Quantization of Complex Tori via Noncommutative Hodge Theory

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Contents

Abstract

In this thesis we use noncommutative Hodge theory to study deformation quantization of Poisson manifolds. The class of torus invariant Poisson structures on noncompact complex tori are analysed in detail. In particular, we define a special class of mixed Hodge structures, called *toric* mixed Hodge structures. Toric mixed Hodge structures are constructed on cohomology rings of Poisson tori with torus invariant Poisson structures and multiparametric quantum tori. We show that a multiparametric quantum torus is determined up to isomorphism by its Hodge structure and use this to give a nonperturbative calculation of the canonical quantization of complex tori.

Résumé

Dans cette thèse nous utilisons la théorie de Hodge noncommutative pour étudier la quantification par déformation des variétés de Poisson. La classe des structures de Poisson sur un tore complexe non-compacte et invariantes par l'action du tore est analysée en détail. En particulier, nous définissons une classe de structures de Hodge mixtes particulières, dites structures de Hodge mixtes toriques. Des structures de Hodge mixtes toriques sont ensuite construites sur les anneaux de cohomologie du tore possédant une structure de Poisson invariante par l'action du tore et du tore quantique multiparamétré. Nous montrons qu'un tore quantique multiparamétré est déterminé à isomorphisme prés par sa structure de Hodge et l'utilisons pour exhiber un calcul non-perturbatif de quantification canonique du tore complexe.

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Selected Notation

$\overset{k}{\cdots}$	occurrence of something k times
S_k	symmetric group on k symbols
$f : V \xrightarrow{\sim} W$	f is an isomorphism from V to W
$k^\times$	group of units in a ring k
$k\langle x^\pm \rangle$	quotient of $k\langle x, x^{-1}\rangle$ by the relations $xx^{-1} = 1$ and $x^{-1}x = 1$
$V^\vee$	linear dual of a vector space V
$f \otimes k$	a morphism f with scalars extended to k
F_∇	curvature of a connection ∇

Introduction

The past 50 years have witnessed a rejuvenation of the interplay between mathematics and theoretical physics. Not only has there been increased interest in understanding the mathematical foundations of physical theories (especially quantum field theories), but ideas from physics have been used to make great strides in mathematical research. A particularly confounding physical process is that of quantization. This is, generally speaking, the process of taking the data of a classical system and producing a theory resembling quantum mechanics or quantum field theory from this data. Mathematically, a classical system is the data of a smooth manifold with a Poisson bracket on its algebra of functions. The observables of classical physics such as position, momenta, and energy are then functions on this manifold. In a quantum system, the observables are no longer functions on a smooth manifold. Instead, the collection of quantum observables forms a noncommutative algebra which reduces to the usual algebra of functions on a manifold as some quantum parameters tend to zero.

To express this more precisely, let X be a smooth manifold with sheaf of smooth functions $\mathcal{O}_{\mathsf{X}}$. A *quantization* of $\mathcal{O}_{\mathsf{X}}$ should be an associative product $\star$, called a *star product*, on $\mathcal{O}_{\mathsf{X}} \otimes \mathbb{R}[[\hbar]]$ giving it the structure of a sheaf of $\mathbb{R}[[\hbar]]$-algebras such that for functions $f, g \in \mathcal{O}_{\mathsf{X}}$, one has

$$\lim_{\hbar \to 0} f \star g = fg.$$

Moreover, it can be shown that

$$\lim_{\hbar \to 0} \frac{f \star g - g \star f}{\hbar} = \{f, g\},$$

where $\{f, g\}$ is the action of a biderivation on (f, g). This biderivation can be shown to satisfy the Jacobi identity, so it must be induced by a Poisson structure. We say that the Poisson bracket $\{\cdot, \cdot\}$ is this star product's *semi-classical limit*. From this we see that a Poisson structure should be viewed as the best linear approximation to the commutative product on $\mathcal{O}_{\mathsf{X}}$ in the space of star products. With these observations, it is natural to ask the following question:

Question 0.0.1. *Given a Poisson manifold* (X, π)*, does there exist a star product on* $\mathcal{O}_{\mathsf{X}}$ *having the Poisson bracket associated to* π *as its semi-classical limit?*

This answer to this question was given by Maxim Kontsevich in 1997 in his paper [Kon03]. Kontsevich proved that the space of Poisson structures close to the zero Poisson structure is canonically isomorphic to the space of star products on $\mathcal{O}_{\mathsf{X}}$ (both taken modulo an appropriate notion of equivalence). A remarkable feature of this proof is that it is constructive: Kontsevich gave an explicit formula for the star product quantizing a Poisson structure π. In fact, this star product formula can be viewed as a perturbative expansion in a 2D topological quantum field theory [CF00]. While explicit, such expansions are incredibly difficult to calculate directly, even for relatively simple Poisson structures. It is natural to ask: is there a more direct way of calculating the quantum algebras which does not use perturbative expansions?

It was proposed by Kontsevich in [Kon08] that one could determine these quantum algebras using noncommutative Hodge theory. Recall that, generally speaking, classical Hodge theory seeks to associate a linear algebraic invariant, called a *Hodge structure*, to a space X. Hodge structures contain a surprising amount of information about the original space. In fact, there are many spaces for which the Hodge structure determines the space up to isomorphism. Theorems proving a space is determined by its Hodge theoretic data are often called *Torelli theorems*. What makes such theorems possible is that Hodge structures can often detect continuous parameters associated to families of spaces, such as the parameter τ associated to an elliptic curve. Kontsevich proposed that the continuous parameters appearing in families of Poisson structures and quantum algebras could be detected by noncommutative analogues of Hodge structures. Instead of working with a formal expansion in $\hbar$, one could then study quantization via this linear algebraic data.

In this thesis, this programme is carried out for families of torus-invariant Poisson structures on noncompact complex tori; i.e. algebraic varieties X isomorphic to $(\mathbb{C}^\times)^n$. We will write G for the rank n torus viewed as a Lie group. In coordinates $(z_1, \ldots, z_n)$ a G-invariant Poisson structure on X has the form

$$\pi = \sum_{1 \leq i < j \leq n} \lambda_{ij} z_i z_j \frac{\partial}{\partial z_i} \wedge \frac{\partial}{\partial z_j}.$$

for some complex numbers $\lambda_{ij} \in \mathbb{C}$. Note that these Poisson structures are *quadratic* in the coordinates z_i, or equivalently, the associated Poisson brackets are of the form

$$\{z_i, z_j\} = \lambda_{ij} z_i z_j.$$

These Poisson structures are extremely difficult to quantize using Kontsevich's formula. In [BPP20] the leading terms of Kontsevich's star product expansion were calculated for these Poisson brackets and (conjecturally) transcendental numbers such as $\zeta(3)^2/\pi^6$ appeared as coefficients. Note, however, that such a Poisson structure is determined up to isomorphism by the skew-symmetric matrix of paramters $\lambda = (\lambda_{ij})_{1 \leq i < j \leq n}$. Following Kontsevich's suggestion it is natural to ask the following questions:

1. Is there a Hodge structure that one can associate to the Poisson variety (X, π) which can detect the parameters λ_{ij}, and
2. Is there a class of associative algebras determined up to isomorphism by an equal number of parameters which are possible quantizations of X? If so, then
3. Is there a Hodge structure one may associate to these algebras which detects these parameters?

The answer to question 2) is *yes*, and such algebras have been studied for many years (though not in the context of Hodge theory). These algebras are called *multiparametric quantum tori*, or simply *quantum tori*, and are of the form

$$A_q = \frac{\mathbb{C}\langle x_1^{\pm}, \ldots, x_n^{\pm} \rangle}{(x_i x_j = q_{ij} x_j x_i)}$$

where $q = (q_{ij})$ is a matrix with entries in $\mathbb{C}^\times$ such that $q_{ij} = q_{ji}^{-1}$. In particular, if one has $q_{ij} = 1$ for all i and j then this is just the algebra of function on $(\mathbb{C}^\times)^n$. Multiparametric quantum tori are reasonable candidates for the quantizations of torus invariant Poisson structures for the following reason: one can make a logarithmic change of coordinates which would transform these Poisson structures to constant Poisson structures, at which point one can apply the standard Moyal-Weyl quantization. Note that if u and v are operators such that $[u, v] = \hbar$, then by changing coordinates to $U = e^u$ and $V = e^v$, one has $UV = e^\hbar VU$ by the Baker-Campbell-Hausdorff formula. As described, this approach is not rigorous, as the logarithmic coordinates are multi-valued when working over $\mathbb{C}$ and Kontsevich's formula will depend on this coordinate change in a non-trivial way. We do not attempt to calculate the quantizations of tori using these coordinate changes, though it would be interesting to know how this relates to our methods.

In [Kon08, Section 1.30], Kontsevich writes:

Statement/Conjecture 0.0.2. *The quantization of a G-invariant Poisson structure π with respect to Kontsevich's formula on a non-compact complex torus X is the multiparametric quantum torus algebra A_q where $q_{ij} = \exp(\hbar\lambda_{ij})$.*

Statement/Conjecture 0.0.2 is our Theorem 4.1.4 and is proven as follows. First, we answer questions 1) and 3) above in the affirmative. That is, we associate canonical Hodge structures to a Poisson structure π on X and to a quantum torus A_q such that these Hodge structures detect the parameters λ and q, respectively. Next, using an extension of Kontsevich's formula known as *cyclic formality* [Wil11], we are able to conclude that if A_q arises as the quantization of π then their Hodge structures are isomorphic. To conclude, we use a Torelli theorem: quantum tori A_q are determined up to isomorphism by their Hodge structures. Let us now describe in more detail the contents of this thesis.

In chapter one, we recall the definitions of Hodge structures, mixed Hodge structures, and variations of mixed Hodge structures and define a subclass of these objects which we call 'toric'. Toric mixed Hodge structures are, in particular, iterated extensions of the Tate structures $\mathbb{Z}(j)$, $j \geq 0$, and from the extension classes we extract a multiplicatively skew-symmetric matrix. We refer to this matrix as the *extension data* of the toric mixed Hodge structure and we will later use extension to detect the parameters described above.

In chapter two, we recall the rudiments of Poisson geometry. An invariant called the *periodic Poisson homology* is defined and shown to be canonically isomorphic to the usual de Rham cohomology. We then recall the definition of the topological K-theory of a space X and compute the topological K-theory of a complex torus. After this, the image of the Chern character in the de Rham cohomology of X is described. While the de Rham cohomology and periodic Poisson homology are isomorphic vector spaces, the lattices given by K-theory vary in families of Poisson structures. We use this varying lattice to construct a toric mixed Hodge structure on the periodic Poisson homology of a Poisson torus. Using this toric mixed Hodge structure, we calculate the extension class described above and show it is determined up to isomorphism by the multiplicatively skew-symmetric matrix $\exp(\lambda) = (\exp(\lambda_{ij}))$.

In chapter three, we turn to noncommutative geometry. We first recall the necessary notions from Hochschild theory: the Hochschild homology, Hochschild cohomology, and the periodic cyclic homology of an associative algebra. We then prove that the periodic cyclic homology of a quantum torus A_q carries a toric mixed Hodge structure. Letting q vary in $(\mathbb{C}^\times)^{\binom{n}{2}}$, we obtain a toric variation of mixed Hodge structures with connection given by the Getzler-Gauss-Manin connection as defined by [Get93]. We use this connection to obtain a lattice in the periodic cyclic homology of A_q by parallel transporing the lattice at the central fibre, which is known. We then calculate the extension data of the toric mixed Hodge structure of a quantum torus A_q. This extension data is shown to be the matrix q itself. Thus, A_q is determined up to isomorphism by its Hodge structure. In a formal neighbourhood of any point, we prove that this lattice agrees with the noncommutative topological K-theory, thus proving a case of the *lattice conjecture* of Blanc [Bla16]. While we are able to prove this in a formal neighbourhood of any quantum torus, we do not know this holds globally. We believe this to be true, and state this as Conjecture 3.3.8.

In chapter 4, we prove Statement/Conjecture 0.0.2 as follows: we first recall several extensions of formality morphisms and prove that these morphisms induce an isomorphism of the toric mixed Hodge structures we have constructed. That is, the toric Hodge structure associated to (X, π) is isomorphic to that obtained from the quantization of π (which we show must be a quantum torus). Combining this statement with the Torelli theorem of chapter 3, we are able to prove Statement/Conjecture 0.0.2.

Conventions

We list here the conventions that will be used throughout the thesis.

- *Graded*, without any other qualifiers, will always mean $\mathbb{Z}$-graded.
- Filtrations are assumed to be indexed by $\mathbb{Z}$.
- If a field is implied not specified, it is assumed to be $\mathbb{C}$.
- Unless stated otherwise, all algebras are associative and unital.
- We will use homological gradings. A lower index is to be interpreted as negative an upper index and vice versa.
- With the exception of H, which denotes (co)homology, mathsf font $(\mathsf{X}, \mathsf{Y}, \dots)$ is used for spaces.
- A diamond $\Diamond$ denotes the end of a definition or remark.

Chapter 1

Hodge structures

In this chapter we introduce the primary objects of study in this thesis: complex tori and Hodge structures.

1.1 Mixed Hodge structures

1.1.1 Complex tori

Definition 1.1.1. Let n be a positive integer. A *complex torus of rank n* is an algebraic variety isomorphic to $(\mathbb{C}^\times)^n$. $\diamondsuit$

One can view theses spaces in many different ways: for example, a complex torus X in its analytic topology is a Lie group under component-wise multiplication. When we wish to consider a complex torus as a *group* we will write G. We will write $\mathfrak{g}$ for the Lie algebra $T_{(1,\overset{n}{\dots},1)}G$ of G and $\mathfrak{g}^\vee$ for its dual as a $\mathbb{C}$-vector space.

In the global coordinates $z_1, \dots, z_n$ on X, a basis of global sections for the tangent bundle is

$$z_1\frac{\partial}{\partial z_1}, \dots, z_n\frac{\partial}{\partial z_n}.$$

We will often write ∂_i for $\frac{\partial}{\partial z_i}$. Upon evaluation at the identity, these vector fields give a basis for $\mathfrak{g}$ which we denote by $\delta_1, \dots, \delta_n$. Note that, by definition of $\mathfrak{g}$, these vector fields are G-invariant. The dual basis of one-forms is

$$\frac{dz_1}{z_1}, \dots, \frac{dz_n}{z_n}$$

and we will often write $d\log z_i$ for $\frac{dz_i}{z_i}$. When regarding these forms as a basis for

$\mathfrak{g}^\vee$, we will often write $e_1, \dots, e_n$, where $e_i := d\log z_i$. Note that these forms are also G-invariant.

Definition 1.1.2. Let G be a complex torus of rank n. The abelian group $\mathrm{Hom}(G, \mathbb{C}^\times)$ is called the *character lattice* of G and is denoted L. ◊

By picking coordinates on G we obtain an isomorphism $L \cong \mathbb{Z}^n$. Taking the logarithmic derivative

$$\begin{aligned} d\log : L &\to \Omega^1(G) \\ z_i &\mapsto d\log z_i \end{aligned}$$

we obtain an isomorphism

$$\wedge^\bullet \left(\frac{1}{2\pi i} L\right) \cong \mathsf{H}^\bullet(G; \mathbb{Z}). \tag{1.1.1}$$

In coordinates, this states that the forms

$$\frac{1}{2\pi i}\frac{dz_1}{z_1}, \dots, \frac{1}{2\pi i}\frac{dz_n}{z_n}$$

are a set of algebra generators for the integral cohomology of G (under the de Rham isomorphism).

1.1.2 Mixed Hodge structures and variations

In this section we introduce the various Hodge theoretic structures we will use in later sections. In this thesis we do not use noncommutative Hodge structures as defined in [KKP08] (in particular, we do not work with $\mathbb{Z}/2$-gradings). However, the invariants we study and use to construct mixed Hodge structures and their variations are those used in noncommutative Hodge theory; for example, the periodic cyclic homology of an algebra. For more details on Hodge structures, consult [PS08].

Definition 1.1.3. An *integral pure Hodge structure of weight k* is a tuple (Λ, V, F, c) where

- Λ is an abelian group of finite rank,
- V is a $\mathbb{C}$-vector space,
- F is a finite decreasing filtration on V by $\mathbb{C}$-vector subspaces called the *Hodge filtration*,

- $c : \Lambda \otimes \mathbb{C} \xrightarrow{\sim} V$ is an isomorphism of $\mathbb{C}$-vector spaces, called the *comparison map*,

such that for all $i, j \in \mathbb{Z}$ with $i+j = k+1$, one has $F^iV \oplus \overline{F^jV} = V$, where conjugation in V is induced by conjugation in $\mathbb{C}$ via the map c. $\Diamond$

This is equivalent to the more common definition in which there is no filtration $F^\bullet V$, but rather one defines a bigraded decomposition

$$V = \bigoplus_{p+q=k} V^{p,q}$$

where $V^{p,q}$ are complex subspaces of V such that $V^{p,q} = \overline{V^{q,p}}$. To see the equivalence of these definitions, note that we can obtain a bigraded decomposition from the Hodge filtration by setting

$$V^{p,q} = F^pV \cap \overline{F^qV}$$

for each p, q such that $p+q = k$. Conversely, given a decomposition $V = \bigoplus_{p+q=k} V^{p,q}$, we obtain a Hodge filtration by defining

$$F^pV = \bigoplus_{j \geq p} V^{j,n-j}.$$

We will often drop the adjectives 'pure' and 'integral' and simply refer to these objects as Hodge structures. The abelian group Λ in this definition is called the *lattice* of the Hodge structure.

Remark 1.1.4. One can define a $\mathbb{Q}$-Hodge structure in an analogous way by choosing Λ to be a finite dimensional $\mathbb{Q}$-vector space. $\Diamond$

Example 1.1.5. The fundamental example of a Hodge structure of weight k is the kth cohomology group $\mathsf{H}^k(\mathsf{X}; \mathbb{C})$ of a compact Kähler manifold X. Indeed, the Hodge decomposition theorem states there is a direct sum decomposition

$$\mathsf{H}^k(\mathsf{X}; \mathbb{C}) \cong \bigoplus_{p+q=k} \mathsf{H}^{p,q}(\mathsf{X})$$

where $\mathsf{H}^{p,q}(\mathsf{X})$ is the sheaf cohomology group $\mathsf{H}^q(\mathsf{X}, \Omega^p_{\mathsf{X}})$ for Ω^p_{X} the sheaf of holomorphic p-forms on X. In this case, the lattice is given by integral cohomology $\mathsf{H}^k(\mathsf{X}; \mathbb{Z})$. $\Diamond$

Note that if H and H' are Hodge structures of weights k and k' respectively then their tensor product $H \otimes H'$ is naturally a Hodge structure of weight $k + k'$.

Definition 1.1.6. The mth *Tate Hodge structure*, denoted $\mathbb{Z}(m)$, is the Hodge structure of weight $-2m$ having lattice

$$\mathbb{Z}(m) = (2\pi i)^m \cdot \mathbb{Z} \subset \mathbb{C}$$

and bidegree decomposition $(\mathbb{Z}(m) \otimes \mathbb{C})^{-m,-m} = \mathbb{Z}(m) \otimes \mathbb{C}$; i.e. this Hodge structure is concentrated in bidegree $(-m, -m)$. $\diamondsuit$

Note that $\mathbb{Z}(m) = \mathbb{Z}(1) \otimes \overset{m}{\cdots} \otimes \mathbb{Z}(1)$.

Definition 1.1.7. Let $H = (\Lambda, V, F, c)$ be a pure Hodge structure of weight k. Its *Tate twist*, denoted $H(1)$, is the Hodge structure $H \otimes \mathbb{Z}(1) = (\Lambda(1), V(1), F(1), c(1))$ of weight $k - 2$. Explicitly, the lattice is $\Lambda(1) := (2\pi i) \cdot \mathbb{Z} \otimes \Lambda$, and

$$c(1) : ((2\pi i) \cdot \mathbb{Z} \otimes \Lambda) \otimes \mathbb{C} \cong \mathbb{C} \otimes \Lambda \xrightarrow{c} V.$$

The Hodge filtration is given by

$$F^p V(1) = F^{p+1} V.$$

$\diamondsuit$

The Hodge structure $V(m) := V \otimes \mathbb{Z}(m)$ is called the m-fold Tate twist, or mth Tate twist of V.

Notation 1.1.8. For Λ an abelian group and k a commutative ring, we will write Λ_k for $\Lambda \otimes_{\mathbb{Z}} k$. If $\psi : \Lambda \to \Lambda'$ is a morphism of abelian groups, we will write ψ_k or $\psi \otimes k$ for the induced morphism of k-modules. $\diamondsuit$

In Example 1.1.5, we saw that one can associate a pure Hodge structure of weight k to the cohomology group $\mathsf{H}^k(\mathsf{X}; \mathbb{C})$ when X is a compact Kähler manifold, such as a complex projective variety. To associate a similar structure to an arbitrary complex algebraic variety, one needs a much more flexible notion of Hodge structure, in which the weights are allowed to vary. The appropriate notion is that of a *mixed Hodge structure*, and is defined below. We recall how one can associate a mixed Hodge structure to any complex algebraic variety in Example 1.1.12.

Definition 1.1.9. A $\mathbb{Z}$-*mixed Hodge structure* (MHS) is a tuple (Λ, W, V, F, c), where

- Λ is a finite rank abelian group,
- W is a finite increasing filtration on Λ via $\mathbb{Z}$-submodules called the *weight filtration*,

- V is a $\mathbb{C}$-vector space,
- F is a finite decreasing filtration on V called the *Hodge filtration*,
- c is an isomorphism $c : \Lambda_{\mathbb{C}} \xrightarrow{\sim} V$, called the *comparison map*,

such that the kth associated graded object of the weight filtration

$$\mathrm{gr}_k^W \Lambda = (W_k\Lambda)/(W_{k-1}\Lambda)$$

is a pure $\mathbb{Z}$-Hodge structure of weight k with Hodge filtration induced by F and comparison isomorphism induced by c. $\diamondsuit$

Remark 1.1.10. One usually defines a mixed Hodge structure so that the weight filtration is defined on $\Lambda_{\mathbb{Q}}$, not Λ. The associated graded objects are then required to be pure $\mathbb{Q}$-Hodge structures. We choose to work with $\mathbb{Z}$-mixed Hodge structures, as the weight filtrations we will be constructing in this thesis are most naturally defined over $\mathbb{Z}$. Unless stated otherwise, all mixed Hodge structures appearing in this thesis are defined with a $\mathbb{Z}$-weight filtration. We drop the adjective $\mathbb{Z}$ from $\mathbb{Z}$-mixed Hodge structure from now on. $\diamondsuit$

There is a natural notion of tensor products of mixed Hodge structures, which we can use to define Tate twists, as in the pure setting.

