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ABSTRACT

A detailed review of the asymptotic matching procedures predicting the formation number of vortex rings is presented. The original studies of Mohseni and Gharib ["A model for universal timescale of vortex ring formation," Phys. Fluids **10**(10), 2436–2438 (1998).], Shusser and Gharib ["A new model for inviscid vortex rings," in 30th AIAA Fluid Dynamical Conference (1999).], and Linden and Turner ["The formation of 'optimal' vortex rings, and the efficiency of propulsion devices," J. Mech. Fluids, **427**, 61–72 (2001).] are applied to the extended slug-flow model for orifice starting jets and the Kaplanski model of isolated vortex rings. A predicted formation number of 3.5 in the modified non-dimensional time frame is found when the closure assumption in terms of the translational ring speed is chosen, which is consistent with experimental evidence. In addition, particle image velocimetry was performed to assess the validity of the closure assumptions of Mohseni and Gharib and Shusser and Gharib. First, it was further demonstrated that the modified slug-flow model provides an appropriate scaling for the kinematics of orifice-generated vortex rings. Second, the measurements provide experimental support to the method of Shusser and Gharib rather than the method of Mohseni and Gharib. This is further demonstrated by data extracted from the literature. To summarize, in order to predict the formation number, it is recommended to use the extended slug-flow model and the Kaplanski model of isolated vortex rings along with the closure assumption of Shusser and Gharib.

I. INTRODUCTION

A starting jet is inevitably accompanied by the formation of a vortex sheet at the lip of the exhaust, which then rolls up and detaches in the form of a vortex ring. When a small amount of fluid is discharged, a single isolated vortex ring is generated which then propagates freely due to its self-induced velocity. When the quantity of fluid expelled from the exhaust exceeds some threshold, the vortex ring is followed by a trailing jet which either takes the form of a continuous shear layer in the case of parallel and converging starting jets or a train of discrete vortices in the case of orifice starting jets. Using a nozzle geometry, Gharib *et al.*[1] determined the transition between these two states to occur at a limiting non-dimensional time of approximately 4, termed the *formation number*. More precisely, the formation number was measured to be the instant at which the circulation of the detached vortex ring equals the total circulation discharged by the apparatus. The non-dimensional time, often referred to as *formation time*, was defined as $t^* = U_0 t/D_0$, where U_0 is the exhaust speed and D_0 is the exhaust diameter; this is also equivalent to the instantaneous stroke-to-diameter ratio $L_0(t)/D_0$, where $L_0(t)$ is the length of the column of fluid discharged at time t. As such, the formation number has been used to refer to both the instant at which the leading vortex starts exhibiting a trailing jet and the maximum stroke ratio allowing for an isolated vortex ring to be formed.

An explanation of the phenomenon was provided invoking the Kelvin-Benjamin variation principle which states, in layman's word, that for a given hydrodynamic impulse and circulation, there exists a maximum energy state among all possible impulse-preserving vorticity distribution.[2–5] Because this variational principle entails the three invariants of the motion, namely, the circulation Γ, the hydrodynamic impulse I, and the kinetic energy E, Gharib *et al.*[1] proposed a semi-empirical model based on a non-dimensional quantity formed with the aforementioned quantities, i.e., $\alpha \equiv E/\rho^{1/2}\Gamma^{3/2}I^{1/2}$. On the one hand, the total integrals of the motion generated by the apparatus were estimated by the classic slug-flow model. On the other hand, the invariants of the motion of the detached vortex ring were measured at a far downstream location. The intersection of these two curves was found to occur at a non-dimensional time of 3.8, consistent with experimental evidence of a formation number ranging from 3.6 to 4.5.[1]

In an attempt to provide a purely analytical estimate of the formation number, studies have proposed an asymptotic matching procedure. First, following the theoretical framework of Gharib *et al.*,[1] the total invariants of the motion delivered by the generator and estimated by the classic slug-flow model were matched to their equivalent asymptotic quantities given by the Fraenkel–Norbury family of vortex rings. Provided one additional assumption, a single analytical value for the formation number could be found; Mohseni and Gharib[6] and Shusser and Gharib[7] proposed to match the ring speed of the isolated vortex ring to an estimated ring speed in the close vicinity of the exhaust, whereas Linden and Turner[8] argued that, instead of the speed, the volume of the ring atmosphere should be taken into account. Overall, the theoretical predictions of Mohseni and Gharib,[6] Shusser and Gharib,[7] and Linden and Turner[8] were close to the experimentally obtained formation number, thus proving this asymptotic matching operation to be a promising methodology to estimate analytically the limiting stroke ratio and the quantities of the detached vortex ring.

Nevertheless, this asymptotic matching procedure rests upon two major assumptions. First, it assumes the classic slug-flow model to be an accurate model for predicting the invariants of the motion delivered by the apparatus. However, if the model provides adequate estimates for parallel starting jets, it was proved to poorly predict the production of the invariants of the motion emanating from orifice geometries.[9–13] An extension to the classic slug-flow model was therefore proposed by Limbourg and Nedić[12,13] to account for the contraction the flow is experiencing when being pushed through an orifice. Second, the asymptotic matching method assumes the detached vortex ring to be a member of the Fraenkel–Norbury family of isolated vortex ring.[14–17] This model, however, assumes a truncated linear vorticity distribution in the core of the vortex ring which is far from physical. In fact, it was shown experimentally that laminar vortex rings exhibit a self-similar Gaussian distribution[18,19] and, recently, Kaplanski and collaborators[20–27] proposed a model of time-dependent viscous laminar vortex rings having a nearly Gaussian distribution of vorticity.

The objective of the paper is twofold. First, the asymptotic matching procedures of Mohseni and Gharib,[6] Shusser and Gharib,[7] and Linden and Turner[8] are revisited and extended to the Kaplanski family of vortex rings in order to assess the influence of the vorticity distribution on these methods predicting the formation number. Although this was first examined by Kaplanski and Rudi,[24] careful examination shows that their arguments rely upon the experimental conclusions of Gharib *et al.*[1] and therefore does not precisely follow the methodology of Mohseni and Gharib[6] and Linden and Turner.[8] Fukumoto and Kaplanski,[25] however, applied successfully the methodology of Shusser and Gharib[7] to the Kaplanski family of vortex rings. In addition, the extended slug-flow model of Limbourg and Nedić[12,13] is applied to estimate the formation number of orifice-generated vortex rings in the exhaust-based non-dimensional time framework $t^* = U_0 t/D_0$. Second, the validity of the closure assumptions of Mohseni and Gharib[6] and Shusser and Gharib[7] is critically reassessed using existing data from the literature and new measurements. More precisely, the kinematics of the leading vortex rings generated with the orifice geometries by Limbourg and Nedić[12,13] are presented in both the exhaust-based non-dimensional time frame $t^* = U_0 t/D_0$ and the modified non-dimensional time frame $T^* = U_\star t/D_\star$. Detailed description of the experimental set-up and conditions for the presented data can be found in Limbourg and Nedić[11] as well as in Limbourg.[28]

The outline of the present study is as follows. First, the slug-flow model predicting the production of the invariants of the motion at an exhaust, nozzle or orifice, is presented in Sec. II. Second, the theoretical models of isolated vortex rings are presented in a succinct manner in Sec. III. Then, the asymptotic matching procedure originally proposed by Gharib *et al.*,[1] Mohseni and Gharib,[6] Shusser and Gharib,[7] and Linden and Turner[8] is presented, regardless of the model of family of vortex rings (Sec. IV). Finally, a critical discussion of the validity of the results is proposed in Sec. V. In particular, the closure assumptions suggested by Mohseni and Gharib[6] and Shusser and Gharib[7] are assessed and compared with experimental data in Sec. V D.

