TRACE-CLASS NORM MULTIPLIERS

by

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ABSTRACT

The multiplier algebra of $L^2(X,\nu) \hat{\otimes} L^2(X,\nu)$ is the space of all functions ϕ defined on $X \times X$ such that $\phi \cdot \psi \in L^2(X,\nu) \hat{\otimes} L^2(X,\nu)$ for all $\psi \in L^2(X,\nu) \hat{\otimes} L^2(X,\nu)$. In this thesis we study the multiplier algebra of $L^2(X,\nu) \hat{\otimes} L^2(X,\nu)$ for different measure spaces (X,ν). For any finite set X and any measure ν on X we prove that ϕ is a multiplier of $L^2(X,\nu) \hat{\otimes} L^2(X,\nu)$ with $\|\phi\|_M \leq 1$ if and only if $\|\iint_{X\times X} \phi(a,b)dP(a)dQ(b)\|_{LH} \leq 1$ for every pair of spectral (not necessarily commuting) measures P and Q on (X,ν). If $B(I)$ denotes the space of bounded Borel functions on the unit interval I, then for any Borel measure ν on I, $\tilde{B}(I\times I,\nu\times\nu)$ denotes the space of all functions ϕ defined on $I \times I$ such that $\phi = \lim \phi_n$ a.e. and $\phi_n \in B(I) \hat{\otimes} B(I)$ with $\|\phi_n\|_M \leq C$ for some constant C and for all n. It is proven that a bounded Borel function ϕ on $I \times I$ is a multiplier of $L^2(I,\nu) \hat{\otimes} L^2(I,\nu)$ if and only if $\phi \in \tilde{B}(I\times I,\nu\times\nu)$. For $X = Z$ and ν is the counting measure, we prove that the multiplier algebra of $L^2(Z) \hat{\otimes} L^2(Z)$ is the space $\tilde{V}(Z) = \tilde{B}(Z\times Z,\nu\times\nu)$. Certain results concerning the maximal ideal space of the multiplier algebra of $L^2(T,m) \hat{\otimes} L^2(T,m)$ are given. Finally, we study certain homomorphisms of the trace-class operators.

RÉSUMÉ

L'algèbre de multiplicateurs de $L^2(X,\nu) \hat{\otimes} L^2(X,\nu)$ est l'ensemble de fonctions ϕ définies sur $X \times X$ telles que $\phi \cdot \psi \in L^2(X,\nu) \hat{\otimes} L^2(X,\nu)$ pour tout $\psi \in L^2(X,\nu) \hat{\otimes} L^2(X,\nu)$. Dans cette thèse, nous étudions l'algèbre de multiplicateurs de $L^2(X,\nu) \hat{\otimes} L^2(X,\nu)$ pour différents espaces mesurés (X,ν). Etant donnés un ensemble fini arbitraire X, et une mesure ν sur X, nous démonstrons que ϕ est un multiplicateur de $L^2(X,\nu) \hat{\otimes} L^2(X,\nu)$ si et seulement si $\|\iint_{X\times X}\phi(a,b)dP(a)dQ(b)\|_{LH} \leq 1$, pour tout couple de mesures spectrales P et Q (qui ne commutent pas nécessairement sur (X,ν). Si $B(I)$ dénote l'espace des fonctions Borèliennes bornées sur l'intervalle unitaire I, alors pour toute mesure Boreliènne ν sur I, $\tilde{B}(I\times I,\nu\times\nu)$ dénote l'espace de toutes les fonctions ϕ définies sur $I \times I$ telles-que $\phi = \lim \phi_n$ presque partout, où $\phi_n \in B(I) \hat{\otimes} B(I)$ et $\|\phi_n\|_M \leq C$ pour tout $n \geq 1$ pour certaine constante C. Il est démontré qu'une fonction Borèlienne bornée ϕ sur $I \times I$ est un multiplicateur de $L^2(I,\nu) \hat{\otimes} L^2(I,\nu)$ si et seulement si $\phi \in \tilde{B}(I\times I,\nu\times\nu)$. Dans le cas où $X = Z$ et ν est la mesure normbrable, nous démontrons que l'algèbre de multiplicateurs de $L^2(Z) \hat{\otimes} L^2(Z)$ est l'espace $\tilde{V}(Z) = \tilde{B}(Z\times Z,\nu\times\nu)$. Certains résultats concernant le spectre de l'algèbre de multiplicateurs de $L^2(T,m) \hat{\otimes} L^2(T,m)$ sont donnés. Finalement, nous étudions certain homomorphismes de l'espace des operateurs de trace finie.

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CONTENTS

NOTATIONS

We attempt to use standard notations and to avoid confusion, let us state some of the more frequent terminology.

T denotes the unit circle and D is the unit disc. We write Z for the integers and N for the set of natural numbers. Z_n denotes the set $\{-n, \ldots, 0, \ldots, n\}$ and Z_n^+ denotes the set $\{1, \ldots, n\}$.

$L^p(T,m)$ denotes the space of p-summable functions on T with respect to the Lebesgue measure m. We write $\hat{f}$ for the Fourier transform of $f \in L^1(T,m)$. $A(T)$ is the space of functions on T which has an absolutely convergent Fourier series. $M(T)$ will denote the space of all Borel measures on T. We write ℓ_n^p for the n-dimensional space of p-summable sequences.

If A and B are Banach spaces, then $A \hat{\otimes} B$ denotes their projective tensor product. $\|a\|_A$ is the norm of a in A. An element in $A \hat{\otimes} B$ of the form $a \otimes b$ will be called an atom (not to be confused with an atom of a measure).

$\mathcal{L}(A,B)$ is the space of all continuous linear mappings from A into B. A^* is the dual space of A with the usual norm.

If $\phi \in L^2(X,\nu) \hat{\otimes} L^2(X,\nu)$ then we write $\|\phi\|_{Tr}$ for the norm of ϕ in such a space. $M(L^2(X,\nu) \hat{\otimes} L^2(X,\nu))$ is the multiplier algebra of $L^2(X,\nu) \hat{\otimes} L^2(X,\nu)$ and $\|\psi\|_M$ is the multiplier norm of ψ. $V(Z_n)$ is the space $L^\infty(Z_n) \hat{\otimes} L^\infty(Z_n)$. $\tilde{V}(Z)$ denotes the space of all ϕ on $Z \times Z$ such that $\phi_n = \phi|_{Z_n \times Z_n} \in V(Z_n)$ and $\sup_n \|\phi_n\|_{V(Z_n)} < \infty$. Finally, H

denotes a Hilbert space, and $\langle a,b \rangle$ is the inner product of a and b in H. $\|a\|$ is the norm of a in H.

Note: $\tilde{V}$ denotes $\tilde{V}(N)$.

PREFACE

Let A be a space of functions defined on a measure space (X,ν). Then a function ϕ defined on X will be called a multiplier of A if $\phi \cdot \psi \in A$ for all $\psi \in A$. The space of all multipliers of A will be called the multiplier algebra of A and we will denote it by $M(A)$. Schur [23], was the first to study this type of problem for the special case $X = N$, ν the counting measure on N and A is the space of all kernel operators on $\ell^2 = \ell^2(N,\nu)$. The purpose of this thesis is to study $M(L^2(X,\nu) \hat{\otimes} L^2(X,\nu))$ for different measure spaces (X,ν).

In section 2.1 we prove the completeness of $M(A)$ under a certain topology. Section 2.2 is devoted to the study of $M(\ell^2 \hat{\otimes} \ell^2)$ where we prove that it is just $\tilde{V}$. The problem of the Hankel multipliers of $L^2(Z) \hat{\otimes} L^2(Z)$ is the object of section 2.3.

Pairs of normal contractions on a Hilbert space have a close relationship to the multiplier algebra of $L^2(X,\nu) \hat{\otimes} L^2(X,\nu)$ for finite set X. Such a relationship is the theme of section 3.1. In section 3.2 we study the relationship between pairs of spectral measures on (X,ν) and $M(L^2(X,\nu) \hat{\otimes} L^2(X,\nu))$ for finite set X. These results are applied in section 3.3 for the study of certain class of multipliers of the space $L^2(I,\nu) \hat{\otimes} L^2(I,\nu)$ for any Borel measure ν on the unit interval I. The study of the maximal ideal space of $M(L^2(T,m) \hat{\otimes} L^2(T,m))$

is the content of section 4.1 where we prove its assymetry. In the last section, 4.2, which is independent of the other sections, we study the problem of homomorphisms of the space $L^2(T,m) \hat{\otimes} L^2(T,m)$.

As far as I know, the work in this thesis is original, except where the text indicates the contrary. In particular, chapter I is purely expository.

CHAPTER I

This chapter presents the concepts and propositions which we shall use in this thesis. All the results that are presented in this chapter are well known.

1.1. Tensor Products of Banach Spaces.

In their work, Schatten [21], and Grothendieck [7], had developed the theory of topological tensor products. Schatten was the first to give a systematic treatment of the ways of norming the algebraic tensor product of two Banach spaces. Simpler expositions of the material in their work can be found in the paper of Amemiya and Shiga [1], and in the recent book of Diestel and Uhl, [6]. In this section we present an outline of the main concepts that are important to our present work.

Let G_1 and G_2 be any two locally compact abelian groups with Haar measures ν_1 and ν_2 respectively. If A and B are two vector spaces of complex valued functions defined on G_1 and G_2 respectively, then $A \otimes B$ will denote their algebraic tensor product, and $f \otimes g$ denotes the tensor product of $f \in A$ and $g \in B$, [11]. If A and B are Banach spaces with norms $\| \|_A$ and $\| \|_B$, then we can define a norm on $A \otimes B$ in different ways. The following two norms are of particular interest to us.

(i) The projective norm. Let $\phi \in A \otimes B$. Then ϕ has a representation of the form: $\phi = \sum_{i=1}^{N} f_i \otimes g_i$, where N is a finite positive integer. The projective-norm of ϕ is defined to be

$$\|\phi\|_\wedge = \inf\{\sum_{i=1}^{N} \|f_i\|_A \cdot \|g_i\|_B\},$$

where the infimum is taken over all the representations of ϕ in $A \otimes B$.

Clearly $\|f \otimes g\|_\wedge = \|f\|_A \cdot \|f\|_B$. If λ is any other norm on $A \otimes B$ such that $\lambda(f \otimes g) \leq \|f\|_A \cdot \|g\|_B$, for all $f \otimes g \in A \otimes B$, then $\lambda(f \otimes g) \leq \|f \otimes g\|_\wedge$ [6]. Some authors call the projective norm, the greatest - crossnorm.

The space $A \otimes B$ with the projective norm need not be complete. Let $A \hat{\otimes} B$ be the completion of $A \otimes B$ with respect to the projective norm. Since A and B are assumed to be complete, then every element $\phi \in A \hat{\otimes} B$ has a representation of the form

$$\phi = \sum_{i=1}^{\infty} f_i \otimes g_i \quad , \sum_{i=1}^{\infty} \|f_i\|_A \cdot \|g_i\|_B < \infty,$$

$$\|\phi\|_\wedge = \inf\{\sum_{i=1}^{\infty} \|f_i\|_A \cdot \|g_i\|_B\},$$

where the infimum is taken over all such representations of ϕ [26].

(ii) The injective norm. Let $\phi \in A \otimes B$ have a representation of the form: $\phi = \sum_{i=1}^{N} f_i \otimes g_i$, $f_i \in A$ and $g_i \in B$. The injective-norm of ϕ is defined as follows:

$$\|\phi\|_v = \sup\{|\sum_{i=1}^{N} \langle f_i,h\rangle\langle g_i,k\rangle|\},$$

where the supremum is taken over all h and k in the unit balls of A^* and B^*, the duals of A and B, respectively. Again one can see that $\|f \otimes g\|_v = \|f\|_A \cdot \|g\|_B$. If η is any other norm on $A \otimes B$ such that $\eta(f \otimes g) \le \|f\|_A \cdot \|g\|_B$ for all $f \otimes g \in A \otimes B$, then $\|f \otimes g\|_v \le \eta(f \otimes g)$, [6]. The injective norm is often called the least-crossnorm.

For $i = 1,2$, let $L^p(G_i,\nu_i)$ denote the space of ν_i-measurable functions f on G_i for which

$$\int_{G_i} |f(x)|^p \, d\nu_i(x) < \infty, \quad 1 \le p < \infty$$

$$\text{essential supremum } |f(x)| < \infty, \quad p = \infty.$$

The spaces $L^p(G_i,\nu_i)$ are Banach spaces under the norm:

$$\|f\|_p = \begin{cases} \left(\int_{G_i} |f(x)|^p d\nu_i(x)\right)^{1/p} & \text{if } 1 \le p < \infty \\ \text{ess. sup}|f(x)| & \text{if } p = \infty. \end{cases}$$

We shall often omit the measure ν_i from the notation and simply write $L^p(G_i)$. Our aim, now, is to have a realization of the space $L^p(G_i) \hat{\otimes} L^p(G_i)$ as a space of functions defined on $G_1 \times G_2$. For this consider the map

$$K : L^p(G_1) \otimes L^p(G_2) \longrightarrow L^p(G_1 \times G_2),$$

which is defined by

$$K(f \otimes g)(x,y) = f(x)g(x).$$

Since $L^p(G_1) \otimes L^p(G_2)$ is dense in $L^p(G_1) \hat{\otimes} L^p(G_2)$, the mapping K can be extended to $L^p(G_1) \hat{\otimes} L^p(G_2)$ and we will continue to write K for the extension. What we must show is that K is a (1-1) mapping. Let $\mathcal{F}$ and $\mathcal{G}$ be the σ-algebras on which ν_1 and ν_2 are defined. Choose two sets of finite σ-algebras $(\mathcal{F}_\alpha)_{\alpha \in X}$ and $(\mathcal{G}_\beta)_{\beta \in Y}$ such that:

$$\mathcal{F}_\alpha \subseteq \mathcal{F}_\gamma \text{ if } \alpha < \gamma \text{ in } X \text{ and } \mathcal{G}_\beta \subseteq \mathcal{G}_e \text{ if } \beta < e \text{ in } Y,$$

and $\mathcal{F}$ is the smallest σ-algebra generated by $(\mathcal{F}_\alpha)$, and $\mathcal{G}$ is the smallest σ-algebra generated by $(\mathcal{G}_\beta)$. For each $\alpha \in X$ and $\beta \in Y$ let E_α and F_β be the conditional expectation operators:

$$E_\alpha : L^p(G_1, \mathcal{F}) \longrightarrow L^p(G_1, \mathcal{F}_\alpha)$$
$$F_\beta : L^p(G_2, \mathcal{G}) \longrightarrow L^p(G_2, \mathcal{G}_\beta).$$

For each $f \in L^p(G_1)$ and $g \in L^p(G_2)$ we have

$$E_\alpha(f) \xrightarrow{\alpha} f \text{ and } F_\beta(g) \xrightarrow{\beta} g, \text{ pointwise.}$$

Consider the following diagram

$$\begin{array}{ccc}
L^p(G_1, \mathcal{F}) \hat{\otimes} L^p(G_2, \mathcal{G}) & \xrightarrow{K} & L^p(G_1 \times G_2, \mathcal{F} \times \mathcal{G}) \\
{\scriptstyle E_\alpha \otimes F_\beta} \downarrow & & \downarrow {\scriptstyle E_\alpha \otimes F_\beta} \\
L^p(G_1, \mathcal{F}_\alpha) \hat{\otimes} L^p(G_2, \mathcal{G}_\beta) & \xrightarrow{\tilde{K}} & L^p(G_1 \times G_2, \mathcal{F}_\alpha \times \mathcal{G}_\beta)
\end{array}$$

where $\tilde{K}$ is the restriction of K on $L^p(G_1, \mathcal{F}_\alpha) \hat{\otimes} L^p(G_2, \mathcal{G}_\beta)$. For $f \times g \in L^p(G_1) \otimes L^p(G_2)$,

$$[(E_\alpha \otimes F_\beta)\circ K](f \otimes g)(x,y) = (E_\alpha \otimes F_\beta)(f\cdot g)(x,y)$$
$$= E_\alpha(f)(x)\cdot F_\beta(g)(y).$$

On the other hand we have

$$[K\circ(E_\alpha \otimes F_\beta)](f \otimes g)(x,y) = \tilde{K}(E_\alpha(f) \otimes F_\beta(g))(x,y)$$
$$= E_\alpha(f)(x)\cdot F_\beta(g)(y).$$

Hence the above diagram commutes. Let $\phi \in L^p(G_1,\mathcal{F}) \otimes L^p(G_2,\mathcal{G})$ be in the kernel of K. From the commutativity of the diagram we get

$$\tilde{K}(E_\alpha \otimes F_\beta)(\phi) = (E_\alpha \otimes F_\beta)(K(\phi)) = 0.$$

Since $L^p(G_1,\mathcal{F}_\alpha)$ and $L^p(G_2,\mathcal{G}_\beta)$ are finite dimensional, then the mapping $\tilde{K}$ is (1-1). Thus $(E_\alpha \otimes F_\beta)(\phi) = 0$. But since this is true for all α and β, and $(E_\alpha \otimes F_\beta)(\phi) \xrightarrow{\alpha,\beta} \phi$, we see that $\phi = 0$ and this proves the claim that K is (1-1).

From the representation we have for the elements in $L^p(G_1) \hat{\otimes} L^p(G_2)$ one can easily see that an element ϕ is in the range of K if and only if ϕ admits a representation

$$\phi(x,y) = \sum_{i=1}^{\infty} f_i(x)g_i(y),$$

where $f_i \in L^p(G_1)$ and $g_i \in L^p(G_2)$ and $\sum_{i=1}^{\infty} \|f_i\|_p\cdot\|g_i\|_p < \infty$, and if $\phi = K(\psi)$, then $\|\psi\|_\wedge = \inf\{\sum_{i=1}^{\infty} \|f_i\|_p\cdot\|g_i\|_p\}$, where the infimum is taken over all representations of ϕ in the range of K.

Let $G_1 = G_2 = T = \{z \in \mathbb{C} \mid |z| = 1\}$ be the circle group. An operator $S : L^2(T) \longrightarrow L^2(T)$ is called a trace-class operator if $\sum_{i=1}^{\infty} |\langle Se_i, e_i\rangle| < \infty$ for every orthonormal basis (e_i) in $L^2(T)$.

Theorem 1.1.1.

(i) $L^2(T) \hat{\otimes} L^2(T)$ can be identified with the space of trace class operators on $L^2(T)$.

(ii) $[L^2(T) \hat{\otimes} L^2(T)]^* \cong \mathcal{L}(L^2(T))$, the space of all continuous linear operators on $L^2(T)$.

(iii) $[L^2(T) \check{\otimes} L^2(T)]^* \cong L^2(T) \hat{\otimes} L^2(T)$.

(iv) $[\mathcal{L}(L^2(T))]^* \cong [L^2(T) \otimes L^2(T)] \oplus \mathcal{C}$

where $\mathcal{C}$ is the space of bounded functionals that vanish on $L^2(T) \check{\otimes} L^2(T)$, and if $F = F_1 + F_2$ with $F_1 \in L^2(T) \hat{\otimes} L^2(T)$ and $F_2 \in \mathcal{C}$, then $\|F\| = \|F_1\| + \|F_2\|$.

Proof.

(i) See [22] theorem 5, page 42.

(ii) See [22] theorem 3, page 48.

(iii) See [22] theorem 1, page 46 .

(iv) See [22] theorem 5, page 50.

The statements of the previous theorem continue to hold if we replace $L^2(T)$ by $\ell^2(Z)$ or $\ell^2 = \ell^2(N)$, where Z is the group of integers and N is the set of natural numbers.

In his development of the theory of tensor products on Banach spaces, Grothendieck [7], stated what he called "the fundamental theorem of the metric theory of tensor products." If ℓ_n^∞ is the n-dimensional space (real or complex) with the supremum norm, then this theorem can be stated as follows:

Theorem 1.1.2. (Grothendieck Inequality).

Let $u_1, \ldots, u_n$ and $v_1, \ldots, v_n$ be any two sets of unit elements of a (complex or real) Hilbert Space H. Let ϕ be the function defined on $Z \times Z$ as follows:

$$\phi(i,j) = \begin{cases} \langle u_i, v_j \rangle & \text{if } i,j \le n \\ 0 & \text{otherwise.} \end{cases}$$

Then $\phi \in \ell_n^\infty \hat{\otimes} \ell_n^\infty$ and $\|\phi\|_\wedge \le K_G$. The constant K_G is known as the Grothendieck constant.

For a proof of the above theorem, one can consult the paper of Lindenstrauss and Pełczynski [14] theorem 2.1, where they prove the theorem in a different setting. Their proof gives a certain bound on K_G. Subsequently, Rietz [17], presented another proof of this theorem, from which he obtained a better bound on K_G. The least possible value of K_G is still unknown. Recently, Stephens and Cohen, [25], proved that $K_G = \sqrt{2}$ if $n = 2$ or $n = 3$ (for the real case).

Theorem 1.1.3. (Littlewood Inequality).

Let $\phi \in \ell_n^\infty \hat{\otimes} \ell_n^\infty$, where n is any positive integer. Then

(i) $\|\phi\|_\wedge \le C \cdot \sup_j \left(\sum_{i=1}^n |\phi(i,j)|^2 \right)^{1/2}$

(ii) $\|\phi\|_\wedge \le C' \cdot \left(\sum_{i,j}^n |\phi(i,j)|^4 \right)^{1/4}$,

where the constants C and C' are independent of n.