Example 1.1.11. The Tate Hodge structures $\mathbb{Z}(m)$ are mixed Hodge structures concentrated in weight $-2m$; i.e.

$$W_k\mathbb{Z}(m) = \begin{cases} \mathbb{Z}(m) & k \geq -2m \\ 0 & k < -2m. \end{cases}$$

If $H = (\Lambda, W, V, F, c)$ is any mixed Hodge structure, then its Tate twist $H(1) = (\Lambda(1), W(1), V(1), F(1), c(1))$ is the mixed Hodge structure having weight filtration

$$W(1)_k\Lambda(1) = W_{k+2}(\Lambda) \otimes 2\pi i \cdot \mathbb{Z}$$

and Hodge filtration

$$F(1)^k(V(1)) = F^{k+1}V.$$

$\diamondsuit$

The following example, due to Deligne [Del71], is of fundamental importance and plays the role of Example 1.1.5 in the mixed setting.

Example 1.1.12. Recall that a divisor D in a complex algebraic variety X is said to be a *normal crossings divisor* if it is locally isomorphic to the union of coordinate hyperplanes. It is said to be a *simple* normal crossings divisor if each connected component of D is smooth. Given a smooth complex algebraic variety U, we say a variety X is a *good compaticfication* of U if X is compact and U is a Zariski open subset of X such that $\mathsf{D} = \mathsf{X} - \mathsf{U}$ is a simple normal crossings divisor. Let X be a good compactification of U and $j : \mathsf{U} \to \mathsf{X}$ the inclusion map. We say a holomorphic form ω on U has *logarithmic poles* along D if ω and $d\omega$ have at most a pole of order one along D. These forms give a subcomplex $\Omega^\bullet_{\mathsf{X}}(\log \mathsf{D}) \subset j_*\Omega^\bullet_{\mathsf{U}}$ called the *logarithmic de Rham complex*. It can be shown that (see [PS08, Section 4.1] for example)

$$\mathsf{H}^k(\mathsf{U};\mathbb{C}) = \mathbb{H}^k(\mathsf{X};\Omega^\bullet_{\mathsf{X}}(\log \mathsf{D}))$$

where $\mathbb{H}^k$ denotes the kth hypercohomology of a complex of sheaves. Moreover, the filtration

$$W_m\Omega^p_{\mathsf{X}}(\log \mathsf{D}) = \begin{cases} 0 & m < 0 \\ \Omega^p_{\mathsf{X}}(\log \mathsf{D}) & m \geq p \\ \Omega^{p-m}_{\mathsf{X}} \wedge \Omega^m_{\mathsf{X}}(\log \mathsf{D}) & 0 \leq m \leq p \end{cases}$$

induces a filtration in cohomology

$$W_m\mathsf{H}^k(\mathsf{U};\mathbb{C}) = \mathrm{image}(\mathbb{H}^k(\mathsf{X};W_{m-k}\Omega^\bullet_{\mathsf{X}}(\log \mathsf{D})) \to \mathsf{H}^k(\mathsf{U};\mathbb{C}))$$

which can be defined over $\mathbb{Q}$. Endowing $\Omega^\bullet_{\mathsf{X}}(\log \mathsf{D})$ with the standard decreasing filtration by degree of forms, we obtain an induced filtration on cohomology

$$F^p\mathsf{H}^k(\mathsf{U};\mathbb{C}) = \mathrm{image}(\mathbb{H}^k(\mathsf{X};F^p\Omega^\bullet_{\mathsf{X}}(\log \mathsf{D})) \to \mathsf{H}^k(\mathsf{U};\mathbb{C})).$$

Collectively this data defines a $\mathbb{Q}$-mixed Hodge structure on $\mathsf{H}^\bullet(\mathsf{U};\mathbb{C})$. ◊

Example 1.1.13. Let X be a complex torus. In this Hodge theoretic language, (1.1.1) is an isomorphism

$$\wedge^\bullet(L(-1)) \cong \mathsf{H}^\bullet(\mathsf{X};\mathbb{Z}),$$

where L is the character lattice of G. $\Diamond$

Notation 1.1.14. Let k be a commutative ring and M an k-module. We will write $\wedge^{\text{even}} M$ to denote the even subalgebra of the exterior algebra on M; i.e.

$$\wedge^{\text{even}} M = \bigoplus_{k \geq 0} \wedge^{2k} M.$$

$\Diamond$

We now define an important subclass of Hodge structures we will use extensively in later sections.

Definition 1.1.15. A *toric mixed Hodge structure* (TMHS) is a tuple $(\Lambda, W, V, F, c, \varphi)$ such that (Λ, W, V, F, c) is a mixed Hodge structure and φ is an isomorphism of graded abelian groups

$$\varphi : \mathrm{gr}^W_\bullet \Lambda \xrightarrow{\sim} \bigoplus_{k \geq 0} (\wedge^{2k} L)(-k),$$

called the *framing map*, where L is the character lattice of a torus G. We say a toric mixed Hodge structure $(\Lambda, W, V, F, c, \varphi)$ is of *rank n* if G is a torus of rank n. $\Diamond$

Let k be a commutative ring and let M and N be k-modules with filtrations $F^\bullet M$ and $G^\bullet N$. A morphism $\varphi : M \to N$ *respects* (or *preserves*) *the filtrations* if

$$\varphi(F^p M) \subset G^p N$$

for any $p \in \mathbb{Z}$. We say that φ *strictly respects the filtrations* if for any $p \in \mathbb{Z}$,

$$\varphi(M) \cap G^p N = \varphi(F^p M).$$

Definition 1.1.16. A *morphism* of mixed Hodge structures $\psi : H \to H'$ is a morphism of the underlying abelian groups such that ψ respects the weight filtration and $\psi_{\mathbb{C}}$ respects the Hodge filtration. If H and H' are toric mixed Hodge structures of the same rank with framing maps φ and φ', respectively, then ψ is a morphism of toric mixed Hodge structures if ψ is a morphism of mixed Hodge structures and

$$\varphi = \varphi' \circ \psi.$$

A morphism of (toric) mixed Hodge structures is an *isomorphism* if it admits a two-sided inverse. $\Diamond$

Remark 1.1.17. It can be shown that a morphism of mixed Hodge structures must *strictly* respect the weight and Hodge filtrations (see [Del71]). $\diamondsuit$

Let $H = (\Lambda, W, V, F, c, \varphi)$ be a TMHS. Using the framing map φ, we see there is an exact sequence

$$0 \longrightarrow \mathbb{Z} \longrightarrow W_2\Lambda \longrightarrow (\wedge^2 L)(-1) \longrightarrow 0$$

so we obtain a class

$$[W_2\Lambda] \in \mathrm{Ext}^1_{\mathrm{MHS}}((\wedge^2 L)(-1), \mathbb{Z})$$

in the Yoneda extension group in the abelian category of mixed Hodge structures (see [PS08, Corollary 3.9] for a proof that mixed Hodge structures form an abelian category). This class determines $W_2\Lambda$ up to isomorphism and will sometimes be referred to as the *extension data* of the Hodge structure. We now state a well known lemma (see [PS08, example 3.34.1]). This lemma will play an important role in the quantization of complex tori, so we provide a partial proof here.

Lemma 1.1.18. *There are canonical group isomorphisms*

$$\mathrm{Ext}^1_{\mathrm{MHS}}(\mathbb{Z}(-1), \mathbb{Z}) \cong \mathrm{Hom}(\mathbb{Z}, \mathbb{C}^\times) \cong \mathbb{C}^\times \tag{1.1.2}$$

with respect to the Yoneda product on $\mathrm{Ext}^1_{\mathrm{MHS}}(\mathbb{Z}(-1), \mathbb{Z})$.

Proof. Let $H \in \mathrm{Ext}^1_{\mathrm{MHS}}(\mathbb{Z}(-1), \mathbb{Z})$; i.e. $H = (\Lambda, W, V, F, c)$ sits in an exact sequence

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\ \beta\ } H \xrightarrow{\ \alpha\ } \mathbb{Z}(-1) \longrightarrow 0.$$

Now, let us split the underlying abelian groups using a section $s : \mathbb{Z}(-1) \to \Lambda$ so that

$$\Lambda = \mathbb{Z} \oplus s(\mathbb{Z}(-1)).$$

Observe that this section preserves the weight filtration, but does not necessarily preserve the Hodge filtration. Note that $F^1(\mathbb{Z} \otimes \mathbb{C}) = 0$ and thus $\beta(\mathbb{Z} \otimes \mathbb{C}) \cap F^1 V = 0$, as morphisms of Hodge structures strictly respect the filtrations. Since α is a surjection, it must be that $F^1 V \xrightarrow{\sim} \mathbb{Z}(-1) \otimes \mathbb{C}$. It follows that $F^1 V$ can be written as the graph of a unique $\mathbb{C}$-linear map $\psi_s : \mathbb{Z}(-1) \otimes \mathbb{C} \to \mathbb{Z} \otimes \mathbb{C}$; i.e.

$$F^1 V = \{\psi_s(x) + s(x) \,|\, x \in \mathbb{Z}(-1) \otimes \mathbb{C}\}.$$

Note that ψ_s is uniquely determined by the complex number $\rho_s := \psi_s(1) \in \mathbb{Z}(-1) \otimes \mathbb{C} \cong \mathbb{C}$. If we were to use a different splitting $t : \mathbb{Z}(-1) \to \Lambda$, then by exactness, $t = s + r$ for some $r : \mathbb{Z}(-1) \to \mathbb{Z}$. Proceeding as above, we find there is a unique map $\psi_t : \mathbb{Z}(-1) \otimes \mathbb{C} \to \mathbb{Z} \otimes \mathbb{C}$ such that F^1V is the graph of ψ_t and obtain a complex number $\rho_t := \psi_t(1)$. Therefore

$$\begin{aligned} F^1V &= \{\psi_t(x) + t(x) \,|\, x \in \mathbb{Z}(-1) \otimes \mathbb{C}\} \\ &= \{\psi_t(x) + (s(x) + r(x)) \,|\, x \in \mathbb{Z}(-1) \otimes \mathbb{C}\} \\ &= \{(\psi_t + r)(x) + s(x) \,|\, x \in \mathbb{Z}(-1) \otimes \mathbb{C}\}, \end{aligned}$$

By uniqueness, we must have that $\psi_s = \psi_t + r$ and therefore

$$\rho_s = \rho_t + r(1).$$

Noting that $r(1) \in 2\pi i \cdot \mathbb{Z}$ we find that

$$\rho_s \equiv \rho_t \mod 2\pi i \cdot \mathbb{Z}.$$

Thus ρ_s is independent of s modulo $2\pi i \cdot \mathbb{Z}$ and we obtain a well defined map $\rho = \exp(\rho_s) \in \mathrm{Hom}(\mathbb{Z}, \mathbb{C}^\times) \cong \mathbb{C}^\times$ where we identify $\mathbb{Z}$ with its image in $\mathbb{Z}(-1) \otimes \mathbb{C}$. One can check that this construction gives a group isomorphism $\mathrm{Ext}^1_{\mathrm{MHS}}(\mathbb{Z}(-1), \mathbb{Z}) \to \mathbb{C}^\times$. $\square$

One can proceed in an identical manner to show

$$\mathrm{Ext}^1_{\mathrm{MHS}}((\wedge^2 L)(-1), \mathbb{Z}) \cong \mathrm{Hom}(\wedge^2 L, (\mathbb{C}^\times)^{\binom{n}{2}}) \cong (\mathbb{C}^\times)^{\binom{n}{2}}.$$

Indeed, given any extension $H = (\Lambda, W, V, F, c) \in \mathrm{Ext}^1_{\mathrm{MHS}}((\wedge^2 L)(-1), \mathbb{Z})$, choose a splitting $s : (\wedge^2 L)(-1) \to \Lambda$. Using exactness and that these maps respect filtrations, we find a unique map $\psi_s : (\wedge^2 L)(-1) \otimes \mathbb{C} \to \mathbb{Z} \otimes \mathbb{C}$ such that F^1V is the graph of ψ_s. Identifying $\wedge^2 L$ with its image in $(\wedge^2 L)(-1) \otimes \mathbb{C}$, we note that ψ_s is determined by its restriction to $\wedge^2 L$ and that the resulting complex number is independent of the choice of s modulo $2\pi i \cdot \mathbb{Z}$. This construction gives an isomorphism

$$\mathrm{Ext}^1_{\mathrm{MHS}}((\wedge^2 L)(-1), \mathbb{Z}) \cong \mathrm{Hom}(\wedge^2 L, \mathbb{C}^\times) \cong (\mathbb{C}^\times)^{\binom{n}{2}}. \tag{1.1.3}$$

Notation 1.1.19. Given a TMHS H we will write $[W_2 H]$ for the elements in $(\mathbb{C}^\times)^{\binom{n}{2}}$

representing its extension class under the isomorphism (1.1.3). $\Diamond$

We will frequently encounter families of Hodge structures that vary in a coherent way over some base space Y. This motivates the following definition.

Definition 1.1.20. Let Y be a smooth $\mathbb{C}$-scheme. A *variation of Hodge structures* (VHS) of weight k over Y is a tuple $(\Lambda, V, F, c, \nabla)$ where

- Λ is a locally constant sheaf of finite rank abelian groups (in the analytic topology),
- V is a finite rank locally free sheaf of $\mathcal{O}_{\mathsf{Y}}$-modules,
- F is a finite decreasing filtration on V via locally constant free $\mathcal{O}_{\mathsf{Y}}$-submodules called the *Hodge filtration*,
- $c : \Lambda \otimes_{\mathbb{Z}} \mathcal{O}_{\mathsf{Y}} \xrightarrow{\sim} V$ is an isomorphism, called the *comparison map*,
- ∇ is a flat connection on V,

such that the following conditions are satisfied:

1. At any point $y \in \mathsf{Y}$, the data (Λ_y, V_y, F_y, c) is a pure Hodge structure of weight k, where V_y is the fibre of V at y, Λ_y is the fibre of Λ at y, and F_y is the induced filtration on the fibre V_y.
2. $\nabla \Lambda_{\mathbb{C}} = 0$. We say Λ is a *lattice of flat sections* or *covariant constant* with respect to ∇, and
3. $\nabla(F^k V) \subset \Omega^1_{\mathsf{Y}} \otimes F^{k-1} V$ for all k. This condition is called *Griffiths transversality*.

$\Diamond$

Remark 1.1.21. Note that a flat connection ∇ on a holomorphic vector bundle V is determined uniquely by its local system of flat sections V^{∇} (see [PS08, Section 10.1] for a proof). $\Diamond$

We will again be interested in a 'toric' subclass of VHS.

Definition 1.1.22. Let Y be a smooth $\mathbb{C}$-scheme. A *variation of mixed Hodge structures* (VMHS) over Y is a tuple $(\Lambda, W, V, F, c, \nabla)$ where

- Λ is a locally constant sheaf of finite rank abelian groups (in the analytic topology),

- W is a finite increasing filtration on Λ by locally constant sheaves of abelian groups, called the *weight filtration*,
- V is a locally free sheaf of finite rank $\mathcal{O}_{\mathsf{Y}}$-modules with flat connection ∇,
- F is a finite decreasing filtration on V by locally constant free $\mathcal{O}_{\mathsf{Y}}$-submodules, called the *Hodge filtration*,
- $c : \Lambda \otimes_{\mathbb{Z}} \mathcal{O}_{\mathsf{Y}} \xrightarrow{\sim} V$ is an isomorphism,

such that $\nabla \Lambda_{\mathbb{C}} = 0$, $\nabla(F^k V) \subset \Omega^1_{\mathsf{Y}} \otimes F^{k-1}V$ for all k and such that for each $y \in \mathsf{Y}$ the tuple $(\Lambda_y, W_y, V_y, F_y, c)$ is a mixed Hodge structure. A *toric variation of mixed Hodge stuctures* (TVMHS) is a tuple $(\Lambda, W, V, F, c, \nabla, \varphi)$ where $(\Lambda, W, V, F, c, \nabla)$ is a variation of mixed Hodge structures and φ is an isomorphism of locally constant $\mathbb{Z}$-graded sheaves

$$\varphi : \mathrm{gr}^W_\bullet \Lambda \xrightarrow{\sim} \bigoplus_{k \geq 0} (\wedge^{2k} L)(-k),$$

where L is the constant sheaf on Y valued in the character lattice of G. $\lozenge$

Note that if H is a TVMHS, then the fibre over any point $y \in \mathsf{Y}$ carries the structure of a TMHS.

Definition 1.1.23. A *morphism* of VMHS is a morphism ψ of the underlying locally constant sheaves of $\mathbb{Z}$-modules such that ψ respects the weight filtrations and $\psi_{\mathbb{C}}$ respects the Hodge filtration. If $\psi : H \to H'$ is a morphism of VMHS such that H, H' are TVMHS with isomorphisms φ, φ', then we say ψ is a morphism of TVMHS if

$$\varphi = \varphi' \circ \psi.$$

$\lozenge$

Formal variations

Since deformation quantization is a problem about formal neighbourhoods of commutative objects, we will construct TVMHS over formal schemes. Let H be a TVMHS over a base Y. Then, the map sending a point $y \in \mathsf{Y}$ to the extension data of the TMHS in the fibre over y gives a section of $\mathcal{H}om(\wedge^2 L, \mathbb{C}^\times)$. In particular, if H is a TVMHS over the formal scheme $\mathrm{Spf}(\mathbb{C}[[\hbar]])$, then this extension data is an element of $[W_2\Lambda] \in \mathrm{Hom}(\wedge^2 L, (\mathbb{C}[[\hbar]])^\times)$.

1.2 Mixed complexes

In this section we recall some concepts from homological algebra which will play a central role in our constructions of Hodge structures. Additional details can be found in [Wei94, Section 9.8].

Definition 1.2.1. Let k be a commutative ring. A *mixed complex* is a non-positively graded k-module (with respect to homological gradings) with a degree -1 endomorphism B and degree $+1$ endomorphism b such that

$$B^2 = b^2 = bB + Bb = 0.$$

$\diamondsuit$

Remark 1.2.2. As mixed complexes are non-positively graded, we will often write the underlying graded k-module with a superscript bullet to avoid introducing too many minus signs. $M^\bullet$ is always to be interpreted as $M_{-\bullet}$. The operator b in a mixed complex is thus a map $b : M^\bullet \to M^{\bullet-1}$, or equivalently, $b : M_{-\bullet} \to M_{-\bullet+1}$, thus of degree $+1$. $\diamondsuit$

Mixed complexes are simultaneously cochain complexes with respect to b and chain complexes with respect to B. The utility of mixed complexes lies partly in the number of different homology theories one can extract from them.

Definition 1.2.3. Let $(M^\bullet, b, B)$ be a mixed complex. The *Hochschild homology* of $M^\bullet$ is the cohomology of the complex $(M^\bullet, b)$ and is denoted $\mathsf{HH}_{-\bullet}(M)$. $\diamondsuit$

Definition 1.2.4. Let $(M^\bullet, b, B)$ and $(N^\bullet, d, D)$ be mixed complexes over a commutative ring k. A *morphism* of mixed complexes is a graded k-module morphism $\varphi : M^\bullet \to N^\bullet$ which is a (co)chain map with respect to both differentials. A morphism of mixed complexes is a *quasi-isomorphism* if it induces an isomorphism on Hochschild homology. $\diamondsuit$

Definition 1.2.5. Let k be a commutative ring and $V^\bullet$ a $\mathbb{Z}$-graded k-module. Its associated 2-*periodic space* is the $\mathbb{Z}$-graded k-module $V((u))$ with graded components

$$V((u))^{-p} = V((u))_p = \prod_{j\in\mathbb{Z}} V^{2j-p} u^j,$$

where u is a variable of homological degree $+2$ (recall our grading conventions: an upper index is negative a lower index). $\diamondsuit$

There is a natural decreasing filtration F on $V((u))$ called the *Hodge filtration* defined for $p \in \mathbb{Z}$ by

$$F^p V((u)) = \left\{ \sum_i v_i u^{q_i} \mid v_i \in V,\ q_i \geq p \ \forall\, i \right\}.$$

This induces a filtration on $V((u))_k$ for all $k \in \mathbb{Z}$.

Definition 1.2.6. Let $(M^\bullet, b, B)$ be a mixed complex. We define the associated *periodic complex* to be $(M((u)), b + uB)$. Explicitly,

$$M((u))_j = \prod_{k \in \mathbb{Z}} M^{2k-j} u^k.$$

The *periodic homology* of $(M^\bullet, b, B)$ is the cohomology of $(M((u)), b + uB)$ and is denoted $\mathsf{HP}_\bullet(M)$. $\Diamond$

For example, a class $[\gamma] \in \mathsf{HP}_0(M)$ is represented by a family $\sum_{k \geq 0} \gamma^{2k} u^k$, with $\gamma^{2k} \in M^{2k}$ for all k, such that

$$B\gamma^{2k} = -b\gamma^{2k+2} \quad \forall\, k \geq 0.$$

The Hodge filtration on $M((u))$ induces a filtration on $\mathsf{HP}_\bullet(M)$ which we continue to call the Hodge filtration. With these definitions, there are vector space isomorphisms $\mathsf{HP}_k(M) \xrightarrow{\cdot u} \mathsf{HP}_{k+2}(M)$ for all k. These are, however, not graded vector space isomorphisms with respect to Hodge filtrations. Indeed, we have

$$uF^p \mathsf{HP}_k(M) = F^{p+1} \mathsf{HP}_{k+2}(M).$$

Because of these isomorphisms, it is often sufficient to consider only the zeroeth and first periodic homology groups. We will do this often, as it considerably reduces notational clutter.

Example 1.2.7. Let k be a commutative ring and M be an k-module concentrated in cohomological degree $+1$. Consider the trivial mixed complex $(\wedge^\bullet M, 0, 0)$. Its periodic complex $\wedge^\bullet M((u))$ in low degrees is

$$\wedge^\bullet M((u))_0 = \sum_{j \geq 0} \omega^{2j} u^j \qquad \omega^{2j} \in \wedge^{2j} M$$
$$\wedge^\bullet M((u))_1 = \sum_{j \geq 1} \omega^{2j-1} u^j \qquad \omega^{2j-1} \in \wedge^{2j-1} M.$$

The Hodge filtration on $\mathsf{HP}_0(\wedge^\bullet M) = \mathsf{HP}_0(\wedge^{\mathrm{even}} M)$ is then

$$F^p \mathsf{HP}_0(\wedge^{\mathrm{even}} M) = \{\omega^{2p} u^p + \omega^{2p+2} u^{p+1} + \dots \mid \omega^{2j} \in \wedge^{2j} M\}.$$

Note that this filtration can be identified with the canonical decreasing filtration on $\wedge^{\mathrm{even}} M$ by degree of exterior powers. $\lozenge$

Periodic homologies will be the vector spaces and bundles in the mixed Hodge structures we construct later. In these mixed Hodge structures, our two seemingly different uses of the term 'Hodge filtration' coincide.

Chapter 2

Hodge theory of Poisson tori

In the first section of this chapter, we define several structures associated to any smooth Poisson variety. We use these structures in the second section of the chapter to construct a toric mixed Hodge structure on the cohomology of any complex torus equipped with a torus-invariant Poisson structure.

2.1 Constructions in Poisson geometry

2.1.1 Poisson structures

In this section we recall standard definitions from Poisson geometry. A more extensive treatment of these topics can be found in the book [LGPV13]. While the language of sheaves is used in many of these definitions, we will quickly specialize to affine varieties, and proceed to work solely with global sections.

