Enumeration of Real Lines on Smooth Cubic Surfaces

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Contents

Abstract

In the nineteenth century, Cayley proved that a smooth cubic surface has exactly 27 lines over the complex field. Schläfli and others showed furthermore that the number of real lines could be 27, 15, 7, or 3. These results were found using the algebraic geometry techniques of the time. In this thesis, we solve this classic real enumeration problem using two different techniques. One method is group theoretic, and the other uses characteristic classes. Part of our goal is to show how these methods are related to each other as well as to the classical results. In particular, we give a description of important signs, which arise in both the classical and characteristic class approaches, in terms of our group theoretic approach.

We also solve two other classic real enumerative problems using the group theoretic approach: the number of bitangent lines to a quartic plane curve, and the number of tritangent planes to a twisted sextic curve.

Résumé

Pendant le dix-neuvième siècle, Cayley a prouvé qu'une surface cubique lisse contient exactement 27 droites sur le corps complexe. De plus, Schläfli et d'autres ont demontré que le nombre de droites réelles pourait être 27, 15, 7, ou 3. Ces résultats ont été découverts à travers des techniques de la géométrie algébrique de l'époque. Dans cette thèse, nous résolvons ce problème classique en utilisant deux techniques différentes. La première méthode utilise la théorie des groupes, et l'autre utilise des classes charactéristique. Notre but est, en partie, de démontrer comment ces méthodes sont reliées entre eux et, de plus, leurs relation aux résultats classiques. En particulier, nous donnons une description de signes importants, provenant de l'approche classique ainsi que l'approche des classes charactéristiques, du point de vu de la théorie des groupes.

De plus, en utilisant l'approche des groupes, nous résolvons deux autres problèmes classiques en geométrie énumérative réelle : l'énumération des droites bitangentes à une courbe quartique du plan, et l'énumération des plans tritangents à une courbe tordu sextique.

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Chapter 1

Introduction

1.1 Motivation and results

The goals of this thesis are the following:

1. Solve real enumerative problems from both a group theoretic approach as well as a characteristic class approach.

2. Make connections between these two approaches.

3. Find a group theoretic interpretation of signs which naturally arise in real enumeration.

The main problem that we deal with is the enumeration of lines on a cubic surface. This is a classic problem which was first studied, and solved, in the nineteenth century by Cayley [3], Schläfli [25], and many others. If we are working over the complex field, the number of lines is 27. When working over the real field, the number of lines is either 27, 15, 7, or 3 [25], [26].

This phenomenon of having different cases for the real enumeration is typical. Consider for example the number of roots of a degree d polynomial. It is well known this number is d if we work over the complex field (assuming

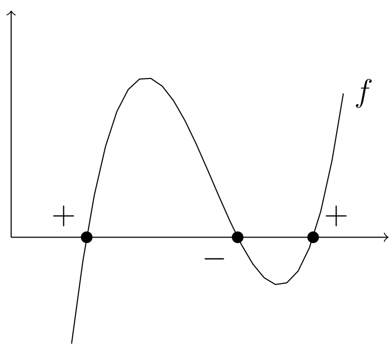


Figure 1.1: Roots with a sign for a cubic polynomial f

the polynomial has no repeated roots). Over the real field, however, the number of roots is $d - 2k$ for some $k \leq d/2$.

There is, at least in some cases, a way to count real objects with a sign to get an invariant number. For the lines on the cubic surface, this real invariant is implicit in the work of Segre [26], as pointed out by [20] and [10]. Segre divided the real lines into two classes, called hyperbolic real lines and elliptic real lines, based on some geometric considerations. The number of hyperbolic real lines minus the number of elliptic real lines is always 3. As another example, for a real polynomial f of degree d, we can count the each real root x_i with the sign of $f'(x_i)$, then we have that $|\sum_i \mathrm{sgn}(f'(x_i))| = d \bmod 2$. See figure 1.1 for an example. More recently, Welschinger has given a method for counting real J-holomorphic curves with a sign to give an invariant number, [28], [19].

We will return to this issue of signs later. But before that, with reference to our first goal, we ask how to find the real enumeration of lines on cubic surfaces. Classically, this was done in a very direct approach by manipulating the equations and by geometric visualization.

Our approach, on the other hand, is to use the fact that there is a group $W(E_6)$ which acts in a natural way on the set of lines. This action can be described as the action of this Weyl group on the vertices of a certain polytope, [5]. The complex conjugation associated to any real surface induces an involution in the Weyl group. The real lines will be the fixed points of this involution. We can therefore compute the number of real lines by counting the number of fixed vertices under all the conjugacy classes of involutions in $W(E_6)$. Furthermore, there is a connection between the groups $W(E_7), W(E_8)$ and two other classic enumerative problems: the number of bitangent lines to a quartic curve; and the number of tritangent planes to a twisted sextic curve. Again, these objects can be represented by certain polytopes, [8]. Using the same principle of finding fixed points, we obtain the same results as Zeuthen [29] for the number of real bitangent lines, and as Comessatti [4] for the number of real tritangent planes, which they obtained using classical algebraic geometry methods. Our first result is therefore the following:

Proposition 1.1. *The real enumeration of lines on the cubic surface; bitangent lines to a quartic curve; and tritangent planes to a twisted sextic curve can be found by counting the fixed vertices of certain polytopes under conjugacy classes of involutions in the groups* $W(E_6)$, $W(E_7)$ *and* $W(E_8)$, *respectively.*

A similar idea was used by Wall to find the number of real lines on Del Pezzo surfaces [27], which are very much related to the real tritangent planes of a sextic curve; the real bitangent lines of a quartic curve; and in fact are exactly the real lines of cubic surfaces (for degree 3 Del Pezzo surfaces), as we will describe in chapter 2. The proof of proposition 1.1 and our method for computing the number of fixed points is the subject of chapter 4.

After we describe how the objects in proposition 1.1 are related to their respective Weyl groups, it will be clear that the complex structure on a cubic surface, quartic curve, or twisted sextic must induce an involution in $W(E_6), W(E_7)$ or $W(E_8)$, respectively. However, that any involution actually comes from a complex structure is not immediate. In the case of cubic surfaces and $W(E_6)$, we can work out from the combinatorics that every involution corresponds to a type of cubic surface as classified by Schläfli (see section 2.1). In the case of the quartic curves and twisted sextic, every involution is also realized, but there turns out to be some redundancy. That is, every involution comes from some complex structure, but there are different topological types of these objects (quartic curves and twisted sextics) which can not be distinguished by their involutions. We cover this in section 4.4.

Next, let us discuss the method of characteristic classes, which are a generalization of the methods Schubert used to solve enumeration problems in projective geometry, [14]. As a motivating example, consider once again the example of counting roots of a polynomial. We consider elements of $\mathbb{C}[x_0, x_1]$ which are homogeneous polynomials. These can be considered as global sections of the bundle $S^d(U^\vee)$ over $\mathbb{CP}$, where U is the tautological bundle, $U^\vee$ is its dual, and S^d refers to the d-th symmetric power. This bundle is commonly referred to as $\mathcal{O}(d)$. The first Chern class of this bundle gives us the class of the zero locus of any section:

$$c_1(S^d(U^\vee)) = c_1(U^\vee \otimes \cdots \otimes U^\vee) = d \cdot c_1(U^\vee). \tag{1.1}$$

After checking that $c_1(U^\vee)([\mathbb{CP}]) = 1$, and assuming our polynomial (and corresponding section) has isolated zeros, we recover the result that there are in fact d zeros.

We also have a real bundle $S^d(U^\vee)$ over $\mathbb{RP} = S^1$. The first Stiefel-Whitney class $w_1(S^d(U^\vee))$ is an element of $H^1(S^1, \mathbb{Z}/2\mathbb{Z})$ which vanishes if there exists nowhere vanishing section, and is nontrivial otherwise. The bundle $U^\vee$ has a twist in it, so we can see that $S^d(U^\vee) = U^\vee \otimes \cdots U^\vee$ has a twist if d is odd and is trivial if d is even. Therefore,

$$w_1(S^d(U^\vee))([S^1]) = d \bmod 2, \tag{1.2}$$

telling us, once again, we must have at least one root when d is odd.

Along the same lines, we can construct a certain vector bundle so that a cubic surface induces a section of it, and the zeros of this section represent the lines on the cubic surface. The top Chern number of this bundle is 27, which gives us the number of complex lines. We can also construct the analogous real bundle and compute its Euler number, giving us 3. The Euler class counts the zeros of a section with a sign, called the Euler index, which is similar to the sign of the derivative in our polynomial example. One would expect that this index corresponds to the hyperbolic and elliptic classification given by Segre, which is in fact the case. We prove these things in chapter 6, which we summarize in the following proposition:

Proposition 1.2. *Let $U^\vee$ denote the dual tautological bundle over either $G_2(\mathbb{C}^4)$ or $G_2(\mathbb{R}^4)$, where G_2 is the Grassmannian of 2-planes. A homogeneous cubic polynomial f, defining a smooth cubic surface, induces a section s_f of $S^3(U^\vee)$. This section has a zero at l, precisely when l is a line of the surface. For the complex bundle we have:*

$$c_4(S^3(U^\vee))[(G_2(\mathbb{C}^4)] = 27. \tag{1.3}$$

For the real bundle we have:

$$e(S^3(U^\vee))[(G_2(\mathbb{R}^4)] = 3, \tag{1.4}$$

and the Euler index at l is positive or negative according to whether l is a real hyperbolic or real elliptic line.

We have now seen that the signs come to play from the classic work of Segre as well as in the modern characteristic class point of view. To link these with our Weyl group approach, we ask how the signs can be described in terms of $W(E_6)$ and the polytope representing the lines. In section 4.3 we show the following:

Proposition 1.3. *Let $r_1, ..., r_4$ be four orthogonal roots in $\Phi(E_6)$. Up to* *conjugation, any involution of* $\tau \in W(E_6)$ *can be written as* $R_1 \cdots R_k$ *for* $k \leq 4$ *where* R_i *is the reflection in the root* r_i*. Let* L *be the 27 vertices of the polytope representing the 27 lines. As previously mentioned, the real lines* L_r *are the fixed points of* τ*. The real hyperbolic lines* L_h *are the elements of* L_r *orthogonal to the root* $\frac{1}{2}(r_1 + r_2 + r_3 + r_4)$*, that is*

$$L_h = L_r \cap < r_1 + r_2 + r_3 + r_4 >^{\perp} . \tag{1.5}$$

This description of the hyperbolic and elliptic lines in terms of roots of E_6 seems to be new.

Overall, we have results connecting ideas from Weyl groups, characteristic classes, and classical geometry: The fixed points of a polytope when acted on by involutions in $W(E_6)$ correspond to zeros of sections in the bundle $S^3(U^\vee) \to G_2(\mathbb{R}^4)$. Furthermore the Euler index of these zeros, whose sign corresponds to a geometric condition given by Segre, depend on the orthogonality to a root in $\Phi(E_6)$.

An extension of this work would be to give a characteristic class calculation for the real enumeration of bitangent lines and tritangent planes and to find some notion of sign. In the complex case, the number bitangent lines to a quartic curve can be calculated using the theory of Gromov-Witten invariants [1], which gives a modern derivation of the classic Plücker formulas [11]. There may be some real analogue of this Gromov-Witten calculation. In the case of the tritangent planes to a sextic curve, we do not know of any characteristic class computation, even in the complex case, to find their number.

1.2 Outline of thesis

In chapter 2, we give the history of the enumeration of lines on cubic surfaces, and explain what are the symmetry groups of the lines on a cubic surface, the bitangent lines to a quartic curve, and the tritangent planes to a twisted sextic.

In chapter 3, we give the reflection group theory needed to study these symmetry groups, and in chapter 4 we find the involutions of these groups and the number of fixed points to determine the real enumeration. We also describe the group theoretic interpretation of Segre's classification of real lines into hyperbolic and elliptic types.

In chapter 5, we give the necessary theory for the Euler and Chern classes. In chapter 6, we construct vector bundles whose zeros correspond to lines on the cubic surface, and we compute the number of zeros in the real and complex cases. Finally, we describe hyperbolic and elliptic real lines geometrically and show this corresponds to the Euler index.

Chapter 2

The group of lines on a cubic surface

Our goal in this chapter is to establish the link between the lines on a cubic surface and the group $W(E_6)$. This group is the group of substitutions among the lines which preserves their intersection properties. The lines can be represented by the vertices of a certain polytope with 27 vertices, and the group $W(E_6)$ acts as the symmetry group of this polytope. Furthermore, we will show how to represent the bitangent lines to a quartic curve by a polytope with $W(E_7)$ symmetry, as well as the tritangent planes to a twisted sextic curve by a polytope with $W(E_8)$ symmetry group. The background concerning reflection groups and polytopes is given in chapter 3 but we assume some familiarity in this chapter.

First we give some history about the enumeration of lines on the cubic surface. Then we will talk about the configuration of the lines so that we can describe the group of substitutions. In section 2.3 we use the more general approach of divisors to describe the groups of the bitangent lines and tritangent planes.

2.1 History of lines on a cubic surface

A cubic surface is an algebraic surface given by a cubic polynomial. For example:

$$x^3 + y^3 + u^3 + v^3 = 0 \tag{2.1}$$

is a cubic surface where $[x : y : u : v]$ are homogeneous coordinates. This polynomial is real, so it defines a real surface in $\mathbb{RP}^3$. If we allow the variables to take on complex values, it also defines a complex surface in $\mathbb{CP}^3$.

In 1849, Cayley (with credit also given to Salmon) found that a nonsingular cubic surface always has 27 lines when working over an algebraically closed field, such as $\mathbb{C}$ [3]. For example, the lines on the surface in the preceding example are given by permutations of

$$\begin{aligned} x + \alpha y &= 0 \\ u + \beta v &= 0, \end{aligned} \tag{2.2}$$

where $\alpha^3 = \beta^3 = 1$. There are 3 choices for α, β, and y (we can switch y with u or v), giving $3^3 = 27$ lines in total. A priori, it may seem that you have to count the algebraic multiplicity of each line to get this number 27, similar to how you must count the algebraic multiplicity of roots of polynomials. However, for nonsingular cubic surfaces (that is, surfaces with well a defined tangent plane at every point) the situation is nice:

Proposition 2.1. *A nonsingular cubic surface has only isolated lines.*

Proof. (This argument comes from [23]) Let l be a line on the cubic surface, and P be a plane containing l. Choose coordinates so that P is given by $v = 0$ and $l \subset P$ is given by $u = v = 0$. Let f be the equation of the surface. For l to have multiplicity greater than 1 implies f is of the form

$$f = u^2 g(x, y, u, v) + vh(x, y, u, v). \tag{2.3}$$

But this surface has a singular point $v = u = h = 0$ where the derivatives

# of real lines	# real tritangent planes	# e	# h
27	45	12	15
15	15	6	9
7	5	2	5
3	7	0	3
3	13	0	3

Table 2.1: Types of smooth real cubic surfaces; their number of real lines; real tritangent planes; and elliptic and hyperbolic lines.

