SOLUTIONS OF TWO MATRIX MODELS FOR THE DIII GENERATOR ENSEMBLE

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Abstract

In this work we solve two new matrix models, using standard and new techniques. The two models are based on matrix ensembles not previously considered. They are represented by special form of antisymmetric matrices and are classified in the DIII generator ensemble. It is shown that, in the double scaling limit, their free energy has the same behavior as previous models describing oriented and unoriented surfaces. We also found an additional solution for the chapter 3 model

Résumé

Dans ce travail, nous avons résolus deux nouveaux modèles de matrices en utilisant les techniques standard ainsi que quelques-unes de notre cru. Les deux modèles sont basés sur des ensembles de matrices qui n'avaient jamais été étudiés. Ils sont représentés par une forme speciale de matrices anti-symmetriques et sont classés dans l'ensemble des générateurs DIII. Il est montré que leur énergie libre, dans la double limite d'échelle, possède le même comportement critique que d'autres modèles déjà étudiés et décrivant des surfaces orientées et non-orientées . Nous avons également trouvé une solution additionnelle pour le modèle du chapitre 3.

Preface

I would like to thank my advisor, Robert C. Myers, for his help and encouragement throughout the preparation of this work, especially when I was convinced that there were no solutions to one of the models (where in fact there was one). I would also like to thank Arley Anderson for his help at the early stages of this work and in the part on symmetric spaces.

Finally, I wish to thank Martin Karmela for carefully reading the manuscript and correcting so many spelling mistakes.

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Introduction

String theory has evolved rapidly in the last five to seven years. This revival, due in part to ref. [1] with the discovery of gauge and gravitational anomaly cancelation, was mainly on the theoretical side.

Unfortunately, string theorists are limited to a perturbative expansion, which is not sufficient to provide vital information such as the true ground state of the theory. In other words, there are many ways to compacify a $D = 10$ string theory to $D = 4$, and a non-perturbative approach would be necessary to eliminate some solutions by showing that they are inconsistent.

The quantity of interest in first quantized string theory is the following path integral (partition function) [2],

$$\mathcal{Z} \sim \sum_{topol} \int \mathcal{D}g\mathcal{D}X \, e^{-S_{poly}} \tag{0.1}$$

where S_{poly} is the Polyakov action,

$$S_{poly} = \frac{1}{2\pi\alpha'} \int d^2\xi \, \sqrt{g} g^{ab} \partial_a X^\mu \partial_b X^\nu G_{\mu\nu}(X) \tag{0.2}$$

In this equation, $G_{\mu\nu}$ is a D-dimensional embedding space-time metric, whereas g_{ab} is the world-sheet metric. One approach is to write the world-sheet metric in the conformal gauge, $g_{ab} = e^{\phi}\hat{g}_{ab}$. This simplifies the action because ϕ vanishes in the action, but there are also new contributions from the path integral measure. This gives rise to the quantum Liouville theory, where the Liouville action is

$$\frac{26 - D}{48\pi} \int d^2\sigma \, \sqrt{g}(\hat{g}^{ab}\partial_a\phi\partial_b\phi + 2R\phi) \tag{0.3}$$

One observes that for $D = 26$, the field ϕ decouples[1] and the quantum theory is conformally invariant. This is an example of a critical string theory. Although these look simple

[1]This is true when one adds local couterterms to cancel the world-sheet cosmological constant as explained in section 1.2

they present technical difficulties when evaluating amplitudes for large genus number. The situation is even more complicated in non-critical string theories where there is no conformal invariance.

An alternative approach, called dynamical triangulation [3, 4], was used for numerical studies of non-critical string theories. The idea is to discretize the string world-sheet with small flat plaquettes, connect them together in all possible ways, and (try to) recover the continuum limit. This corresponds to a sum over all possible deformations for a given genus number and also over all genus. Three years ago some remarkable progress was made [5, 6, 7] on the analytic side, at least for simple theories. However, this method is currently restricted to non-critical string embedded in $D \leq 1$ dimensions -- this is because the analytic methods used for $D \leq 1$ don't apply for higher dimensional string theories.

Even though a string theory in $D = 0$ may appear not to contain much physics, it provides a simple starting point. It also serves as a toy model, which could be useful in understanding, and doing more complicated calculations – in the same way as we acquire some experience with infinite dimensional integrals to better understand their functional analogs.

In this work, chapter one is devoted to a review of matrix models, including the important relation between the topological expansion of 2-D quantum gravity and the perturbative expansion - via Feynman graphs - of matrix models. This idea is central to matrix models. I also discuss the exact solution of matrix models, the double scaling limit, their free energy, and finally a classification scheme for the models within the context of symmetric spaces. In chapter two I show the calculation of a model with complex matrices, introducing the scaling relations and the critical values. Some of the results derived for this model can be applied to the two other models discussed. In chapter three and four, I show the calculations of these two new models in $D = 0$. The reasons for studying them are that they were part of a classification scheme, but also they provided an opportunity to extend the techniques for solving matrix models, namely via skew-orthogonal polynomials [8]. We also had to solve them to compare with previous models to see if the physics revealed by the free energy is the same. Indeed, from its series expansion, we can say if the model describes unoriented surfaces in addition to oriented ones, and, by calculating an appropriate ratio, we can say if it is the same as previous models (discussed in chapter two). The appendix, which presents low-order perturbative calculations for various matrix models, can also help to find whether or not the model includes unorientable surfaces.

Finally, I conclude with a discussion and a comparison of the two models' free energies, and a comparison with other models.

Chapter 1

Review of matrix models

In this chapter I explain the basic concepts of matrix models, starting with the Polyakov path integral in string theory. We then look at two approaches to get some information from the partition function: Liouville theory and dynamical triangulation. The relation between matrix models and dynamical triangulation is explained, and the exact solution of a simple model, using Hermitian matrices, is sketched. Finally, I introduce a classification for matrix models based on symmetric spaces, and discuss the problem of the reduction of the matrix integral.

1.1 Polyakov path integral

In string theory, we generally want to calculate string scattering amplitudes, this is just the string equivalent of the usual calculations that we are doing in quantum field theory (e.g. QED). The fundamental quantity, from which we can derive almost all quatities of interest, is the following partition function, or Polyakov path integral [2],

$$\mathcal{Z} = \sum_{topol} g^{-\chi} \int \mathcal{D}g_{ab}\mathcal{D}X^{\mu}\, e^{-S_{poly}} \tag{1.1}$$

where g_{ab} is the world-sheet metric, and the X^{μ} are the embedding variables of the surfaces. In a scattering amplitude calculation, one also includes various vertex operators in the path integral. The sum over topologies, here, is equivalent to a sum to all orders of loops in two-dimensional quantum field theory. We talk about topologies because string interactions correspond to the surfaces which we may distinguish from one another by their topology (e.g. the number of handles, or the genus). For example, fig 1.1 gives a comparison of graphs in quantum field theory and in string theory.

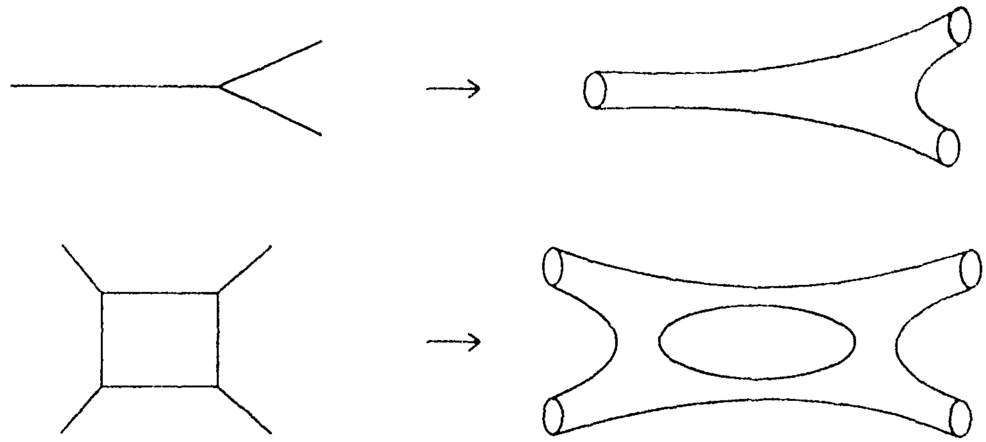


Figure 1.1: Comparison of interactions in quantum field theory and string theory

The sum over topologies is generally written as a sum over χ, the Euler character, where

$$\chi = 2 - 2(\#handles) - (\#holes). \tag{1.2}$$

The number of holes is just the number of asymptotic strings. The number of handles is, for example, 0 in the first graph and 1 in the second. So for a fixed number of asymptotic strings (for a given interaction), we sum over all possible handles.

1.2 Liouville theory

The Polyakov path integral is weighted by the following action [2],

$$S_{poly} = \frac{1}{2\pi\alpha'}\int d^2\xi\, \partial_a X^\mu \partial_b X^\nu G_{\mu\nu}(X) \tag{1.3}$$

One can fix the diffeomorphism freedom of the metric on the world-sheet by choosing the conformal gauge,

$$g_{ab} = e^{\phi}\hat{g}_{ab} \tag{1.4}$$

where we can always choose, locally, $\hat{g}_{ab} = \delta_{ab}$ The action then becomes

$$S_{poly} = \frac{1}{2\pi\alpha'}\int d^2\sigma\, \sqrt{\hat{g}}\hat{g}^{ab}\partial_a X^\mu \partial_b X^\nu G_{\mu\nu}(X). \tag{1.5}$$

So, in this classical action the conformal mode decouples. We would like to keep this property when quantizing the theory. Taking care of the regularization of the path integral measure, one gets,

$$\frac{26-D}{48\pi}\int d^2\sigma\, d^2\sigma'\, \sqrt{g}\mathcal{R}G(\sigma,\sigma')\sqrt{g'}\mathcal{R}' + \int d^2\sigma\sqrt{g}\mu. \tag{1.6}$$

The first term is non-local. It has two contributions, $-D$, coming from the embedding part, $\mathcal{D}X^\mu$, and 26, coming form the metric part $\mathcal{D}g_{ab}$. The second term, involving a bare cosmological constant, can be eliminated with appropriate local counterterms in the action.

In the conformal gauge the previous equation becomes local,

$$\frac{26-D}{48\pi}\int d^2\sigma\, \hat{g}(\hat{g}^{ab}\partial_a\phi\partial_b\phi + 2\hat{R}\phi) \tag{1.7}$$

This is the Liouville action. As was explained in the introduction, the field ϕ decouples for $D = 26$, but even then, there are technical difficulties when evaluating amplitudes for large genus number. String theories with conformal invariance are called critical. Recently, some progress was made in the understanding of non-critical string theories (non-conformally invariant). Some such models were found to be completely soluble, using matrix models, although they are very simple. In the following I discuss in detail the idea of dynamical triangulations as an alternative regularization for non-critical string theories I also explain the relation of this regulator to matrix models, which is the foundation of the present work.

1.3 Dynamical triangulations

We consider, for simplicity, $D = 0$ (i e no embedding spacetime) In that case the path integral reduces to,

$$\mathcal{Z} = \sum_{topol}\int \mathcal{D}g_{ab}\, e^{-\mu A-\kappa\chi}. \tag{1.8}$$

The Euler number, χ, comes from an Einstein term, whereas the other one is a cosmological constant term (A is the area). Obviously, the interesting part of this theory is in defining $\mathcal{D}g_{ab}$. One solution to the problem is through dynamical triangulation [3]. In this procedure we construct surfaces from small, fixed-size plaquettes Suppose that these are squares of side α. Then the functional integral $\int \mathcal{D}g_{ab}$ is replaced by a sum over all numbers of plaquettes n, and a sum over all possible tilings for a given n (the tilings are the possible ways to connect the squares in order to generate surfaces of a given genus number). We write the result in terms of the free energy,

$$\mathcal{F} = -\log \mathcal{Z} = \sum_{topol} \sum_{n} \sum_{tilings} (g_{st})^{-\chi} e^{-\mu n \alpha^2} \qquad (1.9)$$

where $A = n\alpha^2$ is the area, and $g_{st} = e^{\lambda}$ is the string coupling. By using a logarithm we implicitly consider only the connected graphs, as in quantum field theory. Without it, there would be a different sum on the right-hand side to account for the disconnected pieces

1.4 Matrix models

The relation with matrix models can now be seen by considering the following matrix integral,

$$\mathcal{Z} = \int d\phi \, e^{-\mathcal{S}} \qquad (1.10)$$

where $\mathcal{S} = NTr[\frac{1}{2}\phi^2 + \frac{b}{4}\phi^4]$ and ϕ_{ij} is an $N \times N$ hermitian matrix. The flat measure is defined as

$$d\phi = \prod_{i=1}^{N} d\phi_{ii} \prod_{1 \leq i < j \leq N} d(Re\phi_{ij}) d(Im\phi_{ij}). \qquad (1.11)$$

This integration can be evaluated perturbatively in the coupling b, using Feynman diagram techniques. For example, the ϕ^4 term is represented by a cross of two pairs of lines (fig 1.2).

The details can be found in [9] Here we simply point out that the dual graphs of the Feynman diagrams correspond to the tiling of surfaces with small squares The idea of dual graph is not new. Given a general graph, one can construct its dual by tracing, for each line, its perpendicular (fig 1 3). The translation rules are as follows,

$$\begin{aligned} \text{propagators} &\rightarrow \text{edges} \\ \text{matrix interactions} &\rightarrow \text{faces} \\ \text{closed loops} &\rightarrow \text{vertices} \end{aligned}$$

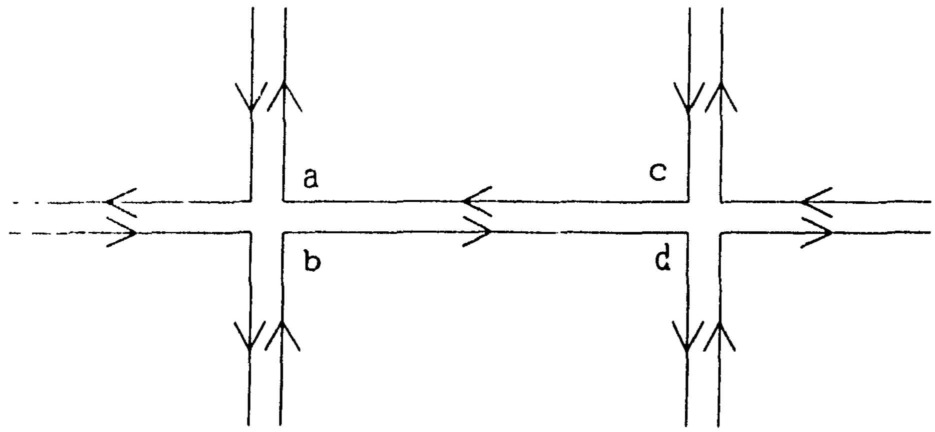


Figure 1.2: Feynman vertex and propagator for a matrix path integral.

The reason why we only get squares (quadrilaterals) is that the interaction is quartic. In general, for a ϕ^M interaction , one would get M-gons (M=3,4,5,...).

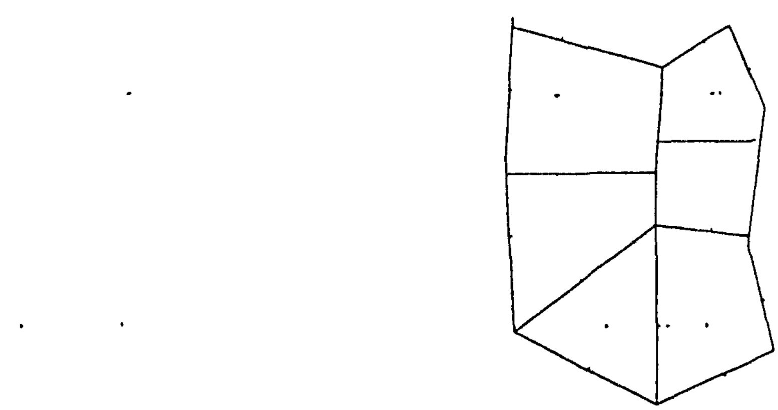


Figure 1.3: A graph and its dual equivalent.