II. THE SLUG-FLOW MODEL

The asymptotic matching operation proposed by Mohseni and Gharib,[6] Shusser and Gharib,[7] and Linden and Turner[8] rests upon the prediction of the invariants of the motion by the classic slug-flow model. For parallel starting jets in which the radial velocity at the exhaust is negligible throughout the fluid ejection, the model was proved to estimate adequately the rate of production of these invariants of the motion. For an orifice starting jet, however, it was shown that the slug-flow model underestimates drastically the invariants of the motion. In particular, Limbourg and Nedić[12,13] showed that the rates of production of the circulation, the hydrodynamic impulse, and the kinetic energy are functions of the orifice-to-tube diameter ratio D_0/D_p, and the discrepancy with the classic slug-flow model is as pronounced as the orifice-to-tube diameter ratio is small. As such, an extension to the classic slug-model was proposed to account for the contraction of the flow at the exhaust of an orifice. The contraction coefficient obtained by Von Mises[29] for a two-dimensional slit in a channel is applied to the equivalent three-dimensional axisymmetric problem and the modified slug-flow model reads

$$\Gamma_\star = \frac{1}{2} U_\star L_\star(t) = \frac{1}{2} U_0 L_0(t) \times 1/C_c^2, \tag{1}$$

$$I_\star = \frac{1}{4}\pi\rho U_\star D_\star^2 L_\star(t) = \frac{1}{4}\pi\rho U_0 D_0^2 L_0(t) \times 1/C_c, \tag{2}$$

$$E_\star = \frac{1}{8}\pi\rho U_\star^2 D_\star^2 L_\star(t) = \frac{1}{8}\pi\rho U_0^2 D_0^2 L_0(t) \times 1/C_c^2, \tag{3}$$

where $L_{0\star}(t)$ is the length of the column of fluid discharged at time t, $D_{0\star}$ is the diameter of the column, and $U_{0\star}$ is the speed of the flow in the slug of fluid. With reference to Fig. 1, the subscript 0 refers to the geometrical exhaust-based quantities and the subscript $\star$ refers to the contracted quantities in the *vena contracta*, both linked by the conservation of volume and the contraction coefficient C_c.[12,13,28] For instance, the (geometrical) exhaust speed is written U_0, whereas the contracted exhaust speed in the *vena contracta* is written as $U_\star$. Similarly, the volume of the discharged column of fluid at the exhaust is written as $\forall_0 \equiv \pi/4 L_0 D_0^2$ and the volume of the respective column of fluid in the *vena contracta* is written as $\forall_\star \equiv \pi/4 L_\star D_\star^2$ although the conservation of volume imposes $\forall_\star = \forall_0$.

In addition to the dimensional invariants of the motion, a set of non-dimensional parameters can be defined based on the critical quantities of the problem and estimated with the above extended slug-flow model,

$$\alpha \equiv \frac{E}{\rho^{1/2}\Gamma^{3/2}I^{1/2}} = \sqrt{\frac{\pi}{2}}\left(\frac{L_\star(t)}{D_\star}\right)^{-1} = \sqrt{\frac{\pi}{2}}\left(\frac{L_0(t)}{D_0}\right)^{-1} \times C_c^{3/2}, \tag{4}$$

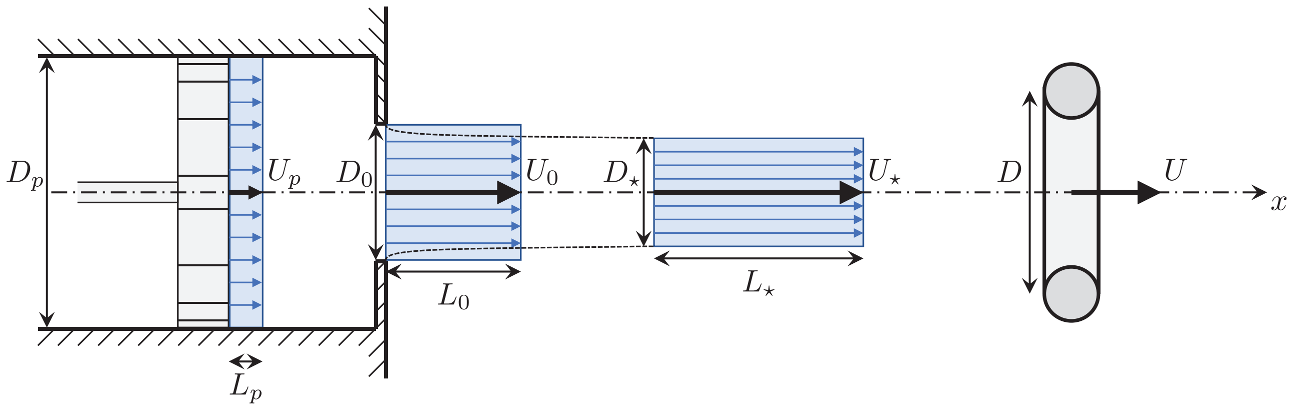

FIG. 1. Schematic of the slug-flow model made to scale for a unit impulse duration.

$$\beta \equiv \frac{\Gamma}{\rho^{-1/3} I^{1/3} U^{2/3}} = \frac{1}{(2\pi)^{1/3}} \left(\frac{L_\star(t)}{D_\star} \right)^{2/3} = \frac{1}{(2\pi)^{1/3}} \left(\frac{L_0(t)}{D_0} \right)^{2/3} \times C_c^{-1}, \tag{5}$$

$$\gamma \equiv \frac{\forall}{\rho^{-3/2} \Gamma^{-3/2} I^{3/2}} = \frac{1}{\sqrt{2\pi}} \left(\frac{L_\star(t)}{D_\star} \right) = \frac{1}{\sqrt{2\pi}} \left(\frac{L_0(t)}{D_0} \right) \times C_c^{-3/2}, \tag{6}$$

given that the β and γ quantities are defined in terms of the contracted quantities $U_\star$ and $\forall_\star$, although the conservation of volume enforces $\forall_\star = \forall_0$. Note that in the publication of Limbourg and Nedić,[12] it is erroneously mentioned that the contraction coefficient cancels out in the expressions of α, β, and γ.

III. MODELS OF ISOLATED VORTEX RINGS

The asymptotic matching operation requires a model for describing the isolated vortex rings obtained far downstream from the generating apparatus. Two models exist to describe isolated vortex rings: the original Fraenkel–Norbury model and the newer Kaplanski model.

A. The Fraenkel-Norbury family of vortex rings

The Fraenkel–Norbury model of vortex rings refers to the class of steady vortex rings having a truncated linear distribution of vorticity in the vortex core and is sometimes referred to as the standard model. The vortex rings are parametrized by the mean core radius ϵ defined as $\epsilon^2 = A/\pi R^2$, where A is the area of the vortex core and $R = D/2$ is the radius of the ring. For small values of ϵ, one is left with a thin-core vortex ring, and asymptotic expressions for the circulation, the hydrodynamic impulse, the kinetic energy, and the translational speed of the vortex ring are furnished by Fraenkel.[15] The mean core radius is bounded on the other end by a value of $\sqrt{2}$ which corresponds to Hill's spherical vortex.[30] For mean core radii between these two limits, one must rely on approximate methods; Norbury[17] computed numerically the resulting streamlines and the boundary of the vortex core for mean core radii in the range of 0.1–1.3, in 0.1 increments. In addition, the invariants of the motion were computed and presented in a table for a set of mean core radii ranging from 0.2 to 1.2, in 0.2 increments, the case $\epsilon = \sqrt{2} \approx 1.4$ corresponding to Hill's spherical vortex. One can therefore write

$$\Gamma_{FN} = \Omega R^3 \epsilon^2 \hat{\Gamma}_{FN}(\epsilon), \tag{7}$$

$$I_{FN} = \rho \Omega R^5 \epsilon^2 \hat{I}_{FN}(\epsilon), \tag{8}$$

$$E_{FN} = \rho \Omega^2 R^7 \epsilon^4 \hat{E}_{FN}(\epsilon), \tag{9}$$

$$U_{FN} = \Omega R^2 \epsilon^2 \hat{U}_{FN}(\epsilon), \tag{10}$$

where Ω is the constant of proportionality of the vorticity distribution in the radial direction and where $\hat{\Gamma}_{FN}$, $\hat{I}_{FN}$, $\hat{E}_{FN}$, and $\hat{U}_{FN}$ are either the non-dimensional quantities tabulated in Norbury[17] or the non-dimensional functions determined analytically by Fraenkel[14,15] for small cross section vortex rings or by Norbury[16,17] for thick Hill's-like vortex rings. In addition, Norbury[17] computed the streamline defining the vortex core, as well as the separating streamline delimiting the ring atmosphere. As such, the volume of the ring atmosphere was computed and tabulated as $\forall_{FN} = R^3 \hat{\forall}_{FN}$. Furthermore, Hill's spherical vortex, the thickest member of the Fraenkel–Norbury family, has analytical expressions for the invariants of the motion, translational speed, and volume from which the non-dimensional parameters can be easily computed: $\alpha_H = (10\pi)^{1/2}/35 \approx 0.160$, $\beta_H = 5/(2\pi)^{1/3} \approx 2.71$, and $\gamma_H = 5/3\,(10/\pi)^{1/2} \approx 2.97$.[30]

