McGill University

Master Thesis

De Sitter solution in the String Landscape

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Declaration of Authorship

The project discussed in this thesis are ideas proposed to me by Professor Keshav Dasgupta and were done in collaboration with him, Maxim Emelin and Radu Tatar. The arXiv preprint number is arXiv: 1908.05288.

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Abstract

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Classical two derivative gravity of type IIB String theory is insufficient to satisfy the no-go theorems in order to have a four dimensional spacetime with positive cosmological constant such as De Sitter. But on the other hand, quantum corrections could allow for de Sitter solutions provided certain constraints are satisfied. But in the time independent background it is found that in order to maintain such constrain an infinite numbers of time-independent corrections are needed. As they have no relative suppression it causes a breakdown in the effective field theory description. Therefore in this study we look for more general time dependent solutions, where both the internal space as well as the background fluxes are all time-dependent with full De Sitter isometry in four dimensional spacetime. We analyse the both the perturbative and non perturbative quantum corrections in such background and determined their corresponding type IIA string coupling g_s scaling. Surprisingly we find out that time dependency allow a finite number of quantum terms at any given order in g_s thus allowing an EFT description. We also show how the no-go theorems and the swampland criteria are avoided in time dependent background. Newton's constant can be kept both time dependent or independent depending upon the ansatz. But the former has a late time singularity which is not present in the later case. We try to present convincing arguments to justify the presence of a late time de Sitter vacuum with time independent Newton's constant to be present in the IIB string landscape and not in the swampland.

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(French Translation) La gravité classique à deux dérivées de type IIB de la théorie des cordes est insuffisante pour satisfaire les théorèmes non-passables, afin de disposer d'un espace-temps à quatre dimensions avec une constante cosmologique positive telle que De Sitter. Par contre, les corrections quantiques pourraient permettre des solutions de Sitter à condition de respecter certaines contraintes. Dans le contexte indépendant du temps, il est évident que pour maintenir une telle contrainte, un nombre infini de corrections indépendantes du temps est nécessaire. Comme ils n'ont pas de suppression relative, cela entraîne une rupture de la description de la théorie du champ effectif. Par conséquent, dans cette étude, nous cherchons des solutions plus générales dépendantes du temps, où l'espace interne, ainsi que les flux de fond, dépendent du temps avec une isométrie De Sitter dans un espace-temps à quatre dimensions. Nous analysons les corrections quantiques perturbative et non-perturbative dans un tel arrière-plan et déterminons la mise à l'échelle de leur couplage de chaînes de type IIA correspondant. De manière surprenante, nous découvrons que la dépendance temporelle permet un nombre fini de termes quantiques pour tout ordre donné, permettant ainsi une description de l'EFT. Nous montrons également comment les théorèmes d'interdiction et les critères de swampland sont évités dans un contexte dépendant du temps. La constante de Newton peut être dépendante du temps ou indépendante en fonction de l'Ansatz. Mais le premier a une singularité tardive qui n'est pas présente dans le dernier cas. Nous essayons de présenter des arguments convaincants pour justifier la présence d'un vide de Sitter à temps tardif avec une constante de Newton indépendante du temps, présent dans le paysage des cordes IIB et non dans les swampland.

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Contents

Chapter 1

Introduction

The concept of the cosmological constant was proposed by Einstein in 1917 to counterbalance the effects of gravity to achieve a static universe - an idea which was the most popular and accepted view at that time. But he eventually dismissed the idea of cosmological constant after Hubble's discovery of the expanding universe in 1931. Quite curiously, from the 1930s until the late 1990s, the cosmological constant was assumed to be zero by most physicists. This viewpoint changed surprisingly in 1998 after people found out that the expansion of the universe is accelerating; implying the possibility of a positive nonzero constant value for the cosmological constant. In the 1990s, after almost sixty years, studies and experiments have confirmed that around 68 percent of the mass–energy density of the universe can be attributed to dark energy. The cosmological constant is the simplest possible explanation for dark energy, and is used in the current standard model of cosmology - referred as the ΛCDM model. The zero point energy due to fluctuations of field, arising from the zero-point energy in their ground state, acts as a contributing factor to the cosmological constant Λ; but calculations considering these fluctuations give rise to an unusually gigantic value of vacuum energy- exceeding the observed value from cosmology by some 120 orders of magnitude. This discrepancy of the calculated value from the observed one is often considered to be one of the worst theoretical prediction in the history of physics (referred to as the cosmological constant problem) and poses one of the greatest theoretical challenges of our time. Possibly we need to have a fully developed theory of quantum gravity, (perhaps superstring theory) before we can predict the smallness of value of the cosmological constant.

It is now well accepted that the late-time behavior of our universe is one of accelerated expansion. De Sitter space is the maximally symmetric vacuum solution of Einstein's field equations with a positive cosmological constant which mimics accelerated expansion. On the other hand, String theory is often described as the leading candidate for the theory of quantum gravity which can give us a nice platform to solve the above mentioned problem. As the late-time behavior of our universe is one of accelerated expansion, we are motivated to look for solutions that exhibit accelerated expansion within string theory. Among the existing proposed constructions, the most prominent one is the KKLT scenario [1], which involves a subtle patchwork of ten-dimensional and four-dimensional phenomena coming from an interplay of supergravity degrees of freedom with stringy effects such as higher derivative corrections, brane instantons or other brane world-volume phenomena. How and whether all the ingredients of any particular construction come together to produce the desired solution is still a matter of some dispute [2, 3, 4, 5].

The lack of full top-down constructions, along with the various objections to existing proposals, has led to recent swampland conjectures[6, 7, 8, 9, 10, 11] regarding the effective potentials that arise in string compactifications; ruling out de Sitter vacua. These conjectures are themselves largely based on the known behavior of effective potentials in regimes of string theory where top-down calculations can be performed. Swampland conjectures favor quintessence models over de Sitter solutions but it comes with the additional problem of time-varying Newton's constant. The problem of finding de Sitter starts with famous Maldacena Nunez No go theorems[12, 13] which basically relates the positivity of four dimensional curvature with energy momentum tensor of the matter fields. Let us consider the an action where gravity is coupled to matter:

$$S_{\text{total}} = \frac{1}{\mathcal{G}_D}\int d^Dx\sqrt{-G_D}R_D + \int d^Dx\mathcal{L}_{\text{int}}, \tag{1.1}$$

Here, $\sqrt{-G_D}$ is the determinant of the D-dimensional metric g_{MN}. $\mathcal{G}_D$ and R_D are the D-dimensional Newton constant and the Ricci scalar in D dimensions respectively. The metric for D dimensional spacetime is g_{MN}, where, M, N etc indices take value from $0, .., D-1$, and $\mathcal{L}_{\text{int}}$ is the interaction lagrangian. The equation of motion for this action is

$$G_{MN} = \frac{\mathcal{G}_D}{2}T_{MN} \tag{1.2}$$

where G_{MN} is the Einstein tensor and T_{MN} the energy momentum tensor. By definition energy momentum tensor is-

$$T_{MN} = -\frac{2}{\sqrt{-G_D}}\frac{\delta\mathcal{L}_{\text{int}}}{\delta g^{MN}}. \tag{1.3}$$

Also Einstein tensor is,

$$G_{MN} = R_{MN} - \frac{1}{2}g_{MN}R \tag{1.4}$$

Re-writing eq. (1.2)

$$R_{MN} = \frac{\mathcal{G}_D}{2}\left(T_{MN} - \frac{1}{D-2}g_{MN}T\right), \tag{1.5}$$

where T is the trace of energy momentum tensor which is defined in the usual way, i.e.

$$T = g^{MN}T_{MN}. \tag{1.6}$$

The D dimensional metric is,

$$ds_D^2 = ds_4^2 + ds_{D-4}^2 \equiv g_{\mu\nu}dx^\mu dx^\nu + g_{mn}dx^m dx^n. \tag{1.7}$$

which could be generalized to include a warp factor which shall be ignored in this section. We have divided the D dimensional spacetime into two parts. The spacetime part M_4 is spanned by coordinates $(x^0, x^1, x^2, x^3) = (t, x, y, z)$, where t is timelike and the rest are spacelike. The rest is internal space $\mathcal{M}^{D-4}$, spanned by spacelike coordinates $x^m, m = 4, .., D-1$. Now as we have considered our D-dimensional spacetime as eq (1.7) we can write the Ricci scalar for M_4 as:

$$R_4 \equiv g^{\mu\nu} R_{\mu\nu}. \tag{1.8}$$

If $R_4 > 0$ we obtain a four dimensional spacetime with positive curvature. De Sitter and FRW metric are prime example. On the other hand if $R_4 < 0$ we will have negative curvature solutions such as anti-de Sitter type geometry, but this is not the state with the current universe. Minkowski space for example has zero curvature, $R_4 = 0$. Taking the trace of (1.5) in the μ, ν directions, we get

$$R_4 = -\frac{\mathcal{K}_D}{2(D-2)} \left[T^\mu_\mu (6 - D) + 4T^m_m \right]. \tag{1.9}$$

Thus to have $R_4 > 0$, we must satisfy the condition:

$$(D-6)\, T^\mu_\mu > 4T^m_m. \tag{1.10}$$

Whatever the content of the Lagrangian, we must satisfy (1.10) if we are to obtain a positively curved four-dimensional universe. But the problem is just using the classical lagrangian of lets say type IIB string theory it is not possible to maintain the above inequality[14, 15]. We definitely need to add quantum corrections[16] but in this setup of time independent compactifications quantum corrections come with their own sets of problems[14, 15]. Please note we are referring ansatz like eq (5) time independent compactification as the internal manifold $\mathcal{M}^{D-4}$ is independent of time. An important upshot of the analysis in ref. [14, 15] is that for a time independent compactification ansatz to de Sitter space, the quantum corrections that needs to be switched on to have a positive cosmological constant in four dimensions, result in the appearance of an infinite tower of additional time-independent corrections. All the quantum corrections come without any clear relative suppression. This was interpreted to indicate a breakdown of an effective field theory description. Therefore, even if a de Sitter compactification ansatz could be realized, the physics in the four dimensional space could not be described by an effective field theory with finite number of fields.

In the previous series of studies by Dasgupta et. al. [14, 15] the construction of de Sitter vacua in type IIB theory were analysed from the M-theory uplift point of view. In M-theory, all the type IIB fluxes can be written into one four-form flux components which makes the equation of motion much simpler to analyse. Additionally, the orientifolds of type IIB become smooth spaces in M-theory. There, all the corrections are built out of various higher order combinations of the curvatures, fluxes and their derivatives can be considered, yielding constraints that the series of quantum corrections have to agree with result in positive 4-dimensional scalar curvature. This proliferation of the number of time-dependent fields does have a slightly simpler representation from the M-theory perspective. However it should be noted that the de Sitter space that we wish to inspect carefully is in the type IIB side. In this study M-theory is used as tracking device to solve the equation of motion in the type IIB setup. It does not imply that we are looking for a de Sitter space in M-theory. So what has been found in previous study is while on one hand all classical sources that could allow for solutions with de Sitter isometries are ruled out by no-go conditions, the quantum corrections, on the other hand, could allow us to have de Sitter solutions in four dimensions provided certain constraints are satisfied. A careful study however reveals that such a constrained system does not allow for an effective field theory description in four-dimensions[14, 15]. In such time-independent compactification there are unfortunately quantum pieces which have no M_p hierarchy and appear in the EOMs without type IIA string coupling any g_s factors.

In this thesis we consider a new ansatz for the internal space geometry as well as the background fluxes where both are time-dependent. We study in detail such a background by including perturbative, nonperturbative as well as local and non-local quantum corrections. Our analysis reveals a possibility of well defined four-dimensional positive constant with de Sitter isometries and time-independent Newton's constant in four dimensions. In our study the quantum contributions obtain a time-dependence and become vanishing at late times, precisely when the type IIB description is expected to be well founded. Also fortunately these hierarchies that we were missing for the time independent cases studied before[14, 15], which in turn lead to the non-existences of four-dimensional EFTs in the type IIB side are present in the time dependent compacification. We find that time dependences of the G-fluxes guarantee a certain level of g_s hierarchies. In time dependent compactification quantum contributions appear as a finite number of quantum terms at any given order in g_s. This consequently allows an EFT description as evident from the g_s scalings for time dependent compactification.

In the next section we first introduce the metric for the time dependent internal manifold. Afterwards, we present an improved classification scheme[17] for the quantum corrections and identify the most general local as well as non-local corrections to M-theory that can be built out of derivatives or integrals of various contractions of the fluxes and curvatures. We evaluate the relative scaling of the quantum corrections with the type IIA string coupling which basically track time dependency of each piece. Here we present two main choices of time dependence for the fluxes and internal geometry, one of which yields time independent Newtons's constant and the other produces time dependent Newton's constant. The former is of course more appealing to us as this is what we observe in nature.

With the classification of the different quantum corrections along with the g_s dependency of each term, we further derive the quantum-corrected equations of motion at every order of g_s in chapter 3. We find out that a solution with positive 4-dimensional curvature can be obtained, i.e. the inequality (1.10) can be maintained provided the leading quantum corrections satisfy constrains similar to those found in reference[18, 19, 20]. For further consistency checking we also investigate the flux quantization as well as anomaly cancellation conditions[21, 22, 23], for our metric ansatz. We further discuss how the no-go condition and the swampland criteria are avoided in generating such a background with the help of quantum corrections. Some recent work on generating de Sitter using different techniques than what we used here are in [24, 25, 26].

Chapter 2

Time dependent compactification

2.1 Time-dependent backgrounds, fluxes and quantum effects

It is difficult to get a four-dimensional effective field theory description with the full set of de Sitter isometries and time-independent internal space ([14] and [15]). The Kasner type solution and dipole type deformation are also seen to be unhelpful ([17]).

These observations may be summarized as follows. Firstly, breaking the de Sitter isometries in four dimensional space for type IIB theory by introducing four-dimensional isometry breaking factors is not useful. It is also unhelpful to keep the metric components of the internal space in type IIB theory time independent by introducing time-independent warp factors. Keeping most of the background G-flux components time-independent, in the M-theory uplift of the type IIB background, also does not help.

We are thus motivated to make the following ansatz for the type IIB metric:

$$ds^2 = \frac{1}{\Lambda(t)\sqrt{h}}(-dt^2 + dx_1^2 + dx_2^2 + dx_3^2) + \sqrt{h}\Big(F_1(t)g_{\alpha\beta}(y)dy^\alpha dy^\beta + F_2(t)g_{mn}(y)dy^m dy^n\Big), \quad (2.1)$$

with $\alpha, \beta = 4, 5$ and $m, n = 6, 7, 8, 9$; h being a function of internal cordinates. Although not natural, this division of the metric components is nevertheless useful. For instance, a time-independent internal space volume can be made by taking the functions $F_i(t)$ as:

$$F_1(t) \equiv \omega^2(t), \qquad F_2(t) \equiv \frac{1}{\omega(t)}, \quad (2.2)$$

where $\omega(t)$ is another arbitrary function of time. Note that with this choice of the metric the internal space is a strict product of a four-dimensional manifold $\mathcal{M}_4$ and a two-dimensional manifold $\mathcal{M}_2$, implying that metric components like $g_{\alpha n}$ will be taken to zero. Generalization of this is easy to achieve simply by switching on $g_{\alpha n}$, so we will not discuss it much here. The division is also reflected in the M-theory uplift of (2.1), which takes the form:

$$ds^2 = e^{2A(y,t)}(-dt^2 + dx_1^2 + dx_2^2) + e^{2B_1(y,t)}g_{\alpha\beta}dy^\alpha dy^\beta + e^{2B_2(y,t)}g_{mn}dy^m dy^n + e^{2C(y,t)}g_{ab}dx^a dx^b, \quad (2.3)$$

where (a, b) are the coordinates of a square two-torus parametrized by coordinates x_3 and x_{11}. The internal eight-manifold in M-theory therefore takes the following form:

$$\mathcal{M}_8 \equiv \mathcal{M}_4 \times \mathcal{M}_2 \times \frac{\mathbb{T}^2}{\mathcal{G}}, \tag{2.4}$$

where locally $\mathcal{G} = 1$ as clear from the metric (2.3). Globally however, as before, we don't want the manifold $\mathcal{M}_8$ to have a vanishing Euler characteristics, so $\mathcal{G}$ will have to be some symmetry group of the internal toroidal space. In terms of the metric (2.3) this is invisible, so we can continue using the local metric. The various warp-factors appearing in (2.3) may now be expressed as:

$$\begin{aligned} e^{2A} &= [\Lambda(t)]^{-\frac{4}{3}}\,[h(y)]^{-\frac{2}{3}}, \quad e^{2C} = [\Lambda(t)]^{\frac{2}{3}}\,[h(y)]^{\frac{1}{3}} \\ e^{2B_1} &= F_1(t)\,[\Lambda(t)]^{-\frac{1}{3}}\,[h(y)]^{\frac{1}{3}}, \quad e^{2B_2} = F_2(t)\,[\Lambda(t)]^{-\frac{1}{3}}\,[h(y)]^{\frac{1}{3}}, \end{aligned} \tag{2.5}$$

where all the parameters appearing above have been defined earlier. The way we have expressed the warp-factors, they appear to be functions of (y^α, y^m) and t, but not functions of the space-time coordinates or of the fibre torus. If we relax the T-duality rules, we could even allow the warp-factors to be functions of the fibre torus, but then the analysis will get more involved. We want to avoid this, and also avoid complicating the space-time geometry by introducing isometry breaking factors.

Again, it may be a concerning the specific procedure of the duality, as the M-theory uplifting requires us to first put the x_3 direction on a circle and then dualize this to M-theory to be eventually combined with the x_{11} circle to form a torus $\mathbf{T}^2$. In this process the special role played by x_3 (or any other chosen space direction) then breaks the De Sitter isometry in the type IIB side converting to a geometry that isn't quite a de Sitter space that we want to study. But we can actually go to the zero volume limit of the M-theory torus $\mathbf{T}^2$ and then slowly increase the type IIA coupling. The latter process is however subtle as the type IIA coupling is proportional to:

$$g_s \propto h^{1/4}\left(\Lambda |t|^2\right)^{1/2}, \tag{2.6}$$

therefore it is only the early time physics that is strongly coupled[1]. Thus the very early times, keeping one of the cycle of $\mathbf{T}^2$ to be of vanishing size, would effectively capture the type IIB background that we want. At late time, since $g_s \to 0$, this can easily be done. The warped eleven-dimensional radius vanishes, and so does the radius of the x_3 circle. Combining them they take us to type IIB.

[1]Recall $-\infty \leq t \leq 0$ because of the flat slicing of the de Sitter space, so $t \to -\infty$ will be early time.

2.1.1 Structure of the warp-factors and the background G-fluxes

F_1 and F_2 for example can maintain this simple relation:

$$F_1(t)F_2^2(t) \equiv e_0 + \frac{e_1 g_s^2}{\sqrt{h}}, \tag{2.7}$$

with specific choices for (e_0, e_1). For example, the choice $(1, 0)$ i.e (2.2) corresponds to the standard de Sitter metric, whereas the choice $(0, 1)$ corresponds to fluctuations over the de Sitter metric. Here we have absorbed the constant type IIB coupling in the definition of h to avoid introducing extra factors and used the IIA coupling g_s to express the RHS. Note that the choice:

$$F_1(t)F_2^2(t) = \frac{g_s^2}{\sqrt{h}}, \tag{2.8}$$

is *not* the volume-preserving choice (2.2). The volume-preserving choice would give us a time-independent overall volume of the internal space. On the other hand (2.8) would give a time-dependent Newton's constant if applied to the standard de Sitter metric. One may then view the two cases from (2.7) as representative of time-independent (i.e (e_0, e_1) = $(1, 0)$) and time-dependent (i.e (e_0, e_1) = $(0, 1)$) cases for the standard de Sitter metric. Interestingly the choice (2.2) resonates well with the condition prescribed for the Newton's constant in [30] (see eq. (2.3) in [30]), so it will be interesting to compare the result of our investigations with the ones in [30]. We will discuss this later.

The functional form for $F_1(t)$ and $F_2(t)$ are still undetermined and the two cases, namely (2.2) and (2.8), differ by having either a constant or g_s^2 on the RHS. For either of these two cases, we can start by defining $F_2(t)$ in the following way:

$$\begin{aligned} F_2(t) &= \sum_{k,n\geq 0} c_{kn} \left(\frac{g_s^2}{\sqrt{h}}\right)^{\Delta k} \exp\left(-\frac{nh^{\Delta/4}}{g_s^\Delta}\right) \\ &= c_{00} + \sum_{k>0} c_{k0}\left(\frac{g_s^2}{\sqrt{h}}\right)^{\Delta k} + \sum_{n>0} c_{0n} \exp\left(-\frac{nh^{\Delta/4}}{g_s^\Delta}\right) + \text{cross terms}, \end{aligned} \tag{2.9}$$

where if c_{00} vanishes then there is no time-independent piece: and c_{kn} are integers with $(k, n) \in \left(\frac{\mathbb{Z}}{2}, \mathbb{Z}\right)$. We have also inserted a constant parameter Δ whose value will be determined later. The above expansion is defined for small g_s in type IIA, and we have assimilated the negative powers of g_s as a non-perturbative sum. The latter is motivated from a resurgent sum of powers of inverse g_s at weak IIA coupling so that all (k, n)-dependent terms in (2.9) are small. However since the type IIA coupling depends on both time and the coordinates of the internal space in the type IIB side, care is needed to interpret what is weak and what is strong coupling here. At a given point y_0 in the internal space, the time interval:

$$|t|^2 < \frac{1}{\Lambda\sqrt{h(y_0)}}, \tag{2.10}$$

should be related to weakly coupled interactions in the type IIA side. For small cosmological constant Λ and small internal warp-factor at any point in the internal space, (2.10) scans a reasonably wide range of time interval provided we can argue for the smallness of both Λ and $h(y)$. The smallness of Λ, in appropriate units, should be viewed as an experimental fact, whereas the smallness of $h(y)$ at all points y^m in the internal space is more non-trivial to establish. We can take this as a requirement and arrange the fluxes etc to suit the equations of motion, but whether this can indeed hold needs to be seen. In any case as long as $h(y) < 1$ and $\Lambda << 1$, (2.10) will assert a wide range of time interval for weakly coupled interactions. With this in mind, we can express $F_1(t)$ as:

$$F_1(t) \equiv \left(\frac{g_s^2}{\sqrt{h}}\right) F_2^{-2}(t) = \sum_{k,n>0} b_{kn} \left(\frac{g_s^2}{\sqrt{h}}\right)^{\Delta k+1} \exp\left(-\frac{nh^{\Delta/4}}{g_s^{\Delta}}\right), \tag{2.11}$$

where b_{kn} are constant coefficients that may be related to the c_{kn} coefficients (for $k > 0, n > 0$) in (2.9) at weak coupling. The way we have expressed (2.11), comparing to (2.9) implies $b_{0n} = b_{1/2,n} = 0$ for $k = 0$ and $k = 1/2$ respectively. Similarly the single and double time derivatives of $F_2(t)$ may be expressed as:

$$\begin{aligned}
\frac{\dot{F}_2}{\sqrt{\Lambda}} &= \sum_{k,n\geq 0} c_{kn}\left[2k\Delta\left(\frac{g_s^2}{\sqrt{h}}\right)^{\Delta k-1/2} + n\Delta\left(\frac{g_s^2}{\sqrt{h}}\right)^{\Delta k-\frac{\Delta}{2}-\frac{1}{2}}\right]\exp\left(-\frac{nh^{\Delta/4}}{g_s^{\Delta}}\right)\\
\frac{\ddot{F}_2}{\Lambda} &= \sum_{k,n\geq 0} c_{kn}\left[2k\Delta(2k\Delta-1)\left(\frac{g_s^2}{\sqrt{h}}\right)^{\Delta k-1} + n^2\Delta^2\left(\frac{g_s^2}{\sqrt{h}}\right)^{\Delta k-\Delta-1}\right]\exp\left(-\frac{nh^{\Delta/4}}{g_s^{\Delta}}\right)\\
&+ \sum_{k,n\geq 0} c_{kn}\left[n\Delta(4k\Delta-\Delta-1)\left(\frac{g_s^2}{\sqrt{h}}\right)^{\Delta k-\Delta/2-1}\right]\exp\left(-\frac{nh^{\Delta/4}}{g_s^{\Delta}}\right),
\end{aligned} \tag{2.12}$$

which shows that the time derivatives of $F_2(t)$ may also be expressed in terms of integer powers of g_s. Needless to say, a similar conclusion also extends to the single and double time derivatives of $F_1(t)$ with the replacement of c_{kn} by b_{kn} in (2.12).

The above discussion pretty much sums up the requirements that we want to impose on the warp-factors so that they solve the equations of motion. Let us take the following configuration:

$$\mathbf{G}_{MNPQ}(y,t) = \sum_{k,n\geq 0} \mathcal{G}^{(k,n)}_{MNPQ}(y)\left(\frac{g_s^2}{\sqrt{h}}\right)^{\Delta k} \exp\left(-\frac{nh^{\Delta/4}}{g_s^{\Delta}}\right), \tag{2.13}$$

with the tensorial coefficient $\mathcal{G}^{(k,n)}_{MNPQ}$ for various choices of $k \in \frac{\mathbb{Z}}{2}$ and $n \in \mathbb{Z}$ being functions of the internal coordinates y^m. Such an expansion guarantees that the flux components are expressed in terms of all positive and negative integer powers of g_s^{Δ}. There could also be a similar expansion for the potential C_{MNP}, but we only use the field strength here as these are the relevant variables for our case. Note also the similarity of the expansion with (2.9) and (2.12). This is intentional as such time dependences should borne out of the time-dependent warp-factors for the internal space, and they in turn will be related to each other via the equations of motion to be satisfied by the corresponding coherent states. All these will be illustrated below, but before we proceed it may be worthwhile to isolate the time dependences of the

G-flux components with all upper indices from the time dependent warp-factors.

The necessity – or more appropriately the usefulness – of such an approach is two-fold. One: isolating the time dependences this way will emphasize the contributions of the warp-factors towards the temporal behavior of the fluxes more succinctly; and two: the time-independent cases would follow simply from the aforementioned expansion by switching off the un-related terms thus forming a single setup to study both time-dependent and time-independent cases. With these in mind, we can isolate the time dependences in the following way:

$$\begin{aligned}
\mathbf{G}^{012\alpha} &= G^{012\alpha}[\Lambda(t)]^{13/3}h^{5/3}F_1^{-1}\\
\mathbf{G}^{012m} &= G^{012m}[\Lambda(t)]^{13/3}h^{5/3}F_2^{-1}\\
\mathbf{G}^{\alpha\beta\gamma\delta} &= G^{\alpha\beta\gamma\delta}[\Lambda(t)]^{4/3}h^{-4/3}F_1^{-4}\\
\mathbf{G}^{\alpha\beta\gamma a} &= G^{\alpha\beta\gamma a}[\Lambda(t)]^{1/3}h^{-4/3}F_1^{-3}\\
\mathbf{G}^{mnpa} &= G^{mnpa}[\Lambda(t)]^{1/3}h^{-4/3}F_2^{-3}\\
\mathbf{G}^{mnpq} &= G^{mnpq}[\Lambda(t)]^{4/3}h^{-4/3}F_2^{-4}\\
\mathbf{G}^{\alpha\beta ab} &= G^{\alpha\beta ab}[\Lambda(t)]^{-2/3}h^{-4/3}F_1^{-2}\\
\mathbf{G}^{mnab} &= G^{mnab}[\Lambda(t)]^{-2/3}h^{-4/3}F_2^{-2}\\
\mathbf{G}^{mnp\alpha} &= G^{mnp\alpha}[\Lambda(t)]^{4/3}h^{-4/3}F_2^{-3}F_1^{-1}\\
\mathbf{G}^{mn\alpha a} &= G^{mn\alpha a}[\Lambda(t)]^{1/3}h^{-4/3}F_2^{-2}F_1^{-1}\\
\mathbf{G}^{m\alpha\beta a} &= G^{m\alpha\beta a}[\Lambda(t)]^{1/3}h^{-4/3}F_1^{-2}F_2^{-1}\\
\mathbf{G}^{mn\alpha\beta} &= G^{mn\alpha\beta}[\Lambda(t)]^{4/3}h^{-4/3}F_2^{-2}F_1^{-2}\\
\mathbf{G}^{m\alpha\beta\gamma} &= G^{m\alpha\beta\gamma}[\Lambda(t)]^{4/3}h^{-4/3}F_2^{-1}F_1^{-3}\\
\mathbf{G}^{m\alpha ab} &= G^{mnab}[\Lambda(t)]^{-2/3}h^{-4/3}F_1^{-1}F_2^{-1},
\end{aligned} \tag{2.14}$$

where the division of the coordinates follow the prescription (2.4) namely, (m, n, p) denote coordinates of $\mathcal{M}_4$; (α, β) denote coordinates of $\mathcal{M}_2$; (a, b) denote coordinates of $\mathbb{T}^2/\mathcal{G}$; and (μ, ν) denote coordinates of the $2+1$ dimensional space-time. It should be clear from (2.14) that the flux components with all upper indices, i.e $G^{MNPQ}(y, t)$ are functions of (y^m, t) and may be got from (2.13) by raising the indices using the un-warped metric components $g_{\alpha\beta}(y), g_{mn}(y)$ and $g_{ab}(y)$ from (2.3). Additionally we can also switch on flux components with at most two legs along the space-time directions. These may be tabulated as:

$$
\begin{aligned}
\mathbf{G}^{\mu\nu ab} &= G^{\mu\nu ab}[\Lambda(t)]^{4/3}h^{2/3}\\
\mathbf{G}^{\mu\nu\alpha a} &= G^{\mu\nu\alpha a}[\Lambda(t)]^{7/3}h^{2/3}F_1^{-1}\\
\mathbf{G}^{\mu\alpha ab} &= G^{\mu\alpha ab}[\Lambda(t)]^{1/3}h^{-1/3}F_1^{-1}\\
\mathbf{G}^{\mu\nu ma} &= G^{\mu\nu mn}[\Lambda(t)]^{7/3}h^{2/3}F_2^{-1}\\
\mathbf{G}^{\mu\nu\alpha\beta} &= G^{\mu\nu\alpha\beta}[\Lambda(t)]^{10/3}h^{2/3}F_1^{-2}\\
\mathbf{G}^{\mu\alpha\beta\gamma} &= G^{\mu\alpha\beta\gamma}[\Lambda(t)]^{7/3}h^{-1/3}F_1^{-3}\\
\mathbf{G}^{\mu\alpha\beta a} &= G^{\mu\alpha\beta a}[\Lambda(t)]^{4/3}h^{-1/3}F_1^{-2}\\
\mathbf{G}^{\mu mab} &= G^{\mu mab}[\Lambda(t)]^{1/3}h^{-1/3}F_2^{-1}\\
\mathbf{G}^{\mu\nu mn} &= G^{\mu\nu mn}[\Lambda(t)]^{10/3}h^{2/3}F_2^{-2}\\
\mathbf{G}^{\mu mna} &= G^{\mu mna}[\Lambda(t)]^{4/3}h^{-1/3}F_2^{-2}\\
\mathbf{G}^{\mu mnp} &= G^{\mu mnp}[\Lambda(t)]^{7/3}h^{-1/3}F_2^{-3}\\
\mathbf{G}^{\mu\nu m\alpha} &= G^{\mu\nu m\alpha}[\Lambda(t)]^{10/3}h^{2/3}F_2^{-1}F_1^{-1}\\
\mathbf{G}^{\mu m\alpha a} &= G^{\mu m\alpha a}[\Lambda(t)]^{4/3}h^{-1/3}F_1^{-1}F_2^{-1}\\
\mathbf{G}^{\mu mn\alpha} &= G^{\mu mn\alpha}[\Lambda(t)]^{7/3}h^{-1/3}F_2^{-2}F_1^{-1}\\
\mathbf{G}^{\mu m\alpha\beta} &= G^{\mu m\alpha\beta}[\Lambda(t)]^{7/3}h^{-1/3}F_2^{-1}F_1^{-2}.
\end{aligned}
\tag{2.15}
$$

Fortunately we will not be required to keep all the flux components in our computations. Some of the G-flux components, such as G_{MNab}, $G_{ma\mu\nu}$ and $G_{mn\mu a}$, have to be put to zero to keep the type IIB solution (2.1) as it is (otherwise cross-terms may develop). The flux components relevant for us are in this study:

$$
\begin{aligned}
&\mathbf{G}_{012m}, \quad \mathbf{G}_{012\alpha}, \quad \mathbf{G}_{mnpa}, \quad \mathbf{G}_{mn\alpha a}, \quad \mathbf{G}_{mnab}\\
&\mathbf{G}_{m\alpha\beta a}, \quad \mathbf{G}_{mnpq}, \quad \mathbf{G}_{mnp\alpha}, \quad \mathbf{G}_{mn\alpha\beta}, \quad \mathbf{G}_{\alpha\beta ab}, \quad \mathbf{G}_{m\alpha ab},
\end{aligned}
\tag{2.16}
$$

whose upper indices may be extracted from (2.14).

2.1.2 Perturbative and non-perturbative quantum corrections

For time-independent Newton's constant, there are two cases to consider for warp-factors $F_1(t)$ and $F_2(t)$ in (2.11) and (2.9). For the first case, we consider vanishing c_{00} for $F_2(t)$ in (2.9). $F_1(t)$ then becomes:

$$
\frac{1}{F_1(t)} = \sum c_{kn}c_{k'n'}\left(\frac{g_s^2}{\sqrt{h}}\right)^{\Delta k+\Delta k'-1}\exp\left[-\frac{(n+n')h^{\Delta/4}}{g_s^\Delta}\right], \tag{2.17}
$$

where $(k, k') = (\frac{\mathbb{Z}}{2}, \frac{\mathbb{Z}}{2})$ and $(n, n') = (\mathbb{Z}, \mathbb{Z})$, and we see that we can equate the inverse of the RHS to the perturbative series (2.11) because of the following limit:

$$
\lim_{g_s\to 0}\frac{1}{g_s^{2n\Delta}}\exp\left(-\frac{1}{g_s^\Delta}\right) = 0, \tag{2.18}
$$

for any finite value of n, implying that for small g_s, both $F_1(t)$ and $F_2(t)$ may be expressed as perturbative series. The difference however is that $F_2(t)$ does not have a time-independent piece whereas $F_1(t)$ does have a time-independent piece for $k = k' = \frac{1}{2}$.

The second case is when we consider non-zero c_{00}, and we take $c_{00} = 1$ without loss of generalities. Clearly $F_2(t)$ now has a time-independent piece, but now $F_1(t)$ takes the following form:

$$F_1(t) = \frac{g_s^2}{\sqrt{h}} - 2\sum_{k,n>0} c_{kn}\left(\frac{g_s^2}{\sqrt{h}}\right)^{\Delta k+1} \exp\left(-\frac{nh^{\Delta/4}}{g_s^{\Delta}}\right) + \mathcal{O}\left(g_s^{4\Delta k+4} e^{-2nh^{\Delta/4}/g_s^{\Delta}}\right), \tag{2.19}$$

where the higher order terms appearing from going beyond quadratic orders for the series sum. We see that (2.19) do not have a time-independent piece, and in fact this could be equated to the perturbative b_{nk} coefficients in (2.9) as alluded to earlier.

Thus it appears that, demanding the fluctuation condition (2.8), allows both $F_1(t)$ and $F_2(t)$ to have a perturbative series but selectively precludes a time-neutral piece in one over the other. This case may be rectified if the demand like (2.8) on Newton's constant is eliminated, wherein the perturbative series for both $F_1(t)$ and $F_2(t)$ may now be unconstrained. For the time being we will take $c_{00} = 1$ in the definition of $F_2(t)$, implying the following relations for the time derivatives of $F_1(t)$:

$$\begin{aligned} \dot{F}_1 &= \frac{2g_s}{h^{1/4}F_2^2}\left(\Lambda^{1/2} - \frac{g_s}{h^{1/4}}.\frac{\partial}{\partial t}\log F_2\right) \propto g_s\Big(1 + \mathcal{O}(g_s^{\Delta})\Big) \\ \ddot{F}_1 &= \frac{2\Lambda}{F_2^2} - \frac{4g_s\Lambda^{1/2}}{h^{1/4}F_2^3} - \frac{4g_s\Lambda^{1/2}\dot{F}_2}{h^{1/4}F_2^3} - \frac{2g_s^2\ddot{F}_2}{h^{1/2}F_2^3} + \frac{6g_s^2\dot{F}_2^2}{h^{1/2}F_2^4} \propto 1 + \mathcal{O}(g_s^{\Delta}), \end{aligned} \tag{2.20}$$

showing that both $\dot{F}_1$ as well as $\ddot{F}_1$ have perturbative expansions in powers of g_s because $1/F_2^n$ has perturbative expansion in terms of g_s for all values of n. However $1/F_1^n$ does not have any perturbative expansion in terms of g_s for $g_s \to 0$, but could have once accompanied by other factors that go as positive powers of g_s. For example the power of g_s that appears from a generic combination of $F_i(t)$ and their time derivatives may be written as:

$$\frac{g_s^m F_2^r \dot{F}_1^n \dot{F}_2^p \ddot{F}_1^l \ddot{F}_2^q}{F_1^k} \sim g_s^{m+n-2k}\Big(1 + \mathcal{O}(g_s^{\Delta})\Big), \tag{2.21}$$

where we only isolate the g_s factor but do not show the perturbative series in the bracket. The latter could be easily ascertained from (2.12) and (2.20). The above analysis shows that as long as

$$k \le \frac{m+n}{2}, \tag{2.22}$$

any series containing terms like (2.20) will have a perturbative g_s expansion in the type IIA side. Our analysis also shows the irrelevancy of the other powers controlled by r, p, l and q as they are always proportional to $1 + \mathcal{O}(g_s^{\Delta})$ and therefore already perturbative.

Product of G-fluxes and g_s expansions

Let us study the following quantum correction, Consider the following series:

$$\mathbb{Q}_1 \equiv \sum_k c_k \left(\frac{\mathbf{G}^{mnpq}\mathbf{G}_{mn}{}^{ab}\mathbf{G}_{abpq}}{M_p^3}\right)^k, \tag{2.23}$$

where c_k are numerical constants, $\mathbf{G}_{MNPQ}$ are the *warped* G-fluxes and M_p is the Planck scale in M-theory. This is an infinite series and clearly every term is time-neutral if we consider time independent compactfication, or its M-theory uplift, as shown in [15]. Plugging the flux and the metric ansatze (2.13) and (2.3) respectively in (2.23), we get:

$$\mathbb{Q}_1 = \sum_k c_k \left[\sum_{\{u_i\}\geq 0} \frac{\left(\mathcal{G}^{(u_1,u_2)}\right)^{mnpq}\left(\mathcal{G}^{(u_3,u_4)}\right)_{mn}{}^{ab}\left(\mathcal{G}^{(u_5,u_6)}\right)_{abpq}}{M_p^3 F_2^4 h^2}\left(\frac{g_s^2}{\sqrt{h}}\right)^{\zeta^s \Delta u_{2s-1}} \exp\left(-\frac{\zeta^r u_{2r} h^{\Delta/4}}{g_s^\Delta}\right)\right]^k, \tag{2.24}$$

where the indices are raised and lowered by the un-warped metric with (m, n) being the coordinates of $\mathcal{M}_4$ and (a, b) being the coordinates of $\mathbb{T}^2/\mathcal{G}$. We have also used ζ^s to denote the sum with both u_{2s-1} as well as u_{2s} with:

$$\zeta^1 = \zeta^2 = \zeta^3 = 1, \quad \zeta^0 = \zeta^k = 0 \ \forall\, k \geq 4, \tag{2.25}$$

such that depending on the value of u_i the series (2.24) may or may not have a time-neutral piece. (The repeated indices are summed over.) From the way we constructed the series, it should be clear that $u_{2s-1} \in \frac{\mathbb{Z}}{2}$ and $u_{2s} \in \mathbb{Z}$, implying that if these parameters start from zero as denoted in (2.24), $\mathbb{Q}_1$ will take the form:

$$\mathbb{Q}_1 = \sum_k c_k \left[\frac{\left(\mathcal{G}^{(0,0)}\right)^{mnpq}\left(\mathcal{G}^{(0,0)}\right)_{mn}{}^{ab}\left(\mathcal{G}^{(0,0)}\right)_{abpq}}{h^2 M_p^3} + \mathcal{O}(g_s^\Delta, e^{-1/g_s^\Delta})\right]^k, \tag{2.26}$$

with the g_s independent term will be the time-neutral piece exactly as we had in [15]. Presence of such a term will create the same hierarchy problem that we encountered in [14, 15], so our attempt here would be to somehow eliminate such a term. This is easily achieved by imposing:

$$\mathcal{G}^{(0,0)}_{MNPQ}(y) = 0, \tag{2.27}$$

which in turn will eliminate all time-neutral pieces that have $\mathbf{G}_{MNPQ}$ in them. The puzzle however is that the condition (2.27) does not preclude terms that were not originally time neutral with the IIB metric, but could now become time-neutral if one chooses the IIB metric (2.1) or it's M-theory uplift (2.3). To see whether this could happen then calls for a more careful analysis.

To begin, let us first concentrate on quantum series constructed exclusively from product of G-fluxes with no extra derivatives. The G-flux may be represented from

(2.13), by including the condition (2.27), in the following way:

$$\begin{aligned}\mathbf{G}_{MNPQ} &= g_s^{2\Delta k}\left[\mathcal{G}_1(y) + G_1(y, g_s^{\Delta})\right]_{MNPQ} + e^{-1/g_s^{\Delta}}\left[\mathcal{G}_2(y) + G_2(y, e^{-1/g_s^{\Delta}})\right]_{MNPQ} \\ &+ g_s^{2\Delta k} e^{-1/g_s^{\Delta}}\left[\mathcal{G}_3 + G_3\left(y, g_s^{\Delta}, e^{-1/g_s^{\Delta}}\right)\right]_{MNPQ}, \end{aligned} \tag{2.28}$$

where $k \in \frac{\mathbb{Z}}{2}$; and $G_i(y, g_s^{\Delta}, e^{-1/g_s^{\Delta}})$ and $\mathcal{G}_i(y)$ for $i = 1, ..., 3$ may be read up from $\mathcal{G}^{(q,n)}$ appearing in (2.13) with or without including the g_s pieces respectively. Note that, compared to (2.13), the smallest power of g_s for the G-flux is $2\Delta k$ whose range of values will be ascertained below[2]. Clearly, once we pull out $g_s^{2\Delta k}$, the series still has a perturbative expansion thanks to the weak coupling limit (2.18).

With this we are now ready to write terms made exclusively with product of G-fluxes. We require two kinds of terms: one, with no free Lorentz indices, and two, with two free Lorentz indices. The one with no free Lorentz indices may be expressed as[3]:

$$\mathbf{g}^{MM'}\mathbf{g}^{NN'}......\mathbf{g}^{DD'}\mathbf{G}_{MQPR}\mathbf{G}_{NUHG}.....\mathbf{G}_{ABCD} \equiv \left[\mathbf{g}^{-1}\right]^{2m}\left[\mathbf{G}\right]^{m}, \tag{2.29}$$

where m is the number of G-flux components and $\mathbf{g}_{MN}$ is the warped M-theory metric components. The indices $M, N, ..$ cover the coordinates of the eight dimensional internal space (2.4), and the RHS of (2.29) is the shortened way of expressing the product of the G-fluxes contracted by the metric indices. The power of the inverse metric is ascertained from the fact that the $4m$ components of the G-flux may be completely contracted by $2m$ inverse metric components. These $2m$ inverse metric components may be divided into l_1 inverse metric components from $\mathbb{T}^2/\mathcal{G}$; l_2 metric components from $\mathcal{M}_2$ and l_3 metric components from $\mathcal{M}_4$ of the internal space (2.4). Using this, the leading order g_s dependence of (2.29) may be written as:

$$\left[\mathbf{g}^{-1}\right]^{2m}\left[\mathbf{G}\right]^{m} \sim g_s^{2\Delta k m - 2(2l_1+2l_2-l_3)/3}\left(1 + \mathcal{O}\left(g_s, e^{-1/g_s}\right)\right), \tag{2.30}$$

where we have used the perturbative series for $F_1(t)$ and $F_2(t)$ given in (2.19) and (2.9) respectively to express their g_s dependences. At this stage it is useful to note that the sum of the (l_1, l_2, l_3) factors should be equal to $2m$, i.e $l_1 + l_2 + l_3 = 2m$ so that (2.29) remains Lorentz invariant. This reproduces our first condition:

$$\left(\frac{6\Delta k - 8}{3}\right)m + 2l_3 \geq 0, \tag{2.31}$$

with the equality leading to the time-neutral case. Clearly for $\Delta k \geq \frac{3}{2}$ there is no constraint as $l_3 \geq 0$. In fact if $m > 1$, l_3 must satisfy $l_3 > 1$, otherwise it will be difficult to have Lorentz invariant terms. For $\Delta k \geq \frac{1}{2}$, we will at least require $l_3 \geq \frac{5m}{6}$, which means for $m = 3$ we require $l_3 = 4$. This is of course consistent with

[2]An erroneous way to proceed would be to expand $\exp\left(-\frac{1}{g_s^{\Delta}}\right)$ as powers of $1/g_s^{\Delta}$ to extract $g_s^{2\Delta k}$ from the series with $k \in \frac{\mathbb{Z}}{2}$, Such an expansion is not valid at any stage of the expansion in the $g_s << 1$ limit that we are working on.

[3]One subtlety that we should keep track of is the fact that the G-fluxes are anti-symmetric whereas the metric components are symmetric in their respective indices.

the simplest case (2.23). Thus for $\frac{1}{2} \leq \Delta k < \frac{3}{2}$ we can avoid the time-neutral series by constraining l_3. However if $\Delta k \geq \frac{3}{2}$, there would be no time-neutral series that can appear from any combinations of pure G-fluxes.

Similarly for the case with two free Lorentz indices with m G-flux components we now require $2m - 1$ number of inverse metric components. The reasoning for this is simple to state. The generic energy-momentum tensor, for either G-fluxes G or quantum terms Q, may be written as:

$$\mathbb{T}_{MN}^{(G,Q)} \equiv -\frac{2}{\sqrt{\mathbf{g}_{11}}}\frac{\delta S_{\rm eff}}{\delta \mathbf{g}^{MN}}, \tag{2.32}$$

where $S_{\rm eff}$ is the effective action at any given scale. Such a procedure either *removes* an inverse metric component or *adds* an inverse-of-an-inverse metric component. In either case, the number of inverse metric components reduces by one. The g_s expansion then remains similar to the RHS of (2.30) but l_i satisfy $l_1 + l_2 + l_3 = 2m - 1$. This gives rise to the following constraint:

$$\left(\frac{6\Delta k - 8}{3}\right) m + \frac{4}{3} + 2l_3 \geq 0, \tag{2.33}$$

which may be compared to (2.31). For $\Delta k = \frac{1}{2}$, l_3 should at least satisfy $l_3 \geq \frac{5m-4}{6}$, implying that for $m = 3$, $l_3 \geq 2$. In general $l_3 \geq 1$ even for $m = 1$, although with $m = 1$ there doesn't appear any simple time-neutral term possible. Again we see that if $\Delta k \geq \frac{3}{2}$, there is no constraint on l_3, and it appears impossible to construct time-neutral series with two free Lorentz indices.

We can also discuss the case when $F_1(t)$ and $F_2(t)$ have inverses that are perturbatively expandable as powers of g_s. Clearly for such a case, (2.8) cannot be satisfied and therefore the Newton's constant has to be defined using (2.2). Nevertheless, one may see that the quantum terms with zero and two free Lorentz indices with only G-fluxes go as $g_s^{k_1}$ and $g_s^{k_2}$ respectively, where k_1 and k_2 are bounded by the following inequalities:

$$\begin{aligned} k_1 &\equiv \left(\frac{6\Delta k + 4}{3}\right) m - 2l_1 \geq 0 \\ k_2 &\equiv \left(\frac{6\Delta k + 4}{3}\right) m - \frac{2}{3} - 2l_1 \geq 0, \end{aligned} \tag{2.34}$$

where we see that the constraints on l_1 are stronger than what we had for l_3 in (2.31) and (2.33) above. However since l_1 captures the metric for the toroidal fibre $\mathbb{T}^2/\mathcal{G}$, we expect l_1 to be small and satisfy the inequalities (2.34). In fact since $l_1 < 2m$, so if $\Delta k \geq \frac{3}{2}$ both the inequalities in (2.34) are easily satisfied. Interestingly when $k = 0$, if we take $m = 3p$ for the scenario with zero Lorentz indices and $m = 3q + 2$ with two free Lorentz indices, we have:

$$\begin{aligned} &l_1 = 2p, \quad l_2 + l_3 = 4p, \quad m = 3p \\ &l_1 = 2q + 1, \quad l_2 + l_3 = 4q + 2, \quad m = 3q + 2. \end{aligned} \tag{2.35}$$

where the combination $l_2 + l_3$ appears because $\mathcal{M}_6$ is not sub-divided into $\mathcal{M}_2$ and

$\mathcal{M}_4$. Thus we see that for $(p,q) \in (\mathbb{Z}, \mathbb{Z})$ there are infinite possible solutions all giving rise to time-neutral series of the form (2.23)[4]. This justifies the claims made in [15] regarding a class of time-neutral quantum series.

G-fluxes with multiple derivatives

We now consider the case where there are derivatives along with G-fluxes, all contracted in two possible ways: one with zero Lorentz indices and two, with two free Lorentz indices. To illustrate this case, let us start with a simple example from [15] that has no free Lorentz indices:

$$\mathbb{Q}_2 \equiv \sum_k b_k \left(\frac{\square^2 \mathbf{G}_{mnab}\mathbf{G}^{mnab}}{M_p^6}\right)^k, \tag{2.36}$$

where $\square$ is the covariant derivative defined on the six-dimensional base $\mathcal{M}_2 \times \mathcal{M}_4$ with the warped metric. With time-independent G-flux, and without any $F_i(t)$ factors in the metric, (2.36) is clearly time-neutral because every term in (2.36) is time-neutral. But now, taking the G-flux as in (2.28), with (m, n) being the coordinates of $\mathcal{M}_4$, $\mathbb{Q}_2$ yields:

$$\mathbb{Q}_2 = \sum_k b_k \left[\sum_{\{u_i\}\geq 0} \frac{\square^2 \left(\mathcal{G}^{(u_1,u_2)}\right)_{mnab}\left(\mathcal{G}^{(u_3,u_4)}\right)^{mnab}}{F_2^4 h^2 M_p^6}\left(\frac{g_s^2}{\sqrt{h}}\right)^{\Delta(u_1+u_3)} \exp\left(-\frac{(u_2+u_4)h^{\Delta/4}}{g_s^\Delta}\right)\right]^k \tag{2.37}$$

where the g_s independent piece will lead to the same issue that we faced in [15], which in turn may be alleviated by imposing (2.27) as before. However the issue plaguing earlier, namely the possibility of generating *new* time-neutral series, now requires a careful assessment of terms of the form (2.36) taking the g_s dependent G-flux (2.28) into account. The kind of term that we want to consider will then be of the form:

$$\mathbf{g}^{MM'}\mathbf{g}^{M_1M_1'}...\mathbf{g}^{DD'}\partial_{M_1}\partial_{M_2}...\partial_{M_n}\left(\mathbf{G}_{MQPR}\mathbf{G}_{NUHG}...\mathbf{G}_{ABCD}\right) \equiv \left[\mathbf{g}^{-1}\right]^{2m+\frac{n}{2}}[\partial]^n[\mathbf{G}]^m, \tag{2.38}$$

where the RHS is a shortened symbolic expression for the derivative expressions. Clearly with only four derivative, contracted appropriately, will reproduce the terms in the series (2.36). Interestingly the form of the g_s expansion is exactly similar to the expression on the RHS of (2.30) i.e $g_s^{k_3}$, except now l_i satisfy $l_1 + l_2 + l_3 = 2m + \frac{n}{2}$. This implies:

$$|k_3| \equiv \left|\left(\frac{6\Delta k - 8}{3}\right)m - \frac{2n}{3} + 2l_3\right| \geq 0 \tag{2.39}$$

where the equality would lead to the time-neutral series. On the other hand, since n appears with a relative *minus* sign, sufficiently large n will reverse the power of k_3 in $g_s^{k_3}$ and make it negative. Such a scenario should make sense if all the inverse

[4] The example in (2.23) is made of $m = 3$ so $p = 1$. Therefore $l_1 = 2, l_2 + l_3 = 4$ with zero free Lorentz indices.

powers of g_s can be rearranged as:

$$\sum_k \frac{\alpha_k h^{\Delta k/4}}{g_s^{2\Delta k}} = \sum_l \beta_l \exp\left(-\frac{n_l h^{\Delta/4}}{g_s^{\Delta}}\right), \tag{2.40}$$

with the integer α_k being related to the integers $(\beta_l.n_l)$. The equality (2.40) is the consequence of summing the series in appropriate way, and should in principle be possible if non-perturbatively the series has to make sense[5]. Assuming this to be the case, the puzzle however is more acute. We show what happens if we take a particular value of n for a given m, i.e n number of derivatives, such that k_3 vanishes. In fact all we require is for n to take the following value:

$$n = 3l_3 + (3\Delta k - 4)\, m, \tag{2.41}$$

to create a new class of time-neutral series with m G-fluxes and n derivatives. One might rewrite (2.40) in a slightly different way that puts the relative minus sign elsewhere as:

$$\left(\frac{6\Delta k + 4}{3}\right) m + \frac{n}{3} - 2\,(l_1 + l_2) \geq 0, \tag{2.42}$$

which simply transfers the puzzle now on the values of l_1 and l_2 instead of on the number of derivatives. This doesn't appear to alleviate the issue because increasing n also increases the metric components. However since l_1 and l_2 denote the metric components along $\mathbb{T}^2/\mathcal{G}$ and $\mathcal{M}_2$ respectively, and if we assume that the G-flux components are functions of the base $\mathcal{M}_4$ *only*, then increasing the number of derivatives will simply increase l_3 without changing l_1 and l_2! This way the constraint (2.42) may be easily satisfied without invoking any extra constraint on k. In fact even if we allow for two free Lorentz indices, the change from (2.42) is minimal:

$$\left(\frac{6\Delta k + 4}{3}\right) m + \frac{n}{3} - \frac{2}{3} - 2\,(l_1 + l_2) \geq 0, \tag{2.43}$$

since $n \geq 2$ in most cases. Thus again with more derivatives, there would be no constraint on k. For small number of derivatives, we expect $l_1 + l_2 < 2m$. Therefore for $\Delta k \geq \frac{3}{2}$, $\left(\frac{6\Delta k+4}{3}\right) m > 4m$ implying that this would dominate over the term $-2(l_1 + l_2)$ making the LHS of both (2.42) as well as (2.43) always positive definite. This brings us to similar conclusion that we had earlier, namely with $\Delta k \geq \frac{3}{2}$, arbitrary flux products with arbitrary number of derivatives do not lead to time-neutral series provided the G-fluxes are functions of the coordinates of the $\mathcal{M}_4$ base only. For F_1 and F_2 satisfying (2.2) instead of (2.8), the constraint equations for zero and

[5] In other words at every order in k, terms on the LHS of (2.40) blow-up, yet the sum on the RHS remains perfectly finite. Thus the representation on the LHS is never the right way to study inverse g_s expansion near $g_s \to 0$. The correct expression will always be the RHS of (2.40).

two free Lorentz indices become respectively:

$$\begin{aligned}\left(\frac{6\Delta k+4}{3}\right)m+\frac{n}{3}-2l_1\geq 0\\ \left(\frac{6\Delta k+4}{3}\right)m+\frac{n}{3}-\frac{2}{3}-2l_1\geq 0,\end{aligned}\tag{2.44}$$

which are readily satisfied by imposing similar conditions on the G-fluxes and on k, because increasing n does not affect l_1 and so $\Delta k\geq\frac{3}{2}$ still controls the positivity of the LHS of both the inequalities in (2.44). We will however soon see that the condition can be relaxed. Again for $k=0$, we expect the following two cases:

$$\begin{aligned}&m=3p_1+p_2,\quad n=2p_2,\quad l_1=2p_1+p_2,\quad l_2+l_3=4p_1+2p_2\\ &m=3q_1+q_2+2,\quad n=2q_2,\quad l_1=2q_1+q_2+1,\quad l_2+l_3=4q_1+2q_2+2\end{aligned}\tag{2.45}$$

with zero and two free Lorentz indices respectively. Clearly since we expect $(p_i,q_i)\in(\mathbb{Z},\mathbb{Z})$, there are infinitely many possible solutions each of which leading to a series like (2.37), and therefore justifying another class of time-neutral quantum series advertised in [15][6].

Curvature algebra and product of curvatures

In general relativity, curvatures may be represented by Riemann tensor, Ricci tensor and Ricci scalar. To simplify the ensuing analysis we develop a curvature algebra.

Curvature tensors are mainly governed by the metric of the internal space. We need to see how everything scales with respect to g_s. For example, in writing the metric components as:

$$\begin{aligned}[\mathbf{g}]\equiv\mathbf{g}_{MN} &= \left(g_s^{4/3}g_{ab},g_s^{4/3}g_{\alpha\beta},g_s^{-2/3}g_{mn}\right)\otimes\left(1+\mathcal{O}(g_s^\Delta,e^{-1/g_s^\Delta})\right)\\ &\equiv \left(g_s^{4/3},g_s^{4/3},g_s^{-2/3}\right)\otimes\left(1+\mathcal{O}(g_s^\Delta,e^{-1/g_s^\Delta})\right)_{MN}\to\left(g_s^{4/3},g_s^{-2/3}\right),\end{aligned}\tag{2.46}$$

the RHS of the second line of (2.46) tells us that the terms in the metric scale as powers of g_s as $\mathcal{O}(g_s^\Delta,e^{-1/g_s^\Delta})$ corrections are irrelevant to the analysis that we want to perform here. In the same vein, we can express the Christoffel symbol in the following way:

$$\begin{aligned}\Gamma^M_{NP}\equiv[\mathbf{g}^{-1}]\partial[\mathbf{g}] &= \left[\left(g_s^{-4/3},g_s^{2/3}\right)\times\left(g_s^{4/3},g_s^{-2/3}\right)\right]\otimes\left(1+\mathcal{O}(\partial,g_s^\Delta,e^{-1/g_s^\Delta})\right)^M_{NP}\\ &= \left(1,g_s^{-2},g_s^2\right)\otimes\left(1+\mathcal{O}(\partial,g_s^\Delta,e^{-1/g_s^\Delta})\right)^M_{NP}\to\left(1,g_s^{-2},g_s^2\right),\end{aligned}\tag{2.47}$$

[6] In fact the term in (2.37) is for $m=2,n=4$, therefore $p_1=0,p_2=2,l_1=2,l_2+l_3=4$ with zero free Lorentz indices.

where again the extreme RHS of the second line denotes the overall scaling of the terms of the Christoffel symbol. Note that the derivative action in the definition of the Christoffel symbol does not act on $g_s/\sqrt{h}$ and therefore directly goes in $\mathcal{O}(\partial, g_s^\Delta, e^{-1/g_s^\Delta})$ implying that it would act on y^M dependent pieces where y^M are in general the coordinates of eight-dimensional internal space in M-theory[7].

The identity element in (2.47) is related to those terms in the Christoffel symbol where the g_s scaling of $[\mathbf{g}^{-1}]$ cancels with the g_s scaling of $\partial[\mathbf{g}]$. This happens when we deal with the metric components of the individual sub-spaces of the eight manifold, namely $\mathcal{M}_2$, $\mathcal{M}_4$ or $\mathbb{T}^2/\mathcal{G}$. Similarly the other powers of g_s may also be explained by looking at various contributions to the Christoffel symbol. For us of course only the g_s scaling matters for the time being.

Christoffel symbols now combine together to create the curvature tensors, namely the Riemann tensor, Ricci tensor and the Ricci scalar. Our symbolic manipulation should again work for these cases. For example the Riemann tensor with one upper index may be expressed in this language, in the following way:

$$\begin{aligned}\mathbf{R}^M{}_{NPQ} &= \partial_{[N}\Gamma^M_{P]Q} + \Gamma^M_{[N|S|}\Gamma^S_{P]Q} \\ &\equiv \left(1, g_s^{-2}, g_s^2\right) \otimes \Big(1 + \mathcal{O}(\partial^2, g_s^\Delta, e^{-1/g_s^\Delta})\Big)^M_{NPQ} + \left(1, g_s^{-2}, g_s^2, g_s^{-4}, g_s^4\right) \otimes \Big(1 + \mathcal{O}(\partial, g_s^\Delta, e^{-1/g_s^\Delta})\Big)^2\Big|^M_{NPQ},\end{aligned} \tag{2.48}$$

where in the first line $|S|$ implies that the index S do not participate in the anti-symmetric operation of its neighboring indices (here it is between indices N and P). The above form of the Riemann tensor implies that, in terms of g_s scalings we can simply express this as:

$$\mathbf{R}^M{}_{NPQ} \equiv \left(1, g_s^2, g_s^{-2}, g_s^4, g_s^{-4}\right), \tag{2.49}$$

which is got by combining the exponents of g_s from the two terms without worrying about the $\mathcal{O}(g_s^\Delta, e^{-1/g_s^\Delta})$ contributions. This shortened form captures the main message and is clearly much more economical to use, but does miss out in distinguishing various components that scale in the same way with g_s. This is not an immediate concern, so we will continue with this formalism unless a more sophisticated analysis is called for. Similarly the Riemann tensor with all lower indices may be expressed as:

$$\begin{aligned}\mathbf{R}_{MNPQ} &= \mathbf{g}_{ML}\mathbf{R}^L{}_{NPQ} \equiv \left(g_s^{-2/3}, g_s^{4/3}, g_s^{-8/3}, g_s^{10/3}, g_s^{-14/3}, g_s^{16/3}\right) \\ &= \left(g_s^{4/3}, g_s^{-2/3}\right) \otimes \left(1, g_s^2, g_s^{-2}\right) + \left(g_s^{4/3}, g_s^{-2/3}\right) \otimes \left(1, g_s^2, g_s^{-2}, g_s^4, g_s^{-4}\right),\end{aligned} \tag{2.50}$$

where the second line shows how the scaling exponents came about by taking products of various terms. It is interesting to note that although the Riemann tensor with one upper index has a g_s independent piece, the Riemann tensor with all lower indices do not seem to have any such piece. Additionally a specific component of

[7]More precisely, defining $h(y) = H^4(y)$, it is easy to infer that $\partial_0\left(\frac{g_s}{H}\right) = \sqrt{\Lambda}$ and $\partial_n\left(\frac{g_s}{H}\right) = 0$. To avoid clutter, we will ignore the $H(y)$ and simply denote the terms with g_s scalings.

Riemann tensor, since it is constructed out of derivatives and products of Christoffel symbols, has at least four terms with leading g_s exponents[8] and therefore may be expressed as:

$$\mathbf{R}_{MNPQ} = \sum_{i=1}^{4} g_s^{a_i} \left[\mathbb{R}_i(y) + \mathcal{R}_i(y, g_s^{\Delta}, e^{-1/g_s^{\Delta}})\right]_{MNPQ} = g_s^{a_k} \left[\mathbb{R}_k + \mathcal{O}(y, g_s^{\Delta}, e^{-1/g_s^{\Delta}})\right]_{MNPQ}, \tag{2.51}$$

where $a_k = \min(a_1, a_2, a_3, a_4)$ will govern the g_s expansion for the particular Riemann tensor. Of course many of the above g_s powers cannot be realized because of the absence of certain cross-terms in the metric. If we ignore these subtleties for the time being, the curvature tensors take the following form:

$$\begin{aligned}
&\mathbf{R}_{MNPQ} = \left(g_s^{-14/3}, g_s^{-8/3}, g_s^{-2/3}, g_s^{4/3}, g_s^{10/3}, g_s^{16/3}\right) \\
&\mathbf{R}_{MP} = \mathbf{g}^{NQ}\mathbf{R}_{MNPQ} = \left(1, g_s^{-6}, g_s^{-4}, g_s^{-2}, g_s^{2}, g_s^{4}, g_s^{6}\right) \\
&\mathbf{R} = \mathbf{g}^{MP}\mathbf{R}_{MP} = \left(g_s^{-22/3}, g_s^{-16/3}, g_s^{-10/3}, g_s^{-4/3}, g_s^{2/3}, g_s^{8/3}, g_s^{14/3}, g_s^{20/3}\right).
\end{aligned} \tag{2.52}$$

All the above g_s scalings got using the curvature algebra assume the generic scenario where the metric components are functions of all the coordinates of the four manifold and, as mentioned earlier, cross-terms exist. However the former cannot be imposed in the flux sector if we want to avoid time-neutral series with derivatives on fluxes. Extending this to the metric components, we can assume that the un-warped metric components and the warp-factors are all functions of the coordinates y^m of $\mathcal{M}_4$ implying that the curvature polynomials will also be functions of y^m.