We will now use the result obtained by 't Hooft [10], that the matrices organize the topological expansion of the dual graphs. This statement requires some explanation. We start by counting the factors of N entering in the evaluation of a Feynman graph. For each interaction we have a factor of N, because there is an N in the action. We also have a factor of $1/N$ for each propagator. And finally, for each closed loops, we have another factor of N. This is because in a closed loop we are summing over indices (i.e. taking a trace), so we get the dimension of the matrix, N. So, for a given Feynman diagram, the overall factor of N is,

$$N^I N^{-P} N^L = N^{F-E+V} = N^{\chi} \tag{1.12}$$

where $\chi = F - E + V$ (Euler character for surfaces made of polygons). We also used the dual graph correspondence. We see, then, that each Feynman diagram can be classified by the Euler character of the surface in its corresponding dual graph.

If we compare then our Feynman diagram expansion of eq.1.10 with the expansion in eq.1.9, we notice that they are the same with $1/N$ playing the role of a bare string coupling constant g_{st}. So we write the free energy for this matrix model as,

$$\mathcal{F} = -\log \mathcal{Z} = \sum_{\chi} \sum_{n} (\frac{1}{N})^{-\chi} (-b)^n [tilings]_{n,\chi} \tag{1.13}$$

where the number of tilings for a given n and χ was evaluated for low genus surface [9],

$$[tilings]_{n,\chi} = (-b_c)^{-n} n^{-\lambda\chi-1} C_{\chi} + O(\frac{1}{N}) \tag{1.14}$$

In this equation, b_c depends on the regularization scheme used and is a negative number[1]. λ is independent of that particular choice, and C_{χ} carries information about the genus expansion. Using this result, the free energy becomes,

$$\begin{aligned} \mathcal{F} &\simeq \sum_{\chi} \sum_{n} (\frac{1}{N})^{-\chi} (\frac{b}{b_c})^n n^{-\lambda\chi-1} C_{\chi} && (1.15) \\ &\simeq \sum_{\chi} \int \frac{dA}{A} (\frac{1}{N})^{-\chi} (\frac{b}{b_c})^{A/\alpha^2} (\frac{A}{\alpha^2})^{-\lambda\chi} C_{\chi} && (1.16) \end{aligned}$$

where the summation over n was replaced by an area integral, and n by A/α^2. Now we want to recover the continuum limit, $\alpha \to 0$. In order to keep the integral finite we have to take $b/b_c \to 1$ while fixing $A_0 = \alpha^2/\log(b_c/b)$. This is to avoid a possible divergence in $(b/b_c)^{A/\alpha^2}$. With these definitions, we have,

$$\mathcal{F} \simeq \sum_{\chi} \int \frac{dA}{A} (\frac{1}{N} \frac{1}{(\log b_c/b)^{\lambda}})^{-\chi} e^{-A/A_0} (\frac{A}{A_0})^{-\lambda\chi} C_{\chi} \tag{1.17}$$

We can define a new renormalization constant via $g_R = N^{-1}(\log b_c/b)^{-\lambda}$. It can be kept finite by taking the so-called double scaling limit $N \to \infty$, and $b/b_c \to 1$. Near the

[1] In fact, we defined our b_c to have an opposite sign to that of ref.[9], an added a minus in front of it. This is for further convenience and doesn't affect the analysis.

critical point, one can write g_R as

$$g_R = \frac{1}{N}(b_c/b - 1)^{-1+\gamma/2} \tag{1.18}$$

where γ is known as the critical index of the string susceptibility. We have the simple relation, $\lambda = 1 - \gamma/2$, and one finds [11] that $\gamma = -1/2$. In this analysis, it is understood that $b \rightarrow b_c$, but b_c is a negative number, and eq.1.10 appears to be divergent. The cure for this problem is to use analytical continuation for negative b.

Although the above calculation was carried out with a factor of N in the action, it should be noted that there are other definitions of this action (and also of the double scaling limit). For example, in the following we will give the action an overall factor of β instead of N, and we will let $\beta \rightarrow N$. We can accomplish this with the following transformation, $\phi \rightarrow (\beta/N)^{1/2}\hat{\phi}$, and rewrite the action as,

$$\begin{aligned}
\hat{S} &= NTr[\frac{1}{2}\phi^2 + \frac{b}{4}\phi^4] \\
&= NTr[\frac{1}{2}\hat{\phi}^2\frac{\beta}{N} + \frac{b}{4}(\frac{\beta}{N})^2\hat{\phi}^4] \\
&= \beta Tr[\frac{1}{2}\hat{\phi}^2 + \frac{b}{4}(\frac{\beta}{N})\hat{\phi}^4].
\end{aligned} \tag{1.19}$$

So, our previous calculation for g_R are essentially unchanged, all that we have to do is to replace N by β, and b by $b(\beta/N)$. We get,

$$g_R = \frac{1}{\beta}\left(\frac{b_c N}{\beta b} - 1\right)^{-1+\gamma/2} \tag{1.20}$$

which is the result found in [12] if we let $b = b_c$. In fact the two approaches are equivalent[2]. Above we first set $\beta = N$, and let $b \rightarrow b_c$. In the other approach used below, we set $b = b_c$ and let $\beta \rightarrow N$. It is just a matter of choosing one parameter or another (in the action)

1.5 Orientability and the double-scaling limit

One interesting feature of the double-line graphs is that they allow us to say if their dual graph equivalent contains unorientable in addition to orientable ones.

To see this, consider the Hermitian matrix model, with propagator $< \bar{\phi}_{ab}\bar{\phi}_{cd} > = \frac{1}{N}\delta_{ad}\delta_{bc}$. This propagator is represented by a double-line, one for each index. One can

[2] We point out, however, that the N-powers counting is slightly different. For further details, see [12].

only connect the indices a-d, and b-c (fig. 1.2). So a given vertex will have connected double lines, each line having an arrow in the opposite direction of the other. Two connected arrows in the same direction implies contraction of indices. This means that one cannot twist a propagator, because the directions of the arrows would not match. On the other hand, ther are no problems with an even number of twists because we can always untwist the lines. In terms of the dual graphs equivalent, we would say that the surface is orientable. A twisted propagator would correspond, in the dual graph, to cutting an edge and reversing its orientation (fig. 1.4). Twisted propagators, and hence unorientable surfaces, would be possible with other matrices, such as antisymmetric ones.

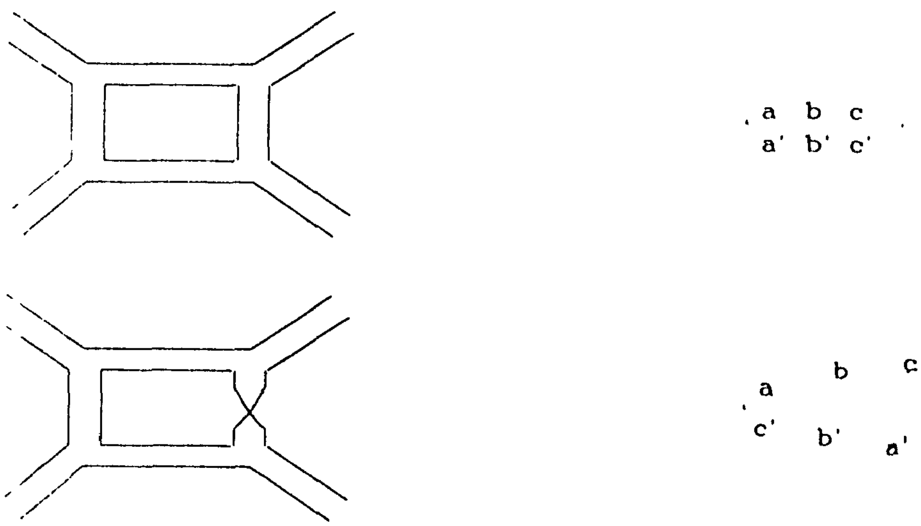


Figure 1.4: Orientable and unorientable surfaces.

One could think here that the Feynman expansion of a given model would tell us if there are unorientable surfaces in addition to the orientable ones in the continuum limit. When one goes to the double-scaling limit, however, the contribution from the unorientable surfaces may not survive. One such example is the antisymmetric matrix model, which has the same free energy as the Hermitian matrix model, despite the fact that its dual graph had unorientable surfaces. Low order calculations are done in the appendix for five different types of matrices including Hermitian matrices. The reason why unorientable contribution disappear in the double-scaling limit is not entirely clear [13]. So even though the dual graph of a Feynman expansion gives us both kinds of contributions, we actually have to solve for the free energy to see which contributions survive in the continuum limit (as explained in the next section). This is one motivation for solving the new matrix models in chapters 3 and 4.

1.6 Exact solution of Hermitian matrix model

The calculation in section 1.4 was carried out to show that it is possible to recover the continuum limit, and also to point out the way to do it (i.e. double scaling limit). We will use the same kind of limit in solving matrix models, but this time we will solve them exactly. To illustrate this procedure, let's consider the following simple model where we use a $N \times N$ Hermitian matrix ϕ,

$$\mathcal{Z}_N = \int d\phi \, e^{-S(\phi)} \tag{1.21}$$

$$= \int dU \int \prod_{i=1}^{N} (dx_i \, e^{-S(x_i)}) \Delta_N^2(x). \tag{1.22}$$

In the second line, ϕ was diagonalized using unitary matrices, and the $x_i's$ are its eigenvalues. The integral over the unitary group (dU) is irrelevant here, because it is only an overall normalization constant. Δ_N is a Vandermonde determinant,

$$\Delta_N(x) = \prod_{1 \le j < i \le N} (x_i - x_j) = \det \begin{bmatrix} 1 & x_1 & x_1^2 & \cdots & x_1^{N-1} \\ 1 & x_2 & x_2^2 & \cdots & x_2^{N-1} \\ \vdots & & & & \vdots \\ 1 & x_N & x_N^2 & \cdots & x_N^{N-1} \end{bmatrix}. \tag{1.23}$$

Above, the factor Δ_N^2 is the Jacobian for the change in variables form those in the measure (1.11) and those in eq 1.22. In solving matrix models, one always uses the technique of going from an integral over (all the elements of) a matrix to an integral over eigenvalues [14], or some simpler set of parameters.

By transforming the determinant (adding multiples of columns), we can get the following form,

$$\Delta_N(x) = \det \begin{bmatrix} P_0(x_1) & P_1(x_1) & P_2(x_1) & \dots & P_{N-1}(x_1) \\ P_0(x_2) & P_1(x_2) & P_2(x_2) & \dots & P_{N-1}(x_2) \\ \vdots & & & & \vdots \\ P_0(x_N) & P_1(x_N) & P_2(x_N) & & P_{N-1}(x_N) \end{bmatrix} \tag{1.24}$$

where the P's are orthogonal polynomials defined as,

$$\int dx \, e^{-S(x)} P_k(x) P_l(x) = \delta_{k,l} h_k \tag{1.25}$$

The partition function is then easily found,

$$\mathcal{Z} = N! \prod_{i=0}^{N-1} h_i. \tag{1.26}$$

Instead of working out all the weights h_i and the polynomials (which would require a lot of work for large N), one generally consider ratios of partition functions,

$$\frac{h_N}{h_{N-1}}\left(1+\frac{1}{N}\right) = \frac{\mathcal{Z}_{N+1}\mathcal{Z}_{N-1}}{\mathcal{Z}_N^2} = \exp[-(\mathcal{F}_{N+1} - 2\mathcal{F}_N + \mathcal{F}_{N-1})] \tag{1.27}$$

$$= \lim_{N\to\infty} \exp\left[-\frac{\partial^2\mathcal{F}}{\partial N^2}\right]. \tag{1.28}$$

In the large N limit, one has $\partial_N^2\mathcal{F} \simeq -\log(h_N/h_{N-1})$. So the problem is reduced to finding the ratio of the last two weights and then integrating twice. This ratio can be found by using a recursion relation for R_k, where $R_k = h_k/h_{k-1}$, and $\lambda P_k = P_{k+1} + S_k P_k + R_k P_{k-1}$. The recursion relation will have the form $\lambda k/N =$ function of R_k, R_{k-1}, R_{k+1}, etc. To solve this equation, for large N with $k \to N$, we will need what is called a scaling ansatz. This comes from the fact that we want to "expand" the region of interest near a critical value of R. The justification for such an ansatz is given in chapter 2. The other details can be found in [7, 15]

The result, known as the Painlevé I equation, is

$$t = f^2 - \frac{1}{3}\partial_t^2 f \tag{1.29}$$

where $\partial_t^2\mathcal{F} = f$, and $t = N^{4/5}|1 - b_c/b|$ is called a scaling parameter. For large t, one finds the following solution for $\mathcal{F}$,

$$\mathcal{F} = \frac{4}{15}t^{5/2} + \frac{1}{24}\log t - \frac{7}{1440}t^{-5/2} + \ldots. \tag{1.30}$$

Each factor differ from the other by $t^{-5/2} = N^{-2}|1 - b_c/b|^{-2+\gamma} = g_R^2$ where $\gamma = -1/2$. This result agrees with the previous analysis, section 1.4.

Concerning the orientability, we remark that the first term in the solution for $\mathcal{F}$ is $t^{5/2} \sim N^2$, while in the second one we have $\log t \sim N^0$. This tells us that the first term corresponds to a sphere and the second to a torus. Doing the same at all orders we would find that only the even Euler numbers surfaces appear. But it does not mean that we only have orientable surfaces. Indeed, the Klein bottle is unorientable, but its Euler number is $\chi = 0$. The analysis of section 1.5, though, tells us that there are only contributions from oriented surfaces for this model. In general we cannot make this assumption based on the solution for the free energy only, and we have to compare with other known models.

1.7 Classification of matrix models

I will now introduce a simple classification scheme for matrix models, based on symmetric spaces The justification of such a classification is that it gives all the models for which the reduction of the matrix integral allows for a solution by a polynomial method (as sketched in the previous section). Indeed, we want to solve various models for different matrix ensemble, but we cannot choose any type of matrices in any combination (the choice of the potential is also restricted), because it could be impossible to reduce the matrix integral to an integral over eigenvalues (or some simpler set of parameters) This reduction is necessary because usually , the matrix integral is impossible to evaluate directly. So we need some kind of classification, or restrictions, on the types of models This is where symmetric spaces comes into play

1.7.1 Symmetric spaces

Symmetric spaces can be considered as an extension to the usual classification of Lie groups. A partial definition is as follows: any symmetric space X can be realized as a set of cosets G/K of a connected Lie group G on one of its subgroups K. This is only a partial definition because homogeneous spaces also satisfy this definition. The difference between the two spaces is that symmetric spaces satisfy some symmetries not found in the homogeneous case The classical types [16, 17], are listed in table 1.1. The two new models, in chapter 3 and 4, are in the DIII ensemble In fact, we generally talk about generator ensembles which are the set of matrices associated to the symmetric spaces[3] They corresponds, for chapter 3, to n odd, and for chapter 4, to n even. The next section deals with the reduction of matrix integrals, and the Jacobians for some of the matrix ensembles are given.