B. The Kaplanski family of vortex rings

Kaplanski and collaborators[20–27] proposed a model of time-dependent viscous laminar vortex rings having a nearly Gaussian distribution of vorticity,

$$\omega = \omega_0 \exp\left[\frac{1}{2} (\sigma^2 + \eta^2 + \tau^2) \right] \mathrm{I}_1(\sigma\tau), \tag{11}$$

with

$$\sigma = \frac{r}{\ell} \quad \nu = \frac{x - x_0(t)}{\ell} \quad \tau = \frac{R}{\ell} \quad \text{and} \quad \ell = \sqrt{2\nu t}$$

and

$$\omega_0 = \frac{2 I_0}{\rho R (4\pi\nu t)^{3/2}},$$

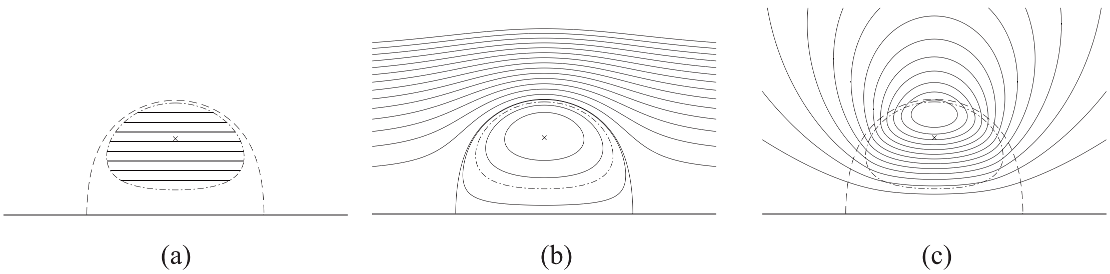

FIG. 2. Fraenkel–Norbury family of vortex rings. (a) Vorticity distribution, (b) streamlines in a frame of reference following the ring, and (c) streamlines in a fixed frame of reference—Streamlines of a vortex ring of mean core radius $\epsilon = 0.8$ for equidistant values of ψ — - —. Extent of the vortex core – – – –. Separating streamline, i.e., $\psi = 0$. The cross corresponds to the center of vorticity.

where ℓ is the viscous core radius, I_0 is the conserved hydrodynamic impulse, and I_1 is the modified Bessel function of the first kind. The parameter τ is essentially the inverse of the non-dimensional mean core radius ϵ defined for the Fraenkel–Norbury family, and one can write for convenience that $\epsilon = 1/\tau$.

The vorticity distribution of Eq. (11) is a solution to the Stokes equations and satisfies the condition of invariance of the hydrodynamic impulse. In fact, the invariants of the motion are

$$\Gamma_K = \frac{I_0}{\rho R^2}\hat{\Gamma}_K(\epsilon), \tag{12}$$

$$I_K = I_0 \hat{I}_K(\epsilon), \tag{13}$$

$$E_K = \frac{I_0^2}{\rho R^3}\hat{E}_K(\epsilon), \tag{14}$$

$$U_K = \frac{I_0}{\rho R^3}\hat{U}_K(\epsilon), \tag{15}$$

where $\hat{\Gamma}_K$, $\hat{I}_K = 1$, $\hat{E}_K$, and $\hat{U}_K$ are functions of the parameter τ only, or equivalently $\epsilon = 1/\tau$. Similarly to Hill's spherical vortex for the Fraenkel–Norbury family, it is possible to define a limiting member of the Kaplanski family. This vortex ring, obtained for $\epsilon \to \infty$, or equivalently for $\tau \to 0$, corresponds to the decaying asymptotic solution of Phillips.[31] The non-dimensional numbers of this limiting vortex are found to be 0.118, 3.32, and 3.49 for the α, β, and γ quantities, respectively.

Unlike the Fraenkel–Norbury family, the Kaplanski vortex rings have a vorticity distribution close to what is observed experimentally. More precisely, in the initial stage, the vorticity distribution has a pronounced toroidal structure with a Gaussian vorticity distribution which agrees well with measurements.[25] Note that although this model stems from the assumption of low Reynolds numbers, Gaussian vorticity distribution in the core of vortex rings has also been observed at moderate Reynolds numbers.[18] For more information on the model, the reader is referred to the original papers of Kaplanski or the review textbook by Danaila *et al.*[32]

C. Comparison of the models

Figure 2 presents the streamlines of a Fraenkel–Norbury vortex ring having a mean core radius of $\epsilon = 0.8$, while Fig. 3 presents the streamlines of a Kaplanski vortex ring having a mean core radius of $\epsilon = 1/\tau = 0.5$. Interestingly, although the linear vorticity distribution of the Frankel–Norbury ring is far from physical [Fig. 2(a)], the resulting streamlines resemble to experimental flow patterns. Besides, the Kaplanski vortex ring of Fig. 3(a) presents a Gaussian-like distribution of vorticity in the core and therefore extends to the centerline; the vorticity distribution is therefore expected to be skewed as the vortex ring grows in size. The streamlines of Figs. 3(b) and 3(c) are found to be very close to the Fraenkel–Norbury ring of Figs. 2(b) and 2(c) although the mean core radius is much smaller. Overall, as highlighted by Danaila and Hélie,[33] the Fraenkel–Norbury family and the Kaplanski family display very similar streamline patterns. Critically, the dividing streamlines, i.e., $\psi = 0$, defining the bounds of the ring atmosphere are found to be almost identical, which thus implies that the volume of the ring can be estimated by fitting the experimental data with either the Fraenkel–Norbury or Kaplanski family of vortex

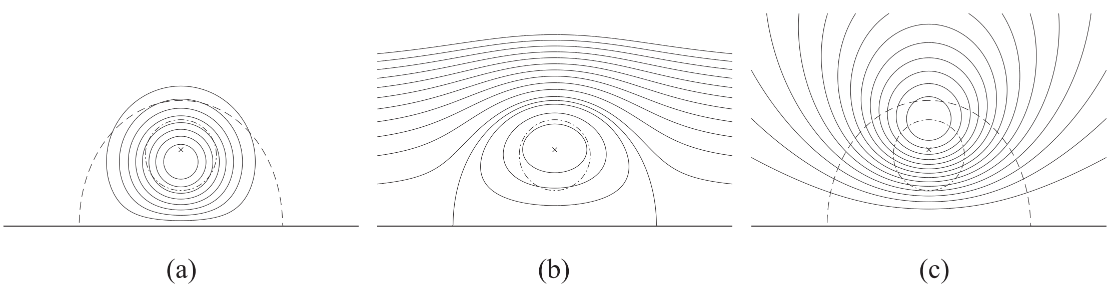

FIG. 3. Kaplanski family of vortex rings. (a) Vorticity distribution, (b) streamlines in a frame of reference following the ring, and (c) streamlines in a fixed frame of reference ——. Streamlines of a vortex ring of mean core radius $\epsilon = 0.5$ for equidistant values of ψ — - —. Extent of the vortex core – – – –. Separating streamline, i.e., $\psi = 0$. The cross corresponds to the center of vorticity.

rings with the corresponding value of ϵ. Nonetheless, Danaila and Hélie[33] noted a 40% underestimation in the modeled volume compared to the measured volume, hence highlighting an apparent bias in the estimate. Finally, the Kaplanski family was shown to have a vortex ring signature, as defined by Moffatt,[34] closer to the numerical results of Danaila and Hélie.[33]

In the remainder of the manuscript, the quantities are written with a subscript i regardless of the family of vortex ring. For instance, the dimensional translational speed of the ring will be written as U_i and the shape function will be written as $\hat{U}_i$.

