

Optimized Schwarz Methods for the Advection-Diffusion Equation

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Abstract

The optimized Schwarz methods were recently introduced to enhance the convergence of the classical Schwarz iteration, by replacing the Dirichlet transmission conditions with different conditions obtained through an optimization of the convergence rate. This is formulated as a min-max problem. These new methods are well-studied for elliptic second order symmetric equations. The purpose of this work is to compute optimized Robin transmission conditions for the advection-diffusion equation in two dimensions, by finding the solution of the min-max problem. The asymptotic expansion, for small mesh size h , of the resulting convergence rate is found: it shows a weak dependence on h , if the overlap is $O(h)$ or no overlap is used. Numerical experiments illustrate the improved convergence of these optimized methods compared to other Schwarz methods, and also justify the continuous Fourier analysis performed on a simple model problem only. The theoretical asymptotic performance is also verified numerically.

Résumé

Les méthodes de Schwarz optimisées ont été introduites récemment pour accélérer la convergence de l'itération classique de Schwarz. Ce but est atteint en remplaçant les conditions de transmissions de type Dirichlet par des conditions différentes, obtenues à l'aide d'une optimisation du taux de convergence. Ceci est formulé avec un problème min-max. Ces nouvelles méthodes ont été bien étudiées pour les opérateurs elliptiques et symétriques de second degré. Le propos de ce travail est de calculer des conditions de Robin optimisées pour l'équation d'advection-diffusion en deux dimensions, en trouvant la solution du problème min-max. L'expansion asymptotique du taux de convergence, pour un petit pas de maillage h , est dérivée: elle montre une faible dépendance sur h , pour un recouvrement de largeur $O(h)$, ou sans recouvrement des sous-domaines. Des calculs numériques illustrent la convergence améliorée des méthodes optimisées, comparé à d'autres méthodes de Schwarz. Les résultats justifient également l'analyse de Fourier continue, performée sur un problème modèle simple. La performance asymptotique théorique est aussi vérifiée numériquement.

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Introduction

Can the idea of domain decomposition, which originated from the 19th century, be made into state-of-the-art, lightning fast numerical solvers for partial differential equations? Current research efforts are directed toward answering positively to this question, to obtain modern, competitive solvers.

In 1870, H. A. Schwarz introduced in [Sch70] an iteration over a decomposition of the domain, for solving Laplace's equation. At the time, the utility of this idea was to prove the existence and uniqueness of the solution on domains that are unions of simple geometries. The application to numerical solvers for partial differential equations was studied only much later. The original algorithm of Schwarz has Dirichlet matching conditions at the interfaces (boundaries of subdomains lying inside the domain). However, nothing forces this choice, and it was noted that one could use different conditions at the interfaces to enhance the convergence of the algorithm, for example in [Lio90]. Recently, a min-max optimization problem was formulated to compute good practical choices for these conditions. This leads to the optimized Schwarz methods. These methods are well-studied for symmetric elliptic equations, like the Poisson and Helmholtz problems.

In this thesis, we focus on computing optimized Robin transmission conditions, for a model advection-diffusion problem, by solving the min-max problem. To compare the performance of the optimized methods with other techniques, the asymptotic expansion of the convergence rate, for small mesh size h , is found. Weaker dependence

on h translates into asymptotically better methods.

This work is organized as follows. In Chapter 1, we introduce more precisely the idea of domain decomposition and Schwarz methods, and state some well-known results in the field. In Chapter 2, we show that optimal transmission conditions exist, but are non-local and thus not very convenient for practical use. In Chapter 3, the optimized Schwarz methods are defined, with the optimization problem we wish to solve. Chapter 4 is the center of interest and main contribution of this work: optimized Robin conditions are computed for an advection-diffusion model problem. In Chapter 5, the min-max problem is solved for conditions with two independent parameters, but only a special case of the equation is considered. Finally, in Chapter 6, various numerical experiments illustrate the good performance of optimized Schwarz methods, compared to other techniques.

Chapter 1

Introduction to Domain Decomposition

1.1 Schwarz Methods

Consider the general boundary value problem

$$\begin{cases} \mathcal{L}u = f & \text{in the open domain } \Omega, \\ \mathcal{B}u = g & \text{on } \partial\Omega \text{ (the boundary of } \Omega\text{)}. \end{cases} \quad (1.1)$$

We construct an iterative method by dividing the original domain Ω into two subdomains Ω_1 and Ω_2 (or into N subdomains in general). Let $\Gamma_i = \partial\Omega_i \setminus \partial\Omega$ be the interior boundaries of the subdomains ($i = 1, 2$), called *interfaces*. Note that the subdomains can be overlapping or not (in the latter case $\Gamma_1 = \Gamma_2$). Sometimes a domain decomposition is suggested by the geometry or the underlying physics of the problem. For example, we might have different materials in different subregions of the domain (making the coefficients of the equation discontinuous), and so it would be convenient in this case to use a non-overlapping decomposition corresponding to these different materials. Figure 1.1 shows simple examples of non-overlapping and overlapping domain decompositions.

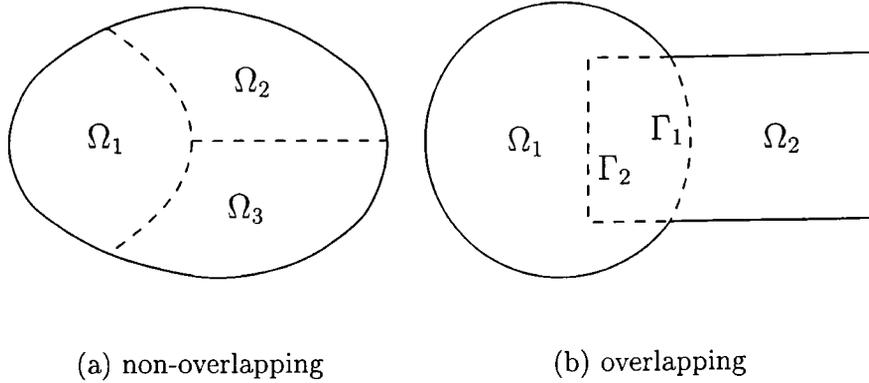


FIGURE 1.1. Examples of domain decomposition.

We construct an iterative procedure as follows. Starting with an initial guess for the solution on the interfaces, problems on the subdomains only are solved. The boundary data at the interfaces is obtained from a previous iterate. First, the subproblems can be solved sequentially: the differential equation is first solved in Ω_1 , and the solution is then used to construct boundary data for the problem in Ω_2

$$\begin{cases} \mathcal{L}u_1^{n+1} = f_1 & \text{in } \Omega_1, \\ \mathcal{B}u_1^{n+1} = g_1 & \text{on } \partial\Omega \cap \partial\Omega_1, \\ \mathcal{B}_1u_1^{n+1} = \mathcal{B}_1u_2^n & \text{on } \Gamma_1, \end{cases}$$

$$\begin{cases} \mathcal{L}u_2^{n+1} = f_2 & \text{in } \Omega_2, \\ \mathcal{B}u_2^{n+1} = g_2 & \text{on } \partial\Omega \cap \partial\Omega_2, \\ \mathcal{B}_2u_2^{n+1} = \mathcal{B}_2u_1^{n+1} & \text{on } \Gamma_2. \end{cases}$$

In the above notation, u_i^n denotes the solution of the subproblem in Ω_i at the n th iteration, f_i is the restriction of f to the subdomain Ω_i and g_i is the restriction of g to $\partial\Omega \cap \partial\Omega_i$. The boundary operators \mathcal{B}_i are called *transmission conditions*, because they control how information is communicated between the subdomains. In other words, they describe the coupling between the subdomains. Throughout this work, it will be assumed that these operators are linear, and are chosen such that the

problem in each subdomain is well-posed. Using Dirichlet transmission conditions in the above iteration (choosing \mathcal{B}_1 and \mathcal{B}_2 to be identity operators), we obtain the well-known *alternating* Schwarz method, or Gauss-Seidel Schwarz (GSS). Note that we could decide to solve in the subdomain Ω_2 first. In general with N subdomains, there are many ways to choose the order for the subdomain solves, leading to different algorithms.

The subproblems can also be solved all in parallel, giving the formulation

$$\begin{cases} \mathcal{L}u_1^{n+1} = f_1 & \text{in } \Omega_1, \\ \mathcal{B}u_1^{n+1} = g & \text{on } \partial\Omega \cap \partial\Omega_1, \\ \mathcal{B}_1u_1^{n+1} = \mathcal{B}_1u_2^n & \text{on } \Gamma_1, \end{cases} \quad (1.2)$$

$$\begin{cases} \mathcal{L}u_2^{n+1} = f_2 & \text{in } \Omega_2, \\ \mathcal{B}u_2^{n+1} = g & \text{on } \partial\Omega \cap \partial\Omega_2, \\ \mathcal{B}_2u_2^{n+1} = \mathcal{B}_2u_1^n & \text{on } \Gamma_2. \end{cases} \quad (1.3)$$

The well-known case is when we use Dirichlet transmission conditions; the resulting method is called the *classical* Schwarz method or Jacobi Schwarz (JS). For a detailed study and comparison between GSS, JSS and their discrete versions, see [Efs03].

These methods were introduced in 1870 by H. A. Schwarz [Sch70] and were used at first to prove existence and uniqueness theorems for the solution of partial differential equations. For example, if the domain Ω is the union of simple geometries where the solution of the problem is known to exist and to be unique (e.g. the Poisson equation on the union of a disk and a rectangle, as in Figure 1.1(b)), one can show existence and uniqueness of the solution on Ω by proving that the classical Schwarz iteration converges. More than a century later, domain decomposition methods came back to interest and are now an active area of research, for developing efficient solvers for partial differential equations. One of the reasons is that these methods are parallel in nature and thus can be implemented efficiently on parallel computers. Another reason

is that one can choose the transmission conditions at the interfaces as to accelerate the convergence of the algorithm. Finding good transmission conditions constitutes the main goal of this work.

Remark 1.1. *In the literature, when the Schwarz method is mentioned, it is often meant as the classical Schwarz method, with Dirichlet transmission conditions. However, in the present discussion, we shall use the name Schwarz for any choice of transmission conditions.*

Some specific choices for the boundary operators lead to other well-known iterative methods. For example, letting $\mathcal{B}_1 u = u$ and $\mathcal{B}_2 u = \frac{\partial u}{\partial n}$, we obtain the *Dirichlet-Neumann* method (DN), which was analyzed for example in [BW86]. Also, we get the *Robin* method by choosing

$$\mathcal{B}_1 u = \frac{\partial u}{\partial n} + \gamma_1 u, \quad \mathcal{B}_2 u = \frac{\partial u}{\partial n} - \gamma_2 u,$$

where γ_i are parameters, often taken to be real, but sometimes complex as in the case of the Helmholtz equation. For the Poisson and modified Helmholtz problems, restricting the γ_i 's to be non-negative and not both 0 guarantees convergence of the Robin method. These two methods (DN and Robin) are usually applied for non-overlapping decompositions.

For any of these iterative methods, we can also introduce an acceleration parameter $\theta \in [0, 1]$, and update one of the interface operators with a weighted sum, using the previous value, for example setting

$$\mathcal{B}_1(u_1^{n+1}) = \theta \mathcal{B}_1(u_2^n) + (1 - \theta) \mathcal{B}_1(u_1^n) \quad \text{on } \Gamma_1,$$

as the boundary condition for the subproblem in Ω_1 .

1.2 Additive and Multiplicative Schwarz

When discretizing problem (1.1), using finite differences or finite elements for example, we obtain a linear system

$$A\mathbf{u} = \mathbf{f}. \quad (1.4)$$

Given a mesh, denote by \mathbf{v} the discrete values of u_1 , and by \mathbf{w} the discrete values of u_2 , for convenience. Also we use \mathbf{v}_i and \mathbf{w}_i for the nodes lying on the interface Γ_i , $i = 1, 2$. When discretizing the subproblems in the Gauss-Seidel Schwarz algorithm, we obtain a linear system for each subproblem

$$\begin{cases} A_1 \mathbf{v}^{n+1} = \mathbf{f}_1, \\ \mathbf{v}_1^{n+1} = \mathbf{w}_1^n, \end{cases} \quad \begin{cases} A_2 \mathbf{w}^{n+1} = \mathbf{f}_2, \\ \mathbf{w}_2^{n+1} = \mathbf{v}_2^{n+1}. \end{cases}$$

Define global approximations of the solution on the whole domain by

$$\mathbf{u}^{n+1} := \begin{cases} \mathbf{w}^{n+1} & \text{in } \bar{\Omega}_2, \\ \mathbf{v}^{n+1} & \text{elsewhere.} \end{cases}$$

Let $R_i \mathbf{u}$ be the restriction of \mathbf{u} to $\bar{\Omega}_i \setminus \Gamma_i$ (i.e. keeping only the entries of \mathbf{u} with indices corresponding to nodes in $\bar{\Omega}_i \setminus \Gamma_i$), and set $\tilde{A}_i^{-1} = R_i^T A_i^{-1} R_i$. Then we can write GSS on the full domain as

$$\mathbf{u}^{n+1} = \mathbf{u}^n + \left(\tilde{A}_1^{-1} + \tilde{A}_2^{-1} - \tilde{A}_2^{-1} A \tilde{A}_1^{-1} \right) (\mathbf{f} - A \mathbf{u}^n), \quad (1.5)$$

or alternatively by

$$\mathbf{u}^{n+1} = \left(I - \tilde{A}_2^{-1} A \right) \left(I - \tilde{A}_1^{-1} A \right) \mathbf{u}^n + \left(\tilde{A}_1^{-1} + \tilde{A}_2^{-1} - \tilde{A}_2^{-1} A \tilde{A}_1^{-1} \right) \mathbf{f}.$$

This discrete method is called the *multiplicative* Schwarz method (MS), because of the product of matrices in front of \mathbf{u}^n in the above equation.

For the Jacobi Schwarz iteration, we can go through a similar argument. Defining

the iterates on the whole domain as

$$\mathbf{u}^{n+1} = \begin{cases} \mathbf{v}^{n+1} & \text{in } \Omega_1 \setminus \Omega_2, \\ \mathbf{v}^{n+1} + \mathbf{w}^{n+1} - \mathbf{u}^n & \text{in } \Omega_1 \cap \Omega_2, \\ \mathbf{w}^{n+1} & \text{in } \Omega_2 \setminus \Omega_1, \end{cases} \quad (1.6)$$

we obtain the iteration

$$\mathbf{u}^{n+1} = \mathbf{u}^n + \left(\tilde{A}_1^{-1} + \tilde{A}_2^{-1} \right) (\mathbf{f} - A\mathbf{u}^n), \quad (1.7)$$

or alternatively written as

$$\mathbf{u}^{n+1} = \left(I - \tilde{A}_1^{-1}A - \tilde{A}_2^{-1}A \right) \mathbf{u}^n + \left(\tilde{A}_1^{-1} + \tilde{A}_2^{-1} \right) \mathbf{f}.$$

This discrete method is called the *additive* Schwarz method (AS), since in this case we only have a sum of the matrices $\tilde{A}_i^{-1}A$. The relation (1.6) may seem like a weird definition for the global iterates in the overlapping region, and indeed this causes convergence problems. We will come back to this issue when we introduce the *restricted* additive Schwarz in Section 1.4. A detailed derivation of equations (1.5) and (1.7) can be found for example in [CM94, SBG96], and a similar representation for the Schwarz algorithms can be obtained without resorting to a discretization of the problem, using continuous operators, see [QV99].

Remark 1.2. *A stationary iterative method to solve the linear system $Ax = b$, with the matrix splitting $A = P - N$, is*

$$x^{n+1} = x^n + P^{-1}(b - Ax^n),$$

where we can think of the matrix P as a preconditioner. The convergence of this method depends on the spectral radius $\rho(P^{-1}N)$. In this section, we have written the multiplicative and additive Schwarz methods in this form, see (1.5) and (1.7).

1.3 The Steklov-Poincaré Equation

Many iterative methods based on nonoverlapping domain decompositions, to solve a boundary value problem in Ω , can be seen as iterative methods for solving an equivalent problem posed on an interface variable appropriately defined. This is the topic of this section. The latter formulation of the method can be useful for proving convergence results, but also for numerical computations since the resulting linear system after discretization has a much smaller dimension, which may be an advantage for solvers that require a lot of storage (e.g. GMRES).

For simplicity, we consider the Dirichlet problem

$$\begin{cases} \mathcal{L}u = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \quad (1.8)$$

where \mathcal{L} is a second order elliptic operator. We follow in this section the notation of Chapter 1 in [QV99]. Only the main steps in the argument are presented and some elementary functional analysis is assumed. Let $a(\cdot, \cdot)$ be the bilinear form associated with the differential operator \mathcal{L} , defined for functions in the Sobolev space

$$H_0^1(\Omega) := \left\{ v \in L^2(\Omega) \mid \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y} \in L^2(\Omega), v|_{\partial\Omega} = 0 \right\}.$$

The weak formulation of problem (1.8) can be written as

$$\text{find } u \in H_0^1(\Omega) : \quad a(u, v) = (f, v)_\Omega \quad \forall v \in H_0^1(\Omega). \quad (1.9)$$

Let $\{\Omega_1, \Omega_2\}$ be a non-overlapping partition of Ω , and Γ be the interface between the two subdomains. We define the local bilinear forms $a_i(\cdot, \cdot)$ as the restriction of $a(\cdot, \cdot)$ on Ω_i , for functions belonging to the space

$$V_i := \{v \in H^1(\Omega_i) : v|_{\partial\Omega_i \setminus \Gamma} = 0\}.$$

Consider an interface function $\lambda \in \Lambda$, where $\Lambda := \{\eta \mid \eta = v|_\Gamma \text{ for } v \in H_0^1(\Omega)\}$ is the *trace space* of $H_0^1(\Omega)$ on Γ . An extension operator $\mathcal{E}_i \lambda$ is defined by solving the

homogeneous subproblem

$$\begin{cases} \mathcal{L}(\mathcal{E}_i\lambda) = 0 & \text{in } \Omega_i, \\ \mathcal{E}_i\lambda = \lambda & \text{on } \Gamma, \\ \mathcal{E}_i\lambda = 0 & \text{on } \partial\Omega \cap \partial\Omega_i. \end{cases}$$

The complementary subproblem, defining the operator \mathcal{G}_if , is

$$\begin{cases} \mathcal{L}(\mathcal{G}_if) = f & \text{in } \Omega_i, \\ \mathcal{G}_if = 0 & \text{on } \partial\Omega_i. \end{cases}$$

Suppose $u \in H_0^1(\Omega)$ solves the weak formulation (1.9) for the differential problem (1.8), and let $u_i := u|_{\Omega_i}$ and $\lambda := u|_{\Gamma}$. Then, it can be seen that

$$\begin{aligned} u_i &= \mathcal{E}_i\lambda + \mathcal{G}_if, \\ a(u, v) &= \sum_{i=1}^2 a_i(u_i, v|_{\Omega_i}) = \sum_{i=1}^2 (f, v|_{\Omega_i})_{\Omega_i}, \quad \forall v \in H_0^1(\Omega). \end{aligned}$$

In particular, choosing $v|_{\Omega_i} := \mathcal{E}_i\mu$ for $\mu \in \Lambda$, we get the equation

$$\langle \mathcal{S}\lambda, \mu \rangle = \langle \chi, \mu \rangle \text{ for } \lambda, \mu \in \Lambda,$$

$$\text{where } \langle \mathcal{S}\lambda, \mu \rangle := \sum_{i=1}^2 a_i(\mathcal{E}_i\lambda, \mathcal{E}_i\mu), \quad \langle \chi, \mu \rangle := \sum_{i=1}^2 [(f, \mathcal{E}_i\mu)_{\Omega_i} - a_i(\mathcal{G}_if, \mathcal{E}_i\mu)].$$

This defines the operator \mathcal{S} , called the *Steklov-Poincaré* operator. In the above, $\langle \cdot, \cdot \rangle$ denotes the duality pairing: it is a bilinear form, from the space $\Lambda' \times \Lambda$ to \mathbb{C} (or \mathbb{R}), defined as $\langle \mathcal{F}, \mu \rangle := \mathcal{F}(\mu)$. We can show that if one solves the variational problem on the interface Γ

$$\text{find } \lambda \in \Lambda : \quad \langle \mathcal{S}\lambda, \mu \rangle = \langle \chi, \mu \rangle \quad \forall \mu \in \Lambda, \quad (1.10)$$

then it follows that $\lambda = u|_{\Gamma}$, where u solves the original differential problem (1.8). Thus, the problem defined over all of Ω can be reduced to a problem on the interface Γ only. After solving problem (1.10) for λ , we can recover the solution u on Ω by solving subproblems with boundary condition $u = \lambda$ on Γ .

We have defined the operator \mathcal{S} in general by using the variational formulation. It is sometimes possible to obtain a differential definition, for simple operators. For example, if $\mathcal{L} = -\Delta$, we can formally write the equation

$$\mathcal{S}\lambda = \chi, \tag{1.11}$$

$$\text{with } \mathcal{S}\lambda := \sum_{i=1}^2 \frac{\partial \mathcal{E}_i \lambda}{\partial n_i}, \quad \chi := - \sum_{i=1}^2 \frac{\partial \mathcal{G}_i f}{\partial n_i},$$

where $\frac{\partial}{\partial n_i}$ is the normal derivative on Γ , in the outward direction with respect to the subdomain Ω_i . The equation (1.11) is called the *Steklov-Poincaré interface equation*. Finally, note that we can define local Steklov-Poincaré operators as

$$\langle \mathcal{S}_i \lambda, \mu \rangle := a_i(\mathcal{E}_i \lambda, \mathcal{E}_i \mu), \quad \mathcal{S} = \mathcal{S}_1 + \mathcal{S}_2.$$

For many iterative methods on subdomains, we can rewrite the iteration and condense it on the interface, to get an iteration on the interface variables only. So, these methods can be seen to be iterations for solving the Steklov-Poincaré interface equation. Taking for example the Dirichlet-Neumann method (with acceleration parameter θ), for the Poisson problem with zero Dirichlet boundary condition, we obtain

$$\lambda^{n+1} = \lambda^n + \theta \mathcal{S}_2^{-1}(\chi - \mathcal{S}\lambda^n), \tag{1.12}$$

which is a stationary iterative method for the Steklov-Poincaré interface system, with *preconditioner* \mathcal{S}_2 .

For symmetric positive definite problems, it was shown that the operators \mathcal{S} and \mathcal{S}_i are symmetric, continuous and coercive. Using the Lax-Milgram lemma, this proves that the variational problem on the interface (1.10) has a unique solution. In addition, these nice properties allow us to prove convergence of a class of methods, including the Dirichlet-Neumann iteration (Chapter 4 in [QV99]).

1.4 Domain Decomposition Preconditioners

In this section, we survey some selected results on preconditioners for the Steklov-Poincaré linear system, for elliptic partial differential equations. We work at the finite-element level, assuming a “nice” triangulation T_h of the domain Ω (no thin triangles). Let h be the maximum diameter of the elements, and H be the maximum diameter of the subdomains. We write the finite-dimensional version of \mathcal{S} as S_h . The goal is to find a preconditioner P_h such that the condition number of $P_h^{-1}S_h$ satisfies (as closely as possible) the properties:

- it is independent of the mesh parameter h ,
- it is independent of the number of subdomains (or H),
- it is independent of the aspect ratio (relative sizes) of the subdomains,
- it is independent of the coefficients of \mathcal{L} (size, discontinuities, etc.).

In addition, we would like a preconditioner such that the matrix-vector product $P_h^{-1}S_h u$ can be computed efficiently; the matrix S_h is dense and very expensive to compute. In two dimensions, we have that $\kappa(S_h) \leq \frac{C}{hH}$, where C is a constant independent of h and H (see [LT94, QV99]), whereas the original stiffness matrix A induced by a discretization of the problem over all of Ω (e.g. using finite differences or finite elements) has a condition number growing like $1/h^2$. In this section, the matrices A , S_h and preconditioners P_h are all symmetric positive definite (unless stated otherwise), and so the spectral condition number is used, which is defined as the ratio of the maximum and minimum magnitude of eigenvalues

$$\kappa(A) = \kappa_2(A) := \frac{\lambda_{max}}{\lambda_{min}}.$$

The criteria mentioned above describe the perfect preconditioner. In the very specific case of the Poisson equation, homogeneous Dirichlet data and with two subdomains

only, it was found that an optimal preconditioner ($\kappa(P_h^{-1}S_h)$ is independent of h) can be efficiently computed using the Fast Fourier Transform. However, in the general case of an elliptic operator \mathcal{L} and N subdomains, the situation is not as easy. See the review article [XZ98] for technical results about well-known preconditioners for the Steklov-Poincaré system.