$\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial u}, \frac{\partial f}{\partial v}$ all vanish, a contradiction. □

Nonsingular algebraic varieties are smooth manifolds, so we will refer to them as smooth cubic surfaces.

In 1863, Schläfli classified all the types of real cubic surfaces based on the their singularities and their number of real lines and real tritangent planes [25]. In particular, he found 5 types of nonsingular cubic surfaces.

In 1942, Segre classified nonsingular cubic surfaces by showing there are 5 connected components of the moduli space [26] (also shown by Klein in 1873 [15]). Each of these connected components corresponds to one of the types described by Schläfli. Segre further classified the real lines into two types, elliptic and hyperbolic, based on a geometric condition (see section 6.3). The difference in the number of these is always 3, which will come into play in the Euler class calculation in chapter 6. We summarize the types of real cubic surfaces in table 2.1.

2.2 Configuration of the lines

An important fact, which we will soon see, is that the way the lines intersect is independent of the cubic surface. We can then talk about the group that substitutes the lines in such a way to preserve their intersection properties. First we will describe this configuration, and then give a way to name the lines so that we can talk about this group of substitutions.

Finding the configuration arises naturally when trying to prove there are 27 lines using classic algebraic geometry. The most difficult part of this proof is actually showing that there exists a least 1 line, see for example [23]. But using this fact, it is not too difficult to show there are in fact 27. First we give a lemma.

Lemma 2.2 (See also [16])**.** *Any line on a smooth cubic surface intersects exactly 10 others.*

Proof. Let x, y, u, v be projective coordinates. Choose them so that a line l is given by $x = y = 0$. Then the equation of the surface must be of the form

$$xf + yg = 0, \tag{2.4}$$

where f and g are quadratic polynomials in x, y, u, v. Take the intersection of the surface and a plane containing l. This plane has equation $y = cx$. The intersection is

$$x(\tilde{f} + c\tilde{g}) = 0 \tag{2.5}$$

where $\tilde{f}, \tilde{g}$ are obtained by substituting $y = cx$ into f, g. Part of the intersection is the line $x = 0$, the other part is a conic section $\tilde{f} + c\tilde{g}$. We want to find values of c so that this conic degenerates into 2 lines. Write the conic in matrix form

$$v^T A v \tag{2.6}$$

where $v = (x, u, v)$. This conic degenerates into two lines when $det(A) = 0$. For a generic $\tilde{g}$, the coefficients of x^2 are quadratic in c, the coefficients of xu, xv are linear in c, and the other coefficients have no c. The entries in A, corresponding to the form $\tilde{f} + c\tilde{g}$, are one greater in degree than those in the matrix for $\tilde{g}$. We write the degree of c in each entry of the matrix A:

$$\begin{pmatrix} 3 & 2 & 2 \\ 2 & 1 & 1 \\ 2 & 1 & 1 \end{pmatrix}. \tag{2.7}$$

So $det(A)$ is of degree five in c. Each of the five roots give a pair of lines intersecting l. So there are 10 in total. Roots with higher multiplicity would lead to lines with higher multiplicity, which is not allowed by proposition 2.1. □

Next we can show that there are in fact 27 lines:

Proposition 2.3. *A smooth cubic surface has 27 lines*

Proof. Take 3 intersecting lines on the surface of the cubic, l_A, l_B, l_C (the existence of such sets of lines was shown in the proof of the lemma). Any other line l on the surface must intersect the plane spanned by these 3 lines. Furthermore, this intersection must occur somewhere on this set of 3 lines, since they are the complete intersection of the plane with the surface. Thus l is either l_A, l_B or l_C, or one of the $10 - 2 = 8$ lines other than l_B or l_C meeting l_A, or one of the 8 lines other than l_A or l_C meeting l_B, or one of the 8 lines other than l_A or l_B meeting l_A. So there are $8 \cdot 3 = 24$ other lines. These 24 others are all distinct since if one of them meets two of l_A, l_B, l_C, it would have to be in the plane spanned by those two, so it would be l_A, l_B, or l_C, a contradiction. We have 3 lines plus 24 more giving a total of 27. □

We have now actually seen a important configuration of lines called a *tritangent plane*. These are planes which are tangent to 3 points on the surface. Since two lines through a point of the surface lie in the tangent plane to that point, and this plane must further intersect the surface in another line, we see that a tritangent plane is just a plane which contains 3 lines of the surface. Conversely, each set of 3 intersecting lines will necessarily give rise to a tritagent plane of the cubic surface, although it is possible that the 3 points of intersection degenerate into one single point, called an Eckardt point. It is not possible, however, to have only two of the points degenerate to one point, because this will give a double line and by proposition 2.1 will violate smoothness. The two possible types of tritangent planes are shown in figure 2.1.

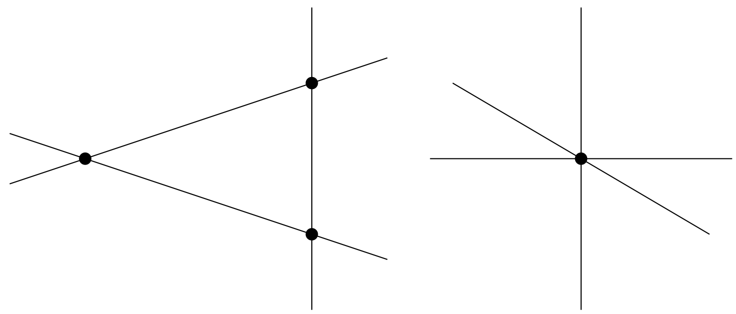


Figure 2.1: The possible configurations of lines in a tritangent planes: 3 lines intersecting at 3 points or 3 lines intersection at an Eckardt point. The plane is tangent to the points of intersection

Implicit in our discussion so far is that some of the lines actually do not intersect, but rather are skew. In fact, any line is skew to $26 - 10 = 16$ other lines. A fact which we will not prove here (although it can be seen from the discussion of divisors in section 2.3 of this chapter) is that the maximum number of mutually skew lines on a cubic surface is six. We call a set of six skew lines a sextuplet. Each sextuplet has a well defined complementary sextuplet (to be explained shortly) and together the sextuplets form what is called a *double six*. These double sixes were discovered by Schläfli and were an important tool in his classification. It is traditional to denote a double six by a Greek letter, and to use the following notation:

$$\alpha = \begin{pmatrix} a_1 & a_2 & a_3 & a_4 & a_5 & a_6 \\ b_1 & b_2 & b_3 & b_4 & b_5 & b_6 \end{pmatrix}. \tag{2.8}$$

Here, α is a double six, and each a_i and b_i represent a line. Together, the a_i's form a sextuplet of skew lines, as do the b_i's, and these sextuplets are complementary to each other. The complementary condition is that a_i is skew to b_i and intersects b_j for $i \neq j$. The pairs $\{a_i, b_i\}$ are called corresponding lines of the double six. Any pair of corresponding lines completely determines the double six to which it belongs. After naming a double six in the preceding way, we can immediately give a name to the other 15 lines: $c_{ij}, i \neq j$. The line c_{ij} is the line of intersection of the planes given by a_ib_j and a_jb_i. It is the unique line which intersects a_i, a_j, b_i, and b_j. The lines c_{ij} and c_{kl} intersect if no indices are the same, and are skew otherwise. For example, c_{12} is skew to c_{13} but intersects c_{34}.

There are 36 double sixes in total. We can denote $\binom{6}{2} = 15$ of them by δ_{ij}, for example:

$$\delta_{12} = \begin{pmatrix} a_1 & b_1 & c_{23} & c_{24} & c_{25} & c_{26} \\ a_2 & b_2 & c_{13} & c_{14} & c_{15} & c_{16} \end{pmatrix}. \tag{2.9}$$

We can name the other $\binom{6}{3} = 20$ of them by γ_{ijk}, for example:

$$\gamma_{123} = \begin{pmatrix} a_1 & a_2 & a_3 & c_{56} & c_{46} & c_{45} \\ c_{23} & c_{13} & c_{12} & b_4 & b_5 & b_6 \end{pmatrix}. \tag{2.10}$$

From this double six construction, we can also deduce the existence of the previously mentioned tritangent planes. Recall that these are planes containing 3 intersecting lines. Using the notation we have developed, we can see that, for example, the set of lines $\{c_{12}, a_1, b_2\}$ or $\{c_{12}, c_{34}c_{56}\}$ each make up tritangent planes. Any two intersecting lines determine a tritangent plane, so there are $(27 \cdot 10)/3! = 45$ in total.

2.2.1 Group of the 27 lines

Now that we have a way to name the 27 lines, a_i, b_i, c_{ij}, we can describe the group G of substitutions among the lines which preserve their configuration. For example, if we send l to l' we must send any line skew to l to a line skew to l'. Since a double six completely determines the naming of the lines, the action of an element in G is completely determined by where it sends the elements of a double six. These new elements must of course form another double six. In other words, a double six must be sent to one of the 36 double sixes. We also have the freedom to permute the 6 columns of a double six in any way we like, or switch the rows. Therefore, $|G| = 36 \cdot 6! \cdot 2 = 51840$.

Let $R(\delta)$ be the group operation which exchanges corresponding lines of a double six α. Since $R(\delta)^2 = 1$, we will call these reflections. The whole group is actually generated by these 36 reflections. Using the double six notation as before, the operator $R(\delta_{12})$ acts like the transposition (12) on the indices naming the lines, since its action on any line is given by switching the index accordingly. For example, $R(\delta_{12})a_1 = a_2, R(\delta_{12})c_{13} = c_{23}$, etc. Likewise, we could write $R(\alpha) = (ab)$ since it switches all the letters a with b. For example, $R(\alpha)a_5 = b_5$.

This group can be in fact generated by certain sets of only 6 of these reflections, for example

$$R(\delta_{12}), R(\delta_{23}), R(\delta_{34}), R(\delta_{45}), R(\delta_{56}), R(\gamma_{123}). \tag{2.11}$$

Proposition 2.4. *The generators in equation 2.11 generate the group G*

Proof. We need to show that these generators allow us to do two things: send a double six, say α, to any of the other 35 double sixes; and permute the columns and rows of this double six α.

First we notice that $R(\gamma_{123})\alpha = \gamma_{456}$, and by further using $R(\delta_{ij}) = (ij)$ we can send α to any γ_{ijk}. Next, we notice that $R(\gamma_{123})\gamma_{124} = \delta_{34}$, and by further application of (ij)'s so we can send α to any of the δ_{ij}. So we have shown that we can send α to any of the other double sixes using these generators. We can permute the columns of α by using the $R(\delta_{ij})$'s. Lastly, we need to show how to get the element $R(\alpha)$ which switches the rows of α. Notice that if η is any double six, and $g \in G$, then

$$R(g\eta) = gR(\eta)g^{-1}. \tag{2.12}$$

This follows from a general property of permutation groups (G being a subgroup of S_{27}), or by a property of reflection groups (see chapter 3). Now since $R(\gamma_{123})\gamma_{456} = \alpha$, we have

$$R(\alpha) = R(\gamma_{123})R(\gamma_{456})R(\gamma_{123}). \tag{2.13}$$

And we have already shown that we can generate a g so that $g\alpha = \gamma_{456}$, therefore

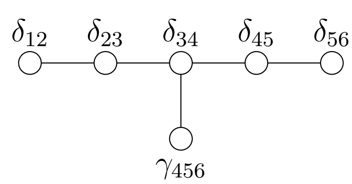
$$R(\alpha) = R(\gamma_{123})gR(\gamma_{123})g^{-1}R(\gamma_{123}). \tag{2.14}$$

So these generators do in fact generate the whole group. □

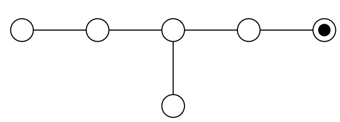
Furthermore, we have the following relations for these generators:

$$\begin{aligned}(R(\delta_{12})R(\delta_{23}))^3 &= 1,\\ (R(\delta_{23})R(\delta_{34}))^3 &= 1,\\ (R(\delta_{34})R(\delta_{45}))^3 &= 1,\\ (R(\delta_{45})R(\delta_{56}))^3 &= 1,\\ (R(\delta_{34})R(\gamma_{123}))^3 &= 1,\end{aligned} \tag{2.15}$$

and all other generators commute. These are the same relations as the generators for $W(E_6)$, which gives further justification to call our generators reflections in the first place. We can draw the Coxeter diagram:

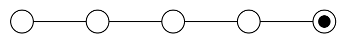


The 72 roots of this reflection group correspond to the 72 sextuplets. The 27 lines are represented by the 27 vertices of the polytope given by



We will call this polytope P_6. The size of the polytope can be normalized so that the distance between any two vertices is either 1 or $\sqrt{2}$.

Two vertices with distance 1 apart are connected by a edge and represent a pair of skew lines. A sextuplet of lines make up a 5-simplex polytope:



For any double six, there is a subgroup of $G = W(E_6)$ isomorphic to $S_6 \times \mathbb{Z}_2$ preserving it. The S_6 permutes the columns and the $\mathbb{Z}_2$ switches rows.

If two vertices are $\sqrt{2}$ distance apart, they are not connected by an edge and they represent intersecting lines. A tritangent plane is represented by a set of 3 vertices a distance $\sqrt{2}$ apart. For any such set, there is a subgroup $W(D_4) \rtimes S_3 \subset W(E_6)$ preserving it, where $W(D_4)$ stabilizes the three vertices, and S_3 permutes them.

2.3 Blow-ups and Lines on Del Pezzo surfaces

Next, we wish to talk about the group acting on the the bitangent lines to a quartic curve, and the tritagent planes to a twisted sextic curve. Since the bitangent lines lie in a plane, they will always intersect, and likewise the twisted sextic lies in 3-space, so the tritangent planes will always intersect as well. Therefore, defining a symmetry group which preserves the intersection of these objects, as we did for the lines on the cubic surface, would seem at first to be pretty meaningless. However, it will make sense if we consider the intersection of divisors in a linear system. A classic reference is from Duval [8] who related the lines, bitangents, and tritangents to certain polytopes.

We first mention that another approach to studying cubic surfaces is by blowing up 6 points on the plane. It can be shown that this blow up is isomorphic to a smooth cubic surface. References for this include [11, Ch. 4], and [12, Ch. 5, section 4]. We will follow the more general treatment of Demazure [7] which considers the blow up of more than 6 points. From this approach, we will also be able to get the polytopes of DuVal. We will discuss blow-ups again in section 4.4 with regards to real surfaces.