IV. ANALYTICAL PREDICTION OF THE FORMATION NUMBER

A. The asymptotic matching procedure originally proposed by Gharib *et al.*[1]

The theoretical model of Gharib *et al.*[1] is semi-empirical as it requires the measurements of the circulation, the hydrodynamic impulse, and the kinetic energy of the isolated vortex ring. Mohseni and Gharib,[6] however, offered a methodology to estimate analytically the non-dimensional quantities of the detached isolated vortex rings. The predicted invariants of the motion delivered by the nozzle generator and derived from the slug-flow model [Eqs. (1)–(3)] are equated to their equivalent analytical expressions derived for the isolated of vortex rings [Eqs. (7)–(9), or equivalently Eqs. (12)–(14)]. This matching procedure of the circulation, the hydrodynamic impulse, and the kinetic energy ultimately results in the curve,

$$\frac{L_\star}{D_\star} = \sqrt{\frac{\pi}{2}}\frac{\rho^{1/2}\Gamma_i^{3/2}I_i^{1/2}}{E_i} = \sqrt{\frac{\pi}{2}}\frac{\hat{\Gamma}_i^{3/2}\hat{I}_i^{1/2}}{\hat{E}_i} = \sqrt{\frac{\pi}{2}}\alpha_i(\epsilon)^{-1}, \quad (16)$$

where α_i is the α quantity of the isolated vortex ring computed from the non-dimensional invariants of the motion following the model of Fraenkel–Norbury or Kaplanski. In short, Eq. (16) provides a correspondence between the generating conditions, i.e., the maximum stroke ratio $L_\star/D_\star$, and the resulting isolated vortex ring parametrized by the mean core radius ϵ.

Interestingly, there exists a maximum stroke ratio above which a single vortex ring cannot be formed. For the Fraenkel–Norbury family of vortex ring, this limiting stroke ratio is 7.83 and would form Hill's spherical vortex, whereas for the Kaplanski family, a limiting stroke ratio of 10.63 is found[24] (Fig. 4).

B. The closure assumption of Mohseni and Gharib[6]

In order to find a single limiting stroke ratio, i.e., a single formation number, and close the system of equations, one additional assumption is required. In addition to the circulation, the hydrodynamic impulse and the kinetic energy, Mohseni and Gharib[6] proposed to match the translational speed of the isolated vortex ring to a hypothetical ring speed during formation. Mohseni and Gharib[6] therefore estimated the speed of the ring at the exhaust by $\partial E/\partial I$, as originally proposed by Roberts[35] following a Hamiltonian formalism, and derived it from the slug-flow model; the translational speed of the ring in the vicinity of the exhaust was therefore estimated to be $\partial E_\star/\partial I_\star = U_\star/2$. Finally, by matching this estimated ring speed to the translational speed of the isolated vortex ring, one is left with

$$\frac{L_\star}{D_\star} = \sqrt{\frac{\pi}{2}}\frac{\Gamma_i^{3/2}}{\rho^{-1/2}I_i^{1/2}U_i} = \sqrt{\frac{\pi}{2}}\frac{\hat{\Gamma}_i^{3/2}}{\hat{I}_i^{1/2}\hat{U}_i} = \sqrt{\frac{\pi}{2}}\beta_i(\epsilon)^{3/2}, \quad (17)$$

and the compatibility equation,

$$E_i = U_i I_i \quad \text{or} \quad \hat{E}_i = \hat{U}_i\hat{I}_i. \quad (18)$$

The limiting stroke ratio, or formation number, is then found to be the intersection between Eqs. (16) and (17), as shown in Fig. 1 of Mohseni and Gharib[6] or in Fig. 4(a). It also corresponds to the mean core radius at which Eq. (18) is satisfied. A limiting stroke ratio of 2.96 is found for a mean core radius of 0.317, and the resulting non-dimensional numbers are $\alpha = 0.442$, $\beta = 1.76$, and $\gamma = 1.24$ (see summary Table I). Note that Mohseni and Gharib[6] argued that experimentally, the speed of the ring close to exhaust is larger than $0.5U_\star$. Repeating the calculations with a estimated ring speed of $0.6U_\star$, which was claimed to be closer to the speed observed experimentally, a limiting stroke ratio of 4.51 was obtained.

If one applies the same procedure to the Kaplanski family, the formation number is found to be 3.50, which is larger than the original value of 2.96 obtained with the Frankel–Norbury family but closer to the values found experimentally for parallel starting jets. Additionally,

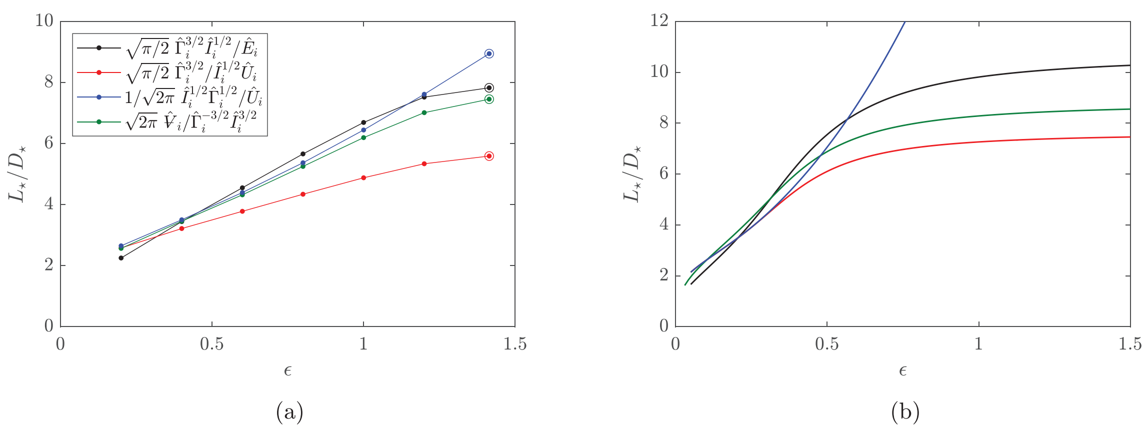

FIG. 4. Theoretical prediction of the limiting stroke-to-diameter ratio, or formation number, using (a) the Fraenkel–Norbury family and (b) the Kaplanski family.

TABLE I. Analytical predictions of the limiting stroke ratio, i.e., the formation number.

Family of vortex rings	Method of	$L_\star/D_\star$	ϵ	α	β	γ[a]
Fraenkel–Norbury	Mohseni and Gharib[6]	2.96	0.318	0.442	1.76	1.24
	Shusser and Gharib[7],[b]	3.75	0.455	0.339	1.93	1.48
	Linden and Turner[8]	3.51	0.411	0.359	1.88	1.40
Kaplanski	Mohseni and Gharib[6],[a]	3.50	0.206	0.358	1.98	1.50
	Shusser and Gharib[7],[c]	3.50	0.206	0.358	1.98	1.50
	Linden and Turner[8],[a]	5.05	0.314	0.248	2.36	2.02

[a]New result.
[b]Result revisited making use of the numerical results of Norbury.[17]
[c]Result originally reported by Fukumoto and Kaplanski.[25]

the predicted non-dimensional quantities of the detached vortex ring are found to be $\alpha = 0.358$, $\beta = 1.98$, and $\gamma = 1.50$. Note that the value of 0.358 for the α quantity is much lower than the value 0.442 obtained using the Frankel–Norbury family, but, again, closer to the value of 0.33 repeatedly found experimentally and numerically for both orifice and nozzle geometries.

C. The closure assumption of Shusser and Gharib[7,36]

Similarly to Mohseni and Gharib,[6] Shusser and Gharib[7,36] hypothesized the detachment of the primary ring from the feeding jet to occur when the speed of the steady isolated vortex ring equates the velocity of the jet. More precisely, Shusser and Gharib[7,36] argued that when detachment occurs, the vortex ring is in the vicinity of the exhaust which forces the diameter and speed of the feeding jet of fluid to be prescribed by the kinematics of the ring. Applying the conservation of mass, along with the slug-flow model, the ring speed at the exhaust is estimated by $U_\star D_\star^2/4R^2$, where R is the radius of the ring. Matching the ring circulation, hydrodynamic impulse, and speed to the exhaust quantities, as proposed by Mohseni and Gharib,[6] one is left with

$$\frac{L_\star}{D_\star} = \frac{1}{\sqrt{2\pi}}\frac{1}{R^2\rho^{1/2}}\frac{I_i^{1/2}\Gamma_i^{1/2}}{U_i} = \frac{1}{\sqrt{2\pi}}\frac{\hat{I}_i^{1/2}\hat{\Gamma}_i^{1/2}}{\hat{U}_i}, \tag{19}$$

and the compatibility equations reads

$$E_i = \pi\rho R^2\Gamma_i U_i \quad \text{or} \quad \hat{E}_i = \pi\hat{\Gamma}_i\hat{U}_i. \tag{20}$$

The derivations of Shusser and Gharib[7] using the Fraenkel–Norbury family are revisited here to make use of the numerical results of Norbury[17] instead of the asymptotic expressions of Fraenkel[15] which are only valid in the thin-ring approximation, i.e., $\epsilon \ll 1$. In combination with Eq. (16), Eq. (19) furnishes a formation number of 3.75 and non-dimensional numbers of $\alpha = 0.339$, $\beta = 1.93$, and $\gamma = 1.48$ (Table I). Note that these values are much larger than the ones originally found by Shusser and Gharib[7] for a thin-core vortex ring.