Definition 2.1.1. Let X be a smooth algebraic variety over $\mathbb{C}$. The sheaf of *polyvector fields* on X, denoted $\mathscr{X}^\bullet_{\mathsf{X}}$, is the exterior algebra of the tangent sheaf $\mathcal{T}_{\mathsf{X}}$ as a $\mathcal{O}_{\mathsf{X}}$-module; i.e. $\mathscr{X}^\bullet_{\mathsf{X}} = \wedge^\bullet \mathcal{T}_{\mathsf{X}}$. ◊

We will often drop the subscript X from $\mathscr{X}^\bullet_{\mathsf{X}}$ and simply write $\mathscr{X}^\bullet$ when no confusion could arise.

Definition 2.1.2. Let X be a smooth algebraic variety and $\mathscr{X}^\bullet$ its sheaf of polyvectors. The *Schouten bracket* is the graded Lie bracket $[\cdot,\cdot]_S : \mathscr{X}^k \otimes \mathscr{X}^\ell \to \mathscr{X}^{k+\ell-1}$ defined by

$$[\gamma,\eta] = \sum_{i=1}^{k}\sum_{j=1}^{\ell}(-1)^{i+j+k}[\gamma^i,\eta^j]\gamma^1 \wedge .\overset{\widehat{i}}{.}. \wedge \gamma^k \wedge \eta^1 \wedge .\overset{\widehat{j}}{.}. \wedge \eta^\ell$$

for $\gamma = \gamma^1 \wedge \cdots \wedge \gamma^k$ and $\eta = \eta^1 \wedge \cdots \wedge \eta^\ell$, where γ^i, η^j are vector fields for all i, j. Here, $.\overset{\widehat{k}}{.}.$ indicates the omission of the kth term in the product. $\Diamond$

Proposition 2.1.3. *Let* X *be a smooth algebraic variety,* $\xi \in \mathscr{X}^k$*, and* $\iota_\xi : \Omega^\bullet_{\mathsf{X}} \to \Omega^{\bullet - k}_{\mathsf{X}}$ *denote the operation of contraction with* ξ*. Then for all* $\omega \in \Omega^\bullet_{\mathsf{X}}$ *and* $\gamma, \eta \in \mathscr{X}^k$*, one has*

$$[[\iota_\gamma, d], \iota_\eta] = \iota_{[\gamma, \eta]}.$$

For a proof, see [LGPV13, page 80].

Definition 2.1.4. A *Poisson structure* on a smooth algebraic variety X is a section $\pi \in \mathsf{H}^0(\mathsf{X}, \wedge^2 \mathcal{T}_{\mathsf{X}})$ such that

$$[\pi, \pi] = 0.$$

A pair (X, π), with X a smooth algebraic variety and π a Poisson structure on X is called a *Poisson variety*. $\Diamond$

From a Poisson structure π on X we obtain a skew-symmetric bilinear map $\{\cdot, \cdot\} : \mathcal{O}_{\mathsf{X}} \times \mathcal{O}_{\mathsf{X}} \to \mathcal{O}_{\mathsf{X}}$ called the *Poisson bracket*, which is defined via

$$(f, g) \mapsto \{f, g\} = \iota_\pi(df \wedge dg).$$

The condition $[\pi, \pi] = 0$ implies the Poisson bracket is a Lie bracket; that is, it satisfies the Jacobi identity. Conversely, given the data of a Poisson bracket, one can recover a unique bivector $\pi \in \mathsf{H}^0(\mathsf{X}, \wedge^2 \mathcal{T}_{\mathsf{X}})$.

Remark 2.1.5. One can take X to be a space in a different category, for example, smooth manifolds, and the same definitions apply. $\Diamond$

Example 2.1.6. Any symplectic manifold (X, ω) is canonically Poisson. Indeed, since ω is a non-degenerate section of $\wedge^2 \mathcal{T}^*_{\mathsf{X}}$ it may be inverted to give a bivector $\omega^{-1} \in \mathsf{H}^0(\mathsf{X}, \wedge^2 \mathcal{T}_{\mathsf{X}})$. Writing $\widetilde{\eta}$ for the image of a one-form η under the inverse of the isomorphism $\mathcal{T}_{\mathsf{X}} \to \mathcal{T}^*_{\mathsf{X}}$ induced by ω, we have

$$\{f, g\} = \omega(\widetilde{df}, \widetilde{dg}).$$

The condition $d\omega = 0$ implies the Jacobi identity for $\{\cdot, \cdot\}$. $\Diamond$

For deformation quantization, we will need the following notion.

Definition 2.1.7. Let X be a Poisson variety. A *formal Poisson structure* on X is a bivector

$$\pi(\hbar) \in \mathsf{H}^0(\mathsf{X}, \wedge^2 \mathcal{T}_{\mathsf{X}} \otimes \mathbb{C}[[\hbar]])$$

such that $[\pi(\hbar), \pi(\hbar)] = 0$, where the Schouten bracket is taken to be linear in $\hbar$. $\Diamond$

We will often view a formal Poisson structure as a Poisson structure on X parametrized by the formal scheme $\mathrm{Spf}(\mathbb{C}[[\hbar]])$.

2.1.2 Periodic Poisson homology

Let (X, π) be a Poisson variety. In this section we will construct a mixed complex $M^\bullet(\mathsf{X}, \pi)$ dependent on both the geometry of X and its Poisson structure.

Let $(\Omega^\bullet(\mathsf{X}), d)$ be the complex of Kähler differentials on X. Contraction with the bivector π gives us a map $\iota_\pi : \Omega^\bullet(\mathsf{X}) \to \Omega^{\bullet-2}(\mathsf{X})$ which we use to define

$$\delta_\pi = [d, \iota_\pi] : \Omega^\bullet(\mathsf{X}) \to \Omega^{\bullet-1}(\mathsf{X}).$$

The operator δ_π was first studied in this context by Brylinski. See [Bry88, Proposition 1.2.3] for a proof of the following proposition.

Proposition 2.1.8. *The operator δ_π is a differential and anti-commutes with the de Rham differential; i.e.*

$$\delta_\pi^2 = 0 \quad \textit{and} \quad d\delta_\pi + \delta_\pi d = 0.$$

Definition 2.1.9. The mixed complex $M^\bullet(\mathsf{X}, \pi) = (\Omega^\bullet(\mathsf{X}), \delta_\pi, d)$ is called the *mixed Poisson complex* of (X, π). $\Diamond$

The Hochschild homology of this complex, as defined in section Section 1.2, is the cohomology of the complex $(\Omega^\bullet(\mathsf{X}), \delta_\pi)$, and is called the *Poisson homology* of (X, π). Following the general procedure described in Section 1.2, we make the following definition.

Definition 2.1.10. The *periodic Poisson complex* of an affine Poisson variety (X, π) is the 2-periodic complex $(M(\mathsf{X}, \pi)((u)), \delta_\pi + ud)$ associated to $M^\bullet(\mathsf{X}, \pi)$. Its cohomology is called the *periodic Poisson homology* and is denoted $\mathsf{HP}_\bullet(\mathsf{X}, \pi)$. $\Diamond$

Note that $\mathsf{HP}_\bullet(\mathsf{X},\pi)$ is endowed with the Hodge filtration described in 1.2. We will view the de Rham complex as a mixed complex

$$(\Omega^\bullet(\mathsf{X}), 0, d)$$

with periodic complex $(\Omega(\mathsf{X})((u)), ud)$. For example, we have

$$\Omega(\mathsf{X})((u))_0 = \prod_{k\geq 0} \Omega^{2k}(\mathsf{X})u^k$$
$$\Omega(\mathsf{X})((u))_1 = \prod_{k\geq 1} \Omega^{2k-1}(\mathsf{X})u^k$$

The associated periodic homology is called the *periodic de Rham cohomology* and is denoted $\mathsf{HP}^{\mathrm{dR}}_\bullet(\mathsf{X})$. The following proposition can be found in [Kon08, Section 1.34] and [CFW11, Section 1.2].

Proposition 2.1.11. *Let* (X,π) *be a Poisson variety. The map*

$$e^{\iota_\pi/u} : M(\mathsf{X},\pi)((u)) \to \Omega(\mathsf{X})((u))$$

is an isomorphism of complexes.

Proof. We first check that $e^{\iota_\pi/u}$ is a chain map; i.e.

$$\mathrm{Ad}_{e^{\iota_\pi/u}}(ud) = e^{-\iota_\pi/u} \circ ud \circ e^{\iota_\pi/u} = \delta_\pi + ud.$$

If so, then this is clearly an isomorphism of complexes. Recalling the identity

$$\mathrm{Ad}_{e^{\iota_\pi/u}}(ud) = \exp([-,\iota_\pi/u])ud,$$

we have that

$$e^{-\iota_\pi/u} \circ ud \circ e^{\iota_\pi/u} = ud + [d,\iota_\pi] + \frac{1}{u}\frac{1}{2!}[\iota_\pi,[\iota_\pi,d]] + \frac{1}{u^2}\frac{1}{3!}[\iota_\pi,[\iota_\pi,[\iota_\pi,d]]] + \dots$$

which simplifies to $ud + \delta_\pi$ by Proposition 2.1.3 since π is Poisson. □

Note that the constructions in this section generalize to a formal Poisson structure on X without complications. In particular, there is an isomorphism of complexes

$$e^{\iota_{\pi(\hbar)}/u} M(\mathsf{X},\pi(\hbar))((u)) \to \Omega(\mathsf{X})[[\hbar]]((u)).$$

2.1.3 Topological K-theory

Now that we have constructed the periodic Poisson homology, which will serve as the vector space in a toric mixed Hodge structure, we must find an appropriate lattice in $\mathsf{HP}_\bullet(\mathsf{X}, \pi)$. While the integral cohomology is a reasonable candidate, the topological K-theory of X, suitably 'twisted' by π, is a much better choice, as we shall see. A good reference for the material in this section is [AH61]. We begin by recalling the definition and first properties of topological K-theory.

Definitions

Assume that X is a topological space homotopic to a finite CW-complex. The isomorphism classes of vector bundles on X form an abelian monoid under $\oplus$. The associated Grothendieck group is a homotopy invariant of X called the *topological K-theory* of X and is denoted $K^0(\mathsf{X})$. The tensor product of vector bundles is well defined on the equivalence classes in $K^0(\mathsf{X})$ and gives $K^0(\mathsf{X})$ the structure of a commutative ring. One defines $K^n(\mathsf{X}) := K^{n+1}(\Sigma\,\mathsf{X})$ and $K^{n+1}(\mathsf{X}) = K^n(\Omega\,\mathsf{X})$, where Σ and Ω are the reduced suspension and loop space functors, respectively. A fundamental result, known as *Bott periodicity*, states that the topological K-theory groups are 2-periodic; that is, we have isomorphisms

$$K^i(\mathsf{X}) = K^{i+2k}(\mathsf{X})$$

for all $k \in \mathbb{Z}$. In order to realize K-theory as a lattice, we must have a map including it into some vector space canonically associated to X. This map is called the *Chern character*:

Definition 2.1.12. Let $E \to \mathsf{X}$ be a C^∞-vector bundle with connection ∇ over a smooth manifold X. The *Chern character* of E, denoted $\mathrm{ch}(E)$ is the cohomology class of

$$[\mathrm{tr}\left(\exp\left(F_\nabla\right)\right)] \in \mathsf{H}^{\mathrm{even}}_{\mathrm{dR}}(\mathsf{X}),$$

where F_∇ is the curvature of ∇. $\Diamond$

If $E \to \mathsf{X}$ is a vector bundle such that $c_k(E) = 0$ for all $k > 0$ then we have $\mathrm{ch}(E) = \exp(c_1(E))$, where $c_k(E)$ denotes the kth Chern class of E.

Remark 2.1.13. Often one defines the Chern character and Chern classes not as functions of F_∇ but of $\frac{1}{2\pi i}F_\nabla$. With this convention, one obtains integral classes. With the definition above, the class $c_1(E)$ is not integral; rather, we obtain an element of

$$\mathsf{H}^2(\mathsf{X};\mathbb{Z}(1)) = \mathsf{H}^2(\mathsf{X};\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Z}(1).$$

By the splitting principle, for any vector bundle $E \to X$ we have

$$\mathrm{ch}(E) \in \bigoplus_{k\geq 0} \mathsf{H}^{2k}(\mathsf{X};\mathbb{Q}(k)) =: \mathsf{H}^{2\bullet}(\mathsf{X};\mathbb{Q}(\bullet)).$$

◊

K-theory as a lattice

Given $E \in K^0(\mathsf{X})$, we can extend the Chern character to a map $\mathrm{ch} : K^0(\mathsf{X}) \to \mathsf{HP}_0^{\mathrm{dR}}(\mathsf{X})$ by sending a class

$$\mathrm{ch}(E) = \sum_{k\geq 0} \omega^{2k}, \qquad \omega^{2k} \in \mathsf{H}^{2k}(\mathsf{X};\mathbb{Q}(k)),$$

to the periodic de Rham class $\sum_{k\geq 0} \omega^{2k} u^k$. Now, let (X, π) be a Poisson variety. We define the *Poisson Chern character* to be the map

$$\mathrm{ch}^{\pi} = e^{-\iota_{\pi}/u} \circ \mathrm{ch} : K^0(\mathsf{X}) \to \mathsf{HP}_0(\mathsf{X}, \pi)$$

using the extension of ch to $\mathsf{HP}_0^{\mathrm{dR}}(\mathsf{X})$ described above. This map is well defined by Proposition 2.1.11, and we define

$$\Lambda_{\pi} = \mathrm{image}(\mathrm{ch}^{\pi}).$$

Lemma 2.1.14. *Let (X, π) be a Poisson variety. Then Λ_{π} is a lattice in $\mathsf{HP}_{\bullet}(\mathsf{X}, \pi)$; i.e.*

$$\Lambda_{\pi} \otimes \mathbb{C} \cong \mathsf{HP}_0(\mathsf{X}, \pi)$$

Proof. This follows from the fact that $e^{\iota_{\pi}/u}$ is an isomorphism of complexes and that image(ch) is a lattice in $\mathsf{HP}_0^{\mathrm{dR}}(\mathsf{X})$ (which follows from [AH61, 2.5]). □

Notation 2.1.15. We define $\Lambda_{\mathrm{dR}} := \mathrm{image}(\mathrm{ch})$; the lattice in $\mathsf{HP}_0^{\mathrm{dR}}(\mathsf{X})$. ◊

Let X be a smooth algebraic variety and $\pi(\hbar)$ a formal Poisson structure on X. We view $\pi(\hbar)$ as a family of Poisson structures over $\mathrm{Spf}(\mathbb{C}[[\hbar]])$. Promoting Λ_{dR} to the constant sheaf over $\mathrm{Spf}(\mathbb{C}[[\hbar]])$, we see that $\Lambda_{\pi(\hbar)}$ is a locally constant sheaf on $\mathrm{Spf}(\mathbb{C}[[\hbar]])$. This will be the locally constant sheaf we use to construct a variation of Hodge structures over $\mathrm{Spf}(\mathbb{C}[[\hbar]])$. It follows from these observations and Remark 1.1.21 that there is a unique flat connection on $\mathsf{HP}_0(\mathsf{X}, \pi(\hbar))$ having $\Lambda_{\pi(\hbar)} \otimes \mathbb{C}$ as its lattice of flat sections.

Definition 2.1.16. The *Poisson Gauss–Manin* connection, denoted $\nabla_{\pi(\hbar)}$, is the unique flat connection on $\mathsf{HP}_0(\mathsf{X}, \pi(\hbar))$ such that $\mathsf{HP}_0(\mathsf{X}, \pi(\hbar))^{\nabla_{\pi(\hbar)}} = \Lambda_{\pi(\hbar)} \otimes \mathbb{C}$. $\Diamond$

The connection $\nabla_{\pi(\hbar)}$ appears in the paper [CFW11]. Their construction produces the same connection as ours, as both connections have the same lattice of flat sections.

2.2 Mixed Hodge structures of Poisson tori

We now study the above constructions in the case of a complex torus X with torus invariant Poisson structure π. We will construct a canonical toric mixed Hodge structure on $\mathsf{HP}_0(\mathsf{X}, \pi)$.

2.2.1 Poisson structures on complex tori

Recall that a complex torus is any complex algebraic variety X such that $\mathsf{X} \cong (\mathbb{C}^\times)^n$. We require the bivectors $\pi \in \mathsf{H}^0(\mathsf{X}, \wedge^2 \mathcal{T}_\mathsf{X})$ under consideration to be invariant under the action of the torus, G. Such bivectors must be of the form

$$\pi = \sum_{i,j} \lambda^{ij} z_i z_j \frac{\partial}{\partial z_i} \wedge \frac{\partial}{\partial z_j}$$

for some $\{\lambda_{ij}\}_{1 \le i < j \le n} \subset \mathbb{C}$. Note that the vector fields appearing in π are of the form $z_i \partial_i$ and thus are invariant under G. Let $\mathfrak{g} = \mathrm{Lie}(G)$ and let $\{\delta_i \wedge \delta_j\}_{1 \le i < j \le n}$ be a basis of $\wedge^2 \mathfrak{g}$. By definition of $\mathfrak{g}$, we see that π is the image of a unique element in $\wedge^2 \mathfrak{g}$ under the canonical map

$$\begin{gathered} \wedge^2 \mathfrak{g} \to \mathsf{H}^0(\mathsf{X}, \wedge^2 \mathcal{T}_\mathsf{X}) \\ \sum_{1 \le i < j \le n} \lambda_{ij} \delta_i \wedge \delta_j \mapsto \sum_{1 \le i < j \le n} \lambda^{ij} z_i z_j \frac{\partial}{\partial z_i} \wedge \frac{\partial}{\partial z_j}. \end{gathered}$$

Clearly this map is injective. Thus, the choice of such a G-invariant bivector field π is equivalent to the choice of an element in $\wedge^2 \mathfrak{g}$. Since the Lie algebra $\mathfrak{g}$ is abelian, it follows that $[\pi, \pi] = 0$ for any choice of $\lambda \in \wedge^2 \mathfrak{g}$. We have shown the following:

Proposition 2.2.1. *The space of torus invariant Poisson structures on a complex torus* X *is isomorphic to* $\wedge^2\mathfrak{g}$.

We call a pair (X,π) a *Poisson torus* if X is a rank n complex torus and π a torus invariant Poisson structure on X.

2.2.2 Periodic Poisson homology of a Poisson torus

Let $\mathsf{X} \cong (\mathbb{C}^\times)^n$ be a rank n complex torus with G-invariant Poisson structure $\pi \in \mathsf{H}^0(\mathsf{X}, \wedge^2\mathcal{T}_{\mathsf{X}})$. We prove that the basis of $\wedge^\bullet\mathfrak{g}^\vee$ given by the forms

$$\omega \in \wedge^\bullet \left\langle \frac{dz_1}{z_1} \dots \frac{dz_n}{z_n} \right\rangle$$

represent periodic Poisson homology classes. To do this, note that the action of G on X by rescaling of coordinates induces an action of $\mathfrak{g}$ on $\mathcal{T}(\mathsf{X})$ and therefore on $\Omega^\bullet(\mathsf{X})$. This action decomposes $\Omega^\bullet(\mathsf{X})$ into a direct sum of weight spaces indexed by $\mathbb{Z}^n$. Under the identification $d\log z_i = e_i \in \mathfrak{g}^\vee$, we see the subalgebra

$$\wedge^\bullet\mathfrak{g}^\vee \subset \Omega^\bullet(\mathsf{X})$$

is precisely the subspace of weight $(0, \overset{n}{\dots}, 0)$, which we will denote by Ω_0.

Lemma 2.2.2. *The projection* $\Omega^\bullet(\mathsf{X}) \to \Omega_0$ *induces a map of mixed complexes* $(\Omega(\mathsf{X}), \delta_\pi, d) \to (\wedge^\bullet\mathfrak{g}^\vee, 0, 0)$ *and induces a filtration preserving isomorphism on periodic homology.*

Proof. The G-action described above yields a direct sum decomposition

$$\Omega^\bullet(\mathsf{X}) = \Omega_0 \oplus \mathfrak{g} \cdot \Omega^\bullet(\mathsf{X}).$$

Note that Ω_0 is a 2^n-dimensional $\mathbb{C}$-vector space. We claim that d and δ_π are identically zero when restricted to Ω_0. Clearly the forms in Ω_0 are d closed, as Ω_0 is a set of representatives for the de Rham cohomology of X. To see these are δ_π closed, it is sufficient to check $\iota_\pi : \Omega_0 \hookrightarrow \Omega_0$. This follows immediately from the definition of Ω_0 and the fact that π is G-invariant. By counting dimensions in the string of isomorphisms

$$\mathsf{HP}^{\mathrm{dR}}_\bullet(\mathsf{X}) \xrightarrow{\sim} \mathsf{HP}_\bullet(\mathsf{X},\pi) \xrightarrow{\sim} \mathsf{H}^\bullet(\Omega_0((u))) \oplus \mathsf{H}^\bullet(\mathfrak{g}\cdot\Omega(\mathsf{X})((u))) \xrightarrow{\sim} \wedge^\bullet\mathfrak{g}^\vee \oplus \mathsf{H}^\bullet(\mathfrak{g}\cdot\Omega(\mathsf{X})((u)))$$

we conclude that $\mathfrak{g}\cdot\Omega(\mathsf{X})((u))$ is acyclic. Since this is a projection map, it is clear that the Hodge filtration is preserved at chain level, and thus on homology as well. $\square$

This yields explicit representatives for $\mathsf{HP}_0(\mathsf{X},\pi)$. Indeed, classes in $\mathsf{HP}_0(\mathsf{X},\pi)$ are represented by elements of the subalgebra of $\Omega(\mathsf{X})((u))_0$ generated by the forms

$$u\frac{dz_i\wedge dz_j}{z_iz_j} \qquad 1\le i<j\le n.$$

Using Example 1.2.7, we see the Hodge filtration on $\mathsf{HP}_0(\mathsf{X},\pi)$ is

$$F^p\mathsf{HP}_0(\mathsf{X},\pi)=\left\{\omega^{2p}u^p+\omega^{2p+2}u^{p+1}+\dots \mid \omega^{2j}\in\wedge^{2j}\left\langle\frac{dz_1}{z_1},\dots,\frac{dz_n}{z_n}\right\rangle\right\}. \tag{2.2.1}$$

2.2.3 Topological K-theory of a complex torus

We now calculate explicit representatives for the lattice $\Lambda_\pi=\mathrm{image}(\mathrm{ch}^\pi)\subset\mathsf{HP}_0(\mathsf{X},\pi)$ for (X,π) a Poisson torus. First we will calculate $\mathrm{image}(\mathrm{ch})\subset\mathsf{H}^{2\bullet}(\mathsf{X};\mathbb{Q}(\bullet))$. This calculation relies on the following fact.