By blowing up a plane in k points (provided the points fall within a certain generic position, specifically: no three on a line, no six on a conic, and no eight of them on a cubic having a node at one of them), we obtain what is called a Del Pezzo surface, X_k. The degree of the Del Pezzo surface is $9-k$. We are most interested in the Del Pezzo surfaces of degree 3, 2, and 1 ($k = 6, 7$, and 8). Let us denote by $Pic(X_k)$ the group of divisors on X_k, otherwise known as the Picard group. The group $Pic(k)$ is generated by $H, E_1, ..., E_k$ where H is the divisor of a line on the plane (not through any k points of the blow up) and the E_i's are the exceptional divisors of the blow ups. The surface X_k has a canonical divisor

$$\omega_k = -3H + E_1 + ... + E_k. \tag{2.16}$$

The intersection of divisors turns $Pic(X_k)$ into a inner product space with signature $(1, k)$. The basis H, E_i satisfies

$$\begin{aligned} H \cdot H &= 1, \\ E_i \cdot E_i &= -1, \\ E_i \cdot E_j &= 0, i \neq j, \\ H \cdot E_i &= 0. \end{aligned} \tag{2.17}$$

And for further reference we have

$$\begin{aligned} \omega_k \cdot H &= -3, \\ \omega_k \cdot E_i &= -1, \\ \omega_k \cdot \omega_k &= 9-k. \end{aligned} \tag{2.18}$$

The set

$$I_k = \{D \in Pic(X_k) : D \cdot D = -1, D \cdot \omega_k = -1\} \tag{2.19}$$

represents the lines (or the (-1)-curves) on X_k. The divisors in the set

$$\Phi_k = \{D \in Pic(X_k) : D \cdot D = -2, D \cdot \omega_k = 0\} \tag{2.20}$$

are called the roots. For $k \leq 8$, the intersection product restricted to the orthogonal complement of ω_k is (negative) definite. This makes it a euclidean space.

Proposition 2.5 (See [7])**.** *The group of automorphisms (isometries) of $Pic(X_k)$ leaving ω_k fixed is the the Weyl group $W(\Phi_k)$.*

In particular, for $k = 6, 7, 8$, the groups are $W(\Phi_k) = W(E_k)$. Next we want an object representing the lines which live in this the Euclidean space of these Weyl groups. We will show shortly how the lines on Del Pezzo surfaces correspond to the lines, bitangent lines, and tritangent planes.

Proposition 2.6. *For $k = 6, 7, 8$, the projection onto the orthogonal complement of ω_k sends the divisors in the set I_k to the vertices of the polytopes P_6, P_7, P_8 described in chapter 4.*

Proof. This is a matter of finding the elements of I_k and then computing their projection with respect to some basis of $\omega_k^{\perp}$. For another way, let $V = \langle \omega_k \rangle$ and decompose

$$Pic(X_k) = V \oplus V^{\perp}, \tag{2.21}$$

we know that $W(\Phi_k)$ leaves everything in V fixed. From the work of [8], the projection onto $V^{\perp}$ must give the polytopes P_k. □

2.3.1 Lines on a cubic surface

The Del Pezzo surfaces of degree 3 are exactly the smooth cubic surfaces, [11], [24],[27]. The 27 elements of I_6 are the following

$$\begin{aligned} a_i &= E_i, \\ c_{ij} &= H - E_i - E_j, \\ b_i &= 2H - \sum_{j \neq i} E_j. \end{aligned} \tag{2.22}$$

The a_i's come from the blowups of each point, the c_{ij}'s come from the blowup of a line through the points marked by i and j, and the b_i's come from a conic through 5 of the points. Our notation here is compatible with section 2.2, that is, a_1 is skew to b_1 (since $a_i \cdot b_i = 0$), etc... We can also see the double six configurations. For example, we have a double six given by $\{a_i, b_i : i \in 1..6\}$.

2.3.2 Bitangent lines to a smooth quartic curve

The surface X_7 is a double cover of the projective plane branched along a quartic curve. The involution of this double cover is called the *Geiser involution* ([24], [27]) and its action on $Pic(X_7)$ is given by

$$D \mapsto (D \cdot \omega_7) w_7 - D. \tag{2.23}$$

pairs of this involution are mapped to bitangent lines of the quartic curve. This can be seen in the following way.

Given a quartic curve with local equation $f(x, y) = 0$, we can write the double cover locally as $z^2 = f(x, y)$. The covering map is given by projecting onto $z = 0$. Now if $l = 0$ is a bitangent line of the quartic, then $f = l \cdot c + q^2$ where c is a cubic and q is a quadratic. The points of bitangency are given by the double roots of q^2. Then on the double cover, we have two curves given by $l = 0, z = \pm q$ which are mapped to the same bitangent line l of the

Real curve	# of bitangents
4 ovals	28
3 ovals	16
2 non-nested ovals	8
1 oval	4
2 nested ovals	4
empty curve	4

Table 2.2: Types of real quartic curves and their number of bitangent lines.

plane quartic curve $f = 0$.

Considering the Picard group again, when we project orthogonally from ω_7 we see that $\pi D \mapsto -\pi D$ under the Geiser involution, and so the bitangent lines are represented by the vertices of $P_7/\pm 1$ (although the quotient isn't really a polytope, we will still refer to its vertices).

The 56 elements of I_7 are $28 = 7 + \binom{7}{2}$ of the form

$$\begin{gathered} E_1, ..., E_7, \\ H - E_i - E_j, \end{gathered} \tag{2.24}$$

and the other 28 given by the action of the Geiser involution on these,

$$\begin{gathered} 3H - 2E_i - \sum_{j \neq i} E_j, \\ 2H - \sum_{k \neq i,j} E_k. \end{gathered} \tag{2.25}$$

Zeuthen classified the possible topological types of real quartic curves and their number of bitangent lines in [29]. We give give a summary these results in table 2.2.

Real twisted sextic	# of tritangent planes
1 ellipse + 4 ovals	120
1 ellipse + 3 ovals	64
1 ellipse + 2 ovals	32
1 ellipse + 2 disjoint ovals	24
3 ellipses	24
1 ellipse + 1 oval	16
1 ellipse	8

Table 2.3: Types of real sextic curves and their number of tritangent planes

2.3.3 Tritangent planes to a smooth twisted sextic curve

The surface X_8 is a double cover of a quadric cone branched along a twisted sextic curve (the intersection of a quadric cone with a cubic). The involution of this cover is called the *Bertini Involution* ([24], [27]), its action on $Pic(X_8)$ is given by

$$D \mapsto 2(D \cdot \omega_8)\omega_8 - D. \tag{2.26}$$

When we project orthogonally from ω_8 we see that $\pi D \mapsto -\pi D$ under the Bertini involution, and so the bitangent lines are represented by the vertices of $P_8/\pm 1$.

Comessatti classified the real sextic curves by their topological type and their number of tritangent planes in [4]. The two topological components of the twisted sextic are: ovals which do not pass around the vertex of the quadric cone, and ellipses, which are the intersection of a plane (not passing through the vertex) and the conic. We summarize in table 2.3.

Chapter 3

Reflection (Weyl) group preliminaries

In this chapter we give the preliminaries concerning reflection groups, Coxeter diagrams and the construction of polytopes. Most of the properties given in this chapter can be found in [13]. For more on polytopes, see [6].

3.1 Basic definitions and properties

Reflection groups are finite groups generated by reflections in a euclidean space. Weyl group is the name traditionally given to reflection groups whose root system is related to a Lie algebra.

Definition 3.1. *Given a euclidean space $(V, \cdot)$, the reflection in the hyperplane orthogonal to v (or simply the reflection in v) is defined as:*

$$R(v)u := u - 2\frac{u \cdot v}{v \cdot v}v. \tag{3.1}$$

Here are some basic properties:

1. $R(v) \in O(V)$.

2. $R(v)u = u$ if and only if $u \cdot v = 0$. We define the reflection hyperplane as $H_v := \{u : v \cdot u = 0\}$.

3. For $T \in GL(V), R(Tv) = TR(v)T^{-1}$. In particular $R(tv) = R(v)$ for all $t \in \mathbb{R}$.

4. If the angle from H_u to H_v is θ, then $R(v)R(u)$ is a rotation (in the plane spanned by u, v) of 2θ.

Definition 3.2. *A root system $\Phi \subset V$ is a finite set of non-zero vectors that satisfies*

1. $\Phi \cap \mathbb{R}v = v, -v, \forall v \in \Phi$,

2. $R(v)\Phi = \Phi, \forall v \in \Phi$,

3. $*$ *The additional requirement that $2\frac{u \cdot v}{v \cdot v} \in \mathbb{Z}$ gives a* crystallographic *root system. This has no effect on the group, but it is a constraint which comes into play when classifying semisimple Lie algebras.*

The elements of Φ are referred to as roots.

The group generated by $\{R(v) : v \in \Phi\}$ is called the reflection group with root system Φ. We may denote it by $W(\Phi)$, or just W if it is clear which root system we are using.

Definition 3.3. *Let Φ be a root system, and W the reflection group generated by Φ. Two roots $v_1, v_2 \in \Phi$ are said to be conjugate if there exists $g \in W$ such that $gv_1 = v_2$.*

Notice that roots are conjugate in Φ if and only if their reflections are conjugate in W.

Definition 3.4. *A subset Δ of Φ is called a simple system if:*

1. *Δ is a basis for V.*

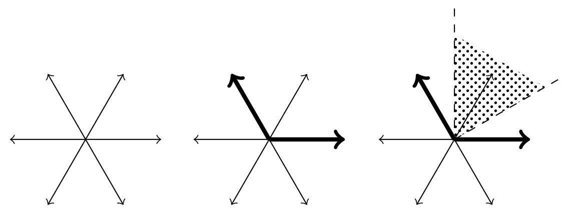


Figure 3.1: Root system, simple roots, and Weyl Chamber

2. *Any vector in* Φ *can be written as a linear combination of vectors in* Δ *with coefficients all the same sign (all non-positive or all non-negative).*

We call the roots in the simple system the simple roots*.*

A simple system exists for any root system. It is not unique, but any two simple systems are conjugate. By the second condition above, we see that a choice of simple system splits the roots into two sets Φ^+ and Φ^- of positive and negative roots. Simple systems have the following properties (see [13]):

1. Let $W(\Delta)$ be the group generated by reflections $R(v), v \in \Delta$. Then $W(\Delta)\Delta = \Phi$, and therefore $W(\Delta) = W(\Phi)$, i.e the group is generated by reflections in the simple roots.

2. Simple systems define a fundamental domain in a natural way (Weyl Chambers, see following paragraph).

3. For any $v \in \Phi$ there exists a g in the reflection group such that $gv \in \Delta$.

4. For any $u, v \in \Delta, u \neq v, u \cdot v \leq 0$.

The fundamental domain, or Weyl chamber, is the open subset of V bounded by the reflection hyperplanes of the simple roots. A different choice of simple roots will give another Weyl chamber. The reflection group acts simply and transitively on these Weyl chambers. A choice of simple roots is equivalent to a choice of Weyl chamber.

3.2 Coxeter diagrams

Coxeter groups are groups with presentations of the form

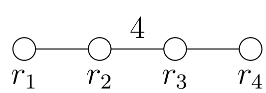
$$\langle r_1, ..., r_k | r_i^2 = (r_i r_j)^{m_{ij}} = 1\rangle, \tag{3.2}$$

with $m_{ii} = 1$. By considering how they are constructed from their simple roots, it is easy to see that reflection groups are Coxeter groups. Let s_i be a simple roots, and $r_i = R(s_i)$. The numbers m_{ij} are the period for the rotations $r_i r_j$.

From the group presentation we can construct a diagram consisting of a node for each generator and an edge labelled by m_{ij} connecting the vertices r_i and r_j. For example:

$$\langle r_1, r_2, r_3, r_4 | (r_1 r_2)^3 = (r_2 r_3)^4 = (r_3 r_4)^3 = 1\rangle, \tag{3.3}$$

with all other products having period 2. This group has the following diagram:



When the period is 2 we don't draw any edge, and since the period 3 occurs so often, we don't write it.

Writing down a Coxeter diagram gives you a group presentation and vice versa. However, not all diagrams give you a finite group. If we assume the diagram comes from a reflection group (with the generator $r_i = R(s_i)$ corresponding to a reflection in the simple root s_i), then writing the euclidean dot product in the basis given by the simple roots will be a positive definite symmetric matrix, i.e. $B_{ij} = s_i \cdot s_j = -|s_i||s_j| \cos(\pi/m_{ij})$ is positive definite. Any diagram whose associated matrix B_{ij} is positive definite defines a finite reflection group (and these are exactly the finite Coxeter groups, [13]). There are 3 families of increasing dimension: A_n, B_n, D_n; a 2-dimensional family

$I_2(m)$; and six exceptional ones: $E_6, E_7, E_8, F_4, H_3, H_4$. Any finite Coxeter group is a direct product of these irreducible ones.

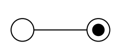
3.2.1 Polytopes

Polytopes are the generalization of polygons and polyhedra to arbitrary dimension. You can give a recipe for making a polytope by putting dots inside nodes of the Coxeter diagram. The construction is to start with a point in the euclidean space that is orthogonal to every simple root on the diagram except the ones with dots. We will call this point a *generator* of the polytope. You then act the the whole reflection group on this point and take the convex hull of these points to be your polytope. We repeat this definition more formally

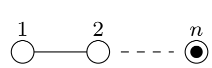
Definition 3.5 (Polytope and generator)**.** *Let a Coxeter diagram coming from a reflection group with simple roots $s_1, ..., s_n$ be given. Let the nodes associated to $s_i, i \in I$ have dots in them. Then the polytope associated to this diagram is constructed as follows: Let v be a vector in $\cap_{i \notin I} H_{s_i}$ (where H_{s_i} is the hyperplane orthogonal to s_i) that is equidistant to H_i for all $i \in I$. We call this vector a* generator*. The set of vertices of the polytope is taken to be $\{gv : g \in W\}$ (where W is the reflection group associated to the roots), and the polytope is the convex hull of this set.*

The equidistant condition ensures that what we get is a *uniform* polytope (that is, all its facets are uniform, with uniform polygons being the regular ones). If we removed this condition, we would still obtain polytopes, only with less symmetry.

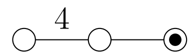
As an example of this construction, take the diagram



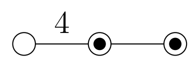
which represents a triangle. More generally, the diagram



represents a regular n-simplex (where the numbers are labelling the nodes, and not the edges in between). As another example, we have



which gives an octahedron. Adding another dot gives what is called a truncated octahedron



Chapter 4

Real enumeration with reflection (Weyl) groups

In this chapter, we will enumerate the real lines on the cubic surface by finding the conjugacy classes of involutions in the group $W(E_6)$ and finding the number of fixed vertices in P_6 under each involution. We do the same thing with E_7 and E_8 for the bitangent lines and tritangent planes.