The latter condition, i.e the presumption that all metric cross-terms exist, again cannot be realized in our case because of the way we expressed the metric (2.3) and the four-manifold (2.4). Thus a more careful considerations of the scalings of the various tensor components are called for. Imposing the two constraints: (a) metric components and the curvature tensors are functions of $\mathcal{M}_4$ only; and (b) only cross-terms satisfying the division (2.4) are allowed, the various curvature tensors scale in the following way:

$$\begin{aligned}
&\mathbf{R}_{mnpq} = g_s^{-2/3}, \quad \mathbf{R}_{abab} = g_s^{10/3}, \quad \mathbf{R}_{abmn} = \mathbf{R}_{ambn} = g_s^{4/3}, \quad \mathbf{R}_{\alpha ab\beta} = g_s^{10/3} \\
&\mathbf{R}_{mn\alpha\beta} = g_s^{4/3}, \quad \mathbf{R}_{\alpha\beta\alpha\beta} = g_s^{10/3}, \quad \mathbf{R}_{\alpha mnp} = \mathbf{R}_{\alpha anp} = \mathbf{R}_{abc\alpha} = \mathbf{R}_{amnp} = \mathbf{R}_{a\alpha\beta n} = 0,
\end{aligned} \tag{2.53}$$

where we do not show the $\mathcal{O}(g_s^{\Delta}, e^{-1/g_s^{\Delta}})$ corrections that accompany all the curvature tensors. Although the above set of tensors and their scalings are considerably simpler than what one would have expected from a generic set-up of (2.52), the generic scalings are nevertheless useful because they do not rely on the way we express the four-manifold. We are searching for a specific cosmological solution with a specific internal space geometry; so we will stick with (2.53) for now and look for

[8] This implies that each of these four terms have a leading g_s exponent followed by higher powers of g_s^{Δ} and $e^{-1/g_s^{\Delta}}$.

quantum series with zero and two free Lorentz indices. A zero free Lorentz index quantum term now takes the following form:

$$\begin{aligned}\mathbb{Q}_3 &= \mathbf{g}^{m_i m'_i}....\mathbf{g}^{\beta_q\beta'_q}\prod_{\{i\}=1}^{\{l_i\}}\mathbf{R}_{m_in_ip_iq_i}\mathbf{R}_{a_jb_ja_jb_j}\mathbf{R}_{p_kq_ka_kb_k}\mathbf{R}_{\alpha_la_lb_l\beta_l}\mathbf{R}_{\alpha_p\beta_pm_pn_p}\mathbf{R}_{\alpha_q\beta_q\alpha_q\beta_q}\\ &\equiv \left[\mathbf{g}^{-1}\right]^{L_1+L_2+L_3}\prod_{\{i\}=1}^{\{l_i\}}\left[\mathbb{R}_i\right],\end{aligned} \tag{2.54}$$

where the set $\{i\}$ denotes the set of $i, j, k...p$ integers that determines the product of all the available Riemann tensors with each set of Riemann tensors (and its various permutations for a given set of indices) occur $l_i, l_j, l_k, ...l_p$ times. The second line is a symbolic way to represent this using inverse metric components. It is clear that:

$$L_1 = 2l_2 + l_3 + l_4, \quad L_2 = 2l_6 + l_4 + l_5, \quad L_3 = 2l_1 + l_3 + l_5, \tag{2.55}$$

with the assumption that $l_1, ..., l_6$ occur in the same order in which the curvature tensors appear in the quantum piece $\mathbb{Q}_3$. In other words $\mathbf{R}_{mnpq}$ occurs l_1 times, $\mathbf{R}_{abab}$ occurs l_2 times, and so on[9]. Similarly, L_1, L_2 and L_3 denote the number of inverse metric components along $\mathbb{T}^2/\mathcal{G}, \mathcal{M}_2$ and $\mathcal{M}_4$ respectively[10]. Using this formalism, and plugging in the appropriate g_s scalings, it is easy to infer that:

$$\mathbb{Q}_3 \equiv \left[\mathbf{g}^{-1}\right]^{L_1+L_2+L_3}\prod_{\{i\}=1}^{\{l_i\}}\left[\mathbb{R}_i\right] = g_s^{2(l_1+l_2+l_3+l_4+l_5+l_6)/3}\left(1 + \mathcal{O}(g_s^\Delta, e^{-1/g_s^\Delta})\right), \tag{2.56}$$

implying that the quantum piece $\mathbb{Q}_3$ can *never* be time-neutral. Such a conclusion is interesting in the light of our earlier discussions with G-fluxes. Therein we had to impose some minimal g_s scalings for the G-flux components to avoid time-neutral series. Here we see that the curvature terms avoid the time-neutrality without any imposition of extra constraints. One would also like to infer what happens when $F_i(t)$ are not constrained by (2.8) but follow (2.2). For such a case the scaling turns

[9] An underlying assumption is that the Riemann tensors are contracted in appropriate ways so that there is no need to explicitly insert the curvature scalar $\mathbf{R}$ or the Ricci tensor $\mathbf{R}_{MN}$ in the expression (2.54) for $\mathbb{Q}_3$. This way we can also avoid differentiating between symmetric or anti-symmetric Ricci tensors, namely $\mathbf{R}_{(MN)}$ or $\mathbf{R}_{[MN]}$ respectively.

[10] The inverse metric components that we are using here have components $\mathbf{g}^{ab}, \mathbf{g}^{\alpha\beta}$ and $\mathbf{g}^{mn}$, and in later sections we will use other space-time components like $\mathbf{g}^{ij}$ and $\mathbf{g}^{00}$. In this language the symbolic representation of the inverse metric components in (2.54), i.e the symbol $\left[\mathbf{g}^{-1}\right]^{L_1+L_2+L_3}$ may be expressed in the following way:

$$\left[\mathbf{g}^{-1}\right]^{L_1+L_2+L_3} \equiv \left(\mathbf{g}^{ab}\right)^{L_1}\left(\mathbf{g}^{\alpha\beta}\right)^{L_2}\left(\mathbf{g}^{mn}\right)^{L_3} \equiv \prod_{i,j,k}^{L_{1,2,3}}\mathbf{g}^{a_ib_i}\mathbf{g}^{\alpha_j\beta_j}\mathbf{g}^{m_kn_k}$$

in other words, $\left(\mathbf{g}^{MN}\right)^{L_k}$ is defined as the following product $\left(\mathbf{g}^{MN}\right)^{L_k} \equiv \prod_{i=1}^{L_k}\mathbf{g}^{M_iN_i}$ where $(M.N)$ = (a, b), (α, β) or (m, n). More generic representations, that include space-time metrics in addition to the internal space metrics, appear in (2.66) and in (2.79).

out to be:

$$\mathbb{Q}'_3 = g_s^{2(l_1+l_2+l_3+l_4+l_5+l_6)/3}\left(1 + \mathcal{O}(g_s^\Delta, e^{-1/g_s^\Delta})\right), \tag{2.57}$$

which is exactly the same scaling as in (2.56) despite that fact that now the metric components have different g_s scalings. The conclusion then remains the same as above: there can be no time-neutral series with zero Lorentz index with only curvature tensors.

What happens when we have two free Lorentz indices? The only changes that can occur are in the values of L_1, L_2 and L_3. This is again easy to quantify: if we want free (a, b) Lorentz indices, all we need is to take (L'_1, L_2, L_3) metric components where $L'_1 = L_1 - 1$, with L_1 being the value quoted in (2.55). Thus generically we need $L'_j = L_j - 1$ with j defining the three possible class of metric choices. Putting everything together, the g_s scaling may be expressed as g_s^κ where κ takes the following two values:

$$\kappa \equiv \frac{2}{3}\sum_{i=1}^{6} l_i + \frac{4}{3}, \quad \kappa \equiv \frac{2}{3}\sum_{i=1}^{6} l_i - \frac{2}{3}, \tag{2.58}$$

where the first one corresponds to indices along $\mathbb{T}^2/\mathcal{G}$ and $\mathcal{M}_2$ and the second one corresponds to indices along $\mathcal{M}_4$. Note that since at least one of the $l_i \geq 1$, $\kappa \geq 0$ where the strict inequality is for the first case. For the second case there is a possibility for $\kappa = 0$ when $l_1 = 1$, implying that the Ricci tensor $\mathbf{R}_{mn}$ is actually time-neutral with or without $F_i(t)$ being constrained by (2.8) as was also evident from our curvature algebra (2.52). This will not be an issue as we will discuss later.

We now elaborate the quantum series with product of curvature tensors and derivatives. As with the G-fluxes we will consider the case where the derivatives are only along the $\mathcal{M}_4$ direction i.e all components of the metric are functions of the internal $\mathcal{M}_4$ coordinates. The quantum terms now take the form:

$$\begin{aligned}
\mathbb{Q}_4 &= \mathbf{g}^{m_i m'_i}....\mathbf{g}^{\beta_q\beta'_q}\partial_{m_r}....\partial_{m_s}\left(\prod_{\{i\}=1}^{\{l_i\}}\mathbf{R}_{m_i n_i p_i q_i}\mathbf{R}_{a_j b_j a_j b_j}\mathbf{R}_{p_k q_k a_k b_k}\mathbf{R}_{\alpha_l a_l b_l \beta_l}\mathbf{R}_{\alpha_p\beta_p m_p n_p}\mathbf{R}_{\alpha_q\beta_q\alpha_q\beta_q}\right)\\
&\equiv \left[\mathbf{g}^{-1}\right]^{L_1+L_2+\hat{L}_3}[\partial]^n\prod_{\{i\}=1}^{\{l_i\}}[\mathbb{R}_i],
\end{aligned} \tag{2.59}$$

where L_1 and L_2 are as given in (2.55) and $\hat{L}_3 = L_3 + \frac{n}{2}$ where n is the number of derivatives. It is now easy to derive the following g_s scalings with zero free Lorentz index:

$$\mathbb{Q}_4 = g_s^{2(l_1+l_2+l_3+l_4+l_5+l_6+n/2)/3}\left(1 + \mathcal{O}(g_s^\Delta, e^{-1/g_s^\Delta})\right), \tag{2.60}$$

showing that there are no time-neutral series possible with curvature tensors and derivatives without imposing any additional constraints. The above scaling remains unchanged even if $F_i(t)$ satisfy volume preserving condition (2.2). On the other hand, if we demand two free Lorentz indices, the change is again minimal in the sense that the two κ values quoted in (2.58) unequivocally change by:

$$\kappa \to \kappa + \frac{n}{3}, \tag{2.61}$$

which is always positive because we expect at least one of the $l_i \geq 1$ and $n > 1$. Thus with derivatives there appears no possibilities of having time-neutral series whether or not $F_i(t)$ are constrained by (2.8).

Adding space-time curvatures with derivatives

The inclusion of space-time curvature contributions is another aspect of the curvatures that is going to change our results. So far we have steered clear of space-time effects, namely fluxes and metric components along the space-time directions, but now it is time to include them in our quantum series. The space-time metric in M-theory scales as $\mathbf{g}_{\mu\nu} \sim g_s^{-8/3}$ which is different from all the metric scalings in the internal space. The g_s scalings of the curvature tensors with legs along the spatial directions are easy to illustrate:

$$\mathbf{R}_{ijij} = g_s^{-14/3}, \quad \mathbf{R}_{ijmn} = g_s^{-8/3}, \quad \mathbf{R}_{iajb} = g_s^{-2/3}, \quad \mathbf{R}_{i\alpha j\beta} = g_s^{-2/3}, \tag{2.62}$$

with other spatial components vanishing. Compared to (2.53), the spatial curvature tensors have predominantly negative powers of g_s scalings.

The curvature tensors with at least one temporal direction is bit more involved because of the time dependences of the various warp-factors creating numerous cross-terms. Nevertheless the g_s scalings can be determined uniquely for each of the curvature tensors. For the present case we have the following tensor components:

$$\begin{aligned} &\mathbf{R}_{0mnp} = g_s^{-5/3}, \quad \mathbf{R}_{0m0n} = g_s^{-8/3}, \quad \mathbf{R}_{0i0j} = g_s^{-14/3}, \quad \mathbf{R}_{0a0b} = g_s^{-2/3} \\ &\mathbf{R}_{0\alpha 0\beta} = g_s^{-2/3}, \quad \mathbf{R}_{0\alpha\beta m} = g_s^{1/3}, \quad \mathbf{R}_{0abm} = g_s^{1/3}, \quad \mathbf{R}_{0ijm} = g_s^{-11/3}, \end{aligned} \tag{2.63}$$

including various possible permutations of each components. The g_s powers are again predominantly negative, and the scalings are computed taken all the earlier considerations of the dependence of the metric components only on the coordinates of $\mathcal{M}_4$. Of course, as before, we have not specified the $\mathcal{O}(g_s^{\Delta}, e^{-1/g_s^{\Delta}})$ corrections that accompany each of the curvature tensors listed in (2.62) and (2.63).

With the curvature scalings at our disposal, let us work out the quantum terms with product of the curvature tensors. Comparing with (2.53), (2.62) and (2.63) we see that there are 18 distinct curvature tensors excluding the allowed permutations of the indices of the individual tensors. Therefore to write the full quantum terms, we resort to some short-hand techniques. We define:

$$(\mathbf{R}_{MNPQ})^{l_i} \equiv \prod_{k=1}^{l_i} \mathbf{R}_{M_k N_k P_k Q_k}, \tag{2.64}$$

where the subscript denote the various possible permutations and products of the curvature tensor for a give set of indices. Using this notation we can express the quantum piece, appearing from the curvature tensors only, in the following way:

$$\begin{aligned}\mathbb{Q}_5 &= \mathbf{g}^{m_i m'_i} \ldots . \mathbf{g}^{j_k j'_k} \left(\mathbf{R}_{mnpq}\right)^{l_1} \left(\mathbf{R}_{abab}\right)^{l_2} \left(\mathbf{R}_{pqab}\right)^{l_3} \left(\mathbf{R}_{\alpha ab\beta}\right)^{l_4} \left(\mathbf{R}_{\alpha\beta mn}\right)^{l_5} \left(\mathbf{R}_{\alpha\beta\alpha\beta}\right)^{l_6} \\ &\times \left(\mathbf{R}_{ijij}\right)^{l_7} \left(\mathbf{R}_{ijmn}\right)^{l_8} \left(\mathbf{R}_{iajb}\right)^{l_9} \left(\mathbf{R}_{i\alpha j\beta}\right)^{l_{10}} \left(\mathbf{R}_{0mnp}\right)^{l_{11}} \left(\mathbf{R}_{0m0n}\right)^{l_{12}} \left(\mathbf{R}_{0i0j}\right)^{l_{13}} \\ &\times \left(\mathbf{R}_{0a0b}\right)^{l_{14}} \left(\mathbf{R}_{0\alpha 0\beta}\right)^{l_{15}} \left(\mathbf{R}_{0\alpha\beta m}\right)^{l_{16}} \left(\mathbf{R}_{0abm}\right)^{l_{17}} \left(\mathbf{R}_{0ijm}\right)^{l_{18}}, \end{aligned} \tag{2.65}$$

where the components of the warped inverse metric are used to contract the indices of the curvature tensors in a suitable way (extra care needs to be implemented to contract the indices because of the anti-symmetry of the first two and the last two indices of a given curvature tensor). In a compact notation, (2.65) may be written as:

$$\mathbb{Q}_5 \equiv \left[\mathbf{g}^{-1}\right]^{E_1+E_2+E_3+E_4+E_5} \prod_{i=1}^{18} \left(\mathbf{R}_{MNPQ}\right)^{l_i}, \tag{2.66}$$

where the term in the bracket is defined in terms of individual components in (2.64) and thus should be expanded accordingly. The powers of the inverse metric components E_i are linear functions of l_i and may be expressed as:

$$\begin{aligned} &E_1 = 2l_7 + l_8 + l_9 + l_{10} + l_{13} + l_{18} \\ &E_2 = \frac{l_{11}}{2} + l_{12} + l_{13} + l_{14} + l_{15} + \frac{l_{16}}{2} + \frac{l_{17}}{2} + \frac{l_{18}}{2} \\ &E_3 = 2l_1 + l_3 + l_5 + l_8 + \frac{3l_{11}}{2} + l_{12} + \frac{l_{16}}{2} + \frac{l_{17}}{2} + \frac{l_{18}}{2} \\ &E_4 = 2l_2 + l_3 + l_4 + l_9 + l_{14} + l_{17}, \quad E_5 = l_4 + l_5 + 2l_6 + l_{10} + l_{15} + l_{16}, \end{aligned} \tag{2.67}$$

where $E_1, E_2, ..., E_5$ count the metric components along (i,j), $(0,0)$, (m,n), (a,b), and (α,β) respectively. Since we are only after the g_s scalings, such a counting of the metric components would make sense. Therefore using the g_s scalings of the metric components as well as the curvature tensors from (2.53), (2.62) and (2.63), it is easy to see that the g_s scaling of $\mathbb{Q}_5$ becomes:

$$\mathbb{Q}_5 = g_s^{2(l_1+l_2+l_3+l_4+.......+l_{17}+l_{18})/3} \left(1 + \mathcal{O}(g_s^\Delta, e^{-1/g_s^\Delta})\right), \tag{2.68}$$

which is a generalization of similar scaling for the part of the product of the curvature tensors in (2.56). The conclusion then is also the same, namely, there is no time-neutral series possible with product of curvature tensors only.

With multiple derivatives we can also work out the quantum terms. The derivatives are going to act only on the internal $\mathcal{M}_4$ coordinates, so the correction to the g_s scaling is easy to ascertain. The derivative action will only change E_3 in (2.67) to

$E_3 \to E_3 + \frac{n}{2}$ where n is the number of derivatives. This implies:

$$\begin{aligned} \mathbb{Q}_6 &\equiv \left[\mathbf{g}^{-1}\right]^{E_1+E_2+E_3+E_4+E_5+n/2}[\partial]^n \left(\prod_{i=1}^{18}(\mathbf{R}_{MNPQ})^{l_i}\right) \\ &= g_s^{2(l_1+l_2+l_3+l_4+.......+l_{17}+l_{18}+n/2)/3}\left(1+\mathcal{O}(g_s^\Delta, e^{-1/g_s^\Delta})\right), \end{aligned} \tag{2.69}$$

with no possibility of any time-neutral series. This is expectedly similar to what we had in (2.60), and thus justifying the genericity of the arguments presented earlier.

With two free Lorentz indices the story should again be similar to what we had earlier. However, because of the possibility of multiple indices, things would be slightly involved. For example if we want free (i, j) Lorentz indices we convert E_1 to $E_1 - 1$ and keep other E_i unchanged. We can quantify such changes by using a simple formalism. Let $k = (k_1, k_2)$ such that k identifies the subscript in E_k and (k_1, k_2) identify the Lorentz indices. For example if $k = 1$ then $k_1 \equiv x_i$ and $k_2 \equiv x_j$. Using this let us define $E_k(w, z)$ as:

$$E_k(w,z) \equiv E_k - \delta_{wk_1}\delta_{zk_2}, \tag{2.70}$$

with E_k as in (2.67). The above form easily gives us the required exponent. For example $E_k(m, n) = E_k$ for $k \neq 3$ and $E_3(m, n) = E_3 - 1$. With this, the quantum terms with two free Lorentz indices will simply be:

$$\mathbb{Q}_7(w,z) \equiv \left[\mathbf{g}^{-1}\right]^{\sum_k E_k(w,z)+n/2}[\partial]^n \left(\prod_{i=1}^{18}(\mathbf{R}_{MNPQ})^{l_i}\right), \tag{2.71}$$

where the choice of (w, z) specify which two Lorentz indices we want to keep free. Some care needs to be imposed in interpreting the results as the derivation of the curvature tensors did not have cross-terms. So indices like $w = a, z = m$ has no meaning here. After the dust settles, the g_s scaling for (2.71) may be expressed as g_s^χ where χ takes the following *three* values:

$$\chi \equiv \frac{2}{3}\sum_{i=1}^{18} l_i + \frac{n}{3} - \frac{8}{3}, \quad \chi \equiv \frac{2}{3}\sum_{i=1}^{18} l_i + \frac{n}{3} - \frac{2}{3}, \quad \chi \equiv \frac{2}{3}\sum_{i=1}^{18} l_i + \frac{n}{3} + \frac{4}{3}, \tag{2.72}$$

where the first one corresponds to two free Lorentz indices (i, j) and $(0, 0)$; the second one corresponds to two free Lorentz indices along $\mathcal{M}_4$, i.e $(m, , n)$; and the third one corresponds to two free Lorentz indices along $\mathcal{M}_2$ and $\mathbb{T}^2/\mathcal{G}$ i.e (α, β) and (a, b) respectively. Note that the relative *minus* signs for the first two values of χ shows the possibility of time-neutral terms. For the first case, looking at E_2 in (2.67), and imposing:

$$l_{12} = l_{13} = l_{14} = l_{15} = 1, \quad n = 0, \tag{2.73}$$

with all other l_i vanishing gives us $\chi = 0$. This exactly leads to a quantum term that appears from the contraction $\mathbf{g}^{AB}\mathbf{R}_{0A0B}$ with (A, B) spanning the four allowed choices, namely, (i, j), (m, n), (a, b) and (α, β), as:

$$\left(\mathbf{g}^{00}\right)^3 \mathbf{g}^{\alpha\beta}\mathbf{g}^{ab}\mathbf{g}^{ij}\mathbf{g}^{mn}\mathbf{R}_{0m0n}\mathbf{R}_{0i0j}\mathbf{R}_{0a0b}\mathbf{R}_{0\alpha0\beta} \in \left(\mathbf{g}^{00}\mathbf{R}_{00}\right)^4 \mathbf{g}_{00}, \tag{2.74}$$

where the LHS is the time-neutral piece in the expansion of the complete term given in the RHS, which for brevity be called the time-neutral $\mathbf{R}_{00}$ term. In a similar vein, one can argue for time-neutral $\mathbf{R}_{ij}$ for the first case and time-neutral $\mathbf{R}_{mn}$ for the second case. In fact the space-time terms appear from expanding $\left(\mathbf{g}^{\mu\nu}\mathbf{R}_{\mu\nu}\right)^4 \mathbf{g}_{MN}$ with (M, N) spanning $(0, 0)$, and (i, j) indices; whereas the (m, n) term simply appears for $\mathbf{R}_{mn}$. Finally, the third case tells us that there are no time-neutral terms possible with either (a, b) or (α, β) indices.

The case with $F_i(t)$ satisfying (2.2) with the inverses having perturbative expansions should in principle be redone in the light of the new g_s scalings to the curvature tensors. At this stage, one might even generalize the story from (2.7) to:

$$F_1(t)F_2^2(t) = \left(\frac{g_s^2}{\sqrt{h}}\right)^{\frac{\gamma}{2}}, \tag{2.75}$$

with $|\gamma| \in \mathbb{Z}$ such that $\gamma = 0, 2$ correspond to (2.2) and (2.8) respectively. Although most others values of γ are not useful for us, it is nevertheless interesting to speculate the fate of our background for generic choice of γ. Incidentally, the only scalings that are affected are:

$$\begin{gathered}\mathbf{R}_{\alpha\beta\alpha\beta} = g_s^{2\gamma-2/3} = g_s^{-2/3}, \;\; \mathbf{R}_{mn\alpha\beta} = g_s^{\gamma-2/3} = g_s^{-2/3}, \;\; \mathbf{R}_{\alpha ab\beta} = g_s^{\gamma+4/3} = g_s^{4/3} \\ \mathbf{R}_{ij\alpha\beta} = g_s^{\gamma-8/3} = g_s^{-8/3}, \;\; \mathbf{R}_{0\alpha\beta m} = g_s^{\gamma-5/3} = g_s^{-5/3}, \;\; \mathbf{R}_{0\alpha0\beta} = g_s^{\gamma-8/3} = g_s^{-8/3},\end{gathered} \tag{2.76}$$

where on the extreme RHS of every equation we have put $\gamma = 0$ to relate the result for (2.2). All these affected components have legs along $\mathcal{M}_2$ but are functions of $\mathcal{M}_4$ only. Once the derivative constraints are removed for the case (2.2), the scalings (2.76) also work perfectly as shown in **Table 2.1**. Putting these curvatures together and introducing n derivatives, lead to exactly the same g_s scalings for the quantum terms that we had in above for both zero and two free Lorentz indices for *any* choice of γ. No extra conditions are needed and thus we share the same conclusion of the non-existence of time-neutral series with curvatures and multiple derivatives as before.

Product of curvatures, G-fluxes and derivatives

We have previously demonstrated how, by choosing G-fluxes and curvature tensors and combining them independently with multiple derivatives, they do not lead to time-neutral quantum terms. Various cases were elaborated exhaustively by allowing $F_1(t)$ and $F_2(t)$ to satisfy either (2.8) or a variant of (2.2) where each of their inverses have perturbative expansions in terms of g_s. It is now time to combine all of these together to write quantum terms as a combinations of G-fluxes, curvature tensors and their covariant derivatives.

Our starting point is of course the G-flux ansatze (2.13) where we will assume that $\Delta k \geq \frac{3}{2}$, so as to comply with earlier constraints (although for certain cases

Riemann tensors for (2.8)	g_s scalings	Riemann tensors for (2.2)
$\mathbf{R}_{mnpq}$	$-\frac{2}{3}$	$\mathbf{R}_{mnpq}, \mathbf{R}_{mnp\alpha}, \mathbf{R}_{mn\alpha\beta}, \mathbf{R}_{m\alpha\alpha\beta}, \mathbf{R}_{\alpha\beta\alpha\beta}$
$\mathbf{R}_{mnab}, \mathbf{R}_{mn\alpha\beta}$	$\frac{4}{3}$	$\mathbf{R}_{mnab}, \mathbf{R}_{m\alpha ab}, \mathbf{R}_{\alpha\beta ab}$
$\mathbf{R}_{abab}, \mathbf{R}_{ab\alpha\beta}, \mathbf{R}_{\alpha\beta\alpha\beta}$	$\frac{10}{3}$	$\mathbf{R}_{abab}$
$\mathbf{R}_{mnp0}$	$-\frac{5}{3}$	$\mathbf{R}_{mnp0}, \mathbf{R}_{mn\alpha 0}, \mathbf{R}_{m\alpha\beta 0}, \mathbf{R}_{0\alpha\alpha\beta}$
$\mathbf{R}_{mnij}, \mathbf{R}_{0m0n}$	$-\frac{8}{3}$	$\mathbf{R}_{mnij}, \mathbf{R}_{m\alpha ij}, \mathbf{R}_{\alpha\beta ij}, \mathbf{R}_{0m0n}, \mathbf{R}_{0\alpha 0\beta}, \mathbf{R}_{0m0\alpha}$
$\mathbf{R}_{m0ij}$	$-\frac{11}{3}$	$\mathbf{R}_{m0ij}, \mathbf{R}_{\alpha 0ij}$
$\mathbf{R}_{ijij}, \mathbf{R}_{0i0j}$	$-\frac{14}{3}$	$\mathbf{R}_{ijij}, \mathbf{R}_{0i0j}$
$\mathbf{R}_{0mab}, \mathbf{R}_{0m\alpha\beta}$	$\frac{1}{3}$	$\mathbf{R}_{0mab}, \mathbf{R}_{0\alpha ab}$
$\mathbf{R}_{abij}, \mathbf{R}_{0a0b}, \mathbf{R}_{\alpha\beta ij}, \mathbf{R}_{0\alpha 0\beta}$	$-\frac{2}{3}$	$\mathbf{R}_{abij}, \mathbf{R}_{0a0b}$

TABLE 2.1: The g_s scalings of the various curvature tensors associated with the two cases (2.2) and (2.8). These curvature tensors form the essential ingredients of the quantum terms (2.94) and (2.78) respectively. The numbers in the middle column, say for example $-\frac{2}{3}$, should be understood as $\left(\frac{g_s}{H}\right)^{-2/3}$ where $H^4(y) \equiv h(y)$ is the warp-factor appearing in (2.1) and (2.3).

we will see that $\Delta k \geq \frac{1}{2}$ suffice). However compared to what we analyzed before, we will now have to take individual components of G-fluxes carefully. The components that we want to consider are listed in (2.16). This way, when we consider the individual components of the curvature tensors in (2.53), (2.62) and (2.63) we will be able to quantify the behave of the quantum terms more accurately.

To start, it is instructive then to specify the product of individual components of G-flux using a notation similar to (2.64) for the product of curvature tensors. This means, we define:

$$(\mathbf{G}_{MNPQ})^{l_i} \equiv \prod_{k=1}^{l_i} \mathbf{G}_{M_k N_k P_k Q_k}, \tag{2.77}$$

the difference now being the complete anti-symmetry of the indices as compared to pair-wise anti-symmetry of the indices for the curvature tensors. Other than this, the two definitions, (2.77) and (2.64), are similar in spirit.

Therefore combining the pieces of the curvature tensors and derivatives as in (2.69) and using the definition (2.77) to insert in the G-fluxes listed from (2.16), we get the following representation of the quantum terms:

$$\begin{aligned}
\mathbb{Q}_{\rm T} &= \mathbf{g}^{m_i m'_i}\mathbf{g}^{m_l m'_l}....\mathbf{g}^{j_k j'_k}\partial_{m_1}\partial_{m_2}.....\partial_{m_n}\left(\mathbf{R}_{mnpq}\right)^{l_1}\left(\mathbf{R}_{abab}\right)^{l_2}\left(\mathbf{R}_{pqab}\right)^{l_3}\left(\mathbf{R}_{\alpha ab\beta}\right)^{l_4}\\
&\times \left(\mathbf{R}_{\alpha\beta mn}\right)^{l_5}\left(\mathbf{R}_{\alpha\beta\alpha\beta}\right)^{l_6}\left(\mathbf{R}_{ijij}\right)^{l_7}\left(\mathbf{R}_{ijmn}\right)^{l_8}\left(\mathbf{R}_{iajb}\right)^{l_9}\left(\mathbf{R}_{i\alpha j\beta}\right)^{l_{10}}\left(\mathbf{R}_{0mnp}\right)^{l_{11}}\\
&\times \left(\mathbf{R}_{0m0n}\right)^{l_{12}}\left(\mathbf{R}_{0i0j}\right)^{l_{13}}\left(\mathbf{R}_{0a0b}\right)^{l_{14}}\left(\mathbf{R}_{0\alpha 0\beta}\right)^{l_{15}}\left(\mathbf{R}_{0\alpha\beta m}\right)^{l_{16}}\left(\mathbf{R}_{0abm}\right)^{l_{17}}\left(\mathbf{R}_{0ijm}\right)^{l_{18}}\\
&\times \left(\mathbf{G}_{mnpq}\right)^{l_{19}}\left(\mathbf{G}_{mnp\alpha}\right)^{l_{20}}\left(\mathbf{G}_{mnpa}\right)^{l_{21}}\left(\mathbf{G}_{mn\alpha\beta}\right)^{l_{22}}\left(\mathbf{G}_{mn\alpha a}\right)^{l_{23}}\left(\mathbf{G}_{m\alpha\beta a}\right)^{l_{24}}\\
&\times \left(\mathbf{G}_{0ijm}\right)^{l_{25}}\left(\mathbf{G}_{0ij\alpha}\right)^{l_{26}}\left(\mathbf{G}_{mnab}\right)^{l_{27}}\left(\mathbf{G}_{ab\alpha\beta}\right)^{l_{28}}\left(\mathbf{G}_{m\alpha ab}\right)^{l_{29}}
\end{aligned} \tag{2.78}$$

where we have inserted in all the available pieces of G-flux and the curvature tensors. Each of the pieces, either from the G-fluxes or curvatures, will have additional components. For example $\mathbf{R}_{mnpq}$ will have 36 components (excluding the permutations), and so on. Additionally each of the components are raised to l_i powers giving rise to an elaborate set of terms. Note that we can now take advantage of the underlying anti-symmetries of the curvatures to contract some of the Riemann tensors to create anti-symmetric Ricci tensors of the form $\mathbf{R}_{[MN]}$. Of course the Ricci scalar $\mathbf{R}$ would also participate in the game as before. We can also express (2.78) in a condensed form as:

$$\mathbb{Q}_{\rm T} \equiv \left[\mathbf{g}^{-1}\right]^{H_1+H_2+H_3+H_4+H_5+n/2}[\partial]^n\left(\prod_{i=1}^{18}\left(\mathbf{R}_{MNPQ}\right)^{l_i}\prod_{k=19}^{29}\left(\mathbf{G}_{RSTU}\right)^{l_k}\right), \tag{2.79}$$

which for a given choice of $\{l_i\}$ determines a specific quantum term with the functional form for $H_k(l_j)$ to be determined soon. Since any such term has zero free Lorentz index, one may take arbitrary linear combinations of powers of this term. Such combinations lead to a complicated structure of the quantum series. Note that a term like (2.79) is suppressed by M_p^σ where:

$$\sigma \equiv \sigma(\{l_i\}, n) = n + 2\sum_{i=1}^{18} l_i + \sum_{k=19}^{29} l_k. \tag{2.80}$$

The above quantum terms (2.78) are generic enough but they could also have powers of metric components along-with the G-fluxes and curvature tensors[11]. However since these metric components will not change the values of H_k functions, we don't specify them here. Additionally all the derivatives should be replaced by covariant derivatives, but since we are taking the fluxes and curvatures, these extra pieces will appear from suitable combinations of these components. One may then

[11]Taking advantage of the underlying pair-wise anti-symmetry of the curvature tensors and full anti-symmetry of the G-fluxes, two other possibilities exist for (2.78) once we remove the derivatives. One: we can suitably contract the indices using eleven-dimensional epsilon tensor (i.e the eleven-dimensional Levi-Civita tensor and *not* tensor density); and two: we can suitable contract the indices using eleven-dimensional Gamma matrices. Since they don't change the g_s scalings (2.84) and (2.86), we will discuss them in the next section.

express the quantum potential as:

$$\mathbb{V}_Q \equiv \sum_{\{l_i\},n} \int d^8y\sqrt{\mathbf{g}_8}\left(\frac{\mathbb{Q}_T^{(\{l_i\},n)}}{M_p^{\sigma(\{l_i\},n)-8}}\right), \tag{2.81}$$

where the superscript on $\mathbb{Q}_T$ denotes the specific choice of l_i and n in (2.78) with σ as in (2.80) to make it dimensionless. The factor of determinant of the eight-dimensional warped metric is same for all terms in the potential (2.81), so we will not count it's g_s contribution in the following, unless mentioned otherwise[12]. However once we go to the non-local contributions to the potential, this determinant will occur multiple times, and then they *will* contribute to the g_s scaling of the potential.

How about other extra components of G-fluxes and curvature tensors that do not appear in the data specifying the background informations? For example various cross-terms in the metric would give rise to extra curvature tensors. Similarly cross-terms in the G-fluxes would contribute extra flux components in (2.78). This is where the Wilsonian viewpoint becomes immensely useful. The quantum terms are indeed specified by all components of fluxes, derivatives and curvature tensors appearing from fluctuations over a given background, but we can *integrate* out the components that are not necessary to specify the background data. Such integrating out modes will result in an infinite series of quantum terms of the form (2.78), thus justifying our approach of expressing the quantum series with arbitrary values for l_i. With this in mind, the H_k functions may be expressed in terms of the following linear combinations of l_i:

$$\begin{aligned} H_1 &= E_1 + l_{25} + l_{26}, \quad H_2 = E_2 + \frac{l_{25}}{2} + \frac{l_{26}}{2} \\ H_4 &= E_4 + \frac{l_{21}}{2} + \frac{l_{23}}{2} + \frac{l_{24}}{2} + l_{27} + l_{28} + l_{29} \\ H_5 &= E_5 + \frac{l_{20}}{2} + l_{22} + \frac{l_{23}}{2} + l_{24} + \frac{l_{26}}{2} + l_{28} + \frac{l_{29}}{2} \\ H_3 &= E_3 + 2l_{19} + \frac{3l_{20}}{2} + \frac{3l_{21}}{2} + l_{22} + l_{23} + \frac{l_{24}}{2} + \frac{l_{25}}{2} + l_{27} + \frac{l_{29}}{2} + \frac{n}{2}, \end{aligned} \tag{2.82}$$

where $E_1, ..., E_5$ functions, which are themselves expressed as linear combinations of l_i, are defined in (2.67); and $(H_1, ..., H_5)$ denote inverse metric components along (i,j), $(0,0)$, (m,n), (a,b) and (α,β) respectively. The story now proceeds in exactly the same way as outlined in the previous section. The g_s scaling of the quantum

[12]In any case the determinant will only contribute $g_s^{-2/3+\gamma}$ to the overall scaling with γ defined in (2.75). Since this does not effect any of the conclusions, we will avoid inserting it in our analysis, unless mentioned otherwise.

piece with zero free Lorentz index may be expressed as:

$$\begin{aligned}\mathbb{Q}_{\rm T} &\equiv g_s^{\theta_k}\left(1+\mathcal{O}(g_s^\Delta, e^{-1/g_s^\Delta})\right)\\ &\equiv \left[\mathbf{g}^{-1}\right]^{H_1+H_2+H_3+H_4+H_5+n/2}[\partial]^n\left(\prod_{i=1}^{18}(\mathbf{R}_{MNPQ})^{l_i}\prod_{k=19}^{29}(\mathbf{G}_{RSTU})^{l_k}\right),\end{aligned} \tag{2.83}$$

where θ_k is the scaling parameter that may now be computed by combining all the information that we have assimilated together, namely from the G-flux scaling in (2.13) to the curvature scalings in (2.63). The result is:

$$\begin{aligned}\theta_k &= \frac{2}{3}\sum_{i=1}^{18} l_i + \frac{n}{3} + \frac{l_{25}}{3} - \frac{2l_{26}}{3} + \left(2\Delta k + \frac{4}{3}\right) l_{19} + \left(2\Delta k + \frac{1}{3}\right)(l_{20}+l_{21})\\ &+ \left(2\Delta k - \frac{2}{3}\right)(l_{22}+l_{23}+l_{27}) + \left(2\Delta k - \frac{8}{3}\right) l_{28} + \left(2\Delta k - \frac{5}{3}\right)(l_{24}+l_{29}),\end{aligned} \tag{2.84}$$

where k specifies the minimum g_s scaling of the G-flux components in (2.13). We expect this to be positive definite if we want the quantum terms in (2.78) to have no time-neutral pieces. Unfortunately the relative minus signs in (2.84) are worrisome, so there should be a way to demonstrate the positivity of (2.84). First, it is easy to see that if $\Delta k > \frac{4}{3}$ most of the terms, except the one with l_{26}, become positive definite[13]. This is where our earlier analysis comes in handy, as we have already argued that $\Delta k \geq \frac{3}{2}$ therein! Secondly, if l_{26} vanishes then we are out of water. Can we make $l_{26} = 0$ here? Looking at (2.78), we see that l_{26} appears with $\mathbf{G}_{0ij\alpha}$. It is clear from [14, 15] that:

$$\mathbf{G}_{0ij\alpha} = -\partial_\alpha\left(\frac{\epsilon_{0ij}}{h(y)\Lambda^2|t|^4}\right) = 0, \tag{2.85}$$

because we have assumed in the earlier sections that all quantities are functions of the $\mathcal{M}_4$ coordinates, and are thus independent of y^α. With these, we now see that $\theta_k > 0$ and therefore $F_i(t)$ satisfying (2.8), there are no time-neutral series altogether.

What happens when $F_i(t)$ satisfy the volume-preserving condition (2.2)? The analysis becomes a bit more tricky because the metric components along (α, β) directions scale differently and so do the curvature tensors. The new scalings of the curvature tensors are now (2.76). After the dust settles, the scaling of the quantum terms (2.78) can be expressed as $g_s^{\theta'_k}$, with additional $\mathcal{O}(g_s^\Delta, e^{-1/g_s^\Delta})$ corrections,

[13] If $\Delta k = \frac{4}{3}$ then the coefficient of l_{28} vanishes, implying that we can insert an arbitrary number of $\mathbf{G}_{ab\alpha\beta}$ components *without* changing the scaling. This will create a hierarchy issue similar to what we encountered in [15].

where θ'_k now takes the following value:

$$\begin{aligned}\theta'_k &= \frac{2}{3}\sum_{i=1}^{18} l_i + \frac{n}{3} + \frac{1}{3}(l_{25}+l_{26}) + \left(2\Delta k + \frac{4}{3}\right)(l_{19}+l_{20}+l_{22}) \\ &+ \left(2\Delta k + \frac{1}{3}\right)(l_{21}+l_{23}+l_{24}) + \left(2\Delta k - \frac{2}{3}\right)(l_{27}+l_{28}+l_{29}).\end{aligned} \tag{2.86}$$

Here we now notice a few important differences from (2.84); one, the coefficient of l_{26} is positive, so the constraint (2.85) is not necessary; and two, we only require $\Delta k > \frac{1}{3}$ for θ'_k to be a positive definite quantity[14]. In addition to that we can relax the derivative constraint, which was originally along $\mathcal{M}_4$, to the full six dimensional internal manifold $\mathcal{M}_4 \times \mathcal{M}_2$ because now both the metric components along (m, n) and (α, β) scale as $g_s^{-2/3}$. (This will lead to some subtleties that we will deal a bit later.) In other words, if there are n_1 derivatives along $\mathcal{M}_4$ and n_2 derivatives along $\mathcal{M}_2$, then n in (2.86) can be replaced for the two cases, (2.2) and (2.8), respectively by:

$$n \to n_1 + n_2, \quad n \to n_1 - 2n_2, \tag{2.87}$$

where the relative minus sign for the second case, i.e for background satisfying (2.8), requires $n_2 = 0$ to preserve the positivity of θ in (2.84). Interestingly for $k = 0$, the condition becomes:

$$\theta'_0 = \frac{2}{3}\sum_{i=1}^{18} l_i + \frac{n}{3} + \frac{1}{3}(l_{21}+l_{23}+l_{24}+l_{25}+l_{26}) + \frac{4}{3}(l_{19}+l_{20}+l_{22}) - \frac{2}{3}(l_{27}+l_{28}+l_{29}),$$

which by construction cannot always be positive definite. In fact the above scaling corresponds precisely to the scalings that we advocated in [15] with time-independent internal space and time-independent G-flux. Of course there were no derivative constraints therein so we could even retain l_{26} which, in turn, also allows us to retain l_{27}, l_{28} and l_{29}, i.e G-fluxes with two indices along (a, b) directions. Since this is important, let us clarify it in some details. To start, we define a scalar function along a compact direction z as

$$\Phi(z) = \sum_k \phi(k) e^{ikz}, \tag{2.88}$$

with $k = \frac{l}{R}$ where $l \in \mathbb{Z}$ and R is the radius of the z-circle. Additionally, we impose $\phi^*(k) = \phi(-k)$ so that $\Phi(z)$ remains real. Using this, we can define a three-form:

$$\mathbf{C}_{MN3}(y^m, y^\alpha, x_{11}) \equiv \mathbb{C}_{MN3}(y^m, y^\alpha) \otimes \Phi(x_{11}), \tag{2.89}$$

[14] As will be clearer later, this condition is exactly equivalent to the condition $\Delta k \geq \frac{1}{2}$. Again imposing $\Delta k = \frac{1}{3}$ would make the coefficients of (l_{27}, l_{28}, l_{29}) vanish, implying the possibility of introducing an infinite possible combinations of $\mathbf{G}_{mnab}, \mathbf{G}_{ab\alpha\beta}$ and $\mathbf{G}_{m\alpha ab}$ components without changing θ'_k in (2.86). As mentioned above, this will create similar problem as in [15].

where (M, N) span coordinates of $\mathcal{M}_4 \times \mathcal{M}_2$ and (x_3, x_{11}) are the periodic coordinates of $\mathbb{T}^2/\mathcal{G}$ such that $\Phi(x_{11})$ is the zero-form on the torus that is not projected out by the $\mathcal{G}$ action. This also implies that the G-flux components are taken to be functions of all the coordinates[15] of the eight manifold except x_3, so components like $\mathbf{G}_{MNab} \equiv \frac{1}{3!}\partial_{[11}\mathbf{C}_{MN3]}$ would lead to, in addition to other possible fields, a RR field $\mathbb{C}^{(2)}_{MN}(y^m, y^\alpha)$ in the type IIB side. For $l \geq 1$, we get KK modes $l/\mathbb{R}_{11}$, with $\mathbb{R}_{11}$ being the warped radius of the eleventh direction (which in turn will be related to g_s as shown in (**??**)). As $\mathbb{R}_{11}$ increases, the modes (2.89) become lighter and we can no longer integrate them out! These light degrees of freedom now contribute to l_{27}, l_{28} and l_{29} in (2.78) and therefore, from [15], time-neutrality for θ'_0 now happens when:

$$l_{27} + l_{28} + l_{29} + \frac{3l_{21}}{2} = \frac{n}{2} + \sum_{i=1}^{18} l_i + 2\sum_{j=19}^{22} l_j + \frac{1}{2}\sum_{k=23}^{26} l_k, \tag{2.90}$$

with n being the number of derivatives that satisfy the first relation in (2.87). Since the l_i's have no additional constraints, (2.90) constitutes one relation between thirty variables, and as such will have infinite number of solutions, leading to the breakdown of an EFT description[16]. A particular set of choice for the l_i numbers, lets call them $\{l_i, r\}$ such that for integer choice of r we can allow different choices for $\{l_i\} = (l_1, l_2, ..., l_{29})$, satisfying (2.90) would constitute a time-neutral quantum term of the form (2.78). Each of these quantum terms may in turn be arranged together as:

$$\mathbb{Q}^{(0)}_{\mathrm{T}\{i\}} \equiv \sum_{k_1, k_2, ..} C_{k_1 k_2 ... k_\infty} \prod_{r=1}^{\infty} \left(\frac{\mathbb{Q}_{\mathrm{T},\{l_i,r\}}}{M_p^{\sigma(\{l_i,r\})}}\right)^{k_r}, \tag{2.91}$$

where the superscript denote time-neutrality and the subscript $\{i\} = (1, 2, ..., 29)$. The power of M_p can be read off from (2.80) for a given choice of $\{l_i, r\}$ and furnish the inverse powers of M_p in the quantum series to keep them dimensionless. The series (2.91) thus constitute the infinite class of time-neutral quantum pieces elaborated in [15].

The above construction gives a satisfying answer to the question of the non-existence of an EFT description in the set-up with time-independent fluxes in [15], although one question could be raised at this point. Since $\mathbb{R}_{11} \to 0$ decouples all the degrees of freedom coming from the KK states of $\mathbf{G}_{MNab}$, and clearly the vanishing of the warped eleven-dimensional radius is also a necessary condition to go to type IIB, couldn't we just decouple all the dangerous states and study the resulting EFT? The answer to this question lies in the three scaling behaviors that we derived earlier, namely (2.84), (2.86) and (2.88). For (2.84) and (2.86), whether or not we

[15]As we saw before, they are also functions of $(g_s^\Delta, e^{-1/g_s^\Delta})$ which we suppress to avoid clutter.

[16]Such a train of thought is particularly consistent with the swampland conjecture as presented in [6]. In particular the swampland distance conjecture should be associated to the distance in the field space where the KK modes in (2.88) and (2.89) start becoming light. Note that one can potentially develop a similar story with three-form field components along x_3 direction as in (2.89). In such a picture, as the x_3 circle increases, the KK modes become lighter and start creating the same issues as above. However the x_3 dependences ruin the Busher's duality employed to convert the type IIB background to type IIA in the first place.

switch on (l_{27}, l_{28}, l_{29}), they are *always* positive definite and therefore cannot create time-neutral series anywhere in the moduli space of M-theory. This is clearly not the case for (2.88), which does create an infinite class of time-neutral series as in (2.90). Thus although $g_s \to 0$ provides a false aura of a healthy EFT with θ'_0 scaling in (2.88), it quickly disappears as we go away from this limit: a property not shared by (2.84) and (2.86) for (2.8) and (2.2) respectively.

All the three scalings discussed above, namely (2.84), (2.86) and (2.88) are related to special choices of γ in (2.75). If we make an arbitrary choice of γ then the g_s scaling of the quantum term (2.78) becomes $g_s^{\theta(k,\gamma)}$, where $\theta(k,\gamma)$ is:

$$\begin{aligned}\theta(k,\gamma) &= \frac{2}{3}\sum_{i=1}^{18} l_i + \frac{n}{3} + \frac{l_{25}}{3} + \left(2\Delta k + \frac{4}{3}\right) l_{19} + \left(2\Delta k + \frac{1}{3}\right) l_{21} + \left(2\Delta k - \frac{2}{3}\right) l_{27} \\ &+ \left(2\Delta k + \frac{4}{3} - \frac{\gamma}{2}\right) l_{20} + \left(2\Delta k + \frac{4}{3} - \gamma\right) l_{22} + \left(2\Delta k + \frac{1}{3} - \frac{\gamma}{2}\right) l_{23} + \left(2\Delta k + \frac{1}{3} - \gamma\right) l_{24} \\ &+ \left(\frac{1}{3} - \frac{\gamma}{2}\right) l_{26} + \left(2\Delta k - \frac{2}{3} - \gamma\right) l_{28} + \left(2\Delta k - \frac{2}{3} - \frac{\gamma}{2}\right) l_{29}, \end{aligned} \tag{2.92}$$

where the first line is generic to all choices of γ, but the second and the third lines specifically depend on what value γ takes. Plugging in $\gamma = 0, 2$ one may easily derive (2.2) and (2.8) respectively. It should also be clear that $\frac{3\gamma+2}{3}$ is the largest attainable value with a relative minus sign, implying that it is only the coefficient of l_{28} that can determine the lower bound on k to avoid time-neutral series. For the present case, this happens when:

$$\Delta k > \frac{1}{3} + \frac{\gamma}{2}, \tag{2.93}$$

from where one may easily derive the two earlier bounds we had. As γ increases the lower bound on k increases. Since Δk determines the *lowest* power of g_s for G-flux in (2.13) or (2.28), it implies that the lowest power is bigger for bigger γ. On the other hand γ from (2.75) also tells us the deviation of the four-dimensional Newton's constant from its standard *constant* value. Consequently, a more un-natural choice for Newton's constant is directly proportional to a more un-natural choice of the g_s dependence (or temporal dependence) of the G-flux components. Additionally, for $\gamma \geq 1$, the coefficient of l_{26} starts becoming negative thus making (2.93) prone to creating time-neutral series. The only way out appears from imposing (2.85). Thus for $\gamma \geq 1$ the fields can only be functions of the $\mathcal{M}_4$ coordinates to avoid the breakdown of a EFT description of the system. This second level of un-naturalness prompts us to ask whether this is the reason why nature chooses the simplest value of $\gamma = 0$ in (2.75) and (2.92). We will speculate on this interesting possibility in section 3.1.

Let us pause for a moment to absorb the consequence of the two lessons that we learnt from generic choice of γ in (2.92). One, larger γ makes k larger from (2.93), and two, larger γ also makes the coefficient of l_{26} negative. Thus $\gamma = 0$ and $\gamma > 0$ share different physics: $\gamma = 0$ no longer requires any derivative constraints so we can assume that all fields are functions of $\mathcal{M}_4 \times \mathcal{M}_2$; whereas $\gamma > 0$ has derivative constraint because of (2.85). For both cases however we will keep the fields independent of $\mathbb{T}^2/\mathcal{G}$. Relaxing the derivative constraints for $\gamma = 0$ will create new components of curvature tensors that should modify (2.78) to the following:

$$\begin{aligned}
\mathbb{Q}_{\rm T} &= \mathbf{g}^{m_i m'_i}\mathbf{g}^{m_l m'_l}....\mathbf{g}^{j_k j'_k}\partial_{m_1}..\partial_{m_{n_1}}\partial_{\alpha_1}..\partial_{\alpha_{n_2}}\left(\mathbf{R}_{mnpq}\right)^{l_1}\left(\mathbf{R}_{abab}\right)^{l_2}\left(\mathbf{R}_{pqab}\right)^{l_3}\left(\mathbf{R}_{\alpha ab\beta}\right)^{l_4}\\
&\times \left(\mathbf{R}_{\alpha\beta mn}\right)^{l_5}\left(\mathbf{R}_{\alpha\beta\alpha\beta}\right)^{l_6}\left(\mathbf{R}_{ijij}\right)^{l_7}\left(\mathbf{R}_{ijmn}\right)^{l_8}\left(\mathbf{R}_{iajb}\right)^{l_9}\left(\mathbf{R}_{i\alpha j\beta}\right)^{l_{10}}\left(\mathbf{R}_{0mnp}\right)^{l_{11}}\\
&\times \left(\mathbf{R}_{0m0n}\right)^{l_{12}}\left(\mathbf{R}_{0i0j}\right)^{l_{13}}\left(\mathbf{R}_{0a0b}\right)^{l_{14}}\left(\mathbf{R}_{0\alpha 0\beta}\right)^{l_{15}}\left(\mathbf{R}_{0\alpha\beta m}\right)^{l_{16}}\left(\mathbf{R}_{0abm}\right)^{l_{17}}\left(\mathbf{R}_{0ijm}\right)^{l_{18}}\\
&\times \left(\mathbf{R}_{mnp\alpha}\right)^{l_{19}}\left(\mathbf{R}_{m\alpha ab}\right)^{l_{20}}\left(\mathbf{R}_{m\alpha\alpha\beta}\right)^{l_{21}}\left(\mathbf{R}_{m\alpha ij}\right)^{l_{22}}\left(\mathbf{R}_{0mn\alpha}\right)^{l_{23}}\left(\mathbf{R}_{0m0\alpha}\right)^{l_{24}}\left(\mathbf{R}_{0\alpha\beta\alpha}\right)^{l_{25}}\\
&\times \left(\mathbf{R}_{0ab\alpha}\right)^{l_{26}}\left(\mathbf{R}_{0ij\alpha}\right)^{l_{27}}\left(\mathbf{G}_{mnpq}\right)^{l_{28}}\left(\mathbf{G}_{mnp\alpha}\right)^{l_{29}}\left(\mathbf{G}_{mnpa}\right)^{l_{30}}\left(\mathbf{G}_{mn\alpha\beta}\right)^{l_{31}}\left(\mathbf{G}_{mn\alpha a}\right)^{l_{32}}\\
&\times \left(\mathbf{G}_{m\alpha\beta a}\right)^{l_{33}}\left(\mathbf{G}_{0ijm}\right)^{l_{34}}\left(\mathbf{G}_{0ij\alpha}\right)^{l_{35}}\left(\mathbf{G}_{mnab}\right)^{l_{36}}\left(\mathbf{G}_{ab\alpha\beta}\right)^{l_{37}}\left(\mathbf{G}_{m\alpha ab}\right)^{l_{38}}, \qquad (2.94)
\end{aligned}$$

where (n_1, n_2) are the number of derivatives along $\mathcal{M}_4$ and $\mathcal{M}_2$ directions respectively. Compared to (2.78), there are now nine extra pieces of curvature tensors, totalling to 38 total pieces of fluxes and curvature tensors. Each of these will have the required copies because of the l_i factors, in addition to the internal permutations as mentioned earlier. Such a quantum term has a M_p suppression of the form M_p^σ, where:

$$\sigma \equiv \sigma(\{l_i\}, n_1, n_2) = n_1 + n_2 + 2\sum_{i=1}^{27} l_i + \sum_{k=28}^{38} l_k, \qquad (2.95)$$

which may be compared to (2.114): the changes coming from new derivatives and new curvature tensors. We also expect both H_i in (2.82) and E_i in (2.67) to change to $\widetilde{H}_i$ and $\widetilde{E}_i$ respectively. The change in the latter may be quantified as:

$$\begin{aligned}
&\widetilde{E}_5 = E_5 + \frac{l_{20}}{2} + \frac{3l_{21}}{2} + \frac{l_{22}}{2} + \frac{l_{23}}{2} + \frac{l_{24}}{2} + \frac{3l_{25}}{2} + \frac{l_{27}}{2}\\
&\widetilde{E}_1 = E_1 + l_{22} + l_{27}, \quad \widetilde{E}_2 = E_2 + \frac{l_{23}}{2} + l_{24} + \frac{l_{25}}{2} + \frac{l_{26}}{2} + \frac{l_{27}}{2}\\
&\widetilde{E}_3 = E_3 + \frac{3l_{19}}{2} + \frac{l_{20}}{2} + \frac{l_{21}}{2} + \frac{l_{22}}{2} + l_{23} + \frac{l_{24}}{2}, \quad \widetilde{E}_4 = E_4 + l_{20} + l_{26},
\end{aligned}$$

with E_n as defined in (2.67). The change in (2.82) is now easy to determine: all the subscript would shift by $+9$ in addition to an extra contribution to $\widetilde{H}_5$ coming from the derivatives. The overall change is:

$$\begin{aligned}
\widetilde{H}_1 &= \widetilde{E}_1 + l_{34} + l_{35}, \quad \widetilde{H}_2 = \widetilde{E}_2 + \frac{l_{34}}{2} + \frac{l_{35}}{2}\\
\widetilde{H}_4 &= \widetilde{E}_4 + \frac{l_{30}}{2} + \frac{l_{32}}{2} + \frac{l_{33}}{2} + l_{36} + l_{37} + l_{38}\\
\widetilde{H}_5 &= \widetilde{E}_5 + \frac{l_{29}}{2} + l_{31} + \frac{l_{32}}{2} + l_{33} + \frac{l_{35}}{2} + l_{37} + \frac{l_{38}}{2} + \frac{n_2}{2}\\
\widetilde{H}_3 &= \widetilde{E}_3 + 2l_{28} + \frac{3l_{29}}{2} + \frac{3l_{30}}{2} + l_{31} + l_{23} + \frac{l_{33}}{2} + \frac{l_{34}}{2} + l_{36} + \frac{l_{38}}{2} + \frac{n_1}{2},
\end{aligned} \qquad (2.96)$$

which expectedly takes the form similar to (2.82), with minor differences. One may also see that the quantum term in (2.94) scale with respect to g_s as $g_s^{\theta'_k}$, with additional $\mathcal{O}(g_s^\Delta, e^{-1/g_s^\Delta})$ corrections,

$$\begin{aligned}\theta'_k &= \frac{2}{3}\sum_{i=1}^{27} l_i + \frac{n_1+n_2}{3} + \frac{1}{3}(l_{34}+l_{35}) + \left(2\Delta k + \frac{4}{3}\right)(l_{28}+l_{29}+l_{31}) \\ &+ \left(2\Delta k + \frac{1}{3}\right)(l_{30}+l_{32}+l_{33}) + \left(2\Delta k - \frac{2}{3}\right)(l_{36}+l_{37}+l_{38}),\end{aligned} \tag{2.97}$$

where the only change from (2.86) is from 2/3 curvature contributions from the additional Riemann tensors and 1/3 derivative contributions from the derivatives along $\mathcal{M}_2$ directions. Note that these additional contributions do not change the sign and therefore the story remains unaltered from what we had earlier. When $k = 0$, we can further relax the derivative contributions to involve derivatives along $\mathbb{T}^2/\mathcal{G}$ directions. This will involve more curvature tensors and additional n_3 derivatives with (a, b) indices. The extra curvature components will again add $+2/3$ to (2.97) whereas the derivatives will add $-4n_3/3$. If $l_i^{(p)}$ denote the proliferation of each l_i components due to the relaxation of the derivative constraints in (2.94), then (2.88) changes to:

$$\theta'_0 = \frac{2}{3}\sum_p\sum_{i=1}^{27} l_i^{(p)} + \frac{n_1+n_2}{3} - \frac{2n_3}{3} + \frac{1}{3}\left(l_{30} + \sum_{p=1}^{4} l_{31+p}\right) + \frac{4}{3}(l_{28}+l_{29}+l_{31}) - \frac{2}{3}\sum_{q=1}^{3} l_{35+q}, \tag{2.98}$$

which as noted above differs from (2.88) by the appearance of another set of relative minus signs from the derivatives along the toroidal direction. This makes it prone to creating additional time neutral series from $\theta'_0 = 0$. The condition for this to happen now becomes:

$$l_{36}+l_{37}+l_{38}+n_3+\frac{3l_{30}}{2} = \frac{n_1+n_2}{2} + \sum_p\sum_{i=1}^{27} l_i^{(p)} + 2\sum_{j=28}^{31} l_j + \frac{1}{2}\sum_{q=32}^{35} l_q,$$

which can be compared to (2.90) and again has more issues as expected leading to the problems with an effective field theory description pointed out in [15]. Interestingly, although the proliferation of curvature tensors do not change much of the story, the proliferation of derivatives along $\mathbb{T}^2/\mathcal{G}$ tends to worsen the problem.

With two free Lorentz indices we need to again discuss the two cases pertaining for (2.8) and (2.2). The second case can be further fine-tuned to discuss the scenario advocated in [15], as we have done so far. The story for either of these cases remain simple as before. For (2.8), it is easy to see that the g_s scaling changes from (2.84) to the following three values[17]:

$$\theta_k \rightarrow \left(\theta_k - \frac{8}{3}, \theta_k - \frac{2}{3}, \theta_k + \frac{4}{3}\right), \tag{2.99}$$

where the first one corresponds to free Lorentz indices along (i,j) and $(0,0)$ directions; the second one corresponds to free Lorentz indices along $\mathcal{M}_4$ i.e along (m,n) directions and the third one corresponds to free Lorentz indices along $\mathbb{T}^2/\mathcal{G}$ and $\mathcal{M}_2$ i.e along (a,b) and (α,β) directions respectively. On the other hand, θ'_k also changes from (2.97) in the aforementioned way:

$$\theta'_k \rightarrow \left(\theta'_k - \frac{8}{3}, \theta'_k - \frac{2}{3}, \theta'_k + \frac{4}{3}\right), \tag{2.100}$$

for both $\Delta k > \frac{1}{3}$ and $k = 0$, with the difference being the second one now corresponds to both (m,n) as well as (α,β) directions as a consequence of identical scalings for the metric components along these directions for the case (2.2) and [15].