1.7.2 Reduction of the matrix integral

There are a few ways to reduce a matrix integral, depending on the type of matrix or the matrix ensemble For example, for the Hermitian matrix model, one can simply

[3] In topological terms, the generator ensemble is identified with the tangent space of a symmetric space Exponentiation of the generators gives us the so-called circular ensemble, which we will not discuss here

Cartan's notation	X=G/K	Rank	System of restricted roots	Multiplicity
A I	$SL(n,R)/SO(n)$	$n-1$	A_{n-1}	$m_\alpha = 1$
A II	$SU(2n)/Sp(n)$	$n-1$	A_{n-1}	$m_\alpha = 4$
A III	$SU(p,q)/S(U(p)\times U(q))$	n=min(p,q)	$p = 1, C_n$ $p > q, BC_n$	$m_\alpha = 2,\ m_{2\beta} = 1$ $m_\alpha = 2,\ m_{2\beta} = 1$ $m_\beta = 2(p-q)$
BD I	$SO(p,q)/SO(p)\times SO(q)$	n=min(p,q)	$p = q, D_n$ $p > q, B_n$	$m_\alpha = 1$ $m_\beta = (p-q)$ $m_\alpha = 1$
D III	$SO(2n)/U(n)$	$n/2$	$n = 2k, C_k$ $n = 2k+1$ BC_k	$m_\alpha = 4,\ m_{2\beta} = 1$ $m_\alpha = 4,\ m_{2\beta} = 1$ $m_\beta = 4$
C I	$Sp(n,R)/U(n)$	n	C_n	$m_\alpha = 1$
C II	$Sp(p,q)/Sp(p)\times Sp(q)$	n=min(p,q)	$p = q,\ C_n$ $p > q,\ BC_n$	$m_\alpha = 1,\ m_{2\beta} = 3$ $m_\alpha = 4,\ m_{2\beta} = 3$ $m_\beta = 4(p-q)$

Table 1.1: Classification of the symmetric spaces

diagonalize the matrix as it is done in Metha's book [14]. The result is that we get an integral over the eigenvalues with a Jacobian in terms of these eigenvalues. The idea behind this reduction is to simplify the matrix integral.

The symmetric spaces introduced in the previous subsection give us a systematic approach to identify the matrix ensemble for which a similar reduction is possible. For these spaces, however, the procedure of reducing the matrix integral is more complex. An analogy can be used to understand how the Jacobian and the set of integration parameters arise. Consider the volume element in flat space written in cartesian coordinates and in spherical coordinates, $\prod_{i=1}^{N} dx_i = d\omega\, dr\, r^{N-1}$. Spherical symmetry would allow us to integrate out the angular part $d\omega$ with r^{N-1} being the Jacobian and r the integration variable. For a symmetric space (matrix ensemble) the Haar measure can be decomposed in the same way with the symmetry provided by the associated group. In doing so we get the Jacobian and also a set of parameters (which are not necessarily the eigenvalues of the matrices). In the example above the symmetry is given by the coset $SO(N+1)/SO(N)$, which is the vector limit of rectangular matrices as studied in [18] (so it really is a vector model). For the technical details the reader is referred to Helgason's book [17]. Let us pr[illegible]

Cartan's notation	System of restricted roots	Jacobian
A I	A_{n-1}	$\prod_{i<j} \lvert x_i - x_j \rvert$
A II	A_{n-1}	$\prod_{i<j} \lvert x_i - x_j \rvert^4$
C I	C_n	$\prod_{i<j} \lvert x_i^2 - x_j^2 \rvert \prod x_i$
D III	$n = 2k$, C_k	$\prod_{i<j} (x_i^2 - x_j^2)^4 \prod x_i$
	$n = 2k + 1$, BC_k	$\prod_{i<j} (x_i^2 - x_j^2)^4 \prod x_i^5$

Table 1 2. Symmetric spaces and their Jacobians

say here that there is a relation between the multiplicities of the roots and the exponents in the Jacobian Table 1 2 gives some symmetric spaces and their associated Jacobian It should be noted here that matrix models with Hermitian matrices, antisymmetric matrices, etc, are not part of this classification. In fact they belong to the more standard groups like $U(n)$,$SO(n)$, $Sp(2n)$, and so on.

The symmetric spaces classification gives not only a broad range of models which are reducible (and a systematic way of reducing them), but also a convenient way to recognize the different matrix models.

Chapter 2

Complex matrix model with special couplings

In this chapter I will show the calculation of a matrix model with complex matrices [19]. In terms of groups, we say that they form the generator ensemble of the coset $\frac{U(N+M)}{U(N)\times U(M)}$, with $N = M$ [18]. The purpose of this model is to introduce some of the ideas that will be used in solving the two other models. Among other things, the ansatz used in the solution of the model here will be used, with slight modifications, in the two others because of the similarities between the polynomials used in both case.

2.1 Partition function

We first write the matrix integral as,

$$\mathcal{Z}_N = \int dM \, e^{-\beta V(M)} \tag{2.1}$$

where,

$$V(M) = aM^\dagger M + b/2\,(M^\dagger M)^2 \tag{2.2}$$

and M is a complex, $N \times N$, matrix. The special couplings $(M^\dagger M)^n$ were used to preserve a $U(N) \times U(N)$ symmetry which allows M to be diagonalized (e.g. no $(M^\dagger)^2$ or $M^{\dagger 2} M^2$ terms). This structure comes out directly of the coset construction [18]. After reduction to an integral over eigenvalues, we find the following Jacobian,

$$\mathcal{J} = \prod_{i=1}^{N} |x_i| \prod_{i<j}^{N} (x_i^2 - x_j^2)^2 \tag{2.3}$$

and the partition function becomes,

$$\begin{aligned} \mathcal{Z}_N &= \int_{-\infty}^{\infty} \prod_{i=1}^{N} dx_i \, \mathcal{J} \, e^{-\beta V(x_i)} \\ &= 2^{-N} \int_0^{\infty} \prod_{i=1}^{N} dy_i \prod_{i<j}^{N} (y_i - y_j)^2 \, e^{-\beta(a y_i + \frac{b}{2} y_i^2)}. \end{aligned} \tag{2.4}$$

Between the first and the second line we made a change of coordinate, $y_i = x_i^2$, $dy_i = 2x_i dx_i$, in order to get a Jacobian expressible in terms of a Vandermonde determinant (more on this below). To do this integral, however, we will need first to define orthogonal polynomials.

2.2 Orthogonal polynomials

Orthogonal polynomials are necessary because the Jacobian will be rewritten as a determinant After some manipulation of the determinant, the entries will be orthogonal polynomials P. So the whole determinant will give us a sum of products of these P's The exponential in the integral will serve as a measure weight for the polynomials and we will be able to evaluate the partition function. Let's define the following polynomials on the half-line,

$$P_k = y^k + lower\ orders \tag{2.5}$$

and

$$\int_0^{\infty} d\mu \, P_k P_l = h_k \delta_{k,l} \tag{2.6}$$

where $d\mu = dy e^{-\beta V(y)}$ We point out that they are different than those for the Hermitian matrix model because in that case they were defined on the whole axis. We see that the partition function, once evaluated, will be written in terms of h's But the free energy can be related to a ratio of $\mathcal{Z}$'s, so at the end of the calculation we will need to know a ratio of two h's. Such ratios are denoted as $R_k = h_k/h_{k-1}$ where R appears in the following recursion relation for the P's,

$$yP_k = P_{k+1} + S_k P_k + R_k P_{k-1} \tag{2.7}$$

which can easily be found from eq.2 5 and eq.2.6. So our problem is reduced to finding an expression for the R's and S's. In most matrix models, including the one that we solved in chapter 3, the evaluation of the partition function is not so hard. The complicated part of the problem comes in finding expressions for the auxiliary values appearing in the recursion relations for the polynomials, such as S in eq 2 7.

2.3 Recursion relations

Here, the approach we will follow is that we are going to try to find recursion relations for the P's other than (2.7). In doing so we'll find a set of two self-consistent equations in terms of R, S, k, a, b, and β. We consider first,

$$\partial_y P_k = kP_{k-1} + l.o. = \sum_l D_{k,l} P_l \tag{2.8}$$

where $D_{k,l} = 0$ for $l > k-1$, and $D_{k,k-1} = k$ The coefficients $D_{k,l}$, for $l < k-1$, can be found using the following integration,

$$\begin{aligned} \int_0^\infty dy \frac{d}{dy}(e^{-\beta V} P_k P_l) &= -P_k(0)P_l(0) && (2.9) \\ &= -\beta \int_0^\infty d\mu(a+by)P_k P_l \\ &\quad + \int d\mu(P_l \partial_y P_k + P_k \partial_y P_l) && (2.10) \end{aligned}$$

to get a general relation for the coefficients,

$$D_{k,l}h_l + D_{l,k}h_k = \beta h_l(a\delta_{k,l} + b[\delta_{k+1,l} + S_k\delta_{k,l} + R_k\delta_{k-1,l}]) - P_k(0)P_l(0) \tag{2.11}$$

Assigning different values to l, we can find some useful relations,

$$\begin{aligned} l &= k & 2D_{k,k}h_k &= 0 = \beta h_k(a+bS_k) - P_k^2(0) && (2.12) \\ l &= k-1 & D_{k,k-1}h_{k-1} + D_{k-1,k}h_k &= kh_{k-1} = \beta h_{k-1}(bR_k) - P_k(0)P_{k-1}(0) && (2.13) \\ l &= k-2 & D_{k,k-2}h_{k-2} &= -P_k(0)P_{k-2}(0) && (2.14) \\ l &< k-1 & D_{k,l}h_l &= -P_k(0)P_l(0) && (2.15) \end{aligned}$$

We see that because of the boundary terms, the $P(0)$'s, the derivative of the polynomials involve all lower order polynomials (i e. the lower bound on l is zero) So, as they stand eq.2.12–2 15 are not useful But, in combining eqs 2 12 and 2 13, we can eliminate the boundary terms,

$$\begin{aligned} P_k^2(0)P_{k-1}^2(0) &= (k - \beta b R_k)^2 h_{k-1}^2 && (2.16) \\ &= \beta^2(a+bS_k)(a+bS_{k-1})h_k h_{k-1} && (2.17) \end{aligned}$$

So, we finally have,

$$(a + bS_k)(a + bS_{k-1})R_k = (\frac{k}{\beta} - bR_k)^2 \tag{2.18}$$

This recursion relation will be slightly modified below to get one of the two relations needed to solve for R and S. To get another we will consider instead,

$$y\partial_y P_k = kP_k + l.o. = \sum_l F_{k,l} P_l \tag{2.19}$$

where $F_{k,l} = 0$ for $l > k$, and $F_{k,k} = k$. The extra y will eliminates boundary terms. Again, to get the coefficients, we consider a particular integral,

$$\begin{aligned}\int_0^\infty dy\, y\partial_y(e^{-\beta V} P_k P_l) &= -\int_0^\infty d\mu\, P_k P_l \\ &= -\delta_{k,l} h_k \qquad (2.20) \\ &= -\beta \int_0^\infty d\mu\, y(a + by)P_k P_l \\ &\quad + \int_0^\infty d\mu\, (P_l y\partial_y P_k + P_k y\partial_y P_l) \qquad (2.21)\end{aligned}$$

from which we get a general relation for the F's,

$$\begin{aligned}F_{k,l}h_l + F_{l,k}h_k &= \beta h_l[a(\delta_{k+1,l} + S_k\delta_{k,l} + R_k\delta_{k-1,l}) \\ &\quad + b\{\delta_{k+2,l} + (S_{k+1} + S_k)\delta_{k+1,l} \\ &\quad + (R_{k+1} + S_k^2 + R_k)\delta_{k,l} \\ &\quad + (S_k + S_{k-1})R_k\delta_{k-1,l} \\ &\quad + R_k R_{k-1}\delta_{k-2,l}\}] - \delta_{k,l}h_k. \qquad (2.22)\end{aligned}$$

As before, we assign different values to l. In fact, $l = k$ is sufficient to get the other recursion relation (there are no boundary terms here, and we could get a recursion relation with a finite number of terms)

$$l = k: \quad 2F_{k,k}h_k = \beta h_k[aS_k + b(R_{k+1} + S_k^2 + R_k)] - h_k \tag{2.23}$$

and using $F_{k,k} = k$, we finally have,

$$\frac{2k+1}{\beta} = aS_k + b(R_k + R_{k+1} + S_k^2) \tag{2.24}$$

This is our first relation. Replacing in eq.2.18, we get a second relation,

$$(a + bS_k)(a + bS_{k-1})R_k = \frac{1}{4}[S_k(a + bS_k) + b(R_{k+1} - R_k) - \frac{1}{\beta}]^2 \tag{2.25}$$

2.4 Choice of the scaling ansatz

Before going on to the planar approximation and the solutions of the recursion relations, we need to find suitable ansatz for R, S. The choice that we will make is not arbitrary. We are not just dealing with power series expansion. In fact this choice is based on the fact that we want to "expand" the region of interest, which will require the introduction of scaling parameters and so on.

Let me start with a generic recursion relation of the form,

$$\frac{n}{\beta} = F(R_n, R_{n\pm1}, \ldots). \tag{2.26}$$

We introduce, for convenience, $\lambda = N/\beta$ and write

$$\frac{\lambda n}{N} = F(R_n, R_{n\pm1}, \ldots). \tag{2.27}$$

In the limit $n = N \to \infty$, we will use the notation $R_{n=N} = R(\lambda)$. Now, for large N and $n \simeq N$, we consider the following Taylor expansion of $R_n(\lambda)$ where $x = n/N \simeq 1$,

$$R_n(\lambda) = R(\lambda) + \frac{n-N}{N}\frac{\partial}{\partial x}R|_{x=1} + \frac{1}{2}\left(\frac{n-N}{N}\right)^2 \frac{\partial^2}{\partial x^2}R|_{x=1} + \tag{2.28}$$

Given that the dependence of $R_n(\lambda)$ on λ occurs as λ/N in eq 2.27, and that in the limit $n = N \to \infty$ this equation yields $R(\lambda)$, we guess that the n dependence can be written as

$$R_n(\lambda) = R(\lambda n/N) \tag{2.29}$$

for large N and $n \simeq N$.