If a preconditioner is constructed by considering only coupling between adjacent subdomains, then the condition number will grow depending on the number of subdomains, like H^{-2} . Heuristically, at least a growth factor of the order of H^{-1} in the condition number is due to the lack of global communication between the subdomains: information can take up to $O(H^{-1})$ iterations to reach another subdomain, where H is a measure of the diameter of each subdomain. To remedy this problem, we can introduce a *coarse grid correction*. For example, suppose we have a non-overlapping decomposition such that the vertices (nodes on the interface adjacent to more than two subdomains) induce a coarse triangulation T_H , i.e. the subdomains form themselves elements. Let A_H be the associated stiffness matrix. We can incorporate this coarse grid into our preconditioner additively

$$(P_h^{new})^{-1} := P_h^{-1} + R_H^T A_H^{-1} R_H,$$

where R_H is a restriction matrix on the coarse grid, with the usual meaning. This preconditioner was first proposed by Bramble, Pasciak and Schatz [BPS86]. As you might have guessed, this idea is closely related to multigrid methods. For more details on multilevel methods in the context of domain decomposition, see [SBG96].

For overlapping domain decompositions, we can use the additive Schwarz preconditioner. It is defined as the sum of the subdomain approximations

$$P_{as}^{-1} = \sum_{i=1}^N R_i^T A_i^{-1} R_i.$$

Recall that in Section 1.2, we have written the additive Schwarz algorithm as a stationary iterative method with preconditioner P_{as} (see (1.7)). For symmetric problems,

P_{as}^{-1} is symmetric positive definite with respect to the Euclidean inner product (for a proof see [QV99]). This implies that one can use the conjugate-gradient algorithm to solve the preconditioned linear system. In contrast, the multiplicative Schwarz preconditioner gives the product

$$P_{ms}^{-1}A = I - (I - R_N^T A_N^{-1} R_N A) \cdots (I - R_1^T A_1^{-1} R_1 A),$$

which is not symmetric, even if A is. If the overlap between two subdomains is of size δH , with $\delta \in (0, 1)$ (i.e. the overlap keeps the same width relative to the size of the subdomain), we get the results [DW89]

$$\begin{aligned} \kappa(P_{as}^{-1}A) &\leq C \frac{1}{\delta^2 H^2}, \\ &\leq C \frac{1}{\delta} \quad \text{with coarse grid correction.} \end{aligned}$$

More recently in 1997, Cai and Sarkis discovered, by accident, that a modified version of the AS preconditioner was performing better and was requiring less communication between subdomains (processors). It was called the *Restricted Additive Schwarz* (RAS) preconditioner [CS99]. It had been noticed before that the original AS method is not always convergent (for example in [SBG96]), and RAS is resolving this problem. Heuristically, looking at the AS preconditioner, we can observe that contributions from different subdomains get added up in the overlapping regions. To define RAS, we assume that we first have a non-overlapping decomposition $\{\Omega_i^0\}$, which is then extended to the overlapping partition $\{\Omega_i\}$ by growing each subdomain. The RAS preconditioner is obtained from AS by changing the way we add the solutions of the subproblems together to get an approximation on Ω (i.e. changing relation (1.6)). Instead of using the restrictions R_i , we use the restrictions on the corresponding non-overlapping subdomains, R_i^0

$$P_{ras}^{-1} = \sum_{i=1}^N (R_i^0)^T A_i^{-1} R_i.$$

This way, when adding the subdomain solutions, nothing gets added up in the overlapping region. In addition, half of the amount of communication is saved by this modification. For a detailed analysis and comparison of AS, MS and RAS, see [Efs03] and references therein.

1.5 A Method for Advection-Diffusion

The convergence proof of certain iterative methods introduced in Section 1.1 only work for symmetric elliptic differential operators (see [QV99] and references therein). For the Dirichlet-Neumann (DN) method for example, we have stated before by equation (1.12) that it can be written as a stationary iterative method for the Steklov-Poincaré system, with preconditioner \mathcal{S}_2 . For symmetric, positive definite, elliptic differential operators, we can show that the \mathcal{S}_i are symmetric, continuous and coercive operators. The convergence of the DN method follows from these facts. However, for non-symmetric equations, \mathcal{S}_i are not symmetric and not necessarily coercive. Indeed, as will be noted later, DN does not always converge for non-symmetric problems.

We consider an advection-diffusion problem with homogeneous Dirichlet boundary condition

$$\begin{cases} \mathcal{L}u & := -\nu\Delta u + \nabla \cdot (\mathbf{a}u) + cu = f & \text{in } \Omega, \\ u & = 0 & \text{on } \partial\Omega. \end{cases}$$

We define two different bilinear forms for $u, v \in H^1(\Omega)$,

$$\begin{aligned} a^0(u, v) &:= \int_{\Omega} \nu \nabla u \cdot \nabla v + \nabla \cdot (\mathbf{a}u)v + cuv, \\ a^*(u, v) &:= \int_{\Omega} \nu \nabla u \cdot \nabla v + \left(\frac{1}{2} \nabla \cdot \mathbf{a} + c \right) uv + \frac{1}{2} \int_{\Omega} \mathbf{a} \cdot (v \nabla u - u \nabla v). \end{aligned}$$

We introduce a non-overlapping decomposition of Ω into Ω_1 and Ω_2 . The local bilinear forms a_i^* are obtained by restricting a^* to Ω_i . Note that $a^0(\cdot, \cdot)$ and $a^*(\cdot, \cdot)$ coincide

on the space $H_0^1(\Omega)$. The weak form of the differential problem can be written as

$$\text{find } u \in H_0^1(\Omega) : \quad a^*(u, v) = (f, v) \quad \forall v \in H_0^1(\Omega). \quad (1.13)$$

We use the particular bilinear form a^* for a reason: even though it is equivalent to a^0 on our Sobolev space, it leads to interface conditions of Robin type in the multidomain formulation (see Chapter 6 in [QV99] for details). Some assumptions are made on the coefficients of \mathcal{L} :

$$\begin{aligned} \mathbf{a} &\in (L^\infty(\Omega))^2, \quad \nabla \cdot \mathbf{a} \in L^\infty(\Omega), \\ c &\in L^\infty(\Omega), \quad f \in L^2(\Omega), \\ \frac{1}{2} \nabla \cdot \mathbf{a}(\mathbf{x}) + c(\mathbf{x}) &\geq 0. \end{aligned}$$

The last condition in particular ensures that the bilinear form $a^*(\cdot, \cdot)$ is coercive. Under these assumptions, using the Lax-Milgram lemma, the weak form (1.13) of the problem has a unique solution.

Our main interest in this section are advection-dominated problems, i.e. for which the viscosity ν is small. When the viscosity is large enough compared to the advection, one can find convergent domain decomposition algorithms for the problem (e.g. DN) using the same methods as in the symmetric case. However, for small viscosities, these methods can produce instabilities. We summarize here only one specific approach relevant to this work.

Let \mathbf{n} be the normal vector to Γ , pointing outward with respect to Ω_1 , and let γ be a function in $L^\infty(\Gamma)$, positive almost everywhere. The γ -Robin/Robin (γ -RR) iterative method reads

$$\begin{cases} \mathcal{L}u_1^{n+1} = f_1 & \text{in } \Omega_1, \\ u_1^{n+1} = 0 & \text{on } \partial\Omega_1 \cap \partial\Omega, \\ \mathcal{B}_1 u_1^{n+1} = \mathcal{B}_1 u_2^n & \text{on } \Gamma, \end{cases}$$

$$\begin{cases} \mathcal{L}u_2^{n+1} = f_2 & \text{in } \Omega_2, \\ u_2^{n+1} = 0 & \text{on } \partial\Omega_2 \cap \partial\Omega, \\ \mathcal{B}_2u_2^{n+1} = \mathcal{B}_2u_1^{n+1} & \text{on } \Gamma, \end{cases}$$

$$\text{where } \mathcal{B}_1u := \nu \frac{\partial u}{\partial n} - \left(\frac{1}{2} \mathbf{a} \cdot \mathbf{n} - \gamma \right) u, \quad \mathcal{B}_2u := \nu \frac{\partial u}{\partial n} - \left(\frac{1}{2} \mathbf{a} \cdot \mathbf{n} + \gamma \right) u.$$

This method was proposed in [ATV98]. Note that this is the alternating version of γ -RR, the parallel version is obtained by replacing u_1^{n+1} by u_1^n in the second subproblem. This method takes into account the direction of the advection across the interface explicitly in the Robin conditions. Let $\lambda^n := \mathcal{B}_1u_2^n$, and $e_i^n := u_i^n - u|_{\Omega_i}$, where u denotes the exact solution of the original problem. Then, on Γ , we have

$$\begin{aligned} \lambda^{n+1} &= \mathcal{B}_1u_2^{n+1}, \\ &= \mathcal{B}_2u_1^{n+1} + 2\gamma u_2^{n+1}, \\ &= \mathcal{B}_1u_2^n + 2\gamma(u_2^{n+1} - u_1^{n+1}), \\ \lambda^{n+1} &= \lambda^n + 2\gamma(e_2^{n+1} - e_1^{n+1}). \end{aligned}$$

Defining the spaces $V_i := \{v \in H^1(\Omega_i) | v = 0 \text{ on } \partial\Omega \cap \partial\Omega_i\}$, let $\mathcal{R}_i : H^{1/2}(\Gamma) \rightarrow V_i$ be any extension operator on the trace space. Using the local bilinear forms a_i^* , we can write the weak formulation of the γ -Robin/Robin iteration as

$$\begin{aligned} \text{find } u_1^{n+1} \in V_1 : \quad & a_1^*(u_1^{n+1}, v_1) + \int_{\Gamma} \gamma u_1^{n+1} v_1 = (f, v_1)_{\Omega_1} + \int_{\Gamma} \lambda^n v_1, \quad \forall v_1 \in V_1, \\ \text{find } u_2^{n+1} \in V_2 : \quad & a_2^*(u_2^{n+1}, v_2) + \int_{\Gamma} \gamma u_2^{n+1} v_2 = (f, v_2)_{\Omega_2} \\ & + \left[(f, \mathcal{R}_1 v_2)_{\Omega_1} - a_1^*(u_1^{n+1}, \mathcal{R}_1 v_2) + \int_{\Gamma} \gamma u_1^{n+1} v_2 \right], \quad \forall v_2 \in V_2. \end{aligned}$$

The value of λ is updated using the relation

$$\lambda^{n+1} = \lambda^n + 2\gamma(u_2^{n+1} - u_1^{n+1})|_{\Gamma}.$$

Note that the term in the square brackets $[\cdot]$ above is the weak form for $\mathcal{B}_2(u_1^{n+1})$. Given some assumptions on γ , the convergence of the method can be proved.

Theorem 1.1 (Convergence of the γ -RR method). *Let Ω be a Lipschitz polygonal domain, and suppose $\mathbf{a}|_{\Gamma} \in L^{\infty}(\Gamma)^2$. For any initial guess $\lambda^0 \in L^2(\Gamma)$, the sequence of iterates u_i^n in the γ -RR method converges in $H^1(\Omega_i)$ to the restrictions $u|_{\Omega_i}$ of the exact solution, for $i = 1, 2$.*

Proof. See [ATV98] or [QV99]. □

In [ATV98], a strategy is proposed for choosing γ . In the proof of convergence, an upper bound is found for an expression involving the error of the iterates. This upper bound is minimized with respect to γ , resulting in a formula that depends on the exact solution u . The choice proposed is to use this formula and replacing u by u^n , to get the parameter γ^{n+1} for the next iteration.

Remark 1.3. *We could also consider a variant of the γ -RR method, by using two different positive functions γ_1 and γ_2 in the definitions of \mathcal{B}_1 and \mathcal{B}_2 . This would allow more freedom in choosing the transmission conditions, and thus can lead to faster convergence. The convergence of such a method has not been proved yet in a general setting similar to Theorem 1.1.*

Chapter 2

Optimal Transmission Conditions

2.1 The Model Problem

We now turn our attention to the advection-diffusion equation in two dimensions

$$\mathcal{L}u := -\nu\Delta u + \mathbf{a} \cdot \nabla u + cu = f,$$

where we assume that the coefficients are all constant in the plane, with $\nu, c > 0$ and $\mathbf{a} = (a, b)$. The coefficient c usually comes from the implicit discretization of the time dependent problem (e.g. $c = 1/\Delta t$), or is given by $c = \nabla \cdot \mathbf{a}$ if the advection is not divergence free. Physically, the equation represents the divergence of a flux. Our model problem will be to solve the advection-diffusion equation in the plane

$$\begin{cases} \mathcal{L}u = 0 \text{ on } \mathbb{R}^2, \\ u \text{ is bounded at infinity.} \end{cases} \quad (2.1)$$

Since the operator \mathcal{L} is linear, it is enough to consider the homogeneous equation and analyze the convergence of the methods to 0, which is the unique solution of problem (2.1) since $c \neq 0$. We divide the plane into two subdomains, as in Figure 2.1, with an overlap of width L

$$\Omega_1 = \left(-\infty, \frac{L}{2}\right) \times \mathbb{R}, \quad \Omega_2 = \left(-\frac{L}{2}, \infty\right) \times \mathbb{R}.$$

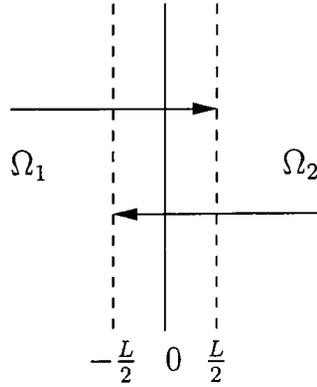


FIGURE 2.1. Decomposition for the model problem.

Since the subdomains are half-planes that are infinite in the y direction, and because the coefficients of the equation are constant, we use the Fourier transform in the y variable to easily obtain explicit formulas for the solution in the subdomains. The Fourier transform is defined, for $L^1(\mathbb{R})$ integrable functions with respect to y , as

$$\begin{aligned}\mathcal{F}_y(u) &= \hat{u}(x, k) = \int_{-\infty}^{\infty} u(x, y) e^{iyk} dy, \\ \mathcal{F}_y^{-1}(\hat{u}) &= u(x, y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{u}(x, k) e^{-iyk} dk, \\ \mathcal{F}_y\left(\frac{\partial u}{\partial y}\right) &= -ik\hat{u}(x, k).\end{aligned}\tag{2.2}$$

The new variable k represents frequencies of the solution in y . Transforming the differential equation to Fourier space, we get

$$\mathcal{F}_y(\mathcal{L}u) = \hat{\mathcal{L}}\hat{u} = -\nu \frac{\partial^2 \hat{u}}{\partial x^2} + a \frac{\partial \hat{u}}{\partial x} + (\nu k^2 - ibk + c)\hat{u} = 0.$$

This is a second order ordinary differential equation in x , with constant coefficients. The corresponding characteristic polynomial has the two roots (possibly complex)

$$\lambda^{\pm}(k) = \frac{a \pm \sqrt{a^2 + 4\nu c - 4i\nu b k + 4\nu^2 k^2}}{2\nu}.\tag{2.3}$$

By inspection, we have that the real part of λ^+ is positive, and the real part of λ^- is negative because $\nu, c > 0$. This explains the choice of notation. Consider solving the

equation $\mathcal{L}u = 0$ on the subdomains, with appropriate conditions at infinity

$$\begin{cases} \mathcal{L}u_1 = 0 & \text{on } \Omega_1, \\ |u_1| < \infty & \text{as } x \rightarrow -\infty, |y| \rightarrow \infty, \end{cases}$$

$$\begin{cases} \mathcal{L}u_2 = 0 & \text{on } \Omega_2, \\ |u_2| < \infty & \text{as } x \rightarrow \infty, |y| \rightarrow \infty. \end{cases}$$

Then, using the Fourier transform in y defined in (2.2), the solutions of these problems in Fourier space are of the form

$$\hat{u}_1(x, k) = C_1(k)e^{\lambda^+(k)(x-\frac{L}{2})}, \quad (2.4)$$

$$\hat{u}_2(x, k) = C_2(k)e^{\lambda^-(k)(x+\frac{L}{2})}, \quad (2.5)$$

where the functions $C_i(k)$ are to be determined by boundary conditions at the interfaces $x = \pm\frac{L}{2}$. We denote by z the square root part of $\lambda^\pm(k)$, namely

$$z(k) = \xi(k) + i\eta(k) := \sqrt{a^2 + 4\nu c - 4i\nu b k + 4\nu^2 k^2}. \quad (2.6)$$

This expression will turn out to be of crucial importance in this study. From now on, to simplify the notation, we omit the boundedness conditions at infinity when writing the subproblems.

The first property of the iteration (1.2) and (1.3) we will seek to have is convergence. In this case, we would like $\{u_i^n(x, y)\}_{n=0}^\infty$ to be convergent sequences for each subdomain Ω_i and at each point $(x, y) \in \Omega_i$. In other words, $u_i^n \rightarrow u_i$ point-wise on Ω_i for some function u_i .

Definition 2.1. The convergence rate of the parallel Schwarz iteration (1.2) and (1.3), applied to our model problem, is defined by

$$\rho(k, L) := \frac{\hat{u}_1^{n+1}(\frac{L}{2}, k)}{\hat{u}_1^{n-1}(\frac{L}{2}, k)}.$$

In the rest of this work, we will say that the iteration is convergent if $|\rho(k, L)| < 1$.

First note that this definition looks only at values of x on the interface Γ_1 . However, convergence of u_1^n on Γ_1 implies convergence on the subdomain Ω_1 , when assuming well-posedness of the subproblems and enough regularity on u_1^n . We cannot use the ratio of consecutive iterates since, for example, u_1^{n+1} is not related to u_1^n but instead depends only on u_2^n . The parallel Schwarz iteration actually contains two separate subsequences, however for our model problem ρ will not depend on n . In addition, we will be able to note later that we always get an equivalent definition for the convergence rate using u_2 on Γ_2

$$\rho(k, L) = \frac{\hat{u}_2^{n+1}(-\frac{L}{2}, k)}{\hat{u}_2^{n-1}(-\frac{L}{2}, k)}.$$

From these observations, it is possible to deduce convergence of u_i^n to the solution u restricted to Ω_i , only from the criterion $|\rho(k, L)| < 1$.

We often get an expression for ρ which is symmetric in k , and so we can look only at nonnegative values for k . Usually in practice, it is sufficient to get convergence only on a bounded range of frequencies $[k_{min}, k_{max}]$ that are relevant to the discretization to be used for solving the subproblems numerically. A discrete mesh cannot capture very high or very low frequencies in the solution. As an approximation for the maximum frequency on a discrete mesh, we take $k_{max} = \pi/h$, where h is the mesh size in the y direction: this corresponds to the frequency of an oscillatory function which is zero at all mesh points. For the minimum frequency, we can use $k_{min} = \pi/H$, where H is the width of the domain Ω in the y direction: this represents the frequency of an oscillatory function which is zero only at the endpoints of an interval. We may also take $k_{min} = 0$ in some situations for simplicity. Note that k_{min} is independent of the discretization.

2.2 Performance of the Classical Schwarz Method

Let us investigate first the performance of the classical Schwarz method (also called Jacobi Schwarz), using Dirichlet transmission conditions, when applied to our model problem,

$$\begin{cases} \mathcal{L}u_1^{n+1} = 0 & \text{on } \Omega_1, \\ u_1^{n+1}(\frac{L}{2}, y) = u_2^n(\frac{L}{2}, y) & \text{for } y \in \mathbb{R}, \\ \mathcal{L}u_2^{n+1} = 0 & \text{on } \Omega_2, \\ u_2^{n+1}(-\frac{L}{2}, y) = u_1^n(-\frac{L}{2}, y) & \text{for } y \in \mathbb{R}. \end{cases}$$

The solutions of the subproblems in Fourier space are of the form given by equations (2.4) and (2.5). Applying the Dirichlet conditions, we obtain

$$\begin{aligned} \hat{u}_1^{n+1}(x, k) &= u_2^n\left(\frac{L}{2}, k\right) e^{\lambda^+(k)(x-\frac{L}{2})}, \\ \hat{u}_2^{n+1}(x, k) &= u_1^n\left(-\frac{L}{2}, k\right) e^{\lambda^-(k)(x+\frac{L}{2})}. \end{aligned}$$

Using these relations, we can easily obtain that

$$\begin{aligned} \hat{u}_1^{n+1}\left(\frac{L}{2}, k\right) &= \hat{u}_2^n\left(\frac{L}{2}, k\right) = \hat{u}_1^{n-1}\left(-\frac{L}{2}, k\right) e^{L\lambda^-(k)}, \\ &= \hat{u}_1^{n-1}\left(\frac{L}{2}, k\right) e^{L(\lambda^-(k)-\lambda^+(k))}. \end{aligned}$$

Hence, the rate of convergence of the classical Schwarz method applied to our model problem is given by

$$\rho_{cs}(k, L) = e^{L(\lambda^-(k)-\lambda^+(k))} = e^{-\frac{L\xi}{\nu}}. \quad (2.7)$$

Proposition 2.1 (Convergence of classical Schwarz). *The classical Schwarz method with overlap ($L > 0$) converges for all non-zero frequencies $k \neq 0$. At $k = 0$, the iteration converges if and only if $|a| > 0$ or $c > 0$.*

Proof. The conditions in the statement guarantee that $L\xi(k) > 0$. Hence,

$$|\rho_{cs}(k, L)| = e^{-\frac{L\xi}{\nu}} < 1.$$

□

We want to study the convergence rate of the *continuous* methods as defined earlier, but by including the dependence of k_{max} and L on the mesh size of an *eventual* discretization. In practice, we usually want to decrease the overlap size proportionally to the mesh size, to limit the computational cost of the solution. It is interesting to compare the performance of different methods (using different transmission conditions) by looking at the asymptotic behavior of the convergence rate as the mesh size decreases to 0. For convenience, we use the notation $\xi_{min} := \xi(k_{min})$.

Proposition 2.2 (Classical Schwarz asymptotics). *For $L = h$, we obtain the asymptotic expansion for small h*

$$\max_{k_{min} \leq k \leq \frac{\pi}{h}} |\rho_{cs}(k, h)| = 1 - \frac{\xi_{min}}{\nu} h + O(h^2).$$

Proof. Recall that we have defined ξ to be the real part of the complex number z . By inspection, $\xi(k)$ is a positive and increasing function for $k \geq 0$. Hence, we get

$$\begin{aligned} |\rho_{cs}(k, h)| &= e^{-\frac{h\xi}{\nu}}, \\ \max_{k_{min} < k < \frac{\pi}{h}} |\rho_{cs}(k, h)| &= e^{-\frac{h\xi(k_{min})}{\nu}}. \end{aligned}$$

Therefore, expanding in a Taylor series around $h = 0$, we get that the convergence rate approaches 1 linearly as $h \rightarrow 0$, as stated. □

We can make a few useful remarks at this point. First, the convergence rate and its asymptotic expansion are independent of scalings of the advection-diffusion operator $\mathcal{L}u$: we can see this by noting that ξ/ν is independent of scalings of the coefficients. This remark holds true for all the remaining results in this work, consequently it will not be noted again. The classical Schwarz algorithm does not converge without overlap, and it converges faster as we increase the size of the overlap. In addition, note that when fixing the overlap size (i.e. using the same value for L as we refine

the mesh), the maximum convergence rate is independent of the mesh parameter h since

$$|\rho_{cs}(k, L)| \leq e^{-\frac{L\xi_{min}}{\nu}}.$$

However, as we mentioned above, we often want to take $L = O(h)$ for computational efficiency. Also, a non-overlapping domain decomposition might be imposed by the physics or geometry of the problem. In these cases, the classical method has poor performance, thus the need to investigate better methods. This will be achieved by changing the transmission conditions in the Schwarz algorithm.

