

Reverse Agmon Estimates and Nodal Intersection Bounds in Forbidden Regions

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ABSTRACT

Let (M, g) be a compact, Riemannian manifold and $V \in C^\infty(M; \mathbb{R})$. Given an energy level $E > \min V$, we consider L^2 -normalized eigenfunctions, u_h , of the Schrödinger operator $P(h) = -h^2 \Delta_g + V - E(h)$ with $P(h)u_h = 0$ in L^2 where $E(h) = E + o(1)$. The well-known Agmon-Lithner estimates [Hel88] are exponential decay estimates (ie. upper bounds) for eigenfunctions in the forbidden region $\{V > E\}$ in terms of the Agmon distance function d_E associated with the degenerate Agmon metric $(V - E)_+ g$ supported in the forbidden region.

The point of this thesis is to prove a partial converse to the Agmon estimates (ie. lower bounds for the eigenfunctions) in terms of Agmon distance in the forbidden region under a control assumption on eigenfunction mass in the allowable region $\{V < E\}$ arbitrarily close to the caustic $\{V = E\}$. We then give some applications to hypersurface restrictions of eigenfunctions in the forbidden region along with applications to nodal intersection bounds.

ABRÉGÉ

Soit (M, g) une variété riemannienne compacte et $V \in C^\infty(M; \mathbb{R})$. Etant donné un niveau en énergie $E > \min V$, on considère fonctions propres L^2 -normalisées, u_h , de l'opérateur de Schrödinger $P(h) = -h^2 \Delta_g + V - E(h)$ avec $P(h)u_h = 0$ dans L^2 où $E(h) = E + o(1)$. Les estimations d'Agmon-Lithner bien connues [Hel88] sont des estimations de décroissance exponentielle (c'est-à-dire des bornes supérieures) pour des fonctions propres dans la région interdite $\{V > E\}$ en terme de la fonction de distance Agmon d_E associée à la métrique Agmon dégénérée $(V - E)_+ g$ supportée dans la région interdite.

Le but de cette thèse est de prouver un inverse partiel des estimations d'Agmon (c'est-à-dire pour les fonctions propres) en terme de distance d'Agmon dans la région interdite sous une hypothèse de contrôle sur la masse propre dans la région admissible $\{V < E\}$ arbitrairement proche du caustique $\{V = E\}$. Nous donnons ensuite quelques applications aux restrictions d'hypersurface des fonctions propres dans la région interdite ainsi qu'aux applications aux limites d'intersection nodale.

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CHAPTER 1

Introduction

1.1 Motivation and background

Let (M, g) be a compact, C^∞ Riemannian manifold and $V \in C^\infty(M; \mathbb{R})$ be a real-valued potential. We assume that E a regular value of V so that $dV|_{V=E} \neq 0$. The corresponding classically allowable region is

$$\Omega_E := \{x \in M; V(x) \leq E\}. \quad (1.1.1)$$

with boundary C^∞ hypersurface (ie. boundary caustic)

$$\Lambda_E := \{x \in M; V(x) = E\}. \quad (1.1.2)$$

The *forbidden region* is the complement $\Omega_E^c = \{x \in M; V(x) > E\}$.

Agmon-Lithner estimates

Let $P(h) : C^\infty(M) \rightarrow C^\infty(M)$ be the Schrödinger operator

$$P(h) := -h^2 \Delta_g + V(x) - E(h)$$

and $u_h \in C^\infty(M)$ be L^2 -normalized eigenfunctions with eigenvalue $E(h)$ so that

$$P(h)u_h = 0 \quad \text{and} \quad E(h) = E + o(1) \text{ as } h \rightarrow 0^+.$$

The *Agmon metric* associated with $P(h)$ is defined by

$$g_E(x) := (V(x) - E)_+ g(x).$$

The degenerate metric g_E is supported in the forbidden region Ω_E^c and we denote the corresponding Riemannian distance function by $d_E : \Omega_E^c \times \Omega_E^c \rightarrow \mathbb{R}^+$. By a slight abuse of notation, we define the associated distance function to Λ_E by

$$d_E(x) := d_E(x, \Lambda_E) = \inf_{y \in \Lambda_E} d_E(x, y), \quad x \in \Omega_E^c. \quad (1.1.3)$$

It is well-known [Hel88] that, $d_E \in Lip(\Omega_E^c)$ and also, $|\nabla_x d_E|_g^2 \leq (V(x) - E)_+$, *a.e.*

Intuitively, in the quantum mechanics the probability of finding the particle in the classically forbidden region Ω_E^c in the semiclassical limit should be quite small. In 1980s, Agmon [Agm82] gave a quantitative estimate for Schrödinger eigenfunction $u(x)$. Given an open subset, U , of the forbidden region Ω_E^c with $\overline{U} \subset \Omega_E^c$, the *Agmon-Lithner estimate* says that for any $\varepsilon > 0$,

$$\|u(x)e^{(1-\varepsilon)\varphi(x)}\|_{L^2(U)} \leq O_\varepsilon(1), \quad (1.1.4)$$

for a suitable positive weight function $\varphi(x)$.

Later, Helffer [[Hel88] Proposition 3.3.4] showed in the semiclassical case,

$$\|e^{(1-\varepsilon)d_E/h} u_h\|_{H_h^1(U)} = O_\varepsilon(1), \quad (1.1.5)$$

where $\|f\|_{H_h^1(U)}^2 := \int_U (|f|^2 + |h\partial f|^2)$. A standard argument with Sobolev estimates then yields corresponding pointwise upper bounds as well. Such estimates have widespread applications to tunnelling problems [Hel88] [CS81] [Sim84] and the theory of Morse-Witten complexes [Wit82].

For the ground states, [CS81] [Sim84] showed (1.1.4) and (1.1.5) cannot be improved by using *Feynman-Kac* path integrals. In [Sim84], the author gave a very explicit lower bound for a ground state eigenfunction along the Agmon geodesic. Indeed the analysis of tunneling for ground states is known in great detail including eigenfunction asymptotics, eigenvalue asymptotics and splitting. These topics were studied extensively by B.Helffer and J. Sjöstrand in a series of papers in the 1980's ([HS84, HS85b, HS85a, HS85c, HS86]).

The purpose of this thesis is as follows: (i) to reveal the tunneling phenomenon (i.e. a partial converse to (1.1.5)) in the case of an excited state E in a Fermi neighborhood of the caustic Λ_E under a suitable control assumption on eigenfunction mass and certain growth restrictions for the potential $V(x)$; (ii) to give lower bounds for L^p -restrictions of eigenfunctions to hypersurfaces in the forbidden region (so-called *goodness* estimates in the terminology of Toth and Zelditch [TZ09]) ; (iii) improve the nodal intersection bounds of [CT16] for a large class of hypersurfaces in forbidden regions. We now describe our results in more detail.

In the following we fix a constant $r_0 \in (0, \frac{\text{inj}(M, g)}{2})$ and let $U_E(r_0)$ be a Fermi neighbourhood of the caustic Λ_E of diameter $2r_0$ with respect to the ambient metric g . We denote the Fermi defining function $y_n : M \rightarrow \mathbb{R}$ with the property that $y_n > 0$ in the forbidden part and $\Lambda_E = \{y_n = 0\}$. In terms of Fermi coordinates,

the collar neighbourhood $U_E(r_0) := \{y; |y_n| < 2r_0\}$. Consider an annular region in $U_E(r_0) \cap \{V > E\}$ given by $A(\delta_1, \delta_2) := \{y \in U_E(r_0); E + \delta_1 < V(y) < E + \delta_2\}$ with $0 < \delta_1 < \delta_2$. Our first result in Theorem 1.2.1 is a partial converse to the Agmon estimates in (1.1.5). First, we introduce a *control* assumption on the eigenfunctions u_h in the allowable region.

Definition 1.1.1. We say that the eigenfunctions u_h satisfy the *control assumption* if for every $\varepsilon > 0$ there exists constants $C(\varepsilon) > 0$ and $h_0(\varepsilon) > 0$ so that for $h \in (0, h_0(\varepsilon)]$,

$$\int_{\{E-\varepsilon/2 < V(x) \leq E\}} |u_h|^2 dv_g \geq C_N(\varepsilon) h^N \quad (1.1.6)$$

for some $N > 0$. When (1.1.6) is satisfied for a *fixed* $\varepsilon = \varepsilon_0 > 0$, we say that the eigenfunction sequence satisfies the ε_0 *control assumption*.

Roughly speaking, the control assumption in Definition 1.1.1 says that in an arbitrarily small (but independent of h) annular neighbourhoods of the caustic in the *allowable* region, eigenfunctions have at least polynomial mass in h . It is easy to see that this assumption is necessary since simple counterexamples can be constructed otherwise by introducing additional effective potentials (see Chapter 6).

We note that the control assumption is automatically satisfied in the 1D case where WKB theory yields asymptotics for the eigenfunctions. In section 6, we give examples of eigenfunction sequences satisfying this condition in arbitrary dimension.

Our second assumption is a convexity condition on the potential V itself; in particular, ruling out tunnelling phenomena in the Fermi neighbourhood. Specifically, we make the following

Definition 1.1.2. We say that V satisfies the *convexity assumption* provided:

$$(i) \text{ Crit}(V) \cap (U_E(r_0) \cap \Omega_E^c) = \emptyset,$$

$$(ii) \text{ } V|_{U_E(r_0) \cap \Omega_E^c} \text{ is convex.}$$

Under the control and convexity conditions, by using Carleman estimates to pass across the caustic hypersurface, in Theorem 1.2.1 we prove that for any $\varepsilon > 0$ and $h \in (0, h_0(\varepsilon)]$,

$$\|e^{\tau_0 d_E/h} u_h\|_{H_h^1(A(\delta_1, \delta_2))} \geq C(\varepsilon, \delta_1, \delta_2) e^{-\beta(\varepsilon)/h}, \quad (1.1.7)$$

where $\beta(\varepsilon) = o(1)$ as $\varepsilon \rightarrow 0^+$ and

$$\tau_0 = \left(\frac{\max_{y \in U_E(r_0)} |\partial_{y_n} V|}{\min_{y \in U_E(r_0)} |\partial_{y_n} V|} \right)^{1/2}.$$

Here the explicit geometric constant $\tau_0 \geq 1$ and the result in (1.1.7) is clearly a partial converse to the Agmon estimates in (1.1.5).

1.2 Statement of results

The main result of this thesis is

Theorem 1.2.1. *Let $r_0 > 0$ define the collar neighbourhood $U_E(r_0)$ of the hypersurface $\{V = E\}$ as above and consider an annular subdomain*

$$A(\delta_1, \delta_2) \subset \left(\{V > E\} \cap U_E(r_0) \right), \quad 0 < \delta_1 < \delta_2 < r_0.$$

Then, under the control and convexity assumptions in Definitions 1.1.1 and 1.1.2, it follows that for any $\varepsilon > 0$ and $h \in (0, h_0(\varepsilon)]$, there exists a constant $C(\varepsilon, \delta_1, \delta_2) > 0$ such that

$$\|e^{\tau_0 d_E/h} u_h\|_{H_h^1(A(\delta_1, \delta_2))} \geq C(\varepsilon, \delta_1, \delta_2) e^{-\beta(\varepsilon)/h},$$

with

$$\tau_0 = \left(\frac{\max_{U_E(r_0)} \partial_{y_n} V}{\min_{U_E(r_0)} \partial_{y_n} V} \right)^{1/2}$$

and where $\beta(\varepsilon) \rightarrow 0$ as $\varepsilon \rightarrow 0^+$.

Remark 1.2.2. We note in the more general case where the eigenfunction sequence satisfies the ε_0 -control assumption, the estimate in Theorem 1.2.1 is still valid (similarly for Theorems 1.2.3 and 1.2.5). In such a case, the constant $\beta(\varepsilon_0)$ can be readily estimated explicitly in terms of the potential from (5.1.17) and (5.1.18) above.

In section 5.2 we use the Carleman bounds in (1.1.7) with shrinking annuli together with a Green's formula argument to get lower bounds for L^p eigenfunction restrictions to hypersurfaces smoothly isotopic in $U_E(r_0) \cap \{V > E\}$ to level sets $H = \{y_n = \text{const.}\}$ (see Definition 5.2.1) to prove the following

Theorem 1.2.3. *Suppose H is an admissible hypersurface in sense of Definition 5.2.1. Then, under the control and convexity conditions and with $E(H)$ in (5.2.4), it follows that for any $\varepsilon > 0$ and with $h \in (0, h_0(\varepsilon)]$,*

$$\|u_h\|_{L^p(H)} \geq C(\varepsilon, p) e^{-[2\tau_0 d_E^H - d_E(H) + \beta(\varepsilon)]/h}, \quad p \geq 1, \quad (1.2.1)$$

where $\beta(\varepsilon) \rightarrow 0^+$ as $\varepsilon \rightarrow 0^+$ and $d_E(H) = \min_{q \in H} d_E(q)$, $d_E^H = \max_{q \in \Lambda_{E(H)}} d_E(q)$.

Remark 1.2.4. We note that since $\tau_0 \geq 1$ and $d_E^H \geq d_E(H)$, it is clear that the constant $2\tau_0(H)d_E^H - d_E(H) > 0$.

The bounds in (1.2.1) are *goodness* estimates in the terminology of Toth and Zelditch [TZ09]; the key novelty here being the rather explicit geometric rate $2\tau_0 d_E^H - d_E(H)$ appearing in (1.2.1).