We now use the following identities for a general function f,

$$\frac{\partial}{\partial x} f(\lambda x)|_{x=1} = \lambda \frac{\partial}{\partial \lambda} f(\lambda) \tag{2.30}$$

$$\frac{\partial^n}{\partial x^n} f(\lambda x)|_{x=1} = \lambda^n \frac{\partial^n}{\partial \lambda^n} f(\lambda) \tag{2.31}$$

and rewrite the R expansion as,

$$R_n(\lambda) \simeq R(\lambda) + \lambda\frac{n-N}{N}R'(\lambda) + \frac{1}{2}\lambda^2\left(\frac{n-N}{N}\right)^2 R''(\lambda) + \tag{2.32}$$

where the prime denotes a derivative with respect to λ

Expanding the recursion relation for $\lambda \simeq \lambda_c$, $R \simeq R_c$ (with $N \to \infty$ and $n \simeq N$), we first have,

$$\lambda_c = F(R_c) \tag{2.33}$$

where $R(\lambda_c) = R_c$. We also know that near these critical values the recursion relation takes the form,

$$\lambda = F(R, R', R'', \ldots). \tag{2.34}$$

So we try a scaling solution for $\lambda \simeq \lambda_c$ (which is the expansion we talked about above),

$$\beta^{\mu}(R(\lambda) - R_c) = f(\beta^{\alpha}(\lambda_c - \lambda)) \tag{2.35}$$

and define the scaling parameter t as,

$$t = \beta^{\alpha}(\lambda_c - \lambda). \tag{2.36}$$

The derivative appearing in the R expansion takes the form,

$$R' = \frac{\partial}{\partial\lambda}R(\lambda) = \beta^{-\mu}(-\beta^{\alpha})\dot{f} \tag{2.37}$$

$$R'' = \frac{\partial^2}{\partial\lambda^2}R(\lambda) = \beta^{-\mu}(-\beta^{\alpha})^2\ddot{f}. \tag{2.38}$$

We now write $R_n(\lambda)$ for $n = N + l$ using the scaling solutions for $\lambda, R, R', \ldots$

$$\begin{aligned}
R_{N+l}(\lambda) &= R_c + \beta^{-\mu}f(t) + \lambda\left(\frac{N+l-N}{N}\right)\beta^{-\mu}(-\beta^{\alpha})\dot{f} \\
&\quad + \frac{1}{2}\lambda^2\left(\frac{N+l-N}{N}\right)^2\beta^{-\mu}(-\beta^{\alpha})^2\ddot{f} + \ldots \\
&= R_c + \beta^{-\mu}\left(f(t) + \frac{l\lambda}{\lambda\beta}(-\beta^{\alpha})\dot{f} + \frac{1}{2}\frac{l^2\lambda^2}{\beta^2\lambda^2}(-\beta^{\alpha})^2\ddot{f} + \ldots\right) \\
&= R_c + \beta^{-\mu}\exp\left(\frac{-l\beta^{\alpha}}{\beta}\frac{\partial}{\partial t}\right)f \\
&= R_c\left(1 + \beta^{-\mu}\exp\left(-l\beta^{-\nu}\frac{\partial}{\partial t}\right)\hat{f}\right).
\end{aligned} \tag{2.39}$$

In the last line we just used a rescaling of f and also made the change $\alpha \to 1 - \nu$ so that the scaling parameter takes the form,

$$t = \beta^{-\nu}(\lambda_c\beta - N). \tag{2.40}$$

2.5 Planar approximation

In the so-called planar approximation, ansatz are reduced to the minimum. That is, they are reduced to constants ($k = N \to \infty$, so, $R_{k\pm l} \to R$ and $S_{k\pm l} \to S$). This, together with the criticality condition, will allow us to determine the critical values appearing in these ansatz. We also let $k \to \infty$, while keeping $k/\beta = \lambda$ fixed. This is just the double scaling limit. Eq.2.24 becomes,

$$2\lambda = aS + 2bR + bS^2. \tag{2.41}$$

Similarly, from eq.2.25 we have,

$$\begin{aligned} (a+bS)^2 R &= S^2(a+bS)^2/4 \\ R &= S^2/4. \end{aligned} \tag{2.42}$$

Inserting in eq.2.41 we find,

$$2\lambda = aS + 3/2\, bS^2. \tag{2.43}$$

We now require, for criticality [5], $\partial_S \lambda = 0 = a + 3bS$. This gives us the critical value $S_c = -a/3b$. Inserting in eq.2.43 and 2.42 we also get $\lambda_c = -a^2/12b$ and $R_c = \frac{1}{36}a^2/b^2$. If we choose $\lambda_c = 1/2$ and $R_c = 1$, then we find all the values,

$$\begin{aligned} \lambda_c = 1/2 \quad R_c = 1 \qquad S_c = 2 \\ a = 1 \quad b = -1/6. \end{aligned} \tag{2.44}$$

2.6 Solutions of the recursion relations

We are now ready to insert the full ansatz in our two recursion relations. We choose the scaling solutions to be,

$$S_{N+l} = 2(1 - \beta^{-\mu}\exp(-l\beta^{-\nu}\frac{\partial}{\partial t})f) \tag{2.45}$$

$$R_{N+l} = 1 - \beta^{-\mu}\exp(-l\beta^{-\nu}\frac{\partial}{\partial t})(g_0 + \beta^{-\nu}g_1 + g_2\beta^{-2\nu} + \dots) \tag{2.46}$$

where t is defined by $t = (\beta/2 - N)\beta^{-\nu}$. We also write N instead of k because we are only interested in the large N behavior of R and S (and so we replace k by N in our recursion relations). In the R ansatz we added some more degrees of freedom by expanding the

order	eq.2 25	eq.2.24
β^0	$4/9 = 4/9$	1=1
$\beta^{-\mu}$	$g_0 = 2f$	$g_0 = 2f$
$\beta^{-\mu-\nu}$	$g_1 = f'$	$g_0' = 2g_1$
$\beta^{-2\mu} = \beta^{-\mu-2\nu}$	$g_2 = 1/4f'' - f^2$	$2t = 2/3\,f^2 - 1/3\,g_2 + 1/6\,g_1' - 1/12\,g_0''$

Table 2 1: Lowest order solutions of the two recursion relations for the complex matrices matrix model

g function in powers of $\beta^{-\nu}$ This will be necessary to get consistent solutions beyond leading order in $\beta^{-\nu}$. Doing the same thing for f would only yield redundant equations

After inserting in the recursion relations, and working out the lowest order equations we see that a consistent solution requires $\mu = 2\nu$. For the two recursion relations, the results are given, order by order, in table 2 1.

From eq 2.24, we obtain the values of the exponents, $\nu = 1/5$, and $\mu = 2/5$, which is consistent with our previous relation between μ and ν. Also, using the relations between the g coefficients and f from the first recursion relation, we get, at order $\beta^{-\nu-2\nu}$ in the second recursion relation, a differential equation for f,

$$2t = f^2 - 1/12\,f'' \tag{2 47}$$

which is the well-known Painlevé I equation, upon renormalization of t. At order $\beta^{-\mu}$ and $\beta^{-\mu-\nu}$, we simply get equalities (e g $0 = 0$), which come from our criticality requirement (first derivative of λ with respect to S is zero)

2.7 Solution for the free energy

We can solve this equation pertubatively for large t. So, the value of R is determined All that we have to do is to solve for the partition function. One easily finds (as in chapter

1),

$$\mathcal{Z}_N = 2^{-N} N! \prod_{i=1}^{N} h_i. \tag{2.48}$$

We calculate, as usual, the ratio,

$$\begin{aligned} \frac{\mathcal{Z}_{N+1}\mathcal{Z}_{N-1}}{\mathcal{Z}_N^2} = \frac{2(N+1)h_{N+1}}{2Nh_N} &= (1+1/N)R_{N+1} \\ &\simeq 1 - \beta^{-2/5} g_0 \\ &= 1 - \beta^{-2/5} 2f \end{aligned} \tag{2.49}$$

The ratio of the $\mathcal{Z}$'s can also be expressed as,

$$\begin{aligned} \ln \frac{\mathcal{Z}_{N+1}\mathcal{Z}_{N-1}}{\mathcal{Z}_N^2} &= -(F_{N+1} - 2F_N + F_{N-1}) \\ &= -\frac{\partial^2 F}{\partial N^2} \\ &= -\left(\frac{\partial t}{\partial N}\right)^2 \frac{\partial^2 F}{\partial t^2} \\ &= -\beta^{-2/5}.\frac{\partial^2 F}{\partial t^2} \end{aligned} \tag{2.50}$$

Finally, by comparing eqs.2.49 and 2.50, we have

$$\begin{aligned} -\beta^{-2/5}\frac{\partial^2 F}{\partial t^2} &= \ln(1 - 2f\beta^{-2/5}) \\ -\beta^{-2/5} F'' &\simeq -2f\beta^{-1/5} \\ F'' &\simeq 2f \end{aligned} \tag{2.51}$$

From the Painlevé I equation, we can find a well-known power series solution for f

$$f = \sqrt{2}t^{1/2} - \frac{1}{96}t^{-2} - \frac{49\sqrt{2}}{36864}t^{-9/2} + \tag{2.52}$$

Replacing in eq.2.51 we get,

$$F'' = 2\sqrt{2}t^{1/2} - \frac{1}{48}t^{-2} - \frac{49\sqrt{2}}{18432}t^{-9/2} + \tag{2.53}$$

From the power series, we recognize that the model describes even Euler number surfaces only, in agreement with the results in the appendix. Indeed, each power of t differ from the other by,

$$\begin{aligned} t^{-5/2} &= (\beta - N)^{-5/2}\beta^{-5/2\times(-1/5)} \\ &= \beta^{-2}(1 - N/\beta)^{-5/2} \\ &\simeq N^{-2}(1 - N/\beta)^{-5/2} \end{aligned} \tag{2.54}$$

in the limit that $\beta \to N$. So we have even powers of N only. And because this is exactly the same series as for the Hermitian matrix model, we conclude that this model describes oriented surfaces only.

In fact, to compare with other models, we generally consider a ratio of coefficients to take into account that F and t have unknown normalization. For example, in the following series,

$$F'' = t^{1/2}(c_0 + \prod_{k=1} c_k t^{-5/2 k}) \tag{2.55}$$

one would consider ratios like,

$$\frac{c_j c_k}{c_l c_m} \tag{2.56}$$

where $j + k = l + m$, so that all powers of t cancels. Here, and throughout the thesis, we will use the ratio $c_0 c_2 / c_1^2$ (involving the first three coefficients). We have,

$$\frac{c_0 c_2}{c_1^2} = -\frac{49}{2}. \tag{2.57}$$

This value will serve as a reference when solving other models.

Chapter 3

First model in the DIII generator ensemble

In this chapter we consider the first of the two models in the DIII generator ensemble matrices are the generators of $SO(2n)/U(n)$ for n odd. As usual, we will define recursion relations for the orthogonal polynomials (section 3.2), but in the evaluation of the determinant (section 3.3) we will also need to define recursion relations for the partition function itself as well as for an auxiliary quantity. This is a variation on a standard technique used for other models. In section 3.5 we will solve these recursion relations with ansatz based on those used in chapter 2. Finally, in section 3.6 we solve for the free energy and compare with previous models.

3.1 Partition function

The Jacobian for this model is,

$$\mathcal{J} = \prod_{i}^{N} |x_i^5| \prod_{i<j}^{N} (x_i^2 - x_j^2)^4 \tag{3.1}$$

And we are going to make the analysis with the potential $V = ax^2 + b/2\, x^4$. The reason why there are no linear or cubic term is that we are dealing with antisymmetric matrices (defined at the end of the appendix). And we know that odd powers of antisymmetric matrices are traceless. We use in fact the simplest form for a potential. Higher order potential would only yield more complicated equations. The partition function can then be written as follows,

$$\mathcal{Z}_N = \int_{-\infty}^{\infty} \prod_i^N dx_i \, \mathcal{J} \, e^{-\beta V(x_i^2)} \tag{3.2}$$

$$= 2^{-N} \int_0^{\infty} \prod_i^N dy_i \prod_i^N y_i^2 \prod_{i<j}^N (y_i - y_j)^4 e^{-\beta(ay_i + b/2y_i^2)}. \tag{3.3}$$

Between the first and the second line, we used the substitution $y_i = x_i^2$, $dy_i = 2x_i dx_i$. The complicated part of this integral is the Jacobian For convenience of evaluation we will rewrite it as a determinant [14]. This is the usual approach to this kind of problem. We have,

$$\prod_i^N y_i^2 \prod_{i<j}^N (y_i - y_j)^4 = \det \begin{bmatrix} P_0(y_1) & P_1(y_1) & \dots & P_{2N-1}(y_1) \\ y^2\partial_y P_0(y_1) & y^2\partial_y P_1(y_1) & \dots & y^2\partial_y P_{2N-1}(y_1) \\ P_0(y_2) & P_1(y_2) & \dots & P_{2N-1}(y_2) \\ \vdots & \vdots & & \vdots \\ y^2\partial_y P_0(y_N) & y^2\partial_y P_1(y_N) & \dots & y^2\partial_y P_{2N-1}(y_N) \end{bmatrix}. \tag{3.4}$$

The P's are the usual orthogonal polynomials on the half-line, defined as

$$P_n(y) = y^n + l.o. \qquad P_0(y) = 1 \tag{3.5}$$

$$\int_0^{\infty} d\mu \, P_n(y) P_m(y) = h_n \delta_{m,n} \quad d\mu = dy \, e^{-\beta V(y)} \tag{3.6}$$

and from which we derive the following recursion relation,

$$yP_n = P_{n+1} + S_n P_n + R_n P_{n-1}. \tag{3.7}$$

3.2 Recursion relations

From the form of the determinant it is clear that we have to find a recursion relation for $y^2\partial_y P_n(y)$ Indeed, the recursion relation will tell us how to decompose the determinant to evaluate it. This idea will be clarified later. So we first write the general form of the recursion relation,

$$y^2\partial_y P_n = nP_{n+1} + \sum_{l=0}^{n} F_{n,l} P_l = \sum_{l=0}^{n+1} F_{n,l} P_l \tag{3.8}$$

The F's are $F_{n,n+1} = n$, $F_{n,l} = 0$ for $l > n + 1$, and $F_{n,l}$ undetermined for $l < n + 1$. One generally starts by considering the following integral,

$$\int_0^\infty dy\, y^2 \frac{d}{dy}(e^{-\beta V} P_n P_m) = -\int_0^\infty dy\, P_n P_m e^{-\beta V} 2y \tag{3.9}$$

$$\begin{aligned} &= -2\int_0^\infty dy\, P_n(P_{m+1} + S_m P_m + R_m P_{m-1})e^{-\beta V} \\ &= -2\delta_{n,m+1}h_n - 2S_m\delta_{n,m}h_n - 2R_m\delta_{n,m-1}h_n \\ &= -2[y]_{n,m}h_m. \end{aligned} \tag{3.10}$$

The bracket notation in the last line requires some explanation. The bracket with the subscripts n, m means all possible paths from P_n to P_m by repeated applications of eq 3.7. In this case, because there is only one y, we apply the recursion relation only once. For a term $[y^k]$, one would apply the recursion relation k times. In other words, the resulting polynomials of $y^k P_m$ will have different degrees and their scalar product with P_n will be different than zero only if it is of the same degree. A picture can help to see what happens (fig 3.1). Each horizontal line represents a polynomial with a different degree. When one of them is multiplied by y^2, for example, there are different paths possible according to each term in the recursion relation. Suppose that we want to calculate $\int dy\, e^{-\beta V} y^2 P_n P_n$. Then we have to collect all P_n terms coming out of $y^2 P_n$. The calculation is as follows,

$$\begin{aligned} h_n[y^2]_{n,n} &= \int_0^\infty dy\, e^{-\beta V} y^2 P_n P_n \\ &= \int_0^\infty dy\, e^{-\beta V} P_n(R_{n+1}P_n + S_n^2 P_n + R_n P_n + \dots) \\ &= (R_{n+1} + S_n^2 + R_n)h_n \end{aligned} \tag{3.11}$$

This result can be read directly from fig 3.1.

We can also do the previous integral directly (without integrating by parts). In this case we get

$$\int_0^\infty dy\, y^2 \frac{d}{dy}(e^{-\beta V} P_n P_m) = \int_0^\infty dy\, y^2(-\beta V')e^{-\beta V} P_n P_m + \int_0^\infty dy\, y^2 e^{-\beta V} P_n' P_m$$

$$+ \int_0^\infty dy\, y^2 e^{-\beta V} P_n P_m' \tag{3.12}$$

$$= -\beta[y^2 V'(y)]_{m,n}h_m + \sum_l (F_{n,l}\delta_{l,m}h_m + F_{m,l}\delta_{l,n}h_n) \tag{3.13}$$

$$= -\beta[y^2 V'(y)]_{n,m}h_m + F_{n,m}h_m + F_{m,n}h_n \tag{3.14}$$

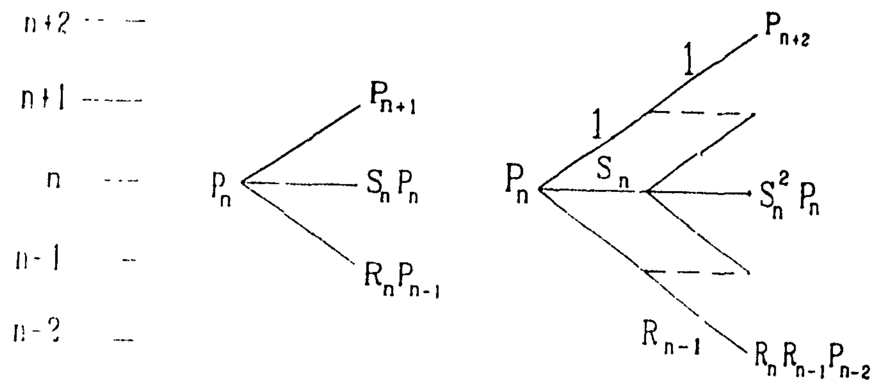


Figure 3.1: Pictorial representation of a scalar product of two polynomials.