2.3 Rate of Convergence of the Schwarz Method

We continue using our model advection-diffusion problem with constant coefficients, on the plane. To compute the rate of convergence, it is enough to consider the homogeneous equation only, since the operator is linear. Recall our decomposition: the original domain $\Omega = \mathbb{R}^2$ is divided into two subdomains: $\Omega_1 = (-\infty, \frac{L}{2}) \times \mathbb{R}$ and $\Omega_2 = (-\frac{L}{2}, \infty) \times \mathbb{R}$, with overlap size $L \geq 0$. We denote the solutions on each of the two subdomains by $u_1^n(x, y)$ and $u_2^n(x, y)$. With the motivation of enhancing the convergence of the classical Schwarz algorithm, we replace the Dirichlet transmission conditions by operators of the form

$$\mathcal{B}_1 u = \frac{\partial u}{\partial x} + \mathcal{S}_1(u), \quad \mathcal{B}_2 u = \frac{\partial u}{\partial x} + \mathcal{S}_2(u), \quad (2.8)$$

and then apply the parallel Schwarz iteration given by equations (1.2) and (1.3). In the above, \mathcal{S}_i are arbitrary linear operators acting in the y variable only, with Fourier symbols $\sigma_i(k)$, i.e.

$$\mathcal{F}_y[\mathcal{S}_i(u)] = \sigma_i(k)\hat{u}(x, k).$$

Lions was the first to suggest replacing the Dirichlet conditions by Robin conditions to obtain a convergent method for non-overlapping decomposition, see [Lio90]. He

also suggested the use of more general conditions:

First of all, it is possible to replace the constants in the Robin conditions by two proportional functions on the interface, or even by local or nonlocal operators.

By taking the Fourier transform, the boundary operators (2.8) become Robin conditions for fixed k . The idea is to find good candidates for the functions σ_i , leading to the best convergence rate possible. For completeness, we rewrite in full the Schwarz algorithm with the new transmission conditions

$$\begin{cases} -\nu\Delta u_1^{n+1} + \mathbf{a} \cdot \nabla u_1^{n+1} + cu_1^{n+1} = 0 & \text{on } (-\infty, \frac{L}{2}) \times \mathbb{R}, \\ \frac{\partial u_1^{n+1}}{\partial x} + \mathcal{S}_1(u_1^{n+1}) = \frac{\partial u_2^n}{\partial x} + \mathcal{S}_1(u_2^n) & \text{at } x = \frac{L}{2}, \\ -\nu\Delta u_2^{n+1} + \mathbf{a} \cdot \nabla u_2^{n+1} + cu_2^{n+1} = 0 & \text{on } (-\frac{L}{2}, \infty) \times \mathbb{R}, \\ \frac{\partial u_2^{n+1}}{\partial x} + \mathcal{S}_2(u_2^{n+1}) = \frac{\partial u_1^n}{\partial x} + \mathcal{S}_2(u_1^n) & \text{at } x = -\frac{L}{2}. \end{cases}$$

Recall that $Re(\lambda^+) > 0$ and $Re(\lambda^-) < 0$ when assuming $\nu, c > 0$. Thus, using the Fourier transform in y and the boundedness conditions at infinity, we obtain that the solutions on the two subdomains are of the form

$$\hat{u}_1^{n+1}(x, k) = C_1^{n+1}(k)e^{\lambda^+(k)(x-\frac{L}{2})}, \quad \hat{u}_2^{n+1}(x, k) = C_2^{n+1}(k)e^{\lambda^-(k)(x+\frac{L}{2})},$$

which are the same as in equations (2.4) and (2.5). The transmission conditions at the interfaces enable us to solve for the constants C_i^{n+1}

$$\begin{aligned} \frac{\partial \hat{u}_1^{n+1}}{\partial x} + \sigma_1 \hat{u}_1^{n+1} &= (\lambda^+ + \sigma_1) \hat{u}_1^{n+1}, \\ &= (\lambda^+ + \sigma_1) C_1^{n+1} && \text{at } x = \frac{L}{2}, \\ &= \frac{\partial \hat{u}_2^n}{\partial x} + \sigma_1 \hat{u}_2^n && \text{at } x = \frac{L}{2}, \\ &= (\lambda^- + \sigma_1) C_2^n e^{L\lambda^-} && \text{at } x = \frac{L}{2}, \\ \Rightarrow C_1^{n+1} &= \frac{(\lambda^- + \sigma_1)}{(\lambda^+ + \sigma_1)} C_2^n e^{L\lambda^-}. \end{aligned}$$

Similarly, we have that $C_2^{n+1} = \frac{(\lambda^+ + \sigma_2)}{(\lambda^- + \sigma_2)} C_1^n e^{-L\lambda^+}$. Combining the last two equations, we obtain that

$$C_1^{n+1} = \frac{(\lambda^- + \sigma_1)(\lambda^+ + \sigma_2)}{(\lambda^+ + \sigma_1)(\lambda^- + \sigma_2)} C_1^{n-1} e^{L(\lambda^- - \lambda^+)}.$$

Finally we can express \hat{u}_1^{n+1} in terms of \hat{u}_1^{n-1} and get the convergence rate

$$\begin{aligned} \hat{u}_1^{n+1} &= \left[\frac{(\lambda^- + \sigma_1)(\lambda^+ + \sigma_2)}{(\lambda^+ + \sigma_1)(\lambda^- + \sigma_2)} C_1^{n-1} e^{L(\lambda^- - \lambda^+)} \right] e^{\lambda^+(x - \frac{L}{2})}, \\ \hat{u}_1^{n+1} &= \left[\frac{(\lambda^- + \sigma_1)(\lambda^+ + \sigma_2)}{(\lambda^+ + \sigma_1)(\lambda^- + \sigma_2)} e^{L(\lambda^- - \lambda^+)} \right] \hat{u}_1^{n-1}, \\ \rho(k, L, \sigma_1, \sigma_2) &= \frac{(\lambda^- + \sigma_1)(\lambda^+ + \sigma_2)}{(\lambda^+ + \sigma_1)(\lambda^- + \sigma_2)} e^{L(\lambda^- - \lambda^+)}. \end{aligned} \quad (2.9)$$

The convergence rate of the iteration depends on the frequency parameter k : given initial guesses $u_1^0(y)$ and $u_2^0(y)$ for the solution at the interfaces Γ_1 and Γ_2 respectively, different frequency components in the error will converge at *different* speeds. Note that the exponential component of (2.9) is exactly the convergence rate of the classical Schwarz method, and it is the only place where the overlap size L appears. In addition, we get a fractional component in front, which we hope is of magnitude less than 1, to obtain better performance. In particular, choosing $\sigma_1 = -\lambda^-$ and $\sigma_2 = -\lambda^+$, the convergence rate becomes uniformly 0. Hence, we have found an optimal choice for the functions σ_i that make the method converge in two iterations only,

$$\begin{aligned} \sigma_1^{opt}(k) &= -\lambda^-(k), \quad \sigma_2^{opt}(k) = -\lambda^+(k), \\ \Rightarrow \rho(k, L, \sigma_1^{opt}, \sigma_2^{opt}) &\equiv 0. \end{aligned}$$

After one iteration, we have that $\mathcal{B}_i(u_i) = \mathcal{B}_i(u_0)$, where u_0 is an initial guess for the solution. Thus, u_i will not be equal to the exact solution restricted to Ω_i , unless we pick an initial guess such that $\mathcal{B}_i(u_0) = \mathcal{B}_i(u)$, which is unlikely if we don't have knowledge of the exact solution. Thus, the parallel version of the Schwarz method cannot converge after only one iteration, no matter the choice of transmission conditions.

Remark 2.1. *The formulas derived in this chapter also apply to some symmetric problems, simply by choosing appropriate values for the coefficients of the advection-diffusion equation. For reference purposes, here are the values of λ^\pm we obtain for well-known symmetric equations*

- *Poisson equation, $\mathcal{L} = -\Delta$*

$$\lambda^\pm(k) = \pm|k|,$$

- *Helmholtz equation, $\mathcal{L} = -\Delta - \omega^2$*

$$\lambda^\pm(k) = \pm\sqrt{k^2 - \omega^2} \text{ for } |k| \geq \omega, \quad \lambda^\pm(k) = \pm i\sqrt{\omega^2 - k^2} \text{ for } |k| < \omega,$$

- *Modified Helmholtz equation, $\mathcal{L} = -\Delta + \omega^2$*

$$\lambda^\pm(k) = \pm\sqrt{k^2 + \omega^2}.$$

2.4 Optimal Operators

In the previous section, we have found optimal symbols in Fourier space for the Schwarz method with transmission conditions of the type given by (2.8). This was achieved simply by inspection of the convergence rate. We now need to transform back the symbols σ_1^{opt} and σ_2^{opt} to real space, to find out what kind of operators \mathcal{S}_1^{opt} and \mathcal{S}_2^{opt} are.

According to Section 2.1, by taking the Fourier transform in y , the advection-diffusion operator in Fourier space can be factorized as

$$\hat{\mathcal{L}} = -\nu \left(\frac{\partial}{\partial x} - \lambda^+(k) \right) \left(\frac{\partial}{\partial x} - \lambda^-(k) \right).$$

Again for simplicity, we will omit the conditions at infinity when writing the subproblems. Define the operator Λ_f^- as

$$\Lambda_f^-[u](\tilde{x}, y) := \frac{\partial w}{\partial x}(\tilde{x}, y), \tag{2.10}$$

where w solves the problem in the right half-plane, for fixed \tilde{x} ,

$$\begin{cases} \mathcal{L}w = f & \text{for } x > \tilde{x}, \\ w(\tilde{x}, y) = u(\tilde{x}, y) & \forall y \in \mathbb{R}. \end{cases}$$

Similarly, define the operator Λ_f^+ as

$$\Lambda_f^+[u](\tilde{x}, y) := \frac{\partial z}{\partial x}(\tilde{x}, y), \quad (2.11)$$

where z solves the problem in the left half-plane, for fixed \tilde{x} ,

$$\begin{cases} \mathcal{L}z = f & \text{for } x < \tilde{x}, \\ z(\tilde{x}, y) = u(\tilde{x}, y) & \forall y \in \mathbb{R}. \end{cases}$$

For the homogeneous problem, we can solve directly for w and z in the definitions of Λ_0^\pm (taking $f \equiv 0$) by using the Fourier transform in y and the boundary conditions, to get the formulas

$$\begin{aligned} \hat{w}(x, k) &= \hat{u}(\tilde{x}, k)e^{\lambda^-(k)(x-\tilde{x})}, \\ \frac{\partial \hat{w}}{\partial x}(\tilde{x}, k) &= \lambda^-(k)\hat{u}(\tilde{x}, k), \\ \mathcal{F}_y[\Lambda_0^-(u)(\tilde{x}, y)] &= \lambda^-(k)\hat{u}(\tilde{x}, k), \\ \hat{z}(x, k) &= \hat{u}(\tilde{x}, k)e^{\lambda^+(k)(x-\tilde{x})}, \\ \frac{\partial \hat{z}}{\partial x}(\tilde{x}, k) &= \lambda^+(k)\hat{u}(\tilde{x}, k), \\ \mathcal{F}_y[\Lambda_0^+(u)(\tilde{x}, y)] &= \lambda^+(k)\hat{u}(\tilde{x}, k). \end{aligned}$$

Hence, we have that $\Lambda_0^\pm(u) = \mathcal{F}_y^{-1}[\lambda^\pm(k)\hat{u}]$. We can thus derive a factorization of the advection-diffusion operator in real space

$$\begin{aligned} \mathcal{L}u &= \mathcal{F}_y^{-1}[\hat{\mathcal{L}}\hat{u}] = \mathcal{F}_y^{-1}\left[-\nu\left(\frac{\partial}{\partial x} - \lambda^+\right)\left(\frac{\partial}{\partial x} - \lambda^-\right)\hat{u}\right], \\ &= -\nu\left(\frac{\partial}{\partial x} - \Lambda^+\right)\mathcal{F}_y^{-1}\left[\left(\frac{\partial}{\partial x} - \lambda^-\right)\hat{u}\right], \\ \mathcal{L}u &= -\nu\left(\frac{\partial}{\partial x} - \Lambda_0^+\right)\left(\frac{\partial}{\partial x} - \Lambda_0^-\right)u. \end{aligned}$$

The operators Λ_f^+ and Λ_f^- are called Steklov-Poincaré operators. They are also referred to as Dirichlet-to-Neumann maps, because they take Dirichlet boundary data and return the Neumann data corresponding to the solution in one half-plane. This factorization was derived in [NR95], and it is also used in [Jap97]. We have shown that the optimal operators for the Schwarz algorithm described in the previous section, for the homogeneous problem, are given by

$$\mathcal{S}_1^{opt} = -\Lambda_0^-, \quad \mathcal{S}_2^{opt} = -\Lambda_0^+.$$

Now that we have the optimal operators for the homogeneous equation, we can guess an optimal choice for the non-homogeneous equation and show explicitly the convergence after two iterations. Suppose that u satisfies $\mathcal{L}u = f$ on \mathbb{R}^2 and is bounded at infinity. Then note that $\Lambda_f^\pm u = \frac{\partial u}{\partial x}$. So, the solutions of the problems

$$\begin{cases} \mathcal{L}(u_1) = f & \text{for } x < \tilde{x}, \\ \left(\frac{\partial}{\partial x} - \Lambda_f^-\right) u_1 = 0 & \text{at } x = \tilde{x}, \end{cases}$$

$$\begin{cases} \mathcal{L}(u_2) = f & \text{for } x > \tilde{x}, \\ \left(\frac{\partial}{\partial x} - \Lambda_f^+\right) u_2 = 0 & \text{at } x = \tilde{x}, \end{cases}$$

are exactly the restrictions of u on the subdomains $x < \tilde{x}$ and $x > \tilde{x}$ respectively. For that reason, the conditions $\left(\frac{\partial}{\partial x} - \Lambda_f^\pm\right) u = 0$ are called *artificial boundary conditions*. Usually, they are used for truncating infinite or large computational domains. Let us write down the Schwarz algorithm using *artificial* transmission conditions,

$$\begin{cases} \mathcal{L}(u_1^{n+1}) = f & \text{for } x < L/2, \\ \left(\frac{\partial}{\partial x} - \Lambda_f^-\right) u_1^{n+1} = \left(\frac{\partial}{\partial x} - \Lambda_f^-\right) u_2^n & \text{at } x = L/2, \end{cases}$$

$$\begin{cases} \mathcal{L}(u_2^{n+1}) = f & \text{for } x > -L/2, \\ \left(\frac{\partial}{\partial x} - \Lambda_f^+\right) u_2^{n+1} = \left(\frac{\partial}{\partial x} - \Lambda_f^+\right) u_1^n & \text{at } x = -L/2. \end{cases}$$

After the first iteration, we have $\mathcal{L}(u_1^1) = f$ for $x < L/2$. This implies that $\Lambda_f^+ u_1^1 = \frac{\partial u_1^1}{\partial x}$ at $x = -L/2$. Hence, we have $\left(\frac{\partial}{\partial x} - \Lambda_f^+\right) u_1^1 = 0$ at $x = -L/2$. So, for the second

iteration on the subdomain Ω_2 , the boundary condition used is $(\frac{\partial}{\partial x} - \Lambda_f^+) u_2^2 = 0$ at $x = -L/2$. Hence, as we have shown above, we must have $u_2^2 = u|_{\Omega_2}$ where u solves the original problem on Ω . By a similar argument, we can show that $u_1^2 = u|_{\Omega_1}$. Therefore, this Schwarz algorithm converges in exactly two iterations, and is thus optimal with this choice of transmission conditions. The optimal operators for the advection-diffusion equation are discussed in [JN00].

Remark 2.2. *This result can be directly extended to a more general situation, where Ω is a bounded domain, with a decomposition into Ω_1 and Ω_2 . The derivative with respect to x in the boundary operators (2.8) needs to be replaced by directional derivatives on the interfaces*

$$\mathcal{B}_1 u = \frac{\partial u}{\partial n_1} + \mathcal{S}_1(u), \quad \mathcal{B}_2 u = \frac{\partial u}{\partial n_2} + \mathcal{S}_2(u),$$

where $\frac{\partial}{\partial n_1}$ is the normal derivative to Γ_1 in the outward direction with respect to Ω_1 , and $\frac{\partial}{\partial n_2}$ is the normal derivative to Γ_2 in the inward direction with respect to Ω_2

Also notice that, when using the optimal operators, the convergence of the method is completely independent of the overlap size (including the case $L = 0$) and of the aspect ratio of the subdomains.

2.5 More Subdomains

So far, we have only considered decompositions into two subdomains, and found optimal transmission conditions. In general, we may want to decompose the domain into more subdomains. One of the known results is the following: if we decompose Ω into N strips (or bands), and use the optimal transmission conditions computed for two subdomains at each interface, the Schwarz algorithm converges in at most N steps (see [NRdS94]). This result is optimal, in the sense that at least N iterations are needed to “spread” the information about the source term f over all subdomains.

To remove the dependence of the convergence on the number of subdomains, a coarse grid correction is used in practice, see Section 1.4.

Chapter 3

Optimized Schwarz Methods

In Chapter 2, an optimal choice for the linear operators \mathcal{S}_i was found, making the Schwarz method converge after only two iterations. One might think that the story has ended. However, the Steklov-Poincaré operators are non-local in y , as the definitions (2.10) and (2.11) demonstrate. This can also be deduced from the fact that the symbols λ^\pm are not polynomials in k . The optimal transmission conditions are thus costly and not convenient to implement in numerical solvers, because of this non-locality property. We would like to obtain “good” transmission conditions (in a sense to be made precise) that are local in y , which implies operators involving only derivatives with respect to y .

For the one-dimensional advection-diffusion problem on the real line

$$-\nu u'' + au' + cu = f \text{ on } \mathbb{R},$$

the optimal operators are constants, and thus we obtain the Robin conditions

$$\mathcal{B}_{1,2}^{\text{opt}} u = \frac{du}{dx} + \frac{-a \pm \sqrt{a^2 + 4\nu c}}{2\nu} u$$

as optimal transmission conditions.

3.1 Approximated Artificial Boundary Conditions

Recall that the optimal transmission conditions for the homogeneous advection-diffusion equation are given by

$$\mathcal{B}_1^{opt}u = \frac{\partial u}{\partial x} - \Lambda_0^-(u), \quad \mathcal{B}_2^{opt}u = \frac{\partial u}{\partial x} - \Lambda_0^+(u),$$

with $\mathcal{F}_y[\Lambda_0^\pm(u)] = \lambda^\pm \hat{u}$. The goal now is to find good local transmission conditions: they will be more convenient in practice. One strategy is to use Taylor series approximations of λ^\pm around $k = 0$, for the symbols σ_i . This gives low degree polynomials in ik , and when transforming back to real space, we obtain low order derivatives in y . In [NR95], Taylor approximations of order 0, 1 and 2 are considered,

$$\lambda_0^+(k) := \frac{a + \sqrt{a^2 + 4\nu c}}{2\nu},$$

$$\lambda_0^-(k) := \frac{a - \sqrt{a^2 + 4\nu c}}{2\nu},$$

$$\lambda_1^+(k) := \frac{a + \sqrt{a^2 + 4\nu c}}{2\nu} - i \frac{b}{\sqrt{a^2 + 4\nu c}} k,$$

$$\lambda_1^-(k) := \frac{a - \sqrt{a^2 + 4\nu c}}{2\nu} + i \frac{b}{\sqrt{a^2 + 4\nu c}} k,$$

$$\lambda_2^+(k) := \frac{a + \sqrt{a^2 + 4\nu c}}{2\nu} - i \frac{b}{\sqrt{a^2 + 4\nu c}} k + \frac{\nu}{\sqrt{a^2 + 4\nu c}} \left(1 + \frac{b^2}{a^2 + 4\nu c}\right) k^2, \quad (3.1)$$

$$\lambda_2^-(k) := \frac{a - \sqrt{a^2 + 4\nu c}}{2\nu} + i \frac{b}{\sqrt{a^2 + 4\nu c}} k - \frac{\nu}{\sqrt{a^2 + 4\nu c}} \left(1 + \frac{b^2}{a^2 + 4\nu c}\right) k^2. \quad (3.2)$$

In contrast to the classical Schwarz method, using the above approximations guarantees the convergence of the iteration, even without overlap. This will be proved in Section 3.3. One immediate drawback of this approach is that the Taylor approximations will be good only for low frequencies. These conditions can be used with an overlap, which has the effect of accelerating the convergence for large frequencies.

However, to cover the whole range of relevant frequencies, it would often be necessary to use a high order Taylor approximation and a large overlap size, which might not be possible in practice.

We now investigate the asymptotic performance of some of these methods, as the mesh size h goes to zero. The results in this section are proved in [Jap97], but we present here different proofs using our notation.

Recall that we use the notation

$$z(k) = \xi(k) + i\eta(k) := \sqrt{a^2 + 4\nu c - 4i\nu b k + 4\nu^2 k^2}.$$

Proposition 3.1 (Zeroth order Taylor asymptotics, no overlap). *Let*

$z_0 := z(0) = \sqrt{a^2 + 4\nu c}$. *Taking $k_{max} = \frac{\pi}{h}$, the asymptotic performance of the zeroth order Taylor transmission conditions without overlap is*

$$\max_{k_{min} \leq k \leq k_{max}} |\rho_{T0}(k, 0)| = 1 - \frac{2z_0}{\nu\pi} h + O(h^2).$$

Proof. The zeroth order Taylor approximation of the optimal symbols $-\lambda^\pm$ are

$$-\lambda^-(0) = \frac{-a + z(0)}{2\nu}, \quad -\lambda^+(0) = \frac{-a - z(0)}{2\nu}.$$

The rate of convergence (2.9) becomes

$$\rho_{T0}(k, 0) = \left| \frac{(z_0 - z)(z - z_0)}{(-z - z_0)(z + z_0)} \right| = \frac{(\xi - z_0)^2 + \eta^2}{(\xi + z_0)^2 + \eta^2},$$

where we have defined earlier that $z = \xi + i\eta$. We have $z_0 > 0$, which implies $|\rho_{T0}(k, 0)| < 1$. Also, taking the limit as $k \rightarrow \infty$, it can be seen that $\xi \rightarrow \infty$ and η remains bounded. Hence,

$$\lim_{k \rightarrow \infty} |\rho_{T0}(k, 0)| = 1.$$

Therefore, for k_{max} large enough, we get that the maximum over the interval is attained at the maximum frequency

$$\max_{k_{min} \leq k \leq k_{max}} |\rho_{T0}(k, 0)| = |\rho_{T0}(k_{max}, 0)|, \quad \text{for } k_{max} \text{ large.}$$

Given that $k_{max} = \frac{\pi}{h}$, the real and complex parts of z behave like

$$\begin{aligned}\xi &= \frac{2\nu\pi}{h} + O(h), \\ \eta^2 &= b^2 - O(h^2).\end{aligned}$$

Thus, inserting these asymptotic terms into the convergence rate, we get

$$\begin{aligned}|\rho_{T0}(k_{max}, 0)| &\sim \frac{\xi^2 - 2z_0\xi}{\xi^2 + 2z_0\xi} \sim \frac{1 - \frac{z_0}{\nu\pi}h}{1 + \frac{z_0}{\nu\pi}h}, \\ \max_{k_{min} \leq k \leq k_{max}} |\rho_{T0}(k, 0)| &= 1 - \frac{2z_0}{\nu\pi}h + O(h^2).\end{aligned}$$

□

The zeroth order Taylor approximation without overlap has the same asymptotic performance as the classical Schwarz method with overlap of size h : the convergence rate approaches 1 linearly as $h \rightarrow 0$.

Proposition 3.2 (Zeroth order Taylor asymptotics, with overlap and $b = 0$).