Lemma 2.2.3. *Suppose X is a finite CW complex and that $\mathsf{H}^{\mathrm{even}}(\mathsf{X};\mathbb{Z})$ has no torsion. If A is a subgroup of $K^0(\mathsf{X})$ such that for every $x\in\mathsf{H}^{2p}(\mathsf{X};\mathbb{Z}(p))$, $p\ge 0$, there exists $\xi\in A$ such that*

$$\mathrm{ch}(\xi)=x+\textit{higher degree terms},$$

then $A=K^0(\mathsf{X})$.

The proof of this lemma is identical to that of [AH61, Section 2.5, Corollary iii].

Proposition 2.2.4. *Let X be a complex torus. Then $K^0(\mathsf{X})$ is generated by pullbacks of line bundles from rank two tori which are quotients of X.*

Proof. Choose global coordinates $(z_1,\dots,z_n)$ on X and define X_{ij} to be the rank-two complex subtorus with coordinates (z_i,z_j). Let $\mathcal{L}'_{ij}$ be a line bundle on X_{ij} such that $c_1(\mathcal{L}'_{ij})$ generates $\mathsf{H}^2(\mathsf{X}_{ij},\mathbb{Z}(1))$ and define $\mathcal{L}_{ij}=p_{ij}^*\mathcal{L}'_{ij}$, where $p_{ij}:\mathsf{X}\to\mathsf{X}_{ij}$ denotes the projection. Note that for such a line bundle $\mathcal{L}_{ij}$, we have

$$c_1(\mathcal{L}_{ij})=\frac{1}{2\pi i}\frac{dz_i\wedge dz_j}{z_iz_j}\in\mathsf{H}^2(\mathsf{X};\mathbb{Z}(1)),$$

thus

$$\mathrm{ch}(\mathcal{L}_{ij}) = 1 + c_1(\mathcal{L}_{ij}) + \frac{c_1(\mathcal{L}_{ij})^2}{2!} + \cdots = \exp(c_1(\mathcal{L}_{ij})) = 1 + c_1(\mathcal{L}_{ij})$$

Therefore, the collection $\{\mathcal{L}_{ij}\}_{i<j}$ gives $\binom{n}{2}$ line bundles such that $\{c_1(\mathcal{L}_{ij})\}_{1\le i<j\le n}$ are linearly independent in $\mathsf{H}^2(\mathsf{X};\mathbb{Z}(1))$, thus a $\mathbb{C}$-basis for $\mathsf{H}^2(\mathsf{X};\mathbb{Z}(1))$.

Let $A^0(\mathsf{X})$ denote the subring of $K^0(\mathsf{X})$ generated by the $\mathcal{L}_{ij}$. Since the classes $c_1(\mathcal{L}_{ij})$ generate $\mathsf{H}^2(\mathsf{X};\mathbb{Z}(1))$, and since the Chern character is a ring homomorphism, we see that the Chern character maps $A^0(\mathsf{X})$ surjectively onto $\mathsf{H}^{2\bullet}(\mathsf{X};\mathbb{Z}(\bullet))$. Using Lemma 2.2.3, we conclude that $K^0(\mathsf{X})$ is generated by the classes of the line bundles $\{\mathcal{L}_{ij}\}_{i<j}$. $\square$

From this calculation, we see that

$$\Lambda_{\mathrm{dR}} = \bigoplus_{k\ge 0} \left(\wedge^{2k}\mathbb{Z}\left\langle \frac{dz_1}{z_1},\dots,\frac{dz_n}{z_n}\right\rangle\right)(-k)$$

which is identified with

$$\bigoplus_{k\ge 0} u^k \left(\wedge^{2k}\mathbb{Z}\left\langle \frac{dz_1}{z_1},\dots,\frac{dz_n}{z_n}\right\rangle\right)(-k)$$

in the periodic space $\Omega(X)((u))$. Since $\Lambda_\pi = e^{-\iota_\pi/u}\Lambda_{\mathrm{dR}}$ by definition, we conclude that

$$\Lambda_\pi = e^{-\iota_\pi/u}\left(\bigoplus_{k\ge 0} u^k \left(\wedge^{2k}\mathbb{Z}\left\langle \frac{dz_1}{z_1},\dots,\frac{dz_n}{z_n}\right\rangle\right)(-k)\right) \subset \mathsf{HP}_0(\mathsf{X},\pi). \tag{2.2.2}$$

This will be the lattice in the toric mixed Hodge structure associated to (X,π). With this in mind, we now discuss an increasing filtration on Λ_π that will serve as the weight filtration in the TMHS constructed later. Define a filtration W^{dR} on $\Lambda_{\mathrm{dR}} = \mathrm{ch}(K^0(\mathsf{X}))$ by

$$W^{\mathrm{dR}}_{2k+1}\Lambda_{\mathrm{dR}} = W^{\mathrm{dR}}_{2k}\Lambda_{\mathrm{dR}} = \bigoplus_{\ell\le k} u^\ell \left(\wedge^{2\ell}\mathbb{Z}\left\langle \frac{dz_1}{z_1},\dots,\frac{dz_n}{z_n}\right\rangle\right)(-\ell).$$

for all $k\in\mathbb{Z}$. Since the operator $e^{-\iota_\pi/u}$ cannot increase the degree of forms, and since $\iota_\pi : \Omega_0 \hookrightarrow \Omega_0$, we obtain an increasing filtration W on Λ_π by setting

$$W_k\Lambda_\pi = e^{-\iota_\pi/u}W^{\mathrm{dR}}_k\Lambda_{\mathrm{dR}}. \tag{2.2.3}$$

Finally, note that the associated graded objects of these filtrations are canonically isomorphic:

$$\mathrm{gr}^W_\bullet(e^{-\iota_\pi/u}) : \mathrm{gr}^{W^{\mathrm{dR}}}_\bullet \Lambda_{\mathrm{dR}} \xrightarrow{\sim} \mathrm{gr}^W_\bullet \Lambda_\pi. \tag{2.2.4}$$

In particular, $\mathrm{gr}^W_{2k+1} \Lambda_\pi = 0$ for all k.

Poisson–Gauss–Manin connection

To conclude this section, we consider a family of G-invariant Poisson structures $\pi(\hbar)$ on X parametrized by $\mathrm{Spf}(\mathbb{C}[[\hbar]])$ and find an explicit form for the connection ∇_{GM}. A torus invariant formal Poisson structure on X must be of the form

$$\pi = \sum_{1 \le i < j \le n} \lambda_{ij}(\hbar) z_i z_j \partial_i \wedge \partial_j$$

for some formal power series $\lambda_{ij}(\hbar) \in \mathbb{C}[[\hbar]]$. Viewing Λ_{dR} as a constant sheaf over the formal disk $\mathrm{Spf}(\mathbb{C}[[\hbar]])$, we find Λ_π as defined above is a locally constant sheaf of $\mathbb{Z}$-modules over $\mathrm{Spf}(\mathbb{C}[[\hbar]])$. Recall that the Poisson–Gauss–Manin connection was defined to be the unique connection $\nabla_{\pi(\hbar)}$ on $\mathsf{HP}_0(\mathsf{X}, \pi(\hbar))$ such that $\nabla_{\pi(\hbar)}(\Lambda_{\pi(\hbar)} \otimes \mathbb{C}) = 0$. Using the explicit form of $\Lambda_{\pi(\hbar)}$ above, we can read off what this must be:

Proposition 2.2.5. *Let X be a complex torus with G-invariant formal Poisson structure*

$$\pi = \sum_{1 \le i < j \le n} \lambda_{ij}(\hbar) z_i z_j \partial_i \wedge \partial_j.$$

Then, in the trivialization $\mathsf{HP}_\bullet(\mathsf{X}, \pi(\hbar)) \cong \wedge^\bullet \mathfrak{g}^\vee \otimes \mathbb{C}[[\hbar]]$, the Poisson-Gauss–Manin connection is the operator

$$\nabla_{\pi(\hbar)} = d + \frac{1}{u} \sum_{1 \le i < j \le n} d\lambda_{ij}(\hbar) \iota_{\delta_i \wedge \delta_j},$$

where

$$d\lambda_{ij}(\hbar) := \frac{d\lambda_{ij}(\hbar)}{d\hbar} d\hbar.$$

Proof. First, note that one can indeed show $\mathsf{HP}_\bullet(\mathsf{X}, \pi(\hbar)) \cong \wedge^\bullet \mathfrak{g}^\vee \otimes \mathbb{C}[[\hbar]]$ using a proof identical to that of Lemma 2.2.2. Now, let $\omega \in \wedge^{\mathrm{even}} \mathfrak{g}^\vee \cong \Lambda_{\mathrm{dR}} \otimes \mathbb{C}$. Then we have $e^{-\iota_{\pi(\hbar)}/u}\omega \in \Lambda_{\pi(\hbar)} \otimes \mathbb{C} \subset \mathsf{HP}_0(\mathsf{X}, \pi(\hbar)) \cong \wedge^{\text{even}}\mathfrak{g}^\vee \otimes \mathbb{C}[[\hbar]]$ and

$$\nabla_{\pi(\hbar)}(e^{-\iota_{\pi(\hbar)}/u}\omega) = -\frac{1}{u}d(\pi(\hbar))\left(e^{-\iota_{\pi(\hbar)}/u}\omega\right) + e^{-\iota_{\pi(\hbar)}/u}\nabla_{\pi(\hbar)}\omega,$$

where we have set

$$d(\pi(\hbar)) := \sum_{1\le i<j\le n} d\lambda_{ij}(\hbar)\iota_{\delta_i\wedge\delta_j}.$$

Thus,

$$\begin{aligned}
&\nabla_{\pi(\hbar)}(e^{-\iota_{\pi(\hbar)}/u}\omega)\\
&\quad= -\frac{1}{u}\left(\sum_{1\le i<j\le n} d\lambda_{ij}(\hbar)\iota_{\delta_i\wedge\delta_j}\right)e^{-\iota_{\pi(\hbar)}/u}\omega + \frac{1}{u}e^{-\iota_\pi/u}\sum_{1\le i<j\le n} d\lambda_{ij}(\hbar)\iota_{\delta_i\wedge\delta_j}\omega\\
&\quad= 0
\end{aligned}$$

where we have used that the operators in this expression are of even degree, and therefore commute. □

2.2.4 Mixed Hodge structure of a Poisson torus

We now have all the ingredients we need to construct the toric mixed Hodge structure associated to a Poisson torus.

Theorem 2.2.6. *Let X be a complex torus of rank n and π a torus invariant Poisson structure on X. Then, the tuple*

$$H_\pi = (\Lambda_\pi, W, \mathsf{HP}_0(\mathsf{X}, \pi), F, c, \mathrm{gr}^W_\bullet(e^{\iota_\pi/u})),$$

where W is the filtration of (2.2.3) and F is the Hodge filtration on $\mathsf{HP}_0(\mathsf{X}, \pi)$, is a toric mixed Hodge structure.

Proof. First, we define W on Λ_π as in (2.2.3). By (2.2.2), we know that $\mathrm{ch}^\pi_{\mathbb{C}} : K^0(\mathsf{X}) \otimes \mathbb{C} \xrightarrow{\sim} \mathsf{HP}_0(\mathsf{X}, \pi)$ and this induces the isomorphism $c : \Lambda_\pi \otimes \mathbb{C} \to \mathsf{HP}_0(\mathsf{X}, \pi)$ in the data above. We endow $\mathsf{HP}_0(\mathsf{X}, \pi)$ with its Hodge filtration (in the sense of periodic complexes) and we take this to be the Hodge filtration F. We saw in (2.2.1) that this agrees with the standard decreasing filtration on $\mathsf{HP}_0(\mathsf{X}, \pi)$ by degree of forms. For all k, (2.2.4) yields an isomorphism

$$\mathrm{gr}^W_\bullet(e^{\iota\pi/u}) : \mathrm{gr}^W_k \Lambda_\pi \xrightarrow{\sim} \mathrm{gr}^{W^{\mathrm{dR}}}_k \Lambda_{\mathrm{dR}} = \begin{cases} u^{k/2}(\wedge^k L)(-\frac{k}{2}) & k \text{ even} \\ 0 & k \text{ odd} \end{cases} \tag{2.2.5}$$

by definition of Λ_{dR} and (1.1.1). Thus $\mathrm{gr}^W_k \Lambda_\pi$ is a pure $\mathbb{Z}$-Hodge structure of weight k for all k, as required. Furthermore, (2.2.5) shows this mixed Hodge structure is toric. $\square$

Given a G-invariant Poisson structure $\pi = \sum \lambda_{ij} z_i z_j \partial_i \wedge \partial_j$ on a complex torus X, we will now study the extension data of H_π; i.e. the exact sequence

$$0 \longrightarrow \mathbb{Z} \longrightarrow W_2 H_\pi \longrightarrow (\wedge^2 L)(-1) \longrightarrow 0.$$

We determine the extension class $[W_2 H_\pi]$ under the isomorphism $\mathrm{Ext}^1_{\mathrm{MHS}}((\wedge^2 L)(-1), \mathbb{Z}) \cong (\mathbb{C}^\times)^{\binom{n}{2}}$. Working in coordinates, we have that

$$W_2 \Lambda_\pi = \mathbb{Z} \cdot \langle 1 \rangle \oplus \bigoplus_{1 \le i < j \le n} \frac{1}{2\pi i} \mathbb{Z} \cdot \left\langle u \frac{dz_i \wedge dz_j}{z_i z_j} - \lambda_{ij} \right\rangle .$$

To split this sequence, we use the section

$$\begin{aligned} s : (\wedge^2 L)(-1) &\to W_2 \Lambda_\pi \\ \frac{1}{2\pi i} \frac{dz_i \wedge dz_j}{z_i z_j} &\mapsto \frac{1}{2\pi i} \left(u \frac{dz_i \wedge dz_j}{z_i z_j} - \lambda_{ij} \right) . \end{aligned}$$

Now,

$$F^1 W_2 \mathsf{HP}_0(\mathsf{X}, \pi) = \bigoplus_{1 \le i < j \le n} \mathbb{C} \left\langle \frac{u}{2\pi i} \frac{dz_i \wedge dz_j}{z_i z_j} \right\rangle$$

We can write this subspace as the graph of a unique map $\psi_s : (\wedge^2 L)(-1) \otimes \mathbb{C} \to \mathbb{Z} \otimes \mathbb{C}$, and it is clear this map is

$$\frac{u}{2\pi i} \frac{dz_i \wedge dz_j}{z_i z_j} \mapsto \lambda_{ij}$$

on basis elements. Recalling from the discussion preceding (1.1.3), we find that $W_2 \Lambda_\pi$ is determined up to isomorphism by the multiplicatively skew-symmetric matrix

$$\exp(\pi) = (\exp(\lambda_{ij}))_{1 \le i < j \le n}. \tag{2.2.6}$$

Note that this means a Poisson torus is not determined up to isomorphism by $[W_2 H_\pi]$. Indeed, by translating $\lambda_{ij} \mapsto \lambda_{ij} + 2\pi i$ we obtain a non-isomorphic Poisson structure, which has the same extension data as H_π.

Formal variation of Hodge structures

To conclude this chapter, we consider the toric variation of mixed Hodge structures associated to a G-invariant formal Poisson structure $\pi(\hbar)$ on a complex torus, viewed as a family of Poisson structures over $\mathrm{Spf}(\mathbb{C}[[\hbar]])$.

Proposition 2.2.7. *Let X be a complex torus with G-invariant formal Poisson structure $\pi(\hbar)$. Then, the tuple*

$$H_{\pi(\hbar)} := (\Lambda_{\pi(\hbar)}, W, \mathsf{HP}_0(\mathsf{X}, \pi(\hbar)), F, c, \nabla_{\pi(\hbar)}, \mathrm{gr}^W_\bullet(e^{\iota_{\pi(\hbar)}/u}))$$

is a TVMHS, where

- *W is the filtration described in (2.2.3),*
- *F is the Hodge filtration on $\mathsf{HP}_0(\mathsf{X}, \pi(\hbar))$,*
- *c is induced by the map*

$$\mathrm{ch}^\pi \otimes \mathbb{C}[[\hbar]] : K^0(\mathsf{X}) \otimes \mathbb{C}[[\hbar]] \xrightarrow{\sim} \mathsf{HP}_0(\mathsf{X}, \pi(\hbar)).$$

Proof. The only compatibility of this data which is not immediately satisfied by our previous constructions is that ∇_π satisfies Griffiths transversality. This is true for the following reason: let $\pi(\hbar) = \sum_{1\leq i<j\leq n} \lambda_{ij}(\hbar) z_i z_j \partial_i \wedge \partial_j$. We have seen that

$$\nabla_\pi = d + \frac{1}{u} \sum_{1\leq i<j\leq n} d\lambda_{ij}(\hbar) \iota_{\delta_i \wedge \delta_j}$$

in the trivialization $\mathsf{HP}_0(\mathsf{X}, \pi(\hbar)) \cong \wedge^{\mathrm{even}} \mathfrak{g}^\vee \otimes \mathbb{C}[[\hbar]]$. Since the filtration F on $\mathsf{HP}_0(\mathsf{X}, \pi(\hbar))$ is given by (2.2.1), or equivalently, by degree of forms, we see that ∇_π can only decrease the degree of forms by 2, and thus $\nabla_\pi(F^k \mathsf{HP}_0(\mathsf{X}, \pi(\hbar))) \subset \Omega^1_{\mathrm{Spf}(\mathbb{C}[[\hbar]])} \otimes F^{k-1}(\mathsf{HP}_0(\mathsf{X}, \pi(\hbar)))$, as required. Thus, this data has the structure of a TVMHS. ☐

Recalling the discussion of Section 1.1.2, we see that the extension data of $H_{\pi(\hbar)}$ is given by a map in $\rho \in \mathcal{H}om(\wedge^2 L, \mathbb{C}[[\hbar]]^\times) \cong (\mathbb{C}[[\hbar]]^\times)^{\binom{n}{2}}$. Repeating the calculations preceding (2.2.6), we find that this extension data ρ is determined up to isomorphism by the multiplicatively skew-symmetric matrix

$$\exp(\pi(\hbar)) = (\exp(\lambda_{ij}(\hbar)))_{1 \leq i < j \leq n}. \tag{2.2.7}$$

Chapter 3

Hodge theory of quantum tori

We now turn to the study of the noncommutative algebras which are candidates for the quantizations of Poisson tori. In the first section of this chapter, we recall the definitions of a number of standard tools from noncommutative geometry. A good reference for this material is [Wei94, Chapter 9] or [Lod98]. These objects are generalizations of common geometric invariants, such as de Rham cohomology, to the realm of noncommutative spaces (i.e. dg-categories). In the second section, we study these invariants in detail and use them to construct a toric mixed Hodge structure on the periodic cyclic homology of a quantum torus. To end this chapter, we discuss the noncommutative topological K-theory of quantum tori.

3.1 Hochschild invariants

In this section, given an associative algebra A over a commutative ring k, we denote by $\overline{A}$ the algebra A/k, where k is identified with the subalgebra $k \cdot 1$.

3.1.1 Homological invariants

Our goal is to construct from an associative algebra A a mixed complex whose various homology theories will reproduce the classical invariants of geometric spaces if A happens to be commutative.

Definition 3.1.1. Let A be an associative algebra. The (*reduced*, or *normalized*) *bar complex* of A is the complex

$$\cdots \xrightarrow{b} A \otimes \overline{A} \otimes \overline{A} \xrightarrow{b} A \otimes \overline{A} \xrightarrow{b} A \longrightarrow 0$$

where

$$b(a_0 \otimes \cdots \otimes a_n) = \sum_{i=0}^{n-1} (-1)^i a_0 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_n + (-1)^n a_n a_0 \otimes \cdots \otimes a_{n-1}.$$

We will write $C_{-\bullet}(A)$ for this complex so that

$$C_{-k}(A) = A \otimes \overline{A} \otimes \overset{k}{\cdots} \otimes \overline{A}$$

$\Diamond$

It is an easy exercise to check that $b^2 = 0$.

Notation 3.1.2. We will write $(a_0, \ldots, a_n)$ for the tensor product $a_0 \otimes a_1 \otimes \cdots \otimes a_n$. $\Diamond$

Definition 3.1.3. Let A be an associative algebra. The *Hochschild homology* of A, denoted $\mathsf{HH}_{-\bullet}(A)$, is the cohomology of the complex $(C_{-\bullet}(A), b)$. $\Diamond$

The following result, known as the HKR theorem, is due to Hochschild, Kostant and Rosenberg.

Theorem 3.1.4. *Let X be a smooth affine scheme of finite type over $\mathbb{C}$. Then there is an isomorphism of $\mathcal{O}(\mathsf{X})$-algebras*

$$\psi : \Omega^{\bullet}_{\mathsf{X}/\mathbb{C}} \xrightarrow{\sim} \mathsf{HH}_{-\bullet}(\mathcal{O}(\mathsf{X})).$$

For a proof, see [Wei94, section 9.4]. It is not obvious that $\mathsf{HH}_{-\bullet}(R)$ has a natural algebra structure; this is only possible when R is commutative. This fundamental result implies that we should think of Hochschild homology classes as the noncommutative analogue of differential forms. Let us study $\mathsf{HH}_{-\bullet}(A)$ in low degrees:

- In degree zero we have

$$\mathsf{HH}_0(A) = \frac{A}{[A, A]},$$

  the abelianization of A. In particular, if A is commutative, then $\mathsf{HH}_0(A) = A$, which agrees with the HKR theorem above.

- The degree -1 classes do not have such a simple interpretation. However, if A is commutative, then the map $b : A \otimes A \to A$ is identically zero. Thus,

$$\mathsf{HH}_{-1}(A) = \frac{A \otimes A}{\langle ab \otimes c - a \otimes bc + ca \otimes b \rangle_{a,b,c,\in A}}.$$

It is easy to see the HKR isomorphism here: $a \otimes b \mapsto adb \in \Omega_{\mathrm{Spec}(A)/\mathbb{C}}$. The above quotient corresponds to the Leibniz property of d : $d(ab) - adb - bda = 0$.

This leads to the following question: if the Hochschild homology is the noncommutative analogue of differential forms, then what is the noncommutative de Rham differential? The following operation was first defined by Rinehart in [Rin63] and was used extensively by Alain Connes in his foundational work on noncommutative differential geometry.

