SPECTRAL THEORY AND FLOWS ON REPRESENTATION VARIETIES

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DEDICATION

To my office plants for providing steady support and company.

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ABSTRACT

In this thesis, a new method for studying the spectral gap of certain averaging operators over the square integrable functions on the 2-sphere is explored. The proposed method is the use of Goldman flows to probe associated spectral radii defined over the moduli space of representations of the genus 2 fundamental group into SU_2. In particular, the critical points of the spectral radii are considered. Using Goldman flows, a description of how the first spectral radius of representations of the genus 2 fundamental group into SU_2 changes locally around a point is given. A numerical study highlights representations with minimal first spectral radius of zero as well as shows that highly symmetric representations need not exhibit this minimum. Focused introductions to smooth manifolds, Lie groups, representation varieties along with their natural symplectic structures, and Goldman flows are provided before this first attempt at using Goldman flows to study spectral gaps is discussed.

ABRÉGÉ

Dans ce mémoire, une nouvelle méthode est utilisée pour explorer la lacune dans le spectre de certains opérateurs de moyennisation sur les fonctions de carré sommable sur la 2-sphère. La méthode proposée est l'utilisation des flots de Goldman pour sonder les spectres associés qui sont définis sur l'espace de modules de représentations du groupe fondamental de genre 2 dans SU_2. Spécifiquement, les points critiques des rayons spectraux sont examinés. A l'aide des flots de Goldman, une description est donnée de la variation locale du premier rayon spectral des représentations du groupe fondamental de genre 2 dans SU_2. Une étude numérique nous indique la présence de représentations avec un premier rayon spectral égal à zéro et démontre ainsi que les représentations hautement symétriques ne sont pas forcément celle qui manifestent ce minimum. Des introductions aux variétés différentielles, groupes de Lie, variétés de représentations et leurs formes symplectiques naturelles, ainsi les flots de Goldman sont fournies en préliminaire à cette étude du trou spectral avec flots de Goldman.

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CHAPTER 1
Introduction

This thesis explores by means of Goldman flows the spectral gap of certain standard averaging operators defined over $L^2(S^2)$. The close ties between the spectral gap of these operators and the representation varieties $\mathrm{Hom}(\pi_1(\Sigma_g), SU_2)/SU_2$, realized as real-valued functions on $\mathrm{Hom}(\pi_1(\Sigma_g), SU_2)/SU_2$ in which each representation class is mapped to an associated spectral gap, allow this application of Goldman flows. Once the effect of such flows on the spectral gap is understood, they will facilitate a study of the spectral gaps local behaviour on the variety. Furthermore, providing agility within the representation variety, these flows provide a means of searching for critical points of the spectral gap.

Before the work in this direction is presented, preliminary material is built up in earlier chapters. With the goal of describing this work in mind, in these early chapters only those topics and examples necessary for building its foundation are described. In the second chapter an introduction to manifolds, quickly narrowing in on smooth manifolds, is given. The chapter culminates with a description of the closed orientable surfaces. Another class of smooth manifolds, Lie groups, are discussed following this in Chapter 3. In particular, information about the Lie groups SU_n, with further details yet about SU_2, is presented. In Chapter 4 representation varieties and their natural symplectic structure are described in terms of both representations and flat connections. The preliminary material is wrapped up in Chapter 5's introduction to Goldman flows. In short, Goldman flows are a generalization of Fenchel-Nielsen twist flows on Teichmüller space to more general representation varieties.

In Chapter 6, the inquiry stated above is introduced in greater depth and its historical context in terms of the Banach-Tarski Paradox and Ruziewicz problem is outlined. Finally, Chapter 7 begins to address the questions posed in chapter six in the lowest dimensional and simplest cases. Explicit descriptions of how a sort of simplified spectral gap of representations of $\pi_1(\Sigma_2)$ into SU_2 evolve under Goldman flows are shown.

The first two chapters rely heavily on John Lee's *Introduction to Smooth Manifolds* [12]. François Labourie's course notes *Lectures on Representation of Surface Groups* are the primary source for the sections building up to and addressing the alternative formulation of the representation variety provided in Chapter 4 [11]. William Goldman's work in [8] and [10] source the information on the representation variety's natural symplectic structure and Goldman flows. Work by Alex Gamburd, Dmitry Jakobson, and Peter Sarnak on the spectral gap of free group representations into SU_2 in [7] provides inspiration for those questions asked in Chapter 6.

CHAPTER 2
Manifold basics

This chapter follows its title in both senses: many of the important basics regarding manifolds, in particular smooth manifolds, are quickly laid out. This brief overview is guided by the considerably more expanded introduction given in Lee's *Introduction to Smooth Manifolds* [12]. The concepts which are included have been chosen because they will be used in later chapters. A natural place to start is with the definition of a manifold.

Definition 1. *An* **n-manifold** *is a second countable, Hausdorff, topological space such that every point of the space has a neighbourhood homeomorphic to an open subset of* $\mathbf{R}^n$*.*

The primary concern of this thesis will be 2-manifolds, or *surfaces.* For this reason, generic manifolds will be denoted by S. According to the last property stated in the definition, one may think of these spaces as locally Euclidean. The local Euclidean nature of manifolds may be made slightly more precise by using coordinate charts. A *coordinate chart* is a homeomorphism φ from an open subset U of an n-manifold S to an open subset $\hat{U}$ of $\mathbf{R}^n$. The component functions φ_i of the homeomorphism, given by $\varphi(x) = (\varphi_1(x), \ldots, \varphi_n(x))$, provide *local coordinates* on U. It follows from the definition that every point of the manifold is contained in a coordinate chart, and in reality many. Between any two charts $(U, \varphi), (V, \psi)$ for which $U \cap V \neq \emptyset$, the homeomorphism $\psi \circ \varphi^{-1} : \varphi(U \cap V) \to \psi(U \cap V)$ provides a *transition map* between the local coordinates. A collection of charts whose domains together cover a surface S

is said to be an *atlas* of S. Two atlases are equivalent if their union is again an atlas. Examples of these objects are given for the sphere in figure 2–1.

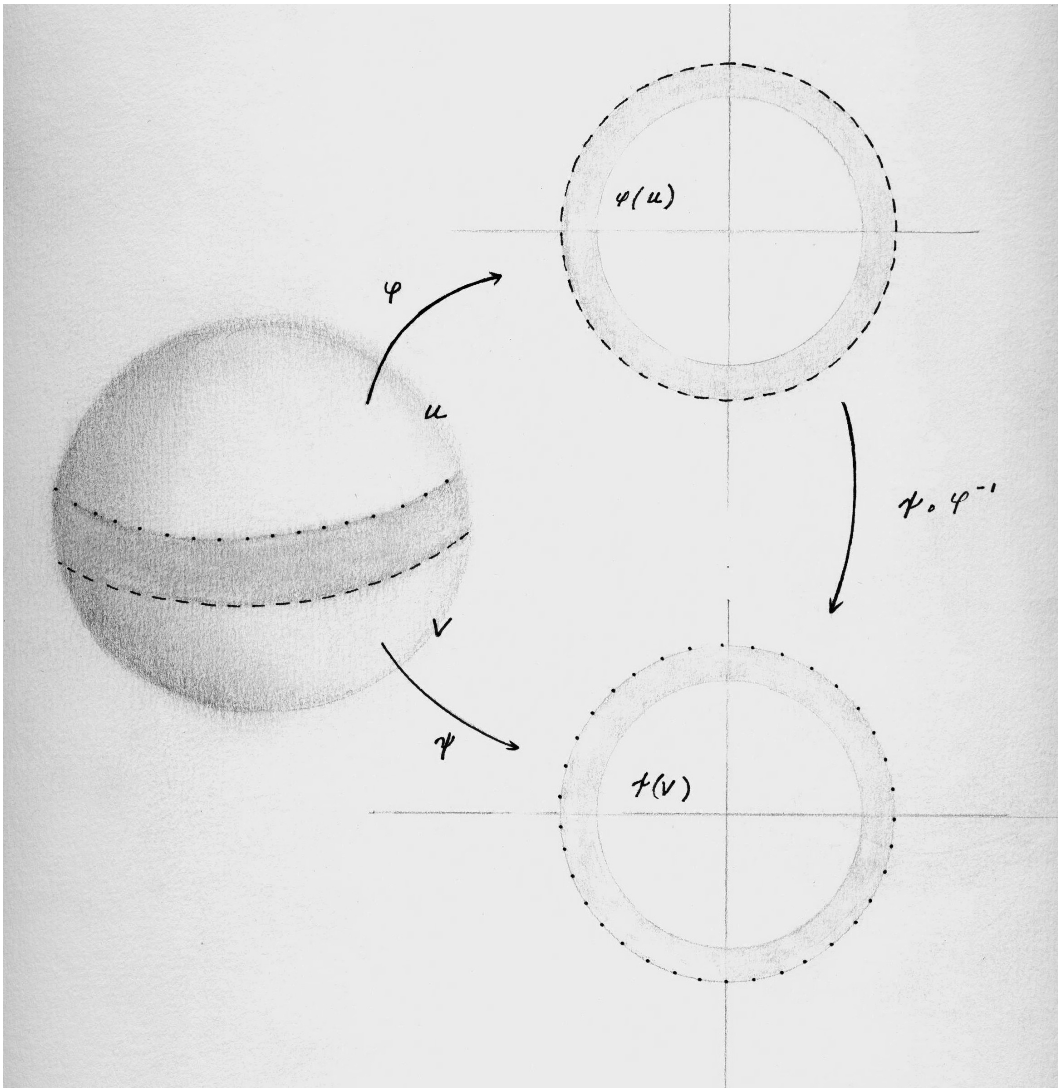

Figure 2–1: Sphere with two coordinate charts comprising an atlas, and a transition map between them.

By placing restrictions on the the coordinate charts, the manifold, at this point a strictly topological entity, is endowed with further structure. In the next section, smooth structures are discussed. G-structures will be encountered later in section 4.1.1.

2.1 Smooth Manifolds

It is desirable, when possible, to build up enough structure on a manifold to enable calculus. Such structure comes from placing certain smooth restrictions on the coordinate charts. In particular, the manifold's atlas of charts must be restricted to a *smooth atlas*: an atlas for which the transition maps between any two overlapping charts of the atlas are diffeomorphisms. A *smooth structure* is an equivalence class of smooth atlases. A topological n-manifold S together with some smooth structure is a *smooth manifold*. It will be assumed that all manifolds discussed from here on will be smooth.

In this context, one may now define *smooth functions*. A smooth function is any function $f : S \to \mathbf{R}^k$ such that for every $x \in S$ there exists a chart (U, φ) whose domain contains x and for which $f \circ \varphi^{-1}$ is smooth on $\varphi(U) \subset \mathbf{R}^n$. As would be expected, the set of all real-valued, smooth functions form a vector space, denoted by $C^\infty(S)$.

Generalizing the previous definition, one may also define *smooth maps* between manifolds. For two smooth manifolds S, M, the map $F : S \to M$ is smooth if for every $x \in S$, there exists charts (U, φ) containing x and (V, ψ) containing $F(x)$ for which $F(U) \subset V$ and $\psi \circ F \circ \varphi^{-1}$ is a smooth map from $\varphi(U)$ to $\psi(V)$. If, moreover, F is a bijection with a smooth inverse, it is said to be a *diffeomorphism* between S and M.

2.1.1 The tangent bundle

With the basic smooth structure defined, the powerful tool in calculus of linear approximation may be brought into the realm of smooth manifolds. The tangent bundle, comprised of local linear approximations of the manifold, is one construction which enables this extension. The basic building block of the tangent bundle is the tangent vector.

Tangent vectors. One of the simplest instances of linear approximation is the familiar notion of geometric tangent vectors and their one-to-one correspondence with directional derivatives. The generalization of directional derivatives to manifolds are derivations.

Definition 2. *For any point x of a manifold S, a* **derivation at** x *is a linear map $X : C^\infty(S) \to \mathbf{R}$ satisfying, for all $f, g \in C^\infty(S)$,*

$$X(fg) = f(x)Xg + g(x)Xf.$$

Each derivation is called a *tangent vector* to S at x. As with directional derivatives, the collection of all derivations forms a vector space. This vector space is called the *tangent space* to S at x and is denoted by T_xS. The correspondence between derivations and geometric tangent vectors becomes apparent when one considers local coordinates about x – see Lee [12, 60 - 70].

Tangent bundle. The tangent spaces at each point together comprise a global object associated to a smooth manifold. For any smooth manifold S, this object is the *tangent bundle* TS of S, defined to be the disjoint union of all the tangent spaces:

$$TS = \coprod_{x \in S} T_xS.$$

The tangent bundle has a natural topology and smooth structure that make it into a 2n-dimensional smooth manifold.

Having considered the tangent structure on individual manifolds, it is interesting to note how tangent vectors behave under smooth maps between them. If S and M are manifolds, and $F : S \to M$ is a smooth map, then the *pushforward* F_* associated with F is a map $F_* : T_xS \to T_{F(x)}M$ acting individually on the tangent spaces as follows

$$(F_*X)(f) = X(f \circ F), \quad \text{for any } X \in T_xS \text{ and } f \in C^\infty(M).$$

Vector fields. Now that the tangent bundle has been established, vector fields on manifolds may be defined. If S is a smooth manifold, a *smooth vector field* on S is a smooth map $X : S \to TS$, such that $X(x)$ is an element of T_xS. Just as for vector fields in $\mathbf{R}^n$, these are, roughly speaking, a smoothly varying choice of a single vector at each point of the manifold. The set of all smooth vector fields, let it be denoted by $\mathfrak{X}(S)$, forms a vector space under pointwise addition and scalar multiplication.

As with individual vectors, it is interesting to consider how vector fields behave under smooth maps between manifolds. In general, the pushforward of an entire vector field need not be defined. This is obvious if one considers maps which are not injective. The pushforward is, however, defined in a natural way in the case of diffeomorphisms.

Of considerable interest for Lie groups, a class of smooth manifolds which will be described in Chapter 3, is how two smooth vector fields may be combined to obtain another via the Lie Bracket.

Definition 3. *The* **Lie bracket** *of two smooth vector fields X, Y is the smooth vector field $[X, Y] : C^\infty(S) \to C^\infty(S)$ defined by*

$$[X, Y]f = XYf - YXf \quad \textit{for all } f \in C^\infty(S).$$

For all vector fields X, Y, Z this map exhibits the following properties:

Bilinearity: for c, d in $\mathbf{R}$,

$$[cX + dY, Z] = c\,[X, Z] + d\,[Y, Z]$$

$$[Z, cX + dY] = c\,[Z, X] + d\,[Z, Y]$$

Antisymmetry:

$$[X, Y] = -[Y, X]$$

Jacobi identity:

$$[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0$$

Another notable aspect of vector fields is their intricate link with flows and integral curves.

Flows and integral curves. Fix any reasonably nice smooth vector field on a smooth manifold. Then, intuitively, an integral curve is the path traced out by an initial point which is placed on the manifold and left to evolve according to the vector field. A flow is a smooth bijective evolution of the entire manifold, and thus a series of diffeomorphisms, in which each point moves along its respective integral curve. Concretely, flows and integral curves may be defined as follows.

Definition 4. *Given a manifold S, smooth vector field X on S, and an open interval $I \subset \mathbf{R}$, a smooth curve $\gamma : I \to S$ is an* **integral curve** *of X if*

$$\gamma'(t) = X_{\gamma(t)} \quad \textit{for all } t \in I.$$

Definition 5. *For a given manifold S, first define a flow domain D to be an open set $D \subset \mathbf{R} \times S$ for which, given any $x \in S$, the set $D^x = \{t \in \mathbf{R} \,|\, (t, x) \in D\}$ is an open interval containing 0. Then a* **(smooth) flow** *on S is a smooth* *map $\theta : D \to S$ satisfying, for all x in S*

$$\theta(0, x) = x,$$

and for all $t \in D^x$ and $s \in D^{\theta(t,x)}$ such that $t + s \in D^x$,

$$\theta(s, \theta(t, x)) = \theta(t + s, x).$$

Integral curves and flows are called *maximal* if their domains cannot be made larger. Both diffeomorphisms of the manifold and integral curves may be extracted from a flow. Indeed, for each t in $\mathbf{R}$ such that it is defined, the map $\theta_t : S \to S$ given by

$$\theta_t(x) = \theta(t, x)$$

is a diffeomorphism. For each x in S, the curve $\theta^x : I \to S$ defined by

$$\theta^{(x)}(t) = \theta(t, x)$$

is an integral curve. For any flow, the vector field defined by $X_x = \theta^{x\prime}(0)$ is called the *infinitesimal generator of θ*. Solidifying the intuitively understood generation of flows and integral curves from vector fields described above, the following theorem establishes that every smooth vector field determines a unique maximal integral curve starting at each point, and also a unique maximal flow.