Interestingly, if one repeats the calculations of Shusser and Gharib[7] with the Kaplanski family of vortex rings, a limiting stroke ratio of 3.50 is obtained which is identical to the prediction obtained with the methodology of Mohseni and Gharib.[6] Naturally, the non-dimensional quantities are also found to be the same (Table I). This was first shown by Fukumoto and Kaplanski.[25] Upon careful examinations of Eqs. (17) and (19), it is found that these two curves approach each other exponentially as $\epsilon = 1/\tau$ decreases. Therefore, at small mean core radii of $\epsilon < 0.3$, the intersections of the latter two curves with the curve given by Eq. (16) result in the same prediction of the formation number and thus the same estimation of the non-dimensional quantities of the detached vortex ring (Table I).

D. The closure assumption of Linden and Turner[8]

Rather than the ring speed, Linden and Turner[8] argued that the limiting constraint is the volume of the ring atmosphere. As such, after estimating the volume of fluid discharged by the generator to that of a cylinder, i.e., $\forall_\star = \forall_0 = \pi/4D_0L_0(t)$, and matching the circulation, the hydrodynamic impulse and the volume, the following relations are obtained:

$$\frac{L_\star}{D_\star} = \sqrt{2\pi}\frac{\forall_i}{\rho^{-3/2}\Gamma_i^{-3/2}I_i^{3/2}} = \sqrt{2\pi}\frac{\hat{\forall}_i}{\hat{\Gamma}_i^{-3/2}\hat{I}_i^{3/2}} = \sqrt{2\pi}\gamma_i(\epsilon) \tag{21}$$

and

$$2E_i\,\forall_i = I_i^2 \quad \text{or} \quad 2\hat{E}_i\hat{\forall}_i = \hat{I}_i^2 \tag{22}$$

When using the Fraenkel–Norbury family of vortex rings, the intersection of Eq. (16) with Eq. (21) gives a limiting stroke ratio of 3.51, and the non-dimension quantities are found to be $\alpha = 0.411$, $\beta = 1.88$, and $\gamma = 1.40$ (Table I).

The methodology of Linden and Turner[8] can be extended to the Kaplanski family of vortex ring provided the volume of the isolated ring. The extent of the ring atmosphere is defined by the separating streamline of the propagating isolated vortex ring. Given the stream function of the Kaplanski ring and the translational speed of the ring, it is possible to find the dividing streamline defined by $\psi = 0$. The volume of revolution of the ring atmosphere is then computed by the disk method. From this, the limiting stroke ratio is found to be 5.05, and the non-dimensional numbers are $\alpha = 0.248$, $\beta = 2.36$, and $\gamma = 2.02$ (Table I). These values differ significantly from the values determined experimentally. In particular, the estimated formation number is larger than the value of about 4 found by Gharib *et al.*[1] Moreover, the non-dimensional quantity α is found to be relatively small compared to the value of 0.33 found experimentally by Gharib, Rambod, and Shariff,[1] Dabiri and Gharib,[37] and Limbourg and

Nedić,[11] yet not so far from the values found by Krueger *et al.*,[38] Rosenfeld *et al.*,[39] or Krieg and Mohseni.[10]

Note that, although Kaplanski and Rudi[24] maintained that they used the methodology of Mohseni and Gharib[6] and Linden and Turner[8] to predict the value of the formation number, their derivations make use of the formation number of $3.6 - 4.5$ found experimentally by Gharib *et al.*[1] to compute the non-dimensional quantities α and β that the Kaplanski vortex ring would have when separation occurs.

E. The influence of the contraction coefficient

In the above derivations, the extended slug-flow model was used to derive the relationships linking the generator conditions, i.e., the stroke ratio $L_\star/D_\star$, to the characteristic quantities of the isolated vortex rings parametrized by the mean core radius ϵ. The expressions are therefore equally valid for both nozzle geometries and orifice geometries. If instead of the contracted quantities, the stroke ratio is defined in terms of the geometrical exhaust quantities, subscript 0, or equivalently, if instead of the corrected non-dimensional time $T^* = U_\star t/D_\star$ defined by Limbourg and Nedić,[12] one uses the exhaust-based non-dimensional time $t^* = U_0 t/D_0$, i.e., the so-called *formation time*, then the predicted formation number is multiplied by $C_c^{3/2}$, where C_c is the contraction coefficient depending on the orifice-to-tube diameter ratio D_0/D_p. For instance, for an orifice geometry having an orifice-to-tube diameter ratio of 0.5, and following the methodology of Mohseni and Gharib[6] (or Shusser and Gharib[7]) in combination with the Kaplanski model, one would obtain a limiting formation number of $3.50 \times C_c^{3/2} = 3.50 \times 0.644^{3/2} = 1.81$ in the exhaust-based non-dimensional time frame, which is consistent with the measurements of Limbourg and Nedić.[11] Again, the contraction coefficient C_c was obtained using the results of Von Mises[29] who found the free streamline of the flow exiting a two-dimensional 90° slit in a channel. For nozzle geometries, however, the contraction coefficient is theoretically identically equal to 1, and the classic slug-flow model is returned, as well as the usual definition of the non-dimensional time. Figure 5 presents the dependency of the predicted limiting stroke ratio as a function of the orifice-to-tube diameter ratio when the former is defined in terms of the geometrical exhaust parameters, denoted by a subscript 0. Note that using the Kaplanski model in combination with the methodologies of Mohseni and Gharib[6] (red solid line in Fig. 5) or Shusser and Gharib[7] (blue solid line in Fig. 5), as well as using the Fraenkel–Norbury family with the closure assumption of Linden and Turner[8] (green dashed line in Fig. 5), all produce the same result. This is in line with the values reported in Table I.

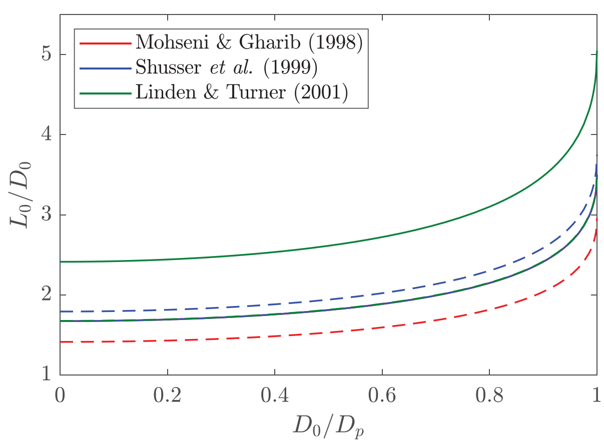

FIG. 5. Evolution of the limiting exhaust-based stroke ratio as a function of the orifice-to-tube diameter ratio using – – – – the Fraenkel-Norbury family and —— the Kaplanski family.

Interestingly, using the modified slug-flow model instead of the classic model does not modify the estimated non-dimensional quantities of the isolated vortex rings, provided that the contracted quantities are used in the definition of β (and γ). Because the modified stroke ratio $L_\star/D_\star$ is related to the exhaust-based stroke ratio L_0/D_0 by a simple factor of C_c, the limiting mean core radius ϵ remains the same, and the non-dimensional quantities presented in Table I are the same whether one uses a nozzle geometry or an orifice geometry. This is also in line with the measurements of Limbourg and Nedić[11] who reported non-dimensional quantities of isolated orifice-generated vortex rings close to the values found by Gharib *et al.*[1] for nozzle-generated vortex rings.

Finally, it was shown by Krieg and Mohseni[9] and Limbourg and Nedić[12,13] that the classic slug-flow slightly underestimates the rate of production of the invariants of the motion discharged by a nozzle geometries. As such, Limbourg and Nedić[12] proposed to use the modified slug-flow model with a contraction coefficient of 0.90 which was shown to better estimate the evolution of the non-dimensional quantities.[12] Therefore, although theoretically there should be no contraction of the flow in the case of a parallel starting jet, a contraction coefficient of 0.90 is applied here. For the Fraenkel-Norbury family, the estimated exhaust-based formation numbers are $2.96 \times 0.90^{3/2} = 2.52$ when using the closure of Mohseni and Gharib,[6] $3.75 \times 0.90^{3/2} = 3.20$ for the closure of Shusser and Gharib[7] and $3.51 \times 0.90^{3/2} = 2.99$ for the closure of Linden and Turner.[8] Similarly, for the Kaplanski family of vortex rings, the values are $3.50 \times 0.90^{3/2} = 2.99$, $3.50 \times 0.90^{3/2} = 2.99$, and $5.05 \times 0.90^{3/2} = 4.31$ for the closure assumptions of Mohseni and Gharib,[6] Shusser and Gharib,[7] and Linden and Turner,[8] respectively.