Let us now elaborate the scaling behavior in bit more details. For the case (2.97) with $\Delta k > \frac{1}{3}$ we first note that switching on any components of G-fluxes or curvature tensors, $\theta'_k \geq 1/3$ and therefore makes every term in (2.97) positive definite, thus ruling out all time-neutral series with zero Lorentz indices along directions $(i,j), (0,0), (m,n), (a,b)$ and (α,β). With two Lorentz indices, there are no time-neutral series at least along the (a,b) directions as is evident from both (2.99) and (2.100). Along (m,n) and (α,β) directions, for (2.97), there are a few cases. Since every Riemann tensor contribute an overall factor of $2/3$ to θ'_k, it is easy to see that we need at most one of:

$$(l_1, l_5, l_8, l_{11}, l_{12}), \quad \text{and} \quad (l_4, l_5, l_6, l_{10}, l_{15}, l_{16}), \tag{2.101}$$

for (m,n) and (α,β) indices respectively, to cancel the factor of $2/3$ in (2.100). In fact it is easy to see that we can only get two time-neutral pieces of the form $\mathbf{R}_{mn}$ and $\mathbf{R}_{\alpha\beta}$, using combinations of curvature tensors. Using G-fluxes, naively either of the three choices $l_{34} = 2$, $l_{35} = 2$ and $l_{34} = l_{35} = 1$ can cancel the $2/3$ factor in (2.100). These are all easily eliminated as they imply either $\widetilde{H}_2$, $\widetilde{H}_5$ or $\widetilde{H}_3$ in (2.96) to be half-integers[18]. If we take $k = 1$ in (2.97), then the only other choices are associated with integer values for (l_{36}, l_{37}, l_{38}). Taking $l_{36} = 2$, $l_{37} = 2$ or $l_{38} = 2$ always make $\widetilde{H}_4 = 2$ and depending on the choices $(\widetilde{H}_3, \widetilde{H}_5) = (0,1)$ or $(1,0)$ from (2.96) respectively give

[17] Although $l_i > 0$ always, H_i from (2.82) or E_i from (2.67), when two free Lorentz indices are allowed, can take integer values starting from -1, i.e $H_i \geq -1$ and $E_i \geq -1$. Similar criteria emerge from (2.96) and (2.96). The negative value implies inserting a metric component, i.e the inverse of an inverse metric component, in either cases.

[18] Subtleties with half-integers will be discussed later.

rise to the following two set of tensors[19]:

$$\Lambda_{mn}^{(11)} \equiv \frac{\mathbf{g}^{bd}\mathbf{g}^{ac}\mathbf{g}^{\alpha\beta}\mathbf{G}_{m\alpha ab}\mathbf{G}_{n\beta cd}}{M_p^2}, \quad \Lambda_{mn}^{(12)} \equiv \frac{\mathbf{g}^{bd}\mathbf{g}^{ac}\mathbf{g}^{lq}\mathbf{G}_{mlab}\mathbf{G}_{nqcd}}{M_p^2}$$
$$\Lambda_{\alpha\beta}^{(21)} \equiv \frac{\mathbf{g}^{bd}\mathbf{g}^{ac}\mathbf{g}^{mn}\mathbf{G}_{m\alpha ab}\mathbf{G}_{n\beta cd}}{M_p^2}, \quad \Lambda_{\alpha\beta}^{(22)} \equiv \frac{\mathbf{g}^{bd}\mathbf{g}^{ac}\mathbf{g}^{\gamma\sigma}\mathbf{G}_{\alpha\gamma ab}\mathbf{G}_{\beta\sigma cd}}{M_p^2}, \tag{2.102}$$

as the sole examples of time-neutral rank two tensors along (m, n) and (α, β) directions. The other choice with $l_{36} = l_{37} = 1$ is eliminated by the anti-symmetry of the G-fluxes. Similarly for $n \geq 1$, there are no additional time-neutral quantum terms with the required indices. Clearly if we demand $\Delta k \geq \frac{3}{2}$, both the examples in (2.102) are no longer allowed. In fact with $\Delta k \geq \frac{3}{2}$, we also eliminate any time-neutral rank two tensors from G-fluxes using (2.84).

Along space-time directions the scenario is more delicate. With $\Delta k \geq \frac{3}{2}$ the only contributions from G-fluxes may appear from (l_{34}, l_{35}) taking integer values in (2.97). Taking $l_{34} = 8$ requires us to pick $\widetilde{H}_1 = 7, \widetilde{H}_2 = 4, \widetilde{H}_3 = 3$ from (2.96). The other choice of $l_{35} = 8$ is similar to the first one because of the identical scalings of the metric components along (m, n) and (α, β) directions. After the dust settles, the generic quantum term along the space-time directions appears to be:

$$\Lambda_{\mu_a\mu_{a+1}}^{(3)} \equiv M_p^{-8} \prod_{k=1}^{8}\prod_{n=1}^{4} \mathbf{G}_{\mu_k\nu_k\rho_k m_k}\mathbf{g}^{m_{2n-1}m_{2n}}\mathbf{g}^{\mu_{2n-1}\mu_{2n}}\mathbf{g}^{\nu_{2n-1}\nu_{2n}}\mathbf{g}^{\rho_{2n-1}\rho_{2n}}\mathbf{g}_{\mu_a\mu_{a+1}}, \tag{2.103}$$

where assuming $1 \leq a \leq 8$ and $\mu_a \in (0, i, j)$ is any one of the three space-time directions in M-theory, (2.103) creates two kind of terms: $\Lambda_{00}^{(3)}$ and $\Lambda_{ij}^{(3)}$. Exactly similar set of terms appear from (2.84) (although $l_{26} = 0$ there). It turns out, since $\mathbf{G}_{\mu\nu\rho m}$ takes the value similar to (2.85), (but now the derivative is with respect to y^m and consequently non-zero), (2.103) is just a function that may be expressed in terms of the warp-factor $h(y)$. Even more generically if we take $l_{34} = 2p$ and $n = 2q$ such that $p + q = 4$ in (2.97), then (2.96) implies $\widetilde{H}_1 = 2p - 1, \widetilde{H}_2 = p$ and $\widetilde{H}_3 = 4$, with (2.103) becoming:

$$\Lambda_{\mu\nu}^{(p,q)} \equiv \partial_{m_1}\partial_{m_2}....\partial_{m_{2q}}\left(\prod_{k=q+1}^{2p+2q}\frac{\partial_{m_k}h}{h^{2p+2}}\right)\frac{\eta_{\mu\nu}}{M_p^8}\prod_{r,s} g^{m_r m_s}, \tag{2.104}$$

where we have expressed everything in terms of regular derivatives and inverse *unwarped* metric g^{mn} so that (2.104) doesn't have to involve covariant derivatives. In fact the way we have written the quantum terms in (2.94), all informations of the internal metrics etc are contained in the definitions of the curvature tensors and

[19]Other possibilities include $\mathbf{g}_{mn}\ \mathbf{g}^{kl}\Lambda_{kl}^{(1j)}$ and $\mathbf{g}_{\alpha\beta}\ \mathbf{g}^{\rho\sigma}\Lambda_{\rho\sigma}^{(2j)}$ that appear from expressing $\widetilde{H}_3 = 1$ alternatively as $\widetilde{H}_3 \equiv 2 + (-1)$ and $\widetilde{H}_5 = 1$ as $\widetilde{H}_5 \equiv 2 + (-1)$ respectively where the minus signs denote inverse of the inverse metric components. Additionally, choices like $\mathbf{g}_{mn}\mathbf{g}^{\alpha\beta}\Lambda_{\alpha\beta}^{(22)}$ etc. are also allowed. All these manipulation don't change θ_k or θ'_k.

the inverse metric components, and not in the derivatives. In this sense (2.104) has all the information in the warp-factor $h(y)$, and since $p + q = 4$, the allowed terns are $(p, q) = (4, 0), (3, 1), (2, 2), (1, 3)$, all being time-neutral by construction; and all suppressed by M_p^8. This M_p suppression remains unchanged even if we add curvature tensors contributions to (2.104). The curvature tensors, at least those that could contribute to the space-time directions, are limited to only four tensors at a time because time-neutrality implies:

$$2\sum_{i=1}^{27} l_i + n_1 + n_2 + l_{34} = 8, \tag{2.105}$$

thus $l_i \le 4$, and where many of the 27 l_i's appearing in (2.94) are irrelevant to (2.105). An example of such a term with only curvature tensors can be taken for $l_8 = l_9 = l_{10} = l_{13} = 1$ in (2.94) which allows us to choose $E_1 = 3, E_2 = E_3 = E_4 = E_5 = 1$ from (2.82) or (2.67). This gives:

$$\Lambda_{ij}^{(4)} \equiv M_p^{-8}\mathbf{R}_{i_1 a j_1 b}\mathbf{R}_{i_2 \alpha j_2 \beta}\mathbf{R}_{i_3 0 i 0}\mathbf{R}_{i_4 m j n}\mathbf{g}^{ab}\mathbf{g}^{\alpha\beta}\mathbf{g}^{mn}\mathbf{g}^{i_1 i_2}\mathbf{g}^{i_3 i_4}\mathbf{g}^{j_1 j_2}\mathbf{g}^{00}, \tag{2.106}$$

which is interestingly not just expressed in terms of the warp-factor $h(y)$ but also in terms of the temporal and spatial derivatives of the internal metric components. One can also mix three curvature tensors and two derivatives or two curvature tensors and four derivatives etc satisfying (2.105) appropriately to generate additional terms. All these quantum terms are finite in number and they are all suppressed by M_p^8 (with $\Delta k > \frac{1}{3}$, the finiteness of quantum terms still remain and can be easily constructed). As we saw earlier, there are *no* time-neutral contributions that can come from (2.97), so the M_p^8 suppression cannot change. In fact exactly similar story could be constructed with (2.84), so we will not discuss this case separately here.

Non-local counter-terms in M-theory and in type IIB

The next set of quantum corrections are a bit unusual from standard quantum field theory, or even supergravity, point of view and are typically christened as non-local counter-terms. Such an umbrella term encompass a broad category of quantum terms in M-theory, for which a detailed analysis is clearly beyond the scope of our work here. As such we will suffice ourselves here with some rudimentary exploration of the subject in the context of M-theory.

Our starting point would be to take the generic quantum terms in (2.78) and (2.94) and construct non-local interactions from them, as we believe that the non-local interactions should still contain powers of curvature tensors, G-fluxes and their covariant-derivatives. To proceed, let us denote the specific quantum term of (2.78) or (2.94) alternatively using the symbol $\mathbb{Q}_{\rm T}^{(\{l_i\},n)}$ so that specific choice of the (l_i, n) integers, the former representing the powers of curvature tensors and G-fluxes and the latter representing the number of derivatives, allow us to specify one quantum term. It is clear that:

$$\left(\mathbb{Q}_{\rm T}^{(\{l_i\},n)}\right) \otimes \left(\mathbb{Q}_{\rm T}^{(\{l_j\},m)}\right) \equiv \mathbb{Q}_{\rm T}^{(\{l_i+l_j\},n+m)}, \tag{2.107}$$

which may be easily derived using the explicit expression from either (2.78) or (2.94). The equality (2.107) tells us that an arbitrary product of any two elements in the set of all the quantum pieces labelled by $\left\{\mathbb{Q}_{\rm T}^{(\{l_k+l_m\},n)}\right\}$ is also an element of the set. This is almost like giving a group structure to the set, except that the set doesn't have an inverse. The elements of the set may even be further generalized by introducing the following notation:

$$\begin{aligned} t^{i_1 i_2 .....i_{2q}} &\equiv \epsilon^{i_1 i_2 ......i_{2q}} + c_1 \left[\left(g^{i_1 i_3} g^{i_2 i_4} - g^{i_1 i_4} g^{i_2 i_3}\right) .... \left(g^{i_{2q-3} i_{2q-1}} g^{i_{2q-2} i_{2q}} - g^{i_{2q-3} i_{2q}} g^{i_{2q-2} i_{2q-3}}\right) + ...\right] \\ &+ \text{ permutations}, \end{aligned} \tag{2.108}$$

where c_1 is a constant and the permutations are between other products of metrics to generate full anti-symmetry, and $\epsilon^{i_1 i_2 ....i_{2q}}$ is the Levi-Civita tensor and *not* a tensor density. As such, with all it's indices lowered, it may be defined with the square root of determinant of metrics and therefore scales in exactly the same way as the product of inverse metrics. However because of the total anti-symmetry of the Levi-Civita tensor (or of the anti-symmetric products of metrics), we cannot have too many of these terms at a given order. This implies that, if we remove all the derivatives in say (2.78), and taking $q = 4$ in (2.108), it is easy to get terms like:

$$\begin{aligned} \mathbb{Q}_1 &\equiv M_p^{-2} t^{i_1 i_2 ......i_8} \mathbf{G}_{i_1 i_2 i_3 i_4} \mathbf{G}_{i_5 i_6 i_7 i_8} \\ \mathbb{Q}_2 &\equiv M_p^{-8} t^{i_1 i_2 ......i_8} t^{j_1 j_2 .....j_8} \mathbf{R}_{i_1 i_2 j_1 j_2} \mathbf{R}_{i_3 i_4 j_3 j_4} \mathbf{R}_{i_5 i_6 j_5 j_6} \mathbf{R}_{i_7 i_8 j_7 j_8}, \end{aligned} \tag{2.109}$$

with i_k denoting coordinates of the internal eight-manifold, and $\mathbb{Q}_2$ can be identified with the famous $t_8 t_8 \mathbf{R}^4$ coupling in string theory [16]. It should be clear that the g_s scalings of $\mathbb{Q}_1$ and $\mathbb{Q}_2$ are identical to the g_s scalings of $\mathbb{Q}_{\rm T}^{(0,..,l_{19}=2,..,0;0)}$ and $\mathbb{Q}_{\rm T}^{(l_1=4,0,..,0;0)}$ respectively in (2.78). Other combinations with curvature tensors and G-fluxes are clearly possible, and their g_s scalings would be identical to the g_s scalings of corresponding terms in $\mathbb{Q}_{\rm T}^{(l_i,n=0)}$ at the same order in curvature tensors and G-fluxes. This story could be elaborated to the same extent as earlier sections[20], but since we are only concerned with the g_s scalings, we will not indulge in further discussions of the topic here.

Thus combining (2.109), with their possible generalizations, and with the set of terms of the form (2.78) or (2.94), we have pretty much all the local (perturbative and non-perturbative) quantum terms at hand. The non-local quantum terms, which we label as non-local counter-terms, are a different class of objects which could nevertheless be related to the local terms (2.109), (2.78) and (2.94). For example we could easily construct the following non-local counter-terms[21]:

$$\mathbb{W}^{(\{l_i\},n)} = \left(\sum_{q=1}^{\infty} \frac{C_q M_p^{2q}}{\square^q}\right) \mathbb{Q}_{\rm T}^{(\{l_i\},n)}, \tag{2.110}$$

where $\square$ is defined over the eight-manifold $\mathcal{M}_2 \times \mathcal{M}_4 \times \mathbb{T}^2/\mathcal{G}$ and C_q could in general

[20] Beyond the possible generalization to $\sum_k d_k \mathbb{Q}_1^k$ and $\sum_l f_l \mathbb{Q}_2^l$ with integer (d_k, f_l).

[21] See also [31] for operators of the form (2.110) and their possible connection to Witten's open string field theory.

be function of the $y \equiv (y^m, y^\alpha, y^a)$ but not functions of $(g_s^\Delta, e^{-1/g_s^\Delta})$. Thus the g_s scalings exclusively appear from the quantum pieces $\mathbb{Q}_{\rm T}^{(\{l_i\},n)}$. The inverse $\Box$ operators may be combined together to create operators of the form $\exp\left(\frac{\Box}{M_p^2}\right), \sin\left(\frac{\Box}{M_p^2}\right)$ etc generating different levels of non-locality. All these operator actions may in turn be re-expressed as integrals which are much easier to handle. To elaborate this, let us first define the non-locality function $\mathbb{F}^{(r)}(y-y') \equiv \mathbb{F}^{(\{l_i\},n;r)}(y-y')$ that is a function of two points (y, y') on the eight-manifold, with r denoting the level of non-locality. By construction the non-locality function should be sharply peaked at low energies so that the low energy physics of M-theory could still be governed by local counter-terms, and hence by eleven-dimensional supergravity. On the other hand, the short distance behavior of this function could be complicated, revealing the full non-local structure of the system. Using this function, let us define our first level of non-locality with zero free Lorentz indices using (2.78) for example as:

$$\mathbb{W}^{(1)}(y) \equiv \mathbb{W}^{(\{l_i\},n;1)} = \int d^8y'\sqrt{\mathbf{g}_8}\left(\frac{\mathbb{F}^{(1)}(y-y')\mathbb{Q}_{\rm T}^{(\{l_i\},n)}(y')}{M_p^{\sigma(\{l_i\},n)-8}}\right), \tag{2.111}$$

where the power of M_p appearing above, i.e $\sigma(\{l_i\},n)$ is defined in (2.80), and the integral captures the first level of non-locality as advertised before. By construction $\mathbb{W}^{(1)}$ is dimensionless, and the non-locality appears from knowing the precise functional form for $\mathbb{F}^{(1)}(y-y')$, which fortunately we won't need to specify. Suffice is to say that the g_s dependence only appears from the quantum terms $\mathbb{Q}^{(\{l_i\},n)}$ defined in (2.78) and (2.109). We can also sum over all allowed choices of $(\{l_i\},n)$ and, using the semi-group structure (2.107), the linear representation of the sum pretty much captures the generic picture. It should be clear that the r-th level of non-locality may be iteratively constructed from:

$$\begin{aligned}\mathbb{W}^{(r)}(y) &= M_p^8\int d^8y'\sqrt{\mathbf{g}_8(y')}\,\mathbb{F}^{(r)}(y-y')\mathbb{W}^{(r-1)}(y') \qquad (2.112)\\ &= M_p^{16}\int d^8y'\sqrt{\mathbf{g}_8(y')}\,\mathbb{F}^{(r)}(y-y')\int d^8y''\sqrt{\mathbf{g}_8(y'')}\,\mathbb{F}^{(r-1)}(y'-y'')\mathbb{W}^{(r-2)}(y''),\end{aligned}$$

thus forming a series of nested integrals that capture the full non-locality of the system, for a given choice of $(\{l_i\},n)$. Clearly as r increases the non-locality becomes more prominent and starts coinciding with the non-locality generated from the operator action (2.110). One expects:

$$\sum_{\{l_i\},n}\sum_{r=1}^{\infty} b_r\mathbb{W}^{(r)}(y) = \sum_{\{l_i\},n} f_{\{l_i\},n}\mathbb{W}^{(\{l_i\},n)}(y), \tag{2.113}$$

with constants b_r and $f_{\{l_i\},n}$, as we can absorb all y-dependent factors in $\mathbb{F}^{(r)}(y)$ of (2.112) and $C_q(y)$ of (2.110) respectively. Such a relation would not only justify the two forms of non-localities (2.110) and (2.112) as one and the same thing, but would also help us relate $C_q(y)$ functions with the $\mathbb{F}^{(r)}(y)$ functions. A formal proof of (2.113) is still lacking, despite evidences pointing towards the veracity of the conjecture. However since we will mostly concentrate on the non-localities of the form (2.112), the exact equivalence depicted in (2.113) will not be used here, and therefore the proof of (2.113) will be relegated to future work. We do note that, $\mathbb{W}^{(\infty)}(y)$ should be related to the $q \to \infty$ value of (2.110) when appropriately summed over $(\{l_i\}, n)$ factors therein as, at a given level of non-locality, the M_p suppression changes from (2.80) or (2.95) to:

$$\sigma(\{l_i\}, n; r) \equiv \sigma_r = \sigma(\{l_i\}, n) - 8r, \tag{2.114}$$

and therefore has both positive and negative values. These additional positive and negative suppressions of the quantum terms were responsible for the loss of M_p hierarchy as discussed in [15]. Here our aim would be to see how the conclusions of [15] may be avoided.

To inquire how the g_s scaling appears now, we will have to work out the non-localities order by order in r. We first work out the lowest level of non-locality from (2.111). Using the metric ansatze (2.3) with the warp-factor as defined in (2.5), the non-local quantum piece (2.111) yields:

$$\begin{aligned}\mathbb{W}^{(1)}(y) &= \int d^8y' F_1(t) F_2^2(t) g_s^{-2/3} h^{3/2} \sqrt{(\det g_{\alpha\beta})(\det g_{mn})(\det g_{ab})} \left(\frac{\mathbb{F}^{(1)}(y-y')\mathbb{Q}_{\rm T}^{(\{l_i\},n)}(y')}{M_p^{\sigma(\{l_i\},n)-8}}\right)\\ &= \int d^8y' \left(e_0 g_s^{-2/3} + \frac{e_1 g_s^{4/3}}{\sqrt{h}}\right) \mathbf{V}_8(y') \left[\frac{\mathbb{F}^{(1)}(y-y') g_s^{\Theta_k}\left(\widetilde{\mathbb{Q}}_{\rm T}^{(\{l_i\},n)}(y') + \mathcal{O}(y', g_s^{\Delta}, e^{-1/g_s^{\Delta}})\right)}{M_p^{\sigma(\{l_i\},n)-8}}\right],\end{aligned} \tag{2.115}$$

where in the second line we have used the relation (2.7) to express the g_s scalings of both the volume-preserving (i.e (2.2) with $(e_0, e_1) = (1, 0)$), and the fluctuating (i.e (2.8) with $(e_0, e_1) = (0, 1)$) cases (special care needs to be used to define the quantum pieces for the two cases (2.8) and (2.2) as the former uses (2.78) and the latter uses (2.94). Apart from this subtlety, everything else remains identical.). The g_s scalings of all the quantum terms in (2.78) and (2.94) are expressed using $\Theta_k \equiv \Theta_k(\{l_i\}, n)$ which would cover for the two cases, (2.97) related to (2.2) and (2.84) related to (2.8). The $\widetilde{\mathbb{Q}}_{\rm T}^{(\{l_i\},n)}(y')$ represent the spatial parts of the quantum terms (2.78) and (2.94) that do not depend on $e^{-1/g_s^{\Delta}}$. Finally $\mathbf{V}_8(y')$ is defined as:

$$\mathbf{V}_8(y') \equiv h^{3/2}(y') \sqrt{(\det g_{\alpha\beta})(\det g_{mn})(\det g_{ab})}, \tag{2.116}$$

which would contribute to the warped volume of the internal space when integrated over the eight-manifold. All the metric components depend on coordinates of the eight-manifold generically, but there are certain constraints that restricted the dependences to certain sub-space of the internal manifold. Such constraints will help us evaluate the quantum terms in (2.115) for the two cases, (2.2) and (2.8), and also compare our results with the generic case discussed in [15].

Let us start by considering the simplified case where $h(y) = h(y_0) \equiv h_0$ where y_0 is a chosen special point inside the eight-manifold. Such a choice allows us to choose the same string coupling g_s at every order of the non-locality. All other variables, for example the metric components, remain functions of y coordinates.

Under such a simplification the g_s scaling of the r-th level of non-locality becomes:

$$\mathbb{W}^{(r)}(y_{r+1}) = \frac{1}{M_p^{\sigma_r}}\left(e_0 g_s^{-2/3} + \frac{e_1 g_s^{4/3}}{\sqrt{h_0}}\right)^r g_s^{\Theta_k}\mathbb{G}_8(y_{r+1}), \tag{2.117}$$

which is defined for a given choice of $(\{l_i\}, n)$, and we have made a judicious coordinate choice of y_{r+1} to label the non-local quantum term with zero Lorentz index[22]. The power of M_p suppression may be read out from (2.114) for the given choice of $(\{l_i\}, n)$, and the functional form for $\mathbb{G}_8(y_{r+1})$ may be expressed in terms of the nested integrals in the following way:

$$\mathbb{G}_8(y_{r+1}) \equiv \prod_{q=0}^{r-1}\int d^8 y_{r-q}\mathbf{V}_8(y_{r-q})\mathbb{F}^{(r-q)}(y_{r-q} - y_{r-q-1})\left(\widetilde{\mathbb{Q}}_{\rm T}^{(\{l_i\},n)}(y_1) + \mathcal{O}(y_1, g_s^{\Delta}, e^{-1/g_s^{\Delta}})\right), \tag{2.118}$$

with $\mathbf{V}_8(y_{r-q})$ being taken from (2.116) with the constant choice of the warp-factor h_0. The nested integrals are expressed in terms of the $\mathbf{V}_8(y')$ and $\mathbb{F}^{(r)}(y - y')$, and this may help us to distinguish between the two choices, (2.2) and (2.8); and also between the generic case discussed in [15]. By construction (2.118) will always be finite because the integrals are over finite domains, and the non-locality functions $\mathbb{F}^{(r)}(y - y')$ are chosen to be normalizable functions.

Case 1: $F_1(t)$ and $F_2(t)$ satisfying the fluctuation condition (2.8)

We start by considering the choice (2.8) where the inverse of $F_2(t)$ has a perturbative expansion but the inverse of $F_1(t)$ does not. This means $e_0 = 0$ and $e_1 = 1$ in (2.117). Additionally because of the derivative constraint there, all variables were taken to be functions of the coordinates of $\mathcal{M}_4$, and were thus independent of both $\mathcal{M}_2$ and $\mathbb{T}^2/\mathcal{G}$ coordinates. We will however take the warp-factor $h(y^m) = h_0$ as before to avoid changing the string coupling g_s to any order in non-locality. Similarly, the non-locality functions will be taken to be functions of $\mathcal{M}_4$ only. Putting everything together, (2.117) changes to:

$$\mathbb{W}_1^{(r)}(y_{r+1}) = \left(\frac{\mathbb{G}_4(y_{r+1}) g_s^{4r/3+\theta_k}}{M_p^{\sigma_r}\sqrt{h_0}}\right)\mathbb{V}_{\mathbb{T}^2}^r \mathbb{V}_{\mathcal{M}_2}^r, \tag{2.119}$$

where the volume elements are defined as: $\mathbb{V}_{\mathbb{T}^2} = \int d^2y^a\sqrt{\det g_{ab}}$ for the volume of the subspace $\mathbb{T}^2/\mathcal{G}$ and $\mathbb{V}_{\mathcal{M}_2} = \int d^2y^\alpha\sqrt{\det g_{\alpha\beta}}$ for the volume of the subspace $\mathcal{M}_2$. The metric components g_{ab} and $g_{\alpha\beta}$ are the un-warped metric coefficients that appear in (2.3). Note that the r-th level of non-locality requires these volume elements to be raised to the r-th powers, as evident from (2.118) above. The g_s scaling for a choice of $(\{l_i\}, n)$ has the expected θ_k dependence from (2.84), but the non-locality adds another $+4r/3$ piece to it. This means that, there are no additional time-neutral pieces generated by non-locality here as θ_k from (2.84) doesn't have

[22] We take $y_0 = 0$ to comply with our choice of coordinates.

any time-neutral solutions with $\Delta k \geq \frac{3}{2}$. Finally, the $\mathbb{G}_4(y^m_{r+1})$ factor has the following nested integral representation as (2.118):

$$\mathbb{G}_4(y_{r+1}) \equiv \prod_{q=0}^{r-1} \int d^4 y_{r-q} \sqrt{g_4}\, \mathbb{F}^{(r-q)}(y_{r-q} - y_{r-q-1}) \left(\widetilde{\mathbb{Q}}_{\rm T}^{(\{l_i\},n)}(y_1) + \mathcal{O}(y_1, g_s^\Delta, e^{-1/g_s^\Delta}) \right), \tag{2.120}$$

where $g_4 = \det g_{mn}$ with the integral defined over the subspace $\mathcal{M}_4$; and we have absorbed the factor of $h_0^{3/2}$ in the definition of g_4. The function $\mathbb{G}_4(y)$ captures the additional $\mathcal{O}(g_s^\Delta, e^{-1/g_s^\Delta})$ corrections and thus responsible for the perturbative and non-perturbative series in g_s. This is as what one would have expected, although a question might be raised on the dependence of the non-locality function $\mathbb{F}^{(r)}(y - y')$ only on $\mathcal{M}_4$ coordinates. This may be justified, beyond declaring it as an imposed condition, by looking at (2.110) in the limit $q = 0$. In this limit $\mathbb{W}^{(\{l_i\},n)}$, i.e for $q = 0$, becomes a local function and therefore the derivative constraints will imply that the coefficients $C_0(y)$ will have to be a function of $\mathcal{M}_4$ coordinates. Similarly taking $q = 1$, $\square\mathbb{W}^{(\{l_i\},n)}$ becomes a local function and therefore $C_1(y)$ will have to be function of $\mathcal{M}_4$ coordinates. Following this chain of logic, C_q for any q becomes a function of $\mathcal{M}_4$ coordinates. Therefore at this stage, using the identification (2.113), the functions $\mathbb{F}^{(r)}(y - y')$ should only depend on the coordinates of $\mathcal{M}_4$, justifying the integral representation (2.120).

Once we allow quantum terms with two free Lorentz indices, the story evolves in the same way as above, so we will suffice ourselves in elaborating the g_s scalings of the various terms. Looking at (2.99), and comparing it with (2.119), the g_s scaling become $g_s^{\tilde{\theta}_k}$, where:

$$\tilde{\theta}_k = \left(\theta_k + \frac{4}{3}(r-2), \theta_k + \frac{2}{3}(2r-1), \theta_k + \frac{4}{3}(r+1) \right), \tag{2.121}$$

with the first one corresponding to free Lorentz indices along (i, j) and $(0, 0)$ directions; the second one corresponds to free Lorentz indices along $\mathcal{M}_4$, i.e along (m, n) directions and the third one corresponds to free Lorentz indices along $\mathbb{T}^2/\mathcal{G}$ and $\mathcal{M}_2$ i.e along (a, b) and (α, β) directions respectively. From (2.121) we see that even with the lowest level of non-locality i.e with $r = 1$, there are no additional time-neutral series along (m, n), (a, b) and (α, β) directions. Even more interestingly, since at the end we have to go to type IIB from M-theory, we can take the limit:

$$\mathbb{V}_{\mathbb{T}^2} \to 0, \tag{2.122}$$

any additional time-neutral series along the (i, j) and $(0, 0)$ directions are heavily suppressed by powers of $\mathbb{V}_{\mathbb{T}^2}$, which in turn should also be the case with zero free Lorentz index in (2.119).

Case 2: $F_1(t)$ and $F_2(t)$ satisfying the volume-preserving condition (2.2)

The story that we elaborated for case 1 pretty much sums up all the procedure that we need for the present case where both $F_1(t)$ and $F_2(t)$ have perturbative expansions, including their inverses. However there are now a few crucial differences that will alter our story in an interesting way. First, the derivative constraints are weakened from case 1 in a way that we no longer restrict the derivatives to be along $\mathcal{M}_4$ only. We do however want the functions to be independent of the (x_3, x_{11}) directions so that components like $\mathbf{G}_{MNab}$ do not complicate our analysis by switching on (l_{36}, l_{37}, l_{38}) in (2.94). Therefore now we can allow all curvature tensors and G-fluxes to be functions of $\mathcal{M}_2 \times \mathcal{M}_4$, implying that, in the type IIB side, all curvature tensors and fluxes would be functions of the six-dimensional internal space. This is good because the derivative constraint for case 1 was a tad bit un-natural in the light of the genericity that we want to impose on the quantum corrections. The r-th level of non-locality may now be read from (2.117) by using $e_0 = 1$ and $e_1 = 0$ and using the quantum terms from (2.94). We will use the same approximation for the warp-factor, namely $h(y) = h_0$ to avoid changing g_s to any order in the non-locality. Putting everything together, (2.117) for the present case becomes:

$$\mathbb{W}_2^{(r)}(y_{r+1}) = \left(\frac{\mathbb{G}_6(y_{r+1})g_s^{-2r/3+\theta'_k}}{M_p^{\sigma_r}\sqrt{h_0}}\right)\mathbb{V}^r_{\mathbb{T}^2}. \tag{2.123}$$

Compared to (2.119) there are a few key differences. First, there is no volume element $\mathbb{V}_{\mathcal{M}_2}$ appearing anymore because this goes inside $\mathbb{G}_4(y)$, as defined in (2.120) to construct $\mathbb{G}_6(y)$. In other words, $\mathbb{G}_6(y)$ takes the following form:

$$\mathbb{G}_6(y_{r+1}) \equiv \prod_{q=0}^{r-1}\int d^6y_{r-q}\sqrt{g_6}\,\mathbb{F}^{(r-q)}(y_{r-q} - y_{r-q-1})\left(\widetilde{\mathbb{Q}}_{\rm T}^{(\{l_i\},n)}(y_1) + \mathcal{O}(y_1, g_s^{\Delta}, e^{-1/g_s^{\Delta}})\right), \tag{2.124}$$

where again we have absorbed a factor of $h_0^{3/2}$ in the definition of g_6 and $\widetilde{\mathbb{Q}}_{\rm T}^{(\{l_i\},n)}(y_1)$ being extracted from (2.94). The second key difference, which is important, is the g_s scaling. Using the original g_s scaling (2.97) with zero Lorentz index for the quantum terms associated with the case (2.2), we now see that the r-th order of non-locality now adds a factor of $-2r/3$ to the original scaling in the local case. Recall that θ'_k as defined in (2.97) for $\Delta k > \frac{1}{3}$ *did not* have any time-neutral series, but now it appears that the non-locality would in fact help to create more time-neutral series. With two free Lorentz indices, the g_s scaling now appears to $g_s^{\tilde{\theta}'_k}$, where:

$$\tilde{\theta}'_k = \left(\theta'_k - \frac{2}{3}(r+4), \theta'_k - \frac{2}{3}(r+1), \theta'_k - \frac{2}{3}(r-2)\right). \tag{2.125}$$

In addition to the difference with the scaling behavior in (2.121), there are a few other differences. The first one is in the ordering of the scaling behavior as it appears in (2.125). The first term in (2.125) corresponds to free Lorentz indices along (i, j) and $(0, 0)$ directions; but the second term corresponds to free Lorentz indices along $\mathcal{M}_4$ as well as $\mathcal{M}_2$, i.e along (m, n) and (α, β) directions respectively. The third term now corresponds to free Lorentz indices along $\mathbb{T}^2/\mathcal{G}$ i.e along (a, b) direction.

The second difference between (2.121) and (2.125) appears from the value of r, i.e from the level of non-locality. While in (2.121) increasing r makes all the three terms there positive definite thus adding no extra time-neutral series, in (2.125) the effect is opposite. Increasing r in (2.125) actually creates more relative minus signs thus making every terms prone to generating new time-neutral series. Fortunately, the degree of non-locality is also suppressed by powers of $\mathbb{V}_{\mathbb{T}^2}$, as may be inferred from (2.123), and in the limit when the volume $\mathbb{V}_{\mathbb{T}^2}$ vanishes, all the additional time-neutral series also decouple completely. The vanishing of $\mathbb{V}_{\mathbb{T}^2}$ is an essential requirement for our M-theory construction to connect it to type IIB theory.

Case 3: Time-independent internal space with $F_1(t) = F_2(t) = 1$

The volume condition (2.122) pretty much saves the day for the two case discussed above despite the fact that, for case 2, new time-neutral series seem to appear from the higher levels of non-localities. The question is what happens when the internal space is time independent i.e when $F_1(t) = F_2(t) = 1$? We expect the story to progress more or less in the same vein as above, and in fact most of the details remain somewhat similar to case 2 above, but with one crucial difference. Since $\mathbf{G}_{MNab}$ features prominently in the discussion concerning this case, as evidenced from (2.88) and (2.89), which in turn are responsible for the time-neutrality condition (2.99) with zero free Lorentz indices, all curvature tensors and G-fluxes in the theory need to be functions of $\mathcal{M}_4 \times \mathcal{M}_2 \times \mathbb{T}^2/\mathcal{G}$ coordinates except the x_3 direction. In addition, there is as such no derivative condition imposed from the dynamics, the non-locality function $\mathbb{F}^{(r)}(y - y')$ could in principle be function of x_3 also. The r-th level of non-locality then becomes:

$$\mathbb{W}_3^{(r)}(y_{r+1}) = \frac{\mathbb{G}_8(y_{r+1}) g_s^{-2r/3+\theta'_0}}{M_p^{\sigma_r}\sqrt{h_0}}. \tag{2.126}$$

where θ'_0 is as given in (2.98), which already allows time-neutral series because there are relative minus signs due to the presence of (l_{36}, l_{37}, l_{38}) as well as n_3. We now see that the r-th level of non-locality creates additional relative minus signs that help in generating more time-neutral series here. Similar picture emerges with two free Lorentz indices, as one may easily derive. Note also the absence of volume components like $\mathbb{V}_{\mathbb{T}^2}$ or $\mathbb{V}_{\mathcal{M}_2}$ as these factors appear in the nested integral (2.118) that defines $\mathbb{G}_8(y)$. It should be clear that in the limit of vanishing volume (2.122), the quantum term (2.126) doesn't have to decouple, thus paving way to the non-local counter-terms as advertised in [15] (see footnote 25 and the example cited in there).

Case 4: Non-locality in time for the various choices of $F_i(t)$

The final case that we want to elaborate is a rather curious one, because it involves non-locality in both (internal) space and time. The temporal non-locality would only make sense as an integral condition. In other words we can take the non-locality function $\mathbb{F}^{(r)}(y - y', t - t')$ to be functions of both (y, t) as well as (y', t'). However since we have identified any temporal dependence with $\frac{g_s^2}{\sqrt{h}}$ (see (2.6)), the non-locality function should now have both y, y' and g_s dependence. Therefore, much in the same vein as before, we can assign the following generic form for the non-locality function:

$$\mathbb{F}^{(r)}(y - y', g_s) \equiv \sum_{l_a, l_b} f_{l_a l_b}^{(r)}(y - y') \left(\frac{g_s^2}{\sqrt{h}}\right)^{\Delta l_a} \exp\left(-\frac{l_b h^{\Delta/4}}{g_s^{\Delta}}\right), \tag{2.127}$$

where $(l_a, l_b) \in (\mathbb{Z}/2, \mathbb{Z})$, the warp-factor $h = h(y - y')$ and $f^{(r)}_{l_a l_b}(y - y')$ to be a highly peaked function at low energies. We can also resort to the simplification $h(y - y') = h_0$ to keep the g_s itself unaltered to all order in the non-locality, as we have done before. Plugging this in (2.117) and (2.118) results in a complicated nested integral form, which would then have to be integrated over time to make sense of the result. In other words, we want:

$$\mathbb{U}^{(r)}(y_{r+1}, g_s(t)) \equiv \int_{-\infty}^{t} \frac{dt' \sqrt{\mathbf{g}_{00}}}{M_p^{\sigma_r}} \left(e_0 g_s^{-2/3}(t') + \frac{e_1 g_s^{4/3}(t')}{\sqrt{h_0}}\right)^r g_s^{\Theta_k}(t') \mathbb{G}_8(y_{r+1}, g_s(t')), \tag{2.128}$$

where the three cases discussed above are described by assigning different values to the triplet (e_0, e_1, Θ_k) i.e $(0, 1, \theta_k), (1, 0, \theta'_k)$ and $(1, 0, \theta'_0)$ with θ_k, θ'_k and θ'_0 as defined in (2.84), (2.97) and (2.98) respectively. The $g_s(t')$ dependence of $\mathbb{G}_8(y_{r+1}, g_s(t'))$ may be determined by plugging in (2.127) in (2.118).

The concern however is the integral (2.128) itself. Since g_s, as defined in (2.6) depends on time itself, so when $t \to -\infty$, $g_s \to +\infty$. The representation (2.127) is not a suitable description at strong coupling. because (2.127) is only defined perturbatively when $g_s \to 0$. We can do a change of variable $t \to 1/t$, or $g_s \to 1/g_s$ to study the strong coupling regime. In either formalism, it then appears that the relevant integral will be:

$$\int_0^{g_s} dg'_s \, g'^{\Delta q_1}_s \exp\left(-\frac{q_2}{g'^{\Delta}_s}\right) = q_2^{q_1 + \frac{1}{\Delta}} \Gamma\left(-q_1 - \frac{1}{\Delta}, \frac{q_2}{g_s^{\Delta}}\right) \tag{2.129}$$

$$= \frac{1}{q_2}\left(g_s^{q_1+1+\frac{1}{\Delta}} + \mathcal{O}(g_s^{q_1+2+\frac{1}{\Delta}})\right) \exp\left(-\frac{q_2}{g_s^{\Delta}} + \mathcal{O}(g_s^{2\Delta})\right), \tag{2.130}$$

with $g_s < 1$ so that the expansion on the second line could be justified. The perturbative expansion then tells us that for any choice q_1 in the g_s expansion, non-locality to any order only adds a $1 + \frac{1}{\Delta}$ factor, and therefore doesn't alter any of our earlier conclusions regarding g_s scalings. Additionally, the decoupling effect for vanishing volume as in (2.122) still persists, so no new subtleties appear at this stage.

Topological quantum terms, curvature forms and fluxes

So far we have dealt with the non-topological quantum terms in terms of curvatures and G-flux components that would contribute to the energy-momentum tensor. However there are also EOMs associated with the G-fluxes that would demand contributions from the quantum terms (2.94), and (2.78) for the cases (2.2) and (2.8) respectively. Once we look at the fluxes, we will have to study both the standard four-form G-fluxes and their dual, the seven-form, flux components. Thus we need to see how the g_s scalings (2.97) and (2.86), respectively for the two cases, would change. Additionally, there would also be topological terms that we have to determine. We first analyze the topological terms.

The topological contributions, as the name suggest, would appear from topological forms that are constructed using the Riemann tensors and the G-flux components by taking advantages of their anti-symmetries. They may be expressed as:

$$\begin{aligned} &\mathbb{R} \equiv \mathbf{R}_{MN}^{a_o b_o} \mathbf{M}_{a_o b_o}\, dy^M \wedge dy^N, \quad \mathbb{G} \equiv \mathbf{G}_{MN}^{a_o b_o} \mathbf{M}_{a_o b_o}\, dy^M \wedge dy^N \\ &\mathbf{R}_{MN}^{a_o b_o} \equiv \mathbf{R}_{MNPQ}\, e^{a_o P} e^{b_o Q}, \qquad \mathbf{G}_{MN}^{a_o b_o} \equiv \mathbf{G}_{MNPQ}\, e^{a_o P} e^{b_o Q}, \end{aligned} \tag{2.131}$$

where $\mathbf{M}_{a_o b_o}$ are the holonomy matrices on the compact manifold over which we will be taking traces. These are just like the generator matrices, for example as the ones appearing like $\mathbf{A}_\mu^a \mathbf{T}^a$, in the definition of a gauge field one-form. Using (2.131), we can construct various higher order forms, one example being the following eight-form:

$$\mathbb{Z}_8 \equiv c_1 \mathrm{tr}\, \mathbb{R}^4 + c_2 \left(\mathrm{tr}\, \mathbb{R}^2\right)^2 + c_3 \left(\mathrm{tr}\, \mathbb{R}^2\right)\left(\mathrm{tr}\, \mathbb{G}^2\right) + c_4 \mathrm{tr}\, \mathbb{G}^4, \tag{2.132}$$

where we have assumed that the holonomy matrices are traceless. For various choices of the c_i coefficients, we can generate certain sub eight-forms. For example with:

$$c_1 = \frac{1}{3 \cdot 2^{10} \cdot \pi^4}, \quad c_2 = -\frac{1}{12 \cdot 2^{10} \cdot \pi^4}, \quad c_3 = c_4 = 0, \tag{2.133}$$

we have our $\mathbf{X}_8$ polynomial which is important to cancel anomalies as we shall see later. However now with non-zero (c_3, c_4) more non-trivial polynomials may be constructed which, in a packaged form, is given as (2.132). In fact polynomials like (2.132) open up the possibility of constructing topological and non-topological interactions in M-theory of the following form:

$$\mathbf{C}_3 \wedge \mathbb{Z}_8, \qquad \mathbf{G}_4 \wedge *_{11} \mathbb{Z}_4, \tag{2.134}$$

where $\mathbf{C}_3$ is the M-theory three-form and the Hodge star is with respect to the full eleven-dimensional *warped* metric (as such it will be a function of g_s). The way we have expressed the non-topological piece, should allow us to extract this from the generalized quantum terms (2.94) and (2.78) for (2.2) and (2.8) respectively. For example the non-topological piece in (2.134) may be expressed as:

$$\begin{aligned} \int \mathbf{G}_4 \wedge *_{11} \mathbb{Z}_4 \;\; &\equiv \;\; \int d^{11}y \sqrt{-\mathbf{g}_{11}} \sum_{\{l_i\}, n_1, n_2} \mathbb{Q}_{\mathrm{T}}\left(\{l_i\}, n_1, n_2\right) \\ &= \;\; \int d^{11}y \sqrt{-\mathbf{g}_{11}} \left(\mathbf{G}_4\right)_{M_1 M_2 M_3 M_4} \left(\mathbb{Z}_4\right)_{N_1 N_2 N_3 N_4} \mathbf{g}^{M_1 N_1} \mathbf{g}^{M_2 N_2} \mathbf{g}^{M_3 N_3} \mathbf{g}^{M_4 N_4}, \end{aligned} \tag{2.135}$$

where we have used the warped metric both as inverses as well as in the definition of the determinant, and the quantum terms $\mathbb{Q}_{\mathrm{T}}\left(\{l_i\}, n_1, n_2\right)$ are defined as in (2.94) for the case (2.2) (changing the quantum terms to (2.78) will provide information for the case (2.8)). The above relation could be used for identifying the $\mathbb{Z}_4$ tensor from the quantum series (2.94) or (2.78) for the two cases (2.2) and (2.8) respectively.

We can then ask the g_s scalings of the following two kinds of quantum terms:

$$(\mathbf{G}_4)_{012M}\,(\mathbf{Z}_4)^{012M}, \quad (\mathbf{G}_4)_{MNPQ}\,(\mathbf{Z}_4)^{MNPQ}, \tag{2.136}$$

where (M, N, P) are the coordinates of the eight-manifold. The g_s scalings of these two interactions may be easily worked out by extracting a $(\mathbf{C}_3)_{012}$ and a $(\mathbf{C}_3)_{MNP}$ out of either (2.94) or (2.78). Since $(\mathbf{G}_4)_{012M}$ and $(\mathbf{G}_4)_{MNPQ}$ scale as $\left(\frac{g_s}{H}\right)^{-4}$ and $\left(\frac{g_s}{H}\right)^{2\Delta k}$ respectively, it is easy to infer the g_s scalings of $(\mathbf{Z}_4)^{012M}$ and $(\mathbf{Z}_4)^{MNPQ}$ respectively as:

$$\theta'_k \;\to\; \theta'_k + 4, \quad\;\; \theta'_k \;\to\; \theta'_k - 2\Delta k, \tag{2.137}$$

with θ'_k as given in (2.97). A similar scaling would work if we replace θ'_k with θ_k from (2.86), as one would expect. On the other hand, $\mathbb{Z}_8$ should be topological. To see this let us first fix the time to $t = t_0$ in the M-theory metric (2.3) and, for simplicity, switch off the G-fluxes. Plugging in the metric ansatze (2.3) at the fixed time, with the choice (2.133), in (2.132) then shows that at any $t = t_0 + \delta t$, (2.132) may in general have δt dependence in addition to a piece that depends on t_0. Since the temporal behavior is traded with g_s, (2.132) will develop g_s dependence. Additionally, because of the underlying non-Kählerity of the internal eight-manifold (at least for the case (2.2)), the integral of $\mathbf{X}_8$ is not exactly the Euler characteristics of the eight-manifold[23]. Switching on the G-fluxes, the integral of $\mathbb{Z}_8$ should also have a g_s dependent pieces. Together all of these would complicate the anomaly cancellation procedure that we have known for the time-independent case, implying a careful study is required in the time-dependent case. More details on this appears in section 3.1.2.

There are other topological contributions possible once we go to the *dual* formalism. Here duality implies a generalized form of electric-magnetic duality, much like the Montonen-Olive one [32]. To implement it here, at least at the level of perturbative and non-perturbative expansions that we have entertained so far, all we need is to express the flux contributions by their dual variables. The dual of a four-form flux is a seven-form flux, and therefore if we can express (2.94) and (2.78) using the dual variables, we should be able to determine their g_s scalings as well. This rather convoluted re-telling of the same story has a deeper purpose: the dual description will not only help us to determine the Bianchi identities later but also help us to ascertain the flux quantization conditions. The dual seven-form $\mathbf{G}_7 = *_{11}\mathbf{G}_4$, may be expressed in terms of components in the following standard way:

$$\mathbf{G}_7 = \frac{1}{7!}\mathbf{G}_{P'Q'R'S'}\sqrt{-\mathbf{g}_{11}}\,\mathbf{g}^{P'P}\mathbf{g}^{Q'Q}\mathbf{g}^{R'R}\mathbf{g}^{S'S}\epsilon_{PQRSM_1M_2.....M_7}dy^{M_1}\wedge dy^{M_2}.......\wedge dy^{M_7}, \tag{2.138}$$

where the metric components as well as the determinant are all defined in terms of the warped metric and $\epsilon_{PQ....M_7}$ is the eleven-dimensional Levi-Civita *symbol*. The above formula is an useful way to determine the g_s scalings of every components of the dual form once the original g_s scalings are known. This will also help us to determine the g_s scalings of the quantum terms, relevant for the case (2.2), that may

[23]We thank Savdeep Sethi for discussions on this point.

now be expressed in the following way:

$$\begin{aligned}
\mathbb{Q}_{\rm T}^{(2)} &= \mathbf{g}^{m_i m'_i}\mathbf{g}^{m_l m'_l}....\mathbf{g}^{j_k j'_k}\partial_{m_1}..\partial_{m_{n_1}}\partial_{\alpha_1}..\partial_{\alpha_{n_2}}\left(\mathbf{R}_{mnpq}\right)^{l_1}\left(\mathbf{R}_{abab}\right)^{l_2}\left(\mathbf{R}_{pqab}\right)^{l_3}\left(\mathbf{R}_{\alpha ab\beta}\right)^{l_4}\\
&\times \left(\mathbf{R}_{\alpha\beta mn}\right)^{l_5}\left(\mathbf{R}_{\alpha\beta\alpha\beta}\right)^{l_6}\left(\mathbf{R}_{ijij}\right)^{l_7}\left(\mathbf{R}_{ijmn}\right)^{l_8}\left(\mathbf{R}_{iajb}\right)^{l_9}\left(\mathbf{R}_{i\alpha j\beta}\right)^{l_{10}}\left(\mathbf{R}_{0mnp}\right)^{l_{11}}\\
&\times \left(\mathbf{R}_{0m0n}\right)^{l_{12}}\left(\mathbf{R}_{0i0j}\right)^{l_{13}}\left(\mathbf{R}_{0a0b}\right)^{l_{14}}\left(\mathbf{R}_{0\alpha 0\beta}\right)^{l_{15}}\left(\mathbf{R}_{0\alpha\beta m}\right)^{l_{16}}\left(\mathbf{R}_{0abm}\right)^{l_{17}}\left(\mathbf{R}_{0ijm}\right)^{l_{18}}\\
&\times \left(\mathbf{R}_{mnp\alpha}\right)^{l_{19}}\left(\mathbf{R}_{m\alpha ab}\right)^{l_{20}}\left(\mathbf{R}_{m\alpha\alpha\beta}\right)^{l_{21}}\left(\mathbf{R}_{m\alpha ij}\right)^{l_{22}}\left(\mathbf{R}_{0mn\alpha}\right)^{l_{23}}\left(\mathbf{R}_{0m0\alpha}\right)^{l_{24}}\left(\mathbf{R}_{0\alpha\beta\alpha}\right)^{l_{25}}\\
&\times \left(\mathbf{R}_{0ab\alpha}\right)^{l_{26}}\left(\mathbf{R}_{0ij\alpha}\right)^{l_{27}}\left(\mathbf{G}_{0ij\alpha\beta ab}\right)^{l_{28}}\left(\mathbf{G}_{0ijq\alpha ab}\right)^{l_{29}}\left(\mathbf{G}_{0ijq\alpha\beta b}\right)^{l_{30}}\left(\mathbf{G}_{0ijmnab}\right)^{l_{31}}\left(\mathbf{G}_{0ijmn\alpha b}\right)^{l_{32}}\\
&\times \left(\mathbf{G}_{0ijnpqb}\right)^{l_{33}}\left(\mathbf{G}_{mnp\alpha\beta ab}\right)^{l_{34}}\left(\mathbf{G}_{mnpq\alpha ab}\right)^{l_{35}}\left(\mathbf{G}_{0ijmn\alpha\beta}\right)^{l_{36}}\left(\mathbf{G}_{0ijmnpq}\right)^{l_{37}}\left(\mathbf{G}_{0ijmnp\alpha}\right)^{l_{38}},
\end{aligned} \tag{2.139}$$

which should now be compared to (2.94) written in terms of the original variables. We could also re-express (2.78), relevant for the case (2.8), in terms of the dual variables, but since the story would be similar to what we have in (2.139) we will avoid this exercise. In fact making the following two-step processes to (2.139), we can convert this to the case corresponding to (2.8): one, make $n_2 = l_{19} = l_{20} = ... = l_{27} = 0$, and two, relabel $l_{28}, ..., l_{38}$ to $l_{19}, ..., l_{29}$. The g_s scalings are easy to determine using the method employed in the earlier sections (see **Table 2.2** for details). Following these footsteps, one may easily verify that the g_s scalings of the quantum terms in (2.139) are *exactly* the same as in (2.97). Needless to say, the g_s scalings of the quantum terms corresponding to the case (2.8), are also exactly the same as in (2.86). This shows that resorting to the dual variables *do not* change the g_s scalings of the quantum terms, and is therefore reassuring to see that the expected equivalences between dual theories are respected at every order in the g_s expansions.

Resorting to the dual fluxes $\mathbf{G}_7$ allow us to define six-form potentials $\mathbf{C}_6$ such that $\mathbf{G}_7 = d\mathbf{C}_6 + ...$, where the dotted terms depend on how the Bianchi identities appear in our set-up. This will be elaborated later when we discuss the EOMs for fluxes. What we want to study here are the various forms of interactions, both topological and non-topological, that may appear when we consider quantum terms like (2.139). Motivated by (2.134), we expect interactions like:

$$\mathbf{C}_6 \wedge \mathbb{Z}_5, \qquad \mathbf{G}_7 \wedge *_{11}\mathbb{Z}_7, \tag{2.140}$$

where $\mathbb{Z}_5$ and $\mathbb{Z}_7$ are five and seven-forms constructed out of the curvature and the flux forms like (2.131). However an odd form like $\mathbb{Z}_5$ cannot be constructed out of the two-forms from (2.131), so can only be expressed as:

$$\mathbb{Z}_5 \equiv \mathbf{\Lambda}_5 + d\hat{\mathbb{Z}}_4, \tag{2.141}$$

where $\mathbf{\Lambda}_5$ is a highly localized form which would represent a M5-brane once wedged with $\mathbf{C}_6$. The other four-form $\hat{\mathbb{Z}}_4$ can be constructed[24] out of the curvature two-form and gauge form coming from localized G-fluxes. Finally, the second term in (2.140)

[24]The two four-forms $\mathbb{Z}_4$ and $\hat{\mathbb{Z}}_4$ are definitely related to each other because they describe similar interactions in M-theory, albeit in the relative dual pictures. We will however not elaborate on their precise equivalence here.

contributes the following non-topological interaction:

$$\begin{aligned}\int \mathbf{G}_7 \wedge *_{11}\mathbb{Z}_7 &\equiv \int d^{11}y\sqrt{-\mathbf{g}_{11}} \sum_{\{l_i\},n_1,n_2} \mathbb{Q}_{\rm T}^{(2)}\left(\{l_i\}, n_1, n_2\right) \\ &= \int d^{11}y\sqrt{-\mathbf{g}_{11}}\,(\mathbf{G}_7)_{M_1.....M_7}\,(\mathbb{Z}_7)_{N_1.....N_7}\,\mathbf{g}^{M_1N_1}......\mathbf{g}^{M_7N_7},\end{aligned} \tag{2.142}$$

which is similar to what we had in (2.135) earlier. Again, the metric components are all taken as the warped ones and therefore involve g_s factors in them, and $\mathbb{Q}_{\rm T}^{(2)}\left(\{l_i\}, n_1, n_2\right)$ are the quantum terms as given in (2.139). The conjectured equality (2.142) is to be used to define the functional form for $\mathbb{Z}_7$ tensor, much like what we had in (2.135) earlier, and basically tells us that that $\mathbb{Z}_7$ is constructed out of products of tensors in such a way that it is an anti-symmetric tensor of rank 7.

Another important thing to notice about (2.78), (2.94) and (2.139) is that , although they contain globally defined tensors like four-form fluxes and the curvature tensors they are *not* globally defined functions. The fact that inverse metric components show up in the definition of the quantum terms, and that metric components are defined only on patches over the compact eight-manifold, render these quantum terms mostly local. Now because the Hodge dual of the forms $\mathbb{Z}_4$ and $\mathbb{Z}_7$ are related to the quantum terms (2.94) and (2.139) via (2.135) and (2.142) respectively, they cannot be globally defined forms. This is much like the form $\mathbf{X}_8 = d\mathbf{X}_7$, where $\mathbf{X}_7$ is not globally defined, and therefore the integral of $\mathbf{X}_8$ over a compact eight-manifold is non-zero.

In the following we will elaborate on all the background EOMs, both for the metric and the G-flux components, that would appear for our case once the effects of the quantum terms are included. The analysis that we presented above will be used once we study the G-flux EOMs and their constraints.

Tensors	Dual Forms	$\frac{g_s}{H}$ scaling for (2.2)	$\frac{g_s}{H}$ scaling for (2.8)
$\mathbb{Z}_7^{npq\alpha\beta ab}$	$\mathbf{G}_{0ijm}$	θ'_k	$\theta_k - 2$
$\mathbb{Z}_7^{mnpq\beta ab}$	$\mathbf{G}_{0ij\alpha}$	θ'_k	θ_k
$\mathbb{Z}_7^{0ij\alpha\beta ab}$	$\mathbf{G}_{mnpq}$	$\theta'_k - 2\Delta k + 2$	$\theta_k - 2\Delta k$
$\mathbb{Z}_7^{0ijq\beta ab}$	$\mathbf{G}_{mnp\alpha}$	$\theta'_k - 2\Delta k + 2$	$\theta_k - 2\Delta k + 2$
$\mathbb{Z}_7^{0ijq\alpha\beta b}$	$\mathbf{G}_{mnpa}$	$\theta'_k - 2\Delta k + 4$	$\theta_k - 2\Delta k + 2$
$\mathbb{Z}_7^{0ijpqab}$	$\mathbf{G}_{mn\alpha\beta}$	$\theta'_k - 2\Delta k + 2$	$\theta_k - 2\Delta k + 4$
$\mathbb{Z}_7^{0ijpq\beta b}$	$\mathbf{G}_{mn\alpha a}$	$\theta'_k - 2\Delta k + 4$	$\theta_k - 2\Delta k + 4$
$\mathbb{Z}_7^{0ijnpqb}$	$\mathbf{G}_{m\alpha\beta a}$	$\theta'_k - 2\Delta k + 4$	$\theta_k - 2\Delta k + 6$
$\mathbb{Z}_7^{0ijpq\alpha\beta}$	$\mathbf{G}_{mnab}$	$\theta'_k - 2\Delta k + 6$	$\theta_k - 2\Delta k + 4$
$\mathbb{Z}_7^{0ijmnpq}$	$\mathbf{G}_{\alpha\beta ab}$	$\theta'_k - 2\Delta k + 6$	$\theta_k - 2\Delta k + 8$
$\mathbb{Z}_7^{0ijnpq\beta}$	$\mathbf{G}_{m\alpha ab}$	$\theta'_k - 2\Delta k + 6$	$\theta_k - 2\Delta k + 6$

TABLE 2.2: The $\frac{g_s}{H}$ scalings of the various components of the seven-form $\mathbb{Z}_7$ represented for the two cases (2.2) and (2.8). We have taken $\Delta = \frac{1}{3}$ and $k \geq \frac{3}{2}$. The other two parameters, θ'_k and θ_k, are defined in (2.97) and (2.86) respectively.

Chapter 3

Equation of motion, Flux quantization and constraints

3.1 Analysis of the quantum equations of motion and constraints

We now have all the ingredients to consider the equations of motion and extract any constraints that may effect the dynamics of the system. The M-theory metric that is relevant for us is (2.3) with the warp-factors appearing there are defined as in (2.5). The $F_i(t)$ factors appearing in the metric are defined either using the volume preserving condition (2.2) or the fluctuating condition (2.8). Although both these forms allow perturbative expansions for $F_i(t)$, the former even allows the inverses to have perturbative expansions. The G-flux components are expressed as in (2.13) except the space-time components $\mathbf{G}_{\mu\nu\rho M}$ with y^M being the internal coordinates of the eight-manifold. Of course not all y^M are allowed, and we will deal with individual cases as we go along.

3.1.1 Einstein's equations and effective field theories

An important aspect of our discussion is the quantum terms as they will be solely responsible to change or alter the course of our analysis. These quantum terms that we will be concerned about right now are the ones that will contribute to the energy-momentum tensors. The other quantum terms that will effect the EOMs for the G-fluxes will be dealt a little later. The former category of quantum terms appear with two free Lorentz indices and whether or not they could create time-neutral series will form the basis of our discussion here. Thus keeping everything in perspective, we can represent the quantum terms in the following way that is a slight variant from what we had in [15]:

$$\mathbb{T}^{Q}_{MN} \equiv \sum_{k_1,k_2} \mathbb{C}^{(k_1,k_2)}_{MN}(y, M_p) \left(\frac{g_s^2}{\sqrt{h}}\right)^{\Delta k_1} \exp\left(-\frac{k_2 h^{\Delta/4}}{g_s^{\Delta}}\right), \tag{3.1}$$

where $(k_1, k_2) = (\mathbb{Z}/2, \mathbb{Z})$ with (M, N) being either of $(m, n), (\alpha, \beta), (a, b), (i, j)$ or $(0, 0)$. The pattern of representation of the quantum terms follow the same pattern of perturbative series expansions employed for the G-fluxes, and the F_i parameters. This is of course intentional and in some sense necessary if we want to balance all the EOMs.

The way we have expressed (3.1), the g_s scalings have been explicitly extracted out. Without pulling out the g_s scalings, (3.1) should be identified with either (2.78) or (2.94) depending on the choice (2.8) or (2.2) respectively for the case when we allow two free Lorentz indices. The g_s scalings should then coincide with either (2.99) or (2.100) respectively. These scalings immediately imply:

$$\Delta = \frac{1}{3}, \quad (k_1, k_2) \in \left(\frac{\mathbb{Z}}{2}, \mathbb{Z}\right), \tag{3.2}$$

for (3.1) and also for scalings of $F_2(t), F_1(t)$ and $\mathbf{G}_{MNPQ}$ in (2.9), (2.11) and (2.13) respectively. Eventually however it all boils down to the question whether $\mathbb{C}_{MN}^{(0,0)}$ exists or not, and if it exists, whether there is a M_p hierarchy or not[1]. For the case (2.8), our study of the scaling behavior (2.99) with θ_k defined as in (2.84), tells us that:

$$\mathbb{C}_{ab}^{(0,0)} = \mathbb{C}_{\alpha\beta}^{(0,0)} = 0, \quad \mathbb{C}_{mn}^{(0,0)} = \mathbf{R}_{mn}, \mathbf{g}_{mn}\, \mathbf{g}^{\alpha\beta}\Lambda_{\alpha\beta}^{(22)}, \tag{3.3}$$

but no $\Lambda_{mn}^{(11)}$ or $\Lambda_{mn}^{(12)}$ terms from (2.102). This is because (2.84) requires $l_{28} = 2$, implying $H_5 = 2, H_4 = 2$ and $H_3 = -1$ from (2.82). This actually vanishes, in the light of both the derivative constraint and the preservation of the type IIB metric form (2.1) as long as we ignore *localized* fluxes. The latter will be useful soon. The other non-zero tensor is the Ricci tensor $\mathbf{R}_{mn}$ that is time-neutral but is *not* a quantum piece. Therefore putting these together, all terms except $\mathbb{C}_{\mu\nu}^{(0,0)}$ vanish for the case (2.8). The non-local counter-terms do not add any extra time-neutral series for this case.