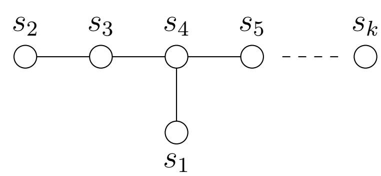
In section 4.3 we identify which vertices correspond to the hyperbolic and elliptic lines of the cubic surface. In doing this, we also give a convenient way to visualize all 27 vertices of P_6.

Finally, we compare our real enumeration results to the classical results given in chapter 2.

4.1 Choice of roots and coordinates for the polytopes

To make things more concrete, we will work with fixed representations of the root systems. We will consider the systems E_6, E_7, E_8 all as subsets of $\mathbb{R}^8$. The roots of $\Phi(E_8)$ are given by all the combinations $\pm e_i \pm e_j, i \neq j$, and $\frac{1}{2}(\pm e_1 \pm ... \pm e_8)$ where e_i are the standard basis vectors. The root system $\Phi(E_7)$ is given by those roots of $\Phi(E_8)$ which are contained in the subspace orthogonal to $e_1+...+e_8$, and the roots of $\Phi(E_6) \subset \Phi(E_7) \subset \Phi(E_8)$ are those contained in the subspace orthogonal to $e_1 + ... + e_6$ and $e_7 + e_8$. Notice that the inclusion of root systems gives us an inclusion of groups $W(E_6) \subset W(E_7) \subset W(E_8)$.

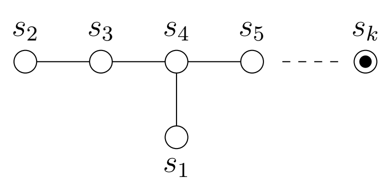
The Coxeter diagram for $E_k, k = 6, 7, 8$ is



with a specific choice of simple roots for $\Phi(E_k)$ given by s_1 to s_k in the following set:

$$\begin{aligned}
s_1 &= \frac{1}{2}(e_1 + e_2 + e_3 - e_4 - e_5 - e_6 - e_7 + e_8),\\
s_2 &= -e_1 + e_2,\\
s_3 &= -e_2 + e_3,\\
s_4 &= -e_3 + e_4,\\
s_5 &= -e_4 + e_5,\\
s_6 &= -e_5 + e_6,\\
s_7 &= -e_6 + e_7,\\
s_8 &= -(e_7 + e_8).
\end{aligned} \tag{4.1}$$

In the next three subsections, we will describe the polytopes P_k corresponding to the diagrams:



4.1.1 The polytope P_6

The polytope P_6 has 27 vertices. Recall the definition of a generator, definition 3.5. A particular generator for this polytope is the vector

$$g_6 = -e_1 - e_2 - e_3 - e_4 - e_5 + 5e_6 - 3e_7 + 3e_8. \tag{4.2}$$

Of these 27 vertices, $6 \cdot 2 = 12$ of these vertices have the same form as g_6 with e_6 permuted with $e_1, ..., e_6$ and e_7 permuted with e_7, e_8. There are an additional $\binom{6}{2} = 15$ of vertices the form

$$2e_1 + 2e_2 + 2e_3 + 2e_4 - 4e_5 - 4e_6 \tag{4.3}$$

under permutations of e_5, e_6 with $e_1, ..., e_6$.

4.1.2 The polytope P_7

The polytope P_7 has 56 vertices. A particular generator for this polytope is the vector

$$g_7 = e_1 + e_2 + e_3 + e_4 + e_5 + e_6 - 3e_7 - 3e_8. \tag{4.4}$$

All the vertices are obtained by the $\binom{8}{2} = 28$ permutations of e_7, e_8 and the negatives of these. There is an important connection between P_7 and P_6 given by the following proposition:

Proposition 4.1. *The polytope P_7 contains two affine copies of the polytope P_6. For any pair of vertices $\{v, -v\}$, there is a subgroup isomorphic to $W(E_6)$ which fixes both vertices of this pair.*

Proof. Let π be the projection onto the $\Phi(E_7)$ to the $\Phi(E_6)$ subspace. The generator g_7 is actually perpendicular to the $\Phi(E_6)$ subspace, so $W(E_6)g_7 = g_7$. The group $W(E_7)$ acts transitively on the vertices of P_7, so this holds for any pair $\{v, -v\} \in \Phi(E_7)$. Furthermore, $g_6 = \left[\frac{1}{2}\pi \circ R(s_7)\right] g_7$, or in other words, $R(s_7)g_7 = 2g_6 + p$, where p is some vector perpendicular to the $\Phi(E_6)$

subspace. Then we see that the orbit of $R(s_7)g_7$ under the action of $W(E_6)$ gives an affine copy of the vertices of the polytope P_6. The negative of these vertices along with $\{g_7, -g_7\}$ give all 56 vertices. □

4.1.3 The polytope P_8

The polytope P_8 has 240 vertices. A particular generator for this polytope is the vector

$$g_7 = e_1 + e_2 + e_3 + e_4 + e_5 + e_6 + e_7 + e_8. \tag{4.5}$$

This is in fact one of the roots of $\Phi(E_8)$, so the vertices of P_8 represent the roots of $\Phi(E_8)$.

4.2 Involutions and fixed points

Our basic tool for finding involutions is the following

Lemma 4.2 ([13]). *Any involution in a reflection group can be written as a product of orthogonal reflections.*

This lemma allows us to find conjugacy classes of involutions by finding conjugacy classes of sets of orthogonal vectors (recall definition 3.3). That is, let $\mathfrak{v} = \{v_1, ..., v_k\}$ be a set of k orthogonal roots, and $\mathfrak{v}' = \{v_1', ..., v_k'\}$ be another. Then $R(v_1) \cdots R(v_k) \sim R(v_1') \cdots R(v_k')$ if and only if there exists a g such that $g\mathfrak{v} = \mathfrak{v}'$.

In finding the involutions, we will also need to use the following lemma:

Lemma 4.3. *For $k = 6, 7, 8$, all the roots in $\Phi(E_k)$ are conjugate. That is, for any two roots $r_1, r_2 \in \Phi(E_k)$ there exists $g \in W(E_k)$ such that $gr_1 = r_2$.*

Proof. By the third property of simple systems given in section 3.1, any root is conjugate to a simple root, so we only need to show that all the simple roots are conjugate. We start with E_6. The simple roots $s_2, ..., s_6$

are conjugate since they form a system A_5 whose group is isomorphic to S_6 (the symmetric group on 6 elements) by identifying the $R(s_i)$'s with the generating permutations. Or, to see the conjugacy more explicitly, take $g_i = R(s_{i-1}) \cdots R(s_2)R(s_i) \cdots R(s_3)$ (for $2 \leq i \leq 6$), then $g_i s_2 = s_i$. Now we only need to show that s_1 is conjugate to one of the five other roots (and hence all of them). The transformation $R(s_4)R(s_1)R(s_4)$ sends s_1 to s_4. This shows all the roots in $\Phi(E_6)$ are conjugate. Now for E_7, all the simple roots $s_1, ..., s_6$ are conjugate in E_6, so they are conjugate in E_7 as well. The transformation g_i extends to include $i = 7$ so we are done. For E_8, the transformation $R(s_7)R(s_8)R(s_7)$ sends s_8 to s_7. □

4.2.1 $W(E_6)$ and P_6

Proposition 4.4. *There are five conjugacy classes of involutions in $W(E_6)$. They are represented by:*

$$\begin{gathered} 1, \\ R_1, \\ R_1R_2, \\ R_1R_2R_3, \\ R_1R_2R_3R_4, \end{gathered} \tag{4.6}$$

where $\{r_1, r_2, r_3, r_4\}$ is a set of orthogonal roots in $\Phi(E_6)$ and $R_i = R(r_i)$.

Proof. The maximum number of orthogonal roots in $\Phi(E_6)$ is four. Any two sets of up to four orthogonal roots are conjugate by the following argument:

- All roots are conjugate in $\Phi(E_6)$ (see lemma 4.3), so we can transform our first orthogonal root into r_1. The roots in $\Phi(E_6)$ orthogonal to r_1 form a root system $\Phi(A_5)$.
- All roots in $\Phi(A_5)$ are conjugate, so we can transform our second orthogonal root into r_2 by an element of $W(A_5)$ all the while fixing r_1. The roots in $\Phi(A_5)$ orthogonal to r_2 form a root system $\Phi(A_3)$.

- All roots in $\Phi(A_3)$ are conjugate, so we can transform our third orthogonal root into r_3 by an element of $W(A_3)$ all the while fixing r_1, r_2. The roots in $\Phi(A_3)$ orthogonal to r_3 form a root system $\Phi(A_1)$.
- $\Phi(A_1)$ has only two roots, which are negatives of each other and conjugate in $\Phi(A_1)$, so we can transform our fourth orthogonal root into r_4 while fixing r_1, r_2, r_3. There are no more orthogonal roots.

The proof can be summarized by saying we have a sequence of root systems,

$$\Phi(E_7) \supset \Phi(A_5) \supset \Phi(A_3) \supset \Phi(A_1), \tag{4.7}$$

where each root system is the orthogonal complement of some root in its parent system, and where each associated reflection group acts transitively on its roots. □

For concreteness, we give an explicit choice of orthogonal roots compatible with our previous descriptions:

$$\begin{aligned} r_1 &= -e_1 + e_2, \\ r_2 &= -e_3 + e_4, \\ r_3 &= -e_5 + e_6, \\ r_4 &= -e_7 + e_8. \end{aligned} \tag{4.8}$$

The roots r_1, r_2, r_3 are in the simple system we chose, r_4 is not.

Corollary 4.5. *The number of fixed vertices of P_6 under the possible involutions described by proposition 4.4 is given by:*

Involution	*# of fixed vertices*
1	*27*
R_1	*15*
R_1R_2	*7*
$R_1R_2R_3$	*3*
$R_1R_2R_3R_4$	*3*

Proof. The proof is simply a matter of using the explicit form of the roots and vertices given and counting how many vertices are orthogonal to a given set of roots. □

4.2.2 $W(E_7)$ and P_7

Proposition 4.6. *There are five conjugacy classes of involutions in* $W(E_7)/\pm 1$*. They are represented by:*

$$\begin{gathered} 1, \\ R_1, \\ R_1R_2, \\ R_1R_2R_3, \\ R_1R_2R_3R_4, \end{gathered} \tag{4.9}$$

where $\{r_1, r_2, r_3, r_4\}$ *is a set of orthogonal roots in* $\Phi(E_7)$ *and* $R_i = R(r_i)$*. These orthogonal roots are actually contained in a root system* $\Phi(E_6)$*.*

Proof. We have a sequence of root systems:

$$\Phi(E_7) \supset \Phi(D_6) \supset \Phi(D_4 + A_1), \tag{4.10}$$

where each subsystem is the orthogonal complement of some root in the parent system. We pick orthogonal roots $r_1 \in \Phi(E_7), r_2 \in \Phi(E_6), r_3 \in \Phi(D_4)$, and $r_7 \in \Phi(A_1)$. The roots in $\Phi(D_4)$ orthogonal to r_3 form a root system $\Phi(D_2 + A_1) = \Phi(A_1 + A_1 + A_1)$. Choose $r_4, r_5, r_6 \in \Phi(A_1 + A_1 + A_1)$. A particular choice of orthogonal roots is the following:

$$\begin{aligned} r_1 &= -e_1 + e_2, \\ r_2 &= -e_3 + e_4, \\ r_3 &= -e_5 + e_6, \\ r_4 &= -e_7 + e_8, \\ r_5 &= \frac{1}{2}(-e_1 - e_2 + e_3 + e_4 + e_5 + e_6 - e_7 - e_8), \\ r_6 &= \frac{1}{2}(e_1 + e_2 - e_3 - e_4 + e_5 + e_6 - e_7 - e_8), \\ r_7 &= \frac{1}{2}(e_1 + e_2 + e_3 + e_4 - e_5 - e_6 - e_7 - e_8). \end{aligned} \tag{4.11}$$

Notice that the first four roots are the same we used for $\Phi(E_6)$. The product of reflections in these (or any other) seven orthogonal roots gives the central inversion, $R_1R_2R_3R_4R_5R_6R_7 = -1$. From this we see that any four orthogonal reflections is conjugate to the negative of three orthogonal reflections.

The reflection group $W(D_4 + A_1)$ does not act transitively on its roots, so we have, at most, two classes of three orthogonal roots, represented by $\{r_1, r_2, r_3\}$ and $\{r_1, r_2, r_7\}$. To see that these classes are indeed disjoint under $W(E_7)$ we notice that a transformation of $\{r_1, r_2, r_3\}$ into $\{r_1, r_2, r_7\}$ would give us a transformation of $\{r_4, r_5, r_6, r_7\}$ into $\{r_3, r_4, r_5, r_6\}$. But the sum $\frac{1}{2}(r_3 + r_4 + r_5 + r_6)$ is a root whereas $\frac{1}{2}(r_4 + r_5 + r_6 + r_7)$ is not, so this is impossible.