The proof of this theorem can be found in the work of Littlewood [15] theorem 1(1), where he proves the dual form of the above theorem.

1.2. Absolutely Summing Operators.

Let A and B be two Banach spaces, and $S \in L(A,B)$, the space of all continuous linear operators from A into B. For $1 \leq p < \infty$, the operator S is called p-absolutely summing if there is a number $C \geq 0$ such that the relation

$$\left(\sum_{i=1}^{n} \|S(a_i)\|_B^p\right)^{1/p} \leq C \sup_{\|a^*\| \leq 1} \left(\sum_{i=1}^{n} |\langle a_i, a^*\rangle|^p\right)^{1/p} : a^* \in A^*\}$$

holds for every finite set $\{a_1, \ldots, a_n\}$ from A, $n = 1,2,\ldots$. Such operators were first introduced in the work of Pietsch [16], and Grothendieck [8]. In their work, [14], Lindenstrauss and Pelczynski studied p-absolutely summing mappings in L^p-spaces.

The space of p-absolutely summing mappings from A into B is a Banach space under the norm

$$\alpha_p(S) = \inf C$$

where the infimum is taken over all C for which the above inequality holds. If $p_1 \leq p_2$, then $\alpha_{p_2}(S) \leq \alpha_{p_1}(S)$, [14]. Hence every p_1-absolutely summing map is also p_2-absolutely summing.

With the aid of the Grothendieck inequality, Lindenstrauss and Pelczynski were able to prove the following

Theorem 1.2.1.

If ν is a σ-finite measure on a measure space X, and H is (real or complex) Hilbert space. Then every operator

$S \in L(L^1(X,\nu),H)$ is 1-absolutely summing.

Not every 2-absolutely summing operator between two Banach spaces is 1-absolutely summing. However, Kwapien [13], proved the following result:

<u>Theorem</u> 1.2.2.

If $1 \le r \le 2$, $1 \le p \le 2$ and $2 \le q < \infty$, then for each Banach space A

(i) Every p-absolutely summing mapping in $L(\ell^r, A)$ is 1-absolutely summing.

(ii) Every q-absolutely summing mapping in $L(X, \ell^r)$ is 2-absolutely summing.

A special case of this theorem that we are going to use in our work is when $r = 2$ and $A = \ell^\infty$.

If A and B are two Banach spaces, then $(A \check{\otimes} B)^*$, the dual of $A \check{\otimes} B$, can be identified with a vector subspace of $L(A, B^*)$ [20]. The operators in such a subspace are called Integral operators. An important property of Integral operators that Grothendieck established is the factorization property: A continuous operator $S : A \longrightarrow B$ is integral if and only if S admits a factorization:

$$\begin{array}{ccccc} A & \xrightarrow{S} & B & \xrightarrow{J} & B^{**} \\ {\scriptstyle T}\downarrow & & & & \uparrow{\scriptstyle Q} \\ L^\infty(\Omega,\nu) & & \xrightarrow{I} & & L^1(\Omega,\nu) \end{array}$$

where ν is a finite regular Borel measure on some compact

Hausdorff space Ω, $J : B \longrightarrow B^{**}$ is the natural embedding, $I : L^{\infty}(\Omega,\nu) \longrightarrow L^{1}(\Omega,\nu)$ is the natural inclusion and $T : A \longrightarrow L^{\infty}(\Omega,\nu)$ and $Q : L^{1}(\Omega,\nu) \longrightarrow B^{**}$ are bounded linear operators.

An interesting result that shows the relationship between Integral operators and p-absolutely summing operators is the following:

<u>Theorem</u> 1.2.3.

If ν is a σ-finite measure on a measure space X, then every 1-absolutely summing operator S in $\mathcal{L}(L^{\infty}(X,\nu),B)$ or in $\mathcal{L}(A,L^{\infty}(X,\nu))$ is integral, for any Banach space A and B.

A proof of this theorem can be found in [20] corollary 2, page 263.

CHAPTER II

The purpose of this chapter is to study the multiplier algebra of the space $\ell^2 \hat{\otimes} \ell^2$ and the Hankel multipliers of $\ell^2(Z) \hat{\otimes} \ell^2(Z)$. In section 2.1 we give the definition of the multiplier algebra of a Banach space of functions, A, defined on some measurable space (X,ν). With the aid of certain conditions on A, we prove the completeness of the multiplier algebra of A under a suitable topology. Section 2.2 contains the characterization of the multiplier algebra of $\ell^2 \hat{\otimes} \ell^2$; and in section 2.3 we apply the results of section 2.2, to characterize the Hankel multipliers of $\ell^2(Z) \hat{\otimes} \ell^2(Z)$.

2.1. The Algebra of Multipliers.

The concept of multipliers of a Banach algebra was introduced by Helgason [9] as follows: Let A be a semisimple Banach algebra considered as an algebra of continuous functions over its maximal ideal space $\Delta(A)$. Then by a multiplier of A is meant a function over $\Delta(A)$ such that $gA \subseteq A$. However, in our present work we introduce the following

Definition 2.1.1.

Let (X,ν) be a measure space and A be a Banach space of complex valued functions on X such that for each $x \in X$ there exists an $f \in A$ which does not vanish at x. By a multiplier of A we mean a complex valued function ϕ defined on X, such that $\phi\cdot f \in A$ for every $f \in A$, where $(\phi\cdot f)(x) = \phi(x)\cdot f(x)$. The set of all multipliers of A is called the multiplier algebra of A and it will be denoted by $M(A)$.

Let ϕ_1 and ϕ_2 be any two multipliers of A, then one can easily check that:

(i) $a\phi_1 + b\phi_2 \in M(A)$ for all a and b in $\mathbb{C}$.

(ii) $\phi_1\cdot\phi_2 \in M(A)$.

Hence $M(A)$ is a vector space of functions over X. Every element $\phi \in M(A)$ can be considered as an operator:

$$\phi : A \longrightarrow A, \quad \phi(f) = \phi\cdot f.$$

Setting $\|\phi\|_M = \sup_{\|f\|_A \le 1} \|\phi(f)\|_A$, the operator norm of ϕ, we then prove

Lemma 2.1.1.

For any Banach space, A, of functions on a measure space (X,ν), $M(A)$ is a Banach space under the operator norm.

Proof.

We need only prove the completeness of $M(A)$. Let (ϕ_α) be a Cauchy net in $M(A)$. That is

$$\lim_{\alpha,\beta} \|\phi_\alpha - \phi_\beta\|_M = 0.$$

Let f be any element in A. The previous identity implies

$$\lim_{\alpha,\beta} \|\phi_\alpha(f) - \phi_\beta(f)\|_A = 0.$$

from which we deduce that $(\phi_\alpha(f))$ is a Cauchy net in A. The completeness of A implies the convergence of $\phi_\alpha(f)$. Let $g = \lim_\alpha \phi_\alpha(f)$. Consider the following mapping:

$$\phi : A \longrightarrow A$$

$$\phi(f) = \lim_\alpha \phi_\alpha(f) = g.$$

Since each ϕ_α is a linear mapping, ϕ is also linear. We claim that $\phi \in M(E)$. For each $x \in X$, choose an $f \in A$ such that $f(x) \neq 0$. Define a function ψ on X as follows:

$$\psi(x) = \frac{\phi(f)(x)}{f(x)} \text{ ------------------- } 1 .$$

For any $f \in A$, $\phi_\alpha^{(f)} = \phi_\alpha \cdot f$ for all α. Hence for $h \in A$,

$$h \cdot \phi_\alpha(f) = f \cdot \phi_\alpha(h) \text{ ------------- } 2 .$$

which implies that

$$\frac{\phi_\alpha(f)}{f} = \frac{\phi_\alpha(h)}{h} .$$

Hence ψ is independent of the choice of f. Furthermore if $f(x) = 0$ for some x in X, then choose an $h \in A$ such that $h(x) \neq 0$. Then relation 2 implies that $\phi(f) = 0$; from this and from relation 1 we obtain

$$\phi(f)(x) = \psi(x) \cdot f(x),$$

for all $f \in A$ and for all $x \in X$. Hence $\phi(f) = \psi \cdot f \in A$ whenever $f \in A$. This implies that $\phi \in M(A)$. Since (ϕ_α) is a Cauchy net in $M(A)$ and $\phi_\alpha(f) \xrightarrow[\alpha]{} \phi(f)$ for every $f \in A$, it follows that $\phi_\alpha \xrightarrow[\alpha]{} \phi$ in $M(A)$. This completes the proof of the lemma.

If the Banach space of functions A contains the constant function 1, then for any $\phi \in M(A)$, one has $\phi \cdot 1 = \phi \in A$. Hence $M(A) \subseteq A$. Now, if we take A to be $L^2(T)$, where T is the unit circle with the Lebesgue measure, then, as it is well known, $M(L^2(T)) = L^\infty(T)$. A more interesting example of $M(A)$ is due to Varopoulos [27]:

Let $V(Z) = \ell^\infty(Z) \hat{\otimes} \ell^\infty(Z)$ and $\tilde{V}(Z)$ be the set of all functions defined on $Z \times Z$ with the following property:

$\phi \in \tilde{V}(Z)$ if and only if for any finite set $F = F_1 \times F_2 \subseteq Z \times Z$ we have

$$\phi_F = \phi|_{F_1 \times F_2} \in \ell^\infty(F_1) \hat{\otimes} \ell^\infty(F_2) = V(F),$$

and the set $\{\|\phi_F\|_{V(F)}\}$ is uniformly bounded. The norm of ϕ in $\tilde{V}(Z)$ is defined to be

$$\|\phi\|_{\tilde{V}(Z)} = \sup \|\phi_F\|_{V(F)},$$

where the supremum is taken over all finite subsets $F = F_1 \times F_2$ in $Z \times Z$. Consider the Banach space $C_0(Z) \hat{\otimes} C_0(Z)$, where $C_0(Z)$ is the space of bounded functions over Z that tend to zero at ∞. Then $M(A)$ can be identified isometrically to $\tilde{V}(Z)$.

Since the constant function $1 = 1 \otimes 1 \in \tilde{V}(Z)$, then $M(\tilde{V}(Z)) \subseteq \tilde{V}(Z)$. But on the other hand, as one can easily see, $\tilde{V}(Z) \subseteq M(\tilde{V}(Z))$. Furthermore, for $\phi \in \tilde{V}(Z)$

$$\|\phi\|_{M} \leq \|\phi\|_{\tilde{V}} = \|\phi \cdot 1 \otimes 1\|_{\tilde{V}} \leq \|\phi\|_{M}$$

Hence $M(\tilde{V}(Z))$ is isometrically isomorphic to $\tilde{V}(Z)$.

2.2. The Multiplier Algebra of $\ell^2 \hat{\otimes} \ell^2$.

Every element $\phi \in \ell^2 \hat{\otimes} \ell^2$ is the kernel of a trace-class operator in $L(\ell^2)$. One of many characterizations of the trace-class operators is given in the following

Lemma 2.2.1.

Let $S \in L(\ell^2)$. Then the following are equivalent:

(i) S is a trace-class operator

(ii) There is a constant K such that

$$\sum_{j=1}^{n} |\langle S(\xi_j),\eta_j\rangle \leq K,$$

for every pair $\{\xi_1,\ldots, \xi_n\}$ and $\{\eta_1,\ldots, \eta_n\}$ of finite orthonormal systems in ℓ^2.

(iii) There is a constant C such that

$$\sup|\tau(S\circ U)| \leq C,$$

where $\tau(S\circ U)$ is the trace of $S\circ U$, and the supremum is taken over all operators $U \in L(\ell^2)$ such that $\|U\| \leq 1$. The operator $S\circ U$ is the composition of the two operators and not the pointwise multiplication.

For a proof of this lemma one can consult the Book of Ringrose [18], or Schatten [22], where an excellent account on trace-class operators is given.

In the following theorem, we see the relationship between $M(\ell^2 \hat{\otimes} \ell^2)$ and absolutely summing operators.

Theorem 2.2.1.

For a function $\phi \in \ell^\infty(N \times N)$, the following are equivalent:

(i) $\phi \in M(\ell^2 \hat{\otimes} \ell^2)$

(ii) $\phi \cdot f \otimes g : \ell^2 \longrightarrow \ell^\infty$ is 2-absolutely summing for all $f \otimes g \in \ell^\infty \hat{\otimes} \ell^2$.

Proof. (ii) → (i). Let $u \otimes v$ be an arbitrary atom in $\ell^2 \hat{\otimes} \ell^2$. Since the linear span of the set of all atoms in $\ell^2 \hat{\otimes} \ell^2$ is dense in $\ell^2 \hat{\otimes} \ell^2$, it is enough to prove that $\phi \cdot u \otimes v \in \ell^2 \hat{\otimes} \ell^2$. Let $\{\xi_1, \ldots, \xi_n\}$ and $\{\eta_1, \ldots, \eta_n\}$ be any pair of finite orthonormal systems in ℓ^2. Then:

$$\begin{aligned} L &= \sum_{j=1}^{n} |< \phi \cdot u \otimes v)(\xi_j), \eta_j>| \\ &= \sum_{j=1}^{n} \Big| \sum_{r,s=1}^{\infty} \phi(r,s)(u(r)v(s)\xi_j(r)\eta_j(s)\Big| \\ &= \sum_{j=1}^{n} \Big|\Big(\sum_{r=1}^{\infty} u(r)\xi_j(r) \cdot \sum_{s=1}^{\infty} \phi(r,s)v(s)\eta_j(s)\Big)\Big| \\ &\le \sum_{j=1}^{n} \Big|\Big(\sum_{r=1}^{\infty} u(r)\xi_j(r)\Big)\Big| \cdot \sup_i \Big| \sum_{s=1}^{\infty} \phi(i,s)v(s)\eta_j(s)\Big| \\ &\le \sum_{j=1}^{n} \Big(\sup_i \Big| \sum_{s=1}^{\infty} \phi(i,s)v(s)\eta_j(s)\Big|\Big)^2\Big)^{1/2} \cdot \Big(\sum_{j=1}^{n} |<u,\xi_j>|^2\Big)^{1/2}, \end{aligned}$$

by the Schwartz inequality. Since $\xi_1, \ldots, \xi_n$ are orthonormal:

$$\sum_{j=1}^{n} |<u,\xi_j>|^2 \le \|u\|_2^2.$$

The function $1 \otimes v \in \ell^\infty \hat{\otimes} \ell^2$, and thus (ii) implies that

$$\phi \cdot 1 \otimes v : \ell^2 \longrightarrow \ell^\infty$$

is 2-absolutely summing. Hence

$$L \leq C \cdot \|u\|_2 \cdot \sup_{\|f\|_2 \leq 1} \left(\sum_{j=1}^{n} |\langle \eta_j, f \rangle|^2 \right)^{1/2},$$

where C is a constant independent of n. The orthonormality of $\eta_1, \ldots, \eta_n$ implies

$$\begin{aligned} L &\leq C \cdot \|u\|_2 \cdot \|f\|_2 \\ &\leq C \cdot \|u\|_2 . \end{aligned}$$

An application of lemma 2.2.1 completes the proof of (ii) → (i). Conversely (i) → (ii). Let $K = \{\xi_1, \ldots, \xi_n\}$ be any finite set of elements in ℓ^2, and $f \otimes g$ be an arbitrary atom in $\ell^\infty \hat{\otimes} \ell^2$. Then

$$\begin{aligned} L &= \sum_{j=1}^{n} \|(\phi \cdot f \otimes g)(\xi_j)\|_\infty^2 \\ &= \sum_{j=1}^{n} \sup_r \left| \sum_{s=1}^{\infty} \phi(r,s) f(r) g(s) \xi_j(s) \right|^2 . \end{aligned}$$

For each $\xi \in K$ there exists r_ξ such that

$$\sup_r \left| \sum_{s=1}^{\infty} \phi(r,s) f(r) g(s) \xi(s) \right|^2 = \left| \sum_{s=1}^{\infty} \phi(r_\xi, s) f(r_\xi) g(s) \xi(s) \right|^2 .$$

The mapping $\lambda : K \longrightarrow N$ such that $\lambda(\xi) = r_\xi$ need not be a (1-1) mapping. Let $K_1, \ldots, K_k$ be a partition of K such that $\lambda(K_1) = r_1, \ldots, \lambda(K_k) = r_k$. Considering ϕ as an infinite matrix, we set ψ to be an infinite matrix obtained from ϕ by repeating the r_1^{th} row n_1 times ,..., r_k^{th} row n_k times, where $n_1,\ldots, n_k$ are the cardinalities of $K_1,\ldots, K_k$. By lemma 3.1.1 in Chapter III of this thesis, the function $\psi \in M(\ell^2 \hat{\otimes} \ell^2)$ and $\|\psi\|_M = \|\phi\|_M$. Since relabelling the indices of the elements of K does not change the value of L, we can assume that $r_1 = 1$, $r_2 = n_1+1,\ldots, r_k = n_1 + \cdots + n_{k-1}+1$. Hence we can write

$$L \le \sum_{j=1}^{n} \Big| \sum_{s=1}^{\infty} \psi(j,s)f(r_j)g(s)\xi_j(s)\Big|^2.$$

Further, since $f \in \ell^\infty$, we can take f to be the constant function 1. Define the function ξ on $N \times N$ as follows:

$$\xi(j,m) = \begin{cases} \xi_j(m) & \text{if} \quad j \le n \\ 0 & \text{if} \quad j > n. \end{cases}$$

Clearly $\xi \in \ell^2 \hat{\otimes} \ell^2$ and so $\psi\cdot\xi \in \ell^2 \hat{\otimes} \ell^2$. One now has

$$\begin{aligned} L &\le \|(\psi\cdot\xi)(g)\|_2^2 \\ &\le \|g\|_2^2 \cdot \|\psi\cdot\xi\|^2_{L(\ell^2)}. \end{aligned}$$

From theorem 1.1.1 (iv) we have

$$\|\psi\cdot\xi\|_{L(\ell^2)} = \sup_{\tau} |\langle\psi\cdot\xi, F\rangle| \ ,$$

where the supremum is taken over all $F \in \ell^2 \hat{\otimes} \ell^2$ such that $\|F\|_{\tau r} \le 1$. Here $\|F\|_{\tau r}$ denotes the norm of F in $\ell^2 \hat{\otimes} \ell^2$. But

$$\begin{aligned}|\langle\psi\cdot\xi, F\rangle| &= |\langle\xi, \psi\cdot F\rangle| \\ &\le \|\xi\|_{L(\ell^2)}\cdot\|\psi\cdot F\|_{\tau r} \\ &\le \|\xi\|_{L(\ell^2)}\cdot\|\psi\|_{M}\cdot\|F\|_{\tau r}.\end{aligned}$$

Hence we conclude

$$\begin{aligned}L &\le \|g\|_2^2\cdot\|\phi\|_M^2\cdot \sup_{\|h\|_2\le 1}\sum_{j=1}^{n}\left|\sum_{s=1}^{\infty}\xi(j,s)h(s)\right|^2 \\ &\le \|g\|_2^2\cdot\|\phi\|_M^2\cdot \sup_{\|h\|_2\le 1}\sum_{j=1}^{n}|\langle\xi_j, h\rangle|^2.\end{aligned}$$

This completes the proof of the theorem.

Let ℓ_n^r be the n-dimensional space of r-summable sequences. Then as a corollary to the previous theorem we have the following

<u>Theorem</u> 2.2.2.

$M(\ell_n^2 \hat{\otimes} \ell_n^2) \subseteq M(\ell_n^\infty \hat{\otimes} \ell_n^2)$ for all $n \in N$.

<u>Proof</u>. Let $\phi \in M(\ell_n^2 \hat{\otimes} \ell_n^2)$ and $f \otimes g$ be any atom in $\ell_n^\infty \hat{\otimes} \ell_n^2$. Set $\phi(r,s) = f(r)g(s) = 0$ for all $r,s > n$. Then $\phi \in \ell^2 \hat{\otimes} \ell^2$ and $f \otimes g \in \ell^\infty \hat{\otimes} \ell^2$. By theorem 2.2.1, the mapping $\phi\cdot f \otimes g : \ell^2 \longrightarrow \ell^\infty$ is 2-absolutely summing. Hence, theorem 1.2.2 implies that $\phi\cdot f \otimes g : \ell^2 \longrightarrow \ell^\infty$ is 1-absolutely summing. Since $\phi\cdot f \otimes g$ has support which is contained in $Z_n^+ \times Z_n^+$, we have

$$\phi\cdot f \otimes g : \ell_n^2 \longrightarrow \ell_n^\infty$$

is 1-absolutely summing. Then it follows from theorem 1.2.3, that $\phi\cdot f\otimes g \in (\ell_n^1 \overset{\vee}{\otimes} \ell_n^2)^*$, the dual of $\ell_n^1 \overset{\vee}{\otimes} \ell_n^2$. However, the spaces ℓ_n^1 and ℓ_n^2 are finite dimensional, which implies $(\ell_n^1 \overset{\vee}{\otimes} \ell_n^2)^* = (\ell_n^\infty \hat{\otimes} \ell_n^2)$, [21]. Since $f\otimes g$ is an arbitrary atom in $\ell_n^\infty \hat{\otimes} \ell_n^2$, and the linear span of the atoms is dense in $\ell_n^\infty \hat{\otimes} \ell_n^2$ (actually it is equal to $\ell_n^\infty \hat{\otimes} \ell_n^2$ since the space is finite dimensional) we obtain $\phi \in M(\ell_n^\infty \hat{\otimes} \ell_n^2)$. This completes the proof of the theorem.