Definition 3.1.5. Let A be an associative algebra and $C_{-\bullet}(A)$ its bar complex. The *Connes differential* is the operator $B : C_{-k}(A) \to C_{-k-1}(A)$ defined by

$$B(a_0 \otimes \cdots \otimes a_k) = \sum_{i=0}^{k} (-1)^{ik} 1 \otimes a_i \otimes \cdots \otimes a_k \otimes a_0 \otimes \cdots \otimes a_{i-1}.$$

$\Diamond$

It is another easy exercise to check that $B^2 = 0$. One can also check that $bB + Bb = 0$, so $M^\bullet(A) := (C_{-\bullet}(A), b, B)$ is a mixed complex. Using the terminology of Section 1.2, we see the Hochschild homology of $M^\bullet(A)$ is precisely the Hochschild homology as defined above. Following Section 1.2, we obtain the *periodic cyclic complex* $(C(A)((u)), b + uB)$, defined in degrees zero and one as

$$C(A)((u))_0 = \prod_{k \geq 0} A \otimes \overline{A}^{\otimes 2k} u^k$$
$$C(A)((u))_1 = \prod_{k \geq 1} A \otimes \overline{A}^{\otimes 2k-1} u^k$$

(recall we can obtain the periodic complex in different degrees by multiplying by an appropriate power of u). The *periodic cyclic homology* of A, denoted $\mathsf{HP}_\bullet(A)$, is the cohomology of the complex $(C(A)((u)), b + uB)$. By restricting to the commutative case, we find this is the appropriate generalization of de Rham cohomology:

Theorem 3.1.6. *Let X be a smooth affine scheme of finite type over $\mathbb{C}$. Then there is an isomorphism of $\mathbb{C}$-vector spaces*

$$\mathsf{HP}_\bullet(\mathcal{O}(\mathsf{X})) \cong \mathsf{HP}^{\mathrm{dR}}_\bullet(\mathsf{X}).$$

Cohomological invariants

In our study of the periodic cyclic homology of a quantum torus, we will need to make use of the Hochschild cochains $C^\bullet(A)$ of an associative algebra A.

Definition 3.1.7. Let A be an associative, unital algebra over a commutative ring k. The *Hochschild cochain complex* $(C^\bullet(A), d_H)$ is the complex with

$$C^p(A) = \mathrm{Hom}_k(A^{\otimes p}, A)$$

in degree p, together with differential d_H called the *Hochschild differential* defined as follows: given $\varphi \in C^p(A)$, let

$$\begin{aligned} d\varphi(a_0, \ldots, a_p) = a_0 \cdot \varphi(a_1, \ldots, a_p) + \sum_{i=0}^{p-1} (-1)^i \varphi(a_0, \ldots, a_i a_{i+1}, \ldots, a_p) \\ + (-1)^p \varphi(a_0, \ldots, a_{p-1}) \cdot a_p \end{aligned}$$

where $a_i \in A$ for all i. ◊

One may check that $d_H^2 = 0$.

Definition 3.1.8. The *Hochschild cohomology* of an associative algebra A is the cohomology of the complex $(C^\bullet(A), d_H)$. ◊

3.2 Hodge theory of quantum tori

In this section, we construct a canonical toric mixed Hodge structure on the periodic cyclic homology of a quantum torus, which we define below. After defining these algebras and describing their basic properties, we trivialize the periodic cyclic homology of a quantum torus, proving that it is isomorphic to $\wedge^\bullet \mathfrak{g}^\vee$. Next, we recall the Getzler-Gauss-Manin connection on periodic cyclic homology and study it in this trivialization. This allows us to parallel transport the lattice from the commutative torus to obtain an integral structure in our mixed Hodge structures. The material of Section 3.2.1 and Section 3.2.2 follows [Yas17, Sections 6 and 7] closely. Many of the results of this section are similar to those obtained by G. Elliott in [Ell84], where noncommutative tori are studied in the context of C^*-algebras.

3.2.1 Periodic cyclic homology

Let $k = \mathbb{C}$ or $k = \mathbb{C}[[\hbar]]$. The following definition is taken from [Wam97, Section 1].

Definition 3.2.1. The *multiparametric quantum torus* is the algebra A_q generated over k by the $2n$ variables $x_1, \ldots, x_n$ and $x_1^{-1}, \ldots, x_n^{-1}$ subject to the relations

$$x_i x_i^{-1} = 1 = x_i^{-1} x_i \quad \text{and} \quad x_i x_j = q_{ij} x_j x_i$$

for every $1 \leq i, j \leq n$, where $\{q_{ij}\}_{1 \leq i,j \leq n}$ is a set of non-zero scalars in $k^\times$ satisfying the relations

$$q_{ii} = 1 = q_{ij} q_{ji}$$

for all $1 \leq i, j \leq n$. We call n the *rank* of A_q. $\Diamond$

We will call these algebras *quantum tori* and will only consider quantum tori over $\mathbb{C}$ for now (though the results of this section hold for quantum tori over any commutative ring of characteristic zero). Note that if we take $q_{ij} = 1$ for all $1 \leq i, j \leq n$, then $A_q = \mathcal{O}((\mathbb{C}^\times)^n)$. There is an action of $\mathfrak{g}$ on A_q by derivations: the classical rank n torus group G acts on the generators x_i by rescaling, inducing a map $\mathfrak{g} \to \mathrm{Der}(A_q)$. Given the basis $\delta_1, \ldots, \delta_n$ of $\mathfrak{g}$ as a $\mathbb{C}$-vector space, the action on an element $x \in A_q$ of the form $x = x_1^{k_1} \cdots x_n^{k_n}$ is

$$\delta_i(x) = k_i x.$$

Remark 3.2.2. This agrees with our earlier use of $\mathfrak{g}$ as the Lie algebra of the torus G with basis $x_1\partial_1, \ldots, x_n\partial_n$, as $x_i\partial_i(x) = k_i x$ for x as above. $\Diamond$

This furnishes A_q with a $\mathbb{Z}^n$-grading, with an element x as above lying in the $(k_1, \ldots, k_n)$-graded direct summand.

Definition 3.2.3. Let A be an associative algebra and $D \in C^k(A)$. The *Lie derivative* with respect to D is the operation $L_D : C_{-p}(A) \to C_{-(p-k+1)}(A)$ defined by

$$\begin{aligned} L_D(a_0, \ldots, a_p) = &\sum_{i=0}^{p-k+1} (-1)^{i(k+1)} a_0 \otimes \cdots \otimes D(a_i, \ldots, a_{i+k-1}) \otimes \cdots \otimes a_p \\ &+ \sum_{i=1}^{k-1} (-1)^{ip} D(a_{p-i+1}, \ldots, a_p, a_0, \ldots, a_{k-i-1}) \otimes a_{k-i} \otimes \cdots \otimes a_{p-i}. \end{aligned}$$

$\Diamond$

In particular, if $D = \delta_i \in \mathfrak{g}$, the above $\mathfrak{g}$ action (and grading) extends to $C_{-\bullet}(A_q)$

via L_{δ_i}. Explicitly, given an element $s \in C_{-p}(A_q)$ of the form

$$s = x_1^{k_1^1} \cdots x_n^{k_n^1} \otimes \cdots \otimes x_1^{k_1^p} \cdots x_n^{k_n^p}$$

we have $L_{\delta_i}(s) = \left(\sum_{j=1}^p k_i^j\right) s$, and s will have multidegree $(\sum_{j=1}^p k_1^j, \ldots, \sum_{j=1}^p k_n^j)$. Note that L_{δ_i} extends to a u-linear derivation on the periodic complex $C(A)((u))$. There is another important operator on chains, given in the following definition.

Definition 3.2.4. Let A be an associative algebra and $D \in C^k(A)$. The *contraction with respect to* D is the map

$$I_D = \iota_D + S_D : C(A)((u))_{-p} \to C(A)((u))_{-(p-k+2)},$$

where

$$\iota_D(a_0 \otimes \cdots \otimes a_p) = \frac{(-1)^{k-1}}{u} a_0 D(a_1, \ldots, a_k) \otimes a_{k+1} \otimes \cdots \otimes a_p$$

and

$$S_D(a_0 \otimes \cdots \otimes a_p) = \sum_{i=1}^{p-k+1} \sum_{j=0}^{p-i-k+1} (-1)^{i(k-1)+j(p-k+1)} \times$$
$$1 \otimes a_{p-j+1} \otimes \cdots \otimes a_p \otimes a_0 \otimes \cdots \otimes a_{i-1} \otimes D(a_i, \ldots, a_{i+k-1}) \otimes a_{i+k} \otimes \cdots \otimes a_{p-j}$$

where we set

$$S_D(1 \otimes a_1 \otimes \cdots \otimes a_p) = 0.$$

$\Diamond$

Note that if $D \in C^2(A)$ then $I_D : C(A)((u))_p \to C(A)((u))_p$. We use this operator to recall the noncommutative Cartan homotopy formula (see [Get93] for a proof):

Lemma 3.2.5 (Cartan homotopy formula)**.** *For any $D \in C^k(A_q)$, one has*

$$[b + uB, I_D] = L_D - I_{d_H D}$$

as maps $C(A)((u))_{-p} \to C(A)((u))_{-(p-k+1)}$.

We define the $\mathfrak{g}$-*invariant bar complex* $C^{\mathfrak{g}}_{-\bullet}(A_q)$ to be the space of coinvariants under the $\mathfrak{g}$-action described above

$$C^{\mathfrak{g}}_{-\bullet}(A_q) = \frac{C_{-\bullet}(A_q)}{\mathfrak{g} \cdot C_{-\bullet}(A_q)}.$$

The operators b and B descend to this complex [Yas17, Section 6.2], thus we define the $\mathfrak{g}$-invariant periodic cyclic homology $\mathsf{HP}^{\mathfrak{g}}_{\bullet}(A_q)$ as the periodic homology of the associated $\mathfrak{g}$-invariant mixed complex $(C^{\mathfrak{g}}_{-\bullet}(A_q), b, B)$. As in the Poisson case, the periodic homology of $(C_{-\bullet}(A_q), b, B)$ only detects the weight-zero part of the grading:

Proposition 3.2.6. *The canonical map*

$$C(A_q)((u)) \to C^{\mathfrak{g}}(A_q)((u))$$

is a quasi-isomorphism; i.e $\mathsf{HP}_{\bullet}(A_q) \cong \mathsf{HP}^{\mathfrak{g}}_{\bullet}(A_q)$. *Furthermore, this isomorphism strictly respects the Hodge filtrations:*

$$F^k\mathsf{HP}^{\mathfrak{g}}_{\bullet}(A_q) \cong F^k\mathsf{HP}_{\bullet}(A_q)$$

for all k.

Proof. First, note that by definition,

$$C(A_q)((u))_i = \prod_{k\in\mathbb{Z}} u^k C_{i-2k}(A_q)$$

and for any $\delta \in \mathfrak{g}$, we have

$$L_\delta \cdot C(A_q)((u))_i = \prod_{k\in\mathbb{Z}} u^k L_\delta \cdot C_{i-2k}(A_q).$$

This action yields a direct sum decomposition of each $C_i(A_q)$ into $\mathbb{Z}^n$-indexed weight spaces such that $C^{\mathfrak{g}}_i(A_q)$ lies in weight $(0, \overset{n}{\ldots}, 0)$. Thus,

$$\begin{aligned} C(A_q)((u))_i &= \prod_{k\in\mathbb{Z}} u^k (C^{\mathfrak{g}}_{i-2k}(A_q) \oplus \mathfrak{g} \cdot C_{i-2k}(A_q)) \\ &= \left(\prod_{k\in\mathbb{Z}} u^k C^{\mathfrak{g}}_{i-2k}(A_q)\right) \oplus \left(\prod_{k\in\mathbb{Z}} u^k \mathfrak{g} \cdot C_{i-2k}(A_q)\right) \\ &= C^{\mathfrak{g}}(A_q)((u))_i \oplus \mathfrak{g} \cdot C(A_q)((u))_i. \end{aligned}$$

Note that since each $\delta \in \mathfrak{g}$ acts by derivations, we have $d_H\delta = 0$. Thus, by the

Cartan homotopy formula, $\mathfrak{g} \cdot C(A_q)((u))_i$ is acyclic with respect to $b + uB$. Since the map $C(A_q)((u)) \to C^{\mathfrak{g}}(A_q)((u))$ is a projection onto a direct summand, it strictly preserves the Hodge filtrations. □

We will now further decompose $C(A_q)((u))$ into an exterior algebra and an acyclic complement. To do this, we define a map of mixed complexes

$$\ell : (C_{-\bullet}(A_q), b, B) \to (\wedge^{\bullet}\mathfrak{g}^{\vee}, 0, 0)$$

and show this map admits a left inverse. To define ℓ, we need a trace function $\tau : A_q \to \mathbb{C}$. Such a function should be $\mathbb{C}$-linear and invariant under cyclic permutations; i.e.

$$\tau(ab) = \tau(ba)$$

for all $a, b \in A_q$.

Proposition 3.2.7. *The function $\tau : A_q \to \mathbb{C}$ defined by*

$$\tau(x) = (x)_{(0,\overset{n}{\dots},0)},$$

(i.e. this function returns the constant term of x) is a trace function on A_q.

Proof. This function is clearly $\mathbb{C}$-linear so we need only show its invariance under cyclic permutations. It is enough to show this on a generating set of elements; to show the constant term of a monomial in A_q

$$x = x_{i_1}^{\alpha_1} \cdots x_{i_M}^{\alpha_M} \qquad i_j \in \{1, \dots, n\},\ \alpha_j \in \mathbb{Z}\ \forall\ j = 1, \dots, M$$

is invariant under cyclic permutations of the x_{i_j}'s. Note that the degree (in the usual $\mathbb{Z}$-grading) of such a monomial is invariant under any permutation of the x_{i_j}'s, so unless x has total degree zero, $\tau(x) = 0$ and in this case τ is invariant under cyclic permutations. Therefore, assume the total degree of x is zero, so that for all $k = 1, \dots, n$,

$$\sum_{i_j = k} \alpha_k = 0. \tag{3.2.1}$$

Then, by using the defining relations of A_q, it is immediate that x can be represented by a single non-zero complex number and the claim is obvious. □

We can now construct our map $C_{-\bullet}(A_q) \to \wedge^\bullet \mathfrak{g}^\vee$. In [Yas17], Yashinski defines a map $\gamma : \wedge^\bullet \mathfrak{g} \to C^\bullet(A_q)$ by

$$\gamma(\delta_1 \wedge \cdots \wedge \delta_k) = \left(a_0 \otimes \cdots \otimes a_k \mapsto \frac{1}{k!} \sum_{\sigma \in S_k} (-1)^\sigma \tau(a_0 \delta_{\sigma(1)}(a_1) \cdots \delta_{\sigma(k)}(a_k)) \right).$$

We define the dual map $\ell : C_{-\bullet}(A_q) \to \wedge^\bullet \mathfrak{g}^\vee$ via

$$\ell(a_0 \otimes \cdots \otimes a_k) = \left(\delta_1 \wedge \cdots \wedge \delta_k \mapsto \frac{1}{k!} \sum_{\sigma \in S_k} (-1)^\sigma \tau(a_0 \delta_{\sigma(1)}(a_1) \cdots \delta_{\sigma(k)}(a_k)) \right).$$

We now recall from [Wam93] a map ψ such that $\psi \circ \ell = \mathrm{Id}$. To define this map, we need some notation to keep track of the q_{ij}'s that will appear in computations.

Notation 3.2.8. For $k \in \mathbb{Z}_{\geq 0}$, let $\sigma \in S_k$ be a permutation. If σ is the simple transposition $\sigma = (i, i+1)$, then we set $q_\sigma = q_{i,i+1} \in \mathbb{C}^\times$. For general $\sigma \in S_k$, write $\sigma = s_1 \cdots s_p$, where s_i is a simple transposition for all i (i.e. each $s_i = (j, j+1)$ for some j) and p is minimal amongst all possible representations of σ as a product of simple transpositions. We then define

$$q_\sigma = q_{s_1} \cdots q_{s_p}.$$

For a proof that this is well defined, see [Wam93, Section 5]. ◊

Notation 3.2.9. Given any $\beta = (i_1, \ldots, i_n) \in \{0,1\}^n := \{0,1\} \times \overset{n}{\cdots} \times \{0,1\}$, we write

$$x^\beta = x_1^{i_1} \cdots x_n^{i_n} \in A_q$$

and

$$x^{-\beta} = x_n^{-i_n} \cdots x_1^{-i_1} \in A_q.$$

Similarly, given $\beta \in \{0,1\}^n$ we write

$$\delta^\beta = \delta_1^{i_1} \wedge \cdots \wedge \delta_n^{i_n} \in \wedge^\bullet \mathfrak{g}.$$

◊

Note that the collection $\{\delta^\beta\}_{\beta \in \{0,1\}^n}$ is a basis for $\wedge^\bullet \mathfrak{g}^\vee$. Given $\beta \in \{0,1\}^n$, let δ^β

act on $x = x_1^{k_1} \cdots x_n^{k_n} \in A_q$ by

$$\delta^\beta(x) = \delta_1^{i_1}(x_1^{k_1}) \cdots \delta_n^{i_n}(x_n^{k_n}).$$

This is well defined if we demand x and δ^β be ordered lexicographically. Extend this to any $a \otimes b \in A_q \otimes A_q$ by $\delta(a \otimes b) = \delta(a) \otimes \delta(b)$. Now, for each $\beta, \beta' \in \{0,1\}^n$, we have

$$\delta^\beta(x^{-\beta'} \otimes x^{\beta'}) = \begin{cases} 1 & \beta = \beta' \\ 0 & \text{else,} \end{cases}$$

thus, the map $x^{-\beta} \otimes x^\beta \mapsto (\delta^\beta)^*$ is an embedding and we can canonically identify $\wedge^\bullet \mathfrak{g}^\vee$ with the vector space spanned by the Hochschild chains $\{x^{-\beta} \otimes x^\beta\}_{\beta \in \{0,1\}^n}$.

We now define (following [Wam97, Section 5]) a map $\widetilde{\psi}_k : \wedge^k \mathfrak{g}^\vee \to C_{-(k-1)}(A_q)$. For $\beta = (i_1, \ldots, i_n) \in \{0,1\}^n$ such that $\sum_{j=1}^n i_j = k$, let $j_1, \ldots, j_k \subset \{1, \ldots, n\}$ be such that $\sum_{p=1}^k i_{j_p} = k$; i.e. the j_p index the non-zero i_j. Then set

$$\widetilde{\psi}_k(x^{-\beta} \otimes x^\beta) := \sum_{\sigma \in S_k} (-1)^\sigma q_\sigma x_{j_{\sigma(1)}} \otimes \cdots \otimes x_{j_{\sigma(k)}}.$$

Example 3.2.10. If $n = 2$ and $\beta = (1,1)$ then we have

$$\widetilde{\psi}_2(x_2^{-1}x_1^{-1} \otimes x_1 x_2) = x_1 \otimes x_2 - q_{12} x_2 \otimes x_1.$$

◊

Given $\beta \in \{0,1\}^n$, we then define $\psi : \wedge^\bullet \mathfrak{g}^\vee \to C^{\mathfrak{g}}_{-\bullet}(A_q)$ by

$$\psi(x^{-\beta} \otimes x^\beta) = x^{-\beta} \otimes \widetilde{\psi}(x^{-\beta} \otimes x^\beta).$$

Note that this indeed lands in the $\mathfrak{g}$-coinvariant chains.

Example 3.2.11. If $n = 2$ and $\beta = (1,1)$ then we have

$$\psi(x_2^{-1}x_1^{-1} \otimes x_1 x_2) = x_2^{-1}x_1^{-1} \otimes x_1 \otimes x_2 - q_{12} x_2^{-1}x_1^{-1} \otimes x_2 \otimes x_1.$$

◊

Theorem 3.2.12. *Let A_q be a quantum torus algebra. Then*

$$\ell \circ \psi = \mathrm{Id}_{\wedge^\bullet \mathfrak{g}^\vee}$$

Proof. Let $x^{-\beta} \otimes x^{\beta} \in \wedge^k \mathfrak{g}^\vee$ for some $\beta \in \{0,1\}^n$ as above, so that $x^\beta = x_{r_1} \cdots x_{r_k}$ for some $r_1, \dots, r_k \in \{1, \dots, n\}$. Then

$$\psi(x^{-\beta} \otimes x^{\beta}) = \sum_{\sigma \in S_k} (-1)^\sigma q_\sigma x^{-\beta} \otimes x_{r_{\sigma(1)}} \otimes \cdots \otimes x_{r_{\sigma(k)}} \in C_k^{\mathfrak{g}}(A_q).$$

Applying ℓ, we obtain the function

$$\begin{aligned} &\left(\delta_{j_1} \wedge \cdots \wedge \delta_{j_k} \mapsto \frac{1}{k!} \sum_{\sigma, \eta \in S_k} (-1)^\sigma (-1)^\eta q_\sigma \tau(x^{-\beta} \delta_{j_{\eta(1)}}(x_{r_{\sigma(1)}}) \dots \delta_{j_{\eta(k)}}(x_{r_{\sigma(k)}}))\right) \\ = &\left(\delta_{j_1} \wedge \cdots \wedge \delta_{j_k} \mapsto \frac{1}{k!} \sum_{\sigma, \eta \in S_k} (-1)^\sigma (-1)^\eta q_\sigma \left(\prod_{\ell=1}^k \delta_{j_{\eta(\ell)}, r_{\sigma(\ell)}}\right) \tau(x^{-\beta} x^{\sigma(\beta)})\right) \end{aligned}$$

where $x^{\sigma(\beta)} := x_{r_{\sigma(1)}} \cdots x_{r_{\sigma(k)}}$. It is not hard to see that $x^{\sigma(\beta)} = q_\sigma^{-1} x^\beta$, so this expression reduces to

$$\left(\delta_{j_1} \wedge \cdots \wedge \delta_{j_k} \mapsto \frac{1}{k!} \sum_{\sigma, \eta \in S_k} (-1)^\sigma (-1)^\eta \left(\prod_{\ell=1}^k \delta_{j_{\eta(\ell)}, r_{\sigma(\ell)}}\right) \tau(1)\right),$$

It is clear at this point that if, $\{r_1, \dots, r_k\} \neq \{j_1, \dots, j_k\}$ as sets, then this expression vanishes identically. We therefore assume that they are equal, and without loss of generality assume that $r_\ell = \ell$ for $\ell = 1, \dots, k$. To simplify matters further, let $\xi \in S_k$ be the unique permutation such that $\xi(1, \dots, k) = (j_1, \dots, j_k)$; that is, $j_i = \xi(i)$. Then $\ell \circ \psi$ is the map

$$\delta_1^* \wedge \cdots \wedge \delta_k^* \mapsto \left(\delta_{\xi(1)} \wedge \cdots \wedge \delta_{\xi(k)} \mapsto \frac{1}{k!} \sum_{\sigma, \eta \in S_k} (-1)^\sigma (-1)^\eta \left(\prod_{\ell=1}^k \delta_{\xi\eta(\ell), \sigma(\ell)}\right)\right),$$

where δ_i^* is the dual of δ_i; i.e. $\delta_i^* = x_i^{-1} \otimes x_i$. The effect of the Kronecker delta in this sum is now clear: unless $\xi\eta = \sigma$, we obtain zero. Using that $\eta = \xi^{-1}\sigma$, this expression further simplifies to

$$\delta_1^* \wedge \cdots \wedge \delta_k^* \mapsto \left((-1)^\xi \delta_1 \wedge \cdots \wedge \delta_k \mapsto \frac{1}{k!} \sum_{\sigma \in S_k} (-1)^\sigma (-1)^{\xi\sigma} \right)$$
$$= \delta_1^* \wedge \cdots \wedge \delta_k^* \mapsto \left((-1)^x i\delta_1 \wedge \cdots \wedge \delta_k \mapsto (-1)^\xi \frac{1}{k!} \sum_{\sigma \in S_k} ((-1)^\sigma)^2 \right)$$
$$= \delta_1^* \wedge \cdots \wedge \delta_k^* \mapsto \left((-1)^\xi \delta_1 \wedge \cdots \wedge \delta_k \mapsto (-1)^\xi \right)$$

which is the identity map. □

Theorem 3.2.13. *The map $\ell : C^{\mathfrak{g}}_{-\bullet}(A_q) \to \wedge^\bullet \mathfrak{g}^\vee$ is a map of mixed complexes*

$$\ell : (C^{\mathfrak{g}}_{-\bullet}(A_q), b, B) \to (\wedge^\bullet \mathfrak{g}^\vee, 0, 0)$$

which induces a filtration preserving isomorphism on periodic cyclic homology.