Theorem 1 (Fundamental Theorem on Flows)**.** [12, 442] *Let S be a smooth manifold, and X be a smooth vector field on S. Then there is a unique maximal smooth flow $\theta : D \to S$ whose infinitesimal generator is X. This flow has the following properties:*

1. *For each $x \in S$, the curve $\theta^{(x)} : D^x \to S$ is the unique maximal integral curve of X starting at x.*

2. *For each t in $\mathbf{R}$, the set $S_t = \{x \in S \,|\, (t,x) \in D\}$ is open in S, and $\theta_t : S_t \to S_{-t}$ is a diffeomorphism with inverse θ_{-t}.*
3. *For each $(t,x) \in D, (\theta_t)_* X_x = X_{\theta_{t(x)}}$.*

2.1.2 Other vector bundles

The tangent bundle is just one example of a vector bundle over a manifold. In general, a vector bundle on a manifold is a collection of vector spaces, one based at each point of the manifold, associated in such a way that locally their union looks like the Cartesian product of the manifold with $\mathbf{R}^n$, however globally may be twisted.

Definition 6. [12, 104] *A* **smooth vector bundle** *of rank k over a smooth manifold S is a smooth manifold E together with a surjective smooth map $\pi : E \to S$ satisfying:*

1. *for each $x \in S$, the set $E_x = \pi^{-1}(x) \subset S$, which is called the* **fibre** *of E over x, has the structure of a k-dimensional real vector space.*
2. *for each $x \in S$, there exists a neighbourhood U of x in S and a diffeomorphism $\Phi : \pi^{-1}(U) \to U \times \mathbf{R}^k$, called a* **local trivialization** *of E over U, such that for each $q \in U$, the restriction Φ to E_q is a linear isomorphism from E_q to $\{q\} \times \mathbf{R}^k \cong \mathbf{R}^k$; and such that the following diagram commutes*

$$\begin{tikzcd} \pi^{-1}(U) \arrow[r, "\Phi"] \arrow[dr, "\pi"'] & U \times \mathbf{R}^k \arrow[d, "\pi_1"] \\ & U \end{tikzcd}$$

where π_1 is the projection on the first coordinate.

The space E is called the *total space* of the bundle, S is called its *base*, and π is its *projection*. Any $E_x = \pi^{-1}(x)$ is a *fibre*. If there exists a trivialization of E over S, then E is referred to as the *trivial bundle*. In any other case, there will be more than one local trivialization on the bundle. Wherever two trivializations overlap, their composition has a simple form as explained by the following theorem.

Theorem 2. [12, 107] *Suppose $\pi : E \to S$ is a smooth vector bundle and that $\Phi : \pi^{-1}(U) \to U \times \mathbf{R}^k$ and $\Psi : \pi^{-1}(V) \to V \times \mathbf{R}^k$ are smooth local trivializations of E for which $U \cap V \neq \emptyset$. Then there exists a smooth map $\tau : U \cap V \to GL(k, \mathbf{R})$, called the* **transition function**, *for which the composition $\Phi \circ \Psi^{-1} : (U \cap V) \times \mathbf{R}^k \to (U \cap V) \times \mathbf{R}^k$ has the form*

$$\Phi \circ \Psi^{-1}(x, \boldsymbol{v}) = (x, \tau(x)\boldsymbol{v}).$$

Given two vector bundles E and E' over S and S' respectively, a *bundle map* is a smooth map $\psi : E \to E'$ which is fibre-preserving and fibre-wise linear. If the bundles E and E' share a base space then an *isomorphism* is a fibre-wise invertible bundle map between them. The group of all automorphisms of a bundle E is called the *gauge group*. It will be denoted by $\mathcal{G}(E)$.

Just as it was interesting to take a single vector from the tangent space at each point to form a vector field, the same procedure is useful on more general vector bundles. The resulting object is a *smooth section*.

Definition 7. *Let $\pi : E \to S$ be a smooth vector bundle over a smooth manifold S. Then a* **smooth section** *is a smooth map $\sigma : S \to E$ satisfying $\pi \circ \sigma = Id_S$.*

Just as the smooth vector fields formed a vector space, so too do the set of smooth sections of a generic vector bundle. According to context, and keeping with standard conventions, this vector space will be denoted either by $C^\infty(S, E)$ or $\Gamma(E)$. If the domain of the section is restricted to some open subset U of S, it is called a *local section.* A first useful application of sections is to provide local bases for vector bundles. Suppose $\pi : E \to S$ is a smooth vector bundle and the sections $\sigma_1, \ldots, \sigma_n$ of E are defined over some open $U \subset S$. These sections are *independent* if their values $\sigma_1(x), \ldots \sigma_n(x)$ are linearly independent elements of E_x for each x in U. If their values span E_x for each x in U, then they are said to *span* E. An ordered k-tuple $(\sigma_1, \ldots, \sigma_k)$ of independent local smooth sections over U that span E is a *smooth local frame* for E. As is clear from their linear independence and spanning nature, for each x in U the vectors $(\sigma_1(x), \ldots, \sigma_k(x))$ form a basis for the fibre E_x.

Cotangent bundle. A vector bundle which exists on every smooth manifold and is of significant importance is the cotangent bundle. Whereas it was shown how the tangent bundle provides a means of describing the derivative of a curve on a given manifold, the cotangent bundle allows a description of the derivative of real-valued functions defined over the manifold.

In general, for any finite-dimensional real vector space V, a *covector* is a real-valued linear functional on V. The set of all convectors forms a vector space with the same dimension as the space V. This space is called the *dual space* and is denoted V^*. Consider now a smooth manifold S. For each point x in S, there exists a *cotangent space* T_x^*S dual to T_xS. Elements of the space T_x^*S are referred to as *tangent covectors.* The disjoint union of the cotangent spaces at each point of a manifold together form the rank-n *cotangent bundle*:

$$T^*S = \coprod_{x \in S} T_x^*S.$$

A second useful application of sections now arises. A smooth section of T^*S is a *covector field*, or a (*differential*) *1-form*. These 1-forms provide the means of describing the first partial derivatives of real-valued smooth functions mentioned above. Given a smooth, real-valued function f on a smooth manifold S, there is an associated smooth 1-form $\mathrm{d}f$, called the *differential of f*, defined by

$$\mathrm{d}f_x(X_x) = X_x f \quad \text{for any } X_x \in T_x S. \tag{2.1}$$

Like the gradient, the differential of f stores the first partial derivatives in each of the tangent directions; given any tangent vector, the differential returns the first partial derivative of the function in that direction.

Dual to the pushforward described earlier for tangent vectors, there exists a pullback of covectors under smooth maps. Precisely, if $F : S \to M$ is a smooth map between manifolds, then for each $x \in S$ there exists a map $F^* : T^*_{F(x)}M \to T^*_x S$ called the *pullback* and defined by

$$F^*(\omega)(X) = \omega(F_* X), \quad \text{for all } \omega \in T^*_{F(x)} \text{ and } X \in T_x S.$$

Tensor bundles. The tangent and cotangent bundles are special cases of a larger class of vector bundles, namely the tensor bundles. The tangent and cotangent bundles are at each point comprised of linear maps. General tensor bundles are pointwise composed of multilinear ones. As vectors and convectors were the basic building blocks of the tangent and cotangent bundles, tensors are the basic elements of tensor bundles.

Definition 8. *Given a vector space V, a* **covariant k-tensor** *on V is a real-valued multilinear function of k elements of V:*

$$\omega : \underbrace{V \times \cdots \times V}_{k \text{ copies}} \to \mathbf{R}.$$

A k-tensor is said to be of *rank* k. The collection of all rank k tensors form a vector space which will be denoted by $T^k(V)$. Tensors of different ranks can be combined via the tensor product to form new tensors. For a vector space V and tensors ω, η in $T^k(V), T^l(V)$ respectively, the *tensor product* of ω and η is the covariant $(k+l)$-tensor

$$\omega \otimes \eta : \underbrace{V \times \cdots \times V}_{k+l \text{ copies}} \to \mathbf{R}$$

given by

$$\omega \otimes \eta(\mathbf{v}_1, \ldots, \mathbf{v}_{k+l}) = \omega(\mathbf{v}_1, \ldots, \mathbf{v}_k)\, \eta(\mathbf{v}_{k+1}, \ldots, \mathbf{v}_{k+l}).$$

With the tensor product, a basis for $T^k(V)$ may easily be described.

Theorem 3. [12, 262] *Let V be a real vector space of dimension n and let (e^i) be its dual basis. The set of all covariant k-tensors of the form $e^{i_1} \otimes \cdots \otimes e^{i_k}$ for $1 \leq i_1 \leq \cdots \leq i_k \leq n$ is a basis for $T^k(V)$.*

That every k-tensor can be written as a linear combination of tensor products of convectors prompts a diversion to an alternative formulation of $T^k(V)$ in terms of tensor products of vector spaces.

Definition 9. *Given vector spaces $V_1, \ldots, V_k$, their* **tensor product** $V_1 \otimes \cdots \otimes V_k$ *is the set*

$$V_1 \otimes \cdots \otimes V_k = \{\textit{multilinear maps } \varphi : V_1^* \times \cdots \times V_k^* \to \mathbf{R}\},$$

which is a vector space.

So each space $T^k(V)$ is the tensor product of k copies of V^*:

$$T^k(V) = \underbrace{V^* \otimes \cdots \otimes V^*}_{\text{n}}.$$

The above definitions may now be expanded to manifolds. The *bundle of covariant k-tensors* on a smooth manifold S is

$$T^k S = \coprod_{x \in S} T^k(T_x S).$$

Note, the bundle of covariant 1-tensors is the cotangent bundle. Smooth sections of tensors bundles are *smooth tensor fields*. The tensor product of vector bundles is the tensor product of the vector spaces at each point. That is, given a manifold S and smooth vector bundles $E^1, \ldots, E^k$ over S, their tensor product is

$$E^1 \otimes \cdots \otimes E^k = \coprod_{x \in S} E^1_x \otimes \cdots \otimes E^k_x. \tag{2.2}$$

An example which will appear later is the *endomorphism bundle* $\mathrm{End}(E)$ associated to a bundle E over S. This bundle, which at every point x contains the endomorphisms of E_x, may be written using the tensor product as $\mathrm{End}(E) = E \otimes E^*$.

Alternating tensors and differential forms. Amongst tensors, the alternating tensors are a noteworthy class. *Alternating tensors* are those tensors for which their value changes sign whenever two arguments are switched. Alternating k-tensors are also referred to as *k-covectors*. As their name suggests, they are the natural extension of covectors to higher rank tensors. For a given space V, the subset of $T^k(V)$ consisting of all alternating tensors is denoted by $\Lambda^k(V)$. Accordingly, for a smooth manifold S, the subset of $T^k S$ comprised of the alternating tensors is $\Lambda^k S$:

$$\Lambda^k S = \coprod_{x \in S} \Lambda^k(T_x S).$$

For any vector space V there exists a projection $\mathrm{Alt} : T^k(V) \to \Lambda^k(V)$ given by

$$\mathrm{Alt}\,\omega(\mathbf{v}_1, \ldots, \mathbf{v}_k) = \frac{1}{k!} \sum_{\sigma \in \mathrm{Sym}_k} (\mathrm{sgn}\ \sigma)\, \omega(\mathbf{v}_{\sigma(1)}, \ldots, \mathbf{v}_{\sigma(k)}), \tag{2.3}$$

where Sym_k is the group of all permutations of k elements.

Alternating tensors are of particular importance to smooth manifold theory because they provide manifolds with objects which can be integrated over. These objects are the smooth sections of $\Lambda^k S$, called *differential k-forms.* The vector space of all such sections is denoted $\Omega^k(S)$. Note the 0-forms are simply smooth functions. The differential 1-forms $\mathrm{d}f$ of smooth functions f already discussed above in the paragraph titled Cotangent bundles are a subset of $\Omega^1(S)$ called the *exact* 1-forms. This subset will be generalized to higher k-forms shortly.

Two important manipulations of forms are the wedge product and the exterior derivative. The wedge product provides a means of combining forms of different rank, while the exterior derivative generalizes the differential operator d already defined in equation 2.1 for smooth functions.

Definition 10. *If ω, η are elements of $\Lambda^k(V), \Lambda^l(V)$ respectively, then the* **wedge product** *of ω and η is the alternating $(k+l)$-tensor*

$$\omega \wedge \eta = \frac{(k+l)!}{k!l!} \mathrm{Alt}(\omega \otimes \eta). \tag{2.4}$$

Definition 11. [12, 306] *Given a smooth manifold S, the* **exterior derivative** *is the unique linear map $d : \Omega^k(S) \to \Omega^{k+1}(S)$ which satisfies the following properties:*

1. *if f is a smooth real-valued function, the df is the differential of f*

2. *if* ω, η *in* $\Omega^k(S), \Omega^l(S)$ *respectively, then*

$$d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^k \omega \wedge d\eta$$

3. $d \circ d = 0$.

Two important subsets of $\Omega^k(S)$ may now be defined. As already seen in the case when $k = 1$, the *exact* k-forms are those forms ω such that $\omega = \mathrm{d}\eta$ for some $(k-1)$-form η. The *closed* forms are those forms ω such that $\mathrm{d}\omega = 0$.

Symplectic structure. A special type of 2-form which will become of particular interest in Chapter 5 covering Goldman flows are the symplectic forms.

Definition 12. *Given a smooth manifold S, a* **symplectic form** *is a closed, nondegenerate 2-form.*

By fixing a symplectic form ω, one endows a smooth manifold with a *symplectic structure*. In this case, the manifold is referred to as a *symplectic manifold*. For any symplectic manifold (S, ω) and smooth function H in $C^\infty(S)$ there exists a particularly interesting smooth vector field. This is the *Hamiltonian vector field* X_H, defined as the vector field satisfying

$$\omega(X_H, Y) = dH(Y) = YH$$

for any vector field Y on S. The function H is referred to as the *Hamiltonian*, and the maximal smooth flow associated to the vector field X_H is the *Hamiltonian flow*.

2.2 Classifying properties

There are several basic properties by which smooth manifolds may be described. For instance, as topological spaces, manifolds may be *compact*, *connected*, and/or *path connected*. Two other important properties of a manifold are the possible existence of boundary and an orientation.

Boundary. The definition of an n-manifold stated above may be slightly expanded to account for manifolds with boundary. This is done by considering *upper half-space*

$$\mathbf{H}^n = \{(x^1, \ldots, x^n) \in \mathbf{R}^n \,|\, x^n \geq 0\}$$

rather than the whole space $\mathbf{R}^n$ itself.

Definition 13. *An* **n-manifold with boundary** *is a second countable, Hausdorff topological space such that every point of the space has a neighbourhood homeomorphic to an open subset of* $\mathbf{H}^n$*.*

The coordinate charts explained above are modified accordingly. Points x on the manifold are *boundary points* if their image under coordinate charts are in $\partial\mathbf{H}^n = \{(x^1, \ldots, x^n) \in \mathbf{R}^n \,|\, x^n = 0\}$, while those points with images in $\mathrm{Int}\mathbf{H}^n = \{(x^1, \ldots, x^n) \in \mathbf{R}^n \,|\, x^n > 0\}$ are *interior points*. Compact manifolds without boundary are referred to as *closed*.

