; \xi_1, ...\xi_n; \varepsilon)$ where

$$V(x_0; w_{\xi_1,\eta_1}, ...w_{\xi_n,\eta_n}; \varepsilon) = \{x \in B(\mathcal{H}) : |\langle (x - x_0)\xi, \eta\rangle| < \varepsilon \, \forall j = 1, ...n\}$$

A sequence $x_n \to x$ in the weak operator topology if and only if $|\langle (x_n - x)\xi, \eta\rangle| \to 0 \forall \xi, \eta \in H$. We note that, fittingly, the weak operator topology is in fact coarser (weaker) than the strong operator topology.

Lemma A.2.1. Let $\varphi : B(\mathcal{H}) \to \mathbb{C}$ be a bounded linear functional. The following are equivalent.

(1) φ is weak operator continuous.

(2) φ is strong operator continuous.

(3) There exist $\xi_1, ...\xi_n, \eta_1, ...\eta_n \in \mathcal{H}$ such that $\varphi(x) = \sum_i \langle x\xi_i, \eta_i\rangle$ for all $x \in B(\mathcal{H})$.

Proof. (1) $\Rightarrow$ (2) is trivial by ordering of topologies, and (3) $\Rightarrow$ (1) is immediate by the definition of the seminorms which generate the WOT. Let's now suppose φ is SOT-continuous. Then the pre-image of the unit ball of $\mathbb{C}$ in $B(\mathcal{H})$ is also open, so we can find a constant $K > 0$ such that $|\varphi(x)|^2 = K \sum_1^n \|x\xi_i\|^2$. Now take the subset

$$\mathcal{H}_0 = \{\oplus_i x\xi_i : x \in B(\mathcal{H})\} \subset \mathcal{H}^{\oplus n}.$$

The map $\oplus_i x\xi_i \mapsto \varphi(x)$ extends to a continuous linear functional on the closure $\overline{\mathcal{H}_0}$. Thus, by Riesz representation theorem, there exist $\eta 1, ...\eta_n$ such that

$$\varphi(x) = \sum_{i=1}^{n} \langle x\xi_i, \eta_i\rangle.$$

□

A major consequence of this theorem is that the continuous dual spaces with respect to the WOT and with respect to the SOT coincide. This gives us the following useful corollary.

Corollary A.2.2. The weak and strong operator closures coincide for convex subsets.

Since the dual spaces coincide for all three topologies, the corollary follows by geometric Hahn-Banach. One can also prove this directly as follows:

Proof. Let $K \subset B(\mathcal{H})$ be convex. Then denote $\overline{K^w}, \overline{K^s}$ the weak and strong operator closures respectively. The first inclusion $\overline{K^w} \supset \overline{K^s}$ follows immediately from the ordering of topologies. Now let $x \in \overline{K^w}$ and we'll show that x is in the strong operator closure. Choose $\xi_1, ...\xi_n \in \mathcal{H}$ and let $\xi = (\xi_i, ..., \xi_n)$. For any $y \in B(\mathcal{H})$, the operator $y^{\oplus n}$ is defined by $y^{\oplus n}(\eta_1, ..., \eta_n) = (y\eta_1, ..., y\eta_n)$. Then $\tilde{K} = \{y^{\oplus n} : y \in K\}$ is a convex subset of $B(\mathcal{H}^{\oplus n})$.

Since $x^{\oplus n}$ is in the weak operator closure of K, $x\xi$ is in the weak operator closure of $\tilde{K}$ in $\mathcal{H}^{\oplus n}$. This means that $x\xi$ is in the norm closure $\Rightarrow$ for some $y \in K$, $\|y\xi_i - x\xi_i\|$ is small for each $i = 1, ...n$. This is precisely what it means for x to be in the strong operator closure. Therefore $\overline{K^w} \subset \overline{K^s}$. □

A.2.1 Kaplansky's Density Theorem

Kaplansky's density theorem is a fundamental result in operator algebras. It is often used in the literature without being mentioned by name. The proof can be found in any textbook on operator algebras, but we follow [35].

Lemma A.2.3. Let $f \in C(\mathbb{C})$. Then f is continuous in the SOT on any set of bounded normal operators.

Proof. We may, by the Stone-Weierstrass theorem, approximate f uniformly by polynomials. Multiplication is SOT continuous and taking adjoint is SOT continuous on normal operators. □

Lemma A.2.4 (Cayley Transform)**.** The map $x \mapsto (x - i)(x + i)^{-1}$ (the Cayley transform on the Riemann sphere) is SOT continuous from $B(\mathcal{H})_{sa}$ to $\mathcal{U}(B(\mathcal{H}))$.