Let $k_{max} = \frac{\pi}{h}$ and $L = h$. The asymptotic performance of the zeroth order Taylor transmission conditions with overlap and $b = 0$ is

$$\max_{k_{min} \leq k \leq k_{max}} |\rho_{T0}(k, h)| = 1 - 4\sqrt{\frac{z_0}{\nu}}h^{\frac{1}{2}} + O(h).$$

Proof. From the previous proof (the conditions are the same here), the convergence rate has an extra exponential component due to the overlap,

$$\begin{aligned}|\rho_{T0}(k, h)| &= \left| \frac{(z - z_0)^2}{(z + z_0)^2} e^{-\frac{hz}{\nu}} \right|, \\ &= \frac{(\xi - z_0)^2}{(\xi + z_0)^2} e^{-\frac{h\xi}{\nu}},\end{aligned}$$

since $z = \xi$ is real because $b = 0$. Now, to find the maximum of the convergence rate, we look at the derivative with respect to ξ , and by simple algebra we obtain that

$$\frac{\partial \rho_{T0}}{\partial \xi} = 0 \iff \xi_1 = z_0 \text{ or } \xi_2 = \sqrt{z_0^2 + \frac{4\nu z_0}{h}}.$$

Note that the critical value ξ_2 will eventually lie in the interval $[\xi(k_{min}), \xi(k_{max})]$: it grows like $h^{-1/2}$ and ξ_{max} grows faster, like h^{-1} . We can easily check that ξ_1 is a minimum and ξ_2 is a maximum. Let k_2 be such that $\xi_2 = \xi(k_2)$. Thus, the maximum of the convergence rate over the interval is attained either at ξ_{min} or at ξ_2 . As $h \rightarrow 0$, $|\rho_{T0}(k_{min}, h)| < 1$, and $|\rho_{T0}(k_2, h)| \rightarrow 1$ because ξ_2 grows like $h^{-1/2}$. Therefore, we get, for h small enough,

$$\max_{k_{min} \leq k \leq k_{max}} |\rho_{T0}(k, h)| = |\rho_{T0}(k_2, h)|.$$

Finally, inserting the asymptotic expansion for ξ_2 into the convergence rate, we obtain

$$\begin{aligned} \xi_2 &= \frac{2\sqrt{\nu z_0}}{h^{1/2}} + O(h^{1/2}), \\ |\rho_{T0}(k_2, h)| &\sim \left(\frac{\frac{2\sqrt{\nu z_0}}{h^{1/2}} - z_0}{\frac{2\sqrt{\nu z_0}}{h^{1/2}} + z_0} \right)^2 e^{-2\sqrt{\frac{z_0}{\nu}} h^{1/2}}, \\ &\sim \left(1 - \sqrt{\frac{z_0}{\nu}} h^{1/2} \right)^2 \left(1 - 2\sqrt{\frac{z_0}{\nu}} h^{1/2} \right), \\ \Rightarrow \max_{k_{min} \leq k \leq k_{max}} |\rho_{T0}(k, h)| &= 1 - 4\sqrt{\frac{z_0}{\nu}} h^{1/2} + O(h). \end{aligned}$$

□

The above result was proved only in the case $b = 0$, but it also holds when b is not zero: the general proof is not presented here since it is more involved. When using an overlap of size $O(h)$, the performance of the zeroth order Taylor approximation is improved, going from $O(h)$ to $O(h^{1/2})$, thus weakening the dependence on the mesh size. What happens when we use higher order Taylor expansions? Let us look at the second order approximations, rewriting them for convenience as

$$\lambda_2^\pm(k) = \frac{1}{2\nu} (a \pm p \pm qik \pm rk^2),$$

where p , q and r are the appropriate constants in the Taylor expansions (3.1) and (3.2). Using this notation, the convergence rate without overlap can be written as

the simple expression

$$|\rho_{T2}(k, 0)| = \left| \frac{p + qik + rk^2 - z}{p + qik + rk^2 + z} \right|^2 = \frac{(p + rk^2 - \xi)^2 + (qk - \eta)^2}{(p + rk^2 + \xi)^2 + (qk + \eta)^2}. \quad (3.3)$$

Proposition 3.3 (Second order Taylor asymptotics, without overlap). *Let $k_{max} = \frac{\pi}{h}$. The asymptotic performance of the second order Taylor transmission conditions, without overlap, is*

$$\max_{k_{min} \leq k \leq k_{max}} |\rho_{T2}(k, 0)| = 1 - \frac{8\nu}{r\pi}h + O(h^2),$$

where the constant r is given by

$$r := \frac{2\nu^2}{\sqrt{a^2 + 4\nu c}} \left(1 + \frac{b^2}{a^2 + 4\nu c} \right).$$

Proof. Again by inspection, the convergence rate approaches 1 as $k \rightarrow \infty$, and $|\rho_{T2}(k, 0)| < 1$ for all relevant frequencies. Thus, for h small enough the maximum of the convergence rate is attained at the maximum frequency

$$\max_{k_{min} \leq k \leq k_{max}} |\rho_{T2}(k, 0)| = |\rho_{T2}(k_{max}, 0)|.$$

Expanding the numerator and denominator of (3.3) and keeping only the leading order terms, we obtain

$$\begin{aligned} |\rho_{T2}(k_{max}, 0)| &\sim \frac{r^2 k^4 - 2r\xi k^2}{r^2 k^4 + 2r\xi k^2} \sim \frac{\frac{r^2 \pi^4}{h^4} - \frac{4r\pi^3 \nu}{h^3}}{\frac{r^2 \pi^4}{h^4} + \frac{4r\pi^3 \nu}{h^3}} \sim \frac{1 - \frac{4\nu}{r\pi}h}{1 + \frac{4\nu}{r\pi}h}, \\ &\Rightarrow \max_{k_{min} \leq k \leq k_{max}} |\rho_{T2}(k, 0)| = 1 - \frac{8\nu}{r\pi}h + O(h^2). \end{aligned}$$

□

Similarly, we expect the convergence rate for the second order Taylor approximation with overlap to have an asymptotic expansion of the form $1 - O(h^{1/2})$: this was proved for the modified Helmholtz equation in [Gan03]. Thus, using higher order Taylor approximations does not improve the asymptotic performance of the method. This will not be the case for the new optimized conditions, that we introduce next.

3.2 The Optimization Idea

Let us summarize quickly the previous ideas. Optimal operators exist for conditions of the form (2.8), however they are non-local in y and thus are not very convenient for practical implementations. Taylor series approximations of the optimal symbols can be used, but they are effective for low frequencies only, and increasing the order of the expansion does not improve the asymptotic performance.

We would like a method that converges fast uniformly for all relevant frequencies. We proceed by fixing a class of local transmission conditions \mathcal{C} , and minimize uniformly the convergence rate over this class. Mathematically, this is formulated as a min-max problem

$$\min_{\mathcal{B}_i \in \mathcal{C}} \left(\max_{k_{min} \leq k \leq k_{max}} |\rho(k, L, \mathcal{B}_1, \mathcal{B}_2)| \right). \quad (3.4)$$

For example, the class \mathcal{C} can consist of Robin conditions, or second order conditions in the tangential derivative to the interface. The maximum of the convergence rate is taken over a range of frequencies. In the cases we consider, $|\rho|$ is always an even function of k , and so we look at positive frequencies only. The minimum frequency is limited by the diameter of the domain or by a coarse grid if one is used. In the latter case, frequencies lower than π/H (approximately) are handled by the coarse grid with mesh size H . The maximum frequency is restrained by the mesh size of the particular discretization. The best transmission conditions in a class are computed by solving the min-max problem (3.4) for the free parameters. This idea was successfully applied to Poisson and Helmholtz problems (see Section 3.4), to the advection-diffusion equation (see Section 3.5), and also in the context of waveform relaxation for RC circuits (see [AK02] and references therein).

We now define a terminology for the different classes of transmission conditions we will refer to.

Definition 3.1 (Optimized Robin conditions). We first use Robin transmission

conditions, but reduced to only one free parameter, by relating the two boundary operators in the following way

$$\mathcal{B}_1 u = \frac{\partial u}{\partial x} + \left(\frac{-a + p}{2\nu} \right) u, \quad \mathcal{B}_2 u = \frac{\partial u}{\partial x} + \left(\frac{-a - p}{2\nu} \right) u.$$

The reason for the specific form of these conditions will become clear later. The associated min-max problem on the single parameter p is given by

$$\min_{p \in \mathbb{R}} \left(\max_{k_{\min} \leq k \leq k_{\max}} |\rho_{OO0}(k, L, p)| \right).$$

The optimized Robin parameter, obtained by solving this min-max problem, will be denoted by p^* . The subscript $OO0$ stands for “Optimized of Order 0” (a notation introduced by Japhet in [Jap97]), since the resulting conditions can be seen as zeroth order polynomial approximations of the optimal symbols λ^\pm .

Definition 3.2 (Optimized two-sided Robin conditions). Again Robin transmission conditions are used, but leaving the two boundary operators separate, giving two distinct parameters

$$\mathcal{B}_1 u = \frac{\partial u}{\partial x} + \left(\frac{-a + p_1}{2\nu} \right) u, \quad \mathcal{B}_2 u = \frac{\partial u}{\partial x} + \left(\frac{-a - p_2}{2\nu} \right) u.$$

The associated min-max problem on the two parameters p_1 and p_2 is given by

$$\min_{p_1, p_2 \in \mathbb{R}} \left(\max_{k_{\min} \leq k \leq k_{\max}} |\rho_{OO0}(k, L, p_1, p_2)| \right).$$

Definition 3.3 (Optimized second order conditions). In this case, we take transmission conditions that are second order in the tangential direction to the interface. For our model problem, this translates to the conditions

$$\begin{aligned} \mathcal{B}_1 u &= \frac{\partial u}{\partial x} + \frac{1}{2\nu} \left[(-a + p)u - q \frac{\partial u}{\partial y} - r \frac{\partial^2 u}{\partial y^2} \right], \\ \mathcal{B}_2 u &= \frac{\partial u}{\partial x} + \frac{1}{2\nu} \left[(-a - p)u + q \frac{\partial u}{\partial y} + r \frac{\partial^2 u}{\partial y^2} \right]. \end{aligned}$$

The associated min-max problem on the three free parameters p , q and r is given by

$$\min_{p,q,r \in \mathbb{R}} \left(\max_{k_{min} \leq k \leq k_{max}} |\rho_{OO2}(k, L, p, q, r)| \right).$$

The optimized parameter values, obtained by solving the above min-max problem, will be denoted by (p^*, q^*, r^*) . Similarly to the optimized Robin conditions, the subscript $OO2$ stands for “Optimized of Order 2”, since the resulting conditions can be seen as second order polynomial approximations of the optimal symbols $-\lambda^\pm$. This method was studied in detail in [Jap97], under simplifying assumptions.

When the tangential component of the advection to the interface is zero ($b = 0$ in our case), then we fix $q = 0$ in the second order conditions. This is justified by the fact that the optimal symbols λ^\pm are even functions of k , thus it is reasonable to drop the first order term. In particular, this is used for symmetric equations (no advection).

Definition 3.4 (Two-sided second order conditions). We can also leave the two boundary operators completely separate, leading to six different parameters

$$\begin{aligned} \mathcal{B}_1 u &= \frac{\partial u}{\partial x} + \frac{1}{2\nu} \left[(-a + p_1)u - q_1 \frac{\partial u}{\partial y} - r_1 \frac{\partial^2 u}{\partial y^2} \right], \\ \mathcal{B}_2 u &= \frac{\partial u}{\partial x} + \frac{1}{2\nu} \left[(-a - p_2)u + q_2 \frac{\partial u}{\partial y} + r_2 \frac{\partial^2 u}{\partial y^2} \right]. \end{aligned}$$

We will not attempt to solve the min-max problem corresponding to these conditions here.

Note that we have left out first order conditions in the y derivative, because we do not use them at all in this work. However, they are relevant for the advection-diffusion equation and should be considered in future work.

3.3 Convergence Results

When using two-sided Robin conditions (Definition 3.2), the rate of convergence of the Schwarz iteration can be simplified to

$$\rho_{OO0}(k, L, p_1, p_2) = \frac{(p_1 - z)(p_2 - z)}{(p_1 + z)(p_2 + z)} e^{-\frac{Lz}{\nu}}.$$

Proposition 3.4 (Convergence of the two-sided Robin method). *For any choice of parameters p_i such that $p_i > 0$, the Schwarz method converges for any initial guess.*

Proof. The expression $z(k)$ is the square root of a complex number, and it can be seen that $\xi = \text{Re}(z) > 0$ for all values of $k \geq 0$. Thus, assuming that $p_i > 0$, we get $p_i \xi > 0$ and hence

$$\begin{aligned} |\rho_{OO0}(k, L, p_1, p_2)| &\leq \prod_{i=1}^2 \left| \frac{p_i - z}{p_i + z} \right| = \prod_{i=1}^2 \left(\frac{p_i^2 + \xi^2 + \eta^2 - 2p_i \xi}{p_i^2 + \xi^2 + \eta^2 + 2p_i \xi} \right)^{\frac{1}{2}}, \\ &\Rightarrow |\rho_{OO0}(k, L, p_1, p_2)| < 1, \end{aligned}$$

proving the proposition. □

Note that this also proves convergence of the “one-sided” Robin method when using $p > 0$, as a special case.

Remark 3.1 (The Dirichlet-Neumann method for non-symmetric operators). *In Chapter 1, we introduced the Dirichlet-Neumann method (DN), in which we use one Dirichlet and one Neumann transmission condition. There are actually two different methods: you can either use Dirichlet matching for the subproblem in Ω_1 or use it for Ω_2 . For symmetric differential operators, this choice does not matter: the convergence rate is the same. However, for our model advection-diffusion equation, we obtain two different behaviors,*

$$\rho_{DN}(k, L) = \frac{\lambda^-}{\lambda^+} e^{L(\lambda^- - \lambda^+)} \quad \text{or} \quad \rho_{ND}(k, L) = \frac{\lambda^+}{\lambda^-} e^{L(\lambda^- - \lambda^+)}.$$

Hence, depending on the sign of $a = \mathbf{a} \cdot \mathbf{n}$, one of the choices gives a convergent iteration and the other leads to a divergent iteration when $L = 0$. This motivates the use of special methods, one of which is introduced in Section 1.5: the direction of the advection across the interface has to be taken into account somehow. This is done with our Robin conditions by the presence of the term $-\frac{a}{2\nu}$ in the transmission conditions.

More generally, we can prove the convergence of the Schwarz method with the two-sided second order transmission conditions from Definition 3.4.

Proposition 3.5 (Convergence of the Schwarz method with two-sided second order conditions). *For any choice of the parameters p_i , q_i and r_i such that*

$$p_i > 0, \quad \text{sign}(q_i) = -\text{sign}(b), \quad r_i \geq 0,$$

the Schwarz method converges for any initial guess.

Proof. Taking the magnitude of the convergence rate, we find the expression

$$\begin{aligned} |\rho_{OO_2}(k, L, p_i, q_i, r_i)| &= \prod_{i=1}^2 \left| \frac{p_i + q_i k + r_i k^2 - z}{p_i + q_i k + r_i k^2 + z} \right| e^{-\frac{L\xi}{\nu}} \\ &= \left(\prod_{i=1}^2 \frac{(p_i + r_i k^2 - \xi)^2 + (q_i k - \eta)^2}{(p_i + r_i k^2 + \xi)^2 + (q_i k + \eta)^2} \right)^{1/2} e^{-\frac{L\xi}{\nu}} \\ &= \left(\prod_{i=1}^2 \frac{A_i - B_i}{A_i + B_i} \right)^{1/2} e^{-\frac{L\xi}{\nu}}, \end{aligned}$$

where

$$A_i := \xi^2 + \eta^2 + (p_i + r_i k^2)^2 + q_i^2 k^2, \quad B_i := 2[\xi(p_i + r_i k^2) + \eta(q_i k)].$$

We know that $A_i - B_i \geq 0$. Under the assumptions of the proposition on the parameters p_i , q_i and r_i , we have $B_i > 0$, and thus

$$\frac{A_i - B_i}{A_i + B_i} < 1, \quad \text{for } i = 1, 2,$$

which implies

$$|\rho_{OO_2}(k, L, p_i, q_i, r_i)| < 1.$$

□

This also proves convergence of the “one-sided” second order method when using $p > 0$, $\text{sign}(q) = -\text{sign}(b)$ and $r \geq 0$. In addition, as a corollary, we get the convergence of the Schwarz method using Taylor approximations of order 0, 1 and 2.

3.4 Results for Symmetric Problems

As a model for symmetric positive definite problems, we consider the modified Helmholtz equation in the plane,

$$\begin{cases} -\Delta u + \omega^2 u = 0 \text{ on } \Omega = \mathbb{R}^2, \\ u \text{ is bounded at infinity.} \end{cases}$$

Applying the Schwarz algorithm with transmission conditions of the type given by (2.8), we obtain the convergence rate (2.9) with $\lambda^\pm(k) := \pm\sqrt{k^2 + \omega^2}$. Optimized Schwarz methods for this problem are well-studied. In [Gan03], the min-max problem is solved to obtain optimized Robin, two-sided Robin, and second order transmission conditions. The convergence is enhanced significantly when using these optimized methods, compared to the classical Schwarz method, as numerical experiments confirm. The asymptotic performance also shows a much weaker dependence on the mesh size h . In addition, the asymptotics do improve as we use higher order transmission conditions, unlike the Taylor approximation methods. Here is a quick summary of the asymptotic results.

Theorem 3.5 (Asymptotics for optimized Schwarz methods). *The asymptotic performance of the optimized Schwarz method for the modified Helmholtz model problem is*

- *Optimized Robin conditions, $L = 0$,*

$$\max_{k_{\min} \leq k \leq \pi/h} |\rho_{OO0}(k, 0, p^*)| = 1 - 4 \frac{(k_{\min}^2 + \omega^2)^{\frac{1}{4}}}{\sqrt{\pi}} h^{\frac{1}{2}} + O(h),$$

- *Optimized Robin conditions, $L = h$,*

$$\max_{k_{\min} \leq k \leq \pi/h} |\rho_{OO0}(k, h, p^*)| = 1 - 4 \cdot 2^{\frac{1}{6}} (k_{\min}^2 + \omega^2)^{\frac{1}{6}} h^{\frac{1}{3}} + O(h^{\frac{2}{3}}),$$

- *Optimized two-sided Robin conditions, $L = 0$,*

$$\max_{k_{\min} \leq k \leq \pi/h} |\rho_{OO0}(k, 0, p_1^*, p_2^*)| = 1 - 2 \frac{\sqrt{2} (k_{\min}^2 + \omega^2)^{\frac{1}{8}}}{\pi^{\frac{1}{4}}} h^{\frac{1}{4}} + O(h^{\frac{1}{2}}),$$

- *Optimized two-sided Robin conditions, $L = h$,*

$$\max_{k_{\min} \leq k \leq \pi/h} |\rho_{OO0}(k, h, p_1^*, p_2^*)| = 1 - 2 \cdot 2^{\frac{4}{5}} (k_{\min}^2 + \omega^2)^{\frac{1}{10}} h^{\frac{1}{5}} + O(h^{\frac{2}{5}}),$$

- *Optimized second order conditions, $L = 0$,*

$$\max_{k_{\min} \leq k \leq \pi/h} |\rho_{OO2}(k, 0, p^*, r^*)| = 1 - 4 \frac{\sqrt{2} (k_{\min}^2 + \omega^2)^{\frac{1}{8}}}{\pi^{\frac{1}{4}}} h^{\frac{1}{4}} + O(h^{\frac{1}{2}}),$$

- *Optimized second order conditions, $L = h$,*

$$\max_{k_{\min} \leq k \leq \pi/h} |\rho_{OO2}(k, h, p^*, r^*)| = 1 - 4 \cdot 2^{\frac{3}{5}} (k_{\min}^2 + \omega^2)^{\frac{1}{10}} h^{\frac{1}{5}} + O(h^{\frac{2}{5}}).$$

Proof. The proofs of these results are given in [Gan03]. □

It was also shown that the optimized parameters computed by solving the min-max problem are close to the values that, numerically, lead to the quickest convergence when using a bounded domain. This justifies the optimization procedure, using a continuous Fourier analysis on a model problem with an infinite domain only.

Optimized methods for the Helmholtz equation have also been studied, with the model problem

$$\begin{cases} -\Delta u - \omega^2 u = 0 \text{ on } \Omega = \mathbb{R}^2, \\ u \text{ is bounded at infinity.} \end{cases}$$

This symmetric differential operator is not positive definite. When applying the Schwarz algorithm with transmission conditions of the form (2.8), we obtain the same convergence rate (2.9), but with $\lambda^\pm(k) := \pm\sqrt{k^2 - \omega^2}$. This causes a problem, since for the special frequency $k = \omega$, the convergence rate is always of magnitude 1 (method does not converge), even with an overlap. This is not affected by changing the transmission conditions. However, when using the Schwarz algorithm as a preconditioner, a Krylov method will take care of the special frequency. A modified min-max problem is considered in [GMN01], avoiding the frequency $k = \omega$

$$\min_{\mathcal{B}_i \in \mathcal{C}} \left(\max_{k \in (k_{min}, \omega_-) \cup (\omega_+, k_{max})} |\rho(k, L, \mathcal{B}_1, \mathcal{B}_2)| \right), \quad (3.5)$$

where ω_\pm are parameters to be chosen. Consider for example the case where the domain Ω is a rectangle of height H , with homogeneous Dirichlet boundary conditions at the top and bottom sides. The solution can be expanded in a Fourier sine series in the y variable, with the basis functions $\sin(\frac{j\pi y}{H})$, $j \in \mathbb{N}$. This shows that the relevant frequencies for the problem are $\frac{j\pi}{H}$, bounded below by $k_{min} = \frac{\pi}{H}$ and above by $k_{max} = \frac{\pi}{h}$. Hence, by choosing $\omega_\pm = \omega \pm \pi/H$, we make sure that all relevant frequencies are treated in the optimization, except for the single one $k = \omega$, left to the Krylov method.

In [GMN01], this idea was introduced and applied to find optimized complex Robin transmission conditions of the form

$$\mathcal{B}_{1,2} = \frac{\partial u}{\partial x} \pm (p + qi)u, \quad p, q \in \mathbb{R}.$$

Also, second order conditions of the form

$$\mathcal{B}_{1,2} = \frac{\partial u}{\partial x} \pm \frac{1}{\alpha + \beta} \left((\alpha\beta - \omega^2)u - \frac{\partial^2 u}{\partial y^2} \right), \quad \alpha \in i\mathbb{R}, \beta \in \mathbb{R},$$

are optimized in [Gan01] for non-overlapping decompositions, and in [GHN01] for the overlapping case.

3.5 Results for the Advection-Diffusion Equation

We summarize here part of the work accomplished by Caroline Japhet in her doctoral thesis [Jap97] and in consequent papers [JNR98, JNR01]. The study is made in the case of two subdomains without overlap only. The transmission conditions used for optimization are second order in the tangential direction to the single interface Γ , in the form of Definition 3.3. In addition, to simplify the minimization problem, the number of free parameters is reduced to two by fixing the value of $p = z_0 = \sqrt{a^2 + 4\nu c}$. This comes from the zeroth order term in the Taylor expansion of $\lambda^\pm(k)$. In addition, the minimum frequency k_{min} is taken to be always 0 in Japhet's work. Let $R_{OO2}(k, q, r) := \rho_{OO2}(k, 0, z_0, q, r)$. The min-max problem is then stated as

$$\min_{q, r \in \mathbb{R}} \left(\max_{|k| \leq k_{max}} |R_{OO2}(k, q, r)| \right).$$

Under a discretization of the domain, an approximation for the maximum frequency that can be used is $k_{max} = \frac{\pi}{h}$, where h is the mesh size in y . First, in the case the advection is normal to the interface ($b = 0$), q is set to zero, because λ^\pm are even functions of k . So, the minimization is done only on one parameter, r , and the resulting min-max problem is solved exactly [Jap97]. In addition, for this case, it was shown that the min-max problem is equivalent to solving

$$\min_{0 < k_{int} \leq k_{max}} \left(\max_{|k| \leq k_{max}} |R_{OO2}(k, 0, r(k_{int}))| \right),$$

where k_{int} are frequencies such that there is a unique value $r(k_{int})$ with $R_{OO2}(k_{int}, 0, r(k_{int})) = 0$.