For the case (2.2) the scenario turns out to be a bit different from (3.3) because now the non-localities do contribute towards creating new time-neutral series as may be inferred from (2.123) with zero Lorentz indices and (2.125) for two free Lorentz indices. This means we should again be looking for $\mathbb{C}_{MN}^{(0,0)}$, which now takes

[1]As cautioned in footnote 2, it will be erroneous to expand (3.1) in inverse powers of g_s to extract g_s independent pieces. For example if one does it, then (3.1) becomes:

$$\mathbb{T}_{MN}^{Q} = \sum_{k_1,k_2,m} \frac{(-1)^m \Delta^m k_2^m \mathbb{C}_{MN}^{(k_1,k_2)}}{m!}\, g_s^{\Delta(2k_1-m)} h^{\Delta(m-2k_1)/4}$$

implying that there are time-neutral pieces whenever $m = 2k_1$. Such an analysis suffers from the problem that for any values of $m > 2k_1$ in the above expansion, the terms are not well defined in the limit $g_s \to 0$. Since all our expansions solely rely on the $g_s << 1$ limit, or more appropriately the $g_s \to 0$ limit, the inverse g_s expansions are not advisable as they will lead to erroneous conclusions.

the following form:

$$
\begin{aligned}
\mathbb{C}_{ab}^{(0,0)} &= 0 + \sum_{\{l_i\},n}\sum_{r=1}^{\infty} M_p^{-\sigma_r}\mathbb{V}_{\mathbb{T}^2}^r \mathbb{G}_{ab}^{(\{l_i\},n)}(y_{r+1})\delta\left(\theta'_k - \frac{2}{3}(r-2)\right)\\
\mathbb{C}_{\mu\nu}^{(0,0)} &= \sum_j \mathbf{C}_{\mu\nu}^{(j)} + \sum_{\{l_i\},n}\sum_{r=1}^{\infty} M_p^{-\sigma_r}\mathbb{V}_{\mathbb{T}^2}^r \mathbb{G}_{\mu\nu}^{(\{l_i\},n)}(y_{r+1})\delta\left(\theta'_k - \frac{2}{3}(r+4)\right)\\
\mathbb{C}_{A_iB_i}^{(0,0)} &= \left\{\mathbf{R}_{A_iB_i}, \Lambda_{A_iB_i}^{(ij)}\right\} + \sum_{\{l_i\},n}\sum_{r=1}^{\infty} M_p^{-\sigma_r}\mathbb{V}_{\mathbb{T}^2}^r \mathbb{G}_{A_iB_i}^{(\{l_i\},n)}(y_{r+1})\delta\left(\theta'_k - \frac{2}{3}(r+1)\right),
\end{aligned}
\tag{3.4}
$$

where (A_1, B_1) and (A_2, B_2) correspond to (m, n) and (α, β) respectively with the superscript notation as in (2.102), θ'_k is defined in (2.97), and the $\mathbb{G}_{MN}^{(\{l_i\},n)}$ may be extracted from the functional form (2.124) by taking care of the Lorentz indices. The M_p power at any degree of non-locality is given in (2.114) by using (2.95). One may easily see that all the three quantum series $\mathbb{C}_{ab}$, $\mathbb{C}_{mn}$ and $\mathbb{C}_{\alpha\beta}$ are suppressed by powers of $\mathbb{V}_{\mathbb{T}^2}$ and in the limit of vanishing volume, i.e (2.122), they decouple. However what survive in this limit are the time-neutral series given by sum over all j in $\mathbf{C}_{\mu\nu}^{(j)}$ because $\Lambda_{A_iB_i}^{(ij)} = 0$ and $\mathbf{R}_{A_iB_i}$ are classical. Again, the vanishings of $\Lambda_{A_iB_i}^{(ij)}$, in the light of both the derivative constraint and the preservation of the type IIB metric form (2.1), are allowed as long as the *localized* fluxes are ignored. Interestingly, the sum over the time-neutral quantum terms $\mathbf{C}_{\mu\nu}^{(j)}$ are now *finite* in number and have well defined hierarchy as evident from (2.103), (2.104), (2.105) and (2.106). This amazing turn of events will help us to find solutions where originally there were none [15].

Einstein equation along (m, n) directions

We can now compute the equations of motion for all the fields and parameters in the theory. We consider first the Einstein's equations. Since there are multiple components in the theory, we consider Einstein's equation along (m, n) directions. The Einstein tensor is given by:

$$
\begin{aligned}
\mathbb{G}_{mn} &= \mathbf{G}_{mn} - \frac{\partial_m h\partial_n h}{2h^2} + g_{mn}\left[3ht\Lambda\dot{F}_2 - 6h\Lambda F_2 + \frac{F_2}{F_1}\frac{\partial_\alpha h\partial^\alpha h}{4h^2} + \frac{\partial_k h\partial^k h}{4h^2}\right]\\
&- g_{mn}\left[\frac{3}{2}ht^2\Lambda\ddot{F}_2 - \frac{ht^2\Lambda\dot{F}_1^2F_2}{4F_1^2} + \frac{3ht^2\Lambda\dot{F}_2\dot{F}_1}{2F_1} - \frac{2ht\Lambda\dot{F}_1F_2}{F_1} + \frac{ht^2\Lambda\ddot{F}_1F_2}{F_1}\right]\\
&= \mathbf{G}_{mn} - \frac{\partial_m h\partial_n h}{2h^2} + g_{mn}\left[3h^{3/4}\Lambda^{1/2}g_s\dot{F}_2 - 6h\Lambda F_2 + \frac{F_2}{F_1}\frac{\partial_\alpha h\partial^\alpha h}{4h^2} + \frac{\partial_k h\partial^k h}{4h^2}\right]\\
&- g_sg_{mn}\sqrt{h}\left[\frac{3}{2}g_s\ddot{F}_2 - \frac{g_s\dot{F}_1^2F_2}{4F_1^2} + \frac{3g_s\dot{F}_2\dot{F}_1}{2F_1} - \frac{2h^{1/4}\sqrt{\Lambda}\dot{F}_1F_2}{F_1} + \frac{g_s\ddot{F}_1F_2}{F_1}\right],
\end{aligned}
\tag{3.5}
$$

where g_{mn} is the un-warped metric from (2.3), which is also the ingredient used in the un-warped Einstein tensor $\mathbf{G}_{mn}$. In the third and the fourth lines, we have replaced the time parameter by g_s. Such a g_s expansion should also be reflected in the definitions of $F_i(t)$ and whose behaviors are governed by either (2.2) or (2.8). Both these cases will be discussed separately as we go along.

The energy-momentum tensor from the G-flux is now given by:

$$\begin{aligned}
\mathbb{T}^G_{mn} &= \frac{1}{4hF_2^2}\left(\mathbf{G}_{mlka}\mathbf{G}_n^{\ lka} - \frac{1}{6}g_{mn}\mathbf{G}_{pkla}\mathbf{G}^{pkla}\right) - \frac{\partial_m h\partial_n h}{2h^2} + g_{mn}\left(\frac{F_2}{F_1}\frac{\partial_\alpha h\partial^\alpha h}{4h^2} + \frac{\partial_{m'} h\partial^{m'} h}{4h^2}\right)\\
&+ \frac{1}{2hF_1F_2}\left(\mathbf{G}_{ml\alpha a}\mathbf{G}_n^{\ l\alpha a} - \frac{1}{4}g_{mn}\mathbf{G}_{pl\alpha a}\mathbf{G}^{pl\alpha a}\right) + \frac{1}{4hF_1^2}\left(\mathbf{G}_{m\alpha\beta a}\mathbf{G}_n^{\ \alpha\beta a} - \frac{1}{2}g_{mn}\mathbf{G}_{p\alpha\beta a}\mathbf{G}^{p\alpha\beta a}\right)\\
&+ \frac{\Lambda(t)}{12hF_2^3}\left(\mathbf{G}_{mlkr}\mathbf{G}_n^{\ lkr} - \frac{1}{8}g_{mn}\mathbf{G}_{pklr}\mathbf{G}^{pklr}\right) + \frac{\Lambda(t)}{4hF_2^2F_1}\left(\mathbf{G}_{mlk\alpha}\mathbf{G}_n^{\ lk\alpha} - \frac{1}{6}g_{mn}\mathbf{G}_{pkl\alpha}\mathbf{G}^{pkl\alpha}\right)\\
&+ \frac{\Lambda(t)}{4hF_2F_1^2}\left(\mathbf{G}_{ml\alpha\beta}\mathbf{G}_n^{\ l\alpha\beta} - \frac{1}{4}g_{mn}\mathbf{G}_{pl\alpha\beta}\mathbf{G}^{pl\alpha\beta}\right) + \frac{1}{4h\Lambda(t)F_2}\left(\mathbf{G}_{mlab}\mathbf{G}_n^{\ lab} - \frac{1}{4}g_{mn}\mathbf{G}_{pkab}\mathbf{G}^{pkab}\right)\\
&+ \frac{1}{4h\Lambda(t)F_1}\left(\mathbf{G}_{m\alpha ab}\mathbf{G}_n^{\ \alpha ab} - \frac{1}{2}g_{mn}\mathbf{G}_{p\alpha ab}\mathbf{G}^{p\alpha ab}\right) - \frac{F_2}{16h\Lambda(t)F_1^2}\left(g_{mn}\mathbf{G}_{\alpha\beta ab}\mathbf{G}^{\alpha\beta ab}\right), \qquad (3.6)
\end{aligned}$$

where one may notice that we have retained components like $\mathbf{G}_{MNab}$. This is just for completeness and, for the cases pertaining to our earlier constraints, we will be dealing with them on an individual basis as we go along. The other ingredients appearing in (3.6) are the $F_i(t)$ functions and the warp-factor $h(y)$. The $F_i(t)$ functions satisfy (2.2) or (2.8) depending on what conditions we want to impose on the Newton's constant for the vanilla de Sitter case; and $h(y)$ is the warp-factor that is not required to be kept as a constant. Our aim in the following would be to study the two cases, (2.2) and (2.8), and ask if solutions exist corresponding to the background (2.3) or (2.1).

Case 1: $F_1(t)$ and $F_2(t)$ satisfying the volume-preserving condition (2.2)

The functional form for $F_2(t)$ has always been fixed to (2.9) for either (2.2) or (2.8). For our purpose however the full form of (2.9) is not useful since we will only be concerned with $g_s \to 0$ limit which incidentally is also the late time limit. For this case, since e^{-1/g_s^Δ} dies off faster than any powers of g_s, we can simplify (2.9) and write it as:

$$F_2(t) = \sum_{k\in\frac{\mathbb{Z}}{2}} C_k\left(\frac{g_s}{H}\right)^{2\Delta k}, \quad F_1(t) = F_2^{-2}(t) = \sum_{k\in\frac{\mathbb{Z}}{2}} \widetilde{C}_k\left(\frac{g_s}{H}\right)^{2\Delta k}, \qquad (3.7)$$

where $H(y) \equiv h^{1/4}(y)$ is used to avoid fractional powers of warp-factors and $C_k \equiv c_{k0}$ in (2.9). Note that we have expressed $F_1(t)$ in the same format as $F_2(t)$, but with coefficients given by $\widetilde{C}_k$. These coefficients[2] may be easily found from (2.2), and here we quote a few of them:

$$\begin{aligned}
&\widetilde{C}_0 = C_0 \equiv 1, \quad \widetilde{C}_{\frac{1}{2}} = -2C_{\frac{1}{2}}, \quad \widetilde{C}_1 = 3C_{\frac{1}{2}}^2 - 2C_1 \qquad (3.8)\\
&\widetilde{C}_{\frac{3}{2}} = -2C_{\frac{3}{2}} + 6C_{\frac{1}{2}}C_1 - 4C_{\frac{1}{2}}^3, \quad \widetilde{C}_2 = -2C_2 + 5C_{\frac{1}{2}}^4 + 3C_1^2 + 6C_{\frac{1}{2}}C_{\frac{3}{2}} - 12C_{\frac{1}{2}}^2C_1.
\end{aligned}$$

[2]The C_k and $\widetilde{C}_k$ coefficients are related by $\sum_{\{k_i\}}\widetilde{C}_{k_1}C_{k_2}C_{k_3}\left(\frac{g_s}{H}\right)^{2\Delta(k_1+k_2+k_3)} = 1$ from where (3.8) may be determined.

These constant coefficients will have to be determined by plugging the ansatze in the supergravity equations of motion in the presence of the quantum terms. Hence, we will need time derivatives of $F_2(t)$ and $F_1(t)$. For $F_2(t)$, they are some variants of (2.12):

$$\dot{F}_2(t) = 2\Delta\sqrt{\Lambda}\sum_{k\in\frac{\mathbb{Z}}{2}} kC_k\left(\frac{g_s}{H}\right)^{2\Delta k-1}, \quad \ddot{F}_2(t) = 2\Delta\Lambda\sum_{k\in\frac{\mathbb{Z}}{2}} k(2\Delta k-1)C_k\left(\frac{g_s}{H}\right)^{2\Delta k-2}, \tag{3.9}$$

arising due to the simplification adopted in (3.7), and Λ is the cosmological constant that appears in (2.3). If we want to work with (2.12) we will have to retain e^{-1/g_s^Δ} pieces, but cannot expand it in inverse powers of g_s^Δ as cautioned in footnotes 2 and 1. The time derivatives of $F_1(t)$ has exactly the same form as (3.9) except the C_k's are replaced by $\widetilde{C}_k$. Plugging these in (3.5) we can express $\mathbb{G}_{mn}$ in powers of g_s in the following way:

$$\begin{aligned}
\mathbb{G}_{mn} &= \mathbf{G}_{mn} + 3\Lambda H^4 g_{mn}\sum_k\left(3\Delta k - 2\Delta^2k^2 - 2\right)C_k\left(\frac{g_s}{H}\right)^{2\Delta k} \\
&+ \Delta^2\Lambda H^4 g_{mn}\sum_{\{k_l\}} k_1k_2\widetilde{C}_{k_1}\widetilde{C}_{k_2}\prod_{i=3}^{7}C_{k_i}\left(\frac{g_s}{H}\right)^{2\Delta(k_1+\ldots+k_7)} - \frac{8\partial_m H\partial_n H}{H^2} \\
&- 2\Delta\Lambda H^4 g_{mn}\sum_{\{k_l\}} k_1(3\Delta k_2 + 2\Delta k_1 - 3)\widetilde{C}_{k_1}\prod_{i=2}^{4}C_{k_i}\left(\frac{g_s}{H}\right)^{2\Delta(k_1+k_2+k_3+k_4)} \\
&+ \frac{4g_{mn}}{H^2}\left(\partial_l H\partial^l H + \partial_\alpha H\partial^\alpha H\sum_{\{k_l\}}C_{k_1}C_{k_2}C_{k_3}\left(\frac{g_s}{H}\right)^{2\Delta(k_1+k_2+k_3)}\right),
\end{aligned} \tag{3.10}$$

where the braces $\{k_l\}$ denote sum over all the $k_l \in \frac{\mathbb{Z}}{2}$ values. It is interesting that only (k_1, k_2) explicitly show up as coefficients which implies summing over all other permutations of k_p for $p \neq 1, 2$. This will be important when we want to extract various powers of g_s to balance the equations.

Let us now consider the energy-momentum tensor for the G-fluxes. The full expression has been given in (3.6). One may note that the last three terms therein are exactly the ones we have in (2.102) (see also footnote 19). In the $g_s \to 0$ limit, we can represent the G-flux from (2.13) as:

$$\mathbf{G}_{MNPQ} = \sum_{k\in\frac{\mathbb{Z}}{2}}\mathcal{G}^{(k)}_{MNPQ}(y)\left(\frac{g_s}{H}\right)^{2\Delta k}, \tag{3.11}$$

where $H = h^{1/4}$ is as defined earlier, and we have used the fact that in the limit of $g_s \to 0$, e^{-1/g_s^Δ} dies-off faster than any powers of g_s. Plugging (3.11) and (3.7) in

(3.6), we get:

$$\begin{aligned}
\mathbb{T}^G_{mn} &= \sum_{\{k_i\}} \frac{\widetilde{C}_{k_1}}{4H^4}\left(\mathcal{G}^{(k_2)}_{mlka}\mathcal{G}^{(k_3)lka}_n - \frac{1}{6}g_{mn}\mathcal{G}^{(k_2)}_{plka}\mathcal{G}^{(k_3)plka}\right)\left(\frac{g_s}{H}\right)^{2\Delta(k_1+k_2+k_3)}\\
&+ \sum_{\{k_i\}} \frac{C_{k_1}}{2H^4}\left(\mathcal{G}^{(k_2)}_{ml\alpha a}\mathcal{G}^{(k_3)l\alpha a}_n - \frac{1}{4}g_{mn}\mathcal{G}^{(k_2)}_{pl\alpha a}\mathcal{G}^{(k_3)pl\alpha a}\right)\left(\frac{g_s}{H}\right)^{2\Delta(k_1+k_2+k_3)}\\
&+ \sum_{\{k_i\}} \frac{C_{k_1}C_{k_2}C_{k_3}C_{k_4}}{4H^4}\left(\mathcal{G}^{(k_5)}_{m\alpha\beta a}\mathcal{G}^{(k_6)\alpha\beta a}_n - \frac{1}{2}g_{mn}\mathcal{G}^{(k_5)}_{p\alpha\beta a}\mathcal{G}^{(k_6)p\alpha\beta a}\right)\left(\frac{g_s}{H}\right)^{2\Delta(k_1+....+k_6)}\\
&+ \sum_{\{k_i\}} \frac{\widetilde{C}_{k_1}\widetilde{C}_{k_2}C_{k_3}}{12H^4}\left(\mathcal{G}^{(k_4)}_{mlkr}\mathcal{G}^{(k_5)lkr}_n - \frac{1}{8}g_{mn}\mathcal{G}^{(k_4)}_{pklr}\mathcal{G}^{(k_5)pklr}\right)\left(\frac{g_s}{H}\right)^{2\Delta(k_1+...+k_5+1/\Delta)}\\
&+ \sum_{\{k_i\}} \frac{1}{4H^4}\left(\mathcal{G}^{(k_1)}_{mlk\alpha}\mathcal{G}^{(k_2)lk\alpha}_n - \frac{1}{6}g_{mn}\mathcal{G}^{(k_1)}_{plk\alpha}\mathcal{G}^{(k_2)plk\alpha}\right)\left(\frac{g_s}{H}\right)^{2\Delta(k_1+k_2+1/\Delta)}\\
&+ \sum_{\{k_i\}} \frac{C_{k_1}C_{k_2}C_{k_3}}{4H^4}\left(\mathcal{G}^{(k_4)}_{ml\alpha\beta}\mathcal{G}^{(k_5)l\alpha\beta}_n - \frac{1}{4}g_{mn}\mathcal{G}^{(k_4)}_{pl\alpha\beta}\mathcal{G}^{(k_5)pl\alpha\beta}\right)\left(\frac{g_s}{H}\right)^{2\Delta(k_1+...+k_5+1/\Delta)}\\
&+ \sum_{\{k_i\}} \frac{\widetilde{C}_{k_1}C_{k_2}}{4H^4}\left(\mathcal{G}^{(k_3)}_{mlab}\mathcal{G}^{(k_4)lab}_n - \frac{1}{4}g_{mn}\mathcal{G}^{(k_3)}_{pkab}\mathcal{G}^{(k_4)pkab}\right)\left(\frac{g_s}{H}\right)^{2\Delta(k_1+...+k_4-1/\Delta)}\\
&+ \sum_{\{k_i\}} \frac{C_{k_1}C_{k_2}}{4H^4}\left(\mathcal{G}^{(k_3)}_{m\alpha ab}\mathcal{G}^{(k_4)\alpha ab}_n - \frac{1}{2}g_{mn}\mathcal{G}^{(k_3)}_{p\alpha ab}\mathcal{G}^{(k_4)p\alpha ab}\right)\left(\frac{g_s}{H}\right)^{2\Delta(k_1+...+k_4-1/\Delta)}\\
&- \frac{g_{mn}}{16H^4}\sum_{\{k_i\}} C_{k_1}.......C_{k_5}\mathcal{G}^{(k_6)}_{\alpha\beta ab}\mathcal{G}^{(k_7)\alpha\beta ab}\left(\frac{g_s}{H}\right)^{2\Delta(k_1+...+k_7-1/\Delta)} - \frac{8\partial_m H\partial_n H}{H^2}\\
&+ \frac{4g_{mn}}{H^2}\left(\partial_l H\partial^l H + \partial_\alpha H\partial^\alpha H\sum_{\{k_l\}} C_{k_1}C_{k_2}C_{k_3}\left(\frac{g_s}{H}\right)^{2\Delta(k_1+k_2+k_3)}\right), \qquad (3.12)
\end{aligned}$$

where note that we have retained components like $\mathcal{G}^{(k)}_{MNab}(y)$, which immediately implies that these components cannot be expressed as (2.89) because for the limit $g_s \to 0$ only the constant zero form survives. We also want to avoid switching on components like $\mathbf{C}_{Mab}$ to avoid developing cross-terms in the type IIB background (2.6). Thus the only option is to view them as *localized* fluxes which, in fact, will also be very useful to resolve other subtle issues surrounding flux quantization etc in the full M-theory framework. By construction, we have:

$$\mathcal{G}^{(0)}_{MNPQ} = 0. \qquad (3.13)$$

With these at hand, we are now ready to discuss all the equations of motion for the system. Our first step would be to study the EOMs at zeroth order in g_s. Looking at (3.10), (3.11) and (3.1), it is easy to infer the following:

$$\mathbf{G}_{mn} - 6\Lambda H^4 g_{mn} = \sum_{\{k_i\}} \left[\frac{\widetilde{C}_{k_1} C_{k_2}}{4H^4} \left(\mathcal{G}^{(k_3)}_{mlab} \mathcal{G}^{(k_4)lab}_n - \frac{1}{4} g_{mn} \mathcal{G}^{(k_3)}_{pkab} \mathcal{G}^{(k_4)pkab} \right) \right.$$
$$\left. + \frac{C_{k_1} C_{k_2}}{4H^4} \left(\mathcal{G}^{(k_3)}_{m\alpha ab} \mathcal{G}^{(k_4)\alpha ab}_n - \frac{1}{2} g_{mn} \mathcal{G}^{(k_3)}_{p\alpha ab} \mathcal{G}^{(k_4)p\alpha ab} \right) \right] \delta(k_1 + k_2 + k_3 + k_4 - 3)$$
$$- \frac{g_{mn}}{16H^4} \sum_{\{k_i\}} C_{k_1} C_{k_2} C_{k_3} C_{k_4} C_{k_5} \mathcal{G}^{(k_6)}_{\alpha\beta ab} \mathcal{G}^{(k_7)\alpha\beta ab} \delta(k_1 + k_2 + .. + k_7 - 3) + \mathbb{C}^{(0,0)}_{mn} \quad (3.14)$$

where the delta function is simply used to fix the condition on k_i. Note that all $k_i \in \mathbb{Z}/2$, and both set of (k_3, k_4) as well as (k_6, k_7) cannot vanish, and take the minimum values of $1/2$, because of (3.13). On the other hand, (2.97) tells us that $\Delta k \geq 1/2$ which, with the delta function constraint above, immediately implies $k_3 = k_4 = 3/2$ in the first two lines and $k_6 = k_7 = 3/2$ in the last line of (3.14) and the rest zero. Thus:

$$\mathbf{G}_{mn} - 6\Lambda H^4 g_{mn} = \mathbb{C}^{(0,0)}_{mn} + \frac{1}{4H^4} \left(\mathcal{G}^{(3/2)}_{mlab} \mathcal{G}^{(3/2)lab}_n - \frac{1}{4} g_{mn} \mathcal{G}^{(3/2)}_{pkab} \mathcal{G}^{(3/2)pkab} \right)$$
$$+ \frac{1}{4H^4} \left(\mathcal{G}^{(3/2)}_{m\alpha ab} \mathcal{G}^{(3/2)\alpha ab}_n - \frac{1}{2} g_{mn} \mathcal{G}^{(3/2)}_{p\alpha ab} \mathcal{G}^{(3/2)p\alpha ab} \right) - \frac{g_{mn}}{16H^4} \mathcal{G}^{(3/2)}_{\alpha\beta ab} \mathcal{G}^{(3/2)\alpha\beta ab}, \quad (3.15)$$

(3.16)

which is actually a set of 10 equations with 31 unknowns. The RHS is completely fixed once we know the functional form for $\mathcal{G}^{(3/2)}_{MNPQ}(y)$ components. All these fluxes appearing above are localized fluxes and according to (3.4), at the zeroth order in g_s, there are no local quantum terms, except classical ones, and contributions to $\mathbb{C}^{(0,0)}_{mn}$ come mostly from the non-local counter-terms. These are suppressed by powers of the torus volume and therefore their contributions are negligible. This is one of the key difference between a similar equation appearing in [15] (see eq (5.25) in [15]). The number of terms appearing in $\mathbb{C}^{(i)}_{mn}$ in eq (5.25) of [15] are the number of solutions of $\theta'_0 = \frac{2}{3}$ in (2.98). There are then an *infinite* number of solutions for (2.98) with no hierarchy, the latter because of the inclusion of the non-local counter-terms. This ruined an EFT description in [15].

Before moving ahead let us clarify few questions that may be asked at this point regarding the two scaling behavior (2.97) for (2.2), and (2.98) for the time-independent case. First, in determining the g_s scaling $g_s^{\theta'_k}$ or $g_s^{\theta'_0}$, what values of the metric and G-flux components should we insert in (2.94)? Recall from (2.3) and (2.5) the metric components are expressed in terms of their g_s scalings as:

$$\mathbf{g}_{\mu\nu} = g_s^{-8/3} \eta_{\mu\nu}, \quad \mathbf{g}_{ab} = g_s^{4/3} \delta_{ab}$$
$$\mathbf{g}_{\alpha\beta} = g_{\alpha\beta} \left[\left(\frac{g_s}{H} \right)^{-\frac{2}{3}} + \widetilde{C}_{\frac{1}{2}} \left(\frac{g_s}{H} \right)^{-\frac{1}{3}} + \widetilde{C}_1 + \widetilde{C}_{\frac{3}{2}} \left(\frac{g_s}{H} \right)^{\frac{1}{3}} + ... \right] H^{4/3}$$
$$\mathbf{g}_{mn} = g_{mn} \left[\left(\frac{g_s}{H} \right)^{-\frac{2}{3}} + C_{\frac{1}{2}} \left(\frac{g_s}{H} \right)^{-\frac{1}{3}} + C_1 + C_{\frac{3}{2}} \left(\frac{g_s}{H} \right)^{\frac{1}{3}} + ... \right] H^{4/3}, \quad (3.17)$$

where the C_k and $\widetilde{C}_k$ are related by (3.8). Coming back, taking a trace on both sides of (3.15) immediately tells us that the internal manifold $\mathcal{M}_4$ cannot be a Calabi-Yau manifold. It cannot generically also be a conformally Calabi-Yau, as the non-Kählerity will be controlled by the localized fluxes as well as the cosmological constant Λ. At this stage one can also count the number of variables we have in the problem. They can be tabulated as:

$$H(y);\;\; g_{mn}(y);\;\; \mathcal{G}^{(3/2)}_{MNPQ}(y),\; \mathcal{G}^{(2)}_{MNPQ}(y),\; \mathcal{G}^{(5/2)}_{MNPQ}(y), ... \tag{3.18}$$

with 10 components for g_{mn}, 1 from $H(y)$ and 70 components from any choice of k in $\mathcal{G}^{(k)}_{MNPQ}$ totalling to at least 81 independent functions for a given k. The g_{mn} EOM connects the metric components with the warp-factor and G-fluxes, which we elucidated to zeroth order in g_s in (3.15). In fact a more precise connection of g_{mn} to the fluxes and the quantum terms appears from the next order in g_s i.e $g_s^{1/3}$. The relation becomes:

$$\begin{aligned} g_{mn} \;=\; & \frac{3}{58\mathbb{A}(y)}\mathbb{C}^{(1/2,0)}_{mn} + \frac{3}{58\mathbb{A}(y)}\sum_{\{k_i\}}\left[\frac{\widetilde{C}_{k_1}C_{k_2}}{4H^4}\left(\mathcal{G}^{(k_3)}_{mlab}\mathcal{G}^{(k_4)lab}_{n} - \frac{1}{4}g_{mn}\mathcal{G}^{(k_3)}_{pkab}\mathcal{G}^{(k_4)pkab}\right)\right. \\ + & \left.\frac{C_{k_1}C_{k_2}}{4H^4}\left(\mathcal{G}^{(k_3)}_{m\alpha ab}\mathcal{G}^{(k_4)\alpha ab}_{n} - \frac{1}{2}g_{mn}\mathcal{G}^{(k_3)}_{p\alpha ab}\mathcal{G}^{(k_4)p\alpha ab}\right)\right]\delta\left(k_1+k_2+k_3+k_4-\frac{7}{2}\right), \end{aligned} \tag{3.19}$$

which is another set of 10 equations with at least 44 unknowns. These would imply the precise connection between the $\mathcal{M}_4$ metric, localized fluxes and the quantum terms. The function[3] $\mathbb{A}(y)$ is again a function of the localized fluxes, and the warp-factor $H(y)$, as:

$$\mathbb{A}(y) \equiv \frac{3}{928H^4}\sum_{\{k_i\}} C_{k_1}C_{k_2}C_{k_3}C_{k_4}C_{k_5}\mathcal{G}^{(k_6)}_{\alpha\beta ab}\mathcal{G}^{(k_7)\alpha\beta ab}\delta\left(k_1+k_2+..+k_7-\frac{7}{2}\right) - C_{\frac{1}{2}}\Lambda H^4, \tag{3.20}$$

where for both (3.19) as well as (3.20) we have to make sure that $(k_3, k_4) \geq (3/2, 3/2)$ as well as $(k_6, k_7) \geq (3/2, 3/2)$ so as to comply with (3.13) as well as the positivity of (2.97). More crucially, note the dependence of g_{mn} on the quantum terms $\mathbb{C}^{(1/2,0)}_{mn}$ from (3.1). Since we are looking at $g_s^{1/3}$, this means the local quantum terms of $\mathbb{C}^{(1/2,0)}_{mn}$ should be extracted from (2.94) and (2.100) with $\theta'_k = 1$ in (2.97), i.e:

$$\begin{aligned} 2\sum_{i=1}^{27} l_i + n_1 + n_2 + l_{34} + l_{35} + 2\,(k+2)\,(l_{28}+l_{29}+l_{31}) + (2k+1)\,(l_{30}+l_{32}+l_{33}) \\ +2\,(k-1)\,(l_{36}+l_{37}+l_{38}) = 3, \end{aligned} \tag{3.21}$$

with $(l_i, n_j) \in (\mathbb{Z}, \mathbb{Z})$ as it appears in (2.94). Again since $k \geq 3/2$, we see that there are only a few quantum terms that can appear from (3.21). These quantum terms

[3] The function (3.20) can never be zero globally because the G-flux components appearing in (3.20) cannot globally cancel the contributions from the warp-factor, as they are by definition localized fluxes.

may be extracted from a sub-class of (3.21) that satisfy:

$$2\sum_{i=1}^{27} l_i + n_1 + n_2 + \sum_{i=0}^{4} l_{34+i} = 3, \tag{3.22}$$

with other l_i not contributing. These clearly select a finite number of local quantum terms from (2.94). The remaining contribution to $\mathbb{C}_{mn}^{(1/2,0)}$ in (3.19) come from the non-local counter-terms, implying that to order g_s^0 and $g_s^{1/3}$, contributions to the metric can only come from the fluxes and curvature tensors satisfying (3.21) and a set of non-local counter-terms (that in turn are heavily suppressed prohibiting us to go beyond a certain level of non-locality). For example, the non-local contributions to r-th order come from:

$$\theta'_k = \frac{2}{3}(r+1), \quad \theta'_k = \frac{2r}{3} + 1, \tag{3.23}$$

for the two cases $\mathbb{C}_{mn}^{(0,0)}$ and $\mathbb{C}_{mn}^{(1/2,0)}$ respectively with θ'_k as in (2.97). Additionally (3.15) is expressed in terms of $\mathcal{G}^{(3/2)}_{MNPQ}(y)$ whereas (3.19) is expressed in terms of $\mathcal{G}^{(3/2)}_{MNPQ}(y)$ and $\mathcal{G}^{(2)}_{MNPQ}(y)$ allowing us to express $\mathcal{G}^{(2)}_{MNPQ}(y)$ in terms of $\mathcal{G}^{(3/2)}_{MNPQ}(y)$ and other variables in the problem, where $y = (y^m, y^\alpha)$ form the coordinates of $\mathcal{M}_4 \times \mathcal{M}_2$.

To elucidate the story further, let us go to the next order in g_s, namely $g_s^{2/3}$. We want to see if there are additional constraints on the metric itself, or whether new degrees of freedom appear. Combining (3.10), (3.11) and (3.1), we get:

$$\begin{aligned} g_{mn} &= \frac{9}{\mathbb{B}(y)}\mathbb{C}_{mn}^{(1,0)} + \frac{9}{\mathbb{B}(y)}\sum_{\{k_i\}}\Bigg[\frac{\widetilde{C}_{k_1}C_{k_2}}{4H^4}\left(\mathcal{G}^{(k_3)}_{mlab}\mathcal{G}^{(k_4)lab}_n - \frac{1}{4}g_{mn}\mathcal{G}^{(k_3)}_{pkab}\mathcal{G}^{(k_4)pkab}\right) \\ &+ \frac{C_{k_1}C_{k_2}}{4H^4}\left(\mathcal{G}^{(k_3)}_{m\alpha ab}\mathcal{G}^{(k_4)\alpha ab}_n - \frac{1}{2}g_{mn}\mathcal{G}^{(k_3)}_{p\alpha ab}\mathcal{G}^{(k_4)p\alpha ab}\right)\Bigg]\delta\left(k_1+k_2+k_3+k_4-4\right), \end{aligned} \tag{3.24}$$

which is somewhat similar to (3.19) but differs in three respects: one, the quantum terms are different; two, the k_i sum over to 4 instead of 7/2 leading to a set of 10 equations with at least 58 unknowns; and three, the denominator is given by $\mathbb{B}(y)$ instead of $\mathbb{A}(y)$. This is defined as:

$$\mathbb{B}(y) \equiv \frac{9}{16H^4}\sum_{\{k_i\}} C_{k_1}C_{k_2}C_{k_3}C_{k_4}C_{k_5}\mathcal{G}^{(k_6)}_{\alpha\beta ab}\mathcal{G}^{(k_7)\alpha\beta ab}\delta\left(k_1+k_2+..+k_7-4\right) - \alpha_a\Lambda H^4, \tag{3.25}$$

which should again be compared to (3.20) (the non-vanishing of this is guaranteed from a similar argument presented in footnote 3). These similarities however do not survive beyond $g_s^{5/3}$ and we will comment on it below. The constant α_a is given by the following expression:

$$\alpha_a \equiv 43C_{\frac{1}{2}}^2 - 61C_1 - 13C_{\frac{1}{2}}, \tag{3.26}$$

with C_k being the constant appearing in the functional form for $F_2(t)$ in (3.7) and (3.8) an should in principle be determined along-with the metric, warp-factor and the G-flux components.

Looking at (3.24) and (3.19) we see that a pattern is emerging: (3.24) is expressed in terms of G-fluxes of the form $\mathcal{G}^{(5/2)}_{MNPQ}(y), \mathcal{G}^{(2)}_{MNPQ}(y)$ and $\mathcal{G}^{(3/2)}_{MNPQ}(y)$. Thus knowing the metric information $g_{mn}(y)$ will enable us to express $\mathcal{G}^{(5/2)}_{MNPQ}(y)$ in terms of $\mathcal{G}^{(2)}_{MNPQ}(y)$, $\mathcal{G}^{(3/2)}_{MNPQ}(y)$ and the warp-factors, as the quantum term in (3.24) is given by l_i in (2.94) satisfying:

$$2\sum_{i=1}^{27} l_i + n_1 + n_2 + l_{34} + l_{35} + 2(k-1)(l_{36} + l_{37} + l_{38}) = 4 + 2r, \tag{3.27}$$

with $r = 0$ producing the local terms. Note that $k \leq 2$ otherwise the terms would be classical, implying that the quantum terms to this order cannot be constructed out of $\mathcal{G}^{(5/2)}_{MNPQ}$ justifying the above pattern.

The form of the Einstein's equations would remain similar till $g_s^{5/3}$. For g_s^2 onwards, other components in the energy-momentum tensor (3.11) would start participating because the $k_i \geq 3/2$ bound for the G-flux components would no longer be prohibitive. Thus for any given component of the G-flux, say for example $\mathcal{G}^{(k)}_{mnab}$, there are infinite number of sub-components classified by k for $k \geq 3/2$. So far we have only dealt with a few G-flux components and their corresponding sub-components (classified above by k_i), but more would appear as we go to order g_s^2 and beyond. In fact 70 new components of G-flux would appear for every choice of k_i, implying that at least 70 new degrees of freedom are added at every order in g_s as we go.

Case 2: $F_1(t)$ and $F_2(t)$ satisfying the fluctuation condition (2.8)

In the above section we discussed in details how the EOMs for the internal space $\mathcal{M}_4$ may be determined from fluxes and the quantum terms. In this section we would like to see how this changes once we impose (2.75) or (2.8) on the metric coefficients $F_1(t)$ and $F_2(t)$. One of the first important distinction is the derivative constraint that appears from looking at the generalized scaling (2.92). This could even prompt us to analyze the whole section using (2.75) instead of the special case (2.8). The generic picture is more technically involved, and since we will not be gaining new physics by looking at (2.75), we will suffice ourselves here with a detailed consequence of imposing the special case (2.8) on the background EOMs. We will however revert to the generic picture whenever possible.

As a start, let us work out the behavior of the metric coefficients $F_1(t)$ and $F_2(t)$. We will keep $F_2(t)$ as in (3.7), but change $F_1(t)$ accordingly. For example, the generic form for $F_i(t)$ may be expressed as:

$$F_2(t) = \sum_k C_k \left(\frac{g_s}{H}\right)^{2\Delta k}, \quad F_1(t) = \sum_k \widetilde{C}_k \left(\frac{g_s}{H}\right)^{2\Delta k+\gamma} \equiv \sum_k \hat{C}_k \left(\frac{g_s}{H}\right)^{2\Delta k}, \tag{3.28}$$

this is almost similar to (3.7), if we define $\hat{C}_k \equiv \widetilde{C}_k \left(\frac{g_s}{H}\right)^{\gamma}$. Note that, in this form the

$(C_k, \widetilde{C}_k)$ coefficients satisfy the same relation as (3.8). However the metric along the (α, β) direction becomes:

$$\mathbf{g}_{\alpha\beta} = g_{\alpha\beta}\left[\left(\frac{g_s}{H}\right)^{-\frac{2}{3}+\gamma} + \widetilde{C}_{\frac{1}{2}}\left(\frac{g_s}{H}\right)^{-\frac{1}{3}+\gamma} + \widetilde{C}_1\left(\frac{g_s}{H}\right)^{\gamma} + \widetilde{C}_{\frac{3}{2}}\left(\frac{g_s}{H}\right)^{\frac{1}{3}+\gamma} + ...\right] H^{4/3}, \tag{3.29}$$

with the other coefficients remaining the same as in (3.17). Choosing $\gamma = 2$ would explain the metric choice that we took earlier in analyzing the g_s scaling (2.86). Again, we could resort to the dominant scalings of the metric coefficient i.e $g_s^{-2/3+\gamma}$, but compared to footnote **??** the inverse will become $g_s^{+2/3-\gamma}$ with the γ exponent picking up a negative sign. This is because F_1^{-1} does not have a perturbative expansion compared to the case explored in footnote **??**. The resulting physics will change as evident from the scaling behavior (2.92) and (2.86).

The time derivatives of $F_2(t)$ will expectedly remain the same as in (3.8), but the time derivatives of $F_1(t)$ will change. The change is easy to quantify:

$$\begin{aligned}
\dot{F}_1(t) &= \sqrt{\Lambda}\sum_{k\in\frac{\mathbb{Z}}{2}} \widetilde{C}_k(2\Delta k + \gamma)\left(\frac{g_s}{H}\right)^{2\Delta k+\gamma-1} \\
\ddot{F}_1(t) &= \Lambda\sum_{k\in\frac{\mathbb{Z}}{2}} \widetilde{C}_k(2\Delta k + \gamma)(2\Delta k + \gamma - 1)\left(\frac{g_s}{H}\right)^{2\Delta k+\gamma-2},
\end{aligned} \tag{3.30}$$

where the inverse powers of g_s will be dealt carefully once we go to the relevant EOMs. These functional form can now be used to determine the Einstein tensor along the (m, n) directions. The result is:

$$\begin{aligned}
\mathbb{G}_{mn} &= \mathbf{G}_{mn} + 3\Lambda H^4 g_{mn}\sum_k \left(3\Delta k - 2\Delta^2 k^2 - 2\right) C_k\left(\frac{g_s}{H}\right)^{2\Delta k} + \frac{4 g_{mn}\partial_l H \partial^l H}{H^2} \\
&+ \frac{1}{4}\Lambda H^4 g_{mn}\sum_{\{k_l\}}(2\Delta k_1 + \gamma)(2\Delta k_2 + \gamma)\widetilde{C}_{k_1}\widetilde{C}_{k_2}\prod_{i=3}^{7} C_{k_i}\left(\frac{g_s}{H}\right)^{2\Delta(k_1+...+k_7)} - \frac{8\partial_m H \partial_n H}{H^2} \\
&- \Lambda H^4 g_{mn}\sum_{\{k_l\}}(2\Delta k_1 + \gamma)(3\Delta k_2 + 2\Delta k_1 + \gamma - 3)\widetilde{C}_{k_1}\prod_{i=2}^{4} C_{k_i}\left(\frac{g_s}{H}\right)^{2\Delta(k_1+k_2+k_3+k_4)},
\end{aligned} \tag{3.31}$$

which in the limit $\gamma = 0$ does *not* reproduce all the terms of (3.10). In particular terms with derivatives with respect to α are missing. This is of course expected because $\gamma = 0$ and $\gamma > 0$ share different physics. Note also that none of the g_s scaling gets effected by the γ factor, although the γ factor does change the some of the coefficients of the terms in a standard way. In a similar vein, the energy-momentum tensor from the G-fluxes may be represented as,

$$\begin{aligned}
\mathbb{T}^G_{mn} &= \sum_{\{k_i\}} \frac{\widetilde{C}_{k_1}}{4H^4}\left(\mathcal{G}^{(k_2)}_{mlka}\mathcal{G}^{(k_3)lka}_n - \frac{1}{6}g_{mn}\mathcal{G}^{(k_2)}_{plka}\mathcal{G}^{(k_3)plka}\right)\left(\frac{g_s}{H}\right)^{2\Delta(k_1+k_2+k_3)} + \frac{4g_{mn}\partial_l H\partial^l H}{H^2}\\
&+ \sum_{\{k_i\}} \frac{C_{k_1}}{2H^4}\left(\mathcal{G}^{(k_2)}_{ml\alpha a}\mathcal{G}^{(k_3)l\alpha a}_n - \frac{1}{4}g_{mn}\mathcal{G}^{(k_2)}_{pl\alpha a}\mathcal{G}^{(k_3)pl\alpha a}\right)\left(\frac{g_s}{H}\right)^{2\Delta(k_1+k_2+k_3-\gamma/2\Delta)}\\
&+ \sum_{\{k_i\}} \frac{C_{k_1}C_{k_2}C_{k_3}C_{k_4}}{4H^4}\left(\mathcal{G}^{(k_5)}_{m\alpha\beta a}\mathcal{G}^{(k_6)\alpha\beta a}_n - \frac{1}{2}g_{mn}\mathcal{G}^{(k_5)}_{p\alpha\beta a}\mathcal{G}^{(k_6)p\alpha\beta a}\right)\left(\frac{g_s}{H}\right)^{2\Delta(k_1+....+k_6-\gamma/\Delta)}\\
&+ \sum_{\{k_i\}} \frac{\widetilde{C}_{k_1}\widetilde{C}_{k_2}C_{k_3}}{12H^4}\left(\mathcal{G}^{(k_4)}_{mlkr}\mathcal{G}^{(k_5)lkr}_n - \frac{1}{8}g_{mn}\mathcal{G}^{(k_4)}_{pklr}\mathcal{G}^{(k_5)pklr}\right)\left(\frac{g_s}{H}\right)^{2\Delta(k_1+...+k_5+1/\Delta)}\\
&+ \sum_{\{k_i\}} \frac{1}{4H^4}\left(\mathcal{G}^{(k_1)}_{mlk\alpha}\mathcal{G}^{(k_2)lk\alpha}_n - \frac{1}{6}g_{mn}\mathcal{G}^{(k_1)}_{plk\alpha}\mathcal{G}^{(k_2)plk\alpha}\right)\left(\frac{g_s}{H}\right)^{2\Delta(k_1+k_2+1/\Delta)}\\
&+ \sum_{\{k_i\}} \frac{C_{k_1}C_{k_2}C_{k_3}}{4H^4}\left(\mathcal{G}^{(k_4)}_{ml\alpha\beta}\mathcal{G}^{(k_5)l\alpha\beta}_n - \frac{1}{4}g_{mn}\mathcal{G}^{(k_4)}_{pl\alpha\beta}\mathcal{G}^{(k_5)pl\alpha\beta}\right)\left(\frac{g_s}{H}\right)^{2\Delta(k_1+...+k_5+1/\Delta-\gamma/\Delta)}\\
&+ \sum_{\{k_i\}} \frac{\widetilde{C}_{k_1}C_{k_2}}{4H^4}\left(\mathcal{G}^{(k_3)}_{mlab}\mathcal{G}^{(k_4)lab}_n - \frac{1}{4}g_{mn}\mathcal{G}^{(k_3)}_{pkab}\mathcal{G}^{(k_4)pkab}\right)\left(\frac{g_s}{H}\right)^{2\Delta(k_1+...+k_4-1/\Delta)}\\
&+ \sum_{\{k_i\}} \frac{C_{k_1}C_{k_2}}{4H^4}\left(\mathcal{G}^{(k_3)}_{m\alpha ab}\mathcal{G}^{(k_4)\alpha ab}_n - \frac{1}{2}g_{mn}\mathcal{G}^{(k_3)}_{p\alpha ab}\mathcal{G}^{(k_4)p\alpha ab}\right)\left(\frac{g_s}{H}\right)^{2\Delta(k_1+...+k_4-1/\Delta-\gamma/2\Delta)}\\
&- \frac{g_{mn}}{16H^4}\sum_{\{k_i\}} C_{k_1}...C_{k_5}\mathcal{G}^{(k_6)}_{\alpha\beta ab}\mathcal{G}^{(k_7)\alpha\beta ab}\left(\frac{g_s}{H}\right)^{2\Delta(k_1+...+k_7-1/\Delta-\gamma/\Delta)} - \frac{8\partial_m H\partial_n H}{H^2}, \quad (3.32)
\end{aligned}$$

where we have used the G-flux ansatze (3.11) to express it in powers of g_s. The above expression is similar to what we had in (3.12) and putting $\gamma = 0$ we get back most of the terms therein. The difference remains the same: terms with derivative with respect to α are missing.

Let us now analyze the EOMs. We equate the Einstein tensor (3.31) with the energy-momentum tensors (3.32), for the G-fluxes and (3.1), for the quantum terms. However, we will have to specify some values for γ to equate (3.31) with the sum of (3.32) and (3.1). Let us take $\gamma = 2$. Such a choice immediately implies, from (2.92) and (2.93), that the *lowest* mode of G-flux that we can take to avoid generating time-neutral series is $9/2$, i.e $\mathcal{G}^{(9/2)}_{MNPQ}$. In other words:

$$\mathbf{G}_{MNPQ} = \mathcal{G}^{(9/2)}_{MNPQ}\left(\frac{g_s}{H}\right)^3 + \mathcal{G}^{(5)}_{MNPQ}\left(\frac{g_s}{H}\right)^{10/3} + ..., \quad (3.33)$$

where we put $\Delta = 1/3$ to illustrate the g_s dependence more precisely. The expansion (3.33) is a bit unnatural in the light of the G-flux behavior for $\gamma = 0$, and in fact increasing γ increases the lower bound from (2.93), but let us carry on to see how this effects the EOMs.

We will analyze the EOMs to order by order in powers of $g_s^{1/3}$. The lowest order is the zeroth power in g_s. Interestingly, because we took $\gamma = 2$, the only flux component that can contribute at this order is $\mathcal{G}^{(9/2)}_{\alpha\beta ab}$. In other words:

$$\mathbf{G}_{mn} - 3\Lambda H^4 g_{mn} = \mathbb{C}^{(0,0)}_{mn} - \frac{g_{mn}}{16H^4}\mathcal{G}^{(9/2)}_{\alpha\beta ab}\mathcal{G}^{(9/2)\alpha\beta ab}, \tag{3.34}$$

where $\mathbb{C}^{(0,0)}_{mn}$ collects all the quantum terms classified by $\theta_k = 2/3$ in (2.84), where the choice of θ_k is governed by the scaling argument in (2.99). The equation (3.34) should be compared to (3.15). The latter has more G-flux components with much lower modes, but the overall story remains somewhat similar, albeit a bit more natural. A degree of freedom counting tells us that we have 10 equations with at least 17 unknowns, thus considerably more constrained than (3.15). Note that the coefficient of Λ, lets call it $\sigma_o\Lambda$, is smaller that what we had in (3.15). This is because γ contributes to the coefficient as:

$$\sigma_o \equiv \frac{3}{4}\left(4\gamma - \gamma^2 - 8\right), \tag{3.35}$$

showing that no real choice of γ can make the cosmological constant term in (3.34) to vanish.

To the next order in g_s the story evolves in a similar way to what we had in (3.19). The metric can be directly related to the G-flux component $\mathcal{G}^{(9/2)}_{\alpha\beta ab}$ and the quantum terms $\mathbb{C}^{(1/2,0)}_{mn}$. The precise expression is:

$$g_{mn} = \frac{144H^8}{\Lambda}\left(\frac{\mathbb{C}^{(1/2,0)}_{mn}}{16H^8\mathbb{J}(y) + 45C_{\frac{1}{2}}\mathcal{G}^{(9/2)}_{\alpha\beta ab}\mathcal{G}^{(9/2)\alpha\beta ab}}\right), \tag{3.36}$$

where the quantum terms are classified, as before, by $\theta_k = 1$, with θ_k defined as in (2.84). The equation (3.36), as also in (3.19), mixes all the un-warped metric components with the G-flux component $\mathcal{G}^{(9/2)}_{\alpha\beta ab}$ as well as the C_k and the $\widetilde{C}_k$ coefficients, so one would need other equations to disantangle everything. The C_k and the $\widetilde{C}_k$ coefficients also appear in the definition of $\mathbb{J}(y)$ which takes the following form:

$$\begin{aligned}\mathbb{J}(y) \;\; \equiv \;\; & -42C_{\frac{1}{2}} + \sum_{\{k_i\}}(k_1+3)(k_2+3)\widetilde{C}_{k_1}\widetilde{C}_{k_2}\prod_{1=3}^{7}C_{k_i}\delta\left(k_1 + ... + k_7 - \frac{1}{2}\right) \\ & -2\sum_{\{k_i\}}(k_1+3)(3k_2+2k_1-3)\widetilde{C}_{k_1}C_{k_2}C_{k_3}C_{k_4}\delta\left(k_1+k_2+k_3+k_4-\frac{1}{2}\right).\end{aligned} \tag{3.37}$$

One could now go to the next order, i.e $g_s^{2/3}$, and analyze the background in a similar way to (3.24), using the same component of G-flux and quantum terms $\mathbb{C}^{(1,0)}_{mn}$ classified by $\theta_k = 4/3$ in (2.84). Compared to our analysis for case 1, only a few new degrees of freedom are added at this stage: the coefficients of the individual quantum terms and the $C_{\frac{1}{2}}$ coefficient. Thus (3.37) is again a set of 10 equations with at least 18 unknowns. Compared to case 1 above, it appears that we have more equations than the number of unknowns, so existence of solution is a question here. Assuming solution exists, we see from (3.34) and (3.36) that the metric on $\mathcal{M}_4$ has to be a non-Kähler metric. The story can then be developed further in a somewhat similar way, but we will not do so here, and instead go with the analysis of the two cases along (α, β) directions.

Einstein equation along (α, β) directions

The Einstein's equations along (m, n) directions have been discussed. We now analyze along the (α, β) directions, namely the directions along $\mathcal{M}_2$. The analysis will proceed more or less in the same way as before, although specific details would differ. In fact these are the differences that we want to illustrate in this section. We will proceed by first studying the volume preserving case (2.2) and then go for the fluctuation case (2.8). However before moving to the specific cases in question, we want to elucidate the general picture starting with the Einstein tensor. This takes the form:

$$\begin{aligned}\mathbb{G}_{\alpha\beta} &= \mathbf{G}_{\alpha\beta} - \frac{8\partial_\alpha H \partial_\beta H}{H^2} + 4g_{\alpha\beta}\left[\frac{1}{4}g_s\sqrt{\Lambda}H^3\dot{F}_1 - \frac{3}{2}\Lambda H^4 F_1 + \frac{\partial_\alpha H\partial^\alpha H}{H^2} + \frac{F_1}{F_2}\left(\frac{\partial_m H\partial^m H}{H^2}\right)\right] \\ &- 4g_{\alpha\beta}\left[\frac{1}{8}g_s^2H^2\ddot{F}_1 - \frac{g_s^2H^2\dot{F}_1^2}{16F_1} + \frac{g_s^2H^2\dot{F}_2^2F_1}{8F_2^2} + \frac{g_s^2H^2\dot{F}_2\dot{F}_1}{4F_2} + \frac{g_s\sqrt{\Lambda}H^3\dot{F}_2F_1}{F_2} + \frac{g_s^2H^2\ddot{F}_2F_1}{2F_2}\right],\end{aligned} \tag{3.38}$$

where $h(y) \equiv H^4(y)$ and $\mathbf{G}_{\alpha\beta}$ is defined with the un-warped metric $g_{\alpha\beta}$. The g_s dependence appearing in (3.38) is clearly not the full story as other g_s dependences hide in the definitions of $F_i(t)$. This will be illustrated for the two case (2.2) and (2.8) soon. The Einstein tensor (3.38) will have to be equated to the sum of the energy-momentum tensors for the G-flux as well as for the quantum terms. The latter is given in (3.1) whereas the former takes the form:

$$\begin{aligned}\mathbb{T}^G_{\alpha\beta} &= \frac{F_1}{H^4F_2^3}\left(-\frac{1}{24}g_{\alpha\beta}G_{mnpa}G^{mnpa}\right) + \frac{\Lambda(t)}{12H^4F_2^3}\left(G_{\alpha lkr}G_\beta^{\ lkr} - \frac{1}{2}g_{\alpha\beta}G_{\gamma klr}G^{\gamma klr}\right) \\ &+ \frac{1}{4H^4F_2^2}\left(G_{\alpha lka}G_\beta^{\ lka} - \frac{1}{2}g_{\alpha\beta}G_{\gamma kla}G^{\gamma kla}\right) + \frac{1}{2H^4F_1F_2}\left(G_{\alpha l\gamma a}G_\beta^{\ l\gamma a} - \frac{1}{4}g_{\alpha\beta}G_{\delta l\gamma a}G^{\delta l\gamma a}\right) \\ &+ \frac{\Lambda(t)}{4H^4F_2F_1^2}\left(G_{\alpha\eta lr}G_\beta^{\ \eta lr} - \frac{1}{4}g_{\alpha\beta}G_{\kappa\eta lr}G^{\kappa\eta lr}\right) - \frac{F_1\Lambda(t)}{12H^4F_2^4}\left(\frac{1}{8}g_{\alpha\beta}G_{mnpq}G^{mnpq}\right) - \frac{8\partial_\alpha H\partial_\beta H}{H^2} \\ &+ \frac{1}{4H^4\Lambda(t)F_2}\left(G_{\alpha lab}G_\beta^{\ lab} - \frac{1}{2}g_{\alpha\beta}G_{\alpha kab}G^{\beta kab}\right) + \frac{1}{4H^4\Lambda(t)F_1}\left(G_{\alpha\gamma ab}G_\beta^{\ \gamma ab} - \frac{1}{4}g_{\alpha\beta}G_{\eta\kappa ab}G^{\eta\kappa ab}\right) \\ &- \frac{F_1}{H^4\Lambda(t)F_2^2}\left(\frac{1}{16}g_{\alpha\beta}G_{mnab}G^{mnab}\right) + 4g_{\alpha\beta}\left[\frac{\partial_\gamma H\partial^\gamma H}{H^2} + \frac{F_1}{F_2}\left(\frac{\partial_m H\partial^m H}{H^2}\right)\right],\end{aligned} \tag{3.39}$$

which captures the contributions to the energy-momentum tensor from the G-fluxes. Interestingly, as in (3.12) all components of G-flux contribute, in addition to the space-time components. With these at hand, let us discuss the individual cases.

Case 1: $F_1(t)$ and $F_2(t)$ satisfying the volume-preserving condition (2.2)

Our starting point would be express both (3.38) and (3.39) using the g_s expansions of $F_i(t)$ as in (3.7) and G-flux as in (3.11). Using these the Einstein tensor becomes:

$$\begin{aligned}
\mathbb{G}_{\alpha\beta} &= \mathbf{G}_{\alpha\beta} - \frac{8\partial_\alpha H \partial_\beta H}{H^2} + \Lambda H^4 g_{\alpha\beta} \sum_{\{k_i\}} \left[2\Delta k \widetilde{C}_k - 6\widetilde{C}_k - \Delta k(2\Delta k - 1)\widetilde{C}_k\right] \left(\frac{g_s}{H}\right)^{2\Delta k} \\
&+ 4g_{\alpha\beta}\left[\frac{\partial_\alpha H \partial^\alpha H}{H^2} + \left(\frac{\partial_m H \partial^m H}{H^2}\right) \sum_{\{k_i\}} \widetilde{C}_{k_1}\widetilde{C}_{k_2} C_{k_3} \left(\frac{g_s}{H}\right)^{2\Delta(k_1+k_2+k_2)}\right] \\
&- \Lambda\Delta H^4 g_{\alpha\beta} \sum_{\{k_i\}} \Big[2\Delta k_1 k_2 C_{k_1} C_{k_2} \widetilde{C}_{k_3} \widetilde{C}_{k_4} - \Delta k_1 k_2 \widetilde{C}_{k_1} \widetilde{C}_{k_2} C_{k_3} C_{k_4} + 4\Delta k_2 k_4 \widetilde{C}_{k_1} \widetilde{C}_{k_2} C_{k_3} C_{k_4} \\
&+ 8k_1 C_{k_1} \widetilde{C}_{k_2} \widetilde{C}_{k_3} C_{k_4} + 4k_1(2\Delta k_1 - 1) C_{k_1} \widetilde{C}_{k_2} \widetilde{C}_{k_3} C_{k_4}\Big] \left(\frac{g_s}{H}\right)^{2\Delta(k_1+k_2+k_3+k_4)}, \qquad (3.40)
\end{aligned}$$

which in turn should be compared to (3.10). As expected, their precise structures are a bit different, but the generic form remains somewhat equivalent. This is also reflected in the form of the energy-momentum tensor, which may be expressed as:

$$\begin{aligned}
\mathbb{T}^G_{\alpha\beta} &= \frac{1}{4H^4} \sum_{\{k_i\}} \widetilde{C}_{k_1} \left(\mathcal{G}^{(k_2)}_{\alpha lka} \mathcal{G}^{(k_3)lka}_{\beta} - \frac{1}{2} g_{\alpha\beta} \mathcal{G}^{(k_2)}_{\gamma lka} \mathcal{G}^{(k_3)\gamma lka}\right) \left(\frac{g_s}{H}\right)^{2\Delta(k_1+k_2+k_3)} \qquad (3.41) \\
&+ \frac{1}{2H^4} \sum_{\{k_i\}} C_{k_1} \left(\mathcal{G}^{(k_2)}_{\alpha l\gamma a} \mathcal{G}^{(k_3)l\gamma a}_{\beta} - \frac{1}{4} g_{\alpha\beta} \mathcal{G}^{(k_2)}_{\delta l\gamma a} \mathcal{G}^{(k_3)\delta l\gamma a}\right) \left(\frac{g_s}{H}\right)^{2\Delta(k_1+k_2+k_3)} \\
&- \frac{g_{\alpha\beta}}{24H^4} \sum_{\{k_i\}} \widetilde{C}_{k_1} \widetilde{C}_{k_2} \widetilde{C}_{k_3} C_{k_4} \mathcal{G}^{(k_5)}_{mnpa} \mathcal{G}^{(k_6)mnpa} \left(\frac{g_s}{H}\right)^{2\Delta(k_1+k_2+k_3+k_4+k_5+k_6)} \\
&- \frac{g_{\alpha\beta}}{96H^4} \sum_{\{k_i\}} \widetilde{C}_{k_1} \widetilde{C}_{k_2} \widetilde{C}_{k_3} \mathcal{G}^{(k_4)}_{mnpq} \mathcal{G}^{(k_5)mnpq} \left(\frac{g_s}{H}\right)^{2\Delta(k_1+k_2+k_3+k_4+k_5+1/\Delta)} \\
&- \frac{g_{\alpha\beta}}{16H^4} \sum_{\{k_i\}} \widetilde{C}_{k_1} \widetilde{C}_{k_2} \mathcal{G}^{(k_3)}_{mnab} \mathcal{G}^{(k_4)mnab} \left(\frac{g_s}{H}\right)^{2\Delta(k_1+k_2+k_3+k_4-1/\Delta)} \\
&+ \frac{1}{4H^4} \sum_{\{k_i\}} \widetilde{C}_{k_1} C_{k_2} \left(\mathcal{G}^{(k_3)}_{\alpha lab} \mathcal{G}^{(k_4)lab}_{\beta} - \frac{1}{2} g_{\alpha\beta} \mathcal{G}^{(k_3)}_{\gamma lab} \mathcal{G}^{(k_4)\gamma lab}\right) \left(\frac{g_s}{H}\right)^{2\Delta(k_1+k_2+k_3+k_4-1/\Delta)} \\
&+ \frac{1}{4H^4} \sum_{\{k_i\}} C_{k_1} C_{k_2} \left(\mathcal{G}^{(k_3)}_{\alpha\gamma ab} \mathcal{G}^{(k_4)\gamma ab}_{\beta} - \frac{1}{4} g_{\alpha\beta} \mathcal{G}^{(k_3)}_{\gamma\eta ab} \mathcal{G}^{(k_4)\gamma\eta ab}\right) \left(\frac{g_s}{H}\right)^{2\Delta(k_1+k_2+k_3+k_4-1/\Delta)} \\
&+ \frac{1}{4H^4} \sum_{\{k_i\}} C_{k_1} C_{k_2} C_{k_3} \left(\mathcal{G}^{(k_4)}_{\alpha\eta lr} \mathcal{G}^{(k_5)\eta lr}_{\beta} - \frac{1}{4} g_{\alpha\beta} \mathcal{G}^{(k_4)}_{\gamma\eta kr} \mathcal{G}^{(k_5)\gamma\eta kr}\right) \left(\frac{g_s}{H}\right)^{2\Delta(k_1+k_2+k_3+k_4+k_5+1/\Delta)} \\
&+ \frac{1}{12H^4} \sum_{\{k_i\}} \widetilde{C}_{k_1} \widetilde{C}_{k_2} C_{k_3} \left(\mathcal{G}^{(k_4)}_{\alpha lkr} \mathcal{G}^{(k_5)lkr}_{\beta} - \frac{1}{2} g_{\alpha\beta} \mathcal{G}^{(k_4)}_{\gamma lkr} \mathcal{G}^{(k_5)\gamma klr}\right) \left(\frac{g_s}{H}\right)^{2\Delta(k_1+k_2+k_3+k_4+k_5+1/\Delta)} \\
&- \frac{8\partial_\alpha H \partial_\beta H}{H^2} + 4g_{\alpha\beta}\left[\frac{\partial_\gamma H \partial^\gamma H}{H^2} + \left(\frac{\partial_m H \partial^m H}{H^2}\right) \sum_{\{k_i\}} \widetilde{C}_{k_1} \widetilde{C}_{k_2} C_{k_3} \left(\frac{g_s}{H}\right)^{2\Delta(k_1+k_2+k_2)}\right],
\end{aligned}$$

which should again be compared to (3.12) and we see that the relevant G-flux components and the warp-factors fall in their rightful places. As expected, the last three terms of (3.41) matches with the three equivalent terms in (3.40). To the zeroth order in g_s, the equation of motion becomes:

$$\begin{aligned}\mathbf{G}_{\alpha\beta} - 6\Lambda H^4 g_{\alpha\beta} &= \mathbb{C}^{(0,0)}_{\alpha\beta} + \frac{1}{4H^4}\left(\mathcal{G}^{(3/2)}_{\alpha\gamma ab}\mathcal{G}^{(3/2)\gamma ab}_{\beta} - \frac{1}{4}g_{\alpha\beta}\mathcal{G}^{(3/2)}_{\gamma\eta ab}\mathcal{G}^{(3/2)\gamma\eta ab}\right) \\ &+ \frac{1}{4H^4}\left(\mathcal{G}^{(3/2)}_{\alpha lab}\mathcal{G}^{(3/2)lab}_{\beta} - \frac{1}{2}g_{\alpha\beta}\mathcal{G}^{(3/2)}_{\gamma lab}\mathcal{G}^{(3/2)\gamma lab}\right) - \frac{g_{\alpha\beta}}{16H^4}\mathcal{G}^{(3/2)}_{mnab}\mathcal{G}^{(3/2)mnab},\end{aligned} \tag{3.42}$$

showing us that the internal space $\mathcal{M}_2$ again cannot be a Calabi-Yau manifold. The non-Kählerity of $\mathcal{M}_2$ is generated by both G-fluxes and the cosmological constant. The G-fluxes entering in (3.42) are the special ones that have legs along the (a, b) directions much like the ones entering in (3.15). As mentioned earlier, these fluxes cannot be of the form (2.89) and therefore will be treated as localized fluxes. The other ingredient is the quantum term $\mathbb{C}^{(0,0)}_{\alpha\beta}$. More details on this will be discussed below.