Finally, in section 5.3, we give an application of (1.2.1) to nodal intersection bounds in forbidden regions. In [CT16], Canzani and Toth prove that for any separating hypersurface H in the forbidden region, with $Z_{u_h} = \{x \in M; u_h(x) = 0\}$,

$$\#\{Z_{u_h} \cap H\} \leq C_H h^{-1}.$$

While this rate in h is easily seen to be sharp in general, there is no explicit estimate for the constant $C_H > 0$ given in [CT16]. Using (1.2.1) with $p = 2$, the bound in (1.2.1) allows to give a rather explicit estimate for C_H in the cases where H is smoothly isotopic to a level set of the defining function y_n in the forbidden region. This is essentially the content of the following result,

Theorem 1.2.5. *Assume that $\dim M = 2$ and (M, g, H, V) are all real-analytic. Then, under the same assumptions as in Theorem 1.2.3, one has*

$$\#\{Z_{u_h} \cap H\} \leq C_H h^{-1},$$

where

$$C_H = 2(\tau_0 d_E^H - d_E(H)) > 0.$$

Finally, we note that while all results are stated here for compact manifolds, the results in Theorems 1.2.1-1.2.5 extend to the case of Schrödinger operators on $\mathbb{R}^n$ and the proofs are the same.

The organization of this thesis is as follows. In Chapter 2, we give a brief introduction of pseudodifferential operator, specialized to the Weyl quantization, in the framework of semiclassical analysis. In Chapter 3, we review Agmon estimates. In Chapter 4, we review the basic theory Carleman estimates along with notation of pseudoellipticity. In Chapter 5, we give the proof of Theorem 1.2.1 and discuss improved nodal intersection bounds (Theorem 1.2.5) using an L^p -restriction lower bound for eigenfunctions (Theorem 1.2.3). In Chapter 6, we discuss some explicit examples.

1.3 Author's Original Contribution

The main original contribution of the author is Theorem 1.2.1. In this result, the author proves lower bounds for Schrödinger eigenfunction in classically forbidden regions that are consistent with the well-known Agmon upper bounds. Theorems 1.2.3 and 1.2.5 are applications of this result to eigenfunction restriction bounds

and nodal intersection bounds respectively. Both of these results are also original contributions by the author.

CHAPTER 2

Pseudodifferential operators

In this chapter, we first give a very brief introduction to pseudodifferential operators ([DS99, Sog17, Zwo12]). we then introduce semiclassical Gårding inequality. The latter result is central to the theory of Carleman estimates which we take up in Chapter 3.

2.1 Semiclassical quantization

Definition 2.1.1. A measurable function $m : \mathbb{R}^{2n} \rightarrow (0, +\infty)$ is called an order function if there exist constants C, N such that

$$m(w) \leq C \langle z - w \rangle^N m(z)$$

for all $w, z \in \mathbb{R}^{2n}$.

$m(z) \equiv 1$ and $m(z) = \langle z \rangle = (1 + |z|^2)^{1/2}$ are standard examples.

Definition 2.1.2. Given an order function m on $\mathbb{R}^{2n}$, we define the corresponding class of symbols:

$$S(m) := \{a \in C^\infty \mid \text{for each multiindex } \alpha$$

$$\text{there exists a constant } C_\alpha \text{ so that } |\partial^\alpha a| \leq C_\alpha m\}.$$

As a special case, the space of Kohn-Nirenberg symbol of $m \in \mathbb{Z}$ is given by

$$S^m := \{a(x, \xi) \in C^\infty(\mathbb{R}^{2n}) \mid |\partial_x^\alpha \partial_\xi^\beta a| \leq C_{\alpha\beta} \langle \xi \rangle^{m-|\beta|} \text{ for all } \alpha, \beta\}.$$

In this thesis, we usually work with the symbol class $S := S(1)$, where

$$S(1) = \{a \in C^\infty(\mathbb{R}^{2n}) \mid |\partial^\alpha a| \leq C_\alpha \text{ for all } \alpha\}.$$

Theorem 2.1.3 ([Zwo12]). *Assume $a_j \in S(m)$ for $j = 0, 1, \dots$. Then there exists a symbol $a \in S(m)$ such that*

$$a \sim \sum_{j=0}^{\infty} h^j a_j \quad \text{in } S(m). \quad (2.1.1)$$

Remark 2.1.4. The notation (2.1.1) means that

$$\forall N \in \mathbb{N}, \quad a - \sum_{j=0}^N h^j a_j \in h^{N+1} S(m).$$

We call a_0 the principal symbol of a .

Definition 2.1.5. For $a \in S(m)$, we define the Weyl quantization on $u \in \mathcal{S}$

$$a^W(x, hD)u(x) := \frac{1}{(2\pi h)^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{\frac{i}{h}\langle x-y, \xi \rangle} a\left(\frac{x+y}{2}, \xi\right) u(y) dy d\xi.$$

Take $a(x, \xi) = \xi^\alpha$ for an example,

$$a^W(x, hD)u(x) = (hD)^\alpha u(x).$$

Proposition 2.1.6 ([Zwo12]). *Assume that*

$$c = (c_1(x), \dots, c_n(x))$$

does not depend on ξ . Then

$$\langle c, hD \rangle^W = \frac{h}{2} \sum_{j=1}^n (D_{x_j} c_j + c_j D_{x_j}).$$

Proof. By definition and integral by parts, for $u \in \mathcal{S}$

$$\begin{aligned}
\langle c, hD \rangle^W u &= \frac{1}{(2\pi h)^n} \sum_{j=1}^n \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{\frac{i}{h} \langle x-y, \xi \rangle} c_j \left(\frac{x+y}{2} \right) \xi_j u(y) dy d\xi \\
&= -\frac{1}{(2\pi h)^n} \sum_{j=1}^n \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \left(h D_{y_j} e^{\frac{i}{h} \langle x-y, \xi \rangle} \right) c_j \left(\frac{x+y}{2} \right) u(y) dy d\xi \\
&= \frac{1}{(2\pi h)^n} \sum_{j=1}^n \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{\frac{i}{h} \langle x-y, \xi \rangle} \left(h D_{y_j} c_j \left(\frac{x+y}{2} \right) \right) u(y) dy d\xi \\
&\quad + \frac{1}{(2\pi h)^n} \sum_{j=1}^n \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{\frac{i}{h} \langle x-y, \xi \rangle} c_j \left(\frac{x+y}{2} \right) (h D_{y_j} u(y)) dy d\xi \\
&= h/2 \sum_{j=1}^n (D_{x_j} c_j)^W u + h \sum_{j=1}^n (c_j)^W D_{x_j} u \\
&= h/2 \sum_{j=1}^n (D_{x_j} c_j) u + h \sum_{j=1}^n c_j D_{x_j} u \\
&= h/2 \sum_{j=1}^n D_{x_j} (c_j u) - h/2 \sum_{j=1}^n c_j D_{x_j} u + h \sum_{j=1}^n c_j D_{x_j} u \\
&= h/2 \sum_{j=1}^n D_{x_j} (c_j u) + h/2 \sum_{j=1}^n c_j D_{x_j} u.
\end{aligned}$$

□

Definition 2.1.7. We say that I_h is an oscillatory integral if it can be written in the form

$$I_h = I_h(a, \varphi) = \int_{\mathbb{R}^n} e^{i\varphi/h} a dx,$$

where $a \in C_c^\infty(\mathbb{R}^n)$, $\varphi \in C^\infty(\mathbb{R}^n)$ are real-valued.

One of most important theorems in semiclassical analysis is the following stationary phase formula,

Theorem 2.1.8 ([Zwo12]). Assume that $a \in C_c^\infty(\mathbb{R}^n)$ and $K := \text{supp}(a)$. Suppose $x_0 \in K$ and

$$\partial\varphi(x_0) = 0, \quad \det \partial^2\varphi(x_0) \neq 0.$$

Assume further that $\partial\varphi(x) \neq 0$ on $K \setminus \{x_0\}$. Then there exist for $k = 0, 1, \dots$ differential operators $A_{2k}(x, D)$ of order less than or equal to $2k$, such that for each N

$$\left| I_h - \left(\sum_{k=1}^{N-1} A_{2k}(x, D) a(x_0) h^{k+\frac{n}{2}} \right) e^{\frac{i\varphi(x_0)}{h}} \right| \leq C_N h^{N+\frac{n}{2}} \sum_{|\alpha| \leq 2N+n+1} \sup_{\mathbb{R}^n} |\partial^\alpha a|.$$

As an important example we have

Theorem 2.1.9 ([Zwo12]). Assume that $a \in C_c^\infty(\mathbb{R}^{2n})$. Then for each positive integers N ,

$$\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{\frac{i}{h}\langle x, y \rangle} a(x, y) dx dy = (2\pi h)^n \left(\sum_{k=0}^{N-1} \frac{h^k}{k!} \left(\frac{\langle D_x, D_y \rangle}{i} \right)^k a(0, 0) + O(h^N) \right).$$

Definition 2.1.10. For $z = (x, \xi)$ and $w = (y, \eta)$, define their symplectic product

$$\sigma(z, w) := \langle \xi, y \rangle - \langle x, \eta \rangle.$$

We have the following product formula for two semiclassical pseudodifferential operators:

Theorem 2.1.11 ([Zwo12]). Suppose $a \in S(m_1)$ and $b \in S(m_2)$. Then

$$a^W(x, hD) b^W(x, hD) = (a \# b)^W(x, hD)$$

for the symbol

$$a\#b(x, \xi) := e^{\frac{1}{2}ih\sigma(D_x, D_\xi, D_y, D_\eta)}(a(x, \xi)b(y, \eta))|_{y=x, \eta=\xi}.$$

Here we have following integral representation formula

$$(a\#b)(x, \xi) = \frac{1}{(\pi h)^{2n}} \int_{\mathbb{R}^{2n}} \int_{\mathbb{R}^{2n}} e^{-\frac{2i}{h}\sigma(w_1, w_2)} a(z + w_1) b(z + w_2) dw_1 dw_2,$$

where $z = (x, \xi)$, and

$$a\#b \in S(m_1 m_2).$$

By a standard stationary phase argument, one can then derive an asymptotic expansion for $a\#b$.

Theorem 2.1.12 ([Zwo12]). Assume $a \in S(m_1)$ and $b \in S(m_2)$.

(i) We have for $N = 0, 1, \dots$,

$$a\#b(x, \xi) = \sum_{k=0}^N \frac{(ih)^k}{k!} \left(\frac{1}{2}\sigma(D_x, D_\xi, D_y, D_\eta) \right)^k (a(x, \xi)b(y, \eta))|_{y=x, \eta=\xi} + O_{S(m_1 m_2)}(h^{N+1})$$

as $h \rightarrow 0$.

(ii) In particular,

$$a\#b = ab + \frac{h}{2i}\{a, b\} + O_{S(m_1 m_2)}(h^2)$$

and

$$[a^W(x, hD), b^W(x, hD)] = \frac{h}{i}\{a, b\}^W(x, hD) + O_{S(m_1 m_2)}(h^3).$$

Theorem 2.1.13 ([Zwo12]). *Suppose $a \in S(1)$. Then*

$$\|a^W(x, hD)\|_{L^2 \rightarrow L^2} \leq C \sup_{\mathbb{R}^{2n}} |a| + O(h^{1/2})$$

as $h \rightarrow 0$.

2.2 Semiclassical Gårding inequality

Definition 2.2.1. Let $U \subset \mathbb{R}^n$ be an open set, $h > 0$, and $k \in \mathbb{N}$. The semiclassical norm of a function $f \in H_h^k(U)$ is

$$\|f\|_{H_h^k(U)} := \left(\sum_{|\alpha| \leq k} \int_U |(hD)^\alpha|^2 dx \right)^{1/2}.$$

Theorem 2.2.2 (Gårding inequality [LRL12]). *Let K be a compact set of $\mathbb{R}^n$. Assume $a(x, \xi, h) \in S^m$, with principal symbol a_m . If there exists $C > 0$ such that*

$$\operatorname{Re} a_m \geq C \langle \xi \rangle^m, \quad x \in K, \xi \in \mathbb{R}^n, h \in (0, h_0),$$

then for $0 < C' < C$ and $h_1 > 0$ sufficiently small we have

$$\operatorname{Re} (a^W(x, hD)u, u) \geq C' \|u\|_{H_h^{m/2}}^2, \quad u \in C_c^\infty(K), 0 < h \leq h_1. \quad (2.2.1)$$

Remark 2.2.3. This theorem tells us that the positivity of the principle symbol of a implies a certain positivity for the operator a^W .

Proof. The symbol $a(x, \xi, h)$ is of the form

$$a(x, \xi, h) = a_m(x, \xi, h) + h a_{m-1}(x, \xi, h), \quad \text{with } a_{m-1} \in S^{m-1}.$$

For sufficiently small $h < h_1$, the full symbol $a(x, \xi, h)$ satisfies

$$\operatorname{Re} a(x, \xi, h) \geq C'' \langle \xi \rangle^m, \quad x \in K, \xi \in \mathbb{R}^n, h \in (0, h_1),$$

with $C' < C'' < C$. Let U be a neighborhood of K such that

$$\operatorname{Re} a(x, \xi, h) \geq C''' \langle \xi \rangle^m, \quad (x, \xi) \in U \times \mathbb{R}^n, h \in (0, h_1),$$

with $C' < C''' < C'' < C$. Let $\chi(x) \in C_c^\infty(U)$ such that $0 \leq \chi \leq 1$ and $\chi = 1$ in a neighborhood of K . One sets

$$\tilde{a}(x, \xi, h) = \chi(x)a(x, \xi, h) + C'''(1 - \chi) \langle \xi \rangle^m,$$

which satisfies

$$\tilde{a} \in S^m \quad \text{and} \quad \operatorname{Re} \tilde{a}(x, \xi, h) \geq C''' \langle \xi \rangle^m, \quad (x, \xi) \in \mathbb{R}^n \times \mathbb{R}^n, h \in (0, h_1). \quad (2.2.2)$$

Moreover note that $\langle \tilde{a}^W u, u \rangle = \langle a^W u, u \rangle$ if $\operatorname{supp}(u) \subset K$. Without loss of generality we may thus consider the symbol a satisfies (2.2.2) in the remaining of the proof.