Proof. We will first show that this is a map of mixed complexes; i.e. that $b \circ \ell = B \circ \ell = 0$. We first prove this for B. Let $(a_0, \ldots, a_k) \in C^{\mathfrak{g}}_{-k}(A_q)$. Then

$$B(a_0, \ldots, a_k) = \sum_{i=0}^{k} (-1)^{ik} (1, a_i, \ldots, a_k, a_0, \ldots, a_{i-1})$$
$$= \sum_{s \in C_{k+1}} (-1)^{|s|k} (1, a_{s(0)}, \ldots, a_{s(k)})$$

where C_{k+1} is the cyclic group on $k+1$ elements. Then for any $(\delta_0 \wedge \cdots \wedge \delta_k) \in \wedge^{k+1}\mathfrak{g}$,

$$\ell(B(a_0, \ldots, a_k))(\delta_0 \wedge \cdots \wedge \delta_k)$$
$$= \frac{1}{(k+1)!} \sum_{\sigma \in S_{k+1}, s \in C_{k+1}} (-1)^{|\sigma|+|s|k} \tau(1 \cdot \delta_{\sigma(0)}(a_{s(0)}) \cdots \delta_{\sigma(k)}(a_{s(k)}))$$
$$= \frac{1}{(k+1)!} \sum_{\sigma \in S_{k+1}, s \in C_{k+1}} (-1)^{|s|k+|\sigma|} \tau(\delta_{\sigma s^{-1}(0)}(a_0) \cdots \delta_{\sigma s^{-1}(k)}(a_k))$$
$$= \frac{1}{(k+1)!} \sum_{i=0}^{k} (-1)^i \left(\sum_{\eta \in S_{k+1}} (-1)^\eta \tau(\delta_{\eta(0)}(a_0) \cdots \delta_{\eta(k)}(a_k)) \right).$$

If $k+1$ is even, then the alternating signs cancel here and this is zero. If not, then all but one of these terms cancel and we are left with a single copy of the inner sum with some sign. Write each a_i in terms of generators: $a_i = x_1^{\alpha_1^i} \cdots x_n^{\alpha_n^i}$, so that

$\delta_i(a_j) = \alpha_i^j a_j$. The sum

$$\sum_{\eta \in S_{k+1}} (-1)^\eta \tau(\delta_{\eta(0)}(x_1^{\alpha_1^0} \cdots x_n^{\alpha_n^0}) \ldots \delta_{\eta(k)}(x_1^{\alpha_1^k} \cdots x_n^{\alpha_n^k}))$$

reduces to

$$\det(\delta_i(x_i^{\alpha_i^j}))_{i,j=0,\ldots,k} \tau(a_0 \cdots a_k).$$

However since we are working in the $\mathfrak{g}$-invariant complex, unless we have $\sum_{\ell=0}^k \alpha_\ell^i = 0$ for all $i = 0, \ldots, k$, this determinant is zero. If this holds, then one of the rows in the matrix $\delta_i(x_i^{\alpha_i^j})$ can be expressed as a linear combination of the others and the determinant is zero. This proves $\ell \circ B = 0$.

Now we show that $\ell \circ b = 0$. For $(a_0, \ldots, a_k) \in C_{-k}^{\mathfrak{g}}(A_q)$ and $\delta_1 \wedge \cdots \wedge \delta_{k-1} \in \wedge^{k-1}\mathfrak{g}$, we have

$$\begin{aligned}
&\ell(b(a_0, \ldots, a_k))(\delta_1 \wedge \cdots \wedge \delta_{k-1}) \\
&= \frac{1}{(k-1)!} \sum_{\sigma \in S_{k-1}} (-1)^\sigma \tau(a_0 a_1 \delta_{\sigma(1)}(a_2) \cdots \delta_{\sigma(k-1)}(a_k)) \\
&+ \frac{1}{(k-1)!} \sum_{i=1}^{k-1} \sum_{\sigma \in S_{k-1}} (-1)^\sigma (-1)^i \tau(a_0 \delta_{\sigma(1)}(a_1) \ldots \delta_{\sigma(i)}(a_i a_{i+1}) \delta_{\sigma(i+1)}(a_{i+2}) \ldots \delta_{\sigma(k-1)}(a_k)) \\
&+ (-1)^k \frac{1}{(k-1)!} \sum_{\sigma \in S_{k-1}} (-1)^\sigma \tau(a_k a_0 \delta_{\sigma(1)}(a_1) \cdots \delta_{\sigma(k-1)}(a_{k-1})).
\end{aligned}$$

Using that the δ_i are derivations, we have

$$\begin{aligned}
\tau(a_0 \delta_{\sigma(1)}(a_1) \ldots &\delta_{\sigma(i)}(a_i a_{i+1}) \ldots \delta_{\sigma(k-1)}(a_k)) \\
&= \tau(a_0 \delta_{\sigma(1)}(a_1) \ldots \delta_{\sigma(i)}(a_i)(a_{i+1}) \ldots \delta_{\sigma(k-1)}(a_k)) \\
&+ \tau(a_0 \delta_{\sigma(1)}(a_1) \ldots \delta_{\sigma(i-1)}(a_{i-1}) a_i \delta_{\sigma(i)}(a_{i+1}) \ldots \delta_{\sigma(k-1)}(a_k)).
\end{aligned}$$

Using this, it is not hard to check that the above expression for $\ell \circ b$ gives a telescoping sum which sums to zero.

To see that this map induces an isomorphism on periodic cyclic homology, note that by Theorem 3.2.12, $\wedge^\bullet \mathfrak{g}^\vee$ is a direct summand of $C_{-\bullet}^{\mathfrak{g}}(A_q)$. We denote its complement by $\overline{C}$. Thus,

$$\begin{aligned}\mathsf{HP}_\bullet(A_q) &= \mathsf{H}^\bullet(C(A_q)((u)), b+uB)\\ &= \mathsf{H}^\bullet(C^{\mathfrak{g}}((u)), b+uB) \qquad (\textit{Proposition } 3.2.6)\\ &= \mathsf{H}^\bullet(\wedge^\bullet \mathfrak{g}^\vee((u)), 0) \oplus \mathsf{H}^\bullet(\overline{C}((u)), b+uB)\\ &\cong \wedge^\bullet \mathfrak{g}^\vee((u)) \oplus \mathsf{H}^\bullet(\overline{C}((u)), b+uB).\end{aligned}$$

However, from [Wam97, Section 3], we know that $\dim \mathsf{HP}_i(A_q) = 2^{n-1}$, thus $\overline{C}((u))$ is acyclic. The induced map

$$\ell : \mathsf{HP}_\bullet(A_q) \to \wedge^\bullet \mathfrak{g}^\vee((u))$$

is thus a surjective map of vector spaces of the same dimension, and therefore an isomorphism. Since ℓ is a map of mixed complexes and both of these vector spaces are endowed with the Hodge filtration, it follows that this map is filtration preserving. □

Remark 3.2.14. Note that the $\mathfrak{g}$-invariance of this complex was only used to show that $\ell \circ B = 0$. ♢

The relative case

Notation 3.2.15. When working with the objects of Section 3.1 such as periodic cyclic homology, Hochschild cochains, etc. over a commutative ring $k \neq \mathbb{C}$, we will use mathcal letters to denote these objects. For example, given a k-algebra E, we will write $\mathcal{HP}_\bullet(E)$ for the periodic cyclic homology of E as a k-algebra. If k is itself a $\mathbb{C}$-algebra, we will write, for example, $\mathsf{HP}_\bullet(E)$ to denote the periodic cyclic homology of E as a $\mathbb{C}$-algebra. ♢

Note that none of the constructions of this section relied on the fact that the parameters q_{ij} were complex numbers. Indeed, we now let the parameters q_{ij} be the coordinates of a torus $\mathsf{Y} = (\mathbb{C}^\times)^{\binom{n}{2}}$, and note that the previous constructions carry through without modification. We will write $\mathcal{A}_q$ for the $\mathcal{O}(\mathsf{Y})$-algebra subject to the same relations defining A_q. Explicitly,

$$\mathcal{A}_q = \frac{\mathcal{O}(\mathsf{Y})\langle x_1^\pm, \ldots, x_n^\pm\rangle}{(x_i x_j = q_{ij} x_j x_i)_{1\leq i<j\leq n}}$$

where we now define $q_{ji} := q_{ij}^{-1}$ and $q_{ii} = 1$ for $1 \leq i < j \leq n$. Note that the collection of monomials $\{x_1^{i_1}\cdots x_n^{i_n}\}_{i_j\in\mathbb{Z}}$ defines a trivialization of $\mathcal{A}_q$. We will refer to this trivialization of $\mathcal{A}_q$ as the *lexicographic trivialization*, as the basis elements of the global sections of the underlying vector bundle are lexicographically ordered. By repeating the calculations of the previous sections, we have an isomorphism

$$\mathcal{HP}_\bullet(\mathcal{A}_q) \cong (\wedge^\bullet \mathfrak{g}^\vee \otimes \mathcal{O}(\mathsf{Y}))((u))$$

of $\mathcal{O}(\mathsf{Y})$-modules. We choose the basis elements of the global sections of $\wedge^\bullet \mathfrak{g}^\vee \otimes \mathcal{O}(\mathsf{Y})$ to be ordered lexicographically. Since this is a finite rank vector bundle over Y, it follows that at each point $q \in \mathsf{Y}$, the fibre of $\mathcal{HP}_\bullet(\mathcal{A}_q)$ at q is $\mathsf{HP}_\bullet(A_q)$.

3.2.2 The Getzler-Gauss-Manin connection

In this section, we recall the construction of the Getzler–Gauss–Manin connection on $\mathcal{HP}_\bullet(\mathcal{A}_q)$, denoted ∇_{GM}. There are several ways of defining this connection, and in this section we will use the original definition given by Getzler in [Get93]. Later, when discussing intrinsic definitions of the lattice in $\mathsf{HP}_\bullet(A_q)$, we will use an alternative (but equivalent) characterization of this connection which can be found in [GS12, Section 4.4]. When using Getzler's model, ∇_{GM} simplifies considerably with each isomorphism

$$\mathcal{HP}_\bullet(\mathcal{A}_q) \to \mathcal{HP}^{\mathfrak{g}}_\bullet(\mathcal{A}_q) \to (\wedge^\bullet \mathfrak{g}^\vee \otimes \mathcal{O}(\mathsf{Y}))((u)).$$

The resulting action of ∇_{GM} on $\wedge^\bullet \mathfrak{g}^\vee((u))$ is simple enough that one can read off the flat sections.

To define the connection, let Y be a smooth affine $\mathbb{C}$-scheme and E an $\mathcal{O}(\mathsf{Y})$-algebra with connection ∇ on the underlying vector bundle. While ∇ is not $\mathcal{O}(\mathsf{Y})$-linear, $d_H\nabla$ is, where d_H is the Hochschild differential on $\mathcal{O}(\mathsf{Y})$-Hochschild cochains extended linearly over $\Omega^\bullet(\mathsf{Y})$.

Definition 3.2.16. Let Y be a smooth affine $\mathbb{C}$-scheme and E an $\mathcal{O}(\mathsf{Y})$-algebra with connection ∇. The *Getzler-Gauss-Manin connection* is defined to be

$$\nabla_{\mathrm{GM}} = L_\nabla - I_{d_H\nabla} \in \mathrm{End}_{\mathbb{C}}(C(E)((u))) \otimes \Omega^1(\mathsf{Y}),$$

where L_D and I_D are defined for any Hochschild cochain D in Section 3.2.1. $\diamondsuit$

In [Get93] it is shown that ∇_{GM} commutes with $b + uB$ and therefore descends to $\mathcal{HP}_\bullet(E)$. Moreover, the induced connection on $\mathcal{HP}_\bullet(E)$ is independent of the choice of ∇, as proven in [Yas17, Proposition 4.1]. In [Get93] Getzler proves that the induced connection on $\mathcal{HP}_\bullet(E)$ is flat if ∇ is flat.

Specialize now to the setting in which $E = \mathcal{A}_q$ is the algebra of quantum tori over $\mathsf{Y} = (\mathbb{C}^\times)^{\binom{n}{2}}$ as described above. We will study how this connection simplifies as we pass to the $\mathfrak{g}$-invariant periodic cyclic complex. We prove the following:

Theorem 3.2.17. *In the trivialization $\mathcal{HP}_\bullet(\mathcal{A}_q) \cong (\wedge^\bullet \mathfrak{g}^\vee \otimes \mathcal{O}_\mathsf{Y})((u))$, one has*

$$\nabla_{\mathrm{GM}} = d + \frac{1}{u} \sum_{1 \le i < j \le n} d \log q_{ij} \iota_{\delta_i \wedge \delta_j}. \tag{3.2.2}$$

To prove this theorem, we need the map

$$\begin{aligned} \chi : \wedge^\bullet \mathfrak{g} &\to \mathcal{E}nd(\mathcal{HP}^{\mathfrak{g}}_\bullet(\mathcal{A}_q)) \\ \delta_1 \wedge \cdots \wedge \delta_k &\mapsto I_{\delta_1} \cdots I_{\delta_k} \end{aligned}$$

where I_D is defined in Definition 3.2.4. This map was shown to be well defined in [Yas17, Theorem 6.3]. We will also need the following operator: for any associative algebra A over a commutative ring k, and for all $X, Y \in C^1(A)$, define a map $L\{X,Y\} : C_{-\bullet}(A) \to C_{-\bullet}(A)$ by

$$\begin{aligned} L\{X,Y\}(a_0 \otimes \cdots \otimes a_n) = &\sum_{i=1}^{n-1} \sum_{j=i+1}^{n} a_0 \otimes \cdots \otimes X(a_i) \otimes \cdots \otimes Y(a_j) \otimes \cdots \otimes a_n \\ &+ \sum_{i=1}^{n} Y(a_0) \otimes a_1 \otimes \cdots \otimes X(a_i) \otimes \cdots \otimes a_n. \end{aligned}$$

Finally, given derivations X, Y of an algebra A, we define their cup-product by

$$X \cup Y(a_1, a_2) = -X(a_1)Y(a_2).$$

We will now use the following proposition, which is proven in [Yas17, Section 6.3].

Proposition 3.2.18. *Suppose $\mathfrak{g}$ is an abelian Lie algebra of derivations acting on $\mathcal{A}_q$. Let ∇ be a $\mathfrak{g}$-invariant connection on $\mathcal{A}_q$ for which*

$$d_H \nabla = \sum_{i=1}^{k} \omega_i X_i \cup Y_i$$

for some $X_i, Y_i \in \mathfrak{g}$ and $\omega_i \in \Omega^1(\mathsf{Y})$. Then, ∇_{GM} descends to the $\mathfrak{g}$-invariant complex

$C^{\mathfrak{g}}(\mathcal{A}_q)((u))$ *and has the form*

$$\nabla_{\mathrm{GM}} = \widetilde{\nabla} + \sum_{i=1}^{k} \omega_i \chi(X_i \wedge Y_i).$$

where

$$\widetilde{\nabla} = L_{\nabla} + \sum_{1=1}^{k} L\{X_i, Y_i\}.$$

We must therefore take a $\mathfrak{g}$-invariant connection, ∇, on $\mathcal{A}_q$ (such as d) and calculate $d_H \nabla$.

Proposition 3.2.19. *Let $\nabla = d$ be the trivial connection on the $\mathcal{O}(\mathsf{Y})$-algebra $\mathcal{A}_q$ in its lexicographic trivialization. Then,*

$$d_H \nabla = \sum_{1 \leq i < j \leq n} d \log q_{ij} \delta_i \cup \delta_j.$$

Proof. Let $a, b \in \mathcal{A}_q$ be of the form $a = x_1^{i_1} \cdots x_n^{i_n}$ and $b = x_1^{j_1} \cdots x_n^{j_n}$ (these generate $\mathcal{A}_q$ as an $\mathcal{O}(\mathsf{Y})$-algebra). Then

$$\begin{aligned} \nabla(ab) &= \nabla(a)b + a\nabla(b) + d_H\nabla(a,b) \\ &= 0 + 0 + d_H \nabla(a,b). \end{aligned}$$

Thus, we need to caclualte $d(ab)$ for a, b as above:

$$\begin{aligned} d(ab) &= d(x_1^{i_1} \ldots x_n^{i_n} x_1^{j_1} \ldots x_n^{j_n}) \\ &= d\left(\left(\prod_{1 \leq k < \ell \leq n} q_{k\ell}^{-i_\ell j_k}\right) x_1^{i_1+i_j} \ldots x_n^{i_n+j_n}\right). \end{aligned}$$

Define $\mathfrak{Q} = \prod_{1 \leq k < \ell \leq n} q_{k\ell}^{-i_\ell j_k}$ and observe that

$$\begin{aligned} d\mathfrak{Q} &= - \sum_{1 \leq k < \ell \leq n} dq_{k\ell}\ i_\ell j_k q_{k\ell}^{-1} \mathfrak{Q} \\ &= - \sum_{1 \leq k < \ell \leq n} d \log q_{k\ell}\ i_\ell j_k \mathfrak{Q}. \end{aligned}$$

Thus,

$$\begin{aligned}
d(ab) &= d\left(\sum_{1\le k<\ell\le n} \mathfrak{Q} x_1^{i_1+j_1}\cdots x_n^{i_n+j_n}\right)\\
&= -\left(\sum_{1\le k<\ell\le n} d\log q_{k\ell}\; i_\ell j_k \mathfrak{Q} x_1^{i_1+j_1}\cdots x_n^{i_n+j_n}\right)\\
&= -\sum_{1\le k<\ell\le n} d\log q_{k\ell}\; i_\ell j_k ab\\
&= -\sum_{1\le k<\ell\le n} d\log q_{k\ell}\; \delta_\ell(a)\delta_k(b)\\
&= \sum_{1\le k<\ell\le n} d\log q_{k\ell}(\delta_k \cup \delta_\ell)(ab),
\end{aligned}$$

which proves the identity. □

We now know the explicit form of ∇_{GM} on the $\mathfrak{g}$-invariant complex:

Proposition 3.2.20. *Let ∇ be the trivial connection on $\mathcal{A}_q$ in the lexicographic trivialization. On the $\mathfrak{g}$-invariant periodic cyclic homology, the induced Getzler-Gauss-Manin connection*

$$\nabla_{\mathrm{GM}} : \mathcal{HP}^{\mathfrak{g}}_\bullet(\mathcal{A}_q) \to \mathcal{HP}^{\mathfrak{g}}_\bullet(\mathcal{A}_q) \otimes \Omega^1(\mathsf{Y})$$

has the form

$$\nabla_{\mathrm{GM}} = \widetilde{\nabla} + \chi(\Omega)$$

where

$$\Omega = \sum_{1\le i<j\le n} d\log q_{ij}\delta_i \wedge \delta_j.$$

In fact, $\widetilde{\nabla}$ is again a flat connection and descends to $\mathcal{HP}^{\mathfrak{g}}_\bullet(\mathcal{A}_q)$, though it is not well defined on $\mathcal{HP}_\bullet(\mathcal{A}_q)$ [Yas17, Section 6]. Our trivialization $\ell : \mathcal{HP}^{\mathfrak{g}}_\bullet(\mathcal{A}_q) \cong (\wedge^\bullet \mathfrak{g}^\vee \otimes \mathcal{O}_{\mathsf{Y}})((u))$ simplifies ∇_{GM} even further:

Lemma 3.2.21. *The map ℓ is flat with respect to $\widetilde{\nabla}$; i.e. for any $\omega \in (\wedge^\bullet \mathfrak{g}^\vee \otimes \mathcal{O}_{\mathsf{Y}})((u))$, one has $\widetilde{\nabla}(\psi(\omega)) = 0$, where ψ is the inverse of ℓ in Theorem 3.2.12.*

Proof. This follows immediately from [Yas17, proposition 7.7] and Theorem 3.2.12. □

It follows that for any $\omega \in (\wedge^\bullet \mathfrak{g}^\vee \otimes \mathcal{O}_{\mathsf{Y}})((u))$, the connection induced by ℓ has the form

$$\nabla_{\mathrm{GM}}(\omega) = d\omega + \chi(\Omega)(\omega).$$

We will abuse notation by writing ∇_{GM} for both the connection on $\mathcal{HP}^{\mathfrak{g}}_\bullet(\mathcal{A}_q)$ and the induced connection on $(\wedge^\bullet \mathfrak{g}^\vee \otimes \mathcal{O}_{\mathsf{Y}})((u))$. One can use [Yas17, theorem 7.8] to prove the following proposition.

Proposition 3.2.22. *Let* $\Omega = \sum_{1\leq i<j\leq n} d\log q_{ij}\delta_i \wedge \delta_j \in \wedge^2\mathfrak{g} \otimes \Omega^1(\mathcal{O}(\mathsf{Y}))$*, and let* $\omega \in (\wedge^\bullet \mathfrak{g}^\vee \otimes \mathcal{O}_{\mathsf{Y}})((u))$*. Then*

$$\begin{aligned}\chi(\Omega)(\omega) &= \frac{1}{u}\iota_\Omega(\omega)\\ &= \frac{1}{u}\sum_{1\leq i<j\leq n} d\log q_{ij}(\iota_{\delta_i\wedge\delta_j}(\omega)).\end{aligned}$$

This completes the proof of Theorem 3.2.17; i.e. we have shown that

$$\nabla_{\mathrm{GM}} = d + \frac{1}{u}\sum_{1\leq i<j\leq n} d\log q_{ij}\iota_{\delta_i\wedge\delta_j}. \tag{3.2.3}$$

Now that we have a flat connection on $\mathcal{HP}^{\mathfrak{g}}_\bullet(\mathcal{A}_q)$, we have parallel transport isomorphisms between the fibres over any q in Y. In particular, obtain a lattice in the fibre $\mathsf{HP}_\bullet(A_q)$ by parallel transporting the lattice at the commutative fibre (i.e. over $q = 1$). At this commutative fibre, we know that in even degrees the lattice is given by $\Lambda_{\mathrm{dR}} \cong \bigoplus_{k\geq 0}(\wedge^{2k}L)(-k)$ (see Theorem 3.1.6). We discuss these parallel transport isomorphisms now.

Notation 3.2.23. We set

$$\widetilde{\Omega} = \sum_{1\leq i<j\leq n} \log q_{ij}\iota_{\delta_i\wedge\delta_j}.$$

Due to the logarithms, this is not in general a single-valued function. $\Diamond$

We can characterize the flat sections of ∇_{GM} as follows.