Let $\phi \in M(\ell_n^\infty \otimes \ell_n^2)$ for all n and $f\otimes g$ be any atom in $\ell^\infty \hat{\otimes} \ell^2$. Define the following function:

$$(f_n \otimes g_n)(r,s) = \begin{cases} (f\otimes g)(r,s) & \text{if } r,s \le n \\ 0 & \text{if } r,s > n \end{cases}$$

Then $f_n \otimes g_n \in \ell_n^\infty \hat{\otimes} \ell_n^2$, and $f_n\otimes g_n \longrightarrow f\otimes g$ pointwise and $\|f_n\|_\infty\cdot\|g_n\|_2 \le \|f\|_\infty\cdot\|g\|_2$. If $F \in (\ell^\infty \hat{\otimes} \ell^2)^* = L(\ell^\infty,\ell^2)$ and $\|F\| \le 1$, then we have

$$|\langle\phi\cdot f\otimes g,F\rangle| = \Big|\sum_{r,s=1}^{\infty} \phi(r,s)f(r)g(s)F(r,s)\Big|$$

$$\lim_n \Big|\sum_{r,s=1}^{n} \phi(r,s)f_n(r)g_n(s)F_n(r,s)\Big|$$

where $F_n(r,s) = F(r,s)$ if $r,s \le n$ and it is equal to zero otherwise. Since $F_n \in (\ell_n^\infty \hat{\otimes} \ell_n^2)^*$ and $\phi \in M(\ell_n^\infty \hat{\otimes} \ell_n^2)$, $\phi\cdot F_n \in (\ell_n^\infty \otimes \ell_n^2)^*$. Hence $h_n(s) = \sum_r \phi(r,s)F_n(r,s)f_n(r) \in \ell^2$ for all $n \in N$, and $\|h_n\|_2 \le \|\phi\|_M\cdot\|f_n\|_\infty \le \|\phi\|_M\cdot\|f\|_\infty$. Thus

$$|\langle\phi\cdot f\otimes g,F\rangle| = \lim_n \left|\sum_{s=1}^{n} g_n(s)h_n(s)\right|$$

$$\leq \lim_n \|g_n\|_2\cdot\|h_n\|_2$$

$$\leq \|g\|_2\cdot\|h\|_2$$

by the Schwartz inequality and the Lebesgue dominated convergence theorem. Hence $M(\ell_n^\infty \otimes \ell_n^2) \subseteq M(\ell^\infty \otimes \ell^2)$ for all n. Similarly one can prove $M(\ell_n^2 \otimes \ell_n^2) \subset M(\ell^2 \otimes \ell^2)$ for all n. This argument proves the following

Theorem 2.2.2'

$$M(\ell^2 \hat{\otimes} \ell^2) \subseteq M(\ell^\infty \otimes \ell^2)$$

We further prove the other inclusion.

Theorem 2.2.3.

$$M(\ell^2 \hat{\otimes} \ell^2) = M(\ell^\infty \hat{\otimes} \ell^2).$$

Remark. The isomorphism here is not an isometry but rather a norm equivalence.

Proof. From the argument used to prove theorem 2.2.2', it is enough to prove that $M(\ell_n^\infty \otimes \ell_n^2) \subseteq M(\ell_n^2 \hat{\otimes} \ell_n^2)$ for all $n \in N$. Let ϕ be any element in $M(\ell_n^\infty \hat{\otimes} \ell_n^2)$ and ψ an arbitrary element in $(\ell_n^2 \hat{\otimes} \ell_n^2)^* = L(\ell_n^2)$. For $u \in \ell_n^2$, consider:

$$f(i) = (\phi\cdot\psi)(u)(i)$$

$$= \sum_{j=1}^{n} \phi(i,j)\psi(i,j)u(j)$$

$$= \sum_{j=1}^{n} \psi(i,j)\phi(i,j)\cdot(1 \otimes u)(i,j).$$

Since $\phi \in M(\ell_n^\infty \hat{\otimes} \ell_n^2)$, then $\phi \cdot 1 \otimes u \in \ell^\infty \hat{\otimes} \ell_n^2$. If

$$\phi \cdot 1 \otimes u = \sum_{r=1}^{\infty} f_r \otimes g_r$$

is a representation of $\phi \cdot 1 \otimes u$ in $\ell_n^\infty \otimes \ell_n^2$, then

$$f(i) = \sum_{j=1}^{n} \psi(i,j) \sum_{r=1}^{\infty} f_r(i) g_r(j)$$
$$= \sum_{r=1}^{\infty} f_r(i) \cdot \sum_{j=1}^{n} \psi(i,j) g_r(j).$$

The function $h_r(i) = \sum_{j=1}^{n} \psi(i,j) g_r(j)$ is in ℓ_n^2 since $g_r \in \ell_n^2$ and $\psi \in L(\ell_n^2)$. Furthermore, $\|h_r\|_2 \le \|\psi\|_{\tau r} \cdot \|g_r\|_2$. Hence

$$\|f\|_2 \le \|\sum_{r=1}^{\infty} f_r(i) h_r(i)\|_2$$
$$\le \sum_{r=1}^{\infty} \|f_r\|_\infty \cdot \|h_r\|_2$$
$$\le \sum_{r=1}^{\infty} \|f_r\|_\infty \cdot \|g_r\|_2 \cdot \|\psi\|_{\tau r}.$$

This implies that $\phi \cdot \psi \in (\ell_n^2 \hat{\otimes} \ell_n^2)^*$, from which we obtain $\phi \in M(\ell_n^2 \hat{\otimes} \ell_n^2)^* = M(\ell_n^2 \hat{\otimes} \ell_n^2)$. This completes the proof of the theorem.

We are now half the way from the full characterization of $M(\ell^2 \hat{\otimes} \ell^2)$. Using theorem 2.2.3, we prove

Theorem 2.2.4.

The following are equivalent:

(i) $\phi \in M(\ell^\infty \hat{\otimes} \ell^2)$

(ii) $\phi \cdot f \otimes g : \ell^1 \longrightarrow \ell^\infty$ is 2-absolutely summing operator for all $f \otimes g \in \ell^\infty \hat{\otimes} \ell^\infty$.

Proof. (ii) → (i). Let $f \otimes g$ be any atom in $\ell^\infty \hat{\otimes} \ell^\infty$. Since $\phi \cdot f \otimes g : \ell^1 \longrightarrow \ell^\infty$ is 2-absolutely summing operator, then theorem 1.2.2 implies that $\phi \cdot f \otimes g : \ell^1 \longrightarrow \ell^\infty$ is 1-absolutely summing. Hence by theorem 1.2.3, $\phi \cdot f \otimes g \in L(\ell^1, \ell^\infty)$ is an integral operator. However, from the definition of intergral operators, we have $\phi \cdot f \otimes g \in (\ell^1 \check{\otimes} \ell^1)^* = \tilde{V}$. Hence for every atom $f \otimes g \in \ell^\infty \hat{\otimes} \ell^\infty$ we have $\phi \cdot f \otimes g \in \tilde{V}$. Since the linear span of the atoms of $\ell^\infty \hat{\otimes} \ell^\infty$ is dense in $\ell^\infty \hat{\otimes} \ell^\infty$, it follows that for every $F \in \ell^\infty \hat{\otimes} \ell^\infty$ $\phi \cdot F \in \tilde{V}$.

Now let $\psi \in \tilde{V}$. If $\psi_n = \psi|_{Z_n^+ \times Z_n^+}$, then from the definition of $\tilde{V}$, section 2.1, we have $\psi_n \in \ell^\infty(Z_n^+) \hat{\otimes} \ell^\infty(Z_n^+)$, and hence $\phi \cdot \psi_n \in \tilde{V}$. Further

$$\|\phi \cdot \psi_n\|_{\tilde{V}} \le \|\phi\| \cdot \|\psi_n\|_{V(Z_n^+)}$$
$$\le \|\phi\| \cdot \|\psi\|_{\tilde{V}} ,$$

where $\|\phi\|$ is the norm of ϕ as an operator from $\ell^\infty \hat{\otimes} \ell^\infty$ into $\tilde{V}$ and $V(Z_n^+) = \ell^\infty(Z_n^+) \hat{\otimes} \ell^\infty(Z_n^+) = \ell_n^\infty \hat{\otimes} \ell_n^\infty$. But this is the sufficient condition (and also necessary) for $\phi \cdot \psi$ to be in $\tilde{V}$. Thus $\phi \in M(\tilde{V}) = \tilde{V}$ as in section 2.1. Since $\phi_n = \phi|_{Z_n^+ \times Z_n^+} \in \ell^\infty \hat{\otimes} \ell^\infty$ for all $n \in N$, it follows that $\phi \in M(\ell_n^\infty \hat{\otimes} \ell_n^2)$ for all $n \in N$. Then, as in the proof of theorem 2.2.2' $\phi \in M(\ell^\infty \hat{\otimes} \ell^2)$.

Conversely (i) → (ii). Let $\phi \in M(\ell^\infty \hat{\otimes} \ell^2)$ and $f \otimes g$ be any atom in $\ell^\infty \hat{\otimes} \ell^\infty$. Let $K = \{\xi_1, \ldots, \xi_n\}$ be any finite set of elements in ℓ^1. Then

$$L = \sum_{j=1}^{n} \|(\phi \cdot f \otimes g)(\xi_j)\|_\infty^2$$
$$= \sum_{j=1}^{n} \sup_r \left| \sum_{s=1}^{\infty} \phi(r,s)f(r)g(s)\xi_j(s) \right|^2 .$$

For each $\xi \in K$, there exists $r_\xi \in N$ such that

$$\sup_r \left| \sum_{s=1}^{\infty} \phi(r,s)f(r)g(s)\xi(s) \right|^2 = \left| \sum_{s=1}^{\infty} \phi(r_j,s)f(r)g(s)\xi(s) \right|^2 .$$

The mapping $\lambda : K \longrightarrow N$ such that $\lambda(\xi) = r_\xi$ need not be a (1-1) mapping. Let $K_1, \ldots, K_k$ be a partition of K such that $\lambda(K_1) = r_1, \ldots, \lambda(K_k) = r_k$. Considering ϕ as an infinite matrix, we set ψ to be a matrix obtained from ϕ by repeating the r^{th} row n_1 times ,..., the r_k^{th} row n_k times, where $n_1, \ldots, n_k$ are the cardinalities of $K_1, \ldots, K_k$. Since, by theorem 2.2.3, $M(\ell^\infty \hat{\otimes} \ell^2) = M(\ell^2 \hat{\otimes} \ell^2)$, an application of lemma 3.1.1 in Chapter III of this thesis implies that $\psi \in M(\ell^\infty \hat{\otimes} \ell^2)$ and $\|\psi\|_m \le c \cdot \|\phi\|_m$, for some constant c. Since relabelling the indices of the elements of K does not change the value of L, we assume that $r_1 = 1$, $r_2 = n+1, \ldots, r_k = n_1 + \cdots + n_{k-1}+1$. Hence we write

$$L \le \sum_{j=1}^{n} \left| \sum_{s=1}^{\infty} \psi(j,s)f(r_{\xi_j})g(s)\xi_j(s) \right|^2 .$$

Further, we can take f to be the constant function 1. Since j belongs to a finite set of N, it follows that

$\psi \cdot \xi \in L(\ell^\infty, \ell^2)$, where ξ is the function

$$\xi(j,s) = \begin{cases} \xi_j(s) & \text{if } j \le n \\ 0 & \text{if } j > n. \end{cases}$$

This implies that

$$\begin{aligned} L &\le \sum_{j=1}^{n} |\psi(j,s)g(s)\xi(j,s)|^2 \\ &\le \|(\psi\cdot\xi)(g)\|_2^2 \\ &\le \|g\|_\infty^2 \cdot \|\psi\cdot\xi\|^2 , \end{aligned}$$

where $\|\psi\cdot\xi\|$ is the norm of $\psi\cdot\xi$ as an element in $L(\ell^\infty,\ell^2)$. Let F be any element in the unit ball of $\ell^\infty \hat{\otimes} \ell^2$. Then, if $\|\xi\|$ denotes the norm of ξ as an element in $L(\ell^\infty,\ell^2)$, we have

$$\begin{aligned} |\langle\psi\cdot\xi,F\rangle| &= |\langle\xi,\psi\cdot F\rangle| \\ &\le \|\xi\|\cdot\|\psi\cdot F\|_{\ell^\infty\hat{\times}\ell^2} \\ &\le \|\xi\|\cdot\|\psi\|_M \\ &\le c\cdot\|\phi\|_M\cdot\|\xi\| . \end{aligned}$$

Since F was arbitrary in the unit ball of $\ell^\infty \hat{\otimes} \ell^2$, it follows

$$\|\psi\cdot\xi\| \le c\cdot\|\phi\|_M\cdot\|\xi\| .$$

Finally we obtain

$$L \leq \|g\|_\infty^2 \cdot c^2 \cdot \|\phi\|_M^2 \cdot \|\xi\|^2$$

$$\leq c^2 \cdot \|g\|_\infty^2 \cdot \|\phi\|_M^2 \cdot \sup_{\|h\|_\infty \leq 1} \left(\sum_{j=1}^{n} |<\xi_j, h>|^2 \right),$$

from which it follows that $\phi \cdot f \otimes g : \ell^1 \longrightarrow \ell^\infty$ is 2-absolutely summing operator. This completes the proof of the theorem.

As a corollary to the previous theorem we obtain the following

Theorem 2.2.5.

$M(\ell^2 \hat{\otimes} \ell^2) = \tilde{V}$.

Proof. Let $\phi \in M(\ell^2 \hat{\otimes} \ell^2)$. Theorem 2.2.3 implies that $\phi \in M(\ell^\infty \hat{\otimes} \ell^2)$. Hence, by theorem 2.2.4, $\phi \cdot f \otimes g : \ell^1 \longrightarrow \ell^\infty$ is 2-absolutely summing operator. The proof of (ii) $\to$ (i) of theorem 2.2.4, then implies that $\phi \in M(\tilde{V}) = \tilde{V}$. It follows that $M(\ell^2 \hat{\otimes} \ell^2) \subseteq \tilde{V}$. On the other hand, if $\phi \in \tilde{V}$, then $\phi_n = \phi|_{Z_n^+ \times Z_n^+} \in M(\ell_n^2 \hat{\otimes} \ell_n^2)$. Hence $\tilde{V} \subset M(\ell^2 \hat{\otimes} \ell^2)$, and the proof is complete.

Remark. Theorem 2.2.5 is implicit in the work of Bennett [3], although he does not state it explicitly. This work and that of Bennett are independent.

We have to remark that the isomorphism in theorem 2.2.5 is not an isometry but a norm equivalence. That is

$$\|\phi\|_M \leq \|\phi\|_{\tilde{V}} \leq \alpha \cdot \|\phi\|_M, \qquad \alpha > 1.$$

Further, the same results are valid for the space $\ell^2(Z) \hat{\otimes} \ell^2(Z)$.

2.3. The Hankel Multipliers of $\ell^2(Z) \hat{\otimes} \ell^2(Z)$.

Let $f \in \ell^\infty(Z)$ and ϕ be a function on $Z \times Z$ defined by $\phi(r,s) = f(r+s)$. If $\phi \in M(\ell^2(Z) \hat{\otimes} \ell^2(Z))$, then ϕ will be called a Hankel multiplier of $\ell^2(Z) \otimes \ell^2(Z)$. It is the purpose of this section to characterize the Hankel multipliers of $\ell^2(Z) \hat{\otimes} \ell^2(Z)$.

Let $M(T)$ denote the space of all complex valued regular bounded Borel measures on T. Set $B(Z)$ to be the set of functions $f \in \ell^\infty(Z)$ such that $f = \hat{\nu}$ for some $\nu \in T$.

Theorem 2.3.1.

Let $\phi \in \ell^\infty(Z \times Z)$ be defined by: $\phi(r,s) = f(r+s)$ for some $f \in \ell^\infty(Z)$ then the following are equivalent:

(i) $\phi \in M(\ell^2(Z) \hat{\otimes} \ell^2(Z))$.

(ii) $f \in B(Z)$.

Furthermore, $\|f\|_{B(Z)} = \|\phi\|_M$.

Proof. (ii) → (i). Let ν be any element in $M(T)$. It is well known, [10], that there exists a sequence of discrete measures in $M(T)$ such that:

$$\hat{\nu}_n(j) \longrightarrow \hat{\nu}(j) \quad \text{for all } j, \text{ and } \|\nu_n\|_{M(T)} \le \|\nu\|_{M(T)}.$$

For any discrete measure ν, we have

$$\nu = \sum_{j=1}^{\infty} \alpha_j \delta_{t_j}, \quad \hat{\nu}(r) = \sum_{j=1}^{\infty} \alpha_j e^{-irt_j}, \text{ and } \|\hat{\nu}\|_{B(Z)} = \sum_{j=1}^{\infty} |\alpha_j| < \infty,$$

where δ_{t_j} is the unit mass at the point t_j. Now, let

$\phi(r,s) = \hat{\nu}(r+s) = f(r+s)$. Then

$$\phi(r,s) = \sum_{j=1}^{\infty} \alpha_j \, e^{-i(r+s)t_j}$$
$$= \sum_{j=1}^{\infty} \alpha_j \, e^{-irt} \; e^{-ist_j} \; .$$

Setting $f_j(r) = \alpha_j \, e^{-irt_j}$ and $g_j(s) = e^{-ist_j}$, we see that $\phi \in \ell^\infty(Z) \hat{\otimes} \ell^\infty(Z)$. Further

$$\|\phi\|_M \le \|\phi\|_{\tilde{V}(Z)} \le \sum_{j=1}^{\infty} |\alpha_j| = \|f\|_{B(Z)} .$$

For $\phi(r,s) = f(r+s)$, where f is any function in $B(Z)$, we have $\phi(r,s) = \lim_n f_n(r+s)$, where $f_n(r+s) = \hat{\nu}_n(r+s)$ for some discrete measure ν_n and $\|f_n\|_{B(Z)} \le \|f\|_{B(Z)}$. Hence the function ϕ is the pointwise limit of a uniformly bounded sequence of elements in $\ell^\infty \hat{\otimes} \ell^\infty$. It follows, [29], that $\phi \in \tilde{V}(Z)$ and $\|\phi\|_{\tilde{V}(Z)} \le \|f\|_{B(Z)}$. Theorem 2.2.5 implies that $\phi \in M(\ell^2(Z) \hat{\otimes} \ell^2(Z))$. Further $\|\phi\|_M \le \|\phi\|_{\tilde{V}(Z)} \le \|f\|_{B(Z)}$.

Conversely (i) → (ii). Let $F : \ell^\infty(Z) \longrightarrow \ell^\infty(Z \times Z)$ be the mapping $F(u)(r,s) = u(r+s)$, and E be the set of functions ϕ in $M(\ell^2(Z) \hat{\otimes} \ell^2(Z))$ such that $\phi = F(u)$ for some u in $\ell^\infty(Z)$. Theorem 2.2.5 implies that $E \subseteq \tilde{V}(Z)$. Hence if $\phi_n = \phi|_{Z_n \times Z_n}$, then $\phi_n \in \ell^\infty(Z_n) \hat{\otimes} \ell^\infty(Z_n)$. Let $\sum_{i=1}^{k} f_i \otimes g_i$ be a representation of ϕ_n in $\ell^\infty(Z_n) \hat{\otimes} \ell^\infty(Z_n)$. Then

$$\phi_n(r,s) = (F(u))_n(r,s)$$
$$= \sum_{i=1}^{k} f_i(r) \cdot g_i(s)$$

$$= \sum_{i=1}^{k} f_i(\alpha)\cdot g_i(\beta) \text{ -------- } *$$

for all α and β in Z such that $\alpha+\beta = r+s$. For each $n \in N$, define a mapping P_n on E as follows:

$$P_n : E \longrightarrow \ell^{\infty}(Z),$$

$$P_n(\phi) = \frac{1}{2n+1} \sum_{i=1}^{k} f_i * g_i$$

The function $P_n(\phi)$ is independent of the representation of ϕ_n, for

$$P_n(\phi)(k) = \frac{1}{2n+1} \sum_{i=1}^{k} (f_i * g_i)(k)$$

$$= \frac{1}{2n+1} \sum_{j=-n}^{n} \left(\sum_{i=1}^{k} f_i(k-j)v_i(j)\right)$$

$$= \frac{1}{2n+1} \sum_{j=-n}^{n} \phi_n(k-j,j).$$

Let $A(Z)$ be the space $\ell^2(Z) * \ell^2(Z)$ which is, by the Plancherel theorem, the same space as $FL^1(T)$, the Fourier transforms of $L^1(T)$. Then $P_n(\phi) \in A(Z) \subseteq B(Z)$. Further, if $\| \ \|_{\tau_r}$ denotes the norm in $\ell^2(Z_n) \hat{\otimes} \ell^2(Z_n)$ and 1_{Z_n} is the characteristic function of Z_n, then

$$\|P_n(\phi)\|_{A(Z)} \leq \frac{1}{2n+1} \cdot \|\phi_n\|_{\tau r}$$

$$\leq \frac{1}{2n+1} \cdot \|\phi_n \cdot 1_{Z_n} \otimes 1_{Z_n}\|_{\tau r}$$

$$\leq \frac{1}{2n+1} \cdot \|\phi_n\|_M \cdot \|1_{Z_n} \otimes 1_{Z_n}\|_{\tau r}$$

$$\leq \|\phi_n\|_M$$

$$\leq \|\phi\|_M \text{ ------------------ } * \ *$$

On the other hand, since $\phi = F(u)$,

$$\begin{aligned} P_n(\phi)(k) &= P_n(F(u))(k) \\ &= \frac{1}{2n+1} \sum_{j=-n}^{n} \phi_n(k-j,j) \\ &= \frac{1}{2n+1} \sum_{j=-n}^{n} u(k) \\ &= u(k). \end{aligned}$$

Hence $P_n(F(u)) \longrightarrow u$ pointwise. Since $(P_n(F(u)))_{n=1}^{\infty}$ is a uniformly bounded sequence in $A(Z)$ which converges pointwise to u, we obtain that $u \in B(Z)$. Furthermore, relation $**$ implies that

$$\|u\|_{B(Z)} \le \|\phi\|_M .$$

This completes the proof of the theorem.