Choose a constant $\beta > 0$ such that $C' < \beta < C''$ and set

$$p(x, \xi, h) := (\operatorname{Re} a(x, \xi, h) - \beta \langle \xi \rangle^m)^{1/2}, \quad \text{and} \quad A = a^W, P = p^W.$$

Theorem 2.1.12 gives

$$P^* \circ P = \operatorname{Re} A - \beta \Lambda^m + hR,$$

with $\Lambda = (1 + (hD)^2)^{1/2}$ and $R \in \Psi^{m-1}$, where $\operatorname{Re} A = (A + A^*)/2$. We then have

$$\begin{aligned}
\operatorname{Re} \langle Au, u \rangle &= \langle \operatorname{Re} A u, u \rangle \geq \beta \langle \Lambda^m u, u \rangle - h \langle Ru, u \rangle \\
&\geq \beta \|\Lambda^{m/2} u\|_{L^2}^2 - h \beta' \|u\|_{H_h^{(m-1)/2}}^2 \\
&\geq (\beta - h \beta') \|u\|_{H_h^{m/2}}^2
\end{aligned}$$

One can conclude the proof by taking h sufficiently small.

□

CHAPTER 3

Agmon estimates

For a bounded open set $U \supset \Omega_E$ with C^2 boundary, let us consider the Dirichlet problem for the Schrödinger operator $P(h)$,

$$\begin{cases} P(h)u_h(x) = E(h)u_h(x), \\ \|u_h(x)\|_{L^2(U)} = 1, \\ u_h(x)|_{\partial U} = 0, \\ E(h) \rightarrow E \quad \text{as } h \rightarrow 0. \end{cases} \quad (3.0.1)$$

The following integration by parts argument is central to the Agmon estimates:

Proposition 3.0.4 ([Hel88]). *Let φ be a real valued Lipschitz function on $\bar{U}$ ($\nabla\varphi$ is well defined in $L^\infty(U)$ by Rademacher's theorem). Then, for any $u(x) \in C^\infty(\bar{U})$ and $u(x)|_{\partial U} = 0$, we have*

$$\begin{aligned} \int_U e^{2\varphi/h} (P(h) - E(h))u \cdot u \, dx &= h^2 \int_U |\nabla(e^{\varphi/h}u)|^2 \, dx \\ &\quad + \int_U (V(x) - E(h) - |\nabla\varphi(x)|^2) e^{2\varphi/h} u^2 \, dx. \end{aligned} \quad (3.0.2)$$

Proof. This identity is just an application of Green's formula:

$$\begin{aligned}
\int_U e^{2\varphi/h} (P(h) - E(h)) u \cdot u &= -h^2 \int_U e^{2\varphi/h} \Delta u \cdot u + \int_U e^{2\varphi/h} (V(x) - E(h)) u^2 \\
&= h^2 \int_U \nabla u \cdot \nabla (e^{2\varphi/h} u) + \int_U e^{2\varphi/h} (V(x) - E(h)) u^2 \\
&= \int_U (h^2 |\nabla u|^2 + 2hu \nabla u \cdot \nabla \varphi + |\nabla \varphi(x)|^2) e^{2\varphi/h} u^2 dx \\
&\quad + \int_U (V(x) - E(h) - |\nabla \varphi(x)|^2) e^{2\varphi/h} u^2 dx \\
&= h^2 \int_U |\nabla (e^{\varphi/h} u)|^2 dx \\
&\quad + \int_U (V(x) - E(h) - |\nabla \varphi(x)|^2) e^{2\varphi/h} u^2 dx.
\end{aligned}$$

□

Let $d_E(x, y)$ denote the distance from point x to y in the Agmon metric g_E .

Proposition 3.0.5. *The Agmon distance function $d_E(x, y)$ is locally Lipschitz and*

$$|\nabla_y d_E(x, y)|_g^2 \leq (V(y) - E)_+, \text{ a.e.}$$

Proof. By triangle inequality,

$$|d_E(x, y) - d_E(x, z)| \leq d_E(y, z).$$

Let $\gamma(t)$, $t \in [0, 1]$ be the geodesic curve from y to z on (M, g) . Since γ is a geodesic,

$|\gamma'(t)|_g$ is a constant. So $d(y, z) = \int_0^1 |\gamma'(t)|_g dt = |\gamma'(t)|_g$ and the length of γ in g_E is

$$\ell(\gamma) = \int_0^1 (V(\gamma(t)) - E)_+^{1/2} |\gamma'(t)|_g dt = |\gamma'(t)|_g \int_0^1 (V(\gamma(t)) - E)_+^{1/2} dt = C_{y,z} d(y, z),$$

where $C_{y,z} = \int_0^1 (V(\gamma(t)) - E)_+^{1/2} dt$. Hence for a fixed point x

$$|d_E(x, y) - d_E(x, z)| \leq d_E(y, z) \leq C_{y,z} d(y, z) \quad (3.0.3)$$

and so, the Lipschitz condition is satisfied.

A direct computation gives

$$\begin{aligned} \limsup_{|h| \rightarrow 0} |h|^{-1} |d_E(x, y+h) - d_E(x, y)| &\leq \limsup_{|h| \rightarrow 0} |h|^{-1} |d_E(y+h, y)| \\ &\leq \lim_{|h| \rightarrow 0} \int_0^1 (V(\gamma(t)) - E)_+^{1/2} dt = (V(y) - E)_+^{1/2}. \end{aligned}$$

The second inequality follows from (3.0.3). Thus, at points y where $d_E(x, y)$ is differentiable, the above result indicates that $|\nabla_y d_E(x, y)|_g^2 \leq (V(y) - E)_+$.

□

Theorem 3.0.6 ([Hel88]). *Suppose $u_h(x)$ satisfies (3.0.1). For each $\delta > 0$, there exists C_δ such that for $h \in (0, h_0]$*

$$\|e^{(1-\delta)d_E(x)/h} u_h\|_{H_h^1(U)} \leq C_\delta.$$

Proof. For $d_E(x)$,

$$|\partial_x d_E(x)|^2 \leq (V - E)_+(x) \quad \text{for almost every } x.$$

Take $\varphi(x) = (1 - \delta)d_E(x)$, so

$$|\partial \varphi(x)|^2 \leq (1 - \delta)^2 (V - E)_+(x). \quad (3.0.4)$$

Define $\Omega_\varepsilon^+ = \{x \in U, V(x) \geq E + \varepsilon\}$ and $\Omega_\varepsilon^- = \{x \in U, V(x) < E + \varepsilon\}$. Applying identity (3.0.2) for $u = u_h(x)$, one has

$$h^2 \int_U |\nabla(e^{\varphi/h} u)|^2 dx + \int_U (V(x) - E(h) - |\nabla\varphi(x)|^2) e^{2\varphi/h} u^2 dx = 0.$$

Then we deduce that

$$\begin{aligned} h^2 \int_U |\nabla(e^{\varphi/h} u)|^2 dx + \int_{\Omega_\varepsilon^+} (V(x) - E - |\nabla\varphi(x)|^2) e^{2\varphi/h} u^2 dx \\ \leq \sup_{x \in \Omega_\varepsilon^-} |V(x) - E(h) - |\nabla\varphi(x)|^2| \int_{\Omega_\varepsilon^-} e^{2\varphi/h} u^2 dx. \end{aligned}$$

From the inequality (3.0.4) for small $h \in (0, h_0]$ we get

$$h^2 \int_U |\nabla(e^{\varphi/h} u)|^2 dx + \delta^2 \int_{\Omega_\varepsilon^+} e^{2\varphi/h} u^2 dx \leq C \int_{\Omega_\varepsilon^-} e^{2\varphi/h} u^2 dx,$$

and then

$$\begin{aligned} h^2 \int_U |\nabla(e^{\varphi/h} u)|^2 dx + \delta^2 \int_U e^{2\varphi/h} u^2 dx \\ \leq (1 + C) \int_{\Omega_\varepsilon^-} e^{2\varphi/h} u^2 dx \\ \leq (1 + C) e^{2 \sup_{x \in \Omega_\varepsilon^-} \varphi/h} \int_U u^2 dx \\ \leq (1 + C) e^{\beta(\delta)/h} \quad \text{with } \beta(\delta) \rightarrow 0 \quad \text{for } \delta \rightarrow 0. \end{aligned}$$

□

CHAPTER 4

Carleman estimates

Given the discussion in Chapter 2, we know the probability of finding a particle in the forbidden region $\{V > E\}$ is exponentially small. However, in this chapter we will show that it can never be superexponentially small

4.1 Carleman estimate in a bounded region

4.1.1 Interior Carleman estimate

We have following useful elliptic estimate,

Proposition 4.1.1 ([Zwo12] Theorem 7.1). *Write*

$$Q(h) := -h^2\Delta + \langle a, hDf \rangle + b,$$

where the coefficients a, b are smooth. Assume also that $U \subset\subset W \subset \mathbb{R}^n$ are open sets. there exists a constant C such that

$$\|f\|_{H_h^2(U)} \leq C(\|Q(h)f\|_{L^2(W)} + \|f\|_{L^2(W)})$$

for all $f \in C^\infty(W)$.

Definition 4.1.2. Given $\varphi \in C^\infty(\mathbb{R}^n)$, we define the conjugation of $P(h) - E(h)$ by $e^{\varphi/h}$:

$$P_\varphi(h) := e^{\varphi/h}(P(h) - E(h))e^{-\varphi/h}.$$

Lemma 4.1.3. *We have*

$$P_\varphi(h) = p_\varphi^W(x, hD)$$

for the symbol

$$p_\varphi(x, \xi) := \langle \xi + i\partial\varphi(x), \xi + i\partial\varphi(x) \rangle + V(x) - E = |\xi|^2 + V - E - |\partial\varphi|^2 + 2i \langle \xi, \partial\varphi \rangle.$$

Proof. For any $u \in C^\infty(\mathbb{R}^n)$ we have

$$\begin{aligned} P_\varphi(h)u &= e^{\varphi/h}(-h^2\Delta + V - E)(e^{-\varphi/h}u) \\ &= -h^2\Delta u + 2h \langle \partial\varphi, \partial u \rangle - |\partial\varphi|^2 u + (V - E)u + h\Delta\varphi u. \end{aligned}$$

By Proposition 2.1.6,

$$p_\varphi^W(x, hD) = -h^2\Delta u + (V - E)u - |\partial\varphi|^2 u + i(\langle \partial\varphi, hDu \rangle + \langle hD, \partial\varphi u \rangle).$$

So this lemma is valid. □

Definition 4.1.4. Hörmander's hypoellipticity (or pseudoellipticity) condition is the requirement for the symbol p_φ that

$$\text{if } p_\varphi = 0, \quad \text{then } i\{p_\varphi, \bar{p}_\varphi\} > 0. \tag{4.1.1}$$

Here $\{, \}$ denotes the Poisson bracket. Such φ is called Carleman weight.

Remark 4.1.5.

$$\text{If } p_\varphi = 0, \quad \text{then } i\{p_\varphi, \bar{p}_\varphi\} = 0.$$

The corresponding φ is called the limiting Carleman weight. Such functions play crucial roles in the applications of complex geometrical optics solutions to inverse problems [KSU07] [DSFKSU09], and on the other hand it satisfies Hörmander's local solvability condition

$$i\{p_\varphi, \bar{p}_\varphi\} \leq 0, \quad \text{when } p_\varphi = 0.$$

In analogy, one has following definition

Definition 4.1.6. [Tat] We say that the C^2 function φ is strongly pseudoconvex at x_0 with respect to P at x_0 if

$$\operatorname{Re}\{\bar{p}, \{p, \varphi\}\}(x_0, \xi) > 0 \quad \text{whenever} \quad p(x_0, \xi) = 0, \xi \neq 0, \quad (4.1.2)$$

$$\overline{\{p(x, \xi + i\tau \nabla \varphi), p(x, \xi + i\tau \nabla \varphi)\}}/i\tau > 0 \quad \text{whenever} \quad (4.1.3)$$

$$\{p(x, \xi + i\tau \nabla \varphi) = 0, \tau > 0, (\xi, \tau) \neq 0\}.$$

Remark 4.1.7. Choosing $\tau = \frac{1}{h}$, (4.1.3) is the Hörmander's hypoellipticity condition.

Remark 4.1.8. (4.1.2) is the limiting case of (4.1.3) as $\tau \rightarrow 0$. Indeed, by Taylor expansion in τ , the first two terms of $p(x, \xi + i\tau \nabla \varphi)$ at $\tau = 0$ is

$$\begin{aligned} p_l(x, \xi, \tau) &= p(x, \xi) + i\tau \frac{\partial p}{\partial \xi} \cdot \frac{\partial \varphi}{\partial x} \\ &= p(x, \xi) + i\tau \left(\frac{\partial p}{\partial \xi} \cdot \frac{\partial \varphi}{\partial x} - \frac{\partial p}{\partial x} \cdot \frac{\partial \varphi(x)}{\partial \xi} \right) \\ &= p(x, \xi) + i\tau \{p, \varphi\}(x, \xi). \end{aligned}$$

Hence, if $p(x_0, \xi) = 0$, then

$$\lim_{\tau \rightarrow 0} \overline{\{p(x, \xi + i\tau \nabla \varphi), p(x, \xi + i\tau \nabla \varphi)\}}/i\tau = 2\operatorname{Re}\{\bar{p}, \{p, \varphi\}\}(x_0, \xi).$$

Remark 4.1.9. Indeed we have a simple geometrical interpretation of the pseudoconvexity condition. If the symbol p is real, one can rewrite the pseudoconvexity condition (4.1.2) as

$$H_p^2 \varphi > 0 \quad \text{whenever} \quad p(x_0, \xi) = 0.$$

Here $H_p^2 \varphi$ represents the second derivative of φ along the Hamiltonian flow of p . Then the above relation says that the φ is ‘curved’ on the energy surface $p(x_0, \xi) = 0$.