Comparing the two results, we have,

$$F_{n,m}h_m + F_{m,n}h_n = \beta[y^2V'(y)]_{n,m}h_m - 2[y]_{n,m}h_m. \tag{3.15}$$

To find the different coefficients we simply have to assign different values to m. We start with $m = n$ and we stop with $m = n+3$. This is because our potential V is of order two. So y^2V' is of order three. When the difference between m and n is four or more, $[y^2V'(y)]_{n,m}$ is obviously zero. The coefficients are,

$$\begin{aligned}
m = n &\rightarrow F_{n,n} = \frac{\beta}{2}[y^2V'(y)]_{n,n} - [y]_{n,n} \\
m = n-1 &\rightarrow F_{n,n-1} = (\beta[y^2V'(y)]_{n-1,n} - 2[y]_{n-1,n} - (n-1))\, R_n \\
m = n-2 &\rightarrow F_{n,n-2} = \beta[y^2V'(y)]_{n-2,n}R_nR_{n-1} \\
m = n-3 &\rightarrow F_{n,n-3} = \beta[y^2V'(y)]_{n-3,n}R_nR_{n-1}R_{n-2} \\
m < n-3 &\rightarrow F_{n,m} = 0.
\end{aligned}$$

For simplicity we will write $C = [y^2V'(y)]$ and $D = [y]$. The recursion relation can finally be written as

$$\begin{aligned}
y^2\partial_y P_n = \;& nP_{n+1} + \left(\frac{\beta}{2}C_{n,n} - D_{n,n}\right) P_n + (\beta C_{n-1,n} - n - 1)R_nP_{n-1} + \\
& \beta C_{n-2,n}R_nR_{n-1}P_{n-2} + \beta C_{n-3,n}R_nR_{n-1}R_{n-2}P_{n-3}.
\end{aligned} \tag{3.16}$$

3.3 Evaluation of the determinant

We can now evaluate the partition function. But instead of doing it directly, we will find a recursion relation for $\mathcal{Z}$. The way to do it is to consider the upper two lines of the determinant, and look at all the possible products of polynomials giving non zero answers. We write the Jacobian of the partition function $\mathcal{Z}_{N+1}$ as,

$$J = \det \begin{bmatrix} P_0(y_1) & P_1(y_1) & \cdots & P_{2N-2}(y_1) & P_{2N-1}(y_1) & P_{2N}(y_1) & P_{2N+1}(y_1) \\ \vdots & & & P'_{2N-2}(y_1) & P'_{2N-1}(y_1) & P'_{2N}(y_1) & P'_{2N+1}(y_1) \\ \cdot & & & P_{2N-2}(y_2) & P_{2N-1}(y_2) & P_{2N}(y_2) & P_{2N+1}(y_2) \\ \vdots & & & P'_{2N-2}(y_2) & P'_{2N-1}(y_2) & P'_{2N}(y_2) & P'_{2N+1}(y_2) \\ & & & & & & \\ & & & & & & \end{bmatrix} \qquad (3.17)$$

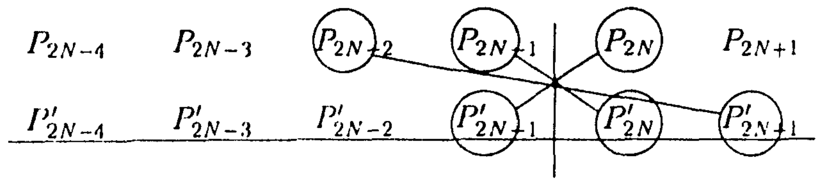
where the $'$ denotes $y^2\partial_y$.

We will now show how to find one of these contributions before listing them all. For example consider the 2×2 block in the upper right corner. In P'_{2N+1} there is a factor P_{2N} coming from the recursion relation that we found. So in the evaluation of the determinant we will have a term like $(N+1) < P'_{2N}(y_1), P_{2N+1}(y_1) > Z_{2N}$. The $(N+1)$ factor comes from the fact that we have (N+1) such terms (the other ones being $P'_{2N-1}(y_2)P_{2N-2}(y_2)$, etc.). The term Z_N is just the subdeterminant that is left after the elimination of the two rows and columns. The notation Z represents the determinant, and $\mathcal{Z}$ the partition function, which is just an integration with appropriate weights of Z. So we see how the evaluation of the partition function arises. The determinant is a sum of products of polynomials, and we integrate over these products using their orthogonality relation. It should be noted here that the factor 2^{-N} in the partition function (3.3) was not taken into account because one can always redefine $\mathcal{Z} = 2^{-N}\hat{\mathcal{Z}}$ and at the end of the calculation, we consider a ratio of $\mathcal{Z}$'s for which these factors are irrelevant.

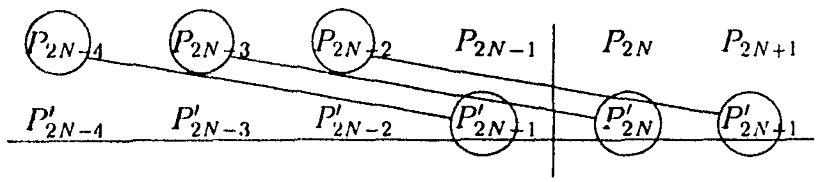
We now make a list of all the contributions with a crude graph of the first two lines of the determinant in each case

$$\begin{array}{cccc|cc} P_{2N-4} & P_{2N-3} & P_{2N-2} & P_{2N-1} & P_{2N} & P_{2N+1} \\ P'_{2N-4} & P'_{2N-3} & P'_{2N-2} & P'_{2N-1} & P'_{2N} & P'_{2N+1} \\ \hline \end{array}$$

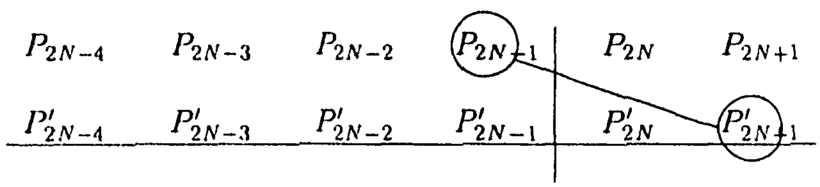
$(< P'_{2N+1}, P_{2N} > - < P'_{2N}, P_{2N+1} >)(N+1)Z_N$



$< P'_{2N+1}, P_{2N-2} > (< P'_{2N}, P_{2N-1} > - < P'_{2N-1}, P_{2N} >)N(N+1)Z_{N-2}$

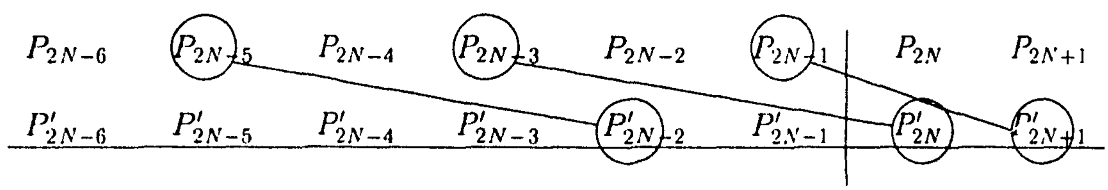


$< P'_{2N+1}, P_{2N-2} >< P'_{2N}, P_{2N-3} >< P'_{2N-1}, P_{2N-4} > (N-1)N(N+1)Z_{N-2}$



$< P'_{2N+1}, P_{2N-1} > (N+1)Y_N$

In the last case we had to define another "partition function" Y_N in order to avoid an infinite recursion relation due to the following pattern,



$< P'_{2N+1}, P_{2N-1} >< P'_{2N}, P_{2N-3} >< P'_{2N-2}, P_{2N-5} > \ldots$

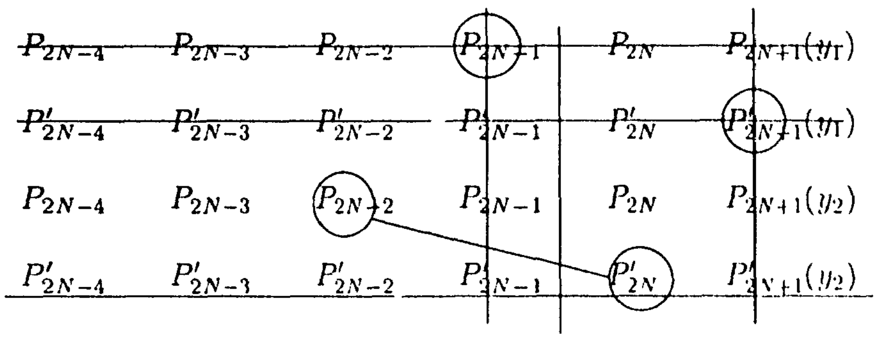
Indeed, in that case, we can go down to the first column using the last term in the recursion relation for P'.

We can now write the recursion relation for the partition function,

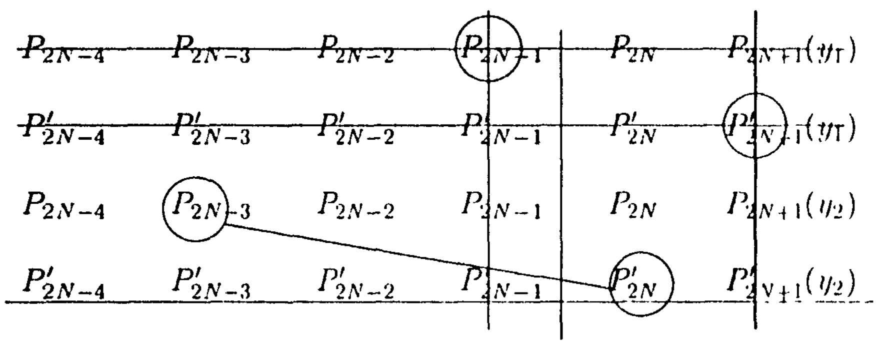
$$\begin{aligned} Z_{N+1} &= (N+1)(< P'_{2N}, P_{2N+1} > - < P'_{2N+1}, P_{2N} >)Z_N + N(N+1) \\ &< P'_{2N-2}, P_{2N+1} > (< P'_{2N-1}, P_{2N} > - < P'_{2N}, P_{2N-1} >)Z_{N-1} \end{aligned}$$

$$-N(N+1)(N-1) < P'_{2N-2}, P_{2N+1} >< P'_{2N-3}, P_{2N} >$$
$$< P'_{2N-4}, P_{2N-1} > \mathcal{Z}_{N-2} - (N+1) < P'_{2N-1}, P_{2N+1} > Y_N \qquad (3.18)$$

But now we must also find a recursion relation for the Y's. By considering the upper right corner of $\mathcal{Z}_{N+1}$ we have the following contributions,



$< P'_{2N}, P_{2N-2} > N Z_{N-1}$



$< P'_{2N}, P_{2N-3} > N Y_{N-1}$

We immediately write down,

$$Y_N = N < P'_{2N-2}, P_{2N} > \mathcal{Z}_{N-1} - N < P'_{2N-3}, P_{2N} > Y_{N-1} \qquad (3.19)$$
$$\downarrow$$
$$Y_{N+1} = (N+1) < P'_{2N}, P_{2N+2} > \mathcal{Z}_N - (N+1) < P'_{2N-1}, P_{2N+2} > Y_N \qquad (3.20)$$

We now define, for later convenience (we want to have a smooth planar limit as $N \to \infty$) the following two ansatz,

$$W_N = \frac{\mathcal{Z}_N}{\mathcal{Z}_{N-1} N \beta b h_{2N-1}} \qquad X_N = \frac{Y_N}{\mathcal{Z}_{N-1} N \beta b h_{2N-1}} \qquad (3.21)$$

and rewrite eqs.3.18 and 3.20 in terms of polynomials in W and X. After some algebraic manipulations, eq 3 18 becomes

$$\begin{aligned}W_{N+1}W_NW_{N-1} - \frac{W_NW_{N-1}}{b}\left([y^2V'(y)]_{2N,2N+1} - \frac{4N}{\beta} - \frac{2}{\beta}\right)\\ - \frac{W_{N-1}R_{2N}}{b}\left([y^2V'(y)]_{2N-1,2N} - \frac{4N}{\beta}\right)\\ + \frac{X_NW_{N-1}}{b}[y^2V'(y)]_{2N-1,2N+1} + R_{2N}R_{2N-1}R_{2N2} = 0.\end{aligned} \qquad (3\ 22)$$

And similarly for eq.3.20,

$$X_{N+1}W_N - \frac{[y^2V'(y)]_{2N,2N+2}}{b}R_{2N+2}W_N + R_{2N+2}X_N = 0. \qquad (3.23)$$

3.4 Planar approximation

We now turn to the planar approximation to get the critical values for W and X. We first have to expand the paths in eqs.3.22, 3.23,

$$[y^2V'(y)]_{2N,2N+2} \rightarrow [ay^2 + by^3]_{2N,2N+2} = a + b(S_{2N} + S_{2N+1} + S_{2N+2}) \qquad (3.24)$$

$$\begin{aligned}[y^2V'(y)]_{2N,2N+1} \rightarrow [ay^2 + by^3]_{2N,2N+1} &= a(S_{2N} + S_{2N+1})\\ &+b(R_{2N+2} + R_{2N+1} + R_{2N} + S_{2N+1}^2 + S_{2N}^2 + S_{2N+1}S_{2N}).\end{aligned} \qquad (3\ 25)$$

Using the critical values found in chapter 2 (except for N/β, more on this in the next section), $a = 1$, $b = -1/6$, $S = 2$, $R = 1$, and $N/\beta \rightarrow 1/4$, eq.3 24 becomes $a + 3bS = 0$, and eq.3 25 becomes $2aS + 3bR + 3bS^2 = 3/2$. Our two recursion relations, eqs 3 22, 3.23, take the simple form,

$$\begin{aligned}W^3 + 3W^2 + 3W + 1 &= 0 \qquad (3.26)\\ XW + X &= 0. \qquad (3.27)\end{aligned}$$

The solutions are $W = -1$ (triple root), while X remains undetermined We don t know the critical value of X, but it turns out that it will be determined when solving the recursion relations

3.5 Scaling ansatz

Let us look now at the full recursion relations that we want to solve,

$$
\begin{aligned}
W_{N+1}W_NW_{N-1} - \frac{W_NW_{N-1}}{b}&(a(S_{2N+1}+S_{2N}) + b(R_{2N}+R_{2N+1}+R_{2N+2}+S_{2N+1}^2 \\
&+S_{2N}^2+S_{2N+1}S_{2N}) - \frac{4N}{\beta} - \frac{2}{\beta}) + \frac{X_NW_{N-1}}{b}(a+b(S_{2N+1}+S_{2N}+S_{2N-1})) \\
&-W_{N-1}R_{2N}(R_{2N+1}+R_{2N}+R_{2N-1}+S_{2N-1}^2+S_{2N}^2+S_{2N-1}S_{2N}) \\
&-\frac{W_{N-1}R_{2N}}{b}\left(a(S_{2N}+S_{2N-1}) - \frac{4N}{\beta}\right) + R_{2N}R_{2N-1}R_{2N-2} = 0
\end{aligned} \tag{3.28}
$$

$$
X_{N+1}W_N - \frac{R_{2N+2}W_N}{b}(a+b(S_{2N+2}+S_{2N+1}+S_{2N})) + R_{2N+2}X_N = 0 \tag{3.29}
$$