Orientation. While orientability is a topological property, it is more easily defined for smooth manifolds. Since the concern here is with smooth manifolds, this route will be taken. Suppose S is a smooth manifold. The tangent space T_x at every point x of the manifold can be given an orientation. In order for these pointwise orientations to have any meaning over the entire manifold, they mush fit together in a certain way. For such a pointwise orientation, a local frame (σ_i) [see the paragraph following definition 7] over some open set U is (positively) *oriented* if $(\sigma_1|_x, \ldots, \sigma_n|_x)$ is a positively oriented basis for each T_xS according to the pointwise orientation at each x in U. If every point of S is in the domain of some such oriented local frame, the pointwise orientation is said to be *continuous*. An *orientation* of S is a continuous pointwise orientation. If a pointwise orientation exists for a manifold, the manifold is said to be *orientable*. If no pointwise orientation exists, it is *nonorientable*. Examples of nonorientable surfaces include the Möbius strip, the Klein bottle, and the real projective plane $\mathbf{R}P^2$.

2.3 Homotopy and the fundamental group

Paths on manifolds provide a precise way of encoding essential aspects of their structure. A *path* in S from x_0 to x_1 is a continuous function from $[0,1] \to S$ such that $f(0) = x_0$ and $f(1) = x_1$. Similar paths may be related through path homotopy.

Definition 14. *Two paths f and g in S are said to be homotopic if they have the same initial and terminal points and if there is a continuous map $H : [0,1] \times [0,1] \to S$ such that*

$$H(t,0) = f(t) \quad \textit{and} \quad H(t,1) = g(t),$$
$$H(0,x) = x_0 \quad \textit{and} \quad H(1,x) = x_1.$$

Intuitively, two paths are homotopic if one can continuously be dragged through the manifold to the other, while keeping the end points fixed. The relation given by path homotopy is an equivalence relation. Considering only the paths with fixed initial and terminal point x_0, the homotopy classes of such paths then form a group with concatenation of paths as the group product. This group is called the *fundamental group* of the manifold with base point

x_0. While fixing a base point is necessary to define the fundamental group, the group is independent of this point. It will therefore simply be denoted by $\pi_1(S)$ without reference to the base point. The fundamental group is a topological invariant, meaning that homeomorphic manifolds have the same fundamental groups.

2.3.1 Closed orientable surfaces

Amongst manifolds, the closed orientable surfaces receive considerable attention. These surfaces are classified according to their *genus*, which may be defined concretely in terms of the Euler characteristic χ via the equation $\chi = 2 - 2g$, or intuitively in terms of the number of 'handles' of the surface. For example, see the genus 2 surface in figure 2–2. A closed orientable surface of genus g will be denoted by Σ_g and generic closed oriented surface will be denoted by Σ. The fundamental groups of such surfaces have a particular

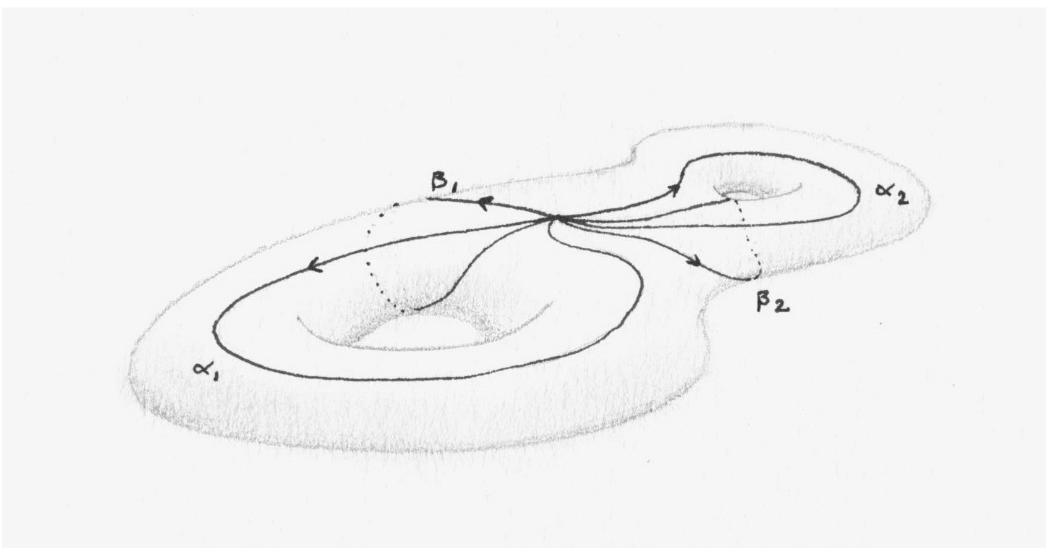

Figure 2–2: Genus 2 surface with generators $\alpha_1, \beta_1, \alpha_2$, and β_2 of its fundamental group.

form:

$$\pi_1(\Sigma_g) = \langle \alpha_1, \beta_1, \ldots, \alpha_g, \beta_g \mid \prod_{i=1}^{g} [\alpha_i, \beta_i] = \mathrm{Id} \rangle.$$

These groups, frequently referred to as *surface groups*, will play an important role in later chapters.

CHAPTER 3
Lie groups

One class of smooth manifolds are Lie groups. These manifolds are characterized by being groups in addition to being smooth manifolds. As in Chapter 2, the brief introduction to Lie groups provided here is guided by Lee [12].

Definition 15. *A* ***Lie group*** *G is a group which is also a smooth manifold and for which the group operation map $o : G \times G \to G$ and inversion map $i : G \to G$, given respectively by*

$$o(g, h) = gh \quad \text{and} \quad i(g) = g^{-1},$$

are both smooth.

One set of examples are the general linear groups $Gl_n(\mathbf{R})$ of invertible $n \times n$ matrices with matrix multiplication as the group operation. Another set of examples which will be of particular importance later in the thesis are the complex, special unitary matrices SU_n. It is shown in Lee [12, 38($Gl_n(\mathbf{R})$); 215(SU_2)] that these groups satisfy the necessary properties. An important class of diffeomorphisms which arise on Lie groups are the left translations. For any $g \in G$, a *left translation* is the map $L_g : G \to G$ defined by $L_g(h) = gh$ for all h in G. Smoothness of the group operation map ensures that these are indeed smooth maps. A vector field X on G is said to be *left-invariant* if it is invariant under all such left translations. That is, more precisely, if for all $g, h \in G$

$$L_{g_*}(X_h) = X_{gh}.$$

Note, this pushforward of the vector field is well-defined since L_g is a diffeomorphism. The set of all left-invariant vector fields forms a real vector subspace of $\mathfrak{X}(G)$ called the *Lie algebra* and denoted by $\mathfrak{g}$. Importantly, the Lie algebra is closed under the Lie bracket defined earlier in definition 3.

The map defined by associating to each vector field in $\mathfrak{g}$ its tangent vector at the group's identity is a vector space isomorphism – see [12, 95]. Therefore the Lie algebra can alternatively be viewed as the identity's tangent space. Within this dual view, there is a canonical smooth map $\exp : \mathfrak{g} \to G$. For any $X \in \mathfrak{g}$, letting $\gamma(t)$ be the integral curve of the smooth vector field X starting at the identity, this map is defined by $\exp X = \gamma(1)$. Simply put, this map takes any tangent direction at the identity, perturbs the identity element in this direction, lets it flow along the integral curve of the corresponding left-invariant vector field, and returns the group element reached after one unit of time. This map is called the *exponential map* and the notation is further generalized so that $\exp tX = \gamma(t)$. The map gets its name from the fact that in the case of matrix Lie groups it is given by the matrix exponential:

$$\exp tX = \sum_{n=0}^{\infty} \frac{1}{n!}(tX)^n,$$

where X is a matrix in $\mathfrak{g}$.

Considering now two Lie groups G and H, of particular interest are the smooth maps between them which preserve their group structure. Such maps are called *Lie group homomorphisms.*

Definition 16. *A* **Lie group homomorphism** *from G to H is a smooth map $F : G \to H$ which is also a group homomorphism.*

3.1 The Lie groups SU_n

The Lie groups which appear in the questions posed in Chapter 6 are the groups SU_n of special unitary matrices. For this reason, basic properties of these Lie groups will be established here.

A complex, square matrix U is unitary if any of the following equivalent conditions hold

1. $U^*U = UU^* = I$, where U^* is the conjugate transpose of U
2. $U^*U = UU^*$ and the eigenvalues of U lie on the unit circle
3. U is invertible with $U^{-1} = U^*$.

For each n, the group SU_n is the group of n-dimensional unitary matrices having determinant 1. Matrix multiplication is the group operation. It is a real compact Lie group of dimension $n^2 - 1$.

3.1.1 Further description of SU_2

The work discussed in Chapter 7 involves, in particular, the group SU_2. This group of matrices SU_2 may be described explicitly as

$$SU_2 = \left\{ \begin{pmatrix} \alpha & -\overline{\nu} \\ \nu & \overline{\alpha} \end{pmatrix} \mid \alpha, \nu \in \mathbf{C}, |\alpha|^2 + |\nu|^2 = 1 \right\}$$

and the Lie algebra of SU_2 as

$$\mathfrak{su}_2 = \left\{ \begin{pmatrix} ix & -\overline{\rho} \\ \rho & -ix \end{pmatrix} \mid x \in \mathbf{R},\ \rho \in \mathbf{C} \right\}.$$

Matrices in the form of those found in the Lie algebra have zero trace and are antihermitian, that is $U^* = -U$.

3.1.2 Correspondence between SU_2 and SO_3

The Lie groups SU_2 and SO_3 are related through a two-to-one surjective Lie group homomorphism of SU_2 onto SO_3. This map can be seen through a representation of SU_2 and $\mathfrak{su}_2$ in terms of unit quaternions.

The norm of a quaternion $q = a + bi + cj + dk$ with $a, b, c, d \in \mathbf{R}$ is $\|q\| = \sqrt{q\overline{q}} = \sqrt{a^2 + b^2 + c^2 + d^2}$. Here $\overline{q}$ is the conjugate of q defined to be $\overline{q} = a - bi - cj - dk$. A unit quaternion is a quaternion of norm 1. The group of all unit quaternions is the set

$$\mathfrak{Q} = \{a + bi + cj + dk \mid a^2 + b^2 + c^2 + d^2 = 1\}$$

along with quaternion multiplication as the group operation. There is an isomorphism between SU_2 and the unit quaternions given by

$$\begin{pmatrix} a+bi & c+di \\ -c+di & a-bi \end{pmatrix} \quad \mapsto \quad a + bi + cj + dk. \tag{3.1}$$

Using the same map, the Lie algebra $\mathfrak{su}_2$ can be associated with the vector space $\mathfrak{I}$ of all purely imaginary quaternions. There exists also a bijective linear map between the purely imaginary quaternions and vectors in $\mathbf{R}^3$ in which the quaternion $bi + cj + dk$ and vector (b, c, d) are associated. Taking the composition of these maps, $\mathfrak{su}_2$ may be viewed as $\mathbf{R}^3$.

Representing SU_2 and $\mathfrak{su}_2$ in terms of quaternions allows for a simple description of the two-to-one surjective Lie group homomorphism of SU_2 onto SO_3. Recall, SO_3 is the Lie group consisting of all rotations of $\mathbf{R}^3$ about the origin, with composition the group operation. In brief, the homomorphism is attained by representing SU_2 as linear transformations of $\mathfrak{su}_2$. This representation is an instance of the more general adjoint representation of a Lie group G as linear transformations of its Lie algebra $\mathfrak{g}$.

Each element q of $\mathfrak{Q} \cong SU_2$ may be represented as a linear transformation of the purely imaginary quaternions, equivalently $\mathfrak{su}_2$ or $\mathbf{R}^3$, via the map $R : \mathfrak{Q} \cong SU_2 \to \mathrm{End}(\mathfrak{I})$ given by quaternion conjugation:

$$R_q(\omega) = q\omega\overline{q}, \quad \omega \in \mathfrak{I}.$$

This map is a homomorphism since $R_{q_1}R_{q_2}(\omega) = q_1q_2\omega\overline{q_1q_2} = R_{q_1q_2}(\omega)$ for all $q_1, q_2 \in \mathfrak{Q}$. From the definition, one easily sees that each R_q is an isometry fixing $\omega = 0$. In addition, it can be shown that each R_q is orientation preserving. Therefore, viewing the purely imaginary quaternions as $\mathbf{R}^3$, it follows that each R_q is a rotation of $\mathbf{R}^3$ about the origin, and hence an element of SO_3. The axis of rotation is the ray through $\mathrm{Im}(q)$ and the origin.

Since the group operation in SU_2 is smooth, the homomorphism $R : SU_2 \to SO_3$ is also a smooth map and is thus a Lie group homomorphism. Finally, the map R is surjective onto SO_3 and, since $R_{\pm q}$ are easily seen to be the same rotation of S^2, we find that R is a two-to-one surjecive Lie group homomorphism from SU_2 onto SO_3.

As a concrete example, the rotation of $\mathbf{R}^3$ associated to $q = \cos\theta + i\sin\theta$ is calculated here. The map R_q fixes i:

$$\begin{aligned} i \mapsto & (\cos\theta + i\sin\theta)i(\cos\theta - i\sin\theta) \\ & = (\cos\theta + i\sin\theta)(\sin\theta + i\cos\theta) \\ & = i, \end{aligned}$$

brings

$$\begin{aligned} j \mapsto (\cos\theta + i\sin\theta)&j(\cos\theta - i\sin\theta) \\ &= (\cos\theta + i\sin\theta)(j\cos\theta + k\sin\theta) \\ &= j\cos^2\theta + 2k\sin\theta\,\cos\theta - j\sin^2\theta \\ &= (\cos 2\theta)j + (\sin 2\theta)k, \end{aligned}$$

and, using similar trigonometric identities, brings

$$k \mapsto\ -(\sin 2\theta)j + (\cos 2\theta)k.$$

Therefore the transformation of the yz-plane under R_q may be described via the matrix multiplication

$$(y, z) \mapsto \begin{pmatrix} \cos 2\theta & -\sin 2\theta \\ \sin 2\theta & \cos 2\theta \end{pmatrix} \begin{pmatrix} y \\ z \end{pmatrix},$$

and it is clear that R_q is a rotation of S^2 about the x axis by an angle of 2θ.

CHAPTER 4
Representation variety

The discussion of manifolds and Lie groups becomes focused in Chapter 4 on the theory of representation varieties. In the last section of the previous chapter, a representation of SU_2 as orthogonal linear transformations of $\mathbf{R}^3$ was outlined. In general, a *representation* of a group G into a vector space V is a homomorphism $\varphi : G \to GL(V)$, where $GL(V)$ is the group of invertible linear transformations of V. For a fixed G and V, the collection of all possible representations forms an object called the representation variety. Whereas the example from the previous chapter provides a representation of the Lie group SU_2, in this chapter, the concern will be with representations into Lie groups. The groups represented will be surface groups. Before the representation variety is discussed further several notions related to representations are presented.

From the definition of a representation stated above one sees that representations can be thought of in terms of actions on the vector space V. If there is a subspace of V, call it W, such that the action induced by the representation fixes W, then there is an induced sub-representation $\varphi_W : G \to GL(W)$ given by the restriction of the image of φ to its action on W. In this case the representation is said to be *reducible*. Conversely, if no such subspace exists, the representation is said to be *irreducible*. If the action on the vector space is discrete, that is the orbit of any point in the vector space under the representation of the group has no accumulation points, the representation is said to be *discrete*. As a final definition concerning representations, if the representation φ is an isomorphism onto its image, the representation is said to be *faithful*.

4.1 Representation variety

A *real analytic variety* is a set $V \subset \mathbf{R}^n$ such that for every x in V there exists a neighbourhood U of x in $\mathbf{R}^n$ and real analytic functions $f_1, \ldots, f_m$ defined in U such that $U \cap V = \{z \mid f_k(z) = 0 \text{ for all } 1 \leq k \leq m\}$. Except for a collection of ill-behaved singular points, analytic varieties are locally real manifolds. Points of V for which there exists a local neighbourhood U such that $V \cap U$ is a manifold are called *regular*. The set of all regular points is denoted by V^-.