V. ASSESSMENT OF THE ASSUMPTIONS AND DISCUSSION

As outlined in Sec. IV, the asymptotic matching methods rest upon the validity of the extended slug-flow model, the theoretical models of isolated vortex rings, and the closure assumptions, which involve either the exhaust ring speed or the volume of the ring. In the present section, each assumption is reviewed in detail.

A. Validity of the extended slug-flow model

The validity of the extended slug-flow model has previously been assessed by Limbourg and Nedić[12,13] who showed that for orifice starting jets and converging nozzles, the extended slug-flow model accurately estimates the production of the invariants of the motion (circulation, hydrodynamic impulse, and kinetic energy). An aspect that was overlooked in the previous work, but plays a pivotal role in validating the closure assumptions, is the scaling of the kinematics of the leading vortex ring in the vicinity of the exhaust.

Figure 6 presents the time history of the axial position of the leading vortex ring for orifice-to-tube diameter ratios ranging from 0.375 to 1.000, in 0.125 increments, and obtained for a fixed exhaust speed

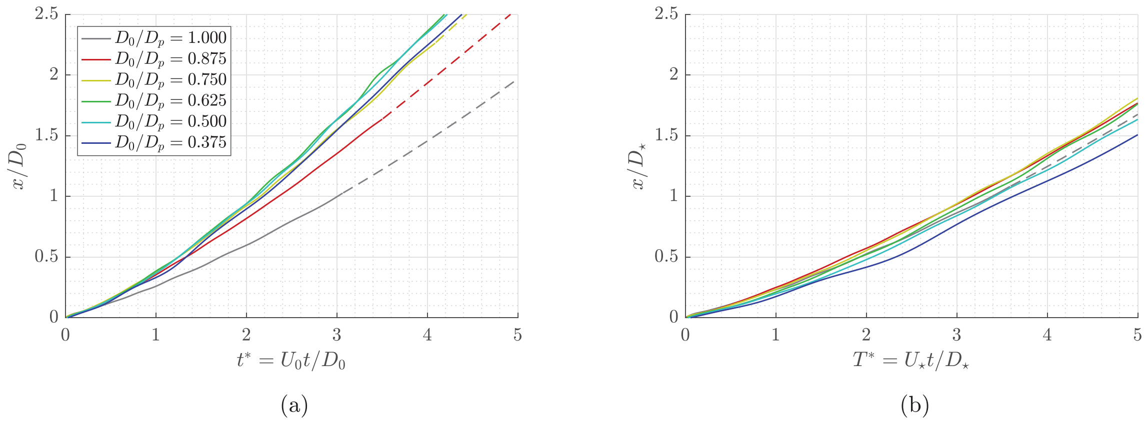

FIG. 6. Time history of the normalized axial position of the core centroid in (a) the geometrical quantities framework and (b) the contracted quantities framework.

of $U_0 = 100\,\text{mm s}^{-1}$ and a tube diameter of $D_p = 101.6\,\text{mm}$. With the present normalization, the slope of the curve corresponds to the normalized translational speed of the ring. The asymptotic translational speed is in fact measured at non-dimensional times between $t^* = 3$ and 5 in Fig. 6(a). For more information on the experimental procedure, the reader is referred to Limbourg and Nedić[11–13] or Limbourg.[28]

With a translational speed of approximately $(0.50 \pm 0.02)U_0$, the nozzle-generated vortex ring propagates slower than the orifice-generated rings, all of which have a propagation speed of about $(0.70 \pm 0.02)U_0$. For an orifice-to-tube diameter ratio of $D_0/D_p = 0.875$, however, the ring has a translational speed in between with a value of about $(0.61 \pm 0.02)U_0$. However, when scaled by the contracted diameter $D_\star$ and when shown as a function of the corrected non-dimensional time $T^* = U_\star t/D_\star$, all curves collapse and the speed of the leading vortex ring is found to be $(0.47 \pm 0.02)U_\star$ for all cases, nozzle and orifices [Fig. 6(b)]. Note that a contraction coefficient of 0.90 was used on the nozzle case to model the contraction effect of the vortex ring on the discharged slug of fluid.[12,13]

The influence of the orifice-to-tube diameter ratio on the kinematics of the leading vortex ring is also visible in Fig. 7(a), which presents the evolution of the ring diameter in the exhaust-based non-dimensional time frame. The nozzle-generated vortex ring is observed to grow larger in diameter compared to the orifice-generated vortex ring. However, similarly to the axial position in Fig. 6(b), all curves collapse when the ring diameter is normalized by the contracted diameter $D_\star$ and presented as a function of the modified non-dimensional time $T^* = U_\star t/D_\star$ [Fig. 7(b)]. Again, a contraction coefficient of $C_c = 0.90$ was used for the nozzle case.

To summarize, the extended slug-flow model introduced in Sec. II not only accurately predicts the production of the invariants of the motion, as shown by Limbourg and Nedić,[12,13] but also appropriately scales the kinematics of the leading vortex ring; this was claimed without justification by Limbourg and Nedić,[12] but now rigorously presented. Finally, note that measurements were also taken for a fixed diameter-based Reynolds number, and identical results were found.[28]

B. Validity of the models of isolated vortex rings

A comparison between the two families of vortex rings is also required. In Sec. IV, the Kaplanski family of isolated vortex ring was used to estimate the limiting stroke ratio, or formation number, using the closure assumptions of Mohseni and Gharib,[6] Shusser and

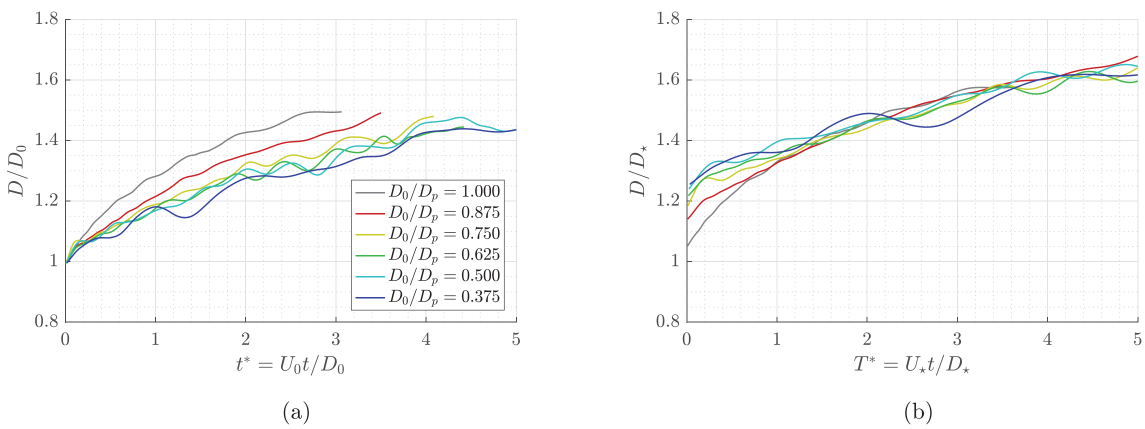

FIG. 7. Time history of the normalized diameter of the vortex ring in (a) the geometrical quantities framework and (b) the contracted quantities framework.

Gharib,[7] and Linden and Turner.[8] Kaplanski vortex rings possess a nearly Gaussian distribution of vorticity inside the vortex core, which is more physical than the linearly distributed vorticity of the Fraenkel–Norbury vortex rings. Although the streamlines of the latter model look physical and somewhat validate the use of this model for estimating the quantities of isolated vortex rings, the Kaplanski family offers the possibility of modeling the vortex core accurately, thus allowing a comparison to the experimental data of Weigand and Gharib[40] or Cater *et al.*,[19] for instance. Nevertheless, this does not necessarily translate into a better estimation of the volume of the ring atmosphere, as shown by Danaila and Hélie.[33] Moreover, the Kaplanski model allows one to incorporate a time-dependent viscous effect and study the entire life-cycle of the vortex ring. For this reason, the Kaplanski family seems to be the most appropriate model for studying the propagation of isolated vortex ring, as opposed to the Fraenkel–Norbury family.