A similar result was proved by Varopoulos [28], where he proved the isometry of $B(Z)$ and its image under F in the tensor algebra norm.

As an application of theorem 2.3.1, we estimate the multiplier norm of the matrix ψ, as an element in $M(\ell^2(Z) \hat{\otimes} \ell^2(Z))$, where

$$\psi(i,j) = \begin{cases} 1 & \text{if } 0 < i+j \le n \\ 0 & \text{otherwise.} \end{cases}$$

<u>Lemma</u> 2.3.1.

$\|\psi\|_M \sim C \cdot \log n$,

where C is a constant independent of n.

Proof. Let f be a function defined on Z as follows:

$$f(i) = \begin{cases} 1 & \text{if } 0 < i \le n \\ 0 & \text{otherwise} \end{cases}$$

Clearly $\psi(i,j) = f(i+j)$. Since f has a finite support in Z, then $f \in B(Z)$. Let $f = \hat{\nu}$ for some $\nu \in M(T)$. By the Riesz-representation theorem, there exists a continuous linear functional $S : C(T) \longrightarrow \mathbb{C}$ such that $S(h) = \int_T h d\nu$ and $\|S\| = \|\nu\|_{M(T)}$, where

$$\|S\| = \sup_h \frac{|S(h)|}{\|h\|}, \quad h \in C(T).$$

It follows from theorem 2.3.1 that

$$\|\psi\|_m = \|f\|_{B(Z)} = \|\nu\|_{M(T)} = \|S\|.$$

Hence it is enough to estimate the norm of S. Further, since the trigonometric polynomials are dense in $C(T)$ under the supremum norm, it is enough to take h, in the definition of $\|S\|$, to be a trigonometric polynomial. Setting $\hat{\nu}(r) = \int_T e^{irt}\, d\nu(t) = f(r)$, we see that

$$S(e^{irt}) = \begin{cases} 1 & \text{if } 0 < r \le n \\ 0 & \text{otherwise.} \end{cases}$$

Thus if $h(t) = \sum_{j=-k}^{k} \alpha_j e^{ijt}$, then

$$S(h) = \begin{cases} \sum_1^n \alpha_j & \text{if } k > n \\ \\ \sum_1^k \alpha_j & \text{if } k < n. \end{cases}$$

Consider the following function in $C(T)$:

$$\tilde{D}_n(t) = \sum_{r=1}^{n} e^{irt}$$

$$= \sum_{r=1}^{n} \cos rt + i \sum_{r=1}^{n} \sin rt$$

$$= (D_n - \tfrac{1}{2})(t) + \bar{D}_n(t),$$

where D_n is the Dirichlet kernel and $\bar{D}_n$ is the conjugate kernel to D_n. A classical result in harmonic analysis, [2], asserts that $\|D_n\|_1 \approx \alpha \log n$ and $\|\bar{D}_n\|_1 \approx \log n$, where $\| \ \|_1$ denotes the norm in $L^1(T)$. Hence $\|\tilde{D}_n\|_1 \approx c \log n$ for some constant c independent of n. Next we observe that

$$\sum_{j=1}^{n} \alpha_j = (\tilde{D}_n * h)(0),$$

from which we conclude

$$|S(h)| = |\sum_{j=1}^{n} \alpha_j|$$

$$= |(\tilde{D}_n * h)(0)|$$

$$\leq \|\tilde{D}_n\|_1 \cdot \|h\|_\infty$$

$$\leq c \log n \cdot \|h\|_\infty .$$

Hence $\|S\| = \sup_h \frac{|S(h)|}{\|h\|_\infty} \leq c \log n$. This completes the proof of the lemma.

The rest of this section is devoted to the study of the Hankel multipliers when Z is replaced by N. Let $f \in \ell^\infty(N)$ such that for each $n \in N$, $f|_{Z_n^+}$ has a representation $f|_{Z_n^+}(r) = \sum_{i=1}^{k} u_i(s)v_i(r-s)$, $0 \le s \le r$. Set

$$\alpha(f,n) = \inf\{\sum_{i=1}^{k} \|u_i\|_\infty \cdot \|v_i\|_\infty\},$$

where the infimum is taken over all the representations of $f|_{Z_n^+}$. Now we introduce the following

<u>Definition</u> 2.3.1.

A function $f \in \ell^\infty(N)$ will be called a tensorial function, if for each $n \in N$, $f|_{Z_n^+}$ has a representation $f|_{Z_n^+}(r) = \sum_{i=1}^{k} u_i(s)v_i(r-s)$, $0 \le s \le r$, such that $(\alpha(f,n))_{n=1}^\infty$ is a bounded sequence. Let J be the space of all tensorial functions. We introduce a norm on J as follows

$$\|f\|_J = \sup_n \alpha(f,n).$$

If $\nu = \sum_{j=1}^\infty c_j \delta_{t_j}$ is any discrete measure on T, such that $\|\nu\| = \sum_{j=1}^\infty |c_j| < \infty$, then

$$\hat{\nu}(r) = \sum_{j=1}^\infty c_j\, e^{irt_j}$$

$$= \sum_{j=1}^\infty c_j\, e^{ist_j}\, e^{i(r-s)t_j}.$$

Furthermore $\|\phi|_{Z_n^+}\|_M \le \sum_{i=1}^{k} \|u_i\|_\infty \cdot \|v_i\|_\infty \le \|f\|_J$. Hence for each $n \in N$, $\phi|_{Z_n^+} \in M(\ell_n^2 \hat{\otimes} \ell_n^2)$. From which we obtain that $\phi|_{Z_n^+} \in \ell^\infty \hat{\otimes} \ell^\infty$, and $\|\phi|_{Z_n^+}\|_{\tilde{V}} \le \|f\|_J$. This implies that $\phi \in \tilde{V}(N)$. Another application of theorem 2.2.5 yields $\phi \in M(\ell^2 \hat{\otimes} \ell^2)$. This completes the proof of the theorem.

The previous theorem shows that there exists a (1-1) mapping F, between the space J and a subspace of $M(\ell^2 \hat{\otimes} \ell^2)$. Let A(N) denote the space $\ell^2 * \ell^2$, which is simply the space of Fourier transforms of $L_+^1(T) = \{f \in L^1(T) | \hat{f}(n) = 0,\ n < 0\}$. We, then, prove the following

Theorem 2.3.3.

Consider $F : \ell^\infty \longrightarrow \ell^\infty(N \times N)$, such that $F(f)(r,s) = f(r+s)$. If $F(f) \in M(\ell^2 \hat{\otimes} \ell^2)$, then $f \in M(A(N))$.

Proof. Let $F(f) \in M(\ell^2 \hat{\otimes} \ell^2)$. Theorem 2.3.2 implies that $f \in J$. From the definition of J and theorem 2.2.5, it is enough to take f of the form $f(r) = \sum_{i=1}^{\infty} u_i(s)v_i(r-s)$, $0 \le s \le r$, and $\sum_{i=1}^{\infty} \|u_i\|_\infty \cdot \|v_i\|_\infty < \infty$. If $\psi = g * h$ is an atom in A(N), then

$$
\begin{aligned}
(f \cdot \psi)(r) &= \Big(\sum_{i=1}^{\infty} u_i(s)v_i(r-s)\Big) \cdot \Big(\sum_{\lambda=0}^{r} g(\lambda)h(r-\lambda)\Big) \\
&= \sum_{\lambda=0}^{r} \Big(g(\lambda)h(r-\lambda) \cdot \sum_{i=1}^{\infty} u_i(\lambda)v_i(r-\lambda)\Big) \\
&= \sum_{i=1}^{\infty} \Big(\sum_{\lambda=0}^{r} g(\lambda)u_i(\lambda) \cdot h(r-\lambda)v_i(r-\lambda)\Big) \\
&= \sum_{i=1}^{\infty} (g_i * h_i)(r),
\end{aligned}
$$

where $g_i = g \cdot u_i$ and $h_i = h \cdot v_i$. Furthermore

$$\|f \cdot \psi\|_{A(N)} \le \sum_{i=1}^{\infty} \|g_i\|_2 \cdot \|h_i\|_2$$

$$\le \|g\|_2 \cdot \|h\|_2 \cdot \sum_{i=1}^{\infty} \|u_i\|_\infty \cdot \|v_i\|_\infty .$$

Hence $f \cdot \psi \in A(N)$. Since ψ was an arbitrary atom in $\ell^2 * \ell^2$, it follows that $f \in M(A(N))$. This completes the proof of the theorem.

CHAPTER III

This chapter is concerned with the relationship between the multiplier algebra of certain function spaces and certain operators on some Hilbert space. Section 3.1, is devoted to the study of the multiplier algebra of function spaces which are defined on finite measure spaces, and their relation to normal contractions defined on some Hilbert space. The relationship between spectral measures and multiplier algebra of certain function spaces is the object of section 3.2. The results of section 3.2 are used, in section 3.3, to study the multiplier algebra of function spaces which are defined on infinite compact measure spaces.

3.1. Multipliers and Normal Contractions

Generally, we denote by H a complex Hilbert space. The set $D = \{Z | Z \in C, |Z| \leq 1\}$ will denote the unit disc and $C(D)$ is the space of continuous functions over D. By a normal contraction on H we mean a linear operator $S : H \to H$ such that $SS^* = S^*S$ and $\|Sx\| \leq \|x\|$ for all $x \in H$, where S^* is the adjoint of S. Fix an $n \in N$, and let H be of dimension n. Since H is finite dimensional, any normal operator on H has a set of orthonormal eigen vectors which span H.

For any pair of normal contractions S and T on H consider the bounded bilinear form

$$F : C(D) \times C(D) \longrightarrow L(H)$$
$$F((f,g)) = f(S) \circ g(T).$$

Then, [24], there exists a bounded linear operator $\tilde{F} : C(D) \hat{\otimes} C(D) \longrightarrow L(H)$ such that the following diagram commutes:

$$\begin{array}{ccc} C(D) \times C(D) & \xrightarrow{\quad F \quad} & L(H) \\ \quad\downarrow i & \nearrow \tilde{F} & \\ C(D) \hat{\otimes} (D) & & \end{array}$$

where i is the inclusion mapping. Hence, for any function ϕ in $C(D) \hat{\otimes} C(D)$, we make sense of $\phi(S,T)$ (since H is finite dimensional $\phi(S,T)$ makes sense for all bounded Borel functions ϕ on $C(D \times D)$.

Let $E_1 = \{\lambda_1, \dots, \lambda_n\}$ and $E_2 = \{\eta_1, \dots, \eta_n\}$ be any two sets of points in D. The sets E_1 and E_2, will be fixed throughout the present section. Since H is finite dimensional, one can find two normal contractions S and T on H whose eigen values are the points of the sets E_1 and E_2 respectively. Now, fix a function ϕ in $C(D) \hat{\otimes} C(D)$. If Z_n^+ is the set $\{1, \dots, n\}$, then consider the function ψ,

$$\psi : Z_n^+ \times Z_n^+ \longrightarrow C$$

$$\psi(i,j) = \phi(\lambda_i, \eta_j), \qquad 1 \le i,j \le n.$$

Then we prove the following theorem.

<u>Theorem</u> 3.1.1.

The following are equivalent:

(i) $\psi \in M(\ell_n^2 \hat{\otimes} \ell_n^2)$ and $\|\psi\|_M \leq 1$.

(ii) $\|\phi(S,T)\|_{L(H)} \leq 1$, for any pair of normal contractions S and T on H whose eigen values are the elements of the sets E_1 and E_2 respectively.

<u>Proof</u>. (i) → (ii). Let $u \otimes v$ be any atom in the unit ball of $\ell_n^2 \hat{\otimes} \ell_n^2$. Since $\psi \in M(\ell_n^2 \otimes \ell_n^2)$ and $\|\psi\|_M \leq 1$, then $\psi \cdot u \otimes v$ is in the unit ball of $\ell_n^2 \hat{\otimes} \ell_n^2$. It follows from lemma 2.2.1(iii) that

$$\sup|\tau r((\psi \cdot u \otimes v) \circ U)| \leq 1,$$

where the supremum is taken over all operators U in the unit ball of $L(\ell_n^2)$, and $\tau r((\psi \cdot u \otimes v) \circ U)$ denotes the trace of $(\psi \cdot u \otimes v) \circ U$. Hence the assumption implies

$$\left| \sum_{i,j=1}^{n} \psi(i,j) u(i) v(j) U(j,i) \right| \leq 1$$

for any atom $u \otimes v$ in the unit ball of $\ell_n^2 \hat{\otimes} \ell_n^2$ and for any U in the unit ball of $L(\ell_n^2)$.

Let S and T be any pair of normal contractions on H whose eigenvalues are the elements of the sets E_1 and E_2 respectively. Again, since H is finite dimensional, then there exists two sets of orthonormal vectors $\{e_1, \ldots, e_n\}$ and $\{f_1, \ldots, f_n\}$ such that

$$Se_i = \lambda_i e_i \quad \text{and} \quad Tf_j = \eta_j f_j, \qquad 1 \leq i,j \leq n.$$

Let $\sum_{k=1}^{\infty} c_k g_k \otimes h_k$ be a representation of ϕ in $C(D) \hat{\otimes} (D)$, and let e and f be any unit vector in H. If $e = \sum_{i=1}^{n} a_i e_i$ and $f = \sum_{j=1}^{n} b_j f_j$, where $a = (a_1, \ldots, a_n)$ and $b = (b_1, \ldots, b_n)$ are unit vectors in ℓ_n^2, then we have

$$\begin{aligned}
|\langle \phi(S,T)f,e\rangle| &= |\langle \sum_{k=1}^{\infty} c_k g_k(S) h_k(T) f, e\rangle| \\
&= |\langle \sum_{k=1}^{\infty} \sum_{j=1}^{n} c_k b_j g_k(S) h_k(T) f_j, \sum_{i=1}^{n} a_i e_i\rangle| \\
&= |\langle \sum_{k=1}^{\infty} \sum_{j=1}^{n} c_k b_j g_k(S) h_k(\eta_j) f_j, \sum_{i=1}^{n} a_i e_i\rangle| \\
&= |\sum_{i,j=1}^{n} a_i b_j \langle f_j, e_i\rangle \sum_{k=1}^{\infty} c_k g_k(\lambda_i) h_k(\eta_j)| \\
&= |\sum_{i,j=1} \phi(\lambda_i, \eta_j) a_i b_j \langle f_j, e_i\rangle|
\end{aligned}$$

Set $u(i) = a_i$, $v(j) = b_j$, and $U(j,i) = \langle f_j, e_i\rangle$. Then $u \otimes v$ is an atom in the unit ball of $\ell_n^2 \hat{\otimes} \ell_n^2$. Further, since H and ℓ_n^2 are isometrically isomorphic, and each of the sets $\{e_1, \ldots, e_n\}$ and $\{f_1, \ldots, f_n\}$ is an orthonormal basis of H, then U is a unitary operator on ℓ_n^2. From this we deduce

$$\begin{aligned}
|\langle \phi(S,T)f,e\rangle| &= |\sum_{i,j=1}^{n} \psi(i,j) u(i) v(j) U(j,i)| \\
&\le \|\psi\|_M \\
&\le 1.
\end{aligned}$$

Since this is true for all unit vectors e and f in H, it follows that $\|\phi(S,T)\|_{L(H)} \le 1$, which proves (ii).

Conversely (ii) → (i). Let $u \otimes v$ be any atom of unit norm in $\ell_n^2 \hat{\otimes} \ell_n^2$, and U be any unitary operator on ℓ_n^2. The identification of H and ℓ_n^2, enables us to choose two sets of vectors $\{e_1, \ldots, e_n\}$ and $\{f_1, \ldots, f_n\}$, each of which is an orthonormal basis of H such that $U(j,i) = \langle f_j, e_i \rangle$. Define linear operators S and T on H as follows:

$$S\left(\sum_{i=1}^{n} a_i e_i\right) = \sum_{i=1}^{n} a_i \lambda_i e_i,$$

$$T\left(\sum_{j=1}^{n} b_j f_j\right) \quad \sum_{j=1}^{n} b_j \eta_j f_j.$$

Then S and T are two normal contractions on H whose eigen values are the elements of the sets E_1 and E_2 respectively and $\{e_1, \ldots, e_n\}$, $\{f_1, \ldots, f_n\}$ are the corresponding eigen vectors. Set $e = \sum_{i=1}^{n} u(i) e_i$ and $f = \sum_{j=1}^{n} v(j) f(j)$, so e and f are unit vectors in ℓ_n^2. Then

$$\begin{aligned} L &= \left| \sum_{i,j=1}^{n} \psi(i,j) u(i) v(j) U(j,i) \right| \\ &= \left| \sum_{i,j=1}^{n} \phi(\lambda_i, \eta_j) u(i) v(j) \langle f_j, e_i \rangle \right| \\ &= \left| \langle \phi(S,T) f, e \rangle \right| \\ &\leq 1 \end{aligned}$$

by assumption. Since $u \otimes v$ is an arbitrary atom in the unit ball of $\ell_n^2 \hat{\otimes} \ell_n^2$, and as it is well known, [18], the unit ball of $L(\ell_n^2)$ is just the closed convex hull of the set of unitary operators on ℓ_n^2, we obtain that $\psi \in M(\ell_n^2 \hat{\otimes} \ell_n^2)$ and $\|\psi\|_M \leq 1$. This completes the proof of the theorem.

Let ϕ be a polynomial in two variables defined on $D \times D$. Then one can have a representation of ϕ in $C(D) \hat{\otimes} (D)$ of the form

$$\phi(z_1,z_2) = \sum_{i=1}^{n} f_i(z_1)g_i(z_2),$$

for some $n \in N$. Let $\|f_i\|_\infty = f_i(\lambda_i)$ and $\|g_j\|_\infty = g_j(\eta_j)$ for $1 \le i,j \le n$. Define a function ψ on $Z_n^+ \times Z_n^+$ by $\psi(i,j) = \phi(\lambda_i,\eta_j)$. In Chapter II, we proved that $M(\ell^2 \hat{\otimes} \ell^2)$ is isomorphic (up to norm equivalence) to $\tilde{V}$. It follows that $M(\ell_n^2 \hat{\otimes} \ell_n^2) = \ell_n^\infty \hat{\otimes} \ell_n^\infty$ (up to norm equivalence). Then one has

$$\|\psi\|_M \le \|\psi\|_{\ell_n^\infty \hat{\otimes} \ell_n^\infty} \le \|\phi\|_{C(D)\hat{\otimes}(D)} \le C_1 \cdot \|\psi\|_{\ell_n^\infty \hat{\otimes} \ell_n^\infty} \le C \cdot \|\psi\|_M .$$

These remarks, together with theorem 3.1.1, imply the following

Theorem 3.1.2.

Let ϕ be a polynomial in two variables defined on $D \times D$. If $\|\phi(S,T\|_{L(H)} \le 1$ for all pair of normal contractions on the Hilbert space H, then there exists a constant C such that $\|\phi\|_{C(D)\hat{\otimes}(D)} \le C$.

3.2 Multipliers and Spectral Measures on Finite Measure Spaces

We start by recalling the definition of spectral measures. Let X be a set and F be a σ-algebra of subsets of X. Given a complex Hilbert space H, then a mapping

$$P : F \longrightarrow L(H)$$

is called a spectral measure on X if the range of P is contained in the set of projections in $L(H)$ and the set function

$$\mu_x : F \longrightarrow \mathbb{C}$$

$$\mu_x(E) = \langle P(E)x,x \rangle$$

is a measure on X for all $x \in H$. Let us adopt the convention that $P(X) = I$, the identity operator on H. An excellent account on spectral measures is given in the book of Berberian [4].

Now take the set X to be $Z_n^+ = \{1,2,\ldots, n\}$, and F to be the family of all subsets of Z_n^+. Then we prove the following

Theorem 3.2.1.

The following are equivalent:

(i) $\phi \in M(\ell_n^2 \hat{\otimes} \ell_n^2)$ and $\|\phi\|_M \le 1$.

(ii) $\left\| \sum_{i,j=1}^{n} \phi(i,j)P(i)Q(j) \right\|_{L(H)} \le 1$ for every pair of (not necessarily commuting) spectral measures P, Q on Z_n^+.