Proof. Let (x_α) be a net of self adjoint operators in $B(\mathcal{H})$ converging to x in the SOT. The spectral mapping theorem (self-adjoint elements are normal) gives that $\|(x_\alpha + 1)^{-1}\| \leq 1$ for each α. Then for each $\xi \in \mathcal{H}$,

$$\begin{aligned} &\|(x-i)(x+i)^{-1}\xi - (x_\alpha - i)(x_\alpha + i)^{-1}\xi\| \\ = &\|(x_\alpha + i)^{-1}[(x_\alpha + i)(x-i)^{-1} - (x_\alpha - i)(x+i)^{-1}](x+i)\xi\| \\ = &\|(x_\alpha + i)^{-1}2i(x - x_\alpha)(x+i)^{-1}\| \\ \leq &2\|(x - x_k)\|\|(x+i)^{-1}\xi\| \to 0 \text{ in the SOT.} \end{aligned}$$

Hence, the Cayley transform is SOT continuous on self adjoint elements. □

Corollary A.2.5. If $f \in C_0(\mathbb{R})$ then f is SOT continuous on self-adjoint operators.

Proof. $f \in C_0$ means it vanishes at infinity so

$$g(t) = \begin{cases} f(i(1+t)/(1-t)), & t \neq 1 \\ 0, & t = 1 \end{cases}$$

defines a continuous function on the unit circle. A.2.3 gives that g is SOT continuous on the unitaries, and A.2.4 gives that the Cayley transform (we can denote it U) is SOT continuous mapping to the unitaries. It then follows that $f = g \circ U$ is SOT continuous. □

Theorem A.2.6 (Kaplansky Density)**.** Let $A \subset B(\mathcal{H})$ be a C*-algebra, denote $\overline{A^s}$ the strong operator closure of A, and $(A)_1$ denote the unit ball of A. Then,

(1) $\overline{(A_{sa})^s} = (\overline{A^s})_{sa}$,
(2) $\overline{(A)_1^s} = (\overline{A^s})_1$.

Proof. In both cases the nontrivial inclusion to prove is $\supset$.Suppose $x_\alpha \to x$ is a net of elements converging to a self-adjoint x in the SOT. Since the involution is continuous in the WOT, $(x_\alpha + x_\alpha^*)/2 \to x$ in the WOT. But since the space of self-adjoint operators is convex, by A.2.2 the WOT and SOT closures coincide, so x is in the strong operator closure of A_{sa}, hence $\overline{(A_{sa})^s} = (\overline{A^s})_{sa}$.

First we show (2) for self adjoint elements. Let $(y_\alpha) \subset A_{sa}$ be s.t. $y_\alpha \to x$ in the SOT. Take a function $f \in C_0(\mathbb{R})$ such that $f(t) = t$ when $|t| \leq \|x\|$ and $|f(t)| \leq \|x\|$ for all $t \in \mathbb{R}$. Then

$|f(y_\alpha)| \leq \|x\|$ for each α, and $f(y_\alpha) \to f(x)$ in the SOT by corollary A.2.5, thus proving the self adjoint case. Now to extend this to the case of an arbitrary element we use a matrix trick:

First we note that $\overline{M_2(A)^s} = M_2(\overline{A^s}) \subset B(\mathcal{H}^2)$. Let $x \in (\overline{A^s})_1$, then

$$\tilde{x} = \begin{pmatrix} 0 & x \\ x^* & 0 \end{pmatrix} \in (M_2(\overline{A^S}))_1$$

is self adjoint. Then we know there is a net of matrices in $(M_2(A))_1$,

$$\tilde{x}_\alpha = \begin{pmatrix} a_\alpha & b_\alpha \\ c_\alpha & d_\alpha \end{pmatrix},$$

converging to $\tilde{x}$. Of course, $\|b_n\| \leq 1$, and $b_n \to x$ in the SOT. □

A.3 Von Neumann Algebras

We first mention that the predual of a W*-algebra is unique, a fact which is nontrivial to prove (see [35, thm. 4.4.4]). It will be useful to note that the sets of self adjoint elements M_{sa} and positive elements M_+ in a W*-algebra are closed in the $\sigma(M, M_*)$ topology (for a simple proof, see [39, lem. 1.7.1]).

Let M be a W*-algebra. The ultraweak topology $\sigma(M, M_*)$ is built so that the multiplication maps $L_a : x \mapsto ax, R_a x \mapsto xa$ are continuous. If p is a projection in M, the left (resp. right) ideal Mp (resp. pM) is closed in the ultraweak topology.