Furthermore, we claim that $R_1R_2R_7 \sim -R_1R_2R_3R_4$. There is an equality $R_1R_2R_3R_4 = -R_5R_6R7$. The only question is whether $R_5R_6R_7$ is conjugate to $R_1R_2R_3$ or to $R_1R_2R_7$. We observe that $\frac{1}{2}(r_1 + r_2 + r_3 + r_4)$ is a root so $R_5R_6R_7 \sim R_1R_2R_7$. □

Corollary 4.7. *The number of fixed vertices of $P_{56}/\pm 1$ under the possible involutions described by proposition 4.6 is given by:*

Involution	*# of fixed vertices*
1	*28*
R_1	*16*
R_1R_2	*8*
$R_1R_2R_3$	*4*
$R_1R_2R_3R_4$	*4*

Proof. Using proposition 4.1 and the fact that the R_i are contained in $W(E_6)$ we immediately find that the enumeration is the same as the E_6 case plus one for the fixed $\pm$ pair of vertices. □

4.2.3 $W(E_8)$ and P_8

Proposition 4.8. *There are six conjugacy classes of involutions in* $W(E_8)/\pm 1$*. They are represented by:*

$$\begin{aligned} &1,\\ &R_0,\\ &R_0R_1,\\ &R_0R_1R_2,\\ &R_0R_1R_2R_3,\\ &R_0R_1R_2R_7, \end{aligned} \tag{4.12}$$

where $\{r_0, r_1, r_2, r_3, r_4, r_7\}$ *is a certain set of orthogonal roots (to be described in the proof) and* $R_i = R(r_i)$*.*

Proof. All the roots in $\Phi(E_8)$ are conjugate. We can pick our first orthogonal root to be

$$r_0 = \frac{1}{2}(e_1 + e_2 + e_3 + e_4 + e_5 + e_6 + e_7 + e_8). \tag{4.13}$$

The roots orthogonal to r_0 form a root system $\Phi(E_7)$. We choose $r_1, ..., r_7$ the same way as we for the E_7 case in equation 4.11. We have the central inversion given by $R_0R_1R_2R_3R_4R_5R_6R_7 = -1$, and so we only need to consider up to four orthogonal roots. In this case, we have potentially two different classes given by $\{r_0, r_1 r_2, r_3\}$ and $\{r_0, r_1, r_2, r_7\}$. That we indeed have two different classes can be seen by the fact that $\frac{1}{2}(r_0 + r_1 + r_2 + r_7)$ is a root, whereas $\frac{1}{2}(r_0 + r_1 + r_2 + r_3)$ is not. □

Corollary 4.9. *The number of fixed vertices of $P_8/\pm 1$ under the possible involutions described by proposition 4.8 is given by:*

Involution	*# of fixed vertices*
1	*120*
R_0	*64*
R_0R_1	*32*
$R_0R_1R_2$	*16*
$R_0R_1R_2R_3$	*8*
$R_0R_1R_2R_7$	*24*

Proof. Recall that the vertices of the polytope P_8 are actually the root vectors of $\Phi(E_8)$. We also recall the sequences of root systems:

$$\begin{gathered}\Phi(E_8) \supset \Phi(E_7) \supset \Phi(D_6) \supset \Phi(D_4 + A_1), \\ \Phi(D_4) \supset \Phi(A_1 + A_1 + A_1),\end{gathered} \tag{4.14}$$

with orthogonal roots $r_0 \in \Phi(E_8)$, $r_1 \in \Phi(E_7)$, $r_2 \in \Phi(D_6)$, $r_3 \in \Phi(D_4)$, and $r_7 \in \Phi(A_1)$. For the identity, the number of fixed points is $|E_8|/2 = 120$. For R_0, the number of fixed points is

$$1 + |E_7|/2 = 1 + 63 = 64, \tag{4.15}$$

where the contribution of 1 comes from r_0 and the contribution of $|E_7|/2$ comes from the roots orthogonal to r_0 modulo ± 1. In a similar fashion, the number of fixed points for R_0R_1 is

$$2 + |D_6|/2 = 2 + 30 = 32. \tag{4.16}$$

For $R_0R_1R_2$, the number of fixed points is

$$3 + |D_4 + A_1|/2 = 3 + (24 + 2)/2 = 16. \tag{4.17}$$

For $R_0R_1R_2R_3$, the number of fixed points is

$$4 + |A_1|/2 + |A_1 + A_1 + A_1|/2 = 4 + 1 + 3 = 8. \tag{4.18}$$

For $R_0R_1R_2R_7$, the calculation is a bit special. The answer is

$$4 + |D_4|/2 + 8 = 4 + 24/2 + 8 = 24. \tag{4.19}$$

The extra contribution of 8 comes from the roots $\frac{1}{2}(\pm r_0 \pm r_1 \pm r_2 \pm r_7)$ which are fixed modulo -1. There are $2^4/2 = 8$ of these. □

4.3 Hyperbolic and elliptic lines on cubic surfaces

Segre divided the lines into two species called hyperbolic and elliptic. We give his definition in section 6.3. Here we describe which vertices of P_6 correspond to hyperbolic and elliptic lines. Recall the involutions given in proposition 4.4.

There is a plane H_0 orthogonal to the 4 roots of the involutions r_1, r_2, r_3, r_4. This plane contains 3 vertices of the polytope P_6. Let H_{ij} be the plane spanned by r_i, r_j. Note that we have a decomposition of our space into $H_0 \oplus_{i<j} H_{ij}$. Every one of the other 24 vertices has a non zero projection onto one and only one H_{ij}. Furthermore, for a given i, j, there are exactly 4 vertices which project non-trivially onto H_{ij} and the projection of these 4 vertices onto P_0 are the same. Call this point of projection p_{ij}. If $\{i, j, k, l\} = \{1, 2, 3, 4\}$, then $p_{ij} = p_{kl}$. To summarize, we have 3 vertices contained in H_0 and 4 others contained in $H_0 \oplus H_{ij}$ for each i, j. See figures

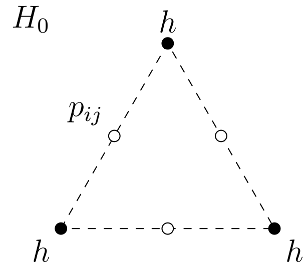


Figure 4.1: The plane H_0 orthogonal to r_1, r_2, r_3, r_4. 3 vertices lie in this plane, and they represent real hyperbolic lines. Theses vertices are $\sqrt{2}$ distance apart and are not connected by an edge. The open circles represent the projections p_{ij} of the vertices in $H_0 \oplus H_{ij}$ not contained in H_0.

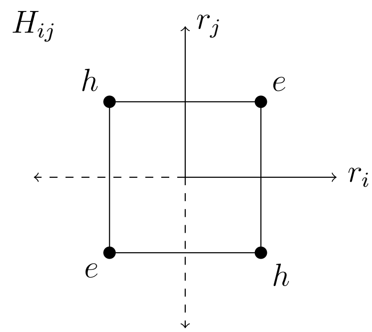


Figure 4.2: The plane H_{ij} spanned by r_i, r_j. The vertices a distance 1 apart are connected by an edge. These lines represented by these vertices are real if the involution does not involve R_i, R_j.

4.1 and 4.2.

If all the lines are real, according to Segre [26], the elliptic lines make up a double six. Up to conjugation, we can take this double six to be represented by the root $\frac{1}{2}(r_1 + r_2 + r_3 + r_4)$, and so the hyperbolic lines are the vertices orthogonal to this root. Thus the 3 vertices in H_0 represent hyperbolic lines. They are $\sqrt{2}$ distance apart and therefore represent intersecting lines. When the involution is nontrivial, let L_r be the set of vertices representing real lines, then hyperbolic lines are represented by the intersection of $(r_1+r_2+r_3+r_4)^\perp$ with L_r.

Involution	Real lines L_r	# Real lines	# Hyperbolic
1	L	27	$3+6\cdot 4/2=15$
R_1	$(H_0\oplus H_{34}\oplus H_{23}\oplus H_{24})\cap L$	$3+3\cdot 4=15$	$3+3\cdot 4/2=9$
R_1R_2	$(H_0\oplus H_{34})\cap L$	$3+4=7$	$3+4/2=5$
$R_1R_2R_3$	$H_0\cap L$	3	3
$R_1R_2R_3R_4$	$H_0\cap L$	3	3

Table 4.1: Vertices representing real lines, where L is the set of vertices of P_6. Half the vertices in each $H_0\oplus H_{ij}$ (not contained in H_0) represent hyperbolic lines.

4.4 Comments on real enumeration

We have found the possible involutions in $W(E_6)$, along with the enumeration of fixed points, but it remains to show the following:

Proposition 4.10. *Every involution in $W(E_6)$ arises from the complex structure of some cubic surface.*

We will do two things. The first is simply comparing our fixed point combinatorics to the table given by Schläfli. Each type of real surface is determined by the number of real lines and real tritangent planes. This shows that all our involutions do indeed come from a certain cubic surface. As a second thing, we recall from section 2.3 the construction of a cubic surface from the blow up of 6 points in a plane (with no 3 in a line, and not all 6 on a conic). We can chose these points in a certain way so that the real part of the blow up is a real cubic surface of the desired type. For the first four types, this is relatively straightforward. For the fifth type, we need to put a special complex structure on the blowup. More details concerning the real blow ups are found in [22, Theorem 1.2].

Proof. We go through each type:

1. The first kind of surface has 27 real lines. The involution in $W(E_6)$ is the identity. This surface can be obtained by blowing up 6 real points in the plane.

2. The second kind of surface has 15 real lines. The involution in $W(E_6)$ is R_1. This surface can be obtained by blowing 4 real points and a pair of complex conjugate points.

3. The third kind of surface has 7 real lines. The involution in $W(E_6)$ is R_1R_2. This surface can be obtained by blowing up 2 real points and two pairs of complex conjugate points.

4. The fourth kind of surface has 3 real lines and 7 real tritangent planes. The involution is $R_1R_2R_3$. Recall that tritangent planes are represented by a set of 3 vertices all a distance $\sqrt{2}$ apart. The 7 real tritangent planes are given by: 1 tritangent plane correspoding to the 3 vertices contained in H_0; The 2 pairs of vertices opposite (distance $\sqrt{2}$ apart) in each of H_{12}, H_{23}, H_{13} (which are exchanged in the involution so that as a set they are preserved) along with the vertex in H_0 a distance $\sqrt{2}$ from each member of the 2 pairs respectively, giving $2 \cdot 3 = 6$ more. This surface can be obtained by blowing up 3 pairs of complex conjugate points.

5. The fifth kind of surface has 3 real lines and 13 tritangent planes. The involution is $R_1R_2R_3R_4$. The 13 tritangent planes are the 7 given in the previous type along with 6 more coming from pairs of opposite vertices in H_{14}, H_{24}, H_{34}. A real cubic surface can be obtained by giving a new complex structure to the blow up of 6 particular points in the plane. These 6 points are chosen in the following way: take five real points $p_1, ..., p_5$. These lie on a real conic. Take the sixth point to be real and lie on the tangent line to the conic at p_5. For any point p in the plane, consider the (unique) conic containing p_1, p_2, p_3, p_4, p. Define an involution by sending this point to the intersection of this conic with the line from p to p_6. This involution extends to an involution on the blow up of the 6 points $p_1, ..., p_6$ and, roughly speaking, the real surface is the quotient of the blow up by that involution. See [22, Theorem

1.2] for the details.

□

We can also explain what is happening in terms of the double-six notation. Let $a_1, ..., a_6$ be a sextuplet associated to the exceptional divisors as in 2.3.1, i.e. associate $a_1, ..., a_6$ to the 6 points in the plane. In the first type of surface, these are all real points. In the second type, we take $a_2 = \overline{a_1}$, so that the involution is the permutation (12). In the third type, we can also take $a_4 = \overline{a_3}$, so that the involution is the permutation (12)(34). For the fourth type, take furthermore $a_6 = \overline{a_5}$, so that the involution is (12)(34)(56). For the fifth type, the only other permutation that is orthogonal to these three is (ab). This switches a point (more correctly, the blow up of a point) with the image of the conic through this point e.g. $a_6 \leftrightarrow b_5$ (where b_5 is the conic through a_1, a_2, a_3, a_4, a_6). This is why that special complex structure was needed.

Next, we discuss real quartic curves. For the number of real bitangent lines, all the possible numbers of fixed points in proposition 4.6 are the ones found in table 2.2. However, in the latter there are 3 different topological types of real quartics with 4 bitangent lines, while there are only 2 involutions which fix 4 lines. The reason for this can be seen from the following. There are other structures associated to the quartic curve called *Cayley octads* and *Steiner complexes* which this group preserves, for definitions see for example [21]. The real enumeration of these is given in the following table. Although the last two curves have different topological types, there is no difference from the point of view of the group action. Steiner octads can be used to define the group of the 28 bitangent lines in a similar way that double sixes were used to define the group of 27 lines, see for example [16].

In the case of sextic curves, once again there are two topological types, each with 24 tritangent planes, which are not distinguished by the group action.

Real curve	# of bitangents	# points in C.O.	# S.C.
4 ovals	28	8	63
3 ovals	16	6	31
2 non-nested ovals	8	4	15
1 oval	4	2	7
2 nested ovals	4	0	15
empty curve	4	0	15

Table 4.2: Number of bitangents, points in the Cayley octad (C.0.), and Steinter complexes (S.C.) for real quartic curves.

All these topological types can be found from the classification of real Del Pezzo surfaces, for example [24].

Chapter 5

Characteristic class preliminaries

We are interested in enumerating the zero locus of generic sections in the bundle. We will see that the Euler and Chern classes allow us to do this. Performing calculations with these classes will require knowing the cohomology ring structure. In the case of Grassmannians, which we are mainly interested in, the manipulation of the cohomology ring is referred to as *Schubert calculus*.

It turns out that in nice situations, the product in cohomology has a useful geometric interpretation. Our approach therefore, following [11] and [9], is to start with the notion of intersection of cycles and of homology classes, and to use that this intersection is Poincare dual to the cohomology product. We will then be able to derive the Schubert calculus for $\mathbb{CP}^n$ and $G_2(\mathbb{C}^4)$ in a relatively intuitively way.

In sections 5.3 and 5.4 we introduce the Euler and Chern classes. Our goal is to describe the geometric interpretation of these classes, making use of the intersection theory we introduced in section 5.1. For the general definitions of these classes, we refer to [2], or [18].

Finally, at the end of the last section, we compute the Chern classes of the dual universal subbundle over a Grassmannian. Our approach makes this a simple matter of linear algebra. The Chern classes of this bundle, along with the Schubert calculus of $G_2(\mathbb{C}^4)$, will be used in chapter 6, to enumerate lines on cubic surfaces.

5.1 Intersection theory

Let X be a smooth oriented n-dimensional manifold. Let A and B be two piecewise smooth cycles.

Definition 5.1 (Transverse Intersection)**.** *We say that A and B are* transverse*, or* intersect transversely *if*

$$dim(A \cap B) = dim(A) + dim(B) - n. \tag{5.1}$$

If the sum of the dimensions of A and B is less than n, then they are transverse if and only if their intersection is empty. If A and B are of complementary dimension ($\dim(A) + \dim(B) = n$), $A \cap B$ is zero dimensional, and there is an isomorphism $T_pA \oplus T_pB \simeq T_pX$ given by $(a, b) \mapsto a + b$. Given the orientations of A and B, we can give a natural orientation to $T_pA \oplus T_pB$ and ask if the pushforward of this orientation gives the orientation for T_pX. This leads us to the following notion of intersection index:

Definition 5.2 (Intersection index)**.** *Let A, B be transverse cycles of complementary dimension. Given an oriented basis $a_1, ..., a_k$ for T_pA, and $b_1, ..., b_{n-k}$ for T_pB, we say the intersection index $I_p(A \cdot B)$ of A with B at the point $p \in A \cap B$ is 1 if $a_1, ..., a_k, b_1, ..., b_{n-k}$ gives an oriented basis for T_pX and is -1 otherwise.*

We can now define an intersection number between homology classes in complementary dimension,

$$H_k(X, \mathbb{Z}) \times H_{n-k}(X, \mathbb{Z}) \to \mathbb{Z}, \tag{5.2}$$

by finding representative cycles which intersect transversely. Let $\alpha \in H_k(X,\mathbb{Z})$ and $\beta \in H_{n-k}(X,\mathbb{Z})$, and let A, B be cycles intersecting transversely such that $[A] = \alpha, [B] = \beta$. The intersection number is defined as

$$\#(\alpha \cdot \beta) := \#(A \cdot B) := \sum_{p \in A \cap B} I_p(A \cdot B). \tag{5.3}$$

The intersection number is well defined on homology classes: If $A = \partial C$, then $\#(A \cdot B) = 0$ for all B.