Proof. (i) → (ii). For a start, let us fix a complex Hilbert space H. Let P and Q be any pair of spectral measures on

Z_n^+ with values in $L(H)$. Set $P(i)H = A_i$ and $Q(j)H = B_j$, $1 \le i,j \le n$. Since $\{P(1),\ldots, P(n)\}$ and $\{Q(1),\ldots, Q(n)\}$ are two sets of orthogonal projections in $L(H)$, then $\{P(1)H,\ldots, P(n)H\}$ and $\{Q(1)H,\ldots, Q(n)H\}$ are two sets of closed orthogonal subspaces in H. Hence

$$H = A_1 + \cdots + A_n$$
$$H = B_1 + \cdots + B_n.$$

We call such decompositions of H, the P-decomposition and the Q-decomposition respectively. Let a and b be any two vectors in the unit ball of H. Let $P(i)a = a_i$ and $Q(j)b = b_j$. Then $a = a_1 + \cdots + a_n$ and $b = b_1 + \cdots + b_n$ in the P-decomposition and the Q-decomposition of H respectively. For the non-zero components of a and b we write

$$a = \|a_1\| \frac{a_1}{\|a_1\|} + \cdots + \|a_n\| \frac{a_n}{\|a_n\|}$$
$$b = \|b_1\| \frac{b_1}{\|b_1\|} + \cdots + \|b_n\| \frac{b_n}{\|b_n\|}.$$

Set $e_i = \frac{a_i}{\|a_i\|}$ if $a_i \neq 0$ and $e_i = 0$ if $a_i = 0$. Similarly $f_j = \frac{b_j}{\|b_j\|}$ if $b_j = 0$ and $f_j = 0$ if $b_j = 0$. Then the non-zero elements of $\{e_1,\ldots, e_n\}$ and $\{f_1,\ldots, f_n\}$ form two sets of orthonormal vectors in H. This implies that the matrix $F(i,j) = \langle f_i, e_j \rangle$ is a contraction on ℓ_n^2.

Set $u(i) = \|a_i\|$ and $v(j) = \|b_j\|$, then since a and b are unit vectors in H, then $u \otimes v$ is an atom in the unit ball of $\ell_n^2 \hat{\otimes} \ell_n^2$. If we notice that $P(i)a = u(i)e_i$ and

$Q(j)b = v(j)f_j$, then we obtain

$$L = \left|\sum_{i,j=1}^{n} \phi(i,j)\langle Q(j)b, P(i)a\rangle\right|$$
$$= \left|\sum_{i,j=1}^{n} \phi(i,j)u(i)v(j)\langle f_j, e_i\rangle\right|.$$

From (i) and lemma 2.2.1(iii) we conclude

$$L \le \|\phi\|_M$$
$$\le 1.$$

Since a and b were arbitrary elements in the unit ball of H, we get $\|\sum_{i,j=1} \phi(i,j)P(i)Q(j)\|_{L(H)} \le 1$. This proves the first half of the theorem.

Conversely (ii) → (i). Let $u \otimes v$ be any atom in the unit ball of $\ell_n^2 \hat{\otimes} \ell_n^2$, and U be any unitary operator in $L(\ell_n^2)$. Since ℓ_n^2 can be identified to a subspace of dimension n in H, then two sets, $\{e_1, \ldots, e_n\}$ and $\{f_1, \ldots, f_n\}$, of orthonormal vectors can be chosen in H such that $U(i,j) = \langle f_i, e_j\rangle$. Set $x = \sum_{i=1}^{n} u(i)e_i$ and $y = \sum_{j=1}^{n} v(j)f_j$. The vectors x and y are in the unit ball of H. If we define

$P(i)$ = Projection on the span of $\{e_i\}$

$Q(j)$ = Projection on the span of $\{f_j\}$,

then $P(i)x = u(i)e_i$, $Q(j)y = v(j)f_j$, and

$$L = \left|\sum_{i,j=1}^{n} \phi(i,j)u(i)v(j)U(j,i)\right|$$

$$= \left| \sum_{i,j=1}^{n} \phi(i,j)u(i)v(j)\langle f_j, e_i\rangle \right|$$

$$= \left| \sum_{i,j=1}^{n} \phi(i,j)\langle Q(j)y, P(i)x\rangle \right|$$

$$\leq \left\| \sum_{i,j=1}^{n} \phi(i,j)P(i)Q(j) \right\|_{L(H)}$$

$$\leq 1.$$

Since $u \otimes v$ is any atom in the unit ball of $\ell_n^2 \hat{\otimes} \ell_n^2$, and since any contraction in $L(\ell_n^2)$ is the convex combination of unitary operators in $L(\ell_n^2)$ [18], it follows, from lemma 2.2.1 (iii) and the definition of the multiplier algebra of $\ell_n^2 \hat{\otimes} \ell_n^2$, that $\phi \in M(\ell_n^2 \hat{\otimes} \ell_n^2)$ and $\|\phi\|_M \leq 1$. This completes the proof of the theorem.

Let x and y be any two vectors in H. Set $P(i)x = x_i$ and $Q(j)y = y_j$. Since P and Q are spectral measures on Z_n^+ then $\{x_1, \ldots, x_n\}$ and $\{y_1, \ldots, y_n\}$ are two sets of orthogonal elements. With this in mind, we can restate theorem 3.2.1 to read

<u>Theorem</u> 3.2.1'

The following are equivalent

(i) $\phi \in M(\ell_n^2 \hat{\otimes} \ell_n^2)$ and $\|\phi\|_M \leq 1$.

(ii) $\left| \sum_{i,j=1}^{n} \phi(i,j)\langle x_i, y_j\rangle \right| \leq \|x\| \cdot \|y\|$, for any pair of sets $\{x_1, \ldots, x_n\}$ and $\{y_1, \ldots, y_n\}$ of orthogonal elements in H.

We obtain from the previous theorem the following

Corollary 3.2.1

(i) $\ell^1 \overset{\vee}{\otimes} \ell^1 \subseteq M(\ell^2 \hat{\otimes} \ell^2)$ and $\|\phi\|_M \leq K_G \cdot \|\phi\|_{\ell^1 \overset{\vee}{\otimes} \ell^1}$, where K_G is the Grothendieck constant.

(ii) $L(\ell^2) = (\ell^2 \hat{\otimes} \ell^2)^* \subset M(\ell^2 \hat{\otimes} \ell^2)$ and $\|\phi\|_M \leq \|\phi\|_{L(H)}$.

Proof. (i) Follows from an application of the Grothendieck inequality (or from the Littlewood inequality).

(ii) Let $\{x_1, \ldots, x_n\}$ and $\{y_1, \ldots, y_n\}$ be two sets of orthogonal elements in H. Without loss of generality we can assume that $\{x_1, \ldots, x_n\}$ and $\{y_1, \ldots, y_n\}$ span the same space, and $\sum_{i=1}^{n} \|x_i\|^2 \leq 1$, $\sum_{j=1}^{n} \|y_j\|^2 \leq 1$. Let $\{e_1, \ldots, e_n\}$ be an orthonormal basis of the span of $\{x_1, \ldots, x_n\}$. Assume $x_i = \sum_{k=1}^{n} \lambda_{ik} e_k$ and $y_j = \sum_{k=1}^{n} \eta_{jk} e_k$. Then

$$L = \left| \sum_{i,j=1}^{n} \phi(i,j) \langle x_i, y_j \rangle \right|$$

$$= \left| \sum_{i,j=1}^{n} \phi(i,j) \sum_{k=1}^{n} \lambda_{ik} \eta_{jk} \right|.$$

Set $f_k(i) = \lambda_{ik}$ and $g_k(j) = \eta_{jk}$. Then the function

$$\psi(i,j) = \sum_{k=1}^{n} f_k(i) g_k(j)$$

is in $\ell_n^2 \hat{\otimes} \ell_n^2$, and $\|\psi\|_{\tau r} \leq \sum_{k=1}^{n} \|f_k\|_2 \cdot \|g_k\|_2 \leq 1$. Hence

$$L = \left| \sum_{i,j=1}^{n} \phi(i,j) \psi(i,j) \right|$$

$$\leq \|\phi\|_{L(H)} \cdot \|\psi\|_{\tau r}$$

$$\leq \|\phi\|_{L(H)}.$$

This completes the proof of the corollary.

Let $\alpha : N \to N$ and $\beta : N \to N$ be any two maps which need not be (1-1). If $\phi \in M(\ell^2 \hat{\otimes} \ell^2)$, then put $\psi = \phi \circ (\alpha \otimes \beta)$, where

$$\phi \circ (\alpha \otimes \beta)(i,j) = \phi(\alpha(i),\beta(j)).$$

Consider any two spectral measures P and Q on N. Set

$$P'(k) = P(\{\alpha^{-1}(k)\})$$
$$Q'(\ell) = Q(\{\beta^{-1}(\ell)\}).$$

Clearly, P' and Q' are two spectral measures on N. The previous remarks together with theorem 3.2.1 enable us to prove the following

Lemma 3.2.1.

Let α and β be as above. If $\phi \in M(\ell^2 \hat{\otimes} \ell^2)$, then the function $\psi = \phi \circ \alpha \otimes \beta \in M(\ell^2 \hat{\otimes} \ell^2)$. Further $\|\phi\|_M = \|\psi\|_M$.

Proof. Fix any finite positive integer n and any two spectral measures P and Q on N. Then

$$\begin{aligned} L &= \sum_{i,j=1}^{n} \psi(i,j)P(i)Q(j) \\ &= \sum_{i,j=1}^{n} (\phi \circ \alpha \otimes \beta)(i,j)P(i)Q(j) \\ &= \sum_{i,j=1}^{n} \phi(\alpha(i),\beta(j))P(i)Q(j) \\ &= \sum_{k,\ell} \phi(k,\ell)P(\{\alpha^{-1}(k)\})Q(\{\beta^{-1}(\ell)\}) \\ &= \sum_{k,\ell} \phi(k,\ell)P'(k)Q'(\ell). \end{aligned}$$

Since $\phi \in M(\ell^2 \hat{\otimes} \ell^2)$, then

$$\|\sum \phi(k,\ell)P'(k)Q'(\ell)\|_{L(H)} \le \|\phi\|_M.$$

This implies that

$$\|\sum_{i,j=1}^{n} \psi(i,j)P(i)Q(j)\|_{L(H)} \le \|\phi\|_M,$$

from which we conclude that $\|\psi\|_M \le \|\phi\|_M$. On the other hand, one can easily see that $\|\phi\|_M \le \|\psi\|_M$. This completes the proof.

In the previous lemma, let us take β to be the identity function on N, and the function α to be finitely many-to 1 on a finite set $E \subset N$, and (1-1) on the complement of E. The function ψ can be considered as an infinite matrix obtained from ϕ by repeating finitely many rows finitely many times. In this setting, lemma 3.2.1 reads that if $\phi \in M(\ell^2 \hat{\otimes} \ell^2)$ and ψ is a matrix obtained from ϕ be repeating finitely many rows finitely many times, then $\psi \in M(\ell^2 \hat{\otimes} \ell^2)$ and $\|\phi\|_M = \|\psi\|_M$.

Let E be a finite set and μ be a discrete measure on E. Assume that $\mu(a) \ne 0$ for all $a \in E$. We write ν for the counting measure on E. Let S be an operator on $L^2(E,\mu)$ such that $\|S\| \le 1$. By theorem 1.1.1(ii), we have $\|S\| = \sup|\langle S, u \otimes v\rangle|$, where the supremum is taken over all atoms $u \otimes v$ in the unit ball of $L^2(E,\mu)$. Then

$$\left|\sum_{a,b} S(a,b)u(a)v(b)\mu(a)\mu(b)\right| \le 1.$$

Set $\tilde{u}(a) = u(a)\sqrt{\mu(a)}$ and $\tilde{v}(b) = v(b)\sqrt{\mu(b)}$. Then $\tilde{u} \otimes \tilde{v}$ is in the unit ball $L^2(E,\nu) \hat{\otimes} L^2(E,\nu)$. This implies that

$$\left|\sum_{a,b} S(a,b)\sqrt{\mu(a)\mu(b)} \cdot \tilde{u}(a)\tilde{v}(b)\right| \le 1.$$

From which we conclude that $\tilde{S}(a,b) = S(a,b)\sqrt{\mu(a)\mu(b)}$ is in the unit ball of $\mathcal{L}(L^2(E,\nu))$. In a similar way one can show that if S is in the unit ball of $\mathcal{L}(L^2(E,\nu))$, then $\tilde{S}(a,b) = \dfrac{S(a,b)}{\sqrt{\mu(a)\mu(b)}}$ is the unit ball of $\mathcal{L}(L^2(E,\mu))$.

Now, let $\phi \in M(L^2(E,\mu) \hat{\otimes} L^2(E,\mu))$ and $\|\phi\|_M \le 1$. This is equivalent to $|\langle\phi\cdot u \otimes v, S\rangle| \le 1$ for all atoms $u \otimes v$ in the unit ball of $L^2(E,\mu) \hat{\otimes} L^2(E,\mu)$ and operators S in the unit ball of $\mathcal{L}(L^2(E,\mu))$. But this is the same as writing

$$\Big|\sum_{a,b} \phi(a,b)u(a)v(b)S(a,b)\mu(a)\mu(b)\Big| \le 1.$$

Let $f \otimes g$ be any atom in the unit ball of $L^2(E,\nu) \hat{\otimes} L^2(E,\nu)$ and W be any operator in the unit ball of $\mathcal{L}(L^2(E,\nu))$. Set $\tilde{f}(a) = \dfrac{f(a)}{\sqrt{\mu(a)}}$, $\tilde{g}(b) = \dfrac{g(b)}{\sqrt{\mu(b)}}$, and $\tilde{W}(a,b) = \dfrac{W(a,b)}{\sqrt{\mu(a)\mu(b)}}$. Then

$$\begin{aligned} L &= \Big|\sum_{a,b} \phi(a,b)f(a)g(b)W(a,b)\Big| \\ &= \Big|\sum_{a,b} \phi(a,b)\tilde{f}(a)\tilde{g}(h)\tilde{W}(a,b)\mu(a)\mu(b)\Big| \\ &= |\langle\phi\cdot\tilde{f}\otimes\tilde{g}, \tilde{W}\rangle| \\ &\le 1. \end{aligned}$$

Hence $\phi \in M(L^2(E,\nu) \hat{\otimes} L^2(E,\nu))$ and $\|\phi\|_M \le 1$. Similarly one can prove that if $\phi \in M(L^2(E,\nu) \hat{\otimes} L^2(E,\nu))$ and $\|\phi\|_M \le 1$ then $\phi \in M(L^2(E,\mu) \otimes L^2(E,\mu))$ and $\|\phi\|_M \le 1$.

We summarize this as

<u>Theorem</u> 3.2.2.

Let E be a finite set. Assume that ν_1 and ν_2 be any two discrete measures on E which are absolutely continuous with respect to each other. Then the following are equivalent:

(i) $\phi \in M(L^2(E,\nu_1) \hat{\otimes} L^2(E,\nu_1))$ and $\|\phi\|_M \leq 1$,

(ii) $\phi \in M(L^2(E,\nu_2) \hat{\otimes} L^2(E,\nu_2))$ and $\|\phi\|_M \leq 1$.

As a corollary of theorem 3.2.1 together with theorem 3.2.2, we have

<u>Theorem</u> 3.2.3.

Let E be a finite set. If $\|\sum_{a,b} \phi(a,b)P(a)Q(b)\|_{L(H)} \leq 1$ for every pair of spectral measures on E, then $\phi \in M(L^2(E,\mu) \hat{\otimes} L^2(E,\mu))$ and $\|\phi\|_M \leq 1$ for every discrete measure μ on E.

So far, we considered multipliers of certain function spaces with respect to a fixed measure. Before we consider non-discrete measure spaces, we prove, in the following, a result concerning the common multipliers of more than one space.

<u>Theorem</u> 3.2.4.

Let X and Y be two finite sets. Then the following are equivalent:

(i) $\phi \in M(L^2(X,\lambda) \hat{\otimes} L^2(Y,\eta))$ and $\|\phi\|_M \leq 1$ for every pair of measures λ and η on X and Y respectively.

(ii) $\phi \in L^2(X,\lambda) \hat{\otimes} L^2(Y,\eta)$ and $\|\phi\|_{\tau r} \leq 1$ for every pair of probability measures λ and η on X and Y respectively.

<u>Proof</u>. (i) $\rightarrow$ (ii). Let λ and η be any two probability measures on X and Y respectively. If $1 \otimes 1$ is the constant

function with range $\{1\}$, then

$$\|1 \otimes 1\|_{L^2(\lambda)\hat{\otimes}L^2(\eta)} = 1.$$

Now, $\phi = \phi \cdot 1 \otimes 1$, from which it follows

$$\begin{aligned} L &= \|\phi\|_{L^2(\lambda)\hat{\otimes}L^2(\eta)} \\ &= \|\phi \cdot 1 \otimes 1\|_{L^2(\lambda)\hat{\otimes}L^2(\eta)} \\ &\le \|\phi\|_M \cdot \|1 \otimes 1\|_{L^2(\lambda)\hat{\otimes}L^2(\eta)} \\ &\le \|\phi\|_M \le 1. \end{aligned}$$

Conversely. (ii) → (i). Let λ and η be any two measures on X and Y respectively. Take $u \otimes v$ to be any atom in the unit ball of $L^2(\lambda) \hat{\otimes} L^2(\eta)$ such that $\|u\|_2 = 1 = \|v\|_2$. Set

$$\tilde{\lambda} = |u|^2\lambda \quad \text{and} \quad \tilde{\eta} = |v|^2\eta$$

Clearly $\tilde{\lambda}$ and $\tilde{\eta}$ are two probability measures on X and Y respectively. From (ii) it follows that ϕ is in the unit ball of $L^2(\tilde{\lambda}) \hat{\otimes} L^2(\tilde{\eta})$. Let $\sum_{i=1}^{\infty} a_i f_i \otimes g_i$ be a representation of ϕ in $L^2(\tilde{\lambda}) \hat{\otimes} L^2(\tilde{\eta})$ such that $\sum_{i=1}^{\infty} |a_i| < 1$, $\|f_i\|_{L^2(\tilde{\lambda})} \le 1$ and $\|g_i\|_{L^2(\tilde{\eta})} \le 1$ for all i. Now consider

$$\phi \cdot u \otimes v = \sum_{i=1}^{\infty} a_i (f_i \cdot u) \otimes (g_i \cdot v),$$

$$\begin{aligned} \|f_i \cdot u\|^2_{L^2(\lambda)} &= \int |f_i|^2 |u|^2 \, d\lambda \\ &= \int |f_i|^2 \, d\tilde{\lambda} \\ &\le 1. \end{aligned}$$

Similarly we have $\|g_i \cdot v\|_{L^2(\eta)} \le 1$. Hence $\phi \cdot u \otimes v$ is in the unit ball of $L^2(\lambda) \hat{\otimes} L^2(\eta)$. Since $u \otimes v$ was an arbitrary atom, and the convex span of the atoms is dense in the unit ball of $L^2(\lambda) \hat{\otimes} L^2(\eta)$, it follows that $\phi \in M(L^2(\lambda) \hat{\otimes} L^2(\eta))$ and $\|\phi\|_M \le 1$. This completes the proof of the theorem.

Now, we consider the multiplier algebra of function spaces defined on finite non-discrete measure spaces. We prove a result that is similar to theorem 3.2.1, namely theorem 3.2.5.

Let E be a finite set and ν be a measure on E. Assume that $E_1, \ldots, E_n$ are the atoms of ν, so $E = \bigcup_{i=1}^{n} E_i$. Take another set X which has the same cardinality as E, and let F be a (1-1) onto mapping from X into E. Now we define a discrete measure μ on X such that $f : (X,\mu) \to (E,\nu)$ is a measure preserving mapping. The mapping F induces an operator

$$U : L^2(E,\nu) \hat{\otimes} L^2(E,\nu) \longrightarrow L^2(X,\mu) \hat{\otimes} L^2(X,\mu),$$

where $U(\psi)(x,y) = (\psi \circ F \otimes F)(x,y) = \psi(F(x),F(y))$.

<u>Lemma</u> 3.2.2.

Let (E,ν), (X,μ) and U be as above. Then the operator U is an isometry.

<u>Proof</u>. For $\psi \in L^2(E,\nu) \hat{\otimes} L^2(E,\nu)$, put $\phi = U(\psi)$. Let $\| \ \|_{\tau r(\nu)}$, $\| \ \|_{\tau r(\mu)}$ denote the norms in $L^2(E,\nu) \hat{\otimes} L^2(E,\nu)$ and $L^2(X,\mu) \hat{\otimes} L^2(X,\mu)$ respectively. By definition we have

$\|\phi\|_{\tau r(\mu)} = \inf\{\sum_i \|f_i\|_2 \cdot \|g_i\|_2,\ \phi = \sum_i f_i \otimes g_i\}$, where the infimum is taken over all representations of ϕ in $L^2(X,\mu) \hat{\otimes} L^2(X,\mu)$. As it is well known (and easy to prove), the operator $S : L^2(E,\nu) \to L^2(X,\mu)$ defined by $S(u) = u \circ F$ is an isometry. It follows that to prove the lemma, it is enough to show that every representation of ϕ in $L^2(X,\mu) \hat{\otimes} L^2(X,\mu)$ is of the form $\sum_i (u_i \circ F) \otimes (v_i \circ F)$, for some representation $\sum u_i \otimes v_i$ of ψ in $L^2(E,\nu) \otimes L^2(E,\nu)$.