Conversely, if $L \triangleleft M$ is a left ideal, there exists a projection p such that $L = Mp$ (and the same goes for a right ideal). To see this we take $N = L \cap L^*$, which is a W*-subalgebra of M (it is clearly a C*-algebra, and the involution is ultraweakly continuous so N is ultraweakly closed). We then take p to be the identity on N, so it is a projection in M with $Mp \subset L$. If $x \in L$, then $x^*x \in N$, and because p is identity on N,

$$px^*xp = px^*x = x^*xp = x^*x.$$

Thus, $(1-p)x^*x(1-p) = 0$, so $a(1-p) = 0$, implying that $Mp \supset L$. It is also not hard to see that this projection must be unique.

If we have an ultraweakly closed two-sided ideal $I \triangleleft M$, there are two projections p, q so that $I = Mp = qM$. However both of these must act as the identity on $I \cap I^*$, so $p = q$. Moreover $px = (px)p = pxp = p(xp) = xp$ for any $x \in M$, so p is central. Thus, we will use that any closed two sided ideal is generated by a central projection.

Proposition A.3.1. If $\varphi : M \to N$ is a W*-homomorphism between two W*-algebras, then the image $\varphi(M)$ is closed in the $\sigma(N, N_*)$ topology (the ultraweak topology on N).

Proof. Note that $I = \ker(\varphi) \triangleleft M$ is an ultrawealky closed two-sided ideal, and so admits a central projection p such that $I = Mp$. Restricting φ to $M(1-p)$ gives us a *-isomorphism, hence an isometry. Thus the image of unit sphere of M is the unit sphere of $\varphi(M)$, and so it is $\sigma(N, N_*)$-compact (Banach-Alaoglu). Thus $\varphi(M)$ is a σ-closed subalgebra. □

A positive linear functional ρ on a C*-algebra A is called *normal* if for any increasing net x_α with least upper bound x, $\rho(x_\alpha)$ has least upper bound $\rho(x)$. Equivalently if we have instead a W*-algebra M, we may define a normal linear functional to be a $\sigma(M, M_*)$-continuous positive linear functional (c.f. [39, p. 1.13.2]). It turns out that the space of normal states on M can be identified with the predual (see [39, p. 1.13.2], and the remarks thereafter).

Lemma A.3.2. Let $x \in M$ be an element in a W*-algebra. If $\rho(x) = 0$ for each σ-continuous positive linear functional ρ, then $x = 0$. In particular, the normal linear functionals form a point-separating family.

Proof. It will be sufficient to show that if x is a self adjoint, non-positive element of M, then there is a σ-continuous positive functional such that $\rho(x) < 0$. Since the positive elements form a convex cone in the space of self-adjoint elements, and are closed in the $\sigma(M, M_*)$-topology. We can find a σ-continuous linear functional g on M_{sa} (separation theorem for locally convex vector spaces) such that $\inf_{y \in M_+} g(y) > g(x)$. But since M_+ is a cone, $0 = \inf_{y \in M_+} g(y) > g(x)$. Therefore, $g(x) < 0$, and $g(y) \geq 0$ for all $y \in M_+$. For arbitrary elements of M, define a linear functional f on sums of self adjoint elements: $f(a + ib) = g(a) + ig(b)$. f is a linear functional on M. Since M_{sa} is closed, the involution is σ-continuous, and so f is σ-continuous and positive such that $f(x) < 0$. □

For a normal state ρ on a W*-algebra M, let $\pi_\rho : M \to B(\mathcal{H}_\rho)$ be associated the GNS representation. For $\xi, \eta \in \mathcal{H}_\rho$, set $f(x) = \langle \pi_\rho(x)\eta, \xi\rangle$. Because the pre-Hilbert space $M/\mathcal{L}_\rho$ is dense, there are sequences a_n, b_n in M for whom the images of $a_n + \mathcal{L}_\rho$ and $b_n + \mathcal{L}_\rho$ in $\mathcal{H}_\rho$ (which we shall denote as $\bar{a_n}, \bar{b_n}$) converge in norm to ξ and η respectively.

$$\begin{aligned}
&|\langle \pi_\rho(x)\xi, \eta\rangle - \langle \pi_\rho(x)\bar{a_n}, \bar{b_n}\rangle| \\
&\leq |\langle \pi_\rho(x)(\xi - \bar{a_n}), \eta\rangle| + |\langle \pi_\rho(x)\bar{a}_n, (\eta - \bar{b}_n)\rangle| \\
&\leq \|x\|\|\xi - \bar{a_n}\|\|\eta\| + \|x\|\|\eta - \bar{b}_n\|\|\bar{a}_n\| \to 0.
\end{aligned}$$

So we see that $f(x)$ is the uniform limit of the sequence

$$f_n(x) = \langle \pi_\rho(x)\bar{a_n}, \bar{b_n}\rangle = \rho(b_n^* x a_n),$$

which lies in the unit sphere of M. Because ρ is normal, $f_n's$ lie in the predual M_* and so does f. We've shown that the map $x \mapsto \pi\rho(x)$ is continuous in the respective ultraweak topologies on M and $B(\mathcal{H}_\rho)$ on bounded spheres, and so π_ρ is a W*-homomorphism. In particular, π_ρ is a W*-representation onto $\mathcal{H}_\rho$.