Now, in the case of arbitrary constant advection ($b \neq 0$), the minimization problem on both q and r is still very complex, so only an approximated problem is solved. From the previous observation in the case $b = 0$, the minimum is taken only over intermediate frequencies k_{int} for which there is a unique choice of constants $q(k_{int})$ and $r(k_{int})$ such that $R_{OO2}(k_{int}, q(k_{int}), r(k_{int})) = 0$. The approximate min-max problem

solved in [Jap97] is

$$\min_{0 < k_{int} \leq k_{max}} \left(\max_{|k| \leq k_{max}} |R_{OO2}(k, q(k_{int}), r(k_{int}))| \right).$$

The transmission conditions resulting from this problem were called ‘‘Optimized Of Order 2’’ (OO2), but the full optimization on the three parameter p , q and r was not solved. To see how robust these conditions are compared with the ones obtained from Taylor expansions, the asymptotic behavior of the convergence rate as $h \rightarrow 0$ is

$$\max_{|k| \leq \frac{\pi}{h}} |R_{OO2}(k, q^*, r^*)| = 1 - 8 \left(\frac{\sqrt{a^2 + 4\nu c}}{4\pi\nu} \right)^{\frac{1}{3}} h^{\frac{1}{3}} + O(h^{\frac{2}{3}}).$$

Recall that these results are for non-overlapping decompositions only. This is a good improvement compared to the Taylor approximations, which have only linear asymptotic performance without overlap (the convergence rate behaves like $1 - O(h)$, see Section 3.1). However, in the symmetric case, a full optimization for second order conditions led to the expansion $1 - O(h^{\frac{1}{4}})$, so we expect that a similar performance can be achieved for the advection-diffusion equation. For more details on these results, including numerical experiments, see [Jap97].

Chapter 4

Optimized Robin Parameter

In this chapter, we compute optimized Robin transmission conditions for the advection-diffusion equation. This is the main contribution of this thesis. Let us recall our model problem

$$\begin{cases} -\nu\Delta u + \mathbf{a} \cdot \nabla u + cu = 0 \text{ in } \mathbb{R}^2, \\ u \text{ is bounded at infinity,} \end{cases}$$

where $\nu, c > 0$ and the advection $\mathbf{a} = (a, b)$ are constant. The plane is decomposed into the two subdomains

$$\Omega_1 = \left(-\infty, \frac{L}{2}\right) \times \mathbb{R}, \quad \Omega_2 = \left(-\frac{L}{2}, \infty\right) \times \mathbb{R}.$$

The Robin transmission conditions we consider here are described in Definition 3.1, namely

$$\mathcal{B}_1 u = \frac{\partial u}{\partial x} + \left(\frac{-a+p}{2\nu}\right) u, \quad \mathcal{B}_2 u = \frac{\partial u}{\partial x} + \left(\frac{-a-p}{2\nu}\right) u.$$

4.1 Without Overlap and Advection Normal to the Interface

We start with the simplest case: the advection is normal to the interface, i.e. $b = 0$, and with no overlap ($L = 0$). This makes the problem much easier, because we have that $z = \xi \in \mathbb{R}$. The convergence rate reduces to

$$\rho_{OOO}(k, 0, p) = \left(\frac{p - \xi}{p + \xi} \right)^2 =: R(\xi, p).$$

Because $\xi(k) = \sqrt{a^2 + 4\nu c + 4\nu^2 k^2}$ is a continuous increasing function of k , we can rewrite the min-max problem by taking the maximum over the variable ξ instead of k , to get

$$\min_{p \in \mathbb{R}} \left(\max_{\xi_{min} \leq \xi \leq \xi_{max}} R(\xi, p) \right), \quad (4.1)$$

where $\xi_{min} = \xi(k_{min})$ and $\xi_{max} = \xi(k_{max})$.

In this section, and also in the next chapter, we will make use of the following theorem, proved by Wachspress in [Wac62].

Theorem 4.1 (Alternance). *Define the quantities*

$$Q_n(\lambda, \boldsymbol{\omega}) := \prod_{i=1}^n \left(\frac{\omega_i - \lambda}{\omega_i + \lambda} \right), \quad Q^{opt} := \min_{\boldsymbol{\omega} \in [a, b]^n} \max_{\lambda \in [a, b]} |Q_n(\lambda, \boldsymbol{\omega})|.$$

Let the set of parameters $\boldsymbol{\omega}^{opt}$ assume the minimum deviation from zero of Q_n (i.e. $|Q_n(\lambda, \boldsymbol{\omega}^{opt})| = Q^{opt}$). Then, the function $Q_n(\lambda, \boldsymbol{\omega}^{opt})$ attains its maximum magnitude exactly $(n+1)$ times for $\lambda \in [a, b]$, with alternating sign. The set of optimal parameters $\boldsymbol{\omega}^{opt}$ is unique. Moreover, they satisfy a useful property: for each i , there is a j such that

$$\frac{ab}{\omega_i^{opt}} = \omega_j^{opt}. \quad (4.2)$$

Remark 4.1. *Note that if we are minimizing for the parameters ω in Theorem 4.1 over all of \mathbb{R}^n , then the optimal parameters necessarily have to be in the interval $[a, b]$: if $\omega_i < a$ then we can decrease the magnitude of Q_n by increasing ω_i . Similarly, if $\omega_i > b$, the magnitude of Q_n decreases as we decrease ω_i ,*

$$\min_{\omega \in \mathbb{R}^n} \max_{\lambda \in [a, b]} |Q_n(\lambda, \omega)| = \min_{\omega \in [a, b]^n} \max_{\lambda \in [a, b]} |Q_n(\lambda, \omega)|.$$

Returning to our problem, it is clear that our min-max formulation (4.1) is a special case of this theorem.

Proposition 4.1 (Optimized Robin parameter, without overlap and $b = 0$).

The solution p^ of the min-max problem (4.1) is given by $p^* = \sqrt{\xi_{\min} \xi_{\max}}$.*

Proof. The optimization problem is equivalent to

$$\min_{p \in \mathbb{R}} \max_{\xi_{\min} \leq \xi \leq \xi_{\max}} \left| \frac{p - \xi}{p + \xi} \right|,$$

by taking the square root of $R(\xi, p)$. This is in the form of Theorem 4.1 with $n = 1$. By property (4.2), and because there is only one parameter, the solution p^* satisfies the equation

$$\frac{\xi_{\min} \xi_{\max}}{p^*} = p^*.$$

Therefore we get the formula $p^* = \sqrt{\xi_{\min} \xi_{\max}}$. \square

Proposition 4.2 (Optimized Robin asymptotics, without overlap and $b = 0$).

Let $k_{\max} = \frac{\pi}{h}$. The asymptotic performance of the Schwarz method with optimized Robin transmission conditions is

$$\max_{k_{\min} \leq k \leq k_{\max}} |\rho_{000}(k, 0, p^*)| = 1 - 2\sqrt{\frac{2\xi_{\min}}{\nu\pi}} h^{\frac{1}{2}} + O(h).$$

Proof. When $k_{max} = \frac{\pi}{h}$, the maximum value for ξ grows like $\frac{2\nu\pi}{h} + O(h)$. Thus, using this in the formula for the optimized parameter p^* we get

$$p^* = \sqrt{2\nu\pi\xi_{min}}h^{-\frac{1}{2}} + O(h^{\frac{3}{2}}).$$

By the Alternance Theorem 4.1, we deduce that the maximum magnitude of the convergence rate is attained at both endpoints of the frequency interval. Thus,

$$\begin{aligned} \max_{\xi_{min} \leq \xi \leq \xi_{max}} R(\xi, p^*) &= R(\xi_{min}, p^*), \\ &\sim \left(\frac{\sqrt{2\nu\pi\xi_{min}}h^{-\frac{1}{2}} - \xi_{min}}{\sqrt{2\nu\pi\xi_{min}}h^{-\frac{1}{2}} + \xi_{min}} \right)^2 \sim \left(1 - \sqrt{\frac{2\xi_{min}}{\nu\pi}}h^{\frac{1}{2}} \right)^2, \\ &= 1 - 2\sqrt{\frac{2\xi_{min}}{\nu\pi}}h^{\frac{1}{2}} + O(h). \end{aligned}$$

□

Figure 4.1 shows the rate of convergence with three different choices for the Robin conditions: using the zeroth order Taylor approximation, the optimized Robin parameter (Proposition 4.1) and finally the optimized two-sided Robin parameters, to be computed in the next chapter. The optimized Robin transmission conditions bring significant improvement for large frequencies compared to the Taylor approximations.

Remark 4.2. *We are assuming that $\nu, c > 0$, hence the minimum value for ξ is strictly positive, $\xi_{min} > 0$. By Proposition 3.4, the Schwarz method with optimized Robin conditions always converges, since $p^* > 0$.*

4.2 With Overlap and Advection Normal to the Interface

We now consider overlapping decompositions ($L > 0$), but we still assume that the advection is normal to the interfaces ($b = 0$). The presence of an overlap adds an

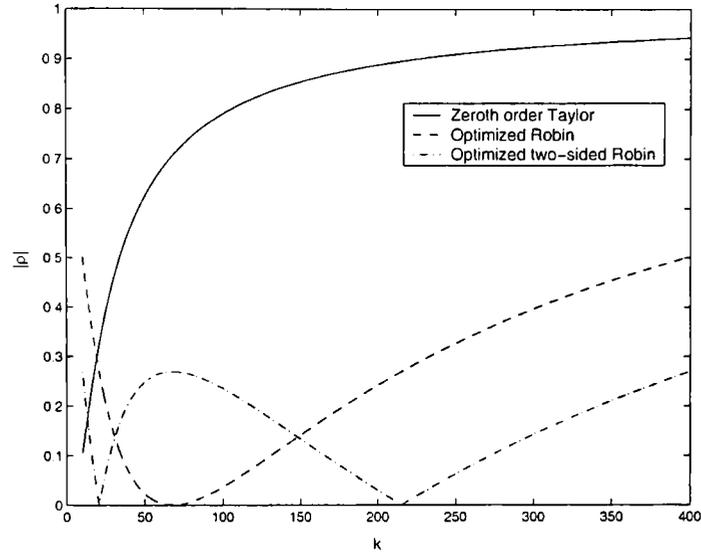


FIGURE 4.1. Convergence rates for $\nu = 0.1$, $a = 1$, $b = 0$, $c = 1$, $L = 0$.

exponential decay to the rate of convergence, affecting large frequencies in the error,

$$\rho_{000}(k, L, p) = \left(\frac{p - \xi}{p + \xi} \right)^2 e^{-\frac{L\xi}{\nu}} =: R(\xi, p).$$

As before, the optimized Robin parameter must be in the interval $[\xi_{min}, \xi_{max}]$. However, the alternance theorem of Wachspress does not apply when a positive overlap is included; an extension of it would be needed. We adopt a different approach instead. The min-max problem we wish to solve is

$$\min_{p \in [\xi_{min}, \xi_{max}]} \left(\max_{\xi \in [\xi_{min}, \xi_{max}]} R(\xi, p) \right). \quad (4.3)$$

Proposition 4.3 (Optimized Robin parameter, with overlap and $b = 0$). *Let $\xi_c := \sqrt{p^2 + \frac{4\nu p}{L}}$, and let the function $g(p)$ be defined for $p \in [\xi_{min}, \xi_{max}]$ by*

$$g(p) := \begin{cases} R(\xi_c, p), & \text{if } \xi_c \leq \xi_{max}, \\ R(\xi_{max}, p), & \text{otherwise.} \end{cases}$$

Then, the solution p^ of the min-max problem (4.3) is the unique root (in $[\xi_{min}, \xi_{max}]$)*

of the equation

$$R(\xi_{min}, p^*) = g(p^*). \quad (4.4)$$

Proof. The critical values of R with respect to ξ are

$$\frac{\partial R}{\partial \xi} = 0 \Rightarrow \xi = p \text{ or } \xi = \sqrt{p^2 + \frac{4\nu p}{L}} =: \xi_c.$$

The value $\xi = p$ is a local minimum and $\xi = \xi_c$ is a local maximum for R . Hence, the maximum magnitude of the convergence rate can be written as

$$\begin{aligned} \max_{\xi \in [\xi_{min}, \xi_{max}]} R(\xi, p) &= \max \left(R(\xi_{min}, p), \left\{ \begin{array}{l} R(\xi_c(p), p), \text{ if } \xi_c \leq \xi_{max} \\ R(\xi_{max}, p), \text{ otherwise} \end{array} \right\} \right), \\ &= \max(\rho_{OO0}(k_{min}, L, p), g(p)). \end{aligned}$$

Now, both $R(\xi_{min}, p)$ and $g(p)$ are continuous functions of p on $[\xi_{min}, \xi_{max}]$. Moreover, note that

$$\begin{aligned} R(\xi_{min}, \xi_{min}) &= 0 < g(\xi_{min}), \\ R(\xi_{min}, \xi_{max}) &> 0 = g(\xi_{max}). \end{aligned}$$

Thus, equation (4.4) has at least a root in the interval $[\xi_{min}, \xi_{max}]$. Now, if we look at the derivatives with respect to p , evaluated at $p \in (\xi_{min}, \xi_{max})$

$$\begin{aligned} \frac{dR(\xi_{min}, p)}{dp} &> 0, \\ \frac{dR(\xi_{max}, p)}{dp} &< 0, \quad \frac{dR(\xi_c(p), p)}{dp} < 0. \end{aligned}$$

Therefore, the root p^* is unique. Also, if $p < p^*$, then $R(\xi_{min}, p) < g(p)$, so the maximum deviation of the convergence rate is $g(p)$, which decreases as we increase p . Similarly, if $p > p^*$, then $R(\xi_{min}, p) > g(p)$, so the maximum deviation of the convergence rate is $R(\xi_{min}, p)$, which decreases as we decrease p . Hence, the minimum is achieved at p^* (when the local maxima of R with respect to ξ are balanced). \square

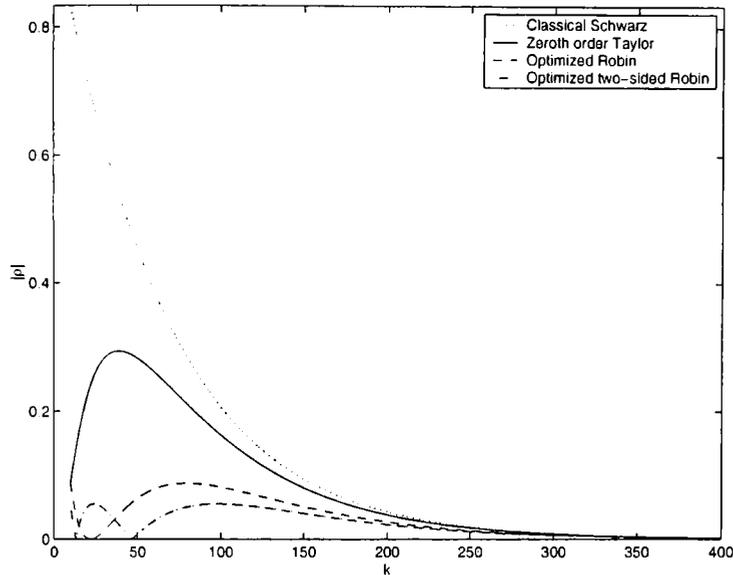


FIGURE 4.2. Convergence rates for the values $\nu = 0.1$, $a = 1$, $b = 0$, $c = 1$, $L = \pi/400$.

Figure 4.2 shows a comparison of the convergence rates using Dirichlet conditions, the zeroth order Taylor approximation, the optimized Robin parameter given by Proposition 4.3, and the optimized two-sided Robin conditions, computed in the next chapter.

Remark 4.3. *The critical value $\xi_c(p^*) := \sqrt{(p^*)^2 + \frac{4\nu p^*}{L}}$ can be on either side of ξ_{max} , so we cannot discard any of the cases. This justifies the need for the definition of $g(p)$ with a condition.*

Remark 4.4. *Again, when using the optimized parameter p^* obtained through Proposition 4.3, we are guaranteed convergence of the Schwarz iteration, since $p^* \geq \xi_{min} > 0$ (see the convergence result stated by Proposition 3.4).*

Proposition 4.4 (Optimized Robin asymptotics, with overlap and $b = 0$). *For $k_{max} = \frac{\pi}{h}$ and $L = h$, the asymptotic performance of the Schwarz method with*

optimized Robin transmission conditions is given by

$$\max_{k_{min} \leq k < \infty} |\rho_{OO0}(k, h, p^*)| = 1 - 4 \left(\frac{\xi_{min}}{\nu} \right)^{\frac{1}{3}} h^{\frac{1}{3}} + O(h^{\frac{2}{3}}). \quad (4.5)$$

Proof. We make the ansatz $p^* = Ch^\alpha$ with $\alpha < 0$ (we know that the optimized parameter increases as h decreases). Recall that p^* is given by the unique solution in $[\xi_{min}, \xi_{max}]$ to the equation $R(\xi_{min}, p^*) = g(p^*)$, and this for all values of $h > 0$. Note that we have the limiting behaviors

$$\lim_{h \rightarrow 0} R(\xi_{min}, p^*) = 1, \quad \lim_{h \rightarrow 0} R(\xi_{max}, p^*) = e^{-2\pi}$$

for any value of $\alpha < 0$, using the expansion $\xi_{max} \sim 2\pi\nu/h + O(h)$. Thus, it is impossible that p^* satisfies $R(\xi_{min}, p^*) = R(\xi_{max}, p^*)$ for any sequence of values of h converging to 0. Hence, we must have $R(\xi_{min}, p^*) = R(\xi_c, p^*)$ asymptotically for small h . Inserting our ansatz into the definition for $\xi_c(p^*)$, we get three different cases:

- If $\alpha = -1$, then $\xi_c = \sqrt{C^2 + 4\nu C} h^{-1}$. Inserting this into the expression for the convergence rate, we observe that $R(\xi_c(p^*), p^*)$ is constant with respect to h . However, $R(\xi_{min}, p^*)$ is not constant, and approaches 1 as $h \rightarrow 0$. Hence, it is impossible for p^* to satisfy equation (4.4) for all h . This case has to be discarded.
- If $\alpha < -1$, then $\xi_c \sim Ch^\alpha$. By inserting into the convergence rate, we can show that $R(\xi_c(p^*), p^*)$ decays exponentially to 0 as $h \rightarrow 0$. Therefore, p^* will not solve equation (4.4) for all h .
- Finally, if $-1 < \alpha < 0$, we get $\xi_c \sim 2\sqrt{\nu C} h^{\frac{\alpha-1}{2}}$. Expanding both sides of equation (4.4) for h small, we obtain

$$\begin{aligned} R(\xi_{min}, p^*) &\sim 1 - 4 \frac{\xi_{min}}{C} h^{-\alpha}, \\ R(\xi_c(p^*), p^*) &\sim 1 - 4 \sqrt{\frac{C}{\nu}} h^{\frac{\alpha+1}{2}}. \end{aligned} \quad (4.6)$$

For the equation to be satisfied for all h , the leading order terms of the above expansions need to match. Equating the coefficients and exponents, we get

$$\alpha = -\frac{1}{3}, \quad C = (\sqrt{\nu}\xi_{min})^{\frac{2}{3}}.$$

Inserting these values back into (4.6), we obtain the asymptotic expansion given by (4.5).

□

4.3 Without Overlap and with Arbitrary Constant Advection

In this section, we deal with the case without overlap ($L = 0$), but with general constant advection ($b \neq 0$), and the aim is still to compute optimized Robin transmission conditions. The main difference is that z is now a complex number, adding significant complexity to the min-max problem.

Recall that the convergence rate of the Schwarz iteration with Robin transmission conditions for our model problem is

$$\rho_{OOO}(k, 0, p) = \left(\frac{p - z}{p + z} \right)^2,$$

where $z(k) := \sqrt{a^2 + 4\nu c + 4\nu^2 k^2 - 4\nu b k i}$ is a complex number if $b \neq 0$. We first make a few additional substitutions

$$\begin{aligned} K &:= 2\nu k, & A &:= a^2 + 4\nu c, \\ z(K) &= \sqrt{A + K^2 - 2bKi}. \end{aligned}$$

To simplify the expression we have for the convergence rate, and to be able to use calculus to solve the min-max problem, we rewrite the imaginary part of z in terms

of the real part ($\eta = \eta(\xi)$) by solving for K . This gives us

$$\begin{aligned}\xi(K) &= \frac{1}{\sqrt{2}}\sqrt{\sqrt{K^4 + 2K^2A + A^2 + 4b^2K^2} + K^2 + A}, \\ \eta(K) &= \frac{-\text{sign}(bK)}{\sqrt{2}}\sqrt{\sqrt{K^4 + 2K^2A + A^2 + 4b^2K^2} - K^2 - A}, \\ K &= \text{sign}(K)\xi\sqrt{\frac{\xi^2 - A}{\xi^2 + b^2}}, \\ \eta(\xi) &= -\text{sign}(K)b\sqrt{\frac{\xi^2 - A}{\xi^2 + b^2}}.\end{aligned}$$

This trick was inspired by [AK02], where it is used to solve a similar min-max problem in the context of waveform relaxation. The imaginary part of z depends on the sign of the frequency k , but only η^2 will appear in the magnitude of the convergence rate, leading to an even function of k . Thus, we are still able to consider positive frequencies only. Substituting back into the convergence rate and taking the modulus, we get

$$|\rho_{OOO}(k, 0, p)| = \frac{(\xi^2 + b^2)(\xi - p)^2 + b^2(\xi^2 - A)}{(\xi^2 + b^2)(\xi + p)^2 + b^2(\xi^2 - A)} =: R(\xi, p), \quad (4.7)$$

$$\min_{p \in \mathbb{R}} \max_{k \in I} |\rho_{OOO}(k, 0, p)| = \min_{p \in \mathbb{R}} \max_{\xi \in J} R(\xi, p), \quad (4.8)$$

where $I = [k_{min}, k_{max}]$ and $J = [\xi(k_{min}), \xi(k_{max})]$. We are able to rewrite the min-max problem with the maximum taken over the variable ξ instead of k , since $\xi(k)$ is a continuous increasing function. By differentiating $R(\xi, p)$ with respect to p , we see that

$$\begin{aligned}\text{sign}\left(\frac{\partial R}{\partial p}\right) &= \text{sign}(p^2 - |z|^2), \\ \frac{\partial R}{\partial p} &= 0 \quad \Leftrightarrow \quad p = |z|.\end{aligned}$$

Note that $|z(k)|$ is an increasing function of k . Thus, if $p < |z_{min}|$, increasing p makes the rate of convergence decrease for all values of $k \in I$. Similarly, if $p > |z_{max}|$, decreasing p also makes the rate of convergence decrease for all values of $k \in I$. Hence, we can restrict the range of p in the min-max problem to the interval $[|z_{min}|, |z_{max}|]$.

Note that $|z_{min}|^2 \geq \xi_{min}^2 \geq A$, which implies that $p^2 \geq A$.

Differentiating $R(\xi, p)$ with respect to ξ , we find that $\frac{\partial R}{\partial \xi}(\xi, p) = 0$ if and only if

$$P_3(X) := X^3 + (b^2 - p^2)X^2 + 2b^2\left(b^2 + \frac{3}{2}A - p^2\right)X + b^4(A - p^2) = 0,$$

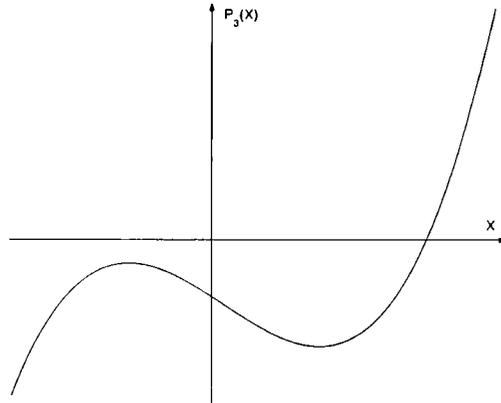


FIGURE 4.3. Example of case 1.

where $X := \xi^2$. Also, the sign of the derivative is given by the sign of this cubic polynomial. As noted above, $X_{min} \geq A$.