In the next order, i.e. $g_s^{1/3}$, we need to be careful because some of the k_i that determine the G-flux components are bounded below as $k_i \geq 3/2$. The others can take any, i.e zero and positive, values lying in $\mathbb{Z}/2$. Keeping this in mind, expanding to $g_s^{1/3}$ gives us:

$$\begin{aligned}g_{\alpha\beta} &= \frac{9}{2\mathbb{C}(y)}\mathbb{C}^{(1/2,0)}_{\alpha\beta} + \frac{9}{8H^4\mathbb{C}(y)}\sum_{\{k_i\}}\Bigg[\widetilde{C}_{k_1}C_{k_2}\left(\mathcal{G}^{(k_3)}_{\alpha lab}\mathcal{G}^{(k_4)lab}_{\beta} - \frac{1}{2}g_{\alpha\beta}\mathcal{G}^{(k_3)}_{\gamma lab}\mathcal{G}^{(k_4)\gamma lab}\right) \\ &+ C_{k_1}C_{k_2}\left(\mathcal{G}^{(k_3)}_{\alpha\gamma ab}\mathcal{G}^{(k_4)\gamma ab}_{\beta} - \frac{1}{4}g_{\alpha\beta}\mathcal{G}^{(k_3)}_{\gamma\eta ab}\mathcal{G}^{(k_4)\gamma\eta ab}\right)\Bigg]\delta\left(k_1+k_2+k_3+k_4-\frac{7}{2}\right) \\ &- \frac{9g_{\alpha\beta}}{32H^4\mathbb{C}(y)}\sum_{\{k_i\}}\left(\widetilde{C}_{k_1}\widetilde{C}_{k_2}\mathcal{G}^{(k_3)}_{mnab}\mathcal{G}^{(k_4)mnab}\right)\delta\left(k_1+k_2+k_3+k_4-\frac{7}{2}\right),\end{aligned} \tag{3.43}$$

where we note that $(k_3, k_4) \geq (3/2, 3/2)$ as alluded to above. This means we are looking at G-flux components with $(k_3, k_4) = (3/2, 3/2), (3/2, 2)$ and $(2, 3/2)$. This, in turn, should be compared to the $(3/2, 3/2)$ distribution that we got in (3.42). The coefficient $\mathbb{C}(y)$ is defined as:

$$\mathbb{C}(y) \equiv 50\Lambda H^2(y)C_{\frac{1}{2}}, \tag{3.44}$$

which is always a non-zero function because $H(y)$ is a non-vanishing real function globally. The other ingredient of (3.43) are the quantum terms. These are classified by $\mathbb{C}^{(1/2,0)}_{\alpha\beta}$ and should be compared to the quantum terms classified by $\mathbb{C}^{(0,0)}_{\alpha\beta}$ in (3.42). Following (2.100), the latter would be classified by $\theta'_k = \frac{2}{3}$ whereas the former would be classified by $\theta'_k = 1$ in (2.97).

The next order is $g_s^{2/3}$, and follows in exactly the same footsteps of the previous case, although details differ. The equation now becomes:

$$\begin{aligned} g_{\alpha\beta} &= \frac{9\mathbb{C}^{(1,0)}_{\alpha\beta}}{\mathbb{E}(y)} + \frac{9}{4H^4\mathbb{E}(y)}\sum_{\{k_i\}}\Bigg[\widetilde{C}_{k_1}C_{k_2}\left(\mathcal{G}^{(k_3)}_{\alpha lab}\mathcal{G}^{(k_4)lab}_{\beta} - \frac{1}{2}g_{\alpha\beta}\mathcal{G}^{(k_3)}_{\gamma lab}\mathcal{G}^{(k_4)\gamma lab}\right) \\ &+ \left(\mathcal{G}^{(k_3)}_{\alpha\gamma ab}\mathcal{G}^{(k_4)\gamma ab}_{\beta} - \frac{1}{4}g_{\alpha\beta}\mathcal{G}^{(k_3)}_{\gamma\eta ab}\mathcal{G}^{(k_4)\gamma\eta ab}\right)\Bigg]\delta\left(k_1+k_2+k_3+k_4-4\right) \\ &- \frac{9g_{\alpha\beta}}{16H^4\mathbb{E}(y)}\sum_{\{k_i\}}\left(\widetilde{C}_{k_1}\widetilde{C}_{k_2}\mathcal{G}^{(k_3)}_{mnab}\mathcal{G}^{(k_4)mnab}\right)\delta\left(k_1+k_2+k_3+k_4-4\right), \quad (3.45) \end{aligned}$$

in exactly the same format as in (3.24). Again k_3 and k_4 are bounded as $(k_3, k_4) \geq (3/2, 3/2)$ so we have G-flux contributions from $\mathcal{G}^{(3/2)}_{MNPQ}, \mathcal{G}^{(2)}_{MNPQ}$ and $\mathcal{G}^{(5/2)}_{MNPQ}$. In the same vein, the quantum terms are classified by an equation of the form (3.27) for local and non-local contributions. Finally the function $\mathbb{E}(y)$ appearing above is defined in the following way:

$$\begin{aligned} \mathbb{E}(y) &\equiv -\Lambda H^4(y)\left[47\widetilde{C}_1 + 3\mathbb{D}(y)\right] \quad (3.46) \\ \mathbb{D}(y) &\equiv \frac{2}{3}\sum_{\{k_i\}}\Big[k_1k_2C_{k_1}C_{k_2}\widetilde{C}_{k_3}\widetilde{C}_{k_4} - \frac{1}{2}k_1k_2\widetilde{C}_{k_1}\widetilde{C}_{k_2}C_{k_3}C_{k_4} + 2k_2k_4\widetilde{C}_{k_1}\widetilde{C}_{k_2}C_{k_3}C_{k_4} \\ &+ 12k_1C_{k_1}\widetilde{C}_{k_2}\widetilde{C}_{k_3}C_{k_4} + 2k_1(2k_1-3)C_{k_1}\widetilde{C}_{k_2}\widetilde{C}_{k_3}C_{k_4}\Big]\delta\left(k_1+k_2+k_3+k_4-1\right), \end{aligned}$$

where we expect both these functions to be non-vanishing globally. All the three EOMs that we listed above, namely (3.42), (3.43) and (3.45), are each a set of three equations with at least 31, 40 and 49 unknowns respectively.

Case 2: $F_1(t)$ and $F_2(t)$ satisfying the fluctuation condition (2.8)

The analysis of (α, β) directions will be a bit more subtle from what we encountered for case 1, partly due to being different modings of the G-flux components and partly due the different scaling behavior of the quantum terms as evident from (2.99). Before we go into these discussions, let us present the Einstein tensor for this case:

$$\begin{aligned} \mathbb{G}_{\alpha\beta} &= \mathbf{G}_{\alpha\beta} + \Lambda H^4 g_{\alpha\beta}\sum_{\{k_i\}}\left[(2\Delta k+\gamma)\widetilde{C}_k - 6\widetilde{C}_k - \frac{1}{2}(2\Delta k+\gamma)(2\Delta k+\gamma-1)\widetilde{C}_k\right]\left(\frac{g_s}{H}\right)^{2\Delta k+\gamma} \\ &+ 4g_{\alpha\beta}\left(\frac{\partial_m H\partial^m H}{H^2}\right)\sum_{\{k_i\}}\widetilde{C}_{k_1}\widetilde{C}_{k_2}C_{k_3}\left(\frac{g_s}{H}\right)^{2\Delta(k_1+k_2+k_2)+\gamma} - \Lambda\Delta H^4 g_{\alpha\beta}\left(\frac{g_s}{H}\right)^{2\Delta(k_1+k_2+k_3+k_4)+\gamma} \\ &\times \sum_{\{k_i\}}\Big[2\Delta k_1k_2C_{k_1}C_{k_2}\widetilde{C}_{k_3}\widetilde{C}_{k_4} - \frac{1}{4\Delta}(2\Delta k_1+\gamma)(2\Delta k_2+\gamma)\widetilde{C}_{k_1}\widetilde{C}_{k_2}C_{k_3}C_{k_4} \\ &+ 2(2\Delta k_2+\gamma)k_4\widetilde{C}_{k_1}\widetilde{C}_{k_2}C_{k_3}C_{k_4} + 8k_1C_{k_1}\widetilde{C}_{k_2}\widetilde{C}_{k_3}C_{k_4} + 4k_1(2\Delta k_1-1)C_{k_1}\widetilde{C}_{k_2}\widetilde{C}_{k_3}C_{k_4}\Big], \quad (3.47) \end{aligned}$$

which may be compared to (3.40). As before, the difference lies in the absence of α dependent terms and the appearance of the γ factor at various places, including the g_s scalings of most of the terms. We will eventually make $\gamma = 2$, but for the time being we shall carry on with the generic picture as far as possible.

The energy-momentum tensor for the G-flux is much easier to compute. All we need is to ask how the g_s scalings of each terms in (3.41) could change. Taking this into account, the expression for the energy-momentum tensor becomes:

$$\begin{aligned}
\mathbb{T}^G_{\alpha\beta} &= \frac{1}{4H^4}\sum_{\{k_i\}}\widetilde{C}_{k_1}\left(\mathcal{G}^{(k_2)}_{\alpha lka}\mathcal{G}^{(k_3)lka}_{\beta} - \frac{1}{2}g_{\alpha\beta}\mathcal{G}^{(k_2)}_{\gamma lka}\mathcal{G}^{(k_3)\gamma lka}\right)\left(\frac{g_s}{H}\right)^{2\Delta(k_1+k_2+k_3)} \\
&+ \frac{1}{2H^4}\sum_{\{k_i\}}C_{k_1}\left(\mathcal{G}^{(k_2)}_{\alpha l\gamma a}\mathcal{G}^{(k_3)l\gamma a}_{\beta} - \frac{1}{4}g_{\alpha\beta}\mathcal{G}^{(k_2)}_{\delta l\gamma a}\mathcal{G}^{(k_3)\delta l\gamma a}\right)\left(\frac{g_s}{H}\right)^{2\Delta(k_1+k_2+k_3-\gamma/2\Delta)} \\
&- \frac{g_{\alpha\beta}}{24H^4}\sum_{\{k_i\}}\widetilde{C}_{k_1}\widetilde{C}_{k_2}\widetilde{C}_{k_3}C_{k_4}\mathcal{G}^{(k_5)}_{mnpa}\mathcal{G}^{(k_6)mnpa}\left(\frac{g_s}{H}\right)^{2\Delta(k_1+k_2+k_3+k_4+k_5+k_6+\gamma/2\Delta)} \\
&- \frac{1}{96H^4}g_{\alpha\beta}\sum_{\{k_i\}}\widetilde{C}_{k_1}\widetilde{C}_{k_2}\widetilde{C}_{k_3}\mathcal{G}^{(k_4)}_{mnpq}\mathcal{G}^{(k_5)mnpq}\left(\frac{g_s}{H}\right)^{2\Delta(k_1+k_2+k_3+k_4+k_5+1/\Delta+\gamma/2\Delta)} \\
&- \frac{g_{\alpha\beta}}{16H^4}\sum_{\{k_i\}}\widetilde{C}_{k_1}\widetilde{C}_{k_2}\mathcal{G}^{(k_3)}_{mnab}\mathcal{G}^{(k_4)mnab}\left(\frac{g_s}{H}\right)^{2\Delta(k_1+k_2+k_3+k_4-1/\Delta+\gamma/2\Delta)} \\
&+ \frac{1}{4H^4}\sum_{\{k_i\}}\widetilde{C}_{k_1}C_{k_2}\left(\mathcal{G}^{(k_3)}_{\alpha lab}\mathcal{G}^{(k_4)lab}_{\beta} - \frac{1}{2}g_{\alpha\beta}\mathcal{G}^{(k_3)}_{\gamma lab}\mathcal{G}^{(k_4)\gamma lab}\right)\left(\frac{g_s}{H}\right)^{2\Delta(k_1+k_2+k_3+k_4-1/\Delta)} \\
&+ \frac{1}{4H^4}\sum_{\{k_i\}}C_{k_1}C_{k_2}\left(\mathcal{G}^{(k_3)}_{\alpha\gamma ab}\mathcal{G}^{(k_4)\gamma ab}_{\beta} - \frac{1}{4}g_{\alpha\beta}\mathcal{G}^{(k_3)}_{\gamma\eta ab}\mathcal{G}^{(k_4)\gamma\eta ab}\right)\left(\frac{g_s}{H}\right)^{2\Delta(k_1+k_2+k_3+k_4-1/\Delta-\gamma/2\Delta)} \\
&+ \frac{1}{4H^4}\sum_{\{k_i\}}C_{k_1}C_{k_2}C_{k_3}\left(\mathcal{G}^{(k_4)}_{\alpha\eta lr}\mathcal{G}^{(k_5)\eta lr}_{\beta} - \frac{1}{4}g_{\alpha\beta}\mathcal{G}^{(k_4)}_{\gamma\eta kr}\mathcal{G}^{(k_5)\gamma\eta kr}\right)\left(\frac{g_s}{H}\right)^{2\Delta(k_1+k_2+k_3+k_4+k_5+1/\Delta-\gamma/\Delta)} \\
&+ \frac{1}{12H^4}\sum_{\{k_i\}}\widetilde{C}_{k_1}\widetilde{C}_{k_2}C_{k_3}\left(\mathcal{G}^{(k_4)}_{\alpha lkr}\mathcal{G}^{(k_5)lkr}_{\beta} - \frac{1}{2}g_{\alpha\beta}\mathcal{G}^{(k_4)}_{\gamma lkr}\mathcal{G}^{(k_5)\gamma klr}\right)\left(\frac{g_s}{H}\right)^{2\Delta(k_1+k_2+k_3+k_4+k_5+1/\Delta)} \\
&+ 4g_{\alpha\beta}\left(\frac{\partial_m H\partial^m H}{H^2}\right)\sum_{\{k_i\}}\widetilde{C}_{k_1}\widetilde{C}_{k_2}C_{k_3}\left(\frac{g_s}{H}\right)^{2\Delta(k_1+k_2+k_2)},
\end{aligned} \tag{3.48}$$

where expectedly the last term matches with an equivalent term in (3.47). Other terms could be compared to (3.41), and here we notice something interesting: to allow for a zeroth power of g_s, the sum of the two modings of the G-flux components, i.e the sum of the two k_i values of the $\mathcal{G}^{(k_i)}_{MNPQ}$ appearing in any term above, should at most be:

$$k_i + k_j = \frac{1}{\Delta}\left(1 + \frac{\gamma}{2}\right), \tag{3.49}$$

where (k_i, k_j) are the modings appearing in the product of two G-flux components in (3.48) that contribute to the energy-momentum tensor. With $\gamma = 2$ and $\Delta = 1/3$, this means the sum in (3.49) should at most be 6. This is unfortunately not possible in the light of (3.33) and (3.13), where $k_i \geq 9/2$ for the G-flux components from (2.93), implying that to zeroth order in g_s, there are no G-flux contributions to the (α, β) EOMs.

What about the quantum terms (3.1)? Here we face another conundrum: according to the scalings of the quantum terms in (2.99), with two free Lorentz indices along (α, β) directions, the g_s expansion should go as:

$$g_s^{\theta_k+4/3} = g_s^2, \;\; g_s^{7/3}, ..., \tag{3.50}$$

with θ_k defined in (2.84), implying that there are no quantum terms to zeroth order in g_s. The *minimum* allowed power of g_s is g_s^2 because terms with $\theta_k = 1/3$ vanishes due to the anti-symmetry of the G-fluxes. The non-local terms cannot contribute anything because it *adds* a factor of $+4r/3$ at r-th level of non-locality to (2.84) as evident from (2.119) and (2.121). This means that at zeroth order in g_s, even the quantum terms cannot contribute. Putting everything together, (3.47), (3.48) and (3.1) with (2.99), gives us:

$$\mathbf{G}_{\alpha\beta} = 0, \tag{3.51}$$

implying that the internal space $\mathcal{M}_2$ can be a conformally Calabi-Yau space[4]. This doesn't imply the metric to be that of a flat torus, because of the warp-factors. On the other hand since $\mathcal{M}_2$ can now have toroidal topology, it's Euler characteristics would vanish, implying the vanishing of the Euler characteristics of the full eight manifold. One might now worry whether non-zero fluxes could be allowed on a manifold with vanishing Euler number [21, 22]. This is a pertinent question and we will analyze this in more details soon, but the short answer is the following. Since the fluxes involved are *time-dependent* the constraints discussed in [21, 22] will have to be modified allowing fluxes to exist on the eight manifold with vanishing Euler number. These fluxes will have to be supported by quantum effects, so there is no contradiction yet[5].

To the next order in g_s, i.e $g_s^{1/3}$, there are no contributions from (3.47), (3.48) and (2.99). In fact the next contributions only come from order g_s^2, and leads to the following EOM:

$$\mathbb{C}_{\alpha\beta}^{(3,0)} + \frac{1}{4H^4}\left(\mathcal{G}_{\alpha\gamma ab}^{(9/2)}\mathcal{G}_{\beta}^{(9/2)\gamma ab} - \frac{1}{4}g_{\alpha\beta}\mathcal{G}_{\gamma\eta ab}^{(9/2)}\mathcal{G}^{(9/2)\gamma\eta ab}\right) + 4\Lambda H^4 g_{\alpha\beta} = 0, \tag{3.52}$$

which is a set of 3 equations with at least 7 unknowns. Note that this is also the first time the quantum terms contribute to the EOM; and here they are classified by $\theta_k = 2/3$ with θ_k given as in (2.84). The above equation however is a bit puzzling in the light of (3.51). In terms of the un-warped metric $g_{\alpha\beta}$ we expect from (3.51) that the internal space be Ricci flat. Putting $g_{\alpha\beta} = \delta_{\alpha\beta}$ then puts a constraint on the form of the quantum terms $\mathbb{C}_{\alpha\beta}^{(3,0)}$ from (3.52). In particular (3.52) tells us that the trace of

[4]A more precise statement is that (3.51) directly implies $R^{(4)} = 0$, i.e the Ricci scalar of $\mathcal{M}_4$ vanishes and we can take the metric g_{mn} to be that of a $K3$ space. Imposing this on (3.51) provides a source-free equation for the metric $g_{\alpha\beta}$ whose solution is a torus. This way the metric for $\mathcal{M}_4 \times \mathcal{M}_2$ can be conformal to $K3 \times \mathbf{T}^2$.

[5]Another possibility is to take the metric of $\mathcal{M}_2$ to be flat everywhere except at one point. Geometrically this is $\mathbf{T}^2/\mathbf{Z}_2$ and therefore doesn't have a vanishing Euler characteristics. However quantum corrections would eventually make this into a smooth space with non-vanishing curvature, so will not be a solution to (3.51). Thus we will continue with $K3 \times \mathbf{T}^2$ as our un-warped background. This will eventually lead to some subtleties that we shall clarify in section 3.1.2.

the quantum terms has to be a negative definite function, i.e:

$$[\mathbb{C}^{\alpha}_{\alpha}]^{(3,0)} = -\frac{1}{8H^4}\left(\mathcal{G}^{(9/2)}_{\alpha\beta ab}\right)^2 - 8\Lambda H^4. \tag{3.53}$$

Whether such a constraint could be satisfied will be discussed later when we analyze all the EOMs together. From here the story progresses in the usual way with the Einstein tensor (3.47) being balanced by the energy-momentum tensors (3.48) and (3.1).

Einstein equation along (a, b) directions

Along the (a, b) directions, i.e directions along $\mathbb{T}^2/\mathcal{G}$ the situation is somewhat moree subtle. Part of the reason is that the variables we took so far are independent of the toroidal direction. This was not the case in [15], which is of course reflected in the scaling expression (2.99). The other main reason is the quantum terms that we will discuss when we study the individual cases, (2.2) and (2.8). For the immediate discussion, we present the expression for the Einstein tensor:

$$\begin{aligned}\mathbb{G}_{ab} &= \delta_{ab}\left(-\frac{R}{2} - 9h\Lambda + \frac{4g^{\alpha\beta}\partial_\alpha H\partial_\beta H}{H^2F_1} + \frac{4g^{mn}\partial_m H\partial_n H}{H^2F_2}\right)\left(\frac{g_s}{H}\right)^2\\ &+ \delta_{ab}H^4\left(\frac{\dot{F}_1^2}{4F_1^2} + \frac{3\dot{F}_1}{tF_1} - \frac{\ddot{F}_1}{F_1} - \frac{\dot{F}_2^2}{2F_2^2} + \frac{6\dot{F}_2}{tF_2} - \frac{2\ddot{F}_2}{F_2} - \frac{2\dot{F}_1\dot{F}_2}{F_1F_2}\right)\left(\frac{g_s}{H}\right)^4,\end{aligned} \tag{3.54}$$

where R is the curvature scalar of the six-dimensional base $\mathcal{M}_4 \times \mathcal{M}_2$ and *not* the full eight-manifold. The reason is of course because we have assigned non-trivial metric to the six-dimensional base, whereas the metric of the toroidal space is governed by the warp-factors only. This is also the reason why δ_{ab} appears in (3.54) above instead of a non-trivial metric g_{ab}. Inclusion of the latter would complicate the dynamics of the system, for example, by changing the coupling constant etc., so we will avoid it here. Note also the absence of g_s independent terms in (3.54). This differs from (3.5) and (3.38), both of which allow g_s neutral terms in the definitions of the Einstein tensors. Similarly the energy-momentum tensor is given by:

$$\begin{aligned}\mathbb{T}^G_{ab} &= \frac{\Lambda(t)}{12H^4F_2^3}\left(G_{amnp}G_b^{mnp} - \frac{1}{2}\delta_{ab}G_{mnpc}G^{mnpc}\right) + \frac{\Lambda(t)}{4H^4F_2^2F_1}\left(G_{amn\alpha}G_b^{mn\alpha} - \frac{1}{2}\delta_{ab}G_{mn\alpha c}G^{mn\alpha c}\right)\\ &+ \frac{\Lambda(t)}{4H^4F_1^2F_2}\left(G_{am\alpha\beta}G_b^{m\alpha\beta} - \frac{1}{2}\delta_{ab}G_{cm\alpha\beta}G^{cm\alpha\beta}\right) + \frac{1}{2H^4F_1F_2}\left(G_{acm\rho}G_b^{cm\rho} - \frac{1}{4}\delta_{ab}G_{m\rho cd}G^{m\rho cd}\right)\\ &+ \frac{1}{4H^4F_2^2}\left(G_{acmn}G_b^{cmn} - \frac{1}{4}\delta_{ab}G_{dcmn}G^{dcmn}\right) + \frac{1}{4H^4F_1^2}\left(G_{ac\alpha\beta}G_b^{c\alpha\beta} - \frac{1}{4}\delta_{ab}G_{cd\alpha\beta}G^{cd\alpha\beta}\right)\\ &- \delta_{ab}\frac{\Lambda(t)^2}{4.4!H^4F_2^4}G_{mnpq}G^{mnpq} - \delta_{ab}\frac{\Lambda(t)^2}{24H^4F_2^3F_1}G_{mnp\alpha}G^{mnp\alpha} - \delta_{ab}\frac{\Lambda(t)^2}{16H^4F_2^2F_1^2}G_{mn\alpha\beta}G^{mn\alpha\beta}\\ &+ \frac{4\Lambda(t)}{H^2F_1}\delta_{ab}g^{\alpha\beta}\partial_\alpha H\partial_\beta H + \frac{4\Lambda(t)}{H^2F_2}\delta_{ab}g^{mn}\partial_m H\partial_n H,\end{aligned} \tag{3.55}$$

where one may note the specific placement of $\Lambda(t) \equiv \left(\frac{g_s}{H}\right)^2$ which will determine the subsequent dynamics of the system once quantum terms are added to the system.

In the following, we proceed with the various cases in consideration.

Case 1: $F_1(t)$ and $F_2(t)$ satisfying the volume-preserving condition (2.2)

Our starting point then is to study the volume preserving case, where now, as mentioned above, some subtleties will arise due to the specific forms of the Einstein and the energy-momentum tensors. The latter for both G-fluxes as well as the quantum terms. The former, i.e the Einstein tensor (3.54), takes the following form:

$$\begin{aligned}
\mathbb{G}_{ab} &= \frac{4\delta_{ab}}{H^2}\sum_{\{k_i\}}\left(C_{k_1}C_{k_2}g^{\alpha\beta}\partial_\alpha H\partial_\beta H + \widetilde{C}_{k_1}C_{k_2}g^{mn}\partial_m H\partial_n H\right)\left(\frac{g_s}{H}\right)^{2\Delta(k_1+k_2+1/\Delta)} \\
&- \frac{\delta_{ab}}{2}\left(R+18H^4\Lambda\right)\left(\frac{g_s}{H}\right)^2 + \Delta^2\Lambda H^4\delta_{ab}\sum_{\{k_i\}}k_1k_2\widetilde{C}_{k_1}\widetilde{C}_{k_2}C_{k_3}....C_{k_6}\left(\frac{g_s}{H}\right)^{2\Delta(k_1+...+k_6+1/\Delta)} \\
&+ 2\Delta\Lambda H^4\delta_{ab}\sum_{\{k_i\}}C_{k_1}C_{k_2}\widetilde{C}_{k_3}\Big(k_1\left(8-\Delta k_2-4\Delta k_1\right)+2k_3\left(2-\Delta k_3-2\Delta k_2\right)\Big)\left(\frac{g_s}{H}\right)^{2\Delta(k_1+k_2+k_3+1/\Delta)},
\end{aligned} \tag{3.56}$$

where expectedly there are no terms to zeroth order in g_s. There is also no curvature term for the toroidal manifold, evident from the δ_{ab} factor appearing from (3.56), presence of which would have altered the coupling constant itself. Similarly, one may represent the energy momentum tensor in the following way:

$$\begin{aligned}
\mathbb{T}^G_{ab} &= \frac{1}{12H^4}\sum_{\{k_i\}}\widetilde{C}_{k_1}\widetilde{C}_{k_2}C_{k_3}\left(\mathcal{G}^{(k_4)}_{amnp}\mathcal{G}_b^{(k_5)mnp} - \frac{1}{2}\delta_{ab}\mathcal{G}^{(k_4)}_{mnpc}\mathcal{G}^{(k_5)mnpc}\right)\left(\frac{g_s}{H}\right)^{2\Delta(k_1+...+k_5+1/\Delta)} \\
&+ \frac{1}{4H^4}\sum_{\{k_i\}}\left(\mathcal{G}^{(k_1)}_{amn\alpha}\mathcal{G}_b^{(k_2)mn\alpha} - \frac{1}{2}\delta_{ab}\mathcal{G}^{(k_1)}_{mn\alpha c}\mathcal{G}^{(k_2)mn\alpha c}\right)\left(\frac{g_s}{H}\right)^{2\Delta(k_1+k_2+1/\Delta)} \\
&+ \frac{1}{4H^4}\sum_{\{k_i\}}C_{k_1}C_{k_2}C_{k_3}\left(\mathcal{G}^{(k_4)}_{am\alpha\beta}\mathcal{G}_b^{(k_5)m\alpha\beta} - \frac{1}{2}\delta_{ab}\mathcal{G}^{(k_4)}_{cm\alpha\beta}\mathcal{G}^{(k_5)cm\alpha\beta}\right)\left(\frac{g_s}{H}\right)^{2\Delta(k_1+...+k_5+1/\Delta)} \\
&+ \frac{1}{4H^4}\sum_{\{k_i\}}\widetilde{C}_{k_1}\left(\mathcal{G}^{(k_2)}_{acmn}\mathcal{G}_b^{(k_3)cmn} - \frac{1}{4}\delta_{ab}\mathcal{G}^{(k_2)}_{dcmn}\mathcal{G}^{(k_3)dcmn}\right)\left(\frac{g_s}{H}\right)^{2\Delta(k_1+k_2+k_3)} \\
&+ \frac{1}{4H^4}\sum_{\{k_i\}}C_{k_1}C_{k_2}C_{k_3}C_{k_4}\left(\mathcal{G}^{(k_5)}_{ac\alpha\beta}\mathcal{G}_b^{(k_6)c\alpha\beta} - \frac{1}{4}\delta_{ab}\mathcal{G}^{(k_5)}_{cd\alpha\beta}\mathcal{G}^{(k_6)cd\alpha\beta}\right)\left(\frac{g_s}{H}\right)^{2\Delta(k_1+...+k_6)} \\
&+ \frac{1}{2H^4}\sum_{\{k_i\}}C_{k_1}\left(\mathcal{G}^{(k_2)}_{acm\rho}\mathcal{G}_b^{(k_3)cm\rho} - \frac{1}{4}\delta_{ab}\mathcal{G}^{(k_2)}_{cdm\rho}\mathcal{G}^{(k_3)cdm\rho}\right)\left(\frac{g_s}{H}\right)^{2\Delta(k_1+k_2+k_3)} \\
&- \frac{\delta_{ab}}{4\cdot 4!H^4}\sum_{\{k_i\}}\widetilde{C}_{k_1}\widetilde{C}_{k_2}\mathcal{G}^{(k_3)}_{mnpq}\mathcal{G}^{(k_4)mnpq}\left(\frac{g_s}{H}\right)^{2\Delta(k_1+k_2+k_3+k_4+2/\Delta)} \\
&- \frac{\delta_{ab}}{4!H^4}\sum_{\{k_i\}}\widetilde{C}_{k_1}C_{k_2}\mathcal{G}^{(k_3)}_{mnp\alpha}\mathcal{G}^{(k_4)mnp\alpha}\left(\frac{g_s}{H}\right)^{2\Delta(k_1+k_2+k_3+k_4+2/\Delta)} \\
&- \frac{\delta_{ab}}{16H^4}\sum_{\{k_i\}}C_{k_1}C_{k_2}\mathcal{G}^{(k_3)}_{mn\alpha\beta}\mathcal{G}^{(k_4)mn\alpha\beta}\left(\frac{g_s}{H}\right)^{2\Delta(k_1+k_2+k_3+k_4+2/\Delta)} \\
&+ \frac{4\delta_{ab}}{H^2}\sum_{\{k_i\}}\left(C_{k_1}C_{k_2}\partial_\alpha H\partial^\alpha H + \widetilde{C}_{k_1}C_{k_2}\partial_m H\partial^m H\right)\left(\frac{g_s}{H}\right)^{2\Delta(k_1+k_2+1/\Delta)},
\end{aligned} \tag{3.57}$$

where as one would expect, the last line of this matches with the first line of the Einstein tensor (3.56). Note also the absence of terms to zeroth order in g_s because of the condition (3.13). This is consistent with what we expect from (3.56), but one may now question whether this also appears from the energy-momentum tensor for the quantum terms in (3.1). From the look of (3.1) it appears that $k_1 = 0$ should be an allowed choice. However, as discussed earlier in (3.50), looking at (2.100) we see that tensors with two free Lorentz indices along (a, b) direction scale as:

$$g_s^{\theta'_k + 4/3} \equiv g_s^{5/3},\ g_s^2,\ g_s^{7/3},\ g_s^{8/3},\ g_s^3, ..., \tag{3.58}$$

as θ'_k defined in (2.97) is bounded below by $\theta'_k \geq 1/3$. Now since the lowest value of $\theta'_k = 1/3$ corresponds to switching on either $(l_{36}, l_{37}, l_{38}) = (1,0,0), (0,1,0)$ or $(0,0,1)$ in (2.97) $-$ and they vanish due to the antisymmetry of the G-flux components $-$ it then appears that the lowest allowed scaling of g_s can only be g_s^2. This seems perfectly consistent with the scalings expected from (3.56) and (3.57), resolving a possible conundrum in our construction[6].

Now that the quantum issues are clarified, we should look at the equations of motion to order g_s^2 by balancing the Einstein tensor in (3.56) with the energy-momentum tensors in (3.57) and (3.1). This produces:

$$\begin{aligned}\left(\frac{R}{2} + 9H^4\Lambda\right)\delta_{ab} = -\mathbb{C}_{ab}^{(3,0)} - \frac{1}{4H^4}\Bigg[&\left(\mathcal{G}_{acmn}^{(3/2)}\mathcal{G}_b^{(3/2)cmn} - \frac{1}{4}\delta_{ab}\mathcal{G}_{dcmn}^{(3/2)}\mathcal{G}^{(3/2)dcmn}\right) \\ + &\left(\mathcal{G}_{ac\alpha\beta}^{(3/2)}\mathcal{G}_b^{(3/2)c\alpha\beta} - \frac{1}{4}\delta_{ab}\mathcal{G}_{cd\alpha\beta}^{(3/2)}\mathcal{G}^{(3/2)cd\alpha\beta}\right) + 2\left(\mathcal{G}_{acm\rho}^{(3/2)}\mathcal{G}_b^{(3/2)cm\rho} - \frac{1}{4}\delta_{ab}\mathcal{G}_{cdm\rho}^{(3/2)}\mathcal{G}^{(3/2)cdm\rho}\right)\Bigg],\end{aligned} \tag{3.59}$$

where the quantum terms manifest themselves as $\mathbb{C}_{ab}^{(3,0)}$ instead of $\mathbb{C}_{ab}^{(0,0)}$, the former being defined for $\theta'_k = 2/3$ in (2.97) exactly as before. It is also interesting to note that, so far all the G-flux energy-momentum tensors appear from $\mathcal{G}_{mnab}^{(k)}, \mathcal{G}_{m\alpha ab}^{(k)}$ and $\mathcal{G}_{\alpha\beta ab}^{(k)}$ for various choices of k satisfying $k \geq 3/2$.

The next order is $g_s^{7/3}$. Interestingly, the Einstein tensor (3.56) cancels out to this order, leaving only the energy-momentum tensor of the G-flux to balance with the energy-momentum tensor of the quantum terms. This gives us:

[6]One may alternatively view the quantum energy-momentum tensor to be represented not as (3.1) but as the following shifted one near $g_s \to 0$:

$$\mathbb{T}_{ab}^Q = \sum_{k\in\mathbb{Z}/2} \mathbb{C}_{ab}^{(k+5/2,0)}\left(\frac{g_s}{H}\right)^{2\Delta(k+5/2)}$$

which would reproduce the correct g_s scalings from (2.94). Such redefinition is possible because (3.1) is conjectured to be equivalent to (2.94), the latter being the main focal point of our analysis.

$$\begin{aligned}4H^4\mathbb{C}_{ab}^{(7/2,0)} &= \sum_{\{k_i\}}\Bigg[\widetilde{C}_{k_1}\left(\mathcal{G}^{(k_2)}_{acmn}\mathcal{G}_b^{(k_3)cnm} - \frac{1}{4}\delta_{ab}\mathcal{G}^{(k_2)}_{dcmn}\mathcal{G}^{(k_3)dcnm}\right)\\ &- 2C_{k_1}\left(\mathcal{G}^{(k_2)}_{acm\rho}\mathcal{G}_b^{(k_3)cm\rho} - \frac{1}{4}\delta_{ab}\mathcal{G}^{(k_2)}_{cdm\rho}\mathcal{G}^{(k_3)cdm\rho}\right)\Bigg]\delta\left(k_1+k_2+k_3-\frac{7}{2}\right)\\ &- \sum_{\{k_i\}}C_{k_1}...C_{k_4}\left(\mathcal{G}^{(k_5)}_{ac\alpha\beta}\mathcal{G}_b^{(k_6)c\alpha\beta} - \frac{1}{4}\delta_{ab}\mathcal{G}^{(k_5)}_{cd\alpha\beta}\mathcal{G}^{(k_6)cd\alpha\beta}\right)\delta\left(k_1+...+k_6-\frac{7}{2}\right),\end{aligned} \tag{3.60}$$

where the quantum terms on the LHS of the above equation is determined for $\theta'_k = 1$ in (2.97). This is similar to the choice of the quantum terms in (3.19) and (3.43). In fact now the story follows the pattern laid out for higher order in g_s as seen previously. For example, the next order in g_s, which is $g_s^{8/3}$, gives us the following equation:

$$\begin{aligned}\delta_{ab} &= \frac{9}{\Lambda\mathbb{F}(y)}\mathbb{C}_{ab}^{(4,0)} + \frac{9}{4\Lambda H^4\mathbb{F}(y)}\sum_{\{k_i\}}\Bigg[\widetilde{C}_{k_1}\left(\mathcal{G}^{(k_2)}_{acmn}\mathcal{G}_b^{(k_3)cmn} - \frac{1}{4}\delta_{ab}\mathcal{G}^{(k_2)}_{dcmn}\mathcal{G}^{(k_3)dcmn}\right)\\ &+ 2C_{k_1}\left(\mathcal{G}^{(k_2)}_{acm\rho}\mathcal{G}_b^{(k_3)cm\rho} - \frac{1}{4}\delta_{ab}\mathcal{G}^{(k_2)}_{cdm\rho}\mathcal{G}^{(k_3)cdm\rho}\right)\Bigg]\delta\left(k_1+k_2+k_3-4\right)\\ &+ \frac{1}{4}\sum_{\{k_i\}}C_{k_1}...C_{k_4}\left(\mathcal{G}^{(k_5)}_{ac\alpha\beta}\mathcal{G}_b^{(k_6)c\alpha\beta} - \frac{1}{4}\delta_{ab}\mathcal{G}^{(k_5)}_{cd\alpha\beta}\mathcal{G}^{(k_6)cd\alpha\beta}\right)\delta\left(k_1+...+k_6-4\right),\end{aligned} \tag{3.61}$$

with the quantum terms being classified by $\theta'_k = 4/3$ as in (3.24) and (3.45). This pattern of fluxes would change eventually as we go higher in g_s, and in fact for g_s^4 we will see new components entering for both G-flux and the quantum energy-momentum tensors. Finally, the function $\mathbb{F}(y)$ appearing in (3.61) is defined as:

$$\mathbb{F}(y) \equiv H^4(y)C^2_{\frac{1}{2}} + 4H^4(y)\sum_{\{k_i\}}C_{k_1}C_{k_2}\widetilde{C}_{k_3}\Big[k_1\left(24-k_2-4k_1\right)+2k_3\left(6-k_3-2k_2\right)\Big]\delta(k_1+k_2+k_3-1), \tag{3.62}$$

which should be compared to (3.20), (3.25), (3.44) and (3.46). The structural similarities of all these functions are of course not a coincidence: they rely on the forms of the EOMs for the various directions analyzed above.

Case 2: $F_1(t)$ and $F_2(t)$ satisfying the fluctuation condition (2.8)

The volume preserving case seems to work rather well, so now we want to see how the story changes once the γ factor is introduced in. We expect changes at all fronts now: the Einstein tensor, the energy-momentum tensors for the G-flux and the quantum terms should all reflect the changes. The subtleties that we encountered with the quantum terms had a nicer resolution here so we will also have to see what happens now. As before we start with the Einstein tensor, that takes the following form:

$$
\begin{aligned}
\mathbb{G}_{ab} &= -\frac{\delta_{ab}}{2}\left(R + 18H^4\Lambda\right)\left(\frac{g_s}{H}\right)^2 + \frac{4\delta_{ab}}{H^2}\sum_{\{k_i\}}\widetilde{C}_{k_1}C_{k_2}g^{mn}\partial_m H\partial_n H\left(\frac{g_s}{H}\right)^{2\Delta(k_1+k_2+1/\Delta)} \\
&+ \frac{1}{4}\Lambda H^4\delta_{ab}\sum_{\{k_i\}}(2\Delta k_1+\gamma)(2\Delta k_2+\gamma)\widetilde{C}_{k_1}\widetilde{C}_{k_2}C_{k_3}....C_{k_6}\left(\frac{g_s}{H}\right)^{2\Delta(k_1+...+k_6+1/\Delta)} \\
&+ \Lambda H^4\delta_{ab}\sum_{\{k_i\}}\Big(2\Delta k_1\left(8-\Delta k_2 - 4\Delta k_1\right) + \left(2\Delta k_3+\gamma\right)\left(4 - 2\Delta k_3 - \gamma - 4\Delta k_2\right)\Big) \\
&\times C_{k_1}C_{k_2}\widetilde{C}_{k_3}\left(\frac{g_s}{H}\right)^{2\Delta(k_1+k_2+k_3+1/\Delta)}, \qquad (3.63)
\end{aligned}
$$

where interestingly none of the g_s scalings get effected by the γ term, but most of the individual terms do have γ dependent coefficients. Similar, the energy-momentum tensor for the G-fluxes changes in an expected way:

$$
\begin{aligned}
\mathbb{T}^G_{ab} &= \frac{1}{12H^4}\sum_{\{k_i\}}\widetilde{C}_{k_1}\widetilde{C}_{k_2}C_{k_3}\left(\mathcal{G}^{(k_4)}_{amnp}\mathcal{G}^{(k_5)mnp}_{b} - \frac{1}{2}\delta_{ab}\mathcal{G}^{(k_4)}_{mnpc}\mathcal{G}^{(k_5)mnpc}\right)\left(\frac{g_s}{H}\right)^{2\Delta(k_1+...+k_5+1/\Delta)} \\
&+ \frac{1}{4H^4}\sum_{\{k_i\}}\left(\mathcal{G}^{(k_1)}_{amn\alpha}\mathcal{G}^{(k_2)mn\alpha}_{b} - \frac{1}{2}\delta_{ab}\mathcal{G}^{(k_1)}_{mn\alpha c}\mathcal{G}^{(k_2)mn\alpha c}\right)\left(\frac{g_s}{H}\right)^{2\Delta(k_1+k_2+1/\Delta-\gamma/2\Delta)} \\
&+ \frac{1}{4H^4}\sum_{\{k_i\}}C_{k_1}C_{k_2}C_{k_3}\left(\mathcal{G}^{(k_4)}_{am\alpha\beta}\mathcal{G}^{(k_5)m\alpha\beta}_{b} - \frac{1}{2}\delta_{ab}\mathcal{G}^{(k_4)}_{cm\alpha\beta}\mathcal{G}^{(k_5)cm\alpha\beta}\right)\left(\frac{g_s}{H}\right)^{2\Delta(k_1+..+k_5+1/\Delta-\gamma/\Delta)} \\
&+ \frac{1}{4H^4}\sum_{\{k_i\}}\widetilde{C}_{k_1}\left(\mathcal{G}^{(k_2)}_{acmn}\mathcal{G}^{(k_3)cmn}_{b} - \frac{1}{4}\delta_{ab}\mathcal{G}^{(k_2)}_{dcmn}\mathcal{G}^{(k_3)dcmn}\right)\left(\frac{g_s}{H}\right)^{2\Delta(k_1+k_2+k_3)} \\
&+ \frac{1}{4H^4}\sum_{\{k_i\}}C_{k_1}C_{k_2}C_{k_3}C_{k_4}\left(\mathcal{G}^{(k_5)}_{ac\alpha\beta}\mathcal{G}^{(k_6)c\alpha\beta}_{b} - \frac{1}{4}\delta_{ab}\mathcal{G}^{(k_5)}_{cd\alpha\beta}\mathcal{G}^{(k_6)cd\alpha\beta}\right)\left(\frac{g_s}{H}\right)^{2\Delta(k_1+...+k_6-\gamma/\Delta)} \\
&+ \frac{1}{2H^4}\sum_{\{k_i\}}C_{k_1}\left(\mathcal{G}^{(k_2)}_{acm\rho}\mathcal{G}^{(k_3)cm\rho}_{b} - \frac{1}{4}\delta_{ab}\mathcal{G}^{(k_2)}_{cdm\rho}\mathcal{G}^{(k_3)cdm\rho}\right)\left(\frac{g_s}{H}\right)^{2\Delta(k_1+k_2+k_3-\gamma/2\Delta)} \\
&- \frac{\delta_{ab}}{4\cdot 4!H^4}\sum_{\{k_i\}}\widetilde{C}_{k_1}\widetilde{C}_{k_2}\mathcal{G}^{(k_3)}_{mnpq}\mathcal{G}^{(k_4)mnpq}\left(\frac{g_s}{H}\right)^{2\Delta(k_1+k_2+k_3+k_4+2/\Delta)} \\
&- \frac{\delta_{ab}}{4!H^4}\sum_{\{k_i\}}\widetilde{C}_{k_1}C_{k_2}\mathcal{G}^{(k_3)}_{mnp\alpha}\mathcal{G}^{(k_4)mnp\alpha}\left(\frac{g_s}{H}\right)^{2\Delta(k_1+k_2+k_3+k_4+2/\Delta-\gamma/2\Delta)} \\
&- \frac{\delta_{ab}}{16H^4}\sum_{\{k_i\}}C_{k_1}C_{k_2}\mathcal{G}^{(k_3)}_{mn\alpha\beta}\mathcal{G}^{(k_4)mn\alpha\beta}\left(\frac{g_s}{H}\right)^{2\Delta(k_1+k_2+k_3+k_4+2/\Delta-\gamma/\Delta)} \\
&+ \frac{4\delta_{ab}}{H^2}\sum_{\{k_i\}}\widetilde{C}_{k_1}C_{k_2}\partial_m H\partial^m H\left(\frac{g_s}{H}\right)^{2\Delta(k_1+k_2+1/\Delta)}, \qquad (3.64)
\end{aligned}
$$

where taking $\gamma = 2$ we see that there are no zeroth order in g_s possible because the lower bound on the moding k_i of any G-flux component has to be $k_i \geq 9/2$. The largest allowed suppression factor is $-\gamma/\Delta = -6$ for the component of G-flux $\mathcal{G}^{(9/2)}_{\alpha\beta ab}$ in (3.64), implying that the lowest power of g_s contribution to the EOM will be g_s^2. This fits rather well with the g_s scaling of the quantum terms in (2.99), which now has a similar form as (3.50) and (3.58) with θ_k defined as in (2.84). Therefore combining (3.63) with (3.64), (3.1) and (3.50) we get, to order g_s^2, the following EOM:

$$\left(\frac{R}{2} + 9H^4\Lambda\right)\delta_{ab} + \frac{1}{4H^4}\left(\mathcal{G}^{(9/2)}_{ac\alpha\beta}\mathcal{G}_b^{(9/2)c\alpha\beta} - \frac{1}{4}\delta_{ab}\mathcal{G}^{(9/2)}_{cd\alpha\beta}\mathcal{G}^{(9/2)cd\alpha\beta}\right) + \mathbb{C}_{ab}^{(3,0)} = 0, \tag{3.65}$$

which may now be compared to (3.59). The quantum terms appearing here is similar to what we had in (3.59), and is classified by $\theta_k = 2/3$ in (2.84). However the number of G-flux components contributing to (3.65) is much smaller; and (3.65) is a set of two equations with at least 7 unknowns.

To the next order in g_s, i.e $g_s^{7/3}$, the Einstein tensor (3.63) does contribute compared to the scenario with (3.56). In fact both the energy-momentum tensors also contribute to this order. The resulting EOM becomes:

$$\delta_{ab} = \frac{1}{4q\Lambda H^8}\sum_{\{k_i\}} C_{k_1}....C_{k_4}\left(\mathcal{G}^{(k_5)}_{ac\alpha\beta}\mathcal{G}_b^{(k_6)c\alpha\beta} - \frac{1}{4}\delta_{ab}\mathcal{G}^{(k_5)}_{cd\alpha\beta}\mathcal{G}^{(k_6)cd\alpha\beta}\right)\delta\left(k_1 + ... + k_6 - \frac{19}{2}\right) + \frac{\mathbb{C}_{ab}^{(7/2,0)}}{q\Lambda H^4}, \tag{3.66}$$

where $q \equiv 4 - 10C_{\frac{1}{2}}$, and one may use this equation to fix the form of the quantum terms classified by $\theta_k = 1$ in (2.84) with the G-flux component appearing above[7]. Once we go to higher orders in g_s new components of G-flux start contributing to the EOM as evident from the form of (3.64). We will not discuss this further here, and instead go to the study of space-time components.

Einstein equation along (μ, ν) directions

The structural similarities of the equations for all the previous cases have some bearings on the choices of G-flux components (at least to some low orders in g_s) enter in the EOMs. The quantum terms are also similar, modulo the subtlety for $\mathbb{T}^Q_{ab}$ requiring some redefinition (see footnote 6).

The story for the space-time components will require additional subtleties that we will illustrate as we go along. First, let us express the Einstein tensor along the two spatial directions in the following way:

$$\begin{aligned}\mathbb{G}_{ij} &= -\frac{\eta_{ij}}{\Lambda(t)}\left(3\Lambda + \frac{R}{2H^4} + \frac{4g^{\alpha\beta}\partial_\alpha H\partial_\beta H}{H^6F_1} + \frac{4g^{mn}\partial_m H\partial_n H}{H^6F_2} - \frac{\Box_{(m)}H^4}{2H^8F_1}\right) \\ &+ \frac{\eta_{ij}}{\Lambda(t)}\left(\frac{\Box_{(\alpha)}H^4}{2H^8F_2}\right) + \eta_{ij}\left(\frac{\dot{F}_1^2}{4F_1^2} + \frac{\dot{F}_1}{tF_1} - \frac{\ddot{F}_1}{F_1} - \frac{\dot{F}_2^2}{2F_2^2} + \frac{2\dot{F}_2}{tF_2} - \frac{2\ddot{F}_2}{F_2} - \frac{2\dot{F}_1\dot{F}_2}{F_1F_2}\right),\end{aligned} \tag{3.67}$$

where, since we identified $\Lambda(t) = \left(\frac{g_s}{H}\right)^2$, the appearance of $\Lambda^{-1}(t)$ is a bit disconcerting for the late time physics where $t \to 0$ or $g_s \to 0$. We will not worry about this

[7] Compared to the (α, β) case the traces of (3.66) and (3.60) do not fix the signs of $[\mathbb{C}]^{(7/2,0)}$ in both cases.

right now and carry on with the Einstein tensor along the temporal direction which, in turn, takes the following form:

$$\mathbb{G}_{00} = \frac{\eta_{00}}{\Lambda(t)}\left(\frac{\Box_{(m)}H^4}{2H^8F_2}\right) - \eta_{00}\left(\frac{\dot{F}_1^2}{4F_1^2} - \frac{3\dot{F}_1}{tF_1} + \frac{3\dot{F}_2^2}{2F_2^2} - \frac{6\dot{F}_2}{tF_2} + \frac{2\dot{F}_1\dot{F}_2}{F_1F_2}\right)$$
$$- \frac{\eta_{00}}{\Lambda(t)}\left(3\Lambda + \frac{R}{2H^4} + \frac{4g^{\alpha\beta}\partial_\alpha H\partial_\beta H}{H^6F_1} + \frac{4g^{mn}\partial_m H\partial_n H}{H^6F_2} - \frac{\Box_{(\alpha)}H^4}{2H^8F_1}\right), \tag{3.68}$$
$$\tag{3.69}$$

where the key difference from (3.67), other than the appearance of η_{00}, is in the terms with derivatives on $F_i(t)$. Other than these, both the Einstein tensors are similar in terms of the appearance of the warp-factor $H(y)$ and the six-dimensional curvature scalar R. In the similar vein, we can express the energy-momentum tensor for the G-flux in the following way:

$$\mathbb{T}^G_{\mu\nu} = -\frac{\eta_{\mu\nu}}{8\Lambda(t)H^8}\left(\frac{1}{3F_2^3}G_{mnpa}G^{mnpa} + \frac{1}{F_2^2F_1}G_{m\alpha pa}G^{m\alpha pa} + \frac{1}{F_1^2F_2}G_{\alpha\beta pa}G^{\alpha\beta pa}\right)$$
$$- \frac{\eta_{\mu\nu}}{24H^8}\left(\frac{1}{4F_2^4}G_{mnpq}G^{mnpq} + \frac{1}{F_2^3F_1}G_{mnp\alpha}G^{mnp\alpha} + \frac{1}{4F_2^2F_1^2}G_{mn\alpha\beta}G^{mn\alpha\beta}\right)$$
$$- \frac{\eta_{\mu\nu}}{8\Lambda^2(t)H^8}\left(\frac{1}{2F_2^2}G_{mnab}G^{mnab} + \frac{1}{F_2F_1}G_{m\alpha ab}G^{m\alpha ab} + \frac{1}{2F_1^2}G_{\beta\alpha ab}G^{\beta\alpha ab}\right)$$
$$- \frac{4\eta_{\mu\nu}}{\Lambda(t)H^6}\left(\frac{g^{mn}\partial_m H\partial_n H}{F_2} + \frac{g^{\alpha\beta}\partial_\alpha H\partial_\beta H}{F_1}\right), \tag{3.70}$$

where again expectedly the last two terms cancel with equivalent terms in both G_{ij} and G_{00} in (3.67) and (3.68) respectively. With these at our disposal, let us go to the individual cases now.

Case 1: $F_1(t)$ and $F_2(t)$ satisfying the volume-preserving condition (2.2)

The inverse $\Lambda(t)$ factors appearing in the expressions of the Einstein tensors as well as the energy-momentum tensors for the G-fluxes are a case of worry in the late time limit that we want to analyze the background. Of course the existence of these factors are expected from the inverse $\Lambda(t)$ factor appearing in the type IIB metric (2.3), but since our construction involve finite values in the $g_s \to 0$ limit, we will need to tread carefully to interpret our answers. To analyze the story further, let us write the Einstein tensor along spatial direction first in the following way:

$$\mathbb{G}_{ij} = -\eta_{ij}\left(3\Lambda + \frac{R}{2H^4}\right)\left(\frac{g_s}{H}\right)^{-2} + \frac{\Lambda\eta_{ij}}{9}\sum_{\{k_i\}} k_1k_2\widetilde{C}_{k_1}\widetilde{C}_{k_2}C_{k_3}...C_{k_6}\left(\frac{g_s}{H}\right)^{2\Delta(k_1+...+k_6-1/\Delta)}$$
$$- \frac{4\eta_{ij}}{H^6}\sum_{\{k_i\}}\left[\left((\partial_\alpha H)^2 - \frac{\Box_{(m)}H^4}{8H^2}\right)C_{k_1}C_{k_2} + \left((\partial_m H)^2 - \frac{\Box_{(\alpha)}H^4}{8H^2}\right)C_{k_1}\widetilde{C}_{k_2}\right]\left(\frac{g_s}{H}\right)^{2\Delta(k_1+k_2-1/\Delta)}$$
$$+ \frac{2\Lambda\eta_{ij}}{9}\sum_{\{k_i\}}\Big[2k_3(3-k_3-2k_2) + k_1(12-4k_1-k_2)\Big]C_{k_1}C_{k_2}\widetilde{C}_{k_3}\left(\frac{g_s}{H}\right)^{2\Delta(k_1+k_2+k_3-1/\Delta)}, \tag{3.71}$$

where we have defined $(\partial_\alpha H)^2 \equiv g^{\alpha\beta}\partial_\alpha H \partial_\beta H$ and the same for $(\partial_m H)^2 \equiv g^{mn}\partial_m H \partial_n H$ with un-warped metrics. It is also easy to read out the form of the $\mathbb{G}_{00}$ tensor:

$$\begin{aligned}
\mathbb{G}_{00} &= -\eta_{00}\left(3\Lambda + \frac{R}{2H^4}\right)\left(\frac{g_s}{H}\right)^{-2} - \frac{\Lambda\eta_{00}}{9}\sum_{\{k_i\}} k_1 k_2 \widetilde{C}_{k_1}\widetilde{C}_{k_2}C_{k_3}...C_{k_6}\left(\frac{g_s}{H}\right)^{2\Delta(k_1+...+k_6-1/\Delta)} \\
&- \frac{4\eta_{00}}{H^6}\sum_{\{k_i\}}\left[\left((\partial_\alpha H)^2 - \frac{\square_{(m)}H^4}{8H^2}\right)C_{k_1}C_{k_2} + \left((\partial_m H)^2 - \frac{\square_{(\alpha)}H^4}{8H^2}\right)C_{k_1}\widetilde{C}_{k_2}\right]\left(\frac{g_s}{H}\right)^{2\Delta(k_1+k_2-1/\Delta)} \\
&+ \frac{2\Lambda\eta_{00}}{9}\sum_{\{k_i\}}\Big[k_3(9-4k_2) + 3k_1(6-k_2)\Big]C_{k_1}C_{k_2}\widetilde{C}_{k_3}\left(\frac{g_s}{H}\right)^{2\Delta(k_1+k_2+k_3-1/\Delta)},
\end{aligned} \tag{3.72}$$

which differs from (3.71) in three respects: presence of η_{00}, sign of the second term, and a different coefficient of the last term. On the other hand, from the various terms of (3.71) and (3.72), it is easy to infer that the lowest power of g_s, which is g_s^{-2}, appears when $k_i = 0$. In the limit $g_s \to 0$, this blows up, so to extract finite terms we have to carefully analyze the other contributions to the EOMs.

The other contributions to the EOM for the spatial components appear from the energy-momentum tensors of the G-flux and the quantum terms. The energy-momentum tensor for the G-fluxes for both spatial and temporal components may be expressed in the following way:

$$\begin{aligned}
\mathbb{T}^G_{\mu\nu} &= \frac{\eta_{\mu\nu}}{4H^8}\left(\frac{1}{6}\widetilde{C}_{k_1}\widetilde{C}_{k_2}C_{k_3}\mathcal{G}^{(k_4)}_{mnap}\mathcal{G}^{(k_5)mnpa} - \frac{1}{2}C_{k_1}C_{k_2}C_{k_3}\mathcal{G}^{(k_4)}_{\alpha\beta pa}\mathcal{G}^{(k_5)\alpha\beta pa}\right)\left(\frac{g_s}{H}\right)^{2\Delta(k_1+...+k_5-1/\Delta)} \\
&- \frac{\eta_{\mu\nu}}{24H^8}\left(\frac{1}{4}\widetilde{C}_{k_1}\widetilde{C}_{k_2}\mathcal{G}^{(k_3)}_{mnpq}\mathcal{G}^{(k_4)mnpq} + \widetilde{C}_{k_1}C_{k_2}\mathcal{G}^{(k_3)}_{mnp\alpha}\mathcal{G}^{(k_4)mnp\alpha} + \frac{1}{4}C_{k_1}C_{k_2}\mathcal{G}^{(k_3)}_{mn\alpha\beta}\mathcal{G}^{(k_4)mn\alpha\beta}\right) \\
&\times \left(\frac{g_s}{H}\right)^{2\Delta(k_1+..+k_4)} - \frac{\eta_{\mu\nu}}{8H^8}\left(\frac{1}{2}\widetilde{C}_{k_1}\mathcal{G}^{(k_2)}_{mnab}\mathcal{G}^{(k_3)mnab} + C_{k_1}\mathcal{G}^{(k_2)}_{m\alpha ab}\mathcal{G}^{(k_3)m\alpha ab}\right)\left(\frac{g_s}{H}\right)^{2\Delta(k_1+k_2+k_3-2/\Delta)} \\
&- \frac{\eta_{\mu\nu}}{H^6}\left(\frac{1}{8H^2}\mathcal{G}^{(k_1)}_{m\alpha pa}\mathcal{G}^{(k_2)m\alpha pa} + 4(\partial_\alpha H)^2 C_{k_1}C_{k_2} + 4(\partial_m H)^2 C_{k_1}\widetilde{C}_{k_2}\right)\left(\frac{g_s}{H}\right)^{2\Delta(k_1+k_2-1/\Delta)} \\
&- \frac{\eta_{\mu\nu}}{16H^8}C_{k_1}C_{k_2}C_{k_3}C_{k_4}\mathcal{G}^{(k_5)}_{\alpha\beta ab}\mathcal{G}^{(k_6)\alpha\beta ab}\left(\frac{g_s}{H}\right)^{2\Delta(k_1+k_2+k_3+k_4+k_5+k_6-2/\Delta)},
\end{aligned} \tag{3.73}$$

where since some of the k_i, accompanying the G-flux components are bounded below as $k_i \geq 3/2$, we would get the g_s^{-2} powers from the $\mathcal{G}^{(3/2)}_{mnab}, \mathcal{G}^{(3/2)}_{m\alpha ab}$ and $\mathcal{G}^{(3/2)}_{\alpha\beta ab}$ components. However this is puzzling in light of the quantum terms (3.1). Our expression from (3.1) allows only g_s^0 as the lowest power of g_s because the negative powers are assimilated to a series in e^{-1/g_s}. In the limit $g_s \to 0$ this dies off faster than any powers of g_s. Additionally as cautioned in footnote 1 it is not advisable to expand e^{-1/g_s} to any finite orders in inverse g_s. One way out of this would be to multiply the Einstein tensor (3.71), the G-flux energy-momentum tensor (3.73) and the quantum energy-momentum tensor (3.1) by $\left(\frac{g_s}{H}\right)^2$. This unfortunately will *not* solve the problem, because now the lowest power of (3.1) will be g_s^2 so cannot be used to balance the g_s^0 terms of (3.71) and (3.73). The quantum terms are essential, to avoid over-constraining the system. Additionally, the g_s scaling along the space-time direction is in fact:

$$g_s^{\theta'_k - 8/3} \equiv g_s^0,\ g_s^{1/3},\ g_s^{2/3},\ g_s,\ g_s^{4/3}, ..., \tag{3.74}$$

as evident from (2.100), implying that the minimum value of θ'_k in (2.97) is $\theta'_k = 8/3$ to account for g_s independent terms. All of these then imply the following way out: redefine the energy-momentum tensor for the quantum pieces along the space-time directions in the $g_s \to 0$ limit as:

$$\mathbb{T}^{Q}_{\mu\nu} \equiv \sum_{\{k\}} \mathbb{C}^{(k,0)}_{\mu\nu} \left(\frac{g_s}{H}\right)^{2\Delta(k-1/\Delta)}, \tag{3.75}$$

instead of (3.1) for (μ, ν) indices. Such a re-definition is similar to the re-definition we did for the (a, b) case (see footnote 6) and is consistent with the scalings employed in [14] and [15] (see eq (5.29) in [14]).