For any connected Lie group G consider the space $\mathrm{Hom}(\pi_1, G)$ of all representations of a surface group $\pi_1(\Sigma)$ into G. This space may be topologized with the *compact-open topology* by specifying that a sequence of representations $\rho_n : \pi_1 \to G$ converges to $\rho \in \mathrm{Hom}(\pi_1, G)$ iff each sequence $\rho_n(\gamma)$ converges to $\rho(\gamma)$ for all $\gamma \in \pi_1$, equivalently iff each $\rho_n(\gamma_i)$ converges to $\rho(\gamma_i)$ for $\gamma_i, i \in I$ a set of generators of π_1. Given this topology the space is a real analytic variety. The group G acts by inner automorphisms on $\mathrm{Hom}(\pi_1, G)$. Using this action to define an equivalence relation, one may construct the moduli space of representations $\mathrm{Hom}(\pi_1, G)/G$. Such a space $\mathrm{Hom}(\pi_1, G)/G$ is called a *representation variety*. They will be the primary concern of the following sections, however, to situate this discussion within the field, it is worthwhile to mention several closely related spaces.

Related spaces

One space related to the representation variety is the character variety. When G is a matrix group, two representations ρ_1 and ρ_2 of π_1 in G may be said to define the same character if $\mathrm{Tr}(\rho_1(\gamma)) = \mathrm{Tr}(\rho_2(\gamma))$ for all $\gamma \in \pi_1$. This defines an equivalence relation on the representations and the quotient of $\mathrm{Hom}(\pi_1, G)$ by this equivalence relation produces what is called the *character* *variety*. Since conjugate representations necessarily have the same character, there exists a projection of the representation variety onto the character variety.

Another related space comes from considering the action of the outer automorphism group $\mathrm{Out}(\pi_1)$ on $\mathrm{Hom}(\pi_1, G)/G$. This action is well defined since the action of inner automorphisms of π_1 on $\mathrm{Hom}(\pi_1, G)$ performs the same manipulations as the G actions, rendering the action of $\mathrm{Inn}(\pi)$ on $\mathrm{Hom}(\pi_1, G)/G$ trivial. *Teichmüller space*, denoted by $\mathfrak{T}_\Sigma$, is the component of discrete, faithful representations in $\mathrm{Hom}(\pi_1(\Sigma), PSL_2(\mathbf{R}))/PSL_2(\mathbf{R})$. In this case, $\mathrm{Out}(\pi_1)$ acts discontinuously and the resulting moduli space $\mathfrak{T}_\Sigma/\mathrm{Out}(\pi_1)$ is the *Riemann moduli space of all complex structures on* Σ.

4.1.1 Representation variety as the space of flat G-connections

Given any surface Σ and Lie group G the associated representation variety may alternatively be formulated in terms of flat G-connections on Σ. While this framework is potentially initially less palatable than representation theory, it provides a more transparent description of the representation variety's symplectic nature, as will be seen later. Before this reformulation is presented, the definitions and basic properties of the necessary structures, namely G-structures, connections, and finally G-connections, are given. This material is based on the lecture notes [11] by François Labourie which provide a more in-depth construction of the representation variety in these terms as well as preliminary material.

G-structures. A given manifold S and Lie group G may be coupled through the consideration of specific structures on a restricted class of vector bundles over S. A vector bundle E over S is said to have a *G-atlas* if there exists a set of trivializations such that the transition functions take values in G. Any two G-atlases are equivalent if their union is again a G-atlas. A

G-structure is an equivalence class of G-atlases. Elsewhere in the literature, G-structures are referred to as *reductions of the structure group to* G. A bundle E together with a G-structure is a G*-bundle*. For a given G-bundle E of over S, it is natural to restrict the gauge group $\mathcal{G}(E)$—that is the group of all bundle automorphisms—to those automorphisms which fibre-wise may be described via elements of G. Due to the canonical association of the gauge group and sections of $\mathrm{End}(E)$ the gauge group may be viewed as $C^\infty(S, G)$. It is the connections defined on these bundles and coherent with these structures which are used to define the representation variety.

Connections and holonomy. Formally, a connection ∇ on the vector bundle E over a manifold S is defined to be a linear map

$$\nabla : \Gamma(E) \to \Gamma(E \otimes T^*S) = \Omega^1(S, E)$$

such that given any vector field X and smooth function f on S the Leibniz rule is satisfied:

$$\nabla_X(f\sigma) = \mathrm{d}f(X) \cdot \sigma + f\, \nabla_X(\sigma),$$

where ∇_X is defined to be the map from $\Gamma(E)$ to itself given by

$$\nabla_X(\sigma) = \nabla(\sigma)(X).$$

In coordinates connections take on the form $\nabla = \mathrm{d} + \Gamma$ with Γ an endomorphism valued 1-form. An important property of each connection is its curvature, defined to be the map

$$R^\nabla : TS \times TS \to \mathrm{End}(E)$$

given by

$$R^\nabla(X, Y)(\sigma) = \nabla_X \nabla_Y(\sigma) - \nabla_Y \nabla_X(\sigma) - \nabla_{[X,Y]}(\sigma)$$

for $X, Y \in \Gamma(TS)$ and $\sigma \in \Gamma(E)$. If R^∇ is identically zero the connection is said to be *flat*.

Later, it will be necessary to consider how connections are pulled back from one bundle to another in coherence with a diffeomorphism between them. First it is necessary to define the pullback of a vector bundle. Let S and M be smooth manifolds and $F : S \to M$ a smooth map between them. For any bundle $\pi : E \to M$ its *pullback bundle* over S is constructed by choosing the fibre at each point $x \in S$ to be $E_{F(x)}$. If F is restricted to diffeomorphisms then any connection ∇ on E may also be pulled back, in particular to the unique connection $F^*\nabla$ which satisfies

$$(F^*\nabla)_X(F^*\sigma) = F^*(\nabla_{F_*(X)}(\sigma))$$

for all $\sigma \in \Gamma(E)$ and $X \in TS$. This connections is called the *pullback connection*. Given a manifold S and vector bundle E, two connections ∇^1 and ∇^2 on E are said to be *gauge equivalent* if there is some element F of the gauge group $\mathcal{G}(E)$ for which $\nabla^1 = F^*(\nabla^2)$. In this case the automorphism of the bundle effects in a nontrivial way the pullback of bundle sections.

Connections, as their name hints, provide a means of transferring information between the fibres at unique points on the manifold. In particular, with a connection one has a means of describing the infinitesimal change of sections over the bundle. More precisely, if σ is a section of E along a path $\gamma(t)$ in S, the derivative of σ along $\gamma(t)$ is given by

$$\nabla_{\dot{\gamma}}(\sigma)$$

where $\dot{\gamma}$ is the tangent vector field along the curve γ. Of importance are the sections σ along γ satisfying

$$\nabla_{\dot{\gamma}}(\sigma) = 0.$$

Such sections are said to be ∇-*parallel along* γ, and are uniquely determined by specifying the initial vector $\sigma(0)$. Let the section uniquely determined by $\sigma(0)$ be denoted by $\sigma_{\sigma(0)}(t)$. By means of this section, ∇ associates to each path γ in S beginning at $\gamma(0)$ and ending at $\gamma(1)$ the linear map

$$\begin{aligned}\mathrm{Hol}^{\nabla}(\gamma) : E_{\gamma(0)} &\to E_{\gamma(1)}\\ \sigma(0) &\mapsto \sigma_{\sigma(0)}(1).\end{aligned}$$

This map $\mathrm{Hol}^{\nabla}(\gamma)$ is referred to as the *holonomy* of ∇ along γ. While for an arbitrary connection there may be many holonomy maps between two fibres $E_{\gamma(0)}$ and $E_{\gamma(1)}$ coming from the plethora of possible paths connecting $\gamma(0)$ and $\gamma(1)$ in E, for flat connections the situation is more simple. When ∇ is flat, the holonomy of ∇ along γ depends only on the homotopy class of γ. Therefore, for a fixed vector bundle E and connection ∇ on E, holonomy may be used to define the group homomorphism

$$\begin{aligned}\pi_1(\Sigma, x_0) &\to GL\left(E_{x_0}\right)\\ [\gamma] &\mapsto \mathrm{Hol}^{\nabla}(\gamma).\end{aligned}$$

This homomorphism is referred to as the *holonomy homomorphism of* ∇.

G-connections and the representation variety. The discussion on G-structures and connections may now be combined to speak of G-connections. When considering a G-bundle one can look for those connections which preserve the G-structure. Such G-connections ∇ are those for which when written in coordinates $\nabla = \mathrm{d}+\Gamma$, the 1-form Γ takes values in $\mathfrak{g}$. When a G-connection is moreover flat, then the associated holonomy homomorphism defined above takes values in G. What is more, when two flat G-connections are gauge equivalent their holonomy homomorphisms are conjugate by an element of G. Thus the link with the representation variety becomes apparent; given a surface Σ and Lie group G there is associated to every class of gauge equivalent flat G-connections on a G-bundle over Σ a class of representations in $\mathrm{Hom}(\pi_1, G)$ given by the holonomy homomorphism as described above. The converse holds also, and therefore the representation variety may be viewed either as the space of flat G-connections modulo action by the gauge group or as the space of group homomorphisms of π_1 into G modulo conjugation by G:

$$\{\text{flat } G\text{-connections}\}/\{\text{gauge group}\} = \mathrm{Hom}(\pi_1, G)/G.$$

It should be noted that although there are singular points on the representation variety the discussion from here on will consider only the regular points. Further information on the singular points of the representation variety may be found in [9].

4.1.2 Natural Symplectic structure of the representation variety

Having understood the representation variety, its symplectic structure may now be explored. In advance of this, there is yet one more step: to understand the local structure. The local structure will be developed within both frameworks of the representation variety outlined above. In both cases the tangent spaces appear as a first cohomology groups.

Tangent spaces in the representation variety as a moduli space of representations. When $\mathrm{Hom}(\pi_1, G)/G$ is viewed as a moduli space of representations, the tangent space to an equivalence class of representations may be expressed in terms of group cohomology. Before the tangent space is discussed, a brief summary of group cohomology is provided.

For a given group H and H-module A for which H acts on A as automorphisms, an associated cochain complex may be defined as follows. For each $n \geq 0$, the group of all functions ϕ from $H^n \to A$ are the n-cochains and the map d^n from n-cochains to $(n+1)$-cochains defined by

$$\begin{aligned}(\mathrm{d}^n\phi)(h_1,\ldots,h_{n+1}) &= h_1\cdot\phi(h_2,\ldots,h_{n+1})\\ &+\sum_{i=1}^{n}(-1)^i\phi(h_1,\ldots,h_{i-1},h_ih_{i+1},h_{i+2},\ldots,h_{n+1})\\ &+(-1)^{n+1}\phi(h_1,\ldots,h_n)\end{aligned}$$

is a coboundary homomorphism. Indeed a routine calculation verifies that $\mathrm{d}^{n+1}\circ\mathrm{d}^n = 0$. Computing cohomology in the standard way, the group of n-cycles, $n\geq 0$, is $Z^n(H,A) = \mathrm{Ker}(\mathrm{d}^n)$ and the group of n-coboundaries is $B^0(H,A) = 0, B^n(H,A) = \mathrm{Im}(\mathrm{d}^{n-1}), n\geq 1$. The n^{th} cohomology group is $H^n(H,A) = Z^n(H,A)/B^n(H,A)$. In particular, a 1-cocycle (also called a crossed homomorphism) is a function $\phi : H\to A$ such that for any $h_1,h_2\in H$

$$h_1\cdot\phi(h_2)-\phi(h_1h_2)+\phi(h_1) = 0.$$

A 1-coboundary (also called a principal crossed homomorphism) is a function $\phi : H\to A$ for which there exists an $a\in A$ such that for all $h\in H$

$$\phi(h) = h\cdot a - a.$$

Now, turning to study the local structure of the representation variety $\mathrm{Hom}(\pi_1,G)/G$ the group and module which will be of interest are π_1 and the π_1-module $\mathfrak{g}_{\mathrm{Ad}\,\rho}$ in which π_1 acts on $\mathfrak{g}$ via the composition $\pi_1\xrightarrow{\rho} G\xrightarrow{\mathrm{Ad}}\mathrm{Aut}(\mathfrak{g})$.

The Zariski tangent space of a representation ρ in $\mathrm{Hom}(\pi_1,G)$ will be established first. In order to do so, fix any variation of ρ. That is a smooth path ρ_t in $\mathrm{Hom}(\pi_1,G)$ for which $\rho_0=\rho$. That ρ_t lies in $\mathrm{Hom}(\pi_1,G)$ implies each ρ_t satisfies the homomorphism condition

$$\rho_t(xy) = \rho_t(x)\rho_t(y)$$

for all $x, y \in \pi_1$. It is the associated infinitesimal variation of ρ which is of importance for understanding the tangent space. Accordingly, writing ρ_t as

$$\rho_t(x) = \exp(t\,u(x) + O(t^2))\rho(x) \tag{4.1}$$

for $x \in \pi_1$ and for t in some interval about 0, it follows that $u : \pi_1 \to \mathfrak{g}_{\mathrm{Ad}\,\rho}$ satisfies the 1-cocycle condition associated with $\mathfrak{g}_{\mathrm{Ad}\,\rho}$:

$$u(xy) = u(x) + \mathrm{Ad}\rho(x)u(y). \tag{4.2}$$

Conversely, suppose $u : \pi_1 \to \mathfrak{g}_{\mathrm{Ad}\,\rho}$ is a 1-cocycle. Then any smooth variation ρ_t satisfying (4.1) is a homomorphism to first order. Therefore the Zariski tangent space to $\mathrm{Hom}(\pi_1, G)$ at ρ is precisely the space $Z^1(\pi_1, \mathfrak{g}_{\mathrm{Ad}\,\rho})$.

Since the desired object is the tangent space to the equivalence class $[\rho]$ in $\mathrm{Hom}(\pi_1, G)/G$ rather than that of ρ in $\mathrm{Hom}(\pi_1, G)$ it must now be determined which infinitesimal variations of ρ are trivial when the moduli space is considered. These are precisely those corresponding to variations of ρ which may be written as

$$\rho_t(x) = g_t^{-1}\rho(x)g_t, \tag{4.3}$$

where g_t is a smooth path in G with $g_0 = \mathrm{Id}$. Assuming g_t may be written as

$$g_t = \exp(t\,u_0 + O(t^2))$$

for some interval of t about 0, it follows that the 1-cocycle u corresponding to ρ_t satisfies

$$u(x) = Ad\rho(x)u_0 - u_0. \tag{4.4}$$

In other words, u is the coboundary of u_0. Working in the other direction, each coboundary corresponds to some smooth variation ρ_t which is trivial in $\mathrm{Hom}(\pi_1, G)/G$. Therefore, the Zariski tangent space to $\mathrm{Hom}(\pi_1, G)/G$ at $[\rho]$

may be represented as the cohomology group $H^1(\pi_1, \mathfrak{g}_{\mathrm{Ad}\,\rho})$. Details of this correspondence are worked out by Goldman in [8].

Tangent spaces of the representation variety as a moduli space of flat G-connections. When $\mathrm{Hom}(\pi_1, G)/G$ is viewed as the space of flat G-connections on Σ modulo gauge equivalence the tangent space to an equivalence class of flat connections may be expressed in terms of twisted de Rham cohomology. As above, a general outline of this variant of cohomology will be given before the tangent space is discussed.