The next step is to find suitable ansatz for R, S, W, and X. Here are some remarks about these ansatz. First, the critical values for a, b, R, and S are the same as those in section 2.5. The reason for this is that we can do the same analysis here as we did in chapter 2. Indeed, the potential is the same, and the polynomials are defined on the half line in both cases. In fact this analysis is independent of the form of the Jacobian. Only the result for the partition function $\mathcal{Z}_N$ is affected as well as the critical value of N/β. The other difference comes from the fact that for R and S, the index is $2N+l = 2(N+l/2)$. So, we will write $l/2$ instead of l in the exponential. W and X have standard ansatz,

$$
\begin{aligned}
R_{2N+l} &= 1-\beta^{-\phi}\exp(-\frac{l}{2}\beta^{-\nu}\frac{\partial}{\partial t})(g_0+g_1\beta^{-\nu}+g_2\beta^{-2\nu}+\ldots) && (3.30)\\
S_{2N+l} &= 2(1-\beta^{-\mu}\exp(-\frac{l}{2}\beta^{-\nu}\frac{\partial}{\partial t})f) && (3.31)\\
W_{N+l} &= -1+\beta^{-\rho}\exp(-l\beta^{-\nu}\frac{\partial}{\partial t})(h_0+\beta^{-\nu}h_1+\beta^{-2\nu}h_2+\ldots) && (3.32)\\
X_{N+l} &= X_c-\beta^{-\sigma}\exp(-l\beta^{-\nu}\frac{\partial}{\partial t})(k_0+\beta^{-\nu}k_1+\beta^{-2\nu}k_2+\ldots) && (3.33)\\
t &= \beta^{-\nu}(\frac{1}{4}\beta-N). && (3.34)
\end{aligned}
$$

After inserting the ansatz in the recursion relations, eqs. 3.28 and 3.29 (at $\beta^{-\nu}$ order), and using $\phi=\mu=2/5$, $\sigma=\rho=\nu=1/5$, we find the results given in table 3.1. We see that a consistent solution requires that $X_c = 0$. In doing so, we get two differential equations defining h_0 and k_0,

$$
\begin{aligned}
0 &= 6f+h_0k_0+k_0' && (3.35)\\
0 &= -6f'-12fh_0+h_0^3+3h_0h_0'+h_0''-6fk_0 && (3.36)
\end{aligned}
$$

order	eq.3 28	eq 3 29
$\beta^{-1/5}$	0	$h_0X_c = 0$
$\beta^{-2/5}$	$6fX_c = 0$	$-6f - g_0X_c + h_1X_c - h_0k_0 - k_0' = 0$
$\beta^{-3/5}$	$-3g_0' - 6g_0h_0 + h_0^3 + 3h_0h_0'$ $+h_0'' - 6fk_0 - 6fh_0X_c = 0$	$3f' + 6fh_0 + g_0k_0 - h_1k_0 + h_0k_0' - h_0k_1$ $k_0''/2 - g_1X_c + g_0'X_c + h_2X_c - k_1' = 0$

Table 3.1: Lowest order solutions of the two recursion relations for the DIII - odd n - matrix model.

We used the fact that $g_0 = 2f$, found from the chapter 2 analysis with R_{2k} and S_{2k} ansatz Using the known solution for f (same as in chapter 2 but with $t \to 2t$), and power series solution for h_0 and k_0,we find a set of algebraic equations that we can solve Finally, we end up with 4 solutions,

$$\begin{aligned}
h_0 &= \pm 2\sqrt{3}t^{1/4} + \frac{7}{8}t^{-1} \mp \frac{5\sqrt{3}}{384}t^{-9/4} + \dots \\
k_0 &= \mp 2\sqrt{3}t^{1/4} + \frac{9}{8}t^{-1} \pm \frac{5\sqrt{3}}{384}t^{-9/4} + \dots
\end{aligned} \tag{3.37}$$

and

$$\begin{aligned}
h_0 &= \pm 2\sqrt{3}t^{1/4} - \frac{1}{8}t^{-1} \mp \frac{5\sqrt{3}}{384}t^{-9/4} + \dots \\
k_0 &= \mp 2\sqrt{3}t^{1/4} + \frac{1}{8}t^{-1} \pm \frac{5\sqrt{3}}{384}t^{-9/4} + \dots .
\end{aligned} \tag{3.38}$$

It turns out that we will only need h_0 in the solution of the free energy.

3.6 Solution of the free energy

We can now look at the free energy. We start with eqs. 3 21 and we invert the one with $\mathcal{Z}$,

$$\frac{\mathcal{Z}_N}{\mathcal{Z}_{N-1}} = N\beta b h_{2N-1} W_N \tag{3.39}$$

$$\frac{\mathcal{Z}_{N+1}}{\mathcal{Z}_N} = (N+1)\beta b h_{2N+1} W_{N+1}. \tag{3.40}$$

Taking the ratio of the two expressions we get,

$$\frac{\mathcal{Z}_{N+1}\mathcal{Z}_{N-1}}{\mathcal{Z}_N^2} = \frac{(N+1)h_{2N+1}W_{N+1}}{Nh_{2N-1}W_N} \simeq R_{2N+1}R_{2N}\frac{W_{N+1}}{W_N} \tag{3.41}$$

The W's can be written,

$$\begin{aligned} W_{N+1} &= -1(1-\beta^{-1/5}(h_0+h_1\beta^{-1/5}+\ldots-\beta^{-1/5}h_0'-\beta^{-2/5}h_1'+h.o.)) \\ &= -1(1-h_0\beta^{-1/5}-(h_1-h_0')\beta^{-2/5})+h.o. \end{aligned} \tag{3.42}$$

$$\begin{aligned} W_N &= -1(1-\beta^{-1/5}(h_0+h_1\beta^{-1/5}+h.o\,)) \\ &= -1(1-h_0\beta^{-1/5}-h_1\beta^{-2/5})+h.o \end{aligned} \tag{3.43}$$

From which we calculate the ratio of W's,

$$\begin{aligned} \frac{W_{N+1}}{W_N} &\simeq \frac{1-h_0\beta^{-1/5}-(h_1-h_0')\beta^{-2/5}}{1-h_0\beta^{-1/5}-h_1\beta^{-2/5}} \\ &= (1-h_0\beta^{-1/5}-(h_1-h_0')\beta^{-2/5})\,(1+h_0\beta^{-1/5}+h_1\beta^{-2/5}+h_0^2\beta^{-2/5}) \\ &= 1-h_0^2\beta^{-2/5}+(h_1+h_0^2)\beta^{-2/5}-(h_1-h_0')\beta^{-2/5} \\ &= 1+h_0'\beta^{-2/5}. \end{aligned} \tag{3.44}$$

Similarly, for the R's,

$$R_{2N+1} \sim R_{2N} = 1-\beta^{-2/5}g_0 = 1-\beta^{-2/5}2f. \tag{3.45}$$

We then have,

$$\frac{\mathcal{Z}_{N+1}\mathcal{Z}_{N-1}}{\mathcal{Z}_N^2} \simeq (1-2f\beta^{-2/5})(1-2f\beta^{-2/5})(1+h_0'\beta^{-2/5}) \simeq 1-(4f-h_0')\beta^{-2/5} \tag{3.46}$$

The ratio of the $\mathcal{Z}$'s can also be expressed as in (2.50),

$$\ln\frac{\mathcal{Z}_{N+1}\mathcal{Z}_{N-1}}{\mathcal{Z}_N^2} = -\beta^{-2/5}\frac{\partial^2 F}{\partial t^2}\,. \tag{3.47}$$

So we finally have, by comparing eqs. 3.46 and 3.47,

$$\begin{aligned} -\beta^{-2/5}\frac{\partial^2 F}{\partial t^2} &= \ln(1-(4f-h_0')\beta^{-2/5}) \\ &\simeq -4f+h_0'\beta^{-2/5} \\ F'' &\simeq 4f-h_0' \end{aligned} \tag{3.48}$$

Between the first and the second line we used the expansion of the natural logarithm Using the solution for f and the solutions (3.37) for h_0, we find,

$$F'' = 8t^{1/2} \mp \frac{\sqrt{3}}{2} t^{-3/4} + \frac{83}{96} t^{-2} + \dots \tag{3.49}$$

In that case, the universal ratio yields 83/9, which is a new result. Doing the same with (3.38), we get,

$$F'' = 8t^{1/2} \mp \frac{\sqrt{3}}{2} t^{-3/4} - \frac{13}{96} t^{-2} + \quad . \tag{3.50}$$

In this second case the ratio is $-13/9$, which is the same result as in [20, 21] although the matrix ensemble in these papers is different. From previous solutions this is the expected ratio for the free energy of pure 2D quantum gravity with oriented and unoriented surfaces. One surprising result is that our two solutions differ only by t^{-2} (verified up to 15th order of F''), which is the term corresponding to the torus and the Klein bottle.

Chapter 4

Second model in the DIII generator ensemble

In this chapter we consider the second of the two models in the DIII generator ensemble – matrices are the generators of $SO(2n)/U(n)$, n even. Again, we will need to define recursion relations, but due to the nature of the Jacobian, we will first find relations between orthogonal polynomials and skew-orthogonal polynomials (section 4.2). This is a new technique. In section 4.3 we will define suitable quantities from which recursion relations will emerge. Standard scaling ansatz are proposed in section 4.6 and a solution for the principal recursion relation is found. Finally, in the last section, we solve for the free energy and , again, compare with previous models.

4.1 The partition function

The Jacobian for this model is,

$$\mathcal{J} = \prod_{i}^{N} |x_i| \prod_{i<j}^{N} (x_i^2 - x_j^2)^4 \tag{4.1}$$

and we are going to make the analysis, as usual, with the potential $V = ax^2 + b/2\, x^4$. The partition function can then be written as follows,

$$\begin{aligned} \mathcal{Z}_N &= \int_{-\infty}^{\infty} \prod_{i}^{N} dx_i\, \mathcal{J}\, e^{-\beta V(x_i^2)} && (4.2) \\ &= 2^{-N} \int_{0}^{\infty} \prod_{i}^{N} dy_i \prod_{i<j}^{N} (y_i - y_j)^4 e^{-\beta(a y_i + b/2 y_i^2)} && (4.3) \end{aligned}$$

Between the first and the second line, we used the substitution $y_i = x_i^2$, $dy_i = 2x_i dx_i$. The complicated part is, again, the Jacobian. For convenience of evaluation we rewrite it as a determinant [8]. We have,

$$\prod_{i<j}^{N}(y_i - y_j)^4 = \det \begin{bmatrix} Q_0(y_1) & Q_1(y_1) & \cdots & Q_{2N-1}(y_1) \\ \partial_y Q_0(y_1) & \partial_y Q_1(y_1) & \cdots & \partial_y Q_{2N-1}(y_1) \\ Q_0(y_2) & Q_1(y_2) & \cdots & Q_{2N-1}(y_2) \\ \vdots & \vdots & & \vdots \\ \partial_y Q_0(y_N) & \partial_y Q_1(y_N) & \cdots & \partial_y Q_{2N-1}(y_N) \end{bmatrix} \tag{4.4}$$

where the Q's are Metha's skew-orthogonal polynomials,

$$Q_i(y) = y^i + l.o. \tag{4.5}$$

and

$$< Q_i, Q_j >_Q = \frac{1}{2}\int dy\, e^{-\beta V(y)}(Q_i Q_j' - Q_i' Q_j) = q_{[i/2]} z_{ij} \tag{4.6}$$

with

$$z_{2i,2i+1} = 1 \tag{4.7}$$

$$z_{2i+1,2i} = -1 \tag{4.8}$$

all others being 0. With these definitions we can easily evaluate $\mathcal{Z}_N$,

$$\mathcal{Z}_N = 2^{-N/2} \prod_{i=0}^{N-1} q_i. \tag{4.9}$$

Here we used skew-orthogonal polynomials because with the orthogonal ones, $\mathcal{Z}_N$ cannot be evaluated (i.e. gives an infinite recursion relation for ∂P, this is what we found in chapter 2). On the other hand, with Q polynomials, $\mathcal{Z}_N$ is easy to find, but the problem is to establish recursion relations for them and the q's. Indeed, the only know recursion relation is infinite [8] The approach that we will follow is to relate the Q polynomials with the P polynomials (these are the usual orthogonal polynomials for which recursion relations are well-known, or at least, easy to find)

4.2 Relation between P and Q polynomials

We start with the general expansion of the P's in term of the Q's.

$$P_{2i} = Q_{2i} + \omega_{i1} Q_{2i-1} + \omega_{i2} Q_{2i-2} + \ldots \tag{4.10}$$

$$P_{2i+1} = Q_{2i+1} + \xi_{i1} Q_{2i} + \xi_{i2} Q_{2i-1} + \qquad (4.11)$$

In order to find odd and even ξ's and ω 's, we will consider four Q products. In the calculation we take the Q product of a P polynomial and a Q polynomial. The first line will show the result using the expansion of P's in terms of Q's. The following ones transform the Q product in a P product (orthogonal polynomial relation). We have,

$$\begin{aligned}
< Q_{2i-2j}, P_{2i+1} >_Q &= \xi_{i,2j} q_{i-j} \\
&= \frac{1}{2} \int dy\, e^{-\beta V} (Q_{2i-2j} P'_{2i+1} - Q'_{2i-2j} P_{2i+1}) \\
&= \frac{1}{2} \int dy\, e^{-\beta V} Q_{2i-2j} \partial_x P_{2i+1} \\
&= \frac{1}{2} e^{-\beta V} Q_{2i-2j} P_{2i+1} |_0^\infty - \frac{1}{2} \int dy (-\beta V') e^{-\beta V} Q_{2i-2j} P_{2i+1} \\
&= \frac{1}{2} < \beta V' Q_{2i-2j}, P_{2i+1} >_P - \frac{1}{2} Q_{2i-2j}(0) P_{2i+1}(0) \qquad (4.12)
\end{aligned}$$

Similarly, for $< Q_{2i-2j}, P_{2i} >_Q$, $< Q_{2i-2j+1}, P_{2i} >_Q$, and $< Q_{2i-2j+1}, P_{2i+1} >_Q$, we respectively have,

$$\begin{aligned}
\omega_{i,2j-1} q_{i-j} &= \frac{1}{2} < \beta V' Q_{2i-2j}, P_{2i} >_P - \frac{1}{2} Q_{2i-2j}(0) P_{2i}(0) \qquad (4.13) \\
-\omega_{i,2j} q_{i-j} &= \frac{1}{2} < \beta V' Q_{2i-2j+1}, P_{2i} >_P - \frac{1}{2} Q_{2i-2j+1}(0) P_{2i}(0) \\
&\quad - \frac{1}{2}(2i+1) \delta_{j0} h_{2i} \qquad (4.14) \\
-\xi_{i,2j+1} q_{i-j} &= \frac{1}{2} < \beta V' Q_{2i-2j+1}, P_{2i+1} >_P - \frac{1}{2} Q_{2i-2j+1}(0) P_{2i+1}(0). \qquad (4.15)
\end{aligned}$$

We will now consider the above equations for specific values of j. For eq. 4.12,

$$\begin{aligned}
&\xi_{i,2j} q_{i-j} &&= \frac{1}{2} < \beta V' Q_{2i-2j}, P_{2i+1} >_P - \frac{1}{2} Q_{2i-2j}(0) P_{2i+1}(0) \\
j = 0 \longrightarrow\ &\xi_{i,0} q_{i-0} &&= \frac{1}{2} < \beta(a + by) Q_{2i}, P_{2i+1} >_P - \frac{1}{2} Q_{2i}(0) P_{2i+1}(0) \\
&q_i &&= \frac{1}{2}\beta < (a + by)(P_{2i} + l.o.), P_{2i+1} >_P - \frac{1}{2} Q_{2i}(0) P_{2i+1}(0) \\
&q_i &&= \frac{1}{2}\beta b h_{2i+1} - \frac{1}{2} Q_{2i}(0) P_{2i+1}(0) \qquad (4.16) \\
j = 1 \longrightarrow\ &\xi_{i,2} q_{i-1} &&= \frac{1}{2} < \beta V' Q_{2i-2}, P_{2i+1} >_P - \frac{1}{2} Q_{2i-2}(0) P_{2i+1}(0) \\
&\xi_{i,2} q_{i-1} &&= -\frac{1}{2} Q_{2i-2}(0) P_{2i+1}(0) \qquad (4.17)
\end{aligned}$$

In the next-to-last line, the P product is 0 because the highest P polynomial in the Q expansion is of order $2i - 2$. When multiplied by the first derivative of the potential (of order y^1), we get at most a $2i - 1$ order polynomial which is not sufficient to produce something in a product with P_{2i+1}.