Theorem A.3.3 (Sakai)**.** Every W*-algebra has a faithful W*-representation on some Hilbert space $\mathcal{H}$. Thus every W*-algebra is *-isomorphic to a weakly closed self adjoint subalgebra of $B(\mathcal{H})$.

Proof. Consider the set of all normal states on M, call it $\mathcal{S}_n(M)$. We build a universal W*-representation much like was done in the Gelfand-Naimark theorem. Let

$$\pi = \bigoplus_{\rho \in \mathcal{S}_n} \pi_\rho : M \to \bigoplus_{\rho \in \mathcal{S}_n} \mathcal{H}_\rho.$$

We must do a little more work to show that π is a W*-representation. Let F be the set consisting of all finite linear combinations of elements in $\bigcup_{\mathcal{S}_n} \mathcal{H}_\rho$. F is dense in $\bigoplus_{\rho \in \mathcal{S}_n} \mathcal{H}_\rho$. Let

$$\xi = \sum_{i=1}^n \xi_i, \quad \eta = \sum_{i=1}^n \eta_i, \text{ where } \xi_i, \eta_i \in \mathcal{H}_{\rho_i}.$$

Then, $f(x) := \langle \pi(x)\xi, \eta\rangle = \sum_{i=1}^n \langle \pi_{\rho_i}(x)\xi_i, \eta_i\rangle$ is the sum of elements of the predual, and so $f \in M_*$. For amy $\xi', \eta' \in \mathcal{H}$ the function $f_{\xi',\eta'} : x \mapsto \langle \pi(x)\xi', \eta'\rangle$ is in M_*. Thus, $\pi : M \to B(\mathcal{H})$ is a W*-representation.

If $x \in \ker(\pi)$, then once again $\pi_\rho(x) = 0$, and thus $\rho(x) = 0$ for each normal state ρ of M. By lemma A.3.2, we see that the normal linear functionals separate points, so $x = 0$. $\square$

Theorem A.3.4. Let $B \subset B(\mathcal{H})$ be a self adjoint set. Then B' is a von Neumann Algebra.

Proof. The self adjointness of B' follows obviously from the self adjointness of B. If $x \in B'$ then for any $y \in B, x^*y = (y^*x)^* = (xy^*)^*$ since B is self adjoint, $= yx^*$. Obviously the identity is in the commutant as well. Now let x_α be any net in B' such that $x_\alpha \to x \in B(\mathcal{H})$. Then for any $y \in B$ and $\xi, \eta \in \mathcal{H}$

$$\langle [x, y]\xi, \eta\rangle = \langle xy\xi, \eta\rangle - \langle x\xi, y^*\eta\rangle = \lim_\alpha \langle x_\alpha y\xi, \eta\rangle - \langle x_\alpha \xi, y^*\eta\rangle = \lim_\alpha \langle [x_\alpha, y]\xi, \eta\rangle = 0$$

where $[x, y] = xy - yx$ is the commutator. This shows that x in the weak operator closure of B' is in fact in B', so in particular that B' is a von Neumann algebra. $\square$

The following nifty little lemma will be helpful in the proof of the main theorem. This lemma, as well as this particular version of the proof of the bicommutant theorem follow the presentation of Adrian Ioana in [18]

Lemma A.3.5. Let M be a unital *-algebra and $K \subset \mathcal{H}$ an M-invariant closed subspace (that is, $xK \subset K$ for $x \in M$). The orthogonal projection p onto K lies in M'.

Proof. If $x \in M$, $xp\mathcal{H} = xK \subset K$. So, $(1-p)xp\mathcal{H} \subset (1-p)K = \{0\} \Rightarrow (1-p)xp = 0$. Note also that $x^* \in H \Rightarrow (1-p)x^*pH = \{0\} \Rightarrow px(1-p) = 0$ (note orthogonal projections are self adjoint). Subtracting the first equation from the second, we get $px - xp = 0$. $\square$

Theorem A.3.6 (von Neumann)**.** Let M be a unital (with identity 1) *-algebra. Then the following conditions are equivalent:

(1) M is weak operator closed.
(2) M is strong operator closed.
(3) $M = M''$

Proof. The implication (1) $\Rightarrow$ (2) is obvious because an algebra is necessarily convex, and implication (3) $\Rightarrow$ (1) follows immediately from theorem A.3.4. We need to show (2) $\Rightarrow$ (3).