C. Compatibility of the extended slug-flow model with the closure assumptions

Now, given the modified slug-flow model to predict the production of the invariants of the motion for any exhaust (nozzle or orifice) and given the Kaplanski model to describe self-propagating vortex rings, one is left with the three closure assumptions of Mohseni and Gharib,[6] Shusser and Gharib,[7] and Linden and Turner.[8] Whereas Linden and Turner[8] assume the volume of the ring to be the limiting factor on the ring growth, both Mohseni and Gharib[6] and Shusser and Gharib[7] assume the ring speed to be the determining constraint, and the separation, or pinch-off, of the vortex ring occurs when the ring translational speed reaches a limiting value.

Mohseni and Gharib[6] estimated the ring speed at the exhaust by $\partial E/\partial I$ and assumed the ring energy and ring impulse during formation to be given by the slug-flow model, hence finding $\partial E_\star/\partial I_\star = U_\star/2$. Mohseni and Gharib[6] made the observation that in practice the translational velocity of the vortex ring is larger than $0.5U_\star$ and they proposed instead to use a value of $0.6U_\star$ which ultimately gives a larger predicted formation number of 4.51 with the Fraenkel–Norbury family and 6.78 with the Kaplanski family. For a straight nozzle, Limbourg and Nedić[12,13] observed that using a contraction coefficient of 0.90 leads to an improved estimation of the invariants of the motion. As such, the estimated ring speed at the exhaust would be $U_\star/2 = U_0/2C_c = 0.56U_0$, in line with the argument of Mohseni and Gharib.[6] Nevertheless, as shown in Fig. 5 and discussed in Sec. IV E, this would lead to a reduced predicted formation number if one uses the modified slug-flow model, or equivalently the corrected non-dimensional time $T^* = U_\star t/D_\star$. Moreover, as shown in Fig. 6(b), the measured translational speed of nozzle-generated vortex rings seems to be closer to $0.45U_\star = 0.50U_0$, which is consistent with the orifice geometry measurements, and provided that a contraction coefficient of 0.90 is used. As such, the latter argument of Mohseni and Gharib[6] does not seem validated by experimental evidence.

On the other hand, Shusser and Gharib[7] argued that the leading vortex ring detaches when the speed of the ring equates the speed of the jet estimated by an assumption of conservation of mass; it is stated that the entire flow rate discharged by the generator is accumulated by the vortex ring when it is forming. Because the extended slug-flow model assumes a conservation of mass (or volume) from the exhaust plane to the *vena contracta*, the closure equation proposed by Shusser and Gharib[7] is identical whether one uses the contracted tube of fluid or the geometrical slug of fluid at the exhaust.

To summarize, the closure assumptions of Mohseni and Gharib[6] and Shusser and Gharib[7] are applicable to the extended slug-flow model, as well as volume argument of Linden and Turner[8] and the asymptotic matching procedure is equally valid for orifice-generated vortex rings.

D. Validity of the translational speed closure assumption of Mohseni and Gharib[6] and Shusser and Gharib[7]

Given the data found in the literature and the measurements presented in Figs. 6 and 7, it is possible to assess the validity of the closure arguments of Mohseni and Gharib[6] and Shusser and Gharib.[7]

Figure 8 presents the normalized axial position of the leading vortex ring as a function of the exhaust-based non-dimensional time $t^* = U_0t/D_0$ obtained by several studies; the slope of the curves thus corresponds to the speed of the ring normalized by the exhaust speed U_0. For a straight nozzle, the experimental results of Didden,[41]

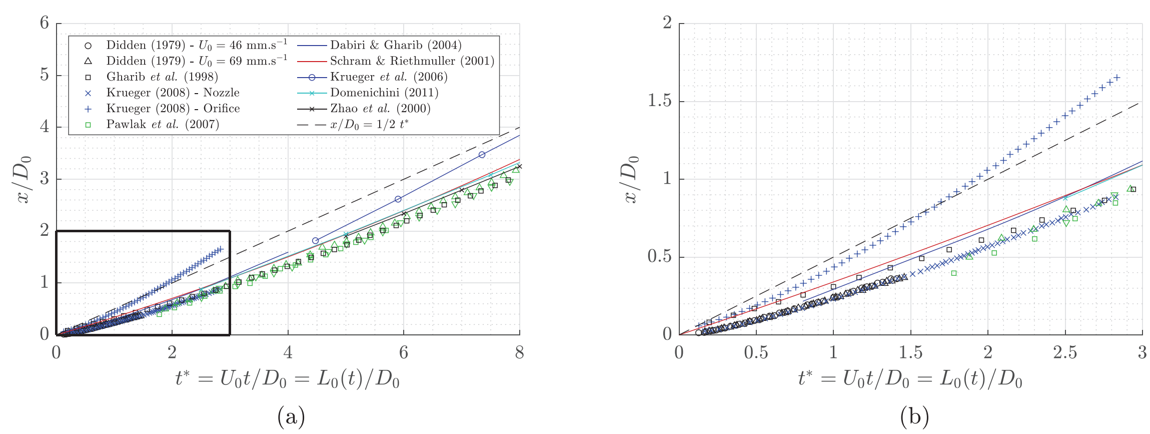

FIG. 8. Axial position of the leading vortex ring for several cases found in the literature.

Gharib *et al.*,[1] Schram and Riethmuller,[42] Dabiri and Gharib[43] and Krueger *et al.*,[44] are presented, as well as the numerical result of Zhao *et al.*,[45] Krueger,[46] and Domenichini.[47] Note that the experiment of Didden[41] was reproduced numerically by Nitsche and Krasny,[48] James and Madnia,[49] Heeg and Riley,[50] and Hettel *et al.*,[51] and identical results were found. In addition, the particle image velocimetry measurements of Pawlak *et al.*[52] obtained for a converging nozzle of 13° conical angle are reported. Pawlak *et al.*[52] also performed simulations and found very close agreement with the experiments. Finally, the numerical results obtained by Krueger[46] for an orifice-generated vortex ring are presented. It is worth noting that Krueger's[46,53] simulations confirm that orifice-generated vortex rings propagate faster than nozzle-generated vortex rings, provided that one places themselves in the exhaust-based non-dimensional time frame $t^* = U_0 t/D_0$.

Interestingly, all straight nozzle curves, except perhaps the one of Krueger *et al.*,[44] collapse together close to the measurements shown in Fig. 6. In particular, the nozzle case of Gharib *et al.*,[1] for which a formation number of about 4.0 was found, presents a maximum asymptotic translational speed of $0.45U_0$, and thus an even lower speed at $t^* = 4.0 - 4.5$, the instant at which pinch-off occurs. Similarly, Pawlak *et al.*[52] reported a measured formation number of $3.5 - 5.0$ although the vortex ring speed was observed to be about $0.45U_0$. Furthermore, Schram and Riethmuller[42] reported a ring speed linearly increasing from $0.36U_0$ at $t^* = 1.4$ to $0.59U_0$ at $t^* = 11.1$; the ring speed is therefore estimated to be $0.44U_0$ at the instant of expected pinch-off of $t^* = 4.0$. These results are in line with the measurements shown in Fig. 6 where orifice and nozzle-generated vortex rings were shown to possess a translational velocity between $0.45U_\star$ and $0.49U_\star$. As a consequence, the assumption of Mohseni and Gharib[6] of having a translational ring velocity larger than $0.50U_\star$ in the vicinity of the exhaust does not seem to be valid. Critically, the methodology of Mohseni and Gharib[6] is strongly dependent to the ring speed at the exhaust. For instance, a reduced exhaust ring speed of $0.40U_\star$ instead of the theoretical speed of $\partial E_\star/\partial I_\star = 0.50U_\star$ would lead to an estimated formation number of 1.82 when using the Kaplanski family, far from the well-accepted value of 4.