Fixing a k cycle in the intersection number gives a map $H_{n-k}(X,\mathbb{Z}) \to \mathbb{Z}$, or in other words, an element of the cohomology $H^{n-k}(X,\mathbb{Z})$. If X is compact, there is an isomorphism $H_k(X) \simeq H^{n-k}(X)$ which is commonly known as the Poincare duality. Using the de Rham isomorphism, another way to state the Poincare duality is: Given a cycle A, there exists a cocyle η_A such that

$$\#(A \cdot B) = \int_B \eta_A. \tag{5.4}$$

Even when X is not compact, there is a version of the Poincare duality using compactly supported cohomolgy. In particular, for the case when X is a vector bundle see [2, p. 59ff].

When the dimensions of the two cycles are not complementary, we can still define an intersection product

$$H_{n-k}(X,\mathbb{Z}) \times H_{n-l}(X,\mathbb{Z}) \to H_{n-k-l}(X,\mathbb{Z}). \tag{5.5}$$

If $a \in H_{n-k}(X,\mathbb{Z}), b \in H_{n-k}(X,\mathbb{Z})$, we find representative cycles A, B intersecting transversely almost everywhere. We give $C = A \cap B$ an orientation such that if $v_1, ...v_{n-k-l}$ is a positively oriented basis for T_pC at a smooth point p, we can complete it to a positively oriented basis

$$a_1, ..., a_l, v_1, ..., v_{n-k-l}, b_1, ..., b_k \tag{5.6}$$

of T_pX so that

$$a_1, ..., a_l, v_1, ..., v_{n-k-l} \tag{5.7}$$

gives a positively oriented basis of T_pA and

$$v_1, ..., v_{n-k-l}, b_1, ..., b_k \tag{5.8}$$

gives a positively oriented basis of T_pB. We denote this product as $A \cdot B$. When A and B are complementary dimension, evaluating $A \cdot B$ by the canonical generator of $H^0(X, \mathbb{Z})$ gives the intersection number.

The wedge product in cohomology is Poincare dual to the intersection product [14], [11, p. 59]. Given cycles A, B and their respective duals η_A, η_B, we have

$$\eta_{(A \cdot B)} = \eta_A \wedge \eta_B. \tag{5.9}$$

When A, B have complementary dimension,

$$\#(A \cdot B) = \int_B \eta_A = \int_A \eta_B = \int_X \eta_A \wedge \eta_B. \tag{5.10}$$

We will often suppress the wedge symbol and use a dot or juxtaposition to represent the cohomology product

5.1.1 Analytic cycles

Let us now consider analytic cycles. These are cycles which are complex analytic subvarieties of a compact complex manifold X. If A is given locally as the zeros of m independent analytic functions then we say A has codimension m, or $\text{codim}(A) = m$. The Poincare dual of A gives an element of $H^{2m}(X, \mathbb{Z})$. Two analytic cycles intersect transversely if $\text{codim}(A \cap B) = \text{codim}(A) + \text{codim}(B)$. Notice that for real cycles, we might have $A \cdot B = -B \cdot A$ and there is no natural way to orient $A \cap B$. For analytic cycles however, we can assign an orientation in a natural way, and with this orientation, transverse cycles always intersect in a positively in the sense given by the following proposition (keep in mind, however, that it is sometimes the case that there are no representative analytic cycles meeting transversely).

Proposition 5.3 (Positive intersection of analytic cycles)**.** *Let A and B be analytic cycles of codimension k and l, respectively, in a compact complex n-dimensional manifold. Suppose they intersect transversely, then $A \cap B$ is an analytic cycle and giving everything their natural orientations, $A \cdot B = A \cap B$.*

Proof. Choose coordinates in a neighbourhood of p so that A is given by $z_{n-k} = ... = z_n = 0$ and B is given by $z_1 = ... = z_l$. Writing $z_i = x_i + iy_i$, the natural orientation of the whole manifold is given by the real basis for $T_p(X)$:

$$\left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial y_1}, ..., \frac{\partial}{\partial x_n}, \frac{\partial}{\partial y_n} \right). \tag{5.11}$$

The coordinates for A are $z_1, ..., z_{n-k}$, and the natural orientation is given by

$$\left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial y_1}, ..., \frac{\partial}{\partial x_{n-k}}, \frac{\partial}{\partial y_{n-k}} \right). \tag{5.12}$$

Similarly, we have the orientation for B given by

$$\left(\frac{\partial}{\partial x_{l+1}}, \frac{\partial}{\partial y_{l+1}}, ..., \frac{\partial}{\partial x_n}, \frac{\partial}{\partial y_n} \right). \tag{5.13}$$

The natural orientation for $A \cap B$ is given by

$$\left(\frac{\partial}{\partial x_{l+1}}, \frac{\partial}{\partial y_{l+1}}, ..., \frac{\partial}{\partial x_{n-k}}, \frac{\partial}{\partial y_{n-k}} \right). \tag{5.14}$$

We these orientations, we have $A \cdot B = B \cdot A = A \cap B$. Basically, this proposition follows from the fact that the real dimension of a complex object is even. □

5.2 Schubert calculus

Our goal in this section is to derive the Schubert calculus for projective space and the Grassmannian $G_2(\mathbb{C}^4)$. Our approach in both cases is to find a cell decomposition of the space, and then to take as generators for the homology the closure of these cells. It turns out that these are analytic cycles called *Schubert cycles*. The cohomology ring is generated by their Poincare duals. We find the ring structure by using the intersection interpretation of the product.

The Schubert calculus has been derived in general for all Grassmannians, see [14], but since we will only actually need it for $G_2(\mathbb{C}^4)$ we will derive it directly.

5.2.1 Schubert calculus for $\mathbb{CP}^n$

Projective space has a cell decomposition

$$\mathbb{CP}^n = \mathbb{C}^n \sqcup \mathbb{C}^{n-1} \sqcup \cdots \sqcup \mathbb{C}^0. \tag{5.15}$$

These cells are only even dimensional, so the boundary maps are all zero and the closures $\sigma_k = \overline{\mathbb{C}^{n-k}} \sim \mathbb{CP}^{n-k}$ give cycles which generate the homology groups $H_{2[n-k]}(\mathbb{CP}^n, \mathbb{Z})$. We will use the same symbol to refer to its Poincare dual, $\sigma_k \in H^{2k}(\mathbb{CP}^n, \mathbb{Z})$. These are analytic cycles since they are the zero set of k linear equations.

The cycle σ_0 represents the whole space so its intersection with any other cycle is just the cycle itself. In this way $\sigma_0 = 1$ in the cohomology ring. The cycle σ_1 represents a hyperplane (projective hyperplane). Next, σ_2 is the intersection of two hyperplanes, so $\sigma_1 \cdot \sigma_1 = \sigma_2$. Continuing in this way, we have $(\sigma_1)^k = \sigma_k$.

Proposition 5.4 (Cohomology ring of $\mathbb{CP}^n$)**.** *The cohomology ring of $\mathbb{CP}^n$ is generated by σ_1 which represents the class of a hyperplane. The only relation* *is* $(\sigma_1)^{n+1} = 0$.

5.2.2 Schubert calculus for $G_2(\mathbb{C}^4)$

Let $G_2(\mathbb{C}^4)$ be the Grassmannian of 2-planes in $\mathbb{C}^4$. Any plane is determined by two linearly independent vectors in $\mathbb{C}^4$. We can make a matrix with two row vectors

$$\begin{pmatrix} v_{11} & v_{12} & v_{13} & v_{14} \\ v_{21} & v_{22} & v_{23} & v_{24} \end{pmatrix}. \tag{5.16}$$

However, this matrix representation is not unique. We reduce our matrix, and generically we expect it to take the form

$$W_{0,0} = \begin{pmatrix} * & * & 1 & 0 \\ * & * & 0 & 1 \end{pmatrix}. \tag{5.17}$$

There are, however, 5 other possibilities:

$$W_{1,0} = \begin{pmatrix} * & 1 & 0 & 0 \\ * & 0 & * & 1 \end{pmatrix}, \tag{5.18}$$

$$W_{1,1} = \begin{pmatrix} * & 1 & 0 & 0 \\ * & 0 & 1 & 0 \end{pmatrix}, W_{2,0} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & * & * & 1 \end{pmatrix}, \tag{5.19}$$

$$W_{2,1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & * & 1 & 0 \end{pmatrix}, \tag{5.20}$$

$$W_{2,2} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}. \tag{5.21}$$

The $W_{i,j}$'s are all disjoint and each one is homeomorphic to $\mathbb{C}^{4-[i+j]}$. This gives a cell decomposition

$$G_2\mathbb{C}^4 = W_{0,0} \sqcup W_{1,0} \sqcup W_{1,1} \sqcup W_{2,0} \sqcup W_{2,1} \sqcup W_{2,2}. \tag{5.22}$$

Since there are only cells in even real dimension, the boundary maps are zero.

Proposition 5.5. *The integral homology of $G_2(\mathbb{C}^4)$ is freely generated in dimension $2k$ for $k = 0...4$ by the Schubert cycles $\sigma_{i,j} = \overline{W_{i,j}}$ with $i+j = 4-k$. It other dimensions, the homology is zero.*

An alternative way to describe the Schubert cycles is as follows. Let V_i be the span of the standard basis vectors $e_1, ..., e_i$, and consider the flag $V = (V_1 \subset \cdots \subset V_4)$. The Schubert cycles can be defined as

$$\sigma_{i,j}(V) = \{\Lambda : \dim(\Lambda \cap V_{3-i}) \geq 1, \dim(\Lambda \cap V_{4-j}) \geq 2\}\,. \tag{5.23}$$

For a general Grassmannian $G_k(\mathbb{C}^n)$, we have Schubert cycles σ_a where $a = (a_1, ..., a_k)$ is a non increasing set of integers between 0 and $n-k$, and a flag $V = (V_1, ..., V_n)$. The cycles are defined as

$$\sigma_a(V) = \{\Lambda : \dim(\Lambda \cap V_{n-k+i-a_i}) \geq i\}. \tag{5.24}$$

For any other flag V', there exists a $g \in GL(\mathbb{C}^4)$ such that $gV' = V$. Furthermore, $GL(\mathbb{C}^4)$ is connected and we can continuously transform V' to V, hence $\sigma_{i,j}(V) \sim \sigma_{i,j}(V')$. In other words, the class of the Schubert cycle is completely determined by the subscripts, see [14, 4.].

We can gain some intuition about these cycles by considering $G_2(\mathbb{C}^4)$ as the set of lines in $\mathbb{C}P^3$. We fix a flag $p \subset l_0 \subset h$, where p is a point, l_0 a line, and h a hyperplane. Then the Schubert cycles are

$$\begin{aligned}
\sigma_{0,0} &= \{l\} = G_2(\mathbb{C}^4),\\
\sigma_{1,0} &= \{l : l \cap l_0 \neq \emptyset\},\\
\sigma_{1,1} &= \{l : l \in h\},\\
\sigma_{2,0} &= \{l : l \cap l_0 \neq \emptyset\},\\
\sigma_{2,1} &= \{l : p \in l \subset h\},\\
\sigma_{2,2} &= \{l_0\}.
\end{aligned} \tag{5.25}$$

Proposition 5.6. *The cohomology ring of $G_2(\mathbb{C}^4)$ has the following relations:*

$$\begin{aligned} \sigma_{1,0}^2 &= \sigma_{1,1} + \sigma_{2,0}, \\ \sigma_1\sigma_{1,1} &= \sigma_{1,0}\sigma_2 = \sigma_{2,1}, \\ \sigma_1\sigma_{2,1} &= \sigma_{1,1}^2 = \sigma_2^2 = \sigma_{2,2}, \\ \sigma_{1,1}\sigma_{2,0} &= 0. \end{aligned} \tag{5.26}$$

By considering the cycles as sets of lines in projective three space, calculating the intersections is rather straightforward in most of the cases. In order to see the generic intersection behaviour we will consider two flags $V = (p \in l_0 \subset h)$ and $V' = (p' \in l_0 \subset h')$, and denote $\sigma_{i,j}(V)$ and $\sigma_{i,j}(V')$ as $\sigma_{i,j}$ and $\sigma'_{i,j}$ respectively. There is a theorem due to Kleiman which tells us that these different flags are generically transverse, [9, 4.2]. Although we defined them from a topologically point of view, it turns out that Schubert cycles are analytic cycles. Grassmannians can be embedded in a higher dimensional projective space (the Plücker embedding), and the Schubert cycles are intersections of this embedding with linear subspaces, [14]. We can then just consider the set theoretic intersection and we know the sign must be positive from the positivity of intersecting analytic cycles in proposition 5.3.

Proof. First of all, $\sigma_{0,0} = 1$ in the ring since intersecting a cycle with the whole space gives you the cycle again. Next we look at cycles in complementary dimensions:

$$\begin{aligned}
\sigma_{1,0} \cdot \sigma_{2,1}^{'} &= \{l : l \cap l_0 \neq \emptyset, p^{'} \in l \subset h^{'}\} \\
&= \{\text{the unique line passing through } p^{'} \text{ and } l_0 \cap h'\} \\
&\sim \sigma_{2,2}, \\
\sigma_{2,0} \cdot \sigma_{2,0}^{'} &= \{l : p \in l, p^{'} \in l\} \\
&= \{\text{the unique line passing through } p \text{ and } p^{'}\} \\
&\sim \sigma_{2,2}, \\
\sigma_{1,1} \cdot \sigma_{1,1}^{'} &= \{l : l \subset h, l \subset h'\} \\
&= \{\text{the unique line } h \cap h'\} \\
&\sim \sigma_{2,2}.
\end{aligned} \tag{5.27}$$

For the remaining complementary dimension product, we have $\sigma_{2,0} \cdot \sigma_{1,1}^{'} = 0$ since their intersection is the set of lines passing through a point p and contained in a plane h', but generically p is not in h' so this is empty.