Lemma 4.1.10. *If $U \subset\subset \mathbb{R}^n$ and Hörmander’s hypoellipticity condition (4.1.1) is valid within $\bar{U}$. Then*

$$M|p_\varphi|^2 + i\{p_\varphi, \bar{p}_\varphi\}(x, \xi) \geq C \langle \xi \rangle^4.$$

Proof. The proof is basic calculus. Write $\mathbb{R}^n = V \cup \mathbb{R}^n \setminus V$, here $V \subset\subset \mathbb{R}^n$.

First consider $(x, \xi) \in U \times V$, the inequality is obvious since $\langle \xi \rangle^4$ is bounded in V and we can take the constant C small enough with the help of condition 4.1.1.

Secondly, for $(x, \xi) \in U \times \mathbb{R}^n \setminus V$.

- (i) Notice $W := \{\xi, p_\varphi(x, \xi) = 0\}$ is a bounded set since $U \subset\subset \mathbb{R}^n$. Similarly, we can take C small enough to get the desired inequality for $(x, \xi) \in U \times W \cap \mathbb{R}^n \setminus V$.
- (ii) For $(x, \xi) \in U \times \mathbb{R}^n \setminus (V \cap W)$, we can take M large enough from comparing the order of ξ since the order is 4 for $|p_\varphi|^2$ and is 2 for $i\{p_\varphi, \bar{p}_\varphi\}(x, \xi)$.

□

Theorem 4.1.11. *Let $U \subset\subset \mathbb{R}^n$, and assume Hörmander’s hypoellipticity condition (4.1.1) is valid within $\bar{U}$.*

Then there exists a constant C such that

$$h^{1/2}\|u\|_{L^2(U)} \leq C\|P_\varphi(h)u\|_{L^2(U)}$$

for all $u \in C_0^\infty(U)$, provided $0 < h \leq h_0$ with h_0 sufficiently small.

Proof. It follows from Theorem 7.5 in [Zwo12], let's sketch the proof,

$$\begin{aligned} \|P_\varphi(h)u\|_{L^2}^2 &= \langle P_\varphi^*(h)P_\varphi(h)u, u \rangle \\ &= \langle P_\varphi(h)P_\varphi^*(h)u, u \rangle + \langle [P_\varphi^*(h), P_\varphi(h)]u, u \rangle \\ &= \|P_\varphi^*(h)u\|^2 + \langle [P_\varphi^*(h), P_\varphi(h)]u, u \rangle. \end{aligned} \tag{4.1.4}$$

For any $M > 1$ and h small enough the calculation above gives

$$\begin{aligned} \|P_\varphi(h)u\|_{L^2}^2 &\geq Mh\|P_\varphi^*(h)u\|^2 + \langle [P_\varphi^*(h), P_\varphi(h)]u, u \rangle \\ &= h \langle (M|p_\varphi|^2 + i\{p_\varphi, \bar{p}_\varphi\})^W u, u \rangle - O(h^2)\|u\|_{H_h^2}^2. \end{aligned}$$

Lemma 4.1.10 implies for M large enough that

$$M|p_\varphi|^2 + i\{p_\varphi, \bar{p}_\varphi\}(x, \xi) \geq C \langle \xi \rangle^4.$$

Then we can apply Gårding inequality (2.2.1) to show that

$$\|P_\varphi(h)u\|_{L^2}^2 \geq Ch\|u\|_{L^2}^2 - O(h^2)\|u\|_{H_h^2}^2.$$

Hence we can complete the proof with the help of Proposition 4.1.1.

□

Remark 4.1.12. For convenience, let us see how the pseudoconvexity of φ plays the crucial role when dimension $n = 1$.

When $n = 1$

$$P_\varphi(h) = -h^2 \frac{d^2}{dx^2} + h \frac{d}{dx}(\varphi' \cdot) + h\varphi' \frac{d}{dx} + V(x) - E - |\varphi'|^2.$$

Let $A = -\frac{d^2}{dx^2}$, $B_1 = \frac{d}{dx}(\varphi' \cdot)$, $B_2 = \varphi' \frac{d}{dx}$ and $g(x) = V(x) - E - |\varphi'|^2$,

$$\begin{aligned} \|P_\varphi(h)u\|_{L^2}^2 &= h^4 \|Au\|^2 + h^2 \|B_1 u\|^2 + h^2 \|B_2 u\|^2 + \|gu\|^2 + \\ &\quad 2h \langle B_1 u, gu \rangle + 2h \langle B_2 u, gu \rangle + 2h^2 \langle Au, gu \rangle + 2h^3 \langle Au, (B_1 + B_2)u \rangle. \end{aligned} \tag{4.1.5}$$

By an integration by parts,

$$\langle B_1 u, gu \rangle + \langle B_2 u, gu \rangle = \langle \varphi' u, (2\varphi' \varphi'' - V')u \rangle.$$

For suitable φ (For an example, one can take φ strictly increasing and convex such that $2\varphi' \varphi'' - V' > 0$), one has

$$\langle B_1 u, gu \rangle + \langle B_2 u, gu \rangle > C \|u\|^2 > 0,$$

for some positive constant C .

In (4.1.5), the last two terms are $o(h)$ and hence for sufficiently small $h > 0$

$$h^{1/2} \|u\|_{L^2(U)} \leq C \|P_\varphi(h)u\|_{L^2(U)}.$$

Remark 4.1.13. For the limiting Carleman weights, one has similar L^2 estimates for the conjugated operator $P_\varphi(h)$ ([KSU07] [DSFKSU09]). One key step is to make

a little perturbation to convexify the limiting Carleman weight to get a Carleman weight satisfying condition 4.1.1.

From the above argument, we can see a key step in proving the Carleman estimate is to construct a weight function φ which satisfies condition (4.1.1). Convexifying a function which has no critical point is the basic idea. It goes back to Hörmander's classic book ([H63]).

Let us start with a Morse function ψ - that is, a smooth real-valued function having no degenerate critical points. We note that Morse functions are dense in $C^\infty(U)$, here U can be any bounded open set in $\mathbb{R}^n$.

Now set $x_1, \dots, x_N$ to be the (necessarily finitely many) critical points of ψ , and $\omega_0 \subset\subset U$.

Lemma 4.1.14 ([Van09] Lemma 2.3). *There exists a diffeomorphism $\varkappa : \overline{U} \rightarrow \overline{U}$ such that $\varkappa(x) = x$ near ∂U and such that $\varkappa(x_j) \in \omega_0$ for $\forall j$.*

Following this lemma, we can assume ψ is such a Morse function which has finitely many critical points, all of which are contained in ω_0 .

Now let

$$\varphi := e^{\gamma\psi},$$

where $\gamma > 0$ is to be determined. We will show such φ is a Carleman weight.

Proposition 4.1.15 ([Van09] Proposition 2.4). *For γ large enough, $\varphi := e^{\gamma\psi}$ is a Carleman weight on $\overline{U} \setminus \omega_0$.*

Proof. We have

$$p_\varphi(x, \xi) = |\xi|^2 + V - E - |\partial\varphi|^2 + 2i \langle \xi, \partial\varphi \rangle.$$

$p_\varphi = 0$ means

$$|\xi|^2 + V - E - |\partial\varphi|^2 = 0 \quad (4.1.6)$$

and

$$\langle \xi, \partial\varphi \rangle = 0. \quad (4.1.7)$$

From $\varphi := e^{\gamma\psi}$ we can get $\partial\varphi = \gamma\partial\psi e^{\gamma\psi}$ and $\partial^2\psi = (\gamma^2\partial\psi \otimes \partial\psi + \gamma\partial^2\psi)e^{\gamma\psi}$.

Hence

$$|\xi| \leq C\gamma e^{\gamma\psi}, \quad (4.1.8)$$

by (4.1.6).

Now we can compute

$$\begin{aligned} \frac{i}{2}\{p_\varphi, \bar{p}_\varphi\}(x, \xi) &= \{\operatorname{Re} p_\varphi, \operatorname{Im} p_\varphi\} \\ &= 4\langle \partial^2\varphi \xi, \xi \rangle + 4\langle \partial^2\varphi \varphi, \varphi \rangle - 2\langle \partial V, \partial\varphi \rangle \\ &= 4(\gamma^2\langle \partial\psi, \xi \rangle^2 + \gamma\langle \partial^2\psi \xi, \xi \rangle)e^{\gamma\psi} + 4(\gamma^4|\partial\psi|^4 \\ &\quad + \gamma^3\langle \partial^2\psi \psi, \psi \rangle)e^{3\gamma\psi} - 2\gamma\langle \partial V, \partial\psi \rangle e^{\gamma\psi} \\ &= 4(\gamma^4|\partial\psi|^4 + O(\gamma^3))e^{3\gamma\psi} - 2\gamma\langle \partial V, \partial\psi \rangle e^{\gamma\psi} \end{aligned}$$

where the last identity follows from (4.1.7), (4.1.8) and $\bar{U}$ is compact. Notice ψ has no critical point in $U \setminus \omega_0$ which means $|\partial\psi| > 0$ there. By comparing the order of γ and using the fact U is compact, we can conclude that $\frac{i}{2}\{p_\varphi, \bar{p}_\varphi\}(x, \xi) > 0$ from the above identity.

□

Remark 4.1.16. $e^{\gamma(2R-|x|)}$ is an example of Carleman weight in $B(0, R) \setminus B(0, r)$ for γ sufficiently large.

Remark 4.1.17. Correspondingly, $\log|x|$ is one example of a limiting Carleman weight in $\mathbb{R}^n \setminus \{0\}$ for the operator $-h^2\Delta$ ([KSU07]). Indeed in [DSFKSU09], the authors determine all the limiting Carleman weights for $-h^2\Delta$ in the Euclidean metric.

Theorem 4.1.18 ([Zwo12]). *Suppose $U \subset\subset \mathbb{R}^n$, and for large constant R_0 , $V \in C^\infty(\mathbb{R}^n, \mathbb{R})$ satisfying*

$$\begin{cases} |\partial^\alpha V(x)| \leq C_\alpha \langle x \rangle^k & \text{for each multiindex } \alpha, |x| \geq R_0, \\ V(x) \geq c \langle x \rangle^k & \text{for } |x| \geq R_0. \end{cases} \quad (4.1.9)$$

If $u(h)$ solves

$$P(h)u(h) = E(h)u(h) \quad \text{in } \mathbb{R}^n,$$

then there exist constants $C, h_0 > 0$ such that

$$\|u(h)\|_{L^2(U)} \geq e^{-\frac{C}{h}} \|u(h)\|_{L^2(\mathbb{R}^n)},$$

for $0 < h < h_0$.

CHAPTER 5

Proof of main results

5.1 Carleman estimates in a Fermi neighbourhood of the caustic

5.1.1 Collar neighbourhood of caustic and Fermi coordinates

Let (M, g) be a compact, C^∞ Riemannian manifold and $V \in C^\infty(M; \mathbb{R})$ be a real-valued potential. For $r_0 < \text{inj}(M, g)/2$ there exists a collar neighbourhood, $U_E(r_0)$, of $\Lambda_E = \{x \in M; V(x) = E\}$ along with Fermi coordinates $(y_n, y') : U_E \rightarrow (-2r_0, 2r_0) \times \mathbb{R}^{n-1}$ for the ambient metric g , so that in terms of these coordinates

$$g = dy_n^2 + h(y', y_n)|dy'|^2, \quad y \in U$$

Here, $y_n \in C^\infty(M; \mathbb{R})$, is an appropriately normalized defining function for Λ_E ; with

$$\Lambda_E = \{y_n = 0\}, \quad dy_n|_{\Lambda_E} \neq 0.$$

where $h(y', y_n) > 0$ and $h(y', 0)|dy'|^2 := \sum_{i,j \neq n} h_{ij}(y', 0) dy_i \wedge dy_j$ is the metric on the hypersurface $\Lambda_E = \{V = E\}$ induced by g . In these coordinates, we choose the sign convention so that

$$\{V > E\} \cap U_E(r_0) = \{y; 0 < y_n < 2r_0\} \text{ and}$$

$$\{V < E\} \cap U_E(r_0) = \{y; -2r_0 < y_n < 0\}.$$

It will also be useful in the following to introduce the following annular domains in the forbidden region defined by

$$A_E(\delta, \delta') := \{x \in M; \delta < y_n < \delta'\}, \quad 0 < \delta < \delta'. \quad (5.1.1)$$

In terms of the Fermi coordinates (y', y_n) , the corresponding Agmon metric has the form

$$g_E = (V(y) - E) (dy_n^2 + h(y', y_n) |dy'|^2), \quad y \in U, y_n > 0, \quad (5.1.2)$$

It follows by first-order Taylor expansion that

$$V(y) - E = y_n F(y', y_n), \quad (5.1.3)$$

where

$$F(y', y_n) = \int_0^1 (\partial_{y_n} V)(y', ty_n) dt.$$

As result, the Agmon metric can also be written in the form

$$g_E = y_n F(y) (dy_n^2 + h(y', y_n) |dy'|^2), \quad y \in U, y_n > 0, \quad (5.1.4)$$

with $F(y)$ in (5.1.3). Since for $y \in \Omega_E^c$, the functions $V(y) - E > 0$ and $y_n > 0$, it is clear from (5.1.3) that $F(y) > 0$.