Twisted de Rham cohomology is defined in terms of a given bundle $\pi : E \to S$ and flat connection ∇ on E. The n-cochains are the bundle valued n-forms $\Omega^n(S, E)$ and the coboundary homomorphisms are linear differential operators

$$\mathrm{d}^n_\nabla : \Omega^n(S, E) \to \Omega^{n+1}(S, E)$$

satisfying the following two properties:

- If $\omega \in \Omega^p(S)$ and $\eta \in \Omega^q(S, E)$, then

  $$\mathrm{d}^{p+q}_\nabla(\omega \wedge \eta) = \mathrm{d}^p\omega \wedge \eta + (-1)^p \omega \wedge \mathrm{d}^p_\nabla \eta$$

  where d^p is the untwisted de Rham differential operator.

- If $\eta \in \Omega^0(S, E) \cong \Gamma(E)$ then

  $$\mathrm{d}^0_\nabla(\eta) = \nabla(\eta).$$

Such operators d^n_∇ exist and are unique. In the case $n = 1$ the operator is defined by

$$\mathrm{d}^1_\nabla(\omega)(X, Y) = \nabla_X(\omega(X)) - \nabla_Y(\omega(X)) - \omega([X, Y]) \tag{4.5}$$

for any vector fields X, Y on S. Again following the standard definitions, the group of n-cycles, $n \geq 0$, is the closed forms $Z^n_\nabla(S) = \mathrm{Ker}(\mathrm{d}^n_\nabla)$, the group of

n-coboundaries is the exact forms $B^0_\nabla(S) = 0, B^n_\nabla(S) = \mathrm{Im}(\mathrm{d}^{n-1}_\nabla), n \geq 1$, and the n^{th} ∇-twisted de Rham cohomology group of S is $H^n_\nabla(S) = Z^n_\nabla(S)/B^n_\nabla(S)$.

The local structure of the representation variety may now be described. Consider any point in the representation variety, and hence any equivalence class $[\nabla]$. The tangent space must be comprised of the infinitesimal variations of some representative ∇ for which the flatness condition is preserved. To account for the quotient of G-connections by gauge equivalence the tangent space will be the quotient of such variations by the subspace of variations which are coming from gauge transformations.

Any connection on the same bundle as ∇ can be written in coordinates in the form $\nabla + \Gamma$, with Γ in $\Omega^1(\Sigma, \mathrm{End}(E))$. So varying ∇ inside the space of smooth connections is the same as choosing a smooth path Γ_t in $\Omega^1(\Sigma, \mathrm{End}(E))$ for which Γ_0 is trivial, and letting $\nabla_t = \nabla + \Gamma_t$. The infinitesimal variation of this path at ∇_0 is then given by $\dot{\nabla} = \frac{d}{dt}\big|_{t=0} \Gamma_t$. The infinitesimal variations which preserve the flatness condition are those for which the infinitesimal change in curvature, defined to be $\dot{R} = \frac{d}{dt}\big|_{t=0} R^{\nabla_t}$, is identically zero. Through the first coboundary homomorphism defined above in equation (4.5) the infinitesimal change in curvature may be expressed in terms of the infinitesimal variation as follows

$$\dot{R} = \mathrm{d}^1_\nabla \dot{\nabla}.$$

A proof of this is provided by Labourie in [11, 70]. Therefore, an infinitesimal variation $\dot{\nabla}$ is flat iff $\mathrm{d}^1_\nabla(\dot{\nabla}) = 0$. In other words, the group of infinitesimal variations preserving the flatness condition is $\mathrm{Ker}(\mathrm{d}^1_\nabla)$.

It is necessary to know now which infinitesimal variations are coming from gauge transformations. Again the coboundary homomorphisms defined above may be used to describe such variations. An infinitesimal variation $\dot{\nabla}$ comes from a gauge transformation iff

$$\dot{\nabla} \in \mathrm{Im}(\mathrm{d}^0_\nabla).$$

The details of this correspondence are also worked out in [11, 72]. Therefore the tangent space at $[\nabla]$ in the representation variety is the quotient

$$\frac{\mathrm{Ker}(\mathrm{d}^1_\nabla)}{\mathrm{Im}(\mathrm{d}^0_\nabla)} = \frac{Z^1_\nabla(\Sigma)}{B^1_\nabla(\Sigma)} = H^1_\nabla(\Sigma),$$

the first ∇-twisted de Rham cohomology group.

Symplectic structure. With the tangent spaces described, the symplectic form on the representation variety may now be defined. Before doing so there is yet one more assumption which must be placed on the Lie group G. It must be assumed that G preserves some non-degenerate, symmetric bilinear form $\mathfrak{B}$ on its Lie algebra $\mathfrak{g}$.

Regard first $H^1(\pi_1, \mathfrak{g}_{\mathrm{Ad}\phi})$ as the tangent space to $\mathrm{Hom}(\pi_1, G)/G$ at $[\rho]$. Then the 2-form defined pointwise as the map

$$\omega^{(\mathfrak{B})} : H^1(\pi_1, \mathfrak{g}_{\mathrm{Ad}\phi}) \times H^1(\pi_1, \mathfrak{g}_{\mathrm{Ad}\phi}) \to H^2(\pi_1, \mathbf{R}) = \mathbf{R}$$

which is the cup-product on π_1 with $\mathfrak{B}$ as the coefficient pairing is a symplectic 2-form on the representation variety.

Alternatively, regard $H^1_\nabla(\Sigma)$ as the the tangent space to the representation variety at $[\nabla]$. Since each tangent vector is an $\mathrm{End}(E)$ valued 1-form on Σ, the wedge product of any two tangent vectors θ and η is a 2-form on Σ taking values in the bundle $\mathrm{End}(E) \otimes \mathrm{End}(E)$. The form $\mathfrak{B}$ on $\mathfrak{g}$ induces in the obvious way a map from $\Omega^2(\Sigma, \mathrm{End}(E) \otimes \mathrm{End}(E))$ to $\Omega^2(\Sigma, \mathbf{R})$. The bilinear map defined pointwise as

$$\omega^{(\mathfrak{B})}_\nabla : H^1_\nabla(\Sigma) \times H^1_\nabla(\Sigma) \to \mathbf{R}$$

by

$$\omega_{\nabla}^{(\mathfrak{B})}(\theta, \eta) = \int_{\Sigma} \mathfrak{B}'(\theta \wedge \eta)$$

is a symplectic 2-form on the representation variety.

Under the same correspondence between the two frameworks for describing the representation variety explained earlier, these two symplectic forms are equivalent. Using a combination of both frameworks, Goldman proves in [8] that each of the essential properties of symplectic forms are satisfied. His approach to showing that the form is closed is based on work of Atiyah and Bott in [1].

CHAPTER 5
Goldman flows: moving around in the representation variety

In Chapter 6 certain real-valued functions defined over the representation variety $\mathrm{Hom}(\pi_1, SU_2)/SU_2$ will be introduced and in Chapter 7 attempts to find their critical points will be described. Goldman flows provide a method of moving around in the representation variety and hence a means of searching for critical points of these functions. These flows will be discussed here, beginning with a simple example motivated by a look to the flow of a single representation in a generic $\mathrm{Hom}(\pi_1, G)$.

From here on, each representation variety will be looked at exclusively as a moduli space of representations. Recall the standard generators of a surface group $\pi_1(\Sigma_g)$ are being denoted by $\alpha_i, \beta_i, i = 1, \ldots, g$. Since each tangent vector to $\mathrm{Hom}(\pi_1, G)$, that is each cocycle in $Z^1(\pi_1, \mathfrak{g}_{\mathrm{Ad}\phi})$, is completely determined by its image on these generators, they will be referred to by the set of these images: $\{a_i, b_i\}, i = 1, \ldots, g$. It will be assumed that the Lie group G is a matrix group and that the symmetric bilinear form $\mathfrak{B}$ on $\mathfrak{g}$ is $\mathfrak{B}(a, b) = \mathrm{Tr}(ab)$. In this case the induced symplectic structure is given by

$$\omega^{(\mathfrak{B})}\left(\{a_i, b_i\}, \{\hat{a}_i, \hat{b}_i\}\right) = \sum_{i=1}^{g} \mathrm{Tr}(a_i\hat{b}_i - \hat{a}_i b_i). \tag{5.1}$$

5.1 Flows of representations: a simple example

Choose any representation $\rho = \{\rho(\alpha_i), \rho(\beta_i)\} = \{A_i, B_i\}$ given by its image on the standard generators and satisfying the relation $\Pi_i[A_i, B_i] = I$. Perturbing this representation in the direction of the cocycle $\{a_i, b_i\}$, the perturbed representation is to first order $\{A_i(1 + t\, a_i), B_i(1 + t\, b_i)\}$ where $\Pi_i[A_i(1+t\ a_i), B_i(1+t\, b_i)] = \Pi_i A_i(1+t\, a_i)B_i(1+t\, b_i)(1-t\, a_i)A_i^{-1}(1-t\, b_i)B_i^{-1} = I$ implies the first order relation

$$\sum_i A_i a_i B_i A_i^{-1} B_i^{-1} + A_i B_i b_i A_i^{-1} B_i^{-1} - A_i B_i a_i A_i^{-1} B_i^{-1} - A_i B_i A_i^{-1} b_i B_i^{-1} = 0. \tag{5.2}$$

In general it is either difficult or unsolvable to determine explicit values of cocycles $\{a_i, b_i\}$ in $\mathfrak{g}$ from the above relation alone. There does, however, exist deformations with simple descriptions. Such may be found by fixing some j, taking $b_i = 0$ for all i and $a_i = 0$ for all $i \neq j$. Then the above relation reduces to

$$A_j a_j B_j A_j^{-1} B_j^{-1} - A_j B_j a_j A_j^{-1} B_j^{-1} = A_j(a_j B_j - B_j a_j) A_j^{-1} B_j^{-1} = 0.$$

Therefore, by taking a_j such that

$$[a_j, B_j] = 0$$

the relation in (5.2) is satisfied. Adhering to these simple restrictions, a perturbation to the representation $\{A_i, B_i\}$ may be defined as follows

$$\mathbf{\Theta}_t(A_j) = A_j \exp(t\, a_j), \quad \mathbf{\Theta}_t(A_i) = A_i, i \neq j, \quad \mathbf{\Theta}_t(B_i) = B_i, \forall i. \tag{5.3}$$

Assigning such a cocycle at each representation in a smooth way forms a tangent vector field on $\mathrm{Hom}(\pi_1, G)$ and thus a flow on this entire representation space. If a_j is a polynomial in B_j then a_j commutes with B_j. So this assignment may be done by taking the same polynomial at each representation. Taking the particular example $b_i = 0$ and $a_i = \delta_{i,j} B_j^m$, $\forall i$ as an initial case study it is relatively easy to see that the associated flows comprised of individual pointwise perturbations such as the one described above are Hamiltonian. Begin by considering the function $H : \mathrm{Hom}(\pi_1, G) \to \mathbf{R}$

given by $H(\{A_i, B_i\}) = \frac{1}{m}\mathrm{Tr}(B_j^m)$, and the symplectic structure given by $\mathfrak{B}(a, b) = \mathrm{Tr}(ab)$. In this case if $\rho = \{A_i, B_i\}$ is perturbed in the tangent direction $\{\hat{a}_i, \hat{b}_i\}$ then

$$dH(\{\hat{a}_i, \hat{b}_i\}) = \frac{d}{dt}\bigg|_{t=0} \frac{1}{m}\mathrm{Tr}(B_j(1 + t\,\hat{b}_j))^m = \mathrm{Tr}(B_j^m \hat{b}_j). \tag{5.4}$$

The last equality above uses the fact that the trace is cyclic and linear. Now looking for the tangent vector at $\{A_i, B_i\}$ of the Hamiltonian vector field $\mathrm{Id}H$ associated to H is equivalent to searching for $\mathrm{Id}H_\rho = \{a_i, b_i\}$ such that

$$dH(\{\hat{a}_i, \hat{b}_i\}) = \omega^{(\mathfrak{B})}(\{a_i, b_i\}, \{\hat{a}_i, \hat{b}_i\}), \quad \forall \{\hat{a}_i, \hat{b}_i\} \in Z^1(\pi_1, \mathfrak{g}_{Ad\rho}).$$

It follows from equations (5.4) and (5.1) that the above equality is equivalent to

$$\mathrm{Tr}(B_j^m \hat{b}_j) = \sum_{i=1}^{g} \mathrm{Tr}(a_i \hat{b}_i - \hat{a}_i b_i) \tag{5.5}$$

which is satisfied iff $b_i = 0$ and $a_i = \delta_{i,j} B_j^m$, $\forall i$. Therefore $\mathrm{Id}H$ is precisely the tangent vector field initially taken and so, as claimed, the associated flows are Hamiltonian. The above demonstration is easily expanded to more elaborate polynomials. For instance $a_j = B_j^m + B_j^n$, where by taking $H : \mathrm{Hom}(\pi_1, G) \to \mathbf{R}$ given by $H(\{A_i, B_i\}) = \frac{1}{m}\mathrm{Tr}(B_j^m) + \frac{1}{n}\mathrm{Tr}(B_j^n)$ and again the symplectic structure $\omega^{(\mathfrak{B})}$ induced by $\mathfrak{B}(a, b) = \mathrm{Tr}(ab)$, one sees that $\{a_i, b_i\}$ is the Hamiltonian vector field $\mathrm{Id}H$.

5.2 Generalization to Goldman flows

Maintaining the key properties of the functions H defined above a larger class of flows may be defined. As these flows have been developed by William Goldman, they are referred to here as Goldman flows. They are referred to by Goldman as Hamiltonian or generalized twist flows. The ideas laid out in this section are based on Goldman's work in [10].

The important property to retain from the simple example above is the invariance of the function $\mathrm{Tr} : G \to \mathbf{R}$ under conjugation. From any invariant function $f : G \to \mathbf{R}$ and fixed simple cycle $\beta \in \pi_1$ on Σ one may define a Goldman flow, which is always Hamiltonian. In the case above β was taken to be one of the standard generators of π_1, in which case it is non-separating, i.e. taking $\Sigma|\beta$ to denote the surface without β, $\Sigma|\beta$ is connected. A simple cycle $\beta \in \pi_1$ on Σ may separate Σ, i.e. $\Sigma|\beta = \Sigma_1 \cup \Sigma_2$, with Σ_1, Σ_2 two connected components. These are the only two possible cases. In the following paragraph some definitions necessary to describe the Goldman flows arising in these two cases are given.

There is associated to any invariant function $f : G \to \mathbf{R}$ and simple cycle $\beta \in \pi_1$ (separating or not) the function $f_\beta : \mathrm{Hom}(\pi_1, G)/G \to \mathbf{R}$ defined by $f_\beta([\rho]) = f(\rho(\beta))$. Invariance of f implies this function is well defined. Then associated to f_β is a unique function $F : G \to \mathfrak{g}$ satisfying

$$\left.\frac{d}{dt}\right|_{t=0} f(B\exp(t\,b)) = \mathfrak{B}(F(B), b), \quad \forall\, B \in G, b \in \mathfrak{g}. \tag{5.6}$$

From F one may define a function $F_\beta : \mathrm{Hom}(\pi_1, G) \to \mathfrak{g}$ by setting $F_\beta(\rho) = F(\rho(\beta))$. As is discussed by Goldman in [10], invariance of f implies the following two properties:

1. $F(gBg^{-1}) = gF(B)g^{-1}, \forall\, B, g \in G$.
2. For any $B \in G, F(B) \in \mathcal{L}(B)$ where $\mathcal{L}(B)$ is the Lie algebra centralizer of B, in other words the subalgebra of $\mathfrak{g}$ which is fixed by $\mathrm{Ad}B$. So $BF(B)B^{-1} = F(B)$.

Considering first the case when β is a non-separating simple loop of Σ, for this is the more direct generalization of the above simple example, there is the following description of the Goldman flow.

Theorem 4. *Let β_j be a non-separating simple loop on Σ, hence a standard generator of π_1. Then the flow defined for each $\rho = \{A_i, B_i\} \in Hom(\pi_1, G)$ by*

$$\mathbf{\Theta}_t(\rho) : \begin{cases} \alpha_j \mapsto A_j \exp(t\, F(B_j)) & \\ \alpha_i \mapsto A_i & \text{for } i \neq j \\ \beta_i \mapsto B_i & \text{for all } i \end{cases} \tag{5.7}$$

is a Goldman flow on $Hom(\pi_1, G)$ which covers the Hamiltonian flow of f_β on $Hom(\pi_1, G)/G$.