We can use the same technique for two other products as well:

$$\begin{aligned}
\sigma_{1,0} \cdot \sigma_{1,1}^{'} &= \{l : l \cap l_0 \neq \emptyset, l \subset h'\} \\
&= \{l : l_0 \cap h' \in l \subset h'\} \\
&\sim \sigma_{2,1}, \\
\sigma_{1,0} \cdot \sigma_{2,0}^{'} &= \{l : p' \in l, l \cap l_0 \neq \emptyset\} \\
&= \{l : p' \in l \subset \text{ the plane } l_0 p'\} \\
&\sim \sigma_{2,1}.
\end{aligned} \tag{5.28}$$

Unfortunately, this approach is not helpful for computing $(\sigma_{1,0})^2$. However, we know purely by the dimension that $(\sigma_{1,0})^2 = \alpha\sigma_{1,1} + \beta\sigma_{2,0}$ for some α, β. We can then multiply this with each of $\sigma_{1,1}$ and $\sigma_{2,0}$ and use our previous results to find the coefficients. On the one hand,

$$\begin{aligned}\sigma_{1,1} \cdot \sigma_{1,0}^2 &= \alpha\sigma_{1,1}\sigma_{1,0} + \beta\sigma_{1,1}\sigma_{2,0} \\ &= \alpha\sigma_{2,2}\end{aligned} \tag{5.29}$$

On the other hand,

$$\sigma_{1,1} \cdot \sigma_{1,0}^2 = (\sigma_{1,1} \cdot \sigma_{1,0}) \cdot \sigma_{1,0} = \sigma_{2,1} \cdot \sigma_{1,0} = \sigma_{2,2} \tag{5.30}$$

so $\alpha = 1$. By the same argument, this time multiplying by $\sigma_{2,0}$, we find $\beta = 1$ as well. □

5.3 Euler class

The Euler class of a rank k oriented real vector bundle $E \to X$ is an element $e(E) \in H^k(X, \mathbb{Z})$. We are primarily interested in this class because it can tell us something about the zero locus of sections.

For the precise construction, we refer to [2, 18], but the idea is the following: The base manifold X is naturally included in the total space E as the zero section, and the image of X under a generic section gives another manifold S. Assume that X is compact and oriented (and hence so is S), then X and S are cycles and we can take their intersection $[X] \cdot [S]$ in E. This intersection is contained in X, and the Euler class is the Poincare dual in X of this intersection. In fact, S is homotopic to X so the Euler class is Poincare dual to the self intersection $[X] \cdot [X]$.

When the rank of the bundle is the same as the dimensional of the manifold, we can evaluate the euler class on the homology class of the manifold, $e(E)([X])$. We will call this the Euler number, and it is nothing but the intersection number $\#(X \cdot X)$. If the intersection is transverse, there is nice way to calculate the intersection index:

Proposition 5.7 (Euler index)**.** *Let p be a point of transverse intersection of* *the section s with the zero section. The intersection index $I_p(s)$ is $+1$ or -1 according to whether the determinant of the Jacobian matrix ds is positive or negative.*

Proof. Let $x_1, ..., x_n$ be local coordinates around p, and $e_1, ..., e_n$ a basis for the local trivialization such that $\frac{\partial}{\partial x_1}, ..., \frac{\partial}{\partial x_n}, e_1, ..., e_n$ is positively oriented in $T_pX \oplus T_p\mathbb{R}^n \simeq T_pE$. Let S be the image of X under s. Write $s = s_ie_1 + ... + s_ne_n$ where the s_i's are functions of $x_1, ..., x_n$. The tangent space T_pS has an oriented basis $\frac{\partial s}{\partial x_1}, ..., \frac{\partial s}{\partial x_1}$. The change of basis matrix from e_i to the $\frac{\partial s}{\partial x_i}$ is just the Jacobian matrix ds given by

$$ds := \begin{pmatrix} \frac{\partial s_1}{\partial x_1} & \cdots & \frac{\partial s_1}{\partial x_n} \\ \vdots & & \vdots \\ \frac{\partial s_n}{\partial x_1} & \cdots & \frac{\partial s_1}{\partial x_n} \end{pmatrix}, \tag{5.31}$$

the determinant of which is positive or negative according to whether ds is orientation preserving or reversing. □

The intersection index of a zero is referred to as the Euler index. The following proposition follows from our intersection theory:

Proposition 5.8 (See, for example, [2, Theorem 11.17])**.** *Let $E \to X$ be an oriented rank n real vector bundle over a compact oriented n-dimensional manifold. Let s be a section of E with isolated zeros z_i. The Euler number is equal to the sum of the indices of the zeros,*

$$e(E)([X]) := \int_X e(E) = \sum_i I_{z_i}(s) \tag{5.32}$$

An Euler index can also be defined for non transverse intersections as long as the zeros are isolated, see [17, p.35 Poincare-Hopf theorem]. In this case, the index can take on other values, but proposition 5.8 still holds with these indices.

The following are some basic properties of the Euler class (see [18, p. 98]):

- $e(f^*E) = f^*e(E)$.
- $e(E \oplus F) = e(E) \wedge e(F)$.
- $e(-E) = -e(E)$ where $-E$ is the vector bundle E with opposite orientation.
- If E has a nowhere zero section, $e(E) = 0$.

5.4 Chern classes

The Chern classes are defined for complex vector bundles. For a bundle E of rank k, there is an i-th Chern class

$$c_i(E) \in H^{2i}(X, \mathbb{Z}). \tag{5.33}$$

The 0-th Chern class is always 1. The k-th Chern class is equal to the Euler class of the underlying rank $2k$ real bundle. Note that the real underlying bundle is always orientable. If $e_1, ..., e_k$ is a complex basis, $e_1, ie_1, ..., e_k, ie_k$ gives an oriented real basis. Furthermore, for a real bundle F of rank k, $F \otimes \mathbb{C}$ is a complex bundle of rank k, and

$$c_k(F \otimes \mathbb{C}) = e(F \otimes \mathbb{C}) = (-1)^r e(F \oplus F) = (-1)^r e(E)^2. \tag{5.34}$$

where $r = k(k-1)/2$ accounts for the difference in orientation. For $i > k$, $c_i(E) = 0$. We call the sum

$$c(E) = 1 + c_1(E) + ... + c_k(E) \tag{5.35}$$

the total Chern class.

For bundles over complex manifolds with enough sections, we have the following characterization of Chern classes, which we will use later to compute the Chern classes of the dual universal subbundle. As we already mentioned, the top Chern class $c_k(E)$ is equal to the Euler class of the underlying real bundle, and so it is Poincare dual to

$$Z(s_1) = \{x \in X : s_1(x) = 0\}. \tag{5.36}$$

Where s_1 is a section transverse to the zero section. If we have another section s_2, the class c_{k-1} is Poincare dual to

$$Z(s_1 \wedge s_2) = \{x \in X : s_1(x) \wedge s_2(x) = 0\}, \tag{5.37}$$

the locus where s_1 and s_2 fail to be linearly independent. To see that this should have (complex) codimension $k-1$, consider the bundle $E/\langle s_1 \rangle$ of rank $k-1$ over $X \setminus Z(s_1)$, and let s_2' be the section in E' induced by s_2. Then we see that $Z(s_1 \wedge s_2)$ is the closure of $Z(s_1')$. We say that a collection $s_1, ..., s_i$ of sections is transverse if $Z_i = Z(s_1 \wedge ... \wedge s_i)$ indeed has codimension $k-i+1$, and in this case c_{k-i+1} is the poincare dual of Z_i, the locus where these sections fail to be linearly independent.

The following properties of Chern classes are important for our subsequent calculations (see [9, 5.3]).

- The Chern classes commute with pullbacks. Let $f : X \to Y$,

$$c(f^*E) = f^*c(E). \tag{5.38}$$

- For an exact sequence of vector bundles

$$0 \longrightarrow E \longrightarrow F \longrightarrow G \longrightarrow 0, \tag{5.39}$$

the total Chern classes are related by $c(F) = c(E)c(G)$. In particular, we have the *Whitney sum formula*

$$c(E \oplus G) = c(E)c(G). \tag{5.40}$$

- If A and B are line bundles, then the line bundle $A \otimes B$ has total Chern class

$$c(A \otimes B) = 1 + c_1(A) + c_1(B). \tag{5.41}$$

An invaluable tool for performing calculations is the splitting principle. In practical terms, this allows us to compute the Chern class of a bundle as if it were a direct sum of line bundles.

Lemma 5.9. *(Splitting Principle) For a vector bundle E over a smooth variety X there exists a smooth variety Y and a morphism $f : Y \to X$ with the properties:*

1. *The induced map $f^* : H^*(X) \to H^*(Y)$ is injective*
2. *The pullback bundle f^*E splits into a direct sum of line bundles $E_1, ..., E_k$.*

The splitting principle is used in the following way: Suppose we have a vector bundle E or rank 2, whose Chern classes we know, but we would like to compute the Chern classes of some construction involving this bundle, say, the second symmetric power S^2E. We proceed as follows. Let f be the morphism described in the theorem. Then

$$f^*E = A \oplus B. \tag{5.42}$$

For some line bundles A and B. Taking the total Chern class gives us

$$1 + f^*c_1(E) + f^*c_2(E) = (1 + a)(1 + b), \tag{5.43}$$

where $a = c_1(A), b = c_1(B)$, and we have commuted the pullback. From this we obtain the relations

$$\begin{aligned} f^*c_1(E) &= a+b, \\ f^*c_2(E) &= ab. \end{aligned} \tag{5.44}$$

There is an isomorphism $S^2(A\oplus B) = (A\otimes A)\oplus(A\otimes B)\oplus(B\otimes B)$, so using the sum and tensor product properties,

$$\begin{aligned} f^*c(S^2E) &= c((A\otimes A)\oplus(A\otimes B)\oplus(B\otimes B)) \\ &= (1+2a)(1+a+b)(1+2b) \\ &= (1+a+b))(1+2a+2b+4ab) \\ &= (1+f^*c_1(E))(1+2f^*c_1(E)+f^*c_2(E)). \end{aligned} \tag{5.45}$$

By expanding this out, and using the fact that f^* is injective, we conclude

$$c(S^2E) = 1+3c_1(E)+[2c_1(E)^2+c_2(E)]+c_1(E)c_2(E). \tag{5.46}$$

In practice, it is not necessary to make reference to the morphism f, rather we can simply assume that our vector bundle splits.

To finish this section, we also include another lemma that will be useful in calculating characteristic classes. First, a definition:

Definition 5.10 (Conjugate Bundle)**.** *If E is a complex vector bundle, then the conjugate bundle $\overline{E}$ is defined to be the complex vector bundle with the same underlying real vector bundle, but with the opposite complex structure. Thus, the identity map $f : E \to \overline{E}$ is conjugate linear,*

$$f(\alpha x) = \overline{\alpha} f(x). \tag{5.47}$$

Lemma 5.11 ([18, Lemma 15.4])**.** *For any complex vector bundle E, let $E_{\mathbb{R}}$ be the underlying real vector bundle. The complexification $E_{\mathbb{R}}\otimes\mathbb{C}$ is canonically isomorphic to $E\oplus\overline{E}$.*

Proof. For a real vector space V, $V\otimes\mathbb{C}$ can be identified with the direct sum

$V \oplus V$. The complex structure on $V \oplus V$ is given by $J(x, y) = (-y, x)$.

Let V $= F_{\mathbb{R}}$ where F is a typical fibre of a complex vector bundle. The map

$$g : x \mapsto (x, -ix) \tag{5.48}$$

from F to $V \oplus V$ is complex linear. In other words, $g(ix) = J(g(x))$. Similarly, the map

$$h : x \mapsto (x, ix) \tag{5.49}$$

is conjugate linear. Every point (x,y) of $V \oplus V = F_{\mathbb{R}} \otimes \mathbb{C}$ can be written uniquely as a sum

$$g\left(\frac{x + iy}{2}\right) + h\left(\frac{x - iy}{2}\right). \tag{5.50}$$

Therefore, the map $F \oplus \overline{F} \to V \oplus V = F_{\mathbb{R}} \otimes \mathbb{C}$ given by

$$(x, \overline{y}) \mapsto \frac{1}{2}\left(g(x) + h(y)\right) \tag{5.51}$$

is a complex linear isomorphism (the inverse of $(x, y) \in F_{\mathbb{R}} \otimes \mathbb{C}$ is given by $(x, \overline{\overline{y}}) \in F \oplus \overline{F}$). To check complex linearity, let $\lambda \in \mathbb{C}$, $(x, \overline{y}) \in F \oplus \overline{F}$. Notice that $\lambda(x, \overline{y}) = (\lambda x, \overline{\overline{\lambda} y})$, and this maps to

$$\frac{1}{2}\left(g(\lambda x) + h(\overline{\lambda} y)\right) = \lambda \frac{1}{2}\left(g(x) + h(y)\right) \tag{5.52}$$

by the conjugate linearity of h. □

5.4.1 Chern classes of the dual universal subbundle

Let U be the universal subbundle (also known as the tautological bundle) over $G_k(\mathbb{C}^n)$,

$$U = \{(p, v) \subset G_k(\mathbb{C}^n) \times \mathbb{C}^n : v \in p\}. \tag{5.53}$$

The dual bundle $U^\vee$ has nontrivial global sections given by the restriction of linear forms in $\mathbb{C}^{n\vee}$ to each plane p. We will compute the Chern classes by finding where a generic collection of sections are linearly dependent.

Let $l_1, ..., l_m \in \mathbb{C}^{n\vee}$. Let $v_1, ..., v_k$ be a basis for the plane p. Let H_i be the hyperplane in $\mathbb{C}^n$ representing the kernel of l_i. Consider the m by k matrix M having entries

$$M_{ij} = l_i(v_j). \tag{5.54}$$

The sections are linearly dependent on p if the null space of this matrix is non trivial. Recall the rank-nullity theorem : $rank(M) + nullity(M) = k$. Said another way, the sections are linearly dependent if at p if $rank(M) \leq m - 1$. Notice also that the null space is equal to $p \cap H_1 \cap ... \cap H_m$, and so

$$dim(p \cap H_1 \cap ... \cap H_m) \geq k - m + 1. \tag{5.55}$$

Lemma 5.12. *The set of planes satisfying equation 5.55, that is, $c_m(U^\vee)$, is the Schubert cycle $\sigma_{1,...,1}$ ($k - m + 1$ subscripts).*

Proof. Recall the definition of Schubert cycles in terms of a flag $V = (V_1, ..., V_n)$:

$$\sigma_a(V) = \{p : dim(p \cap V_{n-k+i-a_i}) \geq i\}, \tag{5.56}$$

where i runs from 1 to k and the a_i are non-increasing. The intersection of hyperplanes $H_1 \cap ... \cap H_m$ is V_{n-m}. When $i = k - m + 1$, we get

$$dim(p \cap V_{n-m+1-a_{k-m+1}}) \geq k - m + 1 \tag{5.57}$$

In order for 5.55 to hold, we must have $a_{k-m+1} = 1$. The largest Schubert cycle satisfying this condition is $\sigma_{1,...,1}$ where there are $k - m + 1$ subscripts (every other Schubert cycle satisfying the condition is contained in this one).