Theorem 3.2.24. *The flat sections of* ∇_{GM} *are precisely those of the form*

$$e^{-\widetilde{\Omega}/u}(\omega)$$

for $\omega \in \Lambda_{\mathrm{dR}}$*.*

Proof. This proof is identical to that of Proposition 2.2.5. We calculate

$$\begin{aligned}\nabla_{\mathrm{GM}}\psi &= \nabla_{\mathrm{GM}}(e^{-\widetilde{\Omega}/u}\omega) \\ &= -\frac{1}{u}d(\widetilde{\Omega})\left(e^{-\widetilde{\Omega}/u}\omega\right) + (e^{-\widetilde{\Omega}/u})\nabla_{\mathrm{GM}}\omega\end{aligned}$$

where

$$\begin{aligned}d(\widetilde{\Omega}) &= \sum_{1\leq i<j\leq n} d\log q_{ij}\iota_{\delta_i\wedge\delta_j} \\ &= \iota_\Omega\end{aligned}$$

Thus,

$$\nabla_{\mathrm{GM}}\psi = e^{-\widetilde{\Omega}/u}\left(-\frac{1}{u}\iota_\Omega\omega + \nabla_{GM}\omega\right)$$

but $\nabla_{\mathrm{GM}}\omega = \frac{1}{u}\iota_\Omega(\omega)$ so this is zero. □

We would like to obtain a lattice in $\mathsf{HP}_\bullet(A_q)$ by parallel transporting the lattice over $q = 1$, but since these sections are multi-valued, it is not clear that this is well defined. To remedy this, we now show that the lattice is invariant under parallel transport. It is enough to show this at the central fibre; i.e. when $q = 1$. Recall that, with the identification $d\log z_i = e_i \in \mathfrak{g}^\vee$, the lattice (in even homology) at the central fibre has the form

$$\Lambda_{\mathrm{dR}} = \bigoplus_{k\geq 0}\frac{u^k}{(2\pi i)^k}\wedge^{2k}\langle e_1,\ldots,e_n\rangle \tag{3.2.4}$$

$$= \bigoplus_{k\geq 0} u^k(\wedge^{2k}L)(-k) \tag{3.2.5}$$

where L is the character lattice of the torus G.

Invariance of the lattice

To keep the notation light, we will work in the case of a 2-torus, thus over the base $\mathbb{C}^\times$, denoting the coordinate in $\mathbb{C}^\times$ by q. The arguments presented pass to the general case with trivial modifications. Given a flat section ψ of ∇_{GM} in the vector bundle $\mathcal{HP}_0(\mathcal{A}_q)$ we want to know what happens as ψ is parallel transported around a positively oriented loop circling the missing point of $\mathbb{C}^\times$. The lattice at $q = 1$ is given by $\Lambda_{\mathrm{dR}} = \mathrm{ch}(K^0(\mathsf{X}))$, and under the identification $d\log z_i = e_i \in \mathfrak{g}^\vee$, this is precisely

$$\mathbb{Z} \cdot \langle 1 \rangle \oplus \mathbb{Z} \left\langle \frac{u}{2\pi i} e_1 \wedge e_2 \right\rangle$$

as a submodule of $\wedge^{\mathrm{even}} \mathfrak{g}^\vee$. Take the loop $q(t) = \exp(2\pi i t)$ for $t \in [0, 1]$ to be the generator of $\pi_1(\mathbb{C}^\times)$. We saw above that the flat sections of ∇_{GM} are of the form $\psi = e^{-\widetilde{\Omega}/u} \omega$ for $\omega \in \Lambda_{\mathrm{dR}}$, where $\widetilde{\Omega} = \log q \, \iota_{\delta_1 \wedge \delta_2}$. Restricting to the loop $q(t)$ this flat section becomes

$$\begin{aligned} \psi(q(t)) &= \exp(-(\log(e^{2\pi i t}) \iota_{\delta_1 \wedge \delta_2})/u)\omega \\ &= \exp(-2\pi i t \, \iota_{\delta_1 \wedge \delta_2}/u)\omega. \end{aligned}$$

Thus, $\psi(q(1))$ at differs from $\psi(q(0))$ by a factor of $\exp(-2\pi i \, \iota_{\delta_1 \wedge \delta_2}/u)$. It follows that the monodromy representation sends $1 \in \pi_1(\mathbb{C}^\times) \cong \mathbb{Z}$ to the automorphism

$$\begin{aligned} 1 &\mapsto 1 \\ \frac{u}{2\pi i} e_1 \wedge e_2 &\mapsto \left(\frac{u}{2\pi i} e_1 \wedge e_2 \right) - 1 \end{aligned}$$

which preserves the lattice. Moreover, we see that the monodromy has contributed a term of lower weight with respect to the weight filtration on Λ. This is a general phenomenon:

Proposition 3.2.25. *Let*

$$\rho : \pi_1(\mathbb{C}^\times)^{\binom{n}{2}} \cong \mathbb{Z}^{\binom{n}{2}} \to \mathrm{Aut}(\wedge^{\mathrm{even}} \mathfrak{g}^\vee)((u))$$

denote the monodromy homomorphism at $q = 1$ obtained from the Gauss-Manin connection ∇_{GM} on the finite rank vector bundle $\mathcal{HP}_0(\mathcal{A}_q)$ over $\mathsf{Y} = (\mathbb{C}^\times)^{\binom{n}{2}}$. Then for all $\underline{n} \in \mathbb{Z}^{\binom{n}{2}}$, $\rho(\underline{n}) \cdot \Lambda_{\mathrm{dR}} \subset \Lambda_{\mathrm{dR}}$ and moreover the induced action on $\mathrm{gr}^{W^{\mathrm{dR}}}_{\bullet} \Lambda_{\mathrm{dR}}$ is the identity.

It follows that, while the flat sections of ∇_{GM} are multi-valued in general, the parallel transport of the lattice between fibres is well defined. Thus we may speak of *the* lattice in $\mathsf{HP}_0(A_q)$.

Corollary 3.2.26. *Let A_q be a quantum torus. The $\mathbb{Z}$-submodule of $\mathsf{HP}_0(A_q)$ given* *by*

$$e^{-\widetilde{\Omega}/u}\Lambda_{\mathrm{dR}} = e^{-\widetilde{\Omega}/u}\left(\bigoplus_{k\geq 0}\frac{u^k}{(2\pi i)^k}\wedge^{2k}\langle e_1,\ldots,e_n\rangle\right)$$

is a lattice in $\mathsf{HP}_0(A_q)$ *and will be denoted* Λ_q.

As in the Poisson case, we define the weight filtration on Λ_q by

$$W_k\Lambda_q = e^{-\widetilde{\Omega}/u}W_k^{\mathrm{dR}}\Lambda_{\mathrm{dR}}. \tag{3.2.6}$$

Notation 3.2.27. We set $\mathrm{ch}^q := e^{-\widetilde{\Omega}/u}\circ\mathrm{ch} : K^0(\mathsf{X})\to\Lambda_q$, where $\mathsf{X} = \mathrm{Spec}(A_1) = (\mathbb{C}^\times)^n$. ♢

3.2.3 Mixed Hodge structure of a quantum torus

Using the results of the previous sections, we construct a toric mixed Hodge structure on the periodic cyclic homology of a quantum torus A_q. It is clear that these constructions globalize to a TVMHS, and we prove this for completeness, although this will not be used for deformation quantization. From the calculation of the extension data of the TMHS associated to A_q, we will find that this TMHS detects the parameter q itself, proving a Torelli theorem for quantum tori.

Theorem 3.2.28. *Let* A_q *be a quantum torus of rank* n*. Then the tuple*

$$(\Lambda_q, W, \mathsf{HP}_0(A_q), F, c, \mathrm{gr}^W_\bullet(e^{\widetilde{\Omega}/u}))$$

is a toric mixed Hodge structure, where F *denotes the Hodge filtration on* $\mathsf{HP}_0(A_q)$.

Proof. First, we define W on Λ_q as in (3.2.6). By Corollary 3.2.26, we have $\mathrm{ch}^q_{\mathbb{C}} : K^0((\mathbb{C}^\times)^n)\otimes\mathbb{C}\xrightarrow{\sim}\mathsf{HP}_0(A_q)$ and this induces the isomorphism c. We endow $\mathsf{HP}_0(A_q)$ with its Hodge filtration (in the sense of periodic complexes) and take this to be the Hodge filtration F. By Proposition 3.2.6 and Theorem 3.2.13 we have a filtration-preserving isomorphism $\mathsf{HP}_0(A_q)\cong(\wedge^{\mathrm{even}}\mathfrak{g}^\vee)((u))$, and therefore F may be identified with the standard decreasing filtration by exterior powers in $(\wedge^{\mathrm{even}}\mathfrak{g}^\vee)((u))$. For all k, we have an isomorphism

$$\mathrm{gr}^W_\bullet(e^{\widetilde{\Omega}/u}) : \mathrm{gr}^W_k\Lambda_q \xrightarrow{\sim} \mathrm{gr}^{W^{\mathrm{dR}}}_k\Lambda_{\mathrm{dR}} = \begin{cases} u^{k/2}(\wedge^k L)(-\frac{k}{2}) & k \text{ even} \\ 0 & k \text{ odd}. \end{cases} \tag{3.2.7}$$

Thus $\mathrm{gr}_k^W \Lambda_q$ is a pure $\mathbb{Z}$-Hodge structure of weight k for all k. By (3.2.7), we see that this mixed Hodge structure is toric. □

We will denote the TMHS associated to A_q by H_q. It is clear from construction that this TMHS is the fibre over $q \in (\mathbb{C}^\times)^{\binom{n}{2}}$ of the following TVMHS.

Theorem 3.2.29. *Let $\mathsf{Y} = (\mathbb{C}^\times)^{\binom{n}{2}}$ and $\mathcal{A}_q$ the family of quantum tori over Y considered above. The tuple*

$$(\Lambda_q, W, \mathcal{HP}_0(\mathcal{A}_q), F, c, \nabla_{\mathrm{GM}}, \mathrm{gr}_\bullet^W(e^{\widetilde{\Omega}/u}))$$

is a toric variation of mixed Hodge structures on Y.

Proof. This is almost immediate from the above constructions. Indeed, we constructed the lattice Λ_q by parallel transport, so it is flat with respect to ∇_{GM} by definition. The only condition which needs checking is that ∇_{GM} satisfies Griffiths transversality, but this follows from the identification of the Hodge filtration F with the standard decreasing filtration on exterior powers in $(\wedge^\bullet \mathfrak{g}^\vee \otimes \mathcal{O}(\mathsf{Y}))((u))$ and the explicit form of ∇_{GM} given in (3.2.2). □

Let us now analyse the extension data of the TMHS H_q. Recall that

$$\Lambda_q = \exp\left(-\frac{1}{u}\left(\sum_{1\le i<j\le n} \log q_{ij}\iota_{e_i\wedge e_j}\right)\right)\Lambda_{\mathrm{dR}}.$$

Thus, we obtain an extension

$$0 \longrightarrow \mathbb{Z} \longrightarrow W_2 H_q \longrightarrow (\wedge^2 L)(-1) \longrightarrow 0.$$

We have that

$$W_2\Lambda_q = \mathbb{Z}\cdot\langle 1\rangle \oplus \bigoplus_{1\le i<j\le n}\left\langle \frac{1}{2\pi i}(ue_i\wedge e_j - \log q_{ij})\right\rangle$$

where we have chosen some branch cut to make sense of $\log q_{ij}$ (we will see that nothing depends on this choice). We split this sequence with the section

$$\begin{aligned} s : (\wedge^2 L)(-1) &\to W_2 H_q \\ \frac{1}{2\pi i}\frac{dz_i\wedge dz_j}{z_i z_j} &\mapsto \frac{1}{2\pi i}(ue_i\wedge e_j - \log q_{ij}) \end{aligned}$$

for all $1 \leq i < j \leq n$. Noting that

$$F^1(W_2H_q \otimes \mathbb{C}) = \bigoplus_{1\leq i<j\leq n} \mathbb{C}\left\langle \frac{u}{2\pi i} e_i \wedge e_j \right\rangle,$$

we find that the unique map $\psi_s : (\wedge^2 L)(-1) \otimes \mathbb{C} \to \mathbb{Z} \otimes \mathbb{C}$ having $F^1(W_2H_q \otimes \mathbb{C})$ as its graph is the function

$$\frac{1}{2\pi i} \frac{dz_i \wedge dz_j}{z_i z_j} \mapsto \log q_{ij}.$$

This map depends on the choice of section, but a different section will only change the resulting map by a summand which is a multiple of $2\pi i$. It follows that under the isomorphism (1.1.3),

$$[W_2H_q] = \exp(\log q) = (\exp(\log q_{ij}))_{1\ \leq i<j\leq n} = (q_{ij})_{1\leq i<j\leq n}.$$

In other words, we have proven the following.

Theorem 3.2.30 (Torelli theorem for quantum tori)**.** *A multiparametric quantum torus A_q is determined up to isomorphism by the extension class $[W_2H_q]$ of its toric mixed Hodge structure H_q.*

Formal quantum tori

The constructions of the previous sections work for formal families of quantum tori without modification. These are quantum tori $A_{q(\hbar)}$ parametrized by $\mathrm{Spf}(\mathbb{C}[[\hbar]])$; i.e. $q_{ij}(\hbar) \in \mathbb{C}[[\hbar]]$ such that $q \not\equiv 0 \mod \hbar$ for all $1 \leq i < j \leq n$. From this, we obtain an isomorphism $(\wedge^{\mathrm{even}}\mathfrak{g} \otimes \mathbb{C}[[\hbar]])((u)) \cong \mathcal{HP}_0(A_{q(\hbar)})$ and a lattice $\Lambda_{q(\hbar)} = \mathrm{ch}^{q(\hbar)}(K^0((\mathbb{C}^\times)^n)$. Then $\Lambda_{q(\hbar)} \otimes \mathbb{C}$ is the lattice of flat sections for the Getzler-Gauss-Manin connection, which, in the above trivialization, has the form

$$\nabla_{\mathrm{GM}} = d + \frac{1}{u} \sum_{1\leq i<j\leq n} d\log q_{ij}(\hbar) \iota_{\delta_i\wedge\delta_j},$$

where

$$d\log q_{ij}(\hbar) := \frac{d\log q_{ij}(\hbar)}{d\hbar} d\hbar.$$

Theorem 3.2.31. *Let $\mathsf{Y} = \mathrm{Spf}(\mathbb{C}[[\hbar]])$ and $A_{q(\hbar)}$ a family of quantum tori over Y.*

The tuple

$$(\Lambda_{\pi(\hbar)}, W, \mathcal{HP}_0(\mathcal{A}_{q(\hbar)}), F, c, \nabla_{\mathrm{GM}}, \mathrm{gr}^W_\bullet(e^{\widetilde{\Omega}/u}))$$

is a toric variation of mixed Hodge structures on Y*.*

We will denote this TVMHS by $H_{q(\hbar)}$. From H_q we obtain a section $[W_2 H_{q(\hbar)}] \in \mathcal{H}om(\wedge^2 L, \mathbb{C}[[\hbar]]^\times)$ which, repeating the above calculations, one can prove is given by the matrix

$$\exp(\log(q(\hbar)) = (q_{ij}(\hbar))_{1\le i<j\le n} \in (\mathbb{C}[[\hbar]]^\times)^{\binom{n}{2}}. \tag{3.2.8}$$

3.3 Intrinsic definition of the lattice

We have now constructed the toric mixed Hodge structure associated to a quantum torus algebra A_q and could, at this time, calculate Kontsevich's canonical quantization for Poisson tori (X, π). A weakness in the above construction is that, while the lattice Λ_q is canonical, it has no intrinsic definition: we constructed it by parallel transport of the lattice at the central fibre. In this section, we show that this lattice has an intrinsic definition in a formal neighbourhood of any point, which is all we need for deformation quantization.

3.3.1 Complex topological K-theory of noncommutative spaces

In [Bla16] Anthony Blanc defines an invariant of $\mathbb{C}$-linear differential graded (dg) categories called the *noncommutative topological K-theory*. This is an extension of the usual topological K-theory of spaces to $\mathbb{C}$-dg-categories, thus, in particular, to associative algebras over $\mathbb{C}$. The noncommutative topological K-theory groups of an associative algebra A will be denoted $K^{\mathrm{top}}_\bullet(A)$. As in the commutative case, we have canonical isomorphisms $K^{\mathrm{top}}_\bullet(A) \cong K^{\mathrm{top}}_{\bullet+2}(A)$. Furthermore, this invariant is defined so that if A is a finitely generated commutative algebra, then $K^{\mathrm{top}}_\bullet(A) \cong K^{-\bullet}(\mathrm{Spec}(A)(\mathbb{C}))$, where $\mathrm{Spec}(A)(\mathbb{C})$ denotes the complex analytic space associated to the affine scheme $\mathrm{Spec}(A)$. In [Bla16, Section 4.4], Blanc defines a Chern character $\mathrm{ch}^{\mathrm{nc}} : K^{\mathrm{top}}_\bullet(A) \to \mathsf{HP}_\bullet(A)$ which, if A is commutative, agrees with the ordinary topological Chern character (see Theorem 3.1.6). The pair $(K^{\mathrm{top}}, \mathrm{ch}^{\mathrm{nc}})$ is important in noncommutative Hodge theory, as it provides a candidate for an integral structure on $\mathsf{HP}_\bullet(A)$. This is formalized in the following conjecture of Blanc's:

Conjecture 3.3.1 (Lattice conjecture). *Let T be a smooth proper $\mathbb{C}$-linear dg-category. Then the map*

$$\mathrm{ch}^{\mathrm{nc}} \wedge_{\mathbb{S}} H\mathbb{C} : \mathbf{K}^{\mathrm{top}}(T) \wedge_{\mathbb{S}} H\mathbb{C} \to \mathbf{HP}(T)$$

is an equivalence of spectra.

See [KKP08, Definition 2.23] for the definitions of smooth and proper in the noncommutative setting. The objects $\mathbf{K}^{\mathrm{top}}(T)$ and $\mathbf{HP}$ are symmetric spectra whose homotopy groups produce the invariants $K^{\mathrm{top}}_{\bullet}(T)$ and $\mathsf{HP}_{\bullet}(T)$. When we take homotopy groups, the smash product $\wedge_{\mathbb{S}}$ becomes the tensor product over $\mathbb{C}$. Our algebras A_q are smooth, but not proper when viewed as dg-categories. This conjecture has been verified in a number of cases, most notably in [Kon21] for nilpotent thickenings of commutative algebras, but remains open in general. We will prove an instance of this conjecture in the following section for a class of associative algebras which include quantum tori.

3.3.2 The lattice conjecture for quantum tori

To prove the lattice conjecture for quantum tori we recall that both HP and K^{top} are $\mathbb{A}^1$*-homotopy invariants*. This was proven in [Tab12] for periodic cyclic homology and [AH18] for noncommutative topological K-theory. An $\mathbb{A}^1$-homotopy invariant is a functor $\mathrm{dgCat}_{\mathbb{C}} \to \mathcal{T}$, where $\mathrm{dgCat}_{\mathbb{C}}$ is the category of small $\mathbb{C}$-linear dg-categories and $\mathcal{T}$ is any triangulated category, subject to a number of conditions that can be found in [TdB21], for example. Now, let T be a dg-category and $\sigma : T \to T$ a dg-functor inducing an isomorphism of homotopy categories $\mathsf{H}^0(T) \xrightarrow{\sim} \mathsf{H}^0(T)$. For our purposes, one may safely assume that T is an associative algebra and σ is an algebra automorphism. The following theorem is due to G. Tabuada [Tab15].

Theorem 3.3.2. *For every $\mathbb{A}^1$-homotopy invariant $E : \mathrm{dgCat}_{\mathbb{C}} \to \mathcal{T}$, there is a distinguished triangle*

$$E(T) \xrightarrow{E(F)-\mathrm{Id}} E(T) \xrightarrow{E(\pi)} E(T/\sigma^{\mathbb{Z}}) \longrightarrow \Sigma E(T)$$

in $\mathcal{T}$.

In this theorem, if T is an associative algebra A, and σ an algebra automorphism, then $T/\sigma^{\mathbb{Z}} = A \rtimes_{\sigma} \mathbb{Z}$ is the *crossed product algebra*. If k is a commutative ring and A

a finitely generated k-algebra with generators $x_1, \ldots, x_m$, then $A \rtimes_\sigma \mathbb{Z}$ is the k-algebra generated by $x_1, \ldots, x_m, y$ subject to the relations of A and

$$x_i y = \sigma(x_i) y x_i.$$

Theorem 3.3.3. *The lattice conjecture holds for quantum torus algebras A_q; i.e.*

$$\mathrm{image}(\mathrm{ch}^{\mathrm{nc}}) \otimes \mathbb{C} \cong \mathsf{HP}_\bullet(A_q).$$

Proof. Let A_q be a rank n quantum torus algebra with generators $x_1^\pm, \ldots, x_n^\pm$ and define $\mathfrak{A}_0 = \mathbb{C}$. Let $\sigma_0 : \mathfrak{A}_0 \to \mathfrak{A}_0$ be the automorphism

$$\sigma_0(1) = 1$$

and define $\mathfrak{A}_1 = \mathfrak{A}_0 \rtimes_{\sigma_0} \mathbb{Z}$. Then,

$$\mathfrak{A}_1 = \mathbb{C}[x^\pm].$$

Note that this is a subalgebra of A_q. We continue inductively. Let $1 < k < n$ and let $\sigma_{k-1} : \mathfrak{A}_{k-1} \to \mathfrak{A}_{k-1}$ be the automorphism

$$\sigma_{k-1}(x_i) = q_{ik} x_i \qquad 1 \leq i \leq k-1$$

and define $\mathfrak{A}_k = \mathfrak{A}_{k-1} \rtimes_{\sigma_{k-1}} \mathbb{Z}$. Explicitly,

$$\mathfrak{A}_k = \frac{\mathbb{C}\langle x_1^\pm, \ldots, x_k^\pm \rangle}{(x_i x_j - q_{ij} x_j x_i)_{1 \leq i < j \leq k}}$$

and is a subalgebra of A_q. For $k = n$ we recover A_q entirely. Note that by Theorem 3.1.6,

$$\mathrm{ch}^{\mathrm{nc}}(K_\bullet^{\mathrm{top}}(\mathfrak{A}_0)) \otimes \mathbb{C} = \mathrm{ch}(K^\bullet(\mathrm{Spec}(\mathbb{C}))) \otimes \mathbb{C} \cong \mathsf{HP}_\bullet(\mathfrak{A}_0).$$

Using Theorem 3.3.2, for $1 \leq k < n$ we obtain a morphism of distinguished triangles

$$\begin{tikzcd}
K_\bullet^{\mathrm{top}}(\mathfrak{A}_k) \otimes \mathbb{C} \arrow[r] \arrow[d, "\mathrm{ch}^{\mathrm{nc}} \otimes \mathbb{C}"] & K_\bullet^{\mathrm{top}}(\mathfrak{A}_k) \otimes \mathbb{C} \arrow[r] \arrow[d, "\mathrm{ch}^{\mathrm{nc}} \otimes \mathbb{C}"] & K_\bullet^{\mathrm{top}}(\mathfrak{A}_{k+1}) \otimes \mathbb{C} \arrow[r] \arrow[d, "\mathrm{ch}^{\mathrm{nc}} \otimes \mathbb{C}"] & \Sigma K_\bullet^{\mathrm{top}}(\mathfrak{A}_k) \otimes \mathbb{C} \arrow[d, "\mathrm{ch}^{\mathrm{nc}} \otimes \mathbb{C}"] \\
\mathsf{HP}_\bullet(\mathfrak{A}_k) \arrow[r] & \mathsf{HP}_\bullet(\mathfrak{A}_k) \arrow[r] & \mathsf{HP}_\bullet(\mathfrak{A}_{k+1}) \arrow[r] & \Sigma\mathsf{HP}_\bullet(\mathfrak{A}_k)
\end{tikzcd}$$

where the vertical maps of the form $K_\bullet^{\mathrm{top}}(\mathfrak{A}_k) \otimes \mathbb{C} \to \mathsf{HP}_\bullet(\mathfrak{A}_k)$ are isomorphisms by induction. This diagram commutes by the naturality of the Chern character[1]. By the five lemma, the map $\mathrm{ch}^{\mathrm{nc}} \otimes \mathbb{C} : K^{\mathrm{top}}_{\bullet}(\mathfrak{A}_{k+1}) \otimes \mathbb{C} \to \mathsf{HP}_{\bullet}(\mathfrak{A}_{k+1})$ is also an isomorphism. Taking $k = n-1$ completes the proof. □

Note that this proves the lattice conjecture for any finitely generated algebra which is constructed via iterated crossed products of a commutative $\mathbb{C}$-algebra. Our goal now is to show that this lattice agrees with the lattice obtained earlier by parallel transport, at least in a formal neighbourhood of a point.