Looking back to the simple example above, one sees that there $f : G \to \mathbf{R}$ was taken to be the function given by $B \mapsto \frac{1}{m}\mathrm{Tr}(B^m)$. Accordingly, $F_{\beta_j}(\rho) = F(B_j) = B_j^m$, $a_j = F(B_j)$, and hence, as intended, $a_j \in \mathcal{L}(B_j)$.

Consider now the second case, that is when β is a simple separating cycle on Σ. For a given invariant function $f : G \to \mathbf{R}$ the functions f_β, F, and F_β may be defined as in the previous case. Here, however, the standard generators of π_1 are not suitable to describe the associated flow. In this case the image of a representation ρ under the flow is described in terms of elements γ from the fundamental groups of Σ_1 and Σ_2. The free product of $\pi_1(\Sigma_1)$ and $\pi_1(\Sigma_2)$ amalgamated over the cyclic subgroup generated by β produces the full fundamental group π_1 of Σ. In this case there is the following theorem.

Theorem 5. *Let β be a separating simple loop and Σ_1, Σ_2 the components of $\Sigma|\beta$. The for each $\rho \in Hom(\pi_1, G)$, letting B denote $\rho(\beta)$, the flow defined by*

$$\mathbf{\Theta}_t(\rho) : \gamma \mapsto \begin{cases} \rho(\gamma) & \text{for } \gamma \in \pi_1(\Sigma_1) \\ \exp(t\, F(B)\rho(\gamma) \exp(-t\, F(B)) & \text{for } \gamma \in \pi_1(\Sigma_2) \end{cases} \tag{5.8}$$

is a flow on $Hom(\pi_1, G)$ which covers the Hamiltonian flow on $Hom(\pi_1, G)/G$ associated to f_β.

Theorems (4) and (5) are proven by Goldman in [10] using intersection theory and Poincaré duality.

CHAPTER 6
New questions: Using Goldman flows to study the spectral gap

The new work presented in this thesis is an application of the theory of Goldman flows presented above to study the eigenvalue spectra of standard averaging operators of L^2 functions on S^2, in particular, their spectral gaps.

6.1 Introduction to the spectral gap

Given m elements $g_1, \ldots, g_m$ of SU_2, in other words m rotations of S^2 or $\mathbf{R}^3$ as seen earlier, one can define an element of the group ring $\mathbf{R}[SU_2]$ by $z = g_1 + g_1^{-1} + \cdots + g_m + g_m^{-1}$ and a corresponding averaging operator on $L^2(S^2)$ as follows

$$zf(x) = \frac{1}{2m} \sum_{i=1}^{m} \left[f(g_i(x)) + f(g_i^{-1}(x)) \right].$$

The set $\{g_1, g_1^{-1}, \ldots, g_m, g_m^{-1}\}$ will be referred to as the support of z, denoted by $\mathrm{supp}(z)$. Since each g_i is self adjoint so is the operator defined by z. It follows that the spectrum of z is real. It is, moreover, contained in the interval [-1,1]. One sees easily that any constant function is an eigenfunction with eigenvalue 1. It is the eigenvalue of second greatest norm which is of interest. Let it be denoted by λ. If $|\lambda| < 1$ then the operator is said to have a *spectral gap*. Following a standard loosening of the nomenclature found in the literature, the group ring element z itself, and even the group generated by the elements of $\mathrm{supp}(z)$, will be referred to as having a spectral gap.

In rough terms, the existence of a spectral gap signifies that the rotations generating z mix the functions of $L^2(S^2)$ exponentially well. In order to establish the significance of the spectral gap more precisely, one may begin by recalling the following well known fact: the space $L^2(S^2)$ decomposes orthogonally under the spherical Laplacian over the spaces H_n of spherical harmonics of degree n. That is, $L^2(S^2) = \mathrm{H}_0 \oplus \mathrm{H}_1 \oplus \cdots \oplus \mathrm{H}_n \oplus \cdots$. Thus any f in $L^2(S^2)$ may be written as $f = h_0 + \cdots + h_n + \cdots$, with each h_n in H_n. The spherical Laplacian commutes with any z operator, and it follows that each z maps $\mathrm{H}_n \to \mathrm{H}_n$. Let the restriction of z to H_n be denoted by z_n. Note z_0 is the identity map. If λ_n is taken to denote the eigenvalue of z_n with greatest norm, then $\|z_n^k h_n\|_2 \leq |\lambda_n|^k \|h_n\|_2$ for any positive integer k and $|\lambda| \geq |\lambda_n|$ for each positive integer n. The eigenvalue λ_n will be referred to as the n^{th} *spectral radius.* So

$$z^k f = z_0^k h_0 + \cdots + z_n^k h_n + \cdots$$

and

$$\begin{aligned}\|z^k f - h_0\|_2 &= \| - h_0 + z_0^k h_0 + z_1^k h_1 + \cdots + z_n^k h_n + \cdots \|_2 \\ &= \sum_{n=1}^{\infty} \|z_n^k h_n\|_2 \;\leq\; \sum_{n=1}^{\infty} |\lambda_n|^k \|h_n\|_2 \;\leq\; \sum_{n=0}^{\infty} |\lambda_n|^k \|h_n\|_2 \;\leq\; |\lambda|^k \|f\|_2.\end{aligned}$$

Hence

$$\frac{\|z^k f - h_0\|_2}{\|f\|_2} \leq |\lambda|^k,$$

and if z has a spectral gap

$$\lim_{k\to\infty} \frac{\|z^k f - h_0\|_2}{\|f\|_2} = 0.$$

Note, having now the decomposition of z in z_n, if it is assumed that z fixes no spherical harmonics of finite degree, then the existence of a spectral gap may be expressed equivalently as

$$\overline{\lim}_{n\to\infty} |\lambda_n| < 1.$$

6.2 An alternative formulation of the spectral gap

Because of their ties to representation varieties, an application of Goldman flows to study the spectral gap occurs more naturally through an alternative formulation of the gap in terms of the N^{th} irreducible representations of SU_2. For $N \geq 0$, the N^{th} irreducible representation is $\text{sym}^N V$, where V is the standard two dimensional matrix representation of SU_2. The N^{th} irreducible representation brings SU_2 into the space of $N+1$ dimensional matrices acting on the space of homogeneous polynomials in (u, v) of degree N via the linear action

$$(u, v) \mapsto (\alpha u + \gamma v, \beta u + \delta v), \qquad \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} \in SU_2.$$

As was discussed above, each operator z decomposes into a series of operators z_n on H_n. Recall that H_n is equivalent to the space of homogeneous harmonic polynomials of degree n restricted to S^2. Consider a fixed z_N. Using the same generic decomposition of f as above,

$$z_N\, h_N(x) = \frac{1}{2m} \sum_{i=1}^{m} h_N(g_i(x)) + h_N(g_i^{-1}(x)).$$

This formula rewritten in terms of coordinates (u, v) in $\mathbf{C}$ becomes

$$\begin{aligned} z_N \mathbf{h}_N &= \frac{1}{2m} \sum_{i=1}^{m} \text{sym}^N(g_i)\, \mathbf{h}_N + \text{sym}^N(g_i^{-1})\, \mathbf{h}_N \\ &= \left[\frac{1}{2m} \sum_{i=1}^{m} \text{sym}^N(g_i) + \text{sym}^N(g_i^{-1}) \right] \mathbf{h}_N \end{aligned}$$

where $\mathbf{h}_N$ is the $N+1$ dimensional coordinate vector of h_N in terms of the standard basis $u^N, u^{N-1}v, \ldots, uv^{N-1}, v^N$ of homogeneous polynomials of degree N. Letting

$$\text{sym}^N(z) = \frac{1}{2m} \sum_{i=1}^{m} \text{sym}^N(g_i) + \text{sym}^N(g_i^{-1}),$$

the eigenvalues of z_N are precisely the eigenvalues of $\text{sym}^N(z)$.

6.3 New questions

Considering side by side the Goldman flow theory and the above definition of the spectral gap, the following question arises: How does the spectral gap of a z whose support is the generators of a surface group representation in SU_2 deform under Goldman flows? Zeroing in on first steps to approaching this question, the work presented in the following chapter is restricted to looking at the eigenvalue spectrum of the first symmetric power $\mathrm{sym}^1(z) = z_1 = z$ and the surface group π_1 coming from a genus two surface Σ_2. In particular, the following questions are explored: How does λ_1, the eigenvalue of z_1 with maximum modulus, evolve under Goldman flows? What are the critical points of λ_1 as a function on the representation variety $\mathrm{Hom}(\pi_1(\Sigma_2), SU_2)/SU_2$? A set of representations in $\mathrm{Hom}(\pi_1(\Sigma_2), SU_2)/SU_2$ which achieve $\lambda_1 = 0$ are discussed. This set is then used to explore whether $\lambda_1 = 0$ is a signature of representations which are (locally) maximal with respect to the spectral gap. These questions emerge from a long history of inquiry into the Ruziewicz problem.

6.4 History: The Ruziewicz problem and the spectral gap

Towards the beginning of the 20th century Ruziewicz posed the following question: Is the Lebesgue measure the unique finitely additive rotation invariant measure defined on the Lebesgue measurable sets of S^n? This question, now referred to the as the (Banach-)Ruziewicz problem, itself was a response to the Banach-Tarski Paradox for S^n. This paradox is that, for $n \geq 2$, the n-dimensional sphere may be decomposed into finitely many components and subsequently reconstructed to form two copies of itself using rotations from the group $O(n+1)$. That there exist no finitely additive rotation invariant measures defined on *all subsets* of S^n for $n \geq 2$ follows quickly from this paradox.

Ruziewicz hence posed his question and, besides Banach's negative solution for the case $n = 1$ in 1921, it lingered.

The eventual solution was found through an alternative formulation of the problem in terms of rotation invariant means on $L^\infty(S^n)$. One can show that if the sort of measure which would answer negatively Ruziewicz's question exists, it must be absolutely continuous with respect to the Lebesgue measure. It could therefore be used to construct in the standard way a rotation invariant integral on $L^\infty(S^n)$. Then Ruziewicz's question becomes one of whether the Lebesgue integral is the unique rotation invariant mean on $L^\infty(S^n)$. In 1979 and 1981 respectively, Del-Junco-Rosenblatt [5] and Rosenblatt [17] produced results giving a condition for when the Lebesgue integral is the unique rotation invariant mean, reinvigorating the question. Shortly after Margulis [14] and Sullivan [18], using this condition, independently provided more of the answer to Ruziewicz's question: the answer for $n \geq 4$ is 'yes'. Drinfeld [6] showed not long after that for $n = 2, 3$ the answer is also in the affirmative, completing the solution. Necessary to achieving his results, Drinfeld established the existence of finitely generated subgroups of SU_2 with spectral gaps.

Concurrently, Lubotzky, Phillips, and Sarnak established for SU_2 lower bounds on the spectral gap and constructed sets of generators which achieve them [13]. Oh has generalized these results, in particular, to rotation groups of all higher dimensional spheres [16]. The existence of finitely generated subgroups of SU_2 with spectral gaps was revisited by Gamburd, Jakobson, and Sarnak, who provided a different and more elementary proof of the existence of the spectral gap than those in [6] and [13]. More recently, Bourgain and Gamburd have expanded these existence results by proving that all free finitely generated subgroups of SU_n for which the generating elements satisfy a noncommutative diophantine property, in particular those z for which the elements of $\mathrm{supp}(z)$ have algebraic entries, have a spectral gap [2], [3].

Sets of elements in SU_2 exhibiting spectral gaps have significance for quantum computing. The 1-qubit, a basic unit of information in quantum computing, may be represented as a two dimensional complex unit vector. Accordingly, the quantum gates which operate on qubits take form as elements of SU_2. In the vocabulary of quantum computing, a finite set of gates, equivalently a finite set in SU_2, is said to be *universal* if it generates a group which is topologically dense in SU_2. Accounting for a discrepancy in the cost of implementing certain gates, an *efficient* universal set is one which approximates any given gate, that is element of SU_2, inexpensively. There exists a present practical concern of designing universal sets which realize the efficiency guaranteed by the Solovay-Kitaev theorem. The design of such efficient universal sets may be aided by an increase in knowledge of sets in SU_2 which exhibit spectral gaps. For further information on quantum computing and the Solovay-Kitaev theorem please see [15] and [4].

Here, a merger with Goldman flows on representation varieties initiates a new approach to studying the spectral gap.

CHAPTER 7
New work

As outlined in Section 6.3 above, this chapter documents the results of some initial exploration into the first spectral radius λ_1 for surface groups π_1 coming from genus two surfaces Σ_2. In section 7.1, computations are given which express the evolution of λ_1 under Goldman flows and provide conditions for its critical points. These same topics are then treated numerically in section 7.2. Initiated by their appearance in the numerical inquiry, the focus shifts in 7.3 to representations for which $\lambda_1 = 0$. A set of examples of such representations is given and then used to explore how much information on the spectral gap is contained in the two dimensional symmetric representations. In particular, initial calculations towards understanding whether those representations which are minimal for $|\lambda_1|$ are (locally) minimal for higher $|\lambda_N|$, and hence (locally) maximal with respect to the spectral gap are given.

7.1 First spectral radius—λ_1—for representations of $\pi_1(\Sigma_2)$ into SU_2 and their evolution under Goldman flows

General information on λ_1 for representations of $\pi_1(\Sigma_2)$ into SU_2.

Consider z whose support is the image of generators of the fundamental group of genus 2 surfaces under irreducible representations ρ into SU_2. Extending the notation established above, let $\lambda_1(\rho)$ denote the function on the space $\mathrm{Hom}(\pi_1(\Sigma_2), SU_2)/SU_2$ which assigns to each representation class $[\rho]$ the eigenvalue of $\mathrm{sym}^1(z) = z_1 = z$ of largest norm. Fix any representation class $[\rho]$ and let $A_i, B_i, i = 1, 2$ denote the images under ρ of the fundamental group generators $\alpha_i, \beta_i, i = 1, 2$ respectively. Let

$$A_i = \begin{bmatrix} a_i & \mathfrak{a}_i \\ -\bar{\mathfrak{a}}_i & \bar{a}_i \end{bmatrix} \quad \text{and} \quad B_i = \begin{bmatrix} b_i & \mathfrak{b}_i \\ -\bar{\mathfrak{b}}_i & \bar{b}_i \end{bmatrix}. \tag{7.1}$$

Then $A_i + A_i^{-1} = 2\mathrm{Re}(a_i)I$ and $B_i + B_i^{-1} = 2\mathrm{Re}(b_i)I$. The simple form of these matrix sums renders the computations in this case of $N = 1$ quite manageable. Indeed,

$$\begin{aligned} z &= \frac{1}{2 \cdot 4}\left[A_1 + A_1^{-1} + B_1 + B_1^{-1} + A_2 + A_2^{-1} + B_2 + B_2^{-1}\right] \\ &= \frac{1}{4}\left[\mathrm{Re}(a_1) + \mathrm{Re}(b_1) + \mathrm{Re}(a_2) + \mathrm{Re}(b_2)\right] I \end{aligned}$$

and therefore

$$\lambda_1(\rho) = \frac{1}{4}\left(\mathrm{Re}(a_1) + \mathrm{Re}(b_1) + \mathrm{Re}(a_2) + \mathrm{Re}(b_2)\right). \tag{7.2}$$

Note, since z is a scalar multiple of the 2×2 identity matrix, λ_1 is the only eigenvalue of z.

Evolution and critical points of λ_1 under Goldman flows.