□

Proposition 5.13. *The total Chern class of $U^\vee$ is*

$$c(U^\vee) = 1 + \sigma_1 + ... + \sigma_{1,...,1} \text{ (up to } k \text{ subscripts).} \tag{5.58}$$

Proof. Add up the cycles from the lemma as m ranges from 1 to k. □

Chapter 6

Real enumeration with characteristic classes

Our goal in this Chapter is to enumerate real lines using characteristic classes. We will first construct a complex vector bundle which allows us to enumerate the complex lines by computing the top Chern class. The real invariant number 3 can be found from the Euler class of the real analogue of this bundle. We will compute this Euler class way by embedding the real Grassmannian in projective space, using the *polar correspondence*. In the last section we give Segre's definition of hyperbolic and elliptic lines and show that this notion corresponds to the Euler index of a zero in the bundle.

6.1 Complex lines

The lines in $\mathbb{CP}^3$ are parametrized by the grassmannian of planes in $\mathbb{C}^4$, $G_2(\mathbb{C}^4)$. That is, any line in $\mathbb{CP}^3$ corresponds to a plane in $\mathbb{C}^4$ and, conversely, any plane in $\mathbb{C}^4$ corresponds to a line in $\mathbb{CP}^3$ so that in fact this parametrization is one-to-one.

Let U be the universal subbundle (also known as the tautological bundle) over $G_2(\mathbb{C}^4)$,

$$U = \{(p, v) \subset G_2(\mathbb{C}^4) \times \mathbb{C}^4 : v \in p\}. \tag{6.1}$$

Now let us consider the third symmetric power of the dual bundle to U, $S^3(U^\vee)$. A homogeneous polynomial $f(x, y, u, v)$ is a section of $S^3(\mathbb{C}^{4\vee})$ and naturally induces a section s_f of $S^3(U^\vee)$ by restriction to each fibre. That is, let $x(s,t), ..., v(s,t)$ be a parametrization for the plane p, then

$$s_f(p) = f(x(s,t), ..., v(s,t)) = (...)s^3 + (...)s^2t + (...)st^2 + (...)t^3. \tag{6.2}$$

The symmetric forms s^3, s^2t, st^2, t^3 form a basis for the fibre. We have a rank 4 bundle over a 4 dimensional manifold.

Lemma 6.1. *Let a cubic surface be given by $f = 0$. This surface contains a line l if and only if $s_f(p) = 0$ where p is the plane in $\mathbb{C}^4$ induced by l.*

Proof. If the line l lies on the surface $f = 0$, then any parametrization of l plugged into the cubic f will satisfy $f = 0$ identically. As mentioned before, the line l corresponds to a plane p, and this (projective) parametrization of l gives a parametrization of the plane p. The condition that $s_f(p) = 0$ is equivalent to the condition that the parametrization identically satisfies $f = 0$. □

Next we will compute the fourth Chern class of $S^3(U^\vee)$ to find out how many zeros we should expect from a section s_f. By the previous lemma, this will tell us how many lines will lie on the surface $f = 0$.

Using the splitting principle we can assume $U^\vee$ splits into a direct sum of line bundles $A \oplus B$ with respective total Chern classes $1 + a, 1 + b$. Setting $c(U^\vee) = c(A)c(B)$, we get the relation

$$1 + \sigma_1 + \sigma_{1,1} = 1 + (a + b) + ab, \tag{6.3}$$

or in other words $\sigma_1 = a + b$, $\sigma_{1,1} = ab$. The symmetric bundle $S^3(A \oplus B)$ is isomorphic to $A^{\otimes 3} \oplus (A^{\otimes 2} \otimes B) \oplus (A \otimes B^{\otimes 2}) \oplus B^{\otimes 3}$, the total Chern class of which is

$$\begin{aligned}
&(1+3a)(1+2a+b)(1+a+2b)(1+3b)\\
=&(1+9ab)(1+3(a+b)+2(a+b)^2+ab)\\
=&(1+9\sigma_{1,1})(1+3\sigma_1+2\sigma_1^2+\sigma_{1,1}) \qquad (6.4)\\
=&(1+9\sigma_{1,1})(1+3\sigma_1+3\sigma_{1,1}+2\sigma_2)\\
=&1+3\sigma_1+12\sigma_{1,1}+2\sigma_2+27\sigma_{2,1}+27\sigma_{2,2}.
\end{aligned}$$

So we see that $c_4(S^3(\mathbb{C}^{4\vee})) = 27\sigma_{2,2}$, or 27 times the class of a point. We are assuming our cubic surface is smooth, so that the lines have multiplicity one (see proposition 2.1), or in terms of the bundle, all the zeros are isolated. Therefore, a smooth cubic surface will have 27 lines.

6.2 Real lines

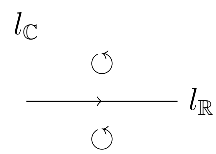
We have a real Grassmannian, $G_2(\mathbb{R}^4)$ parametrizing lines in $\mathbb{RP}^3$. Once again, we have the universal subbundle:

$$U = \{(p, v) \subset G_2(\mathbb{R}^4) \times \mathbb{R}^4 : v \in p\}. \qquad (6.5)$$

We have sections s_f in $S^3(\mathbb{R}^{4\vee})$ induced by homogeneous polynomials. We would like to know how many zeros a section of $S^3(\mathbb{R}^{4\vee})$ by computing its Euler class. We will do this by identifying the Grassmmanian of *oriented* planes, $\widetilde{G}_2(\mathbb{R}^{n+2})$ with a subvariety of $\mathbb{CP}^{n+1}$. This is called the polar correspondence, [10], which we will describe in what follows.

6.2.1 Polar correspondence

Let Q^n be the quadric in $\mathbb{CP}^{n+1}$ defined by $x_0^2 + ... + x_{n+1}^2 = 0$. Note that this quadric has no real points. A plane $p \subset \mathbb{R}^{n+2}$ defines an oriented line in $\mathbb{R}P^{n+1}$. This real line $l_\mathbb{R}$ splits its complexification $l_\mathbb{C}$ into two halves. One of these halves bounds the orientation of the real line (in the following picture, the top half bounds the orientation).



The complexification $l_{\mathbb{C}} \subset \mathbb{CP}^{n+1}$ intersects Q^n in two complex conjugate points. Let q be the point the half which bounds the orientation of $l_{\mathbb{R}}$. The map which sends the plane $p \subset \mathbb{R}^{n+2}$ to the the point $q \in Q^n \subset \mathbb{CP}^{n+1}$ is a diffeomorphism

$$\phi : \widetilde{G}_2(\mathbb{R}^{n+2}) \to Q^n. \tag{6.6}$$

The inverse maps looks like the following: A point $q \in Q^n$ is represented by some vector $v_q \in \mathbb{C}^{n+2}$, and hence by a line $\mathbb{C}v_q$. Project this line to a real plane by the map $\frac{1}{2}(v + \overline{v})$. Note that this map is never zero because a purely imaginary vector in $\mathbb{C}v_q$ implies q is projectively equivalent to a real point (multiply by i), but Q^n has no real points. Finally, to orient the plane we pushforward the orientation of the complex line.

6.2.2 Euler class computation

Now let τ be the restriction to Q^n of the tautological line bundle over $\mathbb{CP}^{n+1}$ (where the fibre over a line in $\mathbb{CP}^{n+1}$ is just the line itself in $\mathbb{C}^{n+1}$). The polar correspondence identifies $\widetilde{U}^\vee$ with the real 2-bundle underlying $L = \phi^\star \tau^\vee$. Therefore, we have $\widetilde{U}^\vee \otimes \mathbb{C} = L \oplus \overline{L}$ (see 5.11). Taking the symmetric powers, we have

$$\begin{aligned} S^k(\widetilde{U}^\vee) \otimes \mathbb{C} &= S^k(\widetilde{U}^\vee \otimes \mathbb{C}) \\ &= S^k(L \otimes \overline{L}) \\ &= L^{\otimes k} \oplus L^{\otimes(k-1)}\overline{L} \oplus \cdots \oplus \overline{L}^{\otimes k}. \end{aligned} \tag{6.7}$$

The total Chern class is therefore:

$$\begin{aligned}&(1+kc_1(L))(1+[k-1]c_1(L)-c_1(L))\cdots(1-kc_1(L))\\=&(1+kc_1(L))(1+[k-2]c_1(L))\cdots(1-kc_1(L)).\end{aligned} \tag{6.8}$$

The Euler class of $S^k(\widetilde{U}^\vee)$ will square to this, see 5.34 (modulo $(-1)^{(k+1)k/2}$ where $k+1$ is the rank of the bundle). Now we take the specific case where $k=3$. The total Chern class is

$$\begin{aligned}c(S^3(\widetilde{U}^\vee)\otimes\mathbb{C}) &= (1+3c_1(L))(1+c_1(L))(1-3c_1(L)(1-c_1(L))\\&=(1-9c_1(L)^2)(1-c_1(L)^2)\end{aligned} \tag{6.9}$$

The top Chern class is $9c_1(L)^4$, therefore $e(S^3(\widetilde{U}^\vee))=3c_1(L)^2$. Now, $c_1(L)$ is an element of $H^2(G_2(\mathbb{R}^{n+2}),\mathbb{Z})$. We want to evaluate this on the fundamental homology class $[G_2(\mathbb{R}^{n+2})]$. But we have $c_1(L)=c_1(\phi^*\tau^\vee)=\phi^*c_1(\tau^\vee)$. If we push everything forward by the polar correspondence ϕ we have

$$c_1(L)[G_2(\mathbb{R}^{n+2})]=c_1(\tau^\vee)[Q^n]. \tag{6.10}$$

Now $c_1(\tau^\vee)$ is just the hyperplane class σ_1. Now taking $n=2$, σ_1^2 is the class of a line in in $\mathbb{CP}^{2+1}$. A line will intersect the quadric Q^2 in two points, so $\sigma_1^2[Q^2]=2$. In other words,

$$\begin{aligned}e(S^k(\widetilde{U}^\vee))[\widetilde{G}_2(\mathbb{R}^4)] &= 3c_1(\tau^\vee)^2[Q^2]\\&=3\sigma_1^2[Q^2]\\&=3\cdot 2.\end{aligned} \tag{6.11}$$

Since $\widetilde{G}_2(\mathbb{R}^4)$ is a double cover of $G_2(\mathbb{R}^4)$, we have

$$e(S^k(U^\vee))[G_2(\mathbb{R}^4)]=3. \tag{6.12}$$

6.3 Hyperbolic and elliptic lines

Consider a real line on the cubic surface. The tangent plane to any point on this line will intersect the surface in the line itself and a further residual conic (perhaps another pair of lines). This residual conic will intersect the line in two points, one of which being the point where we took the tangent plane from. We define an involution on the line by exchanging these two points of intersection. The fixed points of this involution are called parabolic points [26]. It is possible that the parabolic points only exist in the complexification. The real line is called a hyperbolic line if the involution has two real parabolic points. The real line is called an elliptic line if it has a pair of complex conjugate parabolic points.

More concretely, choose projective coordinates x, y, u, v on $\mathbb{RP}^3$ so that the line l is given by $x = y = 0$. Then the defining polynomial of the surface has the form

$$f = u^2L_{11} + 2uvL_{12} + v^2L_{22} + uQ_1 + vQ_2 + C, \tag{6.13}$$

where L_{ij}, Q_i and C are of degree one, two, and three in x, y. Any plane containing l is given by the equation $bx - ay = 0$ for projective coordinates a, b. The pairs of involution for the plane (the intersection of l and residual conic) are given by the roots of the projective quadratic in u, v

$$u^2L_{11} - 2uvL_{12} + v^2L_{22}, \tag{6.14}$$

with L_{ij} being evaluated at $x = a, y = b$. A parabolic point is given by the unique root of this quadratic when its discriminant $L_{12}^2 - L_{11}L_{22}$ is zero. This discriminant is a quadratic form in $x = a$ and $y = b$. Let us call it F. If F is indefinite, there are two real values of $[a : b]$ which make F zero, each of these planes give a real parabolic point by plugging into 6.14 and finding the root. If F is definite, there are two complex conjugate values of $[a : b]$

making F zero, which give the complex conjugate parabolic points. If we let $L_{ij} = l_{ij}x + m_{ij}y$, then the F is explicitly

$$(l_{12}^2 - l_{11}l_{22})x^2 + (2l_{12}m_{12} - l_{11}m_{22} - l_{22}m_{11})xy + (m_{12}^2 - m_{11}l_{22})y^2. \quad (6.15)$$

Write this as $F = Ax^2+Bxy+Cy^2$. F is definite (l is elliptic) if $B^2-4AC < 0$ and indefinite (l hyperbolic) if $B^2 - 4AC > 0$.

Now we compute the index of a zero in in a section of $S^3(U^\vee)$. A projective line $l' \in G_2(\mathbb{R}^4)$ in a neighbourhood of l intersects the projective planes $v = 0$ and $u = 0$ at some points $[x_1 : y_1 : 1 : 0]$ and $[x_2 : y_2 : 0 : 1]$. The values x_1, x_2, y_1, y_2 give local coordinates around l in the Grassmannian. We can parametrize the plane in $\mathbb{R}^4$ corresponding to l' as $(u, v) \mapsto u(x_1, y_1, 1, 0) + v(x_2, y_2, 0, 1)$. The value of a section $s_f(l')$ is defined by substituting $x = ux_1 + v$ and $y = uy_1 + vy_2$ into the polynomial f. We have a frame u^3, u^2v, uv^2, v^3 and base coordinates (x_1, x_2, x_3, x_4). The Jacobi matrix of s_f at l ($x_i = y_i = 0$) is

$$ds_f(l) = \begin{pmatrix} l_{11} & 0 & m_{11} & 0 \\ 2l_{12} & l_{11} & 2m_{12} & m_{11} \\ l_{22} & 2l_{12} & m_{22} & 2m_{12} \\ 0 & l_{22} & 0 & m_{22} \end{pmatrix}. \quad (6.16)$$

The determinant, whose sign gives the index $I_l(s_f)$, is

$$l_{11}^2m_{22}^2 - 4l_{11}m_{22}l_{12}m_{12} - 2l_{11}l_{22}m_{11}m_{22} - 4l_{12}l_{22}m_{11}m_{12} + L_{22}^2m_{11}^2 + 4l_{11}l_{22}m_{12}^2 + 4l_{12}^2m_{11}m_{22}. \quad (6.17)$$

This is equal to $B^2 - 4AC$. Therefore, the index $I_l(s_f)$ for a line l is $+1$ for a hyperbolic line and -1 for an elliptic line. There is the issue of the orientation of the bundle which would reverse the signs of the index, but the important thing is that the index corresponds to Segre's notion of hyperbolic and elliptic lines. Provided the Euler number is $+3$ (and not -3), the indices will correspond as we said.

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