Now, choose any representation of ϕ in $L^2(X,\mu) \hat{\otimes} L^2(X,\mu)$, say $\phi = \sum_i f_i \otimes g_i$. Since the operator S defined above is an isometry, one can write $L^2(X,\mu) = S(L^2(E,\nu)) + (S(L^2(E,\nu)))^{\perp}$, where $\perp$ denotes the orthogonal complement. Consequently we have

$$f_i = f_i^1 + f_i^2 \quad \text{and} \quad g_i = g_i^1 + g_i^2,$$

where $f_i^1, g_i^1 \in S(L^2(E,\nu))$ and $f_i^2, g_i^2 \in (S(L^2(E,\nu)))^{\perp}$. Hence

$$\begin{aligned}
\phi &= \sum_i f_i \otimes g_i \\
&= \sum_i (f_i^1 \otimes g_i^1) + (f_i^1 \otimes g_i^2) + (f_i^2 \otimes g_i^1) + (f_i^2 \otimes g_i^2) \\
&= \sum_i (f_i^1 \otimes g_i^1) + \sum_i (f_i^1 \otimes g_i^2) + (f_i^2 \otimes g_i^1) + (f_i^2 \otimes g_i^2),
\end{aligned}$$

since the sum over i is a finite sum. The operator U considered as a map from $L^2(E\times E,\nu\times\nu)$ into $L^2(X\times X,\mu\times\mu)$ is an isometry. Consequently, since functions of finite rank are dense in $L^2(E\times E,\nu\times\nu)$, it follows that the span of the functions of the form $(a \circ F) \otimes (b \circ F)$ is dense in $U(L^2(E\times E,\nu\times\nu))$. Now consider

$$\langle a\circ F \otimes b\circ F,\ f_i^1 \otimes g_i^2\rangle = \langle a\circ F,\ f_i^1\rangle\cdot\langle b\circ F,\ g_i^2\rangle$$
$$= \langle a\circ F,\ f_i^1\rangle\cdot 0$$
$$= 0$$

since $g_i^2 \in (U(L^2(E)))^{\perp}$. It follows that $f_i^1 \otimes g_i^2 \in (U(L^2(E\times E,\nu\times\nu)))^{\perp}$. Similarly $f_i^2 \otimes g_i^1,\ f_i^2 \otimes g_i^2 \in (U(L^2(E\times E,\nu\times\nu)))^{\perp}$. But the function $\phi = U(\psi) \in U(L^2(E\times E,\nu\times\nu))$, hence $\sum_i (f_i^1 \otimes g_i^2)+(f_i^2 \otimes g_i^1)+(f_i^2 \otimes g_i^2) \equiv 0$. Therefore $\phi = \sum_i f_i^1 \otimes g_i^1 = \sum_i (u_i \circ F) \otimes (v_i \circ F)$ for some $\sum u_i \otimes v_i$ in $L^2(E,\nu) \hat{\otimes} L^2(E,\nu)$. This completes the proof of the lemma.

As a corollary of the previous lemma we have

Lemma 3.2.3.

Let (E,ν), (X,μ), U be as in lemma 3.2.2. If $U(\phi) \in M(L^2(X,\mu) \hat{\otimes} L^2(X,\mu)$ and $\|U(\phi)\|_M \leq 1$, then $\phi \in M(L^2(E,\nu) \hat{\otimes} L^2(E,\nu))$ and $\|\phi\|_M \leq 1$.

Proof. Let ψ be any element of $L^2(E,\nu) \hat{\otimes} L^2(E,\nu)$. Lemma 3.2.2 implies that $\|\phi\cdot\psi\|_{\tau r(\nu)} = \|U(\phi\cdot\psi)\|_{\tau r(\mu)}$. Since

$$U(\phi \cdot \psi) = (\phi \cdot \psi) \circ (F \otimes F)$$
$$= (\phi \circ F \otimes F) \cdot (\psi \circ F \otimes F),$$

it follows that

$$\|\phi \cdot \psi\|_{\tau r(\nu)} = \|(\phi \cdot \psi) \circ F \otimes F\|_{\tau r(\mu)}$$
$$= \|U(\phi) \cdot U(\psi)\|_{\tau r(\mu)}$$
$$\leq \|U(\phi)\|_M \cdot \|U(\psi)\|_{\tau r(\mu)}.$$

Now by assumption we get $\|\phi\cdot\psi\|_{\tau r(\nu)} \le \|U(\psi)\|_{\tau r(\mu)}$. Another application of lemma 3.2.2 implies $\|\phi\cdot\psi\|_{\tau r(\nu)} \le \|\psi\|_{\tau r(\nu)}$. Hence $\phi \in M(L^2(E,\nu) \hat{\otimes} L^2(E,\nu))$ and $\|\phi\|_M \le 1$. This completes the proof.

One can state lemma 3.2.3 in a more general setting as follows: Let (A_1,ν_1), (B_1,μ_1) and (A_2,ν_2), (B_2,μ_2) be finite measure spaces. Assume that $\alpha : A_1 \to A_2$ and $\beta : B_1 \to B_2$ be two (1-1) onto measure-preserving mappings. Then if $(\phi \circ \alpha \otimes \beta) \in M(L^2(A_1,\nu_1) \hat{\otimes} L^2(B_1,\mu_1))$ and $\|\phi \circ \alpha \otimes \beta\|_M \le 1$, then $\phi \in M(L^2(A_2,\nu_2) \hat{\otimes} L^2(B_2,\mu_2))$ and $\|\phi\|_M \le 1$. In case of discrete spaces, then the converse is also true. That is if $\phi \in M(L^2(A_2,\nu_2) \hat{\otimes} L^2(B_2,\mu_2)$ and $\|\phi\|_M \le 1$ then $(\phi \circ \alpha \otimes \beta) \in M(L^2(A_1,\nu_1) \hat{\otimes} L^2(B_1,\mu_1))$ and $\|\phi \circ (\alpha \otimes \beta)\|_M \le 1$. This follows from the fact that the operators

$$S_1 : L^2(A_2,\nu_2) \to L^2(A_1,\nu_1), \quad S_1(f) = f\circ\alpha$$

$$S_2 : L^2(B_2,\mu_2) \to L^2(B_1,\mu_1), \quad S_2(g) = g\circ\beta$$

are onto. So if $u \otimes v$ is an atom in $L^2(A_1,\nu_1) \hat{\otimes} L^2(B_1,\mu_1)$ then there is $f \otimes g$, an atom in $L^2(A_2,\nu_2) \hat{\otimes} L^2(B_2,\mu_2)$ such that $u \otimes v = (f \otimes g) \circ (\alpha \otimes \beta)$.

Now, let E be a finite set and ν a finite measure with atoms $E_1,\dots, E_n$. If f is a measurable function defined on E, then f assumes one value on each E_i. Let $\tilde{E} = \{a_1,\dots,a_n\}$ where $a_i \in E_i$, and put a discrete measure $\tilde{\nu}$ on $\tilde{E}$ such that $\tilde{\nu}(a_i) = \nu(E_i)$. Then we prove

Theorem 3.2.5

The following are equivalent.

(i) $\phi \in M(L^2(E,\nu) \hat{\otimes} L^2(E,\nu))$ and $\|\phi\|_M \le 1$

(ii) $\left\| \iint_{E\times E} \phi(a,b)dP(a)dQ(b) \right\|_{L(H)} \le 1$ for every pair of spectral measures on (E,ν).

Proof. (i) → (ii). If f is an element of the unit ball of $L^2(E,\nu)$ then $\sum_{i=1}^{n} |f(a_i)|^2 \nu(E_i) \le 1$. Hence it is in the unit ball of $L^2(\tilde{E},\tilde{\nu})$. One can then prove that $\tilde{\phi} \in M(L^2(\tilde{E},\tilde{\nu}) \hat{\otimes} L^2(\tilde{E},\tilde{\nu}))$ and $\|\tilde{\phi}\|_M \le 1$, where $\tilde{\phi}(a_i) = \phi(E_i)$. By theorem 3.2.2, $\tilde{\phi} \in M(L^2(\tilde{E},\tilde{\mu}) \hat{\otimes} L^2(\tilde{E},\tilde{\mu}))$ and $\|\tilde{\phi}\|_M \le 1$, where $\tilde{\mu}$ is the counting measure. Theorem 3.2.1, then implies that $\left\| \sum_{i,j=1}^{n} \tilde{\phi}(a_i,a_j)\tilde{P}(a_i)\tilde{Q}(a_j) \right\|_{L(H)} \le 1$ for every pair of spectral measures $\tilde{P}$ and $\tilde{Q}$ on $\tilde{E}$. However,

$$\left\| \iint_{E\times E} \phi(a,b)dP(a)dQ(b) \right\|_{L(H)} = \left\| \sum_{i,j=1}^{n} \phi(a_i,a_j)P(E_i)Q(E_j) \right\|_{L(H)}$$
$$= \left\| \sum_{i,j=1}^{n} \tilde{\phi}(a_i,a_j)\tilde{P}(a_i)\tilde{Q}(a_j) \right\|_{L(H)}$$
$$\le 1,$$

where $\tilde{P}(a_i) = P(E_i)$ and $\tilde{Q}(a_j) = Q(E_j)$.

Conversely (ii) → (i)

$$\iint_{E\times E} \phi(a,b)dP(a)dQ(b) = \sum_{i,j=1}^{n} \phi(a_i,a_j)P(E_i)Q(E_j).$$

Hence (ii) implies that $\left\| \iint_{\tilde{E}\times\tilde{E}} \tilde{\phi}(a,b)d\tilde{P}(a)d\tilde{Q}(b) \right\|_{L(H)} \le 1$ for every pair of spectral measures $\tilde{P}$ and $\tilde{Q}$ on $\tilde{E}$. Theorem 3.2.3 implies that $\tilde{\phi} \in M(L^2(\tilde{E},\tilde{\nu}) \hat{\otimes} L^2(\tilde{E},\tilde{\nu}))$ and $\|\tilde{\phi}\|_M \leq 1$. Hence $\phi \in M(L^2(E,\nu) \hat{\otimes} L^2(E,\nu))$ and $\|\phi\|_M \leq 1$. This completes the proof.

In the same way as in theorem 3.2.2, one can prove that $\phi \in M(L^2(E,\nu) \hat{\otimes} L^2(E,\nu))$ and $\|\phi\|_M \leq 1$ implies that $\phi \in M(L^2(E,\mu) \hat{\otimes} L^2(E,\mu))$ and $\|\phi\|_M \leq 1$ for every measure μ on E with the same atoms as ν.

Let E be a finite set and ν be any finite measure on E. Take X to be a set of the same cardinality as E and μ be a discrete measure on X. If $F : X \to E$ is a (1-1) onto measure-preserving mapping, then

Theorem 3.2.6.

The following are equivalent:

(i) $\phi \in M(L^2(E,\nu) \hat{\otimes} L^2(E,\nu))$ and $\|\phi\|_M \leq 1$

(ii) $(\phi \circ F \otimes F) \in M(L^2(X,\mu) \hat{\otimes} L^2(X,\mu))$ and $\|\phi \circ F \otimes F\|_M \leq 1$.

Proof. (ii) $\to$ (i). This is just lemma 3.2.3.

Conversely (i) $\to$ (ii). By theorem 3.2.2 and the remark after theorem 3.2.5, one can assume without loss of generality that μ is the counting measure on X so if E_i is an atom in E then $\nu(E_i) = |E_i|$, the cardinality of E_i. Theorem 3.2.5 together with (i) implies that $\left\| \int\int_{E\times E} \phi(a,b)dP(a)dQ(b) \right\|_{L(H)} \leq 1$ for every pair of spectral measures P and Q on (E,ν). Thus, using the notation of theorem 3.2.5, we have

$$\|\sum_{i,j=1}^{n} \tilde{\phi}(a_i,a_j)\tilde{P}(a_i)\tilde{Q}(a_j)\|_{L(H)} \le 1.$$

Now, let $u \otimes v$ be any atom in the unit ball of $L^2(X,\mu) \hat{\otimes} L^2(X,\mu)$ and S be any contraction on $L^2(X,\mu)$. Choose two orthonormal sets of vectors in H, $(e_x)_{x\in X}$ and $(f_y)_{y\in X}$ such that $S(x,y) = \langle e_x,f_y\rangle$. Set

$$P(E_i) = \tilde{P}(a_i) = \text{Projection onto the span of } \{e_x | S(x) \in E_i\}$$

$$Q(E_j) = \tilde{Q}(a_j) = \text{Projection onto the span of } \{f_y | S(y) \in E_j\}.$$

Further, put $e = \sum_{x\in X} u(x)e_x$ and $f = \sum_{y\in X} v(y)f_y$. Then

$$\begin{aligned} L &= |\sum_{x,y} (\phi \circ F \times F)(x,y)u(x)v(y)S(y,x)| \\ &= |\sum_{x,y} \phi(F(x),F(y)u(x)v(y)\langle e_y,f_x\rangle| \\ &= |\sum_{i,j=1}^{n} \tilde{\phi}(a_i,a_j)\langle \sum_{F(x)\in E_i} u(x)e_x, \sum_{F(y)\in E_j} v(y)f_y\rangle| \\ &= |\sum_{i,j=1}^{n} \tilde{\phi}(a_i,a_j)\langle \tilde{P}(a_i)e,\tilde{Q}(a_j)f\rangle| \end{aligned}$$

Since e and f are in the unit ball of H, then we obtain $L \le 1$. But $u \otimes v$ was an arbitrary atom in the unit ball of $L^2(X,\mu) \hat{\otimes} L^2(X,\mu)$ and the convex span of such atoms is dense in the unit ball, it follows that $\phi \circ F \otimes F \in \mathcal{M}(L^2(X,\mu) \hat{\otimes} L^2(X,\mu))$ and $\|\phi \circ F \otimes F\|_{\mathcal{M}} \le 1$. This completes the proof of the theorem.

3.3. Multipliers on Infinite Compact Spaces

Let I denote the unit interval $[0,1]$, and $\mathcal{F}$ be the σ-algebra of all Borel sets in I. In this section we will be considering the multiplier algebra of $L^2(I,\mathcal{F},\nu) \hat{\otimes} L^2(I,\mathcal{F},\nu)$ for any Borel measure ν on $(I,\mathcal{F})$. Throughout the section, the σ-algebra $\mathcal{F}$ will be fixed and we will write $L^2(\nu)$ for $L^2(I,\mathcal{F},\nu)$.

Now, for each n let $\mathcal{F}_n$ denote the σ-algebra generated by the 2^n-equal length-intervals of I whose union is I. Clearly we have

$$\mathcal{F}_1 \subseteq \mathcal{F}_2 \subseteq \cdots \subseteq \mathcal{F}_n \subseteq \cdots \subseteq \mathcal{F}.$$

Furthermore $\mathcal{F}$ is just the σ-algebra generated by the $\mathcal{F}_n$'s. Let ϕ be a bounded Borel function on $I \times I$. Then $E_n(\phi)$ will denote the conditional expectation of ϕ with respect to the σ-algebra $\mathcal{F}_n \times \mathcal{F}_n$ in $I \times I$, [5]. So if $E_n(\nu)$ is the restriction of ν to $\mathcal{F}_n$, then

$$\iint_E \phi(x,y)d\nu(x)d\nu(y) = \iint_E E_n(\phi)(x,y)dE_n(\nu)(x)dE_n(\nu)(y),$$

for every set $E \in \mathcal{F}_n \times \mathcal{F}_n$. The following lemma is well known, [5], and the proof will be omitted.

Lemma 3.3.1.

Let ϕ and $E_n(\phi)$ be as above, then

(i) $E_n(\phi) \to \phi$ a.e.

(ii) If $(\phi_i)_{i=1}^{\infty}$ is a sequence of bounded Borel functions on $I \times I$ such that $|\phi_i| \leq K$ for some constant K, and $\phi_i \to \phi$ a.e., then $E_n(\phi_i) \to E_n(\phi)$ a.e.,

(iii) If ψ is a measurable function on $(I \times I, F_n \times F_n)$ and if ϕ, $\phi \cdot \psi$ are both in $L^1(I \times I, F \times F, \nu \times \nu)$, then $E_n(\phi \cdot \psi) = \psi E_n(\phi)$ a.e.

Let $F \in L^2(\nu) \hat{\otimes} L^2(\nu)$. Since $L^2(\nu) \hat{\otimes} L^2(\nu)$ is the dual space $L^2(\nu) \check{\otimes} L^2(\nu)$, [22], it follows that $\|F\|_{\tau r} = \sup|\langle F,\psi\rangle|$, where the supremum is taken over all functions ψ in the unit ball of $L^2(\nu) \check{\times} L^2(\nu)$. However, $L^2(\nu) \check{\times} L^2(\nu)$ is isometrically isomorphic to the space of compact operators on $L^2(\nu)$. Hence, $\|F\|_{\tau r} \leq 1$ implies $|\langle F,\psi\rangle| \leq 1$ for every compact operator ψ of norm ≤ 1. With this in mind, we now prove the following result.

<u>Theorem</u> 3.3.1

Let ϕ, ν, F, $E_n(\phi)$, $E_n(\nu)$ and F_n be as given above. Then the following are equivalent:

(i) $\phi \in M(L^2(\nu) \hat{\otimes} L^2(\nu))$ and $\|\phi\|_M \leq 1$.

(ii) $E_n(\phi) \in M(L^2(I,F_n,E_n(\nu)) \hat{\otimes} L^2(I,F_n,E_n(\nu)))$ and $\|E_n(\phi)\|_M \leq 1$.

<u>Proof</u>. (i) $\to$ (ii). For a start, let $L^2(E_n(\nu)) \hat{\otimes} L^2(E_n(\nu))$ stands for $L^2(I,F_n,E_n(\nu)) \hat{\otimes} L^2(I,F_n,E_n,(\nu))$. Let $f \otimes g$ be an atom in the unit ball of $L^2(E_n(\nu)) \hat{\otimes} L^2(E_n(\nu))$, and K be the kernel of a compact operator on $L^2(E_n(\nu))$ of norm ≤ 1. It follows that $f \otimes g$ is an atom in the unit ball of $L^2(\nu) \hat{\otimes} L^2(\nu)$ and K is the kernel of a compact operator on $L^2(\nu)$ of norm ≤ 1. Since $f \otimes g \cdot K \in L^2(\nu \times \nu)$, it follows

by lemma 3.3.1(iii) that

$$E_n(\phi \cdot f \otimes g \cdot K) = f \otimes g \cdot K \cdot E_n(\phi).$$

This implies

$$\begin{aligned} |\langle E_n(\phi)\cdot f\otimes g, K\rangle| &= \Big|\iint_{I\times I} E_n(\phi)(x,y)f(x)g(y)K(x,y)\,dE_n(\nu)(x)\,dE_n(\nu)(y)\Big| \\ &= \Big|\iint_{I\times I} E_n(\phi\cdot f\otimes g\cdot K)(x,y)\,dE_n(\nu)(x)\,dE_n(\nu)(y)\Big| \\ &= \Big|\iint_{I\times I} (\phi\cdot f\otimes g\cdot K)(x,y)\,d\nu(x)\,d\nu(y)\Big| \\ &\le 1 \end{aligned}$$

by (i). Since $f \otimes g$ was an arbitrary atom in the unit ball of $L^2(E_n(\nu)) \hat{\otimes} L^2(E_n(\nu))$ and K was arbitrary contractive compact operator of norm ≤ 1 on $L^2(E_n(\nu))$, it follows that $E_n(\phi) \in M(L^2(E_n(\nu)) \hat{\otimes} L^2(E_n(\nu)))$.

Conversely (ii) → (i). If K is the kernel of a compact operator on $L^2(\nu)$ of norm ≤ 1, then $\left|\iint (f\otimes g\cdot K)(x,y)\,d\nu(x)\,d\nu(y)\right| \le 1$ for all atoms $f \otimes g$ in the unit ball of $L^2(\nu) \hat{\otimes} L^2(\nu)$. Let $u \otimes v$ be an atom in the unit ball of $L^2(E_n(\nu)) \hat{\otimes} L^2(E_n(\nu))$. Then

$$\begin{aligned} L &= \Big|\iint_{I\times I} u \otimes v\cdot E_n(K)\,dE_n(\nu)(x)\,dE_n(\nu)(y)\Big| \\ &= \Big|\iint_{I\times I} E_n(u \otimes v\cdot K)\,dE_n(\nu)(x)\,dE_n(\nu)(y)\Big|, \end{aligned}$$

by lemma 3.3.1(iii). Considering $u \otimes v$ as an atom in the unit ball of $L^2(\nu) \hat{\otimes} L^2(\nu)$, it follows that

$$L = \iint_{I\times I} (u \otimes v \cdot K)(x,y)d\nu(x)d\nu(y).$$

Hence $|L| \le 1$. This implies that $E_n(K)$ is the kernel of a contractive-compact operator on $L^2(E_n(\nu))$. Now take $f \otimes g$ to be any atom in the unit ball of $L^2(\nu) \hat{\otimes} L^2(\nu)$. Then, [6], $E_n(f \otimes g)$ is in the unit ball of $L^2(E_n(\nu)) \hat{\otimes} L^2(E_n(\nu))$. The assumption (ii) implies that

$$\left|\iint_{I\times I} E_n(\phi) \cdot E_n(f \otimes g) \cdot E_n(K)dE_n(\nu)(x)dE_n(\nu)(y)\right| \le 1.$$

By lemma 3.3.1(i) we have $E_n(\phi) \to \phi$ a.e., $E_n(f \otimes g) \to f \otimes g$ a.e., and $E_n(K) \to K$ a.e. It follows that

$$E_n(\phi)\cdot E_n(f \otimes g)\cdot E_n(K) \to \phi \cdot f \otimes g \cdot K \quad \text{a.e.} \text{ ----- } *$$

Since for each n, $F_n \times F_n$ has finitely many atoms, it follows that $E_n(\phi)$, $E_n(f \otimes g)$, and $E_n(K)$ are bounded. Relation $*$, then, implies that

$$|E_n(\phi)E_n(f \otimes g)E_n(K)| \le C \cdot \phi \cdot f \times g \cdot K, \quad \text{a.e.},$$

for some constant C and for all n. Applying the bounded convergence theorem, one obtains

$$\left|\iint_{I\times I} \phi(x,y)f(x)g(y)K(x,y)d\nu(x)d\nu(y)\right|$$

$$= \lim_n \left|\iint_{I\times I} E_n(\phi)(x,y)E_n(f \otimes g)(x,y)E_n(K)(x,y)dE_n(\nu)(x)dE_n(\nu)(y)\right|$$

$$\le 1.$$

Once again, since $f \otimes g$ and K were arbitrary, it follows that $\phi \in M(L^2(\nu) \hat{\otimes} L^2(\nu))$ and $\|\phi\|_M \leq 1$. This completes the proof.