We know that $M \subset M''$, however we need to show that this embedding is weak operator dense. Fix some $x \in M''$ and some $\xi_1, ...\xi_n \in \mathcal{H}$. Let p be the orthogonal projection onto the subspace

$$\overline{M\xi_1} = \{y\xi_1 : y \in M\}.$$

It's obvious that $\overline{M\xi_1}$ is closed and M-invariant, and so by lemma A.3.5, $p \in M' \Rightarrow px = xp$ since $x \in M''$. This gives

$$x\xi_1 = xp\xi_1 = p(x\xi_1) \in \overline{M\xi_1},$$

which implies that there exists $y \in M$ with $\|x\xi_i - y\xi_i\| < \varepsilon$ for each $i = 1, ...n$ (what we're trying to show), in the case where $n = 1$. To extend this case to larger n, we use a standard matrix trick.

Define a unital *-homomorphism $\pi : B(\mathcal{H}) \to B(\mathcal{H}^n) = M_n(B(\mathcal{H}))$ in the following way:

$$\pi(y) = \begin{pmatrix} y & 0 & \cdots & 0 \\ 0 & y & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & y \end{pmatrix} \qquad \text{In other words, } \pi(y)(\eta_1, ..., \eta_n) = (y\eta_1, ..., y\eta_n)$$

We will need to show the following two facts to finish our proof.

1. $\pi(M'') \subset (M_n(M)')'$
2. $\pi(M') \subset M_n(M)'$

To prove (1), let $\pi(x) \in \pi(M'')$ and $A = (a_{ij}) \in M_n(M')$. $\pi(x)A = (xa_{ij})$ however since $x \in M''$ and each $a_{ij} \in M'$, $(xa_{ij}) = (a_{ij}x = A\pi(x)$. To prove (2) let $A \in \pi(M)'$. So that means that for any $x \in M$, $a_{ij}x = xa_{ij}$, which of course means that $a_{ij} \in M' \Rightarrow A \in M_n(M')$.

Facts (1) and (2) togethter with $(A \subset B \Rightarrow B' \subset A')$ give us that $\pi(M'') \subset \pi(M)''$, and hence that for our fixed $x \in M''$, $\pi(x) \in \pi(M)''$. If $\xi = (\xi_1, ...x_n)$ then the case $n = 1$ implies that there exists a $y \in M$ such that

$$\|\pi(x)\xi - \pi(y)\xi\| < \varepsilon.$$

This shows that x is in the weak operator closure of M. □

Finally, we give a sketch of the proof of the Sherman-Takeda theorem, as presented in [39].

Theorem A.3.7 (Sherman, Takeda)**.** Let A be a C*-algebra. Then the double dual, A^{**}, is a von Neumann algebra. In particular, $A^{**} = \pi_u(A)''$.

Proof. Suppose $\pi_u : A \hookrightarrow B(\mathcal{H})$ is the universal *-representation. Then we can identify A with $\pi_u(A)$ because the representation is an isometric isomorphism onto the image in $B(\mathcal{H})$. Then, the weak operator closure $\overline{\pi_u(A)^w}$ is a von Neumann Algebra and let $\overline{\pi_u(A)^w}_*$ be its Banach space predual. Then for a $\varphi \in \overline{\pi_u(A)^w}_*$,

$$\|\varphi\| = \sup_{\substack{\|x\|\leq 1,\\ x\in\overline{\pi_u(A)^w}}} |\varphi(x)| = \sup_{\substack{\|x\|\leq 1,\\ x\in A}} |\varphi(x)| = \|\varphi|_A\|.$$

Here we used Kaplansky's Density theorem the fact that the unit ball is weakly closed. So the mapping $\varphi \mapsto \varphi|_A$ is isometric. By an application of the Riesz-Markov-Kakutani representation theorem (see [39, p. 1.17.1] for details) any bounded linear functional can be written as the linear combination of states. Then, since $\{\varphi|_A : \varphi \in \overline{\pi_u(A)^w}_*\}$ contains all the states of A, the mapping $\varphi \mapsto \varphi|_A$ defines an isometric isomorphism $\overline{\pi_u(A)^w}_* \to A^*$. Taking the dual, we get A^{**} is isometrically isomorphic to $\overline{\pi_u(A)^w} = \pi_u(A)''$ $\square$