Similarly, eqs 4 13 and 4.14, with $j = 0, 1$, and eq 4 15 with $j = 0$ only, yields similar results Upon dividing these by β and h, we assemble all the useful recursion relations in the following,

$$\frac{q_i}{\beta h_{2i+1}} = \frac{b}{2} - \frac{1}{2}\frac{Q_{2i}(0)P_{2i+1}(0)}{\beta h_{2i+1}} \tag{4 18}$$

$$\frac{\xi_{i2}q_{i-1}}{\beta h_{2i+1}} = -\frac{1}{2}\frac{Q_{2i-2}(0)P_{2i+1}(0)}{\beta h_{2i+1}} \tag{4.19}$$

$$0 = [(a + bS_{2i}) - \omega_{i1}b] - \frac{Q_{2i}(0)P_{2i}(0)}{\beta h_{2i}} \tag{4.20}$$

$$\frac{\omega_{i1}q_{i-1}}{\beta h_{2i}} = -\frac{1}{2}\frac{Q_{2i-2}(0)P_{2i}(0)}{\beta h_{2i}} \tag{4 21}$$

$$\begin{aligned} \frac{-q_i}{\beta h_{2i}} &= \frac{1}{2}[bR_{2i+1} - \xi_{i1}(a + bS_{2i}) + (\xi_{i1}\omega_{i1} - \xi_{i2}b] \\ &\quad -\frac{1}{2}\frac{Q_{2i+1}(0)P_{2i}(0)}{\beta h_{2i}} - \frac{(2i+1)}{\beta} \end{aligned} \tag{4 22}$$

$$\frac{-\omega_{i2}q_{i-1}}{\beta h_{2i}} = \frac{b}{2} - \frac{1}{2}\frac{Q_{2i-1}(0)P_{2i}(0)}{\beta h_{2i}} \tag{4.23}$$

$$\frac{-\xi_{i1}q_i}{\beta h_{2i+1}} = \frac{1}{2}[(a + bS_{2i+1}) - b\xi_{i1}] - \frac{1}{2}\frac{Q_{2i+1}(0)P_{2i+1}(0)}{\beta h_{2i+1}}. \tag{4.24}$$

4.3 Scaling quantities

Instead of working directly with the quantities in the above relations, we will choose suitable variables which have a smooth planar limit. Let us define the following scaling quantities,

$$\begin{aligned} W_i = \frac{q_i}{\beta h_{2i}} \quad X_i = \frac{Q_{2i}(0)}{P_{2i}(0)} \quad Y_i = \frac{Q_{2i+1}(0)}{P_{2i}(0)} \\ Z_k = \frac{P_{k+1}(0)}{P_k(0)} \quad A_k = \frac{P_k(0)^2}{\beta h_k}. \end{aligned} \tag{4 25}$$

Now consider eqs. 4.18 - 4 24 written in terms of these quantities,

$$\frac{W_i}{R_{2i+1}} = \frac{b}{2} - \frac{1}{2}\frac{X_i A_{2i+1}}{Z_{2i}} \tag{4 26}$$

$$\frac{\xi_{i2}W_{i-1}}{R_{2i+1}R_{2i}R_{2i-1}} = -\frac{1}{2}\frac{X_{i-1}A_{2i+1}}{Z_{2i}Z_{2i-1}Z_{2i-2}} \tag{4.27}$$

$$0 = a + bS_{2i} - \omega_{i1}b - X_i A_{2i} \tag{4 28}$$

$$\frac{\omega_{i1} W_{i-1}}{R_{2i} R_{2i-1}} = -\frac{1}{2} \frac{X_{i-1} A_{2i}}{Z_{2i-1} Z_{2i-2}} \tag{4.29}$$

$$\begin{aligned} -2W_i &= bR_{2i+1} - \xi_{i1}(a + bS_{2i}) + (\xi_{i1}\omega_{i1} - \xi_{i2})b \\ &\quad - Y_i A_{2i} - \frac{(2i+1)}{3} \end{aligned} \tag{4.30}$$

$$\frac{-\omega_{i2} W_{i-1}}{R_{2i} R_{2i-1}} = \frac{b}{2} - \frac{1}{2} \frac{Y_{i-1} A_{2i}}{Z_{2i-1} Z_{2i-2}} \tag{4.31}$$

$$\frac{-2\xi_{i1} W_i}{R_{2i+1}} = a + bS_{2i+1} - b\xi_{i1} - \frac{Y_i A_{2i+1}}{Z_{2i}} \tag{4.32}$$

4.4 Planar approximation and critical values

It turns out that we will need to know only the critical values of X and W, because for a, b, etc, we use the same values as in chapter 3. So, we write

$$\begin{array}{ccccc} b \to -1/6 & \lambda = i/3 \to 1/4 & S \to 2 & Z \to -1 & \omega_{i,n} \to \omega_n \\ & a \to 1 & R \to 1 & A \to 2/3 & \xi_{i,n} \to \xi_n \end{array} \tag{4.33}$$

where the values for A and Z can be easily found using the explicit form given in section 4.5. With these critical values, eqs.4.26, 4.28, and 4.29 becomes,

$$12W = -1 + 4X \tag{4.34}$$

$$0 = 4 + \omega_1 - 4X \tag{4.35}$$

$$\omega_1 W = -\frac{1}{3} X \tag{4.36}$$

Combining eqs.4.36 and 4.35, we get a relation with X and W only. Using eq. 4.34 we solve for X and W,

$$\begin{aligned} X &= \frac{1}{2} \\ W &= \frac{1}{12} \end{aligned} \tag{4.37}$$

One also finds that $\omega_1 = -2$. To complete the analysis we will also work out the other critical values. Eqs.4.27, 4.30, 4.31 and 4.32 becomes

$$\frac{\xi_2}{12} = \frac{1}{6} \tag{4.38}$$

$$\frac{1}{6} = \frac{4}{6} + \frac{2}{3}(\xi_1 + Y) - \frac{1}{6}(2\xi_1 + \xi_2) \tag{4.39}$$

$$\frac{1}{12}\omega_2 = \frac{1}{12} + \frac{1}{3} Y \tag{4.40}$$

$$-\frac{1}{6}\xi_1 = \frac{2}{3} + \frac{1}{6}\xi_1 + \frac{2}{3} Y \tag{4.41}$$

from which we get $\xi_2 = 2$, $\xi_1 = -3/2$, $Y = 1/2$, and $\omega_2 = 3$

4.5 Recursion relation for W's

Before going further, we can take a look at the free energy to see exactly what quantities we have to know. The solution for the partition function was found to be,

$$\mathcal{Z}_N = 2^{-N} N! \prod_{i=1}^{N-1} q_i. \tag{4.42}$$

As usual we consider a ratio of $\mathcal{Z}$'s,

$$\frac{\mathcal{Z}_{N+1}\mathcal{Z}_{N-1}}{\mathcal{Z}_N^2} = (1+\frac{1}{N})\frac{q_N}{q_{N-1}} \tag{4.43}$$

but,

$$\frac{W_N}{W_{N-1}} = \left(\frac{q_N}{\beta h_{2N}}\right)\left(\frac{q_{N-1}}{\beta h_{2N-2}}\right)^{-1} = \frac{q_N h_{2N-2} h_{2N-1}}{q_{N-1} h_{2N} h_{2N-1}} = \frac{q_N}{q_{N-1}}\frac{1}{R_{2N}R_{2N-1}} \tag{4.44}$$

so,

$$\frac{\mathcal{Z}_{N+1}\mathcal{Z}_{N-1}}{\mathcal{Z}_N^2} = (1+\frac{1}{N})R_{2N}R_{2N-1}\frac{W_N}{W_{N-1}} \tag{4.45}$$

And the ratio of $\mathcal{Z}$'s is related to F''. So all we have to know is the ratio W_N/W_{N-1}, something that we can easily find with a recursion relation for the W's (of course we also have to find the differential equation satisfied by the function used in the W ansatz)

To find this recursion relation we only need eqs 4.26, 4.28, and 4.29. From (4.28) we have,

$$\rightarrow \omega_{i1} = \frac{a + bS_{2i} - X_i A_{2i}}{b} \tag{4.46}$$

Putting in (4.29),

$$\frac{(a+bS_{2i} - X_i A_{2i})W_{i-1}}{bR_{2i}R_{2i-1}} = -\frac{1}{2}\frac{X_{i-1}A_{2i}}{Z_{2i-1}Z_{2i-2}} \tag{4.47}$$

From (4.26) we also have,

$$\begin{aligned} X_i &= \frac{Z_{2i}}{A_{2i+1}}(b - \frac{2W_i}{R_{2i+1}}) \\ X_{i-1} &= \frac{Z_{2i-2}}{A_{2i-1}}(b - \frac{2W_{i-1}}{R_{2i-1}}) \end{aligned}$$

Replacing in (4.47) we get,

$$2Z_{2i-1}Z_{2i-2}(a+bS_{2i} - \frac{Z_{2i}A_{2i}}{A_{2i+1}}(b-\frac{2W_i}{R_{2i+1}}))W_{i-1} = -bR_{2i}R_{2i-1}\frac{A_{2i}Z_{2i-2}}{A_{2i-1}}(b-\frac{2W_{i-1}}{R_{2i-1}}). \tag{4.48}$$

Multiplying by $A_{2i-1}A_{2i+1}R_{2i+1}$ and rearranging terms, we obtain,

$$4Z_{2i}Z_{2i-1}A_{2i}A_{2i-1}W_iW_{i-1} + [Z_{2i-1}A_{2i-1}A_{2i+1}(a+bS_{2i}) - Z_{2i-1}Z_{2i}A_{2i}A_{2i-1}b$$
$$-A_{2i}A_{2i+1}R_{2i}b]2R_{2i+1}W_{i-1} + b^2R_{2i}R_{2i-1}R_{2i+1}A_{2i}A_{2i+1} = 0. \quad (4.49)$$

At this point we can look at A and Z, and try to express them as functions of known quantities (a, b, S, etc). Starting from,

$$\frac{P_i^2(0)}{\beta h_i} = (a + bS_i) \quad (4.50)$$

and,

$$P_i(0)P_{i-1}(0) = -(i - \beta bR_i)h_{i-1} \quad (4.51)$$

we have,

$$A_i = \frac{P_i^2(0)}{\beta h_i} = (a + bS_i) \quad (4.52)$$

and,

$$Z_i = \frac{(-(i+1)/\beta + bR_{i+1})}{a + bS_i} = \frac{1}{A_i}(-\frac{(i+1)}{\beta} + bR_{i+1}). \quad (4.53)$$

So we can rewrite eq.4.49 as,

$$(-\frac{(2i+1)}{\beta} + bR_{2i+1})(-\frac{2i}{\beta} + bR_{2i})[2W_i - bR_{2i+1}]W_{i-1}$$
$$+R_{2i+1}(a + bS_{2i})(a + bS_{2i+1})[(-\frac{2i}{\beta}W_{i-1}) + \frac{b^2}{2}R_{2i}R_{2i-1}] = 0. \quad (4.54)$$

4.6 Scaling ansatz

We now have to choose suitable ansatz for solving our recursion relation. The choice is similar to the model of chapter 3

$$R_{2N+l} = 1 - \beta^{-\phi}\exp(-\frac{l}{2}\beta^{-\nu}\frac{\partial}{\partial t})(g_0 + g_1\beta^{-\nu} + g_2\beta^{-2\nu} + \dots) \quad (4.55)$$
$$S_{2N+l} = 2(1 - \beta^{-\mu}\exp(-\frac{l}{2}\beta^{-\nu}\frac{\partial}{\partial t})f) \quad (4.56)$$
$$W_{N+l} = \frac{1}{12}(1 - \beta^{-\rho}\exp(-l\beta^{-\nu}\frac{\partial}{\partial t})(h_0 + h_1\beta^{-\nu} + h_2\beta^{-2\nu} + \dots) \quad (4.57)$$

and, as usual,

$$\frac{N}{\beta} = \frac{1}{4} - t\beta^{\nu-1}. \quad (4.58)$$

order	coefficient
$\beta^{-1/5}$	0
$\beta^{-2/5}$	$6f = h_0^2 + h_0'$

Table 4.1: Lowest order solutions of the recursion relation for the DIII - even n - matrix model.

Again, the general index i is replaced by N because we are only interested in the large N limit.

Using some information from the model of chapter 2 (with a factor of 1/2 for each derivative because of a different normalization for t),

$$\begin{aligned} g_0 &= 2f \\ g_1 &= \frac{f'}{2} \\ g_2 &= \frac{1}{16}f'' - f^2 \end{aligned}$$

and using $\phi = \mu = 2/5$, $\rho = \nu = 1/5$, we find the coefficients listed in table 4.1.

So the solution for W can be written,

$$\begin{aligned} W_N &= \frac{1}{12}(1 - \beta^{-1/5}(h_0 + \beta^{-1/5}h_1 + \ldots)) \\ &= \frac{1}{12}(1 - h_0\beta^{-1/5} - h_1\beta^{-2/5} + \ldots) \end{aligned} \tag{4.59}$$

$$\begin{aligned} W_{N-1} &= \frac{1}{12}(1 - \beta^{-1/5}(h_0 + (h_1 + h_0')\beta^{-1/5} + \ldots))) \\ &= \frac{1}{12}(1 - h_0\beta^{-1/5} - (h_1 + h_0')\beta^{-2/5} + \ldots). \end{aligned} \tag{4.60}$$

4.7 Solution for the free energy

We are now ready to calculate the free energy. A simple calculation gives,

$$\frac{\mathcal{Z}_{N+1}\mathcal{Z}_{N-1}}{\mathcal{Z}_N^2} = (1 + \frac{1}{N})R_{2N}R_{2N-1}\frac{W_N}{W_{N-1}}$$

$$
\begin{aligned}
\ln\left(\frac{Z_{N+1}Z_{N-1}}{Z_N^2}\right) &= \ln\left(R_{2N}R_{2N-1}\frac{W_N}{W_{N-1}}\right) \\
-\beta^{-2\nu}\frac{\partial^2 F}{\partial t^2} &= \ln\left[\frac{1-\beta^{-2\nu}4f}{1-h_0'\beta^{-2\nu}}\right] \\
&= -\beta^{-2\nu}4f + h_0'\beta^{-2\nu}
\end{aligned}
\tag{4.61}
$$

where the ratio for the W's was obviously the same as in chapter 3 (their ansatz are identical). So,

$$F'' = 4f - h_0'. \tag{4.62}$$

Now, we know that f satisfy the Painlevé I equation, $t = f^2 - \frac{1}{12}f''$, and a well known power series solution is,

$$f = 2t^{1/2} - \frac{1}{384}t^{-2} - \frac{49}{589824}t^{-9/2} + \tag{4.63}$$

Replacing in $6f = h_0^2 + h_0'$, we find a similar series for h_0,

$$h_0 = \pm 2\sqrt{3}t^{1/4} - \frac{1}{8}t^{-1} \mp \frac{5\sqrt{3}}{384}t^{-9/4} + \ldots . \tag{4.64}$$

Finally,

$$F'' = 4f - h_0' \simeq 8t^{1/2} \mp \frac{\sqrt{3}}{2}t^{-3/4} - \frac{13}{96}t^{-2} + \ldots \tag{4.65}$$

The difference in powers of t is again $t^{-5/4}$, which is the same as in chapter 3. So we can say that this models describes unoriented surfaces for sure, as well as oriented ones. To compare with previous model, we consider the ratio of coefficients,

$$\frac{c_0 c_2}{c_1^2} = \frac{8 \times -\frac{13}{96}}{(\mp\frac{\sqrt{3}}{2})^2} = -\frac{13}{9} \tag{4.66}$$

This is the same ratio as what was found in the first solution of chapter 3, so this model describes exactly the same physics.