Lemma 4.1. *The polynomial $P_3(X)$ has at most one real root larger than X_{min} .*

Proof. A cubic polynomial with real coefficients has either one or three real roots. We will be assuming that there are three, as the other case trivially satisfies the lemma. We have that $P_3(0) \leq 0$, since $p^2 \geq A$. We proceed with a case-by-case analysis.

- Suppose that the coefficient of X is negative ($p^2 \geq b^2 + \frac{3}{2}A$). Then $P'_3(0) \leq 0$, which implies that the cubic will have exactly one root for $X \geq 0$. This case is illustrated in Figure 4.3.
- In the other case where $p^2 < b^2 + \frac{3}{2}A$, we have $p^2 - b^2 < \frac{3}{2}A$. The inflexion point of the cubic polynomial is at $X_i = -(b^2 - p^2)/3$, and hence $X_i < \frac{A}{2}$ using the previous inequality. Suppose in addition that $P_3(X_i)$ is negative, then there can be only one root larger than X_i ($< A \leq X_{min}$). See Figure 4.4 for an example.
- Now, in the remaining case $P_3(X_i) > 0$, the smallest of the three roots is in the interval $[0, X_i)$ (recall that $P_3(0) \leq 0$). We denote the three roots by $X_1 \leq X_2 \leq X_3$. A known property of the cubic polynomial is that the average

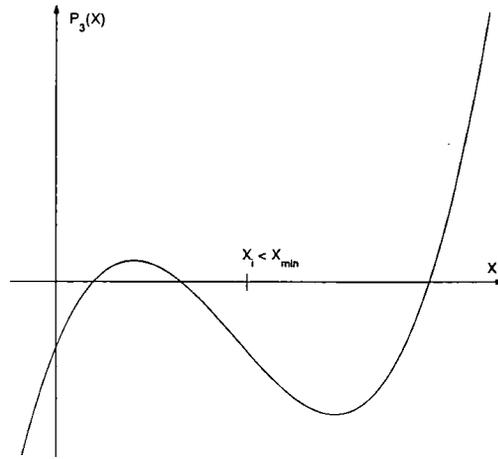


FIGURE 4.4. Example of case 2.

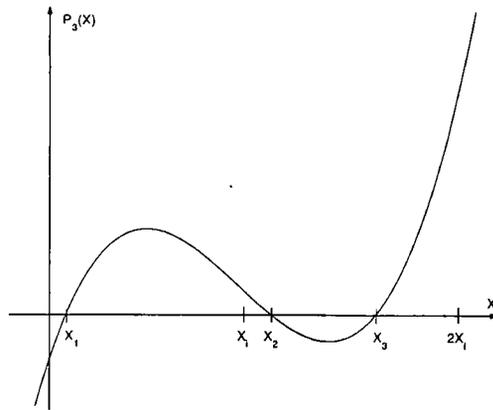


FIGURE 4.5. Example of case 3.

of the roots gives the inflexion point (when all the roots are real). Thus, we get

$$\begin{aligned} X_1 + X_2 + X_3 &= 3X_i < \frac{3}{2}A, \\ \Rightarrow X_2 + X_3 &< \frac{3}{2}A - X_1 \leq \frac{3}{2}A, \\ \Rightarrow X_2 &< \frac{3}{4}A < A \leq X_{min}. \end{aligned}$$

Hence, only the largest root X_3 can possibly be greater than or equal to X_{min} . Figure 4.5 illustrates this last case.

This covers all possible cases and therefore concludes the proof. \square

The largest root of the polynomial represents a local minimum of the function

$R(\xi, p)$. This means that the maximum of R on the interval $[\xi_{min}, \xi_{max}]$ is attained at one or both of the endpoints, i.e.

$$\max_{\xi \in [\xi_{min}, \xi_{max}]} R(\xi, p) = \max \{R(\xi_{min}, p), R(\xi_{max}, p)\}.$$

Consider the function $F(p) := R(\xi_{max}, p) - R(\xi_{min}, p)$. Using the expression (4.7) for R , we have

$$\text{sign}(F(p)) = \text{sign} [-(\xi_{max} - \xi_{min})p^2 + (\xi_{min}|z_{max}|^2 - \xi_{max}|z_{min}|^2)],$$

$$F(p) = 0 \Rightarrow p = \sqrt{\frac{\xi_{min}|z_{max}|^2 - \xi_{max}|z_{min}|^2}{\xi_{max} - \xi_{min}}} =: p_c.$$

We can show that we always have $p_c \geq A$. First suppose that $p_c \in [|z_{min}|, |z_{max}|]$. If $p < p_c$, then $F(p) > 0$, meaning that $R(\xi_{max}, p) > R(\xi_{min}, p)$. Increasing p will make the value of R at $\xi = \xi_{max}$ decrease. If $p > p_c$, then $R(\xi_{min}, p) > R(\xi_{max}, p)$. Decreasing p will make the value of R at $\xi = \xi_{min}$ decrease. Hence, the optimal value for p is p_c . However, if $p_c \notin [|z_{min}|, |z_{max}|]$, which is a case that does occur, then the optimal value for p is the endpoint of the interval closest to p_c . In that case, the optimal convergence rate will not have equal values at $\xi = \xi_{min}$ and $\xi = \xi_{max}$. In this section, we proved

Theorem 4.2 (Optimized Robin parameter, without overlap). *If there is no overlap ($L = 0$), the solution p^* of the min-max problem (4.8) is given by*

$$p^* = \begin{cases} |z_{min}| & \text{if } p_c < |z_{min}|, \\ p_c & \text{if } |z_{min}| \leq p_c \leq |z_{max}|, \\ |z_{max}| & \text{if } p_c > |z_{max}|, \end{cases}$$

$$\text{where } p_c = \sqrt{\frac{\xi_{min}|z_{max}|^2 - \xi_{max}|z_{min}|^2}{\xi_{max} - \xi_{min}}}.$$

Theorem 4.3 (Optimized Robin asymptotics, without overlap). *For $k_{max} = \frac{\pi}{h}$, the asymptotic performance of the Schwarz method without overlap, with optimized Robin transmission conditions is*

$$\max_{k_{min} \leq k \leq \frac{\pi}{h}} |\rho_{OO0}(k, 0, p^*)| = 1 - 2\sqrt{\frac{2\xi_{min}}{\pi\nu}} h^{\frac{1}{2}} + O(h). \quad (4.9)$$

Proof. When $k_{max} = \frac{\pi}{h}$, we have $\xi_{max} = \frac{2\nu\pi}{h} + O(h)$ and $\eta_{max}^2 = b^2 - O(h^2)$, thus the maximum magnitude of z grows like $|z_{max}| = \frac{2\nu\pi}{h} + O(h)$. Inserting these into the formula for p_c , we obtain

$$p_c = \frac{\sqrt{2\nu\xi_{min}}}{h^{\frac{1}{2}}} + O(h^{\frac{1}{2}}).$$

Hence, asymptotically, we have $|z_{min}| \leq p_c \leq |z_{max}|$, and so the optimal parameter is given by $p^* = p_c$. The asymptotic result (4.9) is obtained by expanding $R(\xi_{min}, p_c)$ for small h . \square

Note that by setting $b = 0$, Theorem 4.2 and 4.3 reduce to the results obtained in Section 4.1, as expected. We have also presented the proofs for the simpler cases to get some insight on the effect of introducing an advective term in the equation: solving to the optimization problem is significantly different when $b \neq 0$, and extra care needs to be taken since the optimized parameter is not always given by an equilibrium of maxima. Figure 4.6 shows a comparison of the convergence rates using three different Robin transmission conditions, in the case $b \neq 0$.

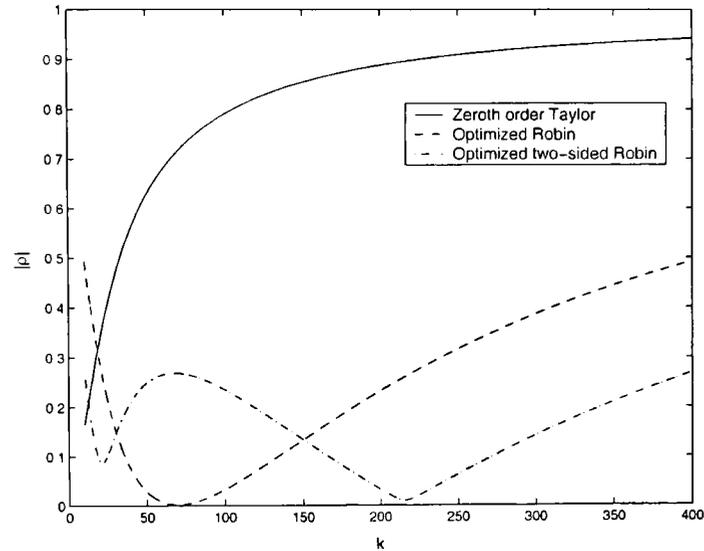


FIGURE 4.6. Convergence rates for the values $\nu = 0.1$, $a = 1$, $b = 1$, $c = 1$, $L = 0$.

4.4 With Overlap and with Arbitrary Constant Advection

We now consider the Schwarz method with overlap. We can write the convergence rate as

$$|\rho_{000}(k, L, p)| = \frac{(p - \xi)^2 + \eta^2}{(p + \xi)^2 + \eta^2} e^{-\frac{L\xi}{\nu}}.$$

As we noted in the previous section, the real part of z , ξ , grows asymptotically like h^{-1} when the maximum frequency is chosen to be $O(h^{-1})$. However, the imaginary part η remains bounded: $|\eta| \in [0, |b|]$, and $\eta_{max}^2 = b^2 - O(h^2)$ as h approaches 0. Thus we have

$$|\rho_{000}(k, L, p)| \leq \frac{(p - \xi)^2 + b^2}{(p + \xi)^2 + b^2} e^{-\frac{L\xi}{\nu}} =: R(\xi, p).$$

This upper bound holds from the observation that functions of the form

$$\frac{A + C}{B + C}, \quad \text{with } A > 0, B, C \geq 0,$$

are increasing in C when $B \geq A$ (here $C = \eta^2$). Instead of using the exact value for ρ , we solve the approximate min-max problem using the upper bound

$$\min_{p \in \mathbb{R}} \max_{\xi_{\min} \leq \xi \leq \xi_{\max}} R(\xi, p). \quad (4.10)$$

We expect that the parameter we obtain from this optimization will be close to the exact optimized parameter, when ξ_{\max} is large compared to $|b|$.

Theorem 4.4 (Approximate Robin parameter, with overlap). *Let $L > 0$ and $k_{\max} = \infty$. Define the critical value*

$$\xi_2(p) := \sqrt{\frac{2\nu p - Lb^2 + Lp^2 + 2\sqrt{\nu^2 p^2 - 2\nu Lpb^2 - L^2 b^2 p^2}}{L}}, \quad (4.11)$$

and let $p_{\min} := \sqrt{\xi_{\min}^2 + b^2}$. If $\xi_2(p_{\min})$ is complex or $\xi_2(p_{\min}) < \xi_{\min}$ or if

$$R(\xi_{\min}, p_{\min}) > R(\xi_2(p_{\min}), p_{\min}),$$

then the solution p^* of the min-max problem (4.10) is $p^* = p_{\min}$. Otherwise, the solution is given by the unique root (larger than p_{\min}) of the equation

$$R(\xi_{\min}, p^*) = R(\xi_2(p^*), p^*).$$

Proof. By differentiating R with respect to p , we find that the derivative has the same sign as $(p^2 - \sqrt{\xi^2 + b^2})$. Thus, we know the solution of the min-max problem p^* will be larger than $p_{\min} = \sqrt{\xi_{\min}^2 + b^2}$. On the other hand, by differentiating R with respect to ξ , we find that R has a maximum at $\xi = \xi_2(p)$, defined in (4.11). So, if $\xi_2(p_{\min})$ is complex or smaller than ξ_{\min} , then R is decreasing in ξ for all $\xi \geq \xi_{\min}$.

The maximum is thus attained at ξ_{min} , and because $\frac{\partial R}{\partial p}(\xi_{min}, p) \geq 0$, it is impossible to move p to improve the convergence rate. Hence, the solution of the min-max problem is $p^* = p_{min}$.

Now, for the case $\xi_2(p_{min}) > \xi_{min}$, note that $p < \sqrt{\xi_2(p) + b^2}$, which implies

$$\begin{aligned} \frac{dR}{dp}(\xi_2(p), p) &= \frac{\partial R}{\partial p}(\xi_2(p), p) + \frac{\partial R}{\partial \xi}(\xi_2(p), p)\xi_2'(p) \\ &= \frac{\partial R}{\partial p}(\xi_2(p), p) < 0. \end{aligned}$$

This means that the value at the interior maximum decreases with p . Hence, if $R(\xi_{min}, p) > R(\xi_2(p), p)$ at $p = p_{min}$, then we again have $p^* = p_{min}$. Finally remains the last case when $R(\xi_{min}, p) < R(\xi_2(p), p)$ at $p = p_{min}$. Note that for p large enough, we will get $R(\xi_{min}, p) > R(\xi_2(p), p)$ since $R(\xi_2(p), p)$ decreases to 0 as $p \rightarrow \infty$. This shows that there exists at least one solution of the equation

$$R(\xi_{min}, p^*) = R(\xi_2(p^*), p^*).$$

In addition, there is exactly one root since $R(\xi_{min}, p)$ is strictly increasing in p and $R(\xi_2(p), p)$ is strictly decreasing, for all $p \geq p_{min}$. Hence, the solution p^* of the min-max problem is given by this unique root. \square

Theorem 4.5 (Approximate Robin asymptotics with overlap). *For $k_{max} = \frac{\pi}{h}$ and $L = h$, the asymptotic performance of the optimized Schwarz method, with the Robin parameter p^* obtained through the approximate min-max problem (4.10) is given by*

$$\max_{\xi_{min} \leq \xi \leq \xi_{max}} |\rho_{OO_0}(k, h, p^*)| = 1 - 4 \left(\frac{\xi_{min}}{\nu} \right)^{\frac{1}{3}} h^{\frac{1}{3}} + O(h^{\frac{2}{3}}). \quad (4.12)$$

Proof. First note that, as $h \rightarrow 0$, $\xi_2(p_{min}) \sim 2\sqrt{\nu p_{min}}h^{-1/2}$, and thus we get

$$\xi_{min} \leq \xi_2(p_{min}) \leq \xi_{max},$$

for h small enough. In addition, we observe by inspection that

$$\lim_{h \rightarrow 0} R(\xi_2(p_{min}), p_{min}) = 1.$$

Hence, we also have $R(\xi_2(p_{min}), p_{min}) > R(\xi_{min}, p_{min})$ for h small enough. So, asymptotically, p^* is given by the solution of the equation

$$R(\xi_{min}, p^*) = R(\xi_2(p^*), p^*). \quad (4.13)$$

We now make the ansatz $p^* = Ch^\alpha$ with $\alpha < 0$ (we know that the optimized parameter increases as h decreases). It will not be possible to satisfy equation (4.13) unless $\alpha > -1$, by an argument similar to the one in the proof of Proposition 4.4. Inserting this in the definition for ξ_2 , we get that $\xi_2(p^*) \sim 2\sqrt{\nu C}h^{\frac{\alpha-1}{2}}$. Inserting our ansatz for p^* in equation (4.13) and expanding for small h , we get the leading order terms

$$1 - \frac{4}{C}\xi_{min}h^{-\alpha} \quad \text{and} \quad 1 - 4\sqrt{\frac{C}{\nu}}h^{\frac{\alpha+1}{2}}.$$

Since the equation is satisfied for all h , we can match the exponents and the coefficients, which yields

$$\alpha = -\frac{1}{3}, \quad C = (\nu\xi_{min}^2)^{\frac{1}{3}}.$$

Finally, using these results and expanding $\rho_{OO0}(k_{min}, h, p^*)$ for small h , we obtain the asymptotic expansion (4.12). \square

Note that this result is the same as the asymptotic expansion we obtained in the case $b = 0$ (Proposition 4.4): even the coefficient matches. The approximation we make by replacing $|\rho_{OO0}(k, L, p)|$ with $R(\xi, p)$ affects only higher order terms in the expansion. Figure 4.7 shows a comparison between the convergence rate using the optimized Robin parameter computed numerically (dotted line), and the convergence rate using the parameter found by solving the approximate min-max problem. Figure 4.8 shows a case where b is larger and the approximation is not as good. Figure 4.9 shows the convergence rate for four different transmission conditions.

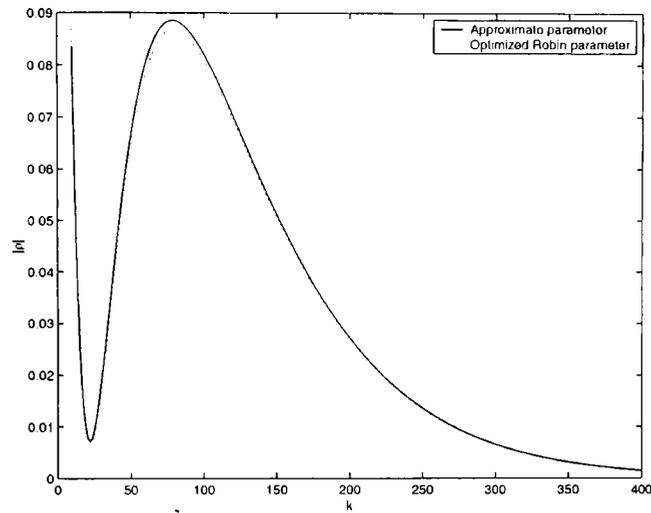


FIGURE 4.7. Comparison of the convergence rate using the approximate and optimized parameters, for $\nu = 0.1$, $a = 1$, $b = 1$, $c = 1$, $L = \pi/400$.

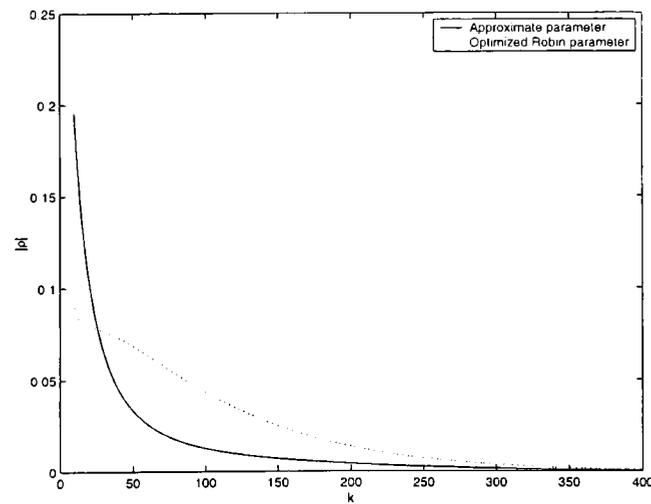


FIGURE 4.8. Comparison of the convergence rate using the approximate and optimized parameters, for $\nu = 0.1$, $a = 1$, $b = 25$, $c = 1$, $L = \pi/400$.

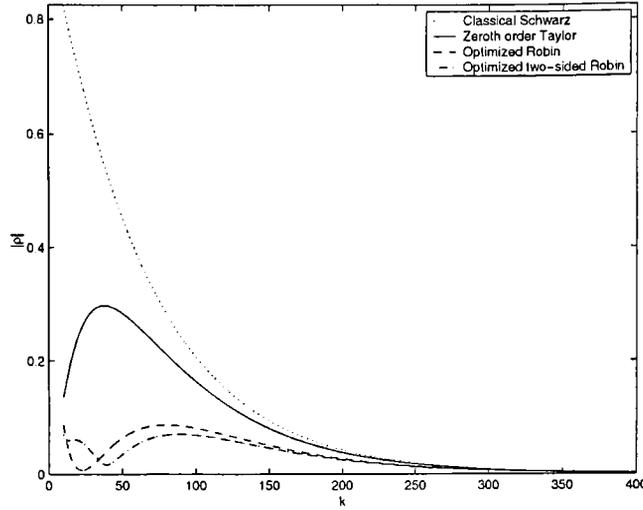


FIGURE 4.9. Convergence rates for the values $\nu = 0.1$, $a = 1$, $b = 1$, $c = 1$, $L = \pi/400$.

4.5 Discussion

To summarize, we computed optimized Robin transmission conditions for the Schwarz method, that have asymptotic performance of $1 - O(h^{\frac{1}{2}})$ without overlap, and of $1 - O(h^{\frac{1}{3}})$ when an overlap of size $O(h)$ is used. Thus, even using a very small overlap, a few grid points wide, improves significantly the convergence of the optimized Schwarz method (for small mesh size h). This improvement is obtained with virtually no additional computational cost. Hence, it is recommended to use overlapping decomposition, when permitted by the geometry and physics of the underlying problem.

Comparing with the optimized methods of second order found previously by C. Japhet, there are two main comments to make. First, we have computed optimized conditions for overlapping and non-overlapping decompositions; previously the overlapping case was not considered. Also, we were able to get an asymptotic performance of the order of $h^{\frac{1}{3}}$ by using simple Robin conditions optimized on one parameter only.

This was the best performance previously known for the advection-diffusion equation, but with second order conditions.

Chapter 5

Optimization on Two Parameters

We are interested in this chapter in finding transmission conditions that are optimized on two separate parameters. This includes the two-sided Robin conditions and the second order conditions as defined in Section 3.2. We show in fact that these two classes of transmission conditions are closely related. We consider only the case where the advection is normal to the interface ($b = 0$): we can prove our results by simply reducing the problems to the modified Helmholtz equation, analyzed in [Gan03]. The general case when $b \neq 0$ is still an open problem.