A potential flaw of the closure assumption of Mohseni and Gharib[6] could be the estimation of the translational speed of the ring during formation and the use of the Hamiltonian formalism to express the ring speed in terms of the variations of kinetic energy and hydrodynamic impulse. In particular, the formula of Roberts[35] is obtained for freely propagating isolated vortex ring. However, during formation, the primary vortex ring strongly interacts with the feeding jet, and the presence of the generator imposes a wall boundary condition. Estimating the ring kinetic energy and the ring hydrodynamic impulse by the slug-flow model, however, seems to be a reasonable assumption as all the fluid discharged is entrained in the rolling motion of the structure at the very first instants. In order to assess the validity of the expression provided by Roberts,[35] the rates of change of the hydrodynamic impulse and the kinetic energy are measured to form the ratio $\partial E/\partial I$. It is then compared to the translational speed presented in Fig. 6 which was measured by tracking the center of vorticity; Fig. 9 presents the measured ring speed and the estimated ring speed before and after the correction for the contraction of the flow is applied. First, as previously shown in Fig. 6(b), the modified slug-flow model, and more precisely the normalization by the contracted exhaust speed $U_\star$, allows one to bring all ring speeds close to the theoretical value given by the slug-flow model of $0.50U_\star$ (hollow symbols in Fig. 9). Now, comparing the measured ring speed with the ratio $\partial E/\partial I$, a close agreement between the two is found, with values ranging between $0.42U_\star$ and $0.49U_\star$. Note that the estimated ring speed obtained by means of the Hamiltonian formalism is observed to underestimate slightly the ring speed for $D_0/D_p = 0.375$ and $D_0/D_p = 0.875$. Nonetheless, overall, the ratio $\partial E/\partial I$ adequately estimates the translational speed of the vortex ring, and this even in the close vicinity of the exhaust. Therefore, rather than the Hamiltonian formalism and the equation of Roberts,[35] this is the closure assumption of Mohseni and Gharib[6] itself that is put into question. Identical results were obtained at a fixed diameter-based Reynolds number which is not a surprise as the exhaust speed was shown to have a minimal impact on the latter ratio [see Fig. 5.4(d) of Limbourg[28]].

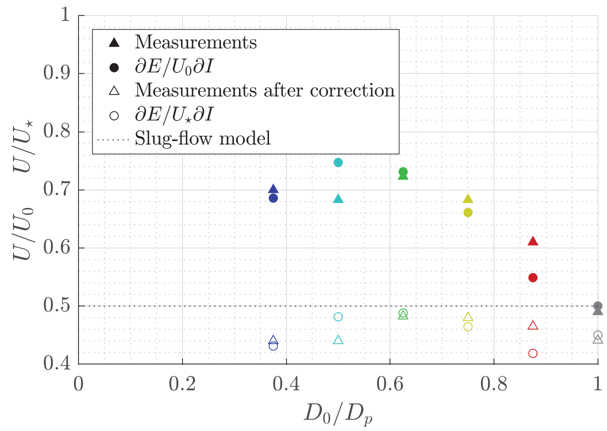

FIG. 9. Measured and estimated translational speed of the leading vortex ring as a function of the orifice-to-tube diameter ratio.

Shusser and Gharib[7,36] made the assumption that the vortex ring completes its formation process when it reaches the jet flow velocity; at early times, the presence of the vortex in the close vicinity of the exhaust prevents one from using the exhaust speed $U_\star$ prescribed by the slug-flow model. However, the mass flow rate discharged by the generator is fully accumulated by the forming vortex ring until the speed of the self-propagating isolated vortex ring is reached. In other words, the vortex ring separates from the feeding jet once the condition $UD^2 = U_\star D_\star^2$ is met. Figure 10(a) presents this ratio $UD^2/U_\star D_\star^2$ as a function of the modified non-dimensional time $T^* = U_\star t/D_\star$ for the experimental cases shown in Figs. 6 and 7. The ratio increases and reaches a value of 1 at a corrected non-dimensional time between 3.5 and 4.5, consistent with the formation number found experimentally by Limbourg and Nedić[11] and Krieg and Mohseni[10] for orifice geometries in the corrected non-dimensional time and with the formation number found by Gharib *et al.*,[1] and others for nozzle-generated vortex rings. Additionally, Fig. 10(b) presents this same ratio for nozzle cases inferred from the literature. Again, the ratio reaches a value of 1 at a corrected non-dimensional time, or an exhaust-based non-dimensional time, of 4–5, which is consistent with the formation number reported by these studies of approximately 4. As such, in consideration of the latter experimental and numerical evidence, the closure assumption of Shusser and Gharib[7] is met when the leading vortex ring detaches from the feeding starting jet, and it is therefore reasonable to

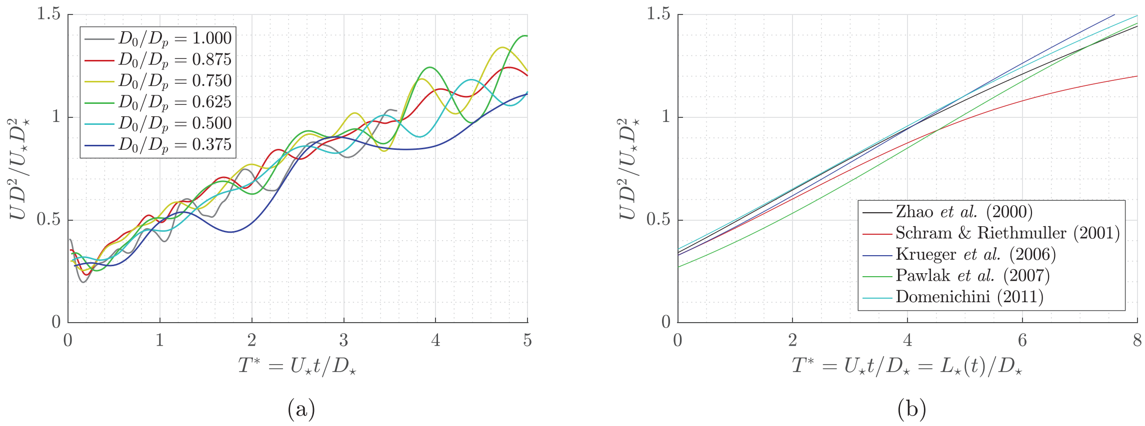

FIG. 10. Ratio $UD^2/U_\star D_\star^2$ as a function of the modified non-dimensional time for (a) the present measurements (b) the data inferred form the literature.

select this closure assumption over the one of Mohseni and Gharib[6] as it is better supported by experimental evidence.

VI. CONCLUSIONS

To conclude, a critical review of the asymptotic matching procedure of Mohseni and Gharib,[6] Shusser and Gharib,[7] and Linden and Turner[8] to predict the formation number was proposed. First, the extended slug-flow model of Limbourg and Nedić[12,13] was presented, as well as the Fraenkel–Norbury and the Kaplanski models of isolated vortex rings. Then, the asymptotic matching methodologies were reviewed and extended upon the Kaplanski family of vortex ring although the methodology of Shusser and Gharib[7,36] had already been applied to the Kaplanski family by Fukumoto and Kaplanski.[25] It was also proved that the extended slug-flow model can be used to predict the formation number of orifice-generated vortex rings in the exhaust-based non-dimensional time frame $t^* = U_0t/D_0$. Moreover, it is encouraged to use the Kaplanski family instead of Fraenkel–Norbury family as it better models time-evolving vortex rings observed experimentally. Finally, a critical discussion of the validity of the closure assumptions of Mohseni and Gharib[6] and Shusser and Gharib[7] was proposed. Particle image velocimetry measurements obtained for different orifice-to-tube diameter ratios, including the nozzle case, were performed. The time histories of the axial position and the ring diameter of the leading vortex rings were presented. Not only do these figures further validate the extended slug-flow model of Limbourg and Nedić,[12,13] but they also show that estimating the ring speed to be half the exhaust speed for all cases is too coarse of an approach, and the asymptotic ring speed is in fact closer to $0.47U_\star$. This is also corroborated by data gathered from the literature. This therefore questions the validity of the closure assumption of Mohseni and Gharib.[6] Besides, the closure assumption of Shusser and Gharib[7,36] which stipulates that the separation of the vortex ring from the feeding jet stops when the translational speed of the ring reaches the one of the jet, however, was corroborated by experimental and numerical evidence. In fact, the measurements show that the flow rate accumulated by the leading vortex ring matches the flow rate discharged by the generator at a non-dimensional time of about $3.5 - 4.5$. To summarize, the closure assumption of Shusser and Gharib[7] finds more experimental support than the methodology of Mohseni and Gharib.[6] The validation of the method of Linden and Turner[8] is left for future study as the measurement of the volume of the ring atmosphere during formation remains a challenging study on its own.[43,54]

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AUTHOR DECLARATIONS

Conflict of Interest

The authors have no conflicts to disclose.

DATA AVAILABILITY

The data that support the findings of this study are available from the corresponding author upon reasonable request.

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