There is yet another contribution that we have ignored so far and has to do with the energy-momentum tensor of an almost *static* set of membranes. These are related to static D3-branes (integer and fractional) in the type IIB side, and we can consider both branes and anti-branes in our picture. For simplicity, let us assume that we have n_b number of coincident membranes at a point on the internal eight-dimensional manifold. These membranes are therefore stretched along the $2+1$ dimensional space-time[8]. The analysis of the energy-momentum tensor proceeds in exactly the same way as given in [14], so we will suffice ourselves by simply quoting the answer:

$$\mathbb{T}^{(B)}_{\mu\nu} \approx -\frac{\kappa^2 T_2 n_b}{H^8\sqrt{g_6}} \left(\frac{g_s}{H}\right)^{-2} \delta^8(y - Y)\, \eta_{\mu\nu}, \tag{3.76}$$

where T_2 is the tension of the individual membranes, κ is a constant related to M_p, g_6 is the determinant of the unwarped metric of the six-dimensional base $\mathcal{M}_4 \times \mathcal{M}_2$, and n_b is the number of membranes located at Y^M in the internal eight-manifold.

With these definitions of the quantum energy-momentum tensor in (3.75) and the membrane energy-momentum tensor in (3.76), we can move ahead with the EOMs. First we multiply all the tensors with $\left(\frac{g_s}{H}\right)^2$ to get rid of any infinities arising in the $g_s \to 0$ limit. Secondly, we compare the zeroth order in g_s for (3.71), (3.73) and (3.75), to get the following EOM:

$$\begin{aligned} &6\Lambda + \frac{R}{H^4} - \frac{\Box H^4}{H^8} + \left[\mathbb{C}^i_i\right]^{(0,0)} - \frac{2\kappa^2 T_2 n_b}{H^8\sqrt{g_6}}\delta^8(y - Y) \\ &= \frac{1}{8H^8}\Big(\mathcal{G}^{(3/2)}_{mnab}\mathcal{G}^{(3/2)mnab} + 2\mathcal{G}^{(3/2)}_{m\alpha ab}\mathcal{G}^{(3/2)m\alpha ab} + \mathcal{G}^{(3/2)}_{\alpha\beta ab}\mathcal{G}^{(3/2)\alpha\beta ab}\Big), \end{aligned} \tag{3.77}$$

showing how the same set of G-flux components appear again to balance the spatial equation of motion. We have also defined $\Box \equiv \Box_{(m)} + \Box_{(\alpha)}$ to avoid clutter. The equation (3.77) is somewhat similar to what we had in eq (5.32) of [14] with two crucial differences. One, the G-flux components are the set $\mathcal{G}^{(3/2)}_{mnab}, \mathcal{G}^{(3/2)}_{m\alpha ab}$ and $\mathcal{G}^{(3/2)}_{\alpha\beta ab}$ of *localized* fluxes and not the globally-defined time-independent flux component

[8]We will consider both integer and fractional M2-branes. The latter being M5-branes wrapped on 3-cycles.

appearing in [14]. Two, the quantum terms $\mathbb{C}_{\mu\nu}^{(0,0)}$ are classified by:

$$2\sum_{i=1}^{27} l_i + n_1 + n_2 + \sum_{i=0}^{4} l_{34+i} = 8, \tag{3.78}$$

i.e with $\theta'_k = 8/3$ in (2.97) (l_i, n_i are defined in (2.94)), compared to $\theta'_0 = 8/3$ in (2.98). The former, i.e (3.78), has a large but *finite* number of solutions, whereas the latter has an *infinite* number of solutions with no g_s or M_p hierarchies. In a similar vein one may work out the $\mathbb{G}_{00}$ EOM, but to this order the result (3.77) will not change.

The next order in g_s, i.e for $g_s^{1/3}$, one may easily find the EOMs by comparing terms of this order from (3.71), (3.72), (3.73) and (3.75) with no contributions from the membranes. The G-flux components contributing now are of the form $\mathcal{G}_{MNab}^{(3/2)}$ and $\mathcal{G}_{MNab}^{(2)}$ with (M,N) spanning the coordinates of $\mathcal{M}_4 \times \mathcal{M}_2$. The quantum terms $\mathbb{C}_{ij}^{(1/2,0)}$ are classified by $\theta'_k = 3$ in (2.97). Combining the two set of equations, one from the (i,j) components, and one from the $(0,0)$ components, we get:

$$2\left[\mathbb{C}_0^0\right]^{(1/2,0)} = \left[\mathbb{C}_i^i\right]^{(1/2,0)}, \tag{3.79}$$

where the quantum terms $\mathbb{C}_{\mu\nu}^{(1/2,0)}$ are the specific linear combinations of all terms classified by $\theta'_k = 3$ for individual components in (2.97). According to the discussions around (3.17) these quantum terms are computed using the dominant scalings of the metric components $\mathbf{g}_{mn}$ and $\mathbf{g}_{\alpha\beta}$. Thus the LHS of (3.79) is fixed in terms of the known components of the metric and the G-fluxes in a way that their sum vanishes. Such an equation can be used to predict the relative coefficient of the various terms to the same order in curvatures and G-fluxes.

One can even go higher orders in g_s, say for example $g_s^{2/3}$ as we have done before, and compare the (i,j) and the $(0,0)$ EOMs. The quantum terms would be of the form $\mathbb{C}_{\mu\nu}^{(1,0)}$ and are classified by $\theta'_k = 10/3$ in (2.97). These could be used to fix the higher order coefficients of $F_i(t)$ in terms of the quantum terms. For example taking the traces of (3.71) and (3.72) appropriately, we get:

$$C_{\frac{1}{2}}^2 = 3\left(2\left[\mathbb{C}_0^0\right]^{(1,0)} - \left[\mathbb{C}_i^i\right]^{(1,0)}\right), \tag{3.80}$$

which tells us that it is only the constant pieces of the quantum terms (2.94) that are responsible in generating the $F_i(t)$ functions. Note that, to this order C_1 and $\widetilde{C}_1$ coefficients cancel out. To determine these, we have to go to the next order in g_s where, in turn the $C_{\frac{3}{2}}$ and $\widetilde{C}_{\frac{3}{2}}$ pieces cancel out, leaving us with C_1 and $\widetilde{C}_1$. We will leave the evaluation of these coefficients for interested readers, and instead go to the discussion of the case with γ switched on.

Case 2: $F_1(t)$ and $F_2(t)$ satisfying the fluctuation condition (2.8)

The analysis along the space-time directions has a few subtleties that we clarified above. Additional subtleties arise when we switch on non-zero γ from the fact that the internal eight-manifold has zero Euler characteristics. This implies that one cannot switch on either non-zero components of G-fluxes that are time-independent, or dynamical M2-branes at least in the supersymmetric limit [21, 22]. Our study is for non-supersymmetric states, plus we take vanishing time-independent component of G-flux (3.13), so the situation is a bit more subtle. Nevertheless the bound considered in [21, 22] does not allow us to take static M2-branes[9]. What happens for dynamical branes will be discussed later.

We will start by elaborating the Einstein tensor for both spatial and temporal directions. The Einstein tensor for the two spatial directions may be expressed in the following way:

$$\begin{aligned}\mathbb{G}_{ij} &= -\eta_{ij}\left(3\Lambda+\frac{R}{2H^4}\right)\left(\frac{g_s}{H}\right)^{-2}+\frac{\Lambda\eta_{ij}}{4}\sum_{\{k_i\}}(2\Delta k_1+\gamma)(2\Delta k_2+\gamma)\widetilde{C}_{k_1}\widetilde{C}_{k_2}C_{k_3}...C_{k_6}\left(\frac{g_s}{H}\right)^{2\Delta(k_1+...+k_6-1/\Delta)}\\ &+ \frac{4\eta_{ij}}{H^6}\sum_{\{k_i\}}C_{k_1}\left[C_{k_2}\left(\frac{\square_{(m)}H^4}{8H^2}\right)\left(\frac{g_s}{H}\right)^{2\Delta(k_1+k_2-1/\Delta-\gamma/2\Delta)}-\widetilde{C}_{k_2}(\partial_m H)^2\left(\frac{g_s}{H}\right)^{2\Delta(k_1+k_2-1/\Delta)}\right]\\ &+ \frac{\Lambda\eta_{ij}}{9}\sum_{\{k_i\}}\Big[(2k_3+3\gamma)(6-2k_3-3\gamma-4k_2)+2k_1(12-4k_1-k_2)\Big]C_{k_1}C_{k_2}\widetilde{C}_{k_3}\left(\frac{g_s}{H}\right)^{2\Delta(k_1+k_2+k_3-1/\Delta)},\end{aligned} \tag{3.81}$$

where we see that only one g_s scaling is effected by the γ factor, although quite a few coefficients do pick up γ dependent factors. In addition to that, derivatives with respect to α are missing compared to (3.71). Similar story also shows up for the Einstein tensor along the temporal directions in the following way:

$$\begin{aligned}\mathbb{G}_{00} &= -\eta_{00}\left(3\Lambda+\frac{R}{2H^4}\right)\left(\frac{g_s}{H}\right)^{-2}-\frac{\Lambda\eta_{00}}{4}\sum_{\{k_i\}}(2\Delta k_1+\gamma)(2\Delta k_2+\gamma)\widetilde{C}_{k_1}\widetilde{C}_{k_2}C_{k_3}...C_{k_6}\left(\frac{g_s}{H}\right)^{2\Delta(k_1+...+k_6-1/\Delta)}\\ &+ \frac{4\eta_{00}}{H^6}\sum_{\{k_i\}}C_{k_1}\left[C_{k_2}\left(\frac{\square_{(m)}H^4}{8H^2}\right)\left(\frac{g_s}{H}\right)^{2\Delta(k_1+k_2-1/\Delta-\gamma/2\Delta)}-\widetilde{C}_{k_2}(\partial_m H)^2\left(\frac{g_s}{H}\right)^{2\Delta(k_1+k_2-1/\Delta)}\right]\\ &+ \frac{\Lambda\eta_{00}}{9}\sum_{\{k_i\}}\Big[(2k_3+3\gamma)(9-4k_2)+6k_1(6-k_2)\Big]C_{k_1}C_{k_2}\widetilde{C}_{k_3}\left(\frac{g_s}{H}\right)^{2\Delta(k_1+k_2+k_3-1/\Delta)},\end{aligned} \tag{3.82}$$

where again, as compared to (3.72), other than the last term and one relative sign difference, the two Einstein tensors are identical. Similarly, the energy-momentum

[9] See however footnote 5.

tensor for the G-flux is given by:

$$\begin{aligned}
\mathbb{T}^G_{\mu\nu} &= \frac{\eta_{\mu\nu}}{24H^8}\left(\widetilde{C}_{k_1}\widetilde{C}_{k_2}C_{k_3}\mathcal{G}^{(k_4)}_{mnap}\mathcal{G}^{(k_5)mnpa}\right)\left(\frac{g_s}{H}\right)^{2\Delta(k_1+...+k_5-1/\Delta)} \\
&- \frac{\eta_{\mu\nu}}{8H^8}\left(C_{k_1}C_{k_2}C_{k_3}\mathcal{G}^{(k_4)}_{\alpha\beta pa}\mathcal{G}^{(k_5)\alpha\beta pa}\right)\left(\frac{g_s}{H}\right)^{2\Delta(k_1+...+k_5-1/\Delta-\gamma/\Delta)} \\
&- \frac{\eta_{\mu\nu}}{96H^8}\left(\widetilde{C}_{k_1}\widetilde{C}_{k_2}\mathcal{G}^{(k_3)}_{mnpq}\mathcal{G}^{(k_4)mnpq}\right)\left(\frac{g_s}{H}\right)^{2\Delta(k_1+k_2+k_3+k_4)} \\
&+ \frac{\eta_{\mu\nu}}{24H^8}\left(\widetilde{C}_{k_1}C_{k_2}\mathcal{G}^{(k_3)}_{mnp\alpha}\mathcal{G}^{(k_4)mnp\alpha}\right)\left(\frac{g_s}{H}\right)^{2\Delta(k_1+k_2+k_3+k_4-\gamma/2\Delta)} \\
&+ \frac{\eta_{\mu\nu}}{96H^8}\left(C_{k_1}C_{k_2}\mathcal{G}^{(k_3)}_{mn\alpha\beta}\mathcal{G}^{(k_4)mn\alpha\beta}\right)\left(\frac{g_s}{H}\right)^{2\Delta(k_1+k_2+k_3+k_4-\gamma/\Delta)} \\
&- \frac{\eta_{\mu\nu}}{16H^8}\left(\widetilde{C}_{k_1}\mathcal{G}^{(k_2)}_{mnab}\mathcal{G}^{(k_3)mnab}\right)\left(\frac{g_s}{H}\right)^{2\Delta(k_1+k_2+k_3-2/\Delta)} \\
&- \frac{\eta_{\mu\nu}}{8H^8}\left(C_{k_1}\mathcal{G}^{(k_2)}_{m\alpha ab}\mathcal{G}^{(k_3)m\alpha ab}\right)\left(\frac{g_s}{H}\right)^{2\Delta(k_1+k_2+k_3-2/\Delta-\gamma/2\Delta)} \\
&- \frac{\eta_{\mu\nu}}{8H^8}\left(\mathcal{G}^{(k_1)}_{m\alpha pa}\mathcal{G}^{(k_2)m\alpha pa}\right)\left(\frac{g_s}{H}\right)^{2\Delta(k_1+k_2-1/\Delta-\gamma/2\Delta)} \\
&+ \frac{4\eta_{\mu\nu}}{H^6}\left(g^{mn}\partial_m H\partial_n H\right)C_{k_1}\widetilde{C}_{k_2}\left(\frac{g_s}{H}\right)^{2\Delta(k_1+k_2-1/\Delta)} \\
&- \frac{\eta_{\mu\nu}}{16H^8}C_{k_1}C_{k_2}C_{k_3}C_{k_4}\mathcal{G}^{(k_5)}_{\alpha\beta ab}\mathcal{G}^{(k_6)\alpha\beta ab}\left(\frac{g_s}{H}\right)^{2\Delta(k_1+k_2+k_3+k_4+k_5+k_6-2/\Delta-\gamma/\Delta)},
\end{aligned} \tag{3.83}$$

where the various shifts of the g_s scalings due to the γ are shown above. Taking $\gamma = 2$, we se that the issue regarding the lowest order g_s scaling appear here too, albeit in a more severe way. When $\gamma = 0$, the lowest order scaling of the Einstein tensor from (3.71) is g_s^{-2}. For $\gamma > 0$, the lowest order scaling from (3.81) becomes $g_s^{-2\Delta\omega_1}$. On the other hand, the lowest order g_s scaling that can emerge from the energy-momentum tensor (3.83) is $g_s^{-2\Delta\omega_2}$, where:

$$\omega_1 \equiv \frac{\gamma+2}{2\Delta}, \quad \omega_2 \equiv \frac{\gamma+2}{\Delta} - 9, \tag{3.84}$$

which for $\gamma = 2$ and $\Delta = \frac{1}{3}$ is g_s^{-4} and g_s^{-2} respectively[10], implying that there cannot be any contributions from the energy-momentum tensor (3.83) to this order. In fact increasing γ only worsens the problem.

Looking at the modified form of the energy-momentum tensor from the quantum terms in (3.75), shows that it also does not contribute terms to order g_s^{-4}. Therefore one of the simplest way out of this could be to demand:

$$\Box_{(m)}H^4(y) \equiv \Box_{(m)}h(y) = 0, \tag{3.85}$$

on $\mathcal{M}_4$ where the Laplacian is computed using the un-warped metric $g_{mn}(y)$. As we saw before, the manifold $\mathcal{M}_4$ is a compact four-dimensional manifold that supports a non-Kähler metric. Thus $H^4(y) = h(y)$ is a harmonic function on the compact non-Kähler manifold $\mathcal{M}_4$. The manifold $\mathcal{M}_2$ is conformally a torus, and the full

[10]The factor of 9 in (3.84) appears from the minimum moding of the G-flux components $\mathcal{G}^{(9/2)}_{\alpha\beta ab}$ that contributes to (3.83).

Ricci scalar of the six-dimensional space $\mathcal{M}_4 \times \mathcal{M}_2$ is then given by:

$$R = \frac{1}{8H^4}\mathcal{G}^{(9/2)}_{\alpha\beta ab}\mathcal{G}^{(9/2)\alpha\beta ab} - H^4\left[\mathbb{C}^i_i\right]^{(0,0)} - 4\Lambda H^4, \tag{3.86}$$

which vanishes when we take the un-warped metric of the six-dimensional space to be that of $K3 \times \mathbf{T}^2$. Additionally, the quantum terms are again classified by $\theta_k = 8/3$ from (2.99), with θ_k defined as in (2.84). Comparing this to (3.77), we notice a few key differences: the brane term is absent and so are some of the G-flux components. The warp-factor is harmonic so naturally decouples out of (3.77). The contribution from the cosmological constant term is smaller because the coefficient of the Λ term, i.e $\sigma_2\Lambda$, changes to:

$$\sigma_2 \equiv \frac{1}{4}\left(8\gamma - 3\gamma^2 - 12\right). \tag{3.87}$$

To the next order in g_s, i.e $g_s^{1/3}$, surprisingly we get exactly the same relation (3.79) that we encountered earlier despite the presence of the γ factor (which we take as $\gamma = 2$). We expect the other coefficient to appear in a way reminiscent of (3.80) and the story follows the path laid out for case 1.

Before moving to the next sub-section, let us ask if there is an alternative to the choice (3.85). The choice (3.85) tells us that the warp-factor $h(y)$ is simply a harmonic function on the non-Kähler manifold $\mathcal{M}_4$, and all information of the fluxes and the quantum corrections enter indirectly. An alternative to this choice would be to modify further the definition of the quantum energy-momentum tensor (3.75) by changing the g_s exponent from:

$$\frac{1}{\Delta} \rightarrow \frac{\gamma + 2}{2\Delta}, \tag{3.88}$$

which would equate the Laplacian of the warp-factor directly to the quantum corrections at zeroth order in g_s. The Einstein's equation can then be realized at second order in g_s equating (3.81) with (3.83) and the quantum terms. To see how this works out, let us rewrite the quantum corrections, using the input (3.88), in the following way:

$$\mathbb{T}^Q_{\mu\nu} \equiv \sum_{\{k\}} \mathbb{C}^{(k,0)}_{\mu\nu}\left(\frac{g_s}{H}\right)^{2\Delta(k-2/\Delta)}, \tag{3.89}$$

instead of (3.75), where we took $\gamma = 2$. This extra $\left(\frac{g_s}{H}\right)^{-4}$ suppression tells us that the warp-factor H^4 is no longer needed to be a harmonic function as in (3.85), rather it can now satisfy the following equation:

$$\square_{(m)}H^4 = H^8\left[\mathbb{C}^i_i\right]^{(0,0)}, \tag{3.90}$$

with the quantum terms being classified by $\theta_k = \frac{8}{3}$ in (2.86), and therefore involve a mixture of terms in fourth powers of curvature, eighth powers of G-fluxes or a combination of both to the relevant powers. Note that there are no G-flux contributions to this order, as we noted earlier. However once we go to the next order, i.e to order $\left(\frac{g_s}{H}\right)^{-2}$, flux contributions get poured in and the equation becomes:

$$\frac{\square_{(m)}H^4}{H^8} = \frac{1}{\gamma_o}\left(4\Lambda + \frac{R}{H^4} - \frac{1}{8H^8}\mathcal{G}^{(9/2)}_{\alpha\beta ab}\mathcal{G}^{(9/2)\alpha\beta ab} + \left[\mathbb{C}^i_i\right]^{(3,0)}\right), \tag{3.91}$$

which has some surprising similarities with (3.77). The similarities being the appearances of equivalent forms of curvature, fluxes and quantum terms on the RHS. However there are also few crucial differences. One, the G-flux components are not as many as in (3.77). Two, the coefficient of the cosmological constant term is now 4 instead of 6 before. Three, the warp-factor H^4 satisfy a much simpler relation (3.90) in addition to (3.91). And four, the quantum terms are classified by $\theta_k = \frac{14}{3}$ with $[\mathbb{C}^i_i]^{(3,0)}$ instead by $\theta_k = \frac{8}{3}$ with $[\mathbb{C}^i_i]^{(0,0)}$ in (2.86). Finally, γ_o is given by:

$$\gamma_o \equiv \sum_{\{k_i\}} C_{k_1}C_{k_2}\delta\left(k_1 + k_2 - 3\right). \tag{3.92}$$

The question now is which of the two descriptions is the correct one. Clearly we will need more constraints to distinguish one from the other, and in section 3.1.2 we will see that the flux EOMs provide the required constraints to justify (3.91), instead of (3.86), to be the correct EOM for this case.

Metric cross-terms and the $F_i(t)$ factors

So far we have studied the equations of motion without considering the cross-terms. However, cross-terms *do* arise in the Einstein tensor because, for one, the internal metric has time-dependent factors (i.e the functions $F_i(t)$), and for another, the warp-factor $H(y)$ is in general a function of all the coordinates of $\mathcal{M}_4 \times \mathcal{M}_2$. Thus at least we expect the following three cross-terms:

$$\mathbb{G}_{0n} = -2\left(\frac{\dot{F}_1}{F_1} + \frac{\dot{F}_2}{F_2}\right)\frac{\partial_n H}{H}, \quad \mathbb{G}_{0\alpha} = -4\left(\frac{\dot{F}_2}{F_2}\right)\frac{\partial_\alpha H}{H}, \quad \mathbb{G}_{\alpha m} = -\frac{8\partial_\alpha H\partial_m H}{H^2}, \tag{3.93}$$

with other cross-components vanishing. For the Einstein tensors $\mathbb{G}_{0n}$ and $\mathbb{G}_{0\alpha}$, it is easy to argue that there are no corresponding energy-momentum tensors from the G-fluxes because we do not allow $G_{mn\mu\nu}$ and $G_{m\alpha\mu\nu}$ components. Allowing them would not only add new complications to the existing EOMs studied earlier, but also break the de-Sitter isometries in the type IIB side. We want to avoid the latter, so it appears that the Einstein tensors with the cross-terms along temporal direction will have to be balanced solely by the quantum terms. If y^M denote the coordinates of $\mathcal{M}_4 \times \mathcal{M}_2$, the energy-momentum tensor associated with the quantum cross-terms may be expressed in the $g_s \to 0$ limit as:

$$\mathbb{T}^Q_{0M} \equiv \sum_{\{k\}} \mathbb{C}^{(k,0)}_{0M}\left(\frac{g_s}{H}\right)^{2\Delta(k-1/2\Delta)}, \tag{3.94}$$

where the specific choice of the g_s scaling is to take care of g_s^{-1} pieces that may arise from $\dot{F}_i(t)$ in (3.93). Taking for example the volume preserving case (2.2), it is easy to see where the g_s^{-1} factor appear from. The Einstein tensors become:

$$\mathbb{G}_{0\alpha} = -8\Delta\sqrt{\Lambda}\left(\frac{\partial_\alpha H}{H}\right)\sum_{\{k_i\}} k_1 C_{k_1} C_{k_2}\widetilde{C}_{k_3}\left(\frac{g_s}{H}\right)^{2\Delta(k_1+k_2+k_3-1/2\Delta)}$$
$$\mathbb{G}_{0n} = -4\Delta\sqrt{\Lambda}\left(\frac{\partial_n H}{H}\right)\sum_{\{k_i\}}(k_1+k_2)\widetilde{C}_{k_1} C_{k_2} C_{k_3}\left(\frac{g_s}{H}\right)^{2\Delta(k_1+k_2+k_3-1/2\Delta)}, \tag{3.95}$$

with the g_s scaling showing the inverse factor, alluded to above, which we can easily get rid of by multiplying all the tensors in (3.95) and (3.94) by g_s. To zeroth order in g_s there are no contributions from either (3.95) or (3.94). To next order in g_s, i.e $g_s^{1/3}$, we get:

$$C_{\frac{1}{2}} = -\frac{\mathbb{C}_{0\alpha}^{(1/2,0)}}{12\sqrt{\Lambda}}\left(\frac{\partial_\alpha H}{H}\right)^{-1} = \frac{\mathbb{C}_{0n}^{(1/2,0)}}{6\sqrt{\Lambda}}\left(\frac{\partial_n H}{H}\right)^{-1}, \tag{3.96}$$

which should be compared to (3.80). The above set of Einstein tensors provide a much easier way to get the C_k and $\widetilde{C}_k$ coefficients of the $F_i(t)$ functions. Expectedly, they are related to the quantum terms, so classically we can only see time-independent internal space. The latter has problems with EFT as we saw before and also in [14, 15].

Switching on the γ factor to study the case (2.8) or (2.75) eliminates $\mathbb{G}_{0\alpha}$ and $\mathbb{G}_{\alpha m}$ because of the derivative constraint. This only leaves $\mathbb{G}_{0n}$ which takes the following form:

$$\mathbb{G}_{0n} = -4\Delta\sqrt{\Lambda}\left(\frac{\partial_n H}{H}\right)\sum_{\{k_i\}}\left(k_1+k_2+\frac{\gamma}{2\Delta}\right)\widetilde{C}_{k_1} C_{k_2} C_{k_3}\left(\frac{g_s}{H}\right)^{2\Delta(k_1+k_2+k_3-1/2\Delta)}, \tag{3.97}$$

which now does allow a term to the zeroth order in g_s. By ignoring the g_s^{-1} piece for the time being – to be reconciled later using the same line of thought as before – the zeroth order in g_s yields the following relation for the quantum term:

$$\mathbb{C}_{0n}^{(0,0)} = -4\sqrt{\Lambda}\left(\frac{\partial_n H}{H}\right), \tag{3.98}$$

which, once combined with (3.85), should determine the functional form of the quantum term when we take $\gamma = 2$. Going to the next order in g_s, i.e $g_s^{1/3}$, we get exactly the same relation that we have in (3.96), i.e:

$$C_{\frac{1}{2}} = \frac{\mathbb{C}_{0n}^{(1/2,0)}}{6\sqrt{\Lambda}}\left(\frac{\partial_n H}{H}\right)^{-1}. \tag{3.99}$$

All these appear to lead to some consistent formulation of the background data, although there is one puzzle that we have kept under the rug so far. This has to do with the computation of the quantum energy-momentum tensor (3.94). How do we interpret this term? If we follow the definition of the energy-momentum tensor in (2.32), then the *absence* of $\mathbf{g}_{0n}$ should tell us that one cannot construct the cross-term energy-momentum tensor at all. In fact even the formulation of the Einstein tensor comes under scrutiny now.

The key point that we are missing here is the Wilsonian viewpoint that we already emphasized earlier (see the discussions between (2.81) and (2.82)). The background that we consider should contain all the components of metric and fluxes and we integrate out all the ones that would potentially ruin the four-dimensional de Sitter isometries in the type IIB side. This amounts to integrating out specific components of metric and G-fluxes in the M-theory side, leading to an effective action. In the following, let us see how this works when we integrate out one component of the metric, say $\mathbf{g}_{0n}$. We define:

$$\exp\left(-iS_{\rm eff}\right) \equiv \int \mathcal{D}\mathbf{g}_{0n} \exp\left[-i\int d^{11}x\sqrt{\mathbf{g}_{11}(\mathbf{g}_{0n})}\left(\mathbf{R}^{(11)} - \mathbf{g}^{0n}\mathbb{T}^G_{0n} - \mathbf{g}^{0n}\mathbb{T}^Q_{0n} + ...\right)\right], \tag{3.100}$$

where the dots denote terms that are independent of $\mathbf{g}_{0n}$, and the bold-faced components are defined with respect to the warped metric. Since $\mathbf{g}_{0n}$ is a dummy variable, we can re-define this to $\mathbf{g}'_{0n}$ without changing the effective action $S_{\rm eff}$. Taking $\mathbf{g}'_{0n} = \mathbf{g}_{0n} + \mathbf{h}_{0n}$, where $\mathbf{h}_{0n}$ is a small shift of the metric component, does not change the measure. This leads us to:

$$\begin{aligned}\exp\left(-iS_{\rm eff}\right) &\equiv \int \mathcal{D}\mathbf{g}'_{0n} \exp\left[-i\int d^{11}x\sqrt{\mathbf{g}_{11}(\mathbf{g}'_{0n})}\left(\mathbf{R}^{(11)}(\mathbf{g}'_{0n}) - \mathbf{g}'^{0n}\mathbb{T}^G_{0n} - \mathbf{g}'^{0n}\mathbb{T}^Q_{0n} - \mu^2\mathbf{g}'^{0n}\mathbf{g}'_{0n} + ...\right)\right]\\ &= \int \mathcal{D}\mathbf{g}_{0n} \exp\left[-i\int d^{11}x\sqrt{\mathbf{g}_{11}(\mathbf{g}_{0n})}\left(\mathcal{L}_0(\mathbf{g}_{0n}) + \mathbf{h}^{0n}\left(\mathbb{R}_{0n} - \frac{1}{2}\mathbf{g}_{0n}\mathbf{R} - \mathbb{T}^G_{0n} - \mathbb{T}^Q_{0n}\right) + ...\right)\right],\end{aligned} \tag{3.101}$$

where in the second line we have expanded to first order in $\mathbf{h}_{0n}$ to express the factor involving Ricci tensor. We have also inserted a small mass to the graviton so as to integrate this out. Note that $\mathbf{g}_{0n}$ does show up with a coefficient $\mathbf{h}^{0n}$, and we have defined:

$$\mathbb{R}_{0n} \equiv \hat{\mathbf{R}}_{0n}(\mathbf{g}_{0n}) + \mathbf{R}_{0n}, \tag{3.102}$$

where only $\hat{\mathbf{R}}_{0n}$ is a function of $\mathbf{g}_{0n}$. Therefore, neither $\mathbf{R}_{0n}$ nor the energy-momentum tensors are functions of $\mathbf{g}_{0n}$. For the latter we could have divided into a piece that depends on $\hat{\mathbf{R}}_{0n}$, i.e indirectly on $\mathbf{g}_{0n}$, and a piece independent of $\mathbf{g}_{0n}$; but since we are eventually going to integrate out $\mathbf{g}_{0n}$, their presence or absence will not change much the generic quantum term (2.78) or (2.94). Finally, the Lagrangian $\mathcal{L}_0(\mathbf{g}_{0n})$ is defined as:

$$\mathcal{L}_0(\mathbf{g}_{0n}) = \mathbf{R}^{(11)}(\mathbf{g}_{0n}) - \mathbf{g}^{0n}\mathbb{T}^G_{0n} - \mathbf{g}^{0n}\mathbb{T}^Q_{0n} - \mu^2\mathbf{g}^{0n}\mathbf{g}_{0n}. \tag{3.103}$$

The above equation, (3.101), combined with (3.103), is a form of the Schwinger-Dyson equation for our case, but is presented in a slightly different way because we want to integrate out $\mathbf{g}_{0n}$. Doing this leads us to the following two conclusions. One, we recover the terms with polynomial powers of $\left(\mathbb{T}^G_{0n}\right)^2$ and $\left(\mathbb{T}^Q_{0n}\right)^2$ (along-with the mixed terms). These are of course contained in (2.78) and (2.94) according to (2.107): a consequence of the semi-group structure of the system. Two, $\mathbf{g}_{0n}$ appears inside the bracket multiplying $\mathbf{h}^{0n}$. This means, once we integrate out $\mathbf{g}_{0n}$, there would be terms with powers of $\mathbf{h}^{0n}$ accompanied with the combination of the Ricci curvature $\mathbf{R}_{0n}$ and the energy-momentum tensors $\mathbb{T}^G_{0n}$ and $\mathbb{T}^Q_{0n}$, *without* the $\mathbf{g}_{0n}\mathbf{R}$ piece. We also expect the effective action S_{eff} to be independent of any arbitrary parameter like $\mathbf{h}^{0n}$. Combining everything together it appears that if we demand at "on-shell" the following two conditions: $\mathbf{g}_{0n} = 0$ and

$$\mathbf{R}_{0n} - \mathbb{T}^G_{0n} - \mathbb{T}^Q_{0n} = 0, \tag{3.104}$$

then there is a well defined effective action S_{eff}, with the latter reproducing the expected EOM for the cross-term. Notice that none of the terms in (3.104) can depend on $\mathbf{g}_{0n}$, because of the procedure that we have adopted to derive the equations and the effective action. In retrospect this is of course consistent with what we have been considering so far.

The short analysis presented above reveals one crucial fact: we can allow energy-momentum tensors of the form $\mathbb{T}^G_{0n}$ and $\mathbb{T}^Q_{0n}$ even if cross-components of the metric, like $\mathbf{g}_{0n}$, do not appear in the background. The point is that it is not necessary for certain components of the metric (or G-flux) to physically appear as long as they appear inside quantum *loops*. The Wilsonian way of course guarantees this by allowing a small mass to these components that would facilitate their *off-shell* appearances. Such a line of thought does lead to consistent picture as we saw from all our earlier analysis, however one question still lingers: how do we actually determine the g_s scalings for these cross-component energy-momentum tensors?

This can be answered using a simple trick. For concreteness let us consider the quantum series (2.94) meant for the volume preserving case (2.2). Before we contract this completely with inverse metric components, let us insert a function t_{0n} with the property $t^{0m}t_{0n} = \delta^m_n$ as $(t_{0n})^{l_{39}}$ in (2.94), where l_{39} can take values $(0, 1)$ only. We can now put back all the inverse metric components to make it Lorentz invariant. We can also assume that t_{0n} has no g_s scaling, i.e it scales as g_s^0. The g_s scaling of the modified (2.94) now becomes $\hat{\theta}'_k$ where:

$$\hat{\theta}'_k \equiv \theta'_k + \left(\frac{5}{3} - \frac{\gamma}{2}\right) l_{39}, \tag{3.105}$$

with θ'_k as defined in (2.97) and we have inserted γ just for the completeness sake (as γ should have been inserted with θ_k in (2.86)). To extract an expression with one free 0 index and one free n index, to account for the energy-momentum tensor $\mathbb{T}^Q_{0n}$, all we need is to *remove* one $\mathbf{g}^{00}$ and one $\mathbf{g}^{nn}$ metric components to create two free indices anywhere inside the modified quantum terms (2.94). This will change the g_s scaling from (3.105) to $\widetilde{\theta}'_k$, where:

$$\widetilde{\theta}'_k \equiv \theta'_k + \left(\frac{5}{3} - \frac{\gamma}{2}\right) l_{39} - \frac{10}{3}, \tag{3.106}$$

with θ'_k as in (2.97). If we replace θ'_k in (3.106) by θ_k of (2.86), we get the result for (2.8). Finally, contracting the resulting expression with t^{0m} will give us the required expression for $\mathbb{T}^Q_{0m}$ with g_s scaling as in (3.106) and $l_{39} = 1$. Clearly for vanishing γ, the g_s scaling is $\theta'_k - 5/3$, whereas for $\gamma = 2$ we get $\theta_k - 8/3$ representing the two cases (2.2) and (2.8) respectively. Our g_s scaling for the quantum terms in (3.94) for (2.2) should be interpreted in the following way:

$$g_s^{\theta'_k - 5/3} \equiv g_s^0, \;\; g_s^{1/3}, \;\; g_s^{2/3}, \;\; g_s, ..., \tag{3.107}$$

so that the zeroth order terms are classified by $\theta'_k = 5/3$ in (2.97). Similarly for (2.8), the zeroth order terms are classified by $\theta_k = 8/3$ in (2.86). As we saw above, the latter do contribute so that $\mathbb{C}^{(0,0)}_{0n}$ are classified as above for the case (2.8). However for the volume preserving case, i.e (2.2), the first non-trivial contributions come from $\mathbb{C}^{(1/2,0)}_{0n}$ and $\mathbb{C}^{(1/2,0)}_{0\alpha}$. They are classified by $\theta'_k = 2$ in (2.97). In a similar vein one could analyze the $\mathbb{G}_{\alpha m}$ equations for the volume preserving case (2.2).

de Sitter vacua from the quantum constraints

In the above sections we managed to assimilate all the possible quantum corrected EOMs that can occur in the system. Many subtleties regarding the distribution of the quantum terms were noticed, but in the end the arrangement of the these terms reflected a certain level of consistencies that were expected in set-up like ours and also of our earlier works [14, 15] with one noticeable difference: the quantum terms could now be precisely classified using the scaling (2.97) for (2.2) and (2.84) for (2.8). Thus the issue of the existence of effective field theories could now be answered in the affirmative provided the EOMs themselves have solutions. In the following therefore we would like to analyze this for the two cases in question.

Case 1: $F_1(t)$ and $F_2(t)$ satisfying the volume-preserving condition (2.2)

We start by analyzing the volume-preserving case (2.2), by first taking the traces of all the EOMs to lowest order in g_s and try to find if certain consistency condition(s) could be generated. Our first equation is for the (m, n) directions. In the zeroth order in g_s, the equation is given in (3.15), which is constructed using un-warped metric and G-flux components. Taking a trace of this equation yields:

$$R^{(4)} - 2R - 24H^4\Lambda = [\mathbb{C}^m_m]^{(0,0)} - \frac{1}{4H^4}\Big(\mathcal{G}^{(3/2)}_{m\alpha ab}\mathcal{G}^{(3/2)m\alpha ab} + \mathcal{G}^{(3/2)}_{\alpha\beta ab}\mathcal{G}^{(3/2)\alpha\beta ab}\Big), \tag{3.108}$$

where $R^{(4)}$ is the Ricci scalar for the four-dimensional manifold $\mathcal{M}_4$ and R remains the Ricci scalar of the full six-dimensional base $\mathcal{M}_4 \times \mathcal{M}_2$. As mentioned above, both are computed using un-warped metric components, including the traces unless mentioned otherwise.

The quantum terms $[\mathbb{C}^m_m]^{(0,0)}$ are classified by $\theta'_k = 2/3$ in (2.97) and one may easily see that with such a small value for θ'_k there are only a few classical terms mostly made of G-fluxes. The classical terms can only renormalize the existing terms that we have from the energy-momentum tensor for the G-fluxes. In fact an exactly similar story unfolds for the EOM along the (α, β) directions. Taking the trace of (3.42), written for the zeroth order in g_s, we get:

$$R^{(2)} - R - 12\Lambda H^4 = [\mathbb{C}_\alpha^\alpha]^{(0,0)} + \frac{1}{8H^4}\Big(\mathcal{G}^{(3/2)}_{\alpha\beta ab}\mathcal{G}^{(3/2)\alpha\beta ab} - \mathcal{G}^{(3/2)}_{mnab}\mathcal{G}^{(3/2)mnab}\Big), \tag{3.109}$$

where $R^{(2)}$ is the un-warped curvature of $\mathcal{M}_2$, and since $\mathcal{M}_2$ is a non-Kähler two-dimensional space, this does not vanish. The quantum terms $[\mathbb{C}_\alpha^\alpha]^{(0,0)}$ are again classified by $\theta'_k = 2/3$ in (2.97), and therefore can at best renormalize the existing classical terms. Compared to (3.108), the relative factors, signs and G-flux components differ but the main message of (3.109) remains similar to (3.108).

The next set of equations are a bit different from what we had so far and the differences appear mostly from the scalings of the quantum terms. For example looking at the EOM for the (a, b) direction, i.e. (3.59) appearing to order g_s^2 instead of the expected zeroth order in g_s, and taking the trace, we get:

$$R + 18\Lambda H^4 = -\,[\mathbb{C}_a^a]^{(3,0)} - \frac{1}{8H^4}\Big(2\mathcal{G}^{(3/2)}_{m\alpha ab}\mathcal{G}^{(3/2)m\alpha ab} + \mathcal{G}^{(3/2)}_{mnab}\mathcal{G}^{(3/2)mnab} + \mathcal{G}^{(3/2)}_{\alpha\beta ab}\mathcal{G}^{(3/2)\alpha\beta ab}\Big), \tag{3.110}$$

where now we see that the quantum terms have different modings than what we had in (3.108) and (3.109). However they are still classified by $\theta'_k = 2/3$ in (2.97), and therefore can only renormalize the existing classical terms. This shared similarities between the three traces, (3.108), (3.109) and (3.110), *do not* imply that the quantum effects are relatively unimportant because we haven't yet analyzed the space-time EOMs. All the EOMs are inter-related so conclusions based on analyzing only parts of the story typically fail to reveal the true picture.

This becomes clear once we look at the space-time EOMs. Looking at the zeroth order in g_s in (3.77) we notice that the quantum effects now play an important role. To facilitate discussion, let us quote (3.77) again:

$$\begin{aligned} &6\Lambda + \frac{R}{H^4} - \frac{\Box H^4}{H^8} + \left[\mathbb{C}_i^i\right]^{(0,0)} - \frac{2\kappa^2 T_2 n_b}{H^8\sqrt{g_6}}\delta^8(y-Y) \\ &= \frac{1}{8H^8}\Big(\mathcal{G}^{(3/2)}_{mnab}\mathcal{G}^{(3/2)mnab} + 2\mathcal{G}^{(3/2)}_{m\alpha ab}\mathcal{G}^{(3/2)m\alpha ab} + \mathcal{G}^{(3/2)}_{\alpha\beta ab}\mathcal{G}^{(3/2)\alpha\beta ab}\Big), \end{aligned} \tag{3.111}$$

where $\Box$ is now over the full six-dimensional space $\mathcal{M}_4 \times \mathcal{M}_2$, and the quantum terms are classified by $\theta'_k = 8/3$ in (2.97), compared to $\theta'_k = 2/3$ for the three traces above. Such a choice of θ'_k will now allow a large number of terms by choosing various combinations of l_i in (2.94), thus mixing curvature terms with the G-flux components.

All the four equations above shows how the Ricci scalar R may be related to the G-fluxes and the quantum terms. The quantum terms are shown to be classified by choosing appropriate values for θ'_k in (2.97), but there are also non-local contributions to them. Fortunately, in the limit of vanishing (a, b) torus these contributions are negligible so may be avoided in the $g_s \to 0$ limit, i.e in the late time limit. Adding (3.108) and (3.109) we get:

$$\begin{aligned} R + 18H^4\Lambda &= -\frac{1}{2}\,[\mathbb{C}^m_m]^{(0,0)} - \frac{1}{2}\,[\mathbb{C}^\alpha_\alpha]^{(0,0)} \\ &+ \frac{1}{16H^4}\Big(\mathcal{G}^{(3/2)}_{\alpha\beta ab}\mathcal{G}^{(3/2)\alpha\beta ab} + 2\mathcal{G}^{(3/2)}_{m\alpha ab}\mathcal{G}^{(3/2)m\alpha ab} + \mathcal{G}^{(3/2)}_{mnab}\mathcal{G}^{(3/2)mnab}\Big), \end{aligned} \tag{3.112}$$

which, in the absence of the G-flux pieces, would be equivalent to a similar equation in [14] for the time-independent internal space (see eq. (6.4) in [14]). It is reassuring to see the emergence of familiar equations once we resort to the time-independent scenario. The time-dependences therefore not only add new fluxes to the time-independent equations, but also allows us to consider a controlled set of quantum corrections. Interestingly, now looking at (3.110), we notice that the LHS is identical to the LHS of (3.112). In the absence of the G-flux pieces, we could have concluded that the quantum corrections in these two set of equations are related to each other; much like eq. (6.6) of [14]. This is *not* the case now. The quantum corrections along (a, b) directions are not related in a simple way to the sum of the quantum corrections along (m, n) and (α, β) directions. The G-fluxes interfere to make this a bit more involved. We could however add (3.112) and (3.110) to get the following equation:

$$\begin{aligned} R + 18H^4\Lambda &= -\frac{1}{2}\,[\mathbb{C}^a_a]^{(3,0)} - \frac{1}{4}\,[\mathbb{C}^m_m]^{(0,0)} - \frac{1}{4}\,[\mathbb{C}^\alpha_\alpha]^{(0,0)} \\ &- \frac{1}{32H^4}\Big(\mathcal{G}^{(3/2)}_{\alpha\beta ab}\mathcal{G}^{(3/2)\alpha\beta ab} + 2\mathcal{G}^{(3/2)}_{m\alpha ab}\mathcal{G}^{(3/2)m\alpha ab} + \mathcal{G}^{(3/2)}_{mnab}\mathcal{G}^{(3/2)mnab}\Big), \end{aligned} \tag{3.113}$$

combining all the quantum terms and the G-fluxes together. Note the difference in the moding of the (a, b) quantum terms, but as mentioned earlier, they are all classified by $\theta'_k = 2/3$ in (2.97). Since $\theta'_k = 2/3$ is almost classical (one may easily see by choosing the appropriate l_i in (2.94)), all they do here is to renormalize the existing classical pieces without introducing any higher order corrections. This was clearly not the case in [14, 15], where $\theta'_0 = 2/3$ in (2.98) would have led to an infinite number of quantum terms without any visible hierarchies. Switching on time-dependences have completely changed the scenario. On the other hand, subtracting (3.112) from (3.110), we get:

$$[\mathbb{C}^m_m]^{(0,0)} + [\mathbb{C}^\alpha_\alpha]^{(0,0)} - 2\,[\mathbb{C}^a_a]^{(3,0)} = \frac{3}{8H^4}\Big(\mathcal{G}^{(3/2)}_{\alpha\beta ab}\mathcal{G}^{(3/2)\alpha\beta ab} + 2\mathcal{G}^{(3/2)}_{m\alpha ab}\mathcal{G}^{(3/2)m\alpha ab} + \mathcal{G}^{(3/2)}_{mnab}\mathcal{G}^{(3/2)mnab}\Big), \tag{3.114}$$

which instead would directly connect the quantum terms to the fluxes. Such an equation immediately confirms the fact that the three quantum terms in (3.113) or (3.114) only renormalize the existing classical data, without introducing any higher order terms. As mentioned above, this is consistent with the fact that they are classified by $\theta'_k = 2/3$ in (2.97).

We can now use the curvature scalar, defined in terms of the quantum terms for the eight-dimensional manifold and the G-fluxes in (3.113), and plug this (3.111).

Doing this yields:

$$
\begin{aligned}
-\Box H^4 &= 12\Lambda H^8 + \frac{5}{32}\Big(\mathcal{G}^{(3/2)}_{mnab}\mathcal{G}^{(3/2)mnab} + 2\mathcal{G}^{(3/2)}_{m\alpha ab}\mathcal{G}^{(3/2)m\alpha ab} + \mathcal{G}^{(3/2)}_{\alpha\beta ab}\mathcal{G}^{(3/2)\alpha\beta ab}\Big) \\
&+ \frac{2\kappa^2 T_2 n_b}{\sqrt{g_6}}\delta^6(y-Y) + \left(\frac{1}{2}\left[\mathbb{C}^a_a\right]^{(3,0)} + \frac{1}{4}\left[\mathbb{C}^m_m\right]^{(0,0)} + \frac{1}{4}\left[\mathbb{C}^\alpha_\alpha\right]^{(0,0)} - H^4\left[\mathbb{C}^i_i\right]^{(0,0)}\right)H^4,
\end{aligned}
\tag{3.115}
$$

where we have made one change: the M2-branes are now restricted to move on the six-dimensional base $\mathcal{M}_4 \times \mathcal{M}_2$ only as this will facilitate as easier interpretation in the type IIB side. Note also that the only *minus* sign appears from the quantum terms in the space-time directions. This equation is somewhat similar to eq. (6.8) in [14]. The differences being in (a) the relative factors, (b) the choice of the G-flux components and (c) the dependence on the full eight-dimensional coordinates instead of only on the six-dimensional base here; but both equations share one similarity regarding the appearance of the relative minus sign. This is crucial because integrating (3.115) over the six-dimensional base gives us:

$$
\begin{aligned}
&12\Lambda\int d^6y\sqrt{g_6}H^8 + \frac{5}{32}\int d^6y\sqrt{g_6}\Big(\mathcal{G}^{(3/2)}_{mnab}\mathcal{G}^{(3/2)mnab} + 2\mathcal{G}^{(3/2)}_{m\alpha ab}\mathcal{G}^{(3/2)m\alpha ab} + \mathcal{G}^{(3/2)}_{mnab}\mathcal{G}^{(3/2)mnab}\Big) \\
&+2\kappa^2 T_2 n_b + \int d^6y\sqrt{g_6}\left(\frac{1}{2}\left[\mathbb{C}^a_a\right]^{(3,0)} + \frac{1}{4}\left[\mathbb{C}^m_m\right]^{(0,0)} + \frac{1}{4}\left[\mathbb{C}^\alpha_\alpha\right]^{(0,0)} - H^4\left[\mathbb{C}^i_i\right]^{(0,0)}\right)H^4 = 0,
\end{aligned}
\tag{3.116}
$$

which should be compared to eq. (6.10) of [14]. The zero on the RHS appears from integrating $\Box H^4$ over the compact base $\mathcal{M}_4 \times \mathcal{M}_2$, and since $H^4(y) \equiv h(y)$ is a smooth function, the integral vanishes. The smoothness of $H^4(y)$ is guaranteed from the series of quantum corrections appearing in (3.115). Clearly, in the absence of the quantum pieces, the system has no solution because the integral involves only positive definite functions and therefore the consistency will demand vanishing fluxes and vanishing Λ. Interestingly *negative* Λ is allowed even if the quantum terms are absent, implying both Minkowski and AdS spaces may be realized in a set-up like ours. The recent swampland conjectures concerning AdS spaces may be overcome by introducing back the quantum corrections, but we don't want to discuss this here. In the presence of the quantum pieces, the consistency condition here differs in a crucial way with the one presented in [14]. The quantum terms in [14] are classified by $\theta'_0 = 2/3$ and $\theta'_0 = 8/3$ for the internal and the space-time respectively with θ'_0 defined in (2.98). These have infinite number of solutions for both cases, implying that an expression like eq. (6.10) in [14] does not have any solution at all and is in the swampland. However now the scenario has changed. The internal and the space-time quantum terms are now classified by $\theta'_k = 2/3$ and $\theta'_k = 8/3$ respectively with θ'_k defined as in (2.97). These have *finite* number of solutions in both cases, and in fact the internal space quantum terms, as we saw earlier, do not contribute much. This means the actual higher order quantum terms appear only from the space-time part, i.e from the $[\mathbb{C}^i_i]^{(0,0)}$ piece in (3.116). These quantum terms appear with an overall *minus* sign in (3.116), and therefore if we can use only the dominant positive contributions from $[\mathbb{C}^i_i]^{(0,0)}$ then surprisingly solutions would exist where there were none before!

We use the details gathered so far to determine the metric of the internal space in terms of the fluxes and the quantum corrections. Let us start by expressing the un-warped metric g_{mn} using (3.19) in the following way:

$$g_{mn} = \frac{3}{58}\left[\frac{\mathbb{C}_{mn}^{(1/2,0)} + \frac{1}{4H^4}\sum_{\{k_i\}} C_{k_2}\left(\widetilde{C}_{k_1}\mathcal{G}^{(k_3)}_{mlab}\mathcal{G}_n^{(k_4)lab} + C_{k_1}\mathcal{G}^{(k_3)}_{m\alpha ab}\mathcal{G}_n^{(k_4)\alpha ab}\right)}{\mathbb{A}(y) + \frac{3}{928H^4}\sum_{\{k_i\}} C_{k_2}\left(\widetilde{C}_{k_1}\mathcal{G}^{(k_3)}_{pkab}\mathcal{G}^{(k_4)pkab} + 2C_{k_1}\mathcal{G}^{(k_3)}_{p\alpha ab}\mathcal{G}^{(k_4)p\alpha ab}\right)}\right], \tag{3.117}$$

where $\mathbb{A}(y)$ is defined in (3.20) and k_i satisfy $\sum_i k_i = 7/2$, with the constraint that $(k_3, k_4) \geq (3/2, 3/2)$. The C_k and the $\widetilde{C}_k$ coefficients can be determined using the cross-term EOMs as we saw in section 3.1.1. Finally, the quantum terms appearing above are governed by $\theta'_k = 1$ in (2.97), i.e by (3.21). For such small values of θ'_k, the quantum terms are mostly expressed as powers of G-flux components instead of curvature tensors as may be easily seen from (3.21). The curvature tensors appearing here only renormalizes the classical terms. This means the RHS of (3.117) is expressed mostly by powers of G-fluxes and the $(C_k, \widetilde{C}_k)$ coefficients (the latter are also determined by fluxes for small values of k). In fact a somewhat similar story repeats for the metric component $g_{\alpha\beta}$ also, which now takes the following form:

$$g_{\alpha\beta} = \frac{9}{2}\left[\frac{\mathbb{C}_{\alpha\beta}^{(1/2,0)} + \frac{1}{4H^4}\sum_{\{k_i\}} C_{k_2}\left(\widetilde{C}_{k_1}\mathcal{G}^{(k_3)}_{\alpha lab}\mathcal{G}_\beta^{(k_4)lab} + C_{k_1}\mathcal{G}^{(k_3)}_{\alpha\gamma ab}\mathcal{G}_\beta^{(k_4)\gamma ab}\right)}{\mathbb{C}(y) + \frac{9}{32H^4}\sum_{\{k_i\}} C_{k_2}\left(2\widetilde{C}_{k_1}\mathcal{G}^{(k_3)}_{\gamma lab}\mathcal{G}^{(k_4)\gamma lab} + C_{k_1}\mathcal{G}^{(k_3)}_{\gamma\eta ab}\mathcal{G}^{(k_4)\gamma\eta ab} + \hat{C}_{k_{1,2}}\mathcal{G}^{(k_3)}_{mnab}\mathcal{G}^{(k_4)mnab}\right)}\right], \tag{3.118}$$

as gathered from (3.43); where $\mathbb{C}(y)$ defined as in (3.44) and $\hat{C}_{k_{1,2}} \equiv \widetilde{C}_{k_1}\widetilde{C}_{k_2}/C_{k_2}$ with k_i satisfying as before $\sum_i k_i = 7/2$ with the standard constraint $(k_3, k_4) \geq (3/2, 3/2)$. The quantum terms are again classified by $\theta'_k = 1$ in (2.97), and therefore are most populated by powers of G-flux components. Both the metric components, (3.117) and (3.118) are non-Kähler, but the un-warped metric along the (a, b) directions is flat as expected[11]. Thus solving for $h(y)$ from (3.115), and $(C_k, \widetilde{C}_k)$ from the cross-term EOMs in section 3.1.1 (see for example (3.96) and (3.80)), we can pretty much

[11] We can also make some general observations regarding the *sign* of the internal curvature term R from (3.110) and (3.112). Let us first assume that the quantum terms in (3.110) and (3.112) are zero. Then the only solution is with vanishing flux components $\mathcal{G}^{(3/2)}_{MNab}$ and $R = -18\Lambda H^4$. It is also clear from (3.116), for vanishing quantum terms and vanishing fluxes, $\Lambda = 0$ and therefore $R = 0$. When the fluxes vanish, but all the quantum terms are non-zero, then the internal quantum terms must satisfy the relation (3.114) with zero on the RHS. The consistency condition (3.116) allows positive Λ if the space-time quantum terms $[\mathbb{C}_i^i]^{(0,0)}$ dominates over all others terms. In this case $\Lambda > 0$ is allowed. However if the internal space quantum terms vanish (which still allows positive Λ in (3.116)), then from (3.110) and (3.112) the internal curvature scalar has to be *negative* i.e $R = -18|\Lambda|H^4$ with the warp-factor $H(y)$ satisfying:

$$\square H^4 = \left([\mathbb{C}_i^i]^{(0,0)} - 12|\Lambda| - \frac{2\kappa^2 T_2 n_b}{H^8\sqrt{g_6}}\delta^8(y - Y)\right)H^4$$

where n_b is the number of M2-branes, T_2 is the tension of a M2-brane and g_6 is the determinant of the six-dimensional internal metric. The six-dimensional base of the eight-manifold now becomes a non-Kähler space with a negative Ricci scalar. Clearly for *vanishing* $[\mathbb{C}_i^i]^{(0,0)}$, and vanishing fluxes, Λ can only be negative from (3.116) if the internal quantum terms are all positive definite. In this case either $R < 0$ or $R < 18H^4|\Lambda|$. If the internal quantum terms are all negative definite, then there can be $\Lambda > 0$ for vanishing fluxes and vanishing space-time quantum terms. In this case $R > 0$ or $R > -18H^4|\Lambda|$. In the same vein, other possible choices can be entertained. It would also be interesting to compare our results with [35].

determine the full background data provided information about the G-flux components are provided. The latter will require us to solve the flux EOMs, that we shall discuss soon.

The miracle that has happened here has its root in the time-dependence of the G-flux components and the internal space. The time dependences of the G-fluxes are responsible for changing the relative signs of the (l_{36}, l_{37}, l_{38}) terms in (2.98) to the k-dependent scaling (2.97). On the other hand, the time-dependences of the internal space i.e the existence of the $F_i(t)$ factors are related to the quantum terms. The quantum terms are classified by θ'_k in (2.97), thus bringing us back full-circle. This interdependency of the temporal behavior of fluxes and the metric components is solely responsible for the generation of a four-dimensional positive curvature space-time in the type IIB side with de Sitter isometries. Switching off time-dependences (or the quantum terms) will immediately ruin the picture and drag us back to the swampland.

Case 2: $F_1(t)$ and $F_2(t)$ satisfying the fluctuation condition (2.8)

Our procedure to study the scenario corresponding to $\gamma > 0$ will essentially be the same: we will take the traces of the various EOMs and from there inquire whether solutions could be constructed. We first take the trace of the EOM along the (m, n) directions. The EOM is given in (3.34) and is defined at the zeroth order in g_s. The trace yields:

$$R = \frac{1}{8H^4}\mathcal{G}^{(9/2)}_{\alpha\beta ab}\mathcal{G}^{(9/2)\alpha\beta ab} - \frac{1}{2}\left[\mathbb{C}^m_m\right]^{(0,0)} - 6\Lambda H^4, \tag{3.119}$$

where we have used the fact that the un-warped Ricci scalar of $\mathcal{M}_4$ vanishes, which in turn appears from looking at (3.51). In fact this led us to choose the un-warped geometry of the six-dimensional base to be that of $K3 \times \mathbf{T}^2$, implying that the cosmological constant Λ in this set-up may be expressed as:

$$\Lambda = \frac{1}{48H^8}\mathcal{G}^{(9/2)}_{\alpha\beta ab}\mathcal{G}^{(9/2)\alpha\beta ab} - \frac{1}{12H^4}\left[\mathbb{C}^m_m\right]^{(0,0)}, \tag{3.120}$$

which at the face value doesn't contradict anything because the quantum terms are classified by $\theta_k = 2/3$ in (2.86) for $\gamma = 2$, and this allows us to choose $l_{28} = 2$ renormalizing the classical flux piece such that the RHS of (3.120) becomes a positive constant. However this puts a tighter constraint on the behavior of the G-flux component $\mathcal{G}^{(9/2)}_{\alpha\beta ab}$. An alternative to this would be to take $R^{(2)} \neq 0$ in (3.51). This however would be a bit difficult to argue because (3.51) is a source-free equation (see also footnote 5). It is also interesting to note that (3.53) provides a relation similar to (3.120), namely:

$$\Lambda = -\frac{1}{64H^8}\left(\mathcal{G}^{(9/2)}_{\alpha\beta ab}\right)^2 - \frac{1}{8H^4}\left[\mathbb{C}^\alpha_\alpha\right]^{(3,0)}, \tag{3.121}$$

which again shows that there has to be a delicate cancellation to allow for the cosmological constant term to appear from the RHS. Of course again the quantum terms are classified by $\theta_k = 2/3$ in (2.86) so we haven't faced a contradiction yet. However the fact that first term in (3.121) is negative definite shows that the quantum terms have to be negative definite also to reproduce the positive Λ from RHS.

We will not worry about whether (3.121) and (3.120) could be mutually consistent, and instead proceed with analyzing the other equations of the system.

Our next equation is the equation along the (a, b) directions. There are some subtleties in the construction of the EOMs, that we explained earlier, and after the dust settles, the EOM to order g_s^2 (which is the lowest order now) is given by (3.65). Taking the trace leads to:

$$\Lambda = -\frac{1}{144H^8}\left(\mathcal{G}^{(9/2)}_{\alpha\beta ab}\right)^2 - \frac{1}{18H^4}\left[\mathbb{C}^a_a\right]^{(3,0)}, \tag{3.122}$$

which is an equation similar to (3.121) above. The concern associated with this equation remains the same as before as the quantum terms are classified by $\theta_k = 2/3$ in (2.86). We should then go to the space-time EOM to see if any of our concerns could be lifted. As we saw before, there are two space-time EOMs given by (3.86) and (3.91), out of which (3.91) will be the correct EOM once we gather all the constraints from flux EOM in section 3.1.2. For the time being there is no way to choose (3.86) over (3.91), so we shall put both to test now and see what comes out from our exercise.

We consider starting with the *wrong* EOM, i.e (3.86). In this case the story, like (3.122), also repeats for the EOM along the space-time direction as may be seen from (3.86), and we reproduce it here again for completeness:

$$\Lambda = \frac{1}{32H^8}\mathcal{G}^{(9/2)}_{\alpha\beta ab}\mathcal{G}^{(9/2)\alpha\beta ab} - \frac{1}{4}\left[\mathbb{C}^i_i\right]^{(0,0)}. \tag{3.123}$$

We now face a possible conundrum. The quantum terms are classified by $\theta_k = 8/3$ in (2.86) and therefore has many more terms compared to the earlier cases where the quantum terms are classified by $\theta_k = 2/3$. None of these terms are as simple as the classical flux term appearing in (3.123), and therefore to reproduce the constant Λ factor, there needs to be strong constraints on all the quantum terms classified by $\theta_k = 8/3$ in (2.86).

There is also no integral constraint like the one in (3.116) for the volume preserving case (2.2) because the warp-factor $h(y)$ is harmonic from (3.85). Combining (3.121) and (3.123) yields:

$$\Lambda = -\frac{1}{12H^4}\Big(\left[\mathbb{C}^\alpha_\alpha\right]^{(3,0)} + H^4\left[\mathbb{C}^i_i\right]^{(0,0)}\Big), \tag{3.124}$$

which relates Λ directly to the quantum terms. Since $\Lambda > 0$, the quantum terms or their sum have to be a negative definite integer. Additionally, they have to be proportional to H^4 (at least from the first term in (3.124)) if (3.123) has to make sense. Also since the square of the flux piece appearing in the above equations is a positive quantity, we expect:

$$H^4\left[\mathbb{C}^i_i\right]^{(0,0)} > \frac{1}{3}\left[\mathbb{C}^m_m\right]^{(0,0)} > \frac{2}{9}\left[\mathbb{C}^a_a\right]^{(3,0)} > \frac{1}{2}\left[\mathbb{C}^\alpha_\alpha\right]^{(3,0)}, \tag{3.125}$$

as a possible hierarchy between all the quantum terms classified by appropriate values of θ_k in (2.86). All these lead to some strong constraints that are unclear if they could be consistently satisfied. Let us then ask whether the correct EOM, namely (3.91), could ease some of the tension here. Combining (3.90) with (3.91), we get:

$$\Lambda = \frac{1}{32H^8}\mathcal{G}^{(9/2)}_{\alpha\beta ab}\mathcal{G}^{(9/2)\alpha\beta ab} - \frac{1}{4}\left(\left[\mathbb{C}^i_i\right]^{(3,0)} - \left[\mathbb{C}^i_i\right]^{(0,0)}\right), \tag{3.126}$$

which is similar to (3.123), so unfortunately this is not going to alleviate any of the issues that we faced above. The only difference between (3.123) and (3.126) is the quantum terms, so (3.125) would remain as before with the sole replacement:

$$\left[\mathbb{C}^i_i\right]^{(0,0)} \longrightarrow \left[\mathbb{C}^i_i\right]^{(3,0)} - \left[\mathbb{C}^i_i\right]^{(0,0)}, \tag{3.127}$$

leading to same sort of strong constraints as before. Furthermore switching on γ leads to an unnatural derivative constraint that is harder to justify. The absence of M2-branes, due to the vanishing Euler characteristics, is also an issue because M2-branes dualize to D3-branes in the type IIB side and account for the color degrees of freedom. Additionally, the late-time behavior, as may be inferred from (3.28), shows that:

$$F_1(t) \to 0, \quad F_2(t) \to 1, \tag{3.128}$$

thus the subspace $\mathcal{M}_2$ shrinks to zero size leading to singularities at late time. However since we are never at $g_s = 0$ point, the quantum EOMs do not show any signs of complications at this stage. Thus although none of the arguments presented here is damning enough to discard the model with non-zero γ, the issues presented here nonetheless show that the late time physics with a four-dimensional de Sitter space-time, i.e with (2.2), is a preferable scenario over the ones with time-varying Newton constants. In **Table 3.1** we summarize the differences between the two choices (2.2) and (2.8).