5.1 Second Order Transmission Conditions

We investigate in this section the use of second order transmission conditions of the form

$$\begin{aligned} \mathcal{B}_1(u) &:= \frac{\partial u}{\partial x} + \frac{1}{2\nu} \left((-a + p)u - q \frac{\partial^2 u}{\partial y^2} \right), \\ \mathcal{B}_2(u) &:= \frac{\partial u}{\partial x} + \frac{1}{2\nu} \left((-a - p)u + q \frac{\partial^2 u}{\partial y^2} \right). \end{aligned}$$

We assume $b = 0$, so the convergence rate for the Schwarz iteration becomes

$$\rho_{OO_2}(k, L, p, q) = \frac{(p + qk^2 - \xi)^2}{(p + qk^2 + \xi)^2} e^{-\frac{L\xi}{\nu}},$$

where $\xi = z = \sqrt{a^2 + 4\nu c + 4\nu^2 k^2}$ is a real number. To obtain optimized values for the parameters p and q , we wish to solve the min-max problem

$$\min_{p,q \in \mathbb{R}} \max_{k \in [k_{min}, k_{max}]} |\rho_{OO2}(k, L, p, q)|. \quad (5.1)$$

Proposition 5.1 (Optimized second order conditions when $b = 0$). *For $L > 0$, $b = 0$ and $k_{max} = \infty$, the solution (p^*, q^*) of the min-max problem (5.1) is given by the unique root of the system of equations*

$$\rho_{OO2}(k_{min}, L, p^*, q^*) = \rho_{OO2}(k_1, L, p^*, q^*) = \rho_{OO2}(k_2, L, p^*, q^*),$$

where the locations of the maxima k_1 and k_2 are given by

$$\begin{aligned} D &:= \nu^2 L^2 + 4\nu^2 Lq - 4\nu^2 L^2 pq + 4\nu^2 q^2 - 16\nu^2 Lpq^2 + 4Lq^3 A + L^2 q^2 A, \\ k_{1,2} &:= \frac{1}{q} \sqrt{\frac{L+2q-2Lpq \mp \nu^{-1} \sqrt{D}}{2L}}. \end{aligned}$$

When $L = 0$ and $k_{max} < \infty$, the optimized parameters are given directly by the formulas

$$\begin{aligned} p^* &= \frac{\sqrt{2\nu}(k_{max}^2 \xi_{min} - k_{min}^2 \xi_{max})}{\sqrt{(k_{max}^2 - k_{min}^2)(\xi_{max} - \xi_{min})(\xi_{min} \xi_{max})}^{\frac{1}{4}}}, \\ q^* &= \sqrt{\frac{2(\xi_{max} - \xi_{min})}{(k_{max}^2 - k_{min}^2)} \frac{\nu}{(\xi_{max} \xi_{min})}^{\frac{1}{4}}}. \end{aligned}$$

Proof. The min-max problem reduces to the one obtained for the modified Helmholtz equation, with the substitutions

$$\begin{aligned} K &= 2\nu k, \quad \bar{L} = \frac{L}{2\nu}, \\ P &= p, \quad Q = \frac{q}{4\nu^2}, \\ \rho_{OO2}(k, L, p, q) &= \frac{(P+QK^2 - \sqrt{A+K^2})^2}{(P+QK^2 + \sqrt{A+K^2})^2} e^{-2\bar{L}\sqrt{A+K^2}}. \end{aligned}$$

Applying Theorem 4.7 in [Gan03] to this expression, and transforming back to the original variables yields the formulas in the statement of the proposition. \square

Proposition 5.2 (OO2 asymptotics, without overlap and $b = 0$). *For the case $L = 0$ and $k_{max} = \frac{\pi}{h}$, we get the asymptotic performance*

$$\max_{k_{min} \leq k \leq \frac{\pi}{h}} |\rho_{OO2}(k, 0, p^*, q^*)| = 1 - 4 \left(\frac{2\xi_{min}}{\pi\nu} \right)^{\frac{1}{4}} h^{\frac{1}{4}} + O(h^{\frac{1}{2}}). \quad (5.2)$$

Proof. Inserting $k_{max} = \frac{\pi}{h}$ and $\xi_{max} = \frac{2\pi\nu}{h} + O(h)$ directly into the formulas for p^* and q^* , we obtain

$$\begin{aligned} p^* &= \frac{\xi_{min}^{3/4} (\nu\pi)^{1/4}}{2^{3/4}} h^{-1/4} + O(h^{3/4}), \\ q^* &= \frac{2^{1/4} \nu^{3/4}}{\pi^{3/4} \xi_{min}^{1/4}} h^{3/4} + O(h^{7/4}). \end{aligned}$$

Then, expanding the expression for $|\rho_{OO2}(k_{min}, 0, p^*, q^*)|$ we obtain the asymptotic result (5.2). \square

Remark 5.1. *Note that this optimized method of order 2 (OO2) has better asymptotic performance than the one computed in [Jap97], where the asymptotic expansion gives a result of the form $1 - O(h^{\frac{1}{3}})$.*

5.2 Two-Sided Robin Transmission Conditions

Now, let us go back to Robin transmission conditions, and use two different parameters for the operators. That is, we consider the conditions

$$\mathcal{B}_1 u = \frac{\partial u}{\partial x} + \left(\frac{-a + p_1}{2\nu} \right) u, \quad \mathcal{B}_2 u = \frac{\partial u}{\partial x} + \left(\frac{-a - p_2}{2\nu} \right) u.$$

The convergence rate of the Schwarz method becomes

$$\rho_{OO0}(k, L, p_1, p_2) = \frac{(p_1 - \xi)(p_2 - \xi)}{(p_1 + \xi)(p_2 + \xi)} e^{-\frac{L\xi}{\nu}}.$$

We start with the non-overlapping case.

Proposition 5.3 (Optimized two-sided Robin conditions, without overlap and $b = 0$). *When $L = 0$, the solution (p_1^*, p_2^*) of the min-max problem*

$$\min_{p_1, p_2 \in \mathbb{R}} \left(\max_{k_{min} \leq k \leq k_{max}} |\rho_{OO0}(k, 0, p_1, p_2)| \right)$$

is given by the direct formulas

$$p_1^* = \omega^* + \sqrt{(\omega^*)^2 - \xi_{\min}\xi_{\max}}, \quad p_2^* = \omega^* - \sqrt{(\omega^*)^2 - \xi_{\min}\xi_{\max}},$$

where

$$\omega^* = \sqrt{\sqrt{\xi_{\min}\xi_{\max}} \left(\frac{\xi_{\min} + \xi_{\max}}{2} \right)}.$$

Proof. We are able to take the maximum over the variable ξ instead of k , since $\xi(k)$ is a continuous increasing function. Also, we can restrict the range of p_1 and p_2 to the interval $[\xi_{\min}, \xi_{\max}]$, as we did before. Thus, we want to find the values p_1^* and p_2^* that achieve the minimum

$$\min_{(p_1, p_2) \in [\xi_{\min}, \xi_{\max}]^2} \left(\max_{\xi \in [\xi_{\min}, \xi_{\max}]} \left| \frac{(p_1 - \xi)(p_2 - \xi)}{(p_1 + \xi)(p_2 + \xi)} \right| \right).$$

This is in the form of the Wachspres alternance result (Theorem 4.1), with $n = 2$. Using the optimal parameters property (4.2), we can divide by two the number of parameters, by using the substitutions

$$\begin{aligned} \hat{\xi} &:= \frac{\sqrt{\xi_{\min}\xi_{\max}}}{2} \left(\frac{\xi}{\sqrt{\xi_{\min}\xi_{\max}}} + \frac{\sqrt{\xi_{\min}\xi_{\max}}}{\xi} \right), \\ \hat{\xi}_{\min} &:= \sqrt{\xi_{\min}\xi_{\max}}, \quad \hat{\xi}_{\max} := \frac{\xi_{\min} + \xi_{\max}}{2}, \\ \omega &:= \frac{1}{2}(p_1 + p_2) = \frac{1}{2} \left(p_1 + \frac{\xi_{\min}\xi_{\max}}{p_1} \right). \end{aligned}$$

After some algebraic manipulations, the rate of convergence becomes

$$\rho_{\infty}(k, 0, p_1, p_2) = -\frac{(\omega - \hat{\xi})}{(\omega + \hat{\xi})}.$$

Hence, the min-max problem reduces to

$$\min_{\omega \in [\hat{\xi}_{\min}, \hat{\xi}_{\max}]} \left(\max_{\hat{\xi} \in [\hat{\xi}_{\min}, \hat{\xi}_{\max}]} \left| \frac{(\omega - \hat{\xi})}{(\omega + \hat{\xi})} \right| \right).$$

We refer to [Wac62] for more details on the general algorithm to reduce the number of parameters by a factor of two in the minimization. The optimized parameter for this problem is $\omega^* = \sqrt{\hat{\xi}_{\min}\hat{\xi}_{\max}}$. Transforming back to the original parameters, we get

$$p_1^* = \omega^* + \sqrt{(\omega^*)^2 - \xi_{\min}\xi_{\max}}, \quad p_2^* = \omega^* - \sqrt{(\omega^*)^2 - \xi_{\min}\xi_{\max}},$$

which leads to the advertised result. \square

Proposition 5.4 (Optimized two-sided Robin asymptotics, without overlap and $b = 0$). *When $b = L = 0$, the asymptotic performance of the Schwarz method with optimized two-sided Robin transmission conditions is*

$$\max_{k_{\min} \leq k \leq \frac{\pi}{h}} |\rho_{OO0}(k, 0, p_1^*, p_2^*)| = 1 - 2 \left(\frac{2\xi_{\min}}{\nu\pi} \right)^{\frac{1}{4}} h^{\frac{1}{4}} + O(h^{\frac{1}{2}}).$$

Proof. We first compute the expansion for ω^*

$$\begin{aligned} \omega^* &= \sqrt{\sqrt{\xi_{\min}\xi_{\max}}(\xi_{\min} + \xi_{\max})/2} \\ &\sim \frac{1}{\sqrt{2}} \left[\sqrt{\frac{2\nu\pi\xi_{\min}}{h}} \left(\xi_{\min} + \frac{2\nu\pi}{h} \right) \right]^{\frac{1}{2}} \\ &= \frac{(2\xi_{\min})^{\frac{1}{4}}(\nu\pi)^{\frac{3}{4}}}{h^{\frac{3}{4}}} + \frac{2^{\frac{1}{4}}\xi_{\min}^{\frac{5}{4}}}{4(\nu\pi)^{\frac{1}{4}}} h^{\frac{1}{4}} + O(h^{\frac{5}{4}}), \\ \sqrt{(\omega^*)^2 - \xi_{\min}\xi_{\max}} &= \frac{(2\xi_{\min})^{\frac{1}{4}}(\nu\pi)^{\frac{3}{4}}}{h^{\frac{3}{4}}} - \frac{\xi_{\min}^{\frac{3}{4}}(\nu\pi)^{\frac{1}{4}}}{2^{\frac{1}{4}}h^{\frac{1}{4}}} + O(h^{\frac{1}{4}}). \end{aligned}$$

Now, the two different parameters give

$$\begin{aligned} p_1^* &= \frac{2(2\xi_{\min})^{\frac{1}{4}}(\nu\pi)^{\frac{3}{4}}}{h^{\frac{3}{4}}} + O(h^{-\frac{1}{4}}) = Ch^{-\frac{3}{4}} + O(h^{-\frac{1}{4}}), \\ p_2^* &= \frac{\xi_{\min}^{\frac{3}{4}}(\nu\pi)^{\frac{1}{4}}}{2^{\frac{1}{4}}h^{\frac{1}{4}}} + O(h^{\frac{1}{4}}) = Dh^{-\frac{1}{4}} + O(h^{\frac{1}{4}}). \end{aligned}$$

Finally, we can use these expressions in the convergence rate to get the expansion

$$\begin{aligned}
\max_{k_{\min} \leq k \leq k_{\max}} |\rho_{OO0}(k, 0, p_1^*, p_2^*)| &= |\rho(k_{\min}, 0, p_1^*, p_2^*)| \\
&= \frac{(p_1^* - \xi_{\min})(p_2^* - \xi_{\min})}{(p_1^* + \xi_{\min})(p_2^* + \xi_{\min})} \\
&\sim \frac{(1 - \xi_{\min} h^{3/4}/C)(1 - \xi_{\min} h^{1/4}/D)}{(1 + \xi_{\min} h^{3/4}/C)(1 + \xi_{\min} h^{1/4}/D)} \\
&\sim \left(1 - \frac{2\xi_{\min}}{C} h^{\frac{3}{4}}\right) \left(1 - \frac{2\xi_{\min}}{D} h^{\frac{1}{4}}\right) \\
&\sim 1 - \frac{2\xi_{\min}}{D} h^{\frac{1}{4}} + O(h^{\frac{1}{2}}).
\end{aligned}$$

This leads to the result

$$\max_{k_{\min} \leq k \leq k_{\max}} |\rho_{OO0}(k, 0, p_1^*, p_2^*)| = 1 - 2 \left(\frac{2\xi_{\min}}{\nu\pi} \right)^{\frac{1}{4}} h^{\frac{1}{4}} + O(h^{\frac{1}{2}}).$$

□

We now continue with the overlapping case, still assuming $b = 0$.

Proposition 5.5 (Optimized two-sided Robin parameters, with overlap and $b = 0$). *The optimized two-sided Robin parameters are given by*

$$\begin{aligned}
p_1^* &= \frac{2\nu^2}{q^*} \left[1 - \sqrt{1 + \frac{A}{4\nu^4} (q^*)^2 - \frac{p^* q^*}{\nu^2}} \right], \\
p_2^* &= \frac{2\nu^2}{q^*} \left[1 + \sqrt{1 + \frac{A}{4\nu^4} (q^*)^2 - \frac{p^* q^*}{\nu^2}} \right],
\end{aligned}$$

where p^* and q^* are the optimized second order parameters, given by Proposition 5.1, with L replaced by $2L$.

Proof. The convergence rate of the Schwarz iteration is given by

$$\rho_{OO0}(k, L, p_1, p_2) = \frac{(p_1 - \xi)(p_2 - \xi)}{(p_1 + \xi)(p_2 + \xi)} e^{-\frac{L\xi}{\nu}}.$$

Multiplying out the two factors in the fractional part, we obtain

$$\rho_{OO0}(k, L, p_1, p_2) = \frac{\frac{p_1 p_2 + A}{p_1 + p_2} + \frac{4\nu^2}{p_1 + p_2} k^2 - \xi}{\frac{p_1 p_2 + A}{p_1 + p_2} + \frac{4\nu^2}{p_1 + p_2} k^2 + \xi} e^{-\frac{L\xi}{\nu}}.$$

Hence, by setting the new parameters

$$p = \frac{p_1 p_2 + A}{p_1 + p_2}, \quad q = \frac{4\nu^2}{p_1 + p_2},$$

our optimization problem is equivalent to the min-max problem for second order conditions, with L replaced by $2L$. Inverting the above transformation, we get the expressions for p_1^* and p_2^* in terms of the optimized second order parameters p^* and q^* as stated in the theorem. \square

This reduction also applies for the case without overlap, $L = 0$, and matches with the result of Proposition 5.3, for which we have given a different but direct proof. The asymptotic performance in the overlapping case for the second order and two-sided Robin conditions still needs to be proved, but we expect it to be of the form $1 - O(h^{\frac{1}{5}})$, as for the modified Helmholtz equation, see [Gan03].

Chapter 6

Numerical Results

In this chapter, numerical results are presented to illustrate the convergence of the Schwarz method with optimized Robin conditions. Comparisons are also made with other methods. The results were obtained using a finite difference solver implemented in Matlab, for rectangular domains. The original region is divided into two symmetric subdomains, with vertical interfaces. The linear systems corresponding to the discretized subproblems are solved using the backslash operator “\” in Matlab.

6.1 An Example with Constant Coefficients

First, we consider an advection-diffusion equation with constant coefficients and homogeneous Dirichlet boundary condition,

$$\begin{cases} -\frac{1}{10}(u_{xx} + u_{yy}) + u_x + u_y + u = 0 & \text{on } \Omega = (0, \pi)^2, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \quad (6.1)$$

The exact solution of this simple problem is $u(x, y) = 0$. The square domain is divided into the subdomains

$$\Omega_1 = \left(0, \frac{\pi + L}{2}\right) \times (0, \pi), \quad \Omega_2 = \left(0, \frac{\pi - L}{2}\right) \times (0, \pi).$$

For the overlapping case, we choose an overlap of size $L = 2h$ (two grid spaces wide), since in applications an overlap larger than $O(h)$ cannot be afforded, because of the computational cost. Also note that taking a constant overlap leads to a constant convergence rate with respect to h , like for the classical Schwarz method. We take $k_{max} = \frac{\pi}{h}$ as an approximation for the maximum frequency. To start the Schwarz iteration, we specify initial guesses u_1^0 and u_2^0 for the solutions in the subdomains Ω_1 and Ω_2 . In fact, what is really needed for the algorithm is only the values of the interface operators

$$\mathcal{B}_1 u_2^0 \text{ on } \Gamma_1, \quad \mathcal{B}_2 u_1^0 \text{ on } \Gamma_2.$$

However, we choose to specify initial approximations over the full subdomains because we want the different methods to start with the same initial error (see error definition below). For this initial data, we use a matrix of random values ranging from -1 to 1 (using `rand` in Matlab), so that the initial error contains a wide range of frequency components.

We compare the convergence properties of the Schwarz method using four different transmission conditions: Dirichlet (classical Schwarz, when there is overlap), zeroth order Taylor approximations, optimized Robin conditions, and optimized two-sided Robin conditions. The latter are computed by solving the min-max problem numerically when no theoretical result is known (using `fminsearch` in Matlab). The error is measured in the infinity norm as the absolute difference

$$e^n := \|u^n - u\|_\infty = \max_{(x,y) \in \Omega} |u^n - u|,$$

where u is the numerical solution obtained by solving directly over the full domain Ω , and the approximations of the solution u^n are defined as in the Restricted Additive Schwarz method (see Section 1.4)

$$u^n = \begin{cases} u_1^n & \text{in } [0, \pi/2] \times [0, \pi], \\ u_2^n & \text{in } (\pi/2, \pi] \times [0, \pi]. \end{cases}$$

We compare the iterates in the Schwarz iteration to the numerical solution u on Ω for the corresponding mesh, since we want to isolate the error due to the discretization from the error due to the iterative method. For these experiments, we used a 2^{nd} order finite-difference discretization, and so the error in u is $O(h^2)$. Figure 6.1 shows the convergence of the different Schwarz methods, using both non-overlapping and overlapping decompositions for problem (6.1). For these graphs, the mesh size of the finite difference discretization is $h = \pi/300$ in both x and y .

Note that the effect of using an overlap on the convergence is significant: by using a small overlap of only two grid spaces wide, the convergence is accelerated by a factor of more than two in this case! In addition, the computational cost of adding a small overlap in the order of the mesh size is very small: it adds only $O(h^{-1})$ grid points to each subproblem, which already contain $O(h^{-2})$ points.

The difference in convergence for the three Robin transmission conditions with overlap (Figure 6.1(b)) may not seem very apparent. However, the difference increases when taking smaller mesh sizes, as predicted by the different asymptotic expansion of each convergence rate.

By varying the mesh size h , the maximum frequency ($k_{max} = \frac{\pi}{h}$) and the overlap size ($L = 2h$ when positive) also change, from our choice. We look at the effect of h on the convergence of the Schwarz iteration. Figures 6.2(a) and 6.2(b) are logarithmic plots of the number of iterations needed to achieve an error of 10^{-6} , for different values of the mesh size h . We took from 50 to 500 grid points (by increments of 50) in each direction for these results; the solver quickly becomes very slow for smaller mesh sizes, because we use the backslash solver “\” of Matlab, instead of a more efficient method for solving large sparse systems. The plots illustrate well the weaker dependence of the convergence on h for optimized transmission conditions, compared to the classical Schwarz method and the Taylor approximations. For h small enough, the slope of the data points should give approximatively the exponent of the leading order term in the

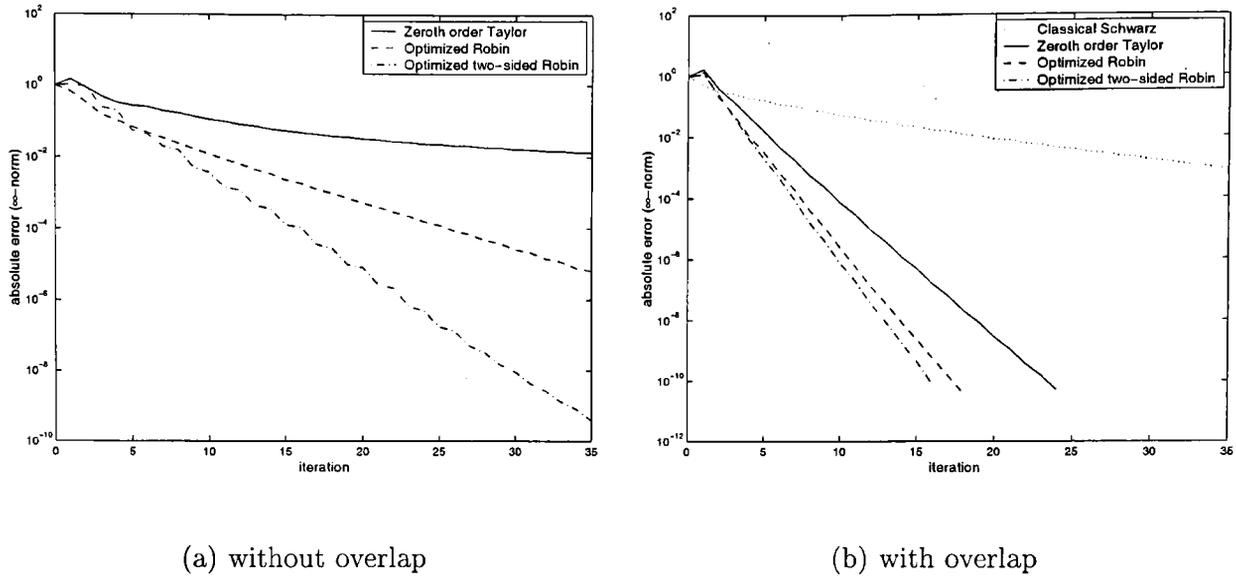
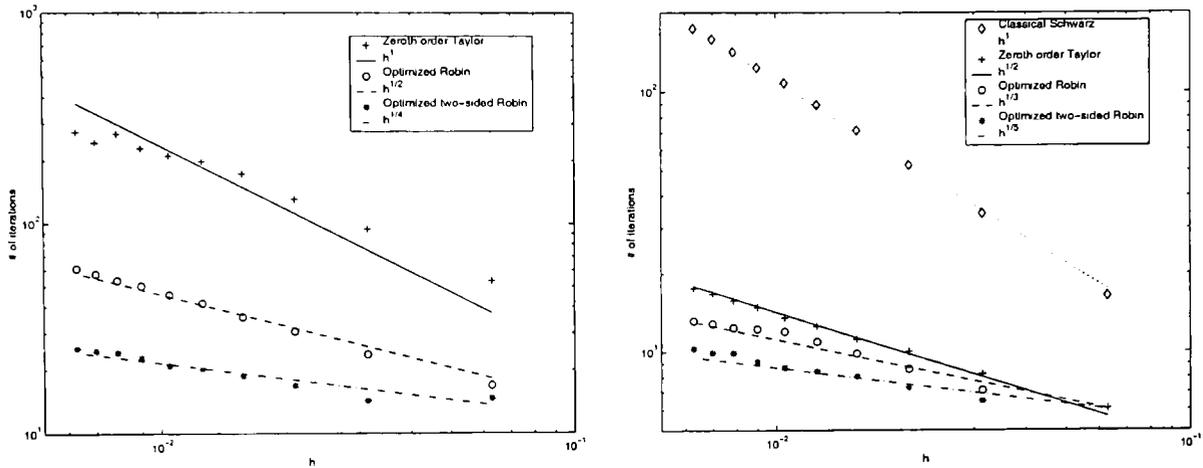


FIGURE 6.1. Comparison of the convergence for different transmission conditions.

asymptotic expansion of the convergence rate. On the plots are also shown lines with the slopes expected from the theoretical expansions for comparison. Note that the zeroth order Taylor method without overlap seems to have better performance than expected for this example, more investigation is needed to understand the reason. Table 6.1 shows the slopes of the best fitted line for the data of Figure 6.2(a) and 6.2(b), along with values predicted by the theoretical asymptotics (in parentheses). We already observe good agreement of these numbers with the theory, even though the smallest mesh size used for this experiment is still fairly large ($\pi/500 \approx 0.0063$). However, these results do not seem to confirm the theory as well as the experiments done for symmetric differential operators in [Gan03].

For the optimized Robin transmission conditions, the optimized parameter is compared to the value leading to the best convergence numerically in Figure 6.3(a) and 6.3(b). The asterisk symbol (*) indicates the optimized parameter, and in the over-



(a) without overlap

(b) with overlap

FIGURE 6.2. Number of iterations needed to achieve an error of 10^{-6} , for different values of h .