We recall (see assumptions (i) and (ii) in the introduction) that by assumption, the collar neighbourhood $U_E(r_0)$ contains no critical points of V and that V is strictly convex in the same neighbourhood. We claim that under these assumptions, not only is $F(y) > 0$, but in fact,

$$\partial_{y_n} V(y) > 0 \quad \text{for all } y \in (U_E(r_0) \cap \Omega_E^c). \quad (5.1.5)$$

To verify (5.1.5), we simply differentiate (5.1.3) in y_n to get

$$\partial_{y_n} V(y) = F(y) + y_n \int_0^1 (\partial_{y_n}^2 V)(y', ty_n) t \, dt,$$

and (5.1.5) follows since $F > 0$, $y_n > 0$ and $\partial^2 V(y) \geq 0$ for all $y \in U_E(r_0) \cap \Omega_E^c$ under the convexity assumption on the potential.

It then follows from (5.1.5) and (5.1.3) that

$$\min_{y \in U_E(r_0)} \partial_{y_n} V(y) \leq F(y) \leq \max_{y \in U_E(r_0)} \partial_{y_n} V(y). \quad (5.1.6)$$

Locally minimal geodesics and Agmon distance

In the collar neighbourhood $U_E(r_0)$, given a point $(y', y_n) \in U_E(r_0) \cap \Omega_E^c$, there is a unique minimal geodesic $\gamma : [0, 1] \times \Lambda_E \rightarrow U_E(r_0)$ for the ambient metric g . Setting $\gamma_t(y) = \gamma(t, y)$ where $\gamma_0 = (y', 0) \in \Lambda_E$ and $\gamma_1 = (y', y)$, the minimal geodesic is

$$\gamma_t(y', 0) = (y', ty_n); \quad 0 \leq t \leq 1.$$

It is easy to see that these “normal” geodesic segments to Λ_E are unfortunately not, in general, minimal geodesics for the conformally rescaled Agmon metric g_E ;

indeed the latter can be quite complicated (see examples in 6.3). Nevertheless, we will need the following elementary estimate for Agmon distance in terms of the natural Fermi defining function $y_n : M \rightarrow \mathbb{R}$ above.

LEMMA 5.1.1. Under the convexity assumption in Definition 1.1.2, it follows that

$$d_E(y) \geq \frac{2}{3} \left(\min_{y \in U_E(r_0)} \partial_{y_n} V(y) \right)^{1/2} y_n^{3/2}; \quad y \in U_E(r_0).$$

Proof. Let $\gamma : [0, 1] \rightarrow \Omega_E^c$ be a piecewise- C^1 minimal geodesic for the Agmon metric g_E joining $y = (y', y_n) \in U_E(r_0) \cap \Omega_E^c$ to Λ_E ; explicitly, $\gamma(0) = (y', y_n)$ and $\gamma(1) = (f(y', y_n), 0) \in \Lambda_E$. Then, writing $\gamma = (\gamma', \gamma_n)$, with $\gamma' = (\gamma_1, \dots, \gamma_{n-1})$,

$$d_E(y) = \int_0^1 |d_t \gamma(t)|_{g_E} dt,$$

and since

$$|d_t \gamma(t)|_{g_E} = \left(F(\gamma(t)) \gamma_n(t) |d_t \gamma_n(t)|^2 + F(\gamma(t)) \gamma_n(t) \langle h(y(t)) d_t \gamma'(t), d_t \gamma'(t) \rangle \right)^{1/2},$$

with $F, \gamma_n > 0$, and $h(\cdot)$ is a positive-definite matrix, it follows that

$$d_E(y) \geq \min F^{1/2} \cdot \int_0^1 \gamma_n(t)^{1/2} |d_t \gamma_n(t)| dt.$$

Finally, by making the change of variables $t \mapsto s = \gamma_n(t)$ in the last integral, one gets

$$d_E(y) \geq \min F^{1/2} \cdot \int_0^{y_n} s^{1/2} ds,$$

and the lemma follows from this last estimate combined with (5.1.6) since $\min F^{1/2} \geq \min (\partial_{y_n} V)^{1/2}$. $\square$

5.1.2 Local control and Carleman bounds near the caustic Λ_E

Model computation

Consider the model Airy operator $P_0(h) := (hD_y)^2 + y$ where $y \in \mathbb{R}$ where $V(y) = y$ and $E = 0$ with the corresponding Airy-type weight function in the forbidden region given by

$$\varphi_0(y) = \frac{2}{3} y^{3/2}, \quad y > 0.$$

Then, the symbol of the conjugated operator $e^{\varphi_0/h} P_0(h) e^{-\varphi_0/h}$ is

$$p_{\varphi_0}(y, \xi) = \xi^2 - |\varphi'_0(y)|^2 + y + 2iy^{1/2}\xi, \quad y > 0$$

and

$$\text{Char}(p_{\varphi_0}) = \{(y, \xi) \in \mathbb{R}^2; \xi = 0, y > 0\}.$$

The latter follows since $(y, \xi) \in \text{Char}(\varphi_{\varphi_0})$ iff $0 = |\xi|^2 - |\varphi'_0(y)|^2 + y + 2iy^{1/2}\xi$ which in turn holds iff $\xi = 0$ since $|\varphi'_0(y)|^2 - y = 0$.

We note that the weight function φ_0 is *borderline* for the Hörmander subelliptic condition in the sense that for $(y, 0) \in \text{Char}(p_{\varphi_0})$, we have

$$\{\text{Re } p_{\varphi_0}, \text{Im } p_{\varphi_0}\} = 4\varphi''_0(y)|\varphi'_0(y)|^2 - 2\varphi'_0(y) \equiv 0, \quad y > 0.$$

Of course, in this case, $\varphi_0(y) = \frac{2}{3}y^{3/2} = \int_0^y \tau^{1/2} d\tau$ is precisely the Agmon distance function $d_E(y)$, where by convention we have set $E = 0$.

Construction of the weight function

Let $P(h) = -h^2\Delta_g + V - E : C^\infty(M) \rightarrow C^\infty(M)$ and consider the conjugated operator $P_\varphi(h) = e^{\varphi/h}P(h)e^{-\varphi/h} : C^\infty(M) \rightarrow C^\infty(M)$ with principal symbol $p_\varphi(x, \xi) = |\xi|_g^2 - |\nabla_x \varphi|_g^2 + V(x) - E + 2i \langle \xi, \nabla_x \varphi \rangle_g$. The model case above suggests that to create subellipticity for $P_\varphi(h)$ in a Fermi neighbourhood of the caustic, it should suffice to slightly modify the model weight function φ_0 in the normal Fermi coordinate y_n . With this in mind, for $\varepsilon > 0$ arbitrarily small (for concreteness, assume $10\varepsilon < r_0$) and constant $\tau > 0$ to be determined later on, we now set in Fermi coordinates $(y', y_n) : U_E \rightarrow \mathbb{R}^{n-1} \times (-2r_0, 2r_0)$,

$$\varphi_\varepsilon(y_n) := \left(\frac{2}{3} + \varepsilon\right) \tau (y_n + 10\varepsilon)^{3/2}, \quad y_n \in (-2\varepsilon, 2r_0). \quad (5.1.7)$$

Remark 5.1.1. We recall here that $r_0 < \text{inj}(M, g)$ is fixed (but not necessarily small), whereas $\varepsilon > 0$ will be chosen arbitrary small (but independent of h) consistent with the control assumption on the eigenfunctions.

We abuse notation somewhat in the following and simply write $\varphi = \varphi_\varepsilon$, the dependence on ε being understood. Then, $\varphi \in C^\infty([-2\varepsilon, 2r_0])$ and plainly $\varphi : [-2\varepsilon, 2r_0] \rightarrow \mathbb{R}^+$ is strictly-convex and monotone increasing with

$$\min(\varphi'(y_n), \varphi''(y_n)) \geq C(\varepsilon) > 0, \quad y_n \in (-2\varepsilon, 2r_0).$$

Moreover, the characteristic variety

$$\begin{aligned} \text{Char}(p_\varphi) \cap \pi^{-1}([-2\varepsilon, 2r_0]) \\ = \{(y, \xi); |\xi|_y^2 - |\partial_{y_n} \varphi|^2 + F(y)y_n = 0, \xi_n = 0, y_n \in (-2\varepsilon, 2r_0)\}. \end{aligned}$$

Since $F(y) > 0$, it follows that this set is non-trivial; indeed for any $-2\varepsilon < y_n < 0$ (ie. a point in the allowable region),

$$\text{Char}(p_\varphi) \cap \pi^{-1}(y_n) \cong S_{y_n}^*(M) \cap \{\xi_n = 0\}.$$

Since $\text{Char}(p_\varphi)$ is non-trivial, global ellipticity over the interval $(-2\varepsilon, 2r_0)$ evidently fails. However, we claim that *subellipticity* is now satisfied in such an interval provided $\tau > 0$ is chosen large enough but depending only on the potential V . Indeed, since the normal Fermi coordinate is y_n and φ is a function of *only* y_n with $g_{n,n} = 1$, a direct computation gives,

$$\begin{aligned} \{\text{Re } p_\varphi, \text{Im } p_\varphi\} &= \{\xi_n^2 + |\xi'_y|^2 - (\partial_{y_n} \varphi)^2 + V - E, 2\partial_{y_n} \varphi \cdot \xi_n\} \\ &= \partial_{\xi_n}(\xi_n^2) \partial_{y_n}(2\partial_{y_n} \varphi \cdot \xi_n) - \partial_{y_n}(-(\partial_{y_n} \varphi)^2 + V - E) \partial_{\xi_n}(2\partial_{y_n} \varphi \cdot \xi_n) \\ &= 4\partial_{y_n}^2 \varphi \left(|\partial_{y_n} \varphi|^2 + \xi_n^2 \right) - 2\partial_{y_n} \varphi \cdot \partial_{y_n} V \\ &\geq 2\partial_{y_n} \varphi (2\partial_{y_n}^2 \varphi \cdot \partial_{y_n} \varphi - \partial_{y_n} V) \\ &\geq 2\tau C(\varepsilon)(2\partial_{y_n}^2 \varphi \cdot \partial_{y_n} \varphi - \partial_{y_n} V), \quad y_n \in (-2\varepsilon, 2r_0). \end{aligned} \tag{5.1.8}$$

From (5.1.7), for any $\varepsilon > 0$ and for all $y_n \in (-2\varepsilon, 2r_0)$,

$$2\partial_{y_n}^2 \varphi \cdot \partial_{y_n} \varphi \equiv \frac{9}{4} \tau^2 \left(\frac{2}{3} + \varepsilon \right)^2 > \tau^2.$$

Choosing

$$\tau = \|\partial_{y_n} V\|_{L^\infty(U_E(r_0))}^{1/2}, \quad (5.1.9)$$

it follows from (5.1.8) that for all (y, ξ) with $y_n \in (-2\varepsilon, 2r_0)$,

$$\{\operatorname{Re} p_\varphi, \operatorname{Im} p_\varphi\}(y, \xi) \geq C(\tau, \varepsilon) > 0.$$

Consequently, $\varphi = \varphi_\varepsilon$ is a Carleman weight for $P(h)$ *globally* in the Fermi neighbourhood of the caustic where $-2\varepsilon < y_n < 2r_0$.

Remark 5.1.2. If $V = V(y_n)$ in $U_E(r_0)$, consider

$$\varphi_\varepsilon(y_n) = (1 + \varepsilon) \int_0^{y_n + 10\varepsilon} \sqrt{V(t) - E} dt, \quad y_n \in (-2\varepsilon, 2r_0).$$

Similarly

$$\begin{aligned} \varphi'(y_n) &= (1 + \varepsilon) \sqrt{V(y_n + 10\varepsilon) - E} \geq C(\varepsilon) > 0, \\ \varphi''(y_n) &= \frac{1 + \varepsilon}{2} \frac{\partial_{y_n} V(y_n + 10\varepsilon)}{\sqrt{V(y_n + 10\varepsilon) - E}} \geq C(\varepsilon) > 0, \quad y_n \in (-2\varepsilon, 2r_0). \end{aligned}$$

Moreover

$$\begin{aligned}
\{\operatorname{Re} p_\varphi, \operatorname{Im} p_\varphi\} &= \{\xi_n^2 + |\xi'_y|^2 - (\partial_{y_n} \varphi)^2 + V - E, 2\partial_{y_n} \varphi \cdot \xi_n\} \\
&= 4\partial_{y_n}^2 \varphi \left(|\partial_{y_n} \varphi|^2 + \xi_n^2 \right) - 2\partial_{y_n} \varphi \cdot \partial_{y_n} V \\
&\geq 2\partial_{y_n} \varphi \left(2\partial_{y_n}^2 \varphi \cdot \partial_{y_n} \varphi - \partial_{y_n} V \right) \\
&\geq 2\partial_{y_n} \varphi \left((1 + \varepsilon)^2 - 1 \right) \partial_{y_n} V \\
&> 0, \quad y_n \in (-2\varepsilon, 2r_0).
\end{aligned}$$

This means $\varphi = \varphi_\varepsilon$ is also a Carleman weight for $P(h)$ *globally* in the Fermi neighbourhood of the caustic where $-2\varepsilon < y_n < 2r_0$.