As is well known, the dimension of the moduli space $\mathrm{Hom}(\pi_1(\Sigma_2), SU_2)/SU_2$ is six. In order to know where $\lambda_1(\rho)$ is critical, one can look to see how it changes in each of ρ's tangent directions. Fixing an invariant function f and an orthogonal structure $\mathfrak{B}$ on $\mathfrak{su}_2$, a basis for the tangent space at ρ may be described in terms of the Hamiltonian flows covered by the Goldman flows induced by the following six curves: each of the standard generators of the fundamental group π_1, the curve γ which separates the surface into two once punctured tori, and the curve $\alpha_1\beta_1$. The curves are drawn on Σ_2 in figure 7–1 below. In what follows, the precise description of the Goldman flow induced by each of these curves, their associated tangent vectors in $H^1(\pi_1, \mathfrak{su}_2)$, and their effect on λ_1 are given. Understanding how λ_1 deforms as any representation is perturbed in each of its tangent directions provides a method for finding critical points of λ_1. All of this is done at the level of the representative $\{A_i, B_i\}$ of $[\rho]$. One should keep in mind the interest at the level of representation classes.

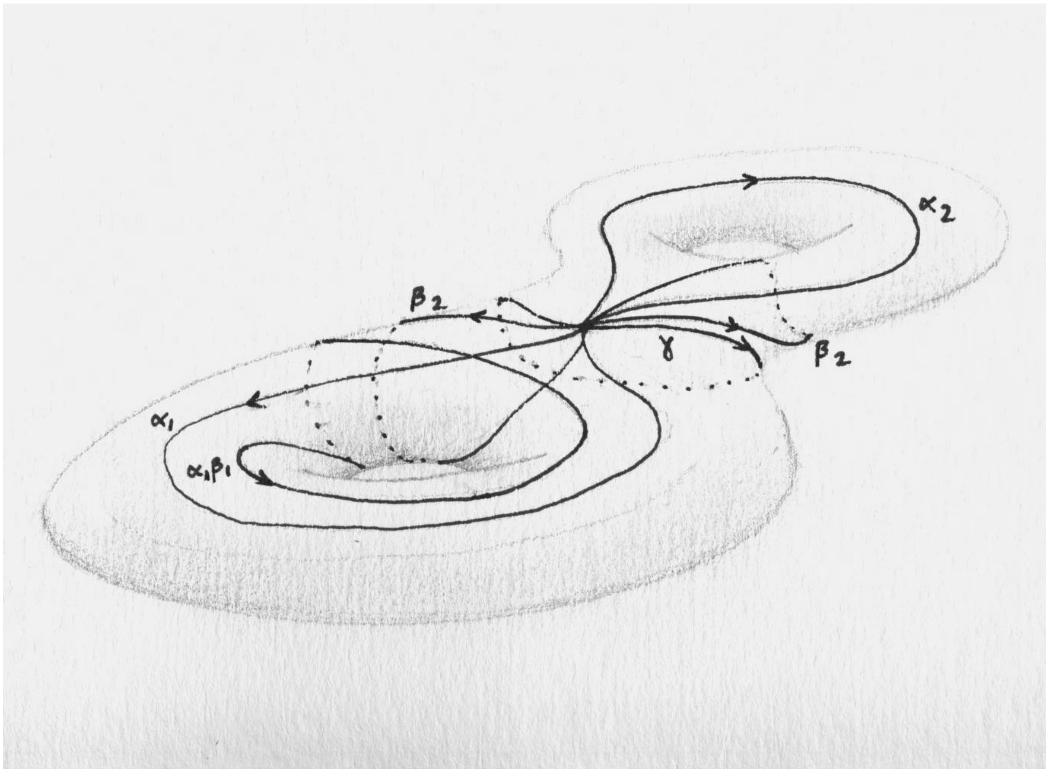

Figure 7–1: Curves on Σ_2 which induce Goldman flows spanning the tangent space at a representation ρ.

The Goldman flows induced from the generators of the fundamental group π_1 are considered first. For the sake of demonstration, the flow induced by β_1 is specifically considered. The other three generators' cases will produce the same results suitably modified. As is stated in Theorem 4, the associated Goldman flow fixes each of the generators except A_1. The deformation of A_1 is

$$\mathbf{\Theta}_t(A_1) = A_1 \exp(tF(B_1)),$$

where, recall, F is a function from SU_2 to $\mathfrak{su}_2$ induced by the fixed invariant function f. The associated cocycle u in $H^1(\pi_1, \mathfrak{su}_2)$ is defined by

$$\alpha_1 \mapsto -\mathrm{Ad}A_1(-F(B_1)) \qquad \alpha_2, \beta_1, \beta_2 \mapsto 0.$$

The representative $\{A_i, B_i\}$ of the representation class $[\rho]$ may be chosen so that the matrix $F(B_1)$ is diagonal:

$$F(B_1) = \begin{bmatrix} i\vartheta & 0 \\ 0 & -i\vartheta \end{bmatrix}.$$

Then $\mathbf{\Theta}_t(a_1) = a_1(t) = e^{ti\vartheta}a_1$ and therefore

$$\mathrm{Re}(a_1(t)) = [\mathrm{Re}(a_1)\cos(t\vartheta) - \mathrm{Im}(a_1)\sin(t\vartheta)].$$

It follows that

$$\frac{d}{dt}\mathrm{Re}(a_1(t)) = -\,\vartheta\,\mathrm{Re}(a_1)\,\sin(t\vartheta) -\,\vartheta\,\mathrm{Im}(a_1)\,\cos(t\vartheta)$$

and hence

$$\left.\frac{d}{dt}\right|_0 \mathrm{Re}(a_1(t)) = -\,\vartheta\,\mathrm{Im}(a_1).$$

Since $\vartheta \neq 0$, a condition on the critical points of λ_1 is that $Im(a_1) = 0$ *in the representative* $\{A_i, B_i\}$ *of* $[\rho]$ *for which* $F(B_1)$ *is diagonal.*

Now consider the effect of the Goldman flow induced from the loop γ which separates the surface into two once punctured tori. Let $\rho(\gamma) = \Gamma$. As is stated in Theorem 5, there is an induced Goldman flow which fixes A_1 and B_1 and which conjugates A_2 and B_2 as follows

$$\mathbf{\Theta}_t(A_2) = \exp(tF(\Gamma))A_2\exp(-tF(\Gamma))$$

$$\mathbf{\Theta}_t(B_2) = \exp(tF(\Gamma))B_2\exp(-tF(\Gamma)).$$

The associated cocycle u in the tangent space of $\{A_i, B_i\}$ is defined by

$$\alpha_1, \beta_1 \mapsto 0 \qquad \alpha_2 \mapsto F(A_2) - \mathrm{Ad}\Gamma F(A_2) \qquad \beta_2 \mapsto F(B_2) - \mathrm{Ad}\Gamma F(B_2)$$

Choosing the representative $\{A_i, B_i\}$ of $[\rho]$ so that $F(\Gamma)$ is diagonal , one sees that both a_2 and b_2, and in particular $\mathrm{Re}(a_2)$ and $\mathrm{Re}(b_2)$, are fixed under this flow: $\mathbf{\Theta}_t(a_2) = a_2(t) = a_2, \mathbf{\Theta}_t(b_2) = b_2(t) = b_2$. Therefore λ_1 remains constant.

Finally, consider the flow induced by the simple curve $\alpha_1\beta_1$. Note such a curve must cross exactly one each of α_1 and β_1 in the positive and negative directions. Again referring to Theorem 4, the associated Goldman flow fixes A_2 and B_2, and deforms A_1 and B_1 as follows

$$\mathbf{\Theta}_t(A_1) = A_1 \exp(tF(A_1B_1))$$
$$\mathbf{\Theta}_t(B_1) = \exp(-tF(A_1B_1))B_1.$$

The associated cocycle u is defined by

$$\alpha_1 \mapsto -\mathrm{Ad}A_1(-F(A_1B_1)) \quad \beta_1 \mapsto -F(A_1B_1) \quad \alpha_2, \beta_2 \mapsto 0.$$

Recall, as is stated in section 5.2, that $F(A_1B_1)$ is in $\mathcal{L}(A_1B_1)$, the Lie algebra centralizer of A_1B_1. Therefore if $[\rho]$ is a class of faithful representations, $F(A_1B_1)$ is not also in $\mathcal{L}(A_1)$ and $\mathcal{L}(B_1)$. So this cocycle is linearly independent from those associated to the flows induced by the cycles α_1 and β_1. As above, the representative $\{A_i, B_i\}$ may be chosen so that the matrix $F(A_1B_1)$ is diagonal:

$$F(A_1B_1) = \begin{bmatrix} i\eta & 0 \\ 0 & -i\eta \end{bmatrix}. \tag{7.3}$$

Then $\mathbf{\Theta}_t(a_1) = a_1(t) = e^{ti\eta}a_1$ and $\mathbf{\Theta}_t(b_1) = b_1(t) = e^{-ti\eta}b_1$. Therefore

$$\begin{aligned}\mathrm{Re}(a_1(t)) + \mathrm{Re}(b_1(t)) =& \mathrm{Re}(a_1)\cos(t\eta) - \mathrm{Im}(a_1)\sin(t\eta) + \mathrm{Re}(b_1)\cos(-t\eta) \\ & - \mathrm{Im}(b_1)\sin(-t\eta) \\ =& [\mathrm{Re}(a_1) + \mathrm{Re}(b_1)]\cos(t\eta) + [\mathrm{Im}(b_1) - \mathrm{Im}(a_1)]\sin(t\eta)\end{aligned}$$

Then

$$\begin{aligned}\frac{d}{dt}\mathrm{Re}(a_1(t)) + \mathrm{Re}(b_1(t)) =& -\eta[\mathrm{Re}(a_1) + \mathrm{Re}(b_1)]\sin(t\eta) + \eta[\mathrm{Im}(b_1) \\ & - \mathrm{Im}(a_1)]\cos(t\eta)\end{aligned}$$

and so

$$\left.\frac{d}{dt}\right|_0 \mathrm{Re}(a_1(t)) + \mathrm{Re}(b_1(t)) = \eta[\mathrm{Im}(b_1) - \mathrm{Im}(a_1)].$$

Since $\eta \neq 0$, it follows that a condition on $[\rho]$ being a critical point for λ_1 is that *Im*(b_1) − *Im*$(a_1) = 0$ *in the representative* $\{A_i, B_i\}$ *of* $[\rho]$ *for which* $F(A_1B_1)$ *is diagonal.*

Having seen that the conditions depend on the representative of ρ, and this on the function F, it is worthwhile to discuss a possible precise form of F. Goldman suggests in [10, 272] that for SU_2 one may take the invariant function f and bilinear form $\mathfrak{B}$ to be the real parts of the character and trace form respectively. Then the variation F of f is given by

$$F(M) = \frac{1}{2}\left(M - M^{-1}\right) - \left(\frac{i}{2}\right)\mathrm{Im}\ \mathrm{tr}M.$$

Since the trace of any matrix M in SU_2 is real, this equation reduces to

$$F(M) = \frac{1}{2}\left(M - M^{-1}\right).$$

For any M:

$$M = \begin{bmatrix} m & \mathfrak{m} \\ -\overline{\mathfrak{m}} & \overline{m} \end{bmatrix},$$

in SU_2,

$$F(M) = \begin{bmatrix} \operatorname{Im} m & \mathfrak{m} \\ -\overline{\mathfrak{m}} & -\operatorname{Im} m \end{bmatrix}.$$

Therefore $F(M)$ is diagonal iff M is diagonal. This variation function F is the one which is used in the numerical inquiry discussed next. The fact that $F(M)$ is diagonal iff M is diagonal is used to conjugate given representations, putting them in the form described above that is more suitable for computations.

Summary of λ_1's oscillation and critical points under Goldman flows.

The following list summarizes how, given a representation class $[\rho]$ in $\mathrm{Hom}(\pi_1(\Sigma_2), SU_2)/SU_2$, λ_1 oscillates under the Goldman flows which take $[\rho]$ in each of its tangent directions. Conditions on the critical points of λ_1 along each of these flows are also summarized. Recall, for a representative $\{A_i, B_i\}$ of any representation class $[\rho]$ the notation used for the entries of matrices A_i, B_i is established in equation (7.1). The precise form of F discussed above has been assumed and its implications incorporated into the descriptions of the oscillations.

Flow induced by α_1:

Taking the representative $\{A_i, B_i\}$ of $[\rho]$ such that A_1 is diagonal, the critical points of λ_1 occur whenever $\mathrm{Im}(b_1) = 0$, and

$$\lambda_1(t) = \frac{1}{4}\left(\mathrm{Re}(a_1) + [\mathrm{Re}(b_1)\cos(t\,\mathrm{Im}(a_1)) - \mathrm{Im}(b_1)\sin(t\,\mathrm{Im}(a_1))] + \mathrm{Re}(a_2) + \mathrm{Re}(b_2)\right). \tag{7.4}$$

Flow induced by β_1:

Taking the representative $\{A_i, B_i\}$ of $[\rho]$ such that B_1 is diagonal, the critical points of λ_1 occur whenever $\mathrm{Im}(a_1) = 0$, and

$$\lambda_1(t) = \frac{1}{4}\left([\mathrm{Re}(a_1)\cos(t\,\mathrm{Im}(b_1)) - \mathrm{Im}(a_1)\sin(t\,\mathrm{Im}(b_1)] + \mathrm{Re}(b_1) + \mathrm{Re}(a_2) + \mathrm{Re}(b_2)\right) \tag{7.5}$$

Flow induced by α_2:

Taking the representative $\{A_i, B_i\}$ of $[\rho]$ such that A_2 is diagonal, the critical points of λ_1 occur whenever $\mathrm{Im}(b_2) = 0$, and

$$\lambda_1(t) = \frac{1}{4}\left(\mathrm{Re}(a_1) + \mathrm{Re}(b_1) + \mathrm{Re}(a_2) + [\mathrm{Re}(b_2)\cos(t\,\mathrm{Im}(a_2)) - \mathrm{Im}(b_2)\sin(t\,\mathrm{Im}(a_2))]\right). \tag{7.6}$$

Flow induced by β_2:

Taking the representative $\{A_i, B_i\}$ of $[\rho]$ such that B_2 is diagonal, the critical points of λ_1 occur whenever $\mathrm{Im}(a_2) = 0$, and

$$\lambda_1(t) = \frac{1}{4}\left(\mathrm{Re}(a_1) + \mathrm{Re}(b_1) + [\mathrm{Re}(a_2)\cos(t\,\mathrm{Im}(b_2)) - \mathrm{Im}(a_2)\sin(t\,\mathrm{Im}(b_2))] + \mathrm{Re}(b_2)\right). \tag{7.7}$$

Flow induced by γ, the curve separating Σ_2 into two once puncture tori:

λ_1 is fixed under this flow.

Flow induced by $\alpha_1\beta_1$:

Taking the representative $\{A_i, B_i\}$ of $[\rho]$ such that A_1B_1 is diagonal, the critical points of λ_1 occur whenever $\mathrm{Im}(b_1) - \mathrm{Im}(a_1) = 0$, and

$$\begin{aligned}\lambda_1(t) =& \frac{1}{4}\left([\mathrm{Re}(a_1) + \mathrm{Re}(b_1)] \cos(t\eta) + [\mathrm{Im}(b_1) - \mathrm{Im}(a_1)] \sin(t\eta) \right. \\ & \left. + \mathrm{Re}(a_2) + \mathrm{Re}(b_2) \right), \end{aligned} \tag{7.8}$$

where η is as specified in equation (7.3).

Geometric rephrasing of λ_1 critical point conditions.