3.1.2 Analysis of the G-flux quantizations and anomaly cancellations

The study of all the Einstein's equation performed above revealed a delicate interconnection between the metric components, the quantum terms and the G-flux components at every order in the g_s expansions. Flux EOMs would introduce yet another layer of interconnections and constraints. We would like to specifically concentrate on two aspects of this: flux quantization and anomaly cancellation. In the process we shall also be able to tie up few of the loose ends from the earlier sections.

Bianchi identities and flux quantizations

Flux quantization is intimately connected to the Bianchi identity. In the time-independent case this was analyzed in details by [33]. Let us first elaborate this using the dual forms $\mathbf{G}_7$ discussed in section 2.1.2. In the absence of the quantum terms, i.e in the absence of $\mathbb{Z}_7$ from (2.140), the M-theory action using the dual variables may be

Time-independent Newton's constant	Time-dependent Newton's constant
No derivative constraint on $\mathcal{M}_4 \times \mathcal{M}_2$	Derivative constraint on $\mathcal{M}_2$
$\mathcal{M}_4$: non-Kähler	$\mathcal{M}_4$: conformally $K3$
$\mathcal{M}_2$: non-Kähler	$\mathcal{M}_2$: conformally $\mathbf{T}^2$
$\chi_8 \neq 0$	$\chi_8 = 0$
Allows static and dynamical M2-branes	Only dynamical M2-branes allowed
No late time singularities	Late time singularities
G-flux components with $k \geq \frac{3}{2}$	G-flux components with $k \geq \frac{9}{2}$

TABLE 3.1: The key differences between backgrounds with time-independent Newton's constant coming from (2.2) and time-dependent Newton's constant coming from (2.8). The Euler characteristics of the eight-manifold (2.4) is denoted by χ_8. The case with dynamical membranes will be discussed in subsection 3.1.2.

written as:

$$\mathbb{S}_{11} \equiv c_1 \int \mathbf{G}_7 \wedge *_{11}\mathbf{G}_7 + N \int \mathbf{C}_6 \wedge \Lambda_5 + c_2 \int \mathbf{C}_6 \wedge d\hat{\mathbb{Z}}_4, \tag{3.129}$$

where N represents the number of M5-branes, c_i are constants that are defined in terms of certain powers of M_p that may be easily specified[12], Λ_5 is a localized five-form that captures the singularities of the M5-branes, the Hodge star is with respect to the warped eleven-dimensional metric and $\mathbf{C}_6$ appears from defining $\mathbf{G}_7 = d\mathbf{C}_6 +$ ... where the dotted terms appears from M2 and M5-branes in appropriate ways. The EOM for $\mathbf{C}_6$ turns out to be:

$$d *_{11} \mathbf{G}_7 = \frac{1}{c_1}\left(N\Lambda_5 + c_2 d\hat{\mathbb{Z}}_4\right) \equiv d\mathbf{G}_4, \tag{3.130}$$

where on the RHS we expressed the equation in terms of the four-form $\mathbf{G}_4$. The above equation represents the Bianchi identity in the absence of any extra contributions from the quantum terms. Integrating the above equation over a five-manifold $\mathbf{\Sigma}_5$ with boundary $\mathbf{\Sigma}_4 = \partial\mathbf{\Sigma}_5$, we get:

$$c_1 \int_{\mathbf{\Sigma}_4} \mathbf{G}_4 = N + c_2 \int_{\mathbf{\Sigma}_4} \hat{\mathbb{Z}}_4, \tag{3.131}$$

where the RHS is expressed in terms of N, the number of *static* M5-branes, and an integral of a four-form over the four-manifold $\mathbf{\Sigma}_4$. In deriving the above equation we have assumed that the integral of Λ_5 over the five-manifold $\mathbf{\Sigma}_5$ is identity. Now

[12]For example $c_1 = M_p^9$ and $c_2 = M_p^6$, but the term with c_2 will involve other powers of M_p.

defining:

$$c_1 = \frac{1}{2\pi}, \quad c_2 = -1, \quad \hat{\mathbb{Z}}_4 = \frac{1}{16\pi^2}\left(\text{tr}\ \mathbb{F}\wedge\mathbb{F} - \frac{1}{2}\text{tr}\ \mathbb{R}\wedge\mathbb{R}\right), \tag{3.132}$$

where the curvature form $\mathbb{R}$ is as defined in (2.131) and the gauge two-form $\mathbb{F}$ will appear from the flux-form $\mathbb{G}$, also defined in (2.131), once we view the G-flux components as *localized* fluxes (this will be elaborated soon). Therefore combining (3.132) with (3.131), we reproduce the G-flux quantization as expressed in [33].

The question now is what happens when the G-flux components become time-dependent? One easy way out would be to introduce moving M5-branes, as the other pieces appearing in (3.131) are topological. These topological pieces could also have time dependences, but as we saw earlier, the time dependences of the G-flux and metric components are correlated to the quantum corrections which in turn are classified by θ'_k in (2.97) or θ_k in (2.86) for (2.2) and (2.8) respectively. This therefore calls for the quantum corrections to the Bianchi identities themselves.

Introducing the quantum corrections here would imply switching on the Hodge dual of $\mathbb{Z}_7$, which in turn implies switching on the second interaction in (2.140). Implementing this, changes the Bianchi identity from (3.130) to the following:

$$d *_{11} \mathbf{G}_7 = \frac{1}{c_1}\left(N\Lambda_5 + c_2 d\hat{\mathbb{Z}}_4 - c_3 d *_{11} \mathbb{Z}_7\right) \equiv d\mathbf{G}_4, \tag{3.133}$$

where c_3 is yet another constant defined in terms of powers of M_p. As discussed in (2.142), the $\mathbb{Z}_7$ interaction should be understood as coming from (2.139) and is therefore non-topological. It is also not globally defined because it involves metric components on the compact space $\mathcal{M}_4 \times \mathcal{M}_2 \times \frac{\mathbb{T}^2}{\mathcal{G}}$, that can only be defined on patches and we will have to specify a function that can take us from one patch to another. Integrating (3.133) in the same way as above, leaves us with the following flux quantization condition:

$$c_1 \int_{\mathbf{\Sigma}_4} \mathbf{G}_4 = N + c_2 \int_{\mathbf{\Sigma}_4} \hat{\mathbb{Z}}_4 - c_3 \int_{\mathbf{\Sigma}_4} *_{11}\hat{\mathbb{Z}}_7, \tag{3.134}$$

where N, the number of M5-branes, would be affected if Λ_5 itself becomes g_s (i.e time) dependent. Recall that Λ_5 in (3.133) is like a delta function and therefore if there are moving M5-branes, it would pick up g_s dependence. Similarly $\hat{\mathbb{Z}}_4$ would also pick up some g_s dependence. However these are all classical, and what we are looking for is more on the quantum side that could account for all *higher order* g_s dependence of the $\mathbf{G}_4$ flux-components $\mathcal{G}^{(k)}_{MNPQ}$ for all $k \geq 3/2$. To see how this would come about, let us express (3.134) in terms of components in the following way:

$$c_1 \sum_{k\in\frac{\mathbb{Z}}{2}} \int_{\mathbf{\Sigma}_4} \mathcal{G}^{(k)}_{N_8N_9N_{10}N_{11}} \left(\frac{g_s}{H}\right)^{2\Delta k} dy^{N_8}\wedge ....\wedge dy^{N_{11}} = N + c_2 \int_{\mathbf{\Sigma}_4} \hat{\mathbb{Z}}_4 \tag{3.135}$$
$$-c_3 \sum_l \int_{\mathbf{\Sigma}_4} \sqrt{-g_{11}} \left(\mathbb{Z}^{(l)}_7\right)_{N'_1....N'_7} g^{N'_1N_1}......g^{N'_7N_7} \left(\frac{g_s}{H}\right)^{\hat{\theta}_l} \epsilon_{N_1....N_7N_8....N_{11}} dy^{N_8}\wedge ....\wedge dy^{N_{11}},$$

where the metric components are all the *un-warped* metric components (including the determinant), and the epsilon is the Levi-Civita symbol (i.e not a tensor). Note also that although the LHS has been expanded in the standard way as in (3.11), the RHS needs some explanation. According to (2.142), the quantum terms (2.139) are expanded by first choosing a particular component from the set of allowed dual forms and then labelling the remaining pieces as the associated seven-form $\mathbb{Z}_7$ accompanying the dual component. This way $\mathbb{Z}_7$ is uniquely identified once the dual G-flux component is chosen. However we expect the dual G-flux component to have a similar expansion as (3.11), albeit with different g_s scalings. The corresponding $\mathbb{Z}_7$ form will then have the g_s scalings as given in **Table 2.2**. The RHS of the (3.135) therefore represents precisely these scalings that we will simply label as $\hat{\theta}_l$. For every choice of $\mathcal{G}^{(k)}_{MNPQ}$ on the LHS, the g_s scalings of the corresponding seven-form $\mathbb{Z}_7^{(l)}$ should match-up[13]. In the following we will do a detailed check of this.

Before delving into this note that if the M5-branes are static, then N will appear with no g_s factor accompanying it in (3.135). Thus if there are no time-neutral G-flux components we cannot allow static M5-branes, although M2-branes can still be allowed[14]. There is however some subtlety that we are hiding under the rug here. Since the $\mathbb{Z}_7$ piece in the Bianchi identity (3.133) should always have g_s dependence, the *static* quantities that can actually appear from the Bianchi identity may be combined as $\mathbb{S}_5$ where:

$$\mathbb{S}_5 \equiv N\Lambda_5 - \frac{c_2}{32\pi^2} d\Big(\mathrm{tr}\, \mathbb{R} \wedge \mathbb{R}\Big), \tag{3.136}$$

where the second term comes from the definition of $\hat{\mathbb{Z}}_4$ in (3.132), and Λ_5 is the localized five-form. The gauge field $\mathbb{F}$ will in general have g_s dependence, but here we will simply put it to zero. Now, clearly if the trace or $\mathbb{R}$ in (3.136) has only g_s dependent terms, then $N = 0$ as $\mathbf{G}_4$ has no g_s independent piece. However if the trace or the curvature form allows a g_s independent piece then we can cancel $\mathbb{S}_5$ locally by identifying Λ_5 with the trace part. The global condition:

$$N = \frac{c_2}{32\pi^2} \int_{\Sigma_4} \mathrm{tr}\, \mathbb{R} \wedge \mathbb{R}, \tag{3.137}$$

over a specific four-cycle $\Sigma_4 \equiv \partial\Sigma_5$ is then automatic. However compared to [33], we now require the integral of the first Pontryagin class to be an integer[15] as we cannot switch on time-independent G-flux components here. Thus time-dependences put some extra constraints that did not exist for the time-independent case. In general, since we will only be concerned about comparing the g_s scalings, N can be effectively taken to zero without altering the flux quantization condition (3.135).

[13] We have been a bit sloppy in defining $\hat{\theta}_l$. The actual g_s scalings of every components of $\mathbb{Z}_7$ may be read from **Table 2.2**. However $\hat{\theta}_l$ will have an additional contribution from $\sqrt{-\mathbf{g}_{11}}$, where the determinant is now expressed in terms of the warped metric components. To avoid all these unnecessary complications we just define $\hat{\theta}_l$ once and for all in (3.135) without worrying too much of its source.

[14] This is a bit more subtle than one would think. Dynamical M2-branes would back-react on the background stirring up corrections to fluxes and the metric. This is however surprisingly tractable, and we will elaborate the story in subsection 3.1.2.

[15] The sign will be determined from the sign of c_2.

There is however no reason to make $c_2 = 0$ because $\hat{\mathbb{Z}}_4$ can have g_s dependences. We will not worry too much about this as we want to match the g_s scalings of the LHS to the g_s scaling of the quantum terms on the RHS of (3.135).

Case 1: $\mathbf{G}_{mnab}$ *component*

We will start by taking $c_2 = 0$ in (3.135) just for simplicity. This will be restored back at the end with appropriate g_s scalings. Such a procedure will help us to compare the LHS and the RHS succinctly. Therefore for a given order in k the matching becomes:

$$c_1 \int_{\mathbf{\Sigma}_4^{(1)}} \mathcal{G}^{(k)}_{mnab} dy^m \wedge .... \wedge dy^b = -c_3 \int_{\mathbf{\Sigma}_4^{(1)}} \sqrt{-g_{11}} \left(\mathbb{Z}_7^{(k)}\right)^{0ijpq\alpha\beta} \epsilon_{0ijpq\alpha\beta mnab} dy^m \wedge ... \wedge dy^b, \tag{3.138}$$

where $\mathbf{\Sigma}_4^{(1)} = \mathcal{C}_2 \times \frac{\mathbb{T}^2}{\mathcal{G}}$, and $\mathcal{C}_2$ is a two-cycle in $\mathcal{M}_4$. The LHS of (3.138) scales as $\left(\frac{g_s}{H}\right)^{2\Delta k}$ with $k \geq 3/2$ for the case (2.2) and $k \geq 9/2$ for the case (2.8). The g_s scaling on the RHS is $\left(\frac{g_s}{H}\right)^{\hat{\theta}_k}$ where $\hat{\theta}_k$ for (2.2) becomes:

$$\hat{\theta}_k = \theta'_k - 2\Delta k + 6 - \frac{14}{3} = \theta'_k - 2\Delta k + \frac{4}{3}, \tag{3.139}$$

where the first three terms in the first equality appears from **Table 2.2** and $-\frac{14}{3}$ comes from $\sqrt{-\mathbf{g}_{11}}$ (note that the determinants in (3.138) and (3.135) have un-bolded metric components). For $k = 3/2$ the g_s scaling of the LHS becomes $2\Delta k = 1$ whereas the g_s scaling of the RHS becomes $\hat{\theta}_k = \theta'_k + \frac{1}{3}$ with θ'_k as in (2.97). This means when $\theta'_k = \frac{2}{3}$ the g_s scalings on both sides of (3.138) matches exactly.

For the case (2.8) there are two changes: the determinant changes to $\sqrt{-\mathbf{g}_{11}} \propto g_s^{-8/3}$ and $k \geq \frac{9}{2}$. Putting the information from **Table 2.2**, we get:

$$\hat{\theta}_k = \theta_k - 2\Delta k + 4 - \frac{8}{3} = \theta_k - 2\Delta k + \frac{4}{3}, \tag{3.140}$$

where θ_k is as in (2.86). The g_s scaling of the LHS for $k = 9/2$ is $2\Delta k = 3$ whereas the g_s scaling of the RHS becomes $\hat{\theta}_k = \theta_k - \frac{5}{3}$, implying that when $\theta_k = \frac{14}{3}$ the g_s scaling on both sides of (3.138) match exactly. Comparing the two cases, we see that the quantization scheme for (2.2) is a bit more natural.

Case 2: $\mathbf{G}_{\alpha\beta ab}$ *component*

Following the same procedure as before we can define the quantization scheme for the G-flux component $\mathbf{G}_{\alpha\beta ab}$ defined over a four-cycle $\mathbf{\Sigma}_4^{(2)} \equiv \mathcal{M}_2 \times \frac{\mathbb{T}^2}{\mathcal{G}}$ in the following way:

$$c_1 \int_{\mathbf{\Sigma}_4^{(2)}} \mathcal{G}^{(k)}_{\alpha\beta ab} dy^\alpha \wedge .... \wedge dy^b = -c_3 \int_{\mathbf{\Sigma}_4^{(2)}} \sqrt{-g_{11}} \left(\mathbb{Z}_7^{(k)}\right)^{0ijmnpq} \epsilon_{0ijmnpq\alpha\beta ab} dy^\alpha \wedge ... \wedge dy^b, \tag{3.141}$$

where now the seven-form has different set of indices. Looking at **Table 2.2** it is easy to see that the g_s scaling of this seven-form component remains the same as earlier and therefore then matching of the g_s scalings on both LHS and RHS of (3.141) happens exactly when $\theta'_k = \frac{2}{3}$ with θ'_k defined as in (2.97). The matching of the higher order terms then follows automatically.

On the other hand, for the case (2.8), the analysis is not similar to what we had before because the g_s scaling of the seven-form changes as should be evident from **Table 2.2**. In fact the scaling becomes:

$$\hat{\theta}_k = \theta_k - 2\Delta k + 8 - \frac{8}{3} = \theta_k - 2\Delta k + \frac{16}{3}, \tag{3.142}$$

implying that for $k = \frac{9}{2}$, we will require $\theta_k = \frac{2}{3}$ in (2.86) to match the lowest powers of g_s on both sides of (3.141). Once matched at the lowest powers, all higher order g_s scalings get matched automatically.

Case 3: $\mathbf{G}_{m\alpha ab}$ *component*

This is an interesting case where the four-cycle on which we define our flux component is chosen from a combination of two one-cycles, one each from $\mathcal{M}_4$ and $\mathcal{M}_2$ respectively, and combined with the existing two-cycle $\frac{\mathbb{T}^2}{\mathcal{G}}$. The one-cycles are possible because neither $\mathcal{M}_4$ nor $\mathcal{M}_2$ are Calabi-Yau manifolds as we saw earlier. We will call this four-cycle as $\mathbf{\Sigma}_4^{(3)}$ and the quantization condition becomes:

$$c_1 \int_{\mathbf{\Sigma}_4^{(3)}} \mathcal{G}^{(k)}_{m\alpha ab} dy^m \wedge .... \wedge dy^b = -c_3 \int_{\mathbf{\Sigma}_4^{(3)}} \sqrt{-g_{11}} \left(\mathbb{Z}_7^{(k)}\right)^{0ijnpq\beta} \epsilon_{0ijnpq\beta m\alpha ab} dy^m \wedge ... \wedge dy^b. \tag{3.143}$$

The g_s scaling of the RHS remains similar to what we had for the two cases above for (2.2). This means that choosing $\theta'_k = \frac{2}{3}$ we can match the lowest order g_s scalings on both sides of (3.143). The higher order terms, as expected, match automatically after that.

The story for the case (2.8) is however a bit different because the g_s scaling of the dual form appearing in (3.143) is different as can be seen from **Table 2.2**. In addition to that, since $\mathcal{M}_4$ and $\mathcal{M}_2$ are conformally CY, *global* one-cycles are non-existent here. Nevertheless local one-cycles are possible and thus $\mathbf{\Sigma}_4^{(3)}$ could only be viewed as a local four-cycle, implying that a relation like (3.143) cannot quite capture the flux quantization scheme for this case. Locally however we can still give some meaning to an equation like (3.143), and if we carry on with such a local quantization condition, it will tell us that the g_s scaling of the RHS of (3.143) becomes:

$$\hat{\theta}_k = \theta_k - 2\Delta k + \frac{10}{3}, \tag{3.144}$$

where $k \geq \frac{9}{2}$. This means that the bound on θ_k from (2.86) is now $\theta_k \geq \frac{8}{3}$, implying that the flux quantization scheme here pits the time variation of the integrated G-flux component with the integrated quantum terms classified by $\theta_k = \frac{8}{3}$ for the case (2.8) and $\theta'_k = \frac{2}{3}$ for the case (2.2).

Case 4: $\mathbf{G}_{mnpq}$ *component*

We now start with components of G-fluxes that do not contribute at lower order in g_s scalings to the EOMs. This means the quantization scheme will involve even higher order quantum corrections that are captured by the dual seven-form. This may be seen from the following quantization condition:

$$c_1 \int_{\mathcal{M}_4} \mathcal{G}^{(k)}_{mnpq} dy^m \wedge .... \wedge dy^q = -c_3 \int_{\mathcal{M}_4} \sqrt{-g_{11}} \left(\mathbb{Z}^{(k)}_7\right)^{0ij\alpha\beta ab} \epsilon_{0ij\alpha\beta abmnpq} dy^m \wedge ... \wedge dy^q. \tag{3.145}$$

where the four-cycle is clearly the manifold $\mathcal{M}_4$. Looking at **Table 2.2** one can easily work out the g_s scaling of the RHS of (3.145). Putting everything together, this gives us:

$$\hat{\theta}_k = \theta'_k - 2\Delta k - \frac{8}{3}, \tag{3.146}$$

with θ'_k as in (2.97) and $k \geq \frac{3}{2}$. The g_s scaling of the LHS of (3.145) remains the same, i.e $2\Delta k$, and therefore to match both sides of (3.145), we need $\theta'_k \geq \frac{14}{3}$ in (2.97). Clearly for this value of θ'_k there are multiple terms which we can easily work out from (2.139).

The case with (2.8) is also different. The g_s scaling of the seven-form may be read from **Table 2.2**, Putting things together, the g_s scaling of the RHS of (3.145) now becomes:

$$\hat{\theta}_k = \theta_k - 2\Delta k - \frac{8}{3}, \tag{3.147}$$

with θ_k as in (2.86), and therefore the only way to match both sides of (3.145) is to impose $\theta_k \geq \frac{26}{3}$ in (2.86). This is a large number and therefore will involve many quantum terms, making the quantization scheme a bit more complicated. Nevertheless, matching of both sides could be made succinctly.

Case 5: $\mathbf{G}_{mnp\alpha}$ *component*

Quantization of flux in this case requires us to find a three-cycle in $\mathcal{M}_4$ and a one-cycle in $\mathcal{M}_2$. This is possible thanks to the non-Kähler nature of $\mathcal{M}_4$ and $\mathcal{M}_2$ for the case (2.2). The quantization scheme now becomes:

$$c_1 \int_{\mathbf{\Sigma}^{(4)}_4} \mathcal{G}^{(k)}_{mnp\alpha} dy^m \wedge .... \wedge dy^\alpha = -c_3 \int_{\mathbf{\Sigma}^{(4)}_4} \sqrt{-g_{11}} \left(\mathbb{Z}^{(k)}_7\right)^{0ijq\beta ab} \epsilon_{0ijq\beta abmnp\alpha} dy^m \wedge ... \wedge dy^\alpha, \tag{3.148}$$

where $\mathbf{\Sigma}^{(4)}_4$ is the corresponding four-cycle. Now according to **Table 2.2**, the g_s scaling of the dual seven-form remains exactly the same as what we had for the $\mathbf{G}_{mnpq}$ component and therefore the analysis will proceed in the same way as before. The net result is that the g_s of the RHS remains (3.146), and therefore the g_s scalings of both sides of (3.148) match when $\theta'_k \geq \frac{14}{3}$ in (2.97).

For the case (2.8), finding a globally defined four-cycle is not possible as both $\mathcal{M}_4$ and $\mathcal{M}_2$ are conformally CY manifolds. Local construction is possible, but that weakens the flux quantization scheme here. Nevertheless if we proceed with a relation like (3.148), but now defined over a local four-cycle $\mathbf{\Sigma}^{(4)}_4$, we could still make some sense of (3.148), at least in identifying the g_s scalings on both sides of the relation. This gives us:

$$\hat{\theta}_k = \theta_k - 2\Delta k - \frac{2}{3}, \tag{3.149}$$

with θ_k as defined in (2.86) and $k \geq \frac{9}{2}$. Thus if $\theta_k \geq \frac{20}{3}$ we can in principle match both sides of (3.148) for the case (2.8). These bigger numbers, for both θ'_k and θ_k, are somewhat consistent with the fact that the corresponding G-flux components do not contrinute at lower values of the g_s to the EOMs.

Case 6: $\mathbf{G}_{mn\alpha\beta}$ *component*

This case is in many sense similar to the one studied for the $\mathbf{G}_{mnpq}$ component, because the g_s scalings of the metric components, for the case (2.2), are similar. Both the metric components, $\mathbf{g}_{mn}$ and $\mathbf{g}_{\alpha\beta}$, scale as $g_s^{-2/3}$ and therefore it is no surprise that the g_s scaling of the dual seven-form is again similar to what we had for the other component. However the flux quantization scheme involve the following components:

$$c_1 \int_{\mathbf{\Sigma}_4^{(5)}} \mathcal{G}^{(k)}_{mn\alpha\beta} dy^m \wedge .... \wedge dy^\beta = -c_3 \int_{\mathbf{\Sigma}_4^{(5)}} \sqrt{-g_{11}} \left(\mathbb{Z}_7^{(k)}\right)^{0ijpqab} \epsilon_{0ijpqabmn\alpha\beta} dy^m \wedge ... \wedge dy^\beta, \tag{3.150}$$

where $\mathbf{\Sigma}_4^{(5)} \equiv \mathcal{C}_2 \times \mathcal{M}_2$, with $\mathcal{C}_2$ is the same two-cycle in $\mathcal{M}_4$ that we had chosen for the case with $\mathbf{G}_{mnab}$ component. The g_s scaling of the RHS of (3.150) remains identical to (3.146) and therefore if $\theta'_k \geq \frac{14}{3}$ in (2.97), we can easily match both sides of (3.150). As mentioned earlier, the higher order terms then match automatically.

For the case (2.8), we are in a better shape now because it is easy to find a two-cycle in $\mathcal{M}_4$ when it is a conformally CY manifold. The four-cycle then becomes a product of the two-cycle in $\mathcal{M}_4$ and the conformally CY manifold $\mathcal{M}_2$ (which is topologically a torus). The g_s scaling of the RHS of (3.150) becomes:

$$\hat{\theta}_k = \theta_k - 2\Delta k + \frac{4}{3}, \tag{3.151}$$

for θ_k as in (2.86). This implies that if $\theta_k \geq \frac{14}{3}$ we should be able to match the g_s scalings of both sides of (3.150) for any order of $k \geq \frac{9}{2}$.

Case 7: $\mathbf{G}_{mnpa}, \mathbf{G}_{mn\alpha a}$ *and* $\mathbf{G}_{m\alpha\beta a}$ *components*

The final three cases are to be defined on four-cycles that are to be constructed with one-cycles from $\frac{\mathbb{T}^2}{\mathcal{G}}$ manifold. By definition such a one-cycle do not exist in $\frac{\mathbb{T}^2}{\mathcal{G}}$ for both cases (2.2) and (2.8). Previously the case with (2.2) did not suffer from any non-existence of global cycles, although the case with (2.8) did have issues with the existence of global cycles. Now we see that for either case, global four-cycles are not possible, and we have to make sense of flux quantization with only local four-cycles. Although the non-existence of global cycles make the quantization scheme questionable, we can nevertheless compare the g_s scalings of flux integrals and the quantum terms using local four-cycles. Allowing this, we now have three set of equations:

$$\begin{aligned}
c_1 \int_{\mathbf{\Sigma}_4^{(6)}} \mathcal{G}^{(k)}_{mnpa} dy^m \wedge .... \wedge dy^a &= -c_3 \int_{\mathbf{\Sigma}_4^{(6)}} \sqrt{-g_{11}} \left(\mathbb{Z}_7^{(k)}\right)^{0ijq\alpha\beta b} \epsilon_{0ijq\alpha\beta bmnpa} dy^m \wedge ... \wedge dy^a, \\
c_1 \int_{\mathbf{\Sigma}_4^{(7)}} \mathcal{G}^{(k)}_{mn\alpha a} dy^m \wedge .... \wedge dy^a &= -c_3 \int_{\mathbf{\Sigma}_4^{(7)}} \sqrt{-g_{11}} \left(\mathbb{Z}_7^{(k)}\right)^{0ijpq\beta b} \epsilon_{0ijpq\beta bmn\alpha a} dy^m \wedge ... \wedge dy^a, \\
c_1 \int_{\mathbf{\Sigma}_4^{(8)}} \mathcal{G}^{(k)}_{m\alpha\beta a} dy^m \wedge .... \wedge dy^a &= -c_3 \int_{\mathbf{\Sigma}_4^{(8)}} \sqrt{-g_{11}} \left(\mathbb{Z}_7^{(k)}\right)^{0ijnpqb} \epsilon_{0ijnpqbmn\alpha a} dy^m \wedge ... \wedge dy^a,
\end{aligned} \tag{3.152}$$

where the four-cycles $\mathbf{\Sigma}_4^{(i)}$ for $i = 6, 7, 8$ respectively are $\mathcal{C}_3 \times S^1_{(3)}, \mathcal{C}_2 \times S^1_{(2)} \times S^1_{(3)}$ and $S^1_{(1)} \times \mathcal{M}_2 \times S^1_{(3)}$, with the subscript denoting which one-cycle is meant. Clearly $S^1_{(1)}$ and $S^1_{(2)}$ are global one-cycles, but $S^1_{(3)}$ is not, as explained earlier. Therefore the set of equations (3.152) can at most help us identify the g_s scalings on both sides of the equalities, but would not serve as flux quantization conditions (as the four-cycles could shrink to zero sizes). From **Table 2.2** we can easily see that, for the case (2.2), the RHS of all the three equations scale in exactly the same way as:

$$\hat{\theta}_k = \theta'_k - 2\Delta k - \frac{2}{3}, \tag{3.153}$$

with θ'_k as in (2.97) and $k \geq \frac{3}{2}$. This means that if we take $\theta'_k \geq \frac{8}{3}$ we can match the g_s scalings of both sides of each individual equalities for all $k \geq \frac{3}{2}$, and to any subsequent orders.

The case for (2.8) is however not as uniform as above. The g_s scalings of the dual seven-forms themselves are different as may be inferred from **Table 2.2**. This directly translates to the g_s scalings of the RHS of the three equations in (3.152) in the following way:

$$\hat{\theta}_k = \theta_k - 2\Delta k - \frac{2}{3}, \quad \hat{\theta}_k = \theta_k - 2\Delta k + \frac{4}{3}, \quad \hat{\theta}_k = \theta_k - 2\Delta k + \frac{10}{3}, \tag{3.154}$$

with θ_k as in (2.86) and $k \geq \frac{9}{2}$. Of course now none of the one-cycles are globally defined, and neither is the three-cycle $\mathcal{C}_3$, so the four-cycles in each of the three cases in (3.152) are local in much weaker sense than what we had earlier. This means the flux-quantization conditions are even more weakly defined than before. Nevertheless we see that the above three scalings in (3.154) puts the following lower bounds on θ_k:

$$\theta_k \geq \frac{20}{3}, \qquad \theta_k \geq \frac{14}{3}, \qquad \theta_k \geq \frac{8}{3}, \tag{3.155}$$

respectively for the three cases in (3.152) for the g_s scalings to match on both sides of the equalities. Once they match at the lowest orders, matchings at higher orders are almost automatic.

Our detailed analysis above should justify how flux quantizations should be understood in the case when the fluxes themselves are varying with respect to time, or alternatively, have g_s dependences (as we packaged all temporal dependences as

g_s scalings). The original time-independent quantization scheme of [33] where:

$$\left[\frac{\mathbf{G}_4}{2\pi}\right] - \frac{p_1(y)}{4} \in \mathbb{H}^4(y, \mathbb{Z}), \tag{3.156}$$

doesn't quite work in the time-dependent case as $\mathbf{G}_4$ is always time-dependent (i.e g_s dependent) in our set-up whereas $p_1(y)$, the first Pontryagin class, may not always be (i.e for some sub-manifold in the internal eight-manifold, $p_1(y)$ may be time, or g_s, independent). Therefore the combination on the LHS of (3.156) being in the fourth cohomology class $\mathbb{H}^4(y, \mathbb{Z})$ doesn't make much sense here, and the quantization scheme now becomes much more involved as we showed above. In principle one would expect both the G-flux components as well as the four-cycles to vary with respect to time. However we have managed to rewrite the flux quantization condition in such a way that all g_s dependences go in the definition of the fluxes, and the cycles themselves are defined using un-warped metric components. Such a procedure then helped us to balance the g_s dependences of the integrated flux components on given four-cycles with the g_s dependences of the corresponding quantum corrections. We have tabulated the results in **Table 3.2**.

There are two other potential contributions to the flux quantization conditions that we only gave cursory attentions. These are the number of dynamical M5-branes, denoted by N, and the integrated four-form, denoted by the integral of $\hat{\mathbb{Z}}_4$, in (3.135). Both these could have potential g_s dependences and would therefore contribute to the flux quantization conditions.

Anomaly cancellations and localized fluxes

In the above section we studied how the flux quantization conditions as well as the Bianchi identities go hand in hand, and how the g_s scalings could be matched for every allowed G-flux components. The results are shown in **Table 3.2**. It is time now to go to the next level of subtleties, namely the interpretation of the flux components that thread the internal manifold, and the cancellations of anomalies that arise from fluxes and branes on compact spaces.

We will start by defining the eleven-dimensional action much like how we described it in (3.129), but now using the fundamental variables and not the dual ones. This means four-form G-flux components will appear instead of the seven-form dual flux components. In this language the action becomes:

$$\mathbb{S}_{11} \equiv b_1 \int \mathbf{G}_4 \wedge *_{11}\mathbf{G}_4 + b_2 \int \mathbf{C}_3 \wedge \mathbf{G}_4 \wedge \mathbf{G}_4 + b_3 \int \mathbf{C}_3 \wedge \mathbb{Z}_8 + b_4 \int \mathbf{G}_4 \wedge *_{11}\mathbb{Z}_4 + n_b \int \mathbf{C}_3 \wedge \mathbf{\Lambda}_8, \tag{3.157}$$

where b_i are all proportional to certain powers of M_p (that may be easily fixed by derivative counting), $\mathbb{Z}_8$ is as defined in (2.132) which contains the $\mathbf{X}_8$ polynomial, and n_b is the number of static M2-branes. The other important ingredient of (3.157) is the $*_{11}\mathbb{Z}_4$ piece that captures the quantum corrections from either (2.78) or (2.94) as elucidated in (2.135). Such a term appearing in (3.157) leads to the non-topological interactions, and by construction $*_{11}\mathbb{Z}_4$ is not a globally defined function on a compact space. The EOM that arises from varying $\mathbf{C}_3$ now takes the following

Forms	Dual Forms	$\hat{\theta}_k$ for (2.2)	$\hat{\theta}_k$ for (2.8)	$[\theta'_k]_{\rm min}$	$[\theta_k]_{\rm min}$
$\mathcal{G}^{(k)}_{mnab}$	$\left(\mathbb{Z}^{(l)}_7\right)^{0ijpq\alpha\beta}$	$\theta'_k - 2\Delta k + \frac{4}{3}$	$\theta_k - 2\Delta k + \frac{4}{3}$	$\frac{2}{3}$	$\frac{14}{3}$
$\mathcal{G}^{(k)}_{\alpha\beta ab}$	$\left(\mathbb{Z}^{(l)}_7\right)^{0ijmnpq}$	$\theta'_k - 2\Delta k + \frac{4}{3}$	$\theta_k - 2\Delta k + \frac{16}{3}$	$\frac{2}{3}$	$\frac{2}{3}$
$\mathcal{G}^{(k)}_{m\alpha ab}$	$\left(\mathbb{Z}^{(l)}_7\right)^{0ijnpq\beta}$	$\theta'_k - 2\Delta k + \frac{4}{3}$	$\theta_k - 2\Delta k + \frac{10}{3}$	$\frac{2}{3}$	$\frac{8}{3}$ *
$\mathcal{G}^{(k)}_{mnpq}$	$\left(\mathbb{Z}^{(l)}_7\right)^{0ij\alpha\beta ab}$	$\theta'_k - 2\Delta k - \frac{8}{3}$	$\theta_k - 2\Delta k - \frac{8}{3}$	$\frac{14}{3}$	$\frac{26}{3}$
$\mathcal{G}^{(k)}_{mnp\alpha}$	$\left(\mathbb{Z}^{(l)}_7\right)^{0ijq\beta ab}$	$\theta'_k - 2\Delta k - \frac{8}{3}$	$\theta_k - 2\Delta k - \frac{2}{3}$	$\frac{14}{3}$	$\frac{20}{3}$ *
$\mathcal{G}^{(k)}_{mn\alpha\beta}$	$\left(\mathbb{Z}^{(l)}_7\right)^{0ijpqab}$	$\theta'_k - 2\Delta k - \frac{8}{3}$	$\theta_k - 2\Delta k + \frac{4}{3}$	$\frac{14}{3}$	$\frac{14}{3}$
$\mathcal{G}^{(k)}_{mnpa}$	$\left(\mathbb{Z}^{(l)}_7\right)^{0ijq\alpha\beta b}$	$\theta'_k - 2\Delta k - \frac{2}{3}$	$\theta_k - 2\Delta k - \frac{2}{3}$	$\frac{8}{3}$ *	$\frac{20}{3}$ *
$\mathcal{G}^{(k)}_{mn\alpha a}$	$\left(\mathbb{Z}^{(l)}_7\right)^{0ijpq\beta b}$	$\theta'_k - 2\Delta k - \frac{2}{3}$	$\theta_k - 2\Delta k + \frac{4}{3}$	$\frac{8}{3}$ *	$\frac{14}{3}$ *
$\mathcal{G}^{(k)}_{m\alpha\beta a}$	$\left(\mathbb{Z}^{(l)}_7\right)^{0ijnpq\beta}$	$\theta'_k - 2\Delta k - \frac{2}{3}$	$\theta_k - 2\Delta k + \frac{10}{3}$	$\frac{8}{3}$ *	$\frac{8}{3}$ *

TABLE 3.2: Flux quantization associated with (3.135) keeping $N = c_2 = 0$. All the integrated flux components scale as $g_s^{2\Delta k}$, and the g_s scalings of the dual forms, that incorporate the quantum corrections, go as $g_s^{\hat{\theta}_k}$. These are tabulated above for the two cases (2.2) and (2.8). The other two parameters, θ'_k and θ_k, are defined in (2.97) and (2.86) respectively. The symbol $*$ denotes the non-existence of global four-cycles.

form:

$$d *_{11} \mathbf{G}_4 = \frac{1}{b_1}\Big(b_2\, \mathbf{G}_4 \wedge \mathbf{G}_4 + b_3\, \mathbb{Z}_8 - b_4\, d *_{11} \mathbb{Z}_4 + n_b \mathbf{\Lambda}_8\Big). \tag{3.158}$$

Since both $\mathbf{G}_4$ and $\mathbf{G}_7 \equiv *_{11}\mathbf{G}_4$ are globally defined forms on the compact eight-manifold $\mathcal{M}_8$, as given in (2.4), integrating the LHS of (3.158) over $\mathcal{M}_8$ would automatically vanish. Doing this on the RHS then reproduces the following anomaly cancellation condition:

$$b_2 \int_{\mathcal{M}_8} \mathbf{G}_4 \wedge \mathbf{G}_4 + b_3 \int_{\mathcal{M}_8} \mathbb{Z}_8 - b_4 \int_{\mathcal{M}_8} d *_{11} \mathbb{Z}_4 + n_b = 0, \tag{3.159}$$

where we have assumed that the integral of the localized form $\mathbf{\Lambda}_8$ over the eight-manifold is identity. This is true of course when the M2-branes are completely *static*. We will discuss more on this later.

On the outset (3.159) looks like the standard anomaly cancellation condition one would get from [21, 22], however a closer inspection reveals a few subtleties. One, the flux integral is now time-dependent because the $\mathbf{G}_4$ fluxes do not have any time-independent parts. Two, we have an integral over the topological 8-form $\mathbb{Z}_8$, whose polynomial form appears in (2.132), instead of just $\mathbf{X}_8$ as in [21, 22]. Three, there appears a *new* contribution coming from the integral of a *locally* exact form

$d *_{11} \mathbb{Z}_4$ over $\mathcal{M}_8$ from the quantum corrections. And four, we have n_b, the number of static M2-branes, that is a time-independent factor. Thus (3.159) is not just a single relation as in [22], rather it is now a mixture of time-dependent and time-independent pieces juxtaposed together. How do we disentangle the various parts of (3.159) to form consistent anomaly cancellation conditions for our case?

The $\mathbf{X}_8$ polynomial and Euler characteristics of the eight-manifold

First let us look at the $\mathbf{X}_8$ part of $\mathbb{Z}_8$. As should be clear from (2.132), the choice (2.133) allows us to construct the $\mathbf{X}_8$ polynomial from $\mathbb{Z}_8$. In the time-independent case, we expect (see the first reference in [21]):

$$\int_{\mathcal{M}_8} \mathbf{X}_8 = -\frac{1}{4!(2\pi)^4}\chi_8, \tag{3.160}$$

where χ_8 is the Euler-characteristics of the eight-manifold $\mathcal{M}_8$ when it has a Calabi-Yau metric on it. In fact, in the time-independent case (3.160) makes sense, but if we now take the metric ansatze (2.3) with the warp-factors as defined in (2.5), how does (3.160) translates to the present case?

To answer this question let us look for the regime of validity of our g_s expansions for all the parameters involved in our analysis. It is easy to see that as long as $0 \leq \left(\frac{g_s}{H}\right)^2 < 1$ we have pretty much controlled quantum series expansions for all the parameters here. Clearly we *cannot* analyze the cases when $\left(\frac{g_s}{H}\right)^2 \geq 1$ because of the way we expressed the G-flux components in (2.13), quantum terms in (3.1) etc. Thus $\left(\frac{g_s}{H}\right)^2 = 1$ forms a kind of *boundary*, below which all the analysis that we performed remains valid. Interestingly when $\left(\frac{g_s}{H}\right)^2 = 1$, the M-theory metric (2.3) takes the following form:

$$ds^2 = H^{-8/3}\Big(-dt^2 + dx_1^2 + dx_2^2\Big) + H^{4/3}\Big(g_{\alpha\beta}dy^\alpha dy^\beta + g_{mn}dy^m dy^n + g_{ab}dy^a dy^b\Big), \tag{3.161}$$

where the metric components appearing above are all the un-warped ones and we have absorbed the $F_i\left(-\frac{1}{\sqrt{\Lambda}}\right)$ in the definition of the internal coordinates (Λ being the cosmological constant). We will not worry about the fluxes and the quantum corrections in this limit as they are any way not well defined according to our g_s expansion scheme. Our present scenario is somewhat similar to the one we encountered earlier, although we do not want to give a coherent state interpretation when comparing (2.3) and (3.161) just yet. What we can say is that as:

$$-\frac{1}{\sqrt{\Lambda}} \,<\, t \,\leq\, 0, \tag{3.162}$$

the metric (3.161) slowly transforms into (2.3), implying that all temporal evolution should be defined for $t \equiv -\frac{1}{\sqrt{\Lambda}} + \delta t$. Such a point of view does not rule out a coherent state formalism for our present background because we can still view the time-dependent evolution for $-\infty < t \leq 0$ to be over a solitonic configuration of the form (3.161). Unfortunately the inaccessibility of the regimes $t \leq -\frac{1}{\sqrt{\Lambda}}$ prohibits us to provide a quantitative analysis of such a scenario.

What it does provide is a way to interpret the integral of $\mathbf{X}_8$ over the eight-manifold. Let us first consider the eight-manifold as given in (3.161). This is not a Calabi-Yau four-fold so the $\mathbf{X}_8$ integral will not necessarily capture the Euler characteristics of the internal eight-manifold $\mathcal{M}_8$ defined as in (2.4). Once we switch on a time interval δt, the warp-factors (2.5) changes to the following:

$$e^{2A} = \left(1 + \frac{8}{3}\sqrt{\Lambda}\delta t\right) H^{-8/3}, \;\; e^{2C} = \left(1 - \frac{4}{3}\sqrt{\Lambda}\delta t\right) H^{4/3}, \;\; \Lambda t^2 \equiv \left(\frac{g_s}{H}\right)^2 = 1 - 2\sqrt{\Lambda}\delta t \quad (3.163)$$
$$e^{2B_1} = F_1\left(-\frac{1}{\sqrt{\Lambda}} + \delta t\right)\left(1 + \frac{2}{3}\sqrt{\Lambda}\delta t\right) H^{4/3}, \;\; e^{2B_2} = F_2\left(-\frac{1}{\sqrt{\Lambda}} + \delta t\right)\left(1 + \frac{2}{3}\sqrt{\Lambda}\delta t\right) H^{4/3},$$

where we see that the temporal evolution of the metric (3.161) appears as additive pieces, each proportional to δt, to every metric components (including the space-time ones) up-to the F_i factors. The F_i factors do not change this observation because:

$$F_2\left(-\frac{1}{\sqrt{\Lambda}} + \delta t\right) = 1 + \sum_k C_k\left(1 - 2\Delta\sqrt{\Lambda}\delta t\right)$$
$$F_1\left(-\frac{1}{\sqrt{\Lambda}} + \delta t\right) = \left[1 + \sum_k \widetilde{C}_k\left(1 - 2\Delta\sqrt{\Lambda}\delta t\right)\right]\left(1 - \gamma\sqrt{\Lambda}\delta t\right), \quad (3.164)$$

where $\gamma = 0, 2$ are related to the two cases (2.2) and (2.8) respectively. The other two set of parameters C_k and $\widetilde{C}_k$ have been determined earlier in terms of the quantum corrections in section 3.1.1.

Therefore combining (3.163) and (3.164), the metric ansatze (2.3) can actually be viewed as a perturbation over the initial metric configuration (3.161). In fact in this language, the late time cosmological evolution may be viewed as evolving from the metric configuration (3.161) via the warp-factors (3.163) and (3.164). It is also easy to replace δt to a finite temporal value by iterating (3.163) and (3.164) or by directly summing over binomial coefficients. All in all, our little exercise above tells us that:

$$\int_{\mathcal{M}_8} \mathbf{X}_8 \equiv \frac{1}{3 \cdot 2^9 \cdot \pi^4}\int_{\mathcal{M}_8}\left(\text{tr}\,\mathbb{R}^4 - \frac{1}{4}\left(\text{tr}\,\mathbb{R}^2\right)^2\right) = -\frac{\omega_o}{4!(2\pi)^4}\chi_8 + g_o(\delta t), \quad (3.165)$$

where $\mathbb{R}$ is the curvature two-form as it appears in (2.131), and ω_o measures the deviation from the Euler characteristics χ_8. This could be integer or fraction depending on our choice of the eight-manifold. Note that the integral (3.165) splits into two pieces: $\omega_o\chi_8$, which is the piece independent of δt, is now only proportional to the Euler characteristics of the eight-manifold appearing in (3.161); and $g_o(\delta t)$ is a factor that depends on our temporal evolution parameter δt. The latter doesn't automatically vanish, at least not for the kind of background that we analyze here, and therefore should contribute to the anomaly cancellation condition (3.159). Exactly how this happens will be illustrated soon.

The Euler characteristics χ_8 can take either values, positive or negative, and both will be useful in analyzing the anomaly cancellation[16]. The case with vanishing Euler is interesting in its own way, but it appears not to be realized at least for the

[16]Thus without loss of generalities we will take $\omega_o > 0$ in (3.165).

case (2.2). One can, however, question the robustness of the interpretation. (3.165). How is the split (3.165) understood in the full cosmological setting? This is where the coherent state interpretation becomes immensely useful. If we assume that the cosmological evolution for $-\infty < t \leq 0$ is via coherent states that evolve over a solitonic background like (3.161) then χ_8 will always be related to the Euler characteristics of the vacuum eight-manifold.

Anomaly cancellation conditions and time-dependent G-fluxes

Let us now come to the anomaly cancellation conditions from (3.159). This equation should now naturally split into at least two parts: one, that is time-independent (i.e independent of g_s), and two, that depends on time, and hence on g_s. It is easy to see that, out of the four set of pieces in (3.159), only two set of pieces are time independent. These are the number n_b of M2-branes and the time independent part of $\mathbb{Z}_8$ that is related to the Euler characteristics of the eight-manifold (3.165). If we take $\chi_8 > 0$, (3.159) immediately gives us the first anomaly cancellation condition:

$$n_b = \frac{b_3}{4!(2\pi)^4}\,\chi_8, \tag{3.166}$$

where b_3 is the factor that depends on ω_o and M_p. Thus we see that, even for a non-Kähler eight-manifold, the Euler characteristics of the internal manifold (3.161) governs the number of *static* M2-branes in our model in some sense. Since the number of M2-branes have to be an integer, the equation (3.166) puts an extra constraint on b_3 and the Euler characteristics of the eight-manifold itself, namely the combination on the RHS of (3.166) should be an integer. Such a condition should be reminiscent of a similar condition in the second reference of [21], and here we see that in a time-dependent background, (3.166) is realized instead of the full anomaly cancellation condition with G-fluxes of [22] (see also the last reference of [21]).

On the other hand, a negative Euler characteristics would be related to anti M2-branes, or to a set-up with dominant number of anti M2-branes. Again the story parallels that of the second reference of [21], albeit now for the time-dependent background. Vanishing Euler characteristics would then mean no M2 or anti M2-branes or equal number of M2 and anti M2-branes (such that global charges cancel).

For the time-dependent parts of (3.159) there are a couple of subtleties. One, we need to tread carefully as various parts of the G-flux components have different g_s scalings; and two, time-dependent contributions now come from both topological and non-topological parts of (3.159). In fact the non-topological piece, given in terms of $*_{11}\mathbb{Z}_4$, is solely time dependent as it is constructed out of the quantum terms (2.78) or (2.94) as shown in (2.135). On the other hand, the topological part does have a time independent piece as seen from (3.165). Combining everything together, our second anomaly cancellation condition may be expressed as:

$$b_2\int_{\mathcal{M}_8}\mathbf{G}_4\wedge\mathbf{G}_4 + b_3\int_{\mathcal{M}_8}\Big(\mathbb{Z}_8 - \mathbf{X}_8\Big) - b_4\int_{\mathcal{M}_8} d *_{11}\mathbb{Z}_4 = \frac{b_3}{4!(2\pi)^4}\,\chi_8, \tag{3.167}$$

which is in fact not a *single* condition, rather it is an infinite number of conditions on various components of the G-fluxes and the quantum terms. To see this, and as we have done before, we will first decouple the b_3 dependent parts of (3.167) to simplify the ensuing analysis. This will be inserted in at the end. Plugging in the G-flux components and the quantum series in (3.167) with $b_3 = 0$, we get:

$$\begin{aligned} & b_1 \sum_{\{k_i\}} \int_{\mathcal{M}_8} \mathcal{G}^{(k_1)}_{N_1N_2N_3N_4} \mathcal{G}^{(k_2)}_{N_5N_6N_7N_8} \left(\frac{g_s}{H}\right)^{2\Delta(k_1+k_2)} dy^{N_1} \wedge .... \wedge dy^{N_8} \\ & = b_4 \sum_k \int_{\mathcal{M}_8} \partial_{N_8} \left(\sqrt{-g_{11}} \left(\mathbb{Z}^{(k)}_4\right)_{M'_8...M'_{11}} g^{M_8M'_8}...g^{M_{11}M'_{11}} \left(\frac{g_s}{H}\right)^{\tilde{\theta}_k}\right) \epsilon_{N_1...N_7M_8...M_{11}} dy^{N_1} \wedge .... \wedge dy^{N_8}, \end{aligned} \tag{3.168}$$

where we see that the RHS is expressed in terms of a total derivative and unwarped metric components. Since $\mathcal{M}_8$ is a compact eight-manifold without a boundary, one might worry that the RHS would vanish. However it doesn't precisely because $d *_{11} \mathbb{Z}_4$ is only a locally-exact form. In other words, $*_{11}\mathbb{Z}_4$ is *not* a globally defined form as it is extracted from the quantum terms in (2.135) and involves metric components that are not globally defined variables on the compact eight-manifold. This is like the $\mathbf{X}_8$ form that is expressed as a locally-exact form $d\mathbf{X}_7$ where $\mathbf{X}_7$ is not a globally defined form on a compact eight-manifold. This renders the RHS non-zero even in the absence of any boundary. Finally, the g_s scaling $\tilde{\theta}_k$ appearing in (3.168) may be defined as:

$$\tilde{\theta}_k \equiv \theta'_k - \frac{2}{3}, \qquad \tilde{\theta}_k \equiv \theta_k + \frac{4}{3}, \tag{3.169}$$

for the two cases, (2.2) and (2.8) respectively where θ'_k and θ_k are defined as in (2.97) and (2.86) respectively. The anomaly cancellation condition then requires us to match the g_s scalings on both sides of the equation (3.168). This gives us:

$$\begin{aligned} \theta'_k &= \frac{2}{3}(k_1 + k_2 + 1), \qquad (k_1, k_2) \geq \left(\frac{3}{2}, \frac{3}{2}\right) \\ \theta_k &= \frac{2}{3}(k_1 + k_2 - 2), \qquad (k_1, k_2) \geq \left(\frac{9}{2}, \frac{9}{2}\right), \end{aligned} \tag{3.170}$$

as the set of anomaly cancellation conditions for the two cases (2.2) and (2.8) respectively. As a check one may see that, for $k_1 = k_2 = \frac{3}{2}$, $\theta'_k = \frac{8}{3}$ and therefore involves the same set of quantum terms that we had for example in (3.77), wherein the quantum terms were classified by (3.78). This makes sense because the equation governing the G-flux components is as in (3.158), and therefore if we restrict the LHS of (3.158) to the G-flux components $\mathbf{G}_{0ijm}$ or $\mathbf{G}_{0ij\alpha}$, then the LHS may be expressed in terms of $\square H^4$ exactly as in (3.77). In fact the similarity goes even deeper: (3.77) has the same number of ingredients as (3.158), for example there are M2-branes, fluxes and quantum corrections almost in one-to-one correspondence to (3.158).

There is however at least one crucial difference between (3.77) and (3.158) apart from the appearance of the b_3 factor in the latter. The difference lies in the choice of the G-flux components themselves: (3.77) is defined in terms of $\mathcal{G}^{(k)}_{MNab}$ components whereas (3.158) involves $*_8\mathcal{G}^{(k)}_{MNab}$ components, with $*_8$ being the Hodge dual over the internal eight-manifold. For the time-independent case this observation has already been registered in [14] (see eq. (7.11) therein), and now we see that such a case happens here too. It is easy to show that in general the G-flux components are no longer self-dual, where the self-duality is defined with respect to the internal eight-dimensional space. In fact presence of self-duality would have been a sign of supersymmetry, but since supersymmetry is broken, it is no surprise that we see non self-dual G-flux components.

For the case (2.8) governed by θ_k in (2.86), there appears to be some mis-match if we compare to (3.86). On one hand, taking $k_1 = k_2 = \frac{9}{2}$ we get $\theta_k = \frac{14}{3}$ from (3.170). On the other hand, (3.86) tells us that the quantum terms are classified by $\theta_k = \frac{8}{3}$ in (3.86). This difference may be attributed to the multiple constraints appearing from (3.85), vanishing Ricci scalar for the six-dimensional base, and vanishing Euler characteristics for the eight-manifold; and therefore a simple comparison between the set of equations cannot be performed.

However a more likely scenario is that (3.86) is *not* the correct EOM, and the correct EOM for this case is actually (3.91). In fact the similarity of (3.91) with (3.77), and the fact that the quantum terms are classified by $\theta_k = \frac{14}{3}$ puts extra confidence in the (3.91) to be the correct EOM. Taking this to be the case, and comparing (3.91) and (3.168), we again observe the non-existence of self-dual fluxes. The number of flux components in (3.168) do not match with the ones in (3.91), but if we only allow components $\mathcal{G}^{(9/2)}_{\alpha\beta ab}$ in (3.168) then the story would be exactly similar to what we had for the case (2.2), reassuring, in turn, the correctness of our procedure so far. Thus we see that the flux EOMs provide powerful consistency checks on our earlier EOMs derived using Einstein's equations[17] .

Dynamical branes, fluxes and additional constraints

The interconnections between the G-flux EOMs and the Einstein's EOMs, in particular the ones that match the quantum terms, do have an additional layer of subtleties. These subtleties arise once we look at the M2 and M5-branes, especially the ones endowed with dynamical motions. To illustrate this, let us first discuss the static M2-branes ignoring, for the time being, the M5-branes[18].

Dynamical membranes and G-fluxes

The subtleties alluded to above arise when the dynamical motions of the membranes tend to stir up additional corrections to the G-flux components, in particular the ones with components along the $2+1$ space-time direction, for example $\mathbf{G}_{M0ij}$. Question then is: how robust is our earlier analysis that we did using the space-time flux components borrowed from [14]? To see this, we will have to re-visit the dynamics of membranes more carefully now. For simplicity however we will only consider single membrane, and ignore M5-branes (as mentioned above). The action

[17] In retrospect this also justifies the locally exact nature of $d *_{11} \mathbb{Z}_4$, because if it were globally exact, it would not have contributed to the RHS of (3.168) resulting in some contradictions with the EOMs from the Einstein's equations.

[18] The M5-branes wrapped on three-cycles of the internal eight-manifold could be viewed as fractional M2-branes. If we ignore the subtleties associated with the KK modes from the wrapped directions, then the dynamics of these will be no different from the M2-branes. In this thesis we will avoid distinguishing between the integer and the fractional M2-branes.

for a *single* membrane can be written as:

$$\mathbb{S}_B = -\frac{T_2}{2}\int d^3\sigma\left\{\sqrt{-\gamma_{(2)}}\Big(\gamma^{\mu\nu}_{(2)}\partial_\mu X^M\partial_\nu X^N\mathbf{g}_{MN} - 1\Big) + \frac{1}{3}\epsilon^{\mu\nu\rho}\partial_\mu X^M\partial_\nu X^N\partial_\rho X^P\mathbf{C}_{MNP}\right\}, \tag{3.171}$$

where $\gamma_{(2)\mu\nu}$ is the world-volume metric, $\epsilon_{\mu\nu\rho}$ is the Levi-Civita *symbol*, $\mathbf{g}_{MN}$ is the warped metric in M-theory, X^M are the coordinates of eleven-dimensional space-time and $\mathbf{C}_{MNP}$ is the three-form potential. The EOM for the world-volume metric easily relates it to the M-theory metric $\mathbf{g}_{MN}$ as the following pull-back:

$$\gamma_{(2)\mu\nu} = \partial_\mu X^M\partial_\nu X^N\mathbf{g}_{MN}, \tag{3.172}$$

which means in the *static-gauge*, we will simply have $\gamma_{(2)\mu\nu} = \mathbf{g}_{\mu\nu}$, i.e the world-volume metric is the $2+1$ dimensional space-time metric. On the other hand, the EOM for the membrane motion takes the following condensed form:

$$\square_{(\sigma)}X^P + \gamma^{\mu\nu}_{(2)}\partial_\mu X^M\partial_\nu X^N\mathbf{\Gamma}^P_{MN} - \frac{\epsilon^{\mu\nu\rho}}{3!\sqrt{-\gamma_{(2)}}}\partial_\mu X^Q\partial_\nu X^N\partial_\rho X^R\mathbf{G}_{SQNR}\mathbf{g}^{SP} = 0, \tag{3.173}$$

with $\square_{(\sigma)}$ forming the Laplacian[19] in $2+1$ dimension described using the world-volume metric $\gamma_{(2)\mu\nu}$, $\mathbf{\Gamma}^P_{MN}$ is the Christoffel symbol described using the warped metric $\mathbf{g}_{MN}$, and $\mathbf{G}_{SQNR}$ is the G-flux components that we have been using so far. in the static-gauge we expect $\square_{(\sigma)}X^P = 0$, and then the remaining two terms of (3.173), simply gives us:

$$\mathbf{G}_{0ijM} = -\frac{3}{2}\sqrt{-\gamma_{(2)}}\,\mathbf{g}^{\mu\nu}\mathbf{g}_{\mu\nu,M}, \tag{3.174}$$

where we identify the world-volume metric to the $2+1$ dimensional space-time warped metric $\mathbf{g}_{\mu\nu}$. Therefore plugging in the metric components from (2.3) and (2.5) we can reproduce the familiar results for $\mathbf{G}_{0ijm}$ and $\mathbf{G}_{0ij\alpha}$ in [14, 15].

All we did above is very standard, but the keen reader must have already noticed the subtlety. The form (3.174) is *only* possible if there are static M2-branes. If the system doesn't have any static M2-branes, or the M2-branes are somehow absent, the result (3.174) doesn't follow naturally. For the case (2.8) all the parameters are independent of y^α so, at least at the face-value, (2.85) makes sense once we compare it with (3.174). However since the Euler characteristics of the internal eight-manifold also vanishes, all static M2-branes are eliminated. How can we then justify the non-zero value of $\mathbf{G}_{0ijm}$ for the case (2.8)?

This is where the difference between time-independent (and also supersymmetric) and time-dependent cases becomes more prominent. In the time-independent

[19] $\square_{(\sigma)}X^P = \frac{1}{\sqrt{-\gamma_{(2)}}}\partial_\mu\Big(\sqrt{-\gamma_{(2)}}\gamma^{\mu\nu}_{(2)}\partial_\nu X^P\Big)$.

supersymmetric case[20], vanishing Euler characteristics for a four-fold implies vanishing fluxes and branes [21, 22]. This is clearly not the case for the time-dependent case where, as we saw above, G-flux components that are time-dependent (i.e g_s dependent) are allowed. This means for vanishing Euler characteristics, *dynamical* M2-branes can be allowed too.