Now, let $\chi \in C_0^\infty(\mathbb{R}; [0, 1])$ be a cutoff satisfying

$$\chi(y_n) = 1; \quad -\frac{3}{2}\varepsilon < y_n < \frac{3}{2}r_0$$

with

$$\chi(y_n) = 0; \quad y_n \in \mathbb{R} \setminus (-2\varepsilon, 2r_0).$$

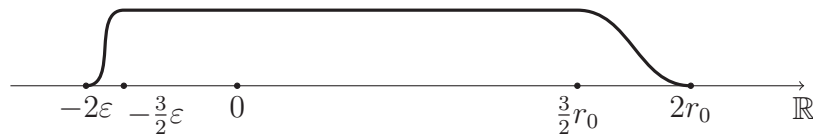


Figure 5–1: Cutoff function χ

In the following, we let $\chi_\pm \in C_0^\infty(\mathbb{R})$ with $0 \leq \chi_\pm \leq 1$. Moreover, writing $f^\pm := f|_{\pm y_n \geq 0}$, we choose $\chi_\pm(y_n)$ so that $\chi_\pm(y_n) = 1$ for $y_n \in \operatorname{supp}(\partial_{y_n} \chi)^\pm$ and $\chi_\pm(y_n) = 0$ for $y_n \in \operatorname{supp}(\partial_{y_n} \chi)^\mp$. More concretely, in terms of Fermi coordinates,

we choose $\chi_{\pm}$ so that

$$\chi_{-}(y_n) = 1; \quad -3\varepsilon < y_n < \frac{\varepsilon}{2},$$

$$\chi_{-}(y_n) = 0; \quad y_n > \varepsilon,$$

and

$$\chi_{+}(y_n) = 1; \quad \frac{3}{2}r_0 - \varepsilon < y_n < 2r_0 + \varepsilon,$$

$$\chi_{+}(y_n) = 0; \quad y_n < \frac{3}{2}r_0 - 2\varepsilon.$$

Set $P_{\varphi}(h) := e^{\varphi/h}P(h)e^{-\varphi/h} : C_0^{\infty}(U) \rightarrow C_0^{\infty}(U)$ and with $\chi = \chi(y_n)$ above,

$$v_h := e^{\varphi/h}\chi u_h$$

where $P(h) := -h^2\Delta_g + V(x) - E(h)$ and

$$P(h)u_h = 0.$$

Moreover, we assume throughout that the eigenfunctions u_h are L^2 -normalized with $\|u_h\|_{L^2(M,g)} = 1$.

In view of the subellipticity estimate in (5.1.8) and the support properties of the cutoff $\chi \in C_0^{\infty}$ it follows by the standard Carleman estimate [Zwo12, Theorem 7.7] that

$$\|P_{\varphi}(h)v_h\|_{L^2}^2 \geq C_1(\varepsilon)h \|v_h\|_{H_h^1}^2. \quad (5.1.10)$$

Since $P(h)u_h = 0$ and $P_\varphi(h)$ is local with $\text{supp } \chi_+ \cap \text{supp } \chi_- = \emptyset$, it follows from (5.1.10) that

$$\begin{aligned} & \|e^{\varphi/h}[P(h), \chi] \chi_+ u_h\|_{L^2}^2 + \|e^{\varphi/h}[P(h), \chi] \chi_- u_h\|_{L^2}^2 \\ & \geq C_1(\varepsilon)h \left(\|e^{\varphi/h} \chi u_h\|_{H_h^1(\text{supp } \chi_+)}^2 + \|e^{\varphi/h} \chi u_h\|_{H_h^1(\text{supp } \chi_-)}^2 \right) \end{aligned} \quad (5.1.11)$$

or equivalently,

$$\begin{aligned} & \|e^{\varphi/h}[P(h), \chi] \chi_+ u_h\|_{L^2}^2 - C_1(\varepsilon)h \|e^{\varphi/h} \chi u_h\|_{H_h^1(\text{supp } \chi_+)}^2 \\ & \geq C_1(\varepsilon)h \|e^{\varphi/h} \chi u_h\|_{H_h^1(\text{supp } \chi_-)}^2 - \|e^{\varphi/h}[P(h), \chi] \chi_- u_h\|_{L^2}^2. \end{aligned} \quad (5.1.12)$$

Then, it follows from (5.1.12) that

$$h^2 \|e^{\varphi/h} u_h\|_{H_h^1(\text{supp } \widetilde{\partial \chi^+})}^2 \geq C_1(\varepsilon)h \|e^{\varphi/h} u_h\|_{H_h^1(\text{supp } \chi_-)}^2 - h^2 C_2(\varepsilon) \|e^{\varphi/h} u_h\|_{H_h^1(\text{supp } \widetilde{\partial \chi^-})}^2, \quad (5.1.13)$$

where, in (5.1.13), the sets $\text{supp } \widetilde{\partial \chi^\pm}$ arbitrarily small neighbourhoods of $\text{supp } (\partial \chi)^\pm$ respectively. Specifically, we can assume that $\text{supp } \chi_\pm \supset \text{supp } \widetilde{\partial \chi^\pm} \supset \text{supp } (\partial \chi)^\pm$ and in addition

$$\text{meas}(\text{supp } \widetilde{\partial \chi^\pm} \setminus \text{supp } (\partial \chi)^\pm) \leq \frac{\varepsilon}{10}.$$

Since $(\partial \chi)^-$ is supported in the classically allowable region where $y_n < 0$, we will now use the control assumption in Definition 1.1.1 to get an effective lower bound for the RHS in (5.1.13).

Computing in Fermi coordinates, the RHS of (5.1.13) is

$$\begin{aligned} &\geq C_1(\varepsilon)h \int_{\{U; y_n \in (-\frac{\varepsilon}{2}, 0)\}} e^{2\varphi(y_n)/h} (|u_h(y)|^2 + |h\partial_y u_h(y)|^2) dy' dy_n \\ &\quad - C_2(\varepsilon)h^2 \int_{\{U; y_n \in (-3\varepsilon, -\varepsilon)\}} e^{2\varphi(y_n)/h} (|u_h(y)|^2 + |h\partial_y u_h(y)|^2) dy' dy_n, \end{aligned} \quad (5.1.14)$$

where the last line in (5.1.14) follows since $\text{supp } \widetilde{\partial\chi^-} \subset \{y \in U; -3\varepsilon < y_n < -\varepsilon\}$.

Next we use strict monotonicity of the weight function $\varphi \in C^\infty([-2\varepsilon, 2r_0])$ in (5.1.7). We set $m(\varepsilon) := \min_{y_n \in (-\frac{\varepsilon}{2}, 0)} \varphi(y_n) > 0$ and $M(\varepsilon) := \max_{y_n \in (-3\varepsilon, -\varepsilon)} \varphi(y_n) > 0$. Then, since φ is strictly increasing,

$$m(\varepsilon) - M(\varepsilon) = C_3(\varepsilon) > 0.$$

So, it follows that (5.1.14) is bounded below by

$$\begin{aligned} &C_1(\varepsilon)e^{2m(\varepsilon)/h} \left(h\|u_h\|_{H_h^1(\{U; y_n \in (-\frac{\varepsilon}{2}, 0)\})}^2 \right. \\ &\quad \left. - C_2(\varepsilon)h^2 e^{2[M(\varepsilon)-m(\varepsilon)]/h} \|u_h\|_{H_h^1(\{U; y_n \in (-3\varepsilon, -\varepsilon)\})}^2 \right). \end{aligned} \quad (5.1.15)$$

Finally, by standard elliptic estimates, $\|u_h\|_{H_h^1} = O(1)$ and by the control assumption in Definition 1.1.1, it follows that for any $\varepsilon > 0$,

$$\|u_h\|_{H_h^1(\{U; y_n \in (-\frac{\varepsilon}{2}, 0)\})}^2 \geq C_{2,N}(\varepsilon)h^N.$$

Consequently, from (5.1.13)-(5.1.15) it follows that with $h \in (0, h_0(\varepsilon)]$, there exist constants $C_j(\varepsilon) > 0, j = 1, \dots, 5$, such that

$$\begin{aligned}
h^2 \|e^{\varphi/h} u_h\|_{H_h^1(\widetilde{\text{supp } \partial \chi^+})}^2 &\geq C_1(\varepsilon) e^{2m(\varepsilon)/h} \left(h^{N+1} C_{2,N}(\varepsilon) + O_\varepsilon(e^{-2C_3(\varepsilon)/h}) \right) \\
&\geq C_{4,N}(\varepsilon) h^{N+1} e^{2m(\varepsilon)/h} \geq C_{5,N}(\varepsilon) e^{m(\varepsilon)/h}.
\end{aligned} \tag{5.1.16}$$

Next, we relate the weight function φ_ε to Agmon distance d_E . From Lemma 5.1.1 we recall that

$$\begin{aligned}
d_E(y) &\geq \frac{2}{3} \left(\min_{U_E(r_0)} \partial_{y_n} V \right)^{1/2} y_n^{3/2} \\
&= \left(\frac{\min_{U_E(r_0)} \partial_{y_n} V}{\max_{U_E(r_0)} \partial_{y_n} V} \right)^{1/2} \varphi_\varepsilon(y_n) + O(\varepsilon).
\end{aligned} \tag{5.1.17}$$

The latter estimate in (5.1.17) follows since in the definition of the weight φ_ε (see (5.1.7)), we choose $\tau = \max_{y \in U_E(r_0)} |\partial_{y_n} V|^{1/2}$. Since from (5.1.5), $\min_{y \in U_E(r_0)} \partial_{y_n} V > 0$, it then follows that

$$\varphi_\varepsilon(y_n) \leq \left(\frac{\max_{y \in U_E(r_0)} \partial_{y_n} V}{\min_{y \in U_E(r_0)} \partial_{y_n} V} \right)^{1/2} d_E(y) + O(\varepsilon) \tag{5.1.18}$$

Thus, in view of (5.1.16) and (5.1.18), we have proved the Theorem 1.2.1.

Remark 5.1.3. Under the same assumptions as in Remark 5.1.2, it's easy to see the piecewise- C^1 minimal geodesic under g_E joining (y', y_n) to Λ_E is just the same minimal geodesic under g (see the section 6.3). Hence

$$d_E(y) = \int_0^{y_n} \sqrt{V(t) - E} dt = \varphi_\varepsilon(y_n) + O(\varepsilon).$$

5.2 L^p restriction lower bounds in forbidden regions

Consider a C^∞ separating hypersurface $H \subset \Omega_E^c$ in the forbidden region that bounds a domain $\Omega_H \subset \Omega_E^c$. The point of this section is to extend Theorem 1.2.1 to lower bounds for L^p -restrictions of eigenfunctions to hypersurfaces H in the forbidden region.

Let ν be the unit exterior normal to H with $\langle \nabla V, \nu \rangle < 0$. Then, under the separation assumption above, by Green's formula,

$$\int_{\Omega_H} |h \nabla u_h|_g^2 dv_g + \int_{\Omega_H} (V - E) |u_h|^2 dv_g = h^2 \int_H \partial_\nu u_h \cdot u_h d\sigma \quad (5.2.1)$$

Using the fact that $V(x) - E \geq C > 0$ for all $x \in \Omega_H$, it follows from (5.2.1) that with a constant $C_\delta = C(V, E, E', \delta) > 0$

$$h^2 \int_H \partial_\nu u_h \cdot u_h d\sigma \geq C_\delta \|u_h\|_{H_h^1(\Omega_H)}^2. \quad (5.2.2)$$

From the pointwise Agmon estimates

$$\|h \partial_\nu u_h\|_{L^\infty(H)} = O_\varepsilon(e^{-d_E(H) + \beta(\varepsilon)/h}), \quad d_E(H) := \min_{q \in H} d_E(q)$$

together with the Hölder inequality,

$$\|u_h\|_{L^p(H)} \geq C_{\delta, \varepsilon}(p) e^{[d_E(H) - \beta(\varepsilon)]/h} \|u_h\|_{H_h^1(\Omega_H)}^2, \quad p \geq 1. \quad (5.2.3)$$

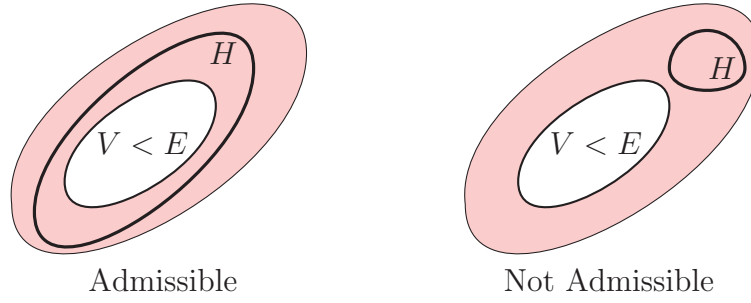
Here, $\beta(\varepsilon) = o(1)$ as $\varepsilon \rightarrow 0^+$.

Definition 5.2.1. We say that the hypersurface $H \subset \{V > E\}$ is *admissible* provided:

- (i) H is a separating hypersurface bounding a domain $\Omega_H \subset \{V > E\}$.
- (ii) There exists $E' > E$ such that the hypersurface $\Lambda_{E'} = \{y_n = E' - E\}$ has

the property that

$$\Lambda_{E'} \subset \Omega_H \cap U_E(r_0).$$



Red region is $\{V > E\} \cap U_E(r_0)$

Figure 5-2: Admissible and not Admissible hypersurfaces

Set

$$E(H) := \inf\{E' > E; \Lambda_{E'} \subset (\Omega_H \cap U_E(r_0))\}. \quad (5.2.4)$$

Since $\Lambda_{E'} \cap \Omega_H = \emptyset$ for any $E' > E$ sufficiently close to E , it follows that $E(H) > E$. Moreover, under the admissibility assumption, it follows that for any $\delta > 0$ sufficiently small,

$$A(E(H), E(H) + \delta) \subset (\Omega_H \cap U_E(r_0))$$

and so,

$$\|u_h\|_{H_h^1(\Omega_H)}^2 \geq \|u_h\|_{H_h^1(A(E(H), E(H) + \delta))}^2. \quad (5.2.5)$$

From the Carleman estimate in Theorem 1.2.1,

$$\|e^{\tau_0 d_E/h} u_h\|_{H_h^1(A(E(H), E(H) + \delta))}^2 \geq C(\delta, \varepsilon) e^{-\beta(\varepsilon, \delta)/h}, \quad (5.2.6)$$

where $\beta(\varepsilon, \delta) \rightarrow 0^+$ as $\varepsilon, \delta \rightarrow 0^+$.