Recall the correspondence between SU_2 and unit quaternions discussed in subsection 3.1.2 and written explicitly in equation (3.1). Through the natural association of unit quaternions and the three sphere, each element of SU_2 is related to point on S^3. Seeing the elements of SU_2 in this way, the conditions on the critical points for λ_1 in the directions of the flows induced by standard generating elements of π_1 summarized above have a simple geometric interpretation. This will be illustrated for the flow induced by α_1. Translated according to the SU_2-unit quaternion correspondence, the conditions for critical points of λ_1 in the direction of this flow are that A_1 and B_1 have the following quaternion forms:

$$\begin{aligned} A_1 &= \mathrm{Re}(a_1) + \mathrm{Im}(a_1)\, i \\ &\stackrel{\text{def}}{=} \alpha_0 + \alpha_1 i \\ B_1 &= \mathrm{Re}(b_1) + \mathrm{Re}(\mathfrak{b}_1)\, j + \mathrm{Im}(\mathfrak{b}_1)\, k \\ &\stackrel{\text{def}}{=} \beta_0 + \beta_2\, j + \beta_3\, k. \end{aligned}$$

These points of S^3 may be shifted along the maximal torus on which they lie via the maps

$$A_1 \mapsto \hat{A}_1(t) = \exp(t \log A_1)$$
$$B_1 \mapsto \hat{B}_1(t) = \exp(t \log B_1).$$

Doing so until they reach the equator produces the orthogonal quaternions

$$\hat{A}_1 = \hat{\alpha}_1 \, i$$
$$\hat{B}_1 = \hat{\beta}_2 \, j + \hat{\beta}_3 \, k.$$

Therefore, for those tangent directions corresponding to Goldman flows induced by standard generators of π_1, conditions on the critical representations of λ_1 correspond to a sort of orthogonal property between pairs of generators A_i, B_i. In searching for representations with minimal λ_1, this property provides in some sense a more efficient approach: as soon as a pair of generators A_i, B_i exhibit this orthogonality property, any representation $\{A_i, B_i\}$ containing them will be critical in the tangent directions corresponding to *both* the α_i and β_i induced flows.

7.2 Numerical inquiry

The above computations have been translated into a computer program. Given any one of the above flows and representation class $[\rho]$, there is a program which computes and plots λ_1 along the flow. This is done via two methods: I. using a direct method of flowing the matrix representation via the definitions given above and computing the eigenvalues at each step, and II. computing the oscillating eigenvalue via the periodic formulas—equations (7.4) through (7.8) in the summary of the previous section. In both methods code has also been written to normalize the periods to 2π.

In each case, when two of the flows commute, a program was also written to plot the values of λ_1 against perturbations in each of these two directions.

Finally, there are functions which, given a representation and any of the flows, find the representation with the minimum and maximum λ_1 along the periodic trajectory.

For example, take the representation $\{A_i, B_i\}$ with

$$A_1 = \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix} \quad B_1 = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$
$$A_2 = B_2 = \begin{bmatrix} \frac{1}{2} - \frac{i}{2} & -\frac{1}{2} + \frac{i}{2} \\ \frac{1}{2} + \frac{i}{2} & \frac{1}{2} + \frac{i}{2} \end{bmatrix}. \tag{7.9}$$

Letting the representation evolve according to each of the discussed Goldman flows produces the oscillations of λ_1 shown in figure 7–2. The variation in λ_1 when $\{A_i, B_i\}$ is deformed according to the commuting Goldman flows induced by α_1 and α_2 is shown in figure 7–3.

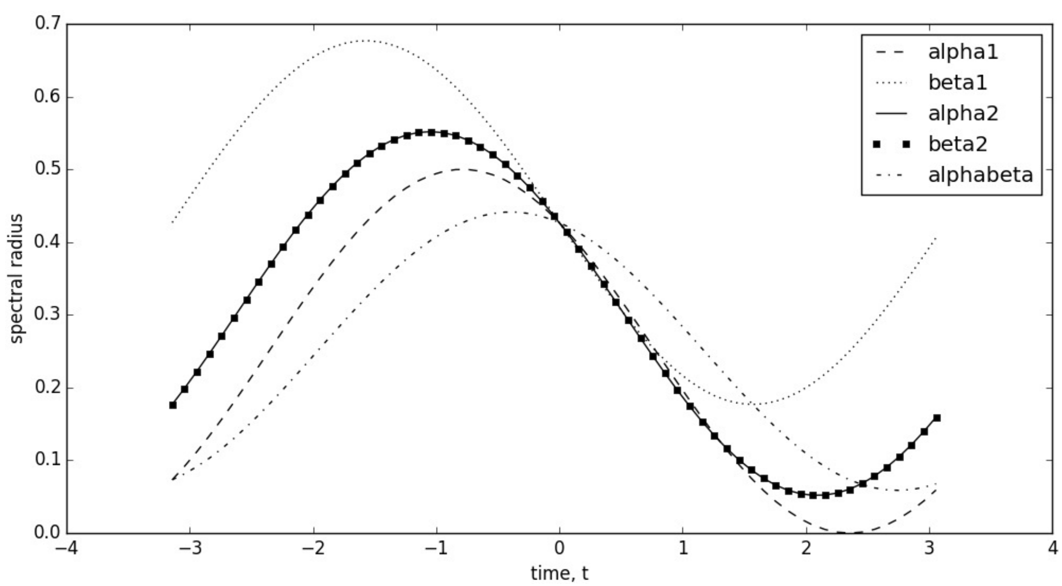

Figure 7–2: Evolution of λ_1 under Goldman flows.

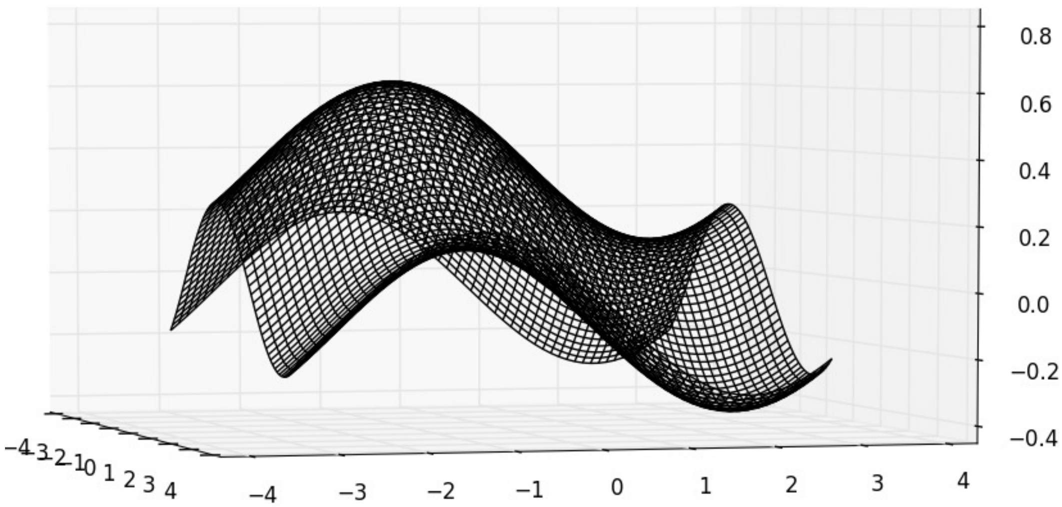

Figure 7–3: Evolution of λ_1 under the commuting Goldman flows induced by α_1 and α_2.

7.2.1 Application to study λ_1 for representations into the octahedral subgroup

Extremal eigenvalues of operators tied to the structure of Riemann surfaces often coincide with a considerable degree of symmetry. For instance, critical values of the first nonzero eigenvalue of the Laplacian Δ on a genus g surface correspond to highly symmetric Riemmanian metrics; the round metric on the sphere is one example. Such connections motivated an application of the above numerical tools to study λ_1 for representations into the octahedral subgroup of SO_3 and hence of SU_2.

The *octahedral group* is the group of symmetries of a regular octahedron, equivalently the group of symmetries of a cube. Situating the centre of a cube at the origin in $\mathbf{R}^3$, this symmetry group is comprised of the following twenty-four elements along with each of these elements composed with inversion:

- the identity rotation;
- $3 \times$ rotations by $180°$ about a proper axis;
- $6 \times$ rotations by $90°$ about a proper axis;
- $6 \times$ rotations by $90°$ about an edge axis;

- $8 \times$ rotations by $120°$ about a body diagonal.

Considering the symmetry present in the octahedral subgroup, one might expect that representations occurring within it to exhibit critical values of λ_1. This was found not to be true. The representation described in equation (7.9) above provides such a counterexample. The image of this representation $\{A_i, B_i\}$ is a subgroup of the octahedral group. It is, however, non-critical for each of the flows in its tangent directions—see figure 7–2.

Looking still at figure 7–2, one sees there is a representation along the periodic trajectory associated to the Goldman flow induced by α_1 for which λ_1 is 0. In the next section, the class of all representations with $\lambda_1 = 0$ in which this representation is situated is discussed.

7.3 Representations with $\lambda_1 = 0$

Interest in the values of the spectral gap lies in particular with those representations for which the gap is maximal. In other words, for those representations which keep $|\lambda_N|$ minimal. This prompts a look to those representations for which $\lambda_1 = 0$.

7.3.1 Examples of representations with $\lambda_1 = 0$

Looking again at equation (7.2) one see that λ_1 occurs precisely when the sum of the real parts of the generators' upper left entry equals 0. Irreducible representations which satisfy such a property can easily be constructed. Indeed, taking any two distinct diagonal matrices $D_1 = \mathrm{diag}(e^{i\vartheta_1}, e^{-i\vartheta_1})$ and $D_2 = \mathrm{diag}(e^{i\vartheta_2}, e^{-i\vartheta_2})$ for which $\cos(\vartheta_1) + \cos(\vartheta_2) = 0$, and a matrix G of SU_2 which does not commute with either D_i, setting $A_1, B_1, A_2, B_2 = D_1, D_2, GD_1G^{-1}, GD_2G^{-1}$ establishes such a representation. In this case, as desired, $\left[D_1 + D_1^{-1}\right] + \left[D_2 + D_2^{-1}\right] = \mathrm{diag}(\cos(\vartheta_1), \cos(\vartheta_1)) +$ $\mathrm{diag}(\cos(\vartheta_2),$

$\cos(\vartheta_2)) = 0$. The D_i commute so that the commutator relation is satisfied, however the group generated remains nonabelian.

7.3.2 Is there a link between $\lambda_1 = 0$ and maximal spectral gaps?

A natural question which arises when looking at representations for which $\lambda_1 = 0$ is whether these representations have maximum spectral gaps. This question is explored by looking at the set of examples described above.

To construct an example, take

$$D_i = \begin{bmatrix} e^{\vartheta_i} & 0 \\ 0 & e^{-\vartheta_i} \end{bmatrix} \quad \text{and} \quad G = \begin{bmatrix} g & \mathfrak{g} \\ -\overline{\mathfrak{g}} & \overline{g} \end{bmatrix}$$

satisfying the conditions of the examples given in the previous section. To determine the values of λ_N for all N the following matrices must be calculated:

$$\begin{aligned}
\operatorname{sym}^N(z) =& \frac{1}{8}\left[\operatorname{sym}^N(D_1) + \operatorname{sym}^N(D_1^{-1}) + \operatorname{sym}^N(D_2) + \operatorname{sym}^N(D_2^{-1}) + \right.\\
& \operatorname{sym}^N(GD_1G^{-1}) + \operatorname{sym}^N(GD_1^{-1}G^{-1}) + \\
& \left. \operatorname{sym}^N(GD_2^{-1}G^{-1}) + \operatorname{sym}^N(GD_2^{-1}G^{-1})\right] \\
=& \frac{1}{8}\left[\operatorname{sym}^N(D_1) + \operatorname{sym}^N(D_1^{-1}) + \operatorname{sym}^N(D_2) + \operatorname{sym}^N(D_2^{-1}) + \right.\\
& \operatorname{sym}^N(G)\left[\operatorname{sym}^N(D_1) + \operatorname{sym}^N(D_1^{-1}) + \operatorname{sym}^N(D_2) + \right.\\
& \left.\left. \operatorname{sym}^N(D_2^{-1})\right] \operatorname{sym}^N(G^{-1})\right] \\
\stackrel{\text{def}}{=}& \frac{1}{8}\left[\operatorname{sym}^N(z_D) + \operatorname{sym}^N(G)\operatorname{sym}^N(z_D)\operatorname{sym}^N(G^{-1})\right].
\end{aligned}$$

The second equality above holds since the operations of conjugation and taking the Nth symmetric power commute. The Nth symmetric power of D_i is

$$\mathrm{sym}^N(D_i) = \begin{bmatrix} e^{\vartheta_i(N)} & 0 & \cdots & 0 & 0 \\ 0 & e^{\vartheta_i(N-2)} & \cdots & 0 & 0 \\ 0 & 0 & \ddots & 0 & 0 \\ 0 & 0 & \cdots & e^{-\vartheta_i(N-2)} & 0 \\ 0 & 0 & \cdots & 0 & e^{-\vartheta_i(N)} \end{bmatrix}.$$

Hence $\mathrm{sym}^N(D_i)+\mathrm{sym}^N(D_i^{-1}) = \mathrm{diag}(\cos(\vartheta_i(N)), \cos(\vartheta_i(N-2)), \cdots, \cos(\vartheta_i(N-2)), \cos(\vartheta_i(N))$ and $\mathrm{sym}^N(z_D) = \mathrm{diag}(\cos(\vartheta_1(N)) + \cos(\vartheta_2(N)), \cos(\vartheta_1(N-2))+\cos(\vartheta_2(N-2)), \cdots, \cos(\vartheta_1(N-2))+\cos(\vartheta_2(N-2)), \cos(\vartheta_1(N))+\cos(\vartheta_2(N)))$

Based on the assumption that $\cos(\vartheta_1)+\cos(\vartheta_2) = 0$ there are two possible cases: $\vartheta_2 = \pi \pm \vartheta_1$. In either case, according to basic trigonometric identities, it follows that for any N,

$$\begin{aligned} \cos(\vartheta_2(N)) &= \cos((\pi \pm \vartheta_1)(N)) = \cos(\pi(N) \pm \vartheta_1(N)) \\ &= \cos(\pi(N))\cos(\vartheta_1(N)) \mp \sin(\pi(N))\sin(\vartheta_1(N)) \\ &= \cos(\pi(N))\cos(\vartheta_1(N)). \end{aligned}$$

Therefore when N is odd

$$\begin{aligned} \mathrm{sym}^N(z_D) &= \mathrm{sym}^N(D_1) + \mathrm{sym}^N(D_1^{-1}) + \mathrm{sym}^N(D_2) + \mathrm{sym}^N(D_2^{-1}) \\ &= 0 \end{aligned}$$

as has already been seen in the case $N = 1$, and when N is even

$$\begin{aligned} \mathrm{sym}^N(z_D) = \mathrm{diag}(&2\cos(\vartheta_1(N)), 2\cos(\vartheta_1(N-2)), \cdots, \\ &2\cos(\vartheta_1(N-2)), 2\cos(\vartheta_1(N))). \end{aligned}$$

This is as far as the computations in this direction have been taken. More work must be done to understand the eigenvalues of $\operatorname{sum}^N(z) = 1/8[\operatorname{sym}^N(z_D) + \operatorname{sym}^N(G)\operatorname{sym}^N(z_D)\operatorname{sum}^N(G^{-1})]$ based on the above form of $\operatorname{sym}^N(z_D)$ when N is even before it will be understood whether $\lambda_1 = 0$ is a signature of critical spectral gaps.

CHAPTER 8
Conclusion

Following earlier chapters building up preliminary material, the relevance of using Goldman flows to study the spectral gaps associated to certain standard averaging operators defined over $L^2(S^2)$ and hence to $\mathrm{Hom}(\pi_1(\Sigma_g), SU_2)/SU_2$ has been shown. Computations in the case of $g = 2$ and at the level of the first spectral radius λ_1 demonstrate how Goldman flows provide both information on the local behaviour of the first spectral radius as well as a means of moving around in the representation variety, and therefore an approach to finding critical points of the first spectral gap.

While the computations in this case are relatively simple, understanding the general case is considerably more difficult. In addition to understanding the generalization to higher genus surface groups, further work must be done before it is understood how the n^{th} spectral radii for $n \geq 2$ and thus the spectral gap itself evolve under Goldman flows. Once it is, these results would allow further research into the critical points of the spectral gap over these representation varieties. Working in another direction—favouring the average over the exceptional—it may interesting to explore how Goldman flows could facilitate the computation of average statistics of the spectral gap over $\mathrm{Hom}(\pi_1(\Sigma_g), SU_2)/SU_2$.

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