Now, let us write $B(I)$ for the space of bounded Borel functions on I. Clearly $B(I) \hat{\otimes} B(I) \subseteq M(L^2(\nu) \hat{\otimes} L^2(\nu))$ for any Borel measure ν on I. We define the space $\tilde{B}(I\times I,\nu\times\nu)$ to be the set of all bounded Borel functions ϕ on $I \times I$ such that there is a sequence in $B(I) \hat{\otimes} B(I)$ which is uniformly bounded in $M(L^2(\nu) \hat{\otimes} L^2(\nu))$ and converges to ϕ a.e. $\nu \times \nu$. We then prove

<u>Theorem</u> 3.3.2

Let ϕ be a bounded Borel function on $I \times I$. Then for any Borel measure ν on I, the following are equivalent:

(i) $\phi \in M(L^2(\nu) \hat{\otimes} L^2(\nu))$

(ii) $\phi \in \tilde{B}(I\times I,\nu\times\nu)$.

<u>Proof</u>. (i) $\to$ (ii). Let $\phi \in M(L^2(\nu) \otimes L^2(\nu))$ and $\|\phi\|_M \leq 1$. Theorem 3.3.1, then, implies that $E_n(\phi) \in M(L^2(E_n(\nu)) \hat{\otimes} L^2(E_n(\nu)))$ and $\|E_n(\nu)\|_M \leq 1$. Since $E_n(\nu)$ is purely atomic and has finite number of atoms, we can consider $(I,F_n,E_n(\nu))$ as a finite discrete measure space. By theorem 3.2.2, we obtain that $E_n(\phi) \in M(L^2(I,F_n,\mu) \hat{\otimes} L^2(I,F_n,\mu))$ and $\|E_n(\phi)\|_M \leq 1$, where μ is the counting measure on (I,F_n). This together with theorem 2.2.5, implies that $E_n(\phi) \in B(I) \hat{\otimes} B(I)$. However, by lemma 3.3.1(i), $E_n(\phi) \to \phi$ a.e. $\nu \times \nu$. Hence $\phi \in \tilde{B}(I\times I,\nu\times\nu)$.

Conversely (ii) $\rightarrow$ (i). Let $(\phi_i)_{i=1}^{\infty}$ be a sequence of functions in $B(I) \hat{\otimes} B(I)$ such that $\|\phi_i\|_M \leq 1$, and $\phi_n \rightarrow \phi$ a.e. $\nu \times \nu$. We would like to prove that $\phi \in M(L^2(\nu) \hat{\otimes} L^2(\nu))$. An application of theorem 3.3.1, implies that $E_n(\phi_i) \in M(L^2(E_n(\nu)) \hat{\otimes} L^2(E_n(\nu))$ and $\|E_n(\phi_i)\|_M \leq 1$ for all $i \geq 1$. Since $\phi_n \rightarrow \phi$ a.e. $\nu\times\nu$, then lemma 3.3.1(ii) implies that $E_n(\phi_i) \rightarrow E_n(\phi)$ a.e.$\nu\times\nu$. Again, since $E_n(\nu)$ is purely atomic and has finite number of atoms, it follows from theorem 3.2.2 together with theorem 2.2.5, that $E_n(\phi_i) \in B(I) \hat{\otimes} B(I)$. Hence $E_n(\phi) \in M(L^2(E_n(\nu)) \hat{\otimes} L^2(E_n(\nu))$ and $\|E_n(\phi)\|_M \leq 1$. Theorem 3.3.1, then, implies that $\phi \in M(L^2(\nu) \hat{\otimes} L^2(\nu))$ and $\|\phi\|_M \leq 1$. This completes the proof of the theorem.

The rest of this section is devoted to state a theorem similar to theorem 3.2.5. To start with, let P and Q be any two spectral measures on (I,F). $E_n(P)$ and $E_n(Q)$ will denote the restriction of P and Q respectively on (I,F_n). Let ϕ be a bounded Borel function on $(I \times I, F \times F)$. Set

$$S_n = \iint_{I\times I} E_n(\phi)(x,y)dE_n(P)(x)dE_n(Q)(y).$$

The sequence of operators $(S_n)_{n=1}^{\infty}$ need not converge in the weak-operator topology. However, if $(S_n)_{n=1}^{\infty}$ converges weakly, then let

$$S = \iint_{I\times I} \phi(x,y)dP(x)dQ(y)$$

denote such a limit.

Theorem 3.3.3.

Let ϕ be a bounded Borel function on $I \times I$. For any Borel measure ν on I, the following are equivalent"

(i) $\phi \in M(L^2(\nu) \hat{\otimes} L^2(\nu))$ and $\|\phi\|_M \leq 1$ for any Borel measure ν on I.

(ii) $\|\int\int_{I\times I} \phi(x,y)dP(x)dQ(y)\|_{L(H)} \leq 1$ for any pair of spectral measures P and Q on (I,F).

Proof. (i) → (ii). Let $\phi \in M(L^2(\nu) \hat{\otimes} L^2(\nu))$ and $\|\phi\|_M \leq 1$. Theorem 3.3.1 implies that $E_n(\phi) \in M(L^2(E_n(\nu) \hat{\otimes} L^2(E_n(\nu)))$ and $\|E_n(\phi)\|_M \leq 1$ for all $n \geq 1$. Since for each n, the σ-algebra F_n is finite, we can apply theorem 3.2.5 to obtain

$$\|S_n\|_{L(H)} = \|\int\int_{I\times I} E_n(\phi)(x,y)dE_n(P)(x)dE_n(Q)(y)\|_{L(H)} \leq 1$$

for all $n \geq 1$. Now, we want to prove that $(<S_n a,b>)_{n=1}^{\infty}$ is a convergent sequence of complex numbers for every a and b in H. We show that such a sequence is a Cauchy sequence.

If $E = A \times B$ is an atom in $F_n \times F_n$, then $E_n(\phi)$ has constant value on E. For $m \geq n$, one has $E \in F_m \times F_m$ but it is not an atom there. Let $E = \sum_{i,j=1}^{k} (A_i \times B_j)$ where A_i and B_j are measurable atoms in F_m , $1 \leq i,j \leq k$. If $E_n(\phi)(E) = \lambda \in C$, then we have

$$E_n(\phi)(E)<P(A)a,Q(B)b>$$

$$= \lambda<P(A)a,Q(B)b>$$

$$= \sum_{i,j=1}^{k} \lambda<P(A_i)a,Q(B_j)b>,$$

where $A = \bigcup_i A_i$ and $B = \bigcup_i B_j$. Repeating the above procedure for every atom E in F_n we could obtain another function

$E_n^{(m)}(\phi)$ which is $F_m \times F_m$-measurable such that $E_n^{(m)}(\phi)(x,y) = E_m(\phi)(x,y)$ for all $(x,y) \in I \times I$. This enables us to write

$$L = |\langle S_n a,b\rangle - \langle S_m a,b\rangle|$$

$$= |\langle \int\int_{I\times I} E_n(\phi)(x,y)dE_n(P)(x)dE_n(Q)(y)a,b\rangle - $$
$$- \langle \int\int_{I\times I} E_m(\phi)(x,y)dE_m(P)(x)dE_m(\phi)(y)a,b\rangle|$$

$$= |\sum_{r,s} \langle (E_n^{(m)}(\phi)(r,s) - E_m(\phi)(r,s))Q(E_s)a, P(E_r)b\rangle|$$

where $(E_r \times E_s)_{r,s=1}^{2^m}$ are the atoms in $F_m \times F_m$ and $(E_n^{(m)}(\phi) - E_m(\phi))(r,s)$ denotes the value that such a function takes on every element of the set $E_r \times E_s$. Now, to estimate the last summation, we need to estimate $\| E_n^{(m)}(\phi) - E_m \|_M$ as an element of $M(L^2(E_m(\nu)) \hat{\otimes} L^2(E_m(\nu)))$.

Let $u \otimes v$ be an atom in the unit ball of $L^2(E_m(\nu)) \hat{\otimes} L^2(E_m(\nu))$ and K be the kernel of a contractive-compact operator on $L^2(E_m(\nu))$. It follows $u \otimes v$ is in the unit ball of $L^2(\nu) \hat{\otimes} L^2(\nu)$ and K is the kernel of a contractive-compact operator on $L^2(\nu)$. Then

$$R(n,m) = \langle E_n^{(m)}(\phi) - E_m(\phi)) \cdot u \otimes v, K\rangle$$

$$= \langle E_n^{(m)}(\phi) \cdot u \otimes v, K\rangle - \langle E_m(\phi) \cdot u \otimes v, K\rangle.$$

Lemma 3.3.1(iii) implies that

$$E_m(\phi \cdot u \otimes v \cdot K) = E_m(\phi) \cdot u \otimes v \cdot K.$$

Hence we obtain

$$R(n,m) = \langle \mathcal{E}_n^{(m)}(\phi) \cdot u \otimes v, K\rangle - \langle \phi \cdot u \otimes v, K\rangle.$$

Another application of lemma 3.3.1(i), we have $\lim\limits_{n,m} \mathcal{E}_n^{(m)}(\phi) = \phi$, where the convergence is a.e. $\nu\times\nu$. This implies that

$$\lim_{n,m} R(n,m) = 0.$$

Since $u \otimes v$ and K were arbitrary in their specific spaces it follows that

$$\lim_{n,m} \| \mathcal{E}_n^{(m)}(\phi) - \mathcal{E}_m(\phi)\|_{\mathcal{M}} = 0.$$

We, again, consider $L = |\langle S_n a, b\rangle - \langle S_m a,b\rangle|$. Theorem 3.2.5 now is applied together with the last estimate on $\| \mathcal{E}_n^{(m)}(\phi) - \mathcal{E}_m(\phi)\|_{\mathcal{M}}$ to yield that

$$\lim_{n,m} |\langle S_n a,b\rangle - \langle S_m a,b\rangle| = 0.$$

Hence $(\langle S_n a,b\rangle)_{n=1}^{\infty}$ is a Cauchy sequence in C for all a and b in $\mathcal{H}$. So $(S_n)_{n=1}^{\infty}$ converges in the weak-operator topology. Further if S is the limit of the sequence then $\|S\|_{L(H)} \le 1$.

Conversely (ii) $\to$ (i). Let $\lim\limits_{n} \|S_n\|_{L(H)} \le 1$. This implies that for each $\delta > 0$ there exists an $r(\delta)$ such that

$$\|S_r\|_{L(H)} \le 1+\delta.$$

Choose a sequence δ_r such that $\delta_r \to 0$ as $r \to \infty$ and $\|S_r\|_{L(H)} \le 1+\delta_r$. By theorem 3.2.5, we have

$$\mathcal{E}_r(\phi) \in \mathcal{M}(L^2(\mathcal{E}_r(\nu)) \hat{\otimes} L^2(\mathcal{E}_r(\nu))) \text{ and } \|\mathcal{E}_r(\phi)\|_{\mathcal{M}} \le 1+\delta_r.$$

Now, let $f \otimes g$ be any atom in the unit ball of $L^2(\nu) \hat{\otimes} L^2(\nu)$ and K be the kernel of a contractive-compact operator on $L^2(\nu)$. As in theorem 3.3.1, $E_n(f \otimes g)$ is an atom in the unit ball of $L^2(E_n(\nu)) \hat{\otimes} L^2(E_n(\nu))$, and $E_n(K)$ is the kernel of a contractive-compact operator on $L^2(E_n(\nu))$. We then have

$$
\begin{aligned}
&|\langle \phi \cdot f \otimes g, K\rangle| \\
&= \lim_n |\langle E_n(\phi) \cdot E_n(f \otimes g), E_n(K)\rangle| \\
&\leq \lim_n (1 + \delta_n) \\
&\leq 1.
\end{aligned}
$$

Since $f \otimes g$ and K were arbitrary in their specific spaces, it follows that $\phi \in M(L^2(\nu) \hat{\otimes} L^2(\nu))$ and $\|\phi\|_M \leq 1$. This completes the proof of the theorem.

CHAPTER IV

This final chapter consists of two sections. In the first section we prove the asymmetry and some other general results about the space $M(L^2(T,m) \hat{\otimes} L^2(T,m))$. The study of the change of variables on the space $L^2(I,m) \hat{\otimes} L^2(I,m)$ is the object of the second section.

4.1. The Asymmetry of $M(L^2(T,m) \hat{\otimes} L^2(T,m))$.

Throughout the whole section T will denote the unit circle and m the Lebesgue measure on T. For the simplicity of the notations, we write L^p for $L^p(T,m)$, $1 \leq p \leq \infty$.

Let $C(T)$ denote the space of continuous functions on T and $A(T)$ be the space of those functions in $C(T)$ that have absolutely convergent Fourier series. Consider the mapping $F : C \longrightarrow C(T \times T)$ defined by $F(f)(x,y) = f(x+y)$. Then

Theorem 4.1.1.

The following are equivalent:

(i) $f \in A(T)$

(ii) $F(f) \in C(T) \hat{\otimes} C(T)$.

Proof. See [12], page 255.

The map F defined above has range in $M(L^2 \hat{\otimes} L^2)$ when it is restricted to $A(T)$. Furthermore we prove the following

Lemma 4.1.1.

$$\|F(f)\|_M = \|f\|_{A(T)}.$$

Proof. Since $f \in A(T)$, then we can write $f(t) = \sum_{r=-\infty}^{\infty} a_r e^{irt}$, $\sum_{r=-\infty}^{\infty} |a_r| < \infty$. Hence $F(f)(x,y) = \sum_{r=-\infty}^{\infty} a_r e^{irx} \cdot e^{iry}$. However, $\|e^{irx}\|_\infty = 1$ for all r, it follows that $\|F(f)\|_M \le \|f\|_{A(T)}$.

To show the other inequality, define a mapping

$$P : C(T \times T) \longrightarrow C(T),$$

such that $P(\phi)(x) = \int_T \phi(x-y,y)dy$. Clearly $P \circ F : C(T) \longrightarrow C(T)$ is just the identity mapping. Let $F(f) \in C(T) \hat{\otimes} C(T)$ and $\sum_{i=1}^{\infty} u_i \otimes v_i$ be any representation of $F(f)$. Then

$$P(F(f)) = \sum_{i=1}^{\infty} u_i * v_i.$$

It follows that

$$\|P(F(f))\|_{A(T)} \le \|F(f)\|_{\tau r}.$$

However, the function $1 \otimes 1 \in L^2 \hat{\otimes} L^2$, so we have

$$\begin{aligned}\|F(f)\|_{\tau r} &= \|F(f) \cdot 1 \otimes 1\|_{\tau r} \\ &\le \|F(f)\|_M \cdot \|1 \otimes 1\|_{\tau r} \\ &= \|F(f)\|_M.\end{aligned}$$

Hence $\|P(F(f))\|_{A(T)} \le \|F(f)\|_M$. Since $P(F(f)) = f$, then $\|f\|_{A(T)} \le \|F(f)\|_M$. This completes the proof.

For the proof of the asymmetry of $M(L^2 \hat{\otimes} L^2)$, we need to prove the following lemma:

Lemma 4.1.2.

Let ϕ_1 and ϕ_2 be any two elements in the unit ball of $M(L^2 \hat{\otimes} L^2)$. Assume, further, that supp $\phi_1 \subseteq \Omega_1 = X_1 \times Y_1$, supp $\phi_2 \subseteq \Omega_2 = X_2 \times Y_2$, where $X_1 \cap X_2 = Y_1 \cap Y_2 = \emptyset$, the empty set. Then there exists a function $\phi \in M(L^2 \hat{\otimes} L^2)$ such that $\phi|_{\Omega_i} = \phi_i$, $i = 1,2$ and $\|\phi\|_M = \max_{i=1,2} \|\phi_i\|_M$.

Proof. Define the following function ϕ on $T \times T$

$$\phi(x,y) = \begin{cases} \phi_1 & \text{if} \quad (x,y) \in \Omega_1 \\ \phi_2 & \text{if} \quad (x,y) \in \Omega_2 \end{cases}$$

and $\phi \equiv 0$ on the complement of $\Omega_1 \cup \Omega_2$. We claim that the function ϕ is the required function. First, since $\phi = \phi_1 + \phi_2$, it follows that $\phi \in M(L^2 \hat{\otimes} L^2)$. Remains to estimate the multiplier-norm of ϕ. To do so, let $f \otimes g$ be any atom in the unit ball of $L^2 \hat{\otimes} L^2$. Since

$$f \otimes g = \frac{f}{\|f\|_2} (\|f\|_2 \cdot \|g\|_2)^{1/2} \cdot \frac{g}{\|g\|_2} (\|f\|_2 \cdot \|g\|_2)^{1/2},$$

we can assume that $\|f\|_2 = \|g\|_2 \leq 1$. Further since the support of ϕ is contained in $\Omega_1 \cup \Omega_2$, we let $\text{supp}(f) \subset X_1 \cup X_2$ and $\text{supp}(g) \subset Y_1 \cup Y_2$. Set $f_i = f|_{X_i}$ and $g_i = g|_{Y_i}$, $i = 1,2$.

Then $f = f_1 + f_2$ and $g = g_1 + g_2$. Further $\|f\|_2^2 = \|f_1\|_2^2 + \|f_2\|_2^2$ and $\|g\|_2^2 = \|g_1\|_2^2 + \|g_2\|_2^2$, since $\bigcap_{i=1}^{2} X_i = \bigcap_{i=1}^{2} Y_i = \emptyset$. Now, consider

$$\phi \cdot f \otimes g = \phi_1 \cdot f_1 \otimes g_1 + \phi_2 \cdot f_2 \otimes g_2.$$

Since $\|\phi_i\|_M \le 1$, $i = 1,2$, we deduce

$$\phi_i \cdot f_i \otimes g_i = \sum_{j=1}^{\infty} u_j^{(i)} \otimes v_j^{(i)}$$

$$\sum_{j=1}^{\infty} \|u_j^{(i)}\|_2 \cdot \|v_j^{(i)}\|_2 \le \|f_i\|_2 \cdot \|g_i\|_2 .$$

Again as above, we can assume that $\|f_i\|_2 = \|g_i\|_2$ and $\|u_j^{(i)}\|_2 = \|v_j^{(i)}\|_2$ for $i = 1,2$ and $j \ge 1$. It follows that

$$\sum_{j=1}^{\infty} \|u_j^{(i)}\|_2^2 \le \|f_i\|_2^2$$

$$\sum_{j=1}^{\infty} \|v_j^{(i)}\|_2^2 \le \|g_i\|_2^2, \qquad i = 1,2.$$

Now define the following functions

$$z_j = u_j^{(1)} + u_j^{(2)}$$

$$w_j = v_j^{(1)} + v_j^{(2)}$$

for all $j \ge 1$. Then

$$\phi \cdot f \times g = \sum_{j=1}^{\infty} (z_j \otimes w_j) \cdot 1_{(X_1 \times Y_1) \cup (X_2 \cup Y_2)},$$

where 1_E denotes the characteristic function of the set E. But since

$$\|z_j\|_2^2 = \|u_j^{(1)}\|_2^2 + \|u_j^{(2)}\|_2^2$$

$$\|w_j\|_2^2 = \|v_j^{(1)}\|_2^2 + \|v_j^{(2)}\|_2^2,$$

it follows that

$$\|\phi \cdot f \otimes g\|_{\tau r} \le \sum_{j=1}^{\infty} \|z_j\|_2 \; \|w_j\|_2$$

$$\le \sum_{j=1}^{\infty} (\|u_j^{(1)}\|_2^2 + \|u_j^{(2)}\|_2^2)^{1/2} \cdot (\|v_j^{(1)}\|_2^2 + \|v_j^{(2)}\|_2^2)^{1/2}$$

$$\le \Big(\sum_{j=1}^{\infty} (\|u_j^{(1)}\|_2^2 + \|u_j^{(2)}\|_2^2)\Big)^{1/2} \cdot \Big(\sum_{j=1}^{\infty} (\|v_j^{(1)}\|_2^2 + \|v_j^{(2)}\|_2^2)\Big)^{1/2}$$

$$\le (\|f_1\|_2^2 + \|f_2\|_2^2)^{1/2} \cdot (\|g_1\|_2^2 + \|g_2\|_2^2)^{1/2}$$

$$\le \|f\|_2 \cdot \|g\|_2$$

$$\le 1.$$

Since $f \otimes g$ was arbitrary atom in the unit ball of $L^2 \hat{\otimes} L^2$, it follows that $\|\phi\|_M \le 1$. This completes the proof of the lemma.