3.3.3 K-theory as flat sections

In this section, we will show that in a formal neighbourhood of the central fibre, the lattice we constructed using the Chern character ch^q; i.e. the lattice constructed via parallel transport with respect to ∇_{GM}, agrees with the image of Blanc's K^{top} under his Chern character $\mathrm{ch}^{\mathrm{nc}}$. Let $(B, \mathfrak{m})$ be a local Artin $\mathbb{C}$-algebra with an isomorphism $\varphi : B/\mathfrak{m} \xrightarrow{\sim} \mathbb{C}$ and let A_b be a deformation of A_1 over B; that is, a B-algebra such that the map $A_b \to A_1$ induced by φ is an isomorphism. Explicitly, this is a quantum torus over B defined by a multiplicatively skew-symmetric matrix $b = (b_{ij})_{1 \le i < j \le n}$ where $b_{ij} \in B^{\times}$ for all i, j and $b_{ij} \equiv 1 \mod \mathfrak{m}$. We will write $\mathsf{HP}_{\bullet}(A_b)$ for the periodic cyclic homology of A_b as a $\mathbb{C}$-algebra, and $\mathcal{HP}_{\bullet}(A_b)$ for the periodic cyclic homology of A_b as a B-algebra.

It is known that HP and K^{top} are *nil-invariant*; i.e. invariant under nilpotent extensions of associative algebras. This was proven for HP and K^{top} in [Goo85] and [Kon21], respectively. Therefore, we have the following lemma.

Lemma 3.3.4. *Let $E = K^{\mathrm{top}}$ or HP. Then, for A_b as above, the canonical map $A_b \to A_b/\mathfrak{m} \cong A_1$ induces an isomorphism*

$$E(A_1) \cong E(A_b).$$

Thus, we have an isomorphism

$$\mathsf{HP}_{\bullet}(A_b) \xrightarrow{\sim} \mathsf{HP}_{\bullet}(A_1)$$

which factors through $\mathcal{HP}_{\bullet}(A_b)$:

[1] We thank Andrei Konovalov for helpful correspondence concerning the commutativity of this diagram.

$$\begin{tikzcd}
& \mathcal{HP}_\bullet(A_b) \arrow[dr, "\varphi_*"] & \\
\mathsf{HP}_\bullet(A_b) \arrow[ur, "\psi"] \arrow[rr, "\sim"] & & \mathsf{HP}_\bullet(A_1).
\end{tikzcd}$$

At this point, we use the characterization of the Getzler-Gauss-Manin connection given in [GS12, Section 4.4]:

Theorem 3.3.5. *Let $\mathsf{Y} = \operatorname{Spec}(k)$ be an affine $\mathbb{C}$-scheme and let E be an associative $\mathcal{O}(\mathsf{Y})$-algebra such that the quotient $E/\mathcal{O}(\mathsf{Y})$ is locally free. Then there is a canonical flat connection ∇_{GM} on $\mathcal{HP}_\bullet(E)$.*

This connection is characterized as follows: write $\mathsf{HP}_\bullet(E)$ for the periodic cyclic homology of E as a $\mathbb{C}$-algebra, and note there is a natural map $\psi : \mathsf{HP}_\bullet(E) \to \mathcal{HP}_\bullet(E)$ induced by $\varphi : B/\mathfrak{m} \xrightarrow{\sim} \mathbb{C}$. The image of ψ is the lattice of flat sections for ∇_{GM} and will be denoted $\mathcal{HP}_\bullet(E)^\nabla$. That is, the lattice of flat sections for ∇_{GM} are those sections in $\mathcal{HP}_\bullet(E)$ which lift to the periodic cyclic homology $\mathsf{HP}_\bullet(E)$ of the 'total space'. This definition of ∇_{GM} is much closer to the original (algebraic) definition given in [KO68], and agrees with Getzler's construction (see [GS12, Remark 4.4.6]).

Returning to our study of tori, we now know that $\operatorname{image}(\psi) = \mathcal{HP}_\bullet(A_b)^\nabla$ is the lattice of flat sections for ∇_{GM}. It follows that we have a commutative diagram

$$\begin{tikzcd}
& \mathcal{HP}_\bullet(A_b) \arrow[dr, "\varphi_*"] & \\
\mathsf{HP}_\bullet(A_b) \arrow[ur, "\psi"] \arrow[r, "\sim"] & \mathcal{HP}_\bullet(A_b)^\nabla \arrow[u, hook] \arrow[r, "\sim"] & \mathsf{HP}_\bullet(A_1).
\end{tikzcd}$$

Using the naturality of the noncommutative topological Chern character and the nil-invariance of K^{top}, we can extend this to the commutative diagram

$$\begin{tikzcd}
K^{\mathrm{top}}_\bullet(A_b) \arrow[rr, "\sim"] \arrow[d, "\mathrm{ch}^{\mathrm{nc}}"] & & K^{\mathrm{top}}_\bullet(A_1) \arrow[d, "\mathrm{ch}^{\mathrm{nc}}"] \\
\mathsf{HP}_\bullet(A_b) \arrow[r, "\sim"] \arrow[dr, "\psi"'] & \mathcal{HP}_\bullet(A_b)^\nabla \arrow[r, "\sim"] \arrow[d, hook] & \mathsf{HP}_\bullet(A_1) \\
& \mathcal{HP}_\bullet(A_b) \arrow[ur, "\varphi_*"'] &
\end{tikzcd}$$

If we tensor this diagram with $\mathbb{C}$ then the vertical map $\mathrm{ch}^{\mathrm{nc}} \otimes \mathbb{C} : K^{\mathrm{top}}_\bullet(A_1) \otimes \mathbb{C} \xrightarrow{\sim} \mathsf{HP}_\bullet(A_1)$ is an isomorphism. Thus, by the commutativity of the diagram, the leftmost vertical map is also an isomorphism after extending scalars to $\mathbb{C}$. This proves the lattice conjecture for such nilpotent extensions. It follows that the image of $K^{\mathrm{top}}_\bullet(A_b)$

in $\mathcal{HP}_\bullet(A_b)^\nabla$ is a lattice that reduces modulo $\mathfrak{m}$ to the standard lattice in $\mathsf{HP}_\bullet(A_1)$. Since these sections are flat, we conclude that the lattice obtained by parallel transport coincides with the image of $\mathrm{ch}^{\mathrm{nc}}$.

Remark 3.3.6. Note that it was not important that we were deforming the central fibre, as Theorem 3.3.3 proves the lattice conjecture for every quantum torus. We could have then chosen a deformation A_b of A_q for any q; i.e. chosen b such that $b \equiv q \mod \mathfrak{m}$ and then applied the same argument. $\lozenge$

We summarize these results in the following theorem.

Theorem 3.3.7. *Let A_q be a quantum torus and A_b a deformation of A_q over an Artin ring $(B, \mathfrak{m})$. Then the noncommutative topological K-theory is covariant constant with respect to the Getzler-Gauss-Manin connection ∇_{GM} on $\mathcal{HP}_\bullet(A_b)$.*

Since deformation quantization is a statement about these formal neighbourhoods, we may identify K^{top} with the lattice $\Lambda_{q(\hbar)}$ when quantizing. Consider the family of quantum tori $\mathcal{A}_q$ over $(\mathbb{C}^\times)^{\binom{n}{2}}$. Theorem 3.3.7 implies that we are able to identify the lattice in the fibre $\mathsf{HP}_\bullet(A_q)$ at every point $q \in (\mathbb{C}^\times)^{\binom{n}{2}}$ with $K_\bullet^{\mathrm{top}}(A_q)$, but we do not have any control over how this lattice varies from point to point. We would like to know that K^{top} is a holomorphic section of the vector bundle $\mathcal{HP}_\bullet(\mathcal{A}_q)$. We state this as a conjecture. We will say that a family of algebras E over a smooth base scheme Y *deforms* a commutative algebra if there is a distinguished point $y_0 \in \mathsf{Y}$ such that $E|_{y_0}$ is commutative.

Conjecture 3.3.8. *Let E be a family of associative algebras over a smooth affine base $\mathbb{C}$-scheme Y deforming a commutative algebra $E|_{y_0}$. Furthermore, assume that $E/\mathcal{O}(\mathsf{Y})$ is free. Then in the analytic topology, the image of $\mathrm{ch}^{\mathrm{nc}} : K^{\mathrm{top}}(E) \to \mathcal{HP}_\bullet(E)$ is a holomorphic section.*

The holomorphicity of such a section would imply K^{top} consists of covariant constant sections of $\mathcal{HP}_\bullet(E)$, as we would be able to check this at the level of Taylor expansions.

Chapter 4

Quantization of complex tori

4.1 Deformation quantization

In this chapter, we use the toric variations of Hodge structures constructed above to directly quantize complex tori and compare this with the quantizations obtained from formality morphisms. For background on deformation theory and formality morphisms, the reader is referred to [Man99] and [Dol06] respectively.

4.1.1 Formality morphisms

Let X be a smooth complex algebraic variety. Recall that a *formality morphism* is an L_∞-quasi-isomorphism

$$\mathscr{F} : (\mathscr{X}_{\mathsf{X}}^{\bullet+1}, 0, [\cdot,\cdot]) \to (C^{\bullet+1}(\mathcal{O}_{\mathsf{X}}), d_H, [\cdot,\cdot]_G)$$

of the DGLAs of polyvector fields on X and Hochschild cochains on $\mathcal{O}_{\mathsf{X}}$, respectively, such that the first component $\mathscr{F}_1$ of $\mathscr{F}$ is the HKR quasi-isomorphism. The bracket $[\cdot,\cdot]_G$ is the Gerstenhaber bracket. We assume now and in the following that $\mathscr{F}$ is Kontsevich's formality morphism as defined in [Kon03] and that X is an open subset of $\mathbb{A}^n$. We will now recall various extensions of $\mathscr{F}$ to the other structures associated with these DGLAs. First, note that the Hochschild chains $(C_{-\bullet}(\mathcal{O}_X), b)$ are an L_∞-module over the DGLA $(C^\bullet(\mathcal{O}_{\mathsf{X}}), d_H)$ (see [Sho03] for a proof). Kontsevich's map $\mathscr{F}$ 'extends scalars' to give $(C_{-\bullet}(\mathcal{O}_{\mathsf{X}}), b)$ the structure of an L_∞-module over $\mathscr{X}^{\bullet+1}$. There is another space carrying a natural module structure over $\mathscr{X}^{\bullet+1}$: the forms $\Omega_{\mathsf{X}}^\bullet$, endowed with the trivial differential. The following theorem was conjectured by Tsygan in [Tsy99] and proven by Shoikhet in [Sho03]:

Theorem 4.1.1 (Formality of chains)**.** *There exists an L_∞-quasi-isomorphism of L_∞-modules over $\mathscr{X}^{\bullet+1}$*

$$\mathcal{U} : (C_{-\bullet}(\mathcal{O}_{\mathsf{X}}), b) \to (\Omega^\bullet_{\mathsf{X}}, 0).$$

One would naturally like to extend this to incorporate the de Rham differential. This extension was proven by Willwacher in [Wil11]:

Theorem 4.1.2 (Cyclic formality)**.** *There is an L_∞-quasi-isomorphism of L_∞-modules over $\mathscr{X}^{\bullet+1}$*

$$\mathcal{V} : (C_{-\bullet}(\mathcal{O}_{\mathsf{X}})((u)), b + uB) \to (\Omega^\bullet_{\mathsf{X}}((u)), ud).$$

In fact, what Willwacher proved is that Shoikhet's morphism $\mathcal{U}$ intertwines the Connes differential with the de Rham differential, so $\mathcal{U}$ is a morphism of mixed complexes.

Recall that $\mathscr{F}$, as well as the extensions of $\mathscr{F}$ described above, are compatible with twists of the DGLAs: given a Maurer–Cartan element in $\mathscr{X}^{\bullet+1}$, i.e. a Poisson structure $\pi \in \mathsf{H}^0(\mathsf{X}, \wedge^2\mathcal{T}_{\mathsf{X}})$, we obtain a Maurer-Cartan element $\mathscr{F}_0^\pi \in C^\bullet(\mathcal{O}_{\mathsf{X}})[[\hbar]]$, which is an associative deformation of the algebra of functions $\mathcal{O}_{\mathsf{X}}$. There is then an L_∞-quasi-isomorphism between the twisted DGLAs

$$\mathscr{F}^\pi : (\mathscr{X}^{\bullet+1}[[\hbar]], [\hbar\pi, -], [\cdot,\cdot]) \to (C^{\bullet+1}(\mathcal{O}_{\hbar\pi})[[\hbar]], d_{\hbar\pi}, [\cdot,\cdot]_G), \tag{4.1.1}$$

where $\mathcal{O}_{\hbar\pi}[[\hbar]]$ is the deformed algebra with product given by $\mathscr{F}_0^\pi$ and $d_{\hbar\pi} := d_H + [\mathscr{F}_0^\pi, -]_G$. By the results of [Wil16, Section 8], these twists are compatible with cyclic formality, so we obtain an isomorphism

$$\mathsf{HP}_\bullet(\mathsf{X}, \hbar\pi)[[\hbar]] \cong \mathsf{HP}_\bullet(\mathcal{O}_{\hbar\pi})[[\hbar]].$$

Since this isomorphism is induced by a quasi-isomorphism of mixed complexes, it preserves the Hodge filtrations. Furthermore, by the results of [CFW11] (see also [Wil16, Remark 9.4]), this isomorphism intertwines the connections $\nabla_{\hbar\pi}$ and ∇_{GM}.

4.1.2 Isomorphism of Hodge structures

We now assume that the variety X being quantized is a complex torus with torus invariant formal Poisson structure $\pi(\hbar) = \sum_{1\le i<j\le n} \lambda_{ij}(\hbar) z_i z_j \partial_i \wedge \partial_j$. Recall from [Kon03] that Kontsevich's formality morphism is GL_n-equivariant. In particular, if

we restrict to the space of G-invariant Poisson structures on X for a torus G, then the resulting algebra must again be torus invariant. In particular, the weight spaces of the G-action are preserved. Thus, for any quantization $\mathcal{O}_{\pi(\hbar)}$ of $(\mathsf{X}, \pi(\hbar))$ with product $\star_{\pi(\hbar)}$, we must have

$$x_i \star_{\pi(\hbar)} x_j = q_{ij}(\hbar) x_j \star_{\pi(\hbar)} x_i \qquad 1 \leq i < j \leq n$$

where $q_{ij}(\hbar)$ is a formal power series in $\hbar$ for all i, j. In other words, we obtain some formal quantum torus, but we do not have an explicit formula for the series $q_{ij}(\hbar)$.

Remark 4.1.3. For a torus invariant formal Poisson structure on $(\mathbb{C}^\times)^2$, these star products were calculated explicitly up to order six in $\hbar$ in [BPP20]. Using a computer, they proved that

$$x_1 \star_{\pi(\hbar)} x_2 = g(\hbar) x_1 x_2,$$

where

$$g(\hbar) = 1 + \frac{\hbar\lambda}{2} + \frac{(\hbar\lambda)^2}{24} - \frac{(\hbar\lambda)^3}{48} - \frac{(\hbar\lambda)^4}{1440} + \frac{(\hbar\lambda)^5}{480} + \left(\frac{251\zeta(3)^2}{2048\pi^6} - \frac{17}{184320} \right) (\hbar\lambda)^6 + \cdots .$$

where $\lambda = \lambda_{12}$. Note that the conjecturally transcendental number $\zeta(3)/\pi^6$ appears at order six in $\hbar$. Even for these relatively simple Poisson brackets, we see that Kontsevich's formula is likely impossible to calculate by hand. $\Diamond$

We now view a formal G-invariant bivector $\pi(\hbar)$ as a family of G-invariant Poisson structures on X parametrized by $\mathrm{Spf}(\mathbb{C}[[\hbar]])$. In Section 2.2.4 we associated to $\pi(\hbar)$ the TVMHS $H_{\pi(\hbar)}$, whose extension data $[W_2 H_{\pi(\hbar)}] \in \mathcal{H}om(\wedge^2 L, \mathbb{C}[[\hbar]]^\times)$ was determined by the multiplicatively skew-symmetric matrix $\exp(\lambda) = (\exp(\lambda_{ij}))_{1 \leq i < j \leq n}$ (see (2.2.7)). We define ψ to be the map assigning this extension data to $\pi(\hbar)$:

$$\psi : \pi(\hbar) \mapsto (\exp(\lambda_{ij}(\hbar)))_{1 \leq i < j \leq n}.$$

Similarly, we now view a formal quantum torus $A_{q(\hbar)}$ over $\mathbb{C}[[\hbar]]$ as a family of quantum tori parametrized by $\mathrm{Spf}(\mathbb{C}[[\hbar]])$. We saw in (3.2.8) the extension data of the associated TVMHS $H_{q(\hbar)}$ is given by the multiplicatively skew-symmetric matrix $q(\hbar)$ itself. We define ξ to be the map assigning to a family of quantum tori $A_{q(\hbar)}$ the extension data $q(\hbar) \in \mathcal{H}om(\wedge^2 L, \mathbb{C}[[\hbar]]^\times)$. Consider the triangle

$$\begin{tikzcd}
\wedge^2\mathfrak{g}\otimes\mathbb{C}[[\hbar]] \arrow[rr, "\mathscr{F}"] \arrow[dr, "\psi"'] & & (\mathbb{C}^\times)^{\binom{n}{2}}\times\mathrm{Spf}(\mathbb{C}[[\hbar]]) \arrow[dl, "\xi"] \\
& \mathcal{H}om(\wedge^2 L, \mathbb{C}[[\hbar]]^\times) &
\end{tikzcd},$$

where we have identified the space of formal torus invariant Poisson structures with $\wedge^2\mathfrak{g}\otimes\mathbb{C}[[\hbar]]$. In this diagram, $(\mathbb{C}^\times)^{\binom{n}{2}}\times\mathrm{Spf}(\mathbb{C}[[\hbar]])$ is the moduli space of formal quantum tori, and the morphisms ψ and ξ are as above. We now use cyclic formality to see that this diagram commutes. Indeed, letting $A_{q(\hbar)} = \mathscr{F}(\pi(\hbar))$, we have

$$\mathsf{HP}_\bullet(\mathsf{X}, \pi(\hbar)) \cong \mathsf{HP}_\bullet(A_{q(\hbar)}).$$

Now, [CFW11, Proposition 1.4] states that the connections ∇_π and ∇_{GM} are intertwined by cyclic formality, so we have an isomorphism of lattices $\Lambda_\pi \cong \Lambda_q$. The isomorphism of lattices induces an isomorphism on the $\mathrm{gr}^W_\bullet$ objects which respects the toric isomorphisms, as these are induced by parallel transport to the commutative fibre. Finally, since the Hodge filtrations were defined to be the natural filtrations obtained from the periodic complex, we conclude these are isomorphic TVMHS; i.e. $H_{\pi(\hbar)} \cong H_{q(\hbar)}$. It follows that we have an isomorphism of extension data

$$[W_2\Lambda_{q(\hbar)}] = [W_2\Lambda_{\pi(\hbar)}],$$

or equivalently,

$$\exp(\lambda_{ij}(\hbar)) = q_{ij}(\hbar) \qquad \forall\, 1 \le i < j \le n.$$

Therefore, the above triangle commutes. Since ξ is an isomorphism, we obtain our main result:

Theorem 4.1.4. *Let $\mathscr{F}$ denote Kontsevich's formality morphism. The quantization of a G-invariant formal Poisson structure $\pi(\hbar)$ on a complex torus X with respect to $\mathscr{F}$ is the multiparametric quantum torus algebra $A_{q(\hbar)}$, where*

$$q_{ij}(\hbar) = \exp(\lambda_{ij}(\hbar)).$$

In particular, given a G-invariant Poisson structure π on X, we can view this as a formal Poisson structure linear in $\hbar$ via $\pi \mapsto \hbar\pi$. Then Theorem 4.1.4 implies that the canonical quantization of π is the quantum torus algebra $A_{\exp \hbar\lambda}$.

Note that, for much of this analysis, it was not essential that we were using Kontsevich's formality morphism. Indeed, we could have let $\mathscr{F}$ be an arbitrary *stable* formality morphism [Dol21], and the results of [Wil17] would then imply that $\mathscr{F}$ lifts to a cyclic formality morphism. Moreover, by the results of [Dol13], each homotopy class of stable formality morphisms has a representative which globalizes. In particular, each homotopy class of stable formality morphisms has a representative which is GL_n invariant. It follows that any torus invariant formal Poisson structure on $(\mathbb{C}^\times)^n$ will quantize to a formal quantum torus algebra $A_{q(\hbar)}$. We believe that these quantized algebras are independent of the (homotopy class of) stable formality morphism chosen. To prove this, it would be enough to show that any stable formality morphism lifts to a *calculus formality morphism.* The operations of the calculus operad axiomatize the operations of contraction of vector fields into differential forms and the Lie derivative. See [DTT09, Section 3] for more information on calculi. Since the Poisson-Gauss-Manin connection and Getzler-Gauss-Manin connection can be constructed with this calculus structure, such a lifting to a calculus formality theorem would imply these connections are intertwined and we would be able to deduce an isomorphism of extension data as we did above for Kontsevich's morphism.

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