It then follows from (5.2.3)-(5.2.6) that for any $\varepsilon' > 0$, and with

$$\tau_0 = \left(\frac{\max_{U_E(r_0)} \partial_{y_n} V}{\min_{U_E(r_0)} \partial_{y_n} V} \right)^{1/2}, \quad d_E^H := \max_{q \in \Lambda_{E(H)}} d_E(q), \quad d_E(H) = \min_{q \in H} d_E(q). \quad (5.2.7)$$

one has the following lower bound for L^p -restrictions of the u_h to H :

$$\|u_h\|_{L^p(H)} \geq C(\varepsilon', p) e^{-2\tau_0 d_E^H/h} \cdot e^{d_E(H)/h} \cdot e^{-\beta(\varepsilon')/h}, \quad p \geq 1$$

where $\beta(\varepsilon') \rightarrow 0^+$ as $\varepsilon' \rightarrow 0^+$. Consequently, we have proved the Theorem 1.2.3

5.3 Nodal intersection bounds in forbidden regions

The main job of this section is to give an improved Theorem 1.2.5 following the steps in [CT16]. Before that, we shall give a very brief review of known results about nodal sets of the eigenfunctions of Laplace and the Schödinger operators.

Theorem 5.3.1 ([Che76]). *Suppose that (M, g) is an n -dim C^∞ Riemannian manifold without boundary (not necessarily compact). If $f \in C^\infty(M)$ satisfies*

$$(\Delta + s(x))f = 0,$$

for $s \in C^\infty(M)$. Then except on a closed set of lower dimension (i.e. $\dim < n - 1$) the nodal set of f forms an $(n - 1)$ -dim C^∞ manifold.

The study of zero sets was historically motivated by the desire to visualize energy states by finding the points where the quantum particle is least likely to be. In fact the nodal sets of the hydrogen atom energy states have become visible to microscopes [SRL⁺13].

In the late 70's, S. T. Yau gave a conjecture for the ‘size’ of the zero set Z_{u_λ} , $Z_{u_\lambda} = \{x, u_\lambda(x) = 0\}$, of eigenfunctions

$$-\Delta_g u_\lambda = \lambda u_\lambda$$

of the laplacian Δ_g on (M, g) . The conjecture is stated as follows,

Conjecture 5.3.2. For general $C^\infty(M, g)$ of any dimension n , there exist positive constants c and C depending only on g so that

$$c\sqrt{\lambda} \leq \mathcal{H}^{n-1}(Z_{u_\lambda}) \leq C\sqrt{\lambda}.$$

Here $\mathcal{H}^{n-1}(Z_{u_\lambda})$ represents the $n - 1$ -dimensional Hausdorff measure.

When (M, g) is analytic, this problem has been extensively studied. In 1988, Donnelly-Fefferman [DF88] proved the conjectured bounds for real analytic Riemannian manifolds (possibly with boundary). In 1991, Lin [Lin91] showed the same upper bound by using the frequency of u_λ .

However, for C^∞ metrics, this conjecture is still not completely solved. A lot of experts made great contributions in this conjecture ([Bru78] [CM11] [Don92] [DF90] [HS89] [SZ12]). We refer to [Zel13] for a list of references. Recently Logunov ([Log] [Log18b] [Log18a]) has some breakthrough results for both the upper and lower bounds.

Motivated by Yau's conjecture, Toth-Zelditch [TZ09] [TZ] considered the distribution of Z_{u_λ} on a piecewise analytic domain in $\mathbb{R}^2$. More specifically, they gave an upper bound for the number of intersection points of Z_{u_λ} with a 'good' analytic curve H (cf Fig. 4-2).

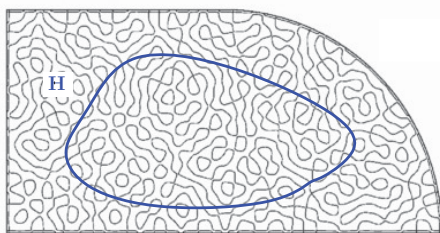


Figure 5-3: Nodal intersetion

One key technique in the proof of Yau's conjecture involves establishing Carleman estimate since the measure of the nodal sets is closely related to the Carleman weight [DF88].

More generally, we can consider the problem of determining the measure of the zero set Z_{u_n} of Schrödinger eigenfunctions. From now on, we only consider the case that $g \in C^\omega$.

Jin [Jin17] extended the known results in the homogeneous case where $V = 0$ to Schrödinger eigenfunctions in Ω_E . Due to quantum tunneling phenomenon, it is natural to study the property of zero set in Ω_E^c , but little is known in this case. In dimension one, we know that the eigenfunctions of the Harmonic Oscillator have no zeros in the forbidden region and Hanin-Zelditch-Zhou [HZZ15] [HZZ17] have proved that in any higher dimension the expected value of the measure of the nodal set of random eigenfunctions of the harmonic oscillator inside any ball is of order $h^{-1/2}$ and the density of zeros is of order $h^{-\frac{2}{3}}$ in an $h^{-\frac{2}{3}}$ -tube around the caustic set $\{x : V(x) = E\}$. Recently Canzani-Toth [CT16] have shown the sharp upper bound for the nodal intersection of Schrödinger eigenfunctions with any simple closed real analytic curve is of order h^{-1} on a compact real analytic surface without boundary.

We now turn to the proof of Theorem 1.2.5, but first review the basic method in [CT16]. One key observation is that the number of nodal intersections with an analytic curve is always upper bounded by the zeros of corresponding complexified eigenfunction in a complex strip. Then the authors apply a crucial inequality [[TZ09], Proposition 10] which says basically that the number of zeros of a holomorphic function is controlled by the logarithm of the holomorphic function.

We now present the details. Consider the special case where $\dim M = 2$ and (M, g, V) are, in addition, real-analytic. Let $H \subset \Omega_E^c$ be a simple, closed, real-analytic curve in the forbidden region. Given the nodal set

$$Z_{u_h} = \{x \in M; u_h(x) = 0\},$$

the problem is to estimate the number of nodal interesections with H ; that is $\#\{H \cap Z_{u_h}\}$ which is just the cardinality of the intersection. Indeed, under an exponential lower bound on the L^2 -restrictions of the eigenfunctions (ie. a *goodness* bound), this intersection consists of a finite set of points.

Let $q : [0, 2\pi] \rightarrow H$ be a C^ω , 2π -periodic, parametrization of H . To bound the number of zeros of $u_h \circ q : [0, 2\pi] \rightarrow \mathbb{R}$ we consider its holomorphic extension $(u_h \circ q)^\mathbb{C} : H_\tau^\mathbb{C} \rightarrow \mathbb{C}$ to the complex strip

$$H_\tau^\mathbb{C} = \{t \in \mathbb{C} : \operatorname{Re} t \in [0, 2\pi], |\operatorname{Im} t| < \tau\}$$

for some $\tau > 0$, and use that $\#\{Z_{u_h} \cap H\} \leq \#\{Z_{(u_h \circ q)^\mathbb{C}} \cap H_\tau^\mathbb{C}\}$. Then, the zeros of $(u_h \circ q)^\mathbb{C}$ are studied using the Poincaré-Lelong formula:

$$\partial \bar{\partial} \log |(u_h \circ q)^\mathbb{C}(z)|^2 = \sum_{z_k \in Z_{(u_h \circ q)^\mathbb{C}}} \delta_{z_k}(z).$$

According to [TZ09, Proposition 10], there exists $C > 0$ so that

$$\#\{Z_{u_h} \cap H\} \leq \#\{Z_{(u_h \circ q)^\mathbb{C}} \cap H_\tau^\mathbb{C}\} \leq C \max_{t \in H_\tau^\mathbb{C}} \log |F_h^\mathbb{C}(t)|, \quad (5.3.1)$$

where $F_h^{\mathbb{C}}(t)$ with $t \in H_\tau^{\mathbb{C}}$ is the holomorphic continuation of the normalized eigenfunction traces

$$F_h(t) := \frac{u_h(q(t))}{\|u_h\|_{L^2(H)}}. \quad (5.3.2)$$

It follows that we shall need to control the complexification $F_h^{\mathbb{C}}(t)$ to obtain upper bounds on $\#\{Z_{\varphi_h} \cap H\}$. Without loss of generality we assume that $H \subset \text{int}(\Omega_\gamma)$ where $\Omega_\gamma \subset \Omega_E^c$ is a domain whose closure is contained in Ω_E^c and whose boundary is a closed C^ω curve that we call γ . Moreover, we choose γ so that for any fixed $\varepsilon > 0$, the distance $d(H, \gamma) < \varepsilon$. Then, in [CT16] (4.9), the authors prove that there exist positive constants C, h_0, d_H and $C_1(\varepsilon)$ so that

$$|F_h^{\mathbb{C}}(t)| \leq C e^{-C_1(\varepsilon)/h} \left(\frac{\|u_h\|_{L^2(\gamma)}}{\|u_h\|_{L^2(H)}} + \frac{\|\partial_\nu u_h\|_{L^2(\gamma)}}{\|u_h\|_{L^2(H)}} \right). \quad (5.3.3)$$

From the Agmon estimates in (1.1.5), one has the upper bounds

$$\max\{\|u_h\|_{L^2(\gamma)}, \|\partial_\nu u_h\|_{L^2(\gamma)}\} \leq C(\varepsilon) e^{-[d_E(H) + \beta_1(\varepsilon)]/h}, \quad d_E(H) = \min_{q \in H} d_E(q),$$

for all $h \in (0, h_0(\varepsilon)]$ with $\beta_1(\varepsilon) = o(1)$ as $\varepsilon \rightarrow 0^+$. On the other hand, from Theorem 1.2.3, we have the lower bound

$$\|u_h\|_{L^2(H)} \geq C(\varepsilon) e^{[-2\tau_0 d_E^H + d_E(H) + \beta_2(\varepsilon)]/h}, \quad d_E^H = \max_{q \in \Lambda_{E(H)}} d_E(q),$$

with $\beta_2(\varepsilon) = o(1)$ as $\varepsilon \rightarrow 0^+$.

Consequently, from (5.3.3) we get that

$$|F_h^{\mathbb{C}}(t)| \leq C(\varepsilon) e^{-C_1(\varepsilon)/h} e^{\beta(\varepsilon)/h} \cdot e^{[2(\tau_0 d_E^H - d_E(H))]/h}, \quad C_1(\varepsilon) > 0, \quad \beta(\varepsilon) = o(1); \quad \varepsilon \rightarrow 0^+. \quad (5.3.4)$$

Then, using the Jensen-type bound in (5.3.1) and letting $\varepsilon > 0$, we complete the proof of Theorem 1.2.5.

CHAPTER 6

Examples and counterexamples

In this chapter, we shall discuss the necessity of the control assumption in Definition 1.1.1 and compare the geodesics under the ambient metric g with those under the Agmon metric g_E .

6.1 Counterexample: Effective potentials and lack of eigenfunction control

Here we show that without the control assumption in Definition 1.1.1, we can establish a Schrödinger model such that the corresponding eigenfunction decays much faster than $e^{-(1-\varepsilon)d_E/h}$ in $A(\delta, \delta')$ for δ' small enough. Such counterexample is essentially inspired by the paper ([CT16]).

Consider a convex surface of revolution generated by rotating a curve $\gamma = \{(r, f(r)), r \in [-1, 1]\}$ about r -axis with $f \in C^\infty([-1, 1], \mathbb{R})$, $f(1) = f(-1) = 0$, and $f''(r) < 0$ for all $r \in [-1, 1]$. Furthermore, one requires $f^{(n)}(-1) = f^{(n)}(1)$ for all n -th derivatives.

Let M be the corresponding convex surface of revolution parametrized by

$$\begin{aligned}\beta &: [-1, 1] \times [0, 2\pi) \rightarrow \mathbb{R}^3, \\ \beta(r, \theta) &= (r, f(r) \cos \theta, f(r) \sin \theta).\end{aligned}$$

Then, M inherits a Riemannian metric g given by

$$g = w^2(r)dr^2 + f^2(r)d\theta^2,$$

where $w(r) = \sqrt{1 + (f'(r))^2}$.

Consider the Schödinger equation on M given by

$$(-h^2\Delta_g + V)\varphi_h = E(h)\varphi_h,$$

where $V \in C^\infty(M)$ and is radial, so that $V(r, \theta) = V(r)$. We also assume that $E(h) = E + o(1)$ and that $f(V^{-1}(E)) > 0$.

We seek eigenfunctions of the form $\varphi_h(r, \theta) = v_h(r)\psi_h(\theta)$. The Laplace operator in the coordinates (r, θ) has the following form

$$\Delta_g = \frac{1}{w(r)f(r)} \frac{\partial}{\partial r} \left(\frac{f(r)}{w(r)} \frac{\partial}{\partial r} \right) + \frac{w^2(r)}{f^2(r)} \frac{\partial^2}{\partial \theta^2}.$$

Making the radial change of variables $s \rightarrow r(s) = \int_0^s \frac{f(\tau)}{w(\tau)} d\tau$, it follows that $v_h(r(s))$ and $\psi_h(\theta)$ must satisfy the ODE

$$-h_k^2 \frac{d^2}{ds^2} \psi_h(\theta) = h_k^2 m_{h_k}^2 \psi_h(\theta) \tag{6.1.1}$$

and

$$\left(-h_k^2 \frac{d^2}{ds^2} + f^2(r(s))(V(r(s)) - E(h)) + w^2(r(s)) \right) v_{h_k}(r(s)) = 0. \tag{6.1.2}$$

for some $m_h \in \mathbb{Z}$. Let $\{h_k\}$ be a decreasing sequence with $h_k \rightarrow 0^+$ as $k \rightarrow +\infty$ and $m_{h_k} = 1/h_k \in \mathbb{Z}$. Then, we choose a particular sequence of solutions of (6.1.1) given

by

$$\psi_{h_k}(\theta) = e^{i\theta/h_k}.$$

Consider the annulus $A(-\varepsilon_0, \varepsilon_0) = \{r(s); -\varepsilon_0 < V(r(s)) - E(h) < \varepsilon_0\}$. Since for $r \in A(-\varepsilon_0, \varepsilon_0)$ we have for $\varepsilon_0 > 0$ sufficiently small

$$f^2(r)(V(r) - E) + w^2(r) > \frac{1}{4}w^2(r),$$

it then follows by the standard Agmon-Lithner estimate applied to (6.1.2) that for any $\delta > 0$, and with $V(r_0) = E$,

$$\|e^{\frac{(1-\delta)}{2} \left(\int_{r_0}^r \frac{w(\tau)}{\partial_s \tau} d\tau \right) / h_k} v_{h_k}(r)\|_{L^2(A(-\varepsilon_0, \varepsilon_0))} = O_\delta(1), \quad (6.1.3)$$

Since $\partial_s r = \frac{f(s)}{w(s)} > 0$ for $r(s) \in A(-\varepsilon_0, \varepsilon_0)$ with $\varepsilon_0 > 0$ sufficiently small, the inequality (6.1.3) contradicts the control condition in Definition 1.1.1; indeed, the eigenfunctions already decay exponentially in h in the allowable region $A(-\varepsilon_0, 0)$.