Conclusion

No one can deny the importance of string theory in today's theoretical physics. The idea of using 1-D objects instead of point-like particles, the hope for quantizing gravity, and the unification of all interactions, are all equally good reasons for studying string theory.

Unfortunately, our perturbative approach to string theory is not sufficient to answer such questions as "what is the true ground state of the theory" A non-perturbative approach is needed.

Matrix models can be viewed as an attempt to provide such non-perturbative information Indeed, by using matrix integral, we can solve some simple string models The two new matrix models studied in this thesis were choosen for many reasons Firstly, they were part of a classification scheme Secondly, they were solved to compare their free energy with other models with the hope that it would confirm the validity of interpretation of previous results and perhaps show new phases of simple string models Finally, looking at the contributions in the free energy and comparing with low-order perturbative analysis could help in determining if even Euler number surfaces includes unoriented surfaces

As an aside, the solution of these models allowed us to extend the techniques used in solving matrix models In chapter 3, we solved a recursion relation for the partition function as well as those for the usual polynomials Moreover, we had to define two kinds of partition functions in order to get a set of two finite recursion relations In chapter 4, we used skew-orthogonal polynomials instead of the usual orthogonal polynomials Due to infinite recursion relations, however, we also had to relate the former to the latter In doing so, we finally found a set of finite recursion relations, with which we could solve the model.

We now summarize our two main results, with their implications. Firstly, for each of the models, the free energy was the same as other models previously studied and using completely different matrix ensembles (ratio $-13/9$) This means the following if the model were to describe gravity coupled to some other system, then we expect that different regulators would introduce a dependence of the free energy on a coupling parameter Using different matrix ensembles would yield different results But this is not the case so our models describe pure gravity, as it was first assumed for previous solutions

Secondly, although both models yield exactly the same result, there is an additional solution for n-odd (ratio $83/9$) This solution differs from the other one only by the coefficient of the torus/Klein bottle term So all ratios that do not involve this term are the same in both solutions A similar effect was found in [22] for QRSD matrices although the physical interpretation was clear in that case Here it remains unclear for the new solution

I will conclude here with an indication of what could be done to develop matrix models further. Another problem with matrix models is that almost all of them require their own particular method of solution There is no standard approach to solving all the models - such an approach would be interesting In fact this is just the same problem as performing ordinary integration, with the difference here that the models are following a classification scheme Concerning the solution of the matrix integral one could also try to solve without diagonalizing the matrices first Finally one could consider not only the tiling of surfaces, but also of volumes, hyper-volumes, etc Such models could yield some insight into regulators for higher dimensionnal, and more realistic theories of gravity

Appendix A

Low-order perturbative calculations

In this appendix, we show the calculation of the low order expansion of some matrix models.

Let us first recall the basic quantity that we want to expand perturbatively. In matrix models we generally start from,

$$\frac{Z(\lambda)}{Z(0)} = \frac{\int dM \exp(-\frac{N}{2}Tr\,M^2 - \lambda N Tr\,M^4)}{\int dM \exp(-\frac{N}{2}Tr\,M^2)} = \sum_{k=0}^{\infty}(-1)^k z_k \lambda^k \tag{A.1}$$

with

$$z_k = \frac{N^k}{k!}\frac{\int dM \exp(-\frac{N}{2}Tr\,M^2)(Tr\,M^4)^k}{\int dM\, \exp(-\frac{N}{2}Tr\,M^2)} \tag{A.2}$$

where the $Z(0)$ factor in the denominator was added for normalization, and M is an $N \times N$ matrix. So, except for k=0, which contains no information (i.e. $z_0 = 1$, the lowest order in this perturbation series comes from the coefficient,

$$z_1 = \frac{N\int dM \exp(-\frac{N}{2}Tr\,M^2)M_{ij}M_{jk}M_{kl}M_{li}}{\int dM\ \exp(-\frac{N}{2}Tr\,M^2)} = N < M_{ij}M_{jk}M_{kl}M_{li} >= N < Tr\,M^4 > \tag{A 3}$$

Before performing the calculations, we state some basic results using Gaussian integrals,

$$\begin{aligned} I &= \int dx\, e^{-\frac{N}{2}x^2} = \left(\frac{2\pi}{N}\right)^{1/2} \\ J &= \int dx\, x^2 e^{-\frac{N}{2}x^2} = \frac{\sqrt{2\pi}}{N^{3/2}} \end{aligned} \tag{A 4}$$

So,

$$K_{ij} = \int d^N x\, x_i\, x_j\, e^{-\frac{N}{2}\sum x_l} = \delta_{ij} \frac{\sqrt{2\pi}}{N^{3/2}} \left(\frac{2\pi}{N}\right)^{\frac{N-1}{2}} \tag{A.5}$$

and we write,

$$< x_i x_j >= \frac{K_{ij}}{I^N} = \frac{\delta_{ij}}{N}. \tag{A.6}$$

We will now calculate the lowest order coefficient for real symmetric, real antisymmetric, Hermitian, complex, and DIII matrices, all $N \times N$. This will give us the lowest order powers of N, and hence the kind of surfaces that appear in each case.

A.1 Real symmetric matrices

It turns out that a four-matrix calculation requires the result of a two matrix calculation, so we start with

$$< S_{ij} S_{kl} >= \frac{\int dS\, e^{-\frac{N}{2} Tr\, S^2} S_{ij} S_{kl}}{\int dS\, e^{-\frac{N}{2} Tr\, S^2}}. \tag{A.7}$$

We first remark that the trace of the matrix squared can be decomposed as follows,

$$Tr\, S^2 = S_{ij} S_{ji} = S_{ij} S_{ij} = S_{ii}^2 + 2 \sum_{i<j} S_{ij}^2. \tag{A.8}$$

So, when we have two different indices (i=k different than j=l), there will be a factor of two in the denominator. The result is $\delta_{ik}\delta_{jl}/2N$. Due to the symmetry of the matrices, there is another term with the indices interchanged. Finally, in the case that i=j=k=l, there should be only one N factor in the denominator. The following result takes into account all possibilities,

$$< S_{ij} S_{kl} >= (\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) \frac{1}{2N} \tag{A.9}$$

We can now easily evaluate the coefficient z_1,

$$\begin{aligned}
N < Tr\, S^4 > &= N < S_{ij} S_{jk} S_{kl} S_{li} > \\
&= N < S_{ij} S_{jk} >< S_{kl} S_{li} > + N < S_{ij} S_{kl} >< S_{jk} S_{li} > \\
&\quad + N < S_{ij} S_{li} >< S_{jk} S_{kl} > \\
&= 2N < S_{ij} S_{jk} >< S_{kl} S_{li} > + N < S_{ij} S_{kl} >< S_{jk} S_{li} > \\
&= 2N[\frac{1}{2N}(\delta_{ij}\delta_{jk} + \delta_{ik}\delta_{jj}) \frac{1}{2N}(\delta_{kl}\delta_{li} + \delta_{ki}\delta_{ll})] + N \frac{1}{4N^2}(\delta_{ij}\delta_{jl} + \delta_{il}\delta_{jk})
\end{aligned}$$

$$
\begin{aligned}
& (\delta_{jl}\delta_{ki} + \delta_{ji}\delta_{hl}) \\
= \; & \frac{1}{4N}[2(N + N^2 + N^2 + N^3) + (N^2 + N + N + N)] \\
= \; & \frac{1}{4}[2N^2 + 5N^1 + 5N^0]. \qquad \text{(A.10)}
\end{aligned}
$$

In that case, we see that there are contributions from oriented and unoriented surfaces.

A.2 Real antisymmetric matrices

Again, we calculate first $< A_{ij} A_{kl} >$. We proceed as above, decomposing the trace of the matrix squared as follows,

$$
Tr\, A^2 = A_{ij} A_{ji} = -A_{ij} A_{ij} = -2 \sum_{i<j} A_{ij}^2. \qquad \text{(A.11)}
$$

So, when we have two different indices (i=k different than j=l), there will be a factor of two in the denominator. The result is $\delta_{ik}\delta_{jl}/2N$. Due to the antisymmetry of the matrices, there is another term with the indices interchanged and a minus sign. Finally, in the case that i=j=k=l, the result should be 0. The following expression, again, takes into account all possibilities,

$$
< A_{ij} A_{kl} > = (\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}) \frac{1}{2N}. \qquad \text{(A.12)}
$$

The evaluation of the coefficient is similar to the symmetric case, except that we have to take into account the minus signs,

$$
\begin{aligned}
N < Tr\, A^4 > \; = \; & N < A_{ij} A_{jk} A_{kl} A_{li} > \\
= \; & 2N < A_{ij} A_{jk} >< A_{kl} A_{li} > + N < A_{ij} A_{kl} >< A_{jk} A_{li} > \\
= \; & 2N[\frac{1}{2N}(\delta_{ij}\delta_{jk} - \delta_{ik}\delta_{jj})\frac{1}{2N}(\delta_{kl}\delta_{li} - \delta_{ki}\delta_{ll})] + N\frac{1}{4N^2}(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}) \\
& (\delta_{jl}\delta_{ki} - \delta_{ji}\delta_{kl}) \\
= \; & \frac{1}{4N}[2(N - N^2 - N^2 + N^3) + (N^2 - N - N + N)] \\
= \; & \frac{1}{4}[2N^2 - 3N^1 + N^0]. \qquad \text{(A.13)}
\end{aligned}
$$

Again, there are contributions from oriented and unoriented surfaces In that case we know that in the continuum limit (double scaling), only oriented surfaces survives.

A.3 Hermitian matrices

Again, we need $< M_{ij}M_{kl} >$. We proceed as above, decomposing the trace of the matrix squared. But we will use the fact that a Hermitian matrix can be written as $M = S + iA$, where S is a real symmetric and A is a real antisymmetric matrix. We have,

$$Tr\, M^2 = Tr\, S^2 - Tr\, A^2 = S_{ii}^2 + 2\sum_{i<j}(S_{ij}^2 + A_{ij}^2) \tag{A.14}$$

This is just a combination of both preceeding results. So we will have, as well,

$$\begin{aligned} < M_{ij}M_{kl} > &= < S_{ij}S_{kl} > - < A_{ij}A_{kl} > \\ &= (\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})\frac{1}{2N} \\ &\quad -(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk})\frac{1}{2N} \\ &= \frac{1}{N}\delta_{il}\delta_{jk}. \end{aligned} \tag{A.15}$$

For the coefficient, we simply have,

$$\begin{aligned} N < Tr\, M^4 > &= 2N < M_{ij}M_{jk} >< M_{kl}M_{li} > + N < M_{ij}M_{kl} >< M_{jk}M_{li} > \\ &= 2\frac{1}{N}\delta_{ik}\delta_{jj}\delta_{ki}\delta_{ll} + \frac{1}{N}\delta_{il}\delta_{jk}\delta_{ji}\delta_{kl} \\ &= (2N^2 + N^0). \end{aligned} \tag{A.16}$$

In that case, there are only contributions from even Euler number surfaces which describes, here, oriented surfaces. This can easily be seen in the products of the delta functions in the next-to-last line.

A.4 Complex matrices

Again, we need to calculate $< C_{ij}C_{kl} >$. We will write $C = A + iB$, where A and B are real matrices, without any particular symmetries[1]. We proceed as above, decomposing the trace of the matrix squared. We have,

$$Tr\, C^2 = Tr\, C^\dagger C \;=\; Tr[(A^T - iB^T)(A + iB)]$$

[1]In [18], a special form of complex matrices is used, but it turn out to be equivalent to the one that we have here except for a factor of 2 in the trace of the matrix to the fourth power.

$$
\begin{aligned}
&= Tr(A^TA - iB^TA + iA^TB + B^TB) \\
&= Tr\, A^TA + Tr\, B^TB \\
&= \sum_{i,j} A_{ij}^2 + \sum_{i,j} B_{ij}^2
\end{aligned}
\tag{A.17}
$$

This factor of 1 for both summations will result in a factor of N in the denominator. The two-matrix product gives us,

$$
\begin{aligned}
< C_{ij}^\dagger C_{kl} > &= < (A_{ij} + iB_{ij})^\dagger (A_{kl} + iB_{kl}) > \\
&= < (A_{ji} - iB_{ji})(A_{kl} + iB_{kl}) > \\
&= < A_{ji}A_{kl} > + < B_{ji}B_{kl} > -i < B_{ji}A_{kl} > +i < A_{ji}B_{kl} > \\
&= \frac{\delta_{jk}\delta_{il}}{N} + \frac{\delta_{jk}\delta_{il}}{N} \\
&= \frac{2}{N}\delta_{jk}\delta_{il}.
\end{aligned}
\tag{A 18}
$$

For the coefficient, we get

$$
\begin{aligned}
N < Tr(C^\dagger C)^2 > &= N < (C^\dagger)_{ij} C_{jk} (C^\dagger)_{kl} C_{li} > \\
&= N < (C^\dagger)_{ij} C_{jk} >< (C^\dagger)_{kl} C_{li} > + \\
&\quad N < (C^\dagger)_{ij} C_{li} >< C_{jk} (C^\dagger)_{kl} > \\
&= N\frac{2}{N}\delta_{ik}\delta_{jj}\frac{2}{N}\delta_{ki}\delta_{ll} + N\frac{2}{N}\delta_{ii}\delta_{jl}\frac{2}{N}\delta_{kk}\delta_{jl} \\
&= 4(2N^2).
\end{aligned}
\tag{A 19}
$$

In this particular calculation, there are only contributions from the sphere. Comparing with Hermitian matrices, we see that the tilings are different on a microscopic scale but the double-scaling continuum is the same.

A.5 DIII generator ensemble matrices

The result, here, will be calculated explicitly using the representation of the matrices for the DIII generator ensemble $SO(2n)/U(n)$,

$$
G = \begin{pmatrix} X_1 & X_2 \\ X_2 & -X_1 \end{pmatrix}
\tag{A 20}
$$

where X_1, X_2 $\in$ $SO(n)$ algebra, and are antisymmetric. We have, first,

$$
Tr\, M^2 = 2(Tr\, X_1^2 + Tr\, X_2^2) = -4\sum_{i<j} X_{1,ij}^2 - 4\sum_{i<j} X_{2,ij}^2
\tag{A.21}
$$

Now, we will calculate the fourth power of the matrix explicitly, and decompose it into products of antisymmetric matrices (i.e. X_1, X_2). We have,

$$\begin{aligned} N < Tr\, G^4 > &= N < Tr(2X_1^4 + 2X_2^4 + 8X_1^2X_2^2 - 4X_1X_2X_1X_2) > \\ &= 2\frac{1}{16}(2N^2 - 3N^1 + N_0) + 2\frac{1}{16}(2N^2 - 3N^1 + N^0) + \\ &\quad 8\frac{1}{16}(N^2 - 2N^1 + N^0) - 4\frac{1}{16}(N^1 - N^0) \\ &= N^2 - 2N^1 + N^0 \end{aligned} \tag{A 22}$$

where we used the results for antisymmetric matrices (both the final result and the two-matrices product). So, again, we have contributions from oriented and unoriented surfaces. Comparing with symmetric matrices, we see that we have different contributions at low order (ratio c_0c_2/c_1^2 is different), but the double-scaling continuum is the same

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