Let us recall that a commutative Banach algebra is called symmetric if, regarded as a function algebra on its maximal ideal space, it is closed under complex conjugation. Clearly $M(L^2 \hat{\otimes} L^2)$ is a commutative Banach algebra, where multiplication is taken to be composition as operators on $L^2 \hat{\otimes} L^2$. Varopoulos, [27], proved the asymmetry of the tensor algebra $C(T) \hat{\otimes} C(T)$. In a similar way we prove the following

<u>Theorem</u> 4.1.2.

The space $M(L^2 \hat{\otimes} L^2)$ is not symmetric.

<u>Proof</u>. To prove the asymmetry of a space it is enough to produce an element in such a space which has independent powers, [30].

Let P be a Cantor independent set which is not Helson in T. The existence of P is illustrated in [19]. Take ν to be a non-negative measure concentrated on $P \cup (-P)$. Then ν has mutually singular convolution powers, and if we choose $\|\nu\|_{M(T)} = 1$, we obtain

$$\left\| \sum_{r=1}^{n} \lambda_r \nu^r \right\|_{M(T)} = \sum_{r=1}^{n} |\lambda_r|,$$

for all $\lambda_r \in C$ and $n \in N$. Since discrete measures on T are dense in M(T) in the weak-* topology [10], then we can find a sequence $(\nu_n)_{n=1}^{\infty}$ of finitely supported discrete measures (the support of each ν_n is a finite subgroup of T) such that

$$\hat{\nu}_n(j) \longrightarrow \hat{\nu}(j)$$

for all $j \in Z$. That P is not Helson enables us to choose ν such that $\|\hat{\nu}\|_\infty$ is as small as we like and $\hat{\nu}$ to be real. If E_n denotes the support of ν_n, then we can find a sequence $(f_n)_{n=1}^{\infty}$ of real functions on T such that

$$\|f_n\|_{A(E_n)} \leq 1 \quad (n \geq 1), \quad \|f_n\|_\infty \longrightarrow 0 \text{ as } n \longrightarrow \infty,$$

$$\sup_n \left\| \sum_{r=1}^{s} \lambda_r f_n^r \right\|_{A(E_n)} = \sum_{r=1} |\lambda_r|,$$

for all $s \in N$ and $\lambda_r \in C$.

Now, let $(X_n^{(i)})_{n=1}^{\infty}$ $i = 1,2$, be two sequences of sets in T such that $X_n^{(i)} \cap X_m^{(i)} = \emptyset$ for $n \neq m$, $i = 1,2$ and $X_n^{(i)}$ has the same cardinality as E_n. Identify, then, $X_n^{(i)}$ with E_n for every $n \geq 1$, and $i = 1,2$. If $F : C(T) \longrightarrow C(T \times T)$ is the function defined in theorem 4.1.1, then set $\phi_n = F(f_n)$,

$n \geq 1$. A simple application of lemma 4.1.1 implies that $\phi_n \in M(L^2 \hat{\otimes} L^2)$ and

$$\|\phi_n\|_M \leq 1 \quad (n \geq 1); \quad \|\phi\|_\infty \longrightarrow 0 \text{ as } n \longrightarrow \infty$$

$$\sup_n \|\sum_{r=0}^{s} \lambda_r \phi_n^r\|_M = \sum_{r=0}^{s} |\lambda_r|;$$

for all $s \in N$ and $\lambda_r \in C$. Using lemma 4.1.2 repeatedly we construct a sequence of real functions $(\psi_n)_{n=1}^{\infty}$ in $M(L^2 \hat{\otimes} L^2)$ such that

$$\|\psi_n\|_M \leq 1 \quad (n \geq 1); \quad \text{supp } \psi_n = \bigcup_{j=1}^{n} X_j^{(1)} \times X_j^{(2)};$$

$$\psi_n|_{X_n^{(1)} \times X_n^{(2)}} = \phi_n, \quad \|\psi_n\|_\infty \longrightarrow 0 \text{ as } n \longrightarrow \infty.$$

Clearly, the sequence $(\psi_n)_{n=1}^{\infty}$ converges uniformly to a function $\psi \in M(L^2 \hat{\otimes} L^2)$. Furthermore $\|\psi\|_M = \sup_n \|\psi_n\|_M$. Hence

$$\|\sum_{r=0}^{s} \lambda_r \psi^r\|_M = \sum_{r=0}^{s} |\lambda_r|.$$

This completes the proof of the theorem.

As a corollary of the previous theorem we have

<u>Theorem</u> 4.1.3.

The space $M(L^2 \hat{\otimes} L^2)$ is not separable.

<u>Proof</u>. The functions $(\psi_n)_{n=1}^{\infty}$ in theorem 4.1.2 have the property that $\|\psi_n - \psi_m\|_M \geq \alpha > 0$ for $n \neq m$. This proves the claim.

The following theorem is similar to that in [31] for

M(T).

Theorem 4.1.4.

Given any $\varepsilon > 0$, there exists a $\phi \in M(L^2 \hat{\otimes} L^2)$ such that

(i) $\|\phi\|_M \le 1$

(ii) ϕ has independent powers

(iii) $0 \le \gamma(\phi) < \varepsilon$ for all self-adjoint character γ in the maximal ideal space of $M(L^2 \hat{\otimes} L^2)$

Proof. Theorem 4.1.2 implies the existence of an $\psi \in M(L^2 \hat{\otimes} L^2)$ such that ψ satisfies (i) and (ii). Set

$$\phi_n = (\|\psi\|_M^2 \cdot 1 \otimes 1 - \psi^2)^n .$$

Clearly ϕ_n satisfies (ii) for all $n \ge 1$. Furthermore

$$\|\phi_n\|_M = 2^n \|\psi\|_M^{2n} .$$

If γ is any element in the maximal ideal space of $M(L^2 \hat{\otimes} L^2)$,

$$|\gamma(\psi)| \le \|\gamma\| \cdot \|\psi\|_M = \|\psi\|_M .$$

Choosing ψ to be real and γ to be self-adjoint, we get

$$-\|\psi\|_M \le \gamma(\psi) \le \|\psi\|_M .$$

Set $\phi = \dfrac{\phi_n}{\|\phi_n\|_M}$ for some $n \ge 1$. Then since $\gamma(1 \otimes 1) = 1$, we have

$$\gamma(\phi_n) = \frac{\gamma(\phi_n)}{\|\phi_n\|_M}$$

$$= \frac{1}{\|\phi_n\|_M} \cdot (\|\psi\|_M^2 - \gamma(\psi)^2)^n .$$

It follows that

$$0 \leq \gamma(\phi_n) \leq \|\psi\|_M^{2n}.$$

Hence one has

$$0 \leq \gamma(\phi) \leq \frac{\gamma(\phi_n)}{\|\phi_n\|_M} = \frac{\|\psi\|_M^{2n}}{2^n \cdot \|\psi\|_M^{2n}} = \frac{1}{2^n} .$$

Taking n to be large enough such that $\frac{1}{2^n} < \varepsilon$ we get $0 \leq \gamma(\phi) < \varepsilon$. This completes the proof of the theorem.

4.2. Certain Homomorphisms of the Trace-Class Operators

This section, in some way, is independent of the rest of this thesis. It deals with the problem of change of variables on the space $L^2(I,m) \hat{\otimes} L^2(I,m)$, where I is the unit interval, and m is the Lebesgue measure on I.

Let $F : I \times I \longrightarrow I \times I$ be a measurable mapping. The mapping F induces an operator $U : L^2(I) \hat{\otimes} L^2(I) \longrightarrow L^2(I) \hat{\otimes} L^2(I)$, where for simplicity of notations we wrote $L^2(I)$ for $L^2(I,m)$. If ψ is an element of $L^2(I) \hat{\otimes} L^2(I)$, then $\|\psi\|_{\tau r}$ will denote the trace-class norm of ψ, and $\|\psi\|_{HS}$ denotes the Hilbert Schmidt norm. Then we prove the following

Theorem 4.2.1.

Let F and U be as above. If $\|U(\psi)\|_{\tau r} = \|\psi \circ F\|_{\tau r} \leq \|\psi\|_{\tau r}$, then F is essentially of the type $F = (F_1, F_2)$, where F_1 and F_2 are measure preserving mappings on I.

Proof. Since the proof is little long, we prove the theorem in steps.

Step I. The mapping F is measure preserving.

Proof. Let X_1 and X_2 be any two disjoint sets in I such that $I = X_1 \cup X_2$. If Y_1 and Y_2 is a similar pair of sets in I, then set $1_{X_i \times Y_j}$ to denote the characteristic function of $X_i \times Y_j$, $1 \leq i,j \leq 2$. From the definition of the Hilbert Schmidt norm we deduce

$$\|1_{X_i \times Y_j} \circ F\|_{HS} = \|1_{X_i \times Y_j} \circ F\|_2$$

$$= [m(F^{-1}(X_i \times Y_j))]^{1/2}$$

$$\leq \| 1_{X_i \times Y_j} \circ F \|_{\tau r}$$

$$\leq \| 1_{X_i \times Y_j} \|_{\tau r}$$

$$= \| 1_{X_i} \otimes 1_{Y_j} \|_{\tau r}$$

$$= [m(X_i \times Y_j)]^{1/2}.$$

On the other hand we have

$$1 = \sum_{i,j=1}^{2} m(F^{-1}(X_i \times Y_j))$$

$$= \sum_{i,j=1}^{2} \| 1_{X_i \times Y_j} \circ F \|_{HS}^2$$

$$\leq \sum_{i,j=1}^{2} \| 1_{X_i \times Y_j} \|_{\tau r}^2$$

$$= \sum_{i,j=1}^{2} m(X_i \times Y_j).$$

Hence, it follows that $m(F^{-1}(X_i \times Y_j)) = m(X_i \times Y_j)$, $1 \leq i,j \leq 2$. Now if $A \times B$ is any rectangle, then setting $X_1 = A$, $X_2 = A^c$, $Y_1 = B$, and $Y_2 = B^c$, the previous argument shows that $m(A \times B) = m(F^{-1}(A \times B))$. So F preserves the measure of any rectangle. However, the set of rectangles in $I \times I$ forms an algebra of sets. Further the σ-algebra of the Lebesgue measurable sets in $I \times I$ is just the completion of the smallest σ-algebra containing the rectangles. It follows that F is measure preserving on $(I \times I, m \times m)$.

Step II. The operator U preserves atoms in $L^2(I) \hat{\otimes} L^2(I)$.

<u>Proof</u>. Let $f \otimes g$ be any atom in $L^2(I) \hat{\otimes} L^2(I)$. If $\|f \otimes g\|_2$ denotes the norm of $f \otimes g$ as an element of $L^2(I \times I)$, then $\|f \otimes g\|_2 = \|f \otimes g\|_{\tau r}$. By step I, as it is well known and easy to prove, the operator U on $L^2(I \times I)$ is an isometry, so $\|f \otimes g\|_2 = \|(f \otimes g) \circ F\|_2$. Then

$$\begin{aligned} \|f \otimes g\|_{HS} &= \|f \otimes g\|_2 \\ &= \|(f \otimes g) \circ F\|_2 \\ &= \|(f \otimes g) \circ F\|_{HS} \\ &\leq \|(f \otimes g) \circ F\|_{\tau r} \\ &\leq \|f \otimes g\|_{\tau r} \\ &= \|f \otimes g\|_2 . \end{aligned}$$

This implies that $\|(f \otimes g) \circ F\|_{HS} = \|(f \otimes g) \circ F\|_{\tau r}$. However, from the definition of the trace-class norm and the Hilbert-Schmidt norm, the two norms coincide only on operators of rank one. Hence $f \otimes g \circ F = u \otimes v$ for some atom $u \otimes v$ in $L^2(I) \hat{\otimes} L^2(I)$.

<u>Step III</u>. Construction of F_1 and F_2.

Let $i : I \longrightarrow I$ be the identity map: $i(x) = x$, and $\pi_1, \pi_2 : I \times I \longrightarrow I$ be the first and the second projections respectively. Set $F_1 = \pi_1 \circ F$ and $F_2 = \pi_2 \circ F$. Then $F = (F_1, F_2)$. Now consider the map $i \otimes 1 \in L^2(I) \hat{\otimes} L^2(I)$, where 1 denotes the constant function with range $\{1\}$. Step II implies that $(i \otimes 1) \circ F = \alpha_1 \otimes \alpha_2$, α_1 and α_2 are in $L^2(I)$. Hence

$$(\alpha_1 \otimes \alpha_2)(x,y) = (i \otimes 1 \circ (F_1,F_2)(x,y)$$

$$\alpha_1(x)\alpha_2(y) = i(F_1(x))$$

$$= F_1(x).$$

Therefore $F_1 = \alpha_1 \otimes \alpha_2$. Similarly $F_2 = \beta_1 \otimes \beta_2$.

<u>Step IV</u>. Each of the functions F_1 and F_2 depend on one of the variables but not both.

Proof. For any function $\psi \in L^2(I) \hat{\otimes} L^2(I)$, set

$$m(\psi) = \iint_{I\times I} \psi(x,y)dxdy.$$

From step I, it follows that

$$m(\psi) = m(U(\psi)) \text{ - - - - } * .$$

For an atom ϕ in $L^2(I) \hat{\otimes} L^2(I)$, set

$$m_1(\phi)(x) = \int_I \phi(x,y)dy$$

$$m_1(\phi)(y) = \int_I \phi(x,y)dx.$$

Hence one can write

$$m(\phi) \cdot \phi = m_1(\phi) \otimes m_2(\phi) \text{ - - - - } **$$

Step II together with $*$ implies

$$m(\phi) \cdot U(\phi) = m_1(U(\phi)) \otimes m_2(U(\phi)),$$

So if $m(\phi) = 0$, then either $m_1(U(\phi)) = 0$ or $m_2(U(\phi)) = 0$. Now, take $\phi = f \otimes 1$, and let us write $U(f)$ for $U(f \otimes 1)$. Set

$$V_j = \{f \mid m_j(U(f-m(f)\cdot 1)) = 0\},$$

where $m(f)$ denotes $m(f \otimes 1)$. Since for any $f \in L^2(I)$ we have

$$m(f-m(f)\cdot 1) = m(f) - m(f) = 0,$$

it follows that for any $f \in L^2$, either $m_1(U(f-m(f)\cdot 1)) = 0$ or $m_2(U(f-m(f)\cdot 1)) = 0$. Hence V_1 and V_2 are closed subspaces of $L^2(I)$ such that $L^2(I) = V_1 \cup V_2$. In this case, as it is well known, either $V_1 = L^2(I)$ or $V_2 = L^2(I)$. That is there exists $j = 1$ or 2 such that

$$m_j(U(f)-m(f)1)) = 0,$$

for all $f \in L^2(I)$. Let us assume that $j = 1$. It follows that

$$m_i(U(f)) = m(f)\cdot 1.$$

Relations $*$ and $**$ then imply

$$m(f)\cdot U(f) = m(f)(1 \otimes m_2(U(f))).$$

Thus if $m(f) \neq 0$, we conclude that

$$U(f) = U(f \otimes 1) = 1 \otimes m_2(U(f)).$$

But $U(f \otimes 1) = (f \otimes 1) \circ (F_1,F_2) = (f \circ F_1) \otimes 1$. So for any $(x,y) \in I \times I$,

$$m_2(U(f))(y) = (f \otimes 1) \circ F(x,y)$$
$$= (f \circ F_1)(x,y)$$
$$= f(\alpha_1(x)\alpha_2(y)),$$

by step III. Hence α_1 is a constant. In case $j = 2$, we obtain α_2 is a constant. This proves that the function F_1 depends either on the first coordinate, or on the second co-ordinate but not on both. Now on considering the atom $1 \otimes f$, we prove the same way as above that either β_1 is a constant or β_2 is a constant, so F_2 also depends on one of the coordinates but not on both. So we conclude that F has one of the following forms:

$$\text{(i)}\quad F = (\alpha_1,\beta_1), \qquad \text{(ii)}\quad F = (\alpha_1,\beta_2)$$
$$\text{(iii)}\quad F = (\alpha_2,\beta_1), \qquad \text{(iv)}\quad F = (\alpha_2,\beta_2).$$

Finally, we have to show that α_1, α_2, β_1, β_2 are all measure preserving maps on I. We prove such a claim only for α_1, since it is the same for the others. Let E be any set in I. Set $f = 1_E$ and consider $f \otimes 1 = 1_E \otimes 1$. Since F is measure preserving we obtain

$$(m(E))^{1/2} = \|1_E \otimes 1\|_2 = \|(1_E \otimes 1) \circ F\|_2$$
$$= \|1_E(\alpha_1) \otimes 1(\beta_1)\|_2$$
$$= \|1_E(\alpha_1)\|_2$$
$$= [m(\alpha^{-1}(E))]^{1/2},$$

where we are considering F to be of the form (i). Before we close the proof of the theorem, we have to remark about the notation in the forms (i), (ii), (iii), and (iv). Here

$$\text{(i)}\quad F(x,y) = (\alpha_1(x),\beta_1(x)), \qquad \text{(ii)}\quad F(x,y) = (\alpha_1(x),\beta_2(y))$$
$$\text{(iii)}\quad F(x,y) = (\alpha_2(y),\beta_1(y)), \qquad \text{(iv)}\quad F(x,y) = (\alpha_2(y),\beta_2(y)).$$

And now, the proof of the theorem is complete.

BIBLIOGRAPHY

[1] Ameiya, I. and Shiga, K., On tensor product of Banach spaces. Kodai Math. Seminar Reports 9 (1957), 161-176.

[2] Bary, N.K., A treatise on Trigonometric Series. Vol. I, Macmillan, N.Y., 1964.

[3] Bennett, G., Schur multipliers, Duke Math. J. 44 (1977), 603-639.

[4] Berberian, S.K., Notes on spectral theory. Van Nostrand, Princeton, 1966.

[5] Billingslely, P., Ergodic theory and information, Wiley, N.Y., 1965.

[6] Diestel, J. and Uhl, JR., Vector measures. Mathematical Surveys 15, 1977.

[7] Grothendieck, A., Résumé de la théorie métrique des produits tensoriels topologiques. Boletin da Sociedade de Mathematica de Sae Paolo 8 (1956), 1-79.

[8] Grothendieck, A., Produits tensoriels topologiques et espaces nucleairés, Memoirs Amer. Math. Soc. 16 (1955).

[9] Helgason, S., Multipliers of Banach algebras, Ann. of Math., 64 (1956), 240-254.

[10] Herz, C.S., Spectral synthesis for the Cantor set. Proc. Natl. Acad. Sci. US 42 (1956), 42-43.

[11] Hewitt, E. and Ross, K., Abstract harmonic analysis. Berlin-Heidelberg-N.Y., 1970.

[12] Katznelson, Y., An introduction to Harmonic analysis, N.Y., 1968.

[13] Kwapien, S., A remark on p-absolutely summing operators in ℓ_r-spaces, Studia Math. 28 (1970), 109-111.

[14] Lindenstrauss, J. and Pelczynski, A., Absolutely summing operators in L^p-spaces and their applications, Studia Math. 29 (1968), 275-326.

[15] Littlewood, J.E., On bounded bilinear forms in an infinite number of variables. Quart. J. Math., Oxford 1 (1930), 164-174.

[16] Pietsch, A., Absolut summierende Abbildungen in lokalkonvexen Räumen, Math. Nachr. 27 (1963), 77-103.

[17] Rietz, R., A proof of the Grothendieck inequality. Israel J. Math. 19 (1974), 271-276.

[18] Ringrose, J.R., Compact non-selfadjoint operators, Van Nostrand, Reinhold, London, 1971.

[19] Rudin, W., Fourier analysis on groups, N.Y., 1962.

[20] Schaefer, H.H., Banach Lattices and Positive Operators, Grundlehren der Math. Wiss., Vol. 215, Springer-Verlag, Berlin and New York, 1974.

[21] Schatten, R., A theory of cross-spaces. Ann. of Math. Stud. no. 26, Princeton Univ. Press, 1950.

[22] Schatten, R., Norm ideals of completely continuous operators, Springer-Verlag, Berlin, 1960.

[23] Schur, J., Bemerkungen zur Theorie der beschränkten Bilinearformen mit unendlich vielen Verandlichen, J. Reine Angew. Math. 140 (1911), 1-28.

[24] Semadeni, Z., Banach spaces, Warszawa, 1971.

[25] Stephens, B. and Cohen, S., Optimization problem for K_G. Manuscripta math. 13 (1974), 131-142.

[26] Varapoulos, N.Th., Tensor algebra and harmonic analysis. Acta Math. 119 (1967), 51-111.

[27] Varapoulos, N.Th., On a problem of A. Beurling. J. Fun. Anal., 2 (1968), 24-30.

[28] Varapoulos, N.Th., Tensor algebra over discrete spaces, J. Fun. Anal., 3 (1968), 321-335.

[29] Varapoulos, N.Th., Some remarks on Q-algebras. Ann. Inst. Fourier, 4 (1972), 1-11.

[30] Williamson, J.H., A theorem on algebras of measures on topological groups. Proc. Edinburgh Math. Soc. 11 (1959), 195-206.

[31] Williamson, J.H., Theorems of Wiener-Pitt type. Symposium on Function Algebra, Tulane Univ., 1965, 186-197.