Method	$L = 0$	$L = 2h$
Classical Schwarz	-	1.0285 (1)
Zeroth order Taylor	0.7097 (1)	0.4624 (1/2)
Optimized Robin	0.5669 (1/2)	0.4283 (1/3)
Optimized two-sided Robin	0.2754 (1/4)	0.3093 (1/5)

TABLE 6.1. Slopes of the asymptotic graphs

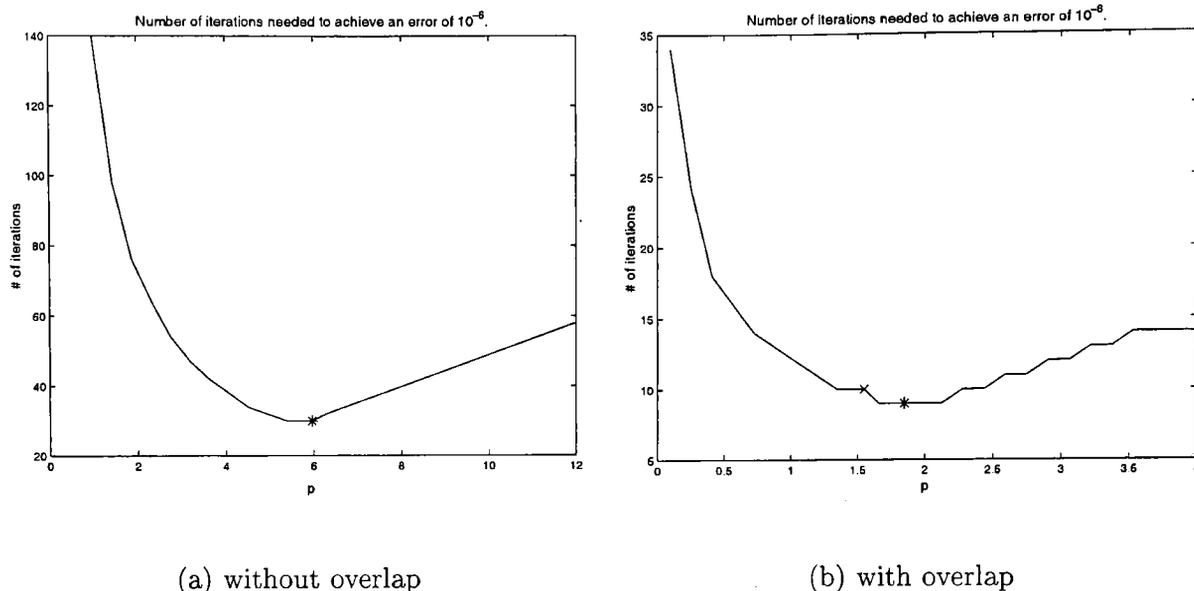


FIGURE 6.3. Comparison of the optimized Robin parameter with other values, for this example with a bounded domain.

lapping case, the cross symbol (\times) shows the approximate Robin parameter resulting from Theorem 4.4. The convergence of the optimized method turns out to be the best that one can achieve with Robin transmission conditions, for this specific example with a bounded domain. This justifies the methodology of the optimized Schwarz methods: the continuous Fourier analysis is done only for a model problem, with an infinite domain. In general, the optimized parameter will be close to the optimal value for a bounded domain, without necessarily achieving the best convergence possible.

6.2 An Example with Variable Advection

The optimized Schwarz methods are constructed in the case of a constant advection. However, when facing an advection varying in the domain, we can still use the optimized conditions. For the non-overlapping situation, the optimized parameters

are given by explicit formulas. So, we can apply these formulas pointwise with the variable advection, where $a = a(y)$ and $b = b(y)$ are now functions on the interfaces. For overlapping decompositions, the optimized parameters are given in general by the solution of a system of non-linear equations. We could solve this system at each grid point on the interfaces, but this can become very costly when using a small mesh size. To reduce the computational cost, we solve the system of equations only for a reasonable number of grid points, and then use interpolation to compute the transmission conditions at intermediate grid points. Note that all of this process is done once, before starting the Schwarz method.

We consider here an example with a varying advection obtained from a Navier-Stokes computation using periodic boundary conditions, see Figure 6.4. Along a vertical line near the center (location of the interface(s)), the advection varies from horizontal to vertical directions, making this a meaningful test case. The domain is the square $\Omega = (0, \pi)^2$, the viscosity is taken to be $\nu = 0.1$, and $c = 1$. The source term is given by $f(x, y) = \sin 5x \sin 5y$. The numerical solution of this problem is illustrated in Figure 6.5. For this example, 300 grid points are used in x and y . To compute the optimized Robin conditions with overlap, the system of equations is solved at only 75 points along each interface, and linear interpolation is applied to obtain parameters for all grid points (using `interp1` in Matlab).

The convergence of the Schwarz method with the different transmission conditions exhibits behavior similar to the case of constant advection, see Figure 6.6(a) and 6.6(b). The optimized Robin conditions bring significant improvement over the Taylor approximations and the classical Schwarz method.

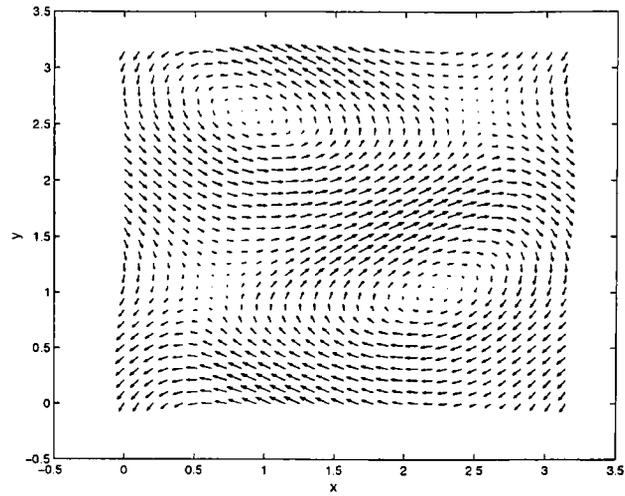


FIGURE 6.4. An example of variable advection.

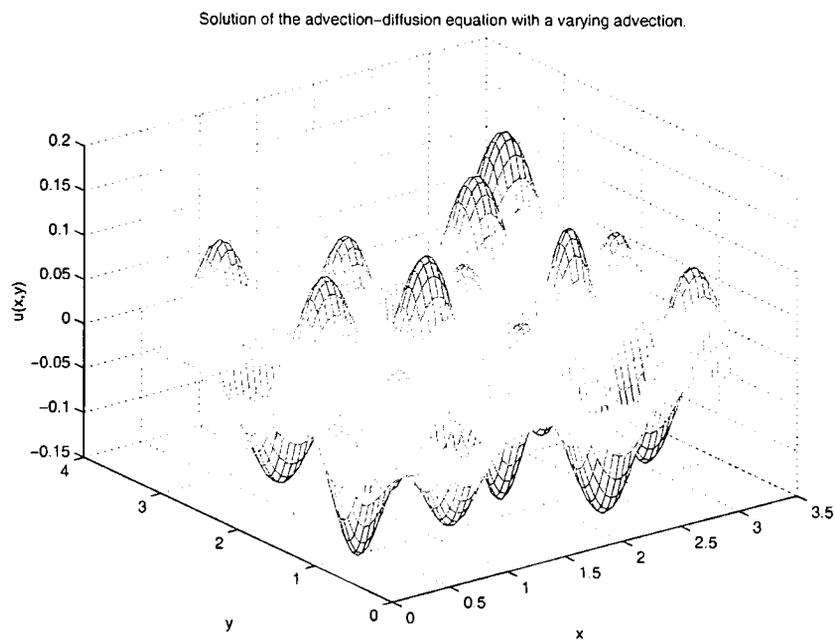
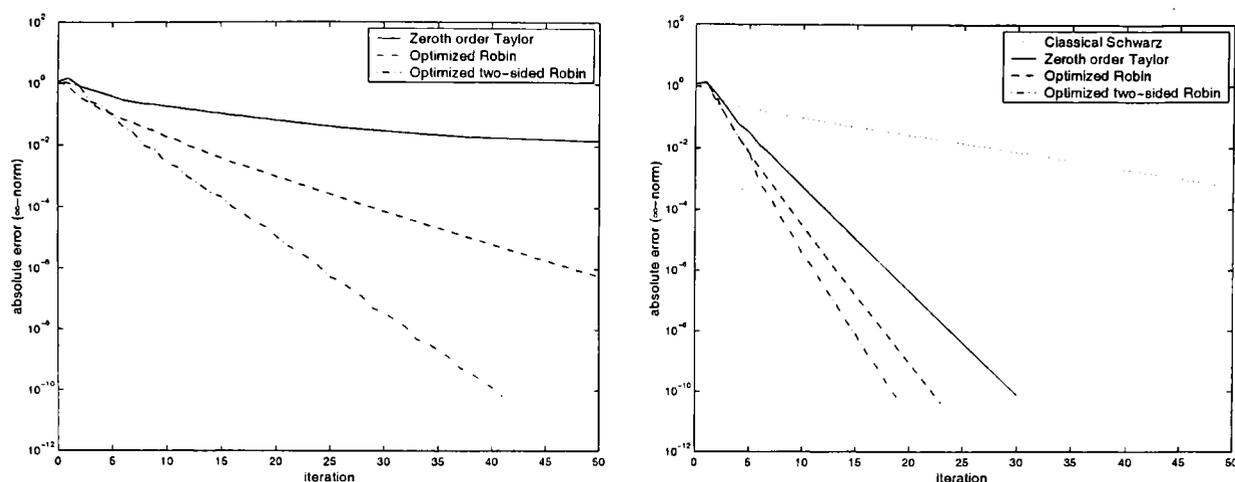


FIGURE 6.5. Numerical solution of the problem.



(a) without overlap

(b) with overlap

FIGURE 6.6. Comparison of different transmission conditions for a varying advection.

6.3 Krylov Acceleration

6.3.1 Condensed Problem on the Interface

We have shown in Chapter 1 that the classical Schwarz method is a stationary iterative method on the interface variable, for the Steklov-Poincaré linear system. We will show in this section that the Schwarz method with Robin transmission conditions can be written as a Richardson iteration for a linear system condensed on the interface.

Consider Robin conditions of the form

$$\mathcal{B}_1 u = \frac{\partial u}{\partial n} + p_1 u, \quad \mathcal{B}_2 u = \frac{\partial u}{\partial n} + p_2 u.$$

Under a particular discretization (e.g. finite difference, finite volume, finite element), we can write the discretized boundary operators in matrix form as

$$\mathcal{B}_1 u_2 \stackrel{h}{\rightarrow} B_1 \mathbf{u}_2 + \mathbf{b}_1, \quad \mathcal{B}_2 u_1 \stackrel{h}{\rightarrow} B_2 \mathbf{u}_1 + \mathbf{b}_2,$$

where \mathbf{b}_i only may depend on f . Note that the matrix B_1 corresponds to the discretized operator \mathcal{B}_1 when applied to *functions on Ω_2 only* (e.g. u_2^n). The operator \mathcal{B}_1 may have a different discretization when applied to functions on Ω_1 ; this discretization will be hidden inside the subdomain solver M_1 , defined below. A similar comment applies to B_2 . Now we define the interface variables as the value of the boundary operators applied to the iterates,

$$\boldsymbol{\mu}_1^n := B_1 \mathbf{u}_2^n + \mathbf{b}_1, \quad \boldsymbol{\mu}_2^n := B_2 \mathbf{u}_1^n + \mathbf{b}_2, \quad \boldsymbol{\mu}^n := \begin{bmatrix} \boldsymbol{\mu}_1^n \\ \boldsymbol{\mu}_2^n \end{bmatrix}.$$

Let M_i be the linear operators representing the subdomain solvers

$$M_i \begin{pmatrix} \mathbf{f}_i \\ \boldsymbol{\mu}_i^n \end{pmatrix} := \mathbf{u}_i^{n+1} \quad \text{where} \quad \begin{cases} A_{ii} \mathbf{u}_i^{n+1} = \mathbf{f}_i \\ \mathcal{B}_i^h \mathbf{u}_i^{n+1} = \boldsymbol{\mu}_i^n \end{cases}.$$

Using these definitions, we can write the parallel Schwarz method given by (1.2) and (1.3) with Robin transmission conditions as

$$\begin{bmatrix} \boldsymbol{\mu}_1^{n+1} \\ \boldsymbol{\mu}_2^{n+1} \end{bmatrix} = \begin{bmatrix} B_1 \mathbf{u}_2^{n+1} + \mathbf{b}_1 \\ B_2 \mathbf{u}_1^{n+1} + \mathbf{b}_2 \end{bmatrix} = \begin{bmatrix} B_1 M_2 \begin{pmatrix} \mathbf{f}_2 \\ \boldsymbol{\mu}_2^n \end{pmatrix} \\ B_2 M_1 \begin{pmatrix} \mathbf{f}_1 \\ \boldsymbol{\mu}_1^n \end{pmatrix} \end{bmatrix} + \begin{bmatrix} \mathbf{b}_1 \\ \mathbf{b}_2 \end{bmatrix}.$$

Now using the linearity of the operators M_i , we obtain an iteration on the interface variables

$$\begin{bmatrix} \boldsymbol{\mu}_1^{n+1} \\ \boldsymbol{\mu}_2^{n+1} \end{bmatrix} = \begin{bmatrix} B_1 M_2 \begin{pmatrix} 0 \\ \boldsymbol{\mu}_2^n \end{pmatrix} \\ B_2 M_1 \begin{pmatrix} 0 \\ \boldsymbol{\mu}_1^n \end{pmatrix} \end{bmatrix} + \begin{bmatrix} B_1 M_2 \begin{pmatrix} \mathbf{f}_2 \\ 0 \end{pmatrix} + \mathbf{b}_1 \\ B_2 M_1 \begin{pmatrix} \mathbf{f}_1 \\ 0 \end{pmatrix} + \mathbf{b}_2 \end{bmatrix} =: T \begin{bmatrix} \boldsymbol{\mu}_1^n \\ \boldsymbol{\mu}_2^n \end{bmatrix} + \mathbf{b}.$$

Hence we get

$$\boldsymbol{\mu}^{n+1} = T \boldsymbol{\mu}^n + \mathbf{b}, \quad \text{where } T = \begin{bmatrix} 0 & T_{12} \\ T_{21} & 0 \end{bmatrix}.$$

The matrix T has the special structure shown above since the value of μ_1^{n+1} depends only on μ_2^n , and vice-versa. Note that the argument demonstrates that the Schwarz algorithm with Robin transmission conditions is a Richardson iteration on the interface variable, for the linear system

$$S\mu = \mathbf{b}, \text{ where } S = I - T. \quad (6.2)$$

Hence, one can apply a Krylov method to the above system instead of the Richardson iteration, to accelerate the convergence. In the continuous formulation, the operator \mathcal{S} is analogous to the Steklov-Poincaré operator. The matrix S is dense and expensive to compute explicitly: each entry involves solving subdomain problems. For second order, symmetric positive definite differential operators (e.g. $-\Delta + \eta$), S is also symmetric positive definite. In our case however, \mathcal{L} is non-symmetric, and as a consequence, so is S . Hence, the conjugate-gradient method cannot be applied to the linear system (6.2). Instead, one can use GMRES or BICGSTAB. For the numerical results that are shown, we use the following implementation of BICGSTAB (from [vdV92]), to solve the linear system $Ax = b$, with initial guess x_0 .

Initial phase:

$$\begin{aligned} z_0 &= Ax_0, & r_0 &= b - z_0, \\ \rho_0 &= \alpha = \omega_0 = 1, & v_0 &= q_0 = 0. \end{aligned}$$

Iteration phase:

$$\begin{aligned}\rho_{k+1} &= (r_0, r_k), & \beta &= \frac{\rho_{k+1}}{\rho_k} \frac{\alpha}{\omega_0}, \\ q_{k+1} &= r_k + \beta(q_k - \omega_k v_k), \\ v_{k+1} &= Aq_{k+1}, & \alpha &= \frac{\rho_{k+1}}{(r_0, v_{k+1})}, \\ s &= r_k - \alpha v_{k+1}, & t &= As, \\ \omega_{k+1} &= \frac{(t, s)}{(t, t)}, \\ x_{k+1} &= x_k + \alpha q_{k+1} + \omega_{k+1} s, & r_{k+1} &= s - \omega_{k+1} t.\end{aligned}$$

Note that each iteration involves two matrix-vector products Ax .

6.3.2 Iterative vs. Krylov Comparison

In Figure 6.7, the convergence of the BICGSTAB method applied to the linear system (6.2) is illustrated for different transmission conditions. Here, we used 300 steps in x and y , a constant advection $\mathbf{a} = (1, 1)$ as in Section 6.1, and a random initial guess for the solution. In Figure 6.8, the convergence of the Schwarz method used iteratively is compared with the Krylov accelerated method on the interface, for different transmission conditions (only the overlapping case is shown). For the comparison to be valid, the x -axis in the graphs corresponds to the number of products Sx , since BICGSTAB does two per iteration.

The graphs show that the Krylov method improves the convergence of the iterative Schwarz method as expected. However, the improvement is less apparent when using more “effective” transmission conditions. When the conditions are modified, the condensed system on the interface (6.2) changes completely, including the definition of the interface variable μ^n . Maybe the fact that the non-symmetry in S changes as we use different conditions affects the Krylov method significantly, and might explain these results. This should be investigated further for a better understanding.

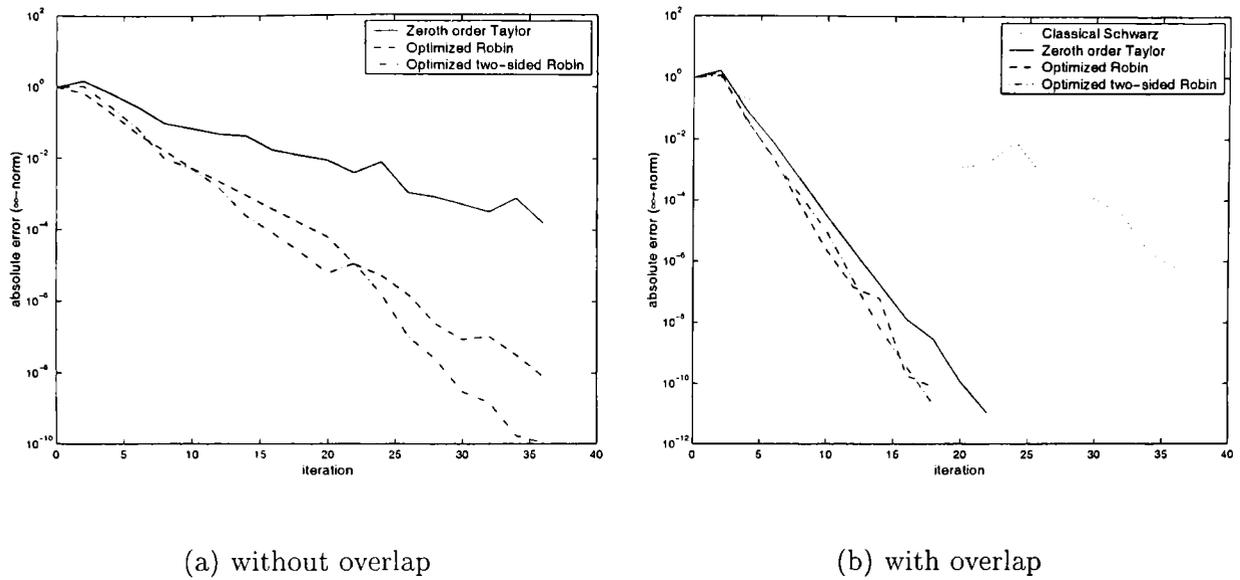


FIGURE 6.7. Convergence of the Krylov accelerated iterations.

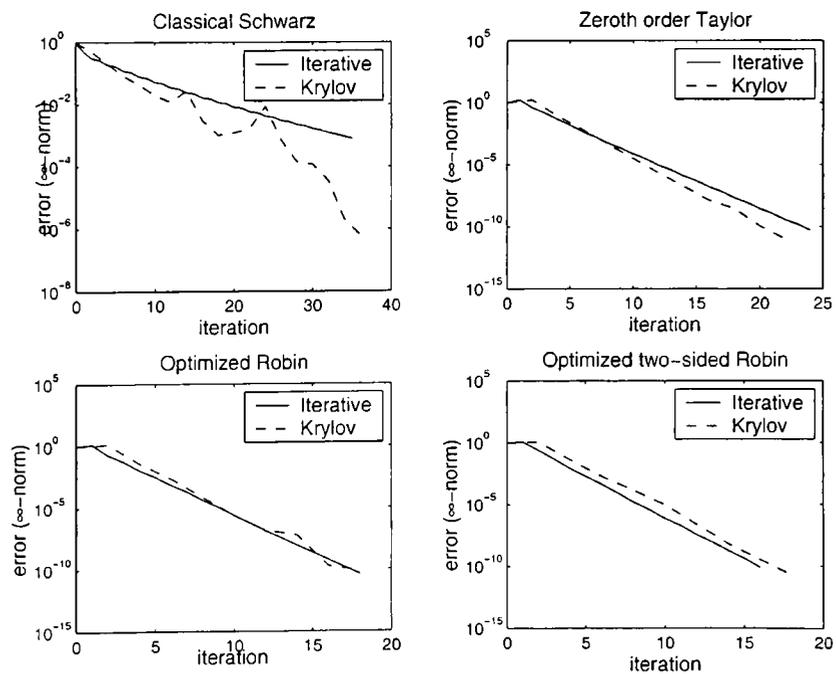


FIGURE 6.8. Comparison of the iterative and Krylov accelerated Schwarz methods, with overlap only.

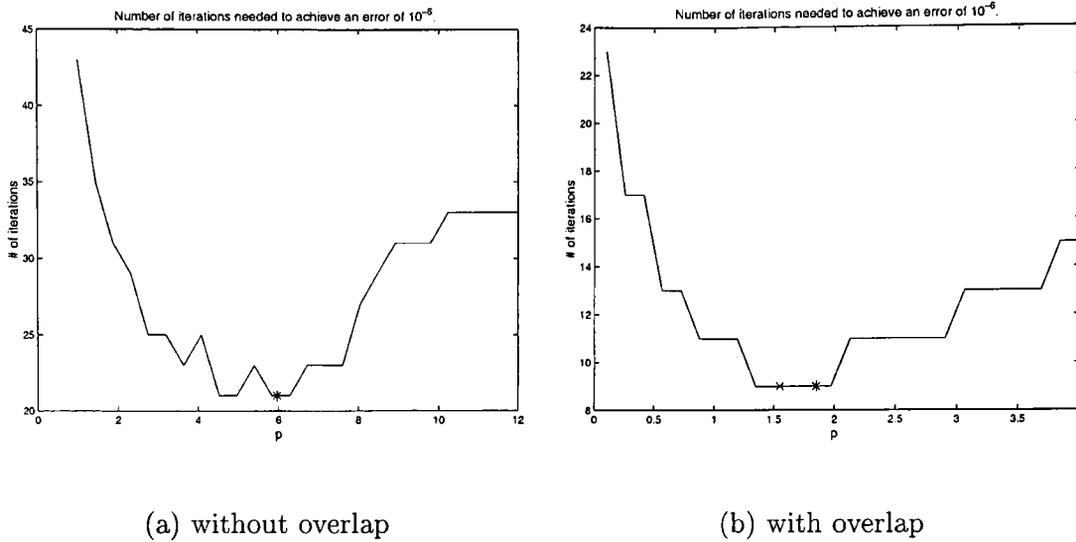


FIGURE 6.9. Comparison of the optimized Robin parameter with other values for the Krylov accelerated method.

The Robin parameter that leads to the quickest convergence in the Krylov accelerated method is very close to the optimized parameter value obtained through the solution of the min-max problem. This is illustrated in Figure 6.9(a) for the non-overlapping case, and in Figure 6.9(b) for the overlapping case. The solver was tested with 25 values of p in the range shown in each case.

Conclusion

In summary, we have computed optimized Robin transmission conditions for the Schwarz method, by solving a min-max problem. An advection-diffusion model problem with constant coefficients on the plane was considered. When the subdomains are not overlapping, the optimized parameter is given by an explicit formula. In the overlapping case, only an approximate min-max problem was considered, which we have shown reduces to solving a non-linear equation for the Robin parameter. The approximation we have made is good asymptotically for small mesh size h and overlap of $O(h)$.

The asymptotic performance of these methods exhibits a weaker dependence on the mesh size than the previously known Robin conditions. Numerical experiments have shown that the optimized conditions improve significantly the convergence of the Schwarz method, compared to other Robin conditions and to the classical Schwarz method.

In addition, a few results were stated for optimized conditions with two parameters: two-sided Robin and second order conditions in the tangential derivative. These partial results only have limited practical use, since they apply only to the special case when the advection is normal to the interfaces. However, they have theoretical interest for future extensions to a general advection.

There are many possibilities for future work. First, finding the exact solution of the min-max problem for the optimized Robin parameter, in the overlapping case,

would be useful: it would always give a good value to the parameter, not only asymptotically for small h as the current result. Also, the results currently known on optimized transmission conditions with two free parameters are very incomplete. Another direction is to consider the case where the coefficients of the advection-diffusion equation are discontinuous across the interface. The physical properties of the material in each subdomain can be different, for example having different viscosities. It would be of interest to compute optimized Schwarz methods in that case and study how the different physics affect the optimization problem.

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