We note that since $d_E(r) = O(|V(r) - E|^{3/2}) = O(\varepsilon_0^{3/2})$ and the additional effective potential term $w(r) = \sqrt{1 + (f'(r))^2} \geq 1$, it follows that for $\varepsilon_0 > 0$ sufficiently small, in the forbidden region where $r \in A(0, \varepsilon_0)$,

$$\frac{(1-\delta)}{2} \int_{r_0}^r \frac{w(\tau)}{\partial_s \tau} d\tau \geq C_0 (V(r) - E) > \tau_0 d_E(r), \quad C_0 > 0.$$

In this case, the exponential decay is therefore more pronounced than in Theorem 1.2.1. This is due to the presence of the effective potential term w^2 which in turn appears because of the particular choice of the sequence of Fourier modes in (6.1.1) with $m_k h_k \sim 1$. This is consistent with our results, since as we have already shown, the control condition is violated for this particular sequence of eigenfunctions.

6.2 Examples of eigenfunction sequences satisfying control

We consider precisely the same example of a Schrödinger operator on a convex surface of rotation as above but choose the quantum number $m = \text{const.}$ so that $mh_k = O(h_k)$ as $h_k \rightarrow 0$. Then, the ODE in (6.1.2) becomes

$$\left(-h_k^2 \frac{d^2}{ds^2} + f^2(r(s))(V(r(s)) - E(h)) + O(h_k^2) w^2(r(s)) \right) v_{h_k}(r(s)) = 0. \quad (6.2.1)$$

The fact that the corresponding eigenfunctions $\varphi_h(r, \theta) = v_h(r)\psi_h(\theta)$ satisfy the control assumption is then an immediate consequence of standard WKB theory applied to the semiclassical ODE (6.2.1). Indeed, writing $\Phi(r) := \int_{r_0}^r \frac{f(r)}{\partial_s r} (E - V(r))^{1/2} dr$, it follows by WKB asymptotics that for $r \in [-1, 1]$ satisfying $E - \varepsilon < V(r) \leq E$,

$$v_h(r) \sim_{h \rightarrow 0^+} e^{i\Phi(r)/h} c_1(h) a_1(r; h) + e^{-i\Phi(r)/h} c_2(h) a_2(r; h), \quad (6.2.2)$$

where for $k = 1, 2$, $a_k(r; h) \sim \sum_{j=0}^{\infty} a_{k,j}(r) h^j$ and

$$|c_1(h)|^2 + |c_2(h)|^2 \geq C_1 > 0, \quad |a_k(r; h)| \geq C_2(\varepsilon) > 0; \quad k = 1, 2.$$

Consequently, from (6.2.2) we get that for any $\varepsilon > 0$,

$$\begin{aligned} \int_{-\varepsilon < V(r) - E \leq 0} \int_0^{2\pi} |\varphi_h(r, \theta)|^2 dr d\theta &= \int_{-\varepsilon < V(r) - E \leq 0} \int_0^{2\pi} |v_h(r)|^2 |e^{im\theta}|^2 dr d\theta \\ &= \int_{-\varepsilon < V(r) - E \leq 0} \int_0^{2\pi} |v_h(r)|^2 dr \geq C(\varepsilon) > 0. \end{aligned}$$

In the last estimate, to control mixed terms, we have used that by an integration by parts,

$$\int_{-\varepsilon < V(r) - E \leq 0} e^{\pm 2i\Phi(r)/h} a_1(r; h) a_2(r; h) dr = O_\varepsilon(h).$$

As a result, this particular sequence clearly satisfies the control assumption in Definition 1.1.1 with $N = 0$.

6.3 Geodesics in the Agmon metric

In this section, we compare and contrast geodesics in background metric g and Agmon metric g_E in some simple cases.

Radial-type Agmon metrics

Let (M, g) be C^∞ compact Riemannian surface ($n = 2$) and $V \in C^\infty(M; \mathbb{R})$ with regular value E . Let $(y_n, y') : U_E(r_0) \rightarrow \mathbb{R}^n$ be Fermi coordinates in a neighbourhood of the caustic set $V = E$. We assume here that V is *radial* in the sense that

$$V = V(y_n), \quad y \in U_E(r_0) \cap \{V > E\}.$$

Recall that in Fermi coordinates

$$g = dy_n^2 + h(y', y_n) |dy'|^2,$$

and the associated Agmon metric is

$$g_E = (V(y_n) - E) \left(dy_n^2 + h(y', y_n) |dy'|^2 \right), \quad V(y) > E.$$

We make the change of radial variables $y_n \mapsto Y_n$ where

$$Y_n(y_n) = \int_0^{y_n} \sqrt{V(u) - E} du, \tag{6.3.1}$$

which is monotone increasing when $V(y_n) > E$. In the (Y_n, y') -coordinates, the Agmon metric has the form

$$g_E = dY_n^2 + H(Y_n, y') |dy'|^2, \quad H > 0.$$

It then easily follows that the curves normal to the caustic given by

$$\gamma(t) = (Y_n(t) = t, y' = y'(0)); \quad |t| < r_0$$

are minimal geodesics in g_E and, modulo the reparametrization in (6.3.1), they are also minimal geodesics in the background metric g . In general, other geodesics of g do not necessarily agree with those of g_E , even in these radial cases.

Non-radial example

When the potential is not radial in the Fermi defining function, it is easy to construct examples where even the minimal geodesics in the background metric g normal to the caustic $V = E$ are no longer geodesics in the Agmon metric, g_E .

To see this, consider the potential $V(r, \theta) = (r + \sin \theta)^2$ and $E = 0$ in the polar coordinates (r, θ) on $\mathbb{R}^2$, so that the background metric in polar form is

$$g = dr^2 + r^2 d\theta^2, \quad (r, \theta) : \mathbb{R}^2 \rightarrow (0, \infty) \times \mathbb{S}^1.$$

Hence, given a constant $\theta_0 \in [0, 2\pi]$, obviously the line segment $\gamma : [0, r_0] \rightarrow \mathbb{R}^2$, $\gamma(t) = (t, \theta_0)$ is the geodesic in the flat metric g joining $(0, 0)$ to (r_0, θ_0) .

Now $g_E = V(r, \theta)dr^2 + r^2V(r, \theta)d\theta^2$ is the Agmon metric. We claim $\gamma(t)$ is not the geodesic under g_E . The length of γ in g_E is

$$\ell(\gamma) = \int_0^{r_0} \sqrt{V(r, \theta_0)} dr = \int_0^{r_0} (r + \sin \theta_0) dr = \frac{1}{2}r_0^2 + r_0 \sin \theta_0.$$

Denote c_1 be the circular arc joining $(r_0, 0)$ to (r_0, θ_0) and c_2 , the straight segment from $(0, 0)$ to $(r_0, 0)$.

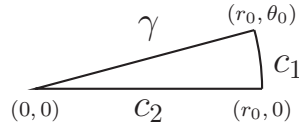


Figure 6-1: Non-radial example

It is easy to compute the lengths of c_1 and c_2 :

$$\ell(c_1) = \int_0^{\theta_0} \sqrt{V(r_0, \theta)} \cdot r_0 d\theta = \int_0^{\theta_0} (r_0 + \sin \theta) \cdot r_0 d\theta = r_0^2 \theta_0 + (1 - \cos \theta_0)r_0,$$

and

$$\ell(c_2) = \int_0^{r_0} \sqrt{V(r, 0)} dr = \int_0^{r_0} r dr = \frac{1}{2}r_0^2.$$

Choosing $r_0 = \frac{1}{2}$ and $\theta_0 = \frac{\pi}{4}$ gives

$$\ell(\gamma) = \frac{1 + 2\sqrt{2}}{8} > \frac{1}{8} + \frac{2 - \sqrt{2}}{4} + \frac{\pi}{4} = \ell(c_1) + \ell(c_2).$$

Consequently, γ does not minimize the distance from $(0, 0)$ to (r_0, θ_0) .

APPENDIX A

Riemannian geometry

In this appendix we include basic definitions and identities which are used throughout this thesis.

The Riemannian metric g on M induces a natural isomorphism between the tangent and cotangent bundles by

$$\begin{aligned} T(M) &\rightarrow T^*(M), \\ (x, X) &\mapsto (x, X^\flat) \end{aligned}$$

where $X^\flat(Y) = \langle X, Y \rangle$, and the corresponding inverse is

$$\begin{aligned} T^*(M) &\rightarrow T(M), \\ (x, \xi) &\mapsto (x, \xi^\sharp) \end{aligned}$$

where $\xi^\sharp$ is defined by $\xi(X) = \langle \xi^\sharp, X \rangle$. Assume the metric g is given by

$$g = g_{jk} dx_j \otimes dx_k$$

in the local coordinates. Then,

$$X^\flat = g_{jk} X_j dx_k, \quad \xi^\sharp = g^{jk} \xi_j \partial_{x_k}.$$

In particular, the gradient field is defined by $\nabla\varphi = d\varphi^\sharp$. The musical isomorphisms allow one to lift the metric to the cotangent bundle. The cotangent bundle is hence naturally endowed with the Riemannian metric g^{-1} given in the local coordinates by

$$g^{-1} = g^{jk} d\xi_j \otimes d\xi_k,$$

here $g_{ij} \cdot g^{jk} = \delta_i^k$.

We denoted D as the uniquely determined Levi-Civita connections on (M, g) which satisfies the following conditions:

- (i) $D_{fX}Y = fD_XY$, and $D_X(fY) = (Xf)Y + fD_XY$ if f is a smooth function on M ,
- (ii) $D_XY - D_YX = [X, Y]$,
- (iii) $X\langle Y, Z \rangle = \langle D_XY, Z \rangle + \langle Y, D_XZ \rangle$.

This connection is determined in local coordinates by

$$D_{\partial_{x_j}}\partial_{x_k} = \Gamma_{jk}^l \partial_{x_l},$$

where the Christoffel symbols Γ_{jk}^l are given by

$$\Gamma_{jk}^l = \frac{1}{2}g^{lm}(\partial_{x_j}g_{km} + \partial_{x_k}g_{jm} - \partial_{x_m}g_{jk}).$$

The Hessian of a smooth function φ is the symmetric $(2, 0)$ -tensor $D^2\varphi = Dd\varphi$. The expression of the Hessian in local coordinates is

$$D^2\varphi = \left(\partial_{x_j x_k}^2 \varphi - \Gamma_{jk}^l \partial_{x_l} \varphi \right) dx_j \otimes dx_k.$$

For the smooth function f on M , we can define the gradient of f , ∇f , to be the vector field on M for which

$$\langle \nabla f, X \rangle = Xf$$

for all $X \in TM$.

APPENDIX B

Symplectic geometry

An even dimensional smooth manifold M endowed with a closed nondegenerate 2-form ω is called a symplectic manifold.

With standard coordinates on $\mathbb{R}^{2n}$ denoted by $(x_1, \dots, x_n, y_1, \dots, y_n)$, has the standard symplectic form

$$\omega = \sum_{i=1}^n dx_i \wedge dy_i.$$

Naturally there is a tautological 1-form, τ , on the cotangent bundle T^*M . It is not hard to prove that $\omega = -d\tau$ is a symplectic form on T^*M . Let (x_i, ξ_i) denote the corresponding natural coordinates on T^*M , one has the canonical symplectic form

$$\omega = \sum_i dx_i \wedge d\xi_i.$$

Theorem B.0.1 (Darboux). *Let (M, ω) be a $2n$ -dimensional symplectic manifold. For any $p \in M$, there are smooth coordinates $(x_1, \dots, x_n, \xi_1, \dots, \xi_n)$ centered at p in which ω has the canonical coordinates representation*

$$\omega = \sum_{i=1}^n dx_i \wedge d\xi_i.$$

Remark B.0.2. Darboux's Theorem states that all symplectic structures are identical locally, in the sense that all are equivalent to that give by ω . It implies that symplectic geometry is at least locally, not as rigid as Riemannian geometry. There

are no local invariants in symplectic geometry, but for example, curvature is a local invariant in Riemannian geometry.

Like the gradient defined for the smooth function f on the Riemannian manifold (M, g) , we can define a Hamiltonian vector field H_f on Symplectic manifold (M, ω) by

$$\omega(H_f, X) = df(X),$$

for all $X \in TM$.

For $f, g \in C^\infty(M)$, their Poisson bracket is defined by

$$\{f, g\} := H_f g.$$

In the canonical coordinates $(x_1, \dots, x_n, \xi_1, \dots, \xi_n)$, that is

$$\{f, g\} = \sum_{i=1}^n \frac{\partial f}{\partial \xi_i} \frac{\partial g}{\partial x_i} - \frac{\partial f}{\partial x_i} \frac{\partial g}{\partial \xi_i}.$$

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