Introducing dynamics open up a new class of subtleties that we have hitherto left unexplored. One of the first subtlety is that the world-volume metric is no longer the $2+1$ dimensional space-time metric. In fact $\gamma_{(2)00}$ becomes:

$$\begin{aligned}\gamma_{(2)00} &= \mathbf{g}_{00} + \dot{y}^m\dot{y}^n\mathbf{g}_{mn} + \dot{y}^\alpha\dot{y}^\beta\mathbf{g}_{\alpha\beta} + \dot{y}^a\dot{y}^b\mathbf{g}_{ab} \\ &= \left(\frac{g_s}{H}\right)^{-8/3}\left(g_{00} + \dot{y}^m\dot{y}^n g_{mn}\left(\frac{g_s}{H}\right)^2 + \dot{y}^\alpha\dot{y}^\beta g_{\alpha\beta}\left(\frac{g_s}{H}\right)^2 + \dot{y}^a\dot{y}^b g_{ab}\left(\frac{g_s}{H}\right)^4\right),\end{aligned} \tag{3.175}$$

where the components are defined, for the case (2.2), using warped M-theory metric and therefore involve g_s dependent terms. The other components of the metric may be taken to be the corresponding space-time metric if $y^M \equiv y^M(t)$. We can now quantify what is meant by slowly moving membrane by specifying the behavior of y^M as:

$$y^M(\mathbf{x}, g_s) = \sum_{k\in\frac{\mathbb{Z}}{2}} y^M_{(k)}(\mathbf{x})\left(\frac{g_s}{H}\right)^{2\Delta k}, \tag{3.176}$$

near $g_s \to 0$ and $y^M_{(k)}(\mathbf{x})$ could in principle depend on the world-volume spatial coordinates, but here we will take it to be a constant as in (3.175). In this representation of y^M, slowly moving membrane means small k at late times, i.e for $g_s << 1$. In the limit $k \to 0$, the membrane is truly static and when $g_s \to 0$, $y^M(\mathbf{x}, 0) \to 0$. This is almost like the end point of an D3-D7 inflationary model [34] where, in IIB, a D3-brane (T-dual of our M2-brane) dissolves in the D7-brane (T-dual of an orbifold point in our eight-manifold). Additionally, the y^M represent the eight scalar fields on the world-volume of the M2-brane, and once we dualize them to type IIB, only six scalar fields would remain. The Laplacian action on y^M then yields:

$$\begin{aligned}\square_{(\sigma)} y^M &= \frac{2\Delta^2\Lambda}{|g_{00}|}\sum_{k_3}\frac{k_3(2k_3-7)}{1+f_o}\left(\frac{g_s}{H}\right)^{2\Delta(k_3+1)} y^M_{(k_3)} \\ &- \frac{8\Lambda^2\Delta^4}{|g_{00}|}\sum_{\{k_i\}}\frac{k_1k_2k_3(k_1+k_2)g_o}{(1+f_o)^2}\left(\frac{g_s}{H}\right)^{2\Delta(k_1+k_2+k_3+1)} y^M_{(k_3)},\end{aligned} \tag{3.177}$$

where note that both the terms are suppressed by positive powers of $\frac{g_s}{H}$, g_{00} is the un-warped metric component, $\Delta = \frac{1}{3}$ as chosen before and Λ is the cosmological constant. We have also assumed no motion along the (a, b) directions and therefore y^M above can either be y^m or y^α. The remaining two factors, (f_o, g_o) are defined in

[20]For the time-independent non-supersymmetric case, as we saw earlier, it is hard to establish an EFT description in lower dimensions with de Sitter isometries. Thus it doesn't make sense to talk about it here and we shall ignore this case altogether.

the following way:

$$
\begin{aligned}
f_o \equiv f_o(y) &= 4\Lambda\Delta^2 \sum_{\{k_i\}} g_o(k_1, k_2; y) k_1 k_2 \left(\frac{g_s}{H}\right)^{2\Delta(k_1+k_2)} \\
g_o \equiv g_o(k_1, k_2; y) &= g^{00}\left(y^m_{(k_1)} y^n_{(k_2)} g_{mn}(y) + y^\alpha_{(k_1)} y^\beta_{(k_2)} g_{\alpha\beta}(y)\right),
\end{aligned} \tag{3.178}
$$

where the metric involved are all the un-warped ones. Note that, since f_o is a series in positive powers in g_s, any series of the form $(1+f_o)^{-|q|}$ for arbitrary q will only contribute *positive* powers of $\frac{g_s}{H}$ to the series (3.177). Thus the generic conclusion of $\square_{(\sigma)}$ being defined in terms of positive powers of $\frac{g_s}{H}$, remains unchanged. In fact this also persists for the second term in the EOM (3.173). To see this, let us take $M = \alpha$ in (3.176) for the case (2.2). We get:

$$
\gamma^{00}_{(2)} \partial_0 X^P \partial_0 X^Q \mathbf{\Gamma}^\alpha_{PQ} = \frac{|g^{00}|}{1+f_o}\left(\frac{g_s}{H}\right)^{2/3}\left[\Gamma^\alpha_{00} + 4\Delta^2\Lambda \sum_{\{k_i\}} k_1 k_2 h^\alpha_o(k_1, k_2; y)\left(\frac{g_s}{H}\right)^{2\Delta(k_1+k_2)}\right], \tag{3.179}
$$

where f_o is defined in (3.178); and $\mathbf{\Gamma}^\alpha_{PQ}$ and Γ^α_{00} are the Christoffel symbols defined with respect to the warped and the un-warped metrics respectively. The other factors, namely Δ and Λ, appearing above have already been defined with (3.177). Finally the factor $h_o(k_1, k_2; y)$ takes the following form:

$$
h^\alpha_o(k_1, k_2; y) \equiv y^m_{(k_1)} y^n_{(k_2)} \Gamma^\alpha_{mn} + y^\sigma_{(k_1)} y^\gamma_{(k_2)} \Gamma^\alpha_{\sigma\gamma} + y^\sigma_{(k_1)} y^m_{(k_2)} \Gamma^\alpha_{\sigma m}, \tag{3.180}
$$

where the Christoffel symbols are again defined with respect to the un-warped metrics. In this form (3.180) should be compared to g_o in (3.178) which was defined using un-warped metric components also. We can also replace α by m in (3.179), but the form would remain unchanged. Therefore putting everything together, the functional form for $\mathbf{G}_{M0ij}$ becomes:

$$
\begin{aligned}
\mathbf{G}_{M0ij} = & \frac{3g_{NM}\sqrt{-\gamma_{(2)}}}{|g_{00}|(1+f_o)}\left[\Gamma^N_{00} + 4\Delta^2\Lambda\sum_{\{k_i\}} k_1 k_2 h^N_o \left(\frac{g_s}{H}\right)^{2\Delta(k_1+k_2)} + \frac{2\Lambda}{9}\sum_{k_3} k_3(2k_3-7) y^N_{(k_3)}\left(\frac{g_s}{H}\right)^{2\Delta k_3}\right. \\
& \left. + |g_{00}|(1+f_o) g^{i'j'}\Gamma^N_{i'j'} - \frac{8\Lambda^2}{81}\sum_{\{k_i\}} \frac{k_1 k_2 k_3 (k_1+k_2) g_o}{1+f_o} y^N_{(k_3)}\left(\frac{g_s}{H}\right)^{2\Delta(k_1+k_2+k_3)}\right],
\end{aligned} \tag{3.181}
$$

where everything is defined with respect to the un-warped metric except $\sqrt{-\gamma_{(2)}}$, which in turn is defined using the warped $2+1$ dimensional space-time metric, implying that the overall g_s scaling of (3.181) is $\left(\frac{g_s}{H}\right)^{-4}$. This *negative* g_s scaling is important because other than that every term in (3.181) scales as *positive* powers of g_s. Therefore with dynamical M2-branes, in the limit $g_s \to 0$, we can express $\mathbf{G}_{M0ij}$ alternatively as the following series:

$$
\mathbf{G}_{0ijM} = -\left(\frac{g_s}{H}\right)^{-4}\partial_M\left(\frac{\epsilon_{0ij}}{H^4}\right) + \sum_{k\in\frac{\mathbb{Z}}{2}} \mathcal{G}^{(k)}_{0ijM}(y, k)\left(\frac{g_s}{H}\right)^{2\Delta(k-2/\Delta)}, \tag{3.182}
$$

which is somewhat similar to the expression for the other G-flux components in

(3.11). Similarities aside, however, the differences between (3.182) and (3.11) are important now. One of the main difference between these two expressions is that in (3.11), $k \geq \frac{3}{2}$ for (2.2) and $k \geq \frac{9}{2}$ for (2.8). However for (3.182), k can be large or small: smaller k implies, according to (3.176), slowly moving M2-brane and for $k = 0$ it is completely static. Another difference is that even if we impose a *lower* bound on k, the k independent piece should always be there as one may infer from the exact expression in (3.181). It should also be clear from (3.181), when $k = 0$, $\mathcal{G}^{(0)}_{0ijM}(y, 0) = 0$. This is important, because it implies that no mater whether we allow dynamical M2-branes or not, the domination of the k independent term in (3.182) over all other terms for $g_s < 1$ puts a strong confidence on our choice of the G-flux components $\mathbf{G}_{0ijm}$ and $\mathbf{G}_{0ij\alpha}$ for both cases (2.2) and (2.8).

Fluxes, seven-branes and additional dynamics

The exact form of the G-flux components $\mathbf{G}_{0ijM}$ for $M = (m, \alpha)$ appearing in (3.181) and (3.182); as well as our ansatze for the other G-flux components in (3.11) pretty much summarize all the background fluxes that could be allowed in the set-up like ours. However, as the patient reader might have noticed, we did not express the G-flux components in terms of their three-form potentials except for the case studied in (3.181). In particular the three crucial G-flux components, namely $\mathbf{G}_{mnab}, \mathbf{G}_{m\alpha ab}$ and $\mathbf{G}_{\alpha\beta ab}$, now require some explanations. It is of course clear that we do not want to express these three G-flux components in terms of the three-form potentials as $\mathbf{C}_{Mab}$ would create metric cross-terms $\mathbf{g}_{M3}$ in the type IIB side. This is not what we need so $\mathbf{G}_{MNab}$ can only appear as *localized* fluxes in M-theory. In other words:

$$\mathbf{G}_{MNab}(y_1, y_2) = \mathbf{F}_{MN}(y_1) \otimes \Omega_{ab}(y_2), \tag{3.183}$$

where we have divided the internal eight-dimensional coordinates y as $y = (y_1, y_2)$, with y_1 parametrizing the coordinates of the four-dimensional base and y_2 parametrizing the coordinates of the remaining four-dimensional space. Such localized fluxes lead to gauge fields – here we express them as $\mathbf{F}_{MN}$ – on D7-branes. In other words, the orbifold points in M-theory lead to seven-branes in the type IIB side wrapping appropriate four-manifolds that we shall specify below. As alluded to earlier, this set-up is then ripe for embedding the D3-D7 inflationary model [34]. The other factor in (3.183), namely $\Omega_{ab}(y_2)$, is the localized normalizable two-form near any of the orbifold singularities.

In the time-independent case, (3.183) is all that we need, but once time-dependences are switched on new subtleties arise. For example, the G-flux components $\mathbf{G}_{MNab}$ have g_s expansions as in (3.11). Question then is how are the g_s expansions for $\mathbf{F}_{MN}$ and Ω_{ab} defined here. To analyze this, let us first consider the G-flux components $\mathbf{G}_{mnab}$. The flux quantization condition is described in (3.138) on a four-cycle $\Sigma_4^{(1)} \equiv \mathcal{C}_2 \times \frac{\mathbf{T}^2}{\mathcal{G}}$, where $\mathcal{C}_2$ is a two-cycle in $\mathcal{M}_4$. The gauge field $\mathbf{F}_{mn}$ will then have to be defined over this two-cycle, and we expect the corresponding D7-brane to wrap the four-cycle $\mathcal{M}_4$.

Since all cycles in the internal eight-manifold is varying with respect to time, it would make sense to endow time-dependences on *both* the gauge flux components $\mathbf{F}_{mn}$ as well as the normalizable two-form Ω_{ab}. The LHS of (3.138) is where we introduce the split (3.183), and the RHS governs the quantization rule with seven-forms, which in turn may be divided into two sub-forms. Such a split doesn't have any new physics other than what we discussed in (3.138), but a new subtlety arises once we express the gauge field $\mathbf{F}_{mn}$ in terms of its potential $\mathbf{A}_m$ because of it's dependence on g_s as well as on (y^m, y^α). Similar subtlety will arise for the gauge potential $\mathbf{A}_\alpha$. Both these potentials will switch on:

$$\partial_0 \mathbf{A}_m(y^m, y^\alpha, g_s) \equiv H\sqrt{\Lambda}\left(\frac{\partial \mathbf{A}_m}{\partial g_s}\right), \quad \partial_0 \mathbf{A}_\alpha(y^m, y^\alpha, g_s) \equiv H\sqrt{\Lambda}\left(\frac{\partial \mathbf{A}_\alpha}{\partial g_s}\right), \tag{3.184}$$

in addition to the existing field strengths. Clearly such components do not arise in the time-independent case and the split (3.183) is all there is to it. The flux quantization conditions (3.141) and (3.143) tell us that the gauge field strengths $\mathbf{F}_{\alpha\beta} = \partial_{[\alpha}\mathbf{A}_{\beta]}$ and $\mathbf{F}_{m\alpha} = \partial_{[m}\mathbf{A}_{\alpha]}$ will have proper quantization schemes when defined over the two-cycles $\mathcal{M}_2$ and $\mathbf{S}^1_{(1)} \times \mathbf{S}^1_{(2)}$ respectively where $\mathbf{S}^1_{(1)} \in \mathcal{M}_4$ and $\mathbf{S}^1_{(2)} \in \mathcal{M}_2$. Both these one-cycles are allowed because neither $\mathcal{M}_4$, nor $\mathcal{M}_2$ are Calabi-Yau manifolds for the case (2.2). For the case (2.8), **Table 3.2** will tell us that the latter is not well-defined. However now we need to deal with new components arising from temporal derivatives, that translate into g_s derivatives, here. A way out this is to switch on electric potential $\mathbf{A}_0(y^m, y^\alpha, g_s)$ satisfying:

$$\partial_m \mathbf{A}_0 \equiv H\sqrt{\Lambda}\left(\frac{\partial \mathbf{A}_m}{\partial g_s}\right), \quad \partial_\alpha \mathbf{A}_0 \equiv H\sqrt{\Lambda}\left(\frac{\partial \mathbf{A}_\alpha}{\partial g_s}\right), \tag{3.185}$$

which in turn will make $\mathbf{F}_{0m} = \mathbf{F}_{0\alpha} = 0$ and would not contribute to the energy-momentum tensors or the quantum terms (2.78) and (2.94). This could be generalized to the non-abelian case also but since we are only dealing with a single D7-brane, (3.185) suffices. However the dependence of $\mathbf{A}_0$ on g_s also switches on $\frac{\partial \mathbf{A}_0}{\partial g_s}$, but this again does not contribute to the energy-momentum tensors or to the quantum terms (2.78) and (2.94).

Interestingly, if we view *all* the G-flux components as localized fluxes of the form (3.183), then we are in principle dealing with only three gauge field components $\mathbf{F}_{mn}, \mathbf{F}_{m\alpha}$ and $\mathbf{F}_{\alpha\beta}$ on D7-branes that are oriented along various directions in the internal space (they all do share the same $3+1$ dimensional space-time directions in the type IIB side). This is an interesting scenario with only seven-brane gauge fluxes and no $\mathbf{H}_3$ and $\mathbf{F}_3$ three-form fluxes as these would require *global* $\mathbf{G}_{mnpa}, \mathbf{G}_{mn\alpha a}$ and $\mathbf{G}_{m\alpha\beta a}$ G-flux components. Such global G-flux components would in turn give rise to components $\mathbf{G}_{0mnp}, \mathbf{G}_{0mn\alpha}$ and $\mathbf{G}_{0m\alpha\beta}$, which are not what we want here. Question then is whether it is possible to retain global *and* local G-flux components without encountering the issues mentioned above.

It appears that there indeed exists a possible way out of this conundrum if we consider the modified Bianchi identity (3.133), i.e the Bianchi identity with the full quantum corrections, carefully. In the absence of M5-branes, i.e when $N = 0$ in (3.133), we can rewrite (3.133) as:

$$d\left(\mathbf{G}_4 - \frac{c_2}{c_1}\hat{\mathbb{Z}}_4 + \frac{c_3}{c_1} *_{11} \mathbb{Z}_7\right) = 0, \tag{3.186}$$

where c_i are constants, and $\mathbb{Z}_7$ and $\hat{\mathbb{Z}}_4$ are defined in (2.140) and (2.141) respectively. Both of these have g_s dependences and in fact $\mathbb{Z}_7$ features prominently in the flux quantization process as discussed earlier. The above equation allows us to introduce an exact form $d\mathbf{C}_3$, and so we can re-write (3.186) as:

$$\mathbf{G}_4 = d\mathbf{C}_3 + \frac{c_2}{c_1}\hat{\mathbb{Z}}_4 - \frac{c_3}{c_1} *_{11} \mathbb{Z}_7, \tag{3.187}$$

where all quantities are functions of g_s as well as of (y^m, y^α). The $\mathbf{C}_3$ could be understood as the potential, but $\mathbf{G}_4$ is not just $d\mathbf{C}_3$ because of the conpiracies of the quantum terms. Note that nothing actually depends explicitly on $\mathbf{C}_3$ (all quantum terms and the energy-momentum tensors, as well as the flux quantization rules and anomaly cancellation conditions, are expressed using $\mathbf{G}_4$), so we have some freedom in the choice of $\mathbf{C}_3$. We can use this freedom to set:

$$\mathbf{G}_{0MNP} \equiv \partial_{[0}\mathbf{C}_{MNP]} + \frac{c_2}{c_1}(\hat{\mathbb{Z}}_4)_{0MNP} - \frac{c_3}{c_1}(*_{11}\mathbb{Z}_7)_{0MNP} = 0, \tag{3.188}$$

which amounts to putting $\mathbf{F}_{0M} = 0$ for the case $\mathbf{G}_{MNab}$, so they are still localized fluxes as (3.183), but the difference is now that we won't need to switch on an electric flux $\mathbf{A}_0$ on the world-volume of the D7-branes[21]. For the other G-flux components, we can now allow global fluxes so type IIB theory can have $\mathbf{H}_3$ and $\mathbf{F}_3$ three-form fluxes. However as discussed in (3.152) the corresponding G-flux components $\mathbf{G}_{MNPa}$ do not have proper quantization schemes because of the absence of global four-cycles in the M-theory side. However in IIB global three-cycles do exist so these fluxes could be properly quantized in the IIB side. The quantization rule will however follow similar trend as in (3.152).

3.1.3 Stability, swampland criteria and the energy conditions

In the following we will provide possible answers to these questions. Firstly, how stable is our background? How do we overcome the swampland criteria? How can we satisfy the null-energy condition, the strong-energy condition and possibly the dominant-energy condition?

Stability of our background and quantum corrections

One of the important question now is the question of stability of our solution. Before going into this, let us answer a related question on what it means to introduce the series of quantum corrections to solve the EOMs. In other words, how do we interpret the quantum corrections here?

To answer this, let us look at the metric components in the (m, n) i.e $\mathcal{M}_4$ direction. The EOM for g_{mn} is given by (3.15). The LHS of this equation has the Einstein tensor parts and the RHS is the sources, including the quantum terms. The quantum

[21] In other words we can keep $\mathbf{C}_{0MN} = 0$ without loss of generalities. Switching on $\mathbf{C}_{0MN}$ will be equivalent to switching on electric flux $\mathbf{A}_0$ on the D7-branes. Here the quantum terms help us *cancel* the $\partial_0\mathbf{C}_{MNP}$ piece without invoking, for example, pieces like $\partial_P\mathbf{C}_{0MN}$ in (3.188). This is the leverage we get using the quantum terms in (3.188).

terms, i.e $\mathbb{C}_{mn}^{(0,0)}$, are classified by $\theta'_k = 2/3$ in (2.97), and they can at best renormalize the existing classical pieces as $\theta'_k = 2/3$ does not allow higher powers of G-flux or curvature components. Thus the RHS of (3.15) is almost classical, and therefore knowing the G-flux components $\mathcal{G}^{(3/2)}_{mnab}, \mathcal{G}^{(3/2)}_{m\alpha ab}$ and $\mathcal{G}^{(3/2)}_{\alpha\beta ab}$ we can express the RHS of (3.15) in terms of the known quantities.

Going to the next order should switch on the quantum terms. How are they interpreted here? The G-flux components that we gather at the zeroth order in g_s, and the metric g_{mn} that comes out of our zeroth order computation[22], now serve as the *input* for the next order, i.e $g_s^{1/3}$, equations. What they do here is rather instructive. The next order equation is (3.19). The LHS of the equation is the g_{mn} that we computed using all the zeroth order equations. The RHS is however made of quantum terms $\mathbb{C}_{mn}^{(1/2,0)}$ as well as *new* G-flux components like $\mathcal{G}^{(2)}_{mnab}, \mathcal{G}^{(2)}_{m\alpha ab}$ and $\mathcal{G}^{(2)}_{\alpha\beta ab}$ generated at this level, including the higher order C_k and $\widetilde{C}_k$ factors from the $F_i(t)$ functions. The quantum terms are now classified by $\theta'_k = 1$ and appear as (3.21), thus clearly allowing at least to third order G-flux terms. All these new components and the quantum terms, with the background data at the zeroth order, balance each other in a precise way so as to preserve the zeroth order metric component g_{mn}. This is the meaning of (3.19).

The quantum terms are therefore computed on the zeroth order background, with additional new data from fluxes and the $(C_k, \widetilde{C}_k)$ coefficients, to balance each other without changing the zeroth order metric and fluxes. Going to next order, i.e $g_s^{2/3}$, the equation is given by (3.24). We see that the story is repeated in exactly the same fashion: the $g_s^{2/3}$ order switches on new quantum terms, i.e $\mathbb{C}_{mn}^{(1,0)}$ classified by (3.27); new G-flux components and higher order $(C_k, \widetilde{C}_k)$ coefficients; but they do not *de stabilize* the existing zeroth order metric g_{mn} and the G-fluxes. The RHS of (3.24) is precisely the statement of balance: at the $g_s^{2/3}$ order the quantum terms use the data at the zeroth and next (i.e $g_s^{1/3}$) order including *new* G-flux components like $\mathcal{G}^{(5/2)}_{mnab}, \mathcal{G}^{(5/2)}_{m\alpha ab}$ and $\mathcal{G}^{(5/2)}_{\alpha\beta ab}$ to balance each other in such a way that LHS of (3.24) still remains g_{mn}.

The story repeats in the same fashion as we go to higher powers of $g_s^{1/3}$. The quantum terms are computed using the data generated at all lowers orders, including new G-flux components at this order along with the higher order $(C_k, \widetilde{C}_k)$ coefficients. All these balance each other so as to keep the zeroth order data, that include metric g_{mn} and G-flux components, unchanged. This delicate balancing act is responsible for keeping our background safe and stable.

Going to the (α, β) directions, the zeroth order in g_s reproduces the un-warped metric information $g_{\alpha\beta}$, once we have the full data on the G-flux components like $\mathcal{G}^{(3/2)}_{\alpha\beta ab}, \mathcal{G}^{(3/2)}_{\alpha mab}$ and $\mathcal{G}^{(3/2)}_{mnab}$, which are of course the same as before (see (3.42)). On *this* background we now compute the quantum terms $\mathbb{C}_{\alpha\beta}^{(1/2,0)}$ classified by $\theta'_k = 1$ in (2.97). The balancing act starts again: new G-flux components like $\mathcal{G}^{(2)}_{mnab}, \mathcal{G}^{(2)}_{m\alpha ab}$ and $\mathcal{G}^{(2)}_{\alpha\beta ab}$ that are required to this order in g_s are added, to be pitted against the quantum

[22]The zeroth order actually mixes $g_{mn}, g_{\alpha\beta}$ as well as $g_{\mu\nu}$ together, so untangling them would require us to use *all* the zeroth order equations. We will avoid this subtlety for the sake of the present argument, but will become clearer as we go along.

terms and the $F_i(t)$ coefficients, such that the metric $g_{\alpha\beta}$ doesn't change in (3.43). Going to order $g_s^{2/3}$, similar argument holds as seen from (3.45).

For the (a, b) directions, there are no zeroth order contributions. The first non-trivial order is g_s^2, and to this order the metric is flat i.e δ_{ab} from (3.59). This flat metric persists to all higher orders in g_s, as may be seen in (3.60) for $g_s^{7/3}$ and (3.61) for $g_s^{8/3}$ where for both cases the quantum terms computed from the lower order data plus new G-flux components to that order, balance against the fluxes and the $(C_k, \widetilde{C}_k)$ coefficients.

The story takes an interesting turn once we look at the space-time directions. The zeroth order in g_s produces the space-time metric with full de Sitter isometries. The EOM is given by (3.77), and one may note that although the flux components appear as before, the quantum terms are now classified by $\theta'_k = 8/3$ in (2.97) as shown in (3.78). Such an equation has the following important implications. For $n_i = l_{34+i} = 0$ in (3.78), the l_i can at best be bounded as $l_i \le 4$. Since l_i for $i = 1, .., 27$ capture the curvature polynomials in (2.94), this implies that at the *classical* level, the space-time EOM should have the fourth-order curvature terms! Not only that, (3.78) predicts that at the classical level all possible eighth-order[23] polynomials with curvature, G-flux components (classified by l_{34+i}) and derivatives (classified by n_i) are *necessary*. It was known for sometime in the literature that classically the fourth-order curvature polynomials (or eighth-order in derivatives) like:

$$J_0 \equiv t_8 t_8 \mathbf{R}^4, \qquad E_8 \equiv \epsilon_{11}\epsilon_{11}\mathbf{R}^4, \tag{3.189}$$

should play a part, and now we not only can confirm this statement but also show that *all* eighth-order polynomials classified by (3.78) should play a part at the classical level. Of course the exact coefficients of these polynomials cannot be predicted from (2.94) or (3.78), but the fact that this comes out naturally from our analysis should suggest that we are on the right track.

A similar pattern follows for the quantum terms as before. To order $g_s^{1/3}$ the quantum terms, classified by $\theta'_k = 3$ in (2.97), balance each other as (3.79) in such a way that the four-dimensional de Sitter metric do not change. To next order in g_s, i.e $g_s^{2/3}$, the quantum terms, now classified by $\theta'_k = 10/3$, balance against the $(C_k, \widetilde{C}_k)$ coefficients as in (3.80) in a way as to again keep the zeroth order de Sitter metric invariant. The story progresses in the same way as we go to higher orders in g_s.

From the above discussions we can now summarize our view of stability here. The classical EOMs, or the EOMs to the lowest order in g_s (which for most cases are to zeroth order in g_s with the exception of one where the lowest order is g_s^2), for all the components are (3.15), (3.42), (3.59) and (3.77). They involve the so-called quantum terms that, for all cases except the space-time ones, renormalize only the existing classical data. The space-time part contributes eight-order (in derivatives) polynomials. Together with the G-flux components they determine the type IIB metric with four-dimensional de Sitter space-time and the un-warped internal six-dimensional non-Kähler metric. The quantum effects on this background, to order-by-order in powers of $g_s^{1/3}$, are balanced against the G-flux components and the coefficients $(C_k, \widetilde{C}_k)$ coefficients, again to order by order in powers of $g_s^{1/3}$, in a way

[23] In derivatives.

so as to preserve the form of the dual type IIB metric to the lowest order in g_s. This is one important criteria of stability here.

Finally we turn our attention to the possible presence of tachyonic modes around our de Sitter background. This is an important question to determine the relationship between our background and the swampland criteria. The presence of tachyonic modes of sufficiently negative mass would be in agreement with the Hessian de Sitter criterion, while the absence of such would call for a re-examination of the criterion in the context of time-dependent backgrounds.

To determine the presence of tachyons we need to perturb our metric ansatze (2.3) (and also the fluxes) and expand the quantum effective action to second order in the perturbations. Of course, the deciding factor is the sign of the various terms. Since we do not know the coefficients of all the quantum corrections, we can not hope to be completely sure of the absence of tachyonic modes using our approach. We do however have some information about the relative signs of some terms, from the requirement of positive four-dimensional curvature, so there may still be a consistency check available. The constraints on the curvature only manifest themselves in the metric equation of motion so we choose the following perturbations:

$$\delta g_{MN}(x,y) = \phi^{(MN)}(x) g_{MN}(y), \tag{3.190}$$

where x is the coordinate along the $2+1$ dimensional space-time directions and y is the internal space coordinates. For the internal components of the metric, $\phi^{(mn)}(x)$ are simply the scalars one obtains from dimensional reduction. For the space-time components these amount to the scalar modes of metric perturbations. The upside to using perturbations proportional to the "background" values of the fields is that the expansion of the quantum potential to second order in the perturbation is the same as calculating the second order variation of the quantum terms with respect to the original fields. The extra x dependence can generate new contributions to the action, if derivatives along the space-time directions act on it. However this will not result in potential terms, but rather will contribute to the kinetic and higher-derivative terms for the scalar, which will have no bearing on the tachyon question. The downside of this choice of fluctuation is that it ignores the fields which are set to zero[24]. Since terms involving these fields don't appear in our background quantum potential, their sign will not be constrained by the curvature conditions anyway. Other subtleties aside, the first variation of the action with respect to the metric is simply given by the equations of motion:

$$\frac{\delta \mathbb{S}_{11}}{\delta \mathbf{g}^{MN}} = \int d^{11}x\sqrt{-\mathbf{g}_{11}}\left(\mathbf{R}^{(11)}_{MN} - \frac{1}{2}\mathbf{R}^{(11)}\mathbf{g}_{MN} - \mathbb{T}^G_{MN} - \mathbb{T}^Q_{MN}\right), \tag{3.191}$$

where the metric components are all taken as the warped ones and the energy-momentum tensors, especially the quantum energy momentum tensor, take the

[24]We have assumed earlier that we have integrated such components out and that the effects of their fluctuations have thus already been incorporated into the quantum potential. This is strictly speaking only possible if their masses are above the scale at which we are studying the theory. Otherwise there are IR modes left over. Note that in either case, these modes are certainly not tachyonic in the ground state of our EFT, so the implicit hope here is simply that they also do not become tachyonic as we move to the coherent de Sitter state.

form that we have used so far. For example the latter would appear from (2.94), say if we consider only the case (2.2). In other words, we can use (2.94) to express the quantum energy-momentum tensor in the following way:

$$\mathbb{T}^Q_{MN} = \frac{1}{2}\mathbf{g}_{MN}\mathcal{L}^{(Q)} - \frac{\delta\mathcal{L}^{(Q)}}{\delta\mathbf{g}^{MN}}, \tag{3.192}$$

where $\mathcal{L}^{(Q)}$ is the the sum of quantum terms in the action (i.e. without Lorenz indices). This is pretty much equivalent to (2.81), with the quantum pieces expressed together as (2.91). Alternatively, we could also express it more directly as (3.1). With these at hand, the second variation takes the form:

$$\begin{aligned}\frac{\delta^2\mathbb{S}_{11}}{\delta\mathbf{g}^{PQ}\delta\mathbf{g}^{MN}} &= \int d^{11}x\sqrt{-\mathbf{g}_{11}}\Bigg[\frac{\delta\mathbf{R}^{(11)}_{MN}}{\delta\mathbf{g}^{PQ}} - \frac{1}{2}\left(\mathbf{R}^{(11)}_{PQ}\mathbf{g}_{MN} - \mathbf{R}^{(11)}\mathbf{g}_{M(P}\mathbf{g}_{Q)N}\right) - \frac{\delta\mathbb{T}^G_{MN}}{\delta\mathbf{g}^{PQ}} \\ &+ \frac{1}{2}\mathcal{L}^{(Q)}\mathbf{g}_{M(P}\mathbf{g}_{Q)N} - \frac{1}{2}\mathbf{g}_{MN}\frac{\delta\mathcal{L}^{(Q)}}{\delta\mathbf{g}^{PQ}} + \frac{\delta^2\mathcal{L}^{(Q)}}{\delta\mathbf{g}^{PQ}\delta\mathbf{g}^{MN}}\Bigg] + \int d^{11}x\sqrt{-\mathbf{g}_{11}}\,\mathbf{g}_{PQ}(\mathrm{EOM})_{\mathrm{MN}}.\end{aligned} \tag{3.193}$$

Stable solutions to the equations of motion are local maxima of the action, so complete stability would require that the above expression is negative.

Note that the first variation of $\mathcal{L}^{(Q)}$ is still present in the expression, and can be re-expressed in terms of the quantum stress tensor $\mathbb{T}^Q_{MN}$, as in (3.1), which contains the quantum corrections $\mathbb{C}^{(k_1,0)}_{MN}$ that appear in the lowest order equations of motion. From here, one approach could be to make a connection with the positivity of the cosmological constant by, for example, taking the same linear combination of diagonal components as was used to obtain (3.116). However, there are still terms involving $\mathcal{L}^Q$ and more importantly its second variation, which does not appear in the equations of motion. These terms have signs that are not fixed by the trace of the metric equations of motion alone as they depend on all the components and fluxes. This means they would need to be determined by solving for all the metric and flux components.

At this stage we could make some general observations. If we restrict the metric variations to be along the six-dimensional base $\mathcal{M}_4 \times \mathcal{M}_2$, and only consider the case (2.2), the second variation of $\mathcal{L}^{(Q)}$ contains quantum terms classified by $\theta'_k - \frac{4}{3}$. This implies that to zeroth order in g_s, which we used to determine the EOMs, the contributions from the second variation of $\mathcal{L}^{(Q)}$ come from the quantum terms[25] classified by $\theta'_k = \frac{4}{3}$ in (2.97). In a similar vein, if one of the metric variation is along $\mathbb{T}^2/\mathcal{G}$ and the other along the six-dimensional base, or if both the variations are along $\mathbb{T}^2/\mathcal{G}$, then the second variations of $\mathcal{L}^{(Q)}$ come from the quantum terms classified by $\theta'_k + \frac{2}{3}$ or $\theta'_k + \frac{4}{3}$ respectively. Clearly, none of them can contribute to the zeroth order in g_s. On the other hand, if both the metric variations are along the $2+1$ dimensional space-time directions, the quantum terms contributing to the second variation of $\mathcal{L}^{(Q)}$ are classified by $\theta'_k = \frac{16}{3}$ in (2.97). In this way, one could go about finding other combinations, but the message should be clear. If all these

[25]In other words, the first variations of the action i.e the EOMs, provide the background values of metric and G-flux components. These values enter inside the quantum terms classified by θ'_k in (2.97) appearing from the second variations of the action.

contributions are such that they make the RHS of (3.193) negative definite, then there would be no tachyonic instability in our background.

Let us compare this to the first variation of $\mathcal{L}^{(Q)}$ contributing to the cosmological constant Λ in (3.116). The internal space quantum terms are classified by $\theta'_k = \frac{2}{3}$ in (2.97) whereas the $2+1$ dimensional space-time quantum terms are classified by $\theta'_k = \frac{8}{3}$. Since the internal space quantum terms simply renormalize the existing classical terms, the burden of getting *positive* cosmological constant rests solely on the space-time quantum terms classified by $\theta'_k = \frac{8}{3}$. We want them to give positive contributions, so that the relative minus sign in (3.116) can make $\Lambda > 0$. Here, in (3.193), we want the opposite (assuming the contributions from the other terms are negligible). It is easy to see that, compared to the case (3.116), there are now quantum terms classified by $\frac{4}{3} \le \theta'_k \le \frac{16}{3}$ in (2.97), so we are no longer restricted only with the quantum terms classified by $\theta'_k = \frac{8}{3}$. We now require these terms to make the RHS of (3.193) negative definite to avoid the tachyonic instability.

There are also second variations of the action with respect to the $\mathbf{C}_{MNP}$ fields, i.e $\frac{\delta^2 \mathbb{S}_{11}}{\delta \mathbf{C}_{MNP} \delta \mathbf{C}_{RSU}}$, that also need to be considered. Most of the three-form potentials scale in an identical way, so we expect the quantum terms contributing at the zeroth order being classified by $\theta'_k = 4\Delta k$ in (2.97) with $k \ge \frac{3}{2}$ for the case (2.2). We have put to zero components like $\mathbf{C}_{0MN}$ using (3.188), and in fact the quantum term $\mathbb{Z}_7$ has enough degrees of freedom to keep these modes from contributing to the tachyonic instability. The space-time potentials $\mathbf{C}_{0ij}$ would contribute quantum terms classified by $\theta'_k + 8$, so they don't change the zeroth order equations. However now there also be *mixed* variations like $\frac{\delta^2 \mathbb{S}_{11}}{\delta \mathbf{C}_{MNP} \delta \mathbf{g}^{RS}}$, and depending on the choice of k and the orientations of the metric components, some of them would contribute to the zeroth order EOMs. Fortunately the quantum terms contributing to this order, or in general any orders, are finite in number so it is not a very difficult exercise to list all these terms appearing from the second variations of (2.94), and see how the tachyonic instability, if any, could be removed. Similar arguments can be given for the case (2.8) but we will not pursue this here.

Stability, landscape and the swampland criteria

So far we have summarized how the quantum corrections do not destabilize the background, and instead tend to stabilize it at every order in $g_s^{1/3}$. Next we see how the stability extends to keeping the background in the *landscape* and out of the *swampland*. That is, we want to see how the swampland criteria are averted by the the time-dependences of the fluxes and the metric components and by our choice of the quantum potential.

The quantum potential, given in (2.81), basically incorporates the information of either (2.78) and (2.94) for the two cases (2.8) and (2.2) respectively. However it is important to note that the cosmological constant Λ appears almost exclusively from the g_s independent, or time independent, parts of the potential (i.e most of the contribution to Λ appears from the g_s independent parts of $\mathbb{V}_Q$ in (2.81)), and it goes without saying that it is truly a constant[26]. The exact form may be expressed as:

[26] In other words, and taking into account the time-independent Newton's constant from (2.2), the late-time cosmology will always be de Sitter in our set-up and *never* quintessence.

$$\begin{aligned}\Lambda &= \frac{1}{12V_6}\left\langle[\mathbb{C}_i^i]^{(0,0)}\right\rangle - \frac{1}{24V_6H^4}\left\langle[\mathbb{C}_a^a]^{(3,0)}\right\rangle - \frac{1}{48V_6H^4}\left\langle[\mathbb{C}_m^m]^{(0,0)}\right\rangle - \frac{1}{48V_6H^4}\left\langle[\mathbb{C}_\alpha^\alpha]^{(0,0)}\right\rangle \\ &- \frac{\kappa^2T_2n_b}{6V_6H^8} - \frac{5}{384V_6H^8}\left[\left\langle\mathcal{G}^{(3/2)}_{mnab}\mathcal{G}^{(3/2)mnab}\right\rangle + \left\langle\mathcal{G}^{(3/2)}_{m\alpha ab}\mathcal{G}^{(3/2)m\alpha ab}\right\rangle + \left\langle\mathcal{G}^{(3/2)}_{\alpha\beta ab}\mathcal{G}^{(3/2)\alpha\beta ab}\right\rangle\right], \end{aligned} \tag{3.194}$$

which may be easily inferred from (3.116), and we have taken, just for simplicity, a very slowly varying function for H. Thus H is essentially a constant and can come out of the integrals in (3.116). V_6 is the volume of the six-dimensional base $\mathcal{M}_4 \times \mathcal{M}_2$, i.e the volume measured using un-warped metric components. The other expectation values are defined in the standard way − we take the functions and integrate over the volume element − namely:

$$\left\langle[\mathbb{C}_M^M]^{(a,0)}\right\rangle \equiv \int d^6y\sqrt{g_6}[\mathbb{C}_M^M]^{(a,0)}, \ \left\langle\mathcal{G}^{(3/2)}_{MNab}\mathcal{G}^{(3/2)MNab}\right\rangle \equiv \int d^6y\sqrt{g_6}\,\mathcal{G}^{(3/2)}_{MNab}\mathcal{G}^{(3/2)MNab}, \tag{3.195}$$

where g_6 is the determinant of the un-warped metric of the six-dimensional base, (M,N) denote the coordinates of the base and the superscript $a = 0,3$ depending on which quantum corrections we choose. In fact as discussed earlier, the most *dominant* quantum terms are the ones classified by $\theta'_k = \frac{8}{3}$ or $\theta_k = \frac{8}{3}$ in (2.97) and (2.86) respectively. These are the quantum terms $[\mathbb{C}_i^i]^{(0,0)}$, and all other quantum terms simply renormalize the existing classical data. Since the fluxes are taken to be small everywhere and n_b is small[27], the cosmological constant Λ can be made positive here, i.e $\Lambda > 0$. The overall volume suppression in (3.194) tells us that for large enough V_6, Λ could indeed be a tiny but a non-zero positive number. The crucial observation however is that the other parts of $\mathbb{V}_Q$ in (2.81) are used to *stabilize* the classical background in a way discussed earlier, but they do not contribute to the cosmological constant here!

One may also ask how the swampland criteria are taken care of here. The fact that new degrees of freedom do not appear when we switch on time-dependences is easy to infer by looking at the g_s scalings θ_k and θ'_k in (2.86) and (2.97) respectively. Putting $k = 0$ is equivalent to switching-off the time-dependences, and we get θ'_0 as in (2.98) which in-turn is defined with relative minus signs. Existence of such relative minus signs lead to an infinite number of states satisfying (2.99) for any given value of θ'_0 in (2.99). This proliferation of states is one sign of the breakdown of an EFT description, and therefore the theory is indeed in the swampland. Switching on time-dependences miraculously cure this problem as both $\theta_k > 0$ and $\theta'_k > 0$ for the cases (2.8) and (2.2) respectively.

The above reasonings do provide a way to overcome the swampland *distance* criterion, namely, switching on time-dependences allows us to avoid inserting arbitrary number of degrees of freedom at any given point in the moduli space of the theory. The question now is how the original swampland criterion [6], namely, $\partial_\phi V > cV$ is taken care of with $c = \mathcal{O}(1)$ number. To see this, let us consider the quantum terms (2.94) for the case (2.2) (similar argument may be given for (2.78) for the case (2.8)). The potential associated to this is (2.81), and we can get scalars from

[27]Note that it doesn't matter whether we take M2 or anti-M2 branes in (3.194). The *sign* of the cosmological constant Λ *cannot* be changed from either of them − a fact reminiscent of the no-go condition of [12, 13].

the G-flux components as well as from the internal metric components. First let us take a simple example where the scalar fields appear from the G-flux components in the following way:

$$\begin{aligned}
\mathbf{C}_3(x,y) &= \langle \mathbf{C}_3(y)\rangle + \sum_i \phi^{(i)}(x)\Omega^{(i)}_{(3)}(y) + \sum_j \mathbf{A}^{(j)}_1(x)\wedge\Omega^{(j)}_{(2)}(y) + \sum_l \mathbf{B}^{(l)}_2(x)\wedge\Omega^{(l)}_{(1)}(y) \\
\mathbf{G}_4(x,y) &= \langle \mathbf{G}_4(y)\rangle + \sum_i \phi^{(i)}(x)d\Omega^{(i)}_{(3)} + \sum_i d\phi^{(i)}(x)\wedge\Omega^{(i)}_{(3)}(y) + \sum_j \mathbf{F}^{(j)}_2(x)\wedge\Omega^{(j)}_{(2)}(y) \\
&- \sum_j \mathbf{A}^{(j)}_1(x)\wedge d\Omega^{(j)}_{(2)}(y) + \sum_l \mathbf{B}^{(l)}_2(x)\wedge d\Omega^{(l)}_{(1)}(y) + \sum_l \mathbf{H}^{(l)}_3\wedge\Omega^{(l)}_{(1)}, \qquad (3.196)
\end{aligned}$$

where $\Omega^{(j)}_{(k)}$ are the k-forms defined over the internal manifold (we can restrict them to the six-dimensional base $\mathcal{M}_4\times\mathcal{M}_2$ with (i,j) representing the number of independent forms), and are not necessarily harmonic functions as the underlying background is non-supersymmetric and the six-dimensional base is non-Kähler. This also explains why we can allow one-forms like $\Omega^{(i)}_{(1)}$. The two-forms $\Omega^{(j)}_{(2)}$ should not be confused with the localized two-form Ω_{ab} in (3.183). Additionally, (3.183) is the decomposition of the background data itself, whereas (3.196) is the decomposition of the *fluctuations* over our background (2.3)[28]. We are also suppressing the g_s dependences, and therefore both the k-forms and the $2+1$ dimensional space-time fields have g_s dependences. In general, for a manifold whose geometry is varying with time, we expect:

$$\int d\Omega^{(i)}_{(k)}\wedge *_6\, d\Omega^{(j)}_{(k)} \equiv \sum_{\{l_i\}}\int d\Omega^{(l_1,i)}_{(k)}\wedge *_6\, d\Omega^{(l_2,j)}_{(k)}\left(\frac{g_s}{H}\right)^{2\Delta(l_1+l_2)} \qquad (3.197)$$

over the six-dimensional base $\mathcal{M}_4\times\mathcal{M}_2$ with the Hodge star defined over this base. Here l_i denotes the mode expansion that we have used so far. In the standard time-independent supersymmetric case this would have vanished, but now we see explicit g_s dependences complicating our analysis. Finally, the expectation values in (3.196) refer to the background values of the three- and the four-forms that we took earlier to solve the background EOMs (and thus they are functions of y^M). We have also given a small x dependences to the *fluctuations* of the three- and the four-forms, and for computational efficiency, let us assume that we take the G-flux component $\mathbf{G}_{mnpq}$. For simplicity then, $i=1$ in (3.196) with $\mathbf{A}^{(j)}_1(x)=\mathbf{B}^{(l)}_2(x)=0$. Plugging (3.196) into (2.94), we get the following form of the potential:

$$\mathbb{V}_Q(x) = \sum_{l_{28}}\phi^{l_{28}}(x)\mathbf{V}\left(\Phi(x)\right), \qquad (3.198)$$

where $\Phi(\mathbf{x})$ are the set of all other scalars in the system, and l_{28} is a positive integer that appears in (2.94). For our discussions we will take $l_{28}\geq 1$, and from the form of the G-flux components (3.11) it is clear that both $\phi(x)$ as well as $\Omega_{(3)}(y)$ should have

[28]We expect $\mathbf{H}^{(l)}_3 = 0$ because it has no dynamics in $2+1$ dimensions.

g_s dependences, confirming the g_s dependence in (3.197). We can then assume:

$$\phi(x) \equiv \phi^{(1)} = \sum_l \phi^{(1,l)}(\mathbf{x}) \left(\frac{g_s}{H}\right)^{2\Delta l}, \tag{3.199}$$

where l has to be bounded below because the k in G-flux components (3.11) are bounded below as $k \geq \frac{3}{2}$ or $k \geq \frac{9}{2}$ for (2.2) and (2.8) respectively. The swampland criterion then gives us:

$$\frac{\partial_\phi \mathbb{V}_Q}{\mathbb{V}_Q} = \frac{\sum_{l_{28}} l_{28} \sum_{\{k_i\}} \phi^{(1,k_1)}......\phi^{(1,k_{l_{28}})} \left(\frac{g_s}{H}\right)^{2\Delta(k_1+...k_{l_{28}})}}{\sum_{\{r,q_i\}} \phi^{(1,r)}\phi^{(1,q_1)}......\phi^{(1,q_{l_{28}})} \left(\frac{g_s}{H}\right)^{2\Delta(r+q_1+...q_{l_{28}})}} = \mathcal{O}\left(\frac{1}{g_s^n}\right) >> 1, \tag{3.200}$$

where $n = \mathcal{O}(2\Delta r) \in \mathbb{Z}$ and $g_s < 1$. The above computation could be easily generalized to all scalar fields coming from the G-flux components in say (2.94), provided of course the decomposition (3.196) is respected. For example taking all the components of $\phi^{(i)}$ in (3.196), we get:

$$\frac{|\nabla \mathbb{V}_Q|}{\mathbb{V}_Q} = \frac{\sqrt{g^{\phi^{(i)}\phi^{(j)}}\partial_{\phi^{(i)}}\mathbb{V}_Q \partial_{\phi^{(i)}}\mathbb{V}_Q}}{\mathbb{V}_Q} = \mathcal{O}\left(\sum_{k=1}^{\dim(\mathcal{M}_\phi)} \frac{1}{g_s^{n_k}}\right) >> 1, \tag{3.201}$$

where $g^{\phi^{(i)}\phi^{(j)}}$ is the metric on the moduli space $\mathcal{M}_\phi$ of all the scalars represented by $\phi^{(i)}$ which, in turn, could be decomposed as (3.199). The subscript k in n_k is summed from 1 to $\dim(\mathcal{M}_\phi)$, i.e dimension of the moduli space of the scalars. None of the scalars appearing from the G-fluxes are related to the inflaton, so the RHS being much bigger than identity is not unreasonable. Under these circumstances, clearly the swampland bound of [6] is easily satisfied.

On the other hand, the scalars coming from the metric components could in principle also be analyzed in a similar vein as (3.201), but the analysis is complicated by the fact that the potentials for these scalars are not as simple as for the scalars from the G-flux components. In any case, the obvious redundancy in indulging in such exercise should already be apparent from our earlier demonstration of the existence of four-dimensional EFT descriptions with de Sitter isometries. Since these conclusions are derived from meticulously studying the g_s scalings of the quantum terms, the swampland criteria are taken care of here, and these theories belong to the landscape of IIB vacua.

It is more relevant to consider how the energy conditions can be taken care of here, because it brings us to the very foundation on which the no-go criteria of [12, 13] are based. To proceed then we will make the assumption of a slowly varying warp-factor $H(y)$ so that the derivatives of the warp-factor do not un-necessarily complicate the ensuing analysis[29]. To zeroth order in g_s the trace of the energy-momentum tensor is defined as:

$$\mathbb{T}^M_M \equiv \left[\mathbb{T}^M_M\right]^G + \left[\mathbb{T}^M_M\right]^Q, \tag{3.202}$$

[29] In other words, the derivatives of the warp-factor $H(y)$ will add irrelevant functions to the traces that we perform below. We can absorb these functions in the quantum terms.

where the superscript G and Q correspond to the G-flux and the quantum energy-momentum tensors respectively. The traces of the individual pieces are taken with respect to the un-warped internal metric components. Restricting (3.202) to the (m, n), (α, β) and (a, b) directions, yield the following traces:

$$\begin{aligned}
\mathbb{T}^\alpha_\alpha &\equiv [\mathbb{C}^\alpha_\alpha]^{(0,0)} + \frac{1}{8H^4}\left(\mathcal{G}^{(3/2)}_{\alpha\beta ab}\mathcal{G}^{(3/2)\alpha\beta ab} - \mathcal{G}^{(3/2)}_{mnab}\mathcal{G}^{(3/2)mnab}\right) \\
\mathbb{T}^m_m &\equiv [\mathbb{C}^m_m]^{(0,0)} - \frac{1}{4H^4}\left(\mathcal{G}^{(3/2)}_{m\alpha ab}\mathcal{G}^{(3/2)m\alpha ab} + \mathcal{G}^{(3/2)}_{\alpha\beta ab}\mathcal{G}^{(3/2)\alpha\beta ab}\right) \\
\mathbb{T}^a_a &\equiv [\mathbb{C}^a_a]^{(3,0)} + \frac{1}{8H^4}\left(2\mathcal{G}^{(3/2)}_{m\alpha ab}\mathcal{G}^{(3/2)m\alpha ab} + \mathcal{G}^{(3/2)}_{mnab}\mathcal{G}^{(3/2)mnab} + \mathcal{G}^{(3/2)}_{\alpha\beta ab}\mathcal{G}^{(3/2)\alpha\beta ab}\right),
\end{aligned} \tag{3.203}$$

where the individual energy-momentum tensors are defined in subsections 3.1.1, 3.1.1 and 3.1.1 respectively for the case (2.2). A similar construction could be done for the case (2.8) too but we will not pursue this here. Note that, as an interesting fact, if we sum up all the three traces in (3.203), we will get:

$$\mathbb{T}^m_m + \mathbb{T}^\alpha_\alpha + \mathbb{T}^a_a = [\mathbb{C}^m_m]^{(0,0)} + [\mathbb{C}^\alpha_\alpha]^{(0,0)} + [\mathbb{C}^a_a]^{(3,0)}, \tag{3.204}$$

with no contributions from the G-flux components. Thus the total trace of the energy-momentum tensor in the internal space is only given by the quantum terms. These quantum terms are classified by $\theta'_k = \frac{2}{3}$, so they are in turn related to the G-flux components as in (3.114), and therefore only renormalizes the existing classical data. On the other hand, the trace along the $2+1$ dimensional space-time direction yields:

$$\begin{aligned}
&\mathbb{T}^i_i = [\mathbb{C}^i_i]^{(0,0)} - \mathbb{A}^i_i, \quad \mathbb{T}^0_0 = \left[\mathbb{C}^0_0\right]^{(0,0)} - \mathbb{A}^0_0 \\
&\mathbb{A}^i_i = \mathbb{A}^0_0 \equiv \frac{2\kappa^2 T_2 n_b}{H^8\sqrt{g_6}}\delta^8(y-Y) + \frac{1}{8H^8}\left(\mathcal{G}^{(3/2)}_{mnab}\mathcal{G}^{(3/2)mnab} + 2\mathcal{G}^{(3/2)}_{m\alpha ab}\mathcal{G}^{(3/2)m\alpha ab} + \mathcal{G}^{(3/2)}_{\alpha\beta ab}\mathcal{G}^{(3/2)\alpha\beta ab}\right),
\end{aligned} \tag{3.205}$$

where by construction $\mathbb{A}^i_i > 0$ and $\mathbb{A}^0_0 > 0$; and both the quantum terms are classified by $\theta'_k = \frac{8}{3}$ in (2.97). They therefore involve eight-derivative terms as we saw in subsection 3.1.1 for the case (2.2). What we now need is:

$$\begin{aligned}
\mathbb{T}^i_i + \mathbb{T}^0_0 \; &> \; \mathbb{T}^m_m + \mathbb{T}^\alpha_\alpha + \mathbb{T}^a_a \\
[\mathbb{C}^i_i]^{(0,0)} + \left[\mathbb{C}^0_0\right]^{(0,0)} - \mathbb{A}^i_i - \mathbb{A}^0_0 \; &> \; [\mathbb{C}^m_m]^{(0,0)} + [\mathbb{C}^\alpha_\alpha]^{(0,0)} + [\mathbb{C}^a_a]^{(3,0)},
\end{aligned} \tag{3.206}$$

which would be the null energy condition. Consistent with the no-go conditions of [12, 13] and [14] when the quantum terms vanish, the inequality (3.206) cannot be satisfied, . However once we allow the quantum terms, and the very fact that the $[\mathbb{C}^\mu_\mu]^{(0,0)}$ terms are classified by higher order polynomials of curvatures and fluxes, the inequality (3.206) can in principle be satisfied. To see this, let us recall that the $\theta'_k = \frac{2}{3}$ in (2.97) for the internal quantum terms allow us to choose (l_{36}, l_{37}, l_{38}) as (2, 0, 0), (0, 2, 0) or (0, 0, 2) in (2.94), implying at most quadratic in these G-flux components. Additionally, the internal quantum terms, to zeroth order in g_s are constrained as (3.114). Combining these two, one possible solution could be that the internal quantum terms cancel the $\mathbb{A}^\mu_\mu$ terms in (3.206). This could happen for:

$$[\mathbb{C}^a_a]^{(3,0)} = -\frac{1}{6H^8}\Big(\mathcal{G}^{(3/2)}_{mnab}\mathcal{G}^{(3/2)mnab} + 2\mathcal{G}^{(3/2)}_{m\alpha ab}\mathcal{G}^{(3/2)m\alpha ab} + \mathcal{G}^{(3/2)}_{\alpha\beta ab}\mathcal{G}^{(3/2)\alpha\beta ab}\Big) \tag{3.207}$$
$$[\mathbb{C}^m_m]^{(0,0)} + [\mathbb{C}^\alpha_\alpha]^{(0,0)} = \frac{1}{24H^8}\Big(\mathcal{G}^{(3/2)}_{mnab}\mathcal{G}^{(3/2)mnab} + 2\mathcal{G}^{(3/2)}_{m\alpha ab}\mathcal{G}^{(3/2)m\alpha ab} + \mathcal{G}^{(3/2)}_{\alpha\beta ab}\mathcal{G}^{(3/2)\alpha\beta ab}\Big),$$

which still leaves enough freedom to determine $[\mathbb{C}^m_m]^{(0,0)}$ and $[\mathbb{C}^\alpha_\alpha]^{(0,0)}$ individually. The viability of the choice (3.207) is guaranteed from the analysis of the EOMs in subsections 3.1.1, 3.1.1 and 3.1.1, where the input (3.207) could determine what kind of internal non-Kähler manifold we get. Note however that, in determining (3.207), we have ignored the M2-brane contribution. Since $n_b \neq 0$ from (3.166), this can be justified from the fact that for $y^M \neq Y^M$ the M2-brane contributions vanish in $\mathbb{A}^\mu_\mu$ from (3.205). Therefore combining (3.207) with (3.206), we see that as long as:

$$[\mathbb{C}^i_i]^{(0,0)} + \left[\mathbb{C}^0_0\right]^{(0,0)} \; > \; 0, \tag{3.208}$$

the null energy condition may be easily satisfied. Since, and as mentioned repeatedly earlier, the $[\mathbb{C}^\mu_\mu]^{(0,0)}$ are classified by eight derivative polynomials in G-flux and curvature tensors, (3.208) can be satisfied for our background, giving us a precise procedure to satisfy the null energy condition. Under special choices of the higher order polynomials, we can even ask for stronger conditions like (see also [30]):

$$\mathbb{T}^i_i + \mathbb{T}^0_0 \; > \; 0 \quad \text{and/or} \quad \mathbb{T}^0_0 \; > \; 0, \tag{3.209}$$

leading to the strong and the dominant energy conditions respectively. Of course all our discussions have been on the M-theory side, but we could also construct similar criteria in the dual IIB side also as all M-theory ingredients have the corresponding IIB dual in our framework. Note that going beyond zeroth order in g_s is not very meaningful here, at least in demonstrating the null, strong or dominant energy conditions, because the Ricci curvature terms in the Einstein tensors (3.71) and (3.72) only appear to the lowest order in g_s. Once we go to higher orders in g_s, the quantum terms, including higher order G-flux and metric terms, simply stabilize the zeroth order classical background in the way discussed in subsection 3.1.3.

The consistent picture evolving from our analysis points to the fact that four-dimensional de Sitter vacua should be in the IIB string landscape and not in the swampland. The swampland criteria were developed, using the data of time-independent backgrounds, to tackle backgrounds that only made sense with inherent time dependences. As we have showed this cannot work. The unsuitability of such an approach is probably one of the main reasons of its failure to predict backgrounds with positive cosmological constants.

Chapter 4

Discussion and conclusions

The time independent ansatz discussed in [14] was the following.

$$ds^2 = \frac{1}{\Lambda(t)\sqrt{h}}(-dt^2 + dx_1^2 + dx_2^2 + dx_3^2) + \sqrt{h}g_{mn}dy^m dy^n \tag{4.1}$$

In previous studies with time-independent warped product compactification of type IIB strings, it was found that in the presence of fluxes, D (anti) branes and/or Op (anti) planes, classical two derivative gravity is not adequate to give four dimensional de Sitter; i.e. Maldacena-Nunez condition is not satisfied . We must look for quantum corrections via higher-derivative gravity terms arising in string theory in order to get De-Sitter solution in four dimensions. But in this setup quantum correction actually do not help us to get a De-Sitter. Because as discussed in [15] a IIB background with de Sitter isometries in four dimensions and time-independent internal space of the form above together with time-independent background fluxes *cannot* be a solution to the string equation of motion irrespective of how much quantum corrections are added. In fact the g_s scalings of the quantum terms, i.e. equation (2.98), show that when we are taking time independent compactification we need to take into account an infinite number of quantum terms for any given order in g_s. This results in the breakdown of effective field theory description and thus they truly belong to the swampland [6] as discussed in [14, 15, 17].

Once time-dependences are allowed in flux components and internal space, our results change significantly. Depending on the choice of ansatz we can make the four-dimensional Newton's constant time independent, or otherwise. Sections 2.1 and 3.1 contain the main results of the thesis where we present our approach to a time-dependent compactification (2.1), i.e a background where there are de Sitter isometries in four dimensions and the compact internal six-dimensional space has time dependent warp-factors (in the flat slicing of De Sitter we choose $-\infty < t \leq 0$). The simplest example can be the choice of our ansatz (2.8). In this case a IIB background of the form (2.1) when uplifted as (2.3) to M-theory in presence of time dependent G-flux components allows to have an EFT description. But the resulting theory has time dependent Newton's constant in four dimensions. This model has a valid EFT description which is evident from the g_s scalings (2.86) of the quantum contributions (2.78). The time dependence only allows a *finite* number of quantum terms at any given order in g_s. Beside the G-fluxes we can use metric and curvature components (properly contracted) to make quantum terms. We first analysed the curvature terms by themselves and tried to figure out whether polynomial powers of the curvature terms can induce hierarchies to the two cases (2.2) and (2.8). The

g_s scaling of the various curvature tensors associated with time dependent internal manifold are shown in **Table 2.1**. Although time dependent Newton's constant is something not desirable to us for the present purpose, there also appears to be a late time singularity, that prohibits such a configuration to be a viable model of late-time cosmology.

It is to be noted that the quantum corrections are computed near weak flux backgrounds, so any generic quantum term could be expressed simply as a polynomial functions of our four index flux components. Also the background fluxes are also taken time-dependent in such a way so that type IIB coupling constant remain time-independent. This is very significant step in order to make sense of any computations that we performed in our study. The quantum terms must be contracted appropriately with warped inverse metric components in M-theory. We have studied such generic polynomial functions made out of the G-flux components subsection 2.1.2. But there is now a significant change which is the type IIA coupling g_s now becomes a function of time. So we can identify the temporal dependences with g_s dependences. Therefore way we can simply evaluate for g_s dependences of the quantum terms to find out the time dependency of each quantum term.

We have also discussed regarding the quantum terms which are topological in nature in subsection 2.1.2. Just like before these quantum terms are constructed out of curvature forms and different G-flux components. But we can also build up some of the non-topological interactions out of them just using Hodge star operations on them. Similarly we can also construct *dual* forms and therefore also the corresponding quantum terms from them. The quantum terms associated with these dual forms, namely (2.140), and their g_s scalings, appear in **Table 2.2**. We have also calculated the g_s scalings of the quantum terms with dual variables. To our surprise we have found that they are exactly the same as that of before (2.94). After classifying g_s dependency of all sort of quantum terms we further go to the detailed study of the equations of motions (EOMs) in section 3.1. We calculate equation of motion in subsection 3.1.1 by incorporating the energy-momentum tensors. The energy momentum tensors take contributions from the G-fluxes and the quantum terms whose g_s dependencies are already known. The internal eight-dimensional manifold is of the form (2.4) with $\mathcal{M}_4$ parametrized by coordinates (m, n); $\mathcal{M}_2$ parametrized by (α, β) and $\frac{\mathbb{T}^2}{\mathcal{G}}$ parametrized by (a, b). We can easily go from M theory to type IIB just by shrinking the (a, b) torus to zero size. A point to note is that this can also be achieved by taking late time limit $(t \to 0)$ to our M-theory background.

In the next case, we take an ansatz eq. (2.2) which results in a time independent Newton's constant in four dimensions despite the internal manifold as well as the fluxes being time dependent. When this ansatz is uplifted to M-theory, we find that it admits an EFT description. This can be easily evident from the g_s scalings (2.97) of the relevant quantum contributions (2.94). And the disappearance of late time singularity in this background makes it preferable. Also as this background successfully overcomes both the no-go and the swampland criteria, this ansatz is the desired late-time cosmological model in the landscape of string vacua. Therefore we can see how time-dependences of internal manifold and flux components are essential to generate a four-dimensional space-time with de Sitter isometries in the IIB landscape which also has an EFT description. As we recall from the previous studies[14] Minkowski or AdS space can be constructed classically, but de Sitter space requires quantum contributions. So both the quantum terms and time-dependences in flux and metric are equally significant to solve the equation of motion and have an EFT at the every order of g_s. Time dependent compactfication guarantees existence of g_s and M_p hierarchies which evidently allows us to have four-dimensional EFT descriptions as tabulated above. Furthermore we also discuss the quantizations of the G-flux components and anomaly cancellations in a time-dependent background in a time dependent background which further confirms the validity of our work.

In our current study we have not put any detail about the fermions. But we could introduce components of gravitino and their interactions with the bosonic degrees of freedom in M-theory. We can give a small mass to the gravitino components and integrate out all the fermionic degrees of freedom in our setup. This will generate the quantum contributions in the polynomial forms. Therefore, the two sets of quantum corrections (2.78) and (2.94) can be viewed as new degrees of freedom when we integrate out both the fermionic as well other bosonic degrees of freedom. It has also been suggested in many previous studies that the cosmological constant can be derived as expectation value of scalar field or fluxes or quantum corrections. We have derived an exact relation for the cosmological constant Λ, completely in terms of the background fluxes and quantum corrections. Our analysis provides a strong indication that a solution with positive cosmological constants with time-independent Newton's constants can exist in the landscape of string theory.

Beside this we have also derived an exact expression for the cosmological constant Λ, completely in terms of the background fluxes and quantum corrections, which can be expressed as (3.194). We have also determined how the G-flux components, appearing from the back-reaction of a dynamical M2-brane, can be expressed as (3.181). We have also demonstrated quantizations of the G-flux components and anomaly cancellations can be achieved even when time-dependences are switched on. And at last we make further comments about energy condition. For example, the null-energy condition can be shown to be satisfied with the choice of fluxes and quantum corrections. In fact it appears that the $2 + 1$ dimensional quantum corrections play a significant role in satisfying the null-energy condition as shown in (3.208). But for certain special choice of these quantum corrections, one can find out that satisfying the strong and the dominant energy conditions (3.209) is also possible.

In this thesis we have mostly focused on late time physics but we are also interested in early time physics in this framework. One of the most popular theory of early time early time physics is inflation. We should note that dynamical membranes, which become dynamical D3-branes in the IIB side, lead to the possibility of realizing inflation in our set-up. In fact, in the presence of seven-branes our setup could be mapped to the D3-D7 inflationary model. We should be able to access certain levels of e-folds from our set-up. And this is one of the challenges in research we would like to take in near future. But this is not the only difficulty. The other difficulty is at early time physics is strongly coupled. In our language $\Lambda|t|^2 \to 0$ or $g_s \to 0$ is late time in the sense that $\Lambda|t|^2 \to \infty$ is the big-bang time and $\Lambda|t|^2 \to 1$ is the inflationary time. So we need to find a appropriate duality transformations which maps the late time physics to early times. Hopefully we have provided convincing arguments to justify the presence of a late time de Sitter solution in the IIB string landscape and in near future we can predict more about the early time